MTH 104 INTERMEDIATE ALGEBRA



Monroe Community College

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MTH 104 Intermediate Algebra

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Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Foundations

In this chapter, you will review the language of algebra and take your first steps toward working with algebraic concepts.

1.1: Use the Language of Algebra

1.1E: Exercises

1.2: Properties of Real Numbers

1.2E: Exercises

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1.1: Use the Language of Algebra

Learning Objectives

By the end of this section, you will be able to:

- Find factors, prime factorizations, and least common multiples
- Use variables and algebraic symbols
- Simplify expressions using the order of operations
- Evaluate an expression
- Identify and combine like terms
- Translate an English phrase to an algebraic expression

This chapter is intended to be a brief review of concepts that will be needed in an Intermediate Algebra course. A more thorough introduction to the topics covered in this chapter can be found in the *Elementary Algebra* chapter, Foundations.

Find Factors, Prime Factorizations, and Least Common Multiples

The numbers 2, 4, 6, 8, 10, 12 are called multiples of 2. A **multiple** of 2 can be written as the product of a counting number and 2.

Multiples of 2:

2,	4,	6,	8,	10,	12, ...
$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$	$2 \cdot 6$

Similarly, a multiple of 3 would be the product of a counting number and 3.

Multiples of 3:

3,	6,	9,	12,	15,	18, ...
$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 3$	$3 \cdot 4$	$3 \cdot 5$	$3 \cdot 6$

We could find the multiples of any number by continuing this process.

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108

MULTIPLE OF A NUMBER

A number is a **multiple** of n if it is the product of a counting number and n.

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

DIVISIBLE BY A NUMBER

If a number m is a multiple of n, then m is **divisible** by n.

If we were to look for patterns in the multiples of the numbers 2 through 9, we would discover the following divisibility tests:

DIVISIBILITY TESTS

A number is divisible by:

- 2 if the last digit is 0, 2, 4, 6, or 8.
- 3 if the sum of the digits is divisible by 3.
- 5 if the last digit is 5 or 0.
- 6 if it is divisible by both 2 and 3.
- 10 if it ends with 0.

Example 1.1.1

Is 5,625 divisible by

a. 2?
b. 3?
c. 5 or 10?
d. 6?

Answer

a.

Is 5,625 divisible by 2?

Does it end in 0, 2, 4, 6 or 8?	No. 5,625 is not divisible by 2.

b.

5,625 divisible by 3?

What is the sum of the digits?	$5+6+2+5=18$
Is the sum divisible by 3?	Yes. 5,625 is divisible by 3.

c.

Is 5,625 divisible by 5 or 10?

What is the last digit? It is 5.	5,625 is divisible by 5 but not by 10.

d.

Is 5,625 divisible by 6?

Is it divisible by both 2 and 3?	No, 5,625 is not divisible by 2, so 5,625 is not divisible by 6.

? Example 1.1.2

Is 4,962 divisible by a. 2? b. 3? c. 5? d. 6? e. 10?

Answer

a. yes b. yes c. no d. yes e. no

? Example 1.1.3

Is 3,765 divisible by a. 2? b. 3? c. 5? d. 6? e. 10?

Answer

a. no b. yes c. yes d. no e. no

In mathematics, there are often several ways to talk about the same ideas. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. For example, since 72 is a multiple of 8, we say 72 is divisible by 8. Since 72 is a multiple of 9, we say 72 is divisible by 9. We can express this still another way.

Since $8 \cdot 9 = 72$, we say that 8 and 9 are factors of 72. When we write $72 = 8 \cdot 9$, we say we have factored 72.

$$\underbrace{8 \cdot 9}_{\text{factors}} = \underbrace{72}_{\text{product}}$$

Other ways to factor 72 are $1 \cdot 72$, $2 \cdot 36$, $3 \cdot 24$, $4 \cdot 18$, and $6 \cdot 12$. The number 72 has many factors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72.

FACTORS

If a and b are counting numbers, and $a \cdot b = m$, then a and b are **factors** of m.

Some numbers, such as 72, have many factors. Other numbers have only two factors. A **prime number** is a counting number greater than 1 whose only factors are 1 and itself.

PRIME NUMBER AND COMPOSITE NUMBER

A **prime number** is a counting number greater than 1 whose only factors are 1 and the number itself.

A **composite number** is a counting number that is not prime. A composite number has factors other than 1 and the number itself.

The counting numbers from 2 to 20 are listed in the table with their factors. Make sure to agree with the "prime" or "composite" label for each!

Number	Factors	Prime or Composite?
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite
11	1,11	Prime

Number	Factors	Prime or Composite?
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime
20	1,2,4,5,10,20	Composite

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Notice that the only even prime number is 2.

A composite number can be written as a unique product of primes. This is called the **prime factorization** of the number. Finding the prime factorization of a composite number will be useful in many topics in this course.

PRIME FACTORIZATION

The **prime factorization** of a number is the product of prime numbers that equals the number.

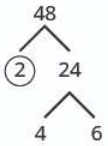
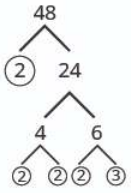
To find the prime factorization of a composite number, find any two factors of the number and use them to create two branches. If a factor is prime, that branch is complete. Circle that prime. Otherwise it is easy to lose track of the prime numbers.

If the factor is not prime, find two factors of the number and continue the process. Once all the branches have circled primes at the end, the factorization is complete. The composite number can now be written as a product of prime numbers.

Example 1.1.4: How to Find the Prime Factorization of a Composite Number

Factor 48.

Answer

Step 1. Find two factors whose product is the given number. Use these numbers to create two branches.	$48 = 2 \cdot 24$	
Step 2. If a factor is prime, that branch is complete. Circle the prime.	2 is prime. Circle the prime.	
Step 3. If a factor is not prime, write it as the product of two factors and continue the process.	24 is not prime. Break it into 2 more factors.	
	4 and 6 are not prime. Break them each into two factors.	
	2 and 3 are prime, so circle them.	

We say $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer.$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer.

If we first factored 48 in a different way, for example as $6 \cdot 8$, the result would still be the same. Finish the prime factorization and verify this for yourself.

Exercise 1.1.4

Find the prime factorization of 80.

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

Example 1.1.6

Find the prime factorization of 60.

Answer

$2 \cdot 2 \cdot 3 \cdot 5$

FIND THE PRIME FACTORIZATION OF A COMPOSITE NUMBER

1. Find two factors whose product is the given number, and use these numbers to create two branches.
2. If a factor is prime, that branch is complete. Circle the prime, like a leaf on the tree.
3. If a factor is not prime, write it as the product of two factors and continue the process.
4. Write the composite number as the product of all the circled primes.

One of the reasons we look at primes is to use these techniques to find the **least common multiple** of two numbers. This will be useful when we add and subtract fractions with different denominators.

LEAST COMMON MULTIPLE

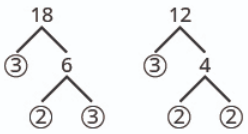
The **least common multiple (LCM)** of two numbers is the smallest number that is a multiple of both numbers.

To find the least common multiple of two numbers we will use the Prime Factors Method. Let's find the LCM of 12 and 18 using their prime factors.

Example 1.1.7: How to Find the Least Common Multiple Using the Prime Factors Method

Find the least common multiple (LCM) of 12 and 18 using the prime factors method.

Answer

Step 1. Write each number as a product of primes.		
Step 2. List the primes of each number. Match primes vertically when possible.	List the primes of 12. List the primes of 18. Line up with the primes of 12 when possible. If not create a new column.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
Step 3. Bring down the number from each column.		$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$
Step 4. Multiply the factors.		LCM = 36

Notice that the prime factors of 12 $(2 \cdot 2 \cdot 3)$ and the prime factors of 18 $(2 \cdot 3 \cdot 3)$ are included in the LCM $(2 \cdot 2 \cdot 3 \cdot 3)$. So 36 is the least common multiple of 12 and 18.

By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the *least* common multiple.

Find the LCM of 9 and 12 using the Prime Factors Method.

Answer

36

Example 1.1.9

Find the LCM of 18 and 24 using the Prime Factors Method.

Answer

72

FIND THE LEAST COMMON MULTIPLE USING THE PRIME FACTORS METHOD

1. Write each number as a product of primes.
2. List the primes of each number. Match primes vertically when possible.
3. Bring down the columns.
4. Multiply the factors.

Use Variables and Algebraic Symbols

In algebra, we use a letter of the alphabet to represent a number whose value may change. We call this a **variable** and letters commonly used for variables are x, y, a, b, and c.

VARIABLE

A **variable** is a letter that represents a number whose value may change.

A number whose value always remains the same is called a **constant**.

CONSTANT

A **constant** is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using. There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We'll list the symbols used to indicate these operations below.

OPERATION SYMBOLS

Operation	Notation	Say:	The result is…
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a \cdot b$, ab, $(a)(b)$, $(a)b$, $a(b)$	a times b	the product of a and b
Division	$a \div b$, a/b, $\frac{a}{b}$, $b\overline{)a}$	a divided by b	the quotient of a and b; a is called the dividend, and b is called the divisor

When two quantities have the same value, we say they are equal and connect them with an equal sign.

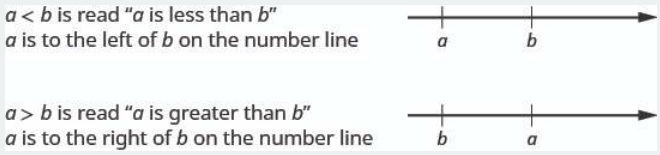
EQUALITY SYMBOL

$a = b$ is read "a is equal to b."

The symbol "$=$" is called the equal sign.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols "$<$" and "$>$".

INEQUALITY



The expressions $a < b$ or $a > b$ can be read from left to right or right to left, though in English we usually read from left to right. In general,

$$a < b \text{ is equivalent to } b > a. \text{ For example, } 7 < 11 \text{ is equivalent to } 11 > 7. \tag{1.1.1}$$

$$a > b \text{ is equivalent to } b < a. \text{ For example, } 17 > 4 \text{ is equivalent to } 4 < 17. \tag{1.1.2}$$

INEQUALITY SYMBOLS

Inequality Symbols	Words
$a \neq b$	a is *not equal to* b.
$a < b$	a is *less than* b.
$a \leq b$	a is *less than or equal to* b.
$a > b$	a is *greater than* b.
$a \geq b$	a is *greater than or equal to* b.

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help identify an expression, which can be made up of number, a variable, or a combination of numbers and variables using operation symbols. We will introduce three types of grouping symbols now.

GROUPING SYMBOLS

$$\begin{array}{ll} \text{Parentheses} & () \\ \text{Brackets} & [] \\ \text{Braces} & \{\} \end{array} \tag{1.1.3}$$

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24 \div \{13-2[1(6-5)+4]\} \tag{1.1.4}$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. A sentence has a subject and a verb. In algebra, we have *expressions* and *equations*.

EXPRESSION

An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

Expression	**Words**	**English Phrase**
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6 \cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

(1.1.5)

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb.

An equation is two expressions linked by an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

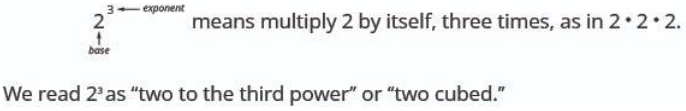
EQUATION

An **equation** is two expressions connected by an equal sign.

Equation	English Sentence
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

(1.1.6)

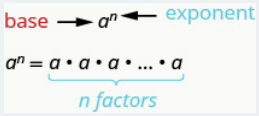
Suppose we need to multiply 2 nine times. We could write this as $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2$. This is tedious and it can be hard to keep track of all those 2s, so we use exponents. We write $2\cdot 2\cdot 2$ as 2^3 and $2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2$ as 2^9. In expressions such as 2^3, the 2 is called the *base* and the 3 is called the *exponent*. The **exponent** tells us how many times we need to multiply the **base**.



EXPONENTIAL NOTATION

We say 2^3 is in *exponential notation* and $2\cdot 2\cdot 2$ is in *expanded notation*.

a^n means multiply n factors of the number a.



The expression a^n is read a to the nth power.

While we read a^n as "a to the nth power", we usually read:

a^2 "a squared"
a^3 "a cubed"

(1.1.7)

We'll see later why a^2 and a^3 have special names.

Table shows how we read some expressions with exponents.

Expression	In Words	
7^2	7 to the second power or	7 squared
5^3	5 to the third power or	5 cubed
9^4	9 to the fourth power	
12^5	12 to the fifth power	

Simplify Expressions Using the Order of Operations

To **simplify an expression** means to do all the math possible. For example, to simplify $4\cdot 2+1$ we would first multiply $4\cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$\begin{aligned} &4 \cdot 2+1 \\ &8+1 \\ &9 \end{aligned} \tag{1.1.8}$$

By not using an equal sign when you simplify an expression, you may avoid confusing expressions with equations.

SIMPLIFY AN EXPRESSION

To **simplify an expression**, do all operations in the expression.

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the **order of operations**. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression $4+3 \cdot 7$. Some students simplify this getting 49, by adding $4+3$ and then multiplying that result by 7. Others get 25, by multiplying $3 \cdot 7$ first and then adding 4.

The same expression should give the same result. So mathematicians established some guidelines that are called the order of operations.

USE THE ORDER OF OPERATIONS.

1. Parentheses and Other Grouping Symbols
   - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. Exponents
   - Simplify all expressions with exponents.
3. Multiplication and Division
   - Perform all multiplication and division in order from left to right. These operations have equal priority.
4. Addition and Subtraction
   - Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?" Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase "Please Excuse My Dear Aunt Sally".

$$\begin{array}{ll} \text{Parentheses} & \text{Please} \\ \text{Exponents} & \text{Excuse} \\ \text{Multiplication Division} & \text{My Dear} \\ \text{Addition Subtraction} & \text{Aunt Sally} \end{array} \tag{1.1.9}$$

It's good that "**M**y **D**ear" goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, "**A**unt **S**ally" goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

? Example 1.1.10

Simplify: $18 \div 6+4(5-2)$.

Answer

	$18 \div 6+4(5-2)$
Parentheses? Yes, subtract first.	$18 \div 6+4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3+4(3)$

Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication of division? No.	
Any addition or subtraction? Yes.	
Add.	15

Example 1.1.11

Simplify: $30 \div 5 + 10(3 - 2)$.

Answer

16

Example 1.1.12

Simplify: $70 \div 10 + 4(6 - 2)$.

Answer

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example 1.1.13

Simplify: $5 + 2^3 + 3[6 - 3(4 - 2)]$.

Answer

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbols)? Yes.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Focus on the parentheses that are inside the brackets. Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes. Simplify exponents.	$5 + 8 + 3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5 + 8 + 0$
Is there any addition of subtraction? Yes.	
Add.	$13 + 0$
Add.	13

Example 1.1.14

Simplify: $9+5^3-[4(9+3)]$.

Answer

86

Example 1.1.15

Simplify: $7^2-2[4(5+1)]$.

Answer

1

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

EVALUATE AN EXPRESSION

To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Evaluate when $x=4$: a. x^2 b. 3^x c. $2x^2+3x+8$.

Answer

a.

	x^2
Replace x with 4.	4^2
Use definition of exponent.	$4 \cdot 4$
Simplify.	16

b.

	3^x
Replace x with 4.	3^4
Use definition of exponent.	$3 \cdot 3 \cdot 3 \cdot 3$
Simplify.	81

c.

	$2x^2+3x+8$
Replace x with 4.	$2(4)^2+3(4)+8$
Follow the order of operations.	$2(16)+3(4)+8$
	$32+12+8$

52

Evaluate when $x = 3$, a. x^2 b. 4^x c. $3x^2 + 4x + 1$.

Answer

a. 9
b. 64
c. 40

Example 1.1.18

Evaluate when $x = 6$, a. x^3 b. 2^x c. $6x^2 - 4x - 7$.

Answer

a. 216
b. 64
c. 185

Identify and Combine Like Terms

Algebraic expressions are made up of terms. A **term** is a constant, or the product of a constant and one or more variables.

TERM

A **term** is a constant or the product of a constant and one or more variables.

Examples of terms are 7, y, $5x^2$, $9a$, and b^5.

The constant that multiplies the variable is called the **coefficient**.

COEFFICIENT

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**.

Look at the following 6 terms. Which ones seem to have traits in common?

$$5x \quad 7 \quad n^2 \quad 4 \quad 3x \quad 9n^2 \tag{1.1.10}$$

We say,

7 and 4 are like terms.

$5x$ and $3x$ are like terms.

n^2 and $9n^2$ are like terms.

LIKE TERMS

Terms that are either constants or have the same variables raised to the same powers are called **like terms.**

If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

$$\begin{array}{ll} \text{Simplify.} & 4x+7x+x \\ \text{Add the coefficients.} & 12x \end{array} \tag{1.1.11}$$

ExAMPLE 1.1.19: How To Combine Like Terms

Simplify: $2x^2+3x+7+x^2+4x+5$.

Answer

Step 1. Identify the like terms.	$2x^2+3x+7+x^2+4x+5$ $2x^2+3x+7+x^2+4x+5$ $2x^2+3x+7+x^2+4x+5$
Step 2. Rearrange the expression so the like terms are together.	$2x^2+x^2+3x+4x+7+5$ $2x^2+x^2+3x+4x+7+5$
Step 3. Combine like terms.	$3x^2+7x+12$

Simplify: $3x^2+7x+9+7x^2+9x+8$.

Answer

$10x^2+16x+17$

ExAMPLE 1.1.21

Simplify: $4y^2+5y+2+8y^2+4y+5$.

Answer

$12y^2+9y+7$

COMBINE LIKE TERMS.

1. Identify like terms.
2. Rearrange the expression so like terms are together.
3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Translate an English Phrase to an Algebraic Expression

We listed many operation symbols that are used in algebra. Now, we will use them to translate English phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. Table summarizes them.

Operation	Phrase	Expression

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$
Multiplication	a times b the product of a and b twice a	$a \cdot b,\ ab,\ a(b),\ (a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b,\ a/b,\ \frac{a}{b},\ b\overline{)a}$

Look closely at these phrases using the four operations:

the **sum** *of a and b*

the **difference** *of a and b*

the **product** *of a and b*

the **quotient** *of a and b*

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

? Example 1.1.22

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

Translate each English phrase into an algebraic expression:

a. the difference of $14x$ and 9

b. the quotient of $8y^2$ and 3

c. twelve more than y

d. seven less than $49x^2$

Answer

a. The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

the *difference of* 14*x* *and* 9

14*x* minus 9

14*x* – 9

b. The key word is *quotient*, which tells us the operation is division.

the *quotient of* $8y^2$ *and* 3

divide $8y^2$ by 3

$8y^2 \div 3$

This can also be written $8y^2/3$ or $\frac{8y^2}{3}$.

c. The key words are *more than*. They tell us the operation is addition. *More than* means "added to."

$$\begin{gathered} \text{twelve more than } y \\ \text{twelve added to } y \\ y+12 \end{gathered} \tag{1.1.12}$$

d. The key words are *less than*. They tell us to subtract. *Less than* means "subtracted from."

$$\begin{gathered} \text{seven less than } 49x^2 \\ \text{seven subtracted from } 49x^2 \\ 49x^2-7 \end{gathered} \tag{1.1.13}$$

? Exercise 1.1.23

Translate the English phrase into an algebraic expression:

a. the difference of $14x^2$ and 13

b. the quotient of $12x$ and 2

c. 13 more than z

d. 18 less than $8x$

Answer

a. $14x^2-13$ b. $12x \div 2$

c. $z+13$ d. $8x-18$

? Example 1.1.24

Translate the English phrase into an algebraic expression:

a. the sum of $17y^2$ and 19

b. the product of 7 and y

c. Eleven more than x

d. Fourteen less than $11a$

Answer

a. $17y^2+19$ b. $7y$

c. $x+11$ d. $11a-14$

We look carefully at the words to help us distinguish between multiplying a sum and adding a product.

? Example 1.1.25

Translate the English phrase into an algebraic expression:

a. eight times the sum of x and y

b. the sum of eight times x and y

Answer

There are two operation words—*times* tells us to multiply and *sum* tells us to add.

a. Because we are multiplying 8 times the sum, we need parentheses around the sum of x and y, $(x+y)$. This forces us to determine the sum first. (Remember the order of operations.)

$$\begin{array}{c} \text{eight times the sum of } x \text{ and } y \\ 8(x+y) \end{array} \tag{1.1.14}$$

b. To take a sum, we look for the words *of* and *and* to see what is being added. Here we are taking the sum *of* eight times x and y.

the sum *of* eight times *x and y*

$8x + y$

? Example 1.1.26

Translate the English phrase into an algebraic expression:

a. four times the sum of p and q

b. the sum of four times p and q

Answer

a. $4(p+q)$ b. $4p+q$

? Example 1.1.27

Translate the English phrase into an algebraic expression:

a. the difference of two times x and 8

b. two times the difference of x and 8

Answer

a. $2x-8$ b. $2(x-8)$

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression. We'll see how to do this in the next two examples.

? Example 1.1.28

The length of a rectangle is 14 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer

$$\begin{array}{ll} \text{Write a phrase about the length of the rectangle.} & 14 \text{ less than the width} \\ \text{Substitute } w \text{ for "the width."} & w \\ \text{Rewrite less than as subtracted from.} & 14 \text{ subtracted from } w \\ \text{Translate the phrase into algebra.} & w-14 \end{array} \tag{1.1.15}$$

Example 1.1.29

The length of a rectangle is 7 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Answer

$w-7$

Example 1.1.30

The width of a rectangle is 6 less than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Answer

$l-6$

The expressions in the next example will be used in the typical coin mixture problems we will see soon.

Example 1.1.31

June has dimes and quarters in her purse. The number of dimes is seven less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer

Write a phrase about the number of dimes.	7 less than 4 times q	
Translate 4 times q.	7 less than $4q$	(1.1.16)
Translate the phrase into algebra.	$4q-7$	

Example 1.1.32

Geoffrey has dimes and quarters in his pocket. The number of dimes is eight less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Answer

$4q-8$

Example 1.1.33

Lauren has dimes and nickels in her purse. The number of dimes is three more than seven times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Answer

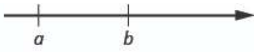
$7n+3$

Key Concepts

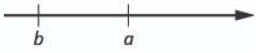
- **Divisibility Tests**
  A number is divisible by:
  2 if the last digit is 0, 2, 4, 6, or 8.
  3 if the sum of the digits is divisible by 3.
  5 if the last digit is 5 or 0.

6 if it is divisible by both 2 and 3.
10 if it ends with 0.

- **How to find the prime factorization of a composite number.**
  1. Find two factors whose product is the given number, and use these numbers to create two branches.
  2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
  3. If a factor is not prime, write it as the product of two factors and continue the process.
  4. Write the composite number as the product of all the circled primes.
- **How To Find the least common multiple using the prime factors method.**
  1. Write each number as a product of primes.
  2. List the primes of each number. Match primes vertically when possible.
  3. Bring down the columns.
  4. Multiply the factors.
- **Equality Symbol**
  $a = b$ is read " a is equal to b." The symbol "=" is called the equal sign.
- $a < b$ is read "a is less than b"
  a is to the left of b on the number line



- $a > b$ is read "a is greater than b"
  a is to the right of b on the number line



- **Inequality Symbols**

Inequality Symbols	Words
$a \neq b$	a is *not equal to* b.
$a < b$	a is *less than* b.
$a \leq b$	a is *less than or equal to* b.
$a > b$	a is *greater than* b.
$a \geq b$	a is *greater than or equal to* b.

- **Grouping Symbols**

Parentheses	$()$
Brackets	$[]$
Braces	$\{\}$

- **Exponential Notation** a^n means multiply a by itself, n times. The expression an is read a to the nth power.
- **Simplify an Expression**
  To simplify an expression, do all operations in the expression.
- **How to use the order of operations.**
  1. Parentheses and Other Grouping Symbols
     - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
  2. Exponents
     - Simplify all expressions with exponents.
  3. Multiplication and Division
     - Perform all multiplication and division in order from left to right. These operations have equal priority.
  4. Addition and Subtraction
     - Perform all addition and subtraction in order from left to right. These operations have equal priority.
- **How to combine like terms.**
  1. Identify like terms.
  2. Rearrange the expression so like terms are together.
  3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Operation	Phrase	Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$
Multiplication	a times b the product of a and b twice a	$a \cdot b,\ ab,\ a(b),\ (a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b,\ a/b,\ \frac{a}{b},\ b\overline{)a}$

Glossary

coefficient

The coefficient of a term is the constant that multiplies the variable in a term.

composite number

A composite number is a counting number that is not prime. It has factors other than 1 and the number itself.

constant

A constant is a number whose value always stays the same.

divisible by a number

If a number m is a multiple of n, then m is divisible by n.

equation

An equation is two expressions connected by an equal sign.

evaluate an expression

To evaluate an expression means to find the value of the expression when the variables are replaced by a given number.

expression

An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

factors

If $a \cdot b = m$, then a and b are factors of m.

least common multiple

The least common multiple (LCM) of two numbers is the smallest number that is a multiple of both numbers.

like terms

Terms that are either constants or have the same variables raised to the same powers are called like terms.

multiple of a number

A number is a multiple of n if it is the product of a counting number and n.

order of operations

The order of operations are established guidelines for simplifying an expression.

prime factorization

The prime factorization of a number is the product of prime numbers that equals the number.

prime number

A prime number is a counting number greater than 1 whose only factors are 1 and the number itself.

simplify an expression

To simplify an expression means to do all the math possible.

term

A term is a constant, or the product of a constant and one or more variables.

variable

A variable is a letter that represents a number whose value may change.

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1.1E: Exercises

Practice Makes Perfect

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

1. 84

Answer

Divisible by 2, 3, 6

2. 96

3. 896

Answer

Divisible by 2

4. 942

5. $22,335$

Answer

Divisible by 3, 5

6. $39,075$

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

7. 86

Answer

$2 \cdot 43$

8. 78

9. 455

Answer

$5 \cdot 7 \cdot 13$

10. 400

11. 432

Answer

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

12. 627

In the following exercises, find the least common multiple of each pair of numbers using the prime factors method.

13. $8,\ 12$

Answer

24

14. $12,\ 16$

15. $28,\ 40$

Answer

280

16. $84,\ 90$

17. $55,\ 88$

Answer

440

18. $60,\ 72$

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

19. $2^3 - 12 \div (9 - 5)$

Answer

5

20. $3^2 - 18 \div (11 - 5)$

21. $2 + 8(6 + 1)$

Answer

58

22. $4 + 6(3 + 6)$

23. $20 \div 4 + 6(5 - 1)$

Answer

29

24. $33 \div 3 + 4(7 - 2)$

25. $3(1 + 9 \cdot 6) - 4^2$

Answer

149

26. $5(2+8\cdot 4)-7^2$

27. $2[1+3(10-2)]$

Answer

50

28. $5[2+4(3-2)]$

29. $8+2[7-2(5-3)]-3^2$

Answer

5

30. $10+3[6-2(4-2)]-2^4$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

31. When $x=2$,

a. x^6

b. 4^x

c. $2x^2+3x-7$

Answer

a. 64
b. 16
c. 7

32. When $x=3$,

a. x^5

b. $5x$

c. $3x^2-4x-8$

33. When $x=4$ and $y=1$

$x^2+3xy-7y^2$

Answer

21

34. When $x=3$ and $y=2$

$6x^2+3xy-9y^2$

35. When $x=10$ and $y=7$

$(x-y)^2$

Answer

9

36. When $a=3$ and $b=8$

a^2+b^2

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

37. $7x+2+3x+4$

Answer

$10x+6$

38. $8y+5+2y-4$

39. $10a+7+5a-2+7a-4$

Answer

$22a+1$

40. $7c+4+6c-3+9c-1$

41. $3x^2+12x+11+14x^2+8x+5$

Answer

$17x^2+20x+16$

42. $5b^2+9b+10+2b^2+3b-4$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.

43. a. the difference of $5x^2$ and $6xy$

b. the quotient of $6y^2$ and $5x$

c. Twenty-one more than y^2

d. $6x$ less than $81x^2$

Answer

a. $5x^2-6xy$ b. $\frac{6y^2}{5x}$

c. y^2+21 d. $81x^2-6x$

44. a. the difference of $17x^2$ and $17x^2$ and $5xy$

b. the quotient of $8y^3$ and $3x$

c. Eighteen more than a^2;

d. $11b$ less than $100b^2$

45. a. the sum of $4ab^2$ and $3a^2b$

b. the product of $4y^2$ and $5x$

c. Fifteen more than m

d. $9x$ less than $121x^2$

Answer

a. $4ab^2+3a^2b$ b. $20xy^2$

c. $m+15$ d. $121x^2-9x$

46. a. the sum of $3x^2y$ and $7xy^2$

b. The product of $6xy^2$ and $4z$

c. Twelve more than $3x^2$

d. $7x^2$ less than $63x^3$

47. a. eight times the difference of y and nine

b. the difference of eight times y and 9

Answer

a. $8(y-9)$
b. $8y-9$

48. a. seven times the difference of y and one

b. the difference of seven times y and 1

49. a. five times the sum of $3x$ and y

b. the sum of five times $3x$ and y

Answer

a. $5(3x+y)$
b. $15x+y$

50. a. eleven times the sum of $4x^2$ and $5x$

b. the sum of eleven times $4x^2$ and $5x$

51. Eric has rock and country songs on his playlist. The number of rock songs is 14 more than twice the number of country songs. Let c represent the number of country songs. Write an expression for the number of rock songs.

Answer

$2c-14$

52. The number of women in a Statistics class is 8 more than twice the number of men. Let m represent the number of men. Write an expression for the number of women.

53. Greg has nickels and pennies in his pocket. The number of pennies is seven less than three times the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Answer

$3n-7$

54. Jeannette has $\$5$ and $\$10$ bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Writing Exercises

55. Explain in your own words how to find the prime factorization of a composite number.

Answer

Answers will vary.

56. Why is it important to use the order of operations to simplify an expression?

57. Explain how you identify the like terms in the expression $8a^2+4a+9-a^2-1$.

Answer

Answers will vary.

58. Explain the difference between the phrases "4 times the sum of x and y" and "the sum of 4 times x and y".

Self Check

a. Use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify multiples and apply divisibility tests.			
find prime factorizations and least common multiples.			
use variables and algebraic symbols.			
simplify expressions using the order of operations.			
evaluate an expression.			
identify and combine like terms.			
translate English phrases to algebraic expressions.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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1.2: Properties of Real Numbers

Learning Objectives

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the properties of identity, inverse, and zero
- Simplify expressions using the Distributive Property

Use the Commutative and Associative Properties

The order we add two numbers doesn't affect the result. If we add $8+9$ or $9+8$, the results are the same—they both equal 17. So, $8+9=9+8$. The order in which we add does not matter!

Similarly, when multiplying two numbers, the order does not affect the result. If we multiply $9 \cdot 8$ or $8 \cdot 9$ the results are the same —they both equal 72. So, $9 \cdot 8=8 \cdot 9$. The order in which we multiply does not matter! These examples illustrate the **Commutative Property**.

COMMUTATIVE PROPERTY

of Addition	If a and $b$are real numbers, then	$a+b=b+a$.	(1.2.1)
of Multiplication	If a and $b$are real numbers, then	$a \cdot b=b \cdot a$.	

When adding or multiplying, changing the *order* gives the same result.

The Commutative Property has to do with order. We subtract $9-8$ and $8-9$, and see that $9-8 \neq 8-9$. Since changing the order of the subtraction does not give the same result, we know that *subtraction is not commutative*.

Division is not commutative either. Since $12 \div 3 \neq 3 \div 12$, changing the order of the division did not give the same result. The commutative properties apply only to addition and multiplication!

- Addition and multiplication *are* commutative.
- Subtraction and division are *not* commutative.

When adding three numbers, changing the grouping of the numbers gives the same result. For example, $(7+8)+2=7+(8+2)$, since each side of the equation equals 17.

This is true for multiplication, too. For example, $\left(5 \cdot \frac{1}{3}\right) \cdot 3=5 \cdot\left(\frac{1}{3} \cdot 3\right)$, since each side of the equation equals 5.

These examples illustrate the **Associative Property**.

ASSOCIATIVE PROPERTY

of Addition	If $a, b,$ and c are real numbers, then	$(a+b)+c=a+(b+c)$.	(1.2.2)
of Multiplication	If $a, b,$ and c are real numbers, then	$(a \cdot b) \cdot c=a \cdot(b \cdot c)$.	

When adding or multiplying, changing the *grouping* gives the same result.

The Associative Property has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.

$$\begin{array}{ll} (10-3)-2 \neq 10-(3-2) & (24 \div 4) \div 2 \neq 24 \div(4 \div 2) \\ 7-2 \neq 10-1 & 6 \div 2 \neq 24 \div 2 \\ 5 \neq 9 & 3 \neq 12 \end{array} \quad (1.2.3)$$

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the Commutative Property of addition to write the like terms together.

? Example 1.2.1

Simplify: $18p+6q+15p+5q$.

Answer

	$18p+6q+15p+5q$	
Use the Commutative Property of addition to reorder so that like terms are together.	$18p+15p+6q+5q$	(1.2.4)
Add like terms.	$33p+11q$	

? Example 1.2.2

Simplify: $23r+14s+9r+15s$.

Answer

$32r+29s$

? Example 1.2.3

Simplify: $37m+21n+4m-15n$.

Answer

$41m+6n$

When we have to simplify algebraic expressions, we can often make the work easier by applying the Commutative Property or Associative Property first.

? EXAMPLE 1.2.4

Simplify: $(\frac{5}{13}+\frac{3}{4})+\frac{1}{4}$.

Answer

	$(\frac{5}{13}+\frac{3}{4})+\frac{1}{4}$
Notice that the last 2 terms have a common denominator, so change the grouping.	$\frac{5}{13}+(\frac{3}{4}+\frac{1}{4})$
Add in parentheses first.	$\frac{5}{13}+(\frac{4}{4})$
Simplify the fraction.	$\frac{5}{13}+1$
Add.	$1\frac{5}{13}$
Convert to an improper fraction.	$\frac{18}{13}$

? EXAMPLE 1.2.5

Simplify: $(\frac{7}{15}+\frac{5}{8})+\frac{3}{8}$.

Answer

$1\frac{7}{15}$

EXAMPLE 1.2.6

Simplify: $(\frac{2}{9}+\frac{7}{12})+\frac{5}{12}$.

Answer

$1\frac{2}{9}$

Use the Properties of Identity, Inverse, and Zero

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**. The **Identity Property of Addition** that states that for any real number $a, a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the multiplicative identity. The **Identity Property of Multiplication** that states that for any real number $a, a\cdot 1=a$ and $1\cdot a=a$.

We summarize the Identity Properties here.

IDENTITY PROPERTY

$$\begin{array}{ll} \textbf{of Addition } \text{For any real number } a: a+0=a & 0+a=a \\ \\ \textbf{0 is the additive identity} & \\ \textbf{of Multiplication } \text{For any real number } a: a\cdot 1=a & 1\cdot a=a \\ \\ \textbf{1 is the multiplicative identity} & \end{array} \tag{1.2.5}$$

What number added to 5 gives the additive identity, 0? We know

$$5+(-5)=0$$

The missing number was the opposite of the number!

We call $-a$ the additive inverse of a. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the **Inverse Property of Addition** that states for any real number $a, a+(-a)=0$.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1? We know

$$\frac{2}{3}\cdot\frac{3}{2}=1$$

The missing number was the reciprocal of the number!

We call $\frac{1}{a}$ the **multiplicative inverse** of a. *The reciprocal of a number is its multiplicative inverse.* This leads to the **Inverse Property of Multiplication** that states that for any real number $a, a\neq 0, a\cdot\frac{1}{a}=1$.

We'll formally state the inverse properties here.

INVERSE PROPERTY

of addition For any real number a, $a+(-a)=0$

$-a$ is the **additive inverse** of a

A number and its *opposite* add to zero.

of multiplication For any real number $a, a \neq 0$ $a \cdot \frac{1}{a}=1$

$\frac{1}{a}$ is the **multiplicative inverse** of a

A number and its *reciprocal* multiply to one. (1.2.6)

The Identity Property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the product equal zero.

What about division involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So, $0 \div 3=0$.

We can check division with the related multiplication fact. So we know $0 \div 3=0$ because $0 \cdot 3=0$.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact:

$4 \div 0=\square$ means $\square \cdot 0=4$

Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4. We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is **undefined**.

We summarize the properties of zero here.

PROPERTIES OF ZERO

Multiplication by Zero: For any real number a,

$a \cdot 0=0 \quad 0 \cdot a=0$ The product of any number and 0 is 0. (1.2.7)

Division by Zero: For any real number a, $a \neq 0$

$\frac{0}{a}=0$ Zero divided by any real number, except itself, is zero.

$\frac{a}{0}$ is undefined Division by zero is undefined. (1.2.8)

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

EXAMPLE 1.2.7

Simplify: $-84n+(-73n)+84n$.

Answer

	$-84n+(-73n)+84n$
Notice that the first and third terms are opposites; use the Commutative Property of addition to re-order the terms.	$-84n+84n+(-73n)$
Add left to right.	$0+(-73n)$
Add.	$-73n$

? EXAMPLE 1.2.8

Simplify: $-27a+(-48a)+27a$.

Answer

$-48a$

? EXAMPLE 1.2.9

Simplify: $39x+(-92x)+(-39x)$.

Answer

$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

? EXAMPLE 1.2.10

Simplify: $\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$.

Answer

	$\frac{7}{15}\cdot\frac{8}{23}\cdot\frac{15}{7}$
Notice the first and third terms are reciprocals, so use the Commutative Property of multiplication to re-order the factors.	$\frac{7}{15}\cdot\frac{15}{7}\cdot\frac{8}{23}$
Multiply left to right.	$1\cdot\frac{8}{23}$
Multiply.	$\frac{8}{23}$

? EXAMPLE 1.2.11

Simplify: $\frac{9}{16}\cdot\frac{5}{49}\cdot\frac{16}{9}$.

Answer

$\frac{5}{49}$

Simplify: $\frac{6}{17}\cdot\frac{11}{25}\cdot\frac{17}{6}$.

Answer

$\frac{11}{25}$

The next example makes us aware of the distinction between dividing 0 by some number or some number being divided by 0.

Simplify: a. $\frac{0}{n+5}$, where $n\neq -5$ b. $\frac{10-3p}{0}$ where $10-3p\neq 0$.

Answer

a.

	$\frac{0}{n+5}$
Zero divided by any real number except itself is 0.	0

b.

	$\frac{10-3p}{0}$
Division by 0 is undefined.	undefined

EXAMPLE 1.2.14

Simplify: a. $\frac{0}{m+7}$, where $m \neq -7$ b. $\frac{18-6c}{0}$, where $18-6c \neq 0$.

Answer

a. 0
b. undefined

EXAMPLE 1.2.15

Simplify: a. $\frac{0}{d-4}$, where $d \neq 4$ b. $\frac{15-4q}{0}$, where $15-4q \neq 0$.

Answer

a. 0
b. undefined

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need $9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times $9 so $27 and 3 times 1 quarter, so 75 cents. In total, they need $27.75. If you think about doing the math in this way, you are using the Distributive Property.

DISTRIBUTIVE PROPERTY

If $a, b,$ and c are real numbers, then

$$a(b+c) = ab+ac$$
$$(b+c)a = ba+ca$$
$$a(b-c) = ab-ac$$
$$(b-c)a = ba-ca$$

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions.

EXAMPLE 1.2.16

Simplify: $3(x+4)$.

Answer

	$3(x+4)$
Distribute.	$3 \cdot x + 3 \cdot 4$
Multiply.	$3x+12$

Simplify: $4(x+2)$.

Answer

$4x8$

EXAMPLE 1.2.18

Simplify: $6(x+7)$.

Answer

$6x42$

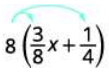
Some students find it helpful to draw in arrows to remind them how to use the Distributive Property. Then the first step in Example would look like this:



EXAMPLE 1.2.19

Simplify: $8\left(\frac{3}{8}x+\frac{1}{4}\right)$.

Answer

	
Distribute.	$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$
Multiply.	$3x+2$

EXAMPLE 1.2.20

Simplify: $6\left(\frac{5}{6}y+\frac{1}{2}\right)$.

Answer

$5y+3$

EXAMPLE 1.2.21

Simplify: $12\left(\frac{1}{3}n+\frac{3}{4}\right)$

Answer

$4n+9$

Using the Distributive Property as shown in the next example will be very useful when we solve money applications in later chapters.

EXAMPLE 1.2.22

Simplify: $100(0.3+0.25q)$.

Answer

Distribute.	$100(0.3) + 100(0.25q)$
Multiply.	$30 + 25q$

EXAMPLE 1.2.23

Simplify: $100(0.7+0.15p)$.

Answer

$70+15p$

EXAMPLE 1.2.24

Simplify: $100(0.04+0.35d)$.

Answer

$4+35d$

When we distribute a negative number, we need to be extra careful to get the signs correct!

EXAMPLE 1.2.25

Simplify: $-11(4-3a)$.

Answer

	$-11(4-3a)$
Distribute.	$-11\cdot 4-(-11)\cdot 3a$
Multiply.	$-44-(-33a)$
Simplify.	$-44+33a$

Notice that you could also write the result as $33a-44$. Do you know why?

Simplify: $-5(2-3a)$.

Answer

$-10+15a$

EXAMPLE 1.2.27

Simplify: $-7(8-15y)$.

Answer

$-56+105y$

In the next example, we will show how to use the Distributive Property to find the opposite of an expression.

Simplify: $-(y+5)$.

Answer

	$-(y+5)$
Multiplying by -1 results in the opposite.	$-1(y+5)$
Distribute.	$-1\cdot y+(-1)\cdot 5$
Simplify.	$-y+(-5)$
Simplify.	$-y-5$

? EXAMPLE 1.2.29

Simplify: $-(z-11)$.

Answer

$-z+11$

? EXAMPLE 1.2.30

Simplify: $-(x-4)$.

Answer

$-x+4$

There will be times when we'll need to use the Distributive Property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the Distributive Property, which removes the parentheses. The next two examples will illustrate this.

? EXAMPLE 1.2.31

Simplify: $8-2(x+3)$

Answer

We follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

	$8-2(x+3)$
Distribute.	$8-2\cdot x-2\cdot 3$
Multiply.	$8-2x-6$
Combine like terms.	$-2x+2$

? EXAMPLE 1.2.32

Simplify: $9-3(x+2)$.

Answer

$3-3x$

? EXAMPLE 1.2.33

Simplify: $7x-5(x+4)$.

Answer

$2x-20$

? EXAMPLE 1.2.34

Simplify: $4(x-8)-(x+3)$.

Answer

	$4(x-8)-(x+3)$
Distribute.	$4x-32-x-3$
Combine like terms.	$3x-35$

? EXAMPLE 1.2.35

Simplify: $6(x-9)-(x+12)$.

Answer

$5x-66$

? EXAMPLE 1.2.36

Simplify: $8(x-1)-(x+5)$.

Answer

$7x-13$

All the properties of real numbers we have used in this chapter are summarized here.

Commutative Property

When adding or multiplying, changing the *order* gives the same result

of Addition	If a and $b$are real numbers, then	$a+b=b+a$.	(1.2.9)
of Multiplication	If a and $b$are real numbers, then	$a\cdot b=b\cdot a$.	

Associative Property

When adding or multiplying, changing the *grouping* gives the same result.

of Addition	If $a, b,$ and c are real numbers, then	$(a+b)+c=a+(b+c)$.	(1.2.10)
of Multiplication	If $a, b,$ and c are real numbers, then	$(a\cdot b)\cdot c=a\cdot(b\cdot c)$.	

Distributive Property

If $a, b,$and $c$are real numbers, then

$$a(b+c)=ab+ac \quad (1.2.11)$$
$$(b+c)a=ba+ca$$
$$a(b-c)=ab-ac$$
$$(b-c)a=ba-ca$$

Identity Property

of Addition For any real number a : $a+0=a$	$0+a=a$	(1.2.12)
0 is the **additive identity**		
of Multiplication For any real number a : $a\cdot 1=a$	$1\cdot a=a$	
1 is the **multiplicative identity**		

Inverse Property

of addition For any real number a, $a+(-a)=0$

$-a$ is the **additive inverse** of a

A number and its *opposite* add to zero.

of multiplication For any real number $a, a \neq 0$ $a \cdot \frac{1}{a}=1$

$\frac{1}{a}$ is the **multiplicative inverse** of a

A number and its *reciprocal* multiply to one. (1.2.13)

Properties of Zero

For any real number a, $a \cdot 0=0$

$0 \cdot a=0$

For any real number $a, a \neq 0$, $\frac{0}{a}=0$

For any real number a, $\frac{a}{0}$ is undefined (1.2.14)

Key Concepts

Commutative Property

When adding or multiplying, changing the *order* gives the same result

of Addition If a and $b$are real numbers, then $a+b=b+a$.

of Multiplication If a and $b$are real numbers, then $a \cdot b=b \cdot a$. (1.2.15)

Associative Property When adding or multiplying, changing the *grouping* gives the same result.

of Addition If $a, b,$ and c are real numbers, then $(a+b)+c=a+(b+c)$.

of Multiplication If $a, b,$ and c are real numbers, then $(a \cdot b) \cdot c=a \cdot(b \cdot c)$. (1.2.16)

Distributive Property

If $a, b,$and $c$are real numbers, then

$a(b+c)=ab+ac$

$(b+c)a=ba+ca$

$a(b-c)=ab-ac$

$(b-c)a=ba-ca$ (1.2.17)

Identity Property

of Addition For any real number $a: a+0=a$ $0+a=a$

0 is the **additive identity**

of Multiplication For any real number $a: a \cdot 1=a$ $1 \cdot a=a$

1 is the **multiplicative identity** (1.2.18)

Inverse Property

of addition For any real number a, $a+(-a)=0$

$-a$ is the **additive inverse** of a

A number and its *opposite* add to zero.

of multiplication For any real number $a, a \neq 0$ $a \cdot \frac{1}{a} = 1$

$\frac{1}{a}$ is the **multiplicative inverse** of a

A number and its *reciprocal* multiply to one.

(1.2.19)

Properties of Zero

For any real number a, $a \cdot 0 = 0$

$0 \cdot a = 0$

For any real number $a, a \neq 0$, $\frac{0}{a} = 0$

For any real number a, $\frac{a}{0}$ is undefined

(1.2.20)

Glossary

additive identity

The number 0 is the additive identity because adding 0 to any number does not change its value.

additive inverse

The opposite of a number is its additive inverse.

multiplicative identity

The number 1 is the multiplicative identity because multiplying 1 by any number does not change its value.

multiplicative inverse

The reciprocal of a number is its multiplicative inverse.

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1.2E: Exercises

Practice Makes Perfect

Use the Commutative and Associative Properties

In the following exercises, simplify.

1. $43m+(-12n)+(-16m)+(-9n)$

Answer

$27m+(-21n)$

2. $-22p+17q+(-35p)+(-27q)$

3. $\frac{3}{8}g+\frac{1}{12}h+\frac{7}{8}g+\frac{5}{12}h$

Answer

$\frac{5}{4}g+\frac{1}{2}h$

4. $\frac{5}{6}a+\frac{3}{10}b+\frac{1}{6}a+\frac{9}{10}b$

5. $6.8p+9.14q+(-4.37p)+(-0.88q)$

Answer

$2.43p+8.26q$

6. $9.6m+7.22n+(-2.19m)+(-0.65n)$

7. $-24\cdot 7\cdot \frac{3}{8}$

Answer

-63

8. $-36\cdot 11\cdot \frac{4}{9}$

9. $\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15}$

Answer

$1\frac{5}{6}$

10. $\left(\frac{11}{12}+\frac{4}{9}\right)+\frac{5}{9}$

11. $17(0.25)(4)$

Answer

17

12. $36(0.2)(5)$

13. $[2.48(12)](0.5)$

Answer

14.88

14. $[9.731(4)](0.75)$

15. $12\left(\frac{5}{6}p\right)$

Answer

$10p$

16. $20\left(\frac{3}{5}q\right)$

Use the Properties of Identity, Inverse and Zero

In the following exercises, simplify.

17. $19a+44-19a$

Answer

44

18. $27c+16-27c$

19. $\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$

Answer

$\frac{7}{8}$

20. $\frac{2}{5}+\frac{5}{12}+\left(-\frac{2}{5}\right)$

21. $10(0.1d)$

Answer

d

22. $100(0.01p)$

23. $\frac{3}{20}\cdot\frac{49}{11}\cdot\frac{20}{3}$

Answer

$\frac{49}{11}$

24. $\frac{13}{18}\cdot\frac{25}{7}\cdot\frac{18}{13}$

25. $\frac{0}{u-4.99}$, where $u\neq 4.99$

Answer

0

26. $0 \div (y - \frac{1}{6})$, where $x \neq 16$

27. $\frac{32-5a}{0}$, where $32 - 5a \neq 0$

Answer

undefined

28. $\frac{28-9b}{0}$, where $28 - 9b \neq 0$

29. $(\frac{3}{4} + \frac{9}{10}m) \div 0$, where $\frac{3}{4} + \frac{9}{10}m \neq 0$

Answer

undefined

30. $(\frac{5}{16}n - \frac{3}{7}) \div 0$, where $\frac{5}{16}n - \frac{3}{7} \neq 0$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

31. $8(4y + 9)$

Answer

$32y + 72$

32. $9(3w + 7)$

33. $6(c - 13)$

Answer

$6c - 78$

34. $7(y - 13)$

35. $\frac{1}{4}(3q + 12)$

Answer

$\frac{3}{4}q + 3$

36. $\frac{1}{5}(4m + 20)$

37. $9(\frac{5}{9}y - \frac{1}{3})$

Answer

$5y - 3$

38. $10(\frac{3}{10}x - \frac{2}{5})$

39. $12(\frac{1}{4}+\frac{2}{3}r)$

Answer

$3+8r$

40. $12(\frac{1}{6}+\frac{3}{4}s)$

41. $15\cdot\frac{3}{5}(4d+10)$

Answer

$36d+90$

42. $18\cdot\frac{5}{6}(15h+24)$

43. $r(s-18)$

Answer

$rs-18r$

44. $u(v-10)$

45. $(y+4)p$

Answer

$yp+4p$

46. $(a+7)x$

47. $-7(4p+1)$

Answer

$-28p-7$

48. $-9(9a+4)$

49. $-3(x-6)$

Answer

$-3x+18$

50. $-4(q-7)$

51. $-(3x-7)$

Answer

$-3x+7$

52. $-(5p-4)$

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53. $16-3(y+8)$

Answer

$-3y-8$

54. $18-4(x+2)$

55. $4-11(3c-2)$

Answer

$-33c+26$

56. $9-6(7n-5)$

57. $22-(a+3)$

Answer

$-a+19$

58. $8-(r-7)$

59. $(5m-3)-(m+7)$

Answer

$4m-10$

60. $(4y-1)-(y-2)$

61. $9(8x-3)-(-2)$

Answer

$72x-25$

62. $4(6x-1)-(-8)$

63. $5(2n+9)+12(n-3)$

Answer

$22n+9$

64. $9(5u+8)+2(u-6)$

65. $14(c-1)-8(c-6)$

Answer

$6c+34$

66. $11(n-7)-5(n-1)$

67. $6(7y+8)-(30y-15)$

Answer

$12y+63$

68. $7(3n+9)-(4n-13)$

Writing Exercises

69. In your own words, state the Associative Property of addition.

Answer

Answers will vary.

70. What is the difference between the additive inverse and the multiplicative inverse of a number

71. Simplify $8(x-\frac{1}{4})$ using the Distributive Property and explain each step.

Answer

Answers will vary.

72. Explain how you can multiply $4(\$5.97)$ without paper or calculator by thinking of $\$5.97$ as $6-0.03$ and then using the Distributive Property.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the commutative and associative properties.			
use the properties of identity, inverse and zero.			
simplify expressions using the distributive property.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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CHAPTER OVERVIEW

2: Solving Linear Equations

In this chapter, you will explore linear equations, develop a strategy for solving them, and relate them to real-world situations.

2.1: Use a General Strategy to Solve Linear Equations

2.1E: Exercises

2.2: Use a Problem Solving Strategy

2.2E: Exercises

2.3: Solve a Formula for a Specific Variable

2.3E: Exercises

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2.1: Use a General Strategy to Solve Linear Equations

Summary

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the properties of identity, inverse, and zero
- Simplify expressions using the Distributive Property

Before you get started, take this readiness quiz.

1. Simplify: $\frac{3}{2}(12x+20)$.
2. Simplify: $5-2(n+1)$.
3. Find the LCD of $\frac{5}{6}$ and $\frac{1}{4}$.

Solve Linear Equations Using a General Strategy

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that makes it a true statement. Any value of the variable that makes the equation true is called a **solution** to the equation. It is the answer to the puzzle!

Solution OF AN EQUATION

A **solution** of an equation is a value of a variable that makes a true statement when substituted into the equation.

To determine whether a number is a solution to an equation, we substitute the value for the variable in the equation. If the resulting equation is a true statement, then the number is a solution of the equation.

DETERMINE WHETHER A NUMBER IS A Solution TO AN EQUATION

1. Substitute the number for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true.
   - If it is true, the number is a solution.
   - If it is not true, the number is not a solution.

Example 2.1.1

Determine whether the values are solutions to the equation: $5y+3=10y-4$.

a. $y=\frac{3}{5}$
b. $y=\frac{7}{5}$

Solution

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

a.

	$5y+3=10y-4$
Substitute $\frac{3}{5}$ for y	$5\left(\frac{3}{5}\right)+3\stackrel{?}{=}10\left(\frac{3}{5}\right)-4$
Multiply.	$3+3\stackrel{?}{=}6-4$
Simplify.	$6\neq 2$

Since $y=\frac{3}{5}$ does not result in a true equation, $y=\frac{3}{5}$ is not a solution to the equation $5y+3=10y-4$.

b.

	$5y+3=10y-4$
Substitute $\frac{7}{5}$ for y	$5\left(\frac{7}{5}\right)+3\stackrel{?}{=}10\left(\frac{7}{5}\right)-4$
Multiply.	$7+3\stackrel{?}{=}14-4$
Simplify.	$10=10\checkmark$

Since $y=\frac{7}{5}$ results in a true equation, $y=\frac{7}{5}$ is a solution to the equation $5y+3=10y-4$.

Exercise 2.1.1A

Determine whether the values are solutions to the equation: $9y+2=6y+3$.

a. $y=\frac{4}{3}$
b. $y=\frac{1}{3}$

Answer a

no

Answer b

yes

Exercise 2.1.1B

Determine whether the values are solutions to the equation: $4x-2=2x+1$.

a. $x=\frac{3}{2}$
b. $x=-\frac{1}{2}$

Answer a

yes

Answer b

no

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation.**

LINEAR EQUATION

A **linear equation** is an equation in one variable that can be written, where a and b are real numbers and $a\neq 0$, as:

$$ax+b=0 \tag{2.1.1}$$

To solve a linear equation it is a good idea to have an overall strategy that can be used to solve any linear equation. In the next example, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

EXAMPLE 2.1.2

Solve: $7(n-3)-8=-15$

Answer

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do—all n's are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n-29+29=-15+29$ $7n=14$
Step 4. Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7}=\frac{14}{7}$ $n=2$
Step 5. Check the solution.	Let $n=2$ Subtract.	**Check:** $7(n-3)-8=-15$ $7(2-3)-8\stackrel{?}{=}-15$ $7(-1)-8\stackrel{?}{=}-15$ $-7-8\stackrel{?}{=}-15$ $-15=-15$ ✓

Solve: $2(m-4)+3=-1$.

Answer

$m=2$

? Exercise 2.1.2B

Solve: $5(a-3)+5=-10$.

Answer

$a=0$

These steps are summarized in the **General Strategy for Solving Linear Equations** below.

SOLVE LINEAR EQUATIONS USING A GENERAL STRATEGY

1. Simplify each side of the equation as much as possible.
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. Collect all the variable terms on one side of the equation.
   Use the Addition or Subtraction Property of Equality.
3. Collect all the constant terms on the other side of the equation.
   Use the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable term equal to 1.
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. Check the solution.
   Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 2.1.3

Solve: $\frac{2}{3}(3m-6)=5-m$.

Answer

	$\frac{2}{3}(3m-6)=5-m$
Distribute.	$2m-4=5-m$
Add m to both sides to get the variables only on the left.	$2m+m-4=5-m+m$
Simplify.	$3m-4=5$
Add 4 to both sides to get constants only on the right.	$3m-4+4=5+4$
Simplify.	$3m=9$
Divide both sides by three.	$\frac{3m}{3}=\frac{9}{3}$
Simplify.	$m=3$

Check:	$\frac{2}{3}(3m-6)=5-m$
Let $m=3$.	$\frac{2}{3}(3\cdot 3-6)\stackrel{?}{=}5-3$
	$\frac{2}{3}(9-6)\stackrel{?}{=}2$
	$\frac{2}{3}(3)\stackrel{?}{=}2$
	$2=2$ ✓

Exercise 2.1.3A

Solve: $\frac{1}{3}(6u+3)=7-u$.

Answer

$u=2$

Exercise 2.1.3B

Solve: $\frac{2}{3}(9x-12)=8+2x$.

Answer

$x=4$

We can solve equations by getting all the **variable** terms to either side of the **equal sign**. By collecting the variable terms on the side where the **coefficient** of the variable is larger, we avoid working with some negatives. This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

EXAMPLE 2.1.4

Solve: $4(x-1)-2=5(2x+3)+6$.

Answer

	$4(x-1)-2=5(2x+3)+6$

Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract $4x$ from each side to get the variables only on the right since $10>4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 from each side to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide both sides by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6 \quad \backslash)$
Let $x=-92$.	$4\left(-\frac{9}{2}-1\right)-2\stackrel{?}{=}5\left(2\left(-\frac{9}{2}\right)+3\right)+6$
	$4\left(-\frac{11}{2}\right)-2\stackrel{?}{=}5(-9+3)+6$
	$-22-2\stackrel{?}{=}5(-6)+6$
	$-24\stackrel{?}{=}-30+6$
	$-24=-24$ ✓

Exercise 2.1.4A

Solve: $6(p-3)-7=5(4p+3)-12$.

Answer

$p=-2$

Exercise 2.1.4B

Solve: $8(q+1)-5=3(2q-4)-1$.

Answer

$q=-8$

EXAMPLE 2.1.5

Solve: $10[3-8(2s-5)]=15(40-5s)$.

Answer

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add $160s$ to both sides to get the $160s$ to both sides to get the variables to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$

Subtract 600 from both sides to get the constants to the left.		$430 - 600 = 600 + 85s - 600$
Simplify.		$-170 = 85s$
Divide both sides by 85.		$\frac{-170}{85} = \frac{85s}{85}$
Simplify.		$-2 = s$, so $s = -2$
Check:	$10[3 - 8(2s - 5)] = 15(40 - 5s)$	
Let $s = -2$.	$10[3 - 8(2(-2) - 5)] \stackrel{?}{=} 15(40 - 5(-2))$	
	$10[3 - 8(-4 - 5)] \stackrel{?}{=} 15(40 + 10)$	
	$10[3 - 8(-9)] \stackrel{?}{=} 15(50)$	
	$10(3 + 72) \stackrel{?}{=} 750$	
	$10(75) \stackrel{?}{=} 750$	
	$750 = 750$ ✓	

Exercise 2.1.5A

Solve: $6[4 - 2(7y - 1)] = 8(13 - 8y)$.

Answer

$y = -\frac{17}{5}$

Exercise 2.1.5B

Solve: $12[1 - 5(4z - 1)] = 3(24 + 11z)$.

Answer

$z = 0$

Classify Equations

Whether or not an equation is true depends on the value of the variable. The equation $7x + 8 = -13$ is true when we replace the variable, *x*, with the value -3, but not true when we replace *x* with any other value. An equation like this is called a **conditional equation**. All the equations we have solved so far are conditional equations.

CONDITIONAL EQUATION

An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let's consider the equation $7y + 14 = 7(y + 2)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

Solve:

	$7y + 14 = 7(y + 2)$
Distribute.	$7y + 14 = 7y + 14$
Subtract $7y$ to each side to get the y's to one side.	$7y - 7y + 14 = 7y - 7y + 14$

Simplify—the y's are eliminated.	$14 = 14$
	But $14 = 14$ is true.

This means that the equation $7y + 14 = 7(y + 2)$ is true for any value of y. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable is called an **identity**.

IDENTITY

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is valid for all real numbers.

What happens when we solve the equation $-8z = -8z + 9$?

Solve:

	$-8z = -8z + 9$
Add $8z$ to both sides to leave the constant alone on the right.	$-8z + 8z = -8z + 8z + 9$
Simplify—the z's are eliminated.	$0 \neq 9$
	But $0 \neq 9$.

Solving the equation $-8z = -8z + 9$ led to the false statement $0 = 9$. The equation $-8z = -8z + 9$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a contradiction.

CONTRADICTION

An equation that is false for all values of the variable is called a **contradiction**.

A contradiction has no solution.

The next few examples will ask us to classify an equation as conditional, an identity, or as a contradiction.

? EXAMPLE 2.1.6

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6(2n - 1) + 3 = 2n - 8 + 5(2n + 1)$.

Answer

	$6(2n - 1) + 3 = 2n - 8 + 5(2n + 1)$
Distribute.	$12n - 6 + 3 = 2n - 8 + 10n + 5$
Combine like terms.	$12n - 3 = 12n - 3$
Subtract $12n$ from each side to get the n's to one side.	$12n - 12n - 3 = 12n - 12n - 3$
Simplify.	$-3 = -3$
This is a true statement.	The equation is an identity.
	The solution is all real numbers.

Exercise 2.1.6A

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $4+9(3x-7)=-42x-13+23(3x-2)$.

Answer

identity; all real numbers

Exercise 2.1.6B

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8(1-3x)+15(2x+7)=2(x+50)+4(x+3)+1$.

Answer

identity; all real numbers

EXAMPLE 2.1.7

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8+3(a-4)=0$.

Answer

	$8+3(a-4)=0$
Distribute.	$8+3a-12=0$
Combine like terms.	$3a-4=0$
Add 4 to both sides.	$3a-4+4=0+4$
Simplify.	$3a=4$
Divide.	$\frac{3a}{3}=\frac{4}{3}$
Simplify.	$a=\frac{4}{3}$
The equation is true when $a=\frac{4}{3}$.	This is a conditional equation.
	The solution is $a=\frac{4}{3}$.

Exercise 2.1.7A

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3)-5=19$.

Answer

conditional equation; $q=-\frac{9}{11}$

Exercise 2.1.7B

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14(k-8)=95$.

Answer

conditional equation; $k=\frac{201}{14}$

? EXAMPLE 2.1.8

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $5m+3(9+3m)=2(7m-11)$.

Answer

	$5m+3(9+3m)=2(7m-11)$
Distribute.	$5m+27+9m=14m-22$
Combine like terms.	$14m+27=14m-22$
Subtract $14m$ from both sides.	$14m+27-14m=14m-22-14m$
Simplify.	$27\neq -22$
But $27\neq -22$.	The equation is a contradiction.
	It has no solution.

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $12c+5(5+3c)=3(9c-4)$.

Answer

contradiction; no solution

? Exercise 2.1.8B

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $4(7d+18)=13(3d-2)-11d$.

Answer

contradiction; no solution

We summarize the methods for classifying equations in the table.

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	True for any value of the variable	All real numbers
Contradiction	False for all values of the variable	No solution

Solve Equations with Fraction or Decimal Coefficients

We could use the General Strategy to solve the next example. This method would work fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the **least common denominator** (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called *clearing* the equation of fractions.

To clear an equation of decimals, we think of all the decimals in their fraction form and then find the LCD of those denominators.

Exercise 2.1.9A

Solve: $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$.

Answer

Step 1. Find the least common denominator of *all* the fractions and decimals in the equation.	What is the LCD of $\frac{1}{12}$, $\frac{5}{6}$, and $\frac{3}{4}$?	$\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ LCD = 12
Step 2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.	Multiply both sides of the equation by the LCD, 12. Use the Distributive Property. Simplify—and notice, no more fractions!	$12\left(\frac{1}{12}x+\frac{5}{6}\right)=12\left(\frac{3}{4}\right)$ $12\cdot\frac{1}{12}x+12\cdot\frac{5}{6}=12\cdot\frac{3}{4}$ $x+10=9$
Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the variable term, subtract 10. Simplify.	$x+10-10=9-10$ $x=-1$ **Check:** $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ $\frac{1}{12}(-1)+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{10}{12}\stackrel{?}{=}\frac{9}{12}$ $\frac{9}{12}=\frac{9}{12}$ ✓

Exercise 2.1.9B

Solve: $\frac{1}{4}x+\frac{1}{2}=\frac{5}{8}$.

Answer

$x=\frac{1}{2}$

Exercise 2.1.9C

Solve: $\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$.

Answer

$x=-2$

Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve. We then used the **General Strategy for Solving Linear Equations**.

SOLVE EQUATIONS WITH FRACTION OR DECIMAL COEFFICIENTS.

1. Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.
2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.
3. Solve using the General Strategy for Solving Linear Equations.

EXAMPLE 2.1.10

Solve: $5 = \frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y$.

Answer

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.	$5 = \frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y$
The LCD is 12.	
Multiply both sides of the equation by 12.	$12(5) = 12 \cdot \left(\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y\right)$
Distribute.	$12(5) = 12 \cdot \frac{1}{2}y + 12 \cdot \frac{2}{3}y - 12 \cdot \frac{3}{4}y$
Simplify—notice, no more fractions.	$60 = 6y + 8y - 9y$
Combine like terms.	$60 = 5y$
Divide by five.	$\frac{60}{5} = \frac{5y}{5}$
Simplify.	$12 = y$
Check:	$5 = \frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y$
Let $y = 12$.	$5 \stackrel{?}{=} \frac{1}{2}(12) + \frac{2}{3}(12) - \frac{3}{4}(12)$
	$5 \stackrel{?}{=} 6 + 8 - 9$
	$5 = 5$ ✓

Exercise 2.1.10A

Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$.

Answer

$x = 12$

Exercise 2.1.10B

Solve: $-1 = \frac{1}{2}u + \frac{1}{4}u - \frac{2}{3}u$.

Answer

$u = -12$

In the next example, we'll distribute before we clear the fractions.

EXAMPLE 2.1.11

Solve: $\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$.

Answer

	$\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2} \cdot y - \frac{1}{2} \cdot 5 = \frac{1}{4} \cdot y - \frac{1}{4} \cdot 1$

Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, four.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
An alternate way to solve this equation is to clear the fractions without distributing first. If you multiply the factors correctly, this method will be easier.	
	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Multiply by the LCD, 4.	$4\cdot\frac{1}{2}(y-5)=4\cdot\frac{1}{4}(y-1)$
Multiply four times the fractions.	$2(y-5)=1(y-1)$
Distribute.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
Check:	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Let $y=9$.	$\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$
Finish the check on your own.	

? Exercise 2.1.11A

Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

Answer

$n=2$

? Exercise 2.1.11B

Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

Answer

$m=-1$

When you multiply both sides of an equation by the LCD of the fractions, make sure you multiply each term by the LCD—even if it does not contain a fraction.

EXAMPLE 2.1.12

Solve: $\frac{4q+3}{2}+6=\frac{3q+5}{4}$

Answer

	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Multiply both sides by the LCD, 4.	$4\left(\frac{4q+3}{2}+6\right)=4\left(\frac{3q+5}{4}\right)$
Distribute.	$4\left(\cdot\frac{4q+3}{2}\right)+4\cdot 6=4\cdot\left(\frac{3q+5}{4}\right)$
Simplify.	$2(4q+3)+24=3q+5$
	$8q+6+24=3q+5$
	$8q+30=3q+5$
Collect the variables to the left.	$8q-3q+30=3q-3q+5$
Simplify.	$5q+30=5$
Collect the constants to the right.	$5q+30-30=5-30$
Simplify.	$5q=-25$
Divide both sides by five.	$\frac{5q}{5}=\frac{-25}{5}$
Simplify.	$q=-5$
Check:	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Let $q=-5$.	$\frac{4(-5)+3}{2}+6\stackrel{?}{=}\frac{3(-5)+5}{4}$
Finish the check on your own.	

Solve: $\frac{3r+5}{6}+1=\frac{4r+3}{3}$.

Answer

$r=3$

Exercise 2.1.12B

Solve: $\frac{2s+3}{2}+1=\frac{3s+2}{4}$.

Answer

$s=-8$

Some equations have decimals in them. This kind of equation may occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.7=\frac{7}{10}$ and $0.29=\frac{29}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the **least common denominator**.

The next example uses an equation that is typical of the ones we will see in the money applications in a later section. Notice that we will clear all decimals by multiplying by the LCD of their fraction form.

Solve: $0.25x+0.05(x+3)=2.85$.

Answer

Look at the decimals and think of the equivalent fractions:

$$0.25 = \frac{25}{100}, \qquad 0.05 = \frac{5}{100}, \qquad 2.85 = 2\frac{85}{100}. \tag{2.1.2}$$

Notice, the LCD is 100. By multiplying by the LCD we will clear the decimals from the equation.

	$0.25x + 0.05(x + 3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$
Distribute.	$30x + 15 = 285$
Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$
Simplify.	$30x = 270$
Divide by 30.	$\frac{30x}{30} = \frac{270}{30}$
Simplify.	$x = 9$
Check it yourself by substituting $x = 9$ into the original equation.	

? Exercise 2.1.13A

Solve: $0.25n + 0.05(n + 5) = 2.95$.

Answer

$n = 9$

? Exercise 2.1.13B

Solve: $0.10d + 0.05(d - 5) = 2.15$.

Answer

$d = 16$

Key Concepts

- **How to determine whether a number is a solution to an equation**
  1. Substitute the number in for the variable in the equation.
  2. Simplify the expressions on both sides of the equation.
  3. Determine whether the resulting equation is true.

     If it is true, the number is a solution.

     If it is not true, the number is not a solution.
- **How to Solve Linear Equations Using a General Strategy**
  1. Simplify each side of the equation as much as possible.

     Use the Distributive Property to remove any parentheses.

     Combine like terms.
  2. Collect all the variable terms on one side of the equation.

Use the Addition or Subtraction Property of Equality.

3. Collect all the constant terms on the other side of the equation.

   Use the Addition or Subtraction Property of Equality.

4. Make the coefficient of the variable term equal to 1.

   Use the Multiplication or Division Property of Equality.

   State the solution to the equation.

5. Check the solution.

   Substitute the solution into the original equation to make sure the result is a true statement.

- **How to Solve Equations with Fraction or Decimal Coefficients**
  1. Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.
  2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.
  3. Solve using the General Strategy for Solving Linear Equations.

Glossary

conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction

An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity

An equation that is true for any value of the variable is called an Identity. The solution of an identity is all real numbers.

linear equation

A linear equation is an equation in one variable that can be written, where a and b are real numbers and $a \neq 0$, as $ax + b = 0$.

solution of an equation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

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2.1E: Exercises

Practice Makes Perfect

Solve Equations Using the General Strategy

In the following exercises, determine whether the given values are solutions to the equation.

1. $6y+10=12y$

a. $y=\frac{5}{3}$

b. $y=-\frac{1}{2}$

Answer

a. yes
b. no

2. $4x+9=8x$

a. $x=-\frac{7}{8}$

b. $x=\frac{9}{4}$

3. $8u-1=6u$

a. $u=-\frac{1}{2}$

b. $u=\frac{1}{2}$

Answer

a. no
b. yes

4. $9v-2=3v$

a. $v=-\frac{1}{3}$

b. $v=\frac{1}{3}$

In the following exercises, solve each linear equation.

5. $15(y-9)=-60$

Answer

$y=5$

6. $-16(3n+4)=32$

7. $-(w-12)=30$

Answer

$w=-18$

8. $-(t-19)=28$

9. $51+5(4-q)=56$

Answer

$q=3$

10. $-6+6(5-k)=15$

11. $3(10-2x)+54=0$

Answer

$x=14$

12. $-2(11-7x)+54=4$

13. $\frac{2}{3}(9c-3)=22$

Answer

$c=4$

14. $\frac{3}{5}(10x-5)=27$

15. $\frac{1}{5}(15c+10)=c+7$

Answer

$c=\frac{5}{2}$

16. $\frac{1}{4}(20d+12)=d+7$

17. $3(4n-1)-2=8n+3$

Answer

$n=2$

18. $9(2m-3)-8=4m+7$

19. $12+2(5-3y)=-9(y-1)-2$

Answer

$y=-5$

20. $-15+4(2-5y)=-7(y-4)+4$

21. $5+6(3s-5)=-3+2(8s-1)$

Answer

$s=10$

22. $-12+8(x-5)=-4+3(5x-2)$

23. $4(p-4)-(p+7)=5(p-3)$

Answer

$p=-4$

24. $3(a-2)-(a+6)=4(a-1)$

25. $4[5-8(4c-3)]=12(1-13c)-8$

Answer

$c=-4$

26. $5[9-2(6d-1)]=11(4-10d)-139$

27. $3[-9+8(4h-3)]=2(5-12h)-19$

Answer

$h=\frac{3}{4}$

28. $3[-14+2(15k-6)]=8(3-5k)-24$

29. $5[2(m+4)+8(m-7)]=2[3(5+m)-(21-3m)]$

Answer

$m=6$

30. $10[5(n+1)+4(n-1)]=11[7(5+n)-(25-3n)]$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

31. $23z+19=3(5z-9)+8z+46$

Answer

identity; all real numbers

32. $15y+32=2(10y-7)-5y+46$

33. $18(5j-1)+29=47$

Answer

conditional equation; $j=\frac{2}{5}$

34. $24(3d-4)+100=52$

35. $22(3m-4)=8(2m+9)$

Answer

conditional equation; $m = 165$

36. $30(2n-1) = 5(10n+8)$

37. $7v+42 = 11(3v+8)-2(13v-1)$

Answer

contradiction; no solution

38. $18u-51 = 9(4u+5)-6(3u-10)$

39. $45(3y-2) = 9(15y-6)$

Answer

contradiction; no solution

40. $60(2x-1) = 15(8x+5)$

41. $9(14d+9)+4d = 13(10d+6)+3$

Answer

identity; all real numbers

42. $11(8c+5)-8c = 2(40c+25)+5$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation with fraction coefficients.

43. $\frac{1}{4}x - \frac{1}{2} = -\frac{3}{4}$

Answer

$x = -1$

44. $\frac{3}{4}x - \frac{1}{2} = \frac{1}{4}$

45. $\frac{5}{6}y - \frac{2}{3} = -\frac{3}{2}$

Answer

$y = -1$

46. $\frac{5}{6}y - \frac{1}{3} = -\frac{7}{6}$

47. $\frac{1}{2}a + \frac{3}{8} = \frac{3}{4}$

Answer

$a = \frac{3}{4}$

48. $\frac{5}{8}b + \frac{1}{2} = -\frac{3}{4}$

49. $2 = \frac{1}{3}x - \frac{1}{2}x + \frac{2}{3}x$

Answer

$x = 4$

50. $2 = \frac{3}{5}x - \frac{1}{3}x + \frac{2}{5}x$

51. $\frac{1}{3}w + \frac{5}{4} = w - \frac{1}{4}$

Answer

$w = \frac{9}{4}$

52. $\frac{1}{2}a - \frac{1}{4} = \frac{1}{6}a + \frac{1}{12}$

53. $\frac{1}{3}b + \frac{1}{5} = \frac{2}{5}b - \frac{3}{5}$

Answer

$b = 12$

54. $\frac{1}{3}x + \frac{2}{5} = \frac{1}{5}x - \frac{2}{5}$

55. $\frac{1}{4}(p-7) = \frac{1}{3}(p+5)$

Answer

$p = -41$

56. $\frac{1}{5}(q+3) = \frac{1}{2}(q-3)$

57. $\frac{1}{2}(x+4) = \frac{3}{4}$

Answer

$x = -\frac{5}{2}$

58. $\frac{1}{3}(x+5) = \frac{5}{6}$

59. $\frac{4n+8}{4} = \frac{n}{3}$

Answer

$n = -3$

60. $\frac{3p+6}{3} = \frac{p}{2}$

61. $\frac{3x+4}{2} + 1 = \frac{5x+10}{8}$

Answer

$x = -2$

62. $\frac{10y-2}{3}+3=\frac{10y+1}{9}$

63. $\frac{7u-1}{4}-1=\frac{4u+8}{5}$

Answer

$u=3$

64. $\frac{3v-6}{2}+5=\frac{11v-4}{5}$

In the following exercises, solve each equation with decimal coefficients.

65. $0.4x+0.6=0.5x-1.2$

Answer

$x=18$

66. $0.7x+0.4=0.6x+2.4$

67. $0.9x-1.25=0.75x+1.75$

Answer

$x=20$

68. $1.2x-0.91=0.8x+2.29$

69. $0.05n+0.10(n+8)=2.15$

Answer

$n=9$

70. $0.05n+0.10(n+7)=3.55$

71. $0.10d+0.25(d+5)=4.05$

Answer

$d=8$

72. $0.10d+0.25(d+7)=5.25$

Everyday Math

73. **Fencing** Micah has 74 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, L, by solving the equation $2L+2(L-2.5)=74$.

Answer

$L=19.75$ feet

74. Stamps Paula bought $22.82 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was eight less than the number of 49-cent stamps. Solve the equation $0.49s+0.21(s-8)=22.82$ for s, to find the number of 49-cent stamps Paula bought.

Writing Exercises

75. Using your own words, list the steps in the general strategy for solving linear equations.

Answer

Answers will vary.

76. Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

77. What is the first step you take when solving the equation $3-7(y-4)=38$? Why is this your first step?

Answer

Answers will vary.

78. If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

79. If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

Answer

Answers will vary.

80. For the equation $0.35x+2.1=3.85$, how do you clear the decimal?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve linear equations using a general strategy.			
classify equations.			
solve equations with fraction or decimal coefficients.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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2.2: Use a Problem Solving Strategy

Learning Objectives

By the end of this section, you will be able to:

- Use a problem solving strategy for word problems
- Solve number word problems
- Solve percent applications
- Solve simple interest applications

Before you get started, take this readiness quiz.

1. Translate "six less than twice x" into an algebraic expression.
   If you missed this problem, review [link].
2. Convert 4.5% to a decimal.
   If you missed this problem, review [link].
3. Convert 0.6 to a percent.
   If you missed this problem, review [link].

Have you ever had any negative experiences in the past with word problems? When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. Realize that your negative experiences with word problems are in your past. To move forward you need to calm your fears and change your negative feelings.

Start with a fresh slate and begin to think positive thoughts. Repeating some of the following statements may be helpful to turn your thoughts positive. Thinking positive thoughts is a first step towards success.

- I think I can! I think I can!
- While word problems were hard in the past, I think I can try them now.
- I am better prepared now—I think I will begin to understand word problems.
- I am able to solve equations because I practiced many problems and I got help when I needed it—I can try that with word problems.
- It may take time, but I can begin to solve word problems.
- You are now well prepared and you are ready to succeed. If you take control and believe you can be successful, you will be able to master word problems.

Use a Problem Solving Strategy for Word Problems

Now that we can solve equations, we are ready to apply our new skills to word problems. We will develop a strategy we can use to solve any word problem successfully.

EXAMPLE 2.2.1

Normal yearly snowfall at the local ski resort is 12 inches more than twice the amount it received last season. The normal yearly snowfall is 62 inches. What was the snowfall last season at the ski resort?

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	What was the snowfall last season?
Step 3. Name what we are looking for and choose a variable to represent it.	Let $s =$ the snowfall last season.
Step 4. Translate. Restate the problem in one sentence with all the important information.	The normal snowfall / was / twelve more than twice the amount last year

Translate into an equation.	$62 = 2s + 12$
Step 5. Solve the equation.	$62 = 2s + 12$
Subtract 12 from each side.	$62 - 12 = 2s + 12 - 12$
Simplify.	$50 = 2s$
Divide each side by two.	$\frac{50}{2} = \frac{2s}{2}$
Simplify.	$25 = s$
Step 6. Check: First, is our answer reasonable? Yes, having 25 inches of snow seems OK. The problem says the normal snowfall is twelve inches more than twice the number of last season. Twice 25 is 50 and 12 more than that is 62.	
Step 7. Answer the question.	The snowfall last season was 25 inches.

Try It! 2.2.1

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was three more than twice the number of notebooks. He bought seven textbooks. How many notebooks did he buy?

Answer

He bought two notebooks

Try It! 2.2.2

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

Answer

He did seven crossword puzzles

We summarize an effective strategy for problem solving.

PROBLEM SOLVING STRATEGY FOR WORD PROBLEMS

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. **Solve** the equation using proper algebra techniques.
6. **Check** the answer in the problem to make sure it makes sense.
7. **Answer** the question with a complete sentence.

Solve Number Word Problems

We will now apply the problem solving strategy to “number word problems.” Number word problems give some clues about one or more numbers and we use these clues to write an equation. Number word problems provide good practice for using the Problem Solving Strategy.

EXAMPLE 2.2.2

The sum of seven times a number and eight is thirty-six. Find the number.

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number
Step 3. Name what you are looking for and choose a variable to represent it.	Let n = the number.
Step 4. Translate: Restate the problem as one sentence. Translate into an equation.	The sum of seven times a number and 8 is 36 $7n+8 = 36$
Step 5. Solve the equation. Subtract eight from each side and simplify. Divide each side by seven and simplify.	$7n+8=36$ $7n=28$ $n=4$
Step 6. Check. Is the sum of seven times four plus eight equal to 36?	$7\cdot 4+8 \stackrel{?}{=} 36$ $28+8 \stackrel{?}{=} 36$ $36=36\checkmark$
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

Try It! 2.2.3

The sum of four times a number and two is fourteen. Find the number.

Answer

3

Try It! 2.2.4

The sum of three times a number and seven is twenty-five. Find the number.

Answer

6

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far, we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

EXAMPLE 2.2.3

The sum of two numbers is negative fifteen. One number is nine less than the other. Find the numbers.

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	two numbers
Step 3. Name what you are looking for by choosing a variable to represent the first number. “One number is nine less than the other.”	Let $n = 1^{\text{st}}$ number. $n - 9 = 2^{\text{nd}}$ number
Step 4. Translate. Write as one sentence. Translate into an equation.	The sum of two numbers is negative fifteen. 1st number + 2nd number is negative fifteen $n + n - 9 = -15$
Step 5. Solve the equation. Combine like terms. Add nine to each side and simplify. Simplify.	$n + n - 9 = -15$ $2n = -6$ $-3 - 9$ -12
Step 6. Check. Is -12 nine less than -3? $-3 - 9 \stackrel{?}{=} -12$ $-12 = -12$✓ Is their sum -15? $-3 + (-12) \stackrel{?}{=} -15$ $-15 = -15$✓	
Step 7. Answer the question.	The numbers are -3 and -12.

? Try It! 2.2.5

The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

Answer

$-15, -8$

? Try It! 2.2.6

The sum of two numbers is negative eighteen. One number is forty more than the other. Find the numbers.

Answer

$-29, 11$

Some number problems involve **consecutive integers**. Consecutive integers are integers that immediately follow each other. Examples of consecutive integers are:

$$\begin{array}{rrrr} 1, & 2, & 3, & 4 \\ -10, & -9, & -8, & -7 \\ 150, & 151, & 152, & 153 \end{array}$$

Notice that each number is one more than the number preceding it. Therefore, if we define the first integer as n, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

$$\begin{array}{ll} n & 1^{\text{st}} \text{integer} \\ n+1 & 2^{\text{nd}} \text{consecutive integer} \\ n+2 & 3^{\text{rd}} \text{consecutive integer} \quad \text{etc.} \end{array}$$

We will use this notation to represent consecutive integers in the next example.

EXAMPLE 2.2.4

Find three consecutive integers whose sum is -54.

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	three consecutive integers
Step 3. Name each of the three numbers	Let $n = 1^{\text{st}}$integer. $n+1 = 2^{\text{nd}}$consecutive integer $n+2 = 3^{\text{rd}}$consecutive integer
Step 4. Translate. Restate as one sentence. Translate into an equation.	The sum of the three integers is -54. $n+n+1+n+2=-54$
Step 5. Solve the equation. Combine like terms. Subtract three from each side. Divide each side by three.	$n+n+1+n+2=-54$ $3n+3=-54$ $3n=-57$ $n=-19$ 1st integer $n+1$ 2nd integer -18 $-19+2$ -17
Step 6. Check. $-19+(-18)+(-17) = -54$ $-54=-54$✓	
Step 7. Answer the question.	The three consecutive integers are −17,−18, and −19.

Try It! 2.2.7

Find three consecutive integers whose sum is -96.

Answer

$-33, -32, -31$

Try It! 2.2.8

Find three consecutive integers whose sum is -36.

Answer

$-13, -12, -11$

Now that we have worked with consecutive integers, we will expand our work to include **consecutive even integers** and **consecutive odd integers**. Consecutive even integers are even integers that immediately follow one another. Examples of consecutive even integers are:

$$24, 26, 28$$

$$-12, -10, -8$$

Notice each integer is two more than the number preceding it. If we call the first one n, then the next one is $n+2$. The one after that would be $n+2+2$ or $n+4$.

n 1^{st}integer
$n+1$ 2^{nd}consecutive integer
$n+2$ 3^{rd}consecutive integer etc.

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 63, 65, and 67.

$$63, 65, 67$$

$$n, n+2, n+4$$

n 1^{st}integer
$n+1$ 2^{nd}consecutive integer
$n+2$ 3^{rd}consecutive integer etc.

Does it seem strange to have to add two (an even number) to get the next odd number? Do we get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you do not have to do anything different. The pattern is still the same—to get to the next odd or the next even integer, add two.

EXAMPLE 2.2.5

Find three consecutive even integers whose sum is 120.

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	three consecutive even integers
Step 3. Name each of the three numbers	Let $n = 1^{\text{st}}$consecutive even integer. $n+2 = 2^{\text{nd}}$consecutive even integer. $n+4 = 3^{\text{rd}}$consecutive even integer.
Step 4. Translate. Restate as one sentence. Translate into an equation.	The sum of the three even integers is 120 $n+n+2+n+4=120$
Step 5. Solve the equation. Combine like terms. Subtract three from each side. Divide each side by three.	$n+n+2+n+4=120$ $3n+6=120$ $3n=114$ $n=38$ 1^{st}integer $n+2$ 2^{nd}integer $38+2$ 40 $n+2$ 3^{rd}integer $38+4$ 42
Step 6. Check. $38+40+42 \stackrel{?}{=} 120$ $120 = 120$✓	
Step 7. Answer the question.	The three consecutive integers are 38, 40, and 42.

Try It! 2.2.9

Find three consecutive even integers whose sum is 102.

Answer

$32, 34, 36$

Try It! 2.2.10

Find three consecutive even integers whose sum is -24.

Answer

$-10, -8, -6$

When a number problem is in a real life context, we still use the same strategies that we used for the previous examples.

EXAMPLE 2.2.6

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution:

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How much does the husband earn?
Step 3. Name each of the three numbers	Let $h =$ the amount the husband earns.
Step 4. Translate. Restate the problem in one sentence with all the important information. Translate into an equation.	$2h - 16,000 =$ the amount the wife earns. Together the husband and wife earn $110,000. $h + 2h - 16,000 = 110,000$
Step 5. Solve the equation. Combine like terms. Add 16,000 to both sides and simplify. Divide each side by three.	$h + 2h - 16,000 = 110,000$ $3h - 16,000 = 110,000$ $3h = 126,000$ $h = 42,000$ amount husband earns $2h - 16,000$ amount wife earns $2(42,000) - 16,000$ $84,000 - 16,000$ $68,000$
Step 6. Check. If the wife earns $68,000 and the husband earns $42,000, is that $110,000? Yes!	
Step 7. Answer the question.	The husband earns $42,000 a year.

According to the National Automobile Dealers Association, the average cost of a car in 2014 was $28,400. This was $1,600 less than six times the cost in 1975. What was the average cost of a car in 1975?

Answer

The average cost was $5,000.

Try It! 2.2.11

US Census data shows that the median price of new home in the U.S. in November 2014 was $280,900. This was $10,700 more than 14 times the price in November 1964. What was the median price of a new home in November 1964?

Answer

The median price was $19,300.

Solve Percent Applications

There are several methods to solve percent equations. In algebra, it is easiest if we just translate English sentences into algebraic equations and then solve the equations. Be sure to change the given percent to a decimal before you use it in the equation.

EXAMPLE 2.2.7

Translate and solve:

a. What number is 45% of 84?
b. 8.5% of what amount is $4.76?
c. 168 is what percent of 112?

Solution:

a.

	What number is 45% of 84?
Translate into algebra. Let n = the number.	$n = 0.45 \cdot 84$
Multiply.	$n = 37.8$
	37.8 is 45% of 84.

b.

	8.5% of what amount is $4.76?
Translate. Let n = the amount.	$0.085 \cdot n = 4.76$
Multiply.	$0.085n = 4.76$
Divide both sides by 0.085 and simplify.	$n = 56$
	8.5% of $56 is $4.76

c.

We are asked to find percent, so we must have our result in percent form.	168 is what percent of 112?
Translate into algebra. Let p = the percent.	$168 = p \cdot 112$
Multiply.	$168 = 112p$
Divide both sides by 112 and simplify.	$1.5 = p$
Convert to percent.	$150\% = p$
	168 is 150% of 112.

Try It! 2.2.12

Translate and solve:

a. What number is 45% of 80?
b. 7.5% of what amount is $1.95?
c. 110 is what percent of 88?

Answer

a. 36 b. $26 c. 125%

Try It! 2.2.13

Translate and solve:

a. What number is 55% of 60?
b. 8.5% of what amount is $3.06?
c. 126 is what percent of 72?

Answer

a. 33 b. $36 c. 175%

Now that we have a problem solving strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we will solve involve everyday situations, you can rely on your own experience.

EXAMPLE 2.2.8

The label on Audrey's yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?

Solution:

What are you asked to find?	What total amount of protein is recommended?
Choose a variable to represent it.	Let a = total amount of protein.
Write a sentence that gives the information to find it.	12 g is 24% of the total amount
Translate into an equation.	$12 = 0.24 \cdot a$
Solve.	$50 = a$
Check: Does this make sense? Yes, 24% is about $\frac{1}{4}$ of the total and 12 is about $\frac{1}{4}$ of 50.	
Write a complete sentence to answer the question.	The amount of protein that is recommended is 50 g.

Try It! 2.2.14

One serving of wheat square cereal has 7 grams of fiber, which is 28% of the recommended daily amount. What is the total recommended daily amount of fiber?

Answer

25 grams

Try It! 2.2.15

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Answer

2,375 mg

Remember to put the answer in the form requested. In the next example we are looking for the percent.

EXAMPLE 2.2.9

Veronica is planning to make muffins from a mix. The package says each muffin will be 240 calories and 60 calories will be from fat. What percent of the total calories is from fat?

Solution:

What are you asked to find?	What percent of the total calories is fat?
Choose a variable to represent it.	Let $p =$ percent of fat.
Write a sentence that gives the information to find it.	What percent of 240 is 60?
Translate the sentence into an equation.	$p \cdot 240 = 60$
Multiply.	$240p = 60$
Divide both sides by 240.	$p = 0.25$
Put in percent form.	$p = 25\%$
Check: does this make sense? Yes, 25% is one-fourth; 60 is one-fourth of 240. So, 25% makes sense.	
Write a complete sentence to answer the question.	Of the total calories in each muffin, 25% is fat.

Try It! 2.2.16

Mitzi received some gourmet brownies as a gift. The wrapper said each 28% brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat? Round the answer to the nearest whole percent.

Answer

50%

Try It! 2.2.17

The mix Ricardo plans to use to make brownies says that each brownie will be 190 calories, and 76 calories are from fat. What percent of the total calories are from fat? Round the answer to the nearest whole percent.

Answer

40%

It is often important in many fields—business, sciences, pop culture—to talk about how much an amount has increased or decreased over a certain period of time. This increase or decrease is generally expressed as a percent and called the **percent change**.

To find the percent change, first we find the amount of change, by finding the difference of the new amount and the original amount. Then we find what percent the amount of change is of the original amount.

FIND PERCENT CHANGE

1. Find the amount of change.

$$\text{change} = \text{new amount} - \text{original amount} \tag{2.2.1}$$

2. Find what percent the amount of change is of the original amount.

change is what percent of the original amount?

EXAMPLE 2.2.10

Recently, the California governor proposed raising community college fees from $36 a unit to $46 a unit. Find the percent change. (Round to the nearest tenth of a percent.)

Solution:

Find the amount of change.	$46 - 36 = 10$
Find the percent.	Change is what percent of the original amount?
Let $p =$ the percent.	10 is what percent of 36?
Translate to an equation.	$10 = p \cdot 36$
Simplify.	$10 = 36p$
Divide both sides by 36.	$0.278 \approx p$
Change to percent form; round to the nearest tenth	$27.8\% \approx p$
Write a complete sentence to answer the question.	The new fees are approximately a 27.8% increase over the old fees.
Remember to round the division to the nearest thousandth in order to round the percent to the nearest tenth.	

Try It! 2.2.18

Find the percent change. (Round to the nearest tenth of a percent.) In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents.

Answer

8.8%

Try It! 2.2.19

Find the percent change. (Round to the nearest tenth of a percent.) In 1995, the standard bus fare in Chicago was $1.50. In 2008, the standard bus fare was 2.25.

Answer

50

Applications of discount and mark-up are very common in retail settings.

When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the **amount of the discount**. To determine the amount of discount, we multiply the discount rate by the original price.

The price a retailer pays for an item is called the **original cost**. The retailer then adds a **mark-up** to the original cost to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

DISCOUNT

$$\text{amount of discount} = \text{discount rate} \cdot \text{original price}$$
$$\text{sale price} = \text{original amount} - \text{discount price}$$

The sale price should always be less than the original price.

MARK-UP

$$\text{amount of mark-up} = \text{mark-up rate} \cdot \text{original price}$$
$$\text{list price} = \text{original cost} - \text{mark-up}$$

The list price should always be more than the original cost.

EXAMPLE 2.2.11

Liam's art gallery bought a painting at an original cost of $750. Liam marked the price up 40%. Find

a. the amount of mark-up and
b. the list price of the painting.

Solution:

a.

Identify what you are asked to find, and choose a variable to represent it.	What is the amount of mark-up? Let $m =$ the amount of mark-up.
Write a sentence that gives the information to find it.	The mark-up is 40% of the $750 original cost
Translate into an equation.	$m = 0.40 \times 750$
Solve the equation.	$m = 300$
Write a complete sentence.	The mark-up on the painting was $300.

b.

Identify what you are asked to find, and choose a variable to represent it.	What is the list price? Let $p =$ the list price.
Write a sentence that gives the information to find it.	The list price is original cost plus the mark-up
Translate into an equation.	$p = 750 + 300$
Solve the equation.	$p = 1{,}050$
Check.	Is the list price more than the original cost? Is $1,050 more than $750? Yes.
Write a complete sentence.	The list price of the painting was $1,050.

Try It! 2.2.20

Find a. the amount of mark-up and b. the list price: Jim's music store bought a guitar at original cost $1,200. Jim marked the price up 50%.

Answer

a. $600 b. $1,800

Try It! 2.2.21

Find a. the amount of mark-up and b. the list price: The Auto Resale Store bought Pablo's Toyota for $8,500. They marked the price up 35%.

Answer

a. $2,975 b. $11,475

Solve Simple Interest Applications

Interest is a part of our daily lives. From the interest earned on our savings to the interest we pay on a car loan or credit card debt, we all have some experience with interest in our lives.

The amount of money you initially deposit into a bank is called the **principal**, P, and the bank pays you interest, I. When you take out a loan, you pay interest on the amount you borrow, also called the principal.

In either case, the interest is computed as a certain percent of the principal, called the **rate of interest**, r. The rate of interest is usually expressed as a percent per year, and is calculated by using the decimal equivalent of the percent. The variable t, (for time) represents the number of years the money is saved or borrowed.

Interest is calculated as simple interest or compound interest. Here we will use simple interest.

SIMPLE INTEREST

If an amount of money, P, called the principal, is invested or borrowed for a period of t years at an annual interest rate r, the amount of interest, I, earned or paid is

$$I = Prt \qquad \text{where} \quad \begin{aligned} I &= \text{interest} \\ P &= \text{principal} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

Interest earned or paid according to this formula is called **simple interest**.

The formula we use to calculate interest is $I = Prt$. To use the formula we substitute in the values for variables that are given, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

EXAMPLE 2.2.12

Areli invested a principal of $950 in her bank account that earned simple interest at an interest rate of 3%. How much interest did she earn in five years?

Solution:

$I = ?$
$P = \$950$
$r = 3\%$
$t = 5$ years

Identify what you are asked to find, and choose a variable to represent it.	What is the simple interest? Let $I =$ interest.
Write the formula.	$I = Prt$
Substitute in the given information.	$I = (950)(0.03)(5)$
Simplify.	$I = 142.5$
Check. Is $142.50 a reasonable amount of interest on $ 950? Yes.	
Write a complete sentence.	The interest is $142.50.

Try It! 2.2.22

Nathaly deposited $12,500 in her bank account where it will earn 4% simple interest. How much interest will Nathaly earn in five years?

Answer

He will earn $2,500.

Try It! 2.2.23

Susana invested a principal of $36,000 in her bank account that earned simple interest at an interest rate of 6.5%. How much interest did she earn in three years?

Answer

She earned $7,020.

There may be times when we know the amount of interest earned on a given principal over a certain length of time, but we do not know the rate.

EXAMPLE 2.2.13

Hang borrowed $7,500 from her parents to pay her tuition. In five years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of simple interest?

Solution:

$I = \$1500$
$P = \$7500$
$r = ?$
$t = 5$ years

Identify what you are asked to find, and choose a variable to represent it.	What is rate of simple interest? Let r = rate of interest
Write the formula.	$I = Prt$
Substitute in the given information.	$1,500 = (7,500)r(5)$
Multiply.	$1,500 = 37,500r$
Divide.	$0.04 = r$
Change to percent form	$r = 4\%$

Check.

$$I = Prt$$
$$1,500 \stackrel{?}{=} (7,500)(0.04)(5)$$
$$1,500 = 1,500\checkmark$$

Write a complete sentence. The rate of interest was 4%.

Try It! 2.2.24

Jim lent his sister $5,000 to help her buy a house. In three years, she paid him the $5,000, plus $900 interest. What was the rate of simple interest?

Answer

The rate of simple interest was 6%.

Try It! 2.2.25

Loren lent his brother $3,000 to help him buy a car. In four years, his brother paid him back the $3,000 plus $660 in interest. What was the rate of simple interest?

Answer

The rate of simple interest was 5.5%.

In the next example, we are asked to find the principal—the amount borrowed.

EXAMPLE 2.2.14

Sean's new car loan statement said he would pay $4,866,25 in interest from a simple interest rate of 8.5% over five years. How much did he borrow to buy his new car?

Solution:

$$\begin{aligned} I &= 4,866.25 \\ P &= ? \\ r &= 8.5\% \\ t &= 5 \text{ years} \end{aligned}$$

Identify what you are asked to find, and choose a variable to represent it.	What is the amount borrowed (the principal)? Let P = principal borrowed
Write the formula.	$I = Prt$
Substitute in the given information.	$4,866.25 = P(0.085)(5)$
Multiply.	$4,866.25 = 0.425P$
Divide.	$11,450 = P$

Check.

$$\begin{aligned} I &= Prt \\ 4,866.25 &\stackrel{?}{=} (11,450)(0.085)(5) \\ 4,866.25 &= 4,866.25\checkmark \end{aligned}$$

Write a complete sentence. The principal was $\$11,450$.

Try It! 2.2.26

Eduardo noticed that his new car loan papers stated that with a 7.5% simple interest rate, he would pay $6,596.25 in interest over five years. How much did he borrow to pay for his car?

Answer

He paid $17,590.

Try It! 2.2.27

In five years, Gloria's bank account earned $2,400 interest at 5% simple interest. How much had she deposited in the account?

Answer

She deposited $9,600.

Access this online resource for additional instruction and practice with using a problem solving strategy.

- Begining Arithmetic Problems

Key Concepts

- **How To Use a Problem Solving Strategy for Word Problems**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what you are looking for.
  3. **Name** what you are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using proper algebra techniques.
  6. **Check** the answer in the problem to make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **How To Find Percent Change**
  1. Find the amount of change

     $$\text{change} = \text{new amount} - \text{original amount}$$

  2. Find what percent the amount of change is of the original amount.

     $$\text{change is what percent of the original amount?}$$

- $$\text{amount of discount} = \text{discount rate} \cdot \text{original price}$$
  $$\text{sale price} = \text{original amount} - \text{discount price}$$
- $$\text{amount of mark-up} = \text{mark-up rate} \cdot \text{original price}$$
  $$\text{list price} = \text{original cost} - \text{mark-up}$$
- If an amount of money, P, called the principal, is invested or borrowed for a period of t years at an annual interest rate r, the amount of interest, I, earned or paid is:

$$I = Prt \qquad \begin{array}{l} I = interest \\ \text{where } P = principal \\ r = rate \\ t = time \end{array}$$

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2.2E: Exercises

Practice Makes Perfect

Use a Problem Solving Strategy for Word Problems

1. List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.

Answer

Answers will vary.

2. List five negative thoughts that you have said to yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

3. There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

Answer

six boys

4. There are 18 Cub Scouts in Troop 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.

5. Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?

Answer

58 hardback books

6. Jeff is lining up children's and adult bicycles at the bike shop where he works. The number of children's bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children's bicycles are there?

Solve Number Word Problems

In the following exercises, solve each number word problem.

7. The difference of a number and 12 is three. Find the number.

Answer

15

8. The difference of a number and eight is four. Find the number.

9. The sum of three times a number and eight is 23. Find the number.

Answer

5

10. The sum of twice a number and six is 14. Find the number.

11. The difference of twice a number and seven is 17. Find the number.

Answer

12

12. The difference of four times a number and seven is 21. Find the number.

13. Three times the sum of a number and nine is 12. Find the number.

Answer

-5

14. Six times the sum of a number and eight is 30. Find the number.

15. One number is six more than the other. Their sum is 42. Find the numbers.

Answer

$18,\ 24$

16. One number is five more than the other. Their sum is 33. Find the numbers.

17. The sum of two numbers is 20. One number is four less than the other. Find the numbers.

Answer

$8,\ 12$

18. The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

19. One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

Answer

$32,\ 46$

20. One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

21. The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

Answer

$4,\ 10$

22. The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

23. The sum of two consecutive integers is 77. Find the integers.

Answer

$38,\ 39$

24. The sum of two consecutive integers is 89. Find the integers.

25. The sum of three consecutive integers is 78. Find the integers.

Answer

$25,\ 26,\ 27$

26. The sum of three consecutive integers is 60. Find the integers.

27. Find three consecutive integers whose sum is -36.

Answer

$-11,\ -12,\ -13$

28. Find three consecutive integers whose sum is -3.

29. Find three consecutive even integers whose sum is 258.

Answer

$84,\ 86,\ 88$

30. Find three consecutive even integers whose sum is 222.

31. Find three consecutive odd integers whose sum is -213.

Answer

$-69,\ -71,\ -73$

32. Find three consecutive odd integers whose sum is -267.

33. Philip pays $\$1,620$ in rent every month. This amount is $\$120$ more than twice what his brother Paul pays for rent. How much does Paul pay for rent?

Answer

$\$750$

34. Marc just bought an SUV for $\$54,000$. This is $\$7,400$ less than twice what his wife paid for her car last year. How much did his wife pay for her car?

35. Laurie has $\$46,000$ invested in stocks and bonds. The amount invested in stocks is $\$8,000$ less than three times the amount invested in bonds. How much does Laurie have invested in bonds?

Answer

$\$13,500$

36. Erica earned a total of $\$50,450$ last year from her two jobs. The amount she earned from her job at the store was $\$1,250$ more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Percent Applications

In the following exercises, translate and solve.

37. a. What number is 45% of 120? b. 81 is 75% of what number? c. What percent of 260 is 78?

Answer

a. 54
b. 108
c. 30%

38. a. What number is 65% of 100? b. 93 is 75% of what number? c. What percent of 215 is 86?

39. a. 250% of 65 is what number? b. 8.2% of what amount is $2.87? c. 30 is what percent of 20?

Answer

a. 162.5
b. $35
c. 150%

40. a. 150% of 90 is what number? b. 6.4% of what amount is $2.88? c. 50 is what percent of 40?

In the following exercises, solve.

41. Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

Answer

$11.88

42. When Hiro and his co-workers had lunch at a restaurant near their work, the bill was $90.50. They want to leave 18% of the total bill as a tip. How much should the tip be?

43. One serving of oatmeal has 8 grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

Answer

24.2 g

44. One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

45. A bacon cheeseburger at a popular fast food restaurant contains 2070 milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

Answer

2407 mg

46. A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

47. The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

Answer

45%

48. The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

49. Emma gets paid $3,000 per month. She pays $750 a month for rent. What percent of her monthly pay goes to rent?

Answer

25%

50. Dimple gets paid $3,200 per month. She pays $960 a month for rent. What percent of her monthly pay goes to rent?

In the following exercises, solve.

51. Tamanika received a raise in her hourly pay, from $15.50 to $17.36. Find the percent change.

Answer

12%

52. Ayodele received a raise in her hourly pay, from $24.50 to $25.48. Find the percent change.

53. Annual student fees at the University of California rose from about $4,000 in 2000 to about $12,000 in 2010. Find the percent change.

Answer

200%

54. The price of a share of one stock rose from $12.50 to $50. Find the percent change.

55. A grocery store reduced the price of a loaf of bread from $2.80 to $2.73. Find the percent change.

Answer

−2.5%

56. The price of a share of one stock fell from $8.75 to $8.54. Find the percent change.

57. Hernando's salary was $49,500 last year. This year his salary was cut to $44,055. Find the percent change.

Answer

−11%

58. In ten years, the population of Detroit fell from 950,000 to about 712,500. Find the percent change.

In the following exercises, find a. the amount of discount and b. the sale price.

59. Janelle bought a beach chair on sale at 60% off. The original price was $44.95.

Answer

a. $26.97
b. $17.98

60. Errol bought a skateboard helmet on sale at 40% off. The original price was $49.95.

In the following exercises, find a. the amount of discount and b. the discount rate (Round to the nearest tenth of a percent if needed.)

61. Larry and Donna bought a sofa at the sale price of $1,344. The original price of the sofa was $1,920.

Answer

a. $576
b. 30%

62. Hiroshi bought a lawnmower at the sale price of $240. The original price of the lawnmower is $300.

In the following exercises, find a. the amount of the mark-up and b. the list price.

63. Daria bought a bracelet at original cost $16 to sell in her handicraft store. She marked the price up 45%. What was the list price of the bracelet?

Answer

a. $7.20
b. $23.20

64. Regina bought a handmade quilt at original cost $120 to sell in her quilt store. She marked the price up 55%. What was the list price of the quilt?

65. Tom paid $0.60 a pound for tomatoes to sell at his produce store. He added a 33% mark-up. What price did he charge his customers for the tomatoes?

Answer

a. $0.20
b. $0.80

66. Flora paid her supplier $0.74 a stem for roses to sell at her flower shop. She added an 85% mark-up. What price did she charge her customers for the roses?

Solve Simple Interest Applications

In the following exercises, solve.

67. Casey deposited $1,450 in a bank account that earned simple interest at an interest rate of 4%. How much interest was earned in two years?

Answer

$116

68. Terrence deposited $5,720 in a bank account that earned simple interest at an interest rate of 6%. How much interest was earned in four years?

69. Robin deposited $31,000 in a bank account that earned simple interest at an interest rate of 5.2%. How much interest was earned in three years?

Answer

$4836

70. Carleen deposited $16,400 in a bank account that earned simple interest at an interest rate of 3.9% How much interest was earned in eight years?

71. Hilaria borrowed $8,000 from her grandfather to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of simple interest?

Answer

3%

72. Kenneth lent his niece $1,200 to buy a computer. Two years later, she paid him back the $1,200, plus $96 interest. What was the rate of simple interest?

73. Lebron lent his daughter $20,000 to help her buy a condominium. When she sold the condominium four years later, she paid him the $20,000, plus $3,000 interest. What was the rate of simple interest?

Answer

3.75%

74. Pablo borrowed $50,000 to start a business. Three years later, he repaid the $50,000, plus $9,375 interest. What was the rate of simple interest?

75. In 10 years, a bank account that paid 5.25% simple interest earned $18,375 interest. What was the principal of the account?

Answer

$35,000

76. In 25 years, a bond that paid 4.75% simple interest earned $2,375 interest. What was the principal of the bond?

77. Joshua's computer loan statement said he would pay $1,244.34 in simple interest for a three-year loan at 12.4%. How much did Joshua borrow to buy the computer?

Answer

$3345

78. Margaret's car loan statement said she would pay $7,683.20 in simple interest for a five-year loan at 9.8%. How much did Margaret borrow to buy the car?

Everyday Math

79. **Tipping** At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

Answer

17.5%

80. **Tipping** Four friends went out to lunch and the bill came to $53.75 They decided to add enough tip to make a total of $64, so that they could easily split the bill evenly among themselves. What percent tip did they leave?

Writing Exercises

81. What has been your past experience solving word problems? Where do you see yourself moving forward?

Answer

Answers will vary.

82. Without solving the problem “44 is 80% of what number” think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

83. After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

Answer

Answers will vary.

84. Because of road construction in one city, commuters were advised to plan that their Monday morning commute would take 150% of their usual commuting time. Explain what this means.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.

I can...	Confidently	With some help	No-I don't get it!
use a problem-solving strategy for word problems.			
solve number problems.			
solve percent applications.			
solve simple interest applications.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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2.3: Solve a Formula for a Specific Variable

Learning Objectives

By the end of this section, you will be able to:

- Solve a formula for a specific variable
- Use formulas to solve geometry applications

Before you get started, take this readiness quiz.

1. Evaluate $2(x+3)$ when $x=5$.
   If you missed this problem, review [link].
2. The length of a rectangle is three less than the width. Let *w* represent the width. Write an expression for the length of the rectangle.
   If you missed this problem, review [link].
3. Evaluate $\frac{1}{2}bh$ when $b=14$ and $h=9$.
   If you missed this problem, review [link].

Solve a Formula for a Specific Variable

We have all probably worked with some geometric formulas in our study of mathematics. Formulas are used in so many fields, it is important to recognize formulas and be able to manipulate them easily.

It is often helpful to solve a formula for a specific variable. If you need to put a formula in a spreadsheet, it is not unusual to have to solve it for a specific variable first. We isolate that variable on one side of the equals sign with a coefficient of one and all other variables and constants are on the other side of the equal sign.

Geometric formulas often need to be solved for another variable, too. The formula $V=\frac{1}{3}\pi r^2 h$ is used to find the **volume** of a right circular cone when given the radius of the base and height. In the next example, we will solve this formula for the height.

EXAMPLE 2.3.1

Solve the formula $V=\frac{1}{3}\pi r^2 h$ for *h*.

Answer

Write the formula.			$V=\frac{1}{3}\pi r^2 h$
Remove the fraction on the right.			$3\cdot V=3\cdot\frac{1}{3}\pi r^2 h$
Simplify.			$3V=\pi r^2 h$
Divide both sides by πr^2.			$\frac{3V}{\pi r^2}=h$

We could now use this formula to find the height of a right circular cone when we know the volume and the radius of the base, by using the formula $h=\frac{3V}{\pi r^2}$.

EXAMPLE 2.3.2:

Use the formula $A=\frac{1}{2}bh$ to solve for *b*.

Answer

$b=\frac{2A}{h}$

? EXAMPLE 2.3.3:

Use the formula $A = \frac{1}{2}bh$ to solve for h.

Answer

$h = \frac{2A}{b}$

In the sciences, we often need to change **temperature** from Fahrenheit to Celsius or vice versa. If you travel in a foreign country, you may want to change the Celsius temperature to the more familiar Fahrenheit temperature.

? EXAMPLE 2.3.4

Solve the formula $C = \frac{5}{9}(F - 32)$ for F.

Answer

Write the formula.			$C = \frac{5}{9}(F - 32)$
Remove the fraction on the right.			$\frac{9}{5}C = \frac{9}{5} \cdot \frac{5}{9}(F - 32)$
Simplify.			$\frac{9}{5}C = (F - 32)$
Add 32 to both sides.			$\frac{9}{5}C + 32 = F$

We can now use the formula $F = \frac{9}{5}C + 32$ to find the Fahrenheit temperature when we know the Celsius temperature.

? EXAMPLE 2.3.5:

Solve the formula $F = \frac{9}{5}C + 32$ for C.

Answer

$C = \frac{5}{9}(F - 32)$

? EXAMPLE 2.3.6:

Solve the formula $A = \frac{1}{2}h(b + B)$ for b.

Answer

$b = \frac{2A - Bh}{h}$

The next example uses the formula for the **surface area** of a right cylinder.

? EXAMPLE 2.3.7

Solve the formula $S = 2\pi r^2 + 2\pi rh$ for h.

Answer

Write the formula.	$S = 2\pi r^2 + 2\pi r h$
Isolate the h term by subtracting $2\pi r^2$ from each side.	$S - 2\pi r^2 = 2\pi r^2 - 2\pi r^2 + 2\pi r h$
Simplify.	$S - 2\pi r^2 = 2\pi r h$

Solve for h by dividing both sides by $2\pi r$.	$\frac{S-2\pi r^2}{2\pi r} = \frac{2\pi r h}{2\pi r}$
Simplify.	$\frac{S-2\pi r^2}{2\pi r} = h$

? EXAMPLE 2.3.8:

Solve the formula $A = P + Prt$ for t.

Answer

$t = \frac{A-P}{Pr}$

? EXAMPLE 2.3.9:

Solve the formula $A = P + Prt$ for r.

Answer

$r = \frac{A-P}{Pt}$

Sometimes we might be given an equation that is solved for y and need to solve it for x, or vice versa. In the following example, we're given an equation with both x and y on the same side and we'll solve it for y.

Solve the formula $8x + 7y = 15$ for y.

Answer

We will isolate y on one side of the equation.	$8x + 7y = 15$
Subtract $6x$ from both sides to isolate the term with y.	$8x - 8x + 7y = 15 - 8x$
Simplify.	$7y = 15 - 8x$
Divide both sides by 7 to make the coefficient of y one.	$\frac{7y}{7} = \frac{15-8x}{7}$
Simplify.	$y = \frac{15-8x}{7}$

Solve the formula $4x + 7y = 9$ for y.

Answer

$y = \frac{9-4x}{7}$

? EXAMPLE 2.3.12

Solve the formula $5x + 8y = 1$ for y.

Answer

$y = \frac{1-5x}{8}$

Use Formulas to Solve Geometry Applications

In this objective we will use some common geometry formulas. We will adapt our problem solving strategy so that we can solve geometry applications. The geometry formula will name the variables and give us the equation to solve.

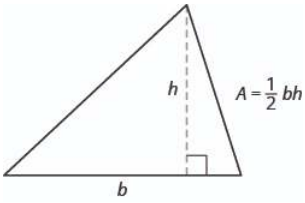
In addition, since these applications will all involve shapes of some sort, most people find it helpful to draw a figure and label it with the given information. We will include this in the first step of the problem solving strategy for geometry applications.

SOLVE GEOMETRY APPLICATIONS.

1. **Read** the problem and make sure all the words and ideas are understood.
2. **Identify** what you are looking for.
3. **Name** what we are looking for by choosing a variable to represent it. Draw the figure and label it with the given information.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

When we solve geometry applications, we often have to use some of the properties of the figures. We will review those properties as needed.

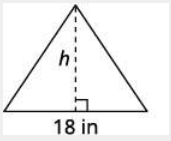
The next example involves the area of a triangle. The **area** of a triangle is one-half the base times the height. We can write this as $A = \frac{1}{2}bh$, where b = length of the base and h = height.



EXAMPLE 2.3.13

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	height of a triangle
Step 3. Name.	
Choose a variable to represent it.	Let $h =$ the height.
Draw the figure and label it with the given information.	Area = 126 sq. in.
	
Step 4. Translate.	
Write the appropriate formula.	$A = \frac{1}{2}bh$
Substitute in the given information.	$126 = \frac{1}{2} \cdot 18 \cdot h$
Step 5. Solve the equation.	$126 = 9h$

Divide both sides by 9.	$14 = h$
Step 6. Check. $A = \frac{1}{2}bh$ $126 \stackrel{?}{=} 12 \cdot 18 \cdot 14$ $126 = 126 \checkmark$	
Step 7. Answer the question.	The height of the triangle is 14 inches.

? EXAMPLE 2.3.14:

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

Answer

The window's height is 12 meters.

? EXAMPLE 2.3.15:

A triangular tent door has area 15 square feet. The height is five feet. What is the length of the base?

Answer

The length of the base is 6 feet.

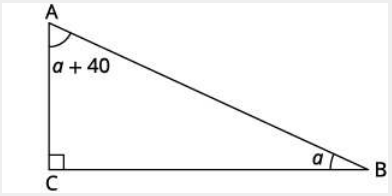
In the next example, we will work with a **right triangle**. To solve for the measure of each angle, we need to use two triangle properties. In any triangle, the sum of the measures of the angles is 180°. We can write this as a formula: $m\angle A + m\angle B + m\angle C = 180$. Also, since the triangle is a right triangle, we remember that a right triangle has one 90° angle.

Here, we will have to define one angle in terms of another. We will wait to draw the figure until we write expressions for all the angles we are looking for.

? EXAMPLE 2.3.16:

The measure of one angle of a right triangle is 40 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles
Step 3. Name. Choose a variable to represent it.	Let a = 1st angle. $a + 40$ = 2nd angle 90 = 3rd angle (the right angle)
Draw the figure and label it with the given information.	
Step 4. Translate.	
Write the appropriate formula.	$m\angle A + m\angle B + m\angle C = 180$

Substitute into the formula.	$a+(a+40)+90=180$
Step 5. Solve the equation.	$2a+130=180$ $2a=50$ $a=25$ first angle $a+40$ second angle $25+40$ 65 90 third angle
Step 6. Check. $25+65+90 \stackrel{?}{=} 180$ $180=180$✓	
Step 7. Answer the question.	The three angles measure 25°, 65°, and 90°.

? EXAMPLE 2.3.17:

The measure of one angle of a right triangle is 50 more than the measure of the smallest angle. Find the measures of all three angles.

Answer

The measures of the angles are 20°, 70°, and 90°.

? EXAMPLE 2.3.18:

The measure of one angle of a right triangle is 30 more than the measure of the smallest angle. Find the measures of all three angles.

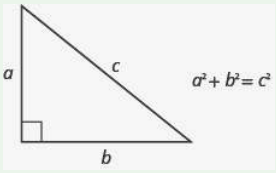
Answer

The measures of the angles are 30°, 60°, and 90°.

The next example uses another important geometry formula. The **Pythagorean Theorem** tells how the lengths of the three sides of a right triangle relate to each other. Writing the formula in every exercise and saying it aloud as you write it may help you memorize the Pythagorean Theorem.

THE PYTHAGOREAN THEOREM

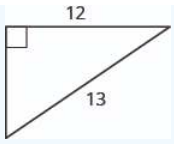
In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



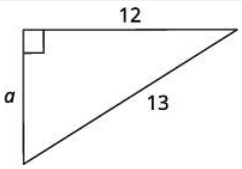
We will use the Pythagorean Theorem in the next example.

? EXAMPLE 2.3.19:

Use the Pythagorean Theorem to find the length of the other leg in

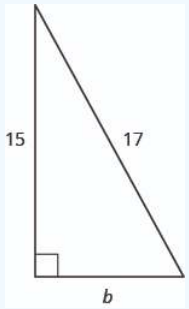


Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the leg of the triangle
Step 3. Name.	
Choose a variable to represent it.	Let a = the leg of the triangle.
Label side a.	
Step 4. Translate.	
Write the appropriate formula. Substitute.	$a^2+b^2=c^2$ $a^2+12^2=13^2$
Step 5. Solve the equation. Isolate the variable term. Use the definition of square root. Simplify.	$a^2+144=169$ $a^2=25$ $a=\sqrt{25}$ $a=5$
Step 6. Check. $5^2+12^2 \stackrel{?}{=} 13^2$ $25+144 \stackrel{?}{=} 169$ $169=169$ ✓	
Step 7. Answer the question.	The length of the leg is 5.

? EXAMPLE 2.3.20:

Use the Pythagorean Theorem to find the length of the leg in the figure.

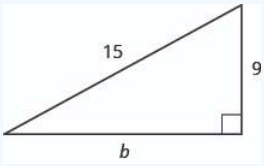


Answer

The length of the leg is 8.

EXAMPLE 2.3.21:

Use the Pythagorean Theorem to find the length of the leg in the figure.



Answer

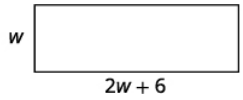
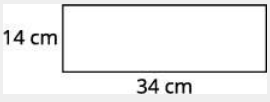
The length of the leg is 12.

The next example is about the **perimeter** of a rectangle. Since the perimeter is just the distance around the rectangle, we find the sum of the lengths of its four sides—the sum of two lengths and two widths. We can write is as $P = 2L + 2W$ where L is the length and W is the width. To solve the example, we will need to define the length in terms of the width.

EXAMPLE 2.3.22:

The length of a rectangle is six centimeters more than twice the width. The perimeter is 96 centimeters. Find the length and width.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and the width
Step 3. Name. Choose a variable to represent the width. The length is six more than twice the width.	Let $w =$ width. $2w + 6 =$ length  $P = 96$ cm
Step 4. Translate.	
Write the appropriate formula.	$P = 2L + 2W$
Substitute in the given information.	$96 = 2(2w + 6) + 2w$
Step 5. Solve the equation.	$96 = 4w + 12 + 2w$ $96 = 6w + 12$ $84 = 6w$ $14 = w$ (width) $2w + 6$ (length) $2(14) + 6$ 34 The length is 34 cm.
Step 6. Check.  $P = 2L + 2W$ $96 \stackrel{?}{=} 2 \cdot 34 + 2 \cdot 14$ $96 = 96$✓	
Step 7. Answer the question.	The length is 34 cm and the width is 14 cm

? EXAMPLE 2.3.23:

The length of a rectangle is seven more than twice the width. The perimeter is 110 inches. Find the length and width.

Answer

The length is 16 inches and the width is 39 inches.

? EXAMPLE 2.3.24:

The width of a rectangle is eight yards less than twice the length. The perimeter is 86 yards. Find the length and width.

Answer

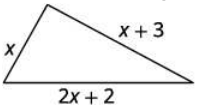
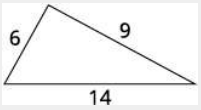
The length is 17 yards and the width is 26 yards.

The next example is about the **perimeter** of a triangle. Since the perimeter is just the distance around the triangle, we find the sum of the lengths of its three sides. We can write this as $P = a + b + c$, where a, b, and c are the lengths of the sides.

? EXAMPLE 2.3.25:

One side of a triangle is three inches more than the first side. The third side is two inches more than twice the first. The perimeter is 29 inches. Find the length of the three sides of the triangle.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the lengths of the three sides of a triangle
Step 3. Name. Choose a variable to represent the length of the first side.	Let $x =$ length of 1st side. $x+3$ = length of 2nd side $2x+2$ = length of 3rd side  Perimeter is 29 in.
Step 4. Translate. Write the appropriate formula. Substitute in the given information.	$P = a + b + c$ $29 = x + (x + 3) + (2x + 2)$
Step 5. Solve the equation.	$29 = 4x + 5$ $24 = 4x$ $6 = x$ length of first side $x + 3$ length of second side $6 + 3$ 9 $2x + 2$ length of second side $2 \cdot 6 + 2$ 14
Step 6. Check.  $29 \stackrel{?}{=} 6 + 9 + 14$ $29 = 29$✓	
Step 7. Answer the question.	The lengths of the sides of the triangle are 6, 9, and 14 inches.

EXAMPLE 2.3.26:

One side of a triangle is seven inches more than the first side. The third side is four inches less than three times the first. The perimeter is 28 inches. Find the length of the three sides of the triangle.

Answer

The lengths of the sides of the triangle are 5, 11 and 12 inches.

EXAMPLE 2.3.27:

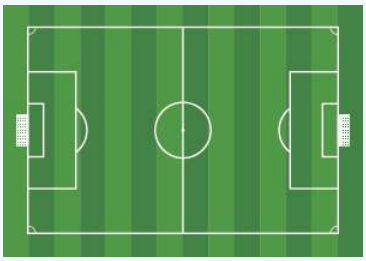
One side of a triangle is three feet less than the first side. The third side is five feet less than twice the first. The perimeter is 20 feet. Find the length of the three sides of the triangle.

Answer

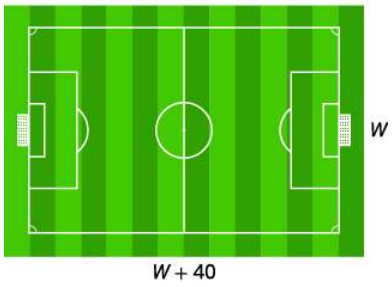
The lengths of the sides of the triangle are 4, 7 and 9 feet.

EXAMPLE 2.3.28:

The perimeter of a rectangular soccer field is 360 feet. The length is 40 feet more than the width. Find the length and width.



Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and width of the soccer field
Step 3. Name. Choose a variable to represent it. The length is 40 feet more than the width. Draw the figure and label it with the given information.	Let w = width. $w + 40 =$ length Perimeter = 360 feet 
Step 4. Translate. Write the appropriate formula and substitute.	$P = 2L + 2W$ $360 = 2(w + 40) + 2w$
Step 5. Solve the equation.	

Step 6. Check. $P = 2L + 2W$ $360 \stackrel{?}{=} 2(110) + 2(70)$ $360 = 360\checkmark$	$360 = 2w + 80 + 2w$ $360 = 4w + 80$ $280 = 4w$ $70 = w$ the width of the field $w + 40$ the length of the field $70 + 40$ 110
Step 7. Answer the question.	The length of the soccer field is 110 feet and the width is 70 feet.

? EXAMPLE 2.3.29:

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Answer

The length of the swimming pool is 70 feet and the width is 30 feet.

? EXAMPLE 2.3.30:

The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.

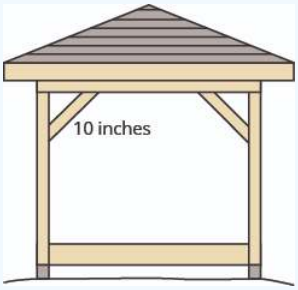
Answer

The length of the garden is 90 yards and the width is 60 yards.

Applications of these geometric properties can be found in many everyday situations as shown in the next example.

? EXAMPLE 2.3.31:

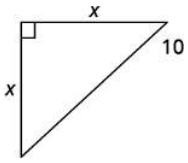
Kelvin is building a gazebo and wants to brace each corner by placing a 10” piece of wood diagonally as shown.



How far from the corner should he fasten the wood if wants the distances from the corner to be equal? Approximate to the nearest tenth of an inch.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the distance from the corner that the bracket should be attached

Step 3. Name. Choose a variable to represent it. Draw the figure and label it with the given information.	Let $x =$ the distance from the corner. 
Step 4. Translate. Write the appropriate formula and substitute.	$a^2+b^2=c^2 \quad x^2+x^2=10^2$
Step 5. Solve the equation. Isolate the variable. Use the definition of square root. Simplify. Approximate to the nearest tenth.	$2x^2=100$ $x^2=50$ $x=\sqrt{50}$ $x\approx 7.1$
Step 6. Check. $a^2+b^2=c^2$ $(7.1)^2+(7.1)^2\approx 10^2$ Yes.	
Step 7. Answer the question.	Kelvin should fasten each piece of wood approximately 7.1" from the corner.

? EXAMPLE 2.3.32:

John puts the base of a 13-foot ladder five feet from the wall of his house as shown in the figure. How far up the wall does the ladder reach?

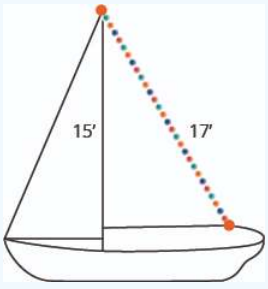


Answer

The ladder reaches 12 feet.

? EXAMPLE 2.3.33:

Randy wants to attach a 17-foot string of lights to the top of the 15 foot mast of his sailboat, as shown in the figure. How far from the base of the mast should he attach the end of the light string?



Answer

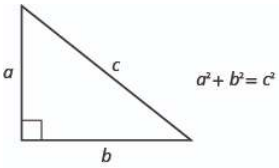
He should attach the lights 8 feet from the base of the mast.

Access this online resource for additional instruction and practice with solving for a variable in literal equations.

- Solving Literal Equations

Key Concepts

- **How To Solve Geometry Applications**
  1. **Read** the problem and make sure all the words and ideas are understood.
  2. **Identify** what you are looking for.
  3. **Name** what you are looking for by choosing a variable to represent it. Draw the figure and label it with the given information.
  4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **The Pythagorean Theorem**
  - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



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2.3E: Exercises

Practice Makes Perfect

Solve a Formula for a Specific Variable

In the following exercises, solve the given formula for the specified variable.

1. Solve the formula $C = \pi d$ for d.

Answer

$$d = \frac{C}{\pi}$$

2. Solve the formula $C = \pi d$ for π.

3. Solve the formula $V = LWH$ for L.

Answer

$$L = \frac{V}{WH}$$

4. Solve the formula $V = LWH$ for H.

5. Solve the formula $A = \frac{1}{2}bh$ for b.

Answer

$$b = \frac{2A}{h}$$

6. Solve the formula $A = \frac{1}{2}bh$ for h.

7. Solve the formula

$A = \frac{1}{2}d_1d_2$ for d_1.

Answer

$$d_1 = \frac{2A}{d_2}$$

8. Solve the formula

$A = \frac{1}{2}d_1d_2$ for d_2.

9. Solve the formula

$A = \frac{1}{2}h(b_1 + b_2)$ for b_1.

Answer

$$b_1 = \frac{2A}{h} - b_2$$

10. Solve the formula

$A = \frac{1}{2}h(b_1 + b_2)$ for b_2.

11. Solve the formula

$h = 54t + \frac{1}{2}at^2$ for a.

Answer

$$a = \frac{2h - 108t}{t^2}$$

12. Solve the formula

$h = 48t + \frac{1}{2}at^2$ for a.

13. Solve $180 = a + b + c$ for a.

Answer

$$a = 180 - b - c$$

14. Solve $180 = a + b + c$ for c.

15. Solve the formula

$A = \frac{1}{2}pI + B$ for p.

Answer

$$p = \frac{2A - 2B}{I}$$

16. Solve the formula

$A = \frac{1}{2}pI + B$ for I.

17. Solve the formula

$P = 2L + 2W$ for L.

Answer

$$L = \frac{P - 2W}{2}$$

18. Solve the formula

$P = 2L + 2W$ for W.

In the following exercises, solve for the formula for y.

19. Solve the formula

$8x + y = 15$ for y.

Answer

$$y = 15 - 8x$$

20. Solve the formula

$9x + y = 13$ for y.

21. Solve the formula

$-4x + y = -6$ for y.

Answer

$y = -6 + 4x$

22. Solve the formula

$-5x + y = -1$ for y.

23. Solve the formula

$x - y = -4$ for y.

Answer

$y = 4 + x$

24. Solve the formula

$x - y = -3$ for y.

25. Solve the formula

$4x + 3y = 7$ for y.

Answer

$y = \frac{7-4x}{3}$

26. Solve the formula

$3x + 2y = 11$ for y.

27. Solve the formula

$2x + 3y = 12$ for y.

Answer

$y = \frac{12-2x}{3}$

28. Solve the formula

$5x + 2y = 10$ for y.

29. Solve the formula

$3x - 2y = 18$ for y.

Answer

$y = \frac{18-3x}{-2}$

30. Solve the formula

$4x - 3y = 12$ for y.

Use Formulas to Solve Geometry Applications

In the following exercises, solve using a geometry formula.

31. A triangular flag has area 0.75 square feet and height 1.5 foot. What is its base?

Answer

1 foot

32. A triangular window has area 24 square feet and height six feet. What is its base?

33. What is the base of a triangle with area 207 square inches and height 18 inches?

Answer

23 inches

34. What is the height of a triangle with area 893 square inches and base 38 inches?

35. The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

Answer

$45^\circ, 45^\circ, 90^\circ$

36. The measure of the smallest angle of a right triangle is 20° less than the measure of the next larger angle. Find the measures of all three angles.

37. The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

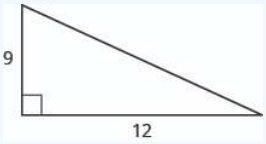
Answer

$30^\circ, 60^\circ, 90^\circ$

38. The angles in a triangle are such that one angle is 20 more than the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.

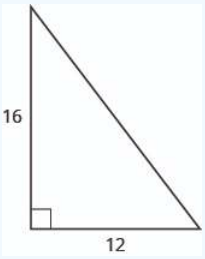
39.



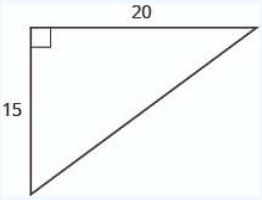
Answer

15

40.



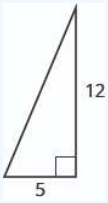
41.



Answer

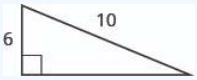
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42.



In the following exercises, use the Pythagorean Theorem to find the length of the unknown leg. Round to the nearest tenth if necessary.

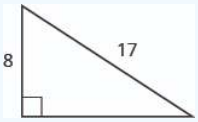
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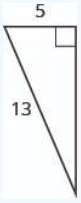
Answer

8

44.



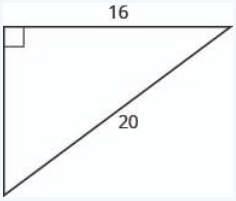
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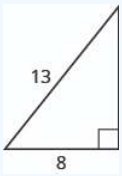
Answer

12

46.



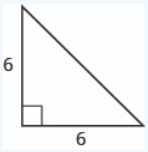
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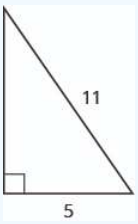
Answer

10.2

48.



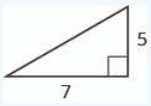
49.



Answer

9.8

50.



In the following exercises, solve using a geometry formula.

51. The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

Answer

18 meters, 11 meters

52. The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

53. The width of the rectangle is 0.7 meters less than the length. The perimeter of a rectangle is 52.6 meters. Find the dimensions of the rectangle.

Answer

13.5m, 12.8m

54. The length of the rectangle is 1.1 meters less than the width. The perimeter of a rectangle is 49.4 meters. Find the dimensions of the rectangle.

55. The perimeter of a rectangle of 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.

Answer

25 ft, 50 ft

56. The length of the rectangle is three times the width. The perimeter of a rectangle is 72 feet. Find the length and width of the rectangle.

57. The length of the rectangle is three meters less than twice the width. The perimeter of a rectangle is 36 meters. Find the dimensions of the rectangle.

Answer

7 m, 11 m

58. The length of a rectangle is five inches more than twice the width. The perimeter is 34 inches. Find the length and width.

59. The perimeter of a triangle is 39 feet. One side of the triangle is one foot longer than the second side. The third side is two feet longer than the second side. Find the length of each side.

Answer

12 ft, 13 ft, 14 ft

60. The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

61. One side of a triangle is twice the smallest side. The third side is five feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.

Answer

3 ft, 6 ft, 8 ft

62. One side of a triangle is three times the smallest side. The third side is three feet more than the shortest side. The perimeter is 13 feet. Find the lengths of all three sides.

63. The perimeter of a rectangular field is 560 yards. The length is 40 yards more than the width. Find the length and width of the field.

Answer

120 yd, 160 yd

64. The perimeter of a rectangular atrium is 160 feet. The length is 16 feet more than the width. Find the length and width of the atrium.

65. A rectangular parking lot has perimeter 250 feet. The length is five feet more than twice the width. Find the length and width of the parking lot.

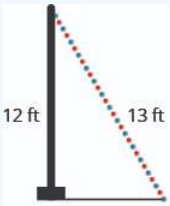
Answer

40 ft, 85 ft

66. A rectangular rug has perimeter 240 inches. The length is 12 inches more than twice the width. Find the length and width of the rug.

In the following exercises, solve. Approximate answers to the nearest tenth, if necessary.

67. A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display as shown. How far from the base of the pole should the end of the string of lights be anchored?



Answer

5 feet

68. Pam wants to put a banner across her garage door diagonally, as shown, to congratulate her son for his college graduation. The garage door is 12 feet high and 16 feet wide. How long should the banner be to fit the garage door?



69. Chi is planning to put a diagonal path of paving stones through her flower garden as shown. The flower garden is a square with side 10 feet. What will the length of the path be?



Answer

14.1 feet

70. Brian borrowed a 20-foot extension ladder to use when he paints his house. If he sets the base of the ladder six feet from the house as shown, how far up will the top of the ladder reach?



Everyday Math

71. Converting temperature While on a tour in Greece, Tatyana saw that the temperature was 40° Celsius. Solve for F in the formula $C = \frac{5}{9}(F - 32)$ to find the Fahrenheit temperature.

Answer

104°F

72. Converting temperature Yon was visiting the United States and he saw that the temperature in Seattle one day was 50° Fahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the Celsius temperature

73. Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are six feet, eight feet and 10 feet. How many feet of fencing will she need to enclose her flowerbed?

Answer

24 ft

74. Jose just removed the children's play set from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep the dog out. He has a 50-foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other side?

Writing Exercises

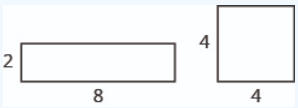
75. If you need to put tile on your kitchen floor, do you need to know the perimeter or the area of the kitchen? Explain your reasoning.

Answer

Answers will vary.

76. If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.

77. Look at the two figures below.



a. Which figure looks like it has the larger area? Which looks like it has the larger perimeter?

b. Now calculate the area and perimeter of each figure. Which has the larger area? Which has the larger perimeter?

c. Were the results of part (b) the same as your answers in part (a)? Is that surprising to you?

Answer

a. Answers will vary. b. The areas are the same. The 2×8 rectangle has a larger perimeter than the 4×4 square.

c. Answers will vary.

78. Write a geometry word problem that relates to your life experience, then solve it and explain all your steps.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve a formula for a specific variable.			
use formulas to solve geometry applications.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

More Practice

Solve a Formula for a Specific Variable

Solve each of the following equations for the indicated variable.

Exercise 79

$x - f = g$ for x

Answer

$x = f + g$

Exercise 80

$a + c = b$ for c

Exercise 81

$S = L + 2B$ for L

Answer

$L = S - 2B$

Exercise 82

$x + 5y = 3$ for x

Exercise 83

$ab = c$ for b

Answer

$b = \frac{c}{a}$

Exercise 84

$rt = d$ for r

Exercise 85

$DS = ds$ for D

Answer

$$D = \frac{ds}{S}$$

Exercise 86

$E = mc^2$ for m

Exercise 87

$3x = \frac{a}{b}$ for x

Answer

$$x = \frac{a}{3b}$$

Exercise 88

$lwh = V$ for w

Exercise 89

$at - bw = s$ for w

Answer

$$w = \frac{at - s}{b}$$

Exercise 90

$R = aT + b$ for T

Exercise 91

$S = \pi rh + \pi r^2$ for h

Answer

$$h = \frac{s - \pi r^2}{\pi r}$$

Exercise 92

$h = vt - 16t^2$ for v

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Exercise 93

$V = \frac{4}{3}\pi r^3$ for π

Answer

$$\pi = \frac{3V}{4r^3}$$

Exercise 94

$V = \frac{\pi Dn}{12}$ for D

Exercise 95

$V = \frac{\pi r^2 h}{3}$ for h

Answer

$$h = \frac{3v}{\pi r^2}$$

Exercise 96

$E = \frac{mv^2}{2}$ for m

Exercise 97

$p = \frac{3y}{q}$ for y

Answer

$$y = \frac{pq}{3}$$

Exercise 98

$\frac{ym}{b} = \frac{c}{d}$ for y

Exercise 99

$g = \frac{h}{i}$ for h

Answer

$$h = gi$$

Exercise 100

$\frac{f}{g}x = b$ for x

Exercise 101

$c = \dfrac{4y}{m+n}$ for y

Answer

$y = \dfrac{cm+cn}{4}$

Exercise 102

$\dfrac{rs}{a-3} = k$ for r

Exercise 103

$\dfrac{k-m}{r} = q$ for k

Answer

$k = qr+m$

Exercise 104

$T = \dfrac{D-d}{L}$ for D

Exercise 105

$I = \dfrac{E_a - E_q}{R}$ for E_a

Answer

$E_a = IR + E_q$

Exercise 106

$C = \dfrac{5}{9}(F-32)$ for F

Exercise 107

$P = n(p-c)$ for n

Answer

$n = \dfrac{P}{p-c}$

Exercise 108

$L = L_0(1+at)$ for L_0

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Exercise 109

$q = 6(L - p)$ for L

Answer

$$L = \frac{q + 6p}{6}$$

Exercise 110

$F = k(R - L)$ for k

Exercise 111

$Q_1 = P(Q_2 - Q_1)$ for Q_2

Answer

$$Q_2 = \frac{Q_1 + PQ_1}{P}$$

Exercise 112

$L = \pi(r_1 + r_2) + 2d$ for r_1

Exercise 113

$P = \frac{V_1(V_2 - V_1)}{g}$ for V_2

Answer

$$V_2 = \frac{Pg + V_1^2}{V_1}$$

Exercise 114

$R = \frac{kA(T_1 + T_2)}{d}$ for T_1

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CHAPTER OVERVIEW

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3.0: Introduction
3.1: Graph Linear Equations in Two Variables
3.1E: Exercises
3.2: Slope of a Line
3.2E: Exercises
3.3: Find the Equation of a Line
3.3E: Exercises
3.4: Graph Linear Inequalities in Two Variables
3.4E: Exercises
3.5: Relations and Functions
3.5E: Exercises
3.6: Graphs of Functions
3.6E: Exercises
Chapter 3 Review Exercises

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3.0: Introduction



Figure 3.0.1: This odd-looking headgear provides the user with a virtual world. (credit: fill/Pixabay)

Imagine visiting a faraway city or even outer space from the comfort of your living room. It could be possible using virtual reality. This technology creates realistic images that make you feel as if you are truly immersed in the scene and even enable you to interact with them. It is being developed for fun applications, such as video games, but also for architects to plan buildings, car companies to design prototypes, the military to train, and medical students to learn.

Developing virtual reality devices requires modeling the environment using graphs and mathematical relationships. In this chapter, you will graph different relationships and learn ways to describe and analyze graphs.

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3.1: Graph Linear Equations in Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Graph a linear equation by plotting points
- Graph vertical and horizontal lines
- Find the x- and y-intercepts
- Graph a line using the intercepts

Before you start

Before you get started, take this readiness quiz.

1. Evaluate $5x-4$ when $x=-1$.
   If you missed this problem, review [link].
2. Evaluate $3x-2y$ when $x=4, y=-3$.
   If you missed this problem, review [link].
3. Solve for y : $8-3y=20$.
   If you missed this problem, review [link].

Plot Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. The rectangular coordinate system is also called the xy-plane or the "coordinate plane."

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the x-axis. The vertical number line is called the y-axis. These axes divide a plane into four regions, called quadrants. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See Figure 3.1.1.

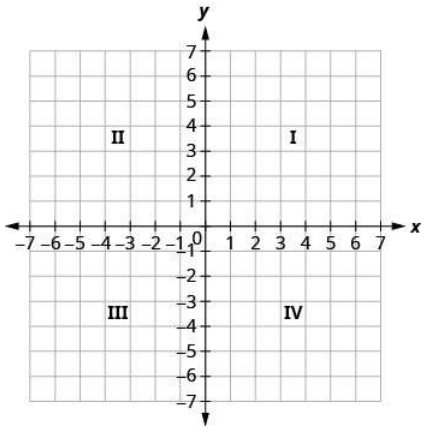
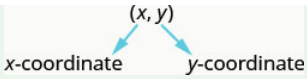


Figure 3.1.1

In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the x-coordinate of the point, and the second number is the y-coordinate of the point. The phrase "ordered pair" means that the order is important.

Ordered Pair

An **ordered pair**, (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the x-coordinate. The second number is the y-coordinate.



What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$.The point $(0, 0)$ has a special name. It is called the **origin**.

The Origin

The point $(0, 0)$ is called the **origin**. It is the point where the x-axis and y-axis intersect.

We use the coordinates to locate a point on the xy-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the x-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the y-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$. See Figure 3.1.2.

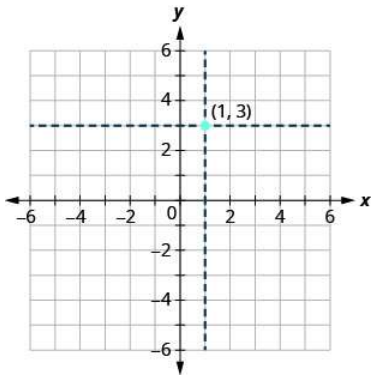


Figure 3.1.2

Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

When one of the coordinate is zero, the point lies on one of the axes. In Figure 3.1.3,the point $(0, 4)$ is on the y-axis and the point $(-2, 0)$ is on the x-axis.

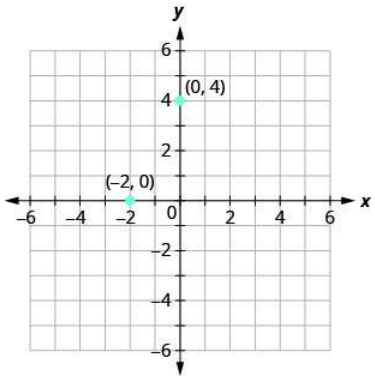


Figure 3.1.3

POINTS ON THE AXES

- Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
- Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.

Example 3.1.1

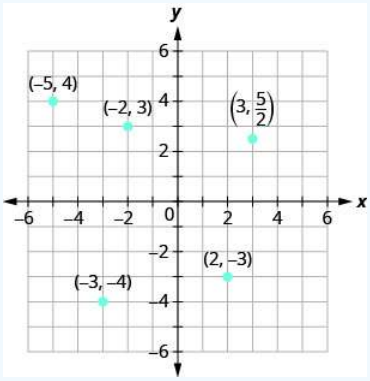
Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-5, 4)$ b. $(-3, -4)$ c. $(2, -3)$ d. $(0, -1)$ e. $(3, \frac{5}{2})$.

Solution

The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate. To plot each point, sketch a vertical line through the x-coordinate and a horizontal line through the y-coordinate. Their intersection is the point.

a. Since $x = -5$, the point is to the left of the y-axis. Also, since $y = 4$, the point is above the x-axis. The point $(-5, 4)$ is in Quadrant II.
b. Since $x = -3$, the point is to the left of the y-axis. Also, since $y = -4$, the point is below the x-axis. The point $(-3, -4)$ is in Quadrant III.
c. Since $x = 2$, the point is to the right of the y-axis. Since $y = -3$, the point is below the x-axis. The point $(2, -3)$ is in Quadrant IV.
d. Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the y-axis.
e. Since $x = 3$, the point is to the right of the y-axis. Since $y = \frac{5}{2}$), the point is above the x-axis. (It may be helpful to write $\frac{5}{2}$) as a mixed number or decimal.) The point $(3, \frac{5}{2})$ is in Quadrant I.

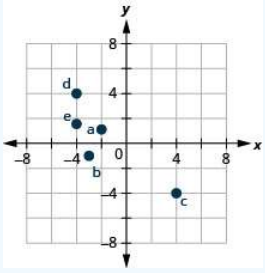


Try It! 3.1.1

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-2, 1)$ b. $(-3, -1)$ c. $(4, -4)$ d. $(-4, 4)$ e. $(-4, \frac{3}{2})$

Answer

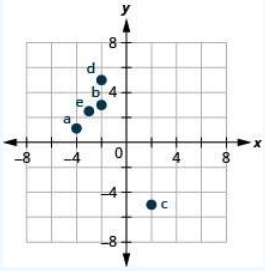


Try It! 3.1.2

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-4, 1)$ b. $(-2, 3)$ c. $(2, -5)$ d. $(-2, 5)$ e. $(-3, \frac{5}{2})$

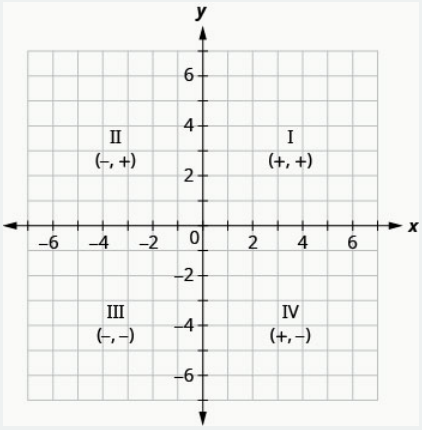
Answer



The signs of the x-coordinate and y-coordinate affect the location of the points. You may have noticed some patterns as you graphed the points in the previous example. We can summarize sign patterns of the quadrants in this way:

QUADRANTS

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$



Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. An equation of this form is called a **linear equation** in two variables.

Linear Equation

An equation of the form $Ax + By = C$, where A and B are not both zero, is called a **linear equation** in two variables.

Here is an example of a linear equation in two variables, x and y.

$$Ax + By = C$$

$$x + 4y = 8$$

$A = 1, B = 4, C = 8$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the Addition Property of Equality and rewrite it in $Ax + By = C$ form.

	y	$=$	$-3x+5$
Add to both sides.	$y+3x$	$=$	$3x+5+3x$
Simplify.	$y+3x$	$=$	5
Use the Commutative Property to put it in $Ax+By=C$ form.	$3x+y$	$=$	5

By rewriting $y=-3x+5$ as $3x+y=5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax+By=C$. When an equation is in the form $Ax+By=C$, we say it is in **standard form of a linear equation**.

Standard Form a of Linear Equation

A linear equation is in **standard form** when it is written $Ax+By=C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y-value. This pair of values is a **solution** to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Solution of a Linear Equation in Two Variables

An ordered pair (x, y) is a **solution** of the linear equation $Ax+By=C$, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.

Linear equations have infinitely many solutions. We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of the equation. It is an example of the saying, "A picture is worth a thousand words." The line shows you *all* the solutions to that equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the graph of the equation. Points *not* on the line are not solutions!

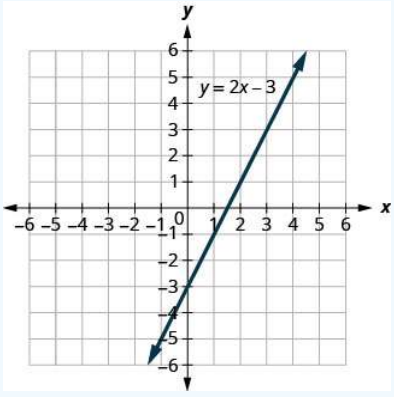
GRAPH OF A LINEAR EQUATION

The graph of a linear equation $Ax+By=C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Example 3.1.2

The graph of $y=2x-3$ is shown.



For each ordered pair, decide:

a. Is the ordered pair a solution to the equation?
b. Is the point on the line?

A: $(0, -3)$ B: $(3, 3)$ C: $(2, -3)$ D: $(-1, -5)$

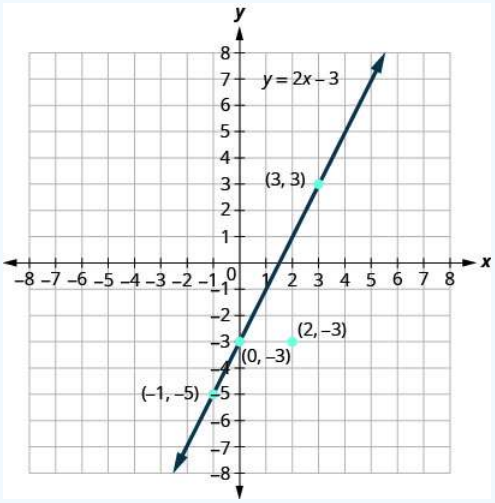
Solution:

Substitute the x- and y-values into the equation to check if the ordered pair is a solution to the equation.

a.

A: (0, –3)	B: (3, 3)	C: (2, –3)	D: (–1, –5)
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \stackrel{?}{=} 2(0) - 3$	$3 \stackrel{?}{=} 2(3) - 3$	$-3 \stackrel{?}{=} 2(2) - 3$	$-5 \stackrel{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
(0, –3) is a solution.	(3, 3) is a solution.	(2, –3) is not a solution.	(–1, –5) is a solution.

b. Plot the points $(0, -3)$, $(3, 3)$, $(2, -3)$, and $(-1, -5)$.



The points $(0, 3)$, $(3, -3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line.

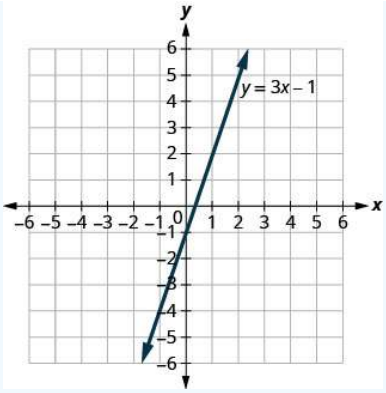
The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

Try It! 3.1.3

Use graph of $y = 3x - 1$. For each ordered pair, decide:

a. Is the ordered pair a solution to the equation?
b. Is the point on the line?

A $(0, -1)$ B $(2, 5)$



Answer

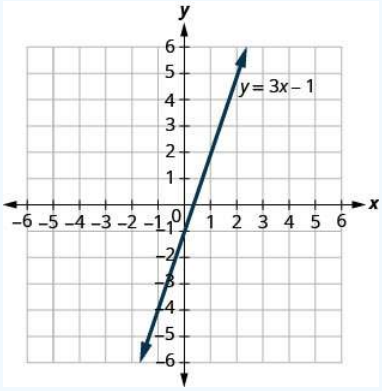
a. yes b. yes

? Try It! 3.1.4

Use graph of $y = 3x - 1$. For each ordered pair, decide:

a. Is the ordered pair a solution to the equation?
b. Is the point on the line?

$A(3, -1)\ B(-1, -4)$



Answer

a. no b. yes

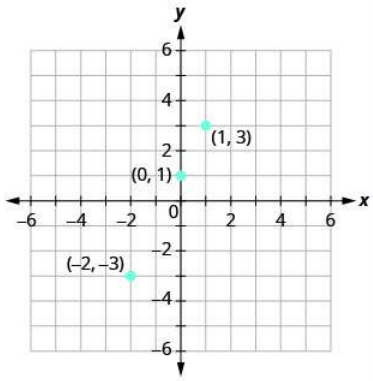
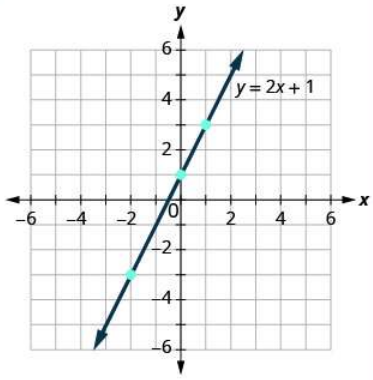
Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The first method we will use is called plotting points, or the Point-Plotting Method. We find three points whose coordinates are solutions to the equation and then plot them in a rectangular coordinate system. By connecting these points in a line, we have the graph of the linear equation.

✓ Example 3.1.3: How to Graph a Linear Equation by Plotting Points

Graph the equation $y = 2x + 1$ by plotting points.

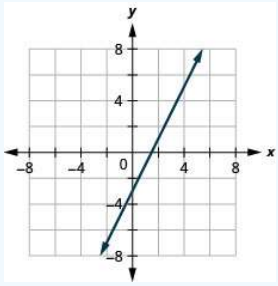
Solution:

Step 1. Find three points whose coordinates are solutions to the equation.	You can choose any values for x or y. In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.	$y = 2x + 1$ $x = 0$ $y = 2x + 1$ $y = 2 \cdot 0 + 1$ $y = 0 + 1$ $y = 1$ $x = 1$ $y = 2x + 1$ $y = 2 \cdot 1 + 1$ $y = 2 + 1$ $y = 3$ $x = -2$ $y = 2x + 1$ $y = 2(-2) + 1$ $y = -4 + 1$ $y = -3$
Organize the solutions in a table.	Put the three solutions in a table.	$y = 2x + 1$ x | y | (x, y) 0 | 1 | (0, 1) 1 | 3 | (1, 3) –2 | –3 | (–2, –3)
Step 2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!	Plot: (0, 1), (1, 3), (–2, –3). Do the points line up? Yes, the points line up.	
Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.	This line is the graph of $y = 2x + 1$.	

Try It! 3.1.5

Graph the equation by plotting points: $y = 2x - 3$.

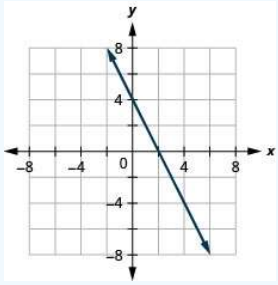
Answer



Try It! 3.1.6

Graph the equation by plotting points: $y = -2x + 4$.

Answer



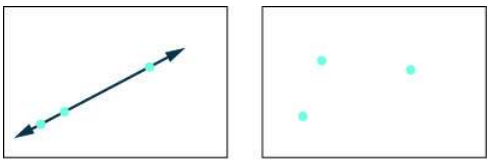
The steps to take when graphing a linear equation by plotting points are summarized here.

GRAPH A LINEAR EQUATION BY PLOTTING POINTS

1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between these illustrations.



(a) (b)

When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the arithmetic is easier if we make "good" choices for the values of x. This way we will avoid fractional answers, which are hard to graph precisely.

Example 3.1.4

Graph the equation: $y = \frac{1}{2}x + 3$.

Solution:

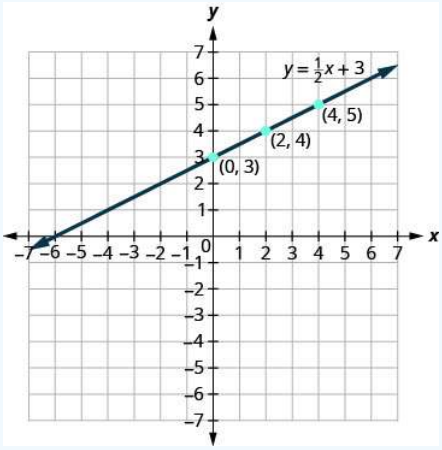
Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of x? By choosing multiples of 2 the multiplication by $\frac{1}{2}$ simplifies to a whole number

$x = 0$	$x = 2$	$x = 4$
$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$
$y = \frac{1}{2}(0) + 3$	$y = \frac{1}{2}(2) + 3$	$y = \frac{1}{2}(4) + 3$
$y = 0 + 3$	$y = 1 + 3$	$y = 2 + 3$
$y = 3$	$y = 4$	$y = 5$

The points are shown in Table.

$y = \frac{1}{2}x + 3$		
x	y	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

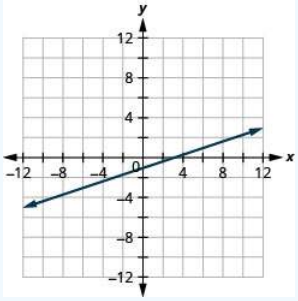
Plot the points, check that they line up, and draw the line.



Try It! 3.1.7

Graph the equation: $y = \frac{1}{3}x - 1$.

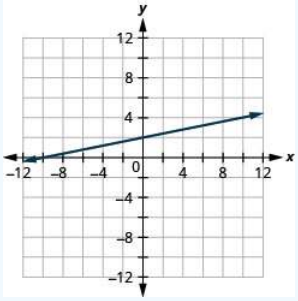
Answer



? Try It! 3.1.8

Graph the equation: $y = \frac{1}{4}x + 2$.

Answer



Graph Vertical and Horizontal Lines

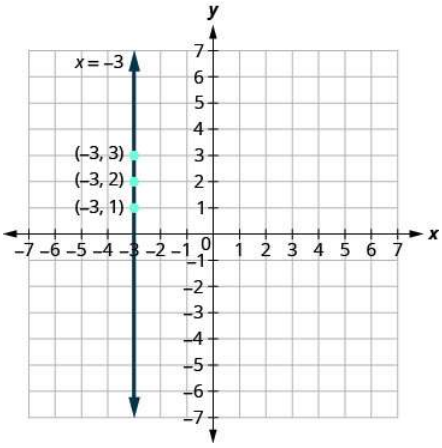
Some linear equations have only one variable. They may have just x and no y, or just y without an x. This changes how we make a table of values to get the points to plot.

Let's consider the equation $x = -3$. This equation has only one variable, x. The equation says that $x$is *always* equal to -3, so its value does not depend on y. No matter what is the value of y, the value of x is always -3.

So to make a table of values, write -3 in for all the x-values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the y-coordinates. See Table.

$x = -3$		
x	y	(x, y)
-3	1	$(-3, 1)$
-3	2	$(-3, 2)$
$(-3,)$	3	$(-3, 3)$

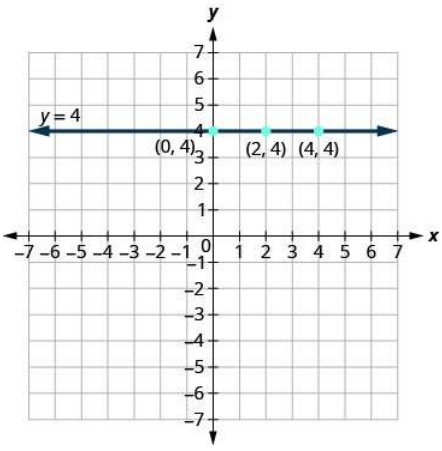
Plot the points from the table and connect them with a straight line. Notice that we have graphed a **vertical line**.



What if the equation has y but no x? Let's graph the equation $y = 4$. This time the y-value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y's in Table and then choose any values for x. We'll use 0, 2, and 4 for the x-coordinates.

$y = 4$		
x	y	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

In this figure, we have graphed a **horizontal line** passing through the y-axis at 4.



VERTICAL AND HORIZONTAL LINES

A **vertical line** is the graph of an equation of the form $x = a$.

The line passes through the x-axis at $(a, 0)$.

A **horizontal line** is the graph of an equation of the form $y = b$.

The line passes through the y-axis at $(0, b)$.

Example 3.1.5

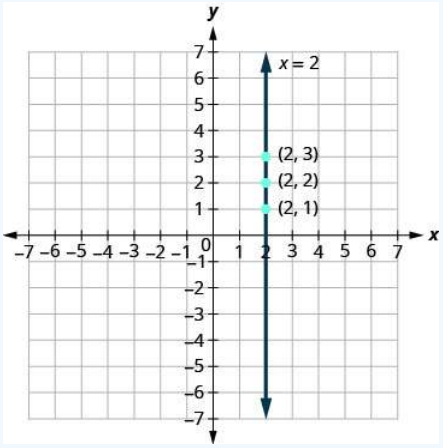
Graph: a. $x = 2$ b. $y = -1$.

Solution

a. The equation has only one variable, x, and x is always equal to 2. We create a table where x is always 2 and then put in any values for y. The graph is a vertical line passing through the x-axis at 2.

$$x = 2$$

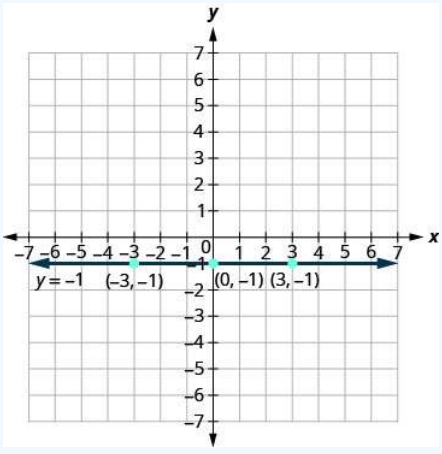
x	y	(x, y)
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$



b. Similarly, the equation $y = -1$ has only one variable, y. The value of y is constant. All the ordered pairs in the next table have the same y-coordinate. The graph is a horizontal line passing through the y-axis at -1.

$$y = -1$$

x	y	(x, y)
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$

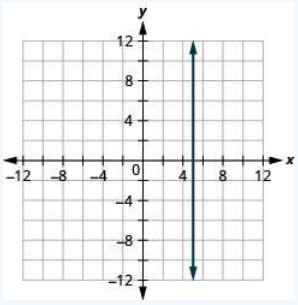


Try It! 3.1.9

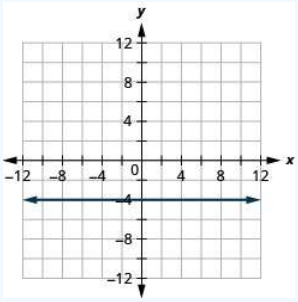
Graph the equations: a. $x = 5$ b. $y = -4$.

Answer

a.



b.

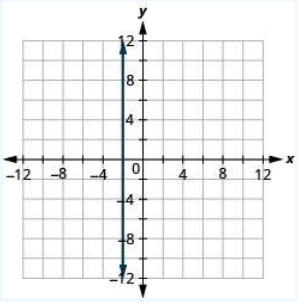


Try It! 3.1.10

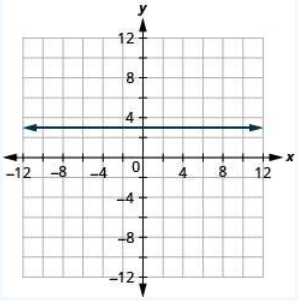
Graph the equations: a. $x = -2$ b. $y = 3$.

Answer

a.

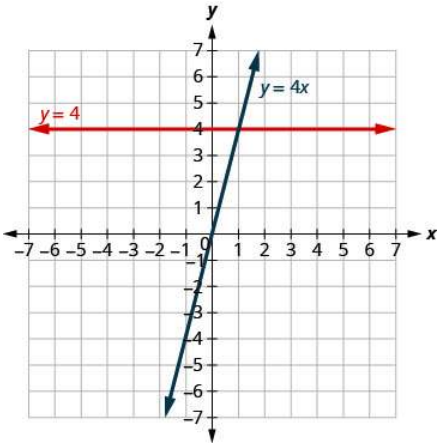


b.



What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both x and y. The value of y depends on the value of x, so the y-coordinate changes according to the value of x. The equation $y = 4$ has only one variable. The value of y is constant, it does not depend on the value of x, so the y-coordinate is always 4.



$y = 4x$		
x	y	(x, y)
0	0	(0, 0)
1	4	(1, 4)
2	8	(2, 8)

$y = 4$		
x	y	(x, y)
0	4	(0, 4)
1	4	(1, 4)
2	4	(2, 4)

Notice, in the graph, the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

Example 3.1.6

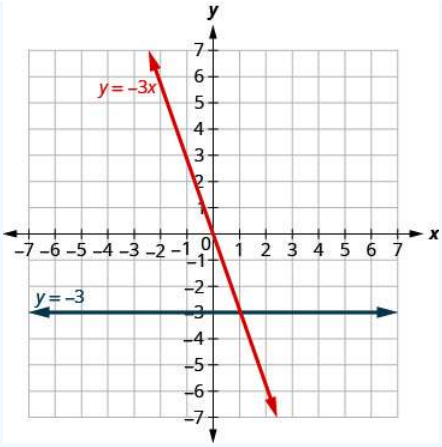
Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.

Solution:

We notice that the first equation has the variable x, while the second does not. We make a table of points for each equation and then graph the lines. The two graphs are shown.

$y = -3x$		
x	y	(x, y)
0	0	(0, 0)
1	–3	(1, –3)
2	–6	(2, –6)

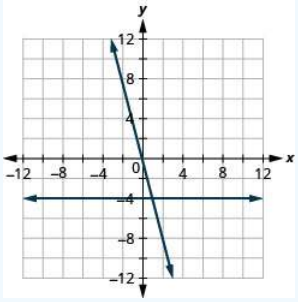
$y = -3$		
x	y	(x, y)
0	–3	(0, –3)
1	–3	(1, –3)
2	–3	(2, –3)



Try It! 3.1.11

Graph the equations in the same rectangular coordinate system: $y = -4x$ and $y = -4$.

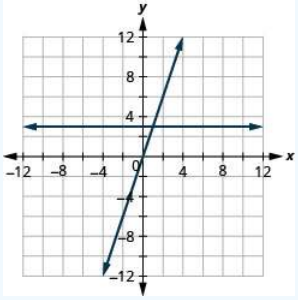
Answer



Try It! 3.1.12

Graph the equations in the same rectangular coordinate system: $y = 3$and $y = 3x$.

Answer



Find x- and y-intercepts

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

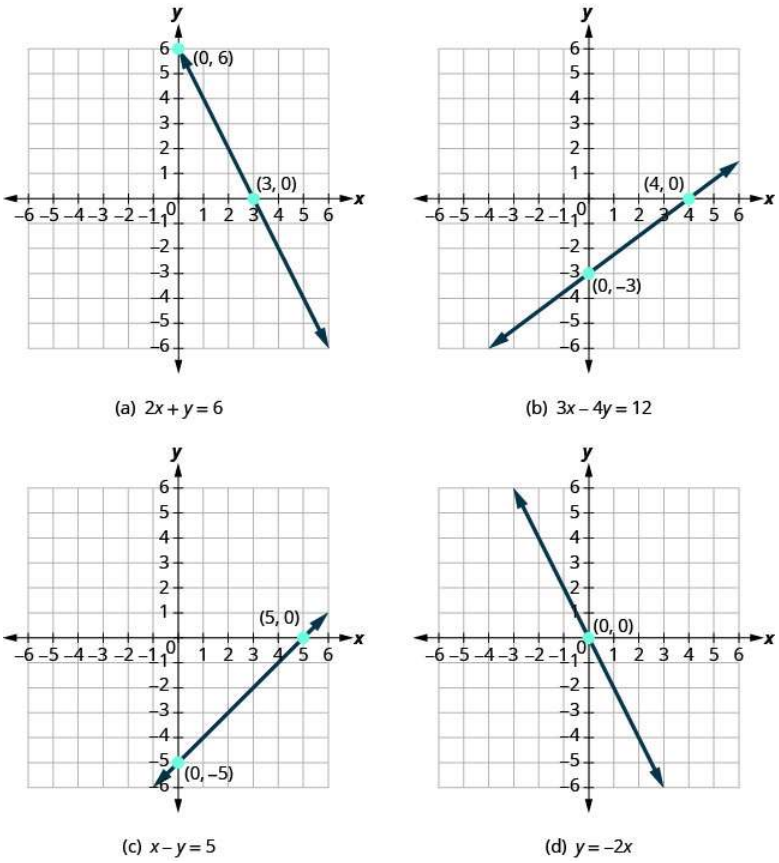
At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at

where the line crosses the x-axis and the y-axis. These points are called the **intercepts of a line**.

INTERCEPTS OF A LINE

The points where a line crosses the x-axis and the y-axis are called the **intercepts of the line**.

Let's look at the graphs of the lines.



(a) $2x + y = 6$ (b) $3x - 4y = 12$

(c) $x - y = 5$ (d) $y = -2x$

First, notice where each of these lines crosses the x-axis. See Table.

Now, let's look at the points where these lines cross the y-axis.

Figure	The line crosses the x-axis at:	Ordered pair for this point	The line crosses the y-axis at:	Ordered pair for this point
Figure (a)	3	$(3, 0)$	6	$(0, 6)$
Figure (b)	4	$(4, 0)$	-3	$(0, -3)$
Figure (c)	5	$(5, 0)$	-5	$(0, 5)$
Figure (d)	0	$(0, 0)$	0	$(0, 0)$
General Figure	a	$(a, 0)$	b	$(0, b)$

Do you see a pattern?

For each line, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form $(a, 0)$ and is called the *x-intercept* of the line. The x-intercept occurs when y is zero.

In each line, the x-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form $(0, b)$ and is called the *y-intercept* of the line. The y-intercept occurs when x is zero.

Intercepts of a Line

The x-intercept is the point $(a, 0)$ where the line crosses the x-axis.

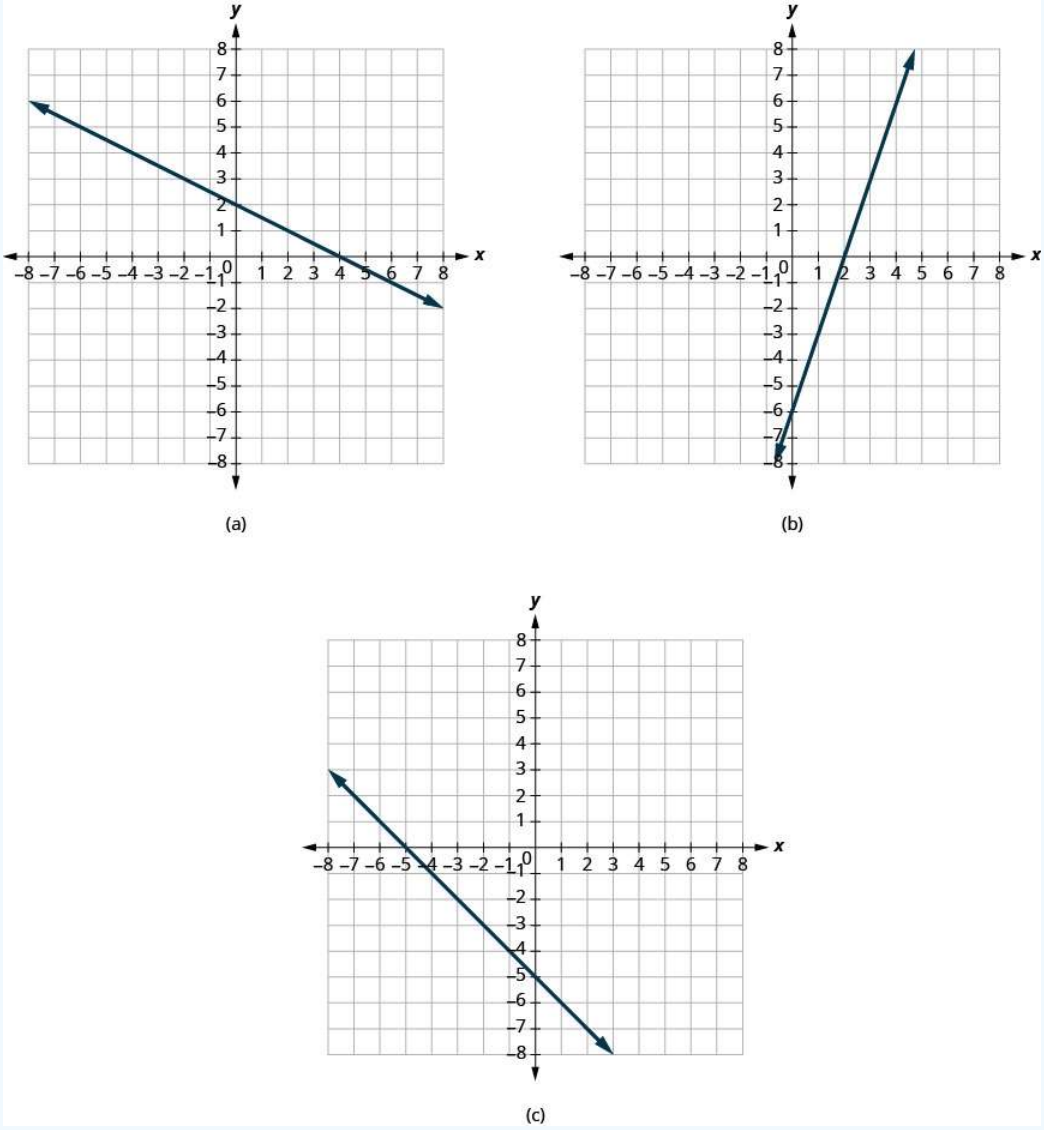
The y-intercept is the point $(0, b)$ where the line crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

x	y
a	0
0	b

Example 3.1.7

Find the x- and y-intercepts on each graph shown.



Solution:

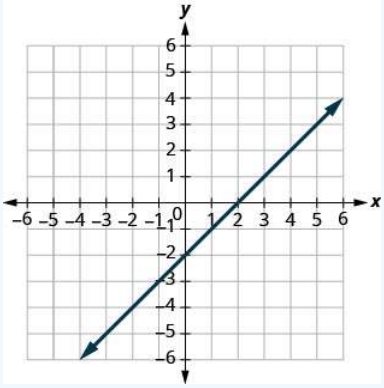
a. The graph crosses the x-axis at the point $(4, 0)$. The x-intercept is $(4, 0)$.
The graph crosses the y-axis at the point $(0, 2)$. The y-intercept is $(0, 2)$.

b. The graph crosses the x-axis at the point $(2, 0)$. The x-intercept is $(2, 0)$.
The graph crosses the y-axis at the point $(0, -6)$. The y-intercept is $(0, -6)$.

c. The graph crosses the x-axis at the point $(-5, 0)$. The x-intercept is $(-5, 0)$.
The graph crosses the y-axis at the point $(0, -5)$. The y-intercept is $(0, -5)$.

Try It! 3.1.13

Find the x- and y-intercepts on the graph.

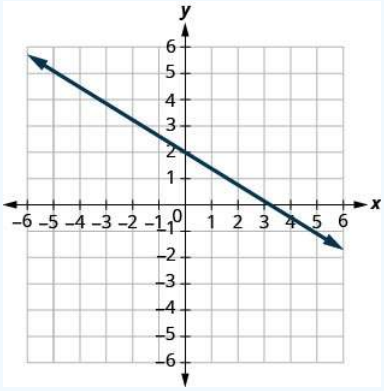


Answer

x-intercept: $(2, 0)$,
y-intercept: $(0, -2)$

Try It! 3.1.14

Find the x- and y-intercepts on the graph.



Answer

x-intercept: $(3, 0)$,
y-intercept: $(0, 2)$

Recognizing that the x-intercept occurs when y is zero and that the y-intercept occurs when x is zero, gives us a method to find the intercepts of a line from its equation. To find the x-intercept, let $y = 0$ and solve for x. To find the y-intercept, let $x = 0$ and solve for y.

Finding Intercepts from the Equation of a Line

Use the equation of the line. To find:

- the x-intercept of the line, let $y = 0$ and solve for x.
- the y-intercept of the line, let $x = 0$ and solve for y.

Example 3.1.8

Find the intercepts of $2x + y = 8$.

Solution:

We will let $y = 0$ to find the x-intercept, and let $x = 0$ to find the y-intercept. We will fill in a table, which reminds us of what we need to find.

$2x + y = 8$	
x	y
	0
0	

x-intercept
y-intercept

To find the x-intercept, let $y = 0$.	
	$2x + y = 8$
Let $y = 0$.	$2x + 0 = 8$
Simplify.	$2x = 8$
	$x = 4$
The x-intercept is:	$(4, 0)$
To find the y-intercept, let $x = 0$.	
	$2x + y = 8$
Let $x = 0$.	$2(0) + y = 8$
Simplify.	$0 + y = 8$
	$y = 8$
The y-intercept is:	$(0, 8)$

The intercepts are the points $(4, 0)$ and $(0, 8)$ as shown in the table.

$\mathbf{2x + y = 8}$	
$\boldsymbol{x}$	$\boldsymbol{y}$
4	0
0	8

Try It! 3.1.15

Find the intercepts: $3x + y = 12$.

Answer

x-intercept: $(4, 0)$,
y-intercept: $(0, 12)$

https://math.libretexts.org/@go/page/30823

Try It! 3.1.16

Find the intercepts: $x + 4y = 8$.

Answer

x-intercept: $(8, 0)$,
y-intercept: $(0, 2)$

Graph a Line Using the Intercepts

To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the *x*- and *y*- intercepts as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

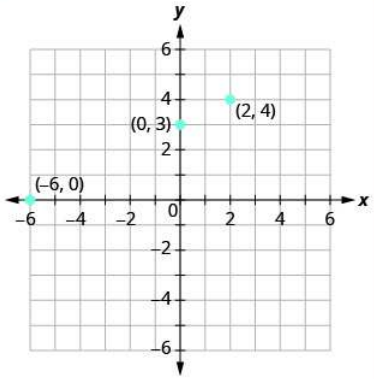
Example 3.1.9: How to Graph a Line Using the Intercepts

Graph $-x + 2y = 6$ using the intercepts.

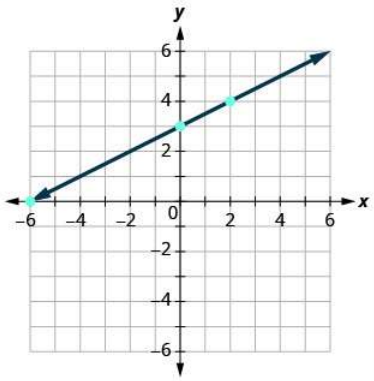
Solution:

Step 1. Find the *x*- and *y*-intercepts of the line.	Find the *x*-intercept.	Let $y = 0$
Let $y = 0$ and solve for *x*.		$-x + 2y = 6$
		$-x + 2(0) = 6$
		$-x = 6$
		$x = -6$
		The *x*-intercept is (–6, 0).
Let $x = 0$ and solve for *y*.	Find the *y*-intercept.	Let $x = 0$.
		$-x + 2y = 6$
		$-0 + 2y = 6$
		$2y = 6$
		$y = 3$
		The *y*-intercept is (0, 3).
Step 2. Find another solution to the equation.	We'll use $x = 2$.	Let $x = 2$.
		$-x + 2y = 6$
		$-2 + 2y = 6$
		$2y = 8$
		$y = 4$
		A third point is (2, 4).

Step 3. Plot the three points. Check that the points line up.	x	y	(x, y)
	-6	0	$(-6, 0)$
	0	3	$(0, 3)$
	2	4	$(2, 4)$



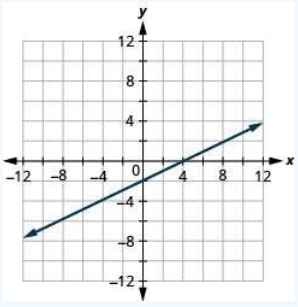
Step 4. Draw the line.	See the graph.



? Try It! 3.1.17

Graph using the intercepts: $x-2y=4$.

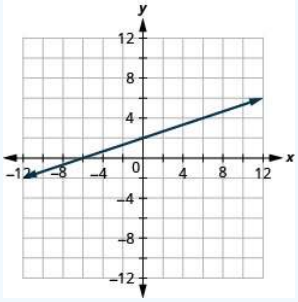
Answer



? Try It! 3.1.18

Graph using the intercepts: $-x+3y=6$.

Answer



The steps to graph a linear equation using the intercepts are summarized here.

GRAPH A LINEAR EQUATION USING THE INTERCEPTS

1. Find the x- and y-intercepts of the line.
   - Let y=0y=0 and solve for x.
   - Let x=0x=0 and solve for y.
2. Find a third solution to the equation.
3. Plot the three points and check that they line up.
4. Draw the line.

Example 3.1.10

Graph $4x - 3y = 12$ using the intercepts.

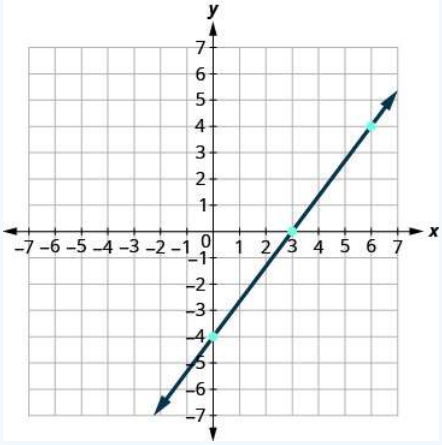
Solution:

Find the intercepts and a third point.

x-intercept, let $y = 0$	y-intercept, let $x = 0$	third point, let $y = 4$
$4x - 3y = 12$	$4x - 3y = 12$	$4x - 3y = 12$
$4x - 3(0) = 12$	$4(0) - 3y = 12$	$4x - 3(4) = 12$
$4x = 12$	$-3y = 12$	$4x - 12 = 12$
$x = 3$	$y = -4$	$4x = 24$
		$x = 6$

We list the points in the table and show the graph.

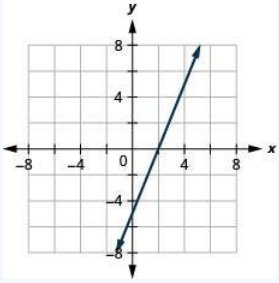
$4x - 3y = 12$		
x	y	(x, y)
3	0	$(3, 0)$
0	-4	$(0, -4)$
6	4	$(6, 4)$



Try It! 3.1.19

Graph using the intercepts: $5x - 2y = 10$.

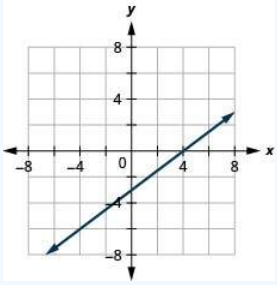
Answer



Try It! 3.1.20

Graph using the intercepts: $3x - 4y = 12$.

Answer



When the line passes through the origin, the x-intercept and the y-intercept are the same point.

Example 3.1.11

Graph $y = 5x$ using the intercepts.

Solution:

x-intercept	y-intercept
Let $y = 0$.	Let $x = 0$.
$y = 5x$	$y = 5x$
$0 = 5x$	$y = 5 \cdot 0$
$0 = x$	$y = 0$
$(0, 0)$	$(0, 0)$

This line has only one intercept. It is the point $(0, 0)$.

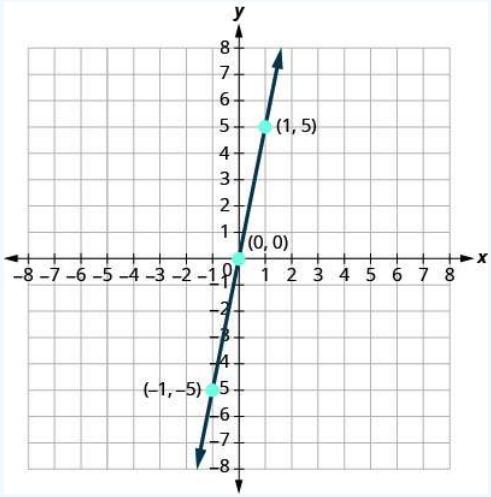
To ensure accuracy, we need to plot three points. Since the x- and y-intercepts are the same point, we need *two* more points to graph the line.

Let $x = 1$.	Let $x = -1$.
$y = 5x$	$y = 5x$
$y = 5 \cdot 1$	$y = 5(-1)$
$y = 5$	$y = -5$

The resulting three points are summarized in the table.

$y = 5x$		
x	y	(x, y)
0	0	$(0, 0)$
1	5	$(1, 5)$
-1	-5	$(-1, -5)$

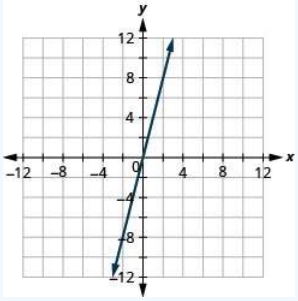
Plot the three points, check that they line up, and draw the line.



? Try It! 3.1.21

Graph using the intercepts: $y = 4x$.

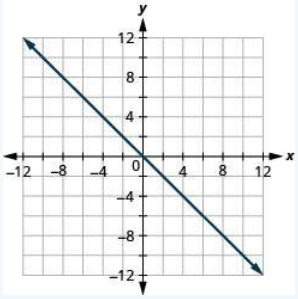
Answer



? Try It! 3.1.22

Graph the intercepts: $y = -x$.

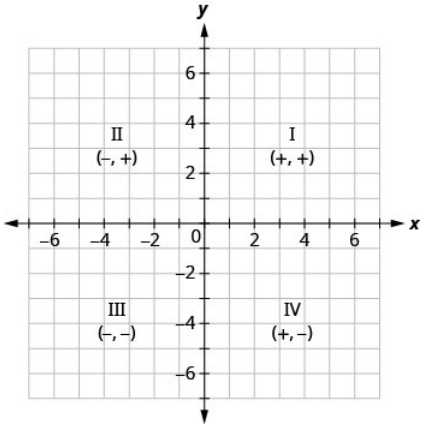
Answer



Key Concepts

- **Points on the Axes**
  - Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
  - Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.
- **Quadrant**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$



- **Graph of a Linear Equation:** The graph of a linear equation $Ax + By = C$ is a straight line. Every point on the line is a solution of the equation.

Every solution of this equation is a point on this line.

- **How to graph a linear equation by plotting points.**
  1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
  2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
  3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.
- **x-intercept and y-intercept of a Line**
  - The x-intercept is the point $(a, 0)$ where the line crosses the x-axis.
  - The y-intercept is the point $(0, b)$ where the line crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

x	y
a	0
0	b

- **Find the x- and y-intercepts from the Equation of a Line**
  - Use the equation of the line. To find:
    the x-intercept of the line, let $y = 0$ and solve for x.
    the y-intercept of the line, let $x = 0$ and solve for y.
- **How to graph a linear equation using the intercepts.**
  1. Find the x- and y-intercepts of the line.
     Let $y = 0$ and solve for x.
     Let $x = 0$ and solve for y.
  2. Find a third solution to the equation.
  3. Plot the three points and check that they line up.
  4. Draw the line.

Glossary

horizontal line

A horizontal line is the graph of an equation of the form $y = b$. The line passes through the y-axis at $(0, b)$.

intercepts of a line

The points where a line crosses the x-axis and the y-axis are called the intercepts of the line.

linear equation

An equation of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation in two variables.

ordered pair

An ordered pair, (x, y), gives the coordinates of a point in a rectangular coordinate system. The first number is the x-coordinate. The second number is the y-coordinate.

origin

The point $(0, 0)$ is called the origin. It is the point where the x-axis and y-axis intersect.

solution of a linear equation in two variables

An ordered pair (x, y) is a solution of the linear equation $Ax + By = C$, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.

standard form of a linear equation

A linear equation is in standard form when it is written $Ax + By = C$.

vertical line

A vertical line is the graph of an equation of the form $x = a$. The line passes through the x-axis at $(a, 0)$.

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3.1E: Exercises

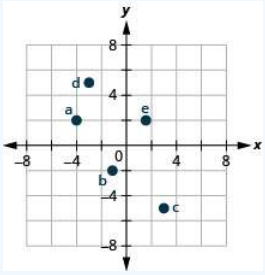
Practice Makes Perfect

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

1. a. $(-4, 2)$ b. $(-1, -2)$ c. $(3, -5)$ d. $(-3, 0)$
e. $(53, 2)$

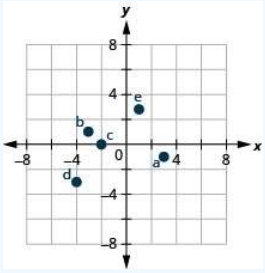
Answer



2. a. $(-2, -3)$ b. $(3, -3)$ c. $(-4, 1)$ d. $(4, -1)$
e. $(32, 1)$

3. a. $(3, -1)$ b. $(-3, 1)$ c. $(-2, 0)$ d. $(-4, -3)$
e. $(1, 145)$

Answer



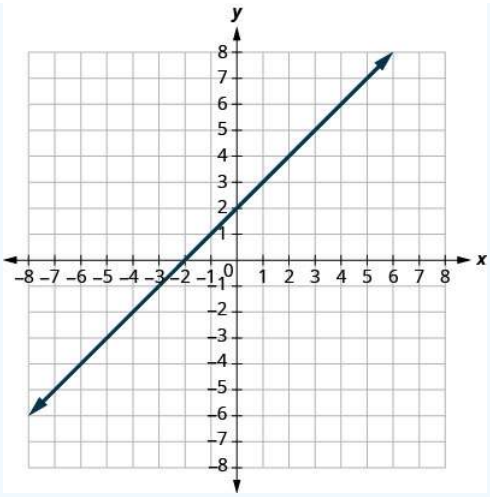
4. a. $(-1, 1)$ b. $(-2, -1)$ c. $(2, 0)$ d. $(1, -4)$
e. $(3, 72)$

In the following exercises, for each ordered pair, decide

a. is the ordered pair a solution to the equation? b. is the point on the line?

5. $y = x + 2$;

A: $(0, 2)$; B: $(1, 2)$; C: $(-1, 1)$; D: $(-3, -1)$.

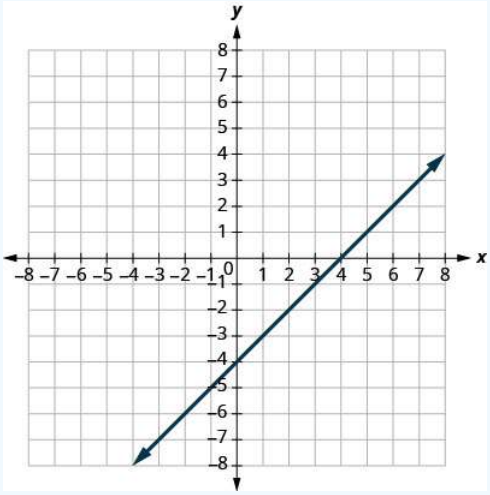


Answer

a. A: yes, B: no, C: yes, D: yes b. A: yes, B: no, C: yes, D: yes

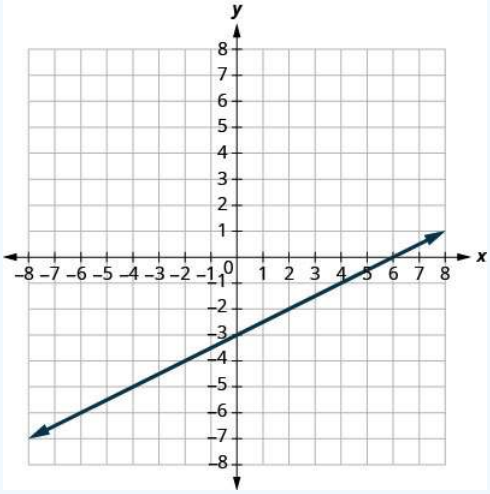
6. $y = x - 4$;

A: $(0, -4)$; B: $(3, -1)$; C: $(2, 2)$; D: $(1, -5)$.



7. $y = 12x - 3$;

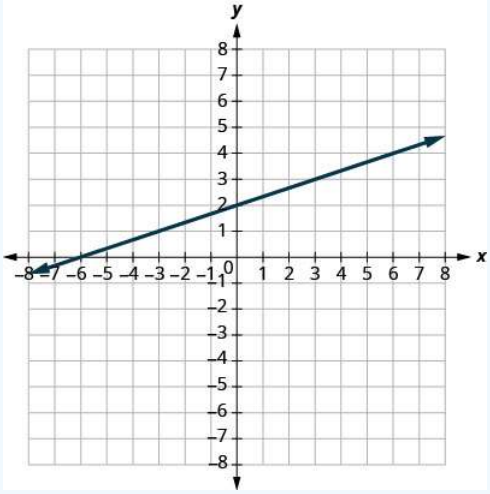
A: $(0, -3)$; B: $(2, -2)$; C: $(-2, -4)$; D: $(4, 1)$.



Answer

a. A: yes, B: yes, C: yes, D: no b. A: yes, B: yes, C: yes, D: no

8. $y = 13x + 2$;
A: $(0, 2)$; B: $(3, 3)$; C: $(-3, 2)$; D: $(-6, 0)$.

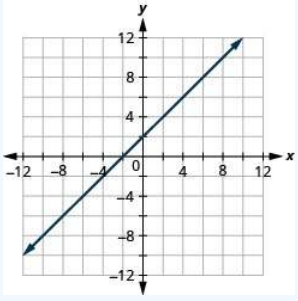


Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

9. $y = x + 2$

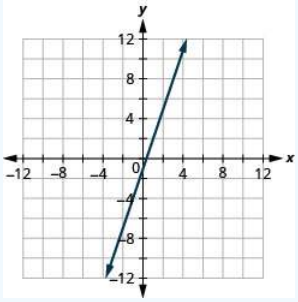
Answer



10. $y = x - 3$

11. $y = 3x - 1$

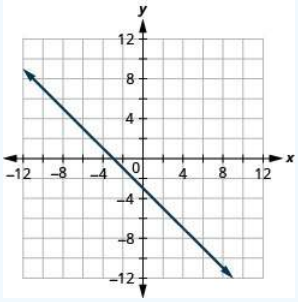
Answer



12. $y = -2x + 2$

13. $y = -x - 3$

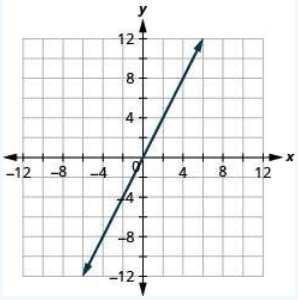
Answer



14. $y = -x - 2$

15. $y = 2x$

Answer

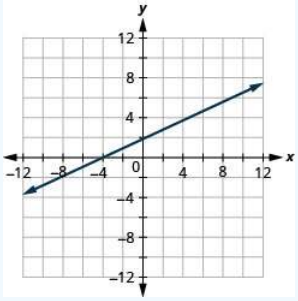


16. $y = -2x$

17. $y = 12x + 2$

Answer

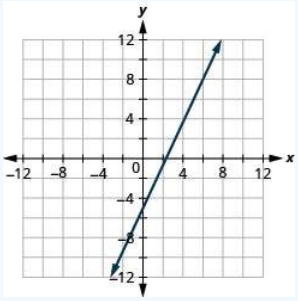
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18. $y = 13x - 1$

19. $y = 43x - 5$

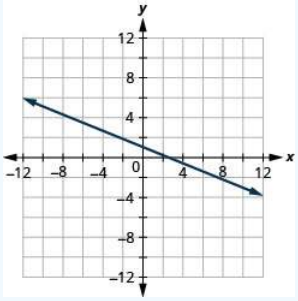
Answer



20. $y = 32x - 3$

21. $y = -25x + 1$

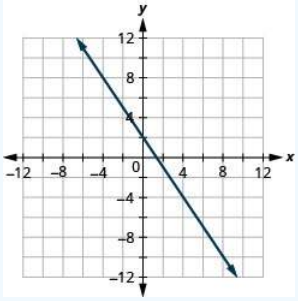
Answer



22. $y = -45x - 1$

23. $y = -32x + 2$

Answer



24. $y = -53x + 4$

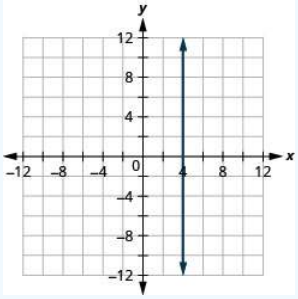
Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

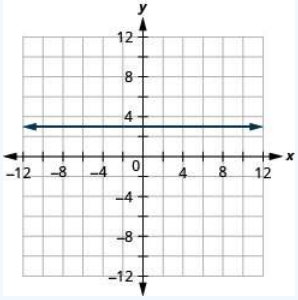
25. a. $x = 4$ b. $y = 3$

Answer

a.



b.



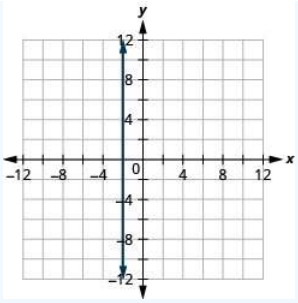
26. a. $x = 3$ b. $y = 1$

27. a. $x = -2$ b. $y = -5$

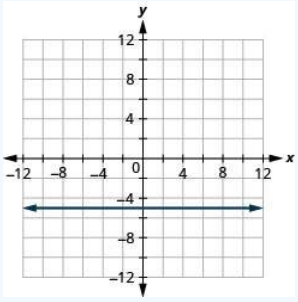
Answer

a.

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b.

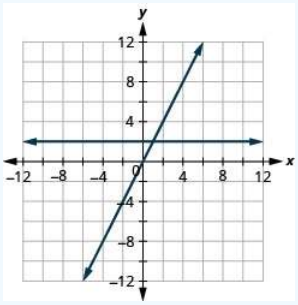


28. a. $x = -5$ b. $y = -2$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

29. $y = 2x$ and $y = 2$

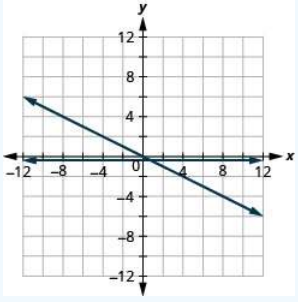
Answer



30. $y = 5x$ and $y = 5$

31. $y = -12x$ and $y = -12$

Answer

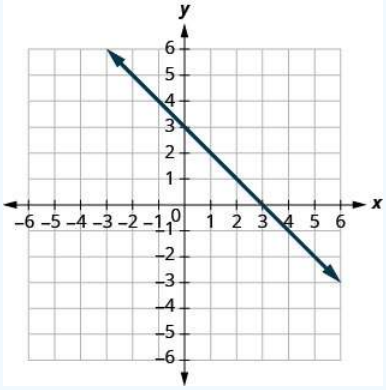


32. $y = -13x$ and $y = -13$

Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts on each graph.

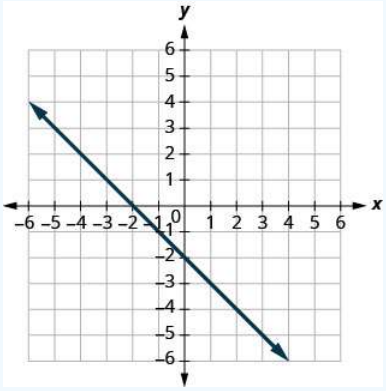
33.



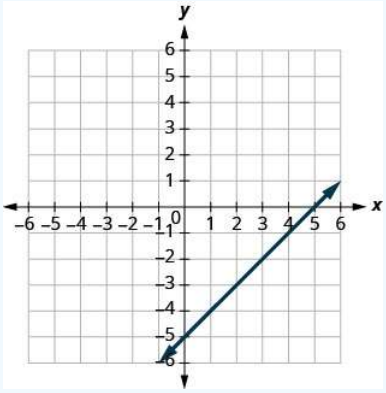
Answer

$(3, 0), (0, 3)$

34.



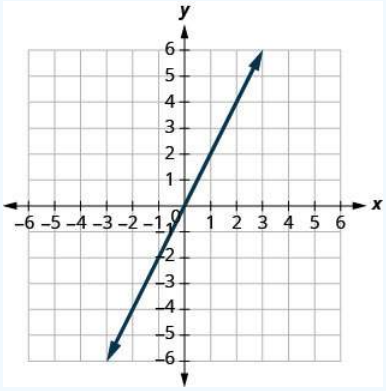
35.



Answer

$(5, 0), (0, -5)$

36.



In the following exercises, find the intercepts for each equation.

37. $x - y = 5$

Answer

x-int: $(5, 0)$, y-int: $(0, -5)$

38. $x - y = -4$

39. $3x + y = 6$

Answer

x-int: $(2, 0)$, y-int: $(0, 6)$

40. $x - 2y = 8$

41. $4x - y = 8$

Answer

x-int: $(2, 0)$, y-int: $(0, -8)$

42. $5x - y = 5$

43. $2x + 5y = 10$

Answer

x-int: $(5, 0)$, y-int: $(0, 2)$

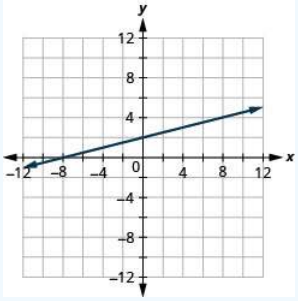
44. $3x - 2y = 12$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

45. $-x + 4y = 8$

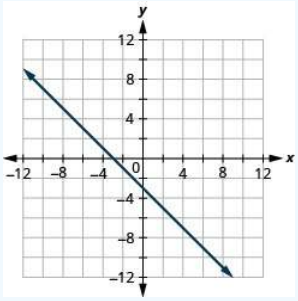
Answer



46. $x + 2y = 4$

47. $x + y = -3$

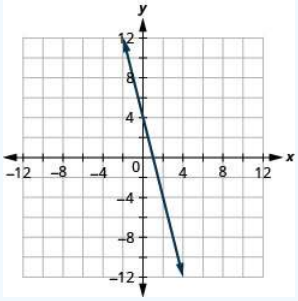
Answer



48. $x - y = -4$

49. $4x + y = 4$

Answer

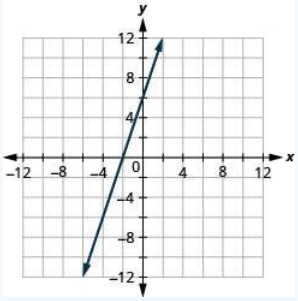


50. $3x + y = 3$

51. $3x - y = -6$

Answer

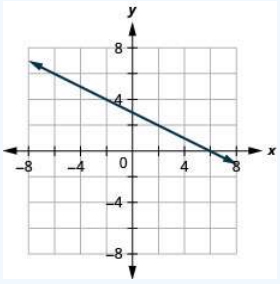
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52. $2x - y = -8$

53. $2x + 4y = 12$

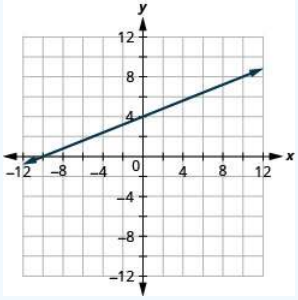
Answer



54. $3x - 2y = 6$

55. $2x - 5y = -20$

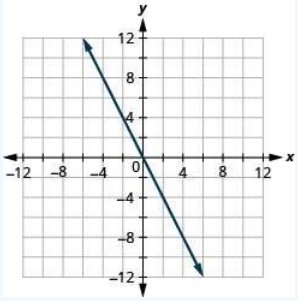
Answer



56. $3x - 4y = -12$

57. $y = -2x$

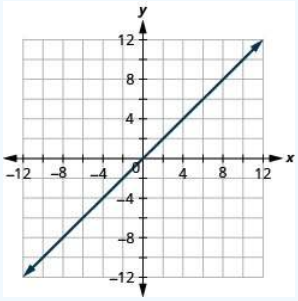
Answer



58. $y = 5x$

59. $y = x$

Answer



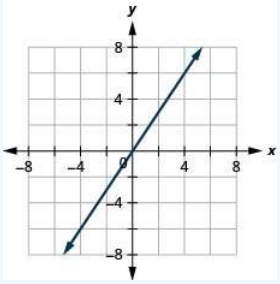
60. $y = -x$

Mixed Practice

In the following exercises, graph each equation.

61. $y = 32x$

Answer

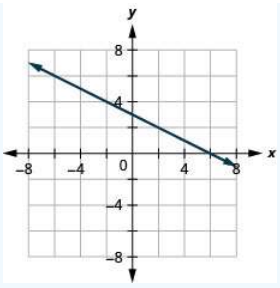


62. $y = -23x$

63. $y = -12x + 3$

Answer

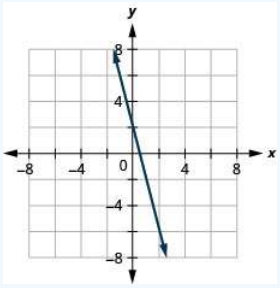
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64. $y = 14x - 2$

65. $4x + y = 2$

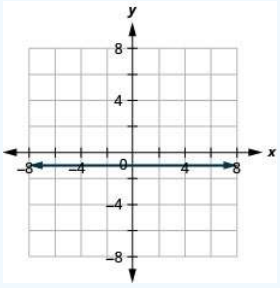
Answer



66. $5x + 2y = 10$

67. $y = -1$

Answer



68. $x = 3$

Writing Exercises

69. Explain how you would choose three *x*-values to make a table to graph the line $y = 15x - 2$.

Answer

Answers will vary.

70. What is the difference between the equations of a vertical and a horizontal line?

71. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $4x + y = -4$? Why?

Answer

Answers will vary.

72. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y = 23x - 2$? Why?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
plot points on a rectangular coordinate system.			
graph a linear equation by plotting points.			
graph vertical and horizontal lines.			
find *x*- and *y*-intercepts.			
graph a line using the intercepts.			

b. If most of your checks were:

Confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

With some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

No, I don't get it. This is a warning sign and you must address it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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3.2: Slope of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel and perpendicular lines

Before you get started, take this readiness quiz.

1. Simplify: $\frac{(1-4)}{(8-2)}$.
   If you missed this problem, review [link].
2. Divide: $\frac{0}{4}$, $\frac{4}{0}$.
   If you missed this problem, review [link].
3. Simplify: $\frac{15}{-3}$, $\frac{-15}{3}$, $\frac{-15}{-3}$.
   If you missed this problem, review [link].

Find the Slope of a Line

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter.

In mathematics, the measure of the steepness of a line is called the *slope* of the line.

The concept of slope has many applications in the real world. In construction the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope. and as you ski or jog down a hill, you definitely experience slope.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The *rise* is the amount the vertical distance changes while the *run* measures the horizontal change, as shown in this illustration. Slope is a rate of change. See Figure.

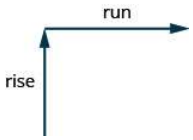


Figure 3.2.1.

SLOPE OF A LINE

The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.

The rise measures the vertical change and the run measures the horizontal change.

To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical.

To find the slope of the line, we measure the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the *rise* and the horizontal distance is called the *run*,

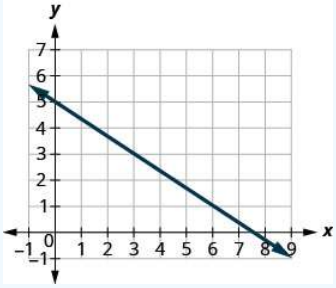
FIND THE SLOPE OF A LINE FROM ITS GRAPH USING $m = \frac{\text{rise}}{\text{run}}$

1. Locate two points on the line whose coordinates are integers.
2. Starting with one point, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.

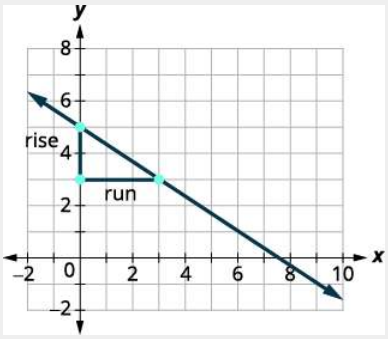
4. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

Example 3.2.1

Find the slope of the line shown.

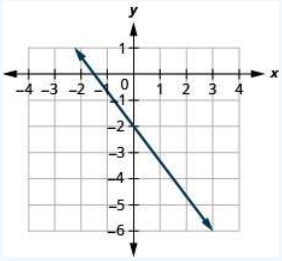


Solution

Locate two points on the graph whose coordinates are integers.	$(0, 5)$ and $(3, 3)$
Starting at $(0, 5)$, sketch a right triangle to $(3, 3)$ as shown in this graph.	
Count the rise— since it goes down, it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-2}{3}$
Simplify.	$m = -\frac{2}{3}$
	The slope of the line is $-\frac{2}{3}$.
	So y decreases by 2 units as x increases by 3 units.

Try It 3.2.2

Find the slope of the line shown.

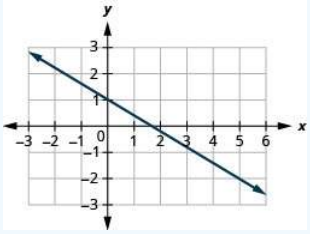


Answer

$-\frac{4}{3}$

Try It 3.2.3

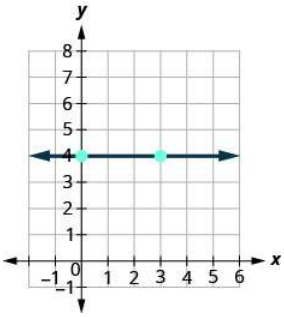
Find the slope of the line shown.



Answer

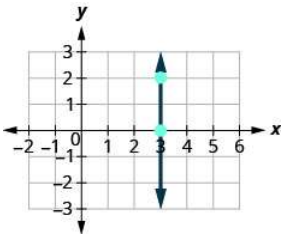
$-\frac{3}{5}$

How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y = 4$, we could graph the line, find two points on it, and count the rise and the run. Let's see what happens when we do this, as shown in the graph below.



What is the rise?	The rise is 0.
What is the run?	The run is 3.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{0}{3}$
	$m = 0$
	The slope of the horizontal line $y = 4$ is 0.

Let's also consider a vertical line, the line $x = 3$, as shown in the graph.



What is the rise?	The rise is 2.
What is the run?	The run is 0.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{0}$

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line $x = 3$ is undefined.

All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

SLOPE OF A HORIZONTAL AND VERTICAL LINE

The slope of a horizontal line, $y = b$, is 0.

The slope of a vertical line, $x = a$, is undefined.

Example 3.2.4

Find the slope of each line: a. $x = 8$ b. $y = -5$.

Solution

a. $x = 8$ This is a vertical line. Its slope is undefined.
b. $y = -5$ This is a horizontal line. It has slope 0.

Try It 3.2.5

Find the slope of the line: $x = -4$.

Answer

undefined

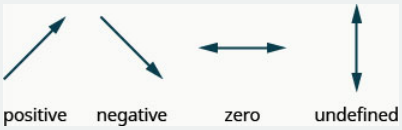
Try It 3.2.6

Find the slope of the line: $y = 7$.

Answer

0

QUICK GUIDE TO THE SLOPES OF LINES



Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

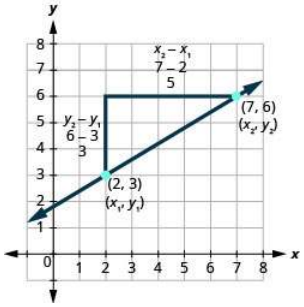
We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

(x_1, y_1) read "x sub 1, y sub 1"
(x_2, y_2) read "x sub 2, y sub 2"

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown in this graph.



Since we have two points, we will use subscript notation. $\begin{pmatrix} x_1, & y_1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ 7 & 6 \end{pmatrix}$

$m = \frac{\text{rise}}{\text{run}}$

On the graph, we counted the rise of 3 and the run of 5. $m = \frac{3}{5}$

Notice that the rise of 3 can be found by subtracting the y-coordinates, 6 and 3, and the run of 5 can be found by subtracting the x-coordinates 7 and 2.

We rewrite the rise and run by putting in the coordinates. $m = \frac{6-3}{7-2}$

But 6 is y_2, the y-coordinate of the second point and 3 is y_1, the y-coordinate of the first point. So we can rewrite the slope using subscript notation. $m = \frac{y_2-y_1}{7-2}$

Also 7 is the x-coordinate of the second point and 2 is the x-coordinate of the first point. So again we rewrite the slope using subscript notation. $m = \frac{y_2-y_1}{x_2-x_1}$

We've shown that $m = \frac{y_2-y_1}{x_2-x_1}$ is really another version of $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

SLOPE OF A LINE BETWEEN TWO POINTS

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is: $m = \frac{y_2-y_1}{x_2-x_1}$.

The slope is:

y of the second point minus y of the first point

over

x of the second point minus x of the first point

Example 3.2.7

Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution

We'll call $(-2,-3)$ point #1and $(-7,4)$ point #2. $\begin{pmatrix} x_1, & y_1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ -7 & 4 \end{pmatrix}$

Use the slope formula. $m = \frac{y_2 - y_1}{x_2 - x_1}$

Substitute the values.

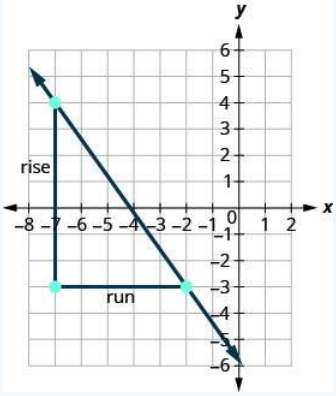
y of the second point minus y of the first point

x of the second point minus x of the first point $m = \frac{4-(-3)}{-7-(-2)}$

Simplify $m = \frac{7}{-5}$

$m = \frac{-7}{5}$

Let's verify this slope on the graph shown.



$$m = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{7}{-5}$$

$$m = \frac{-7}{5}$$

Try It 3.2.8

Use the slope formula to find the slope of the line through the pair of points: $(-3, 4)$ and $(2, -1)$.

Answer

-1

Try It 3.2.9

Use the slope formula to find the slope of the line through the pair of points: $(-2, 6)$ and $(-3, -4)$.

Answer

10

Graph a Line Given a Point and the Slope

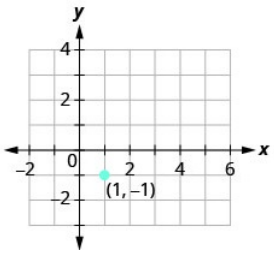
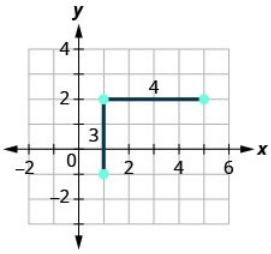
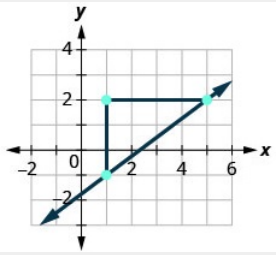
Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

We can also graph a line when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

Example 3.2.10: How to graph a Line Given a Point and the Slope

Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution

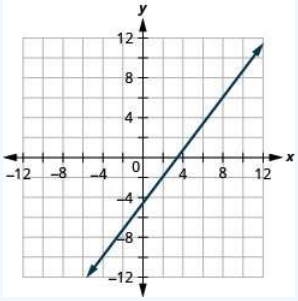
Step 1. Plot the given point.	Plot $(1, -1)$	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ rise = 3 run = 4
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at $(1, -1)$ and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

You can check your work by finding a third point. Since the slope is $m = \frac{3}{4}$, it can also be written as $m = \frac{-3}{-4}$ (negative divided by negative is positive!). Go back to $(1, -1)$ and count out the rise, -3, and the run, -4.

Try It 3.2.11

Graph the line passing through the point $(2, -2)$ with the slope $m = \frac{4}{3}$.

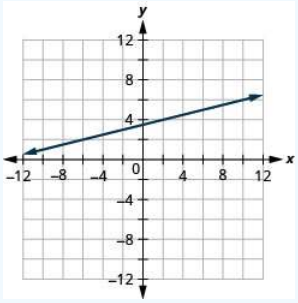
Answer



Try It 3.2.12

Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Answer



GRAPH A LINE GIVEN A POINT AND THE SLOPE.

1. Plot the given point.
2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
3. Starting at the given point, count out the rise and run to mark the second point.
4. Connect the points with a line.

Graph a Line Using its Slope and Intercept

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

See Figure. Let's look at the graph of the equation $y = 12x + 3$ and find its slope and *y*-intercept.

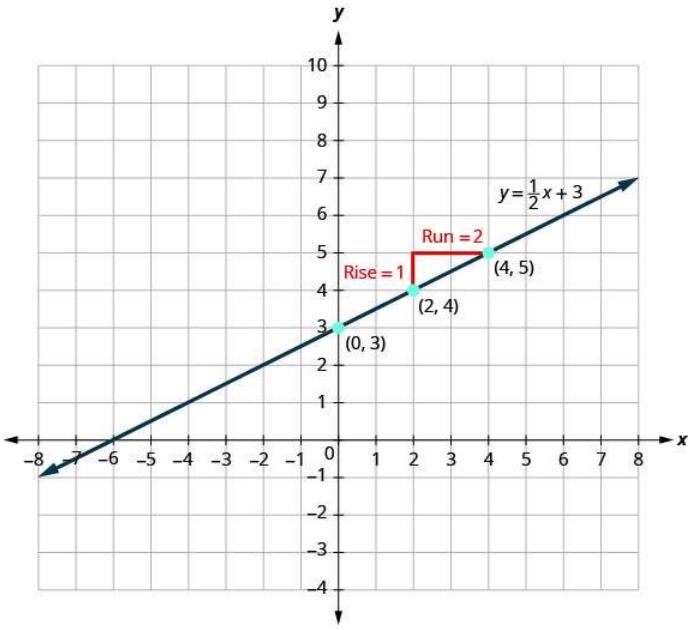


Figure 3.2.2

The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$m = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{1}{2}$$

The *y*-intercept is $(0, 3)$.

Look at the equation of this line. $y = \frac{1}{2}x + 3$

Look at the slope and *y*-intercept. slope $m = \frac{1}{2}$ and y-intercept $(0, 3)$

When a linear equation is solved for *y*, the coefficient of the *x* term is the slope and the constant term is the *y*-coordinate of the *y*-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form. Sometimes the slope–intercept form is called the "*y*-form."

$m = \frac{1}{2}$; *y*-intercept is (0, 3)

$y = \frac{1}{2}x + 3$

$y = mx + b$

SLOPE INTERCEPT FORM OF AN EQUATION OF A LINE

The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, $(0, b)$ is $y = mx + b$.

Let's practice finding the values of the slope and *y*-intercept from the equation of a line.

Example 3.2.13

Identify the slope and *y*-intercept of the line from the equation: a. $y = -\frac{4}{7}x - 2$ b. $x + 3y = 9$

Solution

a. We compare our equation to the slope–intercept form of the equation.

Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{4}{7}x - 2$
Identify the slope.	$m = -\frac{4}{7}$
Identify the y-intercept.	y=intercept is $(0, -2)$

b. When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for y.

Solve for y.	$x + 3y = 9$
Subtract x from each side.	$3y = -x + 9$
Divide both sides by 3.	$\frac{3y}{3} = \frac{-x+9}{3}$
Simplify.	$y = -\frac{1}{3}x + 3$
Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{1}{3}x + 3$
Identify the slope.	$m = -\frac{1}{3}$
Identify the y-intercept.	y=intercept is $(0, 3)$

Try It 3.2.14

Identify the slope and y-intercept from the equation of the line. a. $y = \frac{2}{5}x - 1$ b. $x + 4y = 8$

Answer

a. $m = \frac{2}{5}$; $(0, -1)$
b. $m = -\frac{1}{4}$; $(0, 2)$

Try It 3.2.15

Identify the slope and y-intercept from the equation of the line. a. $y = -\frac{4}{3}x + 1$ b. $3x + 2y = 12$

Answer

a. $m = -\frac{4}{3}$; $(0, 1)$
b. $m = -\frac{3}{2}$; $(0, 6)$

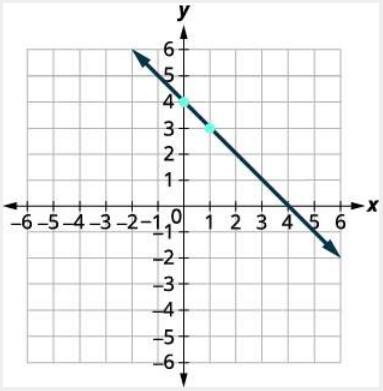
We have graphed a line using the slope and a point. Now that we know how to find the slope and y-intercept of a line from its equation, we can use the y-intercept as the point, and then count out the slope from there.

Example 3.2.16

Graph the line of the equation $y = -x + 4$ using its slope and y-intercept.

Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$

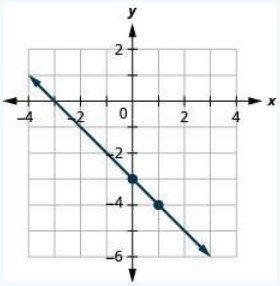
Identify the slope and y-intercept.	$m = -1$ y-intercept is $(0, 4)$
Plot the y-intercept.	See the graph.
Identify the rise over the run.	$m = \frac{-1}{1}$
Count out the rise and run to mark the second point.	rise -1, run 1 

Draw the line as shown in the graph.

Try It 3.2.17

Graph the line of the equation $y = -x - 3$ using its slope and y-intercept.

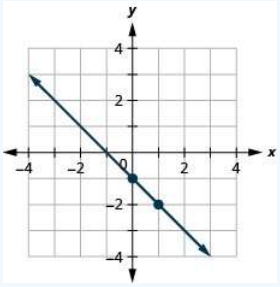
Answer



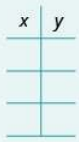
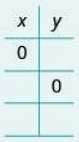
Try It 3.2.18

Graph the line of the equation $y = -x - 1$ using its slope and y-intercept.

Answer



Now that we have graphed lines by using the slope and y-intercept, let's summarize all the methods we have used to graph lines.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
	$y = mx + b$		
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier.

Generally, plotting points is not the most efficient way to graph a line. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are five equations we graphed in this chapter, and the method we used to graph each of them.

	Equation	**Method**
#1	$x = 2$	Vertical line
#2	$y = -1$	Horizontal line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = -x + 4$	Slope–intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both *x* and *y* are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the *x*- intercept and $x = 0$ to find the *y*-intercept, and then found a third point by choosing another value for *x* or *y*.

Equation #5 is written in slope–intercept form. After identifying the slope and *y*-intercept from the equation we used them to graph the line.

This leads to the following strategy.

STRATEGY FOR CHOOSING THE MOST CONVENIENT METHOD TO GRAPH A LINE

Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
  - $x = a$ is a vertical line passing through the *x*-axis at *a*.
  - $y = b$ is a horizontal line passing through the *y*-axis at *b*.
- If *y* is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and *y*-intercept.
  - Identify the slope and *y*-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
  - Find the *x*- and *y*-intercepts, a third point, and then graph.

Example 3.2.19

Determine the most convenient method to graph each line:

ⓐ $y=5$ ⓑ $4x-5y=20$ ⓒ $x=-3$ ⓓ $y=-\frac{5}{9}x+8$

Answer

ⓐ $y=5$
This equation has only one variable, *y*. Its graph is a horizontal line crossing the *y*-axis at 5.
ⓑ $4x-5y=20$
This equation is of the form $Ax+By=C$. The easiest way to graph it will be to find the intercepts and one more point.
ⓒ $x=-3$
There is only one variable, *x*. The graph is a vertical line crossing the *x*-axis at -3.
ⓓ $y=-\frac{5}{9}x+8$
Since this equation is in $y=mx+b$ form, it will be easiest to graph this line by using the slope and *y*-intercepts.

Example 3.2.20

Determine the most convenient method to graph each line:

ⓐ $3x+2y=12$ ⓑ $y=4$ ⓒ $y=\frac{1}{5}x-4$ ⓓ $x=-7$.

Answer

ⓐ intercepts ⓑ horizontal line ⓒ slope-intercept ⓓ vertical line

Example 3.2.21

Determine the most convenient method to graph each line:

ⓐ $x=6$ ⓑ $y=-\frac{3}{4}x+1$ ⓒ $y=-8$ ⓓ $4x-3y=-1$.

Answer

ⓐ vertical line ⓑ slope-intercept ⓒ horizontal line
ⓓ intercepts

Graph and Interpret Applications of Slope-Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so you can see how equations written in slope-intercept form relate to real-world situations.

Usually, when a linear equation models uses real-world data, different letters are used for the variables, instead of using only *x* and *y*. The variable names remind us of what quantities are being measured.

Also, we often will need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

Example 3.2.22

The equation $F=\frac{9}{5}C+32$ is used to convert temperatures, *C*, on the Celsius scale to temperatures, *F*, on the Fahrenheit scale.

ⓐ Find the Fahrenheit temperature for a Celsius temperature of 0.

ⓑ Find the Fahrenheit temperature for a Celsius temperature of 20.

ⓒ Interpret the slope and *F*-intercept of the equation.

ⓓ Graph the equation.

Answer

ⓐ

Find the Fahrenheit temperature for a Celsius temperature of 0.	$F = \frac{9}{5}C + 32$
Find F when C=0.	$F = \frac{9}{5}(0) + 32$
Simplify.	$F = 32$

ⓑ

Find the Fahrenheit temperature for a Celsius temperature of 20.	$F = \frac{9}{5}C + 32$
Find F when C=20.	$F = \frac{9}{5}(20) + 32$
Simplify.	$F = 36 + 32$
Simplify.	$F = 68$

ⓒ

Interpret the slope and *F*-intercept of the equation.

Even though this equation uses *F* and *C*, it is still in slope-intercept form.

$y = mx + b$

$F = mC + b$

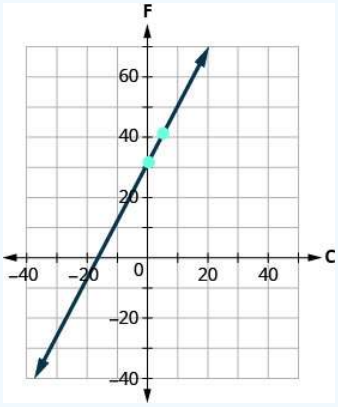
$F = \frac{9}{5}C + 32$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (*F*) increases 9 degrees when the temperature Celsius (*C*) increases 5 degrees.

The *F*-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

ⓓ Graph the equation.

We'll need to use a larger scale than our usual. Start at the *F*-intercept $(0, 32)$, and then count out the rise of 9 and the run of 5 to get a second point as shown in the graph.



Example 3.2.23

The equation \(h=2s+50\) is used to estimate a woman's height in inches, *h*, based on her shoe size, *s*.

ⓐ Estimate the height of a child who wears women's shoe size 0.

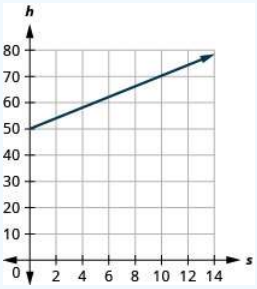
ⓑ Estimate the height of a woman with shoe size 8.

ⓒ Interpret the slope and *h*-intercept of the equation.

ⓓ Graph the equation.

Answer

ⓐ 50 inches
ⓑ 66 inches
ⓒ The slope, 2, means that the height, *h*, increases by 2 inches when the shoe size, *s*, increases by 1. The *h*-intercept means that when the shoe size is 0, the height is 50 inches.
ⓓ



Example 3.2.24

The equation \(T=\frac{1}{4}n+40\) is used to estimate the temperature in degrees Fahrenheit, *T*, based on the number of cricket chirps, *n*, in one minute.

ⓐ Estimate the temperature when there are no chirps.

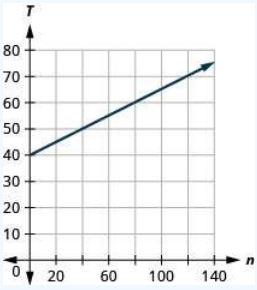
ⓑ Estimate the temperature when the number of chirps in one minute is 100.

ⓒ Interpret the slope and *T*-intercept of the equation.

ⓓ Graph the equation.

Answer

ⓐ 40 degrees
ⓑ 65 degrees
ⓒ The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (*F*) increases 1 degree when the number of chirps, *n*, increases by 4. The *T*-intercept means that when the number of chirps is 0, the temperature is 40°.
ⓓ



The cost of running some types business have two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

Example 3.2.25

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, *C*, in dollars and the number of miles, *m*, that he drives.

ⓐ Find Sam's cost for a week when he drives 0 miles.

ⓑ Find the cost for a week when he drives 250 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

ⓓ Graph the equation.

Answer

ⓐ

Find Sam's cost for a week when he drives 0 miles.	$C = 0.5m + 60$
Find C when m=0.	$C = 0.5(0) + 60$
Simplify.	$C = 60$
	Sam's costs are \$60 when he drives 0 miles.

ⓑ

Find Sam's cost for a week when he drives 250 miles.	$C = 0.5m + 60$
Find C when m=250.	$C = 0.5(250) + 60$
Simplify.	$C = 185$
	Sam's costs are \$185 when he drives 250 miles.

ⓒ Interpret the slope and *C*-intercept of the equation.

$y = mx + b$

$C = 0.5n + 60$

The slope, 0.5, means that the weekly cost, *C*, increases by \$0.50 when the number of miles driven, *n*, increases by 1.
The *C*-intercept means that when the number of miles driven is 0, the weekly cost is \$60.

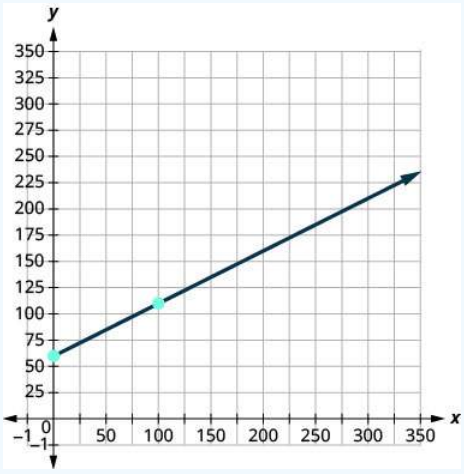
ⓓ Graph the equation.

We'll need to use a larger scale than our usual. Start at the *C*-intercept $(0, 60)$.

To count out the slope $m = 0.5$, we rewrite it as an equivalent fraction that will make our graphing easier.

	$m = 0.5$
Rewrite as a fraction.	$m = \frac{0.5}{1}$
Multiply numerator and denominator by 100	$m = \frac{0.5(100)}{1(100)}$
Simplify.	$m = \frac{50}{100}$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100. The second point will be $(100, 110)$



Example 3.2.26

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, C, in dollars and the number of pizzas, p, that she sells.

ⓐ Find Stella's cost for a week when she sells no pizzas.

ⓑ Find the cost for a week when she sells 15 pizzas.

ⓒ Interpret the slope and C-intercept of the equation.

ⓓ Graph the equation.

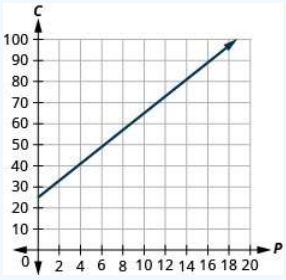
Answer

ⓐ $25

ⓑ $85

ⓒ The slope, 4, means that the weekly cost, C, increases by $4 when the number of pizzas sold, p, increases by 1. The C-intercept means that when the number of pizzas sold is 0, the weekly cost is $25.

ⓓ



Example 3.2.27

Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, C, in dollars and the number of wedding invitations, n, that she writes.

ⓐ Find Loreen's cost for a week when she writes no invitations.

ⓑ Find the cost for a week when she writes 75 invitations.

ⓒ Interpret the slope and C-intercept of the equation.

ⓓ Graph the equation.

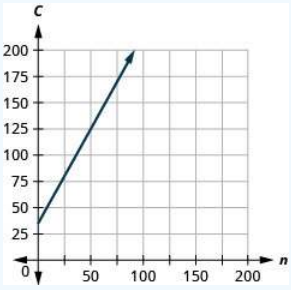
Answer

ⓐ $35

ⓑ $170

ⓒ The slope, 1.8, means that the weekly cost, C, increases by $\$1.80$when the number of invitations, n, increases by 1. The C-intercept means that when the number of invitations is 0, the weekly cost is $35.

ⓓ



Use Slopes to Identify Parallel and Perpendicular Lines

Two lines that have the same slope are called **parallel lines**. Parallel lines have the same steepness and never intersect.

We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different *y*-intercepts are called parallel lines. See Figure.

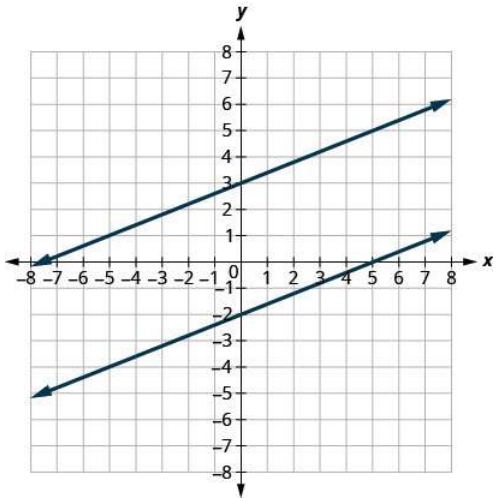


Figure 3.2.3

Verify that both lines have the same slope, $m = \frac{2}{5}$, and different *y*-intercepts.

What about vertical lines? The slope of a vertical line is undefined, so vertical lines don't fit in the definition above. We say that vertical lines that have different *x*-intercepts are parallel, like the lines shown in this graph.

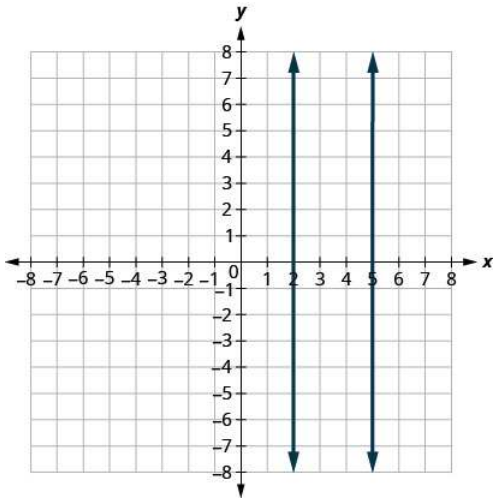


Figure 3.2.4

PARALLEL LINES

Parallel lines are lines in the same plane that do not intersect.

- Parallel lines have the same slope and different *y*-intercepts.
- If $m1$ and $m2$ are the slopes of two parallel lines then $m1 = m2$.
- Parallel vertical lines have different *x*-intercepts

Since parallel lines have the same slope and different *y*-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

Example 3.2.28

Use slopes and y-intercepts to determine if the lines are parallel:

ⓐ $3x-2y=6$ and $y=\frac{3}{2}x+1$ ⓑ $y=2x-3$ and $-6x+3y=-9$.

Answer

ⓐ

	$3x-2y=6$	and	$y=\frac{3}{2}x+1$
	$-2y=-3x+6$		
Solve the first equation for y.	$\frac{-2y}{-2}=\frac{-3x+6}{-2}$		
The equation is now in slope–intercept form.	$y=\frac{3}{2}x-3$		
The equation of the second line is already in slope–intercept form.			$y=\frac{3}{2}x+1$
	$y=\frac{3}{2}x-3$		$y=\frac{3}{2}x+1$
Identify the slope and y-intercept of both lines.	$y=mx+b$		$y=mx+b$
	$m=\frac{3}{2}$		$y=\frac{3}{2}$
	y-intercept is $(0,-3)$		y-intercept is $(0,1)$

The lines have the same slope and different y-intercepts and so they are parallel.
You may want to graph the lines to confirm whether they are parallel.

ⓑ

	$y=2x-3$	and	$-6x+3y=-9$
The first equation is already in slope–intercept form.	$y=2x-3$		
			$-6x+3y=-9$
			$3y=6x-9$
Solve the second equation for y.			$\frac{3y}{3}=\frac{6x-9}{3}$
			$y=2x-3$
The second equation is now in slope–intercept form.			$y=2x-3$
	$y=2x-3$		$y=2x-3$
Identify the slope and y-intercept of both lines.	$y=mx+b$		$y=mx+b$
	$m=2$		$m=2$
	y-intercept is $(0,-3)$		y-intercept is $(0,-3)$

The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line and we say the lines are coincident. They are not parallel; they are the same line.

Example 3.2.29

Use slopes and y-intercepts to determine if the lines are parallel:

ⓐ $2x+5y=5$ and $y=-\frac{2}{5}x-4$ ⓑ $y=-\frac{1}{2}x-1$ and $x+2y=-2$.

Answer

ⓐ parallel ⓑ not parallel; same line

Example 3.2.30

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $4x-3y=6$ and $y=\frac{4}{3}x-1$ ⓑ $y=\frac{3}{4}x-3$ and $3x-4y=12$.

Answer

ⓐ parallel ⓑ not parallel; same line

Example 3.2.31

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y=-4$ and $y=3$ ⓑ $x=-2$ and $x=-5$.

Answer

ⓐ $y=-4$ and $y=3$

We recognize right away from the equations that these are horizontal lines, and so we know their slopes are both 0.
Since the horizontal lines cross the *y*-axis at y=−4y=−4 and at y=3,y=3, we know the *y*-intercepts are (0,−4)(0,−4) and (0,3).
(0,3).
The lines have the same slope and different *y*-intercepts and so they are parallel.

ⓑ $x=-2$ and $x=-5$

We recognize right away from the equations that these are vertical lines, and so we know their slopes are undefined.
Since the vertical lines cross the *x*-axis at $x=-2$ and $x=-5$, we know the *y*-intercepts are $(-2,0)$ and $(-5,0)$.
The lines are vertical and have different *x*-intercepts and so they are parallel.

Example 3.2.32

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y=8$ and $y=-6$ ⓑ $x=1$ and $x=-5$.

Answer

ⓐ parallel ⓑ parallel

Example 3.2.33

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y=1$ and $y=-5$ ⓑ $x=8$ and $x=-6$.

Answer

ⓐ parallel ⓑ parallel

Let's look at the lines whose equations are $y=\frac{1}{4}x-1$ and $y=-4x+2$, shown in Figure.

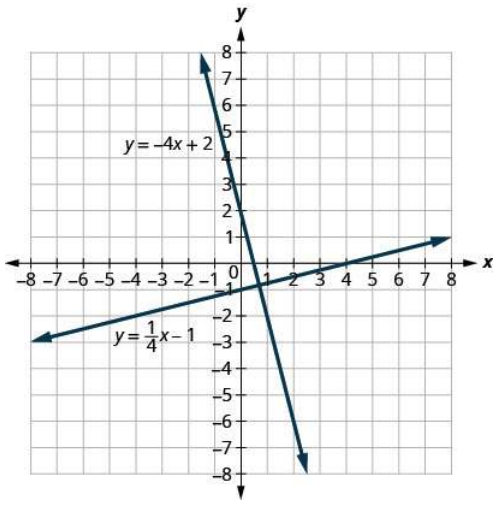


Figure 3.2.5

These lines lie in the same plane and intersect in right angles. We call these lines perpendicular.

If we look at the slope of the first line, $m_1 = \frac{1}{4}$, and the slope of the second line, $m_2 = -4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.

$$\begin{aligned} &m_1 \cdot m_2 \\ &14(-4) \\ &-1 \end{aligned}$$

This is always true for **perpendicular lines** and leads us to this definition.

PERPENDICULAR LINES

Perpendicular lines are lines in the same plane that form a right angle.

- If m_1 and m_2 are the slopes of two perpendicular lines, then:
  - their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$.
  - the product of their slopes is -1, $m_1 \cdot m_2 = -1$.
- A vertical line and a horizontal line are always perpendicular to each other

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are opposite reciprocals. If the product of the slopes is -1, the lines are perpendicular.

Example 3.2.34

Use slopes to determine if the lines are perpendicular:

ⓐ $y = -5x - 4$ and $x - 5y = 5$ ⓑ $7x + 2y = 3$ and $2x + 7y = 5$

Answer

ⓐ

The first equation is in slope–intercept form.Solve the second equation fory.Identify the slope of each line.y=−5x−4yym1=−5x−4=mx+b=−5x−5y−5y−5y=5=−x+5=−x+5−5=15x−1yym2=15x−1=mx+b=15The first equation is in slope–intercept form.y=−5x−4Solve the second equation fory.x−5y=5−5y=−x+5−5y−5=−x+5−5y=15x−1Identify the slope of each line.y=−5x−4y=mx+bm1=−5y=15x−1y=mx+bm2=15

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes, Since

−5(15)=−1,−5(15)=−1, it checks.

ⓑ
Solve the equations fory.Identify the slope of each line.7x+2y2y2y2y=3=−7x+3=−7x+32=−72x+32ym1=mx+b=−722x+7y7y7y7y=5=−2x+5=−2x+57=−27x+57ym1=mx+b=−27Solve the equations fory.7x+2y=32y=−7x+32y2=−7x+32y=−72x+322x+7y=57y=−2x+57y7=−2x+57y=−27x+57Identify the slope of each line.y=mx+bm1=−72y=mx+bm1=−27
The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

? Example 3.2.3

Use slopes to determine if the lines are perpendicular:

ⓐ $y=-3x+2$ and $x-3y=4$ ⓑ $5x+4y=1$ and $4x+5y=3$.

Answer

ⓐ perpendicular ⓑ not perpendicular

? Example 3.2.3

Use slopes to determine if the lines are perpendicular:

ⓐ $y=2x-5$ and $x+2y=-6$ ⓑ $2x-9y=3$ and $9x-2y=1$.

Answer

ⓐ perpendicular ⓑ not perpendicular

Key Concepts

- **Slope of a Line**
  - The slope of a line is $m=\frac{\text{rise}}{\text{run}}$.
  - The rise measures the vertical change and the run measures the horizontal change.

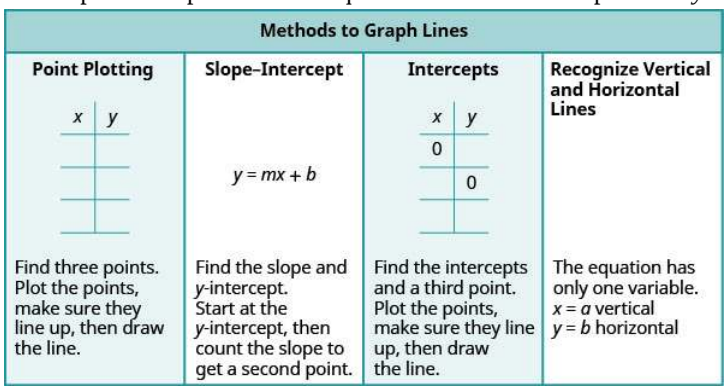
- **How to find the slope of a line from its graph using $m=\frac{\text{rise}}{\text{run}}$.**
  1. Locate two points on the line whose coordinates are integers.
  2. Starting with one point, sketch a right triangle, going from the first point to the second point.
  3. Count the rise and the run on the legs of the triangle.
  4. Take the ratio of rise to run to find the slope: $m=\frac{\text{rise}}{\text{run}}$.
- **Slope of a line between two points.**
  - The slope of the line between two points (x_1,y_1) and (x_2,y_2) is:

$$m=\frac{y_2-y_1}{x_2-x_1}$$

.

- **How to graph a line given a point and the slope.**
  1. Plot the given point.
  2. Use the slope formula $m=\frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  3. Starting at the given point, count out the rise and run to mark the second point.

4. Connect the points with a line.

- **Slope Intercept Form of an Equation of a Line**
  - The slope–intercept form of an equation of a line with slope m and y-intercept, $(0, b)$ is $y = mx + b$
  - 

- **Parallel Lines**
  - Parallel lines are lines in the same plane that do not intersect.
    Parallel lines have the same slope and different y-intercepts.
    If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
    Parallel vertical lines have different x-intercepts.
- **Perpendicular Lines**
  - Perpendicular lines are lines in the same plane that form a right angle.
  - If m_1 and m_2 are the slopes of two perpendicular lines, then:
    their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$.
    the product of their slopes is -1, $m_1 \cdot m_2 = -1$.
  - A vertical line and a horizontal line are always perpendicular to each other.

Glossary

parallel lines

Parallel lines are lines in the same plane that do not intersect.

perpendicular lines

Perpendicular lines are lines in the same plane that form a right angle.

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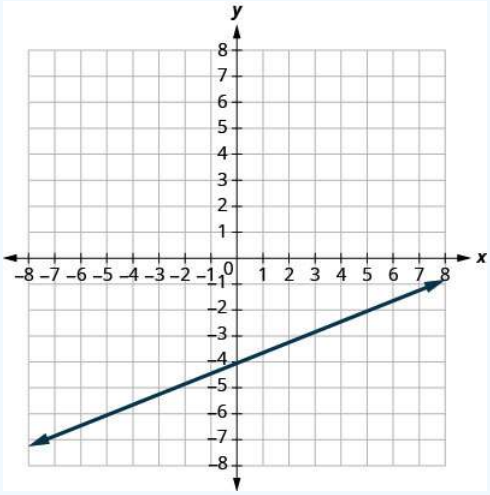
3.2E: Exercises

Practice Makes Perfect

Find the Slope of a Line

In the following exercises, find the slope of each line shown.

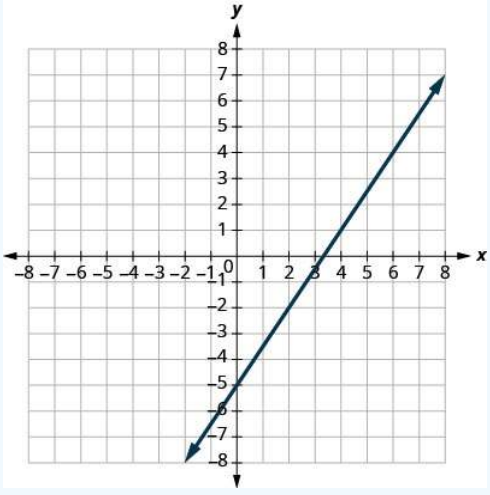
1.



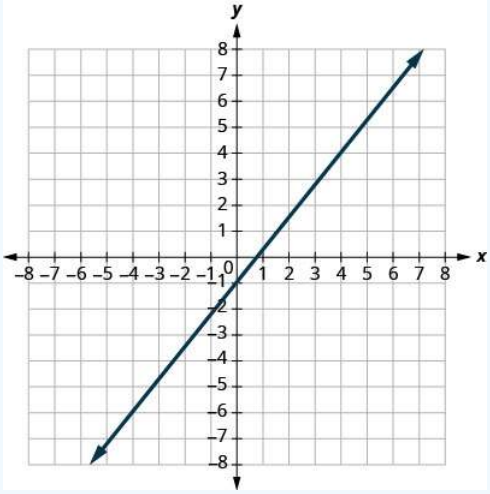
Answer

$m = \frac{2}{5}$

2.



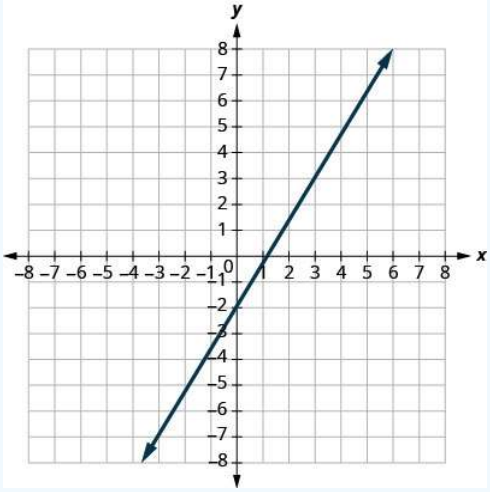
3.



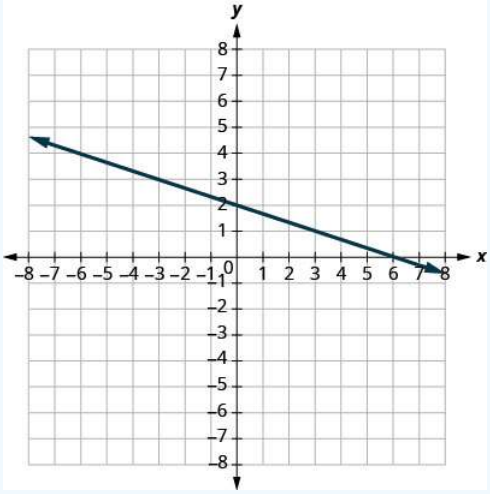
Answer

$m = \frac{5}{4}$

4.



5.

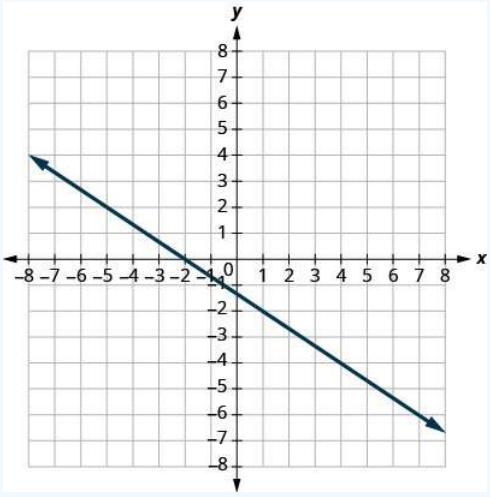


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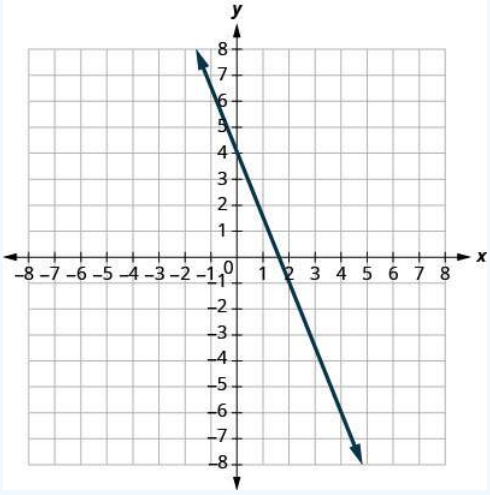
Answer

$m = -\frac{1}{3}$

6.



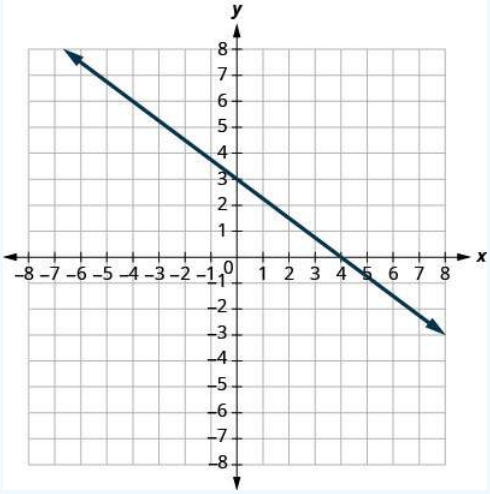
7.



Answer

$m = -\frac{5}{2}$

8.



In the following exercises, find the slope of each line.

9. $y = 3$

Answer

$m = 0$

10. $y = -2$

11. $x = -5$

Answer

undefined

12. $x = 4$

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

13. $(2, 5),\ (4, 0)$

Answer

$m = -\frac{5}{2}$

14. $(3, 6),\ (8, 0)$

15. $(-3, 3),\ (4, -5)$

Answer

$m = -\frac{8}{7}$

16. $(-2, 4),\ (3, -1)$

17. $(-1, -2),\ (2, 5)$

Answer

$m = \frac{7}{3}$

18. $(-2, -1),\ (6, 5)$

19. $(4, -5),\ (1, -2)$

Answer

$m = -1$

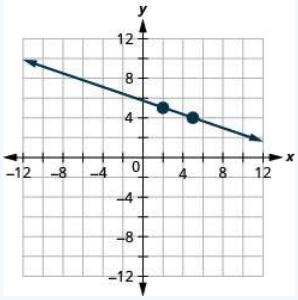
20. $(3, -6),\ (2, -2)$

Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

21. $(2, 5)$; $m = -\frac{1}{3}$

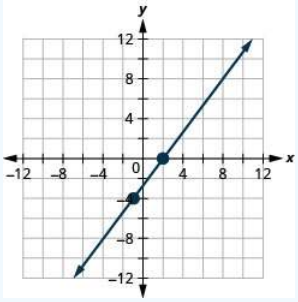
Answer



22. $(1, 4)$; $m = -\frac{1}{2}$

23. $(-1, -4)$; $m = \frac{4}{3}$

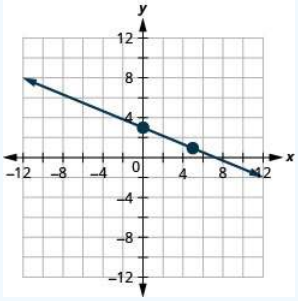
Answer



24. $(-3, -5)$; $m = \frac{3}{2}$

25. y-intercept: $(0, 3)$; $m = -\frac{2}{5}$

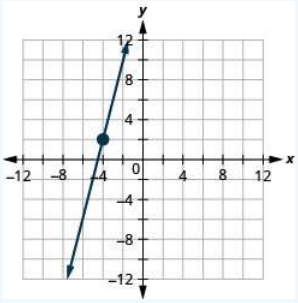
Answer



26. x-intercept: $(-2, 0)$; $m = \frac{3}{4}$

27. $(-4, 2)$; $m = 4$

Answer



28. $(1, 5)$; $m = -3$

Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and *y*-intercept of each line.

29. $y = -7x + 3$

Answer

$m = -7$; $(0, 3)$

30. $y = 4x - 10$

31. $3x + y = 5$

Answer

$m = -3$; $(0, 5)$

32. $4x + y = 8$

33. $6x + 4y = 12$

Answer

$m = -\frac{3}{2}$; $(0, 3)$

34. $8x + 3y = 12$

35. $5x - 2y = 6$

Answer

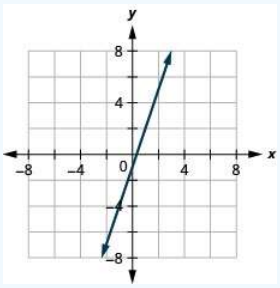
$m = \frac{5}{2}$; $(0, -3)$

36. $7x - 3y = 9$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

37. $y = 3x - 1$

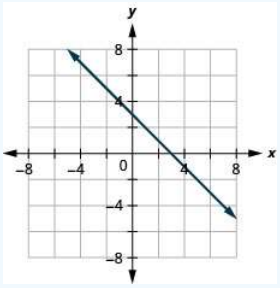
Answer



38. $y = 2x - 3$

39. $y = -x + 3$

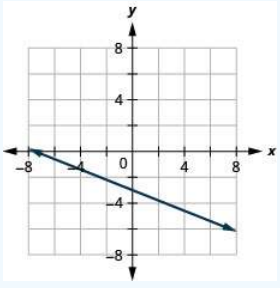
Answer



40. $y = -x - 4$

41. $y = -\frac{2}{5}x - 3$

Answer

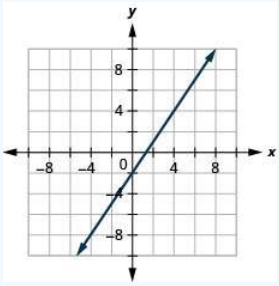


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42. $y = -\frac{3}{5}x + 2$

43. $3x - 2y = 4$

Answer



44. $3x - 4y = 8$

Choose the Most Convenient Method to Graph a Line

In the following exercises, determine the most convenient method to graph each line.

45. $x = 2$

Answer

vertical line

46. $y = 5$

47. $y = -3x + 4$

Answer

slope-intercept

48. $x - y = 5$

49. $x - y = 1$

Answer

intercepts

50. $y = \frac{2}{3}x - 1$

51. $3x - 2y = -12$

Answer

intercepts

52. $2x - 5y = -10$

Graph and Interpret Applications of Slope–Intercept

53. The equation $P = 31 + 1.75w$ models the relation between the amount of Tuyet's monthly water bill payment, P, in dollars, and the number of units of water, w, used.

a. Find Tuyet's payment for a month when 0 units of water are used.

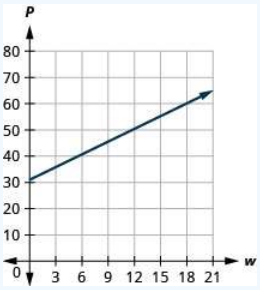
b. Find Tuyet's payment for a month when 12 units of water are used.

c. Interpret the slope and P-intercept of the equation.

d. Graph the equation.

Answer

a. $\$31$
b. $\$52$
c. The slope, 1.75, means that the payment, P, increases by $\$1.75$ when the number of units of water used, w, increases by 1. The P-intercept means that when the number units of water Tuyet used is 0, the payment is $\$31$.
d.



54. The equation $P = 28 + 2.54w$ models the relation between the amount of Randy's monthly water bill payment, P, in dollars, and the number of units of water, w, used.

a. Find the payment for a month when Randy used 0 units of water.

b. Find the payment for a month when Randy used 15 units of water.

c. Interpret the slope and P-intercept of the equation.

d. Graph the equation.

55. Bruce drives his car for his job. The equation $R = 0.575m + 42$ models the relation between the amount in dollars, R, that he is reimbursed and the number of miles, m, he drives in one day.

a. Find the amount Bruce is reimbursed on a day when he drives 0 miles.

b. Find the amount Bruce is reimbursed on a day when he drives 220 miles.

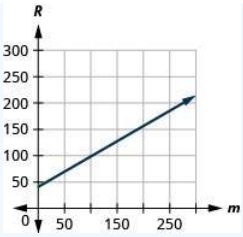
c. Interpret the slope and R-intercept of the equation.

d. Graph the equation.

Answer

a. $\$42$
b. $\$168.50$
c. The slope, 0.575, means that the amount he is reimbursed, R, increases by $\$0.575$ when the number of miles driven, m, increases by 1. The R-intercept means that when the number miles driven is 0, the amount reimbursed is $\$42$.
d.

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56. Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, C, per day and the number of miles, m, she drives in one day.

a. Find the cost if Janelle drives the car 0 miles one day.

b. Find the cost on a day when Janelle drives the car 400 miles.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

57. Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, S, in dollars and the amount of her sales, c, in dollars.

a. Find Cherie's salary for a week when her sales were $\$0$.

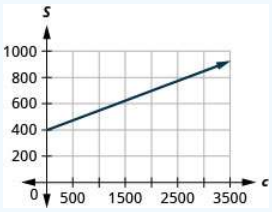
b. Find Cherie's salary for a week when her sales were $\$3,600$

c. Interpret the slope and S-intercept of the equation.

d. Graph the equation.

Answer

a. $\$400$
b. $\$940$
c. The slope, 0.15, means that Cherie's salary, S, increases by $\$0.15$ for every $\$1$ increase in her sales. The S-intercept means that when her sales are $\$0$, her salary is $\$400$
d.



58. Patel's weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, S, in dollars and the amount of his sales, c, in dollars.

a. Find Patel's salary for a week when his sales were 0.

b. Find Patel's salary for a week when his sales were $18,540$

c. Interpret the slope and S-intercept of the equation.

d. Graph the equation.

59. Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, C, of the banquet and the number of guests, g.

a. Find the cost if the number of guests is 40.

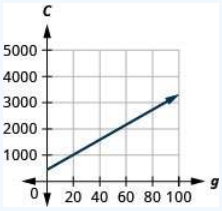
b. Find the cost if the number of guests is 80.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

Answer

a. $\$1570$
b. $\$5690$
c. The slope gives the cost per guest. The slope, 28, means that the cost, C, increases by $\$28$ when the number of guests increases by 1. The C-intercept means that if the number of guests was 0, the cost would be $\$450$
d.



60. Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, C, of the banquet and the number of guests, g.

a. Find the cost if the number of guests is 50.

b. Find the cost if the number of guests is 100.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

61. $y = \frac{3}{4}x - 3\,;\, 3x - 4y = -2$

Answer

parallel

62. $3x - 4y = -2\,;\, y = \frac{3}{4}x - 3$

63. $2x - 4y = 6\,;\, x - 2y = 3$

Answer

neither

64. $8x + 6y = 6\,;\, 12x + 9y = 12$

65. $x = 5\,;\, x = -6$

Answer

parallel

66. $x = -3\,;\, x = -2$

67. $4x-2y=5;\ 3x+6y=8$

Answer

perpendicular

68. $8x-2y=7;\ 3x+12y=9$

69. $3x-6y=12;\ 6x-3y=3$

Answer

neither

70. $9x-5y=4;\ 5x+9y=-1$

71. $7x-4y=8;\ 4x+7y=14$

Answer

perpendicular

72. $5x-2y=11;\ 5x-y=7$

73. $3x-2y=8;\ 2x+3y=6$

Answer

perpendicular

74. $2x+3y=5;\ 3x-2y=7$

75. $3x-2y=1;\ 2x-3y=2$

Answer

neither

76. $2x+4y=3;\ 6x+3y=2$

77. $y=2;\ y=6$

Answer

parallel

78. $y=-1;\ y=2$

Writing Exercises

79. How does the graph of a line with slope $m=12$ differ from the graph of a line with slope $m=2$?

Answer

Answers will vary.

https://math.libretexts.org/@go/page/30826

80. Why is the slope of a vertical line “undefined”?

81. Explain how you can graph a line given a point and its slope.

Answer

Answers will vary.

82. Explain in your own words how to decide which method to use to graph a line.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the slope of a line.			
graph a line given a point and the slope.			
graph a line using its slope and intercept.			
choose the most convenient method to graph a line.			
graph and interpret applications of slope-intercept.			
use slopes to identify parallel and perpendicular lines.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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3.3: Find the Equation of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

Before you get started, take this readiness quiz.

1. Solve: $\frac{2}{5}(x+15)$.
   If you missed this problem, review [link].
2. Simplify: $-3(x-(-2))$.
   If you missed this problem, review [link].
3. Solve for y: $y-3=-2(x+1)$.
   If you missed this problem, review [link].

How do online companies know that “you may also like” a particular item based on something you just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on your commuting time of an increase or decrease in gas prices? It’s all mathematics.

The physical sciences, social sciences, and the business world are full of situations that can be modeled with linear equations relating two variables. To create a mathematical model of a linear relation between two variables, we must be able to find the equation of the line. In this section, we will look at several ways to write the equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of the Line Given the Slope and *y*-Intercept

We can easily determine the slope and intercept of a line if the equation is written in slope-intercept form, $y=mx+b$. Now we will do the reverse—we will start with the slope and *y*-intercept and use them to find the equation of the line.

Example 3.3.1

Find the equation of a line with slope -9 and *y*-intercept $(0,-4)$.

Solution

Since we are given the slope and *y*-intercept of the line, we can substitute the needed values into the slope-intercept form, $y=mx+b$.

Name the slope.	$m=-9$
Name the *y*-intercept.	y=intercept is $(0,-4)$
Substitute the values into $y=mx+b$.	$y=mx+b$
	$y=-9x+(-4)$
	$y=-9x-4$

Try It 3.3.2

Find the equation of a line with slope $\frac{2}{5}$ and y-intercept $(0, 4)$.

Answer

$y = \frac{2}{5}x + 4$

Try It 3.3.3

Find the equation of a line with slope -1 and y-intercept $(0, -3)$.

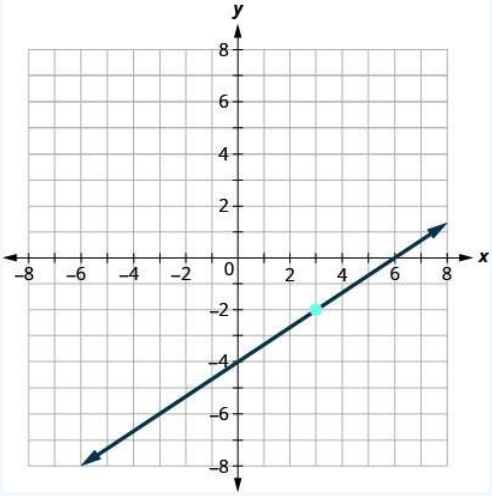
Answer

$y = -x - 3$

Sometimes, the slope and intercept need to be determined from the graph.

Example 3.3.4

Find the equation of the line shown in the graph.



Solution

We need to find the slope and y-intercept of the line from the graph so we can substitute the needed values into the slope-intercept form, $y = mx + b$.

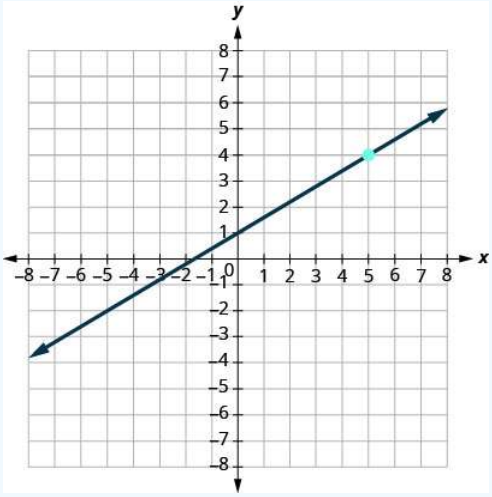
To find the slope, we choose two points on the graph.

The y-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Find the slope, by counting the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{3}$
Find the y-intercept.	y=intercept is $(0, -4)$
Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = \frac{2}{3}x - 4$

Try It 3.3.5

Find the equation of the line shown in the graph.

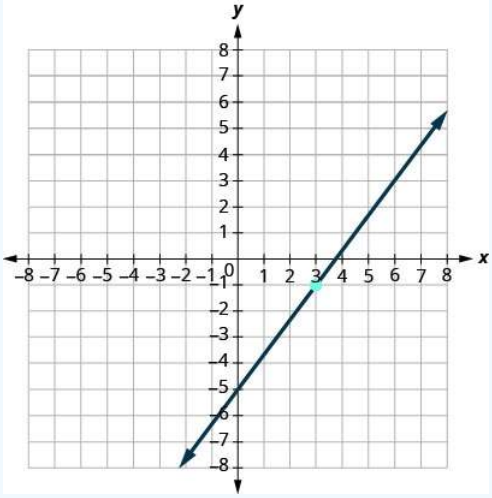


Answer

$y = \frac{3}{5}x + 1$

Try It 3.3.6

Find the equation of the line shown in the graph.



Answer

$y = \frac{4}{3}x - 5$

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope-intercept form of the equation works well when you are given the slope and *y*-intercept or when you read them off a graph. But what happens when you have another point instead of the *y*-intercept?

We are going to use the slope formula to derive another form of an equation of the line.

Suppose we have a line that has slope m and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form.

	m	$=$	$\frac{y-y_1}{x-x_1}$
Multiply both sides of the equation by $x-x_1$.	$m(x-x_1)$	$=$	$\left(\frac{y-y_1}{x-x_1}\right)(x-x_1)$
Simplify.	$m(x-x_1)$	$=$	$y-y_1$
Rewrite the equation with theyterms on the left.	$y-y_1$	$=$	$m(x-x_1)$

This format is called the **point-slope form** of an equation of a line.

POINT-SLOPE FORM OF AN EQUATION OF A LINE

The **point-slope form** of an equation of a line with slope m and containing the point (x_1, y_1) is:

$$y-y_1 = m(x-x_1)$$

We can use the point-slope form of an equation to find an equation of a line when we know the slope and at least one point. Then, we will rewrite the equation in slope-intercept form. Most applications of linear equations use the the slope-intercept form.

How to Find an Equation of a Line Given a Point and the Slope

Example 3.3.7

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope-intercept form.

Solution

Step 1. Identify the slope.	The slope is given.	$m = -\frac{1}{3}$
Step 2. Identify the point.	The point is given.	$\begin{pmatrix} x_1, & y_1 \\ 6 & -4 \end{pmatrix}$
Step 3. Substitute the values into the point-slope form, $y-y_1 = m(x-x_1)$	Simplify.	$y-y_1 = m(x-x_1)$ $y-(-4) = -\frac{1}{3}(x-6)$ $y+4 = -\frac{1}{3}x+2$
Step 4. Write the equation in slope intercept form.		$y = -\frac{1}{3}x-2$

Try It 3.3.8

Find the equation of a line with slope $m = -\frac{2}{5}$ and containing the point $(10, -5)$.

Answer

$y = -\frac{2}{5}x-1$

Try It 3.3.9

Find the equation of a line with slope $m = -\frac{3}{4}$, and containing the point $(4, -7)$.

Answer

$y = -\frac{3}{4}x-4$

We list the steps for easy reference.

TO FIND AN EQUATION OF A LINE GIVEN THE SLOPE AND A POINT.

1. Identify the slope.
2. Identify the point.
3. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
4. Write the equation in slope-intercept form.

Example 3.3.10

Find an equation of a horizontal line that contains the point $(-2,-6)$. Write the equation in slope-intercept form.

Solution

Every horizontal line has slope 0. We can substitute the slope and points into the point-slope form, $y-y_1=m(x-x_1)$.

Identify the slope.	$m=0$
Identify the point.	$\begin{pmatrix} x_1, & y_1 \\ -2 & -6 \end{pmatrix}$
Substitute the values into $y-y_1=m(x-x_1)$. Simplify.	$y-y_1=m(x-x_1)$ $y-(-6)=0(x--2)$ $y+6=0$
Write in slope-intercept form.	It is in y-form, but could be written $y=-6$.

Did we end up with the form of a horizontal line, $y=a$?

Try It 3.3.11

Find the equation of a horizontal line containing the point $(-3,8)$.

Answer

$y=8$

Try It 3.3.12

Find the equation of a horizontal line containing the point $(-1,4)$.

Answer

$y=4$

Find an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we'll see how to find an equation of a line when just two points are given.

So far, we have two options for finding an equation of a line: slope-intercept or point-slope. When we start with two points, it makes more sense to use the point-slope form.

But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find the equation.

How to Find the Equation of a Line Given Two Points

Example 3.3.13

Find an equation of a line that contains the points $(-3, -1)$ and $(2, -2)$ Write the equation in slope-intercept form.

Solution

Step 1. Find the slope using the given points.	Find the slope of the line through $(-3. -1)$ and $(2, -2)$.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{-2-(-1)}{2-(-3)}$ $m = \frac{-1}{5}$ $m = -\frac{1}{5}$
Step 2. Choose one point.	Choose either point.	$\begin{pmatrix} x_1, & y_1 \\ 2 & -2 \end{pmatrix}$
Step 3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - (-2) = -\frac{1}{5}(x - 2)$ $y + 2 = -\frac{1}{5}x + \frac{2}{5}$
Step 4. Write the equation in slope-intercept form.		$y = -\frac{1}{5}x - \frac{8}{5}$

Try It 3.3.14

Find the equation of a line containing the points $(-2, -4)$ and $(1, -3)$.

Answer

$y = \frac{1}{3}x - \frac{10}{3}$

Try It 3.3.15

Find the equation of a line containing the points $(-4, -3)$ and $(1, -5)$.

Answer

$y = -\frac{2}{5}x - \frac{23}{5}$

The steps are summarized here.

TO FIND AN EQUATION OF A LINE GIVEN TWO POINTS.

1. Find the slope using the given points. $m = \frac{y_2 - y_1}{x_2 - x_1}$
2. Choose one point.
3. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.
4. Write the equation in slope-intercept form.

Example 3.3.16

Find an equation of a line that contains the points $(-3, 5)$ and $(-3, 4)$. Write the equation in slope-intercept form.

Solution

Again, the first step will be to find the slope.

Find the slope of the line through$(-3, 5)$and$(-3, 4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4 - 5}{-3 - (-3)}$$

$$m = \frac{-1}{0}$$

The slope is undefined.

This tells us it is a vertical line. Both of our points have an *x*-coordinate of -3. So our equation of the line is $x = -3$. Since there is no *y*, we cannot write it in slope-intercept form.

You may want to sketch a graph using the two given points. Does your graph agree with our conclusion that this is a vertical line?

Try It 3.3.17

Find the equation of a line containing the points $(5, 1)$ and $(5, -4)$.

Answer

$x = 5$

Try It 3.3.18

Find the equation of a line containing the points $(-4, 4)$ and $(-4, 3)$.

Answer

$x = -4$

We have seen that we can use either the slope-intercept form or the point-slope form to find an equation of a line. Which form we use will depend on the information we are given.

To Write an Equation of a Line

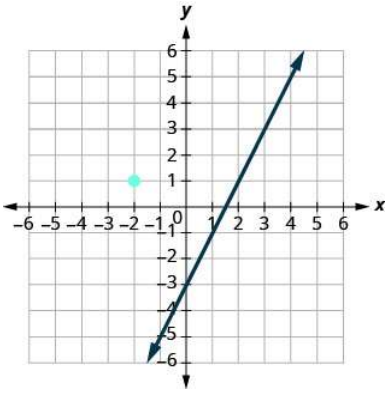
If given:	Use:	Form:
Slope and *y*-intercept	slope-intercept	$y = mx + b$
Slope and a point	point-slope	$y - y_1 = m(x - x_1)$
Two points	point-slope	$y - y_1 = m(x - x_1)$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point-slope equation.

First, let's look at this graphically.

This graph shows $y = 2x - 3$. We want to graph a line parallel to this line and passing through the point $(-2, 1)$.

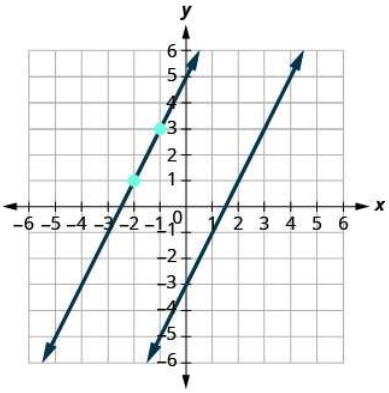


We know that parallel lines have the same slope. So the second line will have the same slope as $y = 2x - 3$. That slope is $m_{\parallel} = 2$. We'll use the notation $m_{\parallel}$ to represent the slope of a line parallel to a line with slope m. (Notice that the subscript $\parallel$ looks like two parallel lines.)

The second line will pass through $(-2, 1)$ and have $m = 2$.

To graph the line, we start at $(-2, 1)$ and count out the rise and run.

With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line, as shown in the graph.



Do the lines appear parallel? Does the second line pass through$(-2, 1)$?

We were asked to graph the line, now let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point-slope form.

How to Find the Equation of a Line Parallel to a Given Line and a Point

Example 3.3.19

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution

Step 1. Find the slope of the given line.	The given line is in slope-intercept form, $y = 2x - 3$	$m = 2$
Step 2. Find the slope of the parallel line.	Parallel lines have the same slope.	$m_{\parallel} = 2$
Step 3. Identify the point.	The given point is $(-2, 1)$.	$\begin{pmatrix} x_1, & y_1 \\ -2 & 1 \end{pmatrix}$

Step 4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-(1)=2(x-(-2))$ $y-1=2(x+2)$ $y-1=2x+4$
Step 5. Write the equation in slope-intercept form.		$y=2x+5$

Look at graph with the parallel lines shown previously. Does this equation make sense? What is the *y*-intercept of the line? What is the slope?

? Try It 3.3.20

Find an equation of a line parallel to the line $y=3x+1$ that contains the point $(4,2)$. Write the equation in slope-intercept form.

Answer

$y=3x-10$

? Try It 3.3.21

Find an equation of a line parallel to the line $y=12x-3$ that contains the point $(6,4)$.

Write the equation in slope-intercept form.

Answer

$y=\frac{1}{2}x+1$

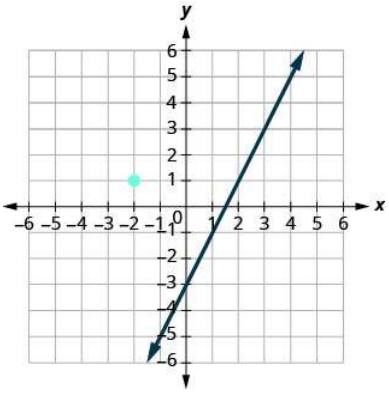
FIND AN EQUATION OF A LINE PARALLEL TO A GIVEN LINE.

1. Find the slope of the given line.
2. Find the slope of the parallel line.
3. Identify the point.
4. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
5. Write the equation in slope-intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let's consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point-slope equation, like we did with parallel lines.

This graph shows $y=2x-3$. Now, we want to graph a line perpendicular to this line and passing through $(-2,1)$.



We know that perpendicular lines have slopes that are negative reciprocals.

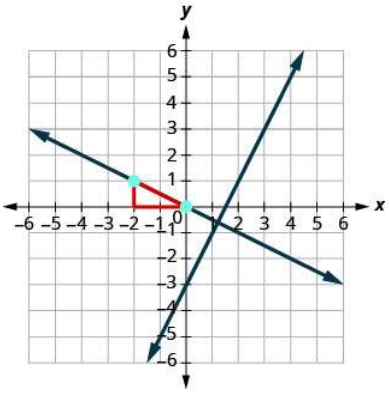
We'll use the notation $m_\perp$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

$$y = 2x - 3 \quad \text{perpendicular line}$$

$$m = 2 \quad m_\perp = -\frac{1}{2}$$

We now know the perpendicular line will pass through $(-2, 1)$ with $m_\perp = -\frac{1}{2}$.

To graph the line, we will start at $(-2, 1)$ and count out the rise -1 and the run 2. Then we draw the line.



Do the lines appear perpendicular? Does the second line pass through$(-2, 1)$?

We were asked to graph the line, now, let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point-slope form.

How to Find the Equation of a Line Perpendicular to a Given Line and a Point

Example 3.3.22

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution

Step 1. Find the slope of the given line.	The given line is in slope-intercept form, $y = 2x - 3$	$m = 2$
Step 2. Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative reciprocals.	$m_\perp = -\frac{1}{2}$

Step 3. Identify the point.	The given point is $(-2, 1)$.	$\begin{pmatrix} x_1, & y_1 \\ -2 & 1 \end{pmatrix}$
Step 4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-(1)=-\frac{1}{2}(x-(-2))$ $y-1=-\frac{1}{2}(x+2)$ $y-1=-\frac{1}{2}x-1$
Step 5. Write the equation in slope-intercept form.		$y=-\frac{1}{2}x$

Try It 3.3.23

Find an equation of a line perpendicular to the line $y=3x+1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

Answer

$y=-\frac{1}{3}x+\frac{10}{3}$

Try It 3.3.24

Find an equation of a line perpendicular to the line $y=12x-3$ that contains the point $(6, 4)$. Write the equation in slope-intercept form.

Answer

$y=-2x+16$

FIND AN EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE.

1. Find the slope of the given line.
2. Find the slope of the perpendicular line.
3. Identify the point.
4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
5. Write the equation in slope-intercept form.

Example 3.3.24

Find an equation of a line perpendicular to $x=5$ that contains the point $(3, -2)$. Write the equation in slope-intercept form.

Solution

Again, since we know one point, the point-slope option seems more promising than the slope-intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x=5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_\perp=0$.

Step 1. Identify the point.	The given point is $(3, -2)$.	$\begin{pmatrix} x_1, & y_1 \\ 3 & -2 \end{pmatrix}$
Step 2. Identify the slope of the perpendicular line.		$m_\perp=0$

Step 3. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-(-2)=0(x-(3))$
Step 4. Simplify		$y+2=0$ $y=-2$

Sketch the graph of both lines. On your graph, do the lines appear to be perpendicular?

Try It 3.3.25

Find an equation of a line that is perpendicular to the line $x=4$ that contains the point $(4,-5)$. Write the equation in slope-intercept form.

Answer

$y=-5$

Try It 3.3.26

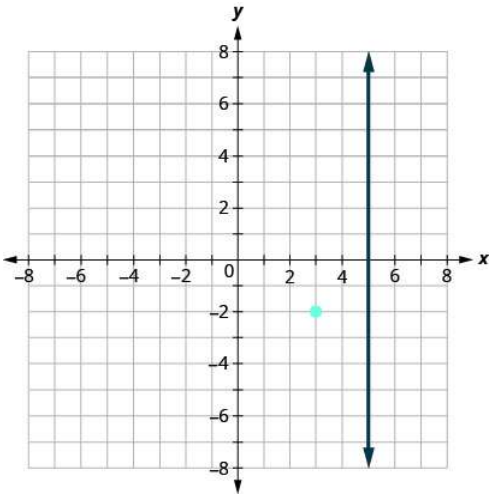
Find an equation of a line that is perpendicular to the line $x=2$ that contains the point $(2,-1)$. Write the equation in slope-intercept form.

Answer

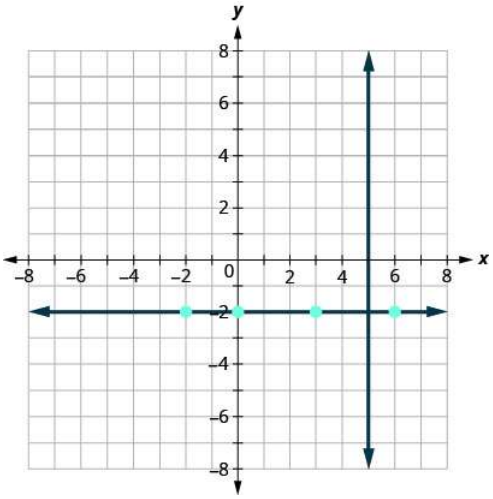
$y=-1$

In the Example above, we used the point-slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x=5$ that contains the point $(3,-2)$. This graph shows us the line $x=5$ and the point $(3,-2)$.



We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3,-2)$.



Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have *y*-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x = 5$ is $y = -2$.

Example 3.3.27

Find an equation of a line that is perpendicular to $y = -3$ that contains the point $(-3, 5)$. Write the equation in slope-intercept form.

Solution

The line $y = -3$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x = a$. Since the perpendicular line is vertical and passes through $(-3, 5)$, every point on it has an *x*-coordinate of -3. The equation of the perpendicular line is $x = -3$.

You may want to sketch the lines. Do they appear perpendicular?

Try It 3.3.28

Find an equation of a line that is perpendicular to the line $y = 1$ that contains the point $(-5, 1)$. Write the equation in slope-intercept form.

Answer

$x = -5$

Try It 3.3.29

Find an equation of a line that is perpendicular to the line $y = -5$ that contains the point $(-4, -5)$. Write the equation in slope-intercept form.

Answer

$x = -4$

Access these online resources for additional instruction and practice with finding the equation of a line.

- Write an Equation of Line Given its slope and Y-Intercept
- Using Point Slope Form to Write the Equation of a Line, Find the equation given slope and point
- Find the equation given two points
- Find the equation of perpendicular and parallel lines

Key Concepts

- **How to find an equation of a line given the slope and a point.**
  1. Identify the slope.
  2. Identify the point.
  3. Substitute the values into the point-slope form,$y-y_1=m(x-x_1)$.
  4. Write the equation in slope-intercept form.

- **How to find an equation of a line given two points.**
  1. Find the slope using the given points. $m=\frac{y_2-y_1}{x_2-x_1}$
  2. Choose one point.
  3. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
  4. Write the equation in slope-intercept form.

To Write an Equation of a Line		
If given:	**Use:**	**Form:**
Slope and *y*-intercept	**slope-intercept**	$y=mx+b$
Slope and a point	**point-slope**	$y-y_1=m(x-x_1)$
Two points	**point-slope**	$y-y_1=m(x-x_1)$

- **How to find an equation of a line parallel to a given line.**
  1. Find the slope of the given line.
  2. Find the slope of the parallel line.
  3. Identify the point.
  4. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
  5. Write the equation in slope-intercept form
- **How to find an equation of a line perpendicular to a given line.**
  1. Find the slope of the given line.
  2. Find the slope of the perpendicular line.
  3. Identify the point.
  4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
  5. Write the equation in slope-intercept form.

Glossary

point-slope form

The point-slope form of an equation of a line with slope *m* and containing the point (x_1,y_1) is $y-y_1=m(x-x_1)$.

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3.3E: Exercises

Practice Makes Perfect

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and *y*-intercept. Write the equation in slope-intercept form.

1. slope 3 and y-intercept $(0, 5)$

Answer

$y = 3x + 5$

2. slope 8 and y-intercept $(0, -6)$

3. slope -3 and y-intercept $(0, -1)$

Answer

$y = -3x - 1$

4. slope -1 and y-intercept $(0, 3)$

5. slope $\frac{1}{5}$ and y-intercept $(0, -5)$

Answer

$y = \frac{1}{5}x - 5$

6. slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

7. slope 0 and y-intercept $(0, -1)$

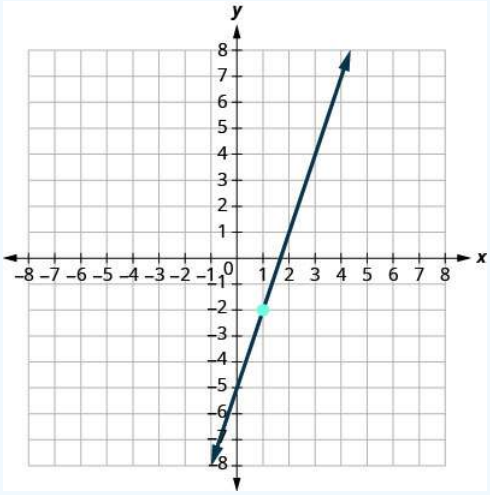
Answer

$y = -1$

8. slope -4 and y-intercept $(0, 0)$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope-intercept form.

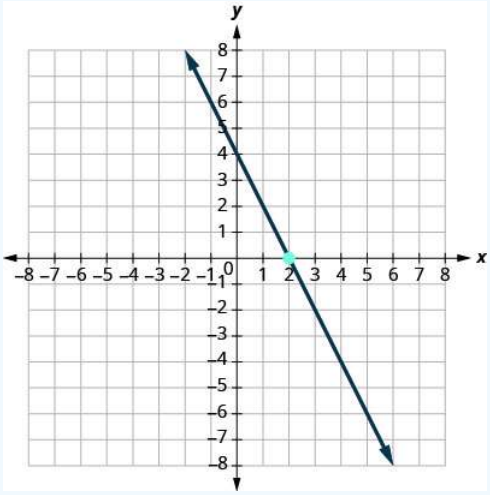
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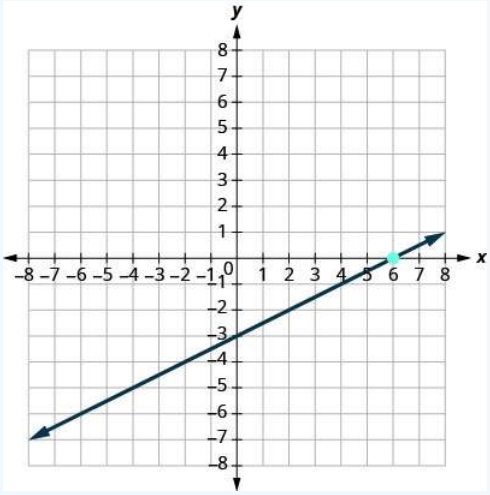
Answer

$y = 3x - 5$

10.



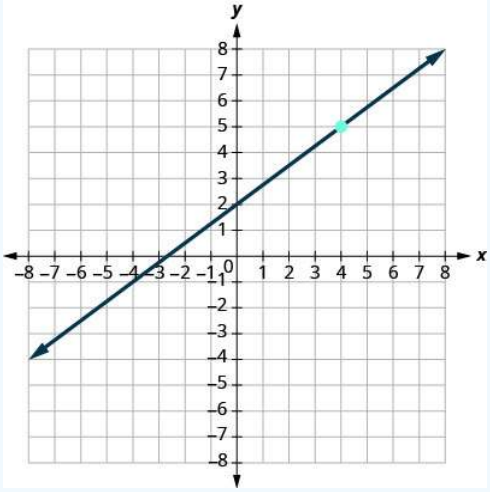
11.



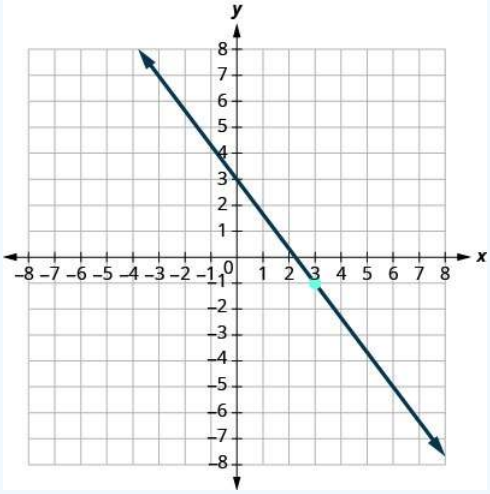
Answer

$y = \frac{1}{2}x - 3$

12.



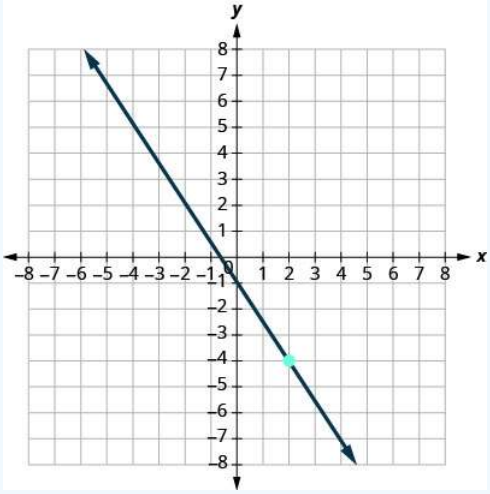
13.



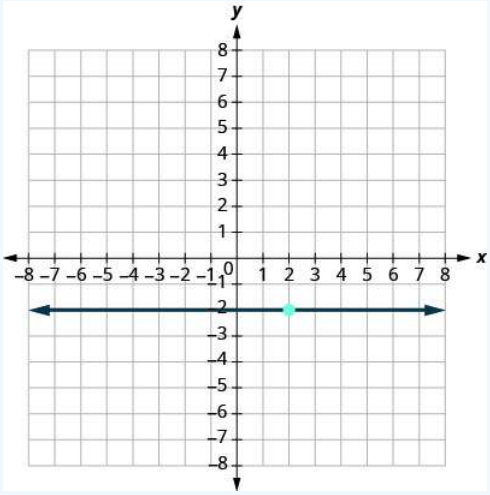
Answer

$y = -\frac{4}{3}x + 3$

14.



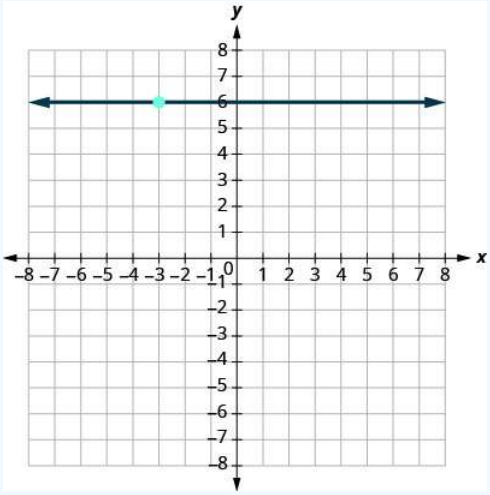
15.



Answer

$y = -2$

16.



Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope-intercept form.

17. $m = \frac{5}{8}$, point $(8, 3)$

Answer

$y = \frac{5}{8}x - 2$

18. $m = \frac{5}{6}$, point $(6, 7)$

19. $m = -\frac{3}{5}$, point $(10, -5)$

Answer

$y = -\frac{3}{5}x + 1$

20. $m = -\frac{3}{4}$, point $(8, -5)$

21. $m = -\frac{3}{2}$, point $(-4, -3)$

Answer

$y = -\frac{3}{2}x + 9$

22. $m = -\frac{5}{2}$, point $(-8, -2)$

23. $m = -7$, point $(-1, -3)$

Answer

$y = -7x - 10$

24. $m = -4$, point $(-2, -3)$

25. Horizontal line containing $(-2, 5)$

Answer

$y = 5$

26. Horizontal line containing $(-2, -3)$

27. Horizontal line containing $(-1, -7)$

Answer

$y = -7$

28. Horizontal line containing $(4, -8)$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope-intercept form.

29. $(2,6)$ and $(5,3)$

Answer

$y=-x+8$

30. $(4,3)$ and $(8,1)$

31. $(-3,-4)$ and $(5-2)$.

Answer

$y=\frac{1}{4}x-\frac{13}{4}$

32. $(-5,-3)$ and $(4,-6)$.

33. $(-1,3)$ and $(-6,-7)$.

Answer

$y=2x+5$

34. $(-2,8)$ and $(-4,-6)$.

35. $(0,4)$ and $(2,-3)$.

Answer

$y=-\frac{7}{2}x+4$

36. $(0,-2)$ and $(-5,-3)$.

37. $(7,2)$ and $(7,-2)$.

Answer

$x=7$

38. $(-2,1)$ and $(-2,-4)$.

39. $(3,-4)$ and $(5,-4)$.

Answer

$y=-4$

40. $(-6,-3)$ and $(-1,-3)$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope-intercept form.

41. line $y=4x+2$, point $(1,2)$

Answer

$y = 4x - 2$

42. line $y = -3x - 1$, point $2, -3)$.

43. line $2x - y = 6$, point $(3, 0)$.

Answer

$y = 2x - 6$

44. line $2x + 3y = 6$, point $(0, 5)$.

45. line $x = -4$, point $(-3, -5)$.

Answer

$x = -3$

46. line $x - 2 = 0$, point $(1, -2)$

47. line $y = 5$, point $(2, -2)$

Answer

$y = -2$

48. line $y + 2 = 0$, point $(3, -3)$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope-intercept form.

49. line $y = -2x + 3$, point $(2, 2)$

Answer

$y = \frac{1}{2}x + 1$

50. line $y = -x + 5$, point $(3, 3)$

51. line $y = \frac{3}{4}x - 2$, point $(-3, 4)$

Answer

$y = -\frac{4}{3}x$

52. line $y = \frac{2}{3}x - 4$, point $(2, -4)$

53. line $2x - 3y = 8$, point $(4, -1)$

Answer

$y = -\frac{3}{2}x + 5$

54. line $4x-3y=5$, point $(-3,2)$

55. line $2x+5y=6$, point $(0,0)$

Answer

$y=\frac{5}{2}x$

56. line $4x+5y=-3$, point $(0,0)$

57. line $x=3$, point $(3,4)$

Answer

$y=4$

58. line $x=-5$, point $(1,-2)$

59. line $x=7$, point $(-3,-4)$

Answer

$y=-4$

60. line $x=-1$, point $(-4,0)$

61. line $y-3=0$, point $(-2,-4)$

Answer

$x=-2$

62. line $y-6=0$, point $(-5,-3)$

63. line y-axis, point $(3,4)$

Answer

$y=4$

64. line y-axis, point $(2,1)$

Mixed Practice

In the following exercises, find the equation of each line. Write the equation in slope-intercept form.

65. Containing the points $(4,3)$ and $(8,1)$

Answer

$y=-\frac{1}{2}x+5$

66. Containing the points $(-2,0)$ and $(-3,-2)$

67. $m=\frac{1}{6}$, containing point $(6,1)$

Answer

$y = \frac{1}{6}x$

68. $m = \frac{5}{6}$, containing point $(6, 7)$

69. Parallel to the line $4x + 3y = 6$, containing point $(0, -3)$

Answer

$y = -\frac{4}{3}x - 3$

70. Parallel to the line $2x + 3y = 6$, containing point $(0, 5)$

71. $m = -\frac{3}{4}$, containing point $(8, -5)$

Answer

$y = -\frac{3}{4}x + 1$

72. $m = -\frac{3}{5}$, containing point $(10, -5)$

73. Perpendicular to the line $y - 1 = 0$, point $(-2, 6)$

Answer

$x = -2$

74. Perpendicular to the line *y*-axis, point $(-6, 2)$

75. Parallel to the line $x = -3$, containing point $(-2, -1)$

Answer

$x = -2$

76. Parallel to the line $x = -4$, containing point $(-3, -5)$

77. Containing the points $(-3, -4)$ and $(2, -5)$

Answer

$y = -\frac{1}{5}x - \frac{23}{5}$

78. Containing the points $(-5, -3)$ and $(4, -6)$

79. Perpendicular to the line $x - 2y = 5$, point $(-2, 2)$

Answer

$y = -2x - 2$

80. Perpendicular to the line $4x + 3y = 1$, point $(0, 0)$

Writing Exercises

81. Why are all horizontal lines parallel?

Answer

Answers will vary.

82. Explain in your own words why the slopes of two perpendicular lines must have opposite signs.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the equation of the line given the slope and y-intercept.			
find an equation of the line given the slope and a point.			
find an equation of the line given two points.			
find an equation of a line parallel to a given line.			
find an equation of a line perpendicular to a given line.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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3.4: Graph Linear Inequalities in Two Variables

SUMMARY

By the end of this section, you will be able to:

- Verify solutions to an inequality in two variables.
- Recognize the relation between the solutions of an inequality and its graph.
- Graph linear inequalities in two variables
- Solve applications using linear inequalities in two variables

Before you get started, take this readiness quiz.

1. Graph $x > 2$ on a number line.
   If you missed this problem, review [link].
2. Solve: $4x + 3 > 23$.
   If you missed this problem, review [link].
3. Translate: $8 < x > 3$.
   If you missed this problem, review [link].

Verify Solutions to an Inequality in Two Variables

Previously we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables. In particular we will look at **linear inequalities** in two variables which are very similar to linear equations in two variables.

Linear inequalities in two variables have many applications. If you ran a business, for example, you would want your revenue to be greater than your costs—so that your business made a profit.

LINEAR INEQUALITY

A **linear inequality** is an inequality that can be written in one of the following forms:

$$Ax + By > C \quad Ax + By \geq C \quad Ax + By < C \quad Ax + By \leq C$$

Where *A* and *B* are not both zero.

Recall that an inequality with one variable had many solutions. For example, the solution to the inequality x>3x>3 is any number greater than 3. We showed this on the number line by shading in the number line to the right of 3, and putting an open parenthesis at 3. See Figure.

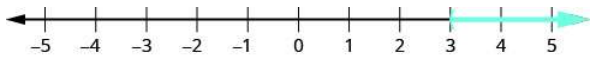


Figure 3.4.1

Similarly, linear inequalities in two variables have many solutions. Any ordered pair (x,y)(x,y) that makes an inequality true when we substitute in the values is a **solution to a linear inequality**.

SOLUTION TO A LINEAR INEQUALITY

An ordered pair (x, y) is a **solution to a linear inequality** if the inequality is true when we substitute the values of *x* and *y*.

Example 3.4.1

Determine whether each ordered pair is a solution to the inequality y>x+4:y>x+4:

ⓐ (0,0)(0,0) ⓑ (1,6)(1,6) ⓒ (2,6)(2,6) ⓓ (−5,−15)(−5,−15) ⓔ (−8,12)(−8,12)

Answer

ⓐ

$(0, 0)$	$y > x + 4$
Substitute 0 for x and 0 for y.	$0 \stackrel{?}{>} 0 + 4$
Simplify.	$0 \not> 4$
	So, $(0, 0)$ is not a solution to $y > x + 4$.

ⓑ

$(1, 6)$	$y > x + 4$
Substitute 1 for x and 6 for y.	$6 \stackrel{?}{>} 1 + 4$
Simplify.	$6 > 5$
	So, $(1, 6)$ is a solution to $y > x + 4$.

ⓒ

$(2, 6)$	$y > x + 4$
Substitute 2 for x and 6 for y.	$6 \stackrel{?}{>} 2 + 4$
Simplify.	$6 \not> 6$
	So, $(2, 6)$ is not a solution to $y > x + 4$.

ⓓ

$(-5, -15)$	$y > x + 4$
Substitute –5 for x and –15 for y.	$-15 \stackrel{?}{>} -5 + 4$
Simplify.	$-15 \not> -1$
	So, $(-5, -15)$ is not a solution to $y > x + 4$.

ⓔ

$(-8, 12)$	$y > x + 4$
Substitute –8 for x and 12 for y.	$12 \stackrel{?}{>} -8 + 4$
Simplify.	$12 > -4$
	So, $(-8, 12)$ is a solution to $y > x + 4$.

Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

ⓐ $(0, 0)$ ⓑ $(4, 9)$ ⓒ $(-2, 1)$ ⓓ $(-5, -3)$ ⓔ $(5, 1)$

Answer

ⓐ yes ⓑ yes ⓒ yes ⓓ yes ⓔ no

Example 3.4.3

Determine whether each ordered pair is a solution to the inequality $y < x + 1$:

ⓐ $(0, 0)$ ⓑ $(8, 6)$ ⓒ $(-2, -1)$ ⓓ $(3, 4)$ ⓔ $(-1, -4)$

Answer

ⓐ yes ⓑ yes ⓒ no ⓓ no ⓔ yes

Recognize the Relation Between the Solutions of an Inequality and its Graph

Now, we will look at how the solutions of an inequality relate to its graph.

Let's think about the number line in shown previously again. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3. On the other side of 3 all the numbers are greater than 3. See Figure.

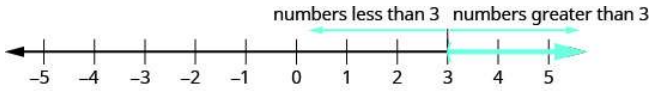


Figure 3.4.2: The solution to $x > 3$ is the shaded part of the number line to the right of $x = 3$.

Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a **boundary line**.

BOUNDARY LINE

The line with equation $Ax + By = C$ is the **boundary line** that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

For an inequality in one variable, the endpoint is shown with a parenthesis or a bracket depending on whether or not a is included in the solution:



Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not it the line is included in the solution.

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is $Ax + By = C$.	Boundary line is $Ax + By = C$.
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	**Boundary line is solid.**

Now, let's take a look at what we found in Example. We'll start by graphing the line $y = x + 4$, and then we'll plot the five points we tested, as shown in the graph. See Figure.

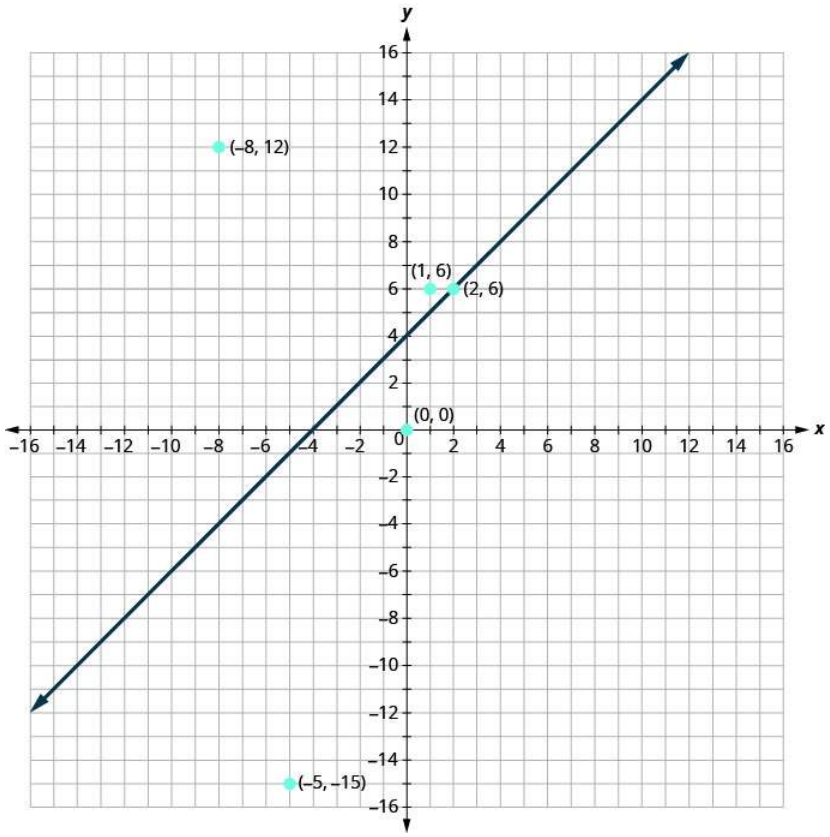


Figure 3.4.3

In Example we found that some of the points were solutions to the inequality $y > x + 4$ and some were not.

Which of the points we plotted are solutions to the inequality $y > x + 4$?

The points $(1, 6)$ and $(-8, 12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y = x + 4$.

The two points $(0, 0)$ and $(-5, -15)$ are on the other side of the boundary line $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points, $y < x + 4$.

What about the point $(2, 6)$? Because $6 = 2 + 4$, the point is a solution to the equation $y = x + 4$, but not a solution to the inequality $y > x + 4$. So the point $(2, 6)$ is on the boundary line.

Let's take another point above the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0, 10)$clearly looks to above the boundary line, doesn't it? Is it a solution to the inequality?

$$\begin{array}{ccc} y & > & x+4 \\ 10 & \stackrel{?}{>} & 0+4 \\ 10 & > & 4 \end{array}$$

So, $(0, 10)$ is a solution to $y > x + 4$.

Any point you choose above the boundary line is a solution to the inequality $y > x + 4$. All points above the boundary line are solutions.

Similarly, all points below the boundary line, the side with $(0, 0)$ and $(-5, -15)$, are not solutions to $y > x + 4$, as shown in Figure.

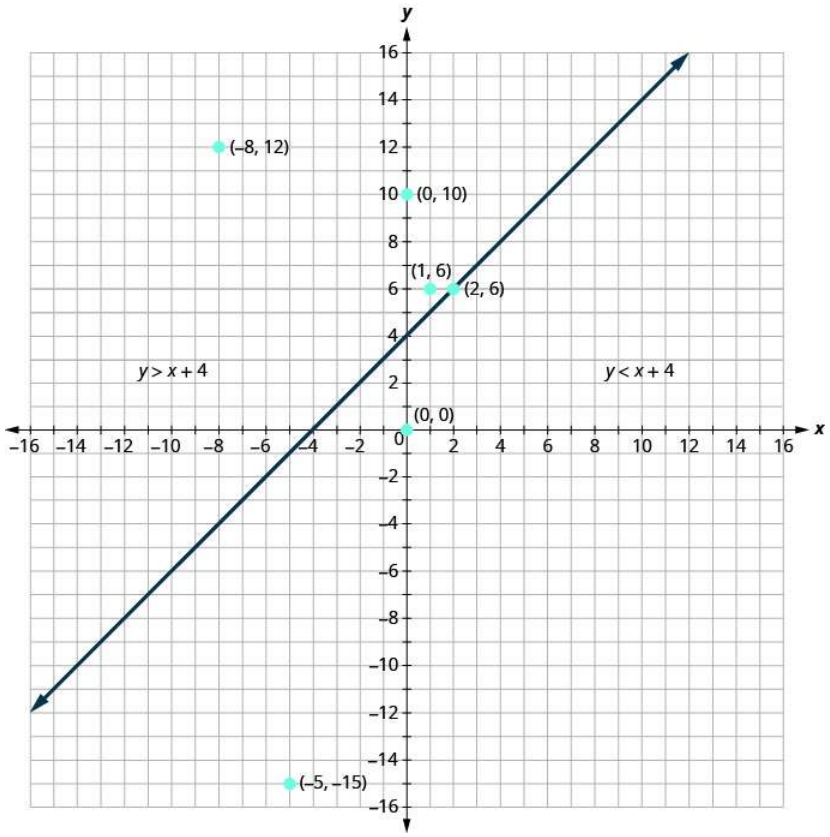
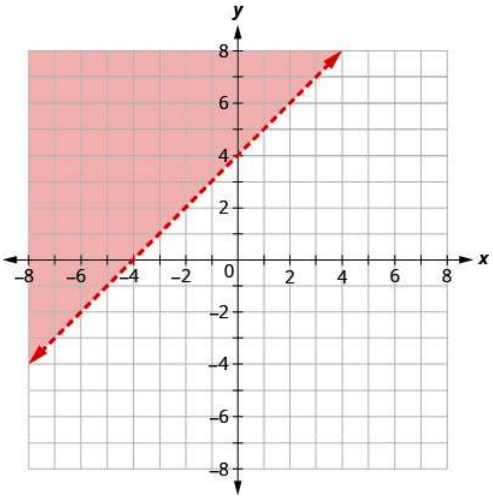


Figure 3.4.4

The graph of the inequality $y > x + 4$ is shown in below.

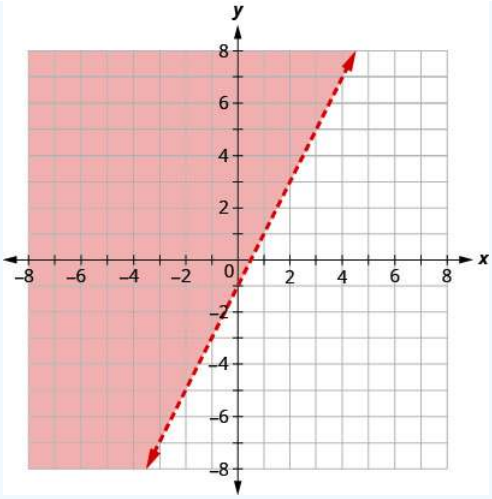
The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$.

The points on the boundary line, those where $y = x + 4$, are not solutions to the inequality $y > x + 4$, so the line itself is not part of the solution. We show that by making the line dashed, not solid.



Example 3.4.4

The boundary line shown in this graph is $y = 2x - 1$. Write the inequality shown by the graph.



Answer

The line $y = 2x - 1$ is the boundary line. On one side of the line are the points with $y > 2x - 1$ and on the other side of the line are the points with $y < 2x - 1$.

Let's test the point $(0, 0)$ and see which inequality describes its position relative to the boundary line.

At $(0, 0)$, which inequality is true: $y > 2x - 1$ or $y < 2x - 1$?

$$\begin{array}{ll} y > 2x - 1 & y < 2x - 1 \\ 0 \overset{?}{>} 2 \cdot 0 - 1 & 0 \overset{?}{<} 2 \cdot 0 - 1 \\ 0 > -1 \text{ True} & 0 < -1 \text{ False} \end{array}$$

Since, $y > 2x - 1$ is true, the side of the line with $(0, 0)$, is the solution. The shaded region shows the solution of the inequality $y > 2x - 1$.

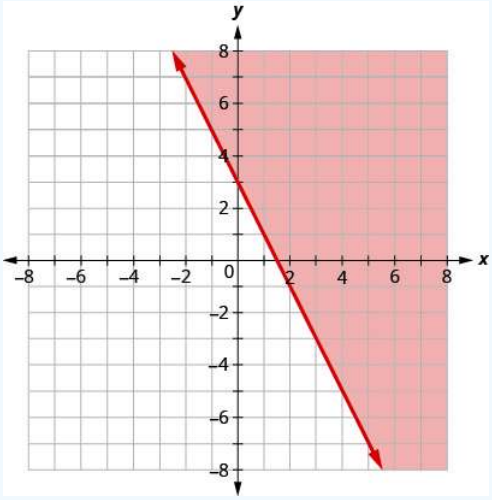
Since the boundary line is graphed with a solid line, the inequality includes the equal sign.

The graph shows the inequality $y \geq 2x - 1$.

We could use any point as a test point, provided it is not on the line. Why did we choose $(0, 0)$? Because it's the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y < 2x - 1$.

Example 3.4.5

Write the inequality shown by the graph with the boundary line $y = -2x + 3$.

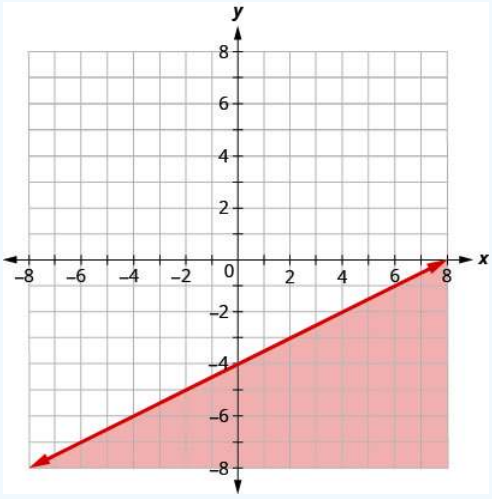


Answer

$y \geq -2x + 3$

Example 3.4.6

Write the inequality shown by the graph with the boundary line $y = \frac{1}{2}x - 4$.

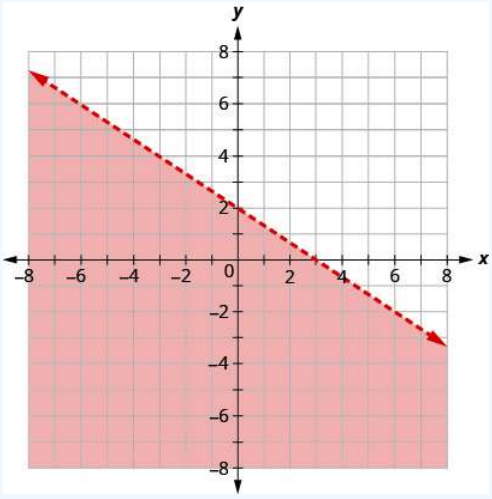


Answer

$y \leq \frac{1}{2}x - 4$

Example 3.4.7

The boundary line shown in this graph is $2x + 3y = 6$. Write the inequality shown by the graph.



Answer

The line $2x + 3y = 6$ is the boundary line. On one side of the line are the points with $2x + 3y > 6$ and on the other side of the line are the points with $2x + 3y < 6$.

Let's test the point $(0, 0)$ and see which inequality describes its side of the boundary line.

At $(0, 0)$, which inequality is true: $2x + 3y > 6$ or $2x + 3y < 6$?

$$\begin{array}{ll} 2x+3y>6 & 2x+3y<6 \\ 2(0)+3(0)\overset{?}{>}6 & 2(0)+3(0)\overset{?}{<}6 \\ 0>6 \text{ False} & 0<6 \text{ True} \end{array}$$

So the side with $(0,0)$ is the side where $2x+3y<6$.

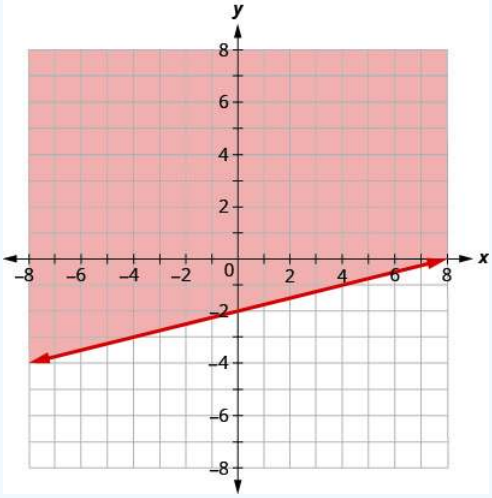
(You may want to pick a point on the other side of the boundary line and check that $2x+3y>6$.)

Since the boundary line is graphed as a dashed line, the inequality does not include an equal sign.

The shaded region shows the solution to the inequality $2x+3y<6$.

Example 3.4.8

Write the inequality shown by the shaded region in the graph with the boundary line $x-4y=8$.

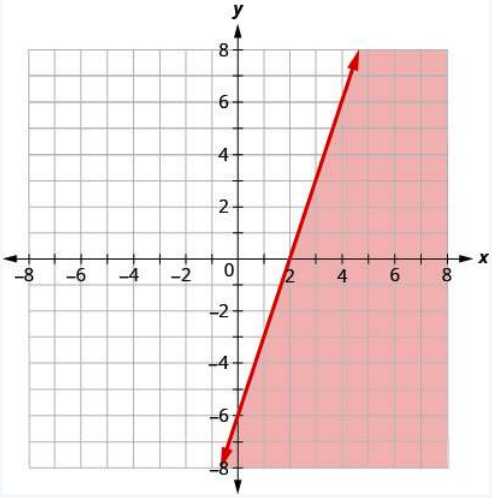


Answer

$x-4y\leq 8$

Example 3.4.9

Write the inequality shown by the shaded region in the graph with the boundary line $3x-y=6$.



Answer

$3x - y \geq 6$

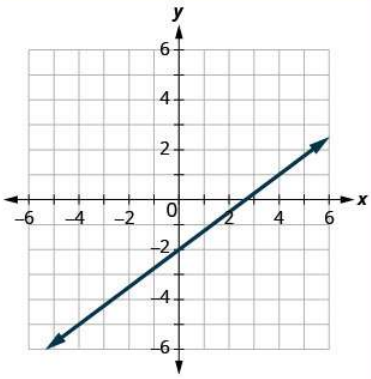
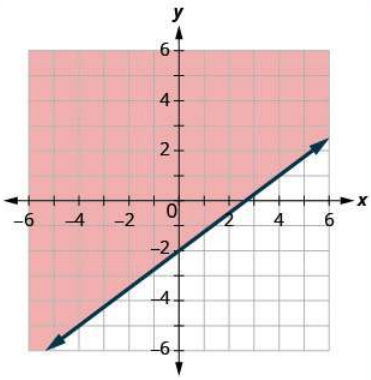
Graph Linear Inequalities in Two Variables

Now that we know what the graph of a linear inequality looks like and how it relates to a boundary equation we can use this knowledge to graph a given linear inequality.

Example 3.4.10: How to Graph a Linear Equation in Two Variables

Graph the linear inequality $y \geq \frac{3}{4}x - 2$.

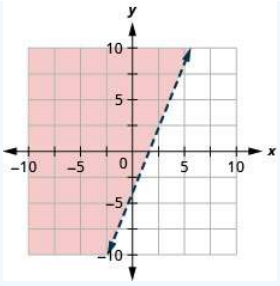
Answer

Step 1. Identify and graph the boundary line. • If the inequality is $\leq$ or $\geq$, the boundary line is solid. • If the inequality is $<$ or $>$, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line $y = \frac{3}{4}x - 2$. The inequality sign is $\geq$, so we draw a solid line.	
Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?	We'll test (0, 0). Is it a solution of the inequality?	At (0, 0), is $y \geq \frac{3}{4}x - 2$? $0 \stackrel{?}{\geq} \frac{3}{4}(0) - 2$ $0 \geq -2$ So, (0, 0) is a solution.
Step 3. Shade in one side of the boundary line. • If the test point is a solution, shade in the side that includes the point. • If the test point is not a solution, shade in the opposite side.	The test point (0, 0) is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.	 All points in the shaded region and on the boundary line represent the solutions to $y \geq \frac{3}{4}x - 2$.

Example 3.4.11

Graph the linear inequality $y > \frac{5}{2}x - 4$.

Answer

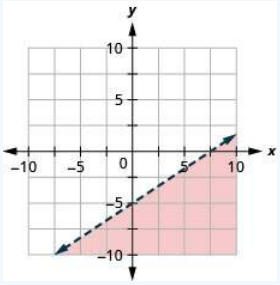


All points in the shaded region and on the boundary line, represent the solutions to $y > \frac{5}{2}x - 4$.

Example 3.4.12

Graph the linear inequality $y < \frac{2}{3}x - 5$.

Answer



All points in the shaded region, but not those on the boundary line, represent the solutions to $y < \frac{2}{3}x - 5$.

The steps we take to graph a linear inequality are summarized here.

GRAPH A LINEAR INEQUALITY IN TWO VARIABLES.

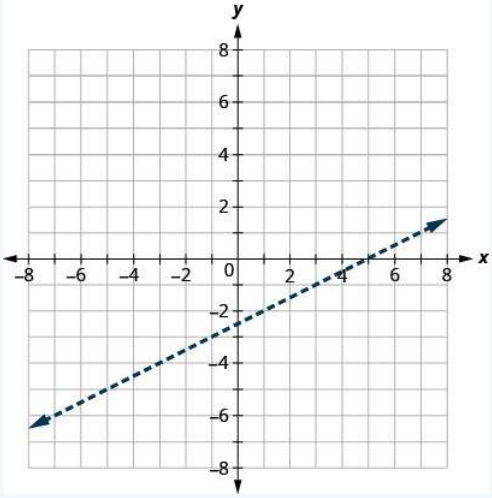
1. Identify and graph the boundary line.
   - If the inequality is \leq or\geq ,\leq or\geq , the boundary line is solid.
   - If the inequality is <or>,<or>, the boundary line is dashed.
2. Test a point that is not on the boundary line. Is it a solution of the inequality?
3. Shade in one side of the boundary line.
   - If the test point is a solution, shade in the side that includes the point.
   - If the test point is not a solution, shade in the opposite side.

Example 3.4.13

Graph the linear inequality $x - 2y < 5$.

Answer

First, we graph the boundary line $x - 2y = 5$. The inequality is $<$ so we draw a dashed line.

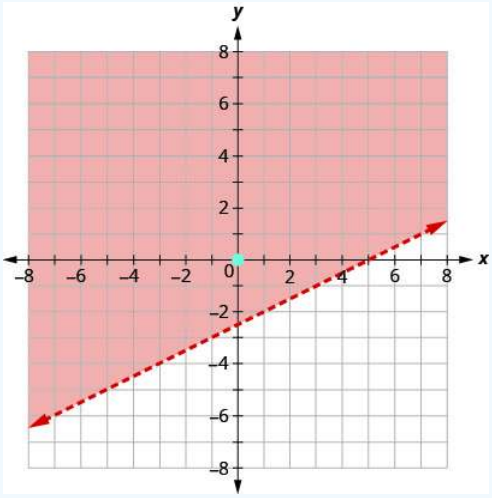


Then, we test a point. We'll use $(0,0)$ again because it is easy to evaluate and it is not on the boundary line.

Is $(0,0)$ a solution of $x-2y<5$?

$$\begin{aligned} 0-2(0) &\stackrel{?}{<} 5 \\ 0-0 &\stackrel{?}{<} 5 \\ 0 &< 5 \end{aligned}$$

The point $(0,0)$ is a solution of $x-2y<5$, so we shade in that side of the boundary line.

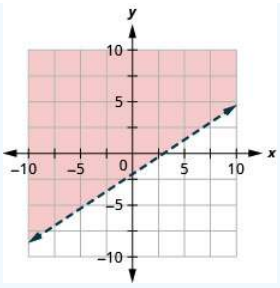


All points in the shaded region, but not those on the boundary line, represent the solutions to $x-2y<5$.

? Example 3.4.14

Graph the linear inequality: $2x-3y<6$.

Answer

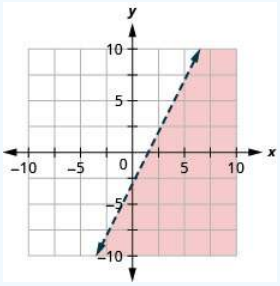


All points in the shaded region, but not those on the boundary line, represent the solutions to $2x - 3y < 6$.

? Example 3.4.15

Graph the linear inequality: $2x - y > 3$.

Answer



All points in the shaded region, but not those on the boundary line, represent the solutions to $2x - y > 3$.

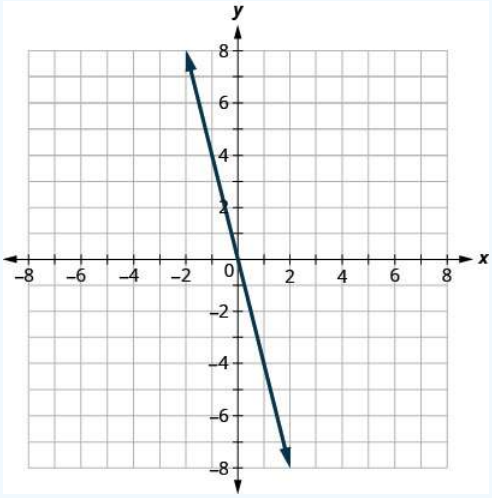
What if the boundary line goes through the origin? Then, we won't be able to use $(0, 0)$ as a test point. No problem—we'll just choose some other point that is not on the boundary line.

? Example 3.4.16

Graph the linear inequality: $y \leq -4x$.

Answer

First, we graph the boundary line $y = -4x$. It is in slope–intercept form, with $m = -4$ and $b = 0$. The inequality is $\leq$ so we draw a solid line.



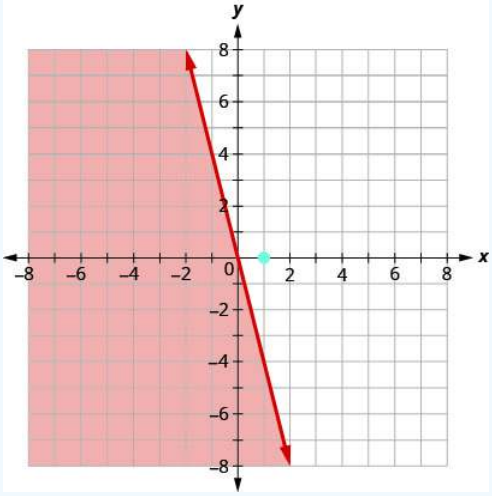
Now we need a test point. We can see that the point (1,0)(1,0) is not on the boundary line.

Is $(1, 0)$ a solution of $y \leq -4x$?

$0 \stackrel{?}{\leq} -4(1)$

$0 \not\leq -4$

The point $(1, 0)$ is not a solution to $y \leq -4x$, so we shade in the opposite side of the boundary line.

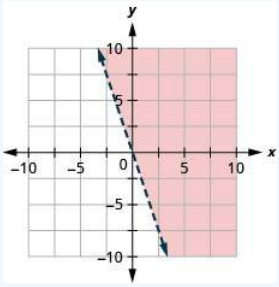


All points in the shaded region and on the boundary line represent the solutions to $y \leq -4x$.

Example 3.4.17

Graph the linear inequality: $y > -3x$.

Answer

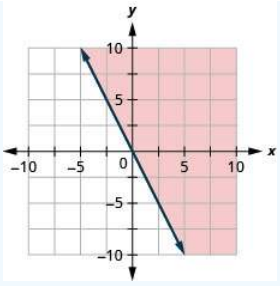


All points in the shaded region, but not those on the boundary line, represent the solutions to $y > -3x$.

Example 3.4.18

Graph the linear inequality: $y \geq -2x$.

Answer



All points in the shaded region and on the boundary line, represent the solutions to $y \geq -2x$.

Some linear inequalities have only one variable. They may have an *x* but no *y*, or a *y* but no *x*. In these cases, the boundary line will be either a vertical or a horizontal line.

Recall that:

$$\begin{array}{ll} x = a & \text{vertical line} \\ y = b & \text{horizontal line} \end{array}$$

? Example 3.4.19

Graph the linear inequality: $y > 3$.

Answer

First, we graph the boundary line $y = 3$. It is a horizontal line. The inequality is $>$ so we draw a dashed line.

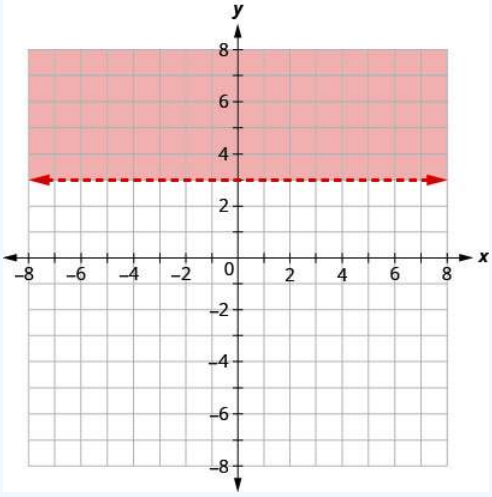
We test the point $(0, 0)$.

$$y > 3$$

$$0 \backslash \text{slashed} > 3$$

So, $(0, 0)$ is not a solution to $y > 3$.

So we shade the side that does not include $(0, 0)$ as shown in this graph.

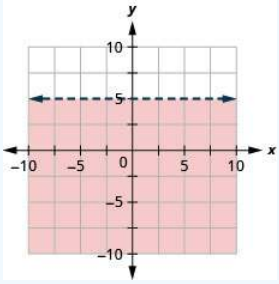


All points in the shaded region, but not those on the boundary line, represent the solutions to $y > 3$.

? Example 3.4.20

Graph the linear inequality: $y < 5$.

Answer

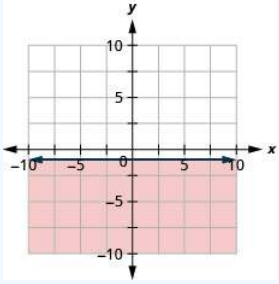


All points in the shaded region, but not those on the boundary line, represent the solutions to $y < 5$.

? Example 3.4.21

Graph the linear inequality: $y \leq -1$.

Answer



All points in the shaded region and on the boundary line represent the solutions to $y \leq -1$.

Solve Applications using Linear Inequalities in Two Variables

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how thay might be used.

? Example 3.4.22

Hilaria works two part time jobs in order to earn enough money to meet her obligations of at least $240 a week. Her job in food service pays $10 an hour and her tutoring job on campus pays $15 an hour. How many hours does Hilaria need to work at each job to earn at least $240?

ⓐ Let xx be the number of hours she works at the job in food service and let *y* be the number of hours she works tutoring. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

Answer

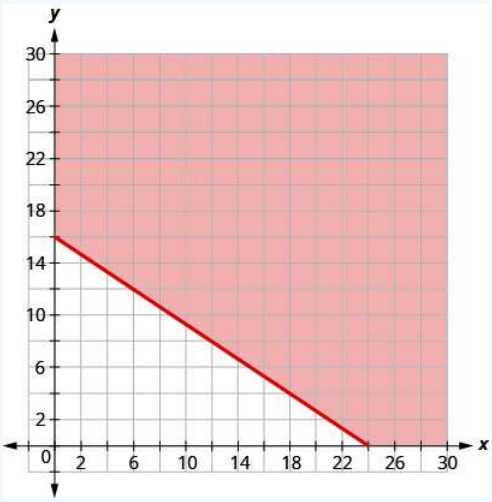
ⓐ We let *x* be the number of hours she works at the job in food service and let *y* be the number of hours she works tutoring.

She earns $10 per hour at the job in food service and $15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will gives the amount earned at that job.

Amount earned at the food service job	plus	the amount earned tutoring	is at least	$240
$10x$	+	$15y$	≥	240

ⓑ To graph the inequality, we put it in slope–intercept form.

$$10x + 15y \geq 240 \tag{3.4.1}$$
$$15y \geq -10x + 240 \tag{3.4.2}$$
$$y \geq -\frac{2}{3}x + 16 \tag{3.4.3}$$



ⓒ From the graph, we see that the ordered pairs $(15, 10)$, $(0, 16)$, $(24, 0)$ represent three of infinitely many solutions. Check the values in the inequality.

(15, 10)	(0, 16)	(24, 0)
$10x + 15y \geq 240$	$10x + 15y \geq 240$	$10x + 15y \geq 240$
$10(15) + 15(10) \stackrel{?}{\geq} 240$	$10(0) + 15(16) \stackrel{?}{\geq} 240$	$10(24) + 15(0) \stackrel{?}{\geq} 240$
$300 \geq 240$ True	$240 \geq 240$ True	$240 \geq 240$ True

For Hilaria, it means that to earn at least $240, she can work 15 hours tutoring and 10 hours at her fast-food job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

? Example 3.4.23

Hugh works two part time jobs. One at a grocery store that pays $10 an hour and the other is babysitting for $13 hour. Between the two jobs, Hugh wants to earn at least $260 a week. How many hours does Hugh need to work at each job to earn at least $260?

ⓐ Let x be the number of hours he works at the grocery store and let y be the number of hours he works babysitting. Write an inequality that would model this situation.

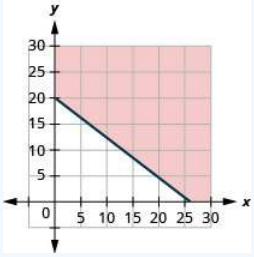
ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hugh.

Answer

ⓐ $10x + 13y \geq 260$

ⓑ



ⓒ Answers will vary..

? Example 3.4.24

Veronica works two part time jobs in order to earn enough money to meet her obligations of at least $280 a week. Her job at the day spa pays $10 an hour and her administrative assistant job on campus pays $17.50 an hour. How many hours does Veronica need to work at each job to earn at least $280?

ⓐ Let x be the number of hours she works at the day spa and let y be the number of hours she works as administrative assistant. Write an inequality that would model this situation.

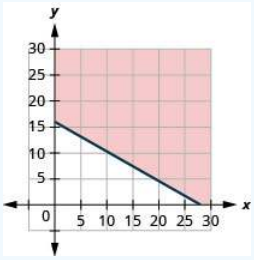
ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Veronica

Answer

ⓐ $10x + 17.5y \geq 280$

ⓑ



ⓒ Answers will vary.

Access this online resource for additional instruction and practice with graphing linear inequalities in two variables.

- Graphing Linear Inequalities in Two Variables

Key Concepts

- **How to graph a linear inequality in two variables.**
  1. Identify and graph the boundary line.
     If the inequality is $\leq$ or $\geq$, the boundary line is solid.
     If the inequality is $<$ or $>$, the boundary line is dashed.
  2. Test a point that is not on the boundary line. Is it a solution of the inequality?
  3. Shade in one side of the boundary line.
     If the test point is a solution, shade in the side that includes the point.
     If the test point is not a solution, shade in the opposite side.

Glossary

boundary line

The line with equation $Ax + By = C$ is the boundary line that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

linear inequality

A linear inequality is an inequality that can be written in one of the following forms: $Ax + By > C$, $Ax + By \geq C$, $Ax + By < C$, or $Ax + By \leq C$, where *A* and *B* are not both zero.

solution to a linear inequality

An ordered pair (x, y) is a solution to a linear inequality if the inequality is true when we substitute the values of *x* and *y*.

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3.4E: Exercises

Practice Makes Perfect

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

1. Determine whether each ordered pair is a solution to the inequality $y > x - 1$:

a. $(0, 1)$
b. $(-4, -1)$
c. $(4, 2)$
d. $(3, 0)$
e. $(-2, -3)$

Answer

a. yes b. yes c. no d. no e. no

2. Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

a. $(0, 0)$
b. $(2, 1)$
c. $(-1, -5)$
d. $(-6, -3)$
e. $(1, 0)$

3. Determine whether each ordered pair is a solution to the inequality $y < 3x + 2$:

a. $(0, 3)$
b. $(-3, -2)$
c. $(-2, 0)$
d. $(0, 0)$
e. $(-1, 4)$

Answer

a. no b. no c. yes d. yes e. no

4. Determine whether each ordered pair is a solution to the inequality $y < -2x + 5$:

a. $(-3, 0)(-3, 0)b.$ \((1, 6)$
c. $(-6, -2)$
d. $(0, 1)$
e. $(5, -4)$

5. Determine whether each ordered pair is a solution to the inequality $3x - 4y > 4$:

a. $(5, 1)$
b. $(-2, 6)$
c. $(3, 2)$
d. $(10, -5)$
e. $(0, 0)$

Answer

a. yes b. no c. no d. no e. no

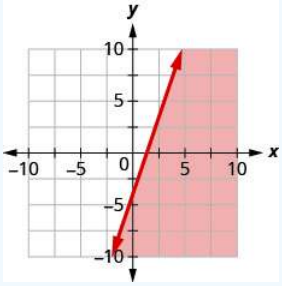
6. Determine whether each ordered pair is a solution to the inequality $2x+3y>2$:

a. $(1,1)$
b. $(4,-3)$
c. $(0,0)$
d. $(-8,12)$
e. $(3,0)$

Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

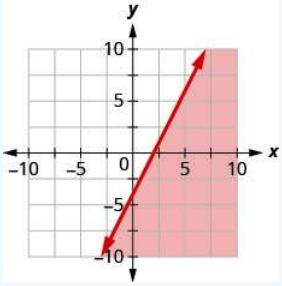
7. Write the inequality shown by the graph with the boundary line $y=3x-4$.



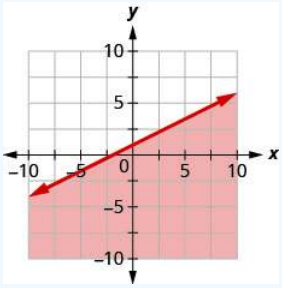
Answer

$y \leq 3x-4$

8. Write the inequality shown by the graph with the boundary line $y=2x-4$.



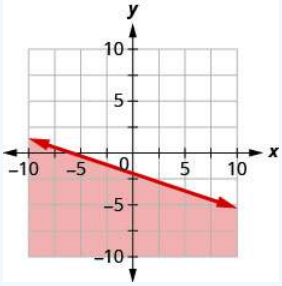
9. Write the inequality shown by the graph with the boundary line $y=-\frac{1}{2}x+1$.



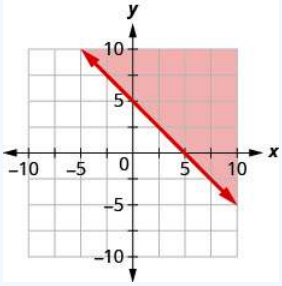
Answer

$y \leq -\frac{1}{2}x+1$

10. Write the inequality shown by the graph with the boundary line $y=-\frac{1}{3}x-2$.



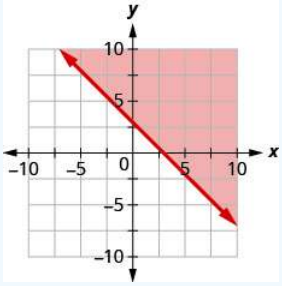
11. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 5$.



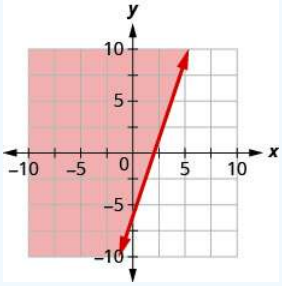
Answer

$x + y \geq 5$

12. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 3$.



13. Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.

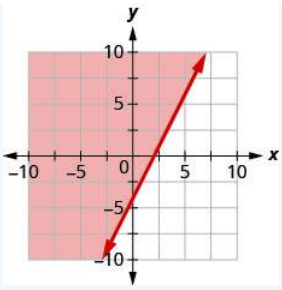


Answer

$3x - y \leq 6$

14. Write the inequality shown by the shaded region in the graph with the boundary line $2x - y = 4$.

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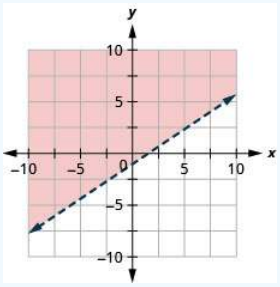


Graph Linear Inequalities in Two Variables

In the following exercises, graph each linear inequality.

15. Graph the linear inequality: $y > \frac{2}{3}x - 1$.

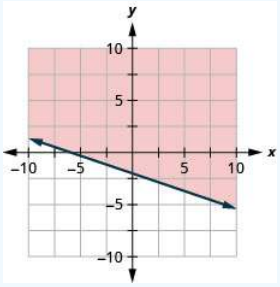
Answer



16. Graph the linear inequality: $y < \frac{3}{5}x + 2$.

17. Graph the linear inequality: $y \leq -\frac{1}{2}x + 4$.

Answer

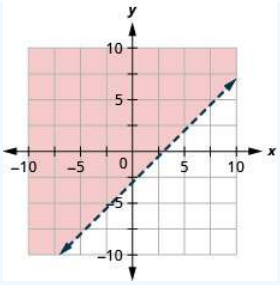


18. Graph the linear inequality: $y \geq -\frac{1}{3}x - 2$.

19. Graph the linear inequality: $x - y \leq 3$.

Answer

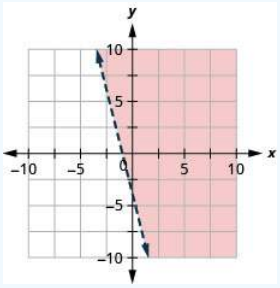
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20. Graph the linear inequality: $x - y \geq -2$.

21. Graph the linear inequality: $4x + y > -4$.

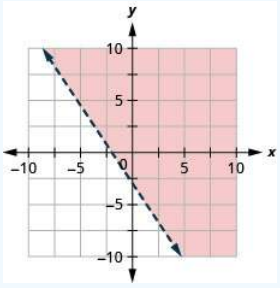
Answer



22. Graph the linear inequality: $x + 5y < -5$.

23. Graph the linear inequality: $3x + 2y \geq -6$.

Answer

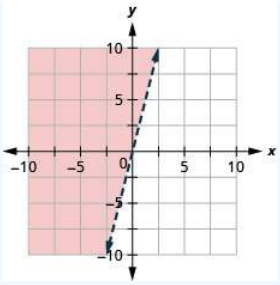


24. Graph the linear inequality: $4x + 2y \geq -8$.

25. Graph the linear inequality: $y > 4x$.

Answer

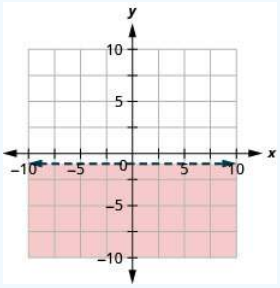
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26. Graph the linear inequality: $y \leq -3x$.

27. Graph the linear inequality: $y < -10$.

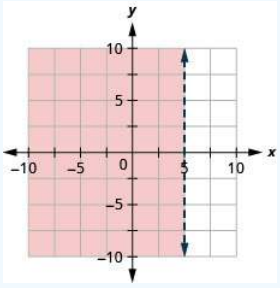
Answer



28. Graph the linear inequality: $y \geq 2$.

29. Graph the linear inequality: $x \leq 5$.

Answer

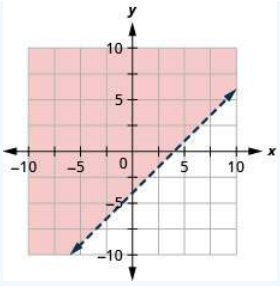


30. Graph the linear inequality: $x \geq 0$.

31. Graph the linear inequality: $x - y < 4$.

Answer

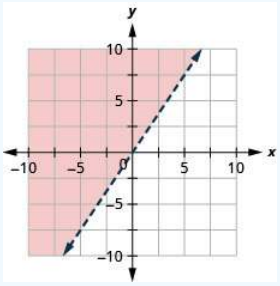
https://math.libretexts.org/@go/page/30830



32. Graph the linear inequality: $x - y < -3$.

33. Graph the linear inequality: $y \geq \frac{3}{2}x$.

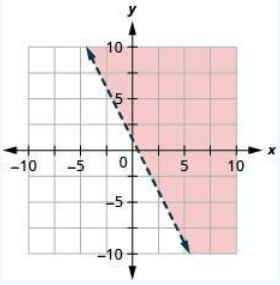
Answer



34. Graph the linear inequality: $y \leq \frac{5}{4}x$.

35. Graph the linear inequality: $y > -2x + 1$.

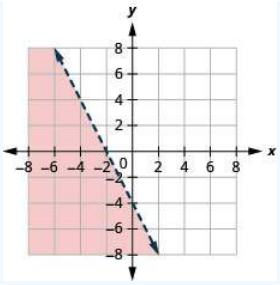
Answer



36. Graph the linear inequality: $y < -3x - 4$.

37. Graph the linear inequality: $2x + y \geq -4$.

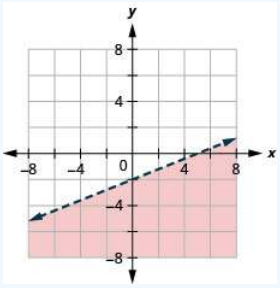
Answer



38. Graph the linear inequality: $x + 2y \leq -2$.

39. Graph the linear inequality: $2x - 5y > 10$.

Answer



40. Graph the linear inequality: $4x - 3y > 12$.

Solve Applications using Linear Inequalities in Two Variables

41. Harrison works two part time jobs. One at a gas station that pays $11 an hour and the other is IT troubleshooting for $16.50$16.50an hour. Between the two jobs, Harrison wants to earn at least $330 a week. How many hours does Harrison need to work at each job to earn at least $330?

a. Let *x* be the number of hours he works at the gas station and let *y* be the number of (hours he works troubleshooting. Write an inequality that would model this situation.

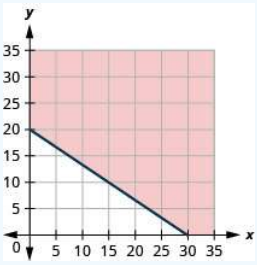
b. Graph the inequality.

c. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Harrison.

Answer

a. $11x + 16.5y \geq 330$

b.



c. Answers will vary.

42. Elena needs to earn at least $450 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $9 an hour and the other as an intern in a genetics lab for $22.50 per hour. How many hours does Elena need to work at each job to earn at least $450 per week?

a. Let x be the number of hours she works teaching swimming and let y be the number of hours she works as an intern. Write an inequality that would model this situation.

b. Graph the inequality.

c. Find three ordered pairs (x,y)(x,y) that would be solutions to the inequality. Then, explain what that means for Elena.

43. The doctor tells Laura she needs to exercise enough to burn 500 calories each day. She prefers to either run or bike and burns 15 calories per minute while running and 10 calories a minute while biking.

a. If x is the number of minutes that Laura runs and y is the number minutes she bikes, find the inequality that models the situation.

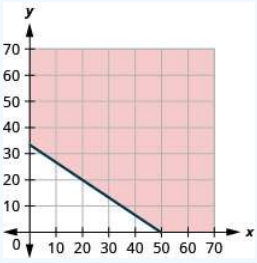
b. Graph the inequality.

c. List three solutions to the inequality. What options do the solutions provide Laura?

Answer

a. $15x + 10y \geq 500$

b.



c. Answers will vary.

44. Armando's workouts consist of kickboxing and swimming. While kickboxing, he burns 10 calories per minute and he burns 7 calories a minute while swimming. He wants to burn 600 calories each day.

a. If x is the number of minutes that Armando will kickbox and y is the number minutes he will swim, find the inequality that will help Armando create a workout for today.

b. Graph the inequality.

c. List three solutions to the inequality. What options do the solutions provide Armando?

Writing Exercises

45. Lester thinks that the solution of any inequality with a >> sign is the region above the line and the solution of any inequality with a << sign is the region below the line. Is Lester correct? Explain why or why not.

Answer

Answers will vary.

46. Explain why, in some graphs of linear inequalities, the boundary line is solid but in other graphs it is dashed.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
verify solutions to an inequality in two variables.			
recognize the relation between the solutions of an inequality and its graph.			
graph linear inequalities.			

b. On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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3.5: Relations and Functions

Learning Objectives

By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function

Before you get started, take this readiness quiz.

1. Evaluate $3x-5$ when $x=-2$.
   If you missed this problem, review [link].
2. Evaluate $2x^2-x-3$ when $x=a$.
   If you missed this problem, review [link].
3. Simplify: $7x-1-4x+5$.
   If you missed this problem, review [link].

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.

$$(\text{Student name},\ \text{Student ID }\#)$$

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Definition: Relation

A **relation** is any set of ordered pairs, (x,y). All the x-values in the ordered pairs together make up the **domain**. All the y-values in the ordered pairs together make up the **range**.

Example 3.5.1

For the relation $(1,1),(2,4),(3,9),(4,16),(5,25)$:

a. Find the domain of the relation.
b. Find the range of the relation.

Answer

$\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$

ⓐ The domain is the set of all x-values of the relation. $\{1,2,3,4,5\}$

ⓑ The range is the set of all y-values of the relation. $\{1,4,9,16,25\}$

Example 3.5.2

For the relation $\{(1,1),(2,8),(3,27),(4,64),(5,125)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer a

$\{1,2,3,4,5\}$

Answer b

$\{1,8,27,64,125\}$

Example 3.5.3

For the relation $\{(1,3),(2,6),(3,9),(4,12),(5,15)\}$

a. Find the domain of the relation.
b. Find the range of the relation.

Answer a

$\{1,2,3,4,5\}$

Answer b

$\{3,6,9,12,15\}$

MAPPING

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

Example 3.5.4

Use the **mapping** of the relation shown to

a. list the ordered pairs of the relation,
b. find the domain of the relation, and
c. find the range of the relation.



Answer

ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person's name as the *x*-value and their birthday as the *y*-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all x-values of the relation.

{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

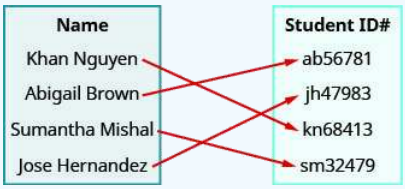
ⓒ The range is the set of all y-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

Example 3.5.5

Use the mapping of the relation shown to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ (Khanh Nguyen, kn68413), (Abigail Brown, ab56781), (Sumantha Mishal, sm32479), (Jose Hern and ez, jh47983)

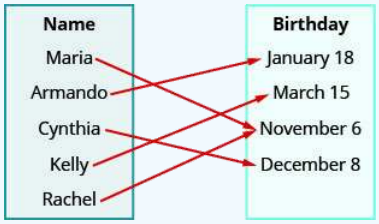
ⓑ {Khanh Nguyen, Abigail Brown, Sumantha Mishal, Jose Hern and ez}

ⓒ {kn68413, ab56781, sm32479, jh47983}

Example 3.5.6

Use the mapping of the relation shown to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ (Maria, November 6), (Arm and o, January 18), (Cynthia, December 8), (Kelly, March 15), (Rachel, November 6)

ⓑ {Maria, Arm and o, Cynthia, Kelly, Rachel}

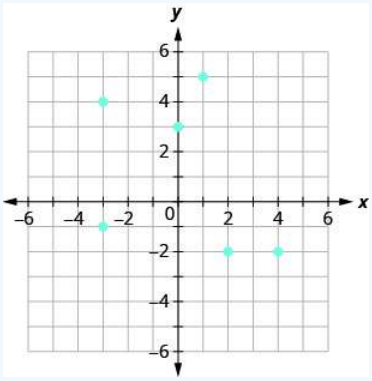
ⓒ{November 6, January 18, December 8, March 15}

A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all x-coordinates is the domain of the relation and the set of all y-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

Example 3.5.7

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ The ordered pairs of the relation are:

$$\{(1,5),(-3,-1),(4,-2),(0,3),(2,-2),(-3,4)\}.$$

ⓑ The domain is the set of all x-values of the relation: $\{-3,0,1,2,4\}$

Notice that while -3 repeats, it is only listed once.

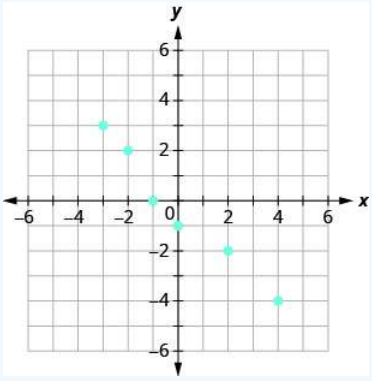
ⓒ The range is the set of all y-values of the relation: $\{-2,-1,3,4,5\}$

Notice that while -2 repeats, it is only listed once.

Example 3.5.8

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

a. $(-3,3),(-2,2),(-1,0),(0,-1),(2,-2),(4,-4)$

b. $\{-3,-2,-1,0,2,4\}$

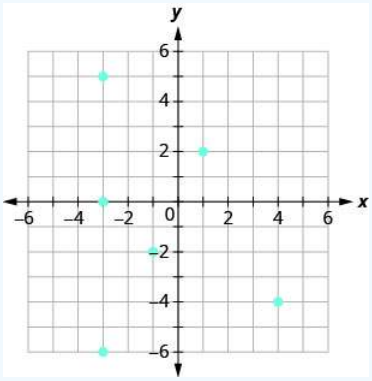
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c. $\{3, 2, 0, -1, -2, -4\}$

Example 3.5.9

Use the graph of the relation to

a. list the ordered pairs of the relation
b. find the domain of the relation
c. find the range of the relation.



Answer

ⓐ $(-3, 0), (-3, 5), (-3, -6), (-1, -2), (1, 2), (4, -4)$
ⓑ $\{-3, -1, 1, 4\}$
ⓒ $\{-6, 0, 5, -2, 2, -4\}$

Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

Definition: Function

A **function** is a relation that assigns to each element in its domain exactly one element in the range.

The birthday example from Example helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation in Example is a function.

The relation shown by the graph in Example includes the ordered pairs $(-3, -1)$ and $(-3, 4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

Example 3.5.10

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

a. $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$
b. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

Answer

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

(i) Each x-value is matched with only one y-value. So this relation is a function.

(ii) The domain is the set of all x-values in the relation.
The domain is: $\{-3, -2, -1, 0, 1, 2, 3\}$

(iii) The range is the set of all y-values in the relation. Notice we do not list range values twice.
The range is: $\{27, 8, 1, 0\}$

ⓑ $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

(i) The x-value 9 is matched with two y-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all x-values in the relation. Notice we do not list domain values twice.
The domain is: $\{0, 1, 2, 4, 9\}$

(iii) The range is the set of all y-values in the relation.
The range is: $\{-3, -2, -1, 0, 1, 2, 3\}$

Example 3.5.11

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the function.

a. $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$
b. $\{(8, -4), (4, -2), (2, -1), (0, 0), (2, 1), (4, 2), (8, 4)\}$

Answer

ⓐ Yes; $\{-3, -2, -1, 0, 1, 2, 3\}$
$\{-6, -4, -2, 0, 2, 4, 6\}$
ⓑ No; $\{0, 2, 4, 8\}$
$\{-4, -2, -1, 0, 1, 2, 4\}$

Example 3.5.12

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

a. $\{(27, -3), (8, -2), (1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$
b. $\{(7, -3), (-5, -4), (8, -0), (0, 0), (-6, 4), (-2, 2), (-1, 3)\}$

Answer

ⓐ No; $\{0, 1, 8, 27\}$
$\{-3, -2, -1, 0, 2, 2, 3\}$
ⓑ Yes; $\{7, -5, 8, 0, -6, -2, -1\}$
$\{-3, -4, 0, 4, 2, 3\}$

Example 3.5.13

Use the mapping to

a. determine whether the relation is a function
b. find the domain of the relation
c. find the range of the relation.

NAME	PHONE

Lydia	321-549-3327 427-658-2314
Eugene	321-964-7324
Janet	684-358-7961
Rick	
Marty	684-369-7231 798-367-8541

Answer

ⓐ Both Lydia and Marty have two phone numbers. So each x-value is not matched with only one y-value. So this relation is not a function.

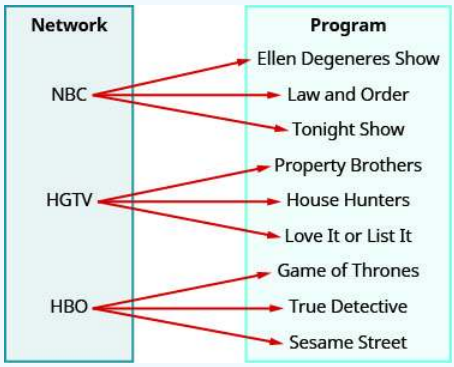
ⓑ The domain is the set of all x-values in the relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

ⓒ The range is the set of all y-values in the relation. The range is:

$\{321-549-3327, 427-658-2314, 321-964-7324, 684-358-7961, 684-369-7231, 798-367-8541\}$

Example 3.5.14

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Answer

ⓐ no ⓑ {NBC, HGTV, HBO} ⓒ {Ellen Degeneres Show, Law and Order, Tonight Show, Property Brothers, House Hunters, Love it or List it, Game of Thrones, True Detective, Sesame Street}

Example 3.5.15

Use the mapping to

a. determine whether the relation is a function
b. find the domain of the relation
c. find the range of the relation.

Neal	123-567-4839 work 231-378-5941 cell
Krystal	743-469-9731 cell
Kelvin	567-534-2970 work
George	684-369-7231 cell
Christa	684-369-7231 cell
Mike	639-847-6971 cell

Answer

ⓐ No ⓑ {Neal, Krystal, Kelvin, George, Christa, Mike} ⓒ {123-567-4839 work, 231-378-5941 cell, 743-469-9731 cell, 567-534-2970 work, 684-369-7231 cell, 798-367-8541 cell, 639-847-6971 cell}

In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for *y*. If each value of *x* results in only one value of *y*, then the equation defines a function.

? Example 3.5.16

Determine whether each equation is a function.

a. $2x+y=7$
b. $y=x^2+1$
c. $x+y^2=3$

Answer

ⓐ $2x+y=7$

For each value of *x*, we multiply it by -2 and then add 7 to get the *y*-value

	$y=-2x+7$
For example, if $x=3$:	$y=-2\cdot 3+7$
	$y=1$

We have that when $x=3$, then $y=1$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓑ $y=x^2+1$

For each value of *x*, we square it and then add 1 to get the *y*-value.

	$y=x^2+1$
For example, if $x=2$:	$y=2^2+1$
	$y=5$

We have that when $x=2$, then $y=5$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓒ

	$x+y^2=3$
Isolate the y term.	$y^2=-x+3$
Let's substitute $x=2$.	$y^2=-2+3$
	$y^2=1$
This give us two values for y.	$y=1\ y=-1$

We have shown that when $x=2$, then $y=1$ and $y=-1$. It would work similarly for any value of x. Since each value of x does not corresponds to only one value of y the equation does not define a function.

Example 3.5.17

Determine whether each equation is a function.

a. $4x+y=-3$
b. $x+y^2=1$
c. $y-x^2=2$

Answer

ⓐ yes ⓑ no ⓒ yes

Example 3.5.18

Determine whether each equation is a function.

a. $x+y^2=4$
b. $y=x^2-7$
c. $y=5x-4$

Answer

ⓐ no ⓑ yes ⓒ yes

Find the Value of a Function

It is very convenient to name a function and most often we name it f, g, h, F, G, or H. In any function, for each x-value from the domain we get a corresponding y-value in the range. For the function f, we write this range value y as $f(x)$. This is called **function notation** and is read f of x or the value of f at x. In this case the parentheses does not indicate multiplication.

Definition: Function Notation

For the function $y=f(x)$

f is the name of the function
x is the domain value
$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as f of x or the value of f at x.

We call x the independent variable as it can be any value in the domain. We call y the dependent variable as its value depends on x.

INDEPENDENT AND DEPENDENT VARIABLES

For the function $y = f(x)$,

x is the independent variable as it can be any value in the domain
y the dependent variable as its value depends on x

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y = 4x - 5$. To find the value of *y* when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let x=2.	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let x=2.	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of *x* is called *evaluating the function.*

Example 3.5.19

For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

a. $f(3)$
b. $f(-2)$
c. $f(a)$

Answer

ⓐ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(3)$, substitute 3 for *x*.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$
	$f(3) = 26$

ⓑ

		$f(x) = 2x^2 + 3x - 1$
To evaluate $f(-2)$, substitute -2 for *x*.		$f(-2) = 2(-2)^2 + 3(-2) - 1$

Simplify.		$f(-2) = 2 \cdot 4 + (-6) - 1$
		$f(-2) = 8 + (-6) - 1$
		$f(-2) = 1$

ⓒ

	$f(x) = 2x^2 + 3x - 1$
To evaluate f(a),f(a), substitute a for x.	$f(a) = 2(a)^2 + 3 \cdot a - 1$
Simplify.	$f(a) = 2a^2 + 3a - 1$

? Example 3.5.20

For the function $f(x) = 3x^2 - 2x + 1$, evaluate the function.

a. $f(3)$
b. $f(-1)$
c. $f(t)$

Answer

ⓐ $f(3) = 22$ ⓑ $f(-1) = 6$ ⓒ $f(t) = 3t^2 - 2t - 1$

? Example 3.5.21

For the function $f(x) = 2x^2 + 4x - 3$, evaluate the function.

a. $f(2)$
b. $f(-3)$
c. $f(h)$

Answer

ⓐ $(2) = 13$ ⓑ $f(-3) = 3$
ⓒ $f(h) = 2h2 + 4h - 3$

In the last example, we found $f(x)$ for a constant value of x. In the next example, we are asked to find $g(x)$ with values of x that are variables. We still follow the same procedure and substitute the variables in for the x.

? Example 3.5.22

For the function $g(x) = 3x - 5$, evaluate the function.

a. $g(h^2)$
b. $g(x + 2)$
c. $g(x) + g(2)$

Answer

ⓐ

			$g(x) = 3x - 5$
To evaluate $g(h^2)$, substitute h^2 for x.			$g(h^2) = 3h^2 - 5$

			$g(h^2) = 3h^2 - 5$

ⓑ

	$g(x) = 3x - 5$
To evaluate $g(x+2)$, substitute $x+2$ for x.	$g(x+2) = 3(x+2) - 5$
Simplify.	$g(x+2) = 3x + 6 - 5$
	$g(x+2) = 3x + 1$

ⓒ

		$g(x) = 3x - 5$
To evaluate $g(x) + g(2)$, first find $g(2)$.		$g(2) = 3 \cdot 2 - 5$
		$g(2) = 1$
Now find $g(x) + g(2)$		$g(x) + g(2) = 3x - 5 + 1$ $g(x) \quad g(2)$
Simplify.		$g(x) + g(2) = 3x - 5 + 1$
		$g(x) + g(2) = 3x - 4$

Notice the difference between part ⓑ and ⓒ. We get $g(x+2) = 3x+1$ and $g(x)+g(2) = 3x-4$. So we see that $g(x+2) \neq g(x)+g(2)$.

Example 3.5.23

For the function $g(x) = 4x - 7$, evaluate the function.

a. $g(m^2)$
b. $g(x-3)$
c. $g(x) - g(3)$

Answer

ⓐ $4m^2 - 7$ ⓑ $4x - 19$
ⓒ $x - 12$

Example 3.5.24

For the function $h(x) = 2x + 1$, evaluate the function.

a. $h(k^2)$
b. $h(x+1)$
c. $h(x) + h(1)$

Answer

ⓐ $2k^2 + 1$ ⓑ $2x + 3$
ⓒ $2x + 4$

Many everyday situations can be modeled using functions.

Example 3.5.25

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(5)$. Explain what this result means.

Answer

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, *N*, depends on the number of days, *t*. Therefore, the variable *N*, is the dependent variable and the variable tt is the independent variable.

ⓑ Find $N(5)$. Explain what this result means.

	$N(t) = 75 + 10t$
Substitute in t=5.t=5.	$N(5) = 75 + 10 \cdot 5$
Simplify.	$N(5) = 75 + 50$
	$N(5) = 125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Example 3.5.26

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t) = 100 + 15t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(7)]$. Explain what this result means.

Answer

ⓐ *t* IND; *N* DEP ⓑ 205; the number of unread emails in Bryan's account on the seventh day.

Example 3.5.27

The number of unread emails in Anthony's account is 110. This number grows by 25 unread emails a day. The function $N(t) = 110 + 25t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(14)$. Explain what this result means.

Answer

ⓐ *t* IND; *N* DEP ⓑ 460; the number of unread emails in Anthony's account on the fourteenth day

Access this online resource for additional instruction and practice with relations and functions.

- Introduction to Functions

Key Concepts

- **Function Notation:** For the function $y = f(x)$
  - f is the name of the function
  - x is the domain value

- $f(x)$ is the range value y corresponding to the value x
  We read $f(x)$ as f of x or the value of f at x.

- **Independent and Dependent Variables:** For the function $y = f(x)$,
  - x is the independent variable as it can be any value in the domain
  - y is the dependent variable as its value depends on x

Glossary

domain of a relation

The domain of a relation is all the x-values in the ordered pairs of the relation.

function

A function is a relation that assigns to each element in its domain exactly one element in the range.

mapping

A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

range of a relation

The range of a relation is all the y-values in the ordered pairs of the relation.

relation

A relation is any set of ordered pairs,(x,y).(x,y). All the x-values in the ordered pairs together make up the domain. All the y-values in the ordered pairs together make up the range.

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3.5E: Exercises

Practice Makes Perfect

Find the Domain and Range of a Relation

In the following exercises, for each relation a. find the domain of the relation b. find the range of the relation.

1. $\{(1,4),(2,8),(3,12),(4,16),(5,20)\}$

Answer

a. $\{1,2,3,4,5\}$b. $\{4,8,12,16,20\}$

2. $\{(1,-2),(2,-4),(3,-6),(4,-8),(5,-10)\}$

3. $\{(1,7),(5,3),(7,9),(-2,-3),(-2,8)\}$

Answer

a. $\{1,5,7,-2\}$b. $\{7,3,9,-3,8\}$

4. $\{(11,3),(-2,-7),(4,-8),(4,17),(-6,9)\}$

In the following exercises, use the mapping of the relation to a. list the ordered pairs of the relation, b. find the domain of the relation, and c. find the range of the relation.

5.



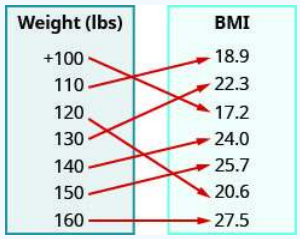
Answer

a. (Rebecca, January 18), (Jennifer, April 1), (John, January 18), (Hector, June 23), (Luis, February 15), (Ebony, April 7), (Raphael, November 6), (Meredith, August 19), (Karen, August 19), (Joseph, July 30)
b. {Rebecca, Jennifer, John, Hector, Luis, Ebony, Raphael, Meredith, Karen, Joseph}
c. {January 18, April 1, June 23, February 15, April 7, November 6, August 19, July 30}

6.



7. For a woman of height $5'4''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$is considered healthy.



Answer

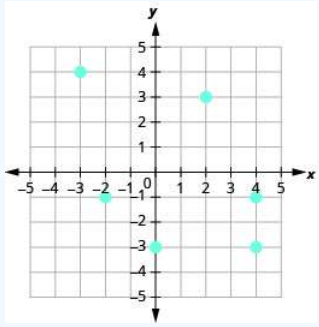
a. $(+100, 17.2), (110, 18.9), (120, 20.6), (130, 22.3), (140, 24.0), (150, 25.7), (160, 27.5)$ b. $\{+100, 110, 120, 130, 140, 150, 160\}$c. $\{17.2, 18.9, 20.6, 22.3, 24.0, 25.7, 27.5\}$

8. For a man of height $5'11''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$is considered healthy.



In the following exercises, use the graph of the relation to a. list the ordered pairs of the relation b. find the domain of the relation c. find the range of the relation.

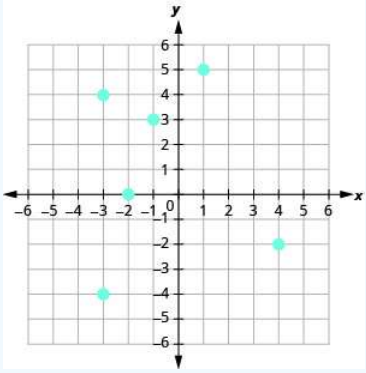
9.



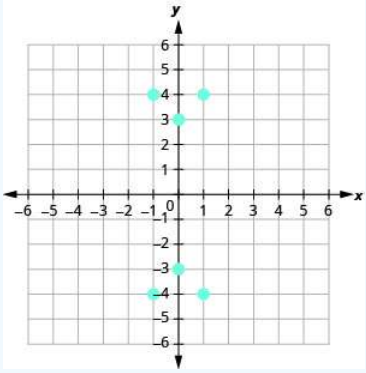
Answer

a. $(2, 3), (4, -3), (-2, -1), (-3, 4), (4, -1), (0, -3)$b. $\{-3, -2, 0, 2, 4\}$
c. $\{-3, -1, 3, 4\}$

10.



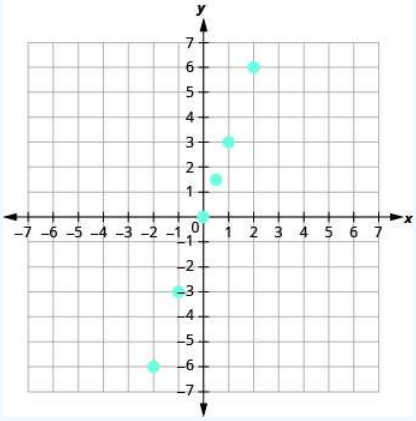
11.



Answer

a. $(1,4),(1,-4),(-1,4),(-1,-4),(0,3),(0,-3)$b. $\{-1,0,1\}$c. $\{-4,-3,3,4\}$

12.



Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to a. determine whether the relation is a function, b. find the domain of the relation, and c. find the range of the relation.

13. $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{9,4,1,0\}$

14. $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

15. $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

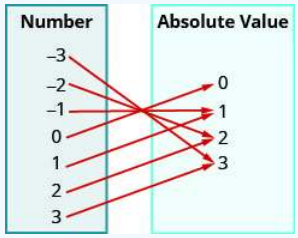
Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{0,1,8,27\}$

16. $\{(-3,-27),(-2,-8),(-1,-1),(0,0),(1,1),(2,8),(3,27)\}$

In the following exercises, use the mapping to a. determine whether the relation is a function, b. find the domain of the function, and c. find the range of the function.

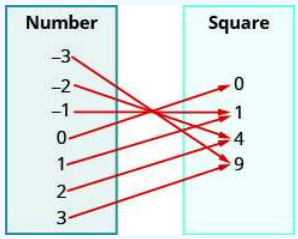
17.



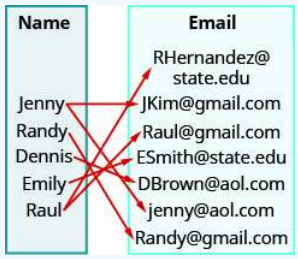
Answer

a. yes b. $\{-3,-2,-1,0,1,2,3\}$c. $\{0,1,2,3\}$

18.



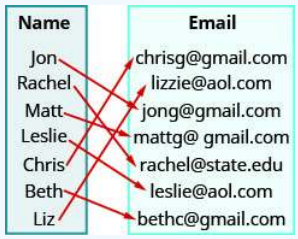
19.



Answer

a. no b. {Jenny, R and y, Dennis, Emily, Raul} c. {RHern and ez@state.edu, JKim@gmail.com, Raul@gmail.com, ESmith@state.edu, DBroen@aol.com, jenny@aol.cvom, R and y@gmail.com}

20.



In the following exercises, determine whether each equation is a function.

21. a. $2x+y=-3$
b. $y=x^2$
c. $x+y^2=-5$

Answer

a. yes b. yes c. no

22. a. $y=3x-5$
b. $y=x^3$
c. $2x+y^2=4$

23. a. $y-3x^3=2$
b. $x+y^2=3$
c. $3x-2y=6$

Answer

a. yes b. no c. yes

24. a. $2x-4y=8$
b. $-4=x^2-y$
c. $y^2=-x+5$

Find the Value of a Function

In the following exercises, evaluate the function: a. $f(2)$ b. $f(-1)$ c. $f(a)$.

25. $f(x)=5x-3$

Answer

a. $f(2)=7$ b. $f(-1)=-8$ c. $f(a)=5a-3$

26. $f(x)=3x+4$

27. $f(x)=-4x+2$

Answer

a. $f(2)=-6$ b. $f(-1)=6$ c. $f(a)=-4a+2$

28. $f(x)=-6x-3$

29. $f(x)=x^2-x+3$

Answer

a. $f(2)=5$ b. $f(-1)=5$
c. $f(a)=a^2-a+3$

30. $f(x)=x^2+x-2$

31. $f(x)=2x^2-x+3$

Answer

a. $f(2)=9$ b. $f(-1)=6$
c. $f(a)=2a^2-a+3$

32. $f(x)=3x^2+x-2$

In the following exercises, evaluate the function: a. $g(h^2)$ b. $g(x+2)$ c. $g(x)+g(2)$.

33. $g(x)=2x+1$

Answer

a. $g(h^2)=2h^2+1$
b. $g(x+2)=4x+5$
c. $g(x)+g(2)=2x+6$

34. $g(x)=5x-8$

35. $g(x)=-3x-2$

Answer

a. $g(h^2)=-3h^2-2$
b. $g(x+2)=-3x-8$
c. $g(x)+g(2)=-3x-10$

36. $g(x)=-8x+2$

37. $g(x)=3-x$

Answer

a. $g(h^2)=3-h^2$
b. $g(x+2)=1-x$
c. $g(x)+g(2)=4-x$

38. $g(x)=7-5x$

In the following exercises, evaluate the function.

39. $f(x)=3x^2-5x;\ f(2)$

Answer

2

40. $g(x) = 4x^2 - 3x$; $g(3)$

41. $F(x) = 2x^2 - 3x + 1$; $F(-1)$

Answer

6

42. $G(x) = 3x^2 - 5x + 2$; $G(-2)$

43. $h(t) = 2|t - 5| + 4$; $f(-4)$

Answer

22

44. $h(y) = 3|y - 1| - 3$; $h(-4)$

45. $f(x) = x + 2x - 1$; $f(2)$

Answer

4

46. $g(x) = x - 2x + 2$; $g(4)$

In the following exercises, solve.

47. The number of unwatched shows in Sylvia's DVR is 85. This number grows by 20 unwatched shows per week. The function $N(t) = 85 + 20t$ represents the relation between the number of unwatched shows, *N*, and the time, *t*, measured in weeks.

a. Determine the independent and dependent variable.

b. Find $N(4)$. Explain what this result means

Answer

a. *t* IND; *N* DEP
b. $N(4) = 165$ the number of unwatched shows in Sylvia's DVR at the fourth week.

48. Every day a new puzzle is downloaded into Ken's account. Right now he has 43 puzzles in his account. The function $N(t) = 43 + t$ represents the relation between the number of puzzles, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.

b. Find $N(30)$. Explain what this result means.

49. The daily cost to the printing company to print a book is modeled by the function $C(x) = 3.25x + 1500$ where *C* is the total daily cost and *x* is the number of books printed.

a. Determine the independent and dependent variable.

b. Find $N(0)$. Explain what this result means.

c. Find $N(1000)$. Explain what this result means.

Answer

a. x IND; C DEP
b. $N(0) = 1500$ the daily cost if no books are printed
c. $N(1000) = 4750$the daily cost of printing 1000 books

50. The daily cost to the manufacturing company is modeled by the function $C(x) = 7.25x + 2500$ where $C(x)$ is the total daily cost and x is the number of items manufactured.

a. Determine the independent and dependent variable.

b. Find $C(0)$. Explain what this result means.

c. Find $C(1000)$. Explain what this result means.

Writing Exercises

51. In your own words, explain the difference between a relation and a function.

52. In your own words, explain what is meant by domain and range.

53. Is every relation a function? Is every function a relation?

54. How do you find the value of a function?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the domain and range of a relation.			
determine if a relation is a function.			
find the value of a function.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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3.6: Graphs of Functions

Learning Objectives

By the end of this section, you will be able to:

- Use the vertical line test
- Identify graphs of basic functions
- Read information from a graph of a function

Before you get started, take this readiness quiz.

1. Evaluate: ⓐ 2^3 ⓑ 3^2.
   If you missed this problem, review [link].
2. Evaluate: ⓐ $|7|$ ⓑ $|-3|$.
   If you missed this problem, review [link].
3. Evaluate: ⓐ $\sqrt{4}$ ⓑ $\sqrt{16}$.
   If you missed this problem, review [link].

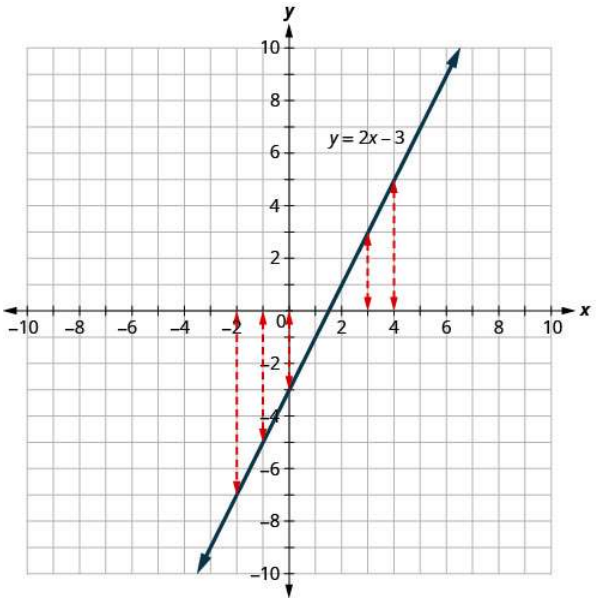
Use the Vertical Line Test

In the last section we learned how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping or an equation. We will now look at how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

The graph of a linear equation is a straight line where every point on the line is a solution of the equation and every solution of this equation is a point on this line.

In Figure, we can see that, in graph of the equation $y = 2x - 3$, for every *x*-value there is only one *y*-value, as shown in the accompanying table.



y = 2x – 3		
x	y	(x, y)
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Figure 3.6.1

A relation is a function if every element of the domain has exactly one value in the range. So the relation defined by the equation $y = 2x - 3$ is a function.

If we look at the graph, each vertical dashed line only intersects the line at one point. This makes sense as in a function, for every *x*-value there is only one *y*-value.

If the vertical line hit the graph twice, the x-value would be mapped to two y-values, and so the graph would not represent a function.

This leads us to the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

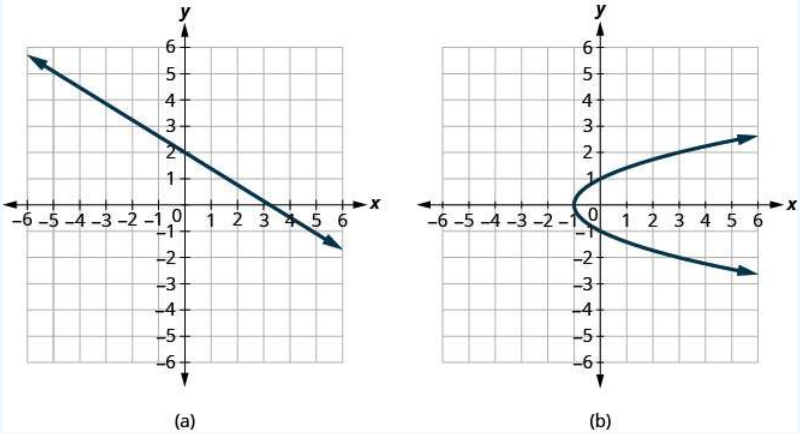
VERTICAL LINE TEST

A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.

If any vertical line intersects the graph in more than one point, the graph does not represent a function.

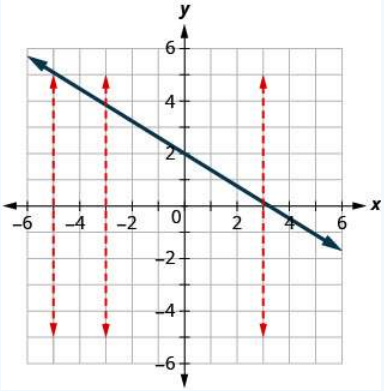
Example 3.6.1

Determine whether each graph is the graph of a function.

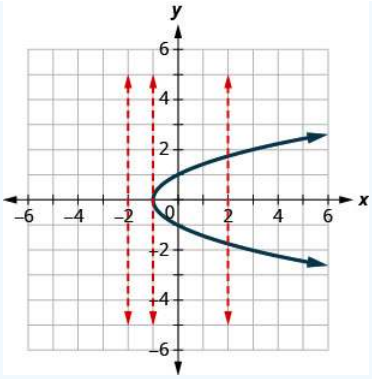


Answer

ⓐ Since any vertical line intersects the graph in at most one point, the graph is the graph of a function.

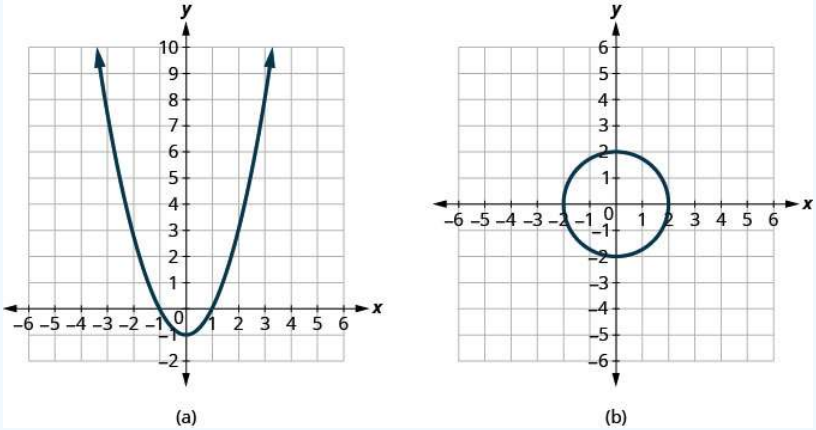


ⓑ One of the vertical lines shown on the graph, intersects it in two points. This graph does not represent a function.



Example 3.6.2

Determine whether each graph is the graph of a function.

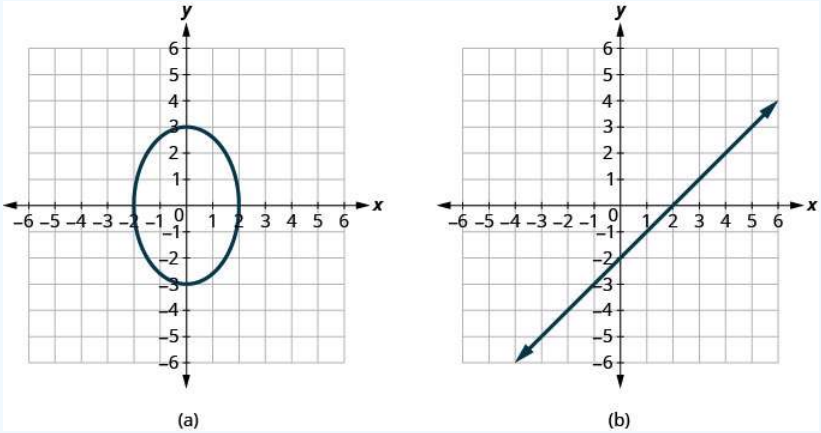


Answer

ⓐ yes ⓑ no

Example 3.6.3

Determine whether each graph is the graph of a function.



Answer

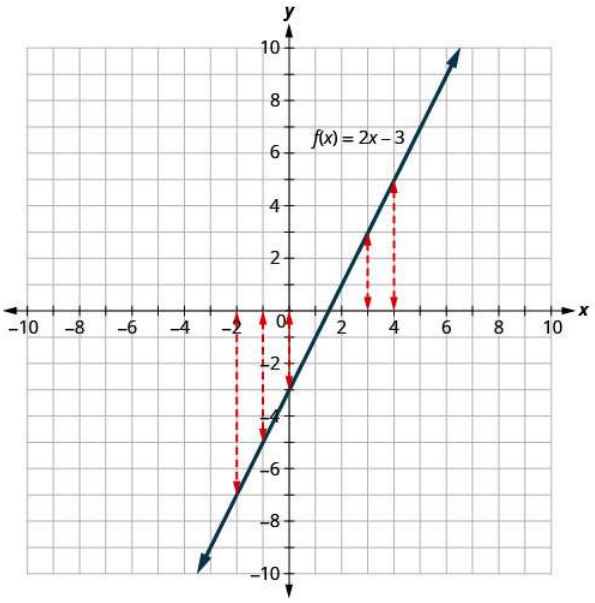
ⓐ no ⓑ yes

Identify Graphs of Basic Functions

We used the equation $y = 2x - 3$ and its graph as we developed the vertical line test. We said that the relation defined by the equation $y = 2x - 3$ is a function.

We can write this as in function notation as $f(x) = 2x - 3$. It still means the same thing. The graph of the function is the graph of all ordered pairs (x, y) where $y = f(x)$. So we can write the ordered pairs as $(x, f(x))$. It looks different but the graph will be the same.

Compare the graph of $y = 2x - 3$ previously shown in Figure with the graph of $f(x) = 2x - 3$ shown in Figure. Nothing has changed but the notation.



$f(x) = 2x - 3$		
x	$f(x)$	$(x, f(x))$
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Figure 3.6.2

GRAPH OF A FUNCTION

The graph of a function is the graph of all its ordered pairs, (x,y)(x,y) or using function notation, (x,f(x))(x,f(x)) where y=f(x).y=f(x).

f	name of function
x	x-coordinate of the ordered pair
$f(x)$	y-coordinate of the ordered pair

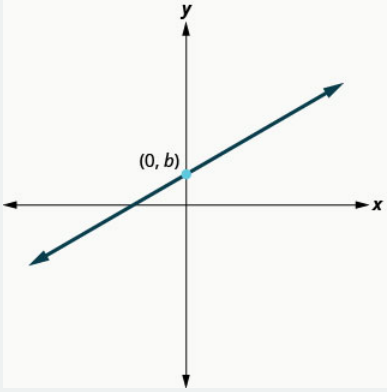
As we move forward in our study, it is helpful to be familiar with the graphs of several basic functions and be able to identify them.

Through our earlier work, we are familiar with the graphs of linear equations. The process we used to decide if $y = 2x - 3$ is a function would apply to all linear equations. All non-vertical linear equations are functions. Vertical lines are not functions as the *x*-value has infinitely many *y*-values.

We wrote linear equations in several forms, but it will be most helpful for us here to use the slope-intercept form of the linear equation. The slope-intercept form of a linear equation is $y = mx + b$. In function notation, this linear function becomes $f(x) = mx + b$ where *m* is the slope of the line and *b* is the *y*-intercept.

The domain is the set of all real numbers, and the range is also the set of all real numbers.

LINEAR FUNCTION



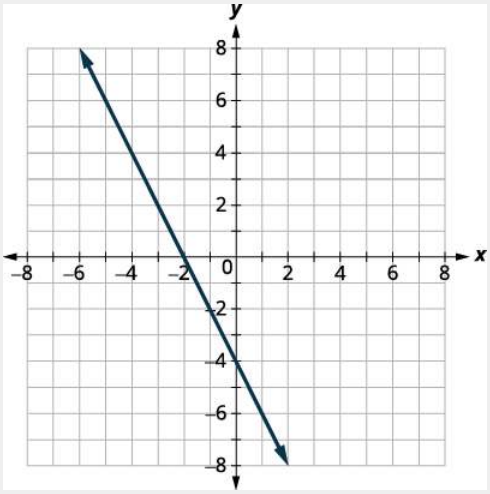
$f(x) = mx + b$
m, b: all real numbers
m: slope of the line
b: y-intercept
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

We will use the graphing techniques we used earlier, to graph the basic functions.

Example 3.6.4

Graph: $f(x) = -2x - 4$.

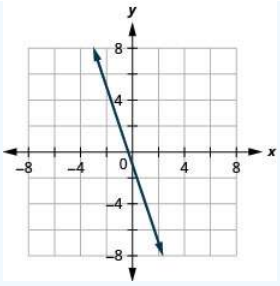
Answer

	$f(x) = -2x - 4$
We recognize this as a linear function.	
Find the slope and y-intercept.	$m = -2$ $b = -4$
Graph using the slope intercept.	

Example 3.6.5

Graph: $f(x) = -3x - 1$

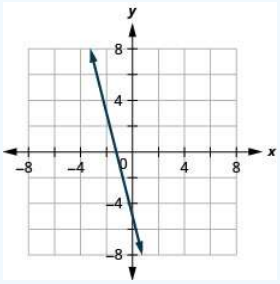
Answer



Example 3.6.6

Graph: $f(x) = -4x - 5$

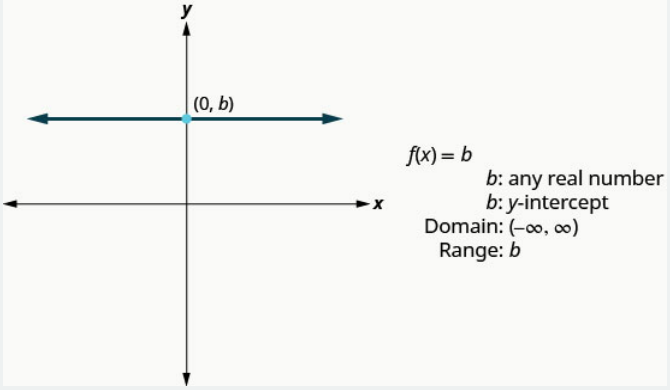
Answer



The next function whose graph we will look at is called the constant function and its equation is of the form $f(x) = b$, where b is any real number. If we replace the $f(x)$ with y, we get $y = b$. We recognize this as the horizontal line whose y-intercept is b. The graph of the function $f(x) = b$, is also the horizontal line whose y-intercept is b.

Notice that for any real number we put in the function, the function value will be b. This tells us the range has only one value, b.

CONSTANT FUNCTION



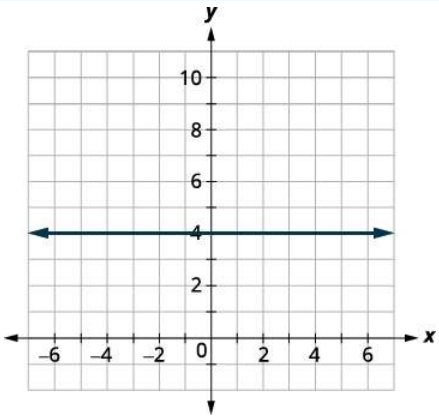
Example 3.6.7

Graph: $f(x) = 4$.

Answer

	$f(x) = 4$
We recognize this as a constant function.	

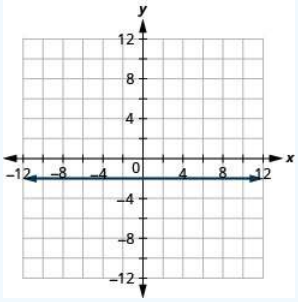
The graph will be a horizontal line through $(0, 4)$.



Example 3.6.8

Graph: $f(x) = -2$.

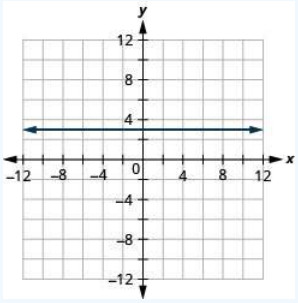
Answer



Example 3.6.9

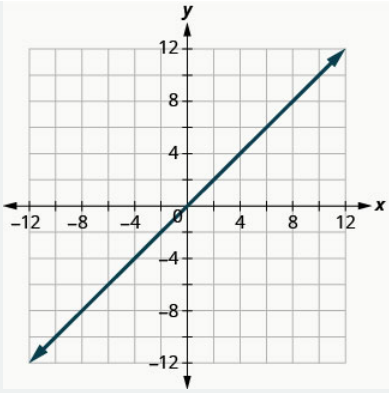
Graph: $f(x) = 3$.

Answer



The identity function, $f(x) = x$ is a special case of the linear function. If we write it in linear function form, $f(x) = 1x + 0$, we see the slope is 1 and the *y*-intercept is 0.

IDENTITY FUNCTION



$f(x) = x$
m: 1
b: 0
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

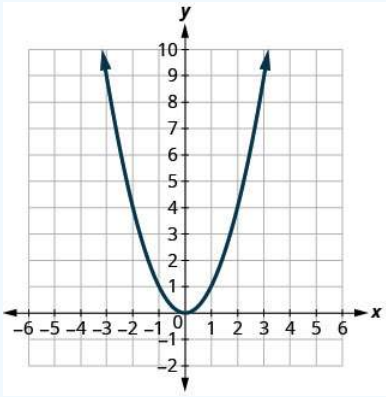
The next function we will look at is not a linear function. So the graph will not be a line. The only method we have to graph this function is point plotting. Because this is an unfamiliar function, we make sure to choose several positive and negative values as well as 0 for our x-values.

?

Graph: $f(x) = x^2$.

Answer

We choose x-values. We substitute them in and then create a chart as shown.

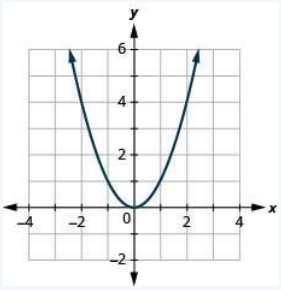


x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

? Example 3.6.11

Graph: $f(x) = x^2$.

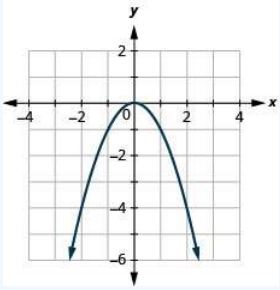
Answer



Example 3.6.12

$f(x) = -x^2$

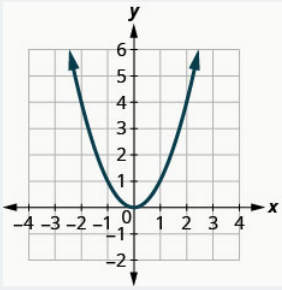
Answer



Looking at the result in Example, we can summarize the features of the square function. We call this graph a parabola. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is not all real numbers. Notice the graph consists of values of *y* never go below zero. This makes sense as the square of any number cannot be negative. So, the range of the square function is all non-negative real numbers.

SQUARE FUNCTION



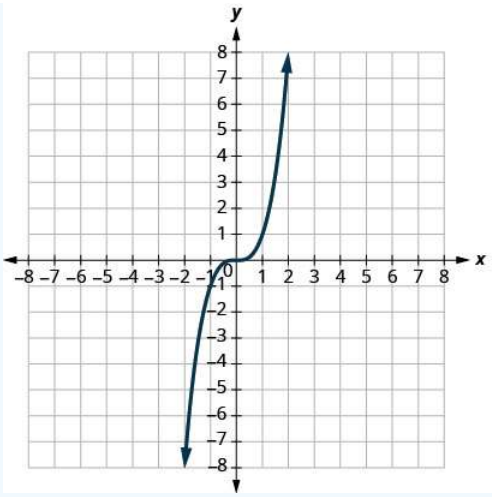
$f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

The next function we will look at is also not a linear function so the graph will not be a line. Again we will use point plotting, and make sure to choose several positive and negative values as well as 0 for our *x*-values.

Graph: $f(x) = x^3$.

Answer

We choose *x*-values. We substitute them in and then create a chart.

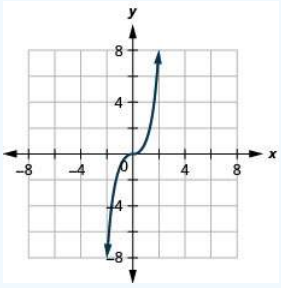


x	$f(x) = x^3$	$(x, f(x))$
–2	–8	(–2, –8)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)

Example 3.6.14

Graph: $f(x) = x^3$.

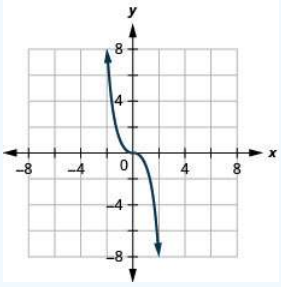
Answer



Example 3.6.15

Graph: $f(x) = -x^3$.

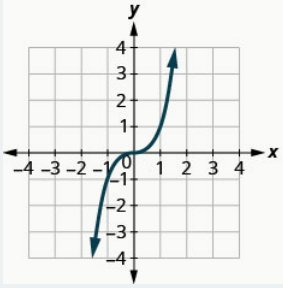
Answer



Looking at the result in Example, we can summarize the features of the cube function. As we consider the domain, notice any real number can be used as an x-value. The domain is all real numbers.

The range is all real numbers. This makes sense as the cube of any non-zero number can be positive or negative. So, the range of the cube function is all real numbers.

CUBE FUNCTION



$f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

The next function we will look at does not square or cube the input values, but rather takes the square root of those values.

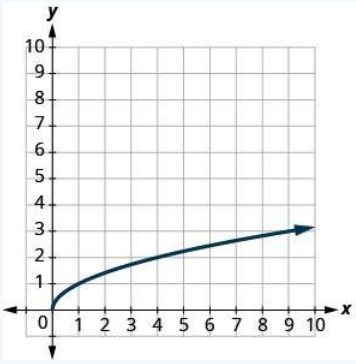
Let's graph the function $f(x) = \sqrt{x}$ and then summarize the features of the function. Remember, we can only take the square root of non-negative real numbers, so our domain will be the non-negative real numbers.

Example 3.6.16

$f(x) = \sqrt{x}$

Answer

We choose *x*-values. Since we will be taking the square root, we choose numbers that are perfect squares, to make our work easier. We substitute them in and then create a chart.

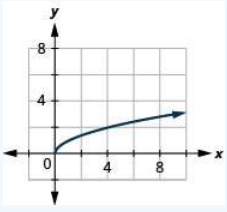


x	$f(x) = \sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Example 3.6.17

Graph: $f(x) = x$.

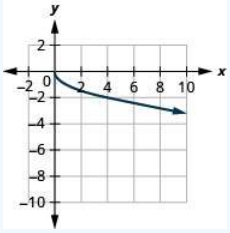
Answer



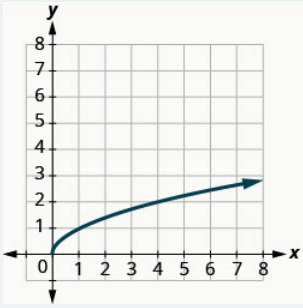
Example 3.6.18

Graph: $f(x) = -\sqrt{x}$.

Answer



SQUARE ROOT FUNCTION



$f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$

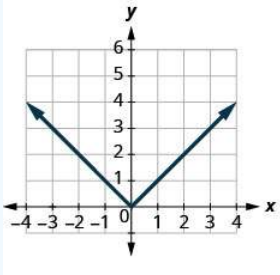
Our last basic function is the absolute value function, $f(x) = |x|$. Keep in mind that the absolute value of a number is its distance from zero. Since we never measure distance as a negative number, we will never get a negative number in the range.

?

Graph: $f(x) = |x|$.

Answer

We choose *x*-values. We substitute them in and then create a chart.

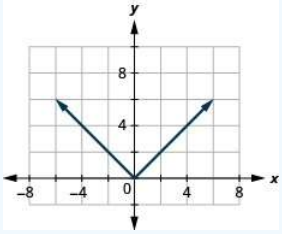


x	$f(x) = \lvert x \rvert$	$(x, f(x))$
–3	3	(–3, 3)
–2	2	(–2, 2)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

? Example 3.6.20

Graph: $f(x) = |x|$.

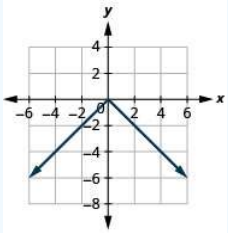
Answer



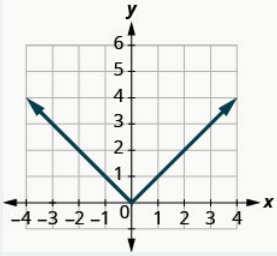
Example 3.6.21

Graph: $f(x) = -|x|$.

Answer



ABSOLUTE VALUE FUNCTION



$f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Read Information from a Graph of a Function

In the sciences and business, data is often collected and then graphed. The graph is analyzed, information is obtained from the graph and then often predictions are made from the data.

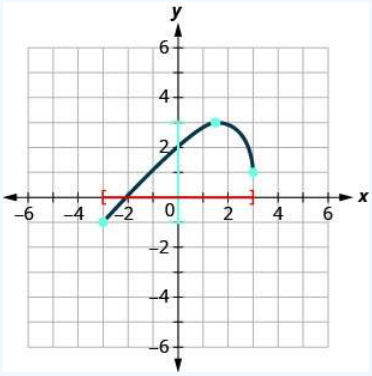
We will start by reading the domain and range of a function from its graph.

Remember the domain is the set of all the *x*-values in the ordered pairs in the function. To find the domain we look at the graph and find all the values of *x* that have a corresponding value on the graph. Follow the value *x* up or down vertically. If you hit the graph of the function then *x* is in the domain.

Remember the range is the set of all the *y*-values in the ordered pairs in the function. To find the range we look at the graph and find all the values of *y* that have a corresponding value on the graph. Follow the value *y* left or right horizontally. If you hit the graph of the function then *y* is in the range.

Example 3.6.22

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



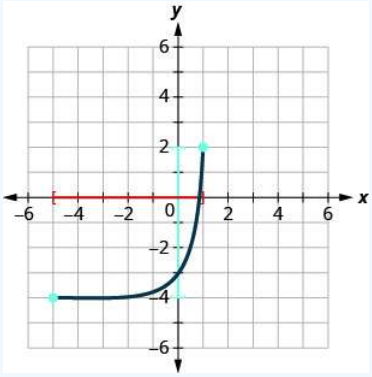
Answer

To find the domain we look at the graph and find all the values of x that correspond to a point on the graph. The domain is highlighted in red on the graph. The domain is $[-3, 3]$.

To find the range we look at the graph and find all the values of y that correspond to a point on the graph. The range is highlighted in blue on the graph. The range is $[-1, 3]$.

Example 3.6.23

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.

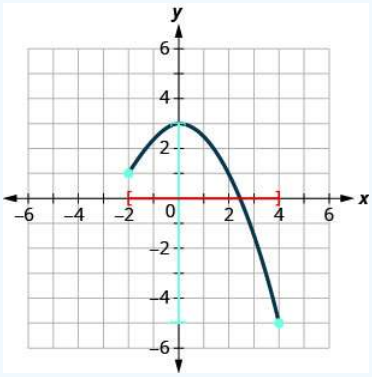


Answer

The domain is $[-5, 1]$. The range is $[-4, 2]$.

Example 3.6.24

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



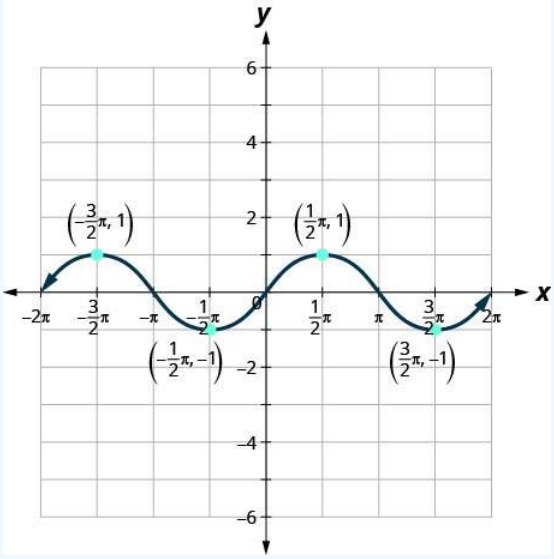
Answer

The domain is $[-2, 4]$. The range is $[-5, 3]$.

We are now going to read information from the graph that you may see in future math classes.

Example 3.6.25

Use the graph of the function to find the indicated values.



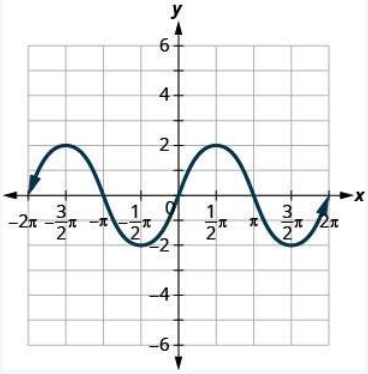
ⓐ Find: $f(0)$.
ⓑ Find: $f(32\pi)$.
ⓒ Find: $f(-12\pi)$.
ⓓ Find the values for x when $f(x) = 0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

ⓐ When $x = 0$, the function crosses the y-axis at 0. So, $f(0) = 0$.
ⓑ When $x = 32\pi$, the y-value of the function is -1. So, $f(32\pi) = -1$.
ⓒ When $x = -12\pi$, the y-value of the function is -1. So, $f(-12\pi) = -1$.
ⓓ The function is 0 at the points, $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$. The x-values when $f(x) = 0$ are $-2\pi, -\pi, 0, \pi, 2\pi$.
ⓔ The x-intercepts occur when $y = 0$. So the x-intercepts occur when $f(x) = 0$. The x-intercepts are $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$.
ⓕ The y-intercepts occur when x=0.x=0. So the y-intercepts occur at $f(0)$. The y-intercept is $(0, 0)$.
ⓖ This function has a value when x is from -2π to 2π. Therefore, the domain in interval notation is $[-2\pi, 2\pi]$.
ⓗ This function values, or y-values go from -1 to 1. Therefore, the range, in interval notation, is $[-1, 1]$.

? Example 3.6.26

Use the graph of the function to find the indicated values.



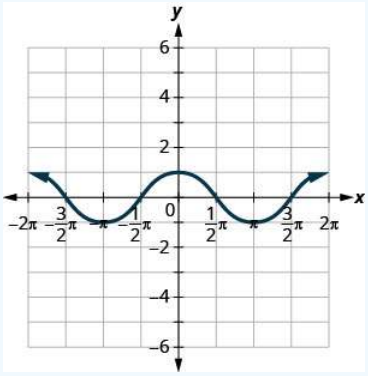
ⓐ Find: f(0).f(0).
ⓑ Find: f(12\pi).f(12\pi).
ⓒ Find: f(−32\pi).f(−32\pi).
ⓓ Find the values for *x* when f(x)=0.f(x)=0.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

ⓐ $f(0)=0$ ⓑ $f=(\pi 2)=2$ ⓒ $f=(-3\pi 2)=2$ ⓓ $f(x)=0$ for $x=-2\pi,-\pi,0,\pi,2\pi$ ⓔ $(-2\pi,0),(-\pi,0),(0,0),(\pi,0),(2\pi,0)$ⓕ (0,0)(0,0) ⓖ $[-2\pi,2\pi]$ ⓗ $[-2,2]$

Example 3.6.27

Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.
ⓑ Find: $f(\pi)$.
ⓒ Find: $f(-\pi)$.
ⓓ Find the values for *x* when $f(x)=0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

ⓐ $f(0)=1$ ⓑ $f(\pi)=-1$ ⓒ $f(-\pi)=-1$ ⓓ $f(x)=0$ for $x=-3\pi 2,-\pi 2,\pi 2,3\pi 2$ ⓔ $(-2pi,0),(-pi,0),(0,0),(pi,0),(2pi,0)$ⓕ $(0,1)$ ⓖ $[-2pi,2pi]$ⓗ $[-1,1]$

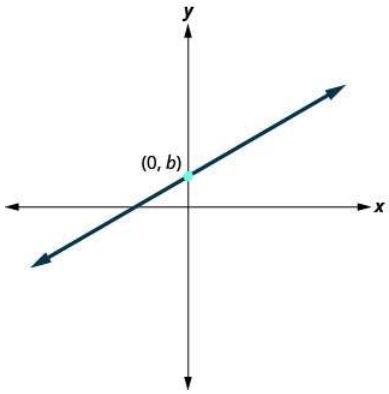
Access this online resource for additional instruction and practice with graphs of functions.

- Find Domain and Range

Key Concepts

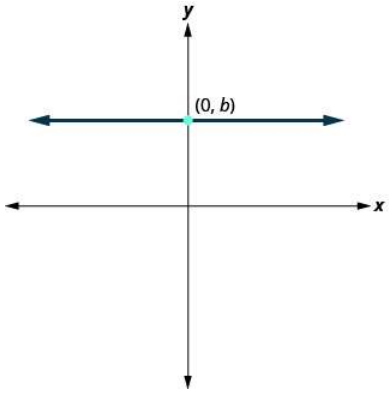
- **Vertical Line Test**
  - A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
  - If any vertical line intersects the graph in more than one point, the graph does not represent a function.
- **Graph of a Function**
  - The graph of a function is the graph of all its ordered pairs, (x,y)(x,y) or using function notation, (x,f(x))(x,f(x)) where y=f(x).y=f(x).

    fxf(x)name of functionx-coordinate of the ordered pairy-coordinate of the ordered pairfname of functionxx-coordinate of the ordered pairf(x)y-coordinate of the ordered pair
- **Linear Function**



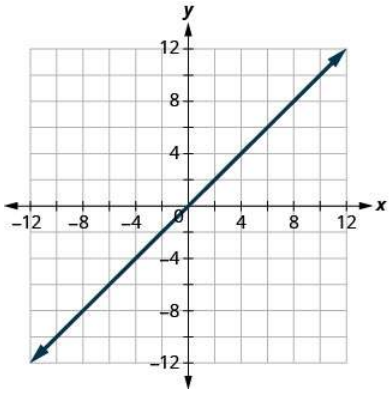
$f(x) = mx + b$
m, b: all real numbers
m: slope of the line
b: y-intercept
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Constant Function**



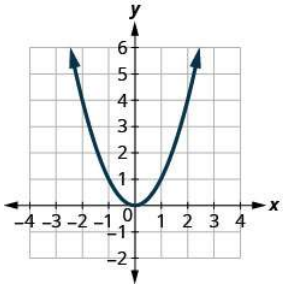
$f(x) = b$
b: any real number
b: y-intercept
Domain: $(-\infty, \infty)$
Range: b

- **Identity Function**



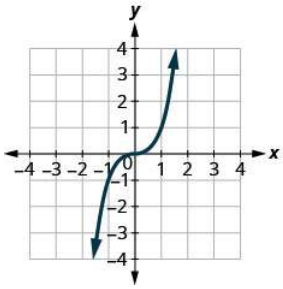
$f(x) = x$
m: 1
b: 0
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Square Function**



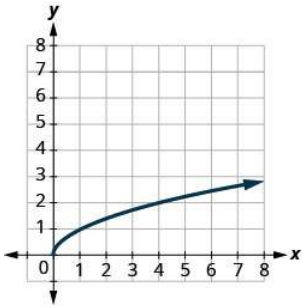
$f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

- **Cube Function**



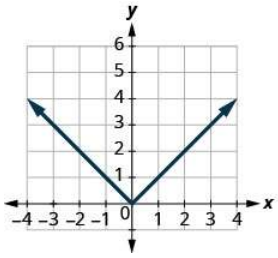
$f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Square Root Function**



$f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$

- **Absolute Value Function**



$f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

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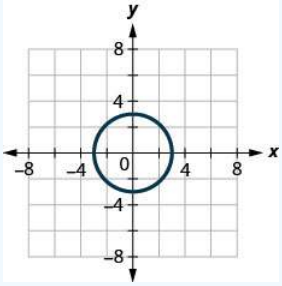
3.6E: Exercises

Practice Makes Perfect

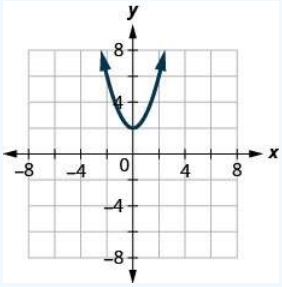
Use the Vertical Line Test

In the following exercises, determine whether each graph is the graph of a function.

1. ⓐ



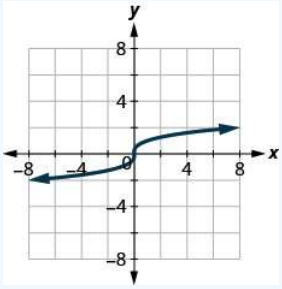
ⓑ



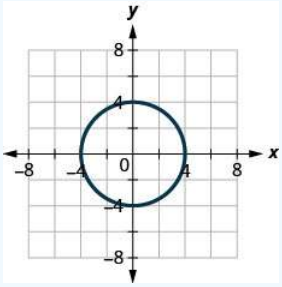
Answer

ⓐ no ⓑ yes

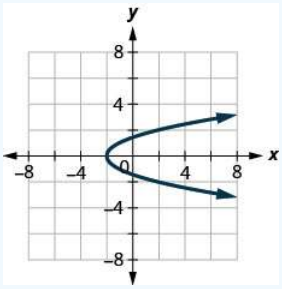
2. ⓐ



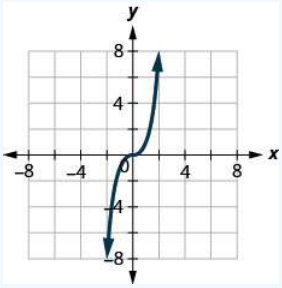
ⓑ



3. ⓐ



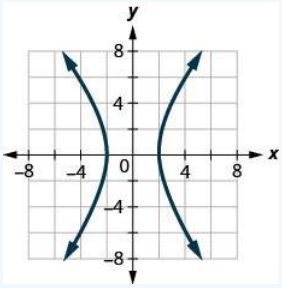
ⓑ



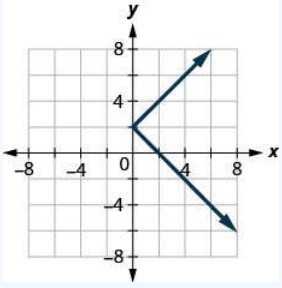
Answer

ⓐ no ⓑ yes

4. ⓐ



ⓑ



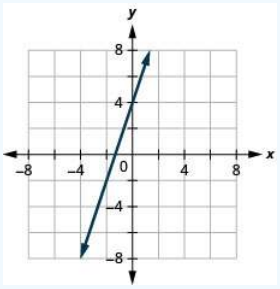
Identify Graphs of Basic Functions

In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

5. $f(x) = 3x + 4$

Answer

ⓐ



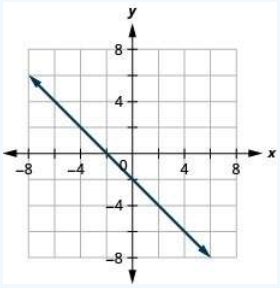
ⓑ $D:(-\text{inf},\text{inf}),\ R:(-\text{inf},\text{inf})$

6. $f(x)=2x+5$

7. $f(x)=-x-2$

Answer

ⓐ



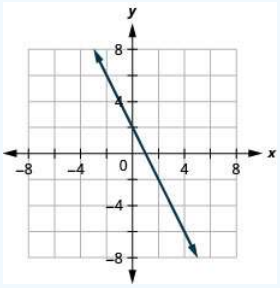
ⓑ $D:(-\text{inf},\text{inf}),\ R:(-\text{inf},\text{inf})$

8. $f(x)=-4x-3$

9. $f(x)=-2x+2$

Answer

ⓐ



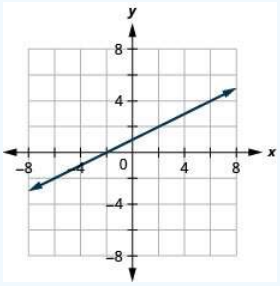
ⓑ $D:(-\text{inf},\text{inf}),\ R:(-\text{inf},\text{inf})$

10. $f(x)=-3x+3$

11. $f(x)=\frac{1}{2}x+1$

Answer

ⓐ



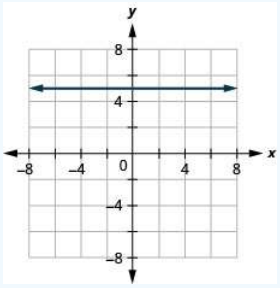
ⓑ $D : (-\text{inf}, \text{inf}),\ R : (-\text{inf}, \text{inf})$

12. $f(x) = \frac{2}{3}x - 2$

13. $f(x) = 5$

Answer

ⓐ



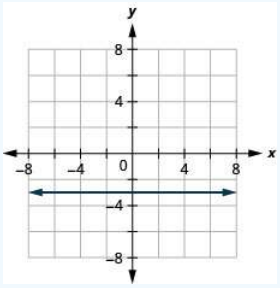
ⓑ $D : (-\text{inf}, \text{inf}), R : 5$

14. $f(x) = 2$

15. $f(x) = -3$

Answer

ⓐ



ⓑ $D : (-\text{inf}, \text{inf}),\ R : -3$

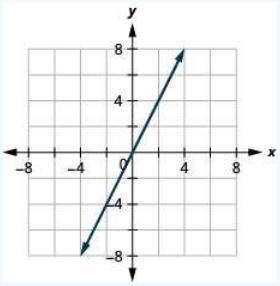
16. $f(x) = -1$

17. $f(x) = 2x$

https://math.libretexts.org/@go/page/30834

Answer

ⓐ



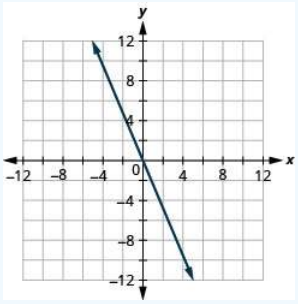
ⓑ $D: (-\text{inf}, \text{inf}),\ R: (-\text{inf}, \text{inf})$

18. $f(x) = 3x$

19. $f(x) = -2x$

Answer

ⓐ



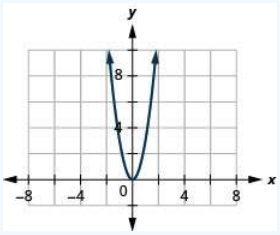
ⓑ $D: (-\text{inf}, \text{inf}), R: (-\text{inf}, \text{inf})$

20. $f(x) = -3x$

21. $f(x) = 3x^2$

Answer

ⓐ



ⓑ $D: (-\text{inf}, \text{inf}),\ R: [0, \text{inf})$

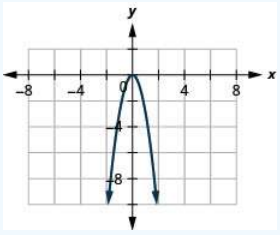
22. $f(x) = 2x^2$

23. $f(x) = -3x^2$

https://math.libretexts.org/@go/page/30834

Answer

ⓐ



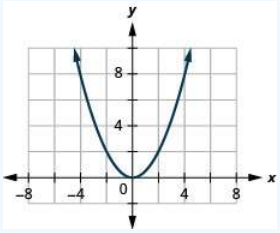
ⓑ $D : (-\text{inf}, \text{inf}),\ R : (-\text{inf}, 0]$

24. $f(x) = -2x^2$

25. $f(x) = 12x^2$

Answer

ⓐ



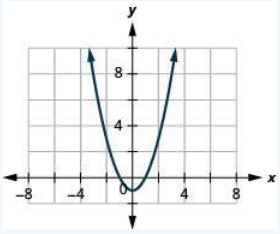
ⓑ $D : (-\text{inf}, \text{inf}),\ R : [-\text{inf}, 0)$

26. $f(x) = \frac{1}{3}x^2$

27. $f(x) = x^2 - 1$

Answer

ⓐ



ⓑ $D : (-\text{inf}, \text{inf}),\ R : [-1, \text{inf})$

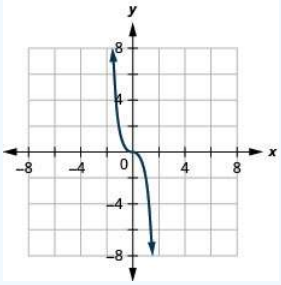
28. $f(x) = x^2 + 1$

29. $f(x) = -2x^3$

Answer

ⓐ

https://math.libretexts.org/@go/page/30834



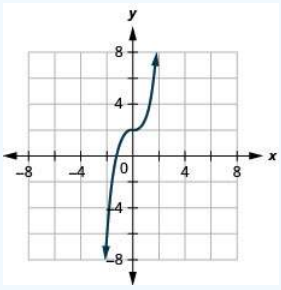
ⓑ $D : (-\text{inf}, \text{inf}),\ R : (-\text{inf}, \text{inf})$

30. $f(x) = 2x^3$

31. $f(x) = x^3 + 2$

Answer

ⓐ



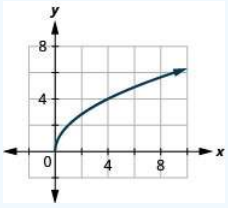
ⓑ $D : (-\text{inf}, \text{inf}), R : (-\text{inf}, \text{inf})$

32. $f(x) = x^3 - 2$

33. $f(x) = 2\sqrt{x}$

Answer

ⓐ



ⓑ $D : [0, \text{inf}), R : [0, \text{inf})$

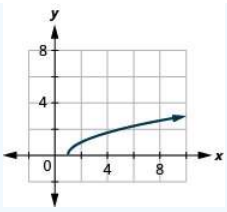
34. $f(x) = -2\sqrt{x}$

35. $f(x) = \sqrt{x-1}$

Answer

ⓐ

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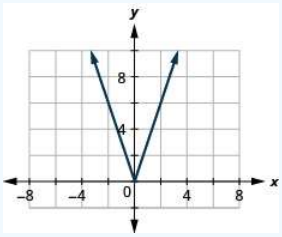
ⓑ $D: [1, \text{inf}), R: [0, \text{inf})$

36. $f(x) = \sqrt{x+1}$

37. $f(x) = 3|x|$

Answer

ⓐ



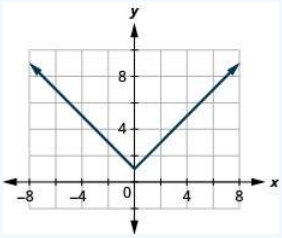
ⓑ $D: [-1, -1, \text{inf}), R: [-\text{inf}, \text{inf})$

38. $f(x) = -2|x|$

39. $f(x) = |x| + 1$

Answer

ⓐ



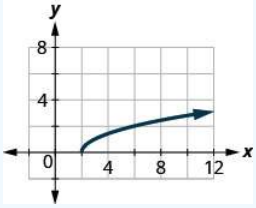
ⓑ $D: (-\text{inf}, \text{inf}), R: [1, \text{inf})$

40. $f(x) = |x| - 1$

Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation.

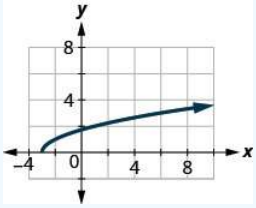
41.



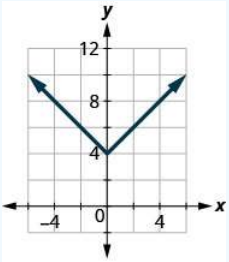
Answer

$D: [2, \text{inf}),\ R: [0, \text{inf})$

42.



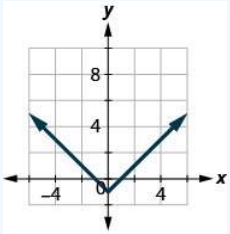
43.



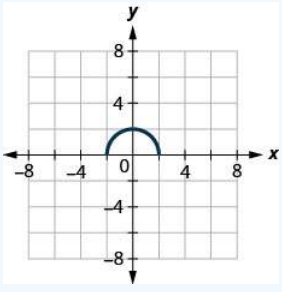
Answer

$D: (-\text{inf}, \text{inf}),\ R: [4, \text{inf})$

44.



45.

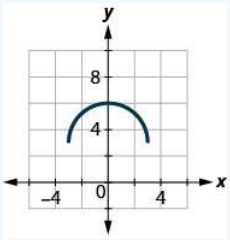


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Answer

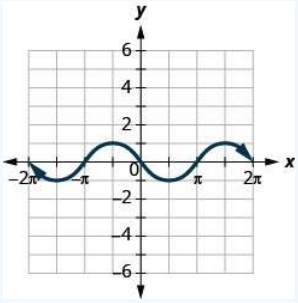
$D: [-2, 2],\ R: [0, 2]$

46.



In the following exercises, use the graph of the function to find the indicated values.

47.

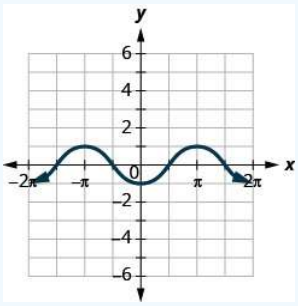


ⓐ Find: $f(0)$.
ⓑ Find: $f(12\pi)$.
ⓒ Find: $f(-32\pi)$.
ⓓ Find the values for x when $f(x) = 0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

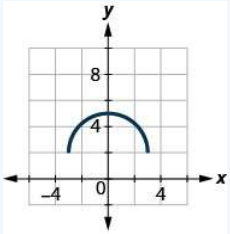
ⓐ $f(0) = 0$ ⓑ $(\pi/2) = -1$
ⓒ $f(-3\pi/2) = -1$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$
ⓔ $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)(f)(0, 0)$
ⓖ $[-2\pi, 2\pi]$ ⓗ $[-1, 1]$

48.



ⓐ Find: $f(0)$.
ⓑ Find: $f(\pi)$.
ⓒ Find: $f(-\pi)$.
ⓓ Find the values for x when $f(x)=0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation

49.

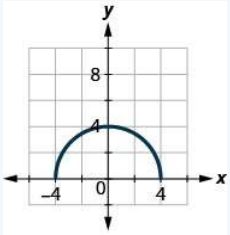


ⓐ Find: $f(0)$.
ⓑ Find: $f(-3)$.
ⓒ Find: $f(3)$.
ⓓ Find the values for x when $f(x)=0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

ⓐ $f(0)=-6$ ⓑ $f(-3)=3$ ⓒ $f(3)=3$ ⓓ $f(x)=0$ for no x ⓔ none ⓕ $y=6$ ⓖ $[-3,3]$ ⓗ $[-3,6]$

50.



ⓐ Find: $f(0)$.
ⓑ Find the values for x when $f(x)=0$.
ⓒ Find the x-intercepts.
ⓓ Find the y-intercepts.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation

Writing Exercises

51. Explain in your own words how to find the domain from a graph.

52. Explain in your own words how to find the range from a graph.

53. Explain in your own words how to use the vertical line test.

54. Draw a sketch of the square and cube functions. What are the similarities and differences in the graphs?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the vertical line test.			
identify graphs of basic functions.			
read information from a graph.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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Chapter 3 Review Exercises

Chapter Review Exercises

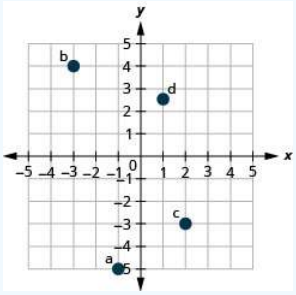
Graph Linear Equations in Two Variables

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

1. ⓐ $(-1,-5)$
ⓑ $(-3,4)$
ⓒ $(2,-3)$
ⓓ $(1,\frac{5}{2})$

Answer



2. ⓐ $(-2,0)$
ⓑ $(0,-4)$
ⓒ $(0,5)$
ⓓ $(3,0)$

In the following exercises, determine which ordered pairs are solutions to the given equations.

3. $5x+y=10$;

ⓐ $(5,1)$
ⓑ $(2,0)$
ⓒ $(4,-10)$

Answer

ⓑ, ⓒ

4. $y=6x-2$;

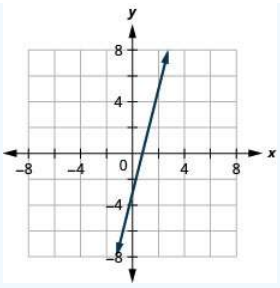
ⓐ $(1,4)$
ⓑ $(13,0)$
ⓒ $(6,-2)$

Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

5. $y=4x-3$

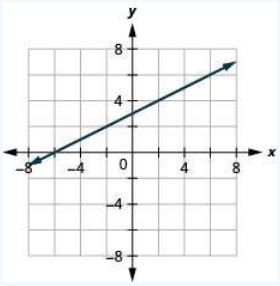
Answer



6. $y = -3x$

7. $y = \frac{1}{2}x + 3$

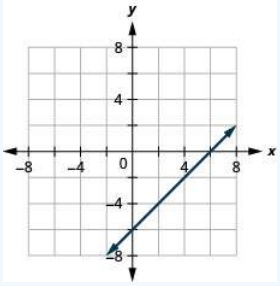
Answer



8. $y = -\frac{4}{5}|x - 1$

9. $x - y = 6$

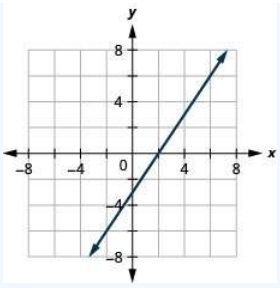
Answer



10. $2x + y = 7$

11. $3x - 2y = 6$

Answer



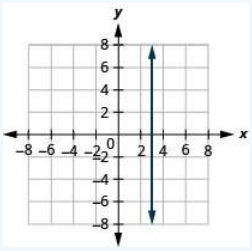
Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

12. $y = -2$

13. $x = 3$

Answer

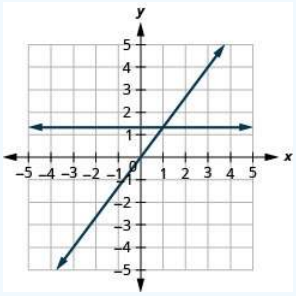


In the following exercises, graph each pair of equations in the same rectangular coordinate system.

14. $y = -2x$ and $y = -2$

15. $y = \frac{4}{3}x$ and $y = \frac{4}{3}$

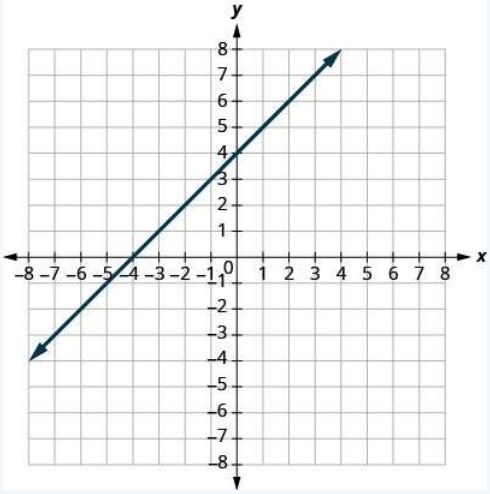
Answer



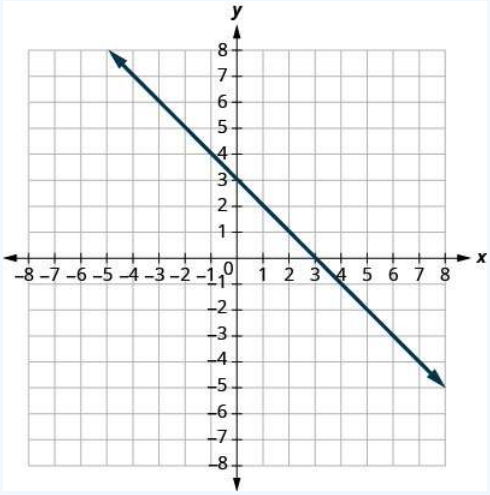
Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts.

16.



17.



Answer

$(0, 3)(3, 0)$

In the following exercises, find the intercepts of each equation.

18. $x - y = -1$

19. $x + 2y = 6$

Answer

$(6, 0),\ (0, 3)$

20. $2x + 3y = 12$

21. $y = \frac{3}{4}x - 12$

Answer

$(16, 0),\ (0, -12)$

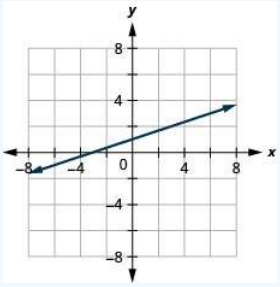
22. $y = 3x$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

23. $-x + 3y = 3$

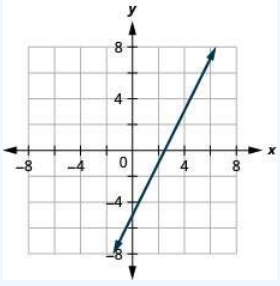
Answer



24. $x - y = 4$

25. $2x - y = 5$

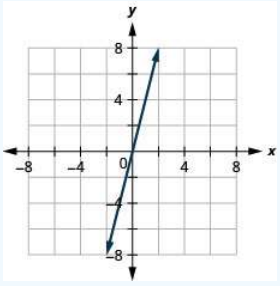
Answer



26. $2x - 4y = 8$

27. $y = 4x$

Answer

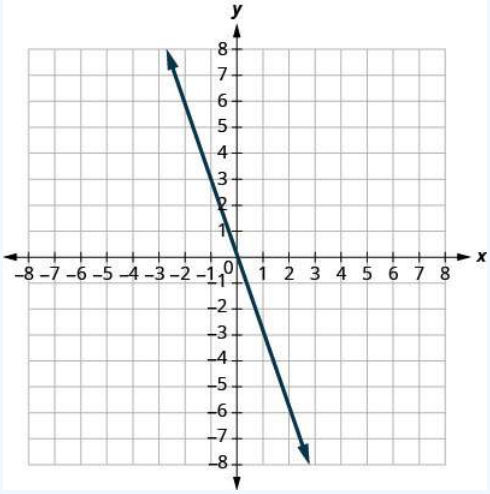


Slope of a Line

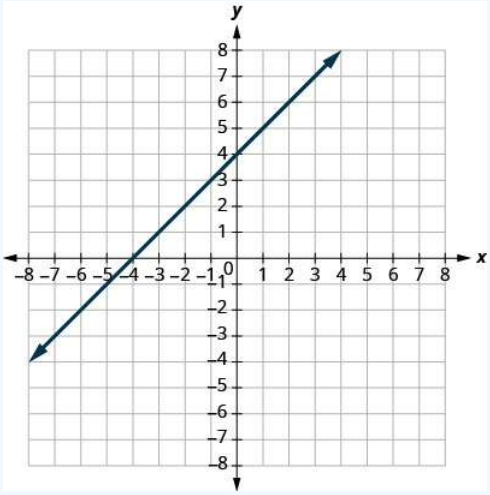
Find the Slope of a Line

In the following exercises, find the slope of each line shown.

28.



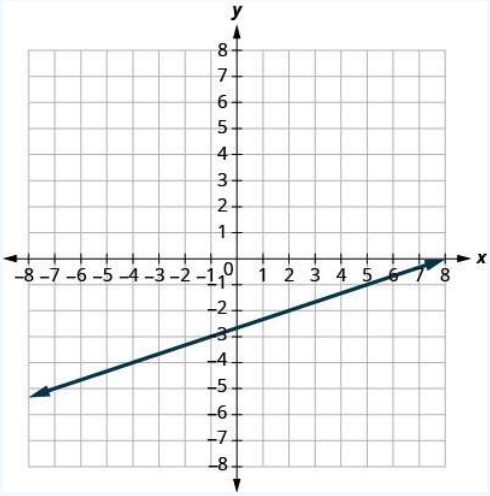
29.



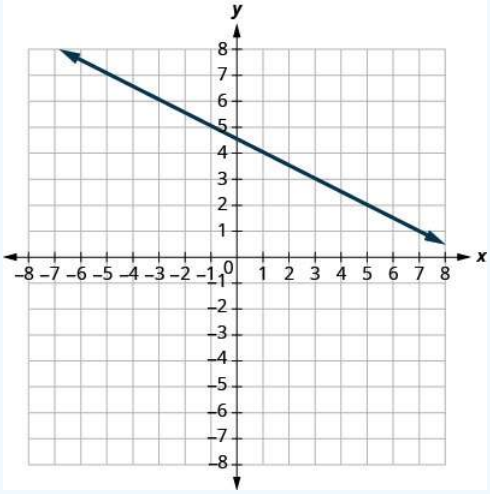
Answer

1

30.



31.



Answer

-12

In the following exercises, find the slope of each line.

32. $y = 2$

33. $x = 5$

Answer

undefined

34. $x = -3$

35. $y = -1$

Answer

0

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

36. $(-1, -1), (0, 5)$

37. $(3.5), (4, -1)$

Answer

-6

38. $(-5, -2), (3, 2)$

39. $(2, 1), (4, 6)$

Answer

52

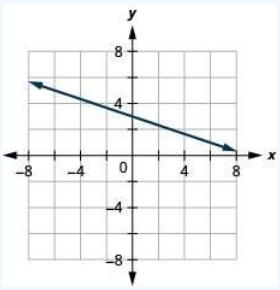
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

40. $(2, -2);\ m = 52$

41. $(-3, 4);\ m = -13$

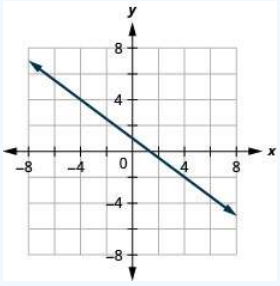
Answer



42. x-intercept $-4; m = 3$

43. y-intercept $1; m = -34$

Answer



Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and y-intercept of each line.

44. $y = -4x + 9$

45. $y = 53x - 6$

Answer

$m = 53;\ (0, -6)$

46. $5x + y = 10$

47. $4x - 5y = 8$

Answer

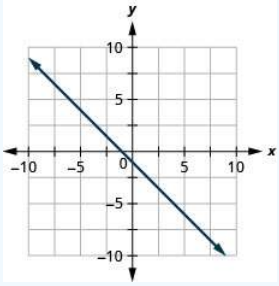
$m = \frac{4}{5};\ (0, -\frac{8}{5})$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

48. $y = 2x + 3$

49. $y = -x - 1$

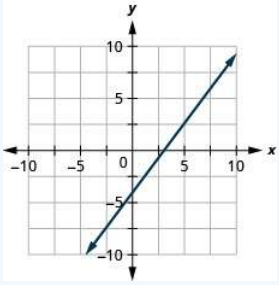
Answer



50. $y = -25x + 3$

51. $4x - 3y = 12$

Answer



In the following exercises, determine the most convenient method to graph each line.

52. $x = 5$

53. $y = -3$

Answer

horizontal line

54. $2x + y = 5$

55. $x - y = 2$

Answer

intercepts

56. $y = 22x + 2$

57. $y = 34x - 1$

Answer

plotting points

Graph and Interpret Applications of Slope-Intercept

58. Katherine is a private chef. The equation $C = 6.5m + 42$ models the relation between her weekly cost, C, in dollars and the number of meals, m, that she serves.

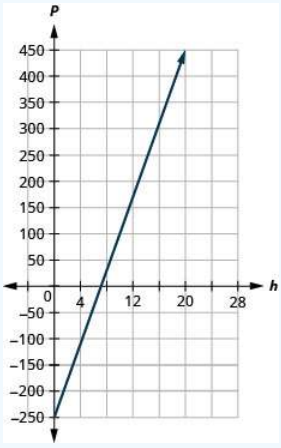
ⓐ Find Katherine's cost for a week when she serves no meals.
ⓑ Find the cost for a week when she serves 14 meals.
ⓒ Interpret the slope and C-intercept of the equation.
ⓓ Graph the equation.

59. Marjorie teaches piano. The equation $P = 35h - 250$ models the relation between her weekly profit, P, in dollars and the number of student lessons, s, that she teaches.

ⓐ Find Marjorie's profit for a week when she teaches no student lessons.
ⓑ Find the profit for a week when she teaches 20 student lessons.
ⓒ Interpret the slope and P-intercept of the equation.
ⓓ Graph the equation.

Answer

ⓐ $-\$250$
ⓑ $\$450$
ⓒ The slope, 35, means that Marjorie's weekly profit, P, increases by \$35 for each additional student lesson she teaches. The P-intercept means that when the number of lessons is 0, Marjorie loses \$250.
ⓓ



Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

60. $4x - 3y = -1; \quad y = 43x - 3$

61. $y = 5x - 1; \quad 10x + 2y = 0$

Answer

neither

62. $3x - 2y = 5; \quad 2x + 3y = 6$

63. $2x - y = 8; \quad x - 2y = 4$

Answer

not parallel

Find the Equation of a Line

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

64. Slope $\frac{1}{3}$ and y-intercept $(0, -6)$

65. Slope -5 and y-intercept $(0, -3)$

Answer

$y = -5x - 3$

66. Slope 0 and y-intercept $(0, 4)$

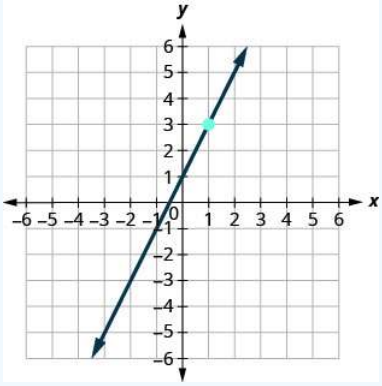
67. Slope -2 and y-intercept $(0, 0)$

Answer

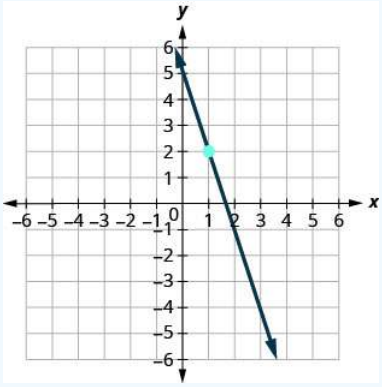
$y = -2x$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

68.



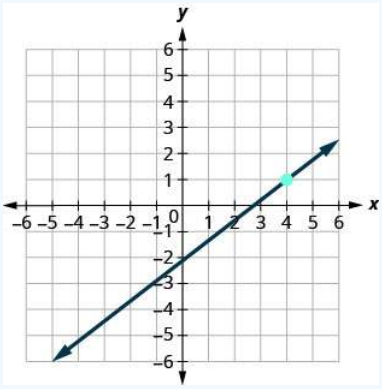
69.



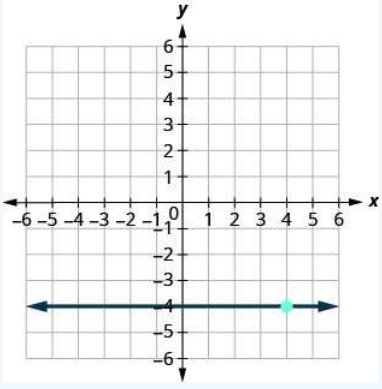
Answer

$y = -3x + 5$

70.



71.



Answer

$y = -4$

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

72. $m = -\frac{1}{4}$, point $(-8, 3)$

73. $m = \frac{3}{5}$, point $(10, 6)$

Answer

$y = \frac{3}{5}x$

74. Horizontal line containing $(-2, 7)$

75. $m = -2$, point $(-1, -3)$

Answer

$y = -2x - 5$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

76. $(2, 10)$ and $(-2, -2)$

77. $(7, 1)$ and $(5, 0)$

Answer

$y = \frac{1}{2}x - \frac{5}{2}$

78. $(3, 8)$ and $(3, -4)$

79. $(5, 2)$ and $(-1, 2)$

Answer

$y = 2$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

80. line $y = -3x + 6$, point $(1, -5)$

81. line $2x + 5y = -10$, point $(10, 4)$

Answer

$y = -\frac{2}{5}x + 8$

82. line $x = 4$, point $(-2, -1)$

83. line $y = -5$, point $(-4, 3)$

Answer

$y = 3$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

84. line $y = -\frac{4}{5}x + 2$, point $(8, 9)$

85. line $2x - 3y = 9$, point $(-4, 0)$

Answer

$y = -\frac{3}{2}x - 6$

86. line $y = 3$, point $(-1, -3)$

87. line $x = -5$ point $(2, 1)$

Answer

$y = 1$

Graph Linear Inequalities in Two Variables

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

88. Determine whether each ordered pair is a solution to the inequality $y < x - 3$:

ⓐ $(0, 1)$ ⓑ $(-2, -4)$ ⓒ $(5, 2)$ ⓓ $(3, -1)$
ⓔ $(-1, -5)$

89. Determine whether each ordered pair is a solution to the inequality $x + y > 4$:

ⓐ $(6, 1)$ ⓑ $(-3, 6)$ ⓒ $(3, 2)$ ⓓ $(-5, 10)$ ⓔ $(0, 0)$

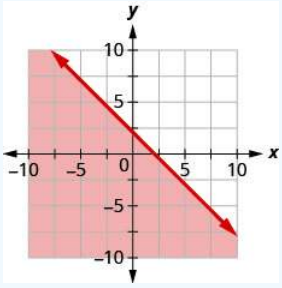
Answer

ⓐ yes ⓑ no ⓒ yes ⓓ yes; ⓔ nom

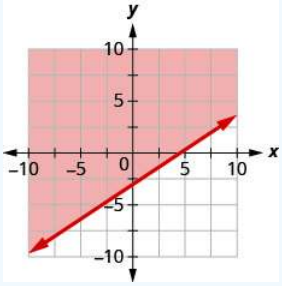
Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

90. Write the inequality shown by the graph with the boundary line $y = -x + 2$.



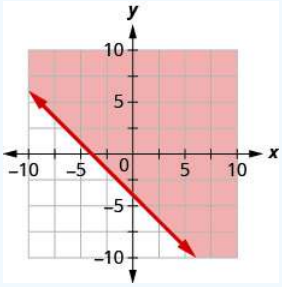
91. Write the inequality shown by the graph with the boundary line $y = \frac{2}{3}x - 3$.



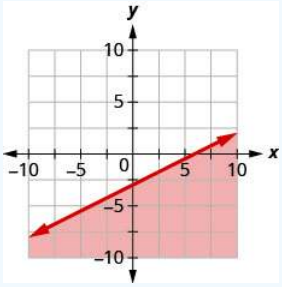
Answer

$y > \frac{2}{3}x - 3$

92. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = -4$.



93. Write the inequality shown by the shaded region in the graph with the boundary line $x - 2y = 6$.



Answer

$x - 2y \geq 6$

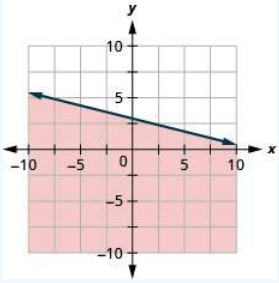
Graph Linear Inequalities in Two Variables

In the following exercises, graph each linear inequality.

94. Graph the linear inequality $y > \frac{2}{5}x - 4$.

95. Graph the linear inequality $y \leq -\frac{1}{4}x + 3$.

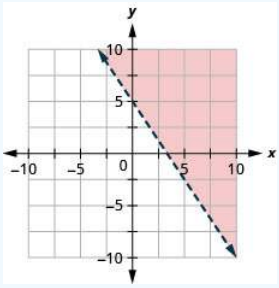
Answer



96. Graph the linear inequality $x - y \leq 5$.

97. Graph the linear inequality $3x + 2y > 10$.

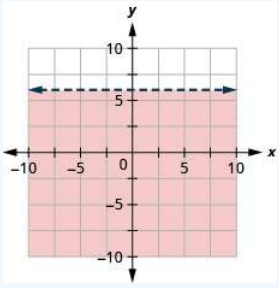
Answer



98. Graph the linear inequality $y \leq -3x$.

99. Graph the linear inequality $y < 6$.

Answer



Solve Applications using Linear Inequalities in Two Variables

100. Shanthie needs to earn at least $500 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $10 an hour and the other as an intern in a law office for $25 hour. How many hours does Shanthie need to work at each job to earn at least $500 per week?

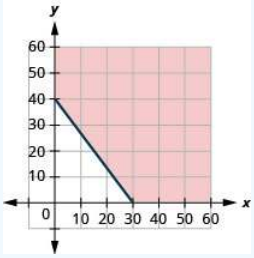
ⓐ Let *x* be the number of hours she works teaching swimming and let *y* be the number of hours she works as an intern. Write an inequality that would model this situation.
ⓑ Graph the inequality.
ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Shanthie.

101. Atsushi he needs to exercise enough to burn 600 calories each day. He prefers to either run or bike and burns 20 calories per minute while running and 15 calories a minute while biking.

ⓐ If x is the number of minutes that Atsushi runs and y is the number minutes he bikes, find the inequality that models the situation.
ⓑ Graph the inequality.
ⓒ List three solutions to the inequality. What options do the solutions provide Atsushi?

Answer

ⓐ $20x+15y\geq 60020x+15y\geq 600$
ⓑ



ⓒ Answers will vary.

Relations and Functions

Find the Domain and Range of a Relation

In the following exercises, for each relation, ⓐ find the domain of the relation ⓑ find the range of the relation.

102. $\{(5,-2),\ (5,-4),\ (7,-6),\ (8,-8),\ (9,-10)\}$

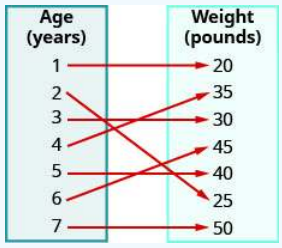
103. $\{(-3,7),\ (-2,3),\ (-1,9),\ (0,-3),\ (-1,8)\}$

Answer

ⓐ $D:-3,-2,-1,0$
ⓑ $R:7,3,9,-3,8$

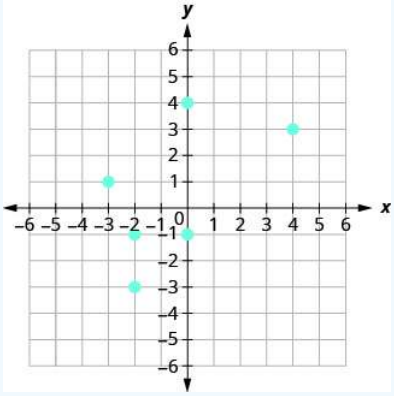
In the following exercise, use the mapping of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

104. The mapping below shows the average weight of a child according to age.



In the following exercise, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

105.



Answer

ⓐ $(4,3),\ (-2,-3),\ (-2,-1),\ (-3,1),\ (0,-1),\ (0,4)$
ⓑ $D: -3,-2,0,4$
ⓒ $R: -3,-1,1,3,4$

Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.

106. $\{(9,-5),\ (4,-3),\ (1,-1),\ (0,0),\ (1,1),\ (4,3),\ (9,5)\}$

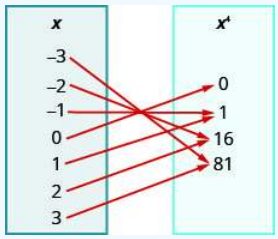
107. $\{(-3,27),\ (-2,8),\ (-1,1),\ (0,0),\ (1,1),\ (2,8),\ (3,27)\}$

Answer

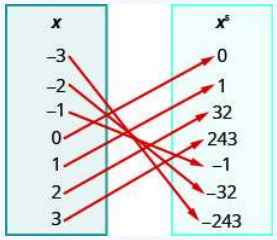
ⓐ yes ⓑ $-3,-2,-1,0,1,2,3$
ⓒ $0,1,8,27$

In the following exercises, use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the function ⓒfind the range of the function.

108.



109.



Answer

ⓐ $-3, -2, -1, 0, 1, 2, 3$
ⓑ $-3, -2, -1, 0, 1, 2, 3$
ⓒ $-243, -32, -1, 0, 1, 32, 243$

In the following exercises, determine whether each equation is a function.

110. $2x + y = -3$

111. $y = x^2$

Answer

yes

112. $y = 3x - 5$

113. $y = x^3$

Answer

yes

114. $2x + y2 = 4$

Find the Value of a Function

In the following exercises, evaluate the function:

ⓐ $f(-2)$ ⓑ $f(3)$ ⓒ $f(a)$.

115. $f(x) = 3x - 4$

Answer

ⓐ $f(-2) = -10$ ⓑ $f(3) = 5$ ⓒ $f(a) = 3a - 4$

116. $f(x) = -2x + 5$

117. $f(x) = x^2 - 5x + 6$

Answer

ⓐ $f(-2) = 20$ ⓑ $f(3) = 0$ ⓒ $f(a) = a^2 - 5a + 6$

118. $f(x) = 3x^2 - 2x + 1$

In the following exercises, evaluate the function.

119. $g(x) = 3x2 - 5x;\ g(2)$

Answer

2

120. $F(x) = 2x2 - 3x + 1;\ F(-1)$

121. $h(t) = 4|t-1|+2;\ h(t) = 4$

Answer

18

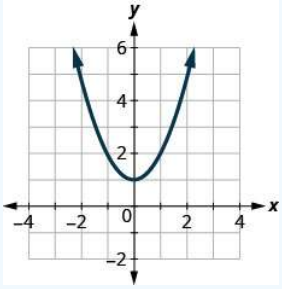
122. $f(x) = x + 2x - 1;\ f(3)$

Graphs of Functions

Use the Vertical line Test

In the following exercises, determine whether each graph is the graph of a function.

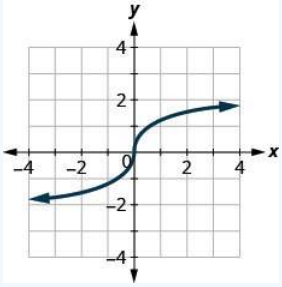
123.



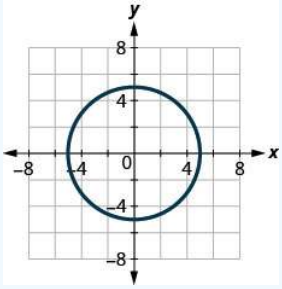
Answer

yes

124.



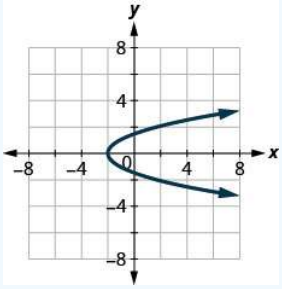
125.



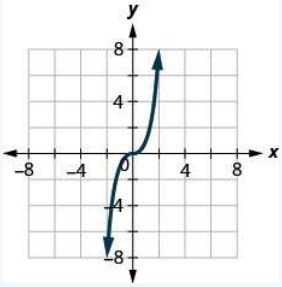
Answer

no

126.



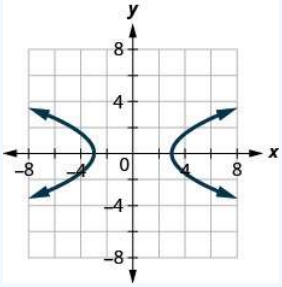
127.



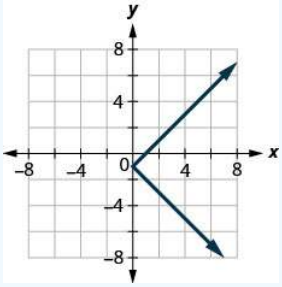
Answer

yes

128.



129.



Answer

no

Identify Graphs of Basic Functions

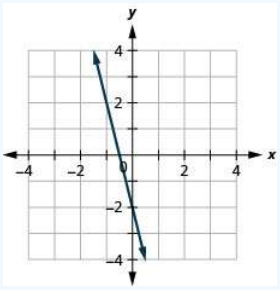
In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

130. $f(x) = 5x + 1$

131. $f(x) = -4x - 2$

Answer

ⓐ



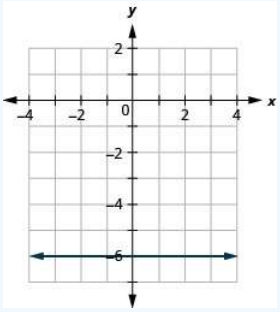
ⓑ $D : (-\text{inf}, \text{inf}), R : (-\text{inf}, \text{inf})$

132. $f(x) = \frac{2}{3}x - 1$

133. $f(x) = -6$

Answer

ⓐ



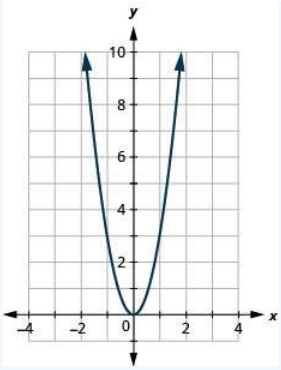
ⓑ $D : (-\text{inf}, \text{inf}), R : (-\text{inf}, \text{inf})$

134. $f(x) = 2x$

135. $f(x) = 3x^2$

Answer

ⓐ



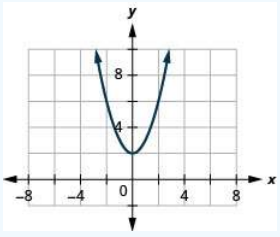
ⓑ $D: (-\text{inf}, \text{inf}), R: (-\text{inf}, 0]$

136. $f(x) = -12x^2$

137. $f(x) = x^2 + 2$

Answer

ⓐ



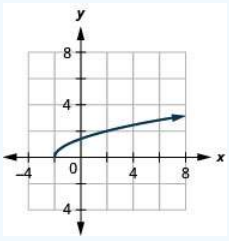
ⓑ $D: (-\text{inf}, \text{inf}), R: (-\text{inf}, \text{inf})$

138. $f(x) = x^3 - 2$

139. $f(x) = \sqrt{x+2}$

Answer

ⓐ



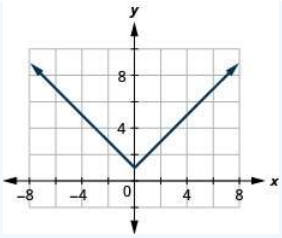
ⓑ $D: [-2, -2, \text{inf}),\ R: [0, \text{inf})$

140. $f(x) = -|x|$

141. $f(x) = |x| + 1$

Answer

ⓐ

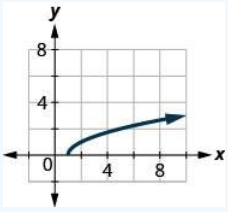


ⓑ $D : (-\text{inf}, \text{inf}),\ R : [1, \text{inf})$

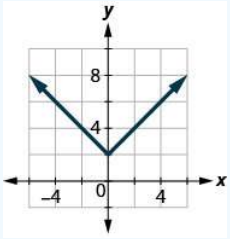
Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation

142.



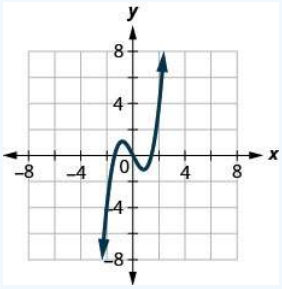
143.



Answer

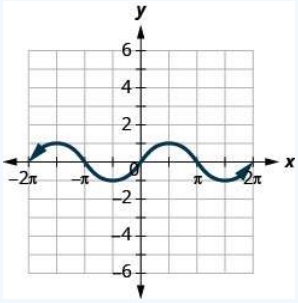
$D : (-\text{inf}, \text{inf}), R : [2, \text{inf})$

144.



In the following exercises, use the graph of the function to find the indicated values.

145.

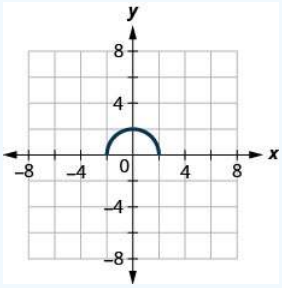


ⓐ Find $f(0)$.
ⓑ Find $f(12\pi)$.
ⓒ Find $f(-32\pi)$.
ⓓ Find the values for x when $f(x)=0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercept(s).
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Answer

ⓐ $f(x)=0$ ⓑ $f(\pi/2)=1$
ⓒ $f(-3\pi/2)=1$ ⓓ $f(x)=0$ for $x=-2\pi,-\pi,0,\pi,2\pi$
ⓔ $(-2\pi,0),(-\pi,0),(0,0),(\pi,0),(2\pi,0)$ ⓕ $(0,0)$
ⓖ $[-2\pi,2\pi]$ ⓗ $[-1,1]$

146.



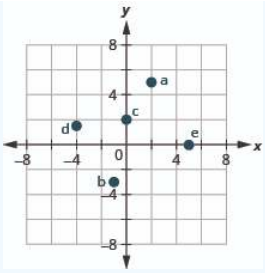
ⓐ Find $f(0)$.
ⓑ Find the values for x when $f(x)=0$.
ⓒ Find the x-intercepts.
ⓓ Find the y-intercept(s).
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation.

Practice Test

1. Plot each point in a rectangular coordinate system.

ⓐ $(2,5)$
ⓑ $(-1,-3)$
ⓒ $(0,2)$
ⓓ $(-4,32)$
ⓔ $(5,0)$

Answer

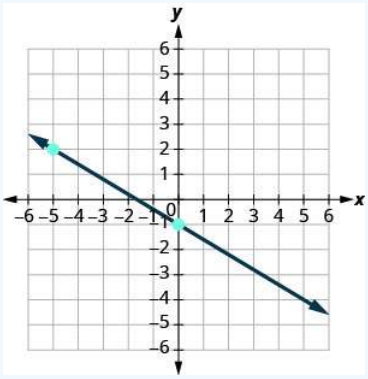


2. Which of the given ordered pairs are solutions to the equation $3x - y = 6$?

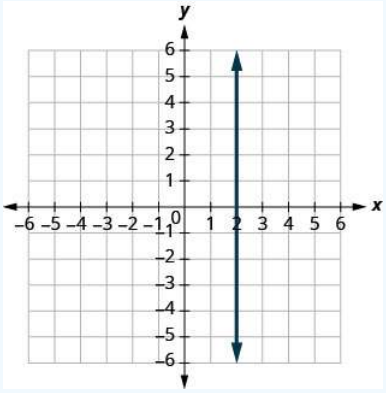
ⓐ $(3, 3)$ ⓑ $(2, 0)$ ⓒ $(4, -6)$

3. Find the slope of each line shown.

ⓐ



ⓑ



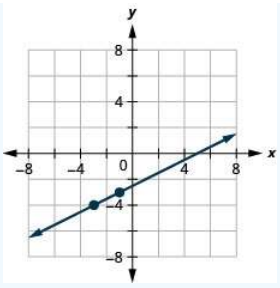
Answer

ⓐ $-\frac{3}{5}$ ⓑ undefined

4. Find the slope of the line between the points $(5, 2)$ and $(-1, -4)$.

5. Graph the line with slope $\frac{1}{2}$ containing the point $(-3, -4)$.

Answer

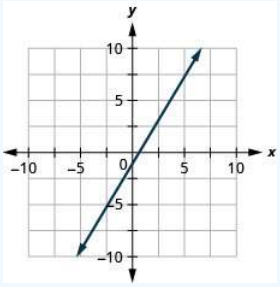


6. Find the intercepts of $4x + 2y = -8$ and graph.

Graph the line for each of the following equations.

7. $y = \frac{5}{3}x - 1$

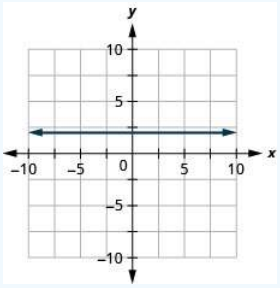
Answer



8. $y = -x$

9. $y = 2$

Answer



Find the equation of each line. Write the equation in slope-intercept form.

10. slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

11. $m = 2$, point $(-3, -1)$

Answer

$y = 2x + 5$

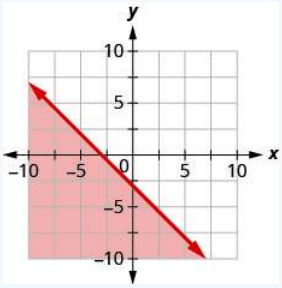
12. containing $(10, 1)$ and $(6, -1)$

13. perpendicular to the line $y = \frac{5}{4}x + 2$, containing the point $(-10, 3)$

Answer

$y = -\frac{4}{5}x - 5$

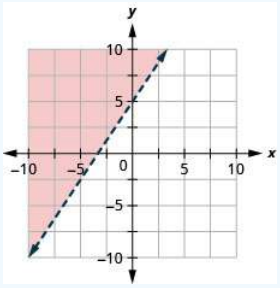
14. Write the inequality shown by the graph with the boundary line $y = -x - 3$.



Graph each linear inequality.

15. $y > \frac{3}{2}x + 5$

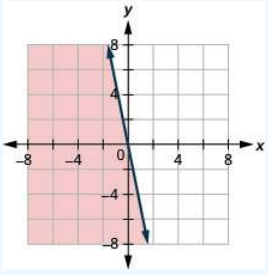
Answer



16. $x - y \geq -4$

17. $y \leq -5x$

Answer



18. Hiro works two part time jobs in order to earn enough money to meet her obligations of at least $450 a week. Her job at the mall pays $10 an hour and her administrative assistant job on campus pays $15 an hour. How many hours does Hiro need to work at each job to earn at least $450?

ⓐ Let *x* be the number of hours she works at the mall and let *y* be the number of hours she works as administrative assistant. Write an inequality that would model this situation.

ⓑ Graph the inequality .
ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then explain what that means for Hiro.

19. Use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒfind the range of the relation.

$\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

Answer

ⓐ yes ⓑ $\{-3, -2, -1, 0, 1, 2, 3\}$ⓒ $\{0, 1, 8, 27\}$

20. Evaluate the function: ⓐ $f(-1)$ ⓑ $f(2)$ ⓒ $f(c)$.

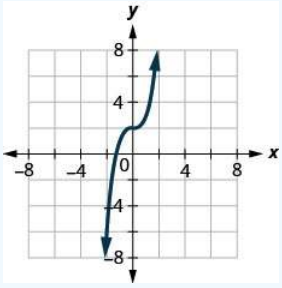
$f(x) = 4x^2 - 2x - 3$

21. For $h(y) = 3|y - 1| - 3$, evaluate $h(-4)$.

Answer

12

22. Determine whether the graph is the graph of a function. Explain your answer.

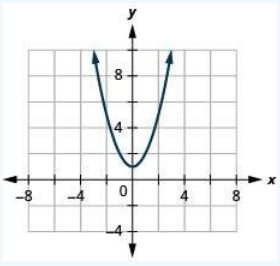


In the following exercises, ⓐ graph each function ⓑ state its domain and range.
Write the domain and range in interval notation.

23. $f(x) = x^2 + 1$

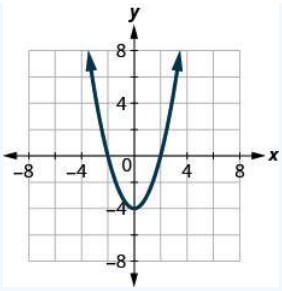
Answer

ⓐ



ⓑ $D : (-\text{inf}, \text{inf}), R : [1, \text{inf})$

24. $f(x) = \sqrt{x + 1}$



ⓑ Find the y-intercepts.
ⓒ Find $f(-1)$.
ⓓ Find $f(1)$.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation.

Answer

ⓐ $x = -2, 2$ ⓑ $y = -4$
ⓒ $f(-1) = -3$ ⓓ $f(1) = -3$
ⓔ $D : (-\text{inf}, \text{inf})$ ⓕ $R : [-4, \text{inf})$

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CHAPTER OVERVIEW

4: Systems of Linear Equations

- 4.0: Introduction
- 4.1: Solve Systems of Linear Equations with Two Variables
  - 4.1E: Exercises
- 4.2: Solve Applications with Systems of Equations
  - 4.2E: Exercises
- 4.3: Solve Mixture Applications with Systems of Equations
  - 4.3E: Exercises
- 4.4: Solve Systems of Equations with Three Variables
  - 4.4E: Exercises
- Chapter 4 Review Exercises

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4.0: Introduction



Figure 4.0.1: In the future, car drivers may become passengers because cars will be able to drive themselves. (credit: jingoba/Pixabay)

Climb into your car. Put on your seatbelt. Choose your destination and then…relax. That's right. You don't have to do anything else because you are in an autonomous car, or one that navigates its way to your destination! No cars are fully autonomous at the moment and so you theoretically still need to have your hands on the wheel. Self-driving cars may help ease traffic congestion, prevent accidents, and lower pollution. The technology is thanks to computer programmers who are developing software to control the navigation of the car. These programmers rely on their understanding of mathematics, including relationships between equations. In this chapter, you will learn how to solve systems of linear equations in different ways and use them to analyze real-world situations.

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4.1: Solve Systems of Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Solve a system of equations by substitution
- Solve a system of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Before you get started, take this readiness quiz.

1. For the equation $y = \frac{2}{3}x - 4$,
   ⓐ Is $(6, 0)$ a solution? ⓑ Is $(-3, -2)$ a solution?
   If you missed this problem, review [link].
2. Find the slope and y-intercept of the line $3x - y = 12$.
   If you missed this problem, review [link].
3. Find the x- and y-intercepts of the line $2x - 3y = 12$.
   If you missed this problem, review [link].

Determine Whether an Ordered Pair is a Solution of a System of Equations

In Solving Linear Equations, we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a **system of linear equations**.

SYSTEM OF LINEAR EQUATIONS

When two or more linear equations are grouped together, they form a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations in two unknowns. We will solve larger systems of equations later in this chapter.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, such as $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to *both* equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the **solutions of a system of equations**.

SolutionS OF A SYSTEM OF EQUATIONS

The **solutions of a system of equations** are the values of the variables that make *all* the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Example 4.1.1

Determine whether the ordered pair is a solution to the system $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$.

ⓐ $(-2, -1)$ ⓑ $(-4, -3)$

Answer

ⓐ

$$\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$$

We substitute $x=-2$ and $y=-1$ into both equations.

$x-y=-1$	$2x-y=-5$
$-2-(-1)\stackrel{?}{=}-1$	$2\cdot -2-(-1)\stackrel{?}{=}-5$
$-1=-1$ ✓	$5\neq -5$
$(-2,-1)$ does not make both equations true.	$(-2,-1)$ is not a solution.

ⓑ

We substitute $x=-4$ and $y=-3$ into both equations.

$x-y=-1$	$2x-y=-5$
$-4-(-3)\stackrel{?}{=}-1$	$2\cdot(-4)-(-3)\stackrel{?}{=}-5$
$-1=-1$ ✓	$-5=-5$ ✓
$(-4,-3)$ does make both equations true.	$(-4,-3)$ is a solution.

? Example 4.1.2

Determine whether the ordered pair is a solution to the system $\begin{cases} x+y=0 \\ x+2y=-5 \end{cases}$.

ⓐ $(1,-3)$ ⓑ $(0,0)$

Answer

ⓐ yes ⓑ no

? Example 4.1.3

Determine whether the ordered pair is a solution to the system $\begin{cases} -3y=-8 \\ -3x-y=4 \end{cases}$.

ⓐ $(2,-2)$ ⓑ $(-2,2)$

Answer

ⓐ no ⓑ yes

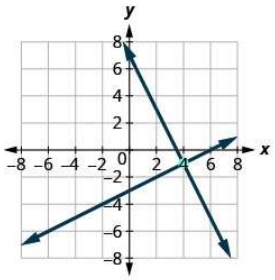
Solve a System of Linear Equations by Graphing

In this section, we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

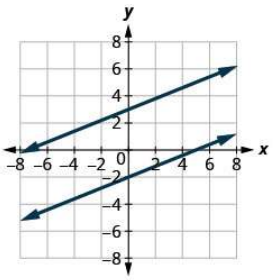
The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

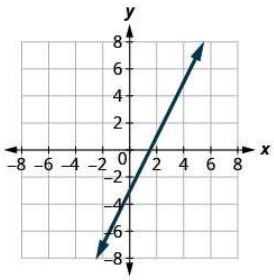
Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.



The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.



Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

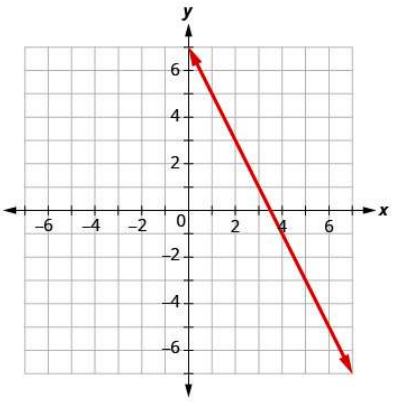
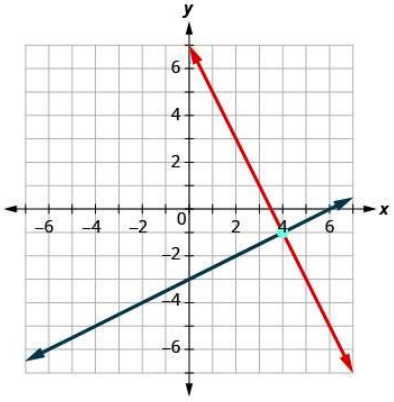
Figure 4.1.1

Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end of the section you'll decide which method was the most convenient way to solve this system.

? Example 4.1.4: How to Solve a System of Equations by Graphing

Solve the system by graphing $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$.

Answer

Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x+y=7$ $y=-2x+7$ $m=-2 \quad b=7$	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ 
Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x-2y=6$ $(0,-3) \quad (6,0)$	

Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4. Identify the solution to the system. • If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. • If the lines are parallel, the system has no solution. • If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at (4, –1). $2x+y=7$ $2(4)+(-1)\stackrel{?}{=}7$ $8-1\stackrel{?}{=}7$ $7=7$ ✓ $x-2y=6$ $4-2(-1)\stackrel{?}{=}6$ $6=6$ ✓ The solution is (4, –1).

Example 4.1.5

Solve the system by graphing: $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$.

Answer

$(3,2)$

Example 4.1.6

Solve the system by graphing: $\begin{cases} -x+y=1 \\ 3x+2y=12 \end{cases}$

Answer

$(2,3)$

The steps to use to solve a system of linear equations by graphing are shown here.

SOLVE A SYSTEM OF LINEAR EQUATIONS BY GRAPHING.

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.
   - If the lines intersect, identify the point of intersection. This is the solution to the system.
   - If the lines are parallel, the system has no solution.
   - If the lines are the same, the system has an infinite number of solutions.
5. Check the solution in both equations.

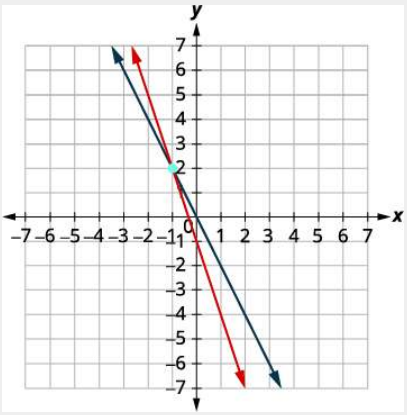
In the next example, we'll first re-write the equations into slope–intercept form as this will make it easy for us to quickly graph the lines.

Example 4.1.7

Solve the system by graphing: $\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$

Answer

We'll solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts.

	$\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$
Solve the first equation for y.	$\begin{aligned} 3x+y&=-1 \\ y&=-3x-1 \end{aligned}$
Find the slope and y-intercept.	$\begin{aligned} m&=-3 \\ b&=-1 \end{aligned}$
Solve the second equation for y.	$\begin{aligned} 2x+y&=0 \\ y&=-2x \end{aligned}$
Find the slope and y-intercept.	$\begin{aligned} m&=-2 \\ b&=0 \end{aligned}$
Graph the lines.	
Determine the point of intersection.	The lines intersect at $(-1, 2)$.
Check the solution in both equations. $\begin{aligned} 3x+y&=-1 \\ 3(-1)+2&\stackrel{?}{=}-1 \\ -1&=-1\ \checkmark \end{aligned}$ $\begin{aligned} 2x+y&=0 \\ 2(-1)+2&\stackrel{?}{=}0 \\ 0&=0\ \checkmark \end{aligned}$	
	The solution is $(-1, 2)$.

? Example 4.1.8

Solve the system by graphing: $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$.

Answer

$(3, 4)$

? Example 4.1.9

Solve the system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$.

Answer

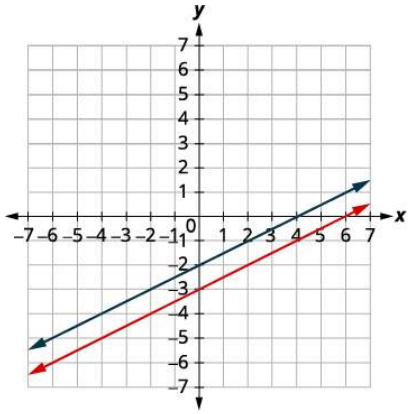
$(5, -4)$

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

? Example 4.1.10

Solve the system by graphing: $\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$.

Answer

	$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$
To graph the first equation, we will use its slope and y-intercept.	$y = \frac{1}{2}x - 3$ $m = \frac{1}{2}$ $b = -3$
To graph the second equation, we will use the intercepts.	$x - 2y = 4$
	x | y 0 | −2 4 | 0
Graph the lines.	
Determine the points of intersection.	The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

? Example 4.1.11

Solve the system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = 4 \end{cases}$.

Answer

no solution

? Example 4.1.12

Solve the system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$.

Answer

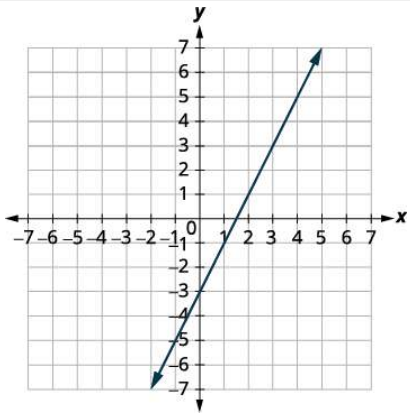
no solution

Sometimes the equations in a system represent the same line. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system.

Example 4.1.13

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = 9 \end{cases}$.

Answer

	$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$
Find the slope and y-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$
Find the intercepts of the second equation.	$-6x + 3y = -9$
	x | y 0 | -3 $\frac{3}{2}$ | 0
Graph the lines.	
	The lines are the same! Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to this system.

If you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same y-intercept.

Example 4.1.14

Solve the system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$.

Answer

infinitely many solutions

Example 4.1.15

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$.

Answer

infinitely many solutions

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are **coincident**. Coincident lines have the same slope and same *y*-intercept.

COINCIDENT LINES

Coincident lines have the same slope and same *y*-intercept.

The systems of equations in Example and Example each had two intersecting lines. Each system had one solution.

In Example, the equations gave coincident lines, and so the system had infinitely many solutions.

The systems in those three examples had at least one solution. A system of equations that has at least one solution is called a *consistent* system.

A system with parallel lines, like Example, has no solution. We call a system of equations like this *inconsistent*. It has no solution.

CONSISTENT AND INCONSISTENT SYSTEMS

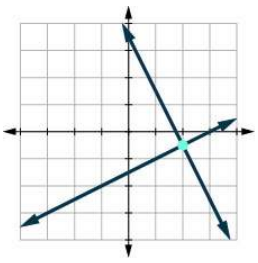
A **consistent system of equations** is a system of equations with at least one solution.

An **inconsistent system of equations** is a system of equations with no solution.

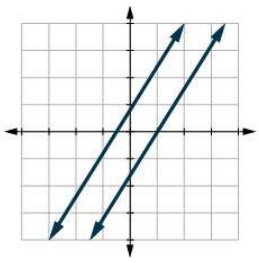
We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

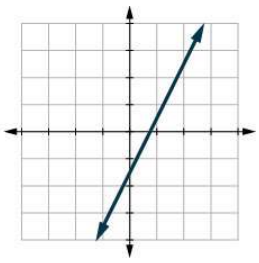
Let's sum this up by looking at the graphs of the three types of systems. See below and Table.



Intersecting



Parallel



Coincident

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/ independent	Independent	Independent	Dependent

Example 4.1.16

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$ ⓑ $\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$

Answer

ⓐ We will compare the slopes and intercepts of the two lines.

$$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$$

$y = 3x - 1$

The first equation is already in slope-intercept form.
Write the second equation in slope-intercept form.

$$\begin{aligned} 6x - 2y &= 12 \\ -2y &= -6x + 12 \\ \tfrac{-2y}{-2} &= \tfrac{-6x+12}{-2} \\ y &= 3x - 6 \end{aligned}$$

$$\begin{aligned} y &= 3x - 1 \\ m &= 3 \\ b &= -1 \end{aligned} \qquad \begin{aligned} y &= 3x - 6 \\ m &= 3 \\ b &= -6 \end{aligned}$$

Find the slope and intercept of each line.

Since the slopes are the same andy-intercepts are different, the lines are parallel.

ⓑ We will compare the slope and intercepts of the two lines.

$$\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$$

Write both equations in slope–intercept form.

$$\begin{aligned} 2x + y &= -3 \\ y &= -2x - 3 \end{aligned} \qquad \begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \tfrac{-5y}{-5} &= \tfrac{-x+5}{-5} \\ y &= \tfrac{1}{5} - 1 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= -2x - 3 \\ m &= -2 \\ b &= -3 \end{aligned} \qquad \begin{aligned} y &= \tfrac{1}{5} - 1 \\ m &= \tfrac{1}{5} \\ b &= -1 \end{aligned}$$

Since the slopes are different, the lines intersect.

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

? Example 4.1.17

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = -2x - 4 \\ 4x + 2y = 9 \end{cases}$ ⓑ $\begin{cases} 3x + 2y = 2 \\ 2x + y = 1 \end{cases}$

Answer

ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

? Example 4.1.18

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\begin{cases} y = \frac{1}{3}x - 5 \\ x - 3y = 6 \end{cases}$ ⓑ $\begin{cases} x + 4y = 12 \\ -x + y = 3 \end{cases}$

Answer

ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

Solve a System of Equations by Substitution

We will now solve systems of linear equations by the substitution method.

We will use the same system we used first for graphing.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

? Example 4.1.19: How to Solve a System of Equations by Substitution

Solve the system by substitution: $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Answer

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for y.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $2x + y = 7$ $y = 7 - 2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$

Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x-2(7-2x)=6$ $x-14+4x=6$ $5x=20$ $x=4$
Step 4. Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x+y=7$ $2(4)+y=7$ $8+y=7$ $y=-1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$, $y=-1$ into both equations and make sure they are both true.	$2x+y=7$ $\quad$ $x-2y=6$ $2(4)+(-1)\stackrel{?}{=}7$ $\quad$ $4-2(-1)\stackrel{?}{=}6$ $7=7\checkmark$ $\quad$ $6=6\checkmark$ Both equations are true. $(4, -1)$ is the solution to the system.

Example 4.1.20

Solve the system by substitution: $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$

Answer

$(6, 1)$

Example 4.1.21

Solve the system by substitution: $\begin{cases} 2x+y=-1 \\ 4x+3y=3 \end{cases}$

Answer

$(-3, 5)$

SOLVE A SYSTEM OF EQUATIONS BY SUBSTITUTION.

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations.

Be very careful with the signs in the next example.

Example 4.1.22

Solve the system by substitution: $\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$

Answer

We need to solve one equation for one variable. We will solve the first equation for y.

	$\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$
Solve the first equation for y. Substitute $-2x+2$ for y in the second equation.	$4x+2y=4$ $2y=-4x+4$ $y=-2x+2$ $6x-y=8$
Replace the y with $-2x+2$.	$6x-(-2x+2)=8$
Solve the equation for x.	$6x+2x-2=8$ $8x-2=8$ $8x=10$ $x=\frac{5}{4}$
Substitute $x=54$ into $4x+2y=4$ to find y.	$4x+2y=4$ $4\left(\frac{5}{4}\right)+2y=4$ $5+2y=4$ $2y=-1$ $y=-\frac{1}{2}$
	The ordered pair is $(54,-12)$.
Check the ordered pair in both equations. $4x+2y=4$ $4\left(\frac{5}{4}\right)+2\left(-\frac{1}{2}\right)\stackrel{?}{=}4$ $5-1\stackrel{?}{=}4$ $4=4$ ✓ $6x-y=8$ $6\left(\frac{5}{4}\right)-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{15}{4}-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{16}{2}\stackrel{?}{=}8$ $8=8$ ✓	
	The solution is $(54,-12)$.

? Example 4.1.23

Solve the system by substitution: $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$

Answer

$(2,32)$

? Example 4.1.24

Solve the system by substitution: $\begin{cases} 4x-y=0 \\ 2x-3y=5 \end{cases}$

Answer

$(-12,-2)$

Solve a System of Equations by Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination

method, too, but we'll have a different way to get there.

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a*, *b*, *c*, and *d*.

$$\begin{aligned} &\text{if} && a=b \\ &\text{and} && c=d \\ &\text{then} && a+c=b+d. \end{aligned}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{cases} 3x+y=5 \\ 2x-y=0 \end{cases}$$
$$5x=5$$

The *y*'s add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases} +4y=2 \\ 2x+5y=-2 \end{cases}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of *x* opposites. We must multiply every term on both sides of the equation by -2.

$$\begin{aligned} -2(x+4y) &= -2(2) \\ 2x+5y &= -2 \end{aligned}$$

Then rewrite the system of equations.

$$\begin{aligned} -2x-8y &= -4 \\ 2x+5y &= -2 \end{aligned}$$

Now we see that the coefficients of the *x* terms are opposites, so *x* will be eliminated when we add these two equations.

$$\begin{array}{r} -2x-8y=-4 \\ 2x+5y=-2 \\ \hline -3y=-6 \end{array}$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

? Exercise 4.1.25: How to Solve a System of Equations by Elimination

Solve the system by elimination: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

Step 1. Write both equations in standard form. • If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2. Make the coefficients of one variable opposites. • Decide which variable you will eliminate. • Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{array}{r} 4x + 2y = 14 \\ x - 2y = 6 \\ \hline 5x \quad = 20 \end{array}$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4$, $y = -1$ into $2x + y = 7$ and $x - 2y = 6$. Do they make both equations true? Yes!	$2x + y = 7$: $2(4) + (-1) \stackrel{?}{=} 7$: $7 = 7$ ✓ $x - 2y = 6$: $4 - 2(-1) \stackrel{?}{=} 6$: $6 = 6$ ✓ The solution is $(4, -1)$.

? Exercise 4.1.26

Solve the system by elimination: $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$

Answer

$(2, -1)$

? Exercise 4.1.27

Solve the system by elimination: $\begin{cases} 4x + y = -5 \\ -2x - 2y = -2 \end{cases}$

Answer

$(-2, 3)$

The steps are listed here for easy reference.

SOLVE A SYSTEM OF EQUATIONS BY ELIMINATION.

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
   - Decide which variable you will eliminate.

- Multiply one or both equations so that the coefficients of that variable are opposites.

3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations.

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Exercise 4.1.28

Solve the system by elimination: $\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$

Answer

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

	$4x-3y=9$ $7x+2y=-6$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$2(4x-3y)=2(9)$ $3(7x+2y)=3(-6)$
Simplify.	$8x-6y=18$ $21x+6y=-18$
Add the two equations to eliminate y.	$8x-6y=18$ $21x+6y=-18$ $29x=0$
Solve for x.	$x=0$ $7x+2y=-6$
Substitute x=0x=0 into one of the original equations.	$7\cdot 0+2y=-6$
Solve for y.	$2y=-6$ $y=-3$
Write the solution as an ordered pair.	The ordered pair is $(0,-3)$.
Check that the ordered pair is a solution to **both** original equations. $4x-3y=9$ $4(0)-3(-3)\stackrel{?}{=}9$ $9=9$ ✓ $7x+2y=-6$ $7(0)+2(-3)\stackrel{?}{=}-6$ $-6=-6$ ✓	
	The solution is $(0,-3)$.

Exercise 4.1.29

Solve the system by elimination: $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Answer

$(1,3)$

Exercise 4.1.30

Solve each system by elimination: $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

Answer

$(4,-3)$

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the LCD of all the fractions in the equation.

Exercise 4.1.31

Solve the system by elimination: $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$

Answer

In this example, both equations have fractions. Our first step will be to multiply each equation by the LCD of all the fractions in the equation to clear the fractions.

	$\begin{aligned} x+\tfrac{1}{2}y&=6 \\ \tfrac{3}{2}x+\tfrac{2}{3}y&=\tfrac{17}{2} \end{aligned}$
To clear the fractions, multiply each equation by its LCD.	$\begin{aligned} 2\left(x+\tfrac{1}{2}y\right)&=2(6) \\ 6\left(\tfrac{3}{2}x+\tfrac{2}{3}y\right)&=6\left(\tfrac{17}{2}\right) \end{aligned}$
Simplify.	$\begin{aligned} 2x+\ y&=12 \\ 9x+4y&=51 \end{aligned}$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.	
We can eliminate y by multiplying the top equation by -4.	$\begin{aligned} -4(2x+\ y)&=-4(12) \\ 9x+4y&=51 \end{aligned}$
Simplify and add. Substitute $x=3$ into one of the original equations.	$\begin{aligned} -8x-4y&=-48 \\ 9x+4y&=\ 51 \\ \hline x\ &=\ 3 \end{aligned}$ $x+\tfrac{1}{2}y=6$
Solve for y.	$\begin{aligned} 3+\tfrac{1}{2}y&=6 \\ \tfrac{1}{2}y&=3 \\ y&=6 \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $(3,6)$.
Check that the ordered pair is a solution to both original equations.	$\begin{aligned} x+\tfrac{1}{2}y&=6 \\ 3+\tfrac{1}{2}(6)&\stackrel{?}{=}6 \\ 3+3&\stackrel{?}{=}6 \\ 6&=6\ \checkmark \end{aligned}$ $\begin{aligned} \tfrac{3}{2}x+\tfrac{2}{3}y&=\tfrac{17}{2} \\ \tfrac{3}{2}(3)+\tfrac{2}{3}(6)&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{9}{2}+4&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{9}{2}+\tfrac{8}{2}&\stackrel{?}{=}\tfrac{17}{2} \\ \tfrac{17}{2}&=\tfrac{17}{2}\ \checkmark \end{aligned}$
	The solution is $(3,6)$.

Exercise 4.1.32

Solve each system by elimination: $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 1 \\ \frac{3}{4}x - y = \frac{5}{2} \end{cases}$

Answer

$(6, 2)$

Exercise 4.1.33

Solve each system by elimination: $\begin{cases} x + \frac{3}{5}y = -\frac{1}{5} \\ -\frac{1}{2}x - \frac{2}{3}y = \frac{5}{6} \end{cases}$

Answer

$(1, -2)$

When we solved the system by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

The same is true using substitution or elimination. If the equation at the end of substitution or elimination is a true statement, we have a consistent but dependent system and the system of equations has infinitely many solutions. If the equation at the end of substitution or elimination is a false statement, we have an inconsistent system and the system of equations has no solution.

Exercise 4.1.34

Solve the system by elimination: $\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$

Answer

$$\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$$

Write the second equation in standard form.

$$\begin{cases} 3x + 4y = 12 \\ \frac{3}{4}x + y = 3 \end{cases}$$

Clear the fractions by multiplying the second equation by 4.

$$\begin{cases} 3x + 4y = 12 \\ 4(\frac{3}{4}x + y) = 4(3) \end{cases}$$

Simplify.

$$\begin{cases} 3x + 4y = 12 \\ 3x + 4y = 12 \end{cases}$$

To eliminate a variable, we multiply the second equation by -1. Simplify and add.

$$\begin{cases} 3x + 4y = 12 \\ -3x - 4y = -12 \end{cases}$$
$$0 = 0$$

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

? Exercise 4.1.35

Solve the system by elimination: $\begin{cases} 5x-3y=15 \\ 5y=-5+\frac{5}{3}x \end{cases}$

Answer

infinitely many solutions

? Exercise 4.1.36

Solve the system by elimination: $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$

Answer

infinitely many solutions

Choose the Most Convenient Method to Solve a System of Linear Equations

When you solve a system of linear equations in in an application, you will not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations a rein standard form.

? Example 4.1.37

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$ ⓑ $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Answer

ⓐ

$$\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$$

Since both equations are in standard form, using elimination will be most convenient.

ⓑ

$$\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$$

Since one equation is already solved for y, using substitution will be most convenient.

? Example 4.1.38

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$ ⓑ $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Answer

ⓐ Since both equations are in standard form, using elimination will be most convenient. ⓑ Since one equation is already solved for x, using substitution will be most convenient.

? Example 4.1.39

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} y = 2x - 1 \\ 3x - 4y = -6 \end{cases}$ ⓑ $\begin{cases} 6x - 2y = 12 \\ 3x + 7y = -13 \end{cases}$

Answer

ⓐ Since one equation is already solved for y, using substitution will be most convenient. ⓑ Since both equations are in standard form, using elimination will be most convenient.

Key Concepts

- **How to solve a system of linear equations by graphing.**
  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations.
- **How to solve a system of equations by substitution.**
  1. Solve one of the equations for either variable.
  2. Substitute the expression from Step 1 into the other equation.
  3. Solve the resulting equation.
  4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
  5. Write the solution as an ordered pair.
  6. Check that the ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
  1. Write both equations in standard form. If any coefficients are fractions, clear them.
  2. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations so that the coefficients of that variable are opposites.
  3. Add the equations resulting from Step 2 to eliminate one variable.
  4. Solve for the remaining variable.
  5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  6. Write the solution as an ordered pair.
  7. Check that the ordered pair is a solution to **both** original equations.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations a rein standard form.

Glossary

coincident lines

Coincident lines have the same slope and same y-intercept.

consistent and inconsistent systems

Consistent system of equations is a system of equations with at least one solution; inconsistent system of equations is a system of equations with no solution.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make *all* the equations true; solution is represented by an ordered pair (x,y).(x,y).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.

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4.1E: Exercises

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Equations

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} 2x-6y=0 \\ 3x-4y=5 \end{cases}$

ⓐ $(3,1)$
ⓑ $(-3,4)$

Answer

ⓐ yes ⓑ no

2. $\begin{cases} -3x+y=8 \\ -x+2y=-9 \end{cases}$

ⓐ $(-5,-7)$
ⓑ $(-5,7)$

3. $\begin{cases} x+y=2 \\ y=\frac{3}{4}x \end{cases}$

ⓐ $(87,67)$
ⓑ $(1,34)$

Answer

ⓐ yes ⓑ no

4. $\begin{cases} 2x+3y=6 \\ y=\frac{2}{3}x+2 \end{cases}$

ⓐ $(-6,2)$
ⓑ $(-3,4)$

Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

5. $\begin{cases} 3x+y=-3 \\ 2x+3y=5 \end{cases}$

Answer

$(-3,2)$

6. $\begin{cases} -x+y=2 \\ 2x+y=-4 \end{cases}$

7. $\begin{cases} y=x+2 \\ y=-2x+2 \end{cases}$

Answer

$(0, 2)$

8. $\begin{cases} y = x - 2 \\ y = -3x + 2 \end{cases}$

9. $\begin{cases} y = \frac{3}{2}x + 1 \\ y = -\frac{1}{2}x + 5 \end{cases}$

Answer

$(2, 4)$

10. $\begin{cases} y = \frac{2}{3}x - 2 \\ y = -\frac{1}{3}x - 5 \end{cases}$

11. $\begin{cases} x + y = -4 \\ -x + 2y = -2 \end{cases}$

Answer

$(-2, 2)$

12. $\begin{cases} -x + 3y = 3 \\ x + 3y = 3 \end{cases}$

13. $\begin{cases} -2x + 3y = 3 \\ x + 3y = 12 \end{cases}$

Answer

$(3, 3)$

14. $\begin{cases} 2x - y = 4 \\ 2x + 3y = 12 \end{cases}$

15. $\begin{cases} x + 3y = -6 \\ y = -\frac{4}{3}x + 4 \end{cases}$

Answer

$(6, -4)$

16. $\begin{cases} -x + 2y = -6 \\ y = -\frac{1}{2}x - 1 \end{cases}$

17. $\begin{cases} -2x + 4y = 4 \\ y = \frac{1}{2}x \end{cases}$

Answer

no solution

18. $\begin{cases} 3x+5y=10 \\ y=-\frac{3}{5}x+1 \end{cases}$

19. $\begin{cases} 4x-3y=8 \\ 8x-6y=14 \end{cases}$

Answer

no solution

20. $\begin{cases} x+3y=4 \\ -2x-6y=3 \end{cases}$

21. $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

Answer

infinite solutions with solution set: $\{(x,y)|2x+6y=8\}$

22. $\begin{cases} 4x=3y+7 \\ 8x-6y=14 \end{cases}$

23. $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$

Answer

infinite solutions with solution set: $\{(x,y)|2x+y=6\}$

24. $\begin{cases} 5x+2y=7 \\ -10x-4y=-14 \end{cases}$

Without graphing, determine the number of solutions and then classify the system of equations.

25. $\begin{cases} y=\frac{2}{3}x+1 \\ -2x+3y=5 \end{cases}$

Answer

1 point, consistent and independent

26. $\begin{cases} y=\frac{3}{2}x+1 \\ 2x-3y=7 \end{cases}$

27. $\begin{cases} 5x+3y=4 \\ 2x-3y=5 \end{cases}$

Answer

1 point, consistent and independent

28. $\begin{cases} y=-12x+5 \\ x+2y=10 \end{cases}$

29. $\begin{cases} 5x-2y=10 \\ y=52x-5 \end{cases}$

Answer

infinite solutions, consistent, dependent

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

30. $\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$

31. $\begin{cases} 2x+y=-2 \\ 3x-y=7 \end{cases}$

Answer

$(1,-4)$

32. $\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$

33. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Answer

$(-3,2)$

34. $\begin{cases} 5x-2y=-6 \\ y=3x+3 \end{cases}$

35. $\begin{cases} -2x+2y=6 \\ y=-3x+1 \end{cases}$

Answer

$(-1/2,5/2)$

36. $\begin{cases} 2x+5y=1 \\ y=\frac{1}{3}x-2 \end{cases}$

37. $\begin{cases} 3x+4y=1 \\ y=-\frac{2}{5}x+2 \end{cases}$

Answer

$(-5,4)$

38. $\begin{cases} 2x+y=5 \\ x-2y=-15 \end{cases}$

39. $\begin{cases} 4x+y=10 \\ x-2y=-20 \end{cases}$

Answer

$(0, 10)$

40. $\begin{cases} y=-2x-1 \\ y=-\frac{1}{3}x+4 \end{cases}$

41. $\begin{cases} y=x-6 \\ y=-\frac{3}{2}x+4 \end{cases}$

Answer

$(4, -2)$

42. $\begin{cases} x=2y \\ 4x-8y=0 \end{cases}$

43. $\begin{cases} 2x-16y=8 \\ -x-8y=-4 \end{cases}$

Answer

$(4, 0)$

44. $\begin{cases} y=\frac{7}{8}x+4 \\ -7x+8y=6 \end{cases}$

45. $\begin{cases} y=-\frac{2}{3}x+5 \\ 2x+3y=11 \end{cases}$

Answer

no solution

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination.

46. $\begin{cases} 5x+2y=2 \\ -3x-y=0 \end{cases}$

47. $\begin{cases} 6x-5y=-1 \\ 2x+y=13 \end{cases}$

Answer

$(4, 5)$

48. $\begin{cases} 2x-5y=7 \\ 3x-y=17 \end{cases}$

49. $\begin{cases} 5x-3y=-1 \\ 2x-y=2 \end{cases}$

Answer

$(7,12)$

50. $\begin{cases} 3x-5y=-9 \\ 5x+2y=16 \end{cases}$

51. $\begin{cases} 4x-3y=3 \\ 2x+5y=-31 \end{cases}$

Answer

$(-3,-5)$

52. $\begin{cases} 3x+8y=-3 \\ 2x+5y=-3 \end{cases}$

53. $\begin{cases} 11x+9y=-5 \\ 7x+5y=-1 \end{cases}$

Answer

$(2,-3)$

54. $\begin{cases} 3x+8y=67 \\ 5x+3y=60 \end{cases}$

55. $\begin{cases} 2x+9y=-4 \\ 3x+13y=-7 \end{cases}$

Answer

$(-11,2)$

56. $\begin{cases} \frac{1}{3}x-y=-3 \\ x+\frac{5}{2}y=2 \end{cases}$

57. $\begin{cases} x+\frac{1}{2}y=\frac{3}{2} \\ \frac{1}{5}x-\frac{1}{5}y=3 \end{cases}$

Answer

$(6/-9,24/7)$

58. $\begin{cases} x+\frac{1}{3}y=-1 \\ \frac{1}{3}x+\frac{1}{2}y=1 \end{cases}$

59. $\begin{cases} \frac{1}{3}x-y=-3 \\ \frac{2}{3}x+\frac{5}{2}y=3 \end{cases}$

Answer

$(-3, 2)$

60. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$

61. $\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$

Answer

infinitely many solutions with solution set: $\{(x, y)|x-4y=-1\}$

62. $\begin{cases} -3x-y=8 \\ 6x+2y=-16 \end{cases}$

63. $\begin{cases} 4x+3y=2 \\ 20x+15y=10 \end{cases}$

Answer

infinitely many solutions with solution set: $\{(x, y)|4x+3y=2\}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

64.

ⓐ $\begin{cases} 8x-15y=-32 \\ 6x+3y=-5 \end{cases}$

ⓑ $\begin{cases} x=4y-3 \\ 4x-2y=-6 \end{cases}$

65.

ⓐ $\begin{cases} y=7x-5 \\ 3x-2y=16 \end{cases}$

ⓑ $\begin{cases} 12x-5y=-42 \\ 3x+7y=-15 \end{cases}$

Answer

ⓐ substitution ⓑ elimination

66.

ⓐ $\begin{cases} y=4x+95 \\ x-2y=-21 \end{cases}$

ⓑ $\begin{cases} 9x-4y=24 \\ 3x+5y=-14 \end{cases}$

67.

ⓐ $\begin{cases} 14x-15y=-30 \\ 7x+2y=10 \end{cases}$

ⓑ $\begin{cases} x = 9y - 11 \\ 2x - 7y = -27 \end{cases}$

Answer

ⓐ elimination ⓑ substitution

Writing Exercises

68. In a system of linear equations, the two equations have the same intercepts. Describe the possible solutions to the system.

69. Solve the system of equations by substitution and explain all your steps in words: $\begin{cases} 3x + y = 1 \\ 2x = y - 8 \end{cases}$

Answer

Answers will vary.

70. Solve the system of equations by elimination and explain all your steps in words: $\begin{cases} 5x + 4y = 10 \\ 2x = 3y + 27 \end{cases}$

71. Solve the system of equations $\begin{cases} x + y = 10 \\ x - y = 6 \end{cases}$

ⓐ by graphing ⓑ by substitution
ⓒ Which method do you prefer? Why?

Answer

Answers will vary.

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered pair is a solution of a system of equations.			
solve a system of linear equations by graphing.			
solve a system of equations by substitution.			
solve a system of equations by elimination.			
choose the most convenient method to solve a system of linear equations.			

If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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4.2: Solve Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Before you get started, take this readiness quiz.

1. The sum of twice a number and nine is 31. Find the number.
   If you missed this problem, review [link].
2. Twins Jon and Ron together earned $96,000 last year. Ron earned $8000 more than three times what Jon earned. How much did each of the twins earn?
   If you missed this problem, review [link].
3. An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in four hours and the local train takes five hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.
   If you missed this problem, review [link].

Solve Direct Translation Applications

Systems of linear equations are very useful for solving applications. Some people find setting up word problems with two variables easier than setting them up with just one variable. To solve an application, we'll first translate the words into a system of linear equations. Then we will decide the most convenient method to use, and then solve the system.

SOLVE APPLICATIONS WITH SYSTEMS OF EQUATIONS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We solved number problems with one variable earlier. Let's see how differently it works using two variables.

?

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let $n =$ the first number. $m =$ the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.
	$n + m = 0$
	One number is nine less than the other.

	$n = m - 9$
The system is:	$n + m = 0$ $n = m - 9$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for n.	
Substitute $m - 9$ for n in the first equation.	$n = (m - 9)$ $n + m = 0$
Solve for m.	$m - 9 + m = 0$
	$2m - 9 = 0$
	$2m = 9$
Substitute $m = \frac{9}{2}$ into the second equation and then solve for n.	$m = \left(\frac{9}{2}\right)$ $n = m - 9$
	$m = \frac{9}{2} - 9$ $m = \frac{9}{2} - \frac{18}{2}$
	$n = -\frac{9}{2}$
Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

? Example 4.2.2

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Answer

$3, 7$

? Example 4.2.3

The sum of two numbers is -6. One number is 10 less than the other. Find the numbers.

Answer

$2, -8$

?

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her \$25,000 plus \$15 for each training session. Option B would pay her $\$10,000 + \$40$ for each training session. How many training sessions would make the salary options equal?

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are looking for.	Let s=s= Heather's salary. n=n= the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her $25,000 plus $15 for each training session.
	$s = 25{,}000 + 15n$
	Option B would pay her $10,000 + $40 for each training session.
	$s = 10{,}000 + 40n$
The system is shown.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Step 5. Solve the system of equations. We will use substitution.	$s = 25{,}000 + 15n$ $s = 10{,}000 + 40n$
Substitute 25,000 +15n for s in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for n.	$25{,}000 = 10{,}000 + 25n$ $15{,}000 = 25n$ $600 = n$
Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when n = 600?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

? Example 4.2.5

Geraldine has been offered positions by two insurance companies. The first company pays a salary of $12,000 plus a commission of $100 for each policy sold. The second pays a salary of $20,000 plus a commission of $50 for each policy sold. How many policies would need to be sold to make the total pay the same?

Answer

160 policies

? Example 4.2.6

Kenneth currently sells suits for company A at a salary of $22,000 plus a $10 commission for each suit sold. Company B offers him a position with a salary of $28,000 plus a $4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Answer

1000 suits

As you solve each application, remember to analyze which method of solving the system of equations would be most convenient.

Example 4.2.7

Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories for each minute of circuit training?

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories burned each minute on the elliptical trainer and each minute of circuit training.
Step 3. Name what we are looking for.	Let e=e= number of calories burned per minute on the elliptical trainer. c=c= number of calories burned per minute while circuit training
Step 4. Translate into a system of equations.	10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories
	$10e + 20c = 278$
	20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories
	$20e + 30c = 473$
The system is:	$\begin{aligned} 10e + 20c &= 278 \\ 20e + 30c &= 473 \end{aligned}$
Step 5. Solve the system of equations.	
Multiply the first equation by −2 to get opposite coefficients of e.	$\begin{aligned} -2(10e + 20c) &= -2(278) \\ 20e + 30c &= 473 \end{aligned}$
Simplify and add the equations. Solve for c.	$\begin{aligned} -20e - 40c &= -556 \\ 20e + 30c &= 473 \\ \hline -10c &= -83 \\ c &= 8.3 \end{aligned}$
Substitute $c = 8.3$ into one of the original equations to solve for e.	$\begin{aligned} 10e + 20c &= 278 \\ 10e + 20(8.3) &= 278 \\ 10e + 166 &= 278 \\ 10e &= 112 \\ e &= 11.2 \end{aligned}$
Step 6. Check the answer in the problem.	Check the math on your own.
	$\begin{aligned} 10(11.2) + 20(8.3) &\stackrel{?}{=} 278 \\ 20(11.2) + 30(8.3) &\stackrel{?}{=} 473 \end{aligned}$
Step 7. Answer the question.	Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.

Example 4.2.8

Translate to a system of equations and then solve:

Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

Answer

Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

Example 4.2.9

Translate to a system of equations and then solve:

Erin spent 30 minutes on the rowing machine and 20 minutes lifting weights at the gym and burned 430 calories. During her next visit to the gym she spent 50 minutes on the rowing machine and 10 minutes lifting weights and burned 600 calories. How many calories did she burn for each minutes on the rowing machine? How many calories did she burn for each minute of weight lifting?

Answer

Erin burned 11 calories for each minute on the rowing machine and 5 calories for each minute of weight lifting.

Solve Geometry Applications

We will now solve geometry applications using systems of linear equations. We will need to add complementary angles and supplementary angles to our list some properties of angles.

The measures of two **complementary angles** add to 90 degrees. The measures of two **supplementary angles** add to 180 degrees.

COMPLEMENTARY AND SUPPLEMENTARY ANGLES

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

Two angles are supplementary if the sum of the measures of their angles is 180 degrees

If two angles are complementary, we say that *one angle is the complement of the other.*

If two angles are supplementary, we say that *one angle is the supplement of the other.*

Example 4.2.10

Translate to a system of equations and then solve.

The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Answer

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for the measure of each angle.

Step 3. Name what we are looking for. Let $x =$ the measure of the first angle.
$y =$ the measure of the second angle

Step 4. Translate into a system of equations. The angles are complementary.

$$x + y = 90$$

The difference of the two angles is 26 degrees.

$$x - y = 26$$

The system is shown.

$$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$$

Step 5. Solve the system of equations by elimination.

$$\begin{cases} x + y = 90 \\ \underline{x - y = 26} \end{cases}$$
$$2x = 116$$
$$x = 58$$

Substitute $x = 58$ into the first equation.

$$x + y = 90$$
$$58 + y = 90$$
$$y = 32$$

Step 6. Check the answer in the problem.

$58 + 32 = 90$✓
$58 - 32 = 26$✓

Step 7. Answer the question. The angle measures are 58 and 32 degrees.

Example 4.2.11

Translate to a system of equations and then solve:

The difference of two complementary angles is 20 degrees. Find the measures of the angles.

Answer

The angle measures are 55 and 35.

Example 4.2.12

Translate to a system of equations and then solve:

The difference of two complementary angles is 80 degrees. Find the measures of the angles.

Answer

The angle measures are 5 and 85.

In the next example, we remember that the measures of supplementary angles add to 180.

Example 4.2.13

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for measure of each angle.
Step 3. Name what we are looking for.	Let x=x= the measure of the first angle. y=y= the measure of the second angle
Step 4. Translate into a system of equations.	The angles are supplementary.
	$x+y=180$
	The larger angle is twelve less than five times the smaller angle.
	$y=5x-12$
The system is shown: **Step 5. Solve** the system of equations substitution.	$x+y=180$ $y=5x-12$ $x+y=180$
Substitute $5x-12$ for y in the first equation. Solve for x.	$x+5x-12=180$ $6x-12=180$
Substitute 32 for x in the second equation, then solve for y.	$6x=192$ $x=32$ $y=5x-12$ $y=5\cdot 32-12$ $y=160-12$ $y=148$
Step 6. Check the answer in the problem.	$32+148=180$ ✓ $5\cdot 32-12=148$ ✓
Step 7. Answer the question.	The angle measures are 148 and 32 degrees.

Example 4.2.14

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

Answer

The angle measures are 42 and 138.

Example 4.2.15

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 18 less than twice the measure of the smaller angle. Find the measures of the angles.

Answer

The angle measures are 66 and 114.

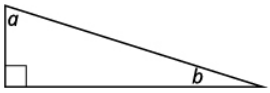
Recall that the angles of a triangle add up to 180 degrees. A right triangle has one angle that is 90 degrees. What does that tell us about the other two angles? In the next example we will be finding the measures of the other two angles.

Example 4.2.16

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Answer

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let a=a= the measure of the first angle. b=b= the measure of the second angle
Step 4. Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is shown.	$a = 3b + 10$ $a + b + 90 = 180$
Step 5. Solve the system of equations. We will use substitution since the first equation is solved for a.	$a = 3b + 10$ $a + b + 90 = 180$
Substitute 3b+103b+10 for a in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for b.	$4b + 100 = 180$ $4b = 80$ $b = 20$ $a = 3b + 10$
Substitute b=20b=20 into the first equation and then solve for a.	$a = 3 \cdot 20 + 10$ $a = 70$
Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70 degrees.

Example 4.2.17

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

$22, 68$

Example 4.2.18

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

$36, 54$

Often it is helpful when solving geometry applications to draw a picture to visualize the situation.

Example 4.2.19

Translate to a system of equations and then solve:

Randall has 125 feet of fencing to enclose the part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
	
Step 3. Name what we are looking for.	Let L=L= the length of the fenced yard. W=W= the width of the fenced yard
Step 4. Translate into a system of equations.	One lenth and two widths equal 125.
	$L + 2W = 125$
	The length will be 5 feet more than four times the width.
	$L = 4W + 5$
The system is shown. **Step 5. Solve** The system of equations by substitution.	$\begin{cases} L + 2W = 125 \\ L = 4W + 5 \end{cases}$ $L + 2W = 125$
Substitute $L = 4W + 5$ into the first equation, then solve for W.	$4W + 5 + 2W = 125$

	$6W + 5 = 125$ $6W = 120$ $W = 20$
Substitute 20 for W in the second equation, then solve for L.	$L = 4W + 5$ $L = 4 \cdot 20 + 5$ $L = 80 + 5$ $L = 85$
Step 6. Check the answer in the problem.	$20 + 85 + 20 = 125$ $85 = 4 \cdot 20 + 5$ $85 = 85$ ✓
Step 7. Answer the equation.	The length is 85 feet and the width is 20 feet.

? Example 4.2.20

Translate to a system of equations and then solve:

Mario wants to put a fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

Answer

The length is 60 feet and the width is 35 feet.

? Example 4.2.21

Translate to a system of equations and then solve:

Alexis wants to build a rectangular dog run in her yard adjacent to her neighbor's fence. She will use 136 feet of fencing to completely enclose the rectangular dog run. The length of the dog run along the neighbor's fence will be 16 feet less than twice the width. Find the length and width of the dog run.

Answer

The length is 60 feet and the width is 38 feet.

Solve uniform motion applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We'll continue using the table here. The basic equation was $D = rt$ where D is the distance traveled, r is the rate, and t is the time.

Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

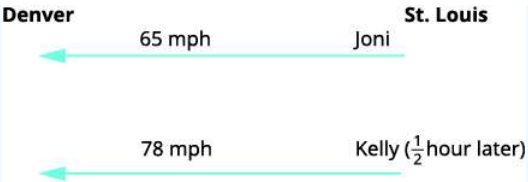
? Example 4.2.22

Translate to a system of equations and then solve:

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

Answer

A diagram is useful in helping us visualize the situation.



Identify and name what we are looking for. A chart will help us organize the data. We know the rates of both Joni and Kelly, and so we enter them in the chart. We are looking for the length of time Kelly, *k*, and Joni, *j*, will each drive.

	Rate · Time = Distance		
Joni	65	j	$65j$
Kelly	78	k	$78k$

Since $D = r \cdot t$ we can fill in the Distance column.

Translate into a system of equations.

To make the system of equations, we must recognize that Kelly and Joni will drive the same distance. So,

$$65j = 78k$$

Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time. So,

$$k = j - \tfrac{1}{2}$$

Now we have the system.

$$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$$

Solve the system of equations by substitution.

Substitute $k = j - 12$ into the second equation, then solve for j.

$$\begin{aligned} 65j &= 78k \\ 65j &= 78(j - \tfrac{1}{2}) \\ 65j &= 78j - 39 \\ -13j &= -39 \\ j &= 3 \end{aligned}$$

To find Kelly's time, substitute $j = 3$ into the first equation, then solve for k.

$$\begin{aligned} k &= j - \tfrac{1}{2} \\ k &= 3 - \tfrac{1}{2} \\ k &= \tfrac{5}{2} \text{ or } k = 2\tfrac{1}{2} \end{aligned}$$

Check the answer in the problem.

Joni $\quad 3$ hours $\quad (65$ mph$) \quad = \quad 195$ miles

Kelly $\quad 2\frac{1}{2}$ hours $\quad (78$ mph$) \quad = \quad 195$ miles

Yes, they will have traveled the same distance when they meet.

Answer the question.

Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have traveled 3 hours.

Example 4.2.23

Translate to a system of equations and then solve:

Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later traveling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

Answer

It will take Clark 4 hours to catch Mitchell.

Example 4.2.24

Translate to a system of equations and then solve:

Charlie left his mother's house traveling at an average speed of 36 miles per hour. His sister Sally left 15 minutes $(\frac{1}{4}\text{ hour})$ later traveling the same route at an average speed of 42 miles per hour. How long before Sally catches up to Charlie?

Answer

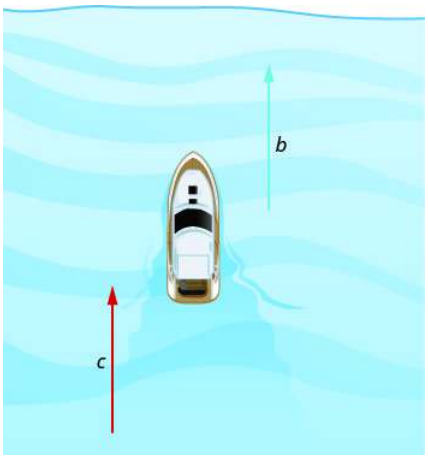
It will take Sally 112 hours to catch up to Charlie.

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights in the United States generally take longer going west than going east because of the prevailing wind currents.

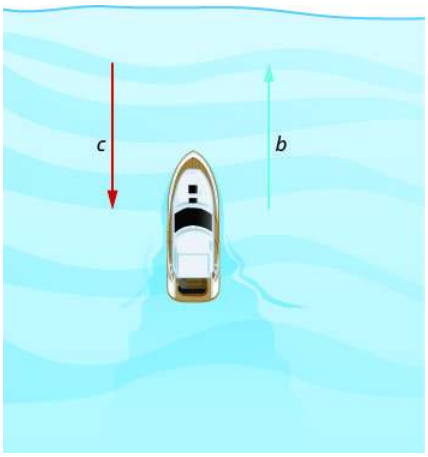
Let's take a look at a boat travelling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

The images below show how a river current affects the speed at which a boat is actually travelling. We'll call the speed of the boat in still water b and the speed of the river current c.

The boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat's actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b+c$.



Now, the boat is going upstream, opposite to the river current. The current is going against the boat, so the boat's actual speed is slower than its speed in still water. The actual speed of the boat is $b-c$.



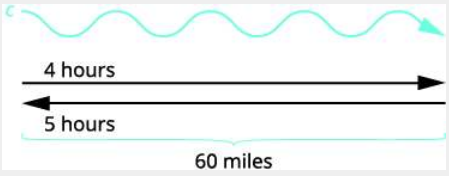
We'll put some numbers to this situation in the next example.

? Example 4.2.25

Translate to a system of equations and then solve.

A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Answer

Read the problem.	This is a uniform motion problem and a picture will help us visualize the situation.
	
Identify what we are looking for.	We are looking for the speed of the ship in still water and the speed of the current.
Name what we are looking for.	Let $s =$ the rate of the ship in still water. $c =$ the rate of the current
A chart will help us organize the information. The ship goes downstream and then upstream. Going downstream, the current helps the ship and so the ship's actual rate is $s + c$. Going upstream, the current slows the ship and so the actual rate is $s - c$.	<table><tr><th></th><th colspan="3">Rate • Time = Distance</th></tr><tr><td>downstream</td><td>$s + c$</td><td>4</td><td>60</td></tr><tr><td>upstream</td><td>$s - c$</td><td>5</td><td>60</td></tr></table>
Downstream it takes 4 hours. Upstream it takes 5 hours. Each way the distance is 60 miles.	
Translate into a system of equations. Since rate times time is distance, we can write the system of equations.	$\begin{cases} 4(s + c) = 60 \\ 5(s - c) = 60 \end{cases}$
Solve the system of equations. Distribute to put both equations in standard form, then solve by elimination.	$\begin{cases} 4s + 4c = 60 \\ 5s - 5c = 60 \end{cases}$

Multiply the top equation by 5 and the bottom equation by 4. Add the equations, then solve for s.	$20s + 20c = 300$ $20s - 20c = 240$ $40s = 540$ $s = 13.5$ $4(s + c) = 60$
Substitute $s = 13.5$ into of the original equations.	$4(13.5 + c) = 60$ $54 + 4c = 60$ $4c = 6$ $c = 1.5$
Check the answer in the problem. The downstream rate would be $13.5 + 1.5 = 15$ mph. In 4 hours the ship would travel $15 \cdot 4 = 60$ miles. The upstream rate would be $13.5 - 1.5 = 12$ mph. In 5 hours the ship would travel $12 \cdot 5 = 60$ miles.	
Answer the question.	The rate of the ship is 13.5 mph and the rate of the current is 1.5 mph.

Example 4.2.26

Translate to a system of equations and then solve:

A Mississippi river boat cruise sailed 120 miles upstream for 12 hours and then took 10 hours to return to the dock. Find the speed of the river boat in still water and the speed of the river current.

Answer

The rate of the boat is 11 mph and the rate of the current is 1 mph.

Example 4.2.27

Translate to a system of equations and then solve:

Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

Answer

The speed of the canoe is 7 mph and the speed of the current is 1 mph.

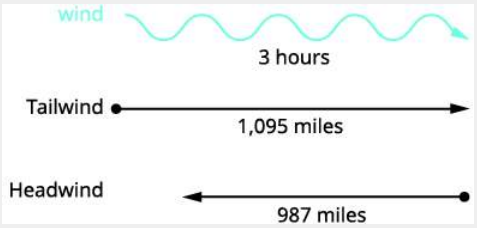
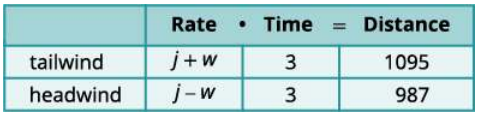
Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We'll see this in the next example. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a *headwind*.

Example 4.2.28

Translate to a system of equations and then solve:

A private jet can fly 1,095 miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Read the problem.	This is a uniform motion problem and a picture will help us visualize.
	
Identify what we are looking for.	We are looking for the speed of the jet in still air and the speed of the wind.
Name what we are looking for.	Let j=j= the speed of the jet in still air. w=w= the speed of the wind.
A chart will help us organize the information. The jet makes two trips—one in a tailwind and one in a headwind. In a tailwind, the wind helps the jet and so the rate is $j + w$. In a headwind, the wind slows the jet and so the rate is $j - w$.	

	Rate •	Time =	Distance
tailwind	$j+w$	3	1095
headwind	$j-w$	3	987

Each trip takes 3 hours. In a tailwind the jet flies 1,095 miles. In a headwind the jet flies 987 miles.	
Translate into a system of equations. Since rate times time is distance, we get the system of equations.	$3(j+w)=1095$ $3(j-w)=\ 987$
Solve the system of equations. Distribute, then solve by elimination. Add, and solve for j.	$3j+3w=1095$ $3j-3w=\ 987$ $6j\ \ =2082$ $j\ \ =347$ $4(j+w)=1095$
Substitute $j = 347$ into one of the original equations, then solve for w.	$3(347+w)=1095$ $1041+3w=1095$ $3w=54$ $w=18$
Check the answer in the problem. With the tailwind, the actual rate of the jet would be $347+18=365$ mph. In 3 hours the jet would travel $365\cdot 3=1,095$ miles Going into the headwind, the jet's actual rate would be $347-18=329$ mph. In 3 hours the jet would travel $329\cdot 3=987$ miles.	
Answer the question.	The rate of the jet is 347 mph and the rate of the wind is 18 mph.

? Example 4.2.29

Translate to a system of equations and then solve:

A small jet can fly 1,325 miles in 5 hours with a tailwind but only 1,035 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

The speed of the jet is 235 mph and the speed of the wind is 30 mph.

? Example 4.2.30

Translate to a system of equations and then solve:

A commercial jet can fly 1,728 miles in 4 hours with a tailwind but only 1,536 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

The speed of the jet is 408 mph and the speed of the wind is 24 mph.

Access this online resource for additional instruction and practice with systems of equations.

- Systems of Equations

Key Concepts

- **How To Solve Applications with Systems of Equations**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose variables to represent those quantities.
  4. **Translate** into a system of equations.
  5. **Solve** the system of equations using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

complementary angles

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

supplementary angles

Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

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4.2E: Exercises

Practice Makes Perfect

Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

1. The sum of two number is 15. One number is 3 less than the other. Find the numbers.

2. The sum of two number is 30. One number is 4 less than the other. Find the numbers.

Answer

13 and 17

3. The sum of two number is −16. One number is 20 less than the other. Find the numbers.

4. The sum of two number is -26. One number is 12 less than the other. Find the numbers.

Answer

-7 and -19

5. The sum of two numbers is 65. Their difference is 25. Find the numbers.

6. The sum of two numbers is 37. Their difference is 9. Find the numbers.

Answer

14 and 23

7. The sum of two numbers is -27. Their difference is -59. Find the numbers.

8. The sum of two numbers is -45. Their difference is -89. Find the numbers.

Answer

22 and -67

9. Maxim has been offered positions by two car companies. The first company pays a salary of $10,000 plus a commission of $1000 for each car sold. The second pays a salary of $20,000 plus a commission of $500 for each car sold. How many cars would need to be sold to make the total pay the same?

10. Jackie has been offered positions by two cable companies. The first company pays a salary of $14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?

Answer

Eighty cable packages would need to be sold to make the total pay the same.

11. Amara currently sells televisions for company A at a salary of $17,000 plus a $100 commission for each television she sells. Company B offers her a position with a salary of $29,000 plus a $20 commission for each television she sells. How televisions would Amara need to sell for the options to be equal?

12. Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to be equal?

Answer

Mitchell would need to sell 120 stoves for the companies to be equal.

13. Two containers of gasoline hold a total of fifty gallons. The big container can hold ten gallons less than twice the small container. How many gallons does each container hold?

14. June needs 48 gallons of punch for a party and has two different coolers to carry it in. The bigger cooler is five times as large as the smaller cooler. How many gallons can each cooler hold?

Answer

8 and 40 gallons

15. Shelly spent 10 minutes jogging and 20 minutes cycling and burned 300 calories. The next day, Shelly swapped times, doing 20 minutes of jogging and 10 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

16. Drew burned 1800 calories Friday playing one hour of basketball and canoeing for two hours. Saturday he spent two hours playing basketball and three hours canoeing and burned 3200 calories. How many calories did he burn per hour when playing basketball? How many calories did he burn per hour when canoeing?

Answer

1000 calories playing basketball and 400 calories canoeing

17. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and thumb drive. Troy bought four notebooks and five thumb drives for $116. Lisa bought two notebooks and three thumb dives for $68. Find the cost of each notebook and each thumb drive.

18. Nancy bought seven pounds of oranges and three pounds of bananas for $17. Her husband later bought three pounds of oranges and six pounds of bananas for $12. What was the cost per pound of the oranges and the bananas?

Answer

Oranges cost $2 per pound and bananas cost $1 per pound

19. Andrea is buying some new shirts and sweaters. She is able to buy 3 shirts and 2 sweaters for $114 or she is able to buy 2 shirts and 4 sweaters for $164. How much does a shirt cost? How much does a sweater cost?

20. Peter is buying office supplies. He is able to buy 3 packages of paper and 4 staplers for $40 or he is able to buy 5 packages of paper and 6 staplers for $62. How much does a package of paper cost? How much does a stapler cost?

Answer

Package of paper $4, stapler $7

21. The total amount of sodium in 2 hot dogs and 3 cups of cottage cheese is 4720 mg. The total amount of sodium in 5 hot dogs and 2 cups of cottage cheese is 6300 mg. How much sodium is in a hot dog? How much sodium is in a cup of cottage cheese?

22. The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?

Answer

Hot dog 150 calories, cup of cottage cheese 220 calories

23. Molly is making strawberry infused water. For each ounce of strawberry juice, she uses three times as many ounces of water as juice. How many ounces of strawberry juice and how many ounces of water does she need to make 64 ounces of strawberry infused water?

24. Owen is making lemonade from concentrate. The number of quarts of water he needs is 4 times the number of quarts of concentrate. How many quarts of water and how many quarts of concentrate does Owen need to make 100 quarts of lemonade?

Answer

Owen will need 80 quarts of water and 20 quarts of concentrate to make 100 quarts of lemonade.

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

25. The difference of two complementary angles is 55 degrees. Find the measures of the angles.

26. The difference of two complementary angles is 17 degrees. Find the measures of the angles.

Answer

53.5degrees and 36.5degree.

27. Two angles are complementary. The measure of the larger angle is twelve less than twice the measure of the smaller angle. Find the measures of both angles.

28. Two angles are complementary. The measure of the larger angle is ten more than four times the measure of the smaller angle. Find the measures of both angles.

Answer

16 degrees and 74 degrees

29. The difference of two supplementary angles is 8 degrees. Find the measures of the angles.

30. The difference of two supplementary angles is 88 degrees. Find the measures of the angles.

Answer

134 degrees and 46 degrees

31. Two angles are supplementary. The measure of the larger angle is four more than three times the measure of the smaller angle. Find the measures of both angles.

32. Two angles are supplementary. The measure of the larger angle is five less than four times the measure of the smaller angle. Find the measures of both angles.

Answer

37 degrees and 143 degrees

33. The measure of one of the small angles of a right triangle is 14 more than 3 times the measure of the other small angle. Find the measure of both angles.

34. The measure of one of the small angles of a right triangle is 26 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

16° and 74°

35. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

36. The measure of one of the small angles of a right triangle is 45 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

45° and 45°

37. Wayne is hanging a string of lights 45 feet long around the three sides of his patio, which is adjacent to his house. The length of his patio, the side along the house, is five feet longer than twice its width. Find the length and width of the patio.

38. Darrin is hanging 200 feet of Christmas garland on the three sides of fencing that enclose his front yard. The length is five feet less than three times the width. Find the length and width of the fencing.

Answer

Width is 41 feet and length is 118 feet.

39. A frame around a family portrait has a perimeter of 90 inches. The length is fifteen less than twice the width. Find the length and width of the frame.

40. The perimeter of a toddler play area is 100 feet. The length is ten more than three times the width. Find the length and width of the play area.

Answer

Width is 10 feet and length is 40 feet.

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

41. Sarah left Minneapolis heading east on the interstate at a speed of 60 mph. Her sister followed her on the same route, leaving two hours later and driving at a rate of 70 mph. How long will it take for Sarah's sister to catch up to Sarah?

42. College roommates John and David were driving home to the same town for the holidays. John drove 55 mph, and David, who left an hour later, drove 60 mph. How long will it take for David to catch up to John?

Answer

11 hours

43. At the end of spring break, Lucy left the beach and drove back towards home, driving at a rate of 40 mph. Lucy's friend left the beach for home 30 minutes (half an hour) later, and drove 50 mph. How long did it take Lucy's friend to catch up to Lucy?

44. Felecia left her home to visit her daughter driving 45 mph. Her husband waited for the dog sitter to arrive and left home twenty minutes (1/3 hour) later. He drove 55 mph to catch up to Felecia. How long before he reaches her?

Answer

1.5 hour

45. The Jones family took a 12-mile canoe ride down the Indian River in two hours. After lunch, the return trip back up the river took three hours. Find the rate of the canoe in still water and the rate of the current.

46. A motor boat travels 60 miles down a river in three hours but takes five hours to return upstream. Find the rate of the boat in still water and the rate of the current.

Answer

Boat rate is 16 mph and current rate is 4 mph.

47. A motor boat traveled 18 miles down a river in two hours but going back upstream, it took 4.54.5 hours due to the current. Find the rate of the motor boat in still water and the rate of the current. (Round to the nearest hundredth.)

48. A river cruise boat sailed 80 miles down the Mississippi River for four hours. It took five hours to return. Find the rate of the cruise boat in still water and the rate of the current.

Answer

Boat rate is 18 mph and current rate is 2 mph.

49. A small jet can fly 1072 miles in 4 hours with a tailwind but only 848 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

50. A small jet can fly 1435 miles in 5 hours with a tailwind but only 1,215 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Jet rate is 265 mph and wind speed is 22 mph.

51. A commercial jet can fly 868 miles in 2 hours with a tailwind but only 792 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

52. A commercial jet can fly 1,320 miles in 3 hours with a tailwind but only 1170 miles in 3 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Jet rate is 415 mph and wind speed is 25 mph.

Writing Exercises

53. Write an application problem similar to Example. Then translate to a system of equations and solve it.\

54. Write a uniform motion problem similar to Example that relates to where you live with your friends or family members. Then translate to a system of equations and solve it.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve direct translation applications.			
solve geometry applications.			
solve uniform motion applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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4.3: Solve Mixture Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications
- Solve applications of cost and revenue functions

Before you get started, take this readiness quiz.

1. Multiply: $4.025(1,562)$
   If you missed this problem, review [link].
2. Write 8.2% as a decimal.
   If you missed this problem, review [link].
3. Earl's dinner bill came to $32.50 and he wanted to leave an 18% tip. How much should the tip be?
   If you missed this problem, review [link].

Solve Mixture Applications

Mixture application involve combining two or more quantities. When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

	Number •	Value =	Total Value
nickels			
dimes			

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we'll just let n be the number of nickels and d be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

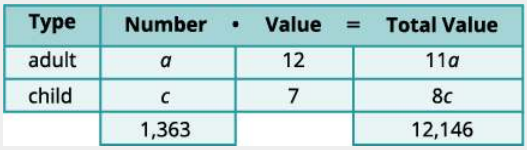
Example 4.3.1

Translate to a system of equations and solve:

A science center sold 1,363 tickets on a busy weekend. The receipts totaled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Answer

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
Step 3. Name what we are looking for.	Let $a =$ the number of adult tickets. $c =$ the number of child tickets
A table will help us organize the data. We have two types of tickets, adult and child.	Write in a and c for the number of tickets.

Write the total number of tickets sold at the bottom of the Number column.	Altogether 1,363 were sold.
Write the value of each type of ticket in the Value column.	The value of each adult ticket is \$12. The value of each child tickets is \$7.
The number times the value gives the total value, so the total value of adult tickets is $a \cdot 12 = 12a$, and the total value of child tickets is $c \cdot 7 = 7c$.	Fill in the Total Value column.
Altogether the total value of the tickets was \$12,146.	

Type	Number	• Value	= Total Value
adult	a	12	$11a$
child	c	7	$8c$
	1,363		12,146

Step 4. Translate into a system of equations.	
The Number column and the Total value column give us the system of equations.	$\begin{cases} a + c = 1{,}363 \\ 12a + 7c = 12{,}146 \end{cases}$
We will use the elimination method to solve this system. Multiply the first equation by -7.	$\begin{cases} -7(a + c) = -7(1{,}363) \\ 12a + 7c = 12{,}146 \end{cases}$
Simplify and add, then solve for a.	$\begin{cases} -7a - 7c = -9{,}541 \\ 12a + 7c = 12{,}146 \end{cases}$ $5a = 2{,}605$ $a = 521$
Substitute $a = 521$ into the first equation, then solve for c.	$a + c = 1{,}363$ $521 + c = 1{,}363$ $c = 842$
Step 6. Check the answer in the problem. 521 adult at \$12 per ticket makes \$ 6,252 842 child at \$7 per ticket makes \$58,994 The total receipts are \$12,146✓	
Step 7. Answer the question.	The science center sold 521 adult tickets and 842 child tickets.

Example 4.3.2

Translate to a system of equations and solve:

The ticket office at the zoo sold 553 tickets one day. The receipts totaled \$3,936. How many \$9 adult tickets and how many \$6 child tickets were sold?

Answer

206 adults, 347 children

Example 4.3.3

Translate to a system of equations and solve:

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled \$1,302. How many \$11 adult and how many \$8 child tickets were sold?

Answer

42 adults, 105 children

In the next example, we'll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the 'number' column will be easy.

Example 4.3.4

Translate to a system of equations and solve:

Juan has a pocketful of nickels and dimes. The total value of the coins is \$8.10. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Answer

Step 1. Read the problem. We will create a table to organize the information.	
Step 2. Identify what we are looking for.	We are looking for the number of nickels and the number of dimes.
Step 3. Name what we are looking for.	Let $n =$ the number of nickels. $d =$ the number of dimes
A table will help us organize the data. We have two types of coins, nickels and dimes.	Write *n* and *d* for the number of each type of coin.
Fill in the Value column with the value of each type of coin.	The value of each nickel is \$0.05. The value of each dime is \$0.10.
The number times the value gives the total value, so, the total value of the nickels is $n(0.05) = 0.05n$ and the total value of dimes is $d(0.10) = 0.10d$. Altogether the total value of the coins is \$8.10.	**Type** | **Number** • **Value** = **Total Value** nickels | n | 0.05 | $0.05n$ dimes | d | 0.10 | $0.10d$ | | | 8.10
Step 4. Translate into a system of equations.	
The Total Value column gives one equation.	$0.05n + 0.10d = 8.10$
We also know the number of dimes is 9 less than twice the number of nickels.	
Translate to get the second equation.	$d = 2n - 9$
Now we have the system to solve.	$0.05n + 0.10d = 8.10$ $d = 2n - 9$
Step 5. Solve the system of equations We will use the substitution method.	
Substitute $d = 2n - 9$ into the first equation.	$0.05n + 0.10d = 8.10$ $0.05n + 0.10(2n - 9) = 8.10$
Simplify and solve for *n*.	$0.05n + 0.2n - 0.90 = 8.10$ $0.25n - 0.90 = 8.10$ $0.25n = 9.00$ $n = 36$
To find the number of dimes, substitute $n = 36$ into the second equation.	$d = 2n - 9$ $d = 2 \cdot 36 - 9$ $d = 63$

Step 6. Check the answer in the problem 63 dimes at $\$0.10 = \$6.30$ 36 nickels at $\$0.05 = \$1.80$ Total $= \$8.10$✓	
Step 7. Answer the question.	Juan has 36 nickels and 63 dimes.

? Example 4.3.5

Translate to a system of equations and solve:

Matilda has a handful of quarters and dimes, with a total value of $8.55. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Answer

13 dimes and 29 quarters

? Example 4.3.6

Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of $7.30. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Answer

19 quarters and 51 nickels

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

? Example 4.3.7

Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $7.60. per pound. Nuts cost $9.00 per pound and chocolate chips cost $2.00 per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Answer

Step 1. Read the problem. We will create a table to organize the information.	
Step 2. Identify what we are looking for.	We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.
Step 3. Name what we are looking for.	Let $n =$ the number of pound of nuts. $c =$ the number of pounds of chips
Carson will mix nuts and chocolate chips to get trail mix. Write in n and c for the number of pounds of nuts and chocolate chips.	

Type	Number of pounds	• Value	= Total Value
Nuts	n	9.00	$9n$
Chocolate chips	c	2.00	$2c$
Trail mix	20	7.60	$7.60(20) = 152$

There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using $\text{Number} \bullet \text{Value} = \text{Total Value}$	
Step 4. Translate into a system of equations. We get the equations from the Number and Total Value columns.	$\begin{aligned} n + c &= 20 \\ 9n + 2c &= 152 \end{aligned}$
Step 5. Solve the system of equations We will use elimination to solve the system. Multiply the first equation by -2 to eliminate c.	$\begin{aligned} -2(n + c) &= -2(20) \\ 9n + 2c &= 152 \end{aligned}$
Simplify and add. Solve for n.	$\begin{aligned} -2n - 2c &= -40 \\ 9n + 2c &= 152 \\ \hline 7n &= 112 \\ n &= 16 \end{aligned}$
To find the number of pounds of chocolate chips, substitute $n = 16$ into the first equation, then solve for c.	$\begin{aligned} n + c &= 20 \\ 16 + c &= 20 \\ c &= 4 \end{aligned}$
Step 6. Check the answer in the problem. $16 + 4 = 20$✓ $9 \cdot 16 + 2 \cdot 4 = 152$✓	
Step 7. Answer the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

? Example 4.3.8

Translate to a system of equations and solve:

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her $6 per pound. Peanuts are $4 per pound and cashews are $9 per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Answer

3 pounds peanuts and 2 pounds cashews

? Example 4.3.9

Translate to a system of equations and solve:

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of $3 per pound. The price of beans is $1 per pound and the price of ground beef is $5 per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Answer

10 pounds of beans, 10 pounds of ground beef

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is

cleanser, and the rest is water. To make 35 ounces of a 20% concentration, you mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

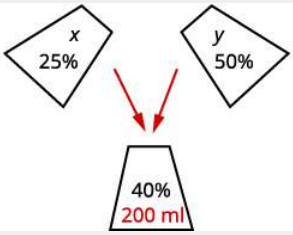
For these kinds of mixture problems, we'll use "percent" instead of "value" for one of the columns in our table.

? Example 4.3.10

Translate to a system of equations and solve:

Sasheena is lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Answer

Step 1. Read the problem. A figure may help us visualize the situation, then we will create a table to organize the information.	Sasheena must mix some of the 25 solution and some of the 50 solution together to get 200 ml of the 40 solution.
	
Step 2. Identify what we are looking for.	We are looking for how much of each solution she needs.
Step 3. Name what we are looking for.	Let $x =$ number of ml of 25 \(y= \text{number of }ml\text{ of }50%\text{ solution}\)
A table will help us organize the data. She will mix x ml of 25 with y ml of 50 to get 200 ml of 40 solution. We write the percents as decimals in the chart. We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.	

Type	Number of units	• Concentration %	= Amount
25%	x	0.25	$0.25x$
50%	y	0.50	$0.50y$
40%	200	0.40	$0.40(200)$

Step 4. Translate into a system of equations. We get the equations from the Number column and the Amount column. Now we have the system.	$$\begin{aligned} x+y &= 200 \\ 0.25x+0.50y &= 0.40(200) \end{aligned}$$
Step 5. Solve the system of equations We will solve the system by elimination. Multiply the first equation by -0.5 to eliminate y.	$$\begin{aligned} -0.5(x+y) &= -0.5(200) \\ 0.25x+0.50y &= 80 \end{aligned}$$
Simplify and add to solve for x.	$$\begin{aligned} -0.5x-0.5y &= -100 \\ 0.25x+0.5y &= 80 \\ \hline -0.25x &= -20 \\ x &= 80 \end{aligned}$$
To solve for y, substitute $x = 80$ into the first equation.	$$\begin{aligned} x+y &= 200 \\ 80+y &= 200 \\ y &= 120 \end{aligned}$$

Step 6. Check the answer in the problem. $80+120 = 200\checkmark$ $0.25(80)+0.50(120) = 200\checkmark$ Yes!	
Step 7. Answer the question.	Sasheena should mix 80 ml of the 25 solution with 120 ml of the 50 solution to get the 200 ml of the 40 solution.

? Example 4.3.11

Translate to a system of equations and solve:

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

Answer

120 ml of 25% solution and 30 ml of 50% solution

? Example 4.3.12

Translate to a system of equations and solve:

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10% and how much of the 40% solutions should he mix to make the 25% solution?

Answer

125 ml of 10% solution and 125 ml of 40% solution

Solve Interest Applications

The formula to model simple interest applications is $I = Prt$. Interest, I, is the product of the principal, P, the rate, r, and the time, t. In our work here, we will calculate the interest earned in one year, so t will be 1.

We modify the column titles in the mixture table to show the formula for interest, as you'll see in the next example.

Translate to a system of equations and solve:

Adnan has $40,000 to invest and hopes to earn 7.1 interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Answer

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the amount to invest in each fund.
Step 3. Name what we are looking for.	Let $s =$ the amount invested in stocks. $b =$ the amount invested in stocks
Write the interest rate as a decimal for each fund. Multiply: Principal · Rate · Time	

Account	Principal •	Rate •	Time =	Interest
Stock fund	s	0.08	1	$0.08s$
Bonds	b	0.03	1	$0.03b$
Total	40,000	0.071		0.071(40,000)

Step 4. Translate into a system of equations. We get our system of equations from the Principal column and the Interest column.	$\begin{cases} s+b=40{,}000 \\ 0.08s+0.03b=0.071(40{,}000) \end{cases}$
Step 5. Solve the system of equations by elimination. Multiply the top equation by -0.03.	$\begin{cases} -0.03(s+b)=-0.03(40{,}000) \\ 0.08s+0.03b=2{,}840 \end{cases}$
Simplify and add to solve for s.	$\begin{array}{r} -0.03s-0.03b=-1{,}200 \\ 0.08s+0.03b=\ \ 2{,}840 \\ \hline 0.05s=\ \ 1{,}640 \\ s=32{,}800 \end{array}$
To find b, substitute $s = 32{,}800$ into the first equation.	$\begin{array}{r} s+b=40{,}000 \\ 32{,}800+b=40{,}000 \\ b=7{,}200 \end{array}$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	Adnan should invest \$32,800 in stock and \$7,200 in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s+b=40,000$, represents the total amount of money invested and the second equation, $0.08s+0.03b=0.071(40,000)$ represents the interest earned.

Example 4.3.14

Translate to a system of equations and solve:

Leon had \$50,000 to invest and hopes to earn 6.2 interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

Answer

\$42,000 in the stock fund and \$8000 in the savings account

Example 4.3.15

Translate to a system of equations and solve:

Julius invested \$7000 into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5 interest on the total investment. How much money did he put in each stock?

Answer

\$1750 at 11% and \$5250 at 13%

The next example requires that we find the principal given the amount of interest earned.

Example 4.3.16

Translate to a system of equations and solve:

Rosie owes \$21,540 on her two student loans. The interest rate on her bank loan is 10.5 and the interest rate on the federal loan is 5.9. The total amount of interest she paid last year was $\$1,669.68$ What was the principal for each loan?

Answer

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the principal of each loan.
Step 3. Name what we are looking for.	Let $b =$ the principal for the bank loan. $f =$ the principal on the federal loan
The total loans are $21,540.	
Record the interest rates as decimals in the chart. Multiply using the formula $I = Prt$ to get the Interest.	

Account	Principal	• Rate	• Time	= Interest
Bank	b	0.105	1	$0.105b$
Federal	f	0.059	1	$0.059f$
Total	21,540			1669.68

Step 4. Translate into a system of equations. The system of equations comes from the Principal column and the Interest column.	$b + f = 21{,}540$ $0.105b + 0.059f = 1669.68$
Step 5. Solve the system of equations We will use substitution to solve. Solve the first equation for b.	$b + f = 21{,}540$ $b = -f + 21{,}540$
	$0.105b + 0.059f = 1669.68$
Substitute $b = -f + 21.540$ into the second equation.	$0.105(-f + 21{,}540) + 0.059f = 1669.68$ $-0.105f + 2261.70 + 0.059f = 1669.68$
Simplify and solve for f.	$-0.046f + 2261.70 = 1669.68$ $-0.046f = -592.02$ $f = 12{,}870$
To find b, substitute $f = 12{,}870$ into the first equation.	$b + f = 21{,}540$ $b + 12{,}870 = 21{,}540$ $b = 8{,}670$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	The principal of the federal loan was $12,870 and the principal for the bank loan was $8,670.

Example 4.3.17

Translate to a system of equations and solve:

Laura owes $18,000 on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9%. The total amount of interest she paid last year was $1,066. What was the principal for each loan?

Answer

Bank $4,000; Federal $14,000

Example 4.3.18

Translate to a system of equations and solve:

Jill's Sandwich Shoppe owes $65,200 on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was $3,582. What was the principal for each loan?

Answer

$41,200 at 4.5%, $24,000 at 7.2%

Solve applications of cost and revenue functions

Suppose a company makes and sells *x* units of a product. The cost to the company is the total costs to produce *x* units. This is the cost to manufacture for each unit times *x*, the number of units manufactured, plus the fixed costs.

The **revenue** is the money the company brings in as a result of selling *x* units. This is the selling price of each unit times the number of units sold.

When the costs equal the revenue we say the business has reached the **break-even point**.

COST AND REVENUE FUNCTIONS

The **cost function** is the cost to manufacture each unit times *x*, the number of units manufactured, plus the fixed costs.

$$C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$$

The **revenue** function is the selling price of each unit times *x*, the number of units sold.

$$R(x) = (\text{selling price per unit}) \cdot x$$

The **break-even point** is when the revenue equals the costs.

$$C(x) = R(x)$$

Example 4.3.19

The manufacturer of a weight training bench spends $105 to build each bench and sells them for $245. The manufacturer also has fixed costs each month of $7,000.

ⓐ Find the cost function *C* when *x* benches are manufactured.

ⓑ Find the revenue function *R* when *x* benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ The manufacturer has $7,000 of fixed costs no matter how many weight training benches it produces. In addition to the fixed costs, the manufacturer also spends $105 to produce each bench. Suppose *x* benches are sold.

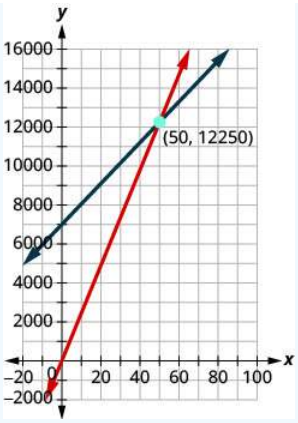
Write the general Cost function formula.	$C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$
Substitute in the cost values.	$C(x) = 105x + 7000$

ⓑ The manufacturer sells each weight training bench for $245. We get the total revenue by multiplying the revenue per unit times the number of units sold.

Write the general Revenue function.	$C(x) = (\text{selling price per unit}) \cdot x$
Substitute in the revenue per unit.	$R(x) = 245x$

ⓒ Essentially we have a system of linear equations. We will show the graph of the system as this helps make the idea of a break-even point more visual.

$$\begin{cases} C(x) = 105x + 7000 \\ R(x) = 245x \end{cases} \quad \text{or} \quad \begin{cases} y = 105x + 7000 \\ y = 245x \end{cases}$$



ⓓ To find the actual value, we remember the break-even point occurs when costs equal revenue.

Write the break-even formula. $C(x) = R(x)$

Solve.
$$105x + 7000 = 245x$$
$$7000 = 140x$$
$$50 = x$$

When 50 benches are sold, the costs equal the revenue.

$C(x) = 105x + 7000$	$R(x) = 245x$
$C(50) = 105(50) + 7000$	$R(50) = 245 \cdot 50$
$C(50) = 12{,}250$	$R(50) = 12{,}250$

When 50 benches are sold, the revenue and costs are both $12,250. Notice this corresponds to the ordered pair $(50, 12250)$.

Example 4.3.20

The manufacturer of a weight training bench spends $15 to build each bench and sells them for $32. The manufacturer also has fixed costs each month of $25,500.

ⓐ Find the cost function C when x benches are manufactured.

ⓑ Find the revenue function R when x benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

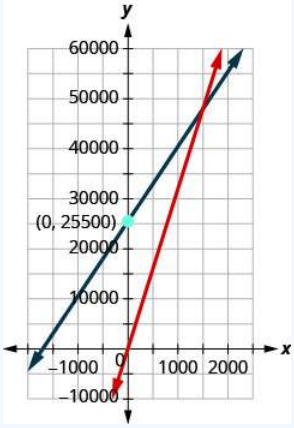
ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 15x + 25,500$

ⓑ $R(x) = 32x$

ⓒ



ⓓ 1,5001,500; when 1,500 benches are sold, the cost and revenue will be both 48,000

? Example 4.3.21

The manufacturer of a weight training bench spends $120 to build each bench and sells them for $170. The manufacturer also has fixed costs each month of $150,000.

ⓐ Find the cost function *C* when *x* benches are manufactured.

ⓑ Find the revenue function *R* when *x* benches are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

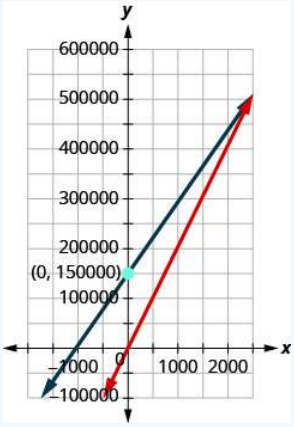
ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 120x + 150,000$

ⓑ $R(x) = 170x$

ⓒ



ⓓ $3,000$; when 3,000 benches are sold, the revenue and costs are both $510,000

Access this online resource for additional instruction and practice with interest and mixtures.

- Interest and Mixtures

Key Concepts

- **Cost function:** The cost function is the cost to manufacture each unit times *x*, the number of units manufactured, plus the fixed costs.

  $C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$

- **Revenue:** The revenue function is the selling price of each unit times *x*, the number of units sold.

  $R(x) = (\text{selling price per unit}) \cdot x$

- **Break-even point:** The break-even point is when the revenue equals the costs.

  $C(x) = R(x)$

Glossary

cost function

The cost function is the cost to manufacture each unit times xx, the number of units manufactured, plus the fixed costs; $C(x)$ = (cost per unit)x + fixed costs.

revenue

The revenue is the selling price of each unit times *x*, the number of units sold; $R(x)$ = (selling price per unit)x.

break-even point

The point at which the revenue equals the costs is the break-even point; C(x)=R(x).

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4.3E: Exercises

Practice Makes Perfect

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

1. Tickets to a Broadway show cost $35 for adults and $15 for children. The total receipts for 1650 tickets at one performance were $47,150. How many adult and how many child tickets were sold?

2. Tickets for the Cirque du Soleil show are $70 for adults and $50 for children. One evening performance had a total of 300 tickets sold and the receipts totaled $17,200. How many adult and how many child tickets were sold?

Answer

110 adult tickets, 190 child tickets

3. Tickets for an Amtrak train cost $10 for children and $22 for adults. Josie paid $1200 for a total of 72 tickets. How many children tickets and how many adult tickets did Josie buy?

4. Tickets for a Minnesota Twins baseball game are $69 for Main Level seats and $39 for Terrace Level seats. A group of sixteen friends went to the game and spent a total of $804 for the tickets. How many of Main Level and how many Terrace Level tickets did they buy?

Answer

6 good seats, 10 cheap seats

5. Tickets for a dance recital cost $15 for adults and $7 dollars for children. The dance company sold 253 tickets and the total receipts were $2771. How many adult tickets and how many child tickets were sold?

6. Tickets for the community fair cost $12 for adults and $5 dollars for children. On the first day of the fair, 312 tickets were sold for a total of $2204. How many adult tickets and how many child tickets were sold?

Answer

92 adult tickets, 220 children tickets

7. Brandon has a cup of quarters and dimes with a total value of $\$3.80$. The number of quarters is four less than twice the number of quarters. How many quarters and how many dimes does Brandon have?

8. Sherri saves nickels and dimes in a coin purse for her daughter. The total value of the coins in the purse is $\$0.95$. The number of nickels is two less than five times the number of dimes. How many nickels and how many dimes are in the coin purse?

Answer

13 nickels, 3 dimes

9. Peter has been saving his loose change for several days. When he counted his quarters and nickels, he found they had a total value $\$13.10$. The number of quarters was fifteen more than three times the number of dimes. How many quarters and how many dimes did Peter have?

10. Lucinda had a pocketful of dimes and quarters with a value of $6.20. The number of dimes is eighteen more than three times the number of quarters. How many dimes and how many quarters does Lucinda have?

Answer

42 dimes, 8 quarters

11. A cashier has 30 bills, all of which are $10 or $20 bills. The total value of the money is $460. How many of each type of bill does the cashier have?

12. A cashier has 54 bills, all of which are $10 or $20 bills. The total value of the money is $910. How many of each type of bill does the cashier have?

Answer

17 $10 bills, 37 $20 bills

13. Marissa wants to blend candy selling for $1.80 per pound with candy costing $1.20 per pound to get a mixture that costs her $1.40 per pound to make. She wants to make 90 pounds of the candy blend. How many pounds of each type of candy should she use?

14. How many pounds of nuts selling for $6 per pound and raisins selling for $3 per pound should Kurt combine to obtain 120 pounds of trail mix that cost him $5 per pound?

Answer

80 pounds nuts and 40 pounds raisins

15. Hannah has to make twenty-five gallons of punch for a potluck. The punch is made of soda and fruit drink. The cost of the soda is $1.79 per gallon and the cost of the fruit drink is $2.49 per gallon. Hannah's budget requires that the punch cost $2.21 per gallon. How many gallons of soda and how many gallons of fruit drink does she need?

16. Joseph would like to make twelve pounds of a coffee blend at a cost of $6 per pound. He blends Ground Chicory at $5 a pound with Jamaican Blue Mountain at $9 per pound. How much of each type of coffee should he use?

Answer

9 pounds of Chicory coffee, 3 pounds of Jamaican Blue Mountain coffee

17. Julia and her husband own a coffee shop. They experimented with mixing a City Roast Columbian coffee that cost $7.80 per pound with French Roast Columbian coffee that cost $8.10 per pound to make a twenty-pound blend. Their blend should cost them $7.92 per pound. How much of each type of coffee should they buy?

18. Twelve-year old Melody wants to sell bags of mixed candy at her lemonade stand. She will mix M&M's that cost $4.89 per bag and Reese's Pieces that cost $3.79 per bag to get a total of twenty-five bags of mixed candy. Melody wants the bags of mixed candy to cost her $4.23 a bag to make. How many bags of M&M's and how many bags of Reese's Pieces should she use?

Answer

10 bags of M&M's, 15 bags of Reese's Pieces

19. Jotham needs 70 liters of a 50% solution of an alcohol solution. He has a 30% and an 80% solution available. How many liters of the 30% and how many liters of the 80% solutions should he mix to make the 50% solution?

20. Joy is preparing 15 liters of a 25% saline solution. She only has 40% and 10% solution in her lab. How many liters of the 40% and how many liters of the 10% should she mix to make the 25% solution?

Answer

7.5 liters of each solution

21. A scientist needs 65 liters of a 15% alcohol solution. She has available a 25% and a 12% solution. How many liters of the 25% and how many liters of the 12% solutions should she mix to make the 15% solution?

22. A scientist needs 120 milliliters of a 20% acid solution for an experiment. The lab has available a 25% and a 10% solution. How many liters of the 25% and how many liters of the 10% solutions should the scientist mix to make the 20% solution?

Answer

80 liters of the 25% solution and 40 liters of the 10% solution

23. A 40% antifreeze solution is to be mixed with a 70% antifreeze solution to get 240 liters of a 50% solution. How many liters of the 40% and how many liters of the 70% solutions will be used?

24. A 90% antifreeze solution is to be mixed with a 75% antifreeze solution to get 360 liters of an 85% solution. How many liters of the 90% and how many liters of the 75% solutions will be used?

Answer

240 liters of the 90% solution and 120 liters of the 75% solution

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

25. Hattie had $3000 to invest and wants to earn 10.6% interest per year. She will put some of the money into an account that earns 12% per year and the rest into an account that earns 10% per year. How much money should she put into each account?

26. Carol invested $2560 into two accounts. One account paid 8% interest and the other paid 6% interest. She earned 7.25% interest on the total investment. How much money did she put in each account?

Answer

$1600 at 8%, 960 at 6%

27. Sam invested $48,000, some at 6% interest and the rest at 10%. How much did he invest at each rate if he received $4000 in interest in one year?

28. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4500 in interest in one year?

Answer

$28,000 at 9%, $36,000 at 5.5%

29. After four years in college, Josie owes $65, 800 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $2878.50 What is the amount of each loan?

30. Mark wants to invest $10,000 to pay for his daughter's wedding next year. He will invest some of the money in a short term CD that pays 12% interest and the rest in a money market savings account that pays 5% interest. How much should he

invest at each rate if he wants to earn $1095 in interest in one year?

Answer

$8500 CD, $1500 savings account

31. A trust fund worth $25,000 is invested in two different portfolios. This year, one portfolio is expected to earn 5.25% interest and the other is expected to earn 4%. Plans are for the total interest on the fund to be $1150 in one year. How much money should be invested at each rate?

32. A business has two loans totaling $85,000. One loan has a rate of 6% and the other has a rate of 4.5% This year, the business expects to pay $4,650 in interest on the two loans. How much is each loan?

Answer

$55,000 on loan at 6% and $30,000 on loan at 4.5%

Solve Applications of Cost and Revenue Functions

33. The manufacturer of an energy drink spends $1.20 to make each drink and sells them for $2. The manufacturer also has fixed costs each month of $8,000.

ⓐ Find the cost function C when x energy drinks are manufactured.

ⓑ Find the revenue function R when x drinks are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

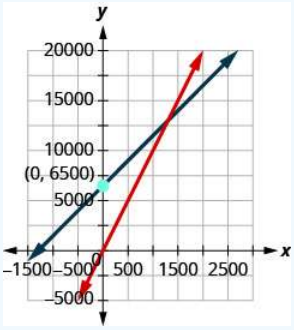
34. The manufacturer of a water bottle spends $5 to build each bottle and sells them for $10. The manufacturer also has fixed costs each month of $6500. ⓐ Find the cost function C when x bottles are manufactured. ⓑ Find the revenue function R when x bottles are sold. ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid. ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 5x + 6500$

ⓑ $R(x) = 10x$

ⓒ



ⓓ 1,500; when 1,500 water bottles are sold, the cost and the revenue equal $15,000

Writing Exercises

35. Take a handful of two types of coins, and write a problem similar to Example relating the total number of coins and their total value. Set up a system of equations to describe your situation and then solve it.

36. In Example, we used elimination to solve the system of equations

$$\begin{cases} s+b=40,000 \\ 0.08s+0.03b=0.071(40,000). \end{cases}$$

Could you have used substitution or elimination to solve this system? Why?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve mixture applications.			
solve interest applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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4.4: Solve Systems of Equations with Three Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered triple is a solution of a system of three linear equations with three variables
- Solve a system of linear equations with three variables
- Solve applications using systems of linear equations with three variables

Before you get started, take this readiness quiz.

1. Evaluate $5x-2y+3z$ when $x=-2, y=-4$, and $z=3$.
   If you missed this problem, review [link].
2. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution.
   $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$
   If you missed this problem, review [link].
3. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution. $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$
   If you missed this problem, review [link].

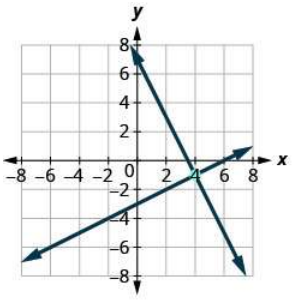
Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In this section, we will extend our work of solving a system of linear equations. So far we have worked with **systems of equations** with two equations and two variables. Now we will work with systems of three equations with three variables. But first let's review what we already know about solving equations and systems involving up to two variables.

We learned earlier that the graph of a **linear equation**, $ax+by=c$, is a line. Each point on the line, an ordered pair (x,y), is a solution to the equation. For a system of two equations with two variables, we graph two lines. Then we can see that all the points that are solutions to each equation form a line. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions

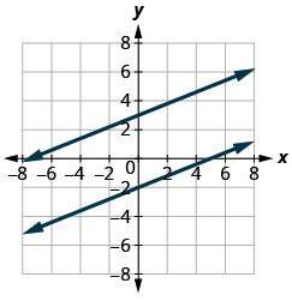
We know when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



One solution

The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.

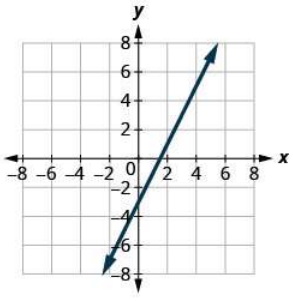
Consistent
Independent



No solution

The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.

Inconsistent



Infinitely many solutions

Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Consistent
Dependent

Similarly, for a linear equation with three variables ax+by+cz=d,ax+by+cz=d, every solution to the equation is an ordered triple, (x,y,z)(x,y,z)that makes the equation true.

LINEAR EQUATION IN THREE VARIABLES

A linear equation with three variables, where a, b, c, and d are real numbers and a, b, and c are not all 0, is of the form

$$ax + by + cz = d$$

Every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.

All the points that are solutions to one equation form a plane in three-dimensional space. And, by finding what the planes have in common, we'll find the solution to the system.

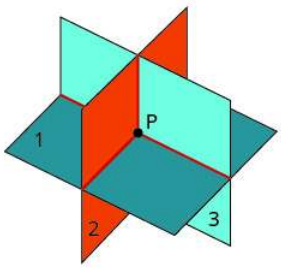
When we solve a system of three linear equations represented by a graph of three planes in space, there are three possible cases.

One solution

Consistent system and Independent equations

The 3 planes intersect.

The three intersecting planes have one point in common.

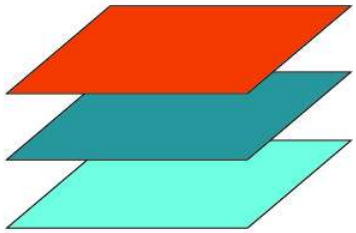


No solution

Inconsistent system

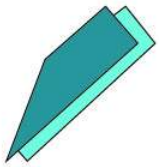
The planes are parallel.

Parallel planes have no points in common.



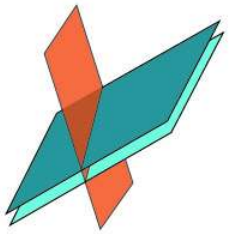
Two planes are coincident and parallel to the third plane.

The planes have no points in common.



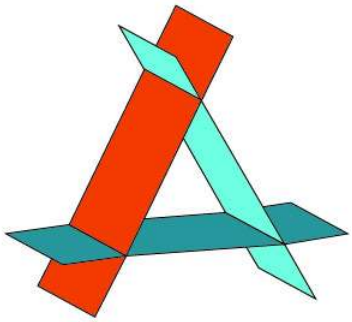
Two planes are parallel and each intersect the third plane.

The planes have no points in common.



Each plane intersects the other two, but all three share no points.

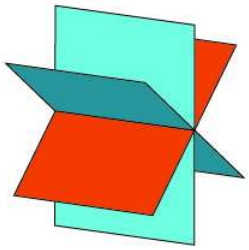
The planes have no points in common.



Infinitely many solutions
Consistent system and dependent equations

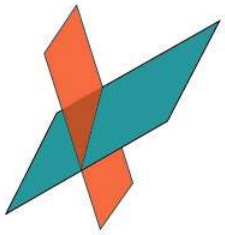
Three planes intersect in one line.

There is just one line, so there are infinitely many solutions.



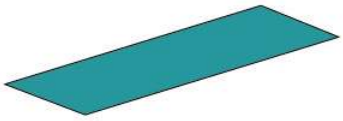
Two planes are coincident and intersect the third plane in a line.

There is just one line, so there are infinitely many solutions.



Three planes are coincident.

There is just one plane, so there are infinitely many solutions.



To solve a system of three linear equations, we want to find the values of the variables that are solutions to all three equations. In other words, we are looking for the ordered triple (x, y, z) that makes all three equations true. These are called the **solutions of the system of three linear equations with three variables**.

SolutionS OF A SYSTEM OF LINEAR EQUATIONS WITH THREE VARIABLES

Solutions of a system of equations are the values of the variables that make all the equations true. A solution is represented by an **ordered triple** (x, y, z).

To determine if an ordered triple is a solution to a system of three equations, we substitute the values of the variables into each equation. If the ordered triple makes all three equations true, it is a solution to the system.

Example 4.4.1

Determine whether the ordered triple is a solution to the system: $\begin{cases} x-y+z=2 \\ 2x-y-z=-6 \\ 2x+2y+z=-3 \end{cases}$

ⓐ $(-2,-1,3)$ ⓑ $(-4,-3,4)$

Answer

ⓐ

$$\begin{cases} x-y+z=2 \\ 2x-y-z=-6 \\ 2x+2y+z=-3 \end{cases}$$

We substitute $x=-2$ and $y=-1$ and $z=3$ into all three equations.

$x-y+z=2$	$2x-y-z=-6$	$2x+2y+z=-3$
$-2-(-1)+3 \stackrel{?}{=} 2$	$2\cdot(-2)-(-1)-3 \stackrel{?}{=} -6$	$2\cdot(-2)+2(-1)+3 \stackrel{?}{=} -3$
$2=2$ ✓	$-6=-6$ ✓	$-3=-3$ ✓
(–2, –1, 3) does make all three equations true.		(–2, –1, 3) is a solution.

ⓑ

$$\begin{cases} x - y + z = 2 \\ 2x - y - z = -6 \\ 2x + 2y + z = -3 \end{cases}$$

We substitute $x = -4$ and $y = -3$ and $z = 4$ into all three equations.

$x - y + z = 2$	$2x - y - z = -6$	$2x + 2y + z = -3$
$-4 - (-3) + 4 \stackrel{?}{=} 2$	$2 \cdot (-4) - (-3) - 4 \stackrel{?}{=} -6$	$2 \cdot (-4) + 2(-3) + 4 \stackrel{?}{=} -3$
$3 \neq 2$	$-9 \neq -6$	$-10 \neq -3$
$(-4, -3, 4)$ does not make all three equations true.		$(-4, -3, 4)$ is not a solution.

Example 4.4.2

Determine whether the ordered triple is a solution to the system: $\begin{cases} 3x + y + z = 2 \\ x + 2y + z = -3 \\ 3x + y + 2z = 4 \end{cases}$

ⓐ $(1, -3, 2)$ ⓑ $(4, -1, -5)$

Answer

ⓐ yes ⓑ no

Example 4.4.3

Determine whether the ordered triple is a solution to the system: $\begin{cases} x - 3y + z = -5 \\ -3x - y - z = 1 \\ 2x - 2y + 3z = 1 \end{cases}$

ⓐ $(2, -2, 3)$ ⓑ $(-2, 2, 3)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

To solve a system of linear equations with three variables, we basically use the same techniques we used with systems that had two variables. We start with two pairs of equations and in each pair we eliminate the same variable. This will then give us a system of equations with only two variables and then we know how to solve that system!

Next, we use the values of the two variables we just found to go back to the original equation and find the third variable. We write our answer as an ordered triple and then check our results.

Example 4.4.4: How to Solve a System of Equations With Three Variables by Elimination

Solve the system by elimination: $\begin{cases} x - 2y + z = 3 \\ 2x + y + z = 4 \\ 3x + 4y + 3z = -1 \end{cases}$

Answer

Step 1. Write the equations in standard form. • If any coefficients are fractions, clear them.	The equations are in standard form. There are no fractions.	$\begin{cases} x-2y+z=3 & (1)\\ 2x+y+z=4 & (2)\\ 3x+4y+3z=-1 & (3)\end{cases}$
Step 2. Eliminate the same variable from two equations. • Decide which variable you will eliminate. • Work with a pair of equations to eliminate the chosen variable. • Multiply one or both equations so that the coefficients of that variable are opposites. • Add the equations to eliminate one variable.	We can eliminate the y's from equations (1) and (2) by multiplying equation (2) by 2. Multiply. We add the x's, y's, and constants. The new equation, (4), has only x and z.	$x-2y+z=3 \quad (1)$ $2(2x+y+z)=(4)2 \quad (2)$ $x-2y+z=3 \quad (1)$ $4x+2y+2z=8 \quad (2)$ $x-2y+z=3 \quad (1)$ $4x+2y+2z=8 \quad (2)$ $5x+3z=11 \quad (4)$
Step 3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.	We can again eliminate the y's using the equations (1) and (3) by multiplying equation (1) by 2. Add the new equations and the result will be equation (5).	$2(x-2y+z)=(3)2 \quad (1)$ $3x+4y+3z=-1 \quad (3)$ $2x-4y+2z=6 \quad (1)$ $3x+4y+3z=-1 \quad (3)$ $5x+5z=5 \quad (5)$
Step 4. The two new equations form a system of two equations with two variables. Solve this system.	Use equations (4) and (5) to eliminate either x or z. Multiply (5) by -1. Use either (4) or (5) to solve for x using substitution.	$\begin{cases} 5x+3z=11 & (4)\\ 5x+5z=5 & (5)\end{cases}$ $5x+3z=11 \quad (4)$ $-1(5x+5z)=5(-1) \quad (5)$ $5x+3z=11 \quad (4)$ $-5x-5z=-5 \quad (5)$ $-2z=6$ $z=-3$ $5x+3z=11 \quad (4)$ $5x+3(-3)=11$ $5x-9=11$ $5x=20$ $x=4$
Step 5. Use the values of the two variables found in Step 4 to find the third variable.	Use an original equation to find y.	$2x+y+z=4 \quad (2)$ $2\cdot 4+y+z=4$ $8+y-3=4$ $y=-1$
Step 6. Write the solution as an ordered triple.	Write it as (x, y, z).	$(4, -1, -3)$
Step 7. Check that the ordered triple is a solution to **all three** original equations.	Substitute $x=4$, $y=-1$ and $z=-3$ into all three equations. Do they make the equations true?	We'll leave the checks to you. The solution is $(4, -1, -3)$.

Example 4.4.5

Solve the system by elimination: $\begin{cases} 3x+y-z=2 \\ 2x-3y-2z=1 \\ 4x-y-3z=0 \end{cases}$

Answer

$(2,-1,3)$

Example 4.4.6

Solve the system by elimination: $\begin{cases} 4x+y+z=-1 \\ -2x-2y+z=2 \\ 2x+3y-z=1 \end{cases}$

Answer

$(-2,3,4)$

The steps are summarized here.

SOLVE A SYSTEM OF LINEAR EQUATIONS WITH THREE VARIABLES.

1. Write the equations in standard form
   - If any coefficients are fractions, clear them.
2. Eliminate the same variable from two equations.
   - Decide which variable you will eliminate.
   - Work with a pair of equations to eliminate the chosen variable.
   - Multiply one or both equations so that the coefficients of that variable are opposites.
   - Add the equations resulting from Step 2 to eliminate one variable
3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
4. The two new equations form a system of two equations with two variables. Solve this system.
5. Use the values of the two variables found in Step 4 to find the third variable.
6. Write the solution as an ordered triple.
7. Check that the ordered triple is a solution to **all three** original equations.

Example 4.4.7

Solve: $\begin{cases} 3x-4z=0 \\ 3y+2z=-3 \\ 2x+3y=-5 \end{cases}$

Answer

$$\begin{cases} 3x-4z=0 & (1) \\ 3y+2z=-3 & (2) \\ 2x+3y=-5 & (3) \end{cases}$$

We can eliminate z from equations (1) and (2) by multiplying equation (2) by 2 and then adding the resulting equations.

$$\begin{array}{l} 3x \quad -4z=0 \quad (1) \\ \quad 3y+2z=-3 \quad (2) \end{array} \longrightarrow \begin{array}{l} 3x \quad -4z=0 \\ \quad 2(3y+2z)=(-3)2 \end{array} \longrightarrow \begin{array}{l} 3x \quad -4z=0 \\ \quad 6y+4z=-6 \\ \hline 3x+6y \quad =-6 \quad (4) \end{array}$$

Notice that equations (3) and (4) both have the variables x and y. We will solve this new system for x and y.

$$\begin{array}{l} 2x+3y=-5 \quad (3) \\ 3x+6y=-6 \quad (4) \end{array} \longrightarrow \begin{array}{l} -2(2x+3y)=-2(-5) \\ 3x+6y=-6 \end{array} \longrightarrow \begin{array}{r} -4x-6y=10 \\ 3x+6y=-6 \\ \hline -x \quad = \quad 4 \\ x=-4 \end{array}$$

To solve for y, we substitute $x=-4$ into equation (3).

$$\begin{array}{r} 2x+3y=-5 \quad (3) \\ 2(-4)+3y=-5 \\ -8+3y=-5 \\ 3y=3 \\ y=1 \end{array}$$

We now have $x=-4$ and $y=1$. We need to solve for z. We can substitute $x=-4$ into equation (1) to find z.

$$\begin{array}{r} 3x-4z=0 \quad (1) \\ 3(-4)-4z=0 \\ -12-4z=0 \\ -4z=12 \\ z=-3 \end{array}$$

We write the solution as an ordered triple. $(-4,1,-3)$

We check that the solution makes all three equations true.

$3x-4z=0\ (1)$ $\quad$ $3y+2z=-3\ (2)$ $\quad$ $2x+3y=-5\ (3)$

$3(-4)-4(-3)\stackrel{?}{=}0$ $\quad$ $3(1)+2(-3)\stackrel{?}{=}-3$ $\quad$ $2(-4)+3(1)\stackrel{?}{=}-5$

$0=0$✓ $\quad$ $-3=-3$✓ $\quad$ $-5=-5$✓

The solution is $(-4,1,-3)$

Example 4.4.8

Solve: $\begin{cases} 3x-4z=-1 \\ 2y+3z=2 \\ 2x+3y=6 \end{cases}$

Answer

$(-3,4,-2)$

Example 4.4.9

Solve: $\begin{cases} 4x-3z=-5 \\ 3y+2z=7 \\ 3x+4y=6 \end{cases}$

Answer

$(-2,3,-1)$

When we solve a system and end up with no variables and a false statement, we know there are no solutions and that the system is inconsistent. The next example shows a system of equations that is inconsistent.

Example 4.4.10

Solve the system of equations: $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$

Answer

$$\begin{cases} x+2y-3z=-1 & (1) \\ x-3y+z=1 & (2) \\ 2x-y-2z=2 & (3) \end{cases}$$

Use equation (1) and (2) to eliminate z.

$$\begin{array}{l} x+2y-3z=-1 \quad (1) \\ x-3y+\ z=\ 1 \quad (2) \end{array} \xrightarrow{\text{multiply by 3}} \begin{array}{r} x+2y-3z=-1 \\ 3x-9y+3z=\ 3 \\ \hline 4x-7y\quad =\ 2 \quad (4) \end{array}$$

Use (2) and (3) to eliminate z again.

$$\begin{array}{l} x-3y+\ z=1 \quad (2) \\ 2x-\ y-2z=2 \quad (3) \end{array} \xrightarrow{\text{multiply by 2}} \begin{array}{r} 2x-6y+2z=2 \\ 2x-\ y-2z=2 \\ \hline 4x-7y\quad =4 \quad (5) \end{array}$$

Use (4) and (5) to eliminate a variable.

$$\begin{array}{l} 4x-7y=2 \quad (4) \\ 4x-7y=4 \quad (5) \end{array} \xrightarrow{\text{multiply by } -1} \begin{array}{r} 4x-7y=2 \\ -4x+7y=-4 \\ \hline 0=-2 \ \textit{false} \end{array}$$

There is no solution.

We are left with a false statement and this tells us the system is inconsistent and has no solution.

Example 4.4.11

Solve the system of equations: $\begin{cases} x+2y+6z=5 \\ -x+y-2z=3 \\ x-4y-2z=1 \end{cases}$

Answer

no solution

Example 4.4.12

Solve the system of equations: $\begin{cases} 2x-2y+3z=6 \\ 4x-3y+2z=0 \\ -2x+3y-7z=1 \end{cases}$

Answer

no solution

When we solve a system and end up with no variables but a true statement, we know there are infinitely many solutions. The system is consistent with dependent equations. Our solution will show how two of the variables depend on the third.

Example 4.4.13

Solve the system of equations: $\begin{cases} x+2y-z=1 \\ 2x+7y+4z=11 \\ x+3y+z=4 \end{cases}$

Answer

$$\begin{cases} x+2y-z=1 & (1) \\ 2x+7y+4z=11 & (2) \\ x+3y+z=4 & (3) \end{cases}$$

Use equation (1) and (3) to eliminate x.

$$\begin{array}{ll} x+2y-z=1 & (1) \\ x+3y+z=4 & (3) \end{array} \xrightarrow{\text{multiply by } -1} \begin{array}{ll} -x-2y+z=-1 & (1) \\ x+3y+z=4 & (3) \\ \hline y+2z=3 & (4) \end{array}$$

Use equation (1) and (2) to eliminate x again.

$$\begin{array}{ll} x+2y-z=1 & (1) \\ 2x+7y+4z=11 & (2) \end{array} \xrightarrow{\text{multiply by } -2} \begin{array}{ll} -2x-4y+2z=-2 & (1) \\ 2x+7y+4z=11 & (2) \\ \hline 3y+6z=9 & (5) \end{array}$$

Use equation (4) and (5) to eliminate y.

$$\begin{array}{ll} y+2z=3 & (4) \\ 3y+6z=9 & (5) \end{array} \xrightarrow{\text{multiply by } -3} \begin{array}{ll} -3y-6z=-9 & (4) \\ 3y+6z=9 & (5) \\ \hline 0=0 & \end{array}$$

	There are infinitely many solutions.
Solve equation (4) for y.	Represent the solution showing how x and y are dependent on z. $y+2z=3$ $y=-2z+3$
Use equation (1) to solve for x.	$x+2y-z=1$
Substitute $y=-2z+3$.	$x+2(-2z+3)-z=1$ $x-4z+6-z=1$ $x-5z+6=1$ $x=5z-5$

The true statement $0=0$ tells us that this is a dependent system that has infinitely many solutions. The solutions are of the form (x,y,z)(x,y,z) where $x=5z-5;\ y=-2z+3$ and z is any real number.

Example 4.4.14

Solve the system by equations: $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6 \\ 3x+6y-3z=9 \end{cases}$

Answer

infinitely many solutions $(x,3,z)$ where $x=z-3;\ y=3;\ z$ is any real number

Example 4.4.15

Solve the system by equations: $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$

Answer

infinitely many solutions (x, y, z) where $x=5z-2;\ y=4z-3;\ z$ is any real number

Solve Applications using Systems of Linear Equations with Three Variables

Applications that are modeled by a systems of equations can be solved using the same techniques we used to solve the systems. Many of the application are just extensions to three variables of the types we have solved earlier.

Example 4.4.16

The community college theater department sold three kinds of tickets to its latest play production. The adult tickets sold for $15, the student tickets for $10 and the child tickets for $8. The theater department was thrilled to have sold 250 tickets and brought in $2,825 in one night. The number of student tickets sold is twice the number of adult tickets sold. How many of each type did the department sell?

Answer

We will use a chart to organize the information.

Type	Number	· Value	= Total Value
adult	x	15	$15x$
student	y	10	$10y$
child	z	8	$8z$
	250		2825

Number of students is twice number of adults.

Rewrite the equation in standard form.

$$\begin{aligned} y &= 2x \\ 2x-y &= 0 \end{aligned}$$

Write the system of equations.

$$\begin{cases} x+y+z=250 & (1) \\ 15x+10y+8z=2825 & (2) \\ -2x+y=0 & (3) \end{cases}$$

Use equations (1) and (2) to eliminate z.

$$\begin{array}{rl} x+y+z=250 & (1) \\ 15x+10y+8z=2825 & (2) \end{array} \xrightarrow{\text{Multiply by } -8} \begin{array}{rl} -8x-8y-8z=250 & (1) \\ 15x+10y+8z=2825 & (2) \\ \hline 7x+2y=825 & (4) \end{array}$$

Use (3) and (4) to eliminate y.

$$\begin{array}{rl} -2x+y=0 & (3) \\ 7x+2y=825 & (4) \end{array} \xrightarrow{\text{Multiply by } -2} \begin{array}{rl} 4x-2y=0 & (3) \\ 7x+2y=825 & (4) \\ \hline 11x=825 & \end{array}$$

Solve for x. $x=75$ adult tickets

Use equation (3) to find y. $-2x+y=0$

Substitute $x=75$.

$$\begin{aligned} -2(75)+y &= 0 \\ -150+y &= 0 \\ y &= 150 \text{ student tickets} \end{aligned}$$

Use equation (1) to find z. $x+y+z=250$

Substitute in the values $x = 75,\ y = 150$.	$\begin{aligned} 75 + 150 + z &= 250 \\ 225 + z &= 250 \\ z &= 25 \text{ child tickets} \end{aligned}$
Write the solution.	The theater department sold 75 adult tickets, 150 student tickets, and 25 child tickets.

Example 4.4.17

The community college fine arts department sold three kinds of tickets to its latest dance presentation. The adult tickets sold for $20, the student tickets for $12 and the child tickets for $10.The fine arts department was thrilled to have sold 350 tickets and brought in $4,650 in one night. The number of child tickets sold is the same as the number of adult tickets sold. How many of each type did the department sell?

Answer

The fine arts department sold 75 adult tickets, 200 student tickets, and 75 child tickets.

Example 4.4.18

The community college soccer team sold three kinds of tickets to its latest game. The adult tickets sold for $10, the student tickets for $8 and the child tickets for $5. The soccer team was thrilled to have sold 600 tickets and brought in $4,900 for one game. The number of adult tickets is twice the number of child tickets. How many of each type did the soccer team sell?

Answer

The soccer team sold 200 adult tickets, 300 student tickets, and 100 child tickets.

Access this online resource for additional instruction and practice with solving a linear system in three variables with no or infinite solutions.

- Solving a Linear System in Three Variables with No or Infinite Solutions
- 3 Variable Application

Key Concepts

- **Linear Equation in Three Variables:** A linear equation with three variables, where *a*, *b*, *c*, and *d* are real numbers and *a*, *b*,and *c* are not all 0, is of the form

$$ax + by + cz = d$$

  Every solution to the equation is an ordered triple, (x, y, z) that makes the equation true.
- **How to solve a system of linear equations with three variables.**
  1. Write the equations in standard form
     If any coefficients are fractions, clear them.
  2. Eliminate the same variable from two equations.
     Decide which variable you will eliminate.
     Work with a pair of equations to eliminate the chosen variable.
     Multiply one or both equations so that the coefficients of that variable are opposites.
     Add the equations resulting from Step 2 to eliminate one variable
  3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
  4. The two new equations form a system of two equations with two variables. Solve this system.
  5. Use the values of the two variables found in Step 4 to find the third variable.
  6. Write the solution as an ordered triple.

7. Check that the ordered triple is a solution to **all three** original equations.

Glossary

solutions of a system of linear equations with three variables

The solutions of a system of equations are the values of the variables that make all the equations true; a solution is represented by an ordered triple (x,y,z).

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4.4E: Exercises

Practice Makes Perfect

Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In the following exercises, determine whether the ordered triple is a solution to the system.

1. $\begin{cases} 2x-6y+z=3 \\ 3x-4y-3z=2 \\ 2x+3y-2z=3 \end{cases}$

ⓐ $(3,1,3)$
ⓑ $(4,3,7)$

2. $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

ⓐ $(-5,-7,4)$
ⓑ $(5,7,4)$

Answer

ⓐ no ⓑ yes

3. $\begin{cases} y-10z=-8 \\ 2x-y=2 \\ x-5z=3 \end{cases}$

ⓐ $(7,12,2)$
ⓑ $(2,2,1)$

4. $\begin{cases} x+3y-z=1 \\ 5y=\frac{2}{3}x \\ -2x-3y+z=-2 \end{cases}$

ⓐ $(-6,5,12)$
ⓑ $(5,\frac{4}{3},-3)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

In the following exercises, solve the system of equations.

5. $\begin{cases} 5x+2y+z=5 \\ -3x-y+2z=6 \\ 2x+3y-3z=5 \end{cases}$

6. $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

Answer

$(4, 5, 2)$

7. $\begin{cases} 2x - 5y + 3z = 8 \\ 3x - y + 4z = 7 \\ x + 3y + 2z = -3 \end{cases}$

8. $\begin{cases} 5x - 3y + 2z = -5 \\ 2x - y - z = 4 \\ 3x - 2y + 2z = -7 \end{cases}$

Answer

$(7, 12, -2)$

9. $\begin{cases} 3x - 5y + 4z = 5 \\ 5x + 2y + z = 0 \\ 2x + 3y - 2z = 3 \end{cases}$

10. $\begin{cases} 4x - 3y + z = 7 \\ 2x - 5y - 4z = 3 \\ 3x - 2y - 2z = -7 \end{cases}$

Answer

$(-3, -5, 4)$

11. $\begin{cases} 3x + 8y + 2z = -5 \\ 2x + 5y - 3z = 0 \\ x + 2y - 2z = -1 \end{cases}$

12. $\begin{cases} 11x + 9y + 2z = -9 \\ 7x + 5y + 3z = -7 \\ 4x + 3y + z = -3 \end{cases}$

Answer

$(2, -3, -2)$

13. $\begin{cases} \frac{1}{3}x - y - z = 1 \\ x + \frac{5}{2}y + z = -2 \\ 2x + 2y + \frac{1}{2}z = -4 \end{cases}$

14. $\begin{cases} x + \frac{1}{2}y + \frac{1}{2}z = 0 \\ \frac{1}{5}x - \frac{1}{5}y + z = 0 \\ \frac{1}{3}x - \frac{1}{3}y + 2z = -1 \end{cases}$

Answer

$(6, -9, -3)$

15. $\begin{cases} x+\frac{1}{3}y-2z=-1 \\ \frac{1}{3}x+y+\frac{1}{2}z=0 \\ \frac{1}{2}x+\frac{1}{3}y-\frac{1}{2}z=-1 \end{cases}$

16. $\begin{cases} \frac{1}{3}x-y+\frac{1}{2}z=4 \\ \frac{2}{3}x+\frac{5}{2}y-4z=0 \\ x-\frac{1}{2}y+\frac{3}{2}z=2 \end{cases}$

Answer

$(3,-4,-2)$

17. $\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$

18. $\begin{cases} 2x+5y=4 \\ 3y-z=\frac{3}{4} \\ x+3z=-3 \end{cases}$

Answer

$(-3,2,3)$

19. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

20. $\begin{cases} 3x-z=-3 \\ 5y+2z=-6 \\ 4x+3y=-8 \end{cases}$

Answer

$(-2,0,-3)$

21. $\begin{cases} 4x-3y+2z=0 \\ -2x+3y-7z=1 \\ 2x-2y+3z=6 \end{cases}$

22. $\begin{cases} x-2y+2z=1 \\ -2x+y-z=2 \\ x-y+z=5 \end{cases}$

Answer

no solution

23. $\begin{cases} 2x+3y+z=1 \\ 2x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

24. $\begin{cases} x+4y+z=-8 \\ 4x-y+3z=9 \\ 2x+7y+z=0 \end{cases}$

Answer

$x=\frac{203}{16};\ y=-\frac{25}{16};\ z=-\frac{231}{16};$

25. $\begin{cases} x+2y+z=4 \\ x+y-2z=3 \\ -2x-3y+z=-7 \end{cases}$

26. $\begin{cases} x+y-2z=3 \\ -2x-3y+z=-7 \\ x+2y+z=4 \end{cases}$

Answer

(x,y,z) where $x=5z+2;\ y=-3z+1;\ z$ is any real number

27. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

28. $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Answer

(x,y,z) where $x=5z-2;\ y=4z-3;\ z$ is any real number

Solve Applications using Systems of Linear Equations with Three Variables

In the following exercises, solve the given problem.

29. The sum of the measures of the angles of a triangle is 180. The sum of the measures of the second and third angles is twice the measure of the first angle. The third angle is twelve more than the second. Find the measures of the three angles.

30. The sum of the measures of the angles of a triangle is 180. The sum of the measures of the second and third angles is three times the measure of the first angle. The third angle is fifteen more than the second. Find the measures of the three angles.

Answer

42, 50, 58

31. After watching a major musical production at the theater, the patrons can purchase souvenirs. If a family purchases 4 t-shirts, the video, and 1 stuffed animal, their total is $135.

A couple buys 2 t-shirts, the video, and 3 stuffed animals for their nieces and spends $115. Another couple buys 2 t-shirts, the video, and 1 stuffed animal and their total is $85. What is the cost of each item?

32. The church youth group is selling snacks to raise money to attend their convention. Amy sold 2 pounds of candy, 3 boxes of cookies and 1 can of popcorn for a total sales of $65. Brian sold 4 pounds of candy, 6 boxes of cookies and 3 cans of popcorn for a total sales of $140. Paulina sold 8 pounds of candy, 8 boxes of cookies and 5 cans of popcorn for a total sales of $250. What is the cost of each item?

Answer

$20, $5, $10

Writing Exercises

33. In your own words explain the steps to solve a system of linear equations with three variables by elimination.

34. How can you tell when a system of three linear equations with three variables has no solution? Infinitely many solutions?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered triple is a solution of a system of three linear equations with three variables.			
solve a system of linear equations with three variables.			
solve applications using systems of linear equations with three variables.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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Chapter 4 Review Exercises

Chapter Review Exercises

Solve Systems of Linear Equations with Two Variables

Determine Whether an Ordered Pair is a Solution of a System of Equations.

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} x+3y=-9 \\ 2x-4y=12 \end{cases}$

ⓐ $(-3,-2)$
ⓑ $(0,-3)$

2. $\begin{cases} x+y=8 \\ y=x-4 \end{cases}$

ⓐ $(6,2)$
ⓑ $(9,-1)$

Answer

ⓐ yes ⓑ no

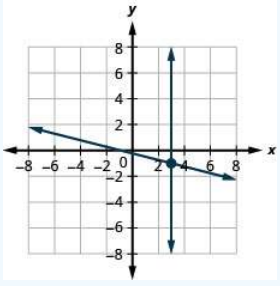
Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

3. $\begin{cases} 3x+y=6 \\ x+3y=-6 \end{cases}$

4. $\begin{cases} x+4y=-1 \\ x=3 \end{cases}$

Answer

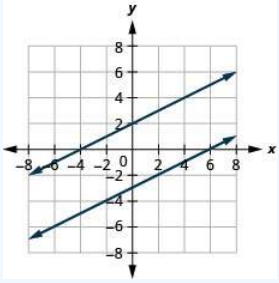


$(3,-1)$

5. $\begin{cases} 2x-y=5 \\ 4x-2y=10 \end{cases}$

6. $\begin{cases} -x+2y=4 \\ y=\frac{1}{2}x-3 \end{cases}$

Answer



no solution

In the following exercises, without graphing determine the number of solutions and then classify the system of equations.

7. $\begin{cases} y = \frac{2}{5}x + 2 \\ -2x + 5y = 10 \end{cases}$

8. $\begin{cases} 3x + 2y = 6 \\ y = -3x + 4 \end{cases}$

Answer

one solution, consistent system, independent equations

9. $\begin{cases} 5x - 4y = 0 \\ y = \frac{5}{4}x - 5 \end{cases}$

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

10. $\begin{cases} 3x - 2y = 2 \\ y = \frac{1}{2}x + 3 \end{cases}$

Answer

$(4, 5)$

11. $\begin{cases} x - y = 0 \\ 2x + 5y = -14 \end{cases}$

12. $\begin{cases} y = -2x + 7 \\ y = \frac{2}{3}x - 1 \end{cases}$

Answer

$(3, 1)$

13. $\begin{cases} y = -5x \\ 5x + y = 6 \end{cases}$

14. $\begin{cases} y = -\frac{1}{3}x + 2 \\ x + 3y = 6 \end{cases}$

Answer

infinitely many solutions

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination

15. $\begin{cases} x+y=12 \\ x-y=-10 \end{cases}$

16. $\begin{cases} 3x-8y=20 \\ x+3y=1 \end{cases}$

Answer

$(4,-1)$

17. $\begin{cases} 9x+4y=2 \\ 5x+3y=5 \end{cases}$

18. $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$

Answer

$(6,2)$

19. $\begin{cases} -x+3y=8 \\ 2x-6y=-20 \end{cases}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

20. $\begin{cases} 6x-5y=27 \\ 3x+10y=-24 \end{cases}$

Answer

elimination

21. $\begin{cases} y=3x-9 \\ 4x-5y=23 \end{cases}$

Solve Applications with Systems of Equations

Solve Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

22. Mollie wants to plant 200 bulbs in her garden, all irises and tulips. She wants to plant three times as many tulips as irises. How many irises and how many tulips should she plant?

Answer

50 irises and 150 tulips

23. Ashanti has been offered positions by two phone companies. The first company pays a salary of $22,000 plus a commission of $100 for each contract sold. The second pays a salary of $28,000 plus a commission of $25 for each contract sold. How many contract would need to be sold to make the total pay the same?

24. Leroy spent 20 minutes jogging and 40 minutes cycling and burned 600 calories. The next day, Leroy swapped times, doing 40 minutes of jogging and 20 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

Answer

10 calories jogging and 10 calories cycling

25. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and calculator. Troy bought four notebooks and five calculators for $116. Lisa bought two notebooks and three calculators for $68. Find the cost of each notebook and each thumb drive.

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

26. The difference of two supplementary angles is 58 degrees. Find the measures of the angles.

Answer

119, 61

27. Two angles are complementary. The measure of the larger angle is five more than four times the measure of the smaller angle. Find the measures of both angles.

28. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

35° and 55°

29. Becca is hanging a 28 foot floral garland on the two sides and top of a pergola to prepare for a wedding. The height is four feet less than the width. Find the height and width of the pergola.

30. The perimeter of a city rectangular park is 1428 feet. The length is 78 feet more than twice the width. Find the length and width of the park.

Answer

the length is 450 feet, the width is 264 feet

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

31. Sheila and Lenore were driving to their grandmother’s house. Lenore left one hour after Sheila. Sheila drove at a rate of 45 mph, and Lenore drove at a rate of 60 mph. How long will it take for Lenore to catch up to Sheila?

32. Bob left home, riding his bike at a rate of 10 miles per hour to go to the lake. Cheryl, his wife, left 45 minutes (34(34 hour) later, driving her car at a rate of 25 miles per hour. How long will it take Cheryl to catch up to Bob?

Answer

$12 an hour

33. Marcus can drive his boat 36 miles down the river in three hours but takes four hours to return upstream. Find the rate of the boat in still water and the rate of the current.

34. A passenger jet can fly 804 miles in 2 hours with a tailwind but only 776 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

the rate of the jet is 395 mph, the rate of the wind is 7 mph

Solve Mixture Applications with Systems of Equations

Solve Mixture Applications with Systems of Equations

For the following exercises, translate to a system of equations and solve.

35. Lynn paid a total of $2,780 for 261 tickets to the theater. Student tickets cost $10 and adult tickets cost $15. How many student tickets and how many adult tickets did Lynn buy?

36. Priam has dimes and pennies in a cup holder in his car. The total value of the coins is $4.21. The number of dimes is three less than four times the number of pennies. How many dimes and how many pennies are in the cup?

Answer

41 dimes and 11 pennies

37. Yumi wants to make 12 cups of party mix using candies and nuts. Her budget requires the party mix to cost her $1.29 per cup. The candies are $2.49 per cup and the nuts are $0.69 per cup. How many cups of candies and how many cups of nuts should she use?

38. A scientist needs 70 liters of a 40% solution of alcohol. He has a 30% and a 60% solution available. How many liters of the 30% and how many liters of the 60% solutions should he mix to make the 40% solution?

Answer

$46\frac{2}{3}$ liters of 30% solution, $23\frac{1}{3}$ liters of 60% solution

Solve Interest Applications

For the following exercises, translate to a system of equations and solve.

39. Jack has $12,000 to invest and wants to earn 7.5% interest per year. He will put some of the money into a savings account that earns 4% per year and the rest into CD account that earns 9% per year. How much money should he put into each account?

40. When she graduates college, Linda will owe $43,000 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $1,585. What is the amount of each loan?

Answer

$29,000 for the federal loan, $14,000 for the private loan

Solve Systems of Equations with Three Variables

Solve Systems of Equations with Three Variables

In the following exercises, determine whether the ordered triple is a solution to the system.

41. $\begin{cases} 3x-4y-3z=2 \\ 2x-6y+z=3 \\ 2x+3y-2z=3 \end{cases}$

ⓐ $(2,3,-1)$
ⓑ $(3,1,3)$

42. $\begin{cases} y=\frac{2}{3}x-2 \\ x+3y-z=15 \\ x-3y+z=-2 \end{cases}$

ⓐ $(-6,5,\frac{1}{2})$
ⓑ $(5,\frac{4}{3},-3)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

In the following exercises, solve the system of equations.

43. $\begin{cases} 3x-5y+4z=5 \\ 5x+2y+z=0 \\ 2x+3y-2z=3 \end{cases}$

44. $\begin{cases} x+\frac{5}{2}y+z=-2 \\ 2x+2y+\frac{1}{2}z=-4 \\ \frac{1}{3}x-y-z=1 \end{cases}$

Answer

$(-3,2,-4)$

45. $\begin{cases} 5x+3y=-6 \\ 2y+3z=-1 \\ 7x+z=1 \end{cases}$

46. $\begin{cases} 2x+3y+z=12 \\ x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

Answer

no solution

47. $\begin{cases} -x-3y+2z=14 \\ -x+2y-3z=-4 \\ 3x+y-2z=6 \end{cases}$

Solve Applications using Systems of Linear Equations with Three Variables

48. After attending a major league baseball game, the patrons often purchase souvenirs. If a family purchases 4 t-shirts, a cap and 1 stuffed animal their total is $135. A couple buys 2 t-shirts, a cap and 3 stuffed animals for their nieces and spends $115. Another couple buys 2 t-shirts, a cap and 1 stuffed animal and their total is $85. What is the cost of each item?

Answer

$25, 20, 15$

Solve Systems of Equations Using Matrices

Write the Augmented Matrix for a System of Equations.

Write each system of linear equations as an augmented matrix.

49. $\begin{cases} 3x - y = -1 \\ -2x + 2y = 5 \end{cases}$

50. $\begin{cases} 4x + 3y = -2 \\ x - 2y - 3z = 7 \\ 2x - y + 2z = -6 \end{cases}$

Answer

$$\begin{bmatrix} 4 & 3 & 0 & -2 \\ 1 & -2 & -3 & 7 \\ 2 & -1 & 2 & -6 \end{bmatrix}$$

Write the system of equations that that corresponds to the augmented matrix.

51. $\left[\begin{array}{cc|c} 2 & -4 & -2 \\ 3 & -3 & -1 \end{array}\right]$

52. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 1 & -2 & 0 & -2 \\ 0 & -1 & 2 & 3 \end{array}\right]$

Answer

$$\begin{cases} x - 3z = -1 \\ x - 2y = -27 \\ -y + 2z = 3 \end{cases}$$

In the following exercises, perform the indicated operations on the augmented matrices.

53. $\left[\begin{array}{cc|c} 4 & -6 & -3 \\ 3 & 2 & 1 \end{array}\right]$

ⓐ Interchange rows 2 and 1.
ⓑ Multiply row 1 by 4.
ⓒ Multiply row 2 by 3 and add to row 1.

54. $\left[\begin{array}{ccc|c} 1 & -3 & -2 & 4 \\ 2 & 2 & -1 & -3 \\ 4 & -2 & -3 & -1 \end{array}\right]$

ⓐ Interchange rows 2 and 3.
ⓑ Multiply row 1 by 2.
ⓒ Multiply row 3 by $-2-2$ and add to row 2.

Answer

ⓐ $\begin{bmatrix} 1 & -3 & -2 & 4 \\ 4 & -2 & -3 & -1 \\ 2 & 2 & -1 & -3 \end{bmatrix}$

ⓑ $\begin{bmatrix} 2 & -6 & -4 & 8 \\ 4 & -2 & -3 & -1 \\ 2 & 2 & -1 & -3 \end{bmatrix}$

ⓒ $\begin{bmatrix} 2 & -6 & -4 & 8 \\ 4 & -2 & -3 & -1 \\ 0 & -6 & -1 & 5 \end{bmatrix}$

Solve Systems of Equations Using Matrices

In the following exercises, solve each system of equations using a matrix.

55. $\begin{cases} 4x+y=6 \\ x-y=4 \end{cases}$

56. $\begin{cases} 2x-y+3z=-3 \\ -x+2y-z=10 \\ x+y+z=5 \end{cases}$

Answer

$(-2, 5, -2)$

57. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

58. $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$

Answer

no solution

59. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

Solve Systems of Equations Using Determinants

Evaluate the Determinant of a 2 × 2 Matrix

In the following exercise, evaluate the determinate of the square matrix.

60. $\begin{bmatrix} 8 & -4 \\ 5 & -3 \end{bmatrix}$

Answer

-4

Evaluate the Determinant of a 3 × 3 Matrix

In the following exercise, find and then evaluate the indicated minors.

61. $\begin{vmatrix} -1 & -3 & 2 \\ 4 & -2 & -1 \\ -2 & 0 & -3 \end{vmatrix}$; Find the minor ⓐ a_1 ⓑ b_1 ⓒ c_2

In the following exercise, evaluate each determinant by expanding by minors along the first row.

62. $\begin{vmatrix} -2 & -3 & -4 \\ 5 & -6 & 7 \\ -1 & 2 & 0 \end{vmatrix}$

Answer

21In the following exercise, evaluate each determinant by expanding by minors.

63. $\begin{vmatrix} 3 & 5 & 4 \\ -1 & 3 & 0 \\ -2 & 6 & 1 \end{vmatrix}$

Use Cramer's Rule to Solve Systems of Equations

In the following exercises, solve each system of equations using Cramer's rule

64. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Answer

$(-3, 2)$

65. $\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$

66. $\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$

Answer

$(-3, 2, 3)$

67. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

68. $\begin{cases} 3x+4y-3z=-2 \\ 2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$

Answer

inconsistent

Solve Applications Using Determinants

In the following exercises, determine whether the given points are collinear.

69. $(0, 2)$, $(-1, -1)$, and $(-2, 4)$

Graphing Systems of Linear Inequalities

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

70. $\begin{cases} 4x + y > 6 \\ 3x - y \leq 12 \end{cases}$

ⓐ $(2, -1)$
ⓑ $(3, -2)$

Answer

ⓐ yes ⓑ no

71. $\begin{cases} y > \frac{1}{3}x + 2 \\ x - \frac{1}{4}y \leq 10 \end{cases}$

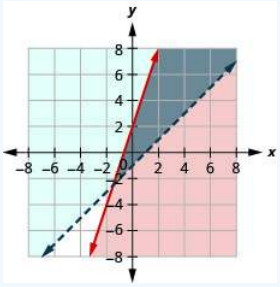
ⓐ $(6, 5)$
ⓑ $(15, 8)$

Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

72. $\begin{cases} y < 3x + 1 \\ y \geq -x - 2 \end{cases}$

Answer

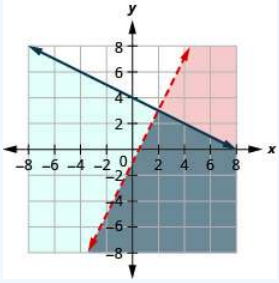


The solution is the grey region.

73. $\begin{cases} x - y > -1 \\ y < \frac{1}{3}x - 2 \end{cases}$

74. $\begin{cases} 2x - 3y < 6 \\ 3x + 4y \geq 12 \end{cases}$

Answer

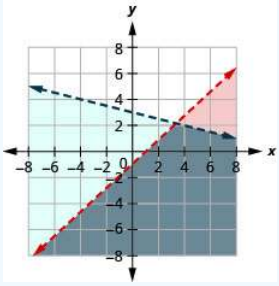


The solution is the grey region.

75. $\begin{cases} y \leq -\frac{3}{4}x + 1 \\ x \geq -5 \end{cases}$

76. $\begin{cases} x + 3y < 5 \\ y \geq -\frac{1}{3}x + 6 \end{cases}$

Answer



No solution.

77. $\begin{cases} y \geq 2x - 5 \\ -6x + 3y > -4 \end{cases}$

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

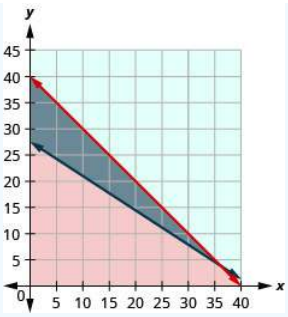
78. Roxana makes bracelets and necklaces and sells them at the farmers' market. She sells the bracelets for $12 each and the necklaces for $18 each. At the market next weekend she will have room to display no more than 40 pieces, and she needs to sell at least $500 worth in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Should she display 26 bracelets and 14 necklaces?
ⓓ Should she display 39 bracelets and 1 necklace?

Answer

ⓐ $\begin{cases} b \geq 0 \\ n \geq 0 \\ b + n \leq 40 \\ 12b + 18n \geq 500 \end{cases}$

ⓑ



ⓒ yes
ⓓ no

79. Annie has a budget of $600 to purchase paperback books and hardcover books for her classroom. She wants the number of hardcover to be at least 5 more than three times the number of paperback books. Paperback books cost $4 each and hardcover books cost $15 each.

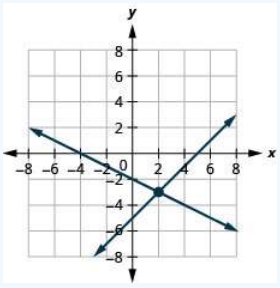
ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 8 paperback books and 40 hardcover books?
ⓓ Can she buy 10 paperback books and 37 hardcover books?

Chapter Practice Test

In the following exercises, solve the following systems by graphing.

1. $\begin{cases} x-y=5 \\ x+2y=-4 \end{cases}$

Answer



$(2,-3)$

2. $\begin{cases} x-y>-2 \\ y\leq 3x+1 \end{cases}$

In the following exercises, solve each system of equations. Use either substitution or elimination.

3. $\begin{cases} x+4y=6 \\ -2x+y=-3 \end{cases}$

Answer

$(2,1)$

4. $\begin{cases} -3x+4y=2 \\ 5x-5y=-23 \end{cases}$

5. $\begin{cases} x+y-z=-1 \\ 2x-y+2z=8 \\ -3x+2y+z=-9 \end{cases}$

Answer

$(2,-2,1)$

Solve the system of equations using a matrix.

6. $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

7. $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

Answer

$(5,7,4)$

Solve using Cramer's rule.

8. $\begin{cases} 3x+y=-3 \\ 2x+3y=6 \end{cases}$

9. Evaluate the determinant by expanding by minors:

$$\begin{vmatrix} 3 & -2 & -2 \\ 2 & -1 & 4 \\ -1 & 0 & -3 \end{vmatrix}$$

Answer

99

In the following exercises, translate to a system of equations and solve.

10. Greg is paddling his canoe upstream, against the current, to a fishing spot 10 miles away. If he paddles upstream for 2.5 hours and his return trip takes 1.25 hours, find the speed of the current and his paddling speed in still water.

11. A pharmacist needs 20 liters of a 2% saline solution. He has a 1% and a 5% solution available. How many liters of the 1% and how many liters of the 5% solutions should she mix to make the 2% solution?

Answer

15 liters of 1% solution, 5 liters of 5% solution

12. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4,500 in interest in one year?

13. The church youth group is selling snacks to raise money to attend their convention. Amy sold 2 pounds of candy, 3 boxes of cookies and 1 can of popcorn for a total sales of $65. Brian sold 4 pounds of candy, 6 boxes of cookies and 3 cans of

popcorn for a total sales of $140. Paulina sold 8 pounds of candy, 8 boxes of cookies and 5 can of popcorn for a total sales of $250. What is the cost of each item?

Answer

The candy cost $20; the cookies cost $5; and the popcorn cost $10.

14. The manufacturer of a granola bar spends $1.20 to make each bar and sells them for $2. The manufacturer also has fixed costs each month of $8,000.

ⓐ Find the cost function C when x granola bars are manufactured
ⓑ Find the revenue function R when x granola bars are sold.
ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.
ⓓ Find the break-even point. Interpret what the break-even point means.

15. Translate to a system of inequalities and solve.

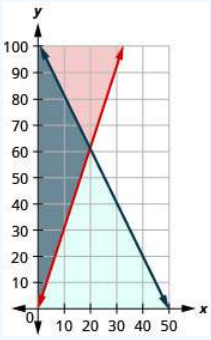
Andi wants to spend no more than $50 on Halloween treats. She wants to buy candy bars that cost $1 each and lollipops that cost $0.50 each, and she wants the number of lollipops to be at least three times the number of candy bars.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 20 candy bars and 40 lollipops?

Answer

ⓐ $$\begin{cases} C \geq 0 \\ L \geq 0 \\ C + 0.5L \leq 50 \\ L \geq 3C \end{cases}$$

ⓑ



ⓒ no
ⓓ yes

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CHAPTER OVERVIEW

5: Polynomial and Polynomial Functions

In this chapter you will investigate polynomials and polynomial functions and learn how to perform mathematical operations on them.

5.0: Prelude to Polynomial and Polynomial Functions
5.1: Add and Subtract Polynomials
5.1E: Exercises
5.2: Properties of Exponents and Scientific Notation
5.2E: Exercises
5.3: Multiply Polynomials
5.3E: Exercises
5.4: Dividing Polynomials
5.4E: Exercises
Chapter 5 Review Exercises

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5.0: Prelude to Polynomial and Polynomial Functions

You may have coins and paper money in your wallet, but you may soon want to acquire a type of currency called bitcoins. They exist only in a digital wallet on your computer. You can use bitcoins to pay for goods at some companies, or save them as an investment. Although the future of bitcoins is uncertain, investment brokers are beginning to investigate ways to make business predictions using this digital currency. Understanding how bitcoins are created and obtained requires an understanding of a type of function known as a polynomial function. In this chapter you will investigate polynomials and polynomial functions and learn how to perform mathematical operations on them.



Figure 1: There are many different kinds of coins in circulation, but a new type of coin exists only in the virtual world. It is the bitcoin.

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5.1: Add and Subtract Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Determine the degree of polynomials
- Add and subtract polynomials
- Evaluate a polynomial function for a given value
- Add and subtract polynomial functions

Note

Before you get started, take this readiness quiz.

1. Simplify: $3x^2+3x+1+8x^2+5x+5$.
   If you missed this problem, review [link].
2. Subtract: $(5n+8)-(2n-1)$.
   If you missed this problem, review [link].
3. Evaluate: $4xy^2$ when $x=-2x$ and $y=5$..
   If you missed this problem, review [link].

Determine the Degree of Polynomials

We have learned that a *term* is a constant or the product of a constant and one or more variables. A **monomial** is an algebraic expression with one term. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial in one variable. Some examples of monomial in one variable are. Monomials can also have more than one variable such as and $-4a^2b^3c^2$.

Definition: MONOMIAL

A *monomial* is an algebraic expression with one term. A monomial in one variable is a term of the form ax^m, where a is a constant and m is a whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a **polynomial**. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a **trinomial** has exactly three terms. There are no special names for polynomials with more than three terms.

Definition: POLYNOMIALS

- *polynomial*—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
- *monomial*—A polynomial with exactly one term is called a monomial.
- *binomial*—A polynomial with exactly two terms is called a binomial.
- *trinomial*—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$y+1$	$4a^2-7ab+2b^2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	$-13a^3b^2c$
Binomial	$a+7ba+7b$	$4x^2-y^2$	y^2-16	$3p^3q-9p^2q$
Trinomial	$x^2-7x+12$	$9m^2+2mn-8n^2$	$6k^4-k^3+8k$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special

polynomials and just call all the rest *polynomials*.

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable. A monomial that has no variable, just a constant, is a special case. The **degree of a constant** is 0.

Definition: DEGREE OF A POLYNOMIAL

- The *degree of a term* is the sum of the exponents of its variables.
- The *degree of a constant* is 0.
- The *degree of a polynomial* is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms. Let's start by looking at a monomial. The monomial $8ab^2$ has two variables a and b. To find the degree we need to find the sum of the exponents. The variable a doesn't have an exponent written, but remember that means the exponent is 1. The exponent of b is 2. The sum of the exponents, 1+2,1+2, is 3 so the degree is 3.



Here are some additional examples.

Monomials	14	$8ab^2$	$-9x^3y^5$	$-13a$
Degree	0	3	8	1
Binomial	$h+7$	$7b^2-3b$	x^2y^2-25	$4n^3-8n^2$
Degree of each term	1 0	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-12x+27$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$y-1$	$3y^2-2y-5$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form of a polynomial**. Get in the habit of writing the term with the highest degree first.

Example 5.1.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $7y2-5y+3$
b. $-2a^4b^2$
c. $3x5-4x3-6x2+x-8$
d. $2y-8xy^3$
e. 15

Answer

	Polynomial	Number of terms	Type	Degree of terms	Degree of polynomial

	Polynomial	Number of terms	Type	Degree of terms	Degree of polynomial
ⓐ	$7y^2-5y+3$	3	Trinomial	2, 1, 0	2
ⓑ	$-2a^4b^2-2a^4b^2$	1	Monomial	4, 2	6
ⓒ	$3x5-4x3-6x2+x-8$	5	Polynomial	5, 3, 2, 1, 0	5
ⓐ	$7y^2-5y+3$	3	Trinomial	2, 1, 0	2
ⓓ	$2y-8xy^3$	2	Binomial	1, 4	4
ⓑ	$-2a^4b^2-2a^4b^2$	1	Monomial	4, 2	6
ⓔ	15	1	Monomial	0	0
ⓒ	$3x5-4x3-6x2+x-8$	5	Polynomial	5, 3, 2, 1, 0	5
ⓓ	$2y-8xy^3$	2	Binomial	1, 4	4
ⓔ	15	1	Monomial	0	0

? Example 5.1.2

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. -5
b. $8y^3-7y^2-y-3$
c. $-3x^2y-5xy+9xy^3$
d. $81m^2-4n^2$
e. $-3x^6y^3z$

Answer a

monomial, 0

Answer b

polynomial, 3

Answer c

trinomial, 3

Answer d

binomial, 2

Answer b

monomial, 10

? Example 5.1.3

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $64k^3-8$
b. $9m^3+4m^2-2$
c. 56
d. $8a^4-7a^3b-6a^2b^2-4ab^3+7b^4$
e. $-p^4q^3$

Answer

ⓐbinomial, 3 ⓑ trinomial, 3 ⓒ monomial, 0 ⓓ polynomial, 4 ⓔ monomial, 7

Add and Subtract Polynomials

We have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficients.

Example 5.1.4

Add or subtract:

a. $25y^2+15y^2$
b. $16pq^3-(-7pq^3)$.

Answer a

	$25y^2+15y^2$
Combine like terms.	$40y^2$

Answer b

	$16pq^3-(-7pq^3)$
Combine like terms.	$23pq^3$

Example 5.1.5

Add or subtract:

a. $12q^2+9q^2$
b. $8mn^3-(-5mn^3)$.

Answer

ⓐ $21q^2$ ⓑ $13mn^3$

Example 5.1.6

Add or subtract:

a. $-15c^2+8c^2$
b. $-15y^2z^3-(-5y^2z^3)$

Answer

ⓐ $-7c^2$ ⓑ $-10y^2z^3$

Remember that like terms must have the same variables with the same exponents.

Example 5.1.7

Simplify:

a. $a^2+7b^2-6a^2$
b. $u^2v+5u^2-3v^2$

Answer

ⓐ Combine like terms.

$$a^2+7b^2-6a^2 \;=\; -5a^2+7b^2$$

ⓑ There are no like terms to combine. In this case, the polynomial is unchanged.

$$u^2v+5u^2-3v^2$$

Example 5.1.8

Add:

a. $8y^2+3z^2-3y^2$
b. $m^2n^2-8m^2+4n^2$

Answer

ⓐ $5y^2+3z^2$
ⓑ $m^2n^2-8m^2+4n^2$

Example 5.1.9

Add:

a. $3m^2+n^2-7m^2$
b. $pq^2-6p-5q^2$

Answer

ⓐ $-4m^2+n^2$
ⓑ $pq^2-6p-5q^2$

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example 5.1.10

Find the sum: $(7y^2-2y+9)+(4y^2-8y-7)$.

Answer

Identify like terms.	$(\underline{\underline{7y^2}}-\underline{2y}+9)+(\underline{\underline{4y^2}}-\underline{8y}-7)$
Rewrite without the parentheses, rearranging to get the like terms together.	$\underline{\underline{7y^2+4y^2}}-\underline{2y-8y}+9-7$
Combine like terms.	$11y^2-10y+2$

Example 5.1.11

Find the sum: $(7x^2-4x+5)+(x^2-7x+3)$

Answer

$8x^2-11x+8$

Example 5.1.12

Find the sum: $(14y^2+6y-4)+(3y^2+8y+5)$

Answer

$17y^2+14y+1$

Be careful with the signs as you distribute while subtracting the polynomials in the next example.

Example 5.1.13

Find the difference: $(9w^2-7w+5)\ -\ (2w^2-4)$

Answer

	$(9w^2-7w+5)\ -\ (2w^2-4)$
Distribute and identify like terms.	$\underline{\underline{9w^2}}-\underline{7w}+5-\underline{\underline{2w^2}}+4$
Rearrange the terms.	$\underline{\underline{9w^2-2w^2}}-\underline{7w}+5+4$
Combine like terms.	$7w^2-7w+9$

Example 5.1.14

Find the difference: $(8x^2+3x-19)\ -\ (7x^2-14)$

Answer

x^2+3x-5

Example 5.1.15

Find the difference: $(9b^2-5b-4)\ -\ (3b^2-5b-7)$

Answer

$6b^2+3$

Example 5.1.16

Subtract $(p^2+10pq-2q^2)$ from (p^2+q^2).

Answer

	$(p^2+q^2)\ -\ (p^2+10pq-2q^2)$
Distribute and identify like terms.	$\underline{\underline{p^2}}+\underline{q^2}-\underline{\underline{p^2}}-10pq+\underline{2q^2}$
Rearrange the terms, putting like terms together.	$\underline{\underline{p^2-p^2}}-10pq+\underline{q^2+2q^2}$
Combine like terms.	$-10pq+3q^2$

Example 5.1.17

Subtract $(a^2+5ab-6b^2)$ from (a^2+b^2)

Answer

$-5ab+7b^2$

Example 5.1.18

Subtract $(m^2-7mn-3n^2)$ from (m^2+n^2).

Answer

7mn+4n^2

Example 5.1.19

Find the sum: $(u^2-6uv+5v^2)+(3u^2+2uv)$

Answer

	$(u^2-6uv+5v^2)+(3u^2+2uv)$
Distribute and identify like terms.	$\underline{\underline{u^2}}-\underline{6uv}+5v^2+\underline{\underline{\underline{3u^2}}}+\underline{2uv}$
Rearrange the terms to put like terms together.	$\underline{\underline{u^2}}+\underline{\underline{\underline{3u^2}}}-\underline{6uv}+\underline{2uv}+5v^2$
Combine like terms.	$4u^2-4uv+5v^2$

Example 5.1.20

Find the sum: $(3x^2-4xy+5y^2)+(2x^2-xy)$

Answer

$5x^2-5xy+5y^2$

Example 5.1.21

Find the sum: $(2x^2-3xy-2y^2)+(5x^2-3xy)$

Answer

$7x^2-6xy-2y^2$

When we add and subtract more than two polynomials, the process is the same.

Example 5.1.22

Simplify: $(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$

Answer

	$(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$
Distribute	$a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Rearrange the terms to put like terms together.	$a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	a^3-b^3

Example 5.1.23

Simplify: $(x^3-x^2y)-(xy^2+y^3)+(x^2y+xy^2)$

Answer

x^3+y^3

Example 5.1.24

Simplify: $(p^3 - p^2q) + (pq^2 + q^3) - (p^2q + pq^2)$

Answer

$p^3 - 3p^2q + q^3$

Evaluate a Polynomial Function for a Given Value

A **polynomial function** is a function defined by a polynomial. For example, $f(x) = x^2 + 5x + 6$ and $g(x) = 3x - 4$ are polynomial functions, because $x^2 + 5x + 6$ and $3x - 4$ are polynomials.

Definition: POLYNOMIAL FUNCTION

A *polynomial function* is a function whose range values are defined by a polynomial.

In Graphs and Functions, where we first introduced functions, we learned that evaluating a function means to find the value of $f(x)$ for a given value of x. To evaluate a polynomial function, we will substitute the given value for the variable and then simplify using the order of operations.

Example 5.1.25

For the function $f(x) = 5x^2 - 8x + 4$ find:

a. $f(4)$
b. $f(-2)$
c. $f(0)$.

Answer

ⓐ

	$f(x) = 5x^2 - 8x + 4$
To find $f(4)$, substitute 4 for x.	$f(4) = 5(4)^2 - 8(4) + 4$
Simplify the exponents.	$f(4) = 5 \cdot 16 - 8(4) + 4$
Multiply.	$f(4) = 80 - 32 + 4$
Simplify.	$f(4) = 52$

ⓑ

	$f(x) = 5x^2 - 8x + 4$
To find $f(-2)$, substitute -2 for x.	$f(-2) = 5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$f(-2) = 5 \cdot 4 - 8(-2) + 4$
Multiply.	$f(-2) = 20 + 16 + 4$
Simplify.	$f(-2) = 40$

ⓒ

	$f(x) = 5x^2 - 8x + 4$
To find $f(0)$, substitute 0 for x.	$f(0) = 5(0)^2 - 8(0) + 4$
Simplify the exponents.	$f(0) = 5 \cdot 0 - 8(0) + 4$

Multiply.	$f(0) = 0 + 0 + 4$

Example 5.1.26

For the function $f(x) = 3x^2 + 2x - 15$, find

a. $f(3)$
b. $f(-5)$
c. $f(0)$.

Answer

ⓐ 18 ⓑ 50 ⓒ -15

Example 5.1.27

For the function $g(x) = 5x^2 - x - 4$, find

a. $g(-2)$
b. $g(-1)$
c. $g(0)$.

Answer

ⓐ 20 ⓑ 2 ⓒ -4

The polynomial functions similar to the one in the next example are used in many fields to determine the height of an object at some time after it is projected into the air. The polynomial in the next function is used specifically for dropping something from 250 ft.

Example 5.1.28

The polynomial function $h(t) = -16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 2$ seconds.

Answer

$$h(t) = -16t^2 + 250$$

To find $h(2)$, substitute $t = 2$.	$h(2) = -16(2)^2 + 250$
Simplify.	$h(2) = -16 \cdot 4 + 250$
Simplify.	$h(2) = -64 + 250$
Simplify.	$h(2) = 186$
	After 2 seconds the height of the ball is 186 feet.

Example 5.1.29

The polynomial function $h(t) = -16t^2 + 150$ gives the height of a stone t seconds after it is dropped from a 150-foot tall cliff. Find the height after $t = 0$ seconds (the initial height of the object).

Answer

The height is 150 feet.

Example 5.1.30

The polynomial function $h(t) = -16t^2 + 175$ gives the height of a ball t seconds after it is dropped from a 175-foot tall bridge. Find the height after $t = 3$ seconds.

Answer

The height is 31 feet.

Add and Subtract Polynomial Functions

Just as polynomials can be added and subtracted, polynomial functions can also be added and subtracted.

Definition: ADDITION AND SUBTRACTION OF POLYNOMIAL FUNCTIONS

For functions $f(x)$ and $g(x)$,

$$(f+g)(x) = f(x) + g(x) \tag{5.1.1}$$

$$(f-g)(x) = f(x) - g(x) \tag{5.1.2}$$

Example 5.1.31

For functions $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$, find:

a. $(f+g)(x)$
b. $(f+g)(3)$
c. $(f-g)(x)$
d. $(f-g)(-2)$.

Answer

ⓐ

	$(f+g)(x) = f(x) + g(x)$
Substitute $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$.	$(f+g)(x) = (3x^2 - 5x + 7) + (x^2 - 4x - 3)$
Rewrite without the parentheses.	$(f+g)(x) = 3x^2 - 5x + 7 + x^2 - 4x - 3$
Put like terms together.	$(f+g)(x) = 3x^2 + x^2 - 5x - 4x + 7 - 3$
Combine like terms.	$(f+g)(x) = 4x^2 - 9x + 4$

ⓑ In part (a) we found $(f+g)(x)$ and now are asked to find $(f+g)(3)$.

$$(f+g)(x) = 4x^2 - 9x + 4$$

To find $(f+g)(3)$, substitute $x = 3$. $(f+g)(3) = 4(3)^2 - 9 \cdot 3 + 4$

$$(f+g)(3) = 4 \cdot 9 - 9 \cdot 3 + 4$$

$$(f+g)(3) = 36 - 27 + 4$$

Notice that we could have found $(f+g)(3)$ by first finding the values of $f(3)$ and $g(3)$ separately and then adding the results.

Find $f(3)$.	$f(x) = 3x^2 - 5x + 7$

	$f(3) = 3(3)^2 - 5(3) + 7$
	$f(3) = 19$
Find $g(3)$.	$g(x) = x^2 - 4x - 3$
	$g(3) = 3^2 - 4(3) - 3$
	$g(3) = -6$
Find $(f+g)(3)$.	$(f+g)(x) = f(x) + g(x)$
	$(f+g)(3) = f(3) + g(3)$
Substitute $f(3) = 19$ and $g(3) = -6$.	$(f+g)(3) = 19 + (-6)$
	$(f+g)(3) = 13$

ⓒ

	$(f-g)(x) = f(x) - g(x)$
Substitute $f(x) = 3x^2 - 5x + 7$ and $g(x) = x^2 - 4x - 3$.	$(f-g)(x) = (3x^2 - 5x + 7) - (x^2 - 4x - 3)$
Rewrite without the parentheses.	$(f-g)(x) = 3x^2 - 5x + 7 - x^2 + 4x + 3$
Put like terms together.	$(f-g)(x) = 3x^2 - x^2 - 5x + 4x + 7 + 3$
Combine like terms.	$(f-g)(x) = 2x^2 - x + 10$

ⓓ

	$(f-g)(x) = 2x^2 - x + 10$
To find $(f-g)(-2)$, substitute $x = -2$.	$(f-g)(-2) = 2(-2)^2 - (-2) + 10$
	$(f-g)(-2) = 2 \cdot 4 - (-2) + 10$
	$(f-g)(-2) = 20$

? Example 5.1.32

For functions $f(x) = 2x^2 - 4x + 3$ and $g(x) = x^2 - 2x - 6$, find: ⓐ $(f+g)(x)$ ⓑ $(f+g)(3)$ ⓒ $(f-g)(x)$ ⓓ $(f-g)(-2)$.

Answer

ⓐ $(f+g)(x) = 3x^2 - 6x - 3$

ⓑ $(f+g)(3) = 6$

ⓒ $(f-g)(x) = x^2 - 2x + 9$

ⓓ $(f-g)(-2) = 17$

? Example 5.1.33

For functions $f(x) = 5x^2 - 4x - 1$ and $g(x) = x^2 + 3x + 8$, find ⓐ $(f+g)(x)$ ⓑ $(f+g)(3)$ ⓒ $(f-g)(x)$ ⓓ $(f-g)(-2)$.

Answer

ⓐ $(f+g)(x) = 6x^2 - x + 7$

ⓑ $(f+g)(3)=58$

ⓒ $(f-g)(x)=4x^2-7x-9$

ⓓ $(f-g)(-2)=21$

Access this online resource for additional instruction and practice with adding and subtracting polynomials.

- Adding and Subtracting Polynomials

Key Concepts

- **Monomial**
  - A **monomial** is an algebraic expression with one term.
  - A monomial in one variable is a term of the form axm,axm, where *a* is a constant and *m* is a whole number.
- **Polynomials**
  - **Polynomial**—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
  - **monomial** —A polynomial with exactly one term is called a monomial.
  - **binomial** — A polynomial with exactly two terms is called a binomial.
  - **trinomial** —A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

Glossary

binomial

A binomial is a polynomial with exactly two terms.

degree of a constant

The degree of any constant is 0.

degree of a polynomial

The degree of a polynomial is the highest degree of all its terms.

degree of a term

The degree of a term is the sum of the exponents of its variables.

monomial

A monomial is an algebraic expression with one term. A monomial in one variable is a term of the form axm,axm, where *a* is a constant and *m* is a whole number.

polynomial

A monomial or two or more monomials combined by addition or subtraction is a polynomial.

standard form of a polynomial

A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial

A trinomial is a polynomial with exactly three terms.

polynomial function

A polynomial function is a function whose range values are defined by a polynomial.

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5.1E: Exercises

Practice Makes Perfect

Determine the Type of Polynomials

In the following exercises, determine if the polynomial is a **monomial**, **binomial**, **trinomial**, or **other polynomial**. Also give the **degree** of each polynomial.

1. ⓐ $47x^5-17x^2y^3+y^2$
ⓑ $5c^3+11c^2-c-8$
ⓒ $59ab+13b$
ⓓ 4
ⓔ $4pq+17$

Answer

ⓐ trinomial, degree 5
ⓑ other polynomial, degree 3
ⓒ binomial, degree 2
ⓓ monomial, degree 0
ⓔ binomial, degree 2

2. ⓐ x^2-y^2
ⓑ $-13c^4$
ⓒ $a^2+2ab-7b^2$
ⓓ $4x^2y^2-3xy+8$
ⓔ 19

3. ⓐ $8y-5x$
ⓑ $y^2-5yz-6z^2$
ⓒ $y^3-8y^2+2y-16$
ⓓ $81ab^4-24a^2b^2+3b$
ⓔ -18

Answer

ⓐ binomial, degree 1
ⓑ trinomial, degree 2
ⓒ other polynomial, degree 3
ⓓ trinomial, degree 5
ⓔ monomial, degree 0

4. ⓐ $11y^2$
ⓑ -73
ⓒ $6x^2-3xy+4x-2y+y^2$
ⓓ $4y^2+17z^2$
ⓔ $5c^3+11c^2-c-8$

5. ⓐ $5a^2+12ab-7b^2$
ⓑ $18xy^2z$
ⓒ $5x+2$
ⓓ $y^3-8y^2+2y-16$
ⓔ -24

Answer

ⓐ trinomial, degree 2
ⓑ monomial, degree 4
ⓒ binomial, degree 1
ⓓ other polynomial, degree 3
ⓔ monomial, degree 0

6. ⓐ $9y^3 - 10y^2 + 2y - 6$
ⓑ $-12p^3q$
ⓒ $a^2 + 9ab + 18b^2$
ⓓ $20x^2y^2 - 10a^2b^2 + 30$
ⓔ 17

7. ⓐ $14s - 29t$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2z + 2yz^2 - 16z^3$
ⓓ $23ab^2 - 14$
ⓔ -3

Answer

ⓐ binomial, degree 1
ⓑ trinomial, degree 2
ⓒ other polynomial, degree 3
ⓓ binomial, degree 3
ⓔ monomial, degree 0

8. ⓐ $15xy$
ⓑ 15
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $10p - 9q$
ⓔ $m^4 + 4m^3 + 6m^2 + 4m + 1$

Add and Subtract Polynomials

In the following exercises, add or subtract the monomials.

9. ⓐ $7x^2 + 5x^2$
ⓑ $4a - 9a$

Answer

ⓐ $12x^2$ ⓑ $-5a$

10. ⓐ $4y^3 + 6y^3$
ⓑ $-y - 5y$

11. ⓐ $-12w + 18w$
ⓑ $7x^2y - (-12x^2y)$

Answer

ⓐ $6w$
ⓑ $19x^2y$

12. ⓐ $-3m+9m$
ⓑ $15yz^2-(-8yz^2)$

13. $7x^2+5x^2+4a-9a$

Answer

$12x^2-5a$

14. $4y^3+6y^3-y-5y$

15. $-12w+18w+7x^2y-(-12x^2y)$

Answer

$6w+19x^2y$

16. $-3m+9m+15yz^2-(-8yz^2)$

17. ⓐ $-5b-17b$
ⓑ $3xy-(-8xy)+5xy$

Answer

ⓐ $-22b$
ⓑ $16xy$

18. ⓐ $-10x-35x$
ⓑ $17mn^2-(-9mn^2)+3mn^2$

19. ⓐ $12a+5b-22a$
ⓑ $pq^2-4p-3q^2$

Answer

ⓐ $-10a+5b$
ⓑ $pq^2-4p-3q^2$

20. ⓐ $14x-3y-13x$
ⓑ $a^2b-4a-5ab^2$

21. ⓐ $2a^2+b^2-6a^2$
ⓑ $x^2y-3x+7xy^2$

Answer

ⓐ $-4a^2+b^2$
ⓑ $x^2y-3x+7xy^2$

22. ⓐ $5u^2+4v^2-6u^2$
ⓑ $12a+8b$

23. ⓐ $xy^2-5x-5y^2$
ⓑ $19y+5z$

Answer

ⓐ $xy^2-5x-5y^2$
ⓑ $19y+5z$

24. $12a+5b-22a+pq^2-4p-3q^2$

25. $14x-3y-13x+a^2b-4a-5ab^2$

Answer

$x-3y+a^2b-4a-5ab^2$

26. $2a^2+b^2-6a^2+x^2y-3x+7xy^2$

27. $5u^2+4v^2-6u^2+12a+8b$

Answer

$-u^2+4v^2+12a+8b$

28. $xy^2-5x-5y^2+19y+5z$

29. Add: $4a, -3b, -8a$

Answer

$-4a-3b$

30. Add: $4x, 3y, -3x$

31. Subtract $5x^6$ from $-12x^6$

Answer

$-7x^6$

32. Subtract $2p^4$ from $-7p^4$

In the following exercises, add the polynomials.

33. $(5y^2+12y+4)+(6y^2-8y+7)$

Answer

$11y^2+4y+11$

34. $(4y^2+10y+3)+(8y^2-6y+5)$

35. $(x^2+6x+8)+(-4x^2+11x-9)$

Answer

$-3x^2+17x-1$

36. $(y^2+9y+4)+(-2y^2-5y-1)$

37. $(8x^2-5x+2)+(3x^2+3)$

Answer

$11x^2-5x+5$

38. $(7x^2-9x+2)+(6x^2-4)$

39. $(5a^2+8)+(a^2-4a-9)$

Answer

$6a^2-4a-1$

40. $(p^2-6p-18)+(2p^2+11)$

In the following exercises, subtract the polynomials.

41. $(4m^2-6m-3)-(2m^2+m-7)$

Answer

$2m^2-7m+4$

42. $(3b^2-4b+1)-(5b^2-b-2)$

43. $(a^2+8a+5)-(a^2-3a+2)$

Answer

$11a+3$

44. $(b^2-7b+5)-(b^2-2b+9)$

45. $(12s^2-15s)-(s-9)$

Answer

$12s^2-14s+9$

46. $(10r^2-20r)-(r-8)$

In the following exercises, subtract the polynomials.

47. Subtract $(9x^2+2)$ from $(12x^2-x+6)$

Answer

$3x^2-x+4$

48. Subtract $(5y^2-y+12)$ from $(10y^2-8y-20)$

49. Subtract $(7w^2-4w+2)$ from $(8w^2-w+6)$

Answer

w^2+3w+4

50. Subtract $(5x^2 - x + 12)$ from $(9x^2 - 6x - 20)$

In the following exercises, find the difference of the polynomials.

51. Find the difference of $(w^2 + w - 42)$ and $(w^2 - 10w + 24)$

Answer

$11w - 64$

52. Find the difference of $(z^2 - 3z - 18)$ and $(z^2 + 5z - 20)$

In the following exercises, add the polynomials.

53. $(7x^2 - 2xy + 6y^2) + (3x^2 - 5xy)$

Answer

$10x^2 - 7xy + 6y^2$

54. $(-5x^2 - 4xy - 3y^2) + (2x^2 - 7xy)$

55. $(7m^2 + mn - 8n^2) + (3m^2 + 2mn)$

Answer

$10m^2 + 3mn - 8n^2$

56. $(2r^2 - 3rs - 2s^2) + (5r^2 - 3rs)$

In the following exercises, add or subtract the polynomials.

57. $(a^2 - b^2) - (a^2 + 3ab - 4b^2)$

Answer

$-3ab + 3b^2$

58. $(m^2 + 2n^2) - (m^2 - 8mn - n^2)$

59. $(p^3 - 3p^2q) + (2pq^2 + 4q^3) - (3p^2q + pq^2)$

Answer

$p^3 - 6p^2q + pq^2 + 4q^3$

60. $(a^3 - 2a^2b) + (ab^2 + b^3) - (3a^2b + 4ab^2)$

61. $(x^3 - x^2y) - (4xy^2 - y^3) + (3x^2y - xy^2)$

Answer

$x^3 + 2x^2y - 5xy^2 + y^3$

62. $(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2)$

Evaluate a Polynomial Function for a Given Value

In the following exercises, find the function values for each polynomial function.

63. For the function $f(x) = 8x^2 - 3x + 2$, find:
ⓐ $f(5)$ ⓑ $f(-2)$ ⓒ $f(0)$

Answer

ⓐ 187 ⓑ 40 ⓒ 2

64. For the function $f(x) = 5x^2 - x - 7$, find:
ⓐ $f(-4)$ ⓑ $f(1)$ ⓒ $f(0)$

65. For the function $g(x) = 4 - 36x$, find:
ⓐ $g(3)$ ⓑ $g(0)$ ⓒ $g(-1)$

Answer

ⓐ -104 ⓑ 4 ⓒ 40

66. For the function $g(x) = 16 - 36x^2$, find:
ⓐ $g(-1)$ ⓑ $g(0)$ ⓒ $g(2)$

In the following exercises, find the height for each polynomial function.

67. A painter drops a brush from a platform 75 feet high. The polynomial function $h(t) = -16t^2 + 75$ gives the height of the brush t seconds after it was dropped. Find the height after $t = 2$ seconds.

Answer

The height is 11 feet.

68. A girl drops a ball off the cliff into the ocean. The polynomial $h(t) = -16t^2 + 200$ gives the height of a ball t seconds after it is dropped. Find the height after $t = 3$ seconds.

69. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of p dollars each is given by the polynomial function $R(p) = -4p^2 + 420p$. Find the revenue received when $p = 60$ dollars.

Answer

The revenue is $10,800.

70. A manufacturer of the latest basketball shoes has found that the revenue received from selling the shoes at a cost of p dollars each is given by the polynomial $R(p) = -4p^2 + 420p$. Find the revenue received when $p = 90$ dollars.

71. The polynomial $C(x) = 6x^2 + 90x$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and height 6 feet. Find the cost of producing a box with $x = 4$ feet.

Answer

The cost is $456.

72. The polynomial $C(x) = 6x^2 + 90x$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and height 4 feet. Find the cost of producing a box with $x = 6$ feet.

Add and Subtract Polynomial Functions

In each example, find ⓐ $(f+g)(x)$ ⓑ $(f+g)(2)$ ⓒ $(f-g)(x)$ ⓓ $(f-g)(3)$.

73. $f(x)=2x^2-4x+1$ and $g(x)=5x^2+8x+3$

Answer

ⓐ $(f+g)(x)=7x^2+4x+4$
ⓑ $(f+g)(2)=40$
ⓒ $(f-g)(x)=-3x^2-12x-2$
ⓓ $(f-g)(-3)=7$

74. $f(x)=4x^2-7x+3$ and $g(x)=4x^2+2x-1$

75. $f(x)=3x^3-x^2-2x+3$ and $g(x)=3x^3-7x$

Answer

ⓐ $(f+g)(x)=6x^3-x^2-9x+3$
ⓑ $(f+g)(2)=29$
ⓒ $(f-g)(x)=-x^2+5x+3$
ⓓ $(f-g)(-3)=-21$

76. $f(x)=5x^3-x^2+3x+4$ and $g(x)=8x^3-1$

Writing Exercises

77. Using your own words, explain the difference between a monomial, a binomial, and a trinomial.

Answer

Answers will vary.

78. Using your own words, explain the difference between a polynomial with five terms and a polynomial with a degree of 5.

79. Ariana thinks the sum $6y^2+5y^4$ is $11y^6$. What is wrong with her reasoning?

Answer

Answers will vary.

80. Is every trinomial a second-degree polynomial? If not, give an example.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
identify polynomials, monomials, binomials, and trinomials.			
determine the degree of polynomials.			
add and subtract monomials.			
add and subtract polynomials.			
evaluate a polynomial for a given value.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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5.2: Properties of Exponents and Scientific Notation

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the properties for exponents
- Use the definition of a negative exponent
- Use scientific notation

Note

Before you get started, take this readiness quiz.

1. Simplify: $(-2)(-2)(-2)$.
   If you missed this problem, review [link].
2. Simplify: $\frac{8x}{24y}$.
   If you missed this problem, review [link].
3. Name the decimal $(-2.6)(4.21)$.
   If you missed this problem, review [link].

Simplify Expressions Using the Properties for Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, in the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.

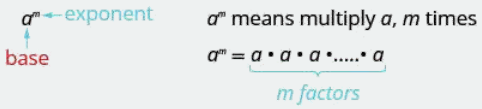
$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{\text{m factors}}$$

For example

$$(-9)^5 = \underbrace{(-9) \cdot (-9) \cdot (-9) \cdot (-9) \cdot (-9)}_{\text{5 factors}}$$

Let's review the vocabulary for expressions with exponents.

DEFINITION: EXPONENTIAL NOTATION



This is read a to the mth power.

In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.

When we combine like terms by adding and subtracting, we need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

First, we will look at an example that leads to the **Product Property**.

		$x^2 \cdot x^3$
What does this mean?		$\underbrace{x \cdot x}_{2 \text{ factors}} \cdot \underbrace{x \cdot x \cdot x}_{3 \text{ factors}}$ $\underbrace{x \cdot x}_{2 \text{factors}} \cdot \underbrace{x \cdot x \cdot x}_{3 \text{factors}}$

		x^5

Notice that 5 is the sum of the exponents, 2 and 3. We see $x_2 \cdot x_3$ is x^{2+3} or x^5.

The base stayed the same and we added the exponents. This leads to the Product Property for Exponents.

DEFINITION: PRODUCT PROPERTY FOR EXPONENTS

If a is a real number and m and n are integers, then

$$a^m \cdot a^n = a^{m+n}$$

To multiply with like bases, add the exponents.

Example 5.2.1

Simplify each expression:

a. $y^5 \cdot y^6$
b. $2^x \cdot 2^{3x}$
c. $2a^7 \cdot 3a$.

Answer

ⓐ

	$y^5 \cdot y^6$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	y^{5+6}
Simplify.	y^{11}

ⓑ

	$2^x \cdot 2^{3x}$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	2^{x+3x}
Simplify.	2^{4x}

ⓒ

	$2a^7 \cdot 3a$
Rewrite, $a = a^1$.	$2a^7 \cdot 3a^1$
Use the Commutative Property and use the Product Property, $a^m \cdot a^n = a^{m+n}$.	$2 \cdot 3 \cdot a^{7+1}$
Simplify.	$6a^8$

ⓓ

	$d^4 \cdot d^5 \cdot d^2$
Add the exponents, since bases are the same.	d^{4+5+2}
Simplify.	d^{11}

Example 5.2.2

Simplify each expression:

a. $b^9 \cdot b^8$
b. $4^{2x} \cdot 4^x$
c. $3p^5 \cdot 4p$
d. $x^6 \cdot x^4 \cdot x^8$.

Answer

ⓐ b^{17} ⓑ 4^{3x} ⓒ $12p^6$
ⓓ x^{18}

Example 5.2.3

Simplify each expression:

a. $x^{12} \cdot x4$
b. $10 \cdot 10^x$
c. $2z \cdot 6z^7$
d. $b^5 \cdot b^9 \cdot b^5$.

Answer a

x^{16}

Answer b

10^{x+1}

Answer c

$12z^8$

Answer d

b^{19}

Now we will look at an exponent property for division. As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$\frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	x^3		$\frac{1}{x}$

Notice, in each case the bases were the same and we subtracted exponents. We see $\frac{x^5}{x^2}$ is x^{5-2} or x^3. We see $\frac{x^2}{x^3}$ is or $\frac{1}{x}$. When the larger exponent was in the numerator, we were left with factors in the numerator. When the larger exponent was in the denominator, we were left with factors in the denominator--notice the numerator of 1. When all the factors in the numerator have been removed, remember this is really dividing the factors to one, and so we need a 1 in the numerator. $\frac{\cancel{x}}{\cancel{x}} = 1$. This leads to the **Quotient Property** for Exponents.

Definition: QUOTIENT PROPERTY FOR EXPONENTS

If a is a real number, $a \neq 0$, and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n}, \quad m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad n > m$$

Example 5.2.4

Simplify each expression:

a. $\frac{x^9}{x^7}$
b. $\frac{3^{10}}{3^2}$
c. $\frac{b^8}{b^{12}}$
d. $\frac{7^3}{7^5}$.

Answer

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since $9 > 7$, there are more factors of x in the numerator.	$\frac{x^9}{x^7}$
Use Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	x^{9-7}
Simplify.	x^2

ⓑ

Since $10 > 2$, there are more factors of 3 in the numerator.	$\frac{3^{10}}{3^2}$
Use Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	3^{10-2}
Simplify.	3^8

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

ⓒ

Since $12 > 8$, there are more factors of bb in the denominator.	$\frac{b^8}{b^{12}}$
Use Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	$\frac{1}{b^{12-8}}$
Simplify.	$\frac{1}{b^4}$

ⓓ

Since $5 > 3$, there are more factors of 3 in the denominator.	$\frac{7^3}{7^5}$
Use Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	$\frac{1}{7^{5-3}}$
Simplify.	$\frac{1}{7^2}$

Simplify.	$\frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

Example 5.2.5

Simplify each expression:

a. $\frac{x^{15}}{x^{10}}$
b. $\frac{6^{14}}{6^5}$
c. $\frac{x^{18}}{x^{22}}$
d. $\frac{12^{15}}{12^{30}}$.

Answer

ⓐ x^5

ⓑ 6^9

ⓒ $\frac{1}{x^4}$

ⓓ $\frac{1}{12^{15}}$

Example 5.2.6

Simplify each expression:

a. $\frac{y^{43}}{y^{37}}$
b. $\frac{10^{15}}{10^7}$
c. $\frac{m^7}{m^{15}}$
d. $\frac{9^8}{9^{19}}$.

Answer

ⓐ y_6

ⓑ 108

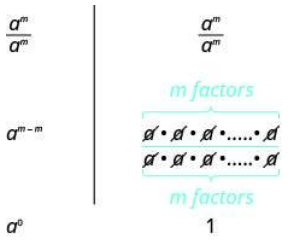
ⓒ $1m8$

ⓓ $\frac{1}{9^{11}}$

A special case of the **Quotient Property** is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. We know, $\frac{x}{x} = 1$, for any $x(x \neq 0)$ since any number divided by itself is 1.

The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^m}$. when $m > n$ and when n<mn<m by subtracting exponents. What if $m = n$? We will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the **Zero Exponent Property**. In general, for

$a \neq 0$:



We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$. Any non-zero base raised to the power of zero equals 1.

DEFINITION: ZERO EXPONENT PROPERTY

If a is a non-zero number, then $a^0 = 1$.

If a is a non-zero number, then a to the power of zero equals 1.

Any non-zero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

EXAMPLE 5.2.7

Simplify each expression: ⓐ 9^0 ⓑ n^0.

Answer

The definition says any non-zero number raised to the zero power is 1.

ⓐ Use the definition of the zero exponent. $9^0 = 1$

ⓑ Use the definition of the zero exponent. $n^0 = 1$

To simplify the expression n raised to the zero power we just use the definition of the zero exponent. The result is 1.

Example 5.2.8

Simplify each expression: ⓐ 11^0 ⓑ q^0.

Answer

ⓐ 1

ⓑ 1

Example 5.2.9

Simplify each expression: ⓐ 23^0 ⓑ r^0.

Answer

ⓐ 1

ⓑ 1

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents has two forms depending on whether the exponent is larger in the numerator or the denominator. What if we just subtract exponents regardless of which is larger?

Let's consider $\frac{x^2}{x^5}$. We subtract the exponent in the denominator from the exponent in the numerator. We see $\frac{x^2}{x^5}$ is x^{2-5} or x^{-3}.

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\frac{\frac{x^2}{x^2}}{\frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x}}$$

$$\frac{1}{x^3}$$

This implies that $x^{-3} = \frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*. If *n* is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

Let's now look at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^{-n}}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$	$\frac{1}{\frac{1}{a^n}}$
Simplify the complex fraction.	$1 \cdot \frac{a^n}{1}$
Multiply.	a^n

This implies $\frac{1}{a^{-n}} = a^n$ and is another form of the definition of **Properties of Negative Exponents**.

DEFINITION: PROPERTIES OF NEGATIVE EXPONENTS

If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$ or $\frac{1}{a^{-n}} = a^n$.

The negative exponent tells us we can rewrite the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Example 5.2.10

Simplify each expression: ⓐ x^{-5} ⓑ 10^{-3} ⓒ $\frac{1}{y^{-4}}$ ⓓ 13^{-2}.

Answer

ⓐ

	x^{-5}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{x^5}$

ⓑ

	10^{-3}
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{10^3}$
Simplify.	$\frac{1}{1000}$

ⓒ

	$\frac{1}{y^{-4}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}}=a^n$.	y^4

ⓓ

	$\frac{1}{3^{-2}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}}=a^n$.	3^2
Simplify.	9

? Example 5.2.11

Simplify each expression: ⓐ z^{-3} ⓑ 10^{-7} ⓒ $\frac{1}{p^{-8}}$ ⓓ $\frac{1}{4^{-3}}$.

Answer

ⓐ $\frac{1}{z^3}$ ⓑ $\frac{1}{10^7}$ ⓒ p^8 ⓓ 64

? Example 5.2.12

Simplify each expression: ⓐ n^{-2} ⓑ 10^{-4} ⓒ $\frac{1}{q^{-7}}$ ⓓ $\frac{1}{2^{-4}}$.

Answer

ⓐ $\frac{1}{n^2}$ ⓑ $\frac{1}{10,000}$ ⓒ q^7
ⓓ 16

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$\frac{16}{9}$

But we know that $\frac{16}{9}$ is $\left(\frac{4}{3}\right)^2$.

This tells us that $\left(\frac{3}{4}\right)^{-2}=\left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the **Quotient to a Negative Power Property**.

QUOTIENT TO A NEGATIVE POWER PROPERTY

If a and b are real numbers, $a\neq 0$, $b\neq 0$ and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^n$$

.

Example 5.2.13

Simplify each expression: ⓐ $\left(\frac{5}{7}\right)^{-2}$ ⓑ $\left(-\frac{x}{y}\right)^{-3}$.

Answer

ⓐ

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^n$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(\frac{7}{5}\right)^2$
Simplify.	$\frac{49}{25}$

ⓑ

	$\left(-\frac{x}{y}\right)^{-3}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(-\frac{y}{x}\right)^{3}$
Simplify.	$-\frac{y^3}{x^3}$

? Example 5.2.14

Simplify each expression: ⓐ $\left(\frac{2}{3}\right)^{-4}$ ⓑ $\left(-\frac{m}{n}\right)^{-2}$.

Answer

ⓐ $\frac{81}{16}$ ⓑ $\frac{n^2}{m^2}$

? Example 5.2.15

Simplify each expression: ⓐ $\left(\frac{3}{5}\right)^{-3}$ ⓑ $\left(-\frac{a}{b}\right)^{-4}$.

Answer

ⓐ $\frac{125}{27}$ ⓑ $\frac{b^4}{a^4}$

Now that we have negative exponents, we will use the **Product Property** with expressions that have negative exponents.

? Example 5.2.16

Simplify each expression: ⓐ $z^{-5}\cdot z^{-3}$ ⓑ $(m^4n^{-3})(m^{-5}n^{-2})$ ⓒ $(2x^{-6}y^8)(-5x^5y^{-3})$.

Answer

ⓐ

	$z^{-5}\cdot z^{-3}$
Add the exponents, since the bases are the same.	z^{-5-3}
Simplify.	z^{-8}
Use the definition of a negative exponent.	$\frac{1}{z^8}$

ⓑ

	$(m^4n^{-3})(m^{-5}n^{-2})$
Use the Commutative Property to get like bases together.	$m^4m^{-5}\cdot n^{-2}n^{-3}$
Add the exponents for each base.	$m^{-1}\cdot n^{-5}$
Take reciprocals and change the signs of the exponents.	$\frac{1}{m^1}\cdot\frac{1}{n^5}$
Simplify.	$\frac{1}{mn^5}$

ⓒ

	$(2x^{-6}y^{8})(-5x^{5}y^{-3})$
Rewrite with the like bases together.	$2(-5)\cdot(x^{-6}x^{5})\cdot(y^{8}y^{-3})$
Multiply the coefficients and add the exponents of each variable.	$-10\cdot x^{-1}\cdot y5$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^{n}}$.	$-10\cdot\frac{1}{x}\cdot y^{5}$
Simplify.	$-10y^{5}x$

? Example 5.2.17

Simplify each expression:

ⓐ $z^{-4}\cdot z^{-5}$ ⓑ $(p^{6}q^{-2})(p^{-9}q^{-1})$ ⓒ $(3u^{-5}v^{7})(-4u^{4}v^{-2})$.

Answer

ⓐ $\frac{1}{z^{9}}$ ⓑ $\frac{1}{p^{3}q^{3}}$ ⓒ $-\frac{12v^{5}}{u}$

? Example 5.2.18

Simplify each expression:

ⓐ $c^{-8}\cdot c^{-7}$ ⓑ $(r^{5}s^{-3})(r^{-7}s^{-5})$ ⓒ $(-6c^{-6}d^{4})(-5c^{-2}d^{-1})$.

Answer

ⓐ $\frac{1}{c^{1}5}$ ⓑ $\frac{1}{r^{2}s^{8}}$ ⓒ $\frac{30d^{3}}{c^{8}}$

Now let's look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

	$(x^{2})^{3}$
What does this mean?	$x^{2}\cdot x^{2}\cdot x^{2}$
How many factors altogether?	$\underbrace{\underbrace{x\cdot x}_{2\text{ factors}}\cdot\underbrace{x\cdot x}_{2\text{ factors}}\cdot\underbrace{x\cdot x}_{2\text{ factors}}}_{6\text{ factors}}$
So we have	x^{6}

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(x^{2})^{3}$ is $x^{2\cdot3}$ or x^{6}.

We multiplied the exponents. This leads to the **Power Property for Exponents.**

DEFINITION: POWER PROPERTY FOR EXPONENTS

If a is a real number and m and n are integers, then

$$(a^{m})^{n}=a^{m\cdot n}$$

To raise a power to a power, multiply the exponents.

? Example 5.2.19

Simplify each expression: ⓐ $(y^{5})^{9}$ ⓑ $(4^{4})^{7}$ ⓒ $(y^{3})^{6}(y^{5})^{4}$.

Answer

ⓐ

	$(y^5)^9$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$y^{5 \cdot 9}$
Simplify.	y^{45}

ⓑ

	$(4^4)^7$
Use the Power Property.	$4^{4 \cdot 7}$
Simplify.	4^{28}

ⓒ

	$(y^3)^6(y^5)^4$
Use the Power Property.	$y^{18} \cdot y^{20}$
Add the exponents.	y^{38}

? Example 5.2.20

Simplify each expression: ⓐ $(b^7)^5$ ⓑ $(5^4)^3$ ⓒ $(a^4)^5(a^7)^4$.

Answer

ⓐ b^{35} ⓑ 5^{12} ⓒ a^{48}

? Example 5.2.21

Simplify each expression: ⓐ $(z^6)^9$ ⓑ $(3^7)^7$ ⓒ $(q^4)^5(q^3)^3$.

Answer

ⓐ z^{54} ⓑ 3^{49} ⓒ q^{29}

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

	$(2x)^3$
What does this mean?	$2x \cdot 2x \cdot 2x$
We group the like factors together.	$2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x$
How many factors of 2 and of x	$2^3 \cdot x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3 \cdot x^3$.

The exponent applies to each of the factors! This leads to the **Product to a Power Property for Exponents**.

DEFINITION: PRODUCT TO A POWER PROPERTY FOR EXPONENTS

If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.

Example 5.2.22

Simplify each expression: ⓐ $(-3mn)^3$ ⓑ $(-4a^2b)^0$ ⓒ $(6k^3)^{-2}$ ⓓ $(5x^{-3})^2$.

Answer

ⓐ

	$(-3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-3)^3 m^3 n^3$
Simplify.	$-27m^3n^3$

ⓑ

	$(-4a^2b)^0$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-4)^0(a^2)^0(b)^0$
Simplify.	$1 \cdot 1 \cdot 1$
Multiply.	1

ⓒ

	$(6k^3)^{-2}$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(6)^{-2}(k^3)^{-2}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$6^{-2}k^{-6}$
Use the Definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{6^2} \cdot \frac{1}{k^6}$
Simplify.	$\frac{1}{36k^6}$

ⓓ

	$(5x^{-3})^2$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$5^2(x^{-3})^2$
Simplify.	$25 \cdot x^{-6}$
Rewrite $x-6$using, $a^{-n} = 1a^n$.	$25 \cdot \frac{1}{x^6}$
Simplify.	$\frac{25}{x^6}$

Example 5.2.23

Simplify each expression: ⓐ $(2wx)^5$ ⓑ $(-11pq3)^0$ ⓒ $(2b^3)^{-4}$ ⓓ $(8a^{-4})^2$.

Answer

ⓐ $32w^5x^5$ ⓑ 1 ⓒ $\frac{1}{16b^{12}}$

ⓓ $\frac{64}{a^8}$

Example 5.2.24

Simplify each expression: ⓐ $(-3y)^3$ ⓑ $(-8m^2n^3)^0$ ⓒ $(-4x^4)^{-2}$ ⓓ $(2c^{-4})^3$.

Answer

ⓐ $-27y^3$ ⓑ 1 ⓒ $\dfrac{1}{16x^8}$
ⓓ $8c^{12}$

Now we will look at an example that will lead us to the Quotient to a Power Property.

	$\left(\dfrac{x}{y}\right)^3$
This means	$\dfrac{x}{y}\cdot\dfrac{x}{y}\cdot\dfrac{x}{y}$
Multiply the fractions.	$\dfrac{x\cdot x\cdot x}{y\cdot y\cdot y}$
Write with exponents.	$\dfrac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $\left(\dfrac{x}{y}\right)^3$ is $\dfrac{x^3}{y^3}$.

This leads to the **Quotient to a Power Property for Exponents**.

DEFINITION: QUOTIENT TO A POWER PROPERTY FOR EXPONENTS

If a and b are real numbers, $b\neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

Example 5.2.25

Simplify each expression:

ⓐ $\left(\dfrac{b}{3}\right)^4$ ⓑ $\left(\dfrac{k}{j}\right)^{-3}$ ⓒ $\left(\dfrac{2xy^2}{z}\right)^3$ ⓓ $\left(\dfrac{4p^{-3}}{q^2}\right)^2$.

Answer

ⓐ

	$\left(\frac{b}{3}\right)^4$
Use Quotient to a Power Property, $(ab)^m=a^mb^m$.	$\frac{b^4}{3^4}$
Simplify.	$\frac{b^4}{81}$

ⓑ

	$\left(\frac{k}{j}\right)^{-3}$
Raise the numerator and denominator to the power.	$\frac{k^{-3}}{j^{-3}}$
Use the definition of negative exponent.	$\frac{1}{k^3}\cdot\frac{j^3}{1}$
Multiply.	$\frac{j^3}{k^3}$

ⓒ

	$\left(\frac{2xy^2}{z}\right)^3$
Use Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{(2xy^2)^3}{z^3}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$\frac{8x^3y^6}{z^3}$

ⓓ

	$\left(\frac{4p^{-3}}{q^2}\right)^2$
Use Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{(4p^{-3})^2}{(q^2)^2}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$\frac{4^2(p^{-3})^2}{(q^2)^2}$
Simplify using the Power Property, $(a^m)^n = a^{m \cdot n}$.	$\frac{16p^{-6}}{q^4}$
Use the definition of negative exponent.	$\frac{16}{q^4} \cdot \frac{1}{p^6}$
Simplify.	$\frac{16}{p^6 q^4}$

? Example 5.2.26

Simplify each expression:

ⓐ $\left(\frac{p}{10}\right)^4$ ⓑ $\left(\frac{m}{n}\right)^{-7}$ ⓒ $\left(\frac{3ab^3}{c^2}\right)^4$ ⓓ $\left(\frac{3x^{-2}}{y^3}\right)^3$.

Answer

ⓐ $\frac{p^4}{10000}$ ⓑ $\frac{n^7}{m^7}$
ⓒ $\frac{81a^4b^{12}}{c^8}$ ⓓ $\frac{27}{x^6y^9}$

? Example 5.2.27

Simplify each expression:

ⓐ $\left(\frac{-2}{q}\right)^3$ ⓑ $\left(\frac{w}{x}\right)^{-4}$ ⓒ $\left(\frac{xy^3}{3z^2}\right)^2$ ⓓ $\left(\frac{2m^{-2}}{n^{-2}}\right)^3$.

Answer

ⓐ $\frac{-8}{q^3}$ ⓑ $\frac{x^4}{w^4}$ ⓒ $\frac{x^2y^6}{9z^4}$
ⓓ $\frac{8n^6}{m^6}$

We now have several properties for exponents. Let's summarize them and then we'll do some more examples that use more than one of the properties.

DEFINITION: SUMMARY OF EXPONENT PROPERTIES

If a and b are real numbers, and m and n are integers, then

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Zero Exponent Property	$a^0 = 1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Example 5.2.28

Simplify each expression by applying several properties:

ⓐ $(3x^2y)^4(2xy^2)^3$ ⓑ $\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$ ⓒ $\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$.

Answer

ⓐ

	$(3x^2y)^4(2xy^2)^3$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$(3^4x^8y^4)(2^3x^3y^6)$
Simplify.	$(81x^8y^4)(8x^3y^6)$
Use the Commutative Property.	$81 \cdot 8 \cdot x^8 \cdot x^3 \cdot y^4 \cdot y^6$
Multiply the constants and add the exponents.	$648x^{11}y^{10}$

ⓑ

	$\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$(x^{12})(x^{-10})(x^{30})$
Add the exponents in the numerator.	$\frac{x^2}{x^{30}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{x^{28}}$

ⓒ

	$\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$
Simplify inside the parentheses first.	$\left(\frac{2y^4}{x^2}\right)^2\left(\frac{12y^4}{x^2}\right)^{-1}$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$.	$\frac{(2y^4)^2}{(x^2)^2}\frac{(12y^4)^{-1}}{(x^2)^{-1}}$
Use the Product to a Power Property, $(ab)^m=a^mb^m$.	$\frac{4y^8}{x^4}\cdot\frac{12^{-1}y^{-4}}{x^{-2}}$
Simplify.	$\frac{4y^4}{12x^2}$
Simplify.	$\frac{y^4}{3x^2}$

? Example 5.2.29

Simplify each expression:

ⓐ $(c^4d^2)^5(3cd^5)^4$ ⓑ $\frac{(a^{-2})^3(a^2)^4}{(a^4)^5}$ ⓒ $\left(\frac{3xy^2}{x^2y^{-3}}\right)^2$

Answer

ⓐ $81c^{24}d^{30}$ ⓑ $\frac{1}{a^{18}}$

ⓒ $\frac{9y^{10}}{x^2}$

? Example 5.2.30

Simplify each expression:

ⓐ $(a^3b^2)^6(4ab^3)^4$ ⓑ $\frac{(p^{-3})^4(p^5)^3}{(p^7)^6}$ ⓒ $\left(\frac{4x^3y^2}{x^2y^{-1}}\right)^2\left(\frac{8xy^{-3}}{x^2y}\right)^{-1}$.

Answer

ⓐ $256a^{22}b^{24}$ ⓑ $\frac{1}{p^{39}}$

ⓒ $2x^3y^{10}$

Use Scientific Notation

Working with very large or very small numbers can be awkward. Since our number system is base ten we can use powers of ten to rewrite very large or very small numbers to make them easier to work with. Consider the numbers 4,000 and 0.004.

Using place value, we can rewrite the numbers 4,000 and 0.004. We know that 4,000 means $4\times1,000$ and 0.004 means $4\times\frac{1}{1,000}$.

If we write the 1,000 as a power of ten in exponential form, we can rewrite these numbers in this way:

4,000	$4\times1,000$	4×103	
0.004	$4\times\frac{1}{1,000}$	$4\times\frac{1}{103}$	4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than ten, and the second factor is a power of 10 written in exponential form, it is said to be in **scientific notation**.

DEFINITION: SCIENTIFIC NOTATION

A number is expressed in **scientific notation** when it is of the form

$$a \times 10^n \quad \text{where} \quad 1 \leq a < 10 \quad \text{and} \quad n \quad \text{is an integer.}$$

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

$4000. = 4 \times 10^3$ — Moved the decimal point 3 places to the left.

$0.004 = 4 \times 10^{-3}$ — Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10.

The power of 10 is positive when the number is larger than 1 : $4,000 = 4 \times 10^3$

The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$

DEFINITION: TO CONVERT A DECIMAL TO SCIENTIFIC NOTATION.

1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
2. Count the number of decimal places, n, that the decimal point was moved.
3. Write the number as a product with a power of 10. If the original number is.
   - greater than 1, the power of 10 will be 10^n.
   - between 0 and 1, the power of 10 will be 10^{-n}.
4. Check.

EXAMPLE 5.2.31

Write in scientific notation: ⓐ $37,000$ ⓑ 0.0052

Answer

ⓐ

The original number, 37,000, is greater than 1 so we will have a positive power of 10.	37,000
Move the decimal point to get 3.7, a number between 1 and 10.	37,000
Count the number of decimal places the point was moved.	4 places
Write as a product with a power of 10.	3.7×10^4
Check: 3.7×10^4 $3.7 \times 10,000$ $37,000$	
	$37,000 = 3.7 \times 10^4$

ⓑ

The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10.	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}
Check: 5.2×10^{-3} $5.2 \times \frac{1}{10^3}$ $5.2 \times \frac{1}{1000}$ 5.2×0.001 0.0052	
	$0.0052 = 5.2 \times 10^{-3}$

? EXAMPLE 5.2.32

Write in scientific notation: ⓐ 96,000 ⓑ 0.0078.

Answer

ⓐ 9.6×10^4 ⓑ 7.8×10^{-3}

? EXAMPLE 5.2.33

Write in scientific notation: ⓐ 48,300 ⓑ 0.0129.

Answer

ⓐ 4.83×10^4
ⓑ 1.29×10^{-2}

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.

9.12×10^4	9.12×10^{-4}
$9.12 \times 10,000$	9.12×0.0001
$91,200$	0.000912

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

$9.12 \times 10^4 = 91,200$	$9.12 \times 10^{-4} = 0.000912$
$9.12 \times 10^4 = 91,200$	$9.12 \times 10^{-4} = 0.000912$
Move the decimal point 4 places to the right.	Move the decimal point 4 places to the left.

In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

DEFINITION: CONVERT SCIENTIFIC NOTATION TO DECIMAL FORM.

1. Determine the exponent, n, on the factor 10.
2. Move the decimal n places, adding zeros if needed.
   - If the exponent is positive, move the decimal point n places to the right.
   - If the exponent is negative, move the decimal point $|n|$ places to the left.
3. Check.

EXAMPLE 5.2.34

Convert to decimal form: ⓐ 6.2×10^3 ⓑ -8.9×10^{-2}.

Answer

ⓐ

	6.2×10^3
Determine the exponent, n, on the factor 10.	
	The exponent is 3.
Since the exponent is positive, move the decimal point 3 places to the right.	6.200
Add zeros as needed for placeholders.	6,200
	$6.2 \times 10^3 = 6{,}200$

ⓑ

	-8.9×10^{-2}
Determine the exponent, n, on the factor 10.	The exponent is -2.-2.
Since the exponent is negative, move the decimal point 2 places to the left.	−8.9
Add zeros as needed for placeholders.	−0.089
	$-8.9 \times 10^{-2} = -0.089$

EXAMPLE 5.2.35

Convert to decimal form: ⓐ 1.3×10^3 ⓑ -1.2×10^{-4}.

Answer

ⓐ 1,300 ⓑ -0.00012

EXAMPLE 5.2.36

Convert to decimal form: ⓐ -9.5×10^4 ⓑ 7.5×10^{-2}.

Answer

ⓐ $-950,000$ⓑ 0.075

When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the **Properties of Exponents** are used to multiply and divide numbers in scientific notation.

? EXAMPLE 5.2.37

Multiply or divide as indicated. Write answers in decimal form: ⓐ $(-4 \times 10^5)(2 \times 10^{-7})$ ⓑ $\frac{9 \times 10^3}{3 \times 10^{-2}}$.

Answer

ⓐ

	$(-4 \times 10^5)(2 \times 10^{-7})$
Use the Commutative Property to rearrange the factors.	$-4 \cdot 2 \cdot 10^5 \cdot 10^{-7}$
Multiply.	-8×10^{-2}
Change to decimal form by moving the decimal two places left.	-0.08

ⓑ

	$\frac{9 \times 10^3}{9 \times 10^{-2}}$
Separate the factors, rewriting as the product of two fractions.	$\frac{9}{3} \times \frac{10^3}{10^{-2}}$
Divide.	3×10^5
Change to decimal form by moving the decimal five places right.	$300,000$

? EXAMPLE 5.2.38

Multiply or divide as indicated. Write answers in decimal form:

ⓐ $(-3 \times 10^5)(2 \times 10^{-8})$ ⓑ $\frac{8 \times 10^2}{4 \times 10^{-2}}$.

Answer

ⓐ -0.006 ⓑ 20,000

? EXAMPLE 5.2.39

Multiply or divide as indicated. Write answers in decimal form:

ⓐ $(-3 \times 10^{-2})(3 \times 10^{-1})$ ⓑ $\frac{8 \times 10^4}{2 \times 10^{-1}}$.

Answer

ⓐ -0.009 ⓑ 400,000

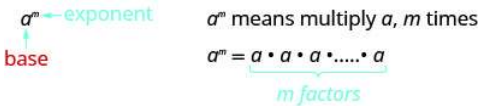
Access these online resources for additional instruction and practice with using multiplication properties of exponents.

- Properties of Exponents
- Negative exponents

- Scientific Notation

Key Concepts

- **Exponential Notation**



  This is read a to the mth power.
  In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.
- **Product Property for Exponents**
  If a is a real number and m and n are integers, then

$$a^m \cdot a^n = a^{m+n}$$

  To multiply with like bases, add the exponents.
- **Quotient Property for Exponents**
  If a is a real number, $a \neq 0$, and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n}, \quad m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad n > m$$

- **Zero Exponent**
  - If a is a non-zero number, then $a^0 = 1$.
  - If a is a non-zero number, then a to the power of zero equals 1.
  - Any non-zero number raised to the zero power is 1.
- **Negative Exponent**
  - If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$ or $\frac{1}{a^{-n}} = a^n$.
- **Quotient to a Negative Exponent Property**
  If a and b are real numbers, $a \neq 0$, $b \neq 0$ and n is an integer, then

$$(ab)^{-n} = (ba)^n$$

- **Power Property for Exponents**
  If a is a real number and m and n are integers, then

$$(a^m)^n = a^{m \cdot n}$$

  To raise a power to a power, multiply the exponents.
- **Product to a Power Property for Exponents**
  If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

  To raise a product to a power, raise each factor to that power.
- **Quotient to a Power Property for Exponents**
  If a and b are real numbers, $b \neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

  To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
  If a and b are real numbers, and m and n are integers, then

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Zero Exponent Property	$a^0 = 1, a \neq 0$
Quotient to a Power Property:	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

- **Scientific Notation**
  A number is expressed in scientific notation when it is of the form

$$a \times 10^n \text{ where } 1 \leq a < 10 \text{ and } n \text{ is an integer.}$$

- **How to convert a decimal to scientific notation.**
  1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
  2. Count the number of decimal places, n, that the decimal point was moved.
  3. Write the number as a product with a power of 10. If the original number is.
     - greater than 1, the power of 10 will be 10^n.
     - between 0 and 1, the power of 10 will be 10^{-n}.
  4. Check.
- **How to convert scientific notation to decimal form.**
  1. Determine the exponent, n, on the factor 10.
  2. Move the decimal n places, adding zeros if needed.
     - If the exponent is positive, move the decimal point n places to the right.
     - If the exponent is negative, move the decimal point $|n|$ places to the left.
  3. Check.

Glossary

Product Property

According to the Product Property, a to the m times a to the a equals a to the m plus n.

Power Property

According to the Power Property, a to the m to the n equals a to the m times n.

Product to a Power

According to the Product to a Power Property, a times b in parentheses to the m equals a to the m times b to the m.

Quotient Property

According to the Quotient Property, a to the m divided by a to the n equals a to the m minus n as long as a is not zero.

Zero Exponent Property

According to the Zero Exponent Property, a to the zero is 1 as long as a is not zero.

Quotient to a Power Property

According to the Quotient to a Power Property, a divided by b in parentheses to the power of m is equal to a to the m divided by b to the m as long as b is not zero.

Properties of Negative Exponents

According to the Properties of Negative Exponents, a to the negative n equals 1 divided by a to the n and 1 divided by a to the negative n equals a to the n.

Quotient to a Negative Exponent

Raising a quotient to a negative exponent occurs when a divided by b in parentheses to the power of negative n equals b divided by a in parentheses to the power of n.

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5.2E: Exercises

Practice Makes Perfect

Simplify Expressions Using the Properties for Exponents

In the following exercises, simplify each expression using the properties for exponents.

1. ⓐ $d^3 \cdot d^6$ ⓑ $4^{5x} \cdot 4^{9x}$ ⓒ $2y \cdot 4y^3$ ⓓ $w \cdot w^2 \cdot w^3$

Answer

ⓐ d^9 ⓑ 4^{14x} ⓒ $8y^4$ ⓓ w^6

2. ⓐ $x^4 \cdot x^2$ ⓑ $8^{9x} \cdot 8^3$ ⓒ $3z^{25} \cdot 5z^8$ ⓓ $y \cdot y^3 \cdot y^5$

3. ⓐ $n^{19} \cdot n^{12}$ ⓑ $3^x \cdot 3^6$ ⓒ $7w^5 \cdot 8w$ ⓓ $a^4 \cdot a^3 \cdot a^9$

Answer

ⓐ n^{31} ⓑ 3^{x+6} ⓒ $56w^6$
ⓓ a^{16}

4. ⓐ $q^{27} \cdot q^{15}$ ⓑ $5^x \cdot 5^{4x}$ ⓒ $9u^{41} \cdot 7u^{53}$
ⓓ $c^5 \cdot c^{11} \cdot c^2$

5. $m^x \cdot m^3$

Answer

m^{x+3}

6. $n^y \cdot n^2$

7. $y^a \cdot y^b$

Answer

y^{a+b}

8. $x^p \cdot x^q$

9. ⓐ $\frac{x^{18}}{x^3}$ ⓑ $\frac{5^{12}}{5^3}$ ⓒ $\frac{q^{18}}{q^{36}}$ ⓓ $\frac{10^2}{10^3}$

Answer

ⓐ x^{15} ⓑ 5^9 ⓒ $\frac{1}{q^{18}}$ ⓓ $\frac{1}{10}$

10. ⓐ $\frac{y^{20}}{y^{10}}$ ⓑ $\frac{7^{16}}{7^2}$ ⓒ $\frac{t^{10}}{t^{40}}$ ⓓ $\frac{8^3}{8^5}$

11. ⓐ $\frac{p^{21}}{p^7}$ ⓑ $\frac{4^{16}}{4^4}$ ⓒ $\frac{b}{b^9}$ ⓓ $\frac{4}{4^6}$

Answer

ⓐ p^{14} ⓑ 4^{12} ⓒ $\frac{1}{b^8}$ ⓓ $\frac{1}{4^5}$

12. ⓐ $\frac{u^{24}}{u^3}$ ⓑ $\frac{9^{15}}{9^5}$ ⓒ $\frac{x}{x^7}$ ⓓ $\frac{10}{10^3}$

13. ⓐ 20^0 ⓑ b^0

Answer

ⓐ 1 ⓑ 1

14. ⓐ 13^0 ⓑ k^0

15. ⓐ -27^0 ⓑ $-(27^0)$

Answer

ⓐ -1 ⓑ -1

16. ⓐ -15^0 ⓑ $-(15^0)$

Use the Definition of a Negative Exponent

In the following exercises, simplify each expression.

17. ⓐ a^{-2} ⓑ 10^{-3} ⓒ $\frac{1}{c^{-5}}$ ⓓ $\frac{1}{3^{-2}}$

Answer

ⓐ $\frac{1}{a^2}$ ⓑ $\frac{1}{1000}$ ⓒ c^5 ⓓ 9

18. ⓐ b^{-4} ⓑ 10^{-2} ⓒ $\frac{1}{c^{-5}}$ ⓓ $\frac{1}{5^{-2}}$

19. ⓐ r^{-3} ⓑ 10^{-5} ⓒ $\frac{1}{q^{-10}}$ ⓓ $\frac{1}{10^{-3}}$

Answer

ⓐ $\frac{1}{r3}$ ⓑ $\frac{1}{100,000}$ ⓒ q^{10} ⓓ $1,000$

20. ⓐ s^{-8} ⓑ 10^{-2} ⓒ $\frac{1}{t^{-9}}$ ⓓ $\frac{1}{10^{-4}}$

21. ⓐ $\left(\frac{5}{8}\right)^{-2}$ ⓑ $\left(-\frac{b}{a}\right)^{-2}$

Answer

ⓐ $\frac{64}{25}$ ⓑ $\frac{a^2}{b^2}$

22. ⓐ $\left(\frac{3}{10}\right)^{-2}$ ⓑ $\left(-\frac{2}{z}\right)^{-3}$

23. ⓐ $\left(\frac{4}{9}\right)^{-3}$ ⓑ $\left(-\frac{u}{v}\right)^{-5}$

Answer

ⓐ $\frac{729}{64}$ ⓑ $-\frac{v^5}{u^5}$

24. ⓐ $\left(\frac{7}{2}\right)^{-3}$ ⓑ $\left(-\frac{3}{x}\right)^{-3}$

25. ⓐ $(-5)^{-2}$ ⓑ -5^{-2} ⓒ $\left(-\frac{1}{5}\right)^{-2}$ ⓓ $-\left(\frac{1}{5}\right)^{-2}$

Answer

ⓐ $\frac{1}{25}$ ⓑ $-\frac{1}{25}$ ⓒ 25 ⓓ -25

26. ⓐ -5^{-3} ⓑ $\left(-\frac{1}{5}\right)^{-3}$ ⓒ $-\left(\frac{1}{5}\right)^{-3}$ ⓓ $(-5)^{-3}$

27. ⓐ $3 \cdot 5^{-1}$ ⓑ $(3 \cdot 5)^{-1}$

Answer

ⓐ $\frac{3}{5}$ ⓑ $\frac{1}{15}$

28. ⓐ $3 \cdot 4^{-2}$ ⓑ $(3 \cdot 4)^{-2}$

In the following exercises, simplify each expression using the Product Property.

29. ⓐ $b^4 b^{-8}$ ⓑ $(w^4 x^{-5})(w^{-2} x^{-4}))$ ⓒ $(-6c^{-3} d^9)(2c^4 d^{-5})$

Answer

ⓐ $\frac{1}{b^4}$ ⓑ $\frac{w^2}{x^9}$ ⓒ $-12cd^4$

30. ⓐ $s^3 \cdot s^{-7}$ ⓑ $(m^3 n^{-3})(m^5 n^{-1})$
ⓒ $(-2j^{-5} k^8)(7j^2 k^{-3})$

31. ⓐ $a^3 \cdot a^{-3}$ ⓑ $(uv^{-2})(u^{-5} v^{-3})$
ⓒ $(-4r^{-2} s^{-8})(9r^4 s^3)$

Answer

ⓐ 1 ⓑ $\frac{1}{u^4 v^5}$ ⓒ $-36\frac{r^2}{j^5}$

32. ⓐ $y^5 \cdot y^{-5}$ ⓑ $(pq^{-4})(p^{-6} q^{-3})$
ⓒ $(-5m^4 n^6)(8m^{-5} n^{-3})$

33. $p^5 \cdot p^{-2} \cdot p^{-4}$

Answer

$\frac{1}{p}$

34. $x^4 \cdot x^{-2} \cdot x^{-3}$

In the following exercises, simplify each expression using the Power Property.

35. ⓐ $(m^4)^2$ ⓑ $(10^3)^6$ ⓒ $(x^3)^{-4}$

Answer

ⓐ m^8 ⓑ 10^{18} ⓒ $\frac{1}{x^{12}}$

36. ⓐ $(b^2)^7$ ⓑ $(3^8)^2$ ⓒ $(k^2)^{-5}$

37. ⓐ $(y^3)^x$ ⓑ $(5^x)^x$ ⓒ $(q^6)^{-8}$

Answer

ⓐ y^{3x} ⓑ 5^{xy} ⓒ $\frac{1}{q^{48}}$

38. ⓐ $(x^2)^y$ ⓑ $(7^a)^b$ ⓒ $(a^9)^{-10}$

In the following exercises, simplify each expression using the Product to a Power Property.

39. ⓐ $(-3xy)^2$ ⓑ $(6a)^0$ ⓒ $(5x^2)^{-2}$ ⓓ $(-4y^{-3})^2$

Answer

ⓐ $9x^2y^2$ ⓑ 1 ⓒ $\frac{1}{25x^4}$ ⓓ $\frac{16}{y^6}$

40. ⓐ $(-4ab)^2$ ⓑ $(5x)^0$ ⓒ $(4y^3)^{-3}$ ⓓ $(-7y^{-3})^2$

41. ⓐ $(-5ab)^3$ ⓑ $(-4pq)^0$ ⓒ $(-6x^3)^{-2}$ ⓓ $(3y^{-4})^2$

Answer

ⓐ $-125a^3b^3$ ⓑ 1 ⓒ $\frac{1}{36x^6}$ ⓓ $\frac{9}{y^8}$

42. ⓐ $(-3xyz)^4$ ⓑ $(-7mn)^0$ ⓒ $(-3x^3)^{-2}$
ⓓ $(2y^{-5})^2$

In the following exercises, simplify each expression using the Quotient to a Power Property.

43. ⓐ $(p^2)^5$ ⓑ $\left(\frac{x}{y}\right)^{-6}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{4p^{-3}}{q^2}\right)^2$

Answer

ⓐ $\frac{p^5}{32}$ ⓑ $\frac{y^6}{x^6}$ ⓒ $\frac{8x^3y^6}{z^3}$ ⓓ $\frac{16}{p^6q^4}$

44. ⓐ $\left(\frac{x}{3}\right)^4$ ⓑ $\left(\frac{a}{b}\right)^{-5}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{x^3y}{z^4}\right)^2$

45. ⓐ $\left(\frac{a}{3b}\right)^4$ ⓑ $\left(\frac{5}{4m}\right)^{-2}$ ⓒ $\left(\frac{3a^{-2}b^3}{c^3}\right)^{-2}$ ⓓ $\left(\frac{p^{-1}q^4}{r^{-4}}\right)^2$

Answer

ⓐ $\frac{a^4}{81b^4}$ ⓑ $\frac{16m^2}{25}$ ⓒ $\frac{a^4c^4}{9b^6}$ ⓓ $\frac{q^8r^8}{p^2}$

46. ⓐ $\left(\frac{x^2}{y}\right)^3$ ⓑ $\left(\frac{10}{3q}\right)^{-4}$ ⓒ $\left(\frac{2x^3y^4}{3z^2}\right)^5$ ⓓ $\left(\frac{5a^3b^{-1}}{2c^4}\right)^{-3}$

In the following exercises, simplify each expression by applying several properties.

47. ⓐ $(5t^2)^3(3t)^2$ ⓑ $\frac{(t^2)^5(t^{-4})^2}{(t^3)^7}$ ⓒ $\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$

Answer

ⓐ $1125t^8$ ⓑ $\frac{1}{t^{19}}$ ⓒ $\frac{y^4}{3x^2}$

48. ⓐ $(10k^4)^3(5k^6)^2$ ⓑ $\frac{(q^3)^6(q^{-2})^3}{(q^4)^8}$

49. ⓐ $(m^2n)^2(2mn^5)^4$ ⓑ $\frac{(-2p^{-2})^4(3p^4)^2}{(-6p^3)^2}$

Answer

ⓐ $16m^8n^{22}$ ⓑ $\frac{4}{p^6}$

50. ⓐ $(3pq^4)^2(6p^6q)^2$ ⓑ $\frac{(-2k^{-3})^2(6k^2)^4}{(9k^4)^2}$

Mixed Practice

In the following exercises, simplify each expression.

51. ⓐ $7n^{-1}$ ⓑ $(7n)^{-1}$ ⓒ $(-7n)^{-1}$

Answer

ⓐ $\frac{7}{n}$ ⓑ $\frac{1}{7n}$ ⓒ $-\frac{1}{7n}$

52. ⓐ $6r^{-1}$ ⓑ $(6r)^{-1}$ ⓒ $(-6r)^{-1}$

53. ⓐ $(3p)^{-2}$ ⓑ $3p^{-2}$ ⓒ $-3p^{-2}$

Answer

ⓐ $\frac{1}{9p^2}$ ⓑ $\frac{3}{p^2}$ ⓒ $-\frac{3}{p^2}$

54. ⓐ $(2q)^{-4}$ ⓑ $2q^{-4}$ ⓒ $-2q^{-4}$

55. $(x^2)^4 \cdot (x^3)^2$

Answer

x^{14}

56. $(y^4)^3 \cdot (y^5)^2$

57. $(a^2)^6 \cdot (a^3)^8$

Answer

a^{30}

58. $(b^7)^5 \cdot (b^2)^6$

59. $(2m^6)^3$

Answer

$2m^{18}$

60. $(3y^2)^4$

61. $(10x^2y)^3$

Answer

$1,000x^6y^3$

62. $(2mn^4)^5$

63. $(-2a^3b^2)^4$

Answer

$16a^{12}b^8$

64. $(-10u^2v^4)^3$

65. $\left(\frac{2}{3}x^2y\right)^3$

Answer

$\frac{8}{27}x^6y^3$

66. $\left(\frac{7}{9}pq^4\right)^2$

67. $(8a^3)^2(2a)^4$

Answer

$1,024a^{10}$

68. $(5r^2)^3(3r)^2$

69. $(10p^4)^3(5p^6)^2$

Answer

$25,000p^{24}$

70. $(4x^3)^3(2x^5)^4$

71. $\left(\frac{1}{2}x^2y^3\right)^4(4x^5y^3)^2$

Answer

$x^{18}y^{18}$

72. $\left(\frac{1}{3}m^3n^2\right)^4(9m^8n^3)^2$

73. $(3m^2n)^2(2mn^5)^4$

Answer

$144m^8n^{22}$

74. $(2pq^4)^3(5p^6q)^2$

75. ⓐ $(3x)^2(5x)$ ⓑ $(2y)^3(6y)$

Answer

ⓐ $45x^3$ ⓑ $48y^4$

76. ⓐ $\left(\frac{1}{2}y^2\right)^3\left(\frac{2}{3}y\right)^2$ ⓑ $\left(\frac{1}{2}j^2\right)^5\left(\frac{2}{5}j^3\right)^2$

77. ⓐ $(2r^{-2})^3(4^{-1}r)^2$ ⓑ $(3x^{-3})^3(3^{-1}x^5)^4$

Answer

ⓐ $12r^4$ ⓑ $13x^{11}$

78. $\left(\frac{k^{-2}k^8}{k^3}\right)^2$

79. $\left(\frac{j^{-2}j^5}{j^4}\right)^3$

Answer

$\frac{1}{j^3}$

80. $\frac{(-4m^{-3})^2(5m^4)^3}{(-10m^6)^3}$

81. $\frac{(-10n^{-2})^3(4n^5)^2}{(2n^8)^2}$

Answer

$-\frac{4000}{n^{12}}$

Use Scientific Notation

In the following exercises, write each number in scientific notation.

82. ⓐ 57,000 ⓑ 0.026

83. ⓐ 340,000 ⓑ 0.041

Answer

ⓐ 34×10^4 ⓑ 41×10^{-3}

84. ⓐ 8,750,000 ⓑ 0.00000871

85. ⓐ 1,290,000 ⓑ 0.00000103

Answer

ⓐ 1.29×10^6

ⓑ 103×10^{-8}

In the following exercises, convert each number to decimal form.

86. ⓐ 5.2×10^2 ⓑ 2.5×10^{-2}

87. ⓐ -8.3×10^2 ⓑ 3.8×10^{-2}

Answer

ⓐ -830 ⓑ 0.038

88. ⓐ 7.5×10^6 ⓑ -4.13×10^{-5}

89. ⓐ 1.6×10^{10} ⓑ 8.43×10^{-6}

Answer

ⓐ 16,000,000,000
ⓑ 0.00000843

In the following exercises, multiply or divide as indicated. Write your answer in decimal form.

90. ⓐ $(3 \times 10^{-5})(3 \times 10^{9})$ ⓑ $\dfrac{7 \times 10^{-3}}{1 \times 10^{-7}}$

91. ⓐ $(2 \times 10^{2})(1 \times 10^{-4})$ ⓑ $\dfrac{5 \times 10^{-2}}{1 \times 10^{-10}}$

Answer

ⓐ 0.02 ⓑ 500,000,000

92. ⓐ $(7.1 \times 10^{-2})(2.4 \times 10^{-4})$ ⓑ $\dfrac{6 \times 10^{4}}{3 \times 10^{-2}}$

93. ⓐ $(3.5 \times 10^{-4})(1.6 \times 10^{-2})$ ⓑ $\dfrac{8 \times 10^{6}}{4 \times 10^{-1}}$

Answer

ⓐ 0.0000056 ⓑ 20,000,000

Writing Exercises

94. Use the Product Property for Exponents to explain why $x \cdot x = x^2$.

95. Jennifer thinks the quotient $\dfrac{a^{24}}{a^{6}}$ simplifies to a^4. What is wrong with her reasoning?

Answer

Answers will vary.

96. Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.

97. When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions using the properties for exponents.			
use the definition of a Negative Exponents.			
use scientific notation.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

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5.3: Multiply Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply monomials
- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a polynomial by a polynomial
- Multiply special products
- Multiply polynomial functions

Before you get started, take this readiness quiz.

1. Distribute: $2(x+3)$.
   If you missed this problem, review [link].
2. Simplify: a. 9^2 b. $(-9)^2$ c. -9^2.
   If you missed this problem, review [link].
3. Evaluate: $2x^2-5x+3$ for $x=-2$.
   If you missed this problem, review [link].

Multiply Monomials

We are ready to perform operations on polynomials. Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials.

Example 5.3.1

Multiply:

a. $(3x^2)(-4x^3)$
b. $\left(\frac{5}{6}x^3y\right)(12xy^2)$.

Answer a

	$(3x^2)(-4x^3)$
Use the Commutative Property to rearrange the terms.	$3\cdot(-4)\cdot x^2\cdot x^3$
	$-12x^5$

Answer b

	$\left(\frac{5}{6}x^3y\right)(12xy^2)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6}\cdot 12\cdot x^3\cdot x\cdot y\cdot y^2$
Multiply.	$10x^4y^3$

Example 5.3.2

Multiply:

a. $(5y^7)(-7y^4)$
b. $(25a^4b^3)(15ab^3)$

Answer a

$-35y^{11}$

Answer b

$375a^5b^6$

Example 5.3.3

Multiply:

a. $(-6b^4)(-9b^5)$
b. $(23r^5s)(12r^6s^7)$.

Answer a

$54b^9$

Answer b

$276r^{11}s^8$

Multiply a Polynomial by a Monomial

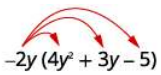
Multiplying a polynomial by a monomial is really just applying the Distributive Property.

Example 5.3.4

Multiply:

a. $-2y(4y^2+3y-5)$
b. $3x^3y(x^2-8xy+y^2)$.

Answer a

	
Distribute.	$-2y \cdot 4y^2 + (-2y) \cdot 3y - (-2y) \cdot 5$
Multiply.	$-8y^3 - 6y^2 + 10y$

Answer b

	$3x^3y(x^2-8xy+y^2)$
Distribute.	$3x^3y \cdot x^2 + (3x^3y) \cdot (-8xy) + (3x^3y) \cdot y^2$
Multiply.	$3x^5y - 24x^4y^2 + 3x^3y^3$

Example 5.3.5

Multiply:

a. $-3y(5y^2+8y^7)$
b. $4x^2y^2(3x^2-5xy+3y^2)$

Answer a

$-15y^3-24y^8$

Answer b

$12x^4y^2-20x^3y^3+12x^2y^4$

Example 5.3.6

Multiply:

a. $4x^2(2x^2-3x+5)$
b. $-6a^3b(3a^2-2ab+6b^2)$

Answer a

$8x^4-12x^3+20x^2$

Answer b

$-18a^5b+12a^4b^2-36a^3b^3$

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Example 5.3.7

Multiply:

a. $(y+5)(y+8)$
b. $(4y+3)(2y-5)$.

Answer

ⓐ

	$(y+5)(y+8)$
Distribute $(y+8)$.	$y(y+8)+5(y+8)$
Distribute again.	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

ⓑ

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

Example 5.3.8

Multiply:

a. $(x+8)(x+9)$
b. $(3c+4)(5c-2)$.

Answer a

$x^2+17x+72$

Answer b

$15c^2+14c-8$

Example 5.3.9

Multiply:

a. $(5x+9)(4x+3)$
b. $(5y+2)(6y-3)$.

Answer a

$20x^2+51x+27$

Answer b

$30y^2-3y-6$

If you multiply binomials often enough you may notice a pattern. Notice that the first term in the result is the product of the *first* terms in each binomial. The second and third terms are the product of multiplying the two *outer* terms and then the two *inner* terms. And the last term results from multiplying the two *last* terms,

We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for 'First, Outer, Inner, Last'. We use this as another method of multiplying binomials. The word FOIL is easy to remember and ensures we find all **four** products.

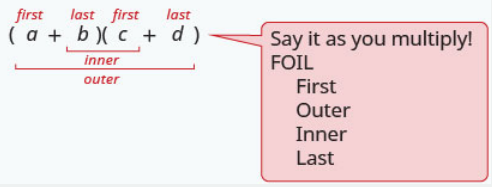
Let's multiply $(x+3)(x+7)$ using both methods.

Distributive Property	FOIL
$(x+3)(x+7)$	$(x+3)(x+7)$
$x(x+7)+3(x+7)$	
$x^2+7x+3x+21$ F O I L	$x^2+7x+3x+21$ F O I L
$x^2+10x+21$	$x^2+10x+21$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

DEFINITION: USE THE FOIL METHOD TO MULTIPLY TWO BINOMIALS.

Step 1. Multiply the *First* terms.
Step 2. Multiply the *Outer* terms.
Step 3. Multiply the *Inner* terms.
Step 4. Multiply the *Last* terms.
Step 5. Combine like terms, when possible.



When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

Example 5.3.10

Multiply:

a. $(y-7)(y+4)$
b. $(4x+3)(2x-5)$.

Answer

a.

$(y-7)(y+4)$

Step 1. Multiply the *First* terms. $y^2 + \_ + \_ + \_$ (F O I L)

Step 2. Multiply the *Outer* terms. $y^2 + 4y + \_ + \_$ (F O I L)

Step 3. Multiply the *Inner* terms. $y^2 + 4y - 7y + \_$ (F O I L)

Step 4. Multiply the *Last* terms. $y^2 + 4y - 7y - 28$ (F O I L)

Step 5. Combine like terms. $y^2 - 3y - 28$

b.

$(4x-3)(2x-5)$

Step 1. Multiply the *First* terms, $4x \cdot 2x$. $8x^2 + \_ + \_ + \_$ (F O I L)

Step 2. Multiply the *Outer* terms, $4x \cdot (-5)$. $8x^2 - 20x + \_ + \_$ (F O I L)

Step 3. Multiply the *Inner* terms, $3 \cdot 2x$. $8x^2 - 2x + 6x + \_$ (F O I L)

Step 4. Multiply the *Last* terms, $3 \cdot (-5)$. $8x^2 - 20x + 6x - 15$ (F O I L)

Step 5. Combine like terms. $8x^2 - 14x - 15$

Exercise 5.3.11

Multiply:

a. $(x-7)(x+5)$
b. $(3x+7)(5x-2)$.

Answer

a. $x^2 - 2x - 35$
b. $15x^2 + 29x - 14$

Exercise 5.3.12

Multiply:

a. $(b-3)(b+6)$
b. $(4y+5)(4y-10)$.

Answer

a. $b^2 + 3b - 18$
b. $16y^2 - 20y - 50$

The final products in the last example were trinomials because we could combine the two middle terms. This is not always the case.

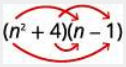
Example 5.3.13

Multiply:

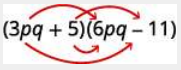
a. $(n^2+4)(n-1)$
b. $(3pq+5)(6pq-11)$.

Answer

a.

	$(n^2+4)(n-1)$
	
Step 1. Multiply the *First* terms.	$n^3+\_\_+\_\_+\_\_$ (F O I L)
Step 2. Multiply the *Outer* terms.	$n^3-n^2+\_\_+\_\_$ (F O I L)
Step 3. Multiply the *Inner* terms.	$n^3-n^2+4n+\_\_$ (F O I L)
Step 4. Multiply the *Last* terms.	n^3-n^2+4n-4 (F O I L)
Step 5. Combine like terms—there are none.	n^3-n^2+4n-4

b.

	$(3pq+5)(6pq-11)$
	
Step 1. Multiply the *First* terms.	$18p^2q^2+\_\_+\_\_+\_\_$ (F O I L)
Step 2. Multiply the *Outer* terms.	$18p^2q^2-33pq+\_\_+\_\_$ (F O I L)
Step 3. Multiply the *Inner* terms.	$18p^2q^2-33pq+30pq+\_\_$ (F O I L)
Step 4. Multiply the *Last* terms.	$18p^2q^2-33pq+30pq-55$ (F O I L)
Step 5. Combine like terms.	$18p^2q^2-3pq-55$

Example 5.3.14

Multiply:

a. $(x^2+6)(x-8)$
b. $(2ab+5)(4ab-4)$.

Answer

a. $x^3-8x^2+6x-48$
b. $8a^2b^2+12ab-20$

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? Example 5.3.15

Multiply:

a. $(y^2+7)(y-9)$
b. $(2xy+3)(4xy-5)$.

Answer

a. $y^3-9y^2+7y-63$
b. $8x^2y^2+2xy-15$

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23 ×46		
138	partial product	Start by multiplying 23 by 6 to get 138.
92	partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058	product	Last you add the partial products.

Now we'll apply this same method to multiply two binomials.

? Example 5.3.16

Multiply using the Vertical Method: $(3y-1)(2y-6)$.

Answer

It does not matter which binomial goes on the top.

$$\begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline \end{array}$$

Multiply $3y-1$ by -6.	$-18y+6$	partial product
Multiply $3y-1$ by $2y$.	$6y^2-2y$	partial product
Add like terms.	$6y^2-20y+6$	

Notice the partial products are the same as the terms in the FOIL method.

$$\begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline -18y+6 \\ 6y^2-2y \\ \hline 6y^2-20x+6 \end{array}$$

$$\begin{array}{l} (3y-1)(2y-6) \\ 6y^2-2y-18y+6 \\ \hline 6y^2-20y+6 \end{array}$$

? Example 5.3.17

Multiply using the Vertical Method: $(5m-7)(3m-6)$.

Answer

$15m^2-51m+42$

Example 5.3.18

Multiply using the Vertical Method: $(6b-5)(7b-3)$.

Answer

$42b^2-53b+15$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

DEFINITION: MULTIPLYING TWO BINOMIALS

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Multiply a Polynomial by a Polynomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a polynomial by a polynomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method.

Example 5.3.19

Multiply $(b+3)(2b^2-5b+8)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

Answer

a.

	$(b+3)(2b^2-5b+8)$
Distribute.	$b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

b. It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2-5b+8)$ by 3. Multiply $(2b^2-5b+8)$ by b.	$\begin{array}{r} 2b^2-5b+8 \\ \times \quad b+3 \\ \hline 6b^2-15b+24 \end{array}$
Add like terms.	$\begin{array}{r} 2b^3-5b^2+8b \\ \hline 2b^3+b^2-7b+24 \end{array}$

Example 5.3.20

Multiply $(y-3)(y^2-5y+2)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

Answer

a. $y^3-8y^2+17y-6$
b. $y^3-8y^2+17y-6$

Example 5.3.21

Multiply $(x+4)(2x^2-3x+5)$ using a) the Distributive Property and b) The Vertical Method.

Answer

a. and b. $2x^3+5x^2-7x+20$

We have now seen two methods you can use to multiply a polynomial by a polynomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

DEFINITION: MULTIPLYING A POLYNOMIAL BY A POLYNOMIAL

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Multiply Special Products

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and multiplying them, there is less work to do if you learn to use a pattern. Let's start by looking at three examples and look for a pattern.

Look at these results. Do you see any patterns?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

What about the number of terms? In each example we squared a binomial and the result was a trinomial.

$$(a+b)^2 = \_\_\_ + \_\_\_ + \_\_\_$$

Now look at the *first term* in each result. Where did it come from?

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2 = a^2 + \_\_\_ + \_\_\_$$

To get the first term of the product, square the first term.

Where did the *last term* come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2 = \_\_\_ + \_\_\_ + b^2$$

To get the last term of the product, square the last term.

Finally, look at the *middle term*. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2 = \_\_\_ + 2ab + \_\_\_$$

$$(a-b)^2 = \_\_\_ - 2ab + \_\_\_$$

To get the middle term of the product, multiply the terms and double their product.

Putting it all together:

definition: BINOMIAL SQUARES PATTERN

If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

To square a binomial, square the first term, square the last term, double their product.

Example 5.3.22

Multiply: a. $(x+5)^2$ b. $(2x-3y)^2$.

Answer

a.

	$(x+5)^2$
Square the first term.	$x^2 + \_\_\_ + \_\_\_$
Square the last term.	$x^2 + \_\_\_ + 5^2$
Double their product.	$x^2 + 2 \cdot x \cdot 5 + 5^2$
Simplify.	$x^2 + 10x + 25$

b.

	$(2x-3y)^2$
Use the pattern.	$(2x)^2 - 2 \cdot 2x \cdot 3y + (3y)^2$
Simplify.	$4x^2 - 12xy + 9y^2$

Example 5.3.23

Multiply: a.$(x+9)^2$ b. $(2c-d)^2$.

Answer

a. $x^2+18x+81$
b. $4c^2-4cd+d^2$

Example 5.3.24

Multiply: a. $(y+11)^2$ b. $(4x-5y)^2$.

Answer

a. $y^2+22y+121$
b. $16x^2-40xy+25y^2$

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a **conjugate pair** and is of the form $(a-b)$, $(a+b)$.

definition: conjugate pair

A **conjugate pair** is two binomials of the form

$$(a-b),(a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier. Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$(x+9)(x-9)$ $\quad$ $(y-8)(y+8)$ $\quad$ $(2x-5)(2x+5)$

$x^2-9x+9x-81$ $\quad$ $y^2+8y-8y-64$ $\quad$ $4x^2+10x-10x-25$

x^2-81 $\quad$ y^2-64 $\quad$ $4x^2-25$

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Each *first term* is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\_\_\_$$

To get the first term, square the first term.

The *last term* came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the last term, square the last term.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form a^2-b^2. This is called a **difference of squares**.

This leads to the pattern:

definition: PRODUCT OF CONJUGATES PATTERN

If a and b are real numbers,

$(a-b)(a+b)=a^2-b^2$ $\quad$ $(a-b)(a+b)=a^2-b^2$ (*conjugates*; *difference*; *squares*)

The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

Example 5.3.25

Multiply using the product of conjugates pattern: a. $(2x+5)(2x-5)$ b. $(5m-9n)(5m+9n)$.

Answer

a.

Are the binomials conjugates?	$(2x+5)(2x-5)$
It is the product of conjugates.	$\left(\begin{smallmatrix}a+b\\2x+5\end{smallmatrix}\right)\left(\begin{smallmatrix}a-b\\2x-5\end{smallmatrix}\right)$
Square the first term, 2x.2x.	$\begin{smallmatrix}a^2-b^2\\(2x)^2-\_\_\end{smallmatrix}$
Square the last term, 5.5.	$\begin{smallmatrix}a^2-b^2\\(2x)^2-5^2\end{smallmatrix}$
Simplify. The product is a difference of squares.	$\begin{smallmatrix}a^2-b^2\\4x^2-25\end{smallmatrix}$

b.

	$(5m-9n)(5m+9n)$
This fits the pattern.	$\left(\begin{smallmatrix}a-b\\5m-9n\end{smallmatrix}\right)\left(\begin{smallmatrix}a+b\\5m+9n\end{smallmatrix}\right)$
Use the pattern.	$\begin{smallmatrix}a^2-b^2\\(5m)^2-(9n)^2\end{smallmatrix}$
Simplify.	$25m^2-81n^2$

Example 5.3.26

Multiply: a. $(6x+5)(6x-5)$ b. $(4p-7q)(4p+7q)$.

Answer

a. $36x^2-25$
b. $16p^2-49q^2$

Example 5.3.27

Multiply: a. $(2x+7)(2x-7)$ b.$(3x-y)(3x+y)$.

Answer

a. $4x^2-49$ b. $9x^2-y^2$

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

COMPARING THE SPECIAL PRODUCT PATTERNS

Binomial Squares	Product of Conjugates
$(a+b)^2=a^2+2ab+b^2$	$(a-b)(a+b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	
• Squaring a binomial	• Multiplying conjugates

Binomial Squares	Product of Conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

? Example 5.3.28

Choose the appropriate pattern and use it to find the product:

a. $(2x-3)(2x+3)$ b. $(8x-5)^2$ c. $(6m+7)^2$ d. $(5x-6)(6x+5)$.

Answer

a. $(2x-3)(2x+3)$

These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

	$(\underset{a}{2x}-\underset{b}{3})(\underset{a}{2x}+\underset{b}{3})$
Use the pattern.	$\underset{a^2}{(2x)^2}-\underset{b^2}{3^2}$
Simplify.	$4x^2-9$

b. $(8x-5)^2$

We are asked to square a binomial. It fits the binomial squares pattern.

	$(\underset{a}{8x}-\underset{b}{5})^2$
Use the pattern.	$\underset{a^2}{(8x)^2}-\underset{2ab}{2\cdot 8x\cdot 5}+\underset{b^2}{5^2}$
Simplify.	$64x^2-80x+25$

c. $(6m+7)^2$

Again, we will square a binomial so we use the binomial squares pattern.

	$(\underset{a}{6m}+\underset{b}{7})^2$
Use the pattern.	$\underset{a^2}{(6m)^2}+\underset{2ab}{2\cdot 6m\cdot 7}+\underset{b^2}{7^2}$
Simplify.	$36m^2+84m+49$

d. $(5x-6)(6x+5)$

This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2+25x-36x-30$
Simplify.	$30x^2-11x-30$

Example 5.3.29

Choose the appropriate pattern and use it to find the product:

a. $(9b-2)(2b+9)$ b. $(9p-4)^2$ c. $(7y+1)^2$ d. $(4r-3)(4r+3)$.

Answer

a. FOIL; $18b^2+77b-18$
b. Binomial Squares; $81p^2-72p+16$
c. Binomial Squares; $49y^2+14y+1$
d. Product of Conjugates; $16r^2-9$

Example 5.3.30

Choose the appropriate pattern and use it to find the product:

a. $(6x+7)^2$ b. $(3x-4)(3x+4)$ c. $(2x-5)(5x-2)$ d. $(6n-1)^2$.

Answer

a. Binomial Squares; $36x^2+84x+49$ b. Product of Conjugates; $9x^2-16$ c. FOIL; $10x^2-29x+10$ d. Binomial Squares; $36n^2-12n+1$

Multiply Polynomial Functions

Just as polynomials can be multiplied, polynomial functions can also be multiplied.

MULTIPLICATION OF POLYNOMIAL FUNCTIONS

For functions $f(x)$ and $g(x)$,

$$(f\cdot g)(x)=f(x)\cdot g(x) \tag{5.3.1}$$

Example 5.3.31

For functions $f(x)=x+2$ and $g(x)=x^2-3x-4$, find:

a. $(f\cdot g)(x)$
b. $(f\cdot g)(2)$.

Answer

a.

	$(f\cdot g)(x)=f(x)\cdot g(x)$
Substitute for $f(x)$ and $g(x)$	$(f\cdot g)(x)=(x+2)(x^2-3x-4)$
Multiply the polynomials.	$(f\cdot g)(x)=x(x^2-3x-4)+2(x^2-3x-4)$
Distribute.	$(f\cdot g)(x)=x3-3x^2-4x+2x^2-6x-8$
Combine like terms.	$(f\cdot g)(x)=x3-x^2-10x-8$

b. In part a. we found $(f\cdot g)(x)$ and now are asked to find $(f\cdot g)(2)$.

	$(f\cdot g)(x)=x^3-x^2-10x-8$
To find $(f\cdot g)(2)$, substitute $x=2$.	$(f\cdot g)(2)=2^3-2^2-10\cdot 2-8$
	$(f\cdot g)(2)=8-4-20-8$
	$(f\cdot g)(2)=-24$

Example 5.3.32

For functions $f(x) = x - 5$ and $g(x) = x^2 - 2x + 3$, find

a. $(f \cdot g)(x)$
b. $(f \cdot g)(2)$.

Answer a

$(f \cdot g)(x) = x^3 - 7x^2 + 13x - 15$

Answer b

$(f \cdot g)(2) = -9$

Example 5.3.33

For functions $f(x) = x - 7$ and $g(x) = x^2 + 8x + 4$, find

a. $(f \cdot g)(x)$
b. $(f \cdot g)(2)$.

Answer a

$(f \cdot g)(x) = x^3 + x^2 - 52x - 28$

Answer a

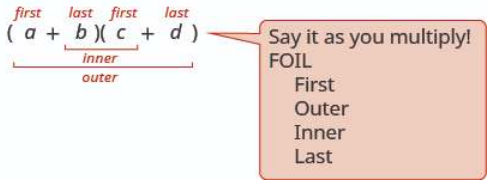
$(f \cdot g)(2) = -120$

Access this online resource for additional instruction and practice with multiplying polynomials.

- Introduction to special products of binomials

Key Concepts

- **How to use the FOIL method to multiply two binomials.**
  Step 1. Multiply the *First* terms.
  Step 2. Multiply the *Outer* terms.
  Step 3. Multiply the *Inner* terms.
  Step 4. Multiply the *Last* terms.
  Step 5. Combine like terms, when possible.



- **Multiplying Two Binomials:** To multiply binomials, use the:
  - Distributive Property
  - FOIL Method
- **Multiplying a Polynomial by a Polynomial:** To multiply a trinomial by a binomial, use the:
  - Distributive Property
  - Vertical Method
- **Binomial Squares Pattern**

$(a + b)^2 = a^2 + 2ab + b^2$

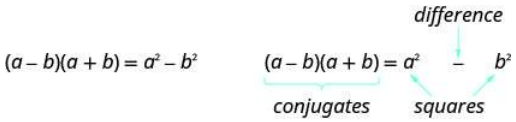
$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

$(a - b)^2 = a^2 - 2ab + b^2$

$\underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$

If a and b are real numbers,

- **Product of Conjugates Pattern**
  If a, b are real numbers



The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

- **Comparing the Special Product Patterns**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)^2 = a^2 - 2ab + b^2$
$(a-b)(a+b) = a^2 - b^2$	
• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

- **Multiplication of Polynomial Functions:**
  - For functions $f(x)$ and $g(x)$,

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Glossary

conjugate pair

A conjugate pair is two binomials of the form $(a-b)$ and $(a+b)$. The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

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5.3E: Exercises

Practice Makes Perfect

Multiply Monomials

In the following exercises, multiply the monomials.

1. ⓐ $(6y^7)(-3y^4)$ ⓑ $(\frac{4}{7}rs^2)(\frac{1}{4}rs^3)$

2. ⓐ $(-10x^5)(-3x^3)$ ⓑ $(58x^3y)(24x^5y)$

Answer

ⓐ $30x^8$ ⓑ $15x^8y^2$

3. ⓐ $(-8u^6)(-9u)$ ⓑ $(\frac{2}{3}x^2y)(\frac{3}{4}xy^2)$

4. ⓐ $(-6c^4)(-12c)$ ⓑ $(\frac{3}{5}m^3n^2)(\frac{5}{9}m^2n^3)$

Answer

ⓐ $72c^5$ ⓑ $\frac{1}{3}m^5n^5$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

5. ⓐ $-8x(x^2+2x-15)$ ⓑ $5pq^3(p^2-2pq+6q^2)$

6. ⓐ $-5t(t^2+3t-18)$ ⓑ $9r^3s(r^2-3rs+5s^2)$

Answer

ⓐ $-5t^3-15t^2+90t$
ⓑ $9sr^5-27s^2r^4+45s^3r^3$

7. ⓐ $-8y(y^2+2y-15)$ ⓑ $-4y^2z^2(3y^2+12yz-z^2)$

8. ⓐ $-5m(m^2+3m-18)$ ⓑ $-3x^2y^2(7x^2+10xy-y^2)$

Answer

ⓐ $-5m^3-15m^2+90m$
ⓑ $-21x^4y^2-30x^3y^3+3x^2y^4$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using ⓐ the Distributive Property; ⓑ the FOIL method; ⓒ the Vertical Method.

9. $(w+5)(w+7)$

10. $(y+9)(y+3)$

Answer

$y^2+12y+27$

11. $(4p+11)(5p-4)$

12. $(7q+4)(3q-8)$

Answer

$21q^2-44q-32$

In the following exercises, multiply the binomials. Use any method.

13. $(x+8)(x+3)$

14. $(y-6)(y-2)$

Answer

$y^2-8y+12$

15. $(2t-9)(10t+1)$

16. $(6p+5)(p+1)$

Answer

$6p^2+11p+5$

17. $(q-5)(q+8)$

18. $(m+11)(m-4)$

Answer

$m^2+7m-44$

19. $(7m+1)(m-3)$

20. $(3r-8)(11r+1)$

Answer

$33r^2-85r-8$

21. $(x^2+3)(x+2)$

22. $(y^2-4)(y+3)$

Answer

$y^3+3y^2-4y-12$

23. $(5ab-1)(2ab+3)$

24. $(2xy+3)(3xy+2)$

Answer

$6x^2y^2+13xy+6$

25. $(x^2+8)(x^2-5)$

26. $(y^2-7)(y^2-4)$

Answer

y^4-11y^2+28

27. $(6pq-3)(4pq-5)$

28. $(3rs-7)(3rs-4)$

Answer

$9r^2s^2-33rs+28$

Multiply a Polynomial by a Polynomial

In the following exercises, multiply using ⓐ the Distributive Property; ⓑ the Vertical Method.

29. $(x+5)(x^2+4x+3)$

30. $(u+4)(u^2+3u+2)$

Answer

$u^3+7u^2+14u+8$

31. $(y+8)(4y^2+y-7)$

32. $(a+10)(3a^2+a-5)$

Answer

$3a^3+31a^2+5a-50$

33. $(y^2-3y+8)(4y^2+y-7)$

34. $(2a^2-5a+10)(3a^2+a-5)$

Answer

$6a^4-13a^3+15a^2+35a-50$

Multiply Special Products

In the following exercises, multiply. Use either method.

35. $(w-7)(w^2-9w+10)$

36. $(p-4)(p^2-6p+9)$

Answer

$p^3-10p^2+33p-36$

37. $(3q+1)(q^2-4q-5)$

38. $(6r+1)(r^2-7r-9)$

Answer

$6r^3-41r^2-61r-9$

In the following exercises, square each binomial using the Binomial Squares Pattern.

39. $(w+4)^2$

40. $(q+12)^2$

Answer

$q^2+24q+144$

41. $(3x-y)^2$

42. $(2y-3z)^2$

Answer

$4y^2-12yz+9z^2$

43. $(y+\frac{1}{4})^2$

44. $(x+\frac{2}{3})^2$

Answer

$x^2+\frac{4}{3}x+\frac{4}{9}$

45. $(\frac{1}{5}x-\frac{1}{7}y)^2$

46. $(\frac{1}{8}x-\frac{1}{9}y)^2$

Answer

$\frac{1}{64}x^2-\frac{1}{36}xy+\frac{1}{81}y^2$

47. $(3x^2+2)^2$

48. $(5u^2+9)^2$

Answer

$25u^4+90u^2+81$

49. $(4y3-2)2$

50. $(8p3-3)2$

Answer

$64p^6-48p^3+9$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

51. $(5k+6)(5k-6)$

52. $(8j+4)(8j-4)$

Answer

$64j^2-16$

53. $(11k+4)(11k-4)$

54. $(9c+5)(9c-5)$

Answer

$81c^2-25$

55. $(9c-2d)(9c+2d)$

56. $(7w+10x)(7w-10x)$

Answer

$49w^2-100x^2$

57. $(m+\frac{2}{3}n)(m-\frac{2}{3}n)$

58. $(p+\frac{4}{5}q)(p-\frac{4}{5}q)$

Answer

$p^2-\frac{16}{25}q^2$

59. $(ab-4)(ab+4)$

60. $(xy-9)(xy+9)$

Answer

x^2y^2-81

61. $(12p^3-11q^2)(12p^3+11q^2)$

62. $(15m^2-8n^4)(15m^2+8n^4)$

Answer

$225m^4-64n^8$

In the following exercises, find each product.

63. $(p-3)(p+3)$

64. $(t-9)^2$

Answer

$t^2 - 18t + 81$

65. $(m+n)^2$

66. $(2x+y)(x-2y)$

Answer

$2x^2 - 3xy - 2y^2$

67. $(2r+12)^2$

68. $(3p+8)(3p-8)$

Answer

$9p^2 - 64$

69. $(7a+b)(a-7b)$

70. $(k-6)^2$

Answer

$k^2 - 12k + 36$

71. $(a^5 - 7b)^2$

72. $(x^2+8y)(8x-y^2)$

Answer

$8x^3 - x^2y^2 + 64xy - 8y^3$

73. $(r^6+s^6)(r^6-s^6)$

74. $(y^4+2z)^2$

Answer

$y^8 + 4y^4z + 4z^2$

75. $(x^5+y^5)(x^5-y^5)$

76. $(m^3-8n)^2$

Answer

$m^6 - 16m^3n + 64n^2$

77. $(9p+8q)^2$

78. $(r^2-s^3)(r^3+s^2)$

Answer

$r^5 + r^2 s^2 - r^3 s^3 - s^5$

Mixed Practice

79. $(10y - 6) + (4y - 7)$

80. $(15p - 4) + (3p - 5)$

Answer

$18p - 9$

81. $(x^2 - 4x - 34) - (x^2 + 7x - 6)$

82. $(j^2 - 8j - 27) - (j^2 + 2j - 12)$

Answer

$-10j - 15$

83. $(\frac{1}{5} f^8)(20f^3)$

84. $(\frac{1}{4} d^5)(36d^2)$

Answer

$9d^7$

85. $(4a^3 b)(9a^2 b^6)$

86. $(6m^4 n^3)(7mn^5)$

Answer

$72m^5 n^8$

87. $-5m(m^2 + 3m - 18)$

88. $5q^3(q^2 - 2q + 6)$

Answer

$5q^5 - 10q^4 + 30q^3$

89. $(s - 7)(s + 9)$

90. $(y^2 - 2y)(y + 1)$

Answer

$y^3 - y^2 - 2y$

91. $(5x - y)(x - 4)$

92. $(6k-1)(k^2+2k-4)$

Answer

$6k^3-11k^2-26k+4$

93. $(3x-11y)(3x-11y)$

94. $(11-b)(11+b)$

Answer

$121-b^2$

95. $(rs-\frac{2}{7})(rs+\frac{2}{7})$

96. $(2x^2-3y^4)(2x^2+3y^4)$

Answer

$4x^4-9y^8$

97. $(m-15)^2$

98. $(3d+1)^2$

Answer

$9d^2+6d+1$

99. $(4a+10)^2$

100. $(3z+15)^2$

Answer

$9z^2-\frac{6}{5}z+\frac{1}{25}$

Multiply Polynomial Functions

101. For functions $f(x)=x+2$ and $g(x)=3x^2-2x+4$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-1)$

102. For functions $f(x)=x-1$ and $g(x)=4x^2+3x-5$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-2)$

Answer

ⓐ $(f\cdot g)(x)=4x^3-x^2-8x+5$
ⓑ $(f\cdot g)(-2)=-15$

103. For functions $f(x)=2x-7$ and $g(x)=2x+7$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-3)$

104. For functions $f(x)=7x-8$ and $g(x)=7x+8$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-2)$

Answer

ⓐ $(f\cdot g)(x)=49x^2-64$
ⓑ $(f\cdot g)(-2)=187$

105. For functions $f(x)=x^2-5x+2$ and $g(x)=x^2-3x-1$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-1)$

106. For functions $f(x)=x^2+4x-3$ and $g(x)=x^2+2x+4$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(1)$

Answer

ⓐ $(f\cdot g)(x)=x^4+6x^3+9x^2+10x-12$ ⓑ $(f\cdot g)(1)=14$

Writing Exercises

107. Which method do you prefer to use when multiplying two binomials: the Distributive Property or the FOIL method? Why? Which method do you prefer to use when multiplying a polynomial by a polynomial: the Distributive Property or the Vertical Method? Why?

108. Multiply the following:

$(x+2)(x-2)$

$(y+7)(y-7)$

$(w+5)(w-5)$

Explain the pattern that you see in your answers.

Answer

Answers will vary.

109. Multiply the following:

$(p+3)(p+3)$

$(q+6)(q+6)$

$(r+1)(r+1)$

Explain the pattern that you see in your answers.

110. Why does $(a+b)^2$ result in a trinomial, but $(a-b)(a+b)$ result in a binomial?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
multiply monomials.			
multiply a polynomial by a monomial.			
multiply a binomial by a binomial.			
multiply a polynomial by a polynomial.			
multiply special products.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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5.4: Dividing Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Divide monomials
- Divide a polynomial by a monomial
- Divide polynomials using long division
- Divide polynomials using synthetic division
- Divide polynomial functions
- Use the remainder and factor theorems

Before you get started, take this readiness quiz.

1. Add: $\frac{3}{d}+\frac{x}{d}$.
   If you missed this problem, review [link].
2. Simplify: $\frac{30xy}{35xy}$.
   If you missed this problem, review [link].
3. Combine like terms: $8a^2+12a+1+3a^2-5a+4$.
   If you missed this problem, review [link].

Dividing Monomials

We are now familiar with all the properties of exponents and used them to multiply polynomials. Next, we'll use these properties to divide monomials and polynomials.

Example 5.4.1

Find the quotient: $54a^2b^3 \div (-6ab^5)$.

Solution

When we divide monomials with more than one variable, we write one fraction for each variable.

	$54a^2b^3 \div (-6ab^5)$
Rewrite as a fraction.	$\frac{54a^2b^3}{-6ab^5}$
Use fraction multiplication.	$\frac{54}{-6}\cdot\frac{a^2}{a}\cdot\frac{b^3}{b^5}$
Simplify and use the Quotient Property.	$-9\cdot a\cdot\frac{1}{b^2}$
Multiply.	$-\frac{9a}{b^2}$

Try It! 5.4.1

Find the quotient: $-72a^7b^3 \div (8a^{12}b^4)$.

Answer

$-\frac{9}{a^5b}$

Try It! 5.4.2

Find the quotient: $-63c^8d^3 \div (7c^{12}d^2)$.

Answer

$$\frac{-9d}{c^4}$$

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

Example 5.4.2

Find the quotient: $\frac{14x^7y^{12}}{21x^{11}y^6}$.

Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\frac{14x^7y^{12}}{21x^{11}y^6}$
Simplify and use the Quotient Property.	$\frac{2y^6}{3x^4}$

Try It! 5.4.3

Find the quotient: $\frac{28x^5y^{14}}{49x^9y^{12}}$.

Answer

$$\frac{4y^2}{7x^4}$$

Try It! 5.4.4

Find the quotient: $\frac{30m^5n^{11}}{48m^{10}n^{14}}$.

Answer

$$\frac{5}{8m^5n^3}$$

Divide a Polynomial by a Monomial

Now that we know how to divide a monomial by a monomial, the next procedure is to divide a polynomial of two or more terms by a monomial. The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition. The sum $\frac{y}{5}+\frac{2}{5}$ simplifies to $\frac{y+2}{5}$. Now we will do this in reverse to split a single fraction into separate fractions. For example, $\frac{y+2}{5}$ can be written $\frac{y}{5}+\frac{2}{5}$.

This is the "reverse" of fraction addition and it states that if *a*, *b*, and *c* are numbers where $c \neq 0$, then $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$. We will use this to divide polynomials by monomials.

definition: DIVISION OF A POLYNOMIAL BY A MONOMIAL

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example 5.4.3

Find the quotient: $(18x^3y - 36xy^2) \div (-3xy)$.

Solution

	$(18x^3y - 36xy^2) \div (-3xy)$
Rewrite as a fraction.	$\frac{18x^3y - 36xy^2}{-3xy}$
Divide each term by the divisor. Be careful with the signs!	$\frac{18x^3y}{-3xy} - \frac{36xy^2}{-3xy}$
Simplify.	$-6x^2 + 12y$

Try It! 5.4.5

Find the quotient: $(32a^2b - 16ab^2) \div (-8ab)$.

Answer

$-4a + 2b$

Try It! 5.4.6

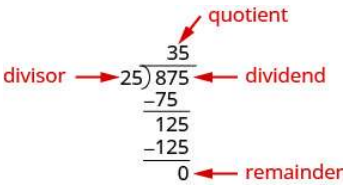
Find the quotient: $(-48a^8b^4 - 36a^6b^5) \div (-6a^3b^3)$.

Answer

$8a^5b + 6a^3b^2$

Divide Polynomials Using Long Division

Divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let's look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.



We check division by multiplying the quotient by the divisor. If we did the division correctly, the product should equal the dividend.

$$35 \cdot 25$$
$$875\checkmark$$

Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

Example 5.4.4

Find the quotient: $(x^2+9x+20)\div(x+5)$.

Solution

	$(x^2+9x+20)\div(x+5)$
Write it as a long division problem. Be sure the dividend is in standard form.	$x+5\overline{)x^2+9x+20}$
Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?"	$\begin{array}{r} x \\ x+5\overline{)x^2+9x+20} \end{array}$
Put the answer, x, in the quotient over the x term. Multiply x times $x+5$. Line up the like terms under the dividend.	$\begin{array}{r} x \\ x+5\overline{)x^2+9x+20} \\ \underline{x^2+5x} \end{array}$
Subtract x^2+5x from x^2+9x . You may find it easier to change the signs and then add. Then bring down the last term, 20.	$\begin{array}{r} x \\ x+5\overline{)x^2+\ \ 9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \end{array}$
Divide $4x$ by x. It may help to ask yourself, "What do I need to multiply x by to get $4x$?" Put the answer, 4, in the quotient over the constant term.	$\begin{array}{r} x+\ 4 \\ x+5\overline{)x^2+\ \ 9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \end{array}$
Multiply 4 times $x+5$.	$\begin{array}{r} x+\ 4 \\ x+5\overline{)x^2+\ \ 9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{4x+20} \end{array}$
Subtract $4x+20$ from $4x+20$.	$\begin{array}{r} x+\ 4 \\ x+5\overline{)x^2+\ \ 9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{-4x+(-20)} \\ 0 \end{array}$
Check: Multiply the quotient by the divisor. $(x+4)(x+5)$ You should get the dividend. $x^2+9x+20$✓	

Try It! 5.4.7

Find the quotient: $(y^2+10y+21)\div(y+3)$.

Answer

$y+7$

Try It! 5.4.8

Find the quotient: $(m^2+9m+20)\div(m+4)$.

Answer

$m+5$

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In the next example, we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

Look back at the dividends in previous examples. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in this example will be $x^4 - x^2 + 5x - 6$. It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

Example 5.4.5

Find the quotient: $(x^4 - x^2 + 5x - 6) \div (x + 2)$.

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

	$(x^4 - x^2 + 5x - 6) \div (x + 2)$
Write it as a long division problem. Be sure the dividend is in standard form with placeholders for missing terms.	$x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$
Divide x^4 by x. Put the answer, x^3, in the quotient over the x^3 term. Multiply x^3 times $x + 2$. Line up the like terms. Subtract and then bring down the next term.	x^3 $x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ It may be helpful to change the signs and add.
Divide $-2x^3$ by x. Put the answer, $-2x^2$, in the quotient over the x^2 term. Multiply $-2x^2$ times $x + 1$. Line up the like terms Subtract and bring down the next term.	$x^3 - 2x^2$ $x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ It may be helpful to change the signs and add.
Divide $3x^2$ by x. Put the answer, $3x$, in the quotient over the x term. Multiply $3x$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	$x^3 - 2x^2 + 3x$ $x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ $-(3x^2 + 6x)$ $-x - 6$ It may be helpful to change the signs and add.
Divide $-x$ by x. Put the answer, -1, in the quotient over the constant term. Multiply -1 times $x + 1$. Line up the like terms. Change the signs, add. Write the remainder as a fraction with the divisor as the denominator.	$x^3 - 2x^2 + 3x - 1 - \frac{4}{x+2}$ $x + 2 \overline{)x^4 + 0x^3 - x^2 + 5x - 6}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ $-(3x^2 + 6x)$ $-x - 6$ $-(-x - 2)$ -4 It may be helpful to change the signs and add.
To check, multiply $(x + 2)(x^3 - 2x^2 + 3x - 1 - 4x + 2)$. The result should be $x^4 - x^2 + 5x - 6$.	

Try It! 5.4.9

Find the quotient: $(x^4 - 7x^2 + 7x + 6) \div (x + 3)$.

Answer

$x^3-3x^2+2x+1+3x+3$

Try It! 5.4.10

Find the quotient: $(x^4-11x^2-7x-6)\div(x+3)$.

Answer

$x^3-3x^2-2x-1-3x+3$

In the next example, we will divide by $2a-3$. As we divide, we will have to consider the constants as well as the variables.

Example 5.4.6

Find the quotient: $(8a^3+27)\div(2a+3)$.

Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.

$(8a^3+27)\div(2a+3)$

$$
\begin{array}{r}
4a^2-6a+9 \\
2a+3\overline{)\,8a^3+0a^2+0a+27} \\
\underline{-(8a^3+12a^2)} \qquad \leftarrow 4a^2(2a+3) \\
-12a^2+0a \\
\underline{-(-12a^2-18a)} \qquad \leftarrow 6a(2a+3) \\
18a+27 \\
\underline{-(18a+27)} \qquad \leftarrow 9(2a+3) \\
0
\end{array}
$$

To check, multiply $(2a+3)(4a^2-6a+9)$.

The result should be $8a^3+27$.

Try It! 5.4.11

Find the quotient: $(x^3-64)\div(x-4)$.

Answer

$x^2+4x+16$

Try It! 5.4.12

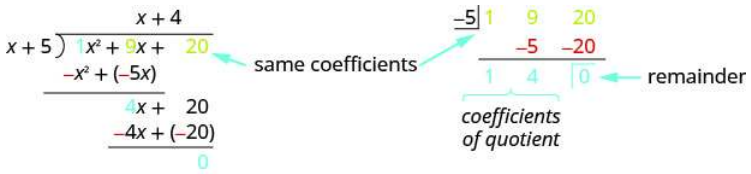
Find the quotient: $(125x^3-8)\div(5x-2)$.

Answer

$25x^2+10x+4$

Divide Polynomials using Synthetic Division

As we have mentioned before, mathematicians like to find patterns to make their work easier. Since long division can be tedious, let's look back at the long division we did in Example and look for some patterns. We will use this as a basis for what is called *synthetic division*. The same problem in the synthetic division format is shown next.



Synthetic division basically just removes unnecessary repeated variables and numbers. Here all the x and x^2 are removed. as well as the $-x^2$ and $-4x$ as they are opposite the term above.

- The first row of the synthetic division is the coefficients of the dividend. The -5 is the opposite of the 5 in the divisor.
- The second row of the synthetic division are the numbers shown in red in the division problem.
- The third row of the synthetic division are the numbers shown in blue in the division problem.

Notice the quotient and remainder are shown in the third row.

Synthetic division only works when the divisor is of the form $x - c$.

The following example will explain the process.

Example 5.4.7

Use synthetic division to find the quotient and remainder when $2x^3 + 3x^2 + x + 8$ is divided by $x + 2$.

Solution

Write the dividend with decreasing powers of x.	$2x^3 + 3x^2 + x + 8$
Write the coefficients of the terms as the first row of the synthetic division.	⌐ | 2 3 1 8
Write the divisor as $x - c$ and place c in the synthetic division in the divisor box.	–2 | 2 3 1 8
Bring down the first coefficient to the third row.	–2 | 2 3 1 8 2
Multiply that coefficient by the divisor and place the result in the second row under the second coefficient.	–2 | 2 3 1 8 –4 2
Add the second column, putting the result in the third row.	–2 | 2 3 1 8 –4 2 –1
Multiply that result by the divisor and place the result in the second row under the third coefficient.	–2 | 2 3 1 8 –4 2 2 –1
Add the third column, putting the result in the third row.	–2 | 2 3 1 8 –4 2 2 –1 3
Multiply that result by the divisor and place the result in the third row under the third coefficient.	–2 | 2 3 1 8 –4 2 –6 2 –1 3
Add the final column, putting the result in the third row.	–2 | 2 3 1 8 –4 2 –6 2 –1 3 | 2
The quotient is $2x^2 - 1x + 3$ and the remainder is 2.	

The division is complete. The numbers in the third row give us the result. The $2 \quad -1 \quad 3$ are the coefficients of the quotient. The quotient is $2x^2 - 1x + 3$. The 2 in the box in the third row is the remainder.

Check:

$$\begin{aligned} (\text{quotient})(\text{divisor}) + \text{remainder} &= \text{dividend} \\ (2x^2 - 1x + 3)(x + 2) + 2 &\stackrel{?}{=} 2x^3 + 3x^2 + x + 8 \\ 2x^3 - x^2 + 3x + 4x^2 - 2x + 6 + 2 &\stackrel{?}{=} 2x^3 + 3x^2 + x + 8 \\ 2x^3 + 3x^2 + x + 8 &= 2x^3 + 3x^2 + x + 8 \checkmark \end{aligned}$$

? Try It! 5.4.13

Use synthetic division to find the quotient and remainder when $3x^3 + 10x^2 + 6x - 2$ is divided by $x + 2$.

Answer

$3x^2 + 4x - 2;\ 2$

? Try It! 5.4.14

Use synthetic division to find the quotient and remainder when $4x^3 + 5x^2 - 5x + 3$ is divided by $x + 2$.

Answer

$4x^2 - 3x + 1; 1$

In the next example, we will do all the steps together.

✓ Example 5.4.8

Use synthetic division to find the quotient and remainder when $x^4 - 16x^2 + 3x + 12$ is divided by $x + 4$.

Solution

The polynomial $x^4 - 16x^2 + 3x + 12$ has its term in order with descending degree but we notice there is no x^3 term. We will add a 0 as a placeholder for the x^3 term. In $x - c$ form, the divisor is $x - (-4)$.

–4	1	0	–16	3	12
		–4	16	0	–12
	1	–4	0	3	0

We divided a 4th degree polynomial by a 1st degree polynomial so the quotient will be a 3rd degree polynomial.

Reading from the third row, the quotient has the coefficients $1 \quad -4 \quad 0 \quad 3$, which is $x^3 - 4x^2 + 3$. The remainder is 0.

? Try It! 5.4.15

Use synthetic division to find the quotient and remainder when $x^4 - 16x^2 + 5x + 20$ is divided by $x + 4$.

Answer

$x^3 - 4x^2 + 5;\ 0$

? Try It! 5.4.16

Use synthetic division to find the quotient and remainder when $x^4 - 9x^2 + 2x + 6$ is divided by $x + 3$.

Answer

$x^3 - 3x^2 + 2;\ 0$

Divide Polynomial Functions

Just as polynomials can be divided, polynomial functions can also be divided.

definition: DIVISION OF POLYNOMIAL FUNCTIONS

For functions $f(x)$ and $g(x)$, where $g(x) \neq 0$,

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example 5.4.9

For functions $f(x) = x^2 - 5x - 14$ and $g(x) = x + 2$, find:

a. $\left(\frac{f}{g}\right)(x)$

b. $\left(\frac{f}{g}\right)(-4)$.

Solution

ⓐ

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$\begin{array}{r} x - 7 \\ x + 2\overline{)x^2 - 5x - 14} \\ \underline{-(x^2 + 2x)} \\ -7x - 14 \\ \underline{-(-7x - 14)} \\ 0 \end{array}$$

Substitute for $f(x)$ and $g(x)$. $\left(\frac{f}{g}\right)(x) = \frac{x^2 - 5x - 14}{x + 2}$

Divide the polynomials. $\left(\frac{f}{g}\right)(x) = x - 7$

ⓑ In part ⓐ we found $\left(\frac{f}{g}\right)(x)$ and now are asked to find $\left(\frac{f}{g}\right)(-4)$.

$\left(\frac{f}{g}\right)(x) = x - 7$

To find $\left(\frac{f}{g}\right)(-4)$, substitute $x = -4$. $\left(\frac{f}{g}\right)(-4) = -4 - 7$

$\left(\frac{f}{g}\right)(-4) = -11$

Try It! 5.4.17

For functions $f(x) = x^2 - 5x - 24$ and $g(x) = x + 3$, find:

a. $\left(\frac{f}{g}\right)(x)$

b. $\left(\frac{f}{g}\right)(-3)$.

Answer a

$\left(\frac{f}{g}\right)(x) = x - 8$

Answer b

$\left(\frac{f}{g}\right)(-3) = -11$

? Try It! 5.4.18

For functions $f(x) = x2 - 5x - 36$ and $g(x) = x + 4$, find:

a. $\left(\frac{f}{g}\right)(x)$

b. $\left(\frac{f}{g}\right)(-5)$.

Answer a

$\left(\frac{f}{g}\right)(x) = x - 9$

Answer b

$\left(\frac{f}{g}\right)(x) = x - 9$

Use the Remainder and Factor Theorem

Let's look at the division problems we have just worked that ended up with a remainder. They are summarized in the chart below. If we take the dividend from each division problem and use it to define a function, we get the functions shown in the chart. When the divisor is written as $x - c$, the value of the function at c, $f(c)$, is the same as the remainder from the division problem.

Dividend	Divisor $x - c$	Remainder	Function	$f(c)$
$x^4 - x^2 + 5x - 6$	$x - (-2)$	-4	$f(x) = x^4 - x^2 + 5x - 6$	-4
$3x^3 - 2x^2 - 10x + 8$	$x - 2$	4	$f(x) = 3x^3 - 2x^2 - 10x + 8$	4
$x^4 - 16x^2 + 3x + 15$	$x - (-4)$	3	$f(x) = x^4 - 16x^2 + 3x + 15$	3

To see this more generally, we realize we can check a division problem by multiplying the quotient times the divisor and add the remainder. In function notation we could say, to get the dividend $f(x)$, we multiply the quotient, $q(x)$ times the divisor, $x - c$, and add the remainder, r.

	$f(x) = q(x)(x - c) + r$
If we evaluate this at c, we get:	$f(c) = q(c)(c - c) + r$
	$f(c) = q(c)(0) + r$
	$f(c) = r$

This leads us to the Remainder Theorem.

Definition: REMAINDER THEOREM

If the polynomial function $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Example 5.4.10

Use the Remainder Theorem to find the remainder when $f(x) = x^3 + 3x + 19$ is divided by $x + 2$.

Solution

To use the Remainder Theorem, we must use the divisor in the $x - c$ form. We can write the divisor $x + 2$ as $x - (-2)$. So, our c is -2.

To find the remainder, we evaluate $f(c)$ which is $f(-2)$.

	$f(x) = x^3 + 3x + 19$
To evaluate $f(-2)$, substitute $x = -2$.	$f(-2) = (-2)^3 + 3(-2) + 19$
Simplify.	$f(-2) = -8 - 6 + 19$
	$f(-2) = 5$
	The remainder is 5 when $f(x) = x^3 + 3x + 19$ is divided by $x + 2$.
Check: Use synthetic division to check.	
$\begin{array}{r\|rrr} -2 & 1 & 0 & 3 & 19 \\ & & -2 & 4 & -14 \\ \hline & 1 & -2 & 7 & \boxed{5} \end{array}$	
The remainder is 5.	

Try It! 5.4.19

Use the Remainder Theorem to find the remainder when $f(x) = x^3 + 4x + 15$ is divided by $x + 2$.

Answer

-1

Try It! 5.4.20

Use the Remainder Theorem to find the remainder when $f(x) = x^3 - 7x + 12$ is divided by $x + 3$.

Answer

6

When we divided $8a^3 + 27$ by $2a + 3$ in Example the result was $4a^2 - 6a + 9$. To check our work, we multiply $4a2 - 6a + 9$ by $2a + 3$ to get $8a^3 + 27$.

$$(4a^2 - 6a + 9)(2a + 3) = 8a^3 + 27$$

Written this way, we can see that $4a^2 - 6a + 9$ and $2a + 3$ are factors of $8a^3 + 27$. When we did the division, the remainder was zero.

Whenever a divisor, $x - c$, divides a polynomial function, $f(x)$, and resulting in a remainder of zero, we say $x - c$ is a factor of $f(x)$.

The reverse is also true. If $x - c$ is a factor of $f(x)$ then $x - c$ will divide the polynomial function resulting in a remainder of zero.

We will state this in the Factor Theorem.

Definition: FACTOR THEOREM

For any polynomial function $f(x)$,

- if $x-c$ is a factor of $f(x)$, then $f(c)=0$
- if $f(c)=0$, then $x-c$ is a factor of $f(x)$

Example 5.4.11

Use the Remainder Theorem to determine if $x-4$ is a factor of $f(x)=x^3-64$.

Solution

The Factor Theorem tells us that $x-4$ is a factor of $f(x)=x^3-64$ if $f(4)=0$.

	$f(x)=x^3-64$
To evaluate $f(4)$ substitute $x=4$.	$f(4)=4^3-64$
Simplify.	$f(4)=64-64$
Subtract.	$f(4)=0$

Since $f(4)=0$, $x-4$ is a factor of $f(x)=x^3-64$.

Try It! 5.4.21

Use the Factor Theorem to determine if $x-5$ is a factor of $f(x)=x^3-125$.

Answer

yes

Try It! 5.4.22

Use the Factor Theorem to determine if $x-6$ is a factor of $f(x)=x^3-216$.

Answer

yes

Access these online resources for additional instruction and practice with dividing polynomials.

- Dividing a Polynomial by a Binomial
- Synthetic Division & Remainder Theorem

Key Concepts

- **Division of a Polynomial by a Monomial**
  - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.
- **Division of Polynomial Functions**
  - For functions $f(x)$ and $g(x)$, where $g(x)\neq 0$,
    $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$
- **Remainder Theorem**
  - If the polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.
- **Factor Theorem:** For any polynomial function $f(x)$,
  - if $x-c$ is a factor of $f(x)$, then $f(c)=0$
  - if $f(c)=0$, then $x-c$ is a factor of $f(x)$

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5.4E: Exercises

Practice Makes Perfect

Divide Monomials

In the following exercises, divide the monomials.

1. $15r^4s^9 \div (15r^4s^9)$

2. $20m^8n^4 \div (30m^5n^9)$

Answer

$\frac{2m^3}{3n^5}$

3. $\frac{18a^4b^8}{-27a^9b^5}$

4. $\frac{45x^5y^9}{-60x^8y^6}$

Answer

$\frac{-3y^3}{4x^3}$

5. $\frac{(10m^5n^4)(5m^3n^6)}{25m^7n^5}$

6. $\frac{(-18p^4q^7)(-6p^3q^8)}{-36p^{12}q^{10}}$

Answer

$\frac{-3q^5}{p^5}$

7. $\frac{(6a^4b^3)(4ab^5)}{(12a^2b)(a^3b)}$

8. $\frac{(4u^2v^5)(15u^3v)}{(12u^3v)(u^4v)}$

Answer

$\frac{5v^4}{u^2}$

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.

9. $(9n^4 + 6n^3) \div 3n$

10. $(8x^3 + 6x^2) \div 2x$

Answer

$4x^2+3x$

11. $(63m^4-42m^3)\div(-7m^2)$

12. $(48y^4-24y^3)\div(-8y^2)$

Answer

$-6y^2+3y$

13. $\dfrac{66x^3y^2-110x^2y^3-44x^4y^3}{11x^2y^2}$

14. $\dfrac{72r^5s^2+132r^4s^3-96r^3s^5}{12r^2s^2}$

Answer

$6r^3+11r^2s-8rs^3$

15. $10x^2+5x-4-5x$

16. $20y^2+12y-1-4y$

Answer

$-5y-3+\dfrac{1}{4y}$

Divide Polynomials using Long Division

In the following exercises, divide each polynomial by the binomial.

17. $(y^2+7y+12)\div(y+3)$

18. $(a^2-2a-35)\div(a+5)$

Answer

$a-7$

19. $(6m^2-19m-20)\div(m-4)$

20. $(4x^2-17x-15)\div(x-5)$

Answer

$4x+3$

21. $(q^2+2q+20)\div(q+6)$

22. $(p^2+11p+16)\div(p+8)$

Answer

$$p+3-\frac{8}{p+8}$$

23. $(3b^3+b^2+4)\div(b+1)$

24. $(2n^3-10n+28)\div(n+3)$

Answer

$$\frac{2n^2-6n+8+4}{n+3}$$

25. $(z^3+1)\div(z+1)$

26. $(m^3+1000)\div(m+10)$

Answer

$m^2-10m+100$

27. $(64x^3-27)\div(4x-3)$

28. $(125y^3-64)\div(5y-4)$

Answer

$25y^2+20x+16$

Divide Polynomials using Synthetic Division

In the following exercises, use synthetic Division to find the quotient and remainder.

29. $x^3-6x^2+5x+14$ is divided by $x+1$

30. $x^3-3x^2-4x+12$ is divided by $x+2$

Answer

$x^2-5x+6;\ 0$

31. $2x^3-11x^2+11x+12$ is divided by $x-3$

32. $2x^3-11x^2+16x-12$ is divided by $x-4$

Answer

$2x^2-3x+4;\ 4$

33. $x^4-5x^2+2+13x+3$ is divided by $x+3$

34. $x^4+x^2+6x-10$ is divided by $x+2$

Answer

$x^3-2x^2+5x-4;\ -2$

35. $2x^4-9x^3+5x^2-3x-6$ is divided by $x-4$

36. $3x^4-11x^3+2x^2+10x+6$ is divided by $x-3$

Answer

$3x^3-2x^2-4x-2;\ 0$

Divide Polynomial Functions

In the following exercises, divide.

37. For functions $f(x)=x^2-13x+36$ and $g(x)=x-4$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-1)$

38. For functions $f(x)=x^2-15x+54$ and $g(x)=x-9$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(-5)$

Answer

ⓐ $\left(\frac{f}{g}\right)(x)=x-6$

ⓑ $\left(\frac{f}{g}\right)(-5)=-11$

39. For functions $f(x)=x^3+x^2-7x+2$ and $g(x)=x-2$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(2)$

40. For functions $f(x)=x^3+2x^2-19x+12$ and $g(x)=x-3$, find ⓐ $\left(\frac{f}{g}\right)(x)$ ⓑ $\left(\frac{f}{g}\right)(0)$

Answer

ⓐ $\left(\frac{f}{g}\right)(x)=x^2+5x-4$

ⓑ $\left(\frac{f}{g}\right)(0)=-4$

41. For functions $f(x)=x^2-5x+2$ and $g(x)=x^2-3x-1$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(-1)$

42. For functions $f(x)=x^2+4x-3$ and $g(x)=x^2+2x+4$, find ⓐ $(f\cdot g)(x)$ ⓑ $(f\cdot g)(1)$

Answer

ⓐ $(f\cdot g)(x)=x^4+6x^3+9x^2+10x-12$; ⓑ $(f\cdot g)(1)=14$

Use the Remainder and Factor Theorem

In the following exercises, use the Remainder Theorem to find the remainder.

43. $f(x)=x^3-8x+7$ is divided by $x+3$

44. $f(x)=x^3-4x-9$ is divided by $x+2$

Answer

-9

45. $f(x) = 2x^3 - 6x - 24$ divided by $x - 3$

46. $f(x) = 7x^2 - 5x - 8$ divided by $x - 1$

Answer

-6

In the following exercises, use the Factor Theorem to determine if x−cx−c is a factor of the polynomial function.

47. Determine whether $x + 3$ a factor of $x^3 + 8x^2 + 21x + 18$

48. Determine whether $x + 4$ a factor of $x^3 + x^2 - 14x + 8$

Answer

no

49. Determine whether $x - 2$ a factor of $x^3 - 7x^2 + 7x - 6$

50. Determine whether $x - 3$ a factor of $x^3 - 7x^2 + 11x + 3$

Answer

yes

Writing Exercises

51. James divides $48y + 6$ by 6 this way: $\frac{48y + 6}{6} = 48y$. What is wrong with his reasoning?

52. Divide $\frac{10x^2 + x - 12}{2x}$ and explain with words how you get each term of the quotient.

Answer

Answer will vary

53. Explain when you can use synthetic division.

54. In your own words, write the steps for synthetic division for $x^2 + 5x + 6$ divided by $x - 2$.

Answer

Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section

I can...	Confidently	With some help	No-I don't get it!
divide monomials.			
divide a polynomial by a monomial.			
divide polynomials using long division.			
divide polynomials synthetic division.			
divide polynomial functions.			
use the Remainder and Factor Theorem.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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Chapter 5 Review Exercises

Chapter Review Exercises

Add and Subtract Polynomials

Determine the Degree of Polynomials

In the following exercises, determine the type of polynomial.

1. $16x^2 - 40x - 25$

2. $5m + 9$

Answer

binomial

3. -15

4. $y^2 + 6y^3 + 9y^4$

Answer

other polynomial

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

5. $4p + 11p$

6. $-8y^3 - 5y^3$

Answer

$-13y^3$

7. $(4a^2 + 9a - 11) + (6a^2 - 5a + 10)$

8. $(8m^2 + 12m - 5) - (2m^2 - 7m - 1)$

Answer

$6m^2 + 19m - 4$

9. $(y^2 - 3y + 12) + (5y^2 - 9)$

10. $(5u^2 + 8u) - (4u - 7)$

Answer

$5u^2 + 4u + 7$

11. Find the sum of $8q^3 - 27$ and $q^2 + 6q - 2$.

12. Find the difference of $x^2 + 6x + 8$ and $x^2 - 8x + 15$.

Answer

$2x^2-2x+23$

In the following exercises, simplify.

13. $17mn^2-(-9mn^2)+3mn^2$

14. $18a-7b-21a$

Answer

$-7b-3a$

15. $2pq^2-5p-3q^2$

16. $(6a^2+7)+(2a^2-5a-9)$

Answer

$8a^2-5a-2$

17. $(3p^2-4p-9)+(5p^2+14)$

18. $(7m^2-2m-5)-(4m^2+m-8)$

Answer

$-3m+3$

19. $(7b^2-4b+3)-(8b^2-5b-7)$

20. Subtract $(8y^2-y+9)$ from $(11y^2-9y-5)$

Answer

$3y^2-8y-14$

21. Find the difference of $(z^2-4z-12)$ and $(3z^2+2z-11)$

22. $(x^3-x^2y)-(4xy^2-y^3)+(3x^2y-xy^2)$

Answer

$x^3+2x^2y-4xy^2$

23. $(x^3-2x^2y)-(xy^2-3y^3)-(x^2y-4xy^2)$

Evaluate a Polynomial Function for a Given Value of the Variable

In the following exercises, find the function values for each polynomial function.

24. For the function $f(x)=7x^2-3x+5$ find:
a. $f(5)$ b. $f(-2)$ c. $f(0)$

Answer

a. 165 b. 39 c. 5

25. For the function $g(x) = 15 - 16x^2$, find:
a. $g(-1)$ b. $g(0)$ c. $g(2)$

26. A pair of glasses is dropped off a bridge 640 feet above a river. The polynomial function $h(t) = -16t^2 + 640$ gives the height of the glasses *t* seconds after they were dropped. Find the height of the glasses when $t = 6$.

Answer

The height is 64 feet.

27. A manufacturer of the latest soccer shoes has found that the revenue received from selling the shoes at a cost of p dollars each is given by the polynomial $R(p) = -5p^2 + 360p$. Find the revenue received when $p = 110$ dollars.

Add and Subtract Polynomial Functions

In the following exercises, find a. $(f+g)(x)$ b. $(f+g)(3)$ c. $(f-g)(x$ d. $(f-g)(-2)$

28. $f(x) = 2x^2 - 4x - 7$ and $g(x) = 2x^2 - x + 5$

Answer

a. $(f+g)(x) = 4x^2 - 5x - 2$
b. $(f+g)(3) = 19$
c. $(f-g)(x) = -3x - 12$
d. $(f-g)(-2) = -6$

29. $f(x) = 4x^3 - 3x^2 + x - 1$ and $g(x) = 8x^3 - 1$

Properties of Exponents and Scientific Notation

Simplify Expressions Using the Properties for Exponents

In the following exercises, simplify each expression using the properties for exponents.

30. $p^3 \cdot p^{10}$

Answer

p^{13}

31. $2 \cdot 2^6$

32. $a \cdot a^2 \cdot a^3$

Answer

a^6

33. $x \cdot x^8$

34. $y^a \cdot y^b$

Answer

y^{a+b}

35. $\frac{2^8}{2^2}$

36. $\frac{a^6}{a}$

Answer

a^5

37. $\frac{n^3}{n^{12}}$

38. $\frac{1}{x^5}$

Answer

$\frac{1}{x^4}$

39. 3^0

40. y^0

Answer

1

41. $(14t)^0$

42. $12a^0 - 15b^0$

Answer

-3

Use the Definition of a Negative Exponent

In the following exercises, simplify each expression.

43. 6^{-2}

44. $(-10)^{-3}$

Answer

$-\frac{1}{1000}$

45. $5 \cdot 2^{-4}$

46. $(8n)^{-1}$

Answer

$\frac{1}{8n}$

47. y^{-5}

48. 10^{-3}

Answer

$\frac{1}{1000}$

49. $\frac{1}{a^{-4}}$

50. $\frac{1}{6^{-2}}$

Answer

36

51. -5^{-3}

52. $\left(-\frac{1}{5}\right)^{-3}$

Answer

$-\frac{1}{25}$

53. $-(12)^{-3}$

54. $(-5)^{-3}$

Answer

$-\frac{1}{125}$

55. $\left(\frac{5}{9}\right)^{-2}$

56. $\left(-\frac{3}{x}\right)^{-3}$

Answer

$\frac{x^3}{27}$

In the following exercises, simplify each expression using the Product Property.

57. $(y^4)^3$

58. $(3^2)^5$

Answer

3^{10}

59. $(a^{10})^y$

60. $x^{-3} \cdot x^9$

Answer

x^5

61. $r^{-5} \cdot r^{-4}$

62. $(uv^{-3})(u^{-4}v^{-2})$

Answer

$\frac{1}{u^3v^5}$

63. $(m^5)^{-1}$

64. $p^5 \cdot p^{-2} \cdot p^{-4}$

Answer

$\frac{1}{m^5}$

In the following exercises, simplify each expression using the Power Property.

65. $(k-2)^{-3}$

66. $\frac{q^4}{q^{20}}$

Answer

$\frac{1}{q^{16}}$

67. $\frac{b^8}{b^{-2}}$

68. $\frac{n^{-3}}{n^{-5}}$

Answer

n^2

In the following exercises, simplify each expression using the Product to a Power Property.

69. $(-5ab)^3$

70. $(-4pq)^0$

Answer

1

71. $(-6x^3)^{-2}$

72. $(3y^{-4})^2$

Answer

$\frac{9}{y^8}$

In the following exercises, simplify each expression using the Quotient to a Power Property.

73. $\left(\frac{3}{5x}\right)^{-2}$

74. $\left(\frac{3xy^2}{z}\right)^4$

Answer

$\frac{81x^4y^8}{z^4}$

75. $(4p-3q^2)^2$

In the following exercises, simplify each expression by applying several properties.

76. $(x^2y)^2(3xy^5)^3$

Answer

$27x^7y^{17}$

77. $(-3a^{-2})^4(2a^4)^2(-6a^2)^3$

78. $\left(\frac{3xy^3}{4x^4y^{-2}}\right)^2\left(\frac{6xy^4}{8x^3y^{-2}}\right)^{-1}$

Answer

$\frac{3y^4}{4x^4}$

In the following exercises, write each number in scientific notation.

79. 2.568

80. $5,300,000$

Answer

5.3×10^6

81. 0.00814

In the following exercises, convert each number to decimal form.

82. 2.9×10^4

Answer

$29,000$

83. 3.75×10^{-1}

84. 9.413×10^{-5}

Answer

0.00009413

In the following exercises, multiply or divide as indicated. Write your answer in decimal form.

85. $(3 \times 10^7)(2 \times 10^{-4})$

86. $(1.5 \times 10^{-3})(4.8 \times 10^{-1})$

Answer

0.00072

87. $\frac{6 \times 10^9}{2 \times 10^{-1}}$

88. $\frac{9 \times 10^{-3}}{1 \times 10^{-6}}$

Answer

$9,000$

Multiply Polynomials

Multiply Monomials

In the following exercises, multiply the monomials.

89. $(-6p^4)(9p)$

90. $\left(\frac{1}{3}c^2\right)\left(30c^8\right)$

Answer

$10c^{10}$

91. $(8x^2y^5)(7xy^6)$

92. $\left(\frac{2}{3}m^3n^6\right)\left(\frac{1}{6}m^4n^4\right)$

Answer

$\frac{m^7n^{10}}{9}$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

93. $7(10-x)$

94. $a^2(a^2-9a-36)$

Answer

$a^4-9a^3-36a^2$

95. $-5y(125y^3-1)$

96. $(4n-5)(2n^3)$

Answer

$8n^4-10n^3$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using:

a. the Distributive Property b. the FOIL method c. the Vertical Method.

97. $(a+5)(a+2)$

98. $(y-4)(y+12)$

Answer

$y^2+8y-48$

99. $(3x+1)(2x-7)$

100. $(6p-11)(3p-10)$

Answer

$18p^2-93p+110$

In the following exercises, multiply the binomials. Use any method.

101. $(n+8)(n+1)$

102. $(k+6)(k-9)$

Answer

$k^2-3k-54$

103. $(5u-3)(u+8)$

104. $(2y-9)(5y-7)$

Answer

$10y^2-59y+63$

105. $(p+4)(p+7)$

106. $(x-8)(x+9)$

Answer

x^2+x-72

107. $(3c+1)(9c-4)$

108. $(10a-1)(3a-3)$

Answer

$30a^2-33a+3$

Multiply a Polynomial by a Polynomial

In the following exercises, multiply using a. the Distributive Property b. the Vertical Method.

109. $(x+1)(x^2-3x-21)$

110. $(5b-2)(3b^2+b-9)$

Answer

$15b^3-b^2-47b+18$

In the following exercises, multiply. Use either method.

111. $(m+6)(m^2-7m-30)$

112. $(4y-1)(6y^2-12y+5)$

Answer

$24y^2-54y^2+32y-5$

Multiply Special Products

In the following exercises, square each binomial using the Binomial Squares Pattern.

113. $(2x-y)^2$

114. $(x+\frac{3}{4})^2$

Answer

$x^2+\frac{3}{2}x+\frac{9}{16}$

115. $(8p^3-3)^2$

116. $(5p+7q)^2$

Answer

$25p^2+70pq+49q^2$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates.

117. $(3y+5)(3y-5)$

118. $(6x+y)(6x-y)$

Answer

$36x^2-y^2$

119. $(a+\frac{2}{3}b)(a-\frac{2}{3}b)$

120. $(12x^3-7y^2)(12x^3+7y^2)$

Answer

$144x^6-49y^4$

121. $(13a^2-8b4)(13a^2+8b^4)$

Divide Monomials

Divide Monomials

In the following exercises, divide the monomials.

122. $72p^{12} \div 8p^3$

Answer

$9p^9$

123. $-26a^8 \div (2a^2)$

124. $\frac{45y^6}{-15y^{10}}$

Answer

$-3y^4$

125. $\frac{-30x^8}{-36x^9}$

126. $\frac{28a^9b}{7a^4b^3}$

Answer

$\frac{4a^5}{b^2}$

127. $\frac{11u^6v^3}{55u^2v^8}$

128. $\dfrac{(5m^9n^3)(8m^3n^2)}{(10mn^4)(m^2n^5)}$

Answer

$\dfrac{4m^9}{n^4}$

129. $\dfrac{(42r^2s^4)(54rs^2)}{(6rs^3)(9s)}$

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial

130. $(54y^4-24y^3)\div(-6y^2)$

Answer

$-9y^2+4y$

131. $\dfrac{63x^3y^2-99x^2y^3-45x^4y^3}{9x^2y^2}$

132. $\dfrac{12x^2+4x-3}{-4x}$

Answer

$-3x-1+\dfrac{3}{4x}$

Divide Polynomials using Long Division

In the following exercises, divide each polynomial by the binomial.

133. $(4x^2-21x-18)\div(x-6)$

134. $(y^2+2y+18)\div(y+5)$

Answer

$y-3+\dfrac{33}{q+6}$

135. $(n^3-2n^2-6n+27)\div(n+3)$

136. $(a^3-1)\div(a+1)$

Answer

a^2+a+1

Divide Polynomials using Synthetic Division

In the following exercises, use synthetic Division to find the quotient and remainder.

137. $x^3-3x^2-4x+12$ is divided by $x+2$

138. $2x^3 - 11x^2 + 11x + 12$ is divided by $x - 3$

Answer

$2x^2 - 5x - 4;\ 0$

139. $x^4 + x^2 + 6x - 10$ is divided by $x + 2$

Divide Polynomial Functions

In the following exercises, divide.

140. For functions $f(x) = x^2 - 15x + 45$ and $g(x) = x - 9$, find a. $\left(\frac{f}{g}\right)(x)$
b. $\left(\frac{f}{g}\right)(-2)$

Answer

a. $\left(\frac{f}{g}\right)(x) = x - 6$
b. $\left(\frac{f}{g}\right)(-2) = -8$

141. For functions $f(x) = x^3 + x^2 - 7x + 2$ and $g(x) = x - 2$, find a. $\left(\frac{f}{g}\right)(x)$
b. $\left(\frac{f}{g}\right)(3)$

Use the Remainder and Factor Theorem

In the following exercises, use the Remainder Theorem to find the remainder.

142. $f(x) = x^3 - 4x - 9$ is divided by $x + 2$

Answer

-9

143. $f(x) = 2x^3 - 6x - 24$ divided by $x - 3$

In the following exercises, use the Factor Theorem to determine if $x - c$ is a factor of the polynomial function.

144. Determine whether $x - 2$ is a factor of $x^3 - 7x^2 + 7x - 6$

Answer

no

145. Determine whether $x - 3$ is a factor of $x^3 - 7x^2 + 11x + 3$

Chapter Practice Test

1. For the polynomial $8y^4 - 3y^2 + 1$
a. Is it a monomial, binomial, or trinomial? b. What is its degree?

Answer

a. trinomial b. 4

2. $(5a^2+2a-12)(9a^2+8a-4)$

3. $(10x^2-3x+5)-(4x^2-6)$

Answer

$6x^2-3x+11$

4. $\left(-\frac{3}{4}\right)^3$

5. $x^{-3}x^4$

Answer

x

6. $5^6 5^8$

7. $(47a^{18}b^{23}c^5)^0$

Answer

1

8. 4^{-1}

9. $(2y)^{-3}$

Answer

$\frac{1}{8y^3}$

10. $p^{-3}\cdot p^{-8}$

11. $\frac{x^4}{x^{-5}}$

Answer

x^9

12. $(3x^{-3})^2$

13. $\frac{24r^3s}{6r^2s^7}$

Answer

$\frac{4r}{s^6}$

14. $(x4y9x-3)2$

15. $(8xy^3)(-6x^4y^6)$

Answer

$-48x^5y^9$

16. $4u(u^2-9u+1)$

17. $(m+3)(7m-2)$

Answer

$21m^2-19m-6$

18. $(n-8)(n^2-4n+11)$

19. $(4x-3)^2$

Answer

$16x^2-24x+9$

20. $(5x+2y)(5x-2y)$

21. $(15xy^3-35x^2y)\div 5xy$

Answer

$3y^2-7x$

22. $(3x^3-10x^2+7x+10)\div(3x+2)$

23. Use the Factor Theorem to determine if $x+3$ a factor of $x^3+8x^2+21x+18$.

Answer

yes

24. a. Convert 112,000 to scientific notation.
b. Convert 5.25×10^{-4} to decimal form.

In the following exercises, simplify and write your answer in decimal form.

25. $(2.4\times 10^8)(2\times 10^{-5})$

Answer

4.4×10^3

26. $\dfrac{9\times 10^4}{3\times 10^{-1}}$

27. For the function $f(x)=6x^2-3x-9$ find:
a. $f(3)$ b. $f(-2)$ c. $f(0)$

Answer

a. 36 b. 21 c. -9

28. For $f(x)=2x^2-3x-5$ and $g(x)=3x^2-4x+1$, find
a. $(f+g)(x)$ b. $(f+g)(1)$
c. $(f-g)(x)$ d. $(f-g)(-2)$

29. For functions $f(x)=3x^2-23x-36$ and $g(x)=x-9$, find
a. $\left(\frac{f}{g}\right)(x)$ b. $\left(\frac{f}{g}\right)(3)$

Answer

a. $\left(\frac{f}{g}\right)(x)=3x+4$

b. $\left(\frac{f}{g}\right)(3)=13$

30. A hiker drops a pebble from a bridge 240 feet above a canyon. The function $h(t)=-16t^2+240$ gives the height of the pebble t seconds after it was dropped. Find the height when $t=3$.

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CHAPTER OVERVIEW

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Chapter 6 Review Exercises

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6.0: Prelude to Factoring



Figure 6.0.1: Scientists use factoring to calculate growth rates of infectious diseases such as viruses. (credit: “FotoshopTofs” / Pixabay)

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving factoring. In this chapter, you will learn how to factor and apply factoring to real-life situations.

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6.1: Greatest Common Factor and Factor by Grouping

Summary

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

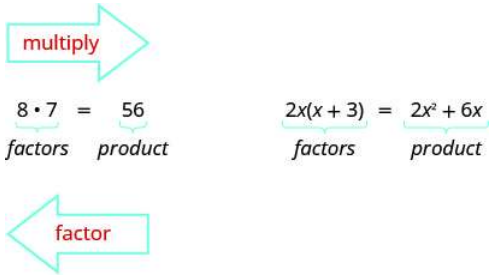
Are you ready?

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review [link].
2. Find the least common multiple (LCM) of 18 and 24.
   If you missed this problem, review [link].
3. Multiply: $-3a(7a+8b)$.
   If you missed this problem, review [link].

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a **product**. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

GREATEST COMMON FACTOR

The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

Example 6.1.1

Find the greatest common factor of $21x^3,\ 9x^2,\ 15x$.

Answer

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$ $\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

Example 6.1.2

Find the greatest common factor: $25m^4,\ 35m^3,\ 20m^2$.

Answer

$5m^2$

Example 6.1.3

Find the greatest common factor: $14x^3,\ 70x^2,\ 105x$.

Answer

$7x$

Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with its factors, $3x(x + 5)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

DISTRIBUTIVE PROPERTY

If *a*, *b*, and *c* are real numbers, then

$$a(b+c) = ab + ac \quad \text{and} \quad ab + ac = a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a **polynomial**? You just find the GCF of all the terms and write the polynomial as a product!

Example 6.1.4: How to Use the Distributive Property to factor a polynomial

Factor: $8m^3 - 12m^2n + 20mn^2$.

Answer

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3, 12m^2n, 20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $8m^3, 12m^2n, 20mn^2$ as products of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$ $12m^2n = 4m \cdot 3mn$ $20mn^2 = 4m \cdot 5n^2$	$8m^3 - 12m^2n + 20mn^2$ $4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4m(2m^2 - 3mn + 5n^2)$
Step 4. Check by multiplying the factors.		$4m(2m^2 - 3mn + 5n^2)$ $4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$ $8m^3 - 12m^2n + 20mn^2$ ✓

Example 6.1.5

Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

Answer

$3y^2(3x + 2x^2 + 7y)$

Example 6.1.6

Factor: $3p^3 - 6p^2q + 9pq^3$.

Answer

$3p(p^2 - 2pq + 3q^3)$

FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the "reverse" Distributive Property to factor the expression.
4. Check by multiplying the factors.

FACTOR AS A NOUN AND A VERB

We use "factor" as both a noun and a verb:

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a + 3$

Example 6.1.7

Factor: $5x^3 - 25x^2$.

Answer

Find the GCF of $5x^3$ and $25x^2$.		$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $GCF = 5 \cdot x \cdot x$
		$GCF = 5x^2$
		$5x^3 - 25x^2$
Rewrite each term.		$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.		$5x^2(x - 5)$
Check: $5x^2(x-5)$ $5x^2 \cdot x - 5x^2 \cdot 5$ $5x^3 - 25x^2$ ✓		

Example 6.1.8

Factor: $2x^3 + 12x^2$.

Answer

$2x^2(x+6)$

Example 6.1.9

Factor: $6y^3 - 15y^2$.

Answer

$3y^2(2y-5)$

Example 6.1.10

Factor: $8x^3y - 10x^2y^2 + 12xy^3$.

Answer

The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.	$8x^3y = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y$ $10x^2y^2 = 2 \cdot 5 \cdot x \cdot x \cdot y \cdot y$ $12xy^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$ $GCF = 2 \cdot x \cdot y$
	$GCF = 2xy$
	$8x^3y - 10x^2y^2 + 12xy^3$
Rewrite each term using the GCF, $2xy$.	$2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$
Factor the GCF.	$2xy(4x^2 - 5xy + 6y^2)$

Check: $$2xy(4x^2-5xy+6y^2)$$ $$2xy\cdot 4x^2-2xy\cdot 5xy+2xy\cdot 6y^2$$ $$8x^3y-10x^2y^2+12xy^3\checkmark$$	

? Example 6.1.11

Factor: $15x^3y-3x^2y^2+6xy^3$.

Answer

$3xy(5x^2-xy+2y^2)$

? Example 6.1.12

Factor: $8a^3b+2a^2b^2-6ab^3$.

Answer

$2ab(4a^2+ab-3b^2)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

? Example 6.1.13

Factor: $-4a^3+36a^2-8a$.

Answer

The leading coefficient is negative, so the GCF will be negative.

	$-4a^3+36a^2-8a$
Rewrite each term using the GCF, $-4a$.	$-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$
Factor the GCF.	$-4a(a^2-9a+2)$
Check: $$-4a(a^2-9a+2)$$ $$-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$$ $$-4a^3+36a^2-8a\checkmark$$	

? Example 6.1.14

Factor: $-4b^3+16b^2-8b$.

Answer

$-4b(b^2-4b+2)$

Example 6.1.15

Factor: $-7a^3+21a^2-14a$.

Answer

$-7a(a^2-3a+2)$

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

Example 6.1.16

Factor: $3y(y+7)-4(y+7)$.

Answer

The GCF is the binomial $y+7$.

		$3y(y+7)-4(y+7)$
Factor the GCF, $(y+7)$.		$(y+7)(3y-4)$
Check on your own by multiplying.		

Example 6.1.17

Factor: $4m(m+3)-7(m+3)$.

Answer

$(m+3)(4m-7)$

Example 6.1.18

Factor: $8n(n-4)+5(n-4)$.

Answer

$(n-4)(8n+5)$

Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the **GCF** in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are **prime**, some polynomials are prime.

Example 6.1.19: How to Factor a Polynomial by Grouping

Factor by grouping: $xy+3y+2x+6$.

Answer

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms?	$xy + 3y + 2x + 6$
	No, so let's separate the first two terms from the second two.	$xy + 3y + 2x + 6$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms.	$y(x + 3) + 2x + 6$
	Factor the GCF from the second two terms.	$y(x + 3) + 2(x + 3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$.	$y(x + 3) + 2(x + 3)$
	Factor out the common factor.	$(x + 3)(y + 2)$
Step 4. Check.	Multiply $(x + 3)(y + 2)$. Is the product the original expression?	$(x + 3)(y + 2)$ $xy + 2x + 3y + 6$ $xy + 3y + 2x + 6$ ✓

? Example 6.1.20

Factor by grouping: $xy+8y+3x+24$.

Answer

$(x+8)(y+3)$

? Example 6.1.21

Factor by grouping: $ab+7b+8a+56$.

Answer

$(a+7)(b+8)$

FACTOR BY GROUPING.

1. Group terms with common factors.
2. Factor out the common factor in each group.
3. Factor the common factor from the expression.
4. Check by multiplying the factors.

? Example 6.1.22

Factor by grouping: ⓐ $x^2+3x-2x-6$ ⓑ $6x^2-3x-4x+2$.

Answer

ⓐ

There is no GCF in all four terms.	$x^2+3x-2x-6$
Separate into two parts.	$x^2+3x \quad -2x-6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x+3)-2(x+3)$
Factor out the common factor.	$(x+3)(x-2)$
Check on your own by multiplying.	

ⓑ

There is no GCF in all four terms.	$6x^2-3x-4x+2$
Separate into two parts.	$6x^2-3x \quad -4x+2$
Factor the GCF from both parts.	$3x(2x-1)-2(2x-1)$
Factor out the common factor.	$(2x-1)(3x-2)$
Check on your own by multiplying.	

? Example 6.1.23

Factor by grouping: ⓐ $x^2+2x-5x-10$ ⓑ $20x^2-16x-15x+12$.

Answer

ⓐ $(x-5)(x+2)$
ⓑ $(5x-4)(4x-3)$

? Example 6.1.24

Factor by grouping: ⓐ $y^2+4y-7y-28$ ⓑ $42m^2-18m-35m+15$.

Answer

ⓐ $(y+4)(y-7)$
ⓑ $(7m-3)(6m-5)$

Key Concepts

- **How to find the greatest common factor (GCF) of two expressions.**
  1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  2. List all factors—matching common factors in a column. In each column, circle the common factors.
  3. Bring down the common factors that all expressions share.
  4. Multiply the factors.
- **Distributive Property:** If a, b and c are real numbers, then

$$a(b+c)=ab+ac \quad \text{and} \quad ab+ac=a(b+c)$$

  The form on the left is used to multiply. The form on the right is used to factor.
- **How to factor the greatest common factor from a polynomial.**
  1. Find the GCF of all the terms of the polynomial.
  2. Rewrite each term as a product using the GCF.
  3. Use the "reverse" Distributive Property to factor the expression.
  4. Check by multiplying the factors.
- **Factor as a Noun and a Verb:** We use "factor" as both a noun and a verb.

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a+3$

- **How to factor by grouping.**
  1. Group terms with common factors.
  2. Factor out the common factor in each group.
  3. Factor the common factor from the expression.
  4. Check by multiplying the factors.

Glossary

factoring

Splitting a product into factors is called factoring.

greatest common factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

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6.1E: Exercises

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $10p^3q, 12pq^2$

Answer

$2pq$

2. $8a^2b^3, 10ab^2$

3. $12m^2n^3, 30m^5n^3$

Answer

$6m^2n^3$

4. $28x^2y^4, 42x^4y^4$

5. $10a^3, 12a^2, 14a$

Answer

$2a$

6. $20y^3, 28y^2, 40y$

7. $35x^3y^2, 10x^4y, 5x^5y^3$

Answer

$5x^3y$

8. $27p^2q^3, 45p^3q^4, 9p^4q^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

9. $6m+9$

Answer

$3(2m+3)$

10. $14p+35$

11. $9n-63$

Answer

$9(n-7)$

12. $45b - 18$

13. $3x^2 + 6x - 9$

Answer

$3(x^2 + 2x - 3)$

14. $4y^2 + 8y - 4$

15. $8p^2 + 4p + 2$

Answer

$2(4p^2 + 2p + 1)$

16. $10q^2 + 14q + 20$

17. $8y^3 + 16y^2$

Answer

$8y^2(y + 2)$

18. $12x^3 - 10x$

19. $5x^3 - 15x^2 + 20x$

Answer

$5x(x^2 - 3x + 4)$

20. $8m^2 - 40m + 16$

21. $24x^3 - 12x^2 + 15x$

Answer

$3x(8x^2 - 4x + 5)$

22. $24y^3 - 18y^2 - 30y$

23. $12xy^2 + 18x^2y^2 - 30y^3$

Answer

$6y^2(2x + 3x^2 - 5y)$

24. $21pq^2 + 35p^2q^2 - 28q^3$

25. $20x^3y - 4x^2y^2 + 12xy^3$

Answer

$4xy(5x^2 - xy + 3y^2)$

26. $24a^3b+6a^2b^2-18ab^3$

27. $-2x-4$

Answer

$-2(x+4)$

28. $-3b+12$

29. $-2x^3+18x^2-8x$

Answer

$-2x(x^2-9x+4)$

30. $-5y^3+35y^2-15y$

31. $-4p^3q-12p^2q^2+16pq^2$

Answer

$-4pq(p^2+3pq-4q)$

32. $-6a^3b-12a^2b^2+18ab^2$

33. $5x(x+1)+3(x+1)$

Answer

$(x+1)(5x+3)$

34. $2x(x-1)+9(x-1)$

35. $3b(b-2)-13(b-2)$

Answer

$(b-2)(3b-13)$

36. $6m(m-5)-7(m-5)$

Factor by Grouping

In the following exercises, factor by grouping.

37. $ab+5a+3b+15$

Answer

$(b+5)(a+3)$

38. $cd+6c+4d+24$

39. $8y^2+y+40y+5$

Answer

$(y+5)(8y+1)$

40. $6y^2+7y+24y+28$

41. $uv-9u+2v-18$

Answer

$(u+2)(v-9)$

42. $pq-10p+8q-80$

43. $u^2-u+6u-6$

Answer

$(u-1)(u+6)$

44. $x^2-x+4x-4$

45. $9p^2-15p+12p-20$

Answer

$(3p-5)(3p+4)$

46. $16q^2+20q-28q-35$

47. $mn-6m-4n+24$

Answer

$(n-6)(m-4)$

48. $r^2-3r-r+3$

49. $2x^2-14x-5x+35$

Answer

$(x-7)(2x-5)$

50. $4x^2-36x-3x+27$

Mixed Practice

In the following exercises, factor.

51. $-18xy^2-27x^2y$

Answer

$-9xy(3x+2y)$

52. $-4x^3y^5-x^2y^3+12xy^4$

53. $3x^3 - 7x^2 + 6x - 14$

Answer

$(x^2 + 2)(3x - 7)$

54. $x^3 + x^2 - x - 1$

55. $x^2 + xy + 5x + 5y$

Answer

$(x + y)(x + 5)$

56. $5x^3 - 3x^2 + 5x - 3$

Writing Exercises

57. What does it mean to say a polynomial is in factored form?

Answer

Answers will vary.

58. How do you check result after factoring a polynomial?

59. The greatest common factor of 36 and 60 is 12. Explain what this means.

Answer

Answers will vary.

60. What is the GCF of y^4, y^5, and y^{10}? Write a general rule that tells you how to find the GCF of y^a, y^b, and y^c.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you

the help you need.

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6.2: Factor Trinomials

6.3: Factor Trinomials

Learning Objectives

By the end of this section, you will be able to:

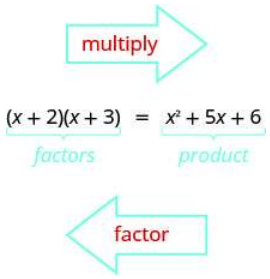
- Factor trinomials of the form x^2+bx+c
- Factor trinomials of the form ax^2+bx+c using trial and error
- Factor trinomials of the form ax^2+bx+c using the 'ac' method
- Factor using substitution

Before you get started, take this readiness quiz.

1. Find all the factors of 72.
   If you missed this problem, review [link].
2. Find the product: $(3y+4)(2y+5)$.
   If you missed this problem, review [link].
3. Simplify: $-9(6);\ -9(-6)$.
   If you missed this problem, review [link].

Factor Trinomials of the Form x^2+bx+c

You have already learned how to multiply binomials using **FOIL**. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a **trinomial** of the form x^2+bx+c, such as x^2+5x+6 and factor it to $(x+2)(x+3)$, let's start with two general binomials of the form $(x+m)$ and $(x+n)$.

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$
Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x^2+bx+c.	$\overbrace{x^2+(m+n)x+mn}^{x^2+bx+c}$

This tells us that to factor a trinomial of the form x^2+bx+c, we need two factors $(x+m)$ and $(x+n)$ where the two numbers m and n multiply to c and add to b.

Example 6.2.1: How to Factor a Trinomial of the form x^2+bx+c

Factor: $x^2+11x+24$.

Answer

Step 1. Write the factors as two binomials with first terms x.	Write two sets of parentheses and put x as the first term.	$x^2 + 11x + 24$ $(x \quad)(x \quad)$
Step 2. Find two numbers m and n that multiply to c, $m \cdot n = c$ add to b, $m + n = b$	Find two numbers that multiply to 24 and add to 11. **Factors of 24** / **Sum of factors** 1, 24 — 1 + 24 = 25 2, 12 — 2 + 12 = 14 3, 8 — 3 + 8 = 11* 4, 6 — 4 + 6 = 10	
Step 3. Use m and n as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x + 3)(x + 8)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 8)$ $x^2 + 8x + 3x + 24$ $x^2 + 11x + 24$ ✓

? Example 6.2.2

Factor: $q^2 + 10q + 24$.

Answer

$(q+4)(q+6)$

? Example 6.2.3

Factor: $t^2 + 14t + 24$.

Answer

$(t+2)(t+12)$

Let's summarize the steps we used to find the factors.

1. Write the factors as two binomials with first terms x. $x^2 + bx + c$ $(x \quad)(x \quad)$
2. Find two numbers m and n that
   - multiply to c, $m \cdot n = c$
   - add to b, $m + n = b$
3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

? Example 6.2.4

Factor: $y^2 - 11y + 28$.

Answer

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

	$y^2-11y+28$
Write the factors as two binomials with first terms y.	$(y\quad)(y\quad)$
Find two numbers that: multiply to 28 and add to -11.	

Factors of 28	Sum of factors
$-1,\ -28$	$-1+(-28)=-29$
$-2,\ -14$	$-2+(-14)=-16$
$-4,\ -7$	$-4+(-7)=-11^*$

Use $-4,\ -7$ as the last terms of the binomials.	$(y-4)(y-7)$
Check:	
	$(y-4)(y-7)$
	$y^2-7y-4y+28$
	$y^2-11y+28\checkmark$

? Example 6.2.5

Factor: $u^2-9u+18$.

Answer

$(u-3)(u-6)$

? Example 6.2.6

Factor: $y^2-16y+63$.

Answer

$(y-7)(y-9)$

Now, what if the last term in the trinomial is negative? Think about **FOIL**. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

? Example 6.2.7

Factor: $2x+x^2-48$.

Answer

	$2x+x^2-48$
First we put the terms in decreasing degree order.	$x^2+2x-48$
Factors will be two binomials with first terms x.	$(x\quad)(x\quad)$

Factors of −48−48	Sum of factors

Factors of -48	Sum of factors
$-1,\ 48$	$-1+48=47$
$-2,\ 24$	$-2+24=22$
$-3,\ 16$	$-3+16=13$
$-4,\ 12$	$-4+12=8$
$-6,\ 8$	$-6+8=2^*$

Use $-6,\ 8$ as the last terms of the binomials. $(x-6)(x+8)$

Check:

$$\begin{aligned}&(x-6)(x+8)\\ &x^2-6q+8q-48\\ &x^2+2x-48\checkmark\end{aligned}$$

Example 6.2.8

Factor: $9m+m^2+18$.

Answer

$(m+3)(m+6)$

Example 6.2.9

Factor: $-7n+12+n^2$.

Answer

$(n-3)(n-4)$

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x\cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain y. To get the coefficients b and c, you use the same process summarized in How To Factor trinomials.

Example 6.2.10

Factor: $r^2-8rs-9s^2$.

Answer

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$\begin{aligned}&r^2-8rs-9s^2\\ &(r\quad s)(r\quad s)\end{aligned}$$

Note that the first terms are r, last terms contain s.

Find the numbers that multiply to -9 and add to -8.

Factors of -9	Sum of factors
$1,\ -9$	$-1+9=8$
$-1,\ 9$	$1+(-9)=-8^*$
$3,\ -3$	$3+(-3)=0$

Use $1,\ -9$ as coefficients of the last terms. $(r+s)(r-9s)$

Check:

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2 \checkmark$$

Example 6.2.11

Factor: $a^2-11ab+10b^2$.

Answer

$(a-b)(a-10b)$

Example 6.2.12

Factor: $m^2-13mn+12n^2$.

Answer

$(m-n)(m-12n)$

Some trinomials are prime. The only way to be certain a trinomial is **prime** is to list all the possibilities and show that none of them work.

Example 6.2.13

Factor: $u^2-9uv-12v^2$.

Answer

We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$u^2-9uv-12v^2$$

Note that the first terms are u, last terms contain v. $(u\quad v)(u\quad v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1, -12$	$1+(-12)=-11$
$-1, 12$	$-1+12=11$
$2, -6$	$2+(-6)=-4$
$-2, 6$	$-2+6=4$
$3, -4$	$3+(-4)=-1$
$-3, 4$	$-3+4=1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

Example 6.2.14

Factor: $x^2-7xy-10y^2$.

Answer

prime

Example 6.2.15

Factor: $p^2+15pq+20q^2$.

Answer

prime

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c .

STRATEGY FOR FACTORING TRINOMIALS OF THE FORM x^2+bx+c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

	x^2+bx+c	
	$(x+m)(x+n)$	
When c is positive, m and n have the same sign.		
b positive		b negative
m, n positive		m, n negative
x^2+5x+6		x^2-6x+8
$(x+2)(x+3)$		$(x-4)(x-2)$
same signs		same signs
When c is negative, m and n have the opposite sign.		
x^2+x-12		$x^2-2x-15$
$(x+4)(x-3)$		$(x-5)(x+3)$
opposite signs		opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

Factor Trinomials of the form $ax^2 + bx + c$ using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c .

Remember to always check for a **GCF** first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

ExAMPLE 6.2.16

Factor completely: $4x^3+16x^2-20x$.

Answer

Is there a greatest common factor? Yes, $GCF = 4x$. Factor it.	$4x^3 + 16x^2 - 20x$ $4x(x^2 + 4x - 5)$
Binomial, trinomial, or more than three terms? It is a trinomial. So "undo FOIL."	$4x(x \quad)(x \quad)$
Use a table like the one shown to find two numbers that multiply to -5 and add to 4.	$4x(x-1)(x+5)$

Factors of -5	Sum of factors
$-1, 5$ $1, -5$	$-1+5=4^*$ $1+(-5)=-4$

Check:

$$4x(x-1)(x+5)$$
$$4x(x^2+5x-x-5)$$
$$4x(x^2+4x-5)$$
$$4x^3+16x2-20x\checkmark$$

? ExAMPLE 6.2.17

Factor completely: $5x^3 + 15x^2 - 20x$.

Answer

$5x(x-1)(x+4)$

? ExAMPLE 6.2.18

Factor completely: $6y^3 + 18y^2 - 60y$.

Answer

$6y(y-2)(y+5)$

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2 + 5x + 2$.

From our earlier work, we expect this will factor into two binomials.

$$3x^2 + 5x + 2$$
$$(\quad)(\quad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x$, $3x$. We can place them in the binomials.

$$3x^2 + 5x + 2$$
$$(x\quad)(3x\quad)$$

Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.

$$3x^2 + 5x + 2 \quad \text{(1x, 3x; 1, 2)} \quad (x+1)(3x+2) \quad \text{or} \quad 3x^2 + 5x + 2 \quad \text{(1x, 3x; 1, 2)} \quad (x+2)(3x+1)$$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$$3x^2 + 5x + 2 \quad \text{(1x, 3x; 1, 2)} \quad (x+1)(3x+2) \quad (3x + 2x = 5x) \quad \text{or} \quad 3x^2 + 5x + 2 \quad \text{(1x, 3x; 1, 2)} \quad (x+2)(6x + 1x = 7x)$$

Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's use FOIL to check.

$$(x+1)(3x+2)$$
$$3x^2 + 2x + 3x + 2$$
$$3x^2 + 5x + 2 \checkmark$$

Our result of the factoring is:

$$3x^2 + 5x + 2$$
$$(x+1)(3x+2)$$

? ExAMPLE 6.2.19: How to Factor a Trinomial Using Trial and Error

Factor completely using trial and error: $3y^2 + 22y + 7$.

Answer

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$
Step 2. Factor any GCF.	There is no GCF.	
Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y, 3y$. Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ (1y, 3y) $3y^2 + 22y + 7$ (1y, 3y) $(y\quad)(3y\quad)$
Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ (1y, 3y; 1, 7) $(y\quad)(3y\quad)$

Step 5. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y + 1)(3y + 7)$ 3y 7y 10y No! We need 22y $3y^2 + 22y + 7$ 1y, 3y 1, 7 $(y + 7)(3y + 1)$ 21y +y 22y	$3y^2 + 22y + 7$ **Possible factors** | **Product** $(y + 1)(3y + 7)$ | $3y^2 + 10y + 7$ $(y + 7)(3y + 1)$ | $3y^2 + 22y + 7$
Step 6. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

? ExAMPLE 6.2.20

Factor completely using trial and error: $2a^2+5a+3$.

Answer

$(a+1)(2a+3)$

? ExAMPLE 6.2.21

Factor completely using trial and error: $4b^2+5b+1$.

Answer

$(b+1)(4b+1)$

FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR.

1. Write the trinomial in descending order of degrees as needed.
2. Factor any GCF.
3. Find all the factor pairs of the first term.
4. Find all the factor pairs of the third term.
5. Test all the possible combinations of the factors until the correct product is found.
6. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

? ExAMPLE 6.2.22

Factor completely using trial and error: $6b^2-13b+5$.

Answer

The trinomial is already in descending order.	$6b^2 - 13b + 5$
Find the factors of the first term.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$

Find the factors of the last term. Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$ $-1, -5$

Consider all the combinations of factors.

$6b^2 - 13b + 5$

Possible factors	Product
$(b-1)(6b-5)$	$6b^2 - 11b + 5$
$(b-5)(6b-1)$	$6b^2 - 31b + 5$
$(2b-1)(3b-5)$	$6b^2 - 13b + 5^*$
$(2b-5)(3b-1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $(2b-1)(3b-5)$

Check by multiplying:

$$(2b-1)(3b-5)$$
$$6b^2 - 10b - 3b + 5$$
$$6b^2 - 13b + 5 \checkmark$$

? ExAMPLE 6.2.23

Factor completely using trial and error: $8x^2 - 14x + 3$.

Answer

$(2x-3)(4x-1)$

? ExAMPLE 6.2.24

Factor completely using trial and error: $10y^2 - 37y + 7$.

Answer

$(2y-7)(5y-1)$

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

? ExAMPLE 6.2.25

Factor completely using trial and error: $18x^2 - 37xy + 15y^2$.

Answer

The trinomial is already in descending order.	$18x^2 - 37xy + 15y^2$
Find the factors of the first term.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $2x \cdot 9x$ $3x \cdot 6x$

Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $\quad -1, -15$ $2x \cdot 9x$ $\quad -3, -5$ $3x \cdot 6x$

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	**Product**
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2x - 3y)(9x - 5y)$

Check by multiplying:

$$(2x - 3y)(9x - 5y)$$
$$18x^2 - 10xy - 27xy + 15y^2$$
$$18x^2 - 37xy + 15y^2 \checkmark$$

? ExAMPLE 6.2.26

Factor completely using trial and error $18x^2 - 3xy - 10y^2$.

Answer

$(3x + 2y)(6x - 5y)$

? ExAMPLE 6.2.27

Factor completely using trial and error: $30x^2 - 53xy - 21y^2$.

Answer

$(3x + y)(10x - 21y)$

Don't forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

? ExAMPLE 6.2.28

Factor completely using trial and error: $-10y^4 - 55y^3 - 60y^2$.

Answer

	$-10y^4 - 55y^3 - 60y^2$
Notice the greatest common factor, so factor it first.	$-5y^2(2y^2 + 11y + 12)$
Factor the trinomial.	$-5y^2\left(\begin{matrix} 2y^2 + 11y + 12 \\ y \cdot 2y \qquad 1 \cdot 12 \\ \qquad\quad 2 \cdot 6 \\ \qquad\quad 3 \cdot 4 \end{matrix}\right)$

Consider all the combinations.

$2y^2 + 11y + 12$	
Possible factors	**Product**
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor -5^y2.

$$-5y^2(y+4)(2y+3)$$

Check by multiplying:

$$-5y^2(y+4)(2y+3)$$
$$-5y^2(2y^2+8y+3y+12)$$
$$-10y^4-55y^3-60y^2\checkmark$$

? ExAMPLE 6.2.29

Factor completely using trial and error: $15n^3 - 85n^2 + 100n$.

Answer

$5n(n-4)(3n-5)$

? ExAMPLE 6.2.30

Factor completely using trial and error: $56q^3 + 320q^2 - 96q$.

Answer

$8q(q+6)(7q-2)$

Factor Trinomials of the Form $ax^2 + bx + c$ using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example 6.2.31: How to Factor Trinomials using the "ac" Method

Factor using the "ac" method: $6x^2+7x+2$.

Answer

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2+7x+2$
Step 2. Find the product ac.	$a \cdot c$ $6 \cdot 2$ 12	ax^2+bx+c $6x^2+7x+2$
Step 3. Find two numbers m and n that: Multiply to ac. $m \cdot n = a \cdot c$ Add to b. $m+n=b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3+4=7$	
Step 4. Split the middle term using m and n. ax^2+bx+c bx $ax^2+mx+nx+c$	Rewrite $7x$ as $3x+4x$. It would also give the same result if we used $4x+3x$. Notice that $6x^2+3x+4x+2$ is equal to $6x^2+7x+2$. We just split the middle term to get a more useful form.	$6x^2+7x+2$ $6x^2+3x+4x+2$
Step 5. Factor by grouping.		$3x(2x+1)+2(2x+1)$ $(2x+1)(3x+2)$
Step 6. Check by multiplying the factors.		$(2x+1)(3x+2)$ $6x^2+4x+3x+2$ $6x^2+7x+2$ ✓

Example 6.2.32

Factor using the "ac" method: $6x^2+13x+2$.

Answer

$(x+2)(6x+1)$

Example 6.2.33

Factor using the "ac" method: $4y^2+8y+3$.

Answer

$(2y+1)(2y+3)$

The "ac" method is summarized here.

FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING THE "ac" METHOD.

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac $m \cdot n = a \cdot c$
   Add to b $m + n = b$
   $ax^2 + bx + c$
4. Split the middle term using m and n. $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

Don't forget to look for a common factor!

Example 6.2.34

Factor using the "'ac" method: $10y^2 - 55y + 70$.

Answer

Is there a greatest common factor?		
Yes. The GCF is 5.		$10y^2 - 55y + 70$
Factor it.		$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.		$ax^2 + bx + c$ $5(2y^2 - 11y + 14)$
Find the product ac.	$ac = 28$	
Find two numbers that multiply to ac	$(-4)(-7) = 28$	
and add to b.	$-4(-7) = -11$	
Split the middle term.		$5(2y^2 - 11y + 14)$
		$5(2y^2 - 7y - 4y + 14)$
Factor the trinomial by grouping.		
		$5(y - 2)(2y - 7)$
Check by multiplying all three factors. $5(y-2)(2y-7)$ $5(2y^2 - 7y - 4y + 14)$ $5(2y^2 - 11y + 14)$ $10y^2 - 55y + 70$✓		

Example 6.2.35

Factor using the "ac" method: $16x^2 - 32x + 12$.

Answer

$4(2x - 3)(2x - 1)$

Example 6.2.36

Factor using the "ac" method: $18w^2 - 39w + 18$.

Answer

$3(3w - 2)(2w - 3)$

Factor Using Substitution

Sometimes a trinomial does not appear to be in the $ax^2 + bx + c$ form. However, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c$ form. This is called **factoring by substitution**. It is standard to use u for the substitution.

In the $ax^2 + bx + c$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

ExAMPLE 6.2.37

Factor by substitution: $x^4 - 4x^2 - 5$.

Answer

The variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $(x^2)^2 = x^4$). If we let $u = x^2$, we can put our trinomial in the $ax^2 + bx + c$ form we need to factor it.

	$x^4 - 4x^2 - 5$
Rewrite the trinomial to prepare for the substitution.	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$(u)^2 - 4(u) - 5$
Factor the trinomial.	$(u + 1)(u - 5)$
Replace u with x^2.	$(x^2 + 1)(x^2 - 5)$
Check: $(x^2 + 1)(x^2 - 5)$ $x^4 - 5x^2 + x^2 - 5$ $x^4 - 4x^2 - 5$✓	

ExAMPLE 6.2.38

Factor by substitution: $h^4 + 4h^2 - 12$.

Answer

$(h^2 - 2)(h^2 + 6)$

ExAMPLE 6.2.39

Factor by substitution: $y^4 - y^2 - 20$.

Answer

$(y^2 + 4)(y^2 - 5)$

Sometimes the expression to be substituted is not a monomial.

ExAMPLE 6.2.40

Factor by substitution: $(x-2)^2+7(x-2)+12$

Answer

The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

	$(x-2)^2+7(x-2)+12$
Rewrite the trinomial to prepare for the substitution.	$(x-2)^2+7(x-2)+12$
Let $u=x-2$ and substitute.	$u^2+7u+12$
Factor the trinomial.	$(u+3)(u+4)$
Replace u with $x-2$.	$((x-2)+3)((x-2)+4)$
Simplify inside the parentheses.	$(x+1)(x+2)$

This could also be factored by first multiplying out the $(x-2)^2$ and the $7(x-2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

ExAMPLE 6.2.41

Factor by substitution: $(x-5)^2+6(x-5)+8$.

Answer

$(x-3)(x-1)$

ExAMPLE 6.2.42

Factor by substitution: $(y-4)^2+8(y-4)+15$.

Answer

$(y-1)(y+1)$

Watch this video for additional instruction and practice with factoring.



Key Concepts

- **How to factor trinomials of the form x^2+bx+c.**
  1. Write the factors as two binomials with first terms x. $l)x^2+bx+c$ $(x\quad)(x\quad)$
  2. Find two numbers m and n that
     multiply to $c,\ m\cdot n=c$
     add to $b,\ m+n=b$
  3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
  4. Check by multiplying the factors.
- **Strategy for Factoring Trinomials of the Form** x^2+bx+c : When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

  For trinomials of the form: $x^2+bx+c=(x+m)(x+n)$

  When c is positive, m and n must have the same sign (and this will be the sign of b).

  Examples: $x^2+5x+6=(x+2)(x+3)$, $x^2-6x+8=(x-4)(x-2)$

  When c is negative, m and n have opposite signs. The larger of m and n will have the sign of b.

  Examples: $x^2+x-12=(x+4)(x-3)$, $x^2-2x-15=(x-5)(x+3)$

  Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.
- **How to factor trinomials of the form ax^2+bx+c using trial and error.**
  1. Write the trinomial in descending order of degrees as needed.
  2. Factor any GCF.
  3. Find all the factor pairs of the first term.
  4. Find all the factor pairs of the third term.
  5. Test all the possible combinations of the factors until the correct product is found.
  6. Check by multiplying.
- **How to factor trinomials of the form ax^2+bx+c using the "ac" method.**

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac. $m \cdot n = a \cdot c$
   Add to b. $m + n = b$
   $ax^2 + bx + c$
4. Split the middle term using m and n. $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

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6.2E: Exercises

Practice Makes Perfect

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form x^2+bx+c .

1. $p^2+11p+30$

Answer

$(p+5)(p+6)$

2. $w^2+10w+21$

3. $n^2+19n+48$

Answer

$(n+3)(n+16)$

4. $b^2+14b+48$

5. $a^2+25a+100$

Answer

$(a+5)(a+20)$

6. $u^2+101u+100$

7. $x^2-8x+12$

Answer

$(x-2)(x-6)$

8. $q^2-13q+36$

9. $y^2-18y+45$

Answer

$(y-3)(y-15)$

10. $m^2-13m+30$

11. x^2-8x+7

Answer

$(x-1)(x-7)$

12. y^2-5y+6

13. $5p-6+p^2$

Answer

$(p-1)(p+6)$

14. $6n-7+n^2$

15. $8-6x+x^2$

Answer

$(x-4)(x-2)$

16. $7x+x^2+6$

17. $x^2-12-11x$

Answer

$(x-12)(x+1)$

18. $-11-10x+x^2$

In the following exercises, factor each trinomial of the form $x^2+bxy+cy^2$.

19. $x^2-2xy-80y^2$

Answer

$(x+8y)(x-10y)$

20. $p^2-8pq-65q^2$

21. $m^2-64mn-65n^2$

Answer

$(m+n)(m-65n)$

22. $p^2-2pq-35q^2$

23. $a^2+5ab-24b^2$

Answer

$(a+8b)(a-3b)$

24. $r^2+3rs-28s^2$

25. $x^2-3xy-14y^2$

Answer

Prime

26. $u^2-8uv-24v^2$

27. $m^2 - 5mn + 30n^2$

Answer

Prime

28. $c^2 - 7cd + 18d^2$

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

29. $p^3 - 8p^2 - 20p$

Answer

$p(p-10)(p+2)$

30. $q^3 - 5q^2 - 24q$

31. $3m^3 - 21m^2 + 30m$

Answer

$3m(m-5)(m-2)$

32. $11n^3 - 55n^2 + 44n$

33. $5x^4 + 10x^3 - 75x^2$

Answer

$5x^2(x-3)(x+5)$

34. $6y^4 + 12y^3 - 48y^2$

35. $2t^2 + 7t + 5$

Answer

$(2t+5)(t+1)$

36. $5y^2 + 16y + 11$

37. $11x^2 + 34x + 3$

Answer

$(11x+1)(x+3)$

38. $7b^2 + 50b + 7$

39. $4w^2 - 5w + 1$

Answer

$(4w-1)(w-1)$

40. $5x^2 - 17x + 6$

41. $4q^2 - 7q - 2$

Answer

$(4q+1)(q-2)$

42. $10y^2 - 53y - 111$

43. $6p^2 - 19pq + 10q^2$

Answer

$(2p-5q)(3p-2q)$

44. $21m^2 - 29mn + 10n^2$

45. $4a^2 + 17ab - 15b^2$

Answer

$(4a-3b)(a+5b)$

46. $6u^2 + 5uv - 14v^2$

47. $-16x^2 - 32x - 16$

Answer

$-16(x+1)(x+1)$

48. $-81a^2 + 153a + 18$

49. $-30q^3 - 140q^2 - 80q$

Answer

$-10q(3q+2)(q+4)$

50. $-5y^3 - 30y^2 + 35y$

Factor Trinomials of the Form $ax^2 + bx + c$ using the 'ac' Method

In the following exercises, factor using the 'ac' method.

51. $5n^2 + 21n + 4$

Answer

$(5n+1)(n+4)$

52. $8w^2 + 25w + 3$

53. $4k^2 - 16k + 15$

Answer

$(2k-3)(2k-5)$

54. $5s^2-9s+4$

55. $6y^2+y-15$

Answer

$(3y+5)(2y-3)$

56. $6p^2+p-22$

57. $2n^2-27n-45$

Answer

$(2n+3)(n-15)$

58. $12z^2-41z-11$

59. $60y^2+290y-50$

Answer

$10(6y-1)(y+5)$

60. $6u^2-46u-16$

61. $48z^3-102z^2-45z$

Answer

$3z(8z+3)(2z-5)$

62. $90n^3+42n^2-216n$

63. $16s^2+40s+24$

Answer

$8(2s+3)(s+1)$

64. $24p^2+160p+96$

65. $48y^2+12y-36$

Answer

$12(4y-3)(y+1)$

66. $30x^2+105x-60$

Factor Using Substitution

In the following exercises, factor using substitution.

https://math.libretexts.org/@go/page/30863

67. $x^4 - x^2 - 12$

Answer

$(x^2 + 3)(x^2 - 4)$

68. $x^4 + 2x^2 - 8$

69. $x^4 - 3x^2 - 28$

Answer

$(x^2 - 7)(x^2 + 4)$

70. $x^4 - 13x^2 - 30$

71. $(x - 3)^2 - 5(x - 3) - 36$

Answer

$(x - 12)(x + 1)$

72. $(x - 2)^2 - 3(x - 2) - 54$

73. $(3y - 2)^2 - (3y - 2) - 2$

Answer

$(3y - 4)(3y - 1)$

74. $(5y - 1)^2 - 3(5y - 1) - 18$

Mixed Practice

In the following exercises, factor each expression using any method.

75. $u^2 - 12u + 36$

Answer

$(u - 6)(u - 6)$

76. $x^2 - 14x - 32$

77. $r^2 - 20rs + 64s^2$

Answer

$(r - 4s)(r - 16s)$

78. $q^2 - 29qr - 96r^2$

79. $12y^2 - 29y + 14$

Answer

$(4y - 7)(3y - 2)$

80. $12x^2+36y-24z$

81. $6n^2+5n-4$

Answer

$(2n-1)(3n+4)$

82. $3q^2+6q+2$

83. $13z^2+39z-26$

Answer

$13(z^2+3z-2)$

84. $5r^2+25r+30$

85. $3p^2+21p$

Answer

$3p(p+7)$

86. $7x^2-21x$

87. $6r^2+30r+36$

Answer

$6(r+2)(r+3)$

88. $18m^2+15m+3$

89. $24n^2+20n+4$

Answer

$4(2n+1)(3n+1)$

90. $4a^2+5a+2$

91. x^4-4x^2-12

Answer

$(x^2+2)(x^2-6)$

92. x^4-7x^2-8

93. $(x+3)^2-9(x+3)-36$

Answer

$(x-9)(x+6)$

94. $(x+2)^2-25(x+2)-54$

Writing Exercises

95. Many trinomials of the form x^2+bx+c factor into the product of two binomials $(x+m)(x+n)$. Explain how you find the values of m and n.

Answer

Answers will vary.

96. Tommy factored x^2-x-20 as $(x+5)(x-4)$. Sara factored it as $(x+4)(x-5)$. Ernesto factored it as $(x-5)(x-4)$. Who is correct? Explain why the other two are wrong.

97. List, in order, all the steps you take when using the "ac" method to factor a trinomial of the form ax^2+bx+c.

Answer

Answers will vary.

98. How is the "ac" method similar to the "undo FOIL" method? How is it different?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form x^2+bx+c			
factor trinomials of the form ax^2+bx+c using trial and error.			
factor trinomials of the form ax^2+bx+c with using the "ac" method.			
factor using substitution.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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6.3: Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Before you get started, take this readiness quiz.

1. Simplify: $(3x^2)^3$.
2. Multiply: $(m+4)^2$.
3. Multiply: $(x-3)(x+3)$.

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$(3x+4)^2$$

$$(3x)^2+2(3x\cdot 4)+4^2$$

$$9x^2+24x+16$$

The trinomial $9x^2+24x+16$ is called a *perfect square trinomial*. It is the square of the binomial $3x+4$.

In this chapter, you will start with a perfect square trinomial and factor it into its **prime** factors. You could factor this **trinomial** using the methods described in the last section, since it is of the form ax^2+bx+c. But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work. Here is the pattern—the reverse of the binomial squares pattern.

PERFECT SQUARE TRINOMIALS PATTERN

If a and b are real numbers

$$a^2+2ab+b^2=(a+b)^2$$

$$a^2-2ab+b^2=(a-b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, you can easily write the factors.

Example 6.3.1: How to Factor Perfect Square Trinomials

Factor: $9x^2+12x+4$.

Answer

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$? • Is the first term a perfect square? Write it as a square, a^2. • Is the last term a perfect square? Write it as a square, b^2. • Check the middle term. Is it $2ab$?	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$. Is 4 a perfect square? Yes—write it as $(2)^2$. Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$9x^2 + 12x + 4$ $(3x)^2$ $(3x)^2$ $(2)^2$ $2(3x)(2)$ $12x$
Step 2. Write the square of the binomial.		$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check.		$(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓

? Example 6.3.2

Factor: $4x^2 + 12x + 9$.

Answer

$(2x + 3)^2$

? Example 6.3.3

Factor: $9y^2 + 24y + 16$.

Answer

$(3y + 4)^2$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

FACTOR PERFECT SQUARE TRINOMIALS

Step 1.	Does the trinomial fit the pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2$ $(b)^2$	$(a)^2$ $(b)^2$
	Check the middle term. Is it $2ab$?	$2 \cdot a \cdot b$	$2 \cdot a \cdot b$
Step 2.	Write the square of the binomial.	$(a + b)^2$	$(a - b)^2$
Step 3.	Check by multiplying.		

We'll work one now where the middle term is negative.

Example 6.3.4

Factor: $81y^2 - 72y + 16$.

Answer

The first and last terms are squares. See if the middle term fits the pattern of a **perfect square** trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \qquad (4)^2$
Check the middle term.	$(9y)^2 \qquad (4)^2$ $2(9y)(4)$ $72y$
Does it match $(a-b)^2$? Yes.	$a^2 \quad -2 \quad a \quad b + b^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$
Write as the square of a binomial.	$(9y-4)^2$
Check by multiplying:	$(9y-4)^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$ $81y^2 - 72y + 16$✓

Example 6.3.5

Factor: $64y^2 - 80y + 25$.

Answer

$(8y-5)^2$

Example 6.3.6

Factor: $16z^2 - 72z + 81$.

Answer

$(4z-9)^2$

The next example will be a perfect square trinomial with two variables.

Example 6.3.7

Factor: $36x^2 + 84xy + 49y^2$.

Answer

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$a^2 \quad +2 \quad a \quad b + b^2$ $(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$
Factor.	$(6x+7y)^2$

Check by multiplying. $(6x+7y)^2$ $(6x)^2+2\cdot 6x\cdot 7y+(7y)^2$ $36x^2+84xy+49y^2$✓	

Example 6.3.8

Factor: $49x^2+84xy+36y^2$.

Answer

$(7x+6y)^2$

Example 6.3.9

Factor: $64m^2+112mn+49n^2$.

Answer

$(8m+7n)^2$

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a **GCF** in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

Example 6.3.10

Factor: $100x^2y-80xy+16y$.

Answer

	$100x^2y-80xy+16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y\left(25x^2-20x+4\right)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[(5x)^2-2\cdot 5x\cdot 2+2^2]$
Factor.	$4y(5x-2)^2$
Remember: Keep the factor $4y$ in the final product.	
Check: $4y(5x-2)^2$ $4y[(5x)2-2\cdot 5x\cdot 2+22]$ $4y(25x2-20x+4)$ 100x2y−80xy+16y\checkmark\]	

Example 6.3.11

Factor: $8x^2y - 24xy + 18y$.

Answer

$2y(2x - 3)^2$

Example 6.3.12

Factor: $27p^2q + 90pq + 75q$.

Answer

$3q(3p + 5)^2$

Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

$$\begin{array}{c} (a-b)(a+b) \\ (3x-4)(3x+4) \\ (a)^2-(b)^2 \\ (3x)^2-(4)^2 \\ 9x^2-16 \end{array}$$

A difference of squares factors to a product of conjugates.

DIFFERENCE OF SQUARES PATTERN

If a and b are real numbers,

$$a^2 - b^2 = (a-b)(a+b)$$

difference: a^2 (squares) $-$ b^2 (squares) $= (a-b)(a+b)$ (conjugates)

Remember, "difference" refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

Example 6.3.13: How to Factor a Trinomial Using the Difference of Squares

Factor: $64y^2 - 1$.

Answer

Step 1. Does the binomial fit the pattern?		$64y^2 - 1$
• Is this a difference?	Yes	$64y^2 - 1$
• Are the first and last terms perfect squares?	Yes	
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(8y)^2 - 1^2$

Step 3. Write the product of conjugates.		$(a-b)(a+b)$ $(8y-1)(8y+1)$
Step 4. Check.		$(8y-1)(8y+1)$ $64y^2-1$ ✓

Example 6.3.14

Factor: $121m^2-1$.

Answer

$(11m-1)(11m+1)$

Example 6.3.15

Factor: $81y^2-1$.

Answer

$(9y-1)(9y+1)$

FACTOR DIFFERENCES OF SQUARES.

Step 1.	Does the binomial fit the pattern?	a^2-b^2
	Is this a difference?	____−____
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2-(b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression a^2+b^2 is prime!

The next example shows variables in both terms.

Example 6.3.16

Factor: $144x^2-49y^2$.

Answer

	$144x^2-49y^2$
Is this a difference of squares? Yes.	$(12x)^2-(7y)^2$
Factor as the product of conjugates.	$(12x-7y)(12x+7y)$
Check by multiplying.	$(12x-7y)(12x+7y)$

Check by multiplying.

$(12x-7y)(12x+7y)$

$144x^2-49y^2$ ✓

Example 6.3.17

Factor: $196m^2 - 25n^2$.

Answer

$(14m - 5n)(14m + 5n)$

Example 6.3.18

Factor: $121p^2 - 9q^2$.

Answer

$(11p - 3q)(11p + 3q)$

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and you won't recognize the perfect squares until you factor the GCF.

Also, to completely factor the binomial in the next example, we'll factor a difference of squares twice!

Example 6.3.19

Factor: $48x^4y^2 - 243y^2$.

Answer

	$48x^4y^2 - 243y^2$
Is there a GCF? Yes, $3y^2$—factor it out!	$3y^2(16x^4 - 81)$
Is the binomial a difference of squares? Yes.	$3y^2\left((4x^2)^2 - (9)^2\right)$
Factor as a product of conjugates.	$3y^2(4x^2 - 9)(4x^2 + 9)$
Notice the first binomial is also a difference of squares!	$3y^2((2x)^2 - (3)^2)(4x^2 + 9)$
Factor it as the product of conjugates.	$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying:

$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

$3y^2(4x^2 - 9)(4x^2 + 9)$

$3y^2(16x^4 - 81)$

$48x^4y^2 - 243y^2$ ✓

Example 6.3.20

Factor: $2x^4y^2 - 32y^2$.

Answer

$2y^2(x - 2)(x + 2)(x^2 + 4)$

Example 6.3.21

Factor: $7a^4c^2 - 7b^4c^2$.

Answer

$7c^2(a-b)(a+b)(a^2+b^2)$

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

Example 6.3.22

Factor: $x^2 - 6x + 9 - y^2$.

Answer

Notice that the first three terms form a perfect square trinomial.

	$x^2 - 6x + 9 - y^2$
Factor by grouping the first three terms.	$\underbrace{x^2 - 6x + 9}\ - y^2$
Use the perfect square trinomial pattern.	$(x-3)^2 - y^2$
Is this a difference of squares? Yes.	
Yes—write them as squares.	$\overset{a^2}{(x-3)^2} - \overset{b^2}{y^2}$
Factor as the product of conjugates.	$((\overset{a}{x-3}) - \overset{b}{y})((\overset{a}{x-3}) + \overset{b}{y})$
	$(x-3-y)(x-3+y)$

You may want to rewrite the solution as $(x-y-3)(x+y-3)$.

Example 6.3.23

Factor: $x^2 - 10x + 25 - y^2$.

Answer

$(x-5-y)(x-5+y)$

Example 6.3.24

Factor: $x^2 + 6x + 9 - 4y^2$.

Answer

$(x+3-2y)(x+3+2y)$

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

We'll check the first pattern and leave the second to you.

	$(a+b)\left(a^2-ab+b^2\right)$
Distribute.	$a\left(a^2-ab+b^2\right)+b\left(a^2-ab+b^2\right)$
Multiply.	$a^3-a^2b+ab^2+a^2b-ab^2+b^3$
Combine like terms.	a^3+b^3

SUM AND DIFFERENCE OF CUBES PATTERN

$$a^3+b^3=(a+b)(a^2-ab+b^2$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
same sign — opposite signs

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
same sign — opposite signs

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in Table.

n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Example 6.3.25: How to Factor the Sum or Difference of Cubes

Factor: x^3+64.

Answer

Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	This is a sum. Yes.	$x^3 + 64$ $x^3 + 64$
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3.	$a^3 + b^3$ $x^3 + 4^3$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$(a + b)(a^2 - ab + b^2)$ $(x + 4)(x^2 - 4x + 4^2)$

Step 4. Simplify inside the parentheses.	It is already simplified.	$(x+4)(x^2-4x+16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2-4x+16 \\ x+4 \\ \hline 4x^2-16x+64 \checkmark \\ x^3-4x^2+16x \\ \hline x^3 \qquad +64 \end{array}$

? Example 6.3.26

Factor: x^3+27.

Answer

$(x+3)(x^2-3x+9)$

? Example 6.3.27

Factor: y^3+8.

Answer

$(y+2)(y^2-2y+4)$

FACTOR THE SUM OR DIFFERENCE OF CUBES.

1. Does the binomial fit the sum or difference of cubes pattern?
   Is it a sum or difference?
   Are the first and last terms perfect cubes?
2. Write them as cubes.
3. Use either the sum or difference of cubes pattern.
4. Simplify inside the parentheses.
5. Check by multiplying the factors.

? Example 6.3.28

Factor: $27u^3-125v^3$.

Answer

	$27u^3-125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3} - \overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{3u} - \overset{b}{5v}\right)\left(\overset{a^2}{(3u)^2} + \overset{ab}{3u \cdot 5v} + \overset{b^2}{(5v)^2}\right)$
Simplify.	$\left(\overset{a}{3u} - \overset{b}{5v}\right)\left(\overset{a^2}{9u^2} + \overset{ab}{15uv} + \overset{b^2}{25v^2}\right)$
Check by multiplying.	We'll leave the check to you.

Example 6.3.29

Factor: $8x^3 - 27y^3$.

Answer

$(2x - 3y)(4x^2 + 6xy + 9y^2)$

Example 6.3.30

Factor: $1000m^3 - 125n^3$.

Answer

$(10m - 5n)(100m^2 + 50mn + 25n^2)$

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

Example 6.3.31

Factor: $6x^3y + 48y^4$.

Answer

	$6x^3y + 48y^4$
Factor the common factor.	$6y\left(x^3 + 8y^3\right)$
This binomial is a sum The first and last terms are perfect cubes.	
Write the terms as cubes.	$6y\left(x^3 + (2y)^3\right)$
Use the sum of cubes pattern.	$6y\left(x + 2y\right)\left(x^2 - x \cdot 2y + (2y)^2\right)$
Simplify.	$6y(x + 2y)(x^2 - 2xy + 4y^2)$

Check:

To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 6y.6y. We'll leave the multiplication for you.

Example 6.3.32

Factor: $500p^3 + 4q^3$.

Answer

$4(5p + q)(25p^2 - 5pq + q^2)$

Example 6.3.33

Factor: $432c^3 + 686d^3$.

Answer

$2(6c + 7d)(36c^2 - 42cd + 49d^2)$

The first term in the next example is a binomial cubed.

Example 6.3.34

Factor: $(x+5)^3-64x^3$.

Answer

	$(x+5)^3-64x^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(x+5)^3}-\overset{b^3}{(4x)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{(x+5)}-\overset{b}{4x}\right)\left(\overset{a^2}{(x+5)^2}+\overset{a}{(x+5)}\cdot\overset{b}{4x}+\overset{b^2}{(4x)^2}\right)$
Simplify.	$(x+5-4x)\left(x^2+10x+25+4x^2+20x+16x^2\right)$
	$(-3x+5)\left(21x^2+30x+25\right)$
Check by multiplying.	We'll leave the check to you.

Example 6.3.35

Factor: $(y+1)^3-27y^3$.

Answer

$(-2y+1)(13y^2+5y+1)$

Example 6.3.36

Factor: $(n+3)^3-125n^3$.

Answer

$(-4n+3)(31n^2+21n+9)$

Access this online resource for additional instruction and practice with factoring special products.

- Factoring Binomials-Cubes #2

Key Concepts

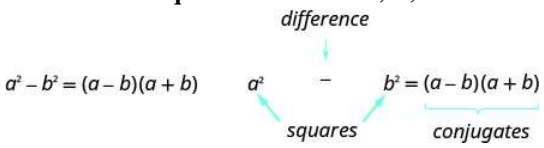
- **Perfect Square Trinomials Pattern:** If *a* and *b* are real numbers,

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

- **How to factor perfect square trinomials.**

Step 1.	Does the trinomial fit the pattern?	$a^2+2ab+b^2$	$a^2-2ab+b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2 \quad (b)^2$	$(a)^2 \quad (b)^2$
	Check the middle term. Is it $2ab$?	$\searrow 2\cdot a\cdot b \swarrow$	$\searrow 2\cdot a\cdot b \swarrow$
Step 2.	Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3.	Check by multiplying.		

- **Difference of Squares Pattern:** If a,ba,b are real numbers,



- **How to factor differences of squares.**

Step 1.	Does the binomial fit the pattern?	$a^2 - b^2$
	Is this a difference?	____−____
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2 - (b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

- **Sum and Difference of Cubes Pattern**
  $a^3 + b3 = (a+b)(a^2 - ab + b^2)$
  $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- **How to factor the sum or difference of cubes.**
  1. Does the binomial fit the sum or difference of cubes pattern?
     Is it a sum or difference?
     Are the first and last terms perfect cubes?
  2. Write them as cubes.
  3. Use either the sum or difference of cubes pattern.
  4. Simplify inside the parentheses
  5. Check by multiplying the factors.

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6.3E: Exercises

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

1. $16y^2+24y+9$

Answer

$(4y+3)^2$

2. $25v^2+20v+4$

3. $36s^2+84s+49$

Answer

$(6s+7)^2$

4. $49s^2+154s+121$

5. $100x^2-20x+1$

Answer

$(10x-1)^2$

6. $64z^2-16z+1$

7. $25n^2-120n+144$

Answer

$(5n-12)^2$

8. $4p^2-52p+169$

9. $49x^2+28xy+4y^2$

Answer

$(7x+2y)^2$

10. $25r^2+60rs+36s^2$

11. $100y^2-52y+1$

Answer

$(50y-1)(2y-1)$

12. $64m^2-34m+1$

13. $10jk^2+80jk+160j$

Answer

$10j(k+4)^2$

14. $64x^2y-96xy+36y$

15. $75u^4-30u^3v+3u^2v^2$

Answer

$3u^2(5u-v)^2$

16. $90p^4+300p^4q+250p^2q^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

17. $25v^2-1$

Answer

$(5v-1)(5v+1)$

18. $169q^2-1$

19. $4-49x^2$

Answer

$(7x-2)(7x+2)$

20. $121-25s^2$

21. $6p^2q^2-54p^2$

Answer

$6p^2(q-3)(q+3)$

22. $98r^3-72r$

23. $24p^2+54$

Answer

$6(4p^2+9)$

24. $20b^2+140$

25. $121x^2-144y^2$

Answer

$(11x-12y)(11x+12y)$

26. $49x^2-81y^2$

27. $169c^2 - 36d^2$

Answer

$(13c - 6d)(13c + 6d)$

28. $36p^2 - 49q^2$

29. $16z^4 - 1$

Answer

$(2z - 1)(2z + 1)(4z^2 + 1)$

30. $m^4 - n^4$

31. $162a^4b^2 - 32b^2$

Answer

$2b^2(3a - 2)(3a + 2)(9a^2 + 4)$

32. $48m^4n^2 - 243n^2$

33. $x^2 - 16x + 64 - y^2$

Answer

$(x - 8 - y)(x - 8 + y)$

34. $p^2 + 14p + 49 - q^2$

35. $a^2 + 6a + 9 - 9b^2$

Answer

$(a + 3 - 3b)(a + 3 + 3b)$

36. $m^2 - 6m + 9 - 16n^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

37. $x^3 + 125$

Answer

$(x + 5)(x^2 - 5x + 25)$

38. $n^6 + 512$

39. $z^6 - 27$

Answer

$(z^2 - 3)(z^4 + 3z^2 + 9)$

40. $v^3 - 216$

41. $8 - 343t^3$

Answer

$(2-7t)(4+14t+49t^2)$

42. $125 - 27w^3$

43. $8y^3 - 125z^3$

Answer

$(2y-5z)(4y^2+10yz+25z^2)$

44. $27x^3 - 64y^3$

45. $216a^3 + 125b^3$

Answer

$(6a+5b)(36a^2-30ab+25b^2)$

46. $27y^3 + 8z^3$

47. $7k^3 + 56$

Answer

$7(k+2)(k^2-2k+4)$

48. $6x^3 - 48y^3$

49. $2x^2 - 16x^2y^3$

Answer

$2x^2(1-2y)(1+2y+4y^2)$

50. $-2x^3y^2 - 16y^5$

51. $(x+3)^3 + 8x^3$

Answer

$9(x+1)(x^2+3)$

52. $(x+4)^3 - 27x^3$

53. $(y-5)^3 - 64y^3$

Answer

$-(3y+5)(21y^2-30y+25)$

54. $(y-5)^3+125y^3$

Mixed Practice

In the following exercises, factor completely.

55. $64a^2-25$

Answer

$(8a-5)(8a+5)$

56. $121x^2-144$

57. $27q^2-3$

Answer

$3(3q-1)(3q+1)$

58. $4p^2-100$

59. $16x^2-72x+81$

Answer

$(4x-9)^2$

60. $36y^2+12y+1$

61. $8p^2+2$

Answer

$2(4p^2+1)$

62. $81x^2+169$

63. $125-8y^3$

Answer

$(5-2y)(25+10y+4y^2)$

64. $27u^3+1000$

65. $45n^2+60n+20$

Answer

$5(3n+2)^2$

66. $48q^3-24q^2+3q$

67. $x^2-10x+25-y^2$

Answer

$(x+y-5)(x-y-5)$

68. $x^2+12x+36-y^2$

69. $(x+1)^3+8x^3$

Answer

$(3x+1)(3x^2+1)$

70. $(y-3)^3-64y^3$

Writing Exercises

71. Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

Answer

Answers will vary.

72. How do you recognize the binomial squares pattern?

73. Explain why $n^2+25 \neq (n+5)^2$. Use algebra, words, or pictures.

Answer

Answers will vary.

74. Maribel factored $y^2-30y+81$ as $(y-9)^2$. Was she right or wrong? How do you know?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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6.4: General Strategy for Factoring Polynomials

Summary

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

GENERAL STRATEGY FOR FACTORING POLYNOMIALS

GCF

Binomial	Trinomial	More than 3 terms
• **Difference of Squares** $a^2 - b^2 = (a - b)(a + b)$	• $x^2 + bx + c$ $(x \quad)(x \quad)$	• grouping
• **Sum of Squares** Sums of squares do not factor.	• $ax^2 + bx + c$ ◦ **'a' and 'c' squares** $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$ ◦ **'ac' method**	
• **Sum of Cubes** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$		
• **Difference of Cubes** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$		

USE A GENERAL STRATEGY FOR FACTORING POLYNOMIALS.

1. Is there a greatest common factor?
   Factor it out.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   If it is a binomial:
   - Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
   - Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.

   If it is a trinomial:
   - Is it of the form $x^2 + bx + c$? Undo FOIL.
   - Is it of the form $ax^2 + bx + c$?
     If a and c are squares, check if it fits the trinomial square pattern.
     Use the trial and error or "ac" method.

   If it has more than three terms:
   - Use the grouping method.
3. Check.
   Is it factored completely?
   Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are **prime**!

Example 6.4.1

Factor completely: $7x^3 - 21x^2 - 70x$.

Solution

$7x^3 - 21x^2 - 70x$	
Is there a GCF? Yes, $7x$.	
Factor out the GCF.	$7x(x^2 - 3x - 10)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	
Trinomial with leading coefficient 1.	
"Undo" FOIL.	$7x(x \quad)(x \quad)$
	$7x(x+2)(x-5)$
Is the expression factored completely? Yes.	
Neither binomial can be factored.	
Check your answer.	
Multiply.	
	$7x(x+2)(x-5)$
	$7x(x^2 - 5x + 2x - 10)$
	$7x(x^2 - 3x - 10)$
	$7x^3 - 21x^2 - 70x$✓

Try It 6.4.1

Factor completely: $8y^3 + 16y^2 - 24y$.

Answer

$8y(y-1)(y+3)$

Try It 6.4.2

Factor completely: $5y^3 - 15y^2 - 270y$.

Answer

$5y(y-9)(y+6)$

Be careful when you are asked to factor a binomial as there are several options!

Example 6.4.2

Factor completely: $24y^2 - 150$

Solution

	$24y^2-150$
Is there a GCF? Yes, 6.	
Factor out the GCF.	$6(4y^2-25)$
In the parentheses, is it a binomial, trinomial or are there more than three terms? Binomial.	
Is it a sum? No.	
Is it a difference? Of squares or cubes? Yes, squares.	$6((2y)^2-(5)^2)$
Write as a product of conjugates.	$6(2y-5)(2y+5)$

Is the expression factored completely?

Neither binomial can be factored.

Check:

Multiply.

$6(2y-5)(2y+5)$

$6(4y^2-25)$

$24y^2-150$✓

? Try It 6.4.3

Factor completely: $16x^3-36x$.

Answer

$4x(2x-3)(2x+3)$

? Try It 6.4.4

Factor completely: $27y^2-48$.

Answer

$3(3y-4)(3y+4)$

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

? Example 6.4.3

Factor completely: $4a^2-12ab+9b^2$.

Solution

$4a^2-12ab+9b^2$

Is there a GCF? No.

Is it a binomial, trinomial, or are there more terms?

Trinomial with $a \neq 1$. But the first term is a perfect square.

Is the last term a perfect square? Yes. $(2a)^2-12ab+(3b)^2$

Does it fit the pattern, $a^2-2ab+b^2$? Yes. $(2a)^2-12ab+(3b)^2$

$\searrow \quad -2(2a)(3b) \quad \swarrow$

Write it as a square. $(2a-3b)^2$

Is the expression factored completely? Yes.

The binomial cannot be factored.

Check your answer.

Multiply.

$$(2a-3b)^2$$
$$(2a)^2-2\cdot 2a\cdot 3b+(3b)^2$$
$$4a^2-12ab+9b^2 \checkmark$$

? Try It 6.4.5

Factor completely: $4x^2+20xy+25y^2$.

Answer

$(2x+5y)^2$

? Try It 6.4.6

Factor completely: $9x^2-24xy+16y^2$.

Answer

$(3x-4y)^2$

Remember, sums of squares do not factor, but sums of cubes do!

? Example 6.4.4

Factor completely $12x^3y^2+75xy^2$.

Solution

	$12x^3y^2+75xy^2$
Is there a GCF? Yes, $3xy^2$.	
Factor out the GCF.	$3xy^2(4x^2+25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms? Binomial.	
Is it a sum? Of squares? Yes.	Sums of squares are prime.
Is the expression factored completely? Yes.	

Check:

Multiply.

$3xy^2(4x^2+25)$

$12x^3y^2+75xy^2$ ✓

Try It 6.4.7

Factor completely: $50x^3y+72xy$.

Answer

$2xy(25x^2+36)$

Try It 6.4.8

Factor completely: $27xy^3+48xy$.

Answer

$3xy(9y^2+16)$

When using the sum or difference of cubes pattern, being careful with the signs.

Example 6.4.5

Factor completely: $24x^3+81y^3$.

Solution

Is there a GCF? Yes, 3.	$24x^3+81y^3$
Factor it out.	$3(8x^3+27y^3)$
In the parentheses, is it a binomial, trinomial, of are there more than three terms? Binomial.	
Is it a sum or difference? Sum.	
Of squares or cubes? Sum of cubes.	$3\left(\overset{a^3}{(2x)^3}+\overset{b^3}{(3y)^3}\right)$
Write it using the sum of cubes pattern.	$3\left(\overset{a}{2x}+\overset{b}{3y}\right)\left(\overset{a^2}{(2x)^2}-\overset{ab}{2x\cdot 3y}+\overset{b^2}{(3y)^3}\right)$
Is the expression factored completely? Yes.	$3(2x+3y)(4x^2-6xy+9y^2)$

Check by multiplying.

Try It 6.4.9

Factor completely: $250m^3+432n^3$.

Answer

$2(5m+6n)(25m^2-30mn+36n^2)$

Try It 6.4.10

Factor completely: $2p^3+54q^3$.

Answer

$2(p+3q)(p^2-3pq+9q^2)$

Example 6.4.6

Factor completely: $3x^5y-48xy$.

Solution

	$3x^5y-48xy$
Is there a GCF? Factor out $3xy$	$3xy(x^4-16)$
Is the binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$3xy\left((x^2)^2-(4)2\right)$
Factor it as a product of conjugates	$3xy(x^2-4)(x^2+4)$
The first binomial is again a difference of squares.	$3xy\left((x)^2-(2)^2\right)(x^2+4)$
Factor it as a product of conjugates.	$3xy(x-2)(x+2)(x^2+4)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
$3xy(x-2)(x+2)(x^2+4)$	
$3xy(x^2-4)(x^2+4)$	
$3xy(x^4-16)$	
$3x^5y-48xy$✓	

Try It 6.4.11

Factor completely: $4a^5b-64ab$.

Answer

$4ab(a^2+4)(a-2)(a+2)$

Try It 6.4.12

Factor completely: $7xy^5-7xy$.

Answer

$7xy(y^2+1)(y-1)(y+1)$

Example 6.4.7

Factor completely: $4x^2+8bx-4ax-8ab$.

Solution

	$4x^2+8bx-4ax-8ab$
Is there a GCF? Factor out the GCF, 4.	$4(x^2+2bx-ax-2ab)$
There are four terms. Use grouping.	$4[x(x+2b)-a(x+2b)]4(x+2b)(x-a)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
	$4(x+2b)(x-a)$
	$4(x^2-ax+2bx-2ab)$
	$4x^2+8bx-4ax-8ab$✓

Try It 6.4.13

Factor completely: $6x^2-12xc+6bx-12bc$.

Answer

$6(x+b)(x-2c)$

Try It 6.4.14

Factor completely: $16x^2+24xy-4x-6y$.

Answer

$2(4x-1)(2x+3y)$

Taking out the complete GCF in the first step will always make your work easier.

Example 6.4.8

Factor completely: $40x^2y+44xy-24y$.

Solution

	$40x^2y+44xy-24y$
Is there a GCF? Factor out the GCF, $4y$.	$4y(10x^2+11x-6)$
Factor the trinomial with $a \neq 1$.	$4y(10x^2+11x-6)$
	$4y(5x-2)(2x+3)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
	$4y(5x-2)(2x+3)$
	$4y(10x^2+11x-6)$
	$40x^2y+44xy-24y$✓

Try It 6.4.15

Factor completely: $4p^2q-16pq+12q$.

Answer

$4q(p-3)(p-1)$

Try It 6.4.16

Factor completely: $6pq^2-9pq-6p$.

Answer

$3p(2q+1)(q-2)$

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

Example 6.4.9

Factor completely: $9x^2-12xy+4y^2-49$.

Solution

	$9x^2-12xy+4y^2-49$
Is there a GCF? No.	
With more than 3 terms, use grouping. Last 2 terms have no GCF. Try grouping first 3 terms.	$9x^2-12xy+4y^2-49$
Factor the trinomial with $a\neq 1$. But the first term is a perfect square.	
Is the last term of the trinomial a perfect square? Yes.	$(3x)^2-12xy+(2y)^2-49$
Does the trinomial fit the pattern, $a^2-2ab+b^2$? Yes.	$(3x)^2-12xy+(2y)^2-49$ $\searrow \quad -2(3x)(2y) \quad \swarrow$
Write the trinomial as a square.	$(3x-2y)^2-49$
Is this binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$(3x-2y)^2-72$
Write it as a product of conjugates.	$((3x-2y)-7)((3x-2y)+7)$ $(3x-2y-7)(3x-2y+7)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
$(3x-2y-7)(3x-2y+7)$	
$9x^2-6xy-21x-6xy+4y^2+14y+21x-14y-49$	
$9x^2-12xy+4y^2-49$✓	

Try It 6.4.17

Factor completely: $4x^2-12xy+9y^2-25$.

Answer

$(2x-3y-5)(2x-3y+5)$

? Try It 6.4.18

Factor completely: $16x^2 - 24xy + 9y^2 - 64$.

Answer

$(4x - 3y - 8)(4x - 3y + 8)$

Key Concepts

General Strategy for Factoring Polynomials

GCF

Binomial	Trinomial	More than 3 terms
• **Difference of Squares** $a^2 - b^2 = (a - b)(a + b)$	• $x^2 + bx + c$ $(x \quad)(x \quad)$	• grouping
• **Sum of Squares** Sums of squares do not factor.	• $ax^2 + bx + c$ ◦ **'a' and 'c' squares** $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$ ◦ **'ac' method**	
• **Sum of Cubes** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$		
• **Difference of Cubes** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$		

- **How to use a general strategy for factoring polynomials.**
  1. Is there a greatest common factor?
     Factor it out.
  2. Is the polynomial a binomial, trinomial, or are there more than three terms?
     If it is a binomial:
     Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
     Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.
     If it is a trinomial:
     Is it of the form $x^2 + bx + c$? Undo FOIL.
     Is it of the form $ax^2 + bx + c$?
     If *a* and *c* are squares, check if it fits the trinomial square pattern.
     Use the trial and error or "ac" method.
     If it has more than three terms:
     Use the grouping method.
  3. Check.
     Is it factored completely?
     Do the factors multiply back to the original polynomial?

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6.4E: Exercises

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

1. $2n^2+13n-7$

Answer

$(2n-1)(n+7)$

2. $8x^2-9x-3$

3. a^5+9a^3

Answer

$a^3(a^2+9)$

4. $75m^3+12m$

5. $121r^2-s^2$

Answer

$(11r-s)(11r+s)$

6. $49b^2-36a^2$

7. $8m^2-32$

Answer

$8(m-2)(m+2)$

8. $36q^2-100$

9. $25w^2-60w+36$

Answer

$(5w-6)^2$

10. $49b^2-112b+64$

11. $m^2+14mn+49n^2$

Answer

$(m+7n)^2$

12. $64x^2+16xy+y^2$

13. $7b^2+7b-42$

Answer

$7(b+3)(b-2)$

14. $30n^2+30n+72$

15. $3x^4y-81xy$

Answer

$3xy(x-3)(x^2+3x+9)$

16. $4x^5y-32x^2y$

17. k^4-16

Answer

$(k-2)(k+2)(k^2+4)$

18. m^4-81

19. $5x5y^2-80xy^2$

Answer

$5xy^2(x^2+4)(x+2)(x-2)$

20. $48x^5y^2-243xy^2$

21. $15pq-15p+12q-12$

Answer

$3(5p+4)(q-1)$

22. $12ab-6a+10b-5$

23. $4x^2+40x+84$

Answer

$4(x+3)(x+7)$

24. $5q^2-15q-90$

25. $4u^5v+4u^2v^3$

Answer

$u^2(u+1)(u^2-u+1)$

26. $5m^4n+320mn^4$

27. $4c^2+20cd+81d^2$

Answer

prime

28. $25x^2+35xy+49y^2$

29. $10m^4-6250$

Answer

$10(m-5)(m+5)(m^2+25)$

30. $3v^4-768$

31. $36x^2y+15xy-6y$

Answer

$3y(3x+2)(4x-1)$

32. $60x^2y-75xy+30y$

33. $8x^3-27y^3$

Answer

$(2x-3y)(4x^2+6xy+9y^2)$

34. $64x^3+125y^3$

35. y^6-1

Answer

$(y+1)(y-1)(y^2-y+1)$

36. y^6+1

37. $9x^2-6xy+y^2-49$

Answer

$(3x-y+7)(3x-y-7)$

38. $16x^2-24xy+9y^2-64$

39. $(3x+1)^2-6(3x-1)+9$

Answer

$(3x-2)2$

40. $(4x-5)^2-7(4x-5)+12$

Writing Exercises

41. Explain what it mean to factor a polynomial completely.

Answer

Answers will vary.

42. The difference of squares $y^4 - 625$ can be factored as $(y^2 - 25)(y^2 + 25)$. But it is not completely factored. What more must be done to completely factor.

43. Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, 'ac' method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

Answer

Answers will vary.

44. Create three factoring problems that would be good test questions to measure your knowledge of factoring. Show the solutions.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize and use the appropriate method to factor a polynomial completely.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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6.5: Polynomial Equations

Summary

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve equations with polynomial functions
- Solve applications modeled by polynomial equations

Are you ready?

Before you get started, take this readiness quiz.

1. Solve: $5y-3=0$.
   If you missed this problem, review [link].
2. Factor completely: n^3-9n^2-22n.
   If you missed this problem, review [link].
3. If $f(x)=8x-16$, find $f(3)$ and solve $f(x)=0$.
   If you missed this problem, review [link].

We have spent considerable time learning how to factor polynomials. We will now look at polynomial equations and solve them using factoring, if possible.

A **polynomial equation** is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

POLYNOMIAL EQUATION

A **polynomial equation** is an equation that contains a polynomial expression.

The **degree of the polynomial equation** is the degree of the polynomial.

We have already solved polynomial equations of **degree one**. Polynomial equations of degree one are linear equations are of the form ax+b=c.ax+b=c.

We are now going to solve polynomial equations of **degree two**. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \quad 3y^2+4y=10 \quad 64u^2-81=0 \quad n(n+1)=42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n2+n.n2+n.

The general form of a quadratic equation is $ax^2+bx+c=0$, with $a\neq 0$. (If $a=0$, then $0\cdot x^2=0$ and we are left with no quadratic term.)

QUADRATIC EQUATION

An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.

$$a, b, \text{ and } c \text{ are real numbers and } a\neq 0$$

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply

by zero itself.

ZERO PRODUCT PROPERTY

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a **quadratic equation**.

Example 6.5.1: How to Solve a Quadratic Equation Using the Zero Product Property

Solve: $(5n-2)(6n-1)=0$.

Answer

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n-2)(6n-1)=0$ $5n-2=0$ or $6n-1=0$
Step 2. Solve the linear equations.	Solve each equation.	$n=\frac{2}{5}$ $n=\frac{1}{6}$
Step 3. Check.	Substitute each solution separately into the original equation.	$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-1\right)\stackrel{?}{=}0$ $0\cdot\frac{7}{5}\stackrel{?}{=}0$ $0=0$ ✓ $n=\frac{1}{6}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0$ ✓

Example 6.5.2

Solve: $(3m-2)(2m+1)=0$.

Answer

$m=\frac{2}{3}$, $m=-\frac{1}{2}$

Example 6.5.3

Solve: $(4p+3)(4p-3)=0$.

Answer

$p=-\frac{3}{4}$, $p=\frac{3}{4}$

USE THE ZERO PRODUCT PROPERTY.

1. Set each factor equal to zero.
2. Solve the linear equations.
3. Check.

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in **standard form**, $ax^2+bx+c=0$. Then we factor the expression on the left.

Solve: $2y^2=13y+45$.

Answer

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	Write the equation in standard form.	$2y^2=13y+45$ $2y^2-13y-45=0$
Step 2. Factor the quadratic expression.	Factor $2y^2-13y+45$ $(2y+5)(y-9)$	$(2y+5)(y-9)=0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y+5=0$ $y-9=0$
Step 4. Solve the linear equations.		$y=-\frac{5}{2}$ $y=9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y=-\frac{5}{2}$ $2y^2=13y+45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right)+45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right)+\frac{90}{2}$ $\frac{25}{2}=\frac{25}{2}$ ✓ $y=9$ $2y^2=13y+45$ $2(9)^2 \stackrel{?}{=} 13(9)+45$ $2(81) \stackrel{?}{=} 117+45$ $162=162$ ✓

Example 6.5.5

Solve: $3c^2=10c-8$.

Answer

$c=2,\ c=\frac{4}{3}$

Example 6.5.6

Solve: $2d^2 - 5d = 3$.

Answer

$d = 3,\ d = -12$

SOLVE A QUADRATIC EQUATION BY FACTORING.

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check. Substitute each solution separately into the original equation.

Before we factor, we must make sure the **quadratic equation** is in **standard form**.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Example 6.5.7

Solve: $169q^2 = 49$.

Answer

	$169x^2 = 49$
Write the quadratic equation in standard form.	$169x^2 - 49 = 0$
Factor. It is a difference of squares.	$(13x - 7)(13x + 7) = 0$
Use the Zero Product Property to set each factor to 0.	
	$13x - 7 = 0 \quad 13x + 7 = 0$
Solve each equation.	$13x = 7 \quad 13x = -7$
	$x = \frac{7}{13} \quad x = -\frac{7}{13}$

Check:

We leave the check up to you.

Example 6.5.8

Solve: $25p^2 = 49$.

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Example 6.5.9

Solve: $36x^2 = 121$.

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the **Zero Product Property**, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Example 6.5.10

Solve: $(3x-8)(x-1)=3x$.

Answer

	$(3x-8)(x-1)=3x$
Multiply the binomials.	$3x^2-11x+8=3x$
Write the quadratic equation in standard form.	$3x^2-14x+8=0$
Factor the trinomial.	$(3x-2)(x-4)=0$
Use the Zero Product Property to set each factor to 0.	$3x-2=0 \quad x-4=0$
Solve each equation.	$3x=2 \quad x=4$
	$x=\frac{2}{3}$
Check your answers.	The check is left to you.

Example 6.5.11

Solve: $(2m+1)(m+3)=12m$.

Answer

$m=1,\ m=\frac{3}{2}$

Example 6.5.12

Solve: $(k+1)(k-1)=8$.

Answer

$k=3,\ k=-3$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example 6.5.13

Solve: $3x^2=12x+63$.

Answer

	$3x^2=12x+63$
Write the quadratic equation in standard form.	$3x^2-12x-63=0$
Factor the greatest common factor first.	$3(x^2-4x-21)=0$
Factor the trinomial.	$3(x-7)(x+3)=0$
Use the Zero Product Property to set each factor to 0.	$3\neq 0 \quad x-7=0 \quad x+3=0$
Solve each equation.	$3\neq 0 \quad x=7 \quad x=-3$
Check your answers.	The check is left to you.

Example 6.5.14

Solve: $18a^2-30=-33a$.

Answer

$a=-\frac{5}{2}, a=\frac{2}{3}$

? Example 6.5.15

Solve: $123b = -6 - 60b^2$

Answer

$b = -2,\ b = -\frac{1}{20}$

The **Zero Product Property** also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

? Example 6.5.16

Solve: $9m^3 + 100m = 60m^2$

Answer

	$9m^3 + 100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m = 0$
Factor the greatest common factor first.	$m(9m^2 - 60m + 100) = 0$
Factor the trinomial.	$m(3m - 10)^2 = 0$
Use the Zero Product Property to set each factor to 0.	
Solve each equation.	$m = 0 \quad 3m - 10 = 0$ $m = 0 \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

? Example 6.5.17

Solve: $8x^3 = 24x^2 - 18x$.

Answer

$x = 0,\ x = \frac{3}{2}$

? Example 6.5.18

Solve: $16y^2 = 32y^3 + 2y$.

Answer

$y = 0,\ y = 14$

Solve Equations with Polynomial Functions

As our study of polynomial functions continues, it will often be important to know when the function will have a certain value or what points lie on the graph of the function. Our work with the **Zero Product Property** will be help us find these answers.

? Example 6.5.19

For the function $f(x) = x^2 + 2x - 2$,

ⓐ find x when $f(x) = 6$

ⓑ find two points that lie on the graph of the function.

Answer

ⓐ

	$f(x)=x^2+2x-2$
Substitute 6 for $f(x)$.	$6=x^2+2x-2$
Put the quadratic in standard form.	$x^2+2x-8=0$
Factor the trinomial.	$(x+4)(x-2)=0$
Use the zero product property.	$x+4=0 \quad \text{or} \quad x-2=0$
Solve.	$x=-4 \quad \text{or} \quad x=2$
Check:	

$f(x)=x^2+2x-2$	$f(x)=x^2+2x-2$
$f(-4)=(-4)^2+2(-4)-2$	$f(2)=2^2+2\cdot 2-2$
$f(-4)=16-8-2$	$f(2)=4+4-2$
$f(-4)=6$✓	$f(2)=6$✓

ⓑ Since $f(-4)=6$ and $f(2)=6$, the points $(-4,6)$ and $(2,6)$ lie on the graph of the function.

? Example 6.5.20

For the function $f(x)=x^2-2x-8$,

ⓐ find x when $f(x)=7$
ⓑ Find two points that lie on the graph of the function.

Answer

ⓐ $x=-3$ or $x=5$
ⓑ $(-3,7)$ $(5,7)$

? Example 6.5.21

For the function $f(x)=x^2-8x+3$,

ⓐ find x when $f(x)=-4$
ⓑ Find two points that lie on the graph of the function.

Answer

ⓐ $x=1$ or $x=7$
ⓑ $(1,-4)$ $(7,-4)$

The **Zero Product Property** also helps us determine where the function is zero. A value of x where the function is 0, is called a **zero of the function**.

ZERO OF A FUNCTION

For any function f, if $f(x)=0$, then x is a **zero of the function**.

When $f(x)=0$, the point $(x,0)$ is a point on the graph. This point is an x-**intercept** of the graph. It is often important to know where the graph of a function crosses the axes. We will see some examples later.

Example 6.5.22

For the function $f(x)=3x^2+10x-8$, find

ⓐ the zeros of the function,
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function

Answer

ⓐ To find the zeros of the function, we need to find when the function value is 0.

	$f(x)=3x^2+10x-8$
Substitute 0 for $f(x)$.	$0=3x^2+10x-8$
Factor the trinomial.	$(x+4)(3x-2)=0$
Use the zero product property.	$x+4=0 \quad \text{or} \quad 3x-2=0$
Solve.	$x=-4 \quad \text{or} \quad x=\frac{2}{3}$

ⓑ An x-intercept occurs when $y=0$. Since $f(-4)=0$ and $f(\frac{2}{3})=0$, the points $(-4,0)$ and $(\frac{2}{3},0)$ lie on the graph. These points are x-intercepts of the function.

ⓒ A y-intercept occurs when $x=0$. To find the y-intercepts we need to find $f(0)$.

	$f(x)=3x^2+10x-8$
Find $f(0)$ by substituting 0 for x.	$f(0)=3\cdot 0^2+10\cdot 0-8$
Simplify.	$f(0)=-8$

Since $f(0)=-8$, the point $(0,-8)$ lies on the graph. This point is the y-intercept of the function.

Example 6.5.23

For the function $f(x)=2x^2-7x+5$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Answer

ⓐ $x=1$ or $x=\frac{5}{2}$
ⓑ $(1,0),\ (\frac{5}{2},0)$ ⓒ $(0,5)$

Example 6.5.24

For the function $f(x)=6x^2+13x-15$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Answer

ⓐ $x=-3$ or $x=\frac{5}{6}$
ⓑ $(-3,0),\ (\frac{5}{6},0)$ ⓒ $(0,-15)$

Solve Applications Modeled by Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.

USE A PROBLEM SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

Example 6.5.25

The product of two consecutive odd integers is 323. Find the integers.

Answer

Step 1. Read the problem.

Step 2. Identify what we are looking for. — We are looking for two consecutive integers.

Step 3. Name what we are looking for. — Let $n=$ the first integer.
$n+2=$ next consecutive odd integer

Step 4. Translate into an equation. Restate the problem in a sentence. — The product of the two consecutive odd integers is 323.
$n(n+2)=323$

Step 5. Solve the equation. $n^2+2n=323$

Bring all the terms to one side. $n^2+2n-323=0$

Factor the trinomial. $(n-17)(n+19)=0$

Use the Zero Product Property. $n-17=0$ $\quad$ $n+19=0$

Solve the equations. $n=17$ $\quad$ $n=-19$

There are two values for n that are solutions to this problem. So there are two sets of consecutive odd integers that will work.

If the first integer is $n=17$ then the next odd integer is

$$\begin{aligned} n+2 \\ 17+2 \\ 19 \\ 17,19 \end{aligned}$$

If the first integer is $n=-19$ then the next odd integer is

$$\begin{aligned} n+2 \\ -19+2 \\ -17 \\ -17,-19 \end{aligned}$$

Step 6. Check the answer.

The results are consecutive odd integers
$17,\ 19$ and $-19,\ -17$.

$17\cdot 19=323$✓ $\quad$ $-19(-17)=323$✓

Both pairs of consecutive integers are solutions.

Step 7. Answer the question — The consecutive integers are $17, 19$ and $-19, -17$.

? Example 6.5.26

The product of two consecutive odd integers is 255. Find the integers.

Answer

$-15, -17$ and $15, 17$

? Example 6.5.27

The product of two consecutive odd integers is 483 Find the integers.

Answer

$-23, -21$ and $21, 23$

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

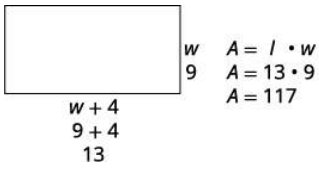
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

? Example 6.5.28

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Answer

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom.
The length is four feet more than the width.	$w + 4 =$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the area of a rectangle.	$A = l \cdot w$
Substitute in the variables.	$117 = (w + 4)w$
Step 5. Solve the equation Distribute first.	$117 = w^2 + 4w$
Get zero on one side.	$117 = w^2 + 4w$
Factor the trinomial.	$0 = w^2 + 4w - 117$
Use the Zero Product Property.	$0 = (w^2 + 13)(w - 9)$
Solve each equation.	$0 = w + 13 \quad 0 = w - 9$
Since w is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for w.	$\cancel{w = -13} \quad w = 9$
	$w = 9$ The width is 9 feet.
Find the value of the length.	$w + 4$ $9 + 4$ 13 The length is 13 feet.

Step 6. Check the answer. Does the answer make sense?  $A = l \cdot w$ $A = 13 \cdot 9$ $A = 117$ Yes, this makes sense.	
Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

Example 6.5.29

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Answer

The width is 5 feet and length is 6 feet.

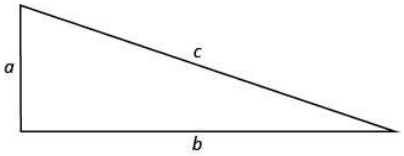
Example 6.5.30

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Answer

The length of the patio is 12 feet and the width 15 feet.

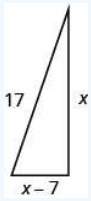
In the next example, we will use the Pythagorean Theorem $(a^2 + b^2 = c^2)$. This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

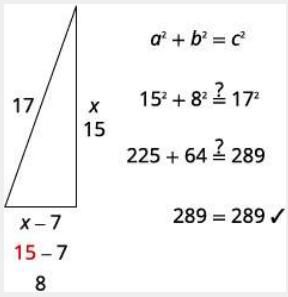
Example 6.5.31

A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.



Answer

Step 1. Read the problem

Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x =$ length of a side of the sail. $x - 7 =$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Substitute in the variables.	$x^2 + (x-7)^2 = 17^2$
Step 5. Solve the equation Simplify.	$x^2 + x^2 - 14x + 49 = 289$
	$2x^2 - 14x + 49 = 289$
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$
Factor the trinomial.	$2(x-15)(x+8) = 0$
Use the Zero Product Property.	$2 \neq 0 \quad x - 15 = 0 \quad x + 8 = 0$
Solve.	$2 \neq 0 \quad x = 15 \quad x = -8$
Since x is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0 \quad x = 15 \quad \cancel{x = -8}$
Find the length of the other side.	
If the length of one side is then the length of the other side is	$x = 15$ $x - 7$ $15 - 7$ 8 is the length of the other side.
Step 6. Check the answer in the problem Do these numbers make sense? 	
Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

? Example 6.5.32

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

Answer

5 feet and 12 feet

? Example 6.5.33

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

Answer

24 feet and 25 feet

The next example uses the function that gives the height of an object as a function of time when it is thrown from 80 feet above the ground.

? Example 6.5.34

Dennis is going to throw his rubber band ball upward from the top of a campus building. When he throws the rubber band ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, h, of the ball above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the ball hits the ground
ⓑ when the ball will be 80 feet above the ground
ⓒ the height of the ball at $t = 2$ seconds.

Answer

ⓐ The zeros of this function are found by solving $h(t) = 0$. This will tell us when the ball will hit the ground.

	$h(t) = 0$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 0$
Factor the GCF, -16.	$-16(t^2 - 4t - 5) = 0$
Factor the trinomial.	$-16(t-5)(t+1) = 0$
Use the Zero Product Property.	$t-5=0 \quad t+1=0$
Solve.	$t = 5 \quad t = -1$

The result $t = 5$ tells us the ball will hit the ground 5 seconds after it is thrown. Since time cannot be negative, the result $t = -1$ is discarded.

ⓑ The ball will be 80 feet above the ground when $h(t) = 80$.

	$h(t) = 80$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 80$
Subtract 80 from both sides.	$-16t^2 + 64t = 0$
Factor the GCF, $-16t$.	$-16t(t-4) = 0$
Use the Zero Product Property.	$-16t = 0 \quad t-4=0$
Solve.	$t = 0 \quad t = 4$
	The ball will be at 80 feet the moment Dennis tosses the ball and then 4 seconds later, when the ball is falling.

ⓒ To find the height ball at $t = 2$ seconds we find $h(2)$.

	$h(t) = -16t^2 + 64t + 80$
To find $h(2)$ substitute 2 for t.	$h(2) = -16(2)^2 + 64 \cdot 2 + 80$
Simplify.	$h(2) = 144$
	After 2 seconds, the ball will be at 144 feet.

? Example 6.5.35

Genevieve is going to throw a rock from the top a trail overlooking the ocean. When she throws the rock upward from 160 feet above the ocean, the function $h(t) = -16t^2 + 48t + 160$ models the height, h, of the rock above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which tell us when the rock will hit the ocean
ⓑ when the rock will be 160 feet above the ocean.
ⓒ the height of the rock at $t = 1.5$ seconds.

Answer

ⓐ 5 ⓑ 0;3 ⓒ 196

? Example 6.5.36

Calib is going to throw his lucky penny from his balcony on a cruise ship. When he throws the penny upward from 128 feet above the ground, the function $h(t) = -16t^2 + 32t + 128$ models the height, h, of the penny above the ocean as a function of time, t. Find:

ⓐ the zeros of this function which is when the penny will hit the ocean
ⓑ when the penny will be 128 feet above the ocean.
ⓒ the height the penny will be at $t = 1$ seconds which is when the penny will be at its highest point.

Answer

ⓐ 4 ⓑ 0;2 ⓒ 144

Access this online resource for additional instruction and practice with quadratic equations.

- Beginning Algebra & Solving Quadratics with the Zero Property

Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a, b, c \text{ are real numbers and } a \neq 0$$

- **Zero Product Property:** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.
- **How to use the Zero Product Property**
  1. Set each factor equal to zero.
  2. Solve the linear equations.
  3. Check.
- **How to solve a quadratic equation by factoring.**
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check. Substitute each solution separately into the original equation.
- **Zero of a Function:** For any function f, if $f(x) = 0$, then x is a zero of the function.
- **How to use a problem solving strategy to solve word problems.**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.

4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

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6.5E: Exercises

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

1. $(3a-10)(2a-7)=0$

Answer

$a=\frac{10}{3},\ a=\frac{7}{2}$

2. $(5b+1)(6b+1)=0$

3. $6m(12m-5)=0$

Answer

$m=0,\ m=\frac{5}{12}$

4. $2x(6x-3)=0$

5. $(2x-1)^2=0$

Answer

$x=\frac{1}{2}$

6. $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

7. $5a^2-26a=24$

Answer

$a=-\frac{4}{5},\ a=6$

8. $4b^2+7b=-3$

9. $4m^2=17m-15$

Answer

$m=\frac{5}{4},\ m=3$

10. $n^2=5-6n$

11. $7a^2+14a=7a$

Answer

$a=-1,\ a=0$

12. $12b^2 - 15b = -9b$

13. $49m^2 = 144$

Answer

$m = \frac{12}{7},\ m = -\frac{12}{7}$

14. $625 = x^2$

15. $16y^2 = 81$

Answer

$y = -\frac{9}{4},\ y = \frac{9}{4}$

16. $64p^2 = 225$

17. $121n^2 = 36$

Answer

$n = -\frac{6}{11},\ n = \frac{6}{11}$

18. $100y^2 = 9$

19. $(x+6)(x-3) = -8$

Answer

$x = 2,\ x = -5$

20. $(p-5)(p+3) = -7$

21. $(2x+1)(x-3) = -4x$

Answer

$x = \frac{3}{2},\ x = -1$

22. $(y-3)(y+2) = 4y$

23. $(3x-2)(x+4) = 12x$

Answer

$x = \frac{3}{2},\ x = -1$

24. $(2y-3)(3y-1) = 8y$

25. $20x^2 - 60x = -45$

Answer

$x = -\frac{2}{3}$

26. $3y^2 - 18y = -27$

27. $15x^2 - 10x = 40$

Answer

$x = 2,\ x = -\frac{4}{3}$

28. $14y^2 - 77y = -35$

29. $18x^2 - 9 = -21x$

Answer

$x = -\frac{3}{2},\ x = \frac{1}{3}$

30. $16y^2 + 12 = -32y$

31. $16p^3 = 24p^2 - 9p$

Answer

$p = 0,\ p = \frac{3}{4}$

32. $m^3 - 2m^2 = -m$

33. $2x^3 + 72x = 24x^2$

Answer

$x = 0,\ x = 6$

34. $3y^3 + 48y = 24y^2$

35. $36x^3 + 24x^2 = -4x$

Answer

$x = 0,\ x = \frac{1}{3}$

36. $2y^3 + 2y^2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

37. For the function, $f(x) = x^2 - 8x + 8$, ⓐ find when $f(x) = -4$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = 2$ or $x = 6$ ⓑ $(2, -4)$ $(6, -4)$

38. For the function, $f(x) = x^2 + 11x + 20$, ⓐ find when $f(x) = -8$ ⓑ Use this information to find two points that lie on the graph of the function.

39. For the function, $f(x) = 8x^2 - 18x + 5$, ⓐ find when $f(x) = -4$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = \frac{3}{2}$ or $x = \frac{3}{4}$
ⓑ $(\frac{3}{2}, -4)$ $(\frac{3}{4}, -4)$

40. For the function, $f(x) = 18x^2 + 15x - 10$, ⓐ find when $f(x) = 15$ ⓑ Use this information to find two points that lie on the graph of the function.

In the following exercises, for each function, find: ⓐ the zeros of the function ⓑ the x-intercepts of the graph of the function ⓒ the y-intercept of the graph of the function.

41. $f(x) = 9x^2 - 4$

Answer

ⓐ $x = \frac{2}{3}$ or $x = -\frac{2}{3}$
ⓑ $(\frac{2}{3}, 0)$, $(-\frac{2}{3}, 0)$
ⓒ $(0, -4)$

42. $f(x) = 25x^2 - 49$

43. $f(x) = 6x^2 - 7x - 5$

Answer

ⓐ $x = \frac{5}{3}$ or $x = -\frac{1}{2}$
ⓑ $(\frac{5}{3}, 0)$, $(-\frac{1}{2}, 0)$
ⓒ $(0, -5)$

44. $f(x) = 12x^2 - 11x + 2$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

45. The product of two consecutive odd integers is 143. Find the integers.

Answer

$-13,\ -11$ and $11,\ 13$

46. The product of two consecutive odd integers is 195. Find the integers.

47. The product of two consecutive even integers is 168. Find the integers.

Answer

$-14,\ -12$ and $12,\ 14$

48. The product of two consecutive even integers is 288. Find the integers.

49. The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

Answer

-4 and 7

50. A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

51. The area of a bulletin board is 55 square feet. The length is four feet less than three times the width. Find the length and the width of the a bulletin board.

Answer

$5,\ 11$

52. A rectangular carport has area 150 square feet. The height of the carport is five feet less than twice its length. Find the height and the length of the carport.

53. A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

Answer

$6,\ 8$

54. A stained glass window is shaped like a right triangle. The hypotenuse is 15 feet. One leg is three more than the other. Find the lengths of the legs.

55. A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

Answer

$8,\ 15,\ 17$

56. A goat enclosure is in the shape of a right triangle. One leg of the enclosure is built against the side of the barn. The other leg is 4 feet more than the leg against the barn. The hypotenuse is 8 feet more than the leg along the barn. Find the three sides of the goat enclosure.

57. Juli is going to launch a model rocket in her back yard. When she launches the rocket, the function $h(t) = -16t^2 + 32t$ models the height, h, of the rocket above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tells us when the rocket will hit the ground. ⓑ the time the rocket will be 16 feet above the ground.

Answer

ⓐ 0, 2 ⓑ 1

58. Gianna is going to throw a ball from the top floor of her middle school. When she throws the ball from 48 feet above the ground, the function $h(t) = -16t^2 + 32t + 48$ models the height, h, of the ball above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 48 feet above the ground. ⓒ the height the ball will be at $t = 1$ seconds which is when the ball will be at its highest point.

Writing Exercises

59. Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

Answer

Answers will vary.

60. Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Additional Factoring Applications

Number problems

61. The product of two consecutive even integers is 624. Find the integers.

Answer

-26 and -24, and 24 and 26

62. The product of two consecutive even integers is 528. Find the integers.

63. The product of two consecutive positive integers is 552. Find the integers.

Answer

23, 24

64. The product of two consecutive positive integers is 756. Find the integers..

65.

The product of two consecutive odd integers is 483. Find the integers.

Answer

-23 and -21, and 21 and 23

66. The product of two consecutive odd integers is 783. Find the integers.

67. You have two positive numbers. The second number is three more than two times the first number. The difference of their squares is 144. Find both positive numbers.

Answer

5 and 13

68. You have two positive numbers. The second number is two more than three times the first number. The difference of their squares is 60. Find both positive numbers.

69.

Two numbers differ by 5. The sum of their squares is 97. Find the two numbers.

Answer

4 and 9, and -4 and -9

70. Two numbers differ by 6. The sum of their squares is 146. Find the two numbers.

Area of Rectangles

71. A rectangular canvas picture measures 14 inches by 36 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 720 square inches. Find the uniform width of the frame.

Answer

2 inches

72. A rectangular canvas picture measures 10 inches by 32 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 504 square inches. Find the uniform width of the frame.

73. A rectangle has perimeter 42 feet and area 104 square feet. Find the dimensions of the rectangle.

Answer

8 feet by 13 feet

74. A rectangle has perimeter 32 feet and area 55 square feet. Find the dimensions of the rectangle.

75. The length of a rectangle is three feet longer than six times its width. If the area of the rectangle is 165 square feet, what is the width of the rectangle?

Answer

5 feet

76. The length of a rectangle is three feet longer than nine times its width. If the area of the rectangle is 90 square feet, what is the width of the rectangle?

77. The ratio of the width to the length of a given rectangle is 2 to 3, or $\frac{2}{3}$. If the width and length are both increased by 4 inches, the area of the resulting rectangle is 80 square inches. Find the width and length of the original rectangle.

Answer

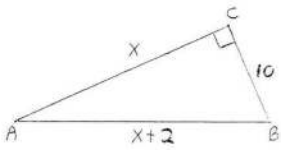
4 inches by 6 inches

78. The ratio of the width to the length of a given rectangle is 3 to 4, or $\frac{3}{4}$. If the width is increased by 3 inches and the length is increased by 6 inches, the area of the resulting rectangle is 126 square inches. Find the width and length of the original

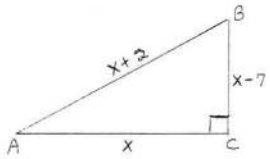
rectangle.

Area of Triangles

79. Find the area of $\triangle ABC$



80. Find the area of $\triangle ABC$

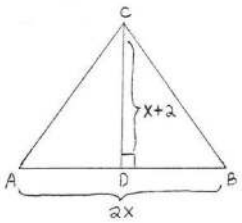


x

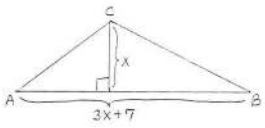
Answer

79. $x = 24$, area is 240 square units

81. Find the value of x given the area of $\triangle ABC$ is 35:



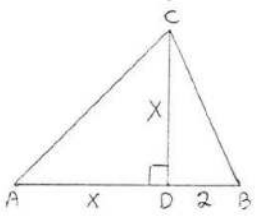
82. Find the value of x given the area of $\triangle ABC$ is 24.



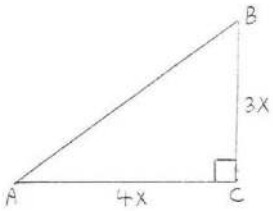
Answer

81. $x = 5$

83. Find the value of x given the area of $\triangle ABC$ is 12:



84. Find the value of x given the area of $\triangle ABC$ is 108:

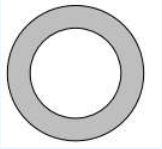


Answer

83. $x = 4$

Area of circles

85. The radius of the outer circle is one inch longer than twice the radius of the inner circle.

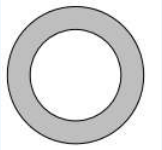


If the area of the shaded region is 40π square inches, what is the length of the inner radius?

Answer

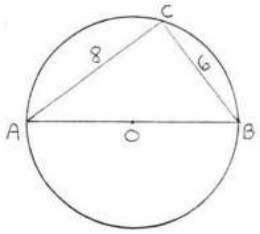
3 inches

86. The radius of the outer circle is two inches longer than three times the radius of the inner circle.

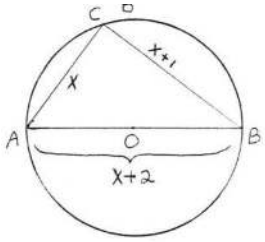


If the area of the shaded region is 180π square inches, what is the length of the inner radius?

87. Find the area of the circle to the nearest tenth



88. Find the area of the circle to the nearest tenth



Answer

87. 78.5 square units

Projectile Problems

89. A projectile is fired at an angle into the air from atop a cliff overlooking the ocean. The projectile's distance (in feet) from the base of the cliff is given by the equation $x = 180t$ and the projectile's height above sea level (in feet) is given by the equation $y = -16t^2 + 352t + 1664$ where t is the amount of time (in seconds) that has passed since the projectile's release. How much time passes before the projectile splashes into the ocean? At that time, how far is the projectile from the base of the cliff?

Answer

26 seconds, $4,680$ feet

90. A projectile is fired at an angle into the air from atop a cliff overlooking the ocean. The projectile's distance (in feet) from the base of the cliff is given by the equation $x = 140t$ and the projectile's height above sea level (in feet) is given by the equation $y = -16t^2 + 288t + 1408$ where t is the amount of time (in seconds) that has passed since the projectile's release. How much time passes before the projectile splashes into the ocean? At that time, how far is the projectile from the base of the cliff?

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Chapter 6 Review Exercises

Chapter Review Exercises

Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

$12a^2b^3,\ 15ab^2$

Answer

$3ab^2$

$12m^2n^3, 42m^5n^3$

$15y^3,\ 21y^2,\ 30y$

Answer

$3y$

$45x^3y^2,\ 15x^4y,\ 10x^5y^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

$35y+84$

Answer

$7(5y+12)$

$6y^2+12y-6$

$18x^3-15x$

Answer

$3x(6x^2-5)$

$15m^4+6m^2n$

$4x^3-12x^2+16x$

Answer

$4x(x^2-3x+4)$

$-3x+24$

$-3x^3+27x^2-12x$

Answer

$-3x(x^2-9x+4)$

$3x(x-1)+5(x-1)$

Factor by Grouping

In the following exercises, factor by grouping.

$ax-ay+bx-by$

Answer

$(a+b)(x-y)$

$x^2y-xy^2+2x-2y$

$x^2+7x-3x-21$

Answer

$(x-3)(x+7)$

$4x^2-16x+3x-12$

m^3+m^2+m+1

Answer

$(m^2+1)(m+1)$

$5x-5y-y+x$

Factor Trinomials

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form x^2+bx+c .

$a^2+14a+33$

Answer

$(a+3)(a+11)$

$k^2-16k+60$

$m^2+3m-54$

Answer

$(m+9)(m-6)$

$x^2-3x-10$

In the following examples, factor each trinomial of the form $x^2+bxy+cy^2$.

$x^2+12xy+35y^2$

Answer

$(x+5y)(x+7y)$

$r^2+3rs-28s^2$

$a^2+4ab-21b^2$

Answer

$(a+7b)(a-3b)$

$p^2-5pq-36q^2$

$m^2-5mn+30n^2$

Answer

Prime

Factor Trinomials of the Form ax2+bx+cax2+bx+c Using Trial and Error

In the following exercises, factor completely using trial and error.

x^3+5x^2-24x

$3y^3-21y^2+30y$

Answer

$3y(y-5)(y-2)$

$5x^4+10x^3-75x^2$

$5y^2+14y+9$

Answer

$(5y+9)(y+1)$

$8x^2+25x+3$

$10y^2-53y-11$

Answer

$(5y+1)(2y-11)$

$6p^2-19pq+10q^2$

$-81a^2+153a+18$

Answer

$-9(9a-1)(a+2)$

Factor Trinomials of the Form ax2+bx+cax2+bx+c using the 'ac' Method

In the following exercises, factor.

$2x^2+9x+4$

$18a^2-9a+1$

Answer

$(3a-1)(6a-1)$

$15p^2+2p-8$

$15x^2+6x-2$

Answer

$(3x-1)(5x+2)$

$8a^2+32a+24$

$3x^2+3x-36$

Answer

$3(x+4)(x-3)$

$48y^2+12y-36$

$18a^2-57a-21$

Answer

$3(2a-7)(3a+1)$

$3n^4-12n^3-96n^2$

Factor using substitution

In the following exercises, factor using substitution.

x^4-13x^2-30

Answer

$(x^2-15)(x^2+2)$

$(x-3)^2-5(x-3)-36$

Factor Special Products

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

$25x^2+30x+9$

Answer

$(5x+3)^2$

$36a^2 - 84ab + 49b^2$

$40x^2 + 360x + 810$

Answer

$10(2x + 9)^2$

$5k^3 - 70k^2 + 245k$

$75u^4 - 30u^3v + 3u^2v^2$

Answer

$3u^2(5u - v)^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

$81r^2 - 25$

$169m^2 - n^2$

Answer

$(13m + n)(13m - n)$

$25p^2 - 1$

$9 - 121y^2$

Answer

$(3 + 11y)(3 - 11y)$

$20x^2 - 125$

$169n^3 - n$

Answer

$n(13n + 1)(13n - 1)$

$6p^2q^2 - 54p^2$

$24p^2 + 54$

Answer

$6(4p^2 + 9)$

$49x^2 - 81y^2$

$16z^4 - 1$

Answer

$(2z-1)(2z+1)(4z^2+1)$

$48m^4n^2-243n^2$

$a^2+6a+9-9b^2$

Answer

$(a+3-3b)(a+3+3b)$

$x^2-16x+64-y^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

a^3-125

Answer

$(a-5)(a^2+5a+25)$

b^3-216

$2m^3+54$

Answer

$2(m+3)(m^2-3m+9)$

$81m^3+3$

General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

$24x^3+44x^2$

Answer

$4x^2(6x+11)$

$24a^4-9a^3$

$16n^2-56mn+49m^2$

Answer

$(4n-7m)^2$

$6a^2-25a-9$

$5u^4-45u^2$

Answer

$5u^2(u+3)(u-3)$

n^4-81

$64j^2+225$

Answer

prime

$5x^2+5x-60$

b^3-64

Answer

$(b-4)(b^2+4b+16)$

m^3+125

$2b^2-2bc+5cb-5c^2$

Answer

$(2b+5c)(b-c)$

$48x^5y^2-243xy^2$

$5q^2-15q-90$

Answer

$5(q+3)(q-6)$

$4u^5v+4u^2v^3$

$10m^4-6250$

Answer

$10(m-5)(m+5)(m^2+25)$

$60x^2y-75xy+30y$

$16x^2-24xy+9y^2-64$

Answer

$(4x-3y+8)(4x-3y-8)$

Polynomial Equations

Use the Zero Product Property

In the following exercises, solve.

$(a-3)(a+7)=0$

$(5b+1)(6b+1)=0$

Answer

$b=-\frac{1}{5},\ b=-\frac{1}{6}$

$6m(12m-5)=0$

$(2x-1)^2=0$

Answer

$x=\frac{1}{2}$

$3m(2m-5)(m+6)=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

$x^2+9x+20=0$

Answer

$x=-4,\ x=-5$

$y^2-y-72=0$

$2p^2-11p=40$

Answer

$p=-\frac{5}{2}, p=8$

$q^3+3q^2+2q=0$

$144m^2-25=0$

Answer

$m=\frac{5}{12},\ m=-\frac{5}{12}$

$4n^2=36$

$(x+6)(x-3)=-8$

Answer

$x=2,\ x=-5$

$(3x-2)(x+4)=12$

$16p^3=24p^2+9p$

Answer

$p = 0,\ p = \frac{3}{4}$

$2y^3 + 2y^2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

For the function, $f(x) = x^2 + 11x + 20$, ⓐ find when $f(x) = -8$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = -7$ or $\backslash x = -4$
ⓑ $(-7, -8)$ $(-4, -8)$

For the function, $f(x) = 9x^2 - 18x + 5$, ⓐ find when $f(x) = -3$ ⓑ Use this information to find two points that lie on the graph of the function.

In each function, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

$f(x) = 64x^2 - 49$

Answer

ⓐ $x = \frac{7}{8}$ or $x = -\frac{7}{8}$
ⓑ $(\frac{7}{8}, 0),\ (-\frac{7}{8}, 0)$ ⓒ $(0, -49)$

$f(x) = 6x^2 - 13x - 5$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

The product of two consecutive numbers is 399. Find the numbers.

Answer

The numbers are -21 and -19 or 19 and 21.

The area of a rectangular shaped patio 432 square feet. The length of the patio is 6 feet more than its width. Find the length and width.

A ladder leans against the wall of a building. The length of the ladder is 9 feet longer than the distance of the bottom of the ladder from the building. The distance of the top of the ladder reaches up the side of the building is 7 feet longer than the distance of the bottom of the ladder from the building. Find the lengths of all three sides of the triangle formed by the ladder leaning against the building.

Answer

The lengths are 8, 15, and 17 ft.

Shruti is going to throw a ball from the top of a cliff. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐthe zeros of this

function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 80 feet above the ground. ⓒ the height the ball will be at $t = 2$ seconds which is when the ball will be at its highest point.

Chapter Practice Test

In the following exercises, factor completely.

$80a^2 + 120a^3$

Answer

$40a^2(2 + 3a)$

$5m(m - 1) + 3(m - 1)$

$x^2 + 13x + 36$

Answer

$(x + 7)(x + 6)$

$p^2 + pq - 12q^2$

$xy - 8y + 7x - 56$

Answer

$(x - 8)(y + 7)$

$40r^2 + 810$

$9s^2 - 12s + 4$

Answer

$(3s - 2)^2$

$6x^2 - 11x - 10$

$3x^2 - 75y^2$

Answer

$3(x + 5y)(x - 5y)$

$6u^2 + 3u - 18$

$x^3 + 125$

Answer

$(x + 5)(x^2 - 5x + 25)$

$32x^5y^2 - 162xy^2$

$6x^4 - 19x^2 + 15$

Answer

$(3x^2-5)(2x^2-3)$

$3x^3-36x^2+108x$

In the following exercises, solve

$5a^2+26a=24$

Answer

$a=\frac{4}{5},\ a=-6$

The product of two consecutive integers is 156. Find the integers.

The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

Answer

The width is 12 inches and the length is 14 inches.

Jing is going to throw a ball from the balcony of her condo. When she throws the ball from 80 feet above the ground, the function $h(t)=-16t^2+64t+80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 128 feet above the ground. ⓒ the height the ball will be at $t=4$ seconds.

For the function, $f(x)=x^2-7x+5$, ⓐ find when $f(x)=-7$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x=3$ or $x=4$ ⓑ $(3,-7)$ $(4,-7)$

For the function $f(x)=25x^2-81$, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒthe *y*-intercept of the graph of the function.

Glossary

degree of the polynomial equation

The degree of the polynomial equation is the degree of the polynomial.

polynomial equation

A polynomial equation is an equation that contains a polynomial expression.

quadratic equation

Polynomial equations of degree two are called quadratic equations.

zero of the function

A value of xx where the function is 0, is called a zero of the function.

Zero Product Property

The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

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CHAPTER OVERVIEW

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7.0: Rational Expressions and Functions

Twelve goals last season. Fifteen home runs. Nine touchdowns. Whatever the statistics, sports analysts know it. Their jobs depend on it. Compiling and analyzing sports data not only help fans appreciate their teams but also help owners and coaches decide which players to recruit, how to best use them in games, how much they should be paid, and which players to trade. Understanding this kind of data requires a knowledge of specific types of expressions and functions. In this chapter, you will work with rational expressions and perform operations on them. And you will use rational expressions and inequalities to solve real-world problems.



Figure 7.0.1: *American football is the most watched spectator sport in the United States. People around the country are constantly tracking statistics for football and other sports. (credit: "keijj44" / Pixabay)*

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7.1: Simplify Rational Expressions

Summary

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Evaluate rational expressions
- Simplify rational expressions
- Simplify rational expressions with opposite factors

Are you Ready?

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{90y}{15y^2}$.
   If you missed this problem, review [link].
2. Factor: $6x^2-7x+2$.
   If you missed this problem, review [link].
3. Factor: n^3+8.
   If you missed this problem, review [link].

In Chapter 1, we reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers, and the denominator is not zero.

In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call these rational expressions.

Definition: RATIONAL EXPRESSION

A **rational expression** is an expression of the form $\frac{p(x)}{q(x)}$, where p and q are polynomials and $q \neq 0$.

Remember, division by 0 is undefined.

Here are some examples of rational expressions:

$$-\frac{13}{42} \quad \frac{7y}{8z} \quad \frac{5x+2}{x^2-7} \quad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{13}{42}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will perform same operations with rational expressions that we do with fractions. We will simplify, add, subtract, multiply, divide, and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

When we work with a numerical fraction, it is easy to avoid dividing by zero, because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

Definition: DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

1. Set the denominator equal to zero.
2. Solve the equation in the set of reals, if possible.

Example 7.1.1

Determine the values for which the rational expression is undefined:

1. $\frac{9y}{x}$
2. $\frac{4b-3}{2b+5}$
3. $\frac{x+4}{x^2+5x+6x}$

Solution

The expression will be undefined when the denominator is zero.

1.	$\frac{9y}{x}$
Set the denominator equal to zero. Solve for the variable.	x=0
	$\frac{9y}{x}$ is undefined for x=0.
2.	$\frac{4b-3}{2b+5}$
Set the denominator equal to zero. Solve for the variable.	2b+5=0
	2b=−5
	$b=-\frac{5}{2}$
	$\frac{4b-3}{2b+5}$ is undefined for $b=-\frac{5}{2}$.
3.	$\frac{x+4}{x^2+5x+6x}$
Set the denominator equal to zero. Solve for the variable.	$x^2+5x+6x=0$
	$(x+2)(x+3)=0$
	x+2=0 or x+3=0
	x=−2 or x=−3
	$\frac{x+4}{x^2+5x+6x}$ is undefined for x=−2 or x=−3.

Saying that the rational expression $\frac{x+4}{x^2+5x+6x}$ is undefined for x=−2orx=−3 is similar to writing the phrase “void where prohibited” in contest rules.

Try It 7.1.2

Determine the values for which the rational expression is undefined:

1. $\frac{3y}{x}$
2. $\frac{8n-5}{3n+1}$
3. $\frac{a+10}{a^2+4a+3a}$

Answer

1. x=0
2. $n=-\frac{1}{3}$
3. a=−1, a=−3

Try It 7.1.3

Determine the values for which the rational expression is undefined:

1. $\frac{4p}{5q}$
2. $\frac{y-1}{3y+2}$
3. $\frac{m-5}{m^2+m-6}$

Answer

1. q=0
2. $y=-\frac{2}{3}$
3. m=2, m=−3

Evaluate Rational Expressions

To evaluate a rational expression, we substitute values of the variables into the expression and simplify, just as we have for many other expressions in this book.

Example 7.1.4

Evaluate $\frac{2x+3}{3x-5}$ for each value:

1. x=0
2. x=2
3. x=−3

Solution

1.	$\frac{2x+3}{3x-5}$
Substitute 0 for x.	$\frac{2(0)+3}{3(0)-5}$
Simplify.	$-\frac{3}{5}$
2.	$\frac{2x+3}{3x-5}$
Substitute 2 for x.	$\frac{2(2)+3}{3(2)-5}$
Simplify.	$\frac{4+3}{6-5}$
	$\frac{7}{1}$
	7
3.	$\frac{2x+3}{3x-5}$
Substitute –3 for x.	$\frac{2(-3)+3}{3(-3)-5}$
Simplify.	$\frac{-6+3}{-9-5}$
	$\frac{-3}{-14}$
	$\frac{3}{14}$

Try It 7.1.5

Evaluate $\frac{y+1}{2y-3}$ for each value:

1. y=1
2. y=−3
3. y=0

Answer

1. −2
2. $\frac{2}{9}$
3. $-\frac{1}{3}$

Simplify Rational Expressions

Just like a fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator, a rational expression is *simplified* if it has no common factors, other than 1, in its numerator and denominator.

Definition: SIMPLIFIED RATIONAL EXPRESSION

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example:

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify **rational expression**s.

Definition: EQUIVALENT FRACTIONS PROPERTY

If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0$, $c \neq 0$ clearly stated. Every time we write a rational expression, we should make a similar statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Let's start by reviewing how we simplify numerical fractions.

Example 7.1.13

Simplify: $-\frac{36}{63}$.

Solution

	$-\frac{36}{63}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4 \cdot 9}{7 \cdot 9}$
Simplify using the Equivalent Fractions Property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

Try It 7.1.14

Simplify: $-\frac{45}{81}$.

Answer

$-\frac{5}{9}$

Try It 7.1.15

Simplify: $-\frac{42}{54}$.

Answer

$-\frac{7}{9}$

Throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

Example 7.1.16

Simplify: $\frac{3xy}{18x^2y^2}$.

Solution

	$\frac{3xy}{18x^2y^2}$
Rewrite the numerator and denominator showing the common factors.	$\frac{1 \cdot 3xy}{6xy \cdot 3xy}$
Simplify using the Equivalent Fractions Property.	$\frac{1}{6xy}$

Did you notice that these are the same steps we took when we divided monomials in Polynomials?

Try It 7.1.17

Simplify: $\frac{4x^2y}{12xy^2}$.

Answer

$\frac{x}{3y}$

Try It $PageIndex18$

Simplify: $\frac{16x^2y}{2xy^2}$.

Answer

$\frac{8x}{y}$

To simplify rational expressions we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$	$\frac{3x(x-9)}{5(x-9)}$	$\frac{x+5}{x}$
$\frac{2}{5}$	$\frac{3x}{5}$	NO COMMON FACTORS
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Note that removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

How to Simplify Rational Binomials

Example 7.1.19

Simplify: $\frac{2x+8}{5x+20}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor $2x+8$ and $5x-20$.	$\frac{2x+8}{5x+20}$ $\frac{2(x+4)}{5(x+4)}$
Step 2. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{2\cancel{(x+4)}}{5\cancel{(x+4)}}$ $\frac{2}{5}$

Try It 7.1.20

Simplify: $\frac{3x-6}{2x-4}$.

Answer

$\frac{3}{2}$

Try It 7.1.21

Simplify: $\frac{7y+35}{5y+25}$.

Answer

$\frac{7}{5}$

We now summarize the steps you should follow to simplify rational expressions.

Definition: SIMPLIFY A RATIONAL EXPRESSION.

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way it is easy to check that we have removed all the common factors!

We'll use the methods we covered in Factoring to factor the polynomials in the numerators and denominators in the following examples.

Example 7.1.22

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution

	$\frac{x^2+5x+6}{x^2+8x+12}$
Factor the numerator and denominator.	$\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Remove the common factor x+2 from the numerator and the denominator.	$\frac{x+3}{x+6}$

Can you tell which values of *x* must be excluded in this example?

Try It 7.1.23

Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}$

Try It 7.1.24

Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Answer

$\frac{x-5}{x-1}$

Example 7.1.25

Simplify: $\frac{y^2+y-42}{y^2-36}$.

Solution

	$\frac{y^2+y-42}{y^2-36}$.
Factor the numerator and denominator.	$\frac{(y+7)(y-6)}{(y+6)(y-6)}$
Remove the common factor y−6 from the numerator and the denominator.	$\frac{y+7}{y+6}$

Try It 7.1.26

Simplify: $\frac{x^2+x-6}{x^2-4}$.

Answer

$\frac{x+3}{x+2}$

Try It 7.1.27

Simplify: $\frac{x^2+8x+7}{x^2-49}$.

Answer

$\frac{x+1}{x-7}$

Example 7.1.28

Simplify: $\frac{p^3-2p^2+2p-4}{p^2-7p+10}$.

Solution

	$\frac{p^3-2p^2+2p-4}{p^2-7p+10}$
Factor the numerator and denominator, using grouping to factor the numerator.	$\frac{p^2(p-2)+2(p-2)}{(p-5)(p-2)}$
	$\frac{(p^2+2)(p-2)}{(p-5)(p-2)}$
Remove the common factor p−2 from the numerator and the denominator.	$\frac{p^2+2}{p-5}$

Try It 7.1.29

Simplify: $\frac{y^3-3y^2+y-3}{y^2-y-6}$.

Answer

$\frac{y^2+1}{y+2}$

Try It 7.1.30

Simplify: $\frac{p^3-p^2+2p-2}{p^2+4p-5}$.

Answer

$\frac{p^2+2}{p+5}$

Example 7.1.31

Simplify: $\frac{2n^2-14n}{4n^2-16n-48}$.

Solution

	$\frac{2n^2-14n}{4n^2-16n-48}$

Factor the numerator and denominator, first factoring out the GCF.	$\frac{2n(n-7)}{4(n^2-4n-12)}$
	$\frac{2n(n-7)}{4(n-6)(n+2)}$
Remove the common factor, 2.	$\frac{n(n-7)}{2(n-6)(n+2)}$

Try It 7.1.32

Simplify: $\frac{2n^2-10n}{4n^2-16n-20}$.

Answer

$\frac{n}{2(n+1)}$

Try It 7.1.33

Simplify: $\frac{4x^2-16x}{8x^2-16x-64}$.

Answer

$\frac{x}{2(x+2)}$

Example 7.1.34

Simplify: $\frac{3b^2-12b+12}{6b^2-24}$.

Solution

	$\frac{3b^2-12b+12}{6b^2-24}$
Factor the numerator and denominator, first factoring out the GCF.	$\frac{3(b^2-4b+4)}{6(b^2-4)}$
	$\frac{3(b-2)(b-2)}{6(b-2)(b+2)}$
Remove the common factors of b−2 and 3.	$\frac{3(b-2)}{2(b+2)}$

Try It 7.1.35

Simplify: $\frac{2x^2-12x+18}{3x^2-27}$.

Answer

$\frac{2(x-3)}{3(x+3)}$

Try It $PageIndex36$

Simplify: $\frac{5y^2-30y+25}{2y^2-50}$.

Answer

$\frac{5(x-1)}{2(x+5)}$

Example 7.1.37

Simplify: $\frac{m^3+8}{m^2-4}$.

Solution

	$\frac{m^3+8}{m^2-4}$
Factor the numerator and denominator, using the formulas for sum of cubes and difference of squares.	$\frac{(m+2)(m^2-2m+4)}{(m+2)(m-2)}$
Remove the common factors of m+2.	$\frac{m^2-2m+4}{m-2}$

Try It 7.1.38

Simplify: $\frac{p^3-64}{p^2-16}$.

Answer

$\frac{p^2+4p+16}{p+4}$

Try It 7.1.39

Simplify: $\frac{x^3+8}{x^2-4}$.

Answer

$\frac{x^2-2x+4}{x-2}$

How to Simplify Rational Binomials

Example 7.1.19

Simplify: $\frac{2x+8}{5x+20}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor $2x + 8$ and $5x - 20$.	$\frac{2x+8}{5x+20}$ $\frac{2(x+4)}{5(x+4)}$
Step 2. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{2\cancel{(x+4)}}{5\cancel{(x+4)}}$ $\frac{2}{5}$

Simplify Rational Expressions with Opposite Factors

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. Let's start with a numerical fraction, say $\frac{7}{-7}$.

We know this fraction simplifies to −1. We also recognize that the numerator and denominator are opposites.

In Foundations, we introduced opposite notation: the opposite of a is −a. We remember, too, that −a=−1·a

We simplify the fraction $\frac{a}{-a}$

	$\frac{a}{-a}$
We could rewrite this.	$\frac{1 \cdot a}{-1 \cdot a}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

So, in the same way, we can simplify the fraction $\frac{x-3}{-(x-3)}$

	$\frac{x-3}{-(x-3)}$
We could rewrite this.	$\frac{1 \cdot (x-3)}{-1 \cdot (x-3)}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

But the opposite of x−3 could be written differently:

	$-(x-3)$
Distribute.	$-x+3$
Rewrite.	$3-x$

This means the fraction $\frac{x-3}{3-x}$ simplifies to −1.

In general, we could write the opposite of a−b as b−a. So the rational expression $\frac{a-b}{b-a}$ simplifies to −1.

Definition: OPPOSITES IN A RATIONAL EXPRESSION

The opposite of a−b is b−a

$\frac{a-b}{b-a} = -1, a \neq b$

An expression and its opposite divide to −1

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators.

Example 7.1.40

Simplify: $\frac{x-8}{8-x}$.

Solution

	$\frac{x-8}{8-x}$.
Recognize that x−8 and 8−x are opposites	−1

Try It 7.1.41

Simplify: $\frac{y-2}{2-y}$.

Answer

−1

Try It 7.1.42

Simplify: $\frac{n-9}{9-n}$.

Answer

-1

Remember, the first step in simplifying a rational expression is to factor the numerator and denominator completely.

Example 7.1.43

Simplify: $\frac{14-2x}{x^2-49}$.

Solution

	$\frac{14-2x}{x^2-49}$
Factor the numerator and denominator.	$\frac{2(7-x)}{(x+7)(x-7)}$
Recognize $7-x$ and $x-7$ are opposites.	$\frac{2(7-x)}{(x+7)(x-7)}(-1)$
Simplify.	$-\frac{2}{x+7}$

Try It 7.1.44

Simplify: $\frac{10-2y}{y^2-25}$.

Answer

$-\frac{2}{y+5}$

Try It 7.1.45

Simplify: $\frac{3y-27}{81-y^2}$.

Answer

$-\frac{3}{9+y}$

Example 7.1.46

Simplify: $\frac{x^2-4x-32}{64-x^2}$.

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Try It 7.1.47

Simplify: $\frac{x^2-4x-5}{25-x^2}$.

Answer

$-\frac{x+1}{x+5}$

Try It 7.1.48

Simplify: $\frac{x^2+x-2}{1-x^2}$.

Answer

$-\frac{x+2}{x+1}$

Key Concepts

- **Determine the Values for Which a Rational Expression is Undefined**
  1. Set the denominator equal to zero.
  2. Solve the equation, if possible.
- **Simplified Rational Expression**
  - A rational expression is considered simplified if there are no common factors in its numerator and denominator.
- **Simplify a Rational Expression**
  1. Factor the numerator and denominator completely.
  2. Simplify by dividing out common factors.
- **Opposites in a Rational Expression**
  - The opposite of a−b is b−a
    $\frac{a-b}{b-a}=-1\ a\neq b, b\neq 0, a\neq b$

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q\neq 0$.

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7.1E: Exercises

Practice Makes Perfect

In the following exercises, determine the values for which the rational expression is undefined.

Example 7.1E.49

1. $\frac{2x}{z}$
2. $\frac{4p-1}{6p-5}$
3. $\frac{n-3}{n^2+2n-8}$

Answer

1. z=0
2. $p=\frac{5}{6}$
3. n=−4, n=2

Example 7.1E.50

1. $\frac{10m}{11n}$
2. $\frac{6y+13}{4y-9}$
3. $\frac{b-8}{b^2-36}$

Example 7.1E.51

1. $\frac{4x^2y}{3y}$
2. $\frac{3x-2}{2x+1}$
3. $\frac{u-1}{u^2-3u-28}$

Answer

1. y=0
2. $x=-\frac{1}{2}$
3. u=−4, u=7

Example 7.1E.52

1. $\frac{5pq^2}{9q}$
2. $\frac{7a-4}{3a+5}$
3. $\frac{1}{x^2-4}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expression for the given values.

Example 7.1E.53

$$\frac{2x}{x-1}$$

1. x=0
2. x=2
3. x=−1

Answer

1. 0
2. 4
3. 1

Example 7.1E.54

$$\frac{4y-1}{5y-3}$$

1. y=0
2. y=2
3. y=−1

Example 7.1E.55

$$\frac{2p+3}{p^2+1}$$

1. p=0
2. p=1
3. p=−2

Answer

1. 3
2. $\frac{5}{2}$
3. $-\frac{1}{5}$

Example 7.1E.56

$$\frac{x+3}{2-3x}$$

1. x=0
2. x=1
3. x=−2

Example 7.1E.57

$$\frac{y^2+5y+6}{y^2-1}$$

1. y=0
2. y=2
3. y=−2

Answer

1. -6
2. $\frac{20}{3}$
3. 0

Example 7.1E.58

$$\frac{z^2+3z-10}{z^2-1}$$

1. z=0
2. z=2
3. z=−2

Example 7.1E.59

$$\frac{a^2-4}{a^2+5a+4}$$

1. a=0
2. a=1
3. a=−2

Answer

1. -1
2. $-\frac{3}{10}$
3. 0

Example 7.1E.60

$$\frac{b^2+2}{b^2-3b-4}$$

1. b=0
2. b=2
3. b=−2

Example 7.1E.61

$$\frac{x^2+3xy+2y^2}{2x^3y}$$

1. x=1, y=−1
2. x=2, y=1
3. x=−1, y=−2

Answer

1. 0
2. $\frac{3}{4}$
3. $\frac{15}{4}$

Example 7.1E.62

$$\frac{c^2+cd-2d^2}{cd^3}$$

1. c=2, d=−1
2. c=1, d=−1
3. c=−1, d=2

Example 7.1E.63

$$\frac{m^2-4n^2}{5mn^3}$$

1. m=2, n=1
2. m=−1, n=−1
3. m=3, n=2

Answer

1. 0
2. $-\frac{3}{5}$
3. $-\frac{7}{20}$

Example 7.1E.64

$$\frac{2s^2t}{s^2-9t^2}$$

1. s=4, t=1
2. s=−1, t=−1
3. s=0, t=2

Simplify Rational Expressions

In the following exercises, simplify.

Example 7.1E.65

$$-\frac{4}{52}$$

Answer

$$-\frac{1}{13}$$

Example 7.1E.66

$$-\frac{44}{55}$$

Example 7.1E.67

$$\frac{56}{63}$$

Answer

$\frac{8}{9}$

Example 7.1E.68

$\frac{65}{104}$

Example 7.1E.69

$\frac{6ab^2}{12a^2b}$

Answer

$\frac{b}{2a}$

Example 7.1E.70

$\frac{15xy^3}{x^3y^3}$

Example 7.1E.71

$\frac{8m^3n}{12mn^2}$

Answer

$\frac{2m^2}{3n}$

Example 7.1E.72

$\frac{36v^3w^2}{27vw^3}$

Example 7.1E.73

$\frac{3a+6}{4a+8}$

Answer

$\frac{3}{4}$

Example 7.1E.74

$\frac{5b+5}{6b+6}$

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Example 7.1E.75

$$\frac{3c-9}{5c-15}$$

Answer

$$\frac{3}{5}$$

Example 7.1E.76

$$\frac{4d+8}{9d+18}$$

Example 7.1E.77

$$\frac{7m+63}{5m+45}$$

Answer

$$\frac{7}{5}$$

Example 7.1E.78

$$\frac{8n-96}{3n-36}$$

Exercise 7.1E.79

$$\frac{12p-240}{5p-100}$$

Answer

$$\frac{12}{5}$$

Example 7.1E.80

$$\frac{6q+210}{5q+175}$$

Example 7.1E.81

$$\frac{a^2-a-12}{a^2-8a+16}$$

Answer

$$\frac{a+3}{a-4}$$

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Example 7.1E.82

$$\frac{x^2+4x-5}{x^2-2x+1}$$

Example 7.1E.83

$$\frac{y^2+3y-4}{y^2-6y+5}$$

Answer

$$\frac{y+4}{y-5}$$

Example 7.1E.84

$$\frac{v^2+8v+15}{v^2-v-12}$$

Example 7.1E.85

$$\frac{x^2-25}{x^2+2x-15}$$

Answer

$$\frac{x-5}{x-3}$$

Example 7.1E.86

$$\frac{a^2-4}{a^2+6a-16}$$

Example 7.1E.87

$$\frac{y^2-2y-3}{y^2-9}$$

Answer

$$\frac{y+1}{y+3}$$

Example 7.1E.88

$$\frac{b^2+9b+18}{b^2-36}$$

Example 7.1E.89

$$\frac{y^3+y^2+y+1}{y^2+2y+1}$$

Answer

$$\frac{y^2+1}{y+1}$$

Example 7.1E.90

$$\frac{p^3+3p^2+4p+12}{p^2+p-6}$$

Example 7.1E.91

$$\frac{x^3-2x^2-25x+50}{x^2-25}$$

Answer

x−2

Example 7.1E.92

$$\frac{q^3+3q^2-4q-12}{q^2-4}$$

Example 7.1E.93

$$\frac{3a^2+15a}{6a^2+6a-36}$$

Answer

$$\frac{a(a+5)}{2(a+3)(a-2)}$$

Example 7.1E.94

$$\frac{8b^2-32b}{2b^2-6b-80}$$

Example 7.1E.95

$$\frac{-5c^2-10c}{-10c^2+30c+100}$$

Answer

$$\frac{c}{2(c-5)}$$

Example 7.1E.96

$$\frac{4d^2-24d}{2d^2-4d-48}$$

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Example 7.1E.97

$$\frac{3m^2+30m+75}{4m^2-100}$$

Answer

$$\frac{3(m+5)}{4(m-5)}$$

Example 7.1E.98

$$\frac{5n^2+30n+45}{2n^2-18}$$

Example 7.1E.99

$$\frac{5r^2+30r-35}{r^2-49}$$

Answer

$$\frac{5(r-1)}{r+7}$$

Example 7.1E.100

$$\frac{3s^2+30s+72}{3s^2-48}$$

Example 7.1E.101

$$\frac{t^3-27}{t^2-9}$$

Answer

$$\frac{t^2+3t+9}{t+3}$$

Example 7.1E.102

$$\frac{v^3-1}{v^2-1}$$

Example 7.1E.103

$$\frac{w^3+216}{w^2-36}$$

Answer

$$\frac{w^2-6w+36}{w-6}$$

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Example 7.1E.104

$$\frac{v^3+125}{v^2-25}$$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify each rational expression.

Example 7.1E.105

$$\frac{a-5}{5-a}$$

Answer

-1

Example 7.1E.106

$$\frac{b-12}{12-b}$$

Example 7.1E.107

$$\frac{11-c}{c-11}$$

Answer

-1

Example 7.1E.108

$$\frac{5-d}{d-5}$$

Example 7.1E.109

$$\frac{12-2x}{x^2-36}$$

Answer

$$-\frac{2}{x+6}$$

Example 7.1E.110

$$\frac{20-5y}{y^2-16}$$

Example 7.1E.111

$$\frac{4v-32}{64-v^2}$$

Answer

$$-\frac{4}{8+v}$$

Example 7.1E.112

$$\frac{7w-21}{9-w^2}$$

Example 7.1E.113

$$\frac{y^2-11y+24}{9-y^2}$$

Answer

$$-\frac{y-8}{3+y}$$

Example 7.1E.114

$$\frac{z^2-9z+20}{16-z^2}$$

Example 7.1E.115

$$\frac{a^2-5a-36}{81-a^2}$$

Answer

$$-\frac{a+4}{9+a}$$

Example 7.1E.116

$$\frac{b^2+b-42}{36-b^2}$$

Everyday Math

Example 7.1E.117

Tax Rates For the tax year 2015, the amount of tax owed by a single person earning between $37,450 and $90,750, can be found by evaluating the formula 0.25x−4206.25, where x is income. The average tax rate for this income can be found by evaluating the formula $\frac{0.25x-4206.25}{x}$. What would be the average tax rate for a single person earning $50,000?

Answer

16.5%

Example 7.1E.118

Work The length of time it takes for two people for perform the same task if they work together can be found by evaluating the formula $\frac{xy}{x+y}$. If Tom can paint the den in x=45 minutes and his brother Bobby can paint it in y=60 minutes, how many

minutes will it take them if they work together?

Writing Exercises

Example 7.1E.119

Explain how you find the values of x for which the rational expression $\frac{x^2-x-20}{x^2-4}$ is undefined.

Answer

Answers will vary, but all should reference setting the denominator function to zero.

Example 7.1E.120

Explain all the steps you take to simplify the rational expression $\frac{p^2+4p-21}{9-p^2}$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
evaluate rational expressions.			
simplify rational expressions.			
simplify rational expressions with opposite factors.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

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7.2: Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Before you get started, take this readiness quiz.

1. Simplify: $\frac{90y}{15y^2}$.
   If you missed this problem, review [link].
2. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review [link].
3. Divide: $\frac{12}{10} \div \frac{8}{25}$.
   If you missed this problem, review [link].

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

RATIONAL EXPRESSION

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

1. Set the denominator equal to zero.
2. Solve the equation.

EXAMPLE 7.2.1

Determine the value for which each rational expression is undefined:

a. $\frac{8a^2b}{3c}$ b. $\frac{4b-3}{2b+5}$ c. $\frac{x+4}{x^2+5x+6}$.

Solution

The expression will be undefined when the denominator is zero.

a.

$$\frac{8a^2b}{3c}$$

Set the denominator equal to zero and solve for the variable.

$$3c=0$$

$$c=0$$

$\frac{8a^2b}{3c}$ is undefined for $c=0$

b.

$$\frac{4b-3}{2b+5}$$

Set the denominator equal to zero and solve for the variable.

$$2b+5=0$$

$$2b=-5$$

$$b=-\frac{5}{2}$$

$\frac{4b-3}{2b+5}$ is undefined for $b=-\frac{5}{2}$

c.

$$\frac{x+4}{x^2+5x+6}$$

Set the denominator equal to zero and solve for the variable.

$$x^2+5x+6=0$$

$$(x+2)(x+3)=0$$

$$x+2=0 \text{ or } x+3=0$$

$$x=-2 \text{ or } x=-3$$

$\frac{x+4}{x^2+5x+6}$ is undefined for $x=-2$ or $x=-3$

Try It 7.2.2

Determine the value for which each rational expression is undefined.

a. $\frac{3y^2}{8x}$ b. $\frac{8n-5}{3n+1}$ c. $\frac{a+10}{a^2+4a+3}$

Answer

a. $x = 0$

b. $n = -\frac{1}{3}$

c. $a = -1, a = -3$

Try It 7.2.3

Determine the value for which each rational expression is undefined.

a. $\frac{4p}{5q}$ b. $\frac{y-1}{3y+2}$ c. $\frac{m-5}{m^2+m-6}$

Answer

a. $q = 0$

b. $y = -\frac{2}{3}$

c. $m = 2, m = -3$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

SIMPLIFIED RATIONAL EXPRESSION

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$\frac{x+2}{x+3}$ is simplified because there are no common factors of $x+2$ and $x+3$.

$\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

EQUIVALENT FRACTIONS PROPERTY

If a, b, and c are numbers where $b \neq 0, c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$	$\frac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$	where $x \neq 9$	$\frac{x+5}{x}$
$\frac{2}{5}$	$\frac{3x}{5}$		NO COMMON FACTORS
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.		While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

How to Simplify a Rational Expression

EXAMPLE 7.2.4

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$

Solution

Step 1. Factor the numerator and denominator completely.	Factor x^2+5x+6 and $x^2+8x+12$	$\frac{x^2+5x+6}{x^2+8x+12}$ $\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Step 2. Simplify by dividing out common factors.	Remove the common factor $x+2$ from the numerator and the denominator.	$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $\frac{(x+3)}{(x+6)}$ $x \neq -2 \quad x \neq -6$

Try It 7.2.5

Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}, x \neq 2, x \neq 1$

Try It 7.2.6

Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Answer

$\frac{x-5}{x-1}, x \neq -2, x \neq 1$

We now summarize the steps you should follow to simplify rational expressions.

SIMPLIFY A RATIONAL EXPRESSION.

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

EXAMPLE 7.2.7

Simplify: $\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$.

Solution

$$\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$$

Factor the numerator and denominator, first factoring out the GCF.

$$\frac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$$

$$\frac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$$

Remove the common factors of $a-2b$ and 3.

$$\frac{\cancel{3}(a-2b)\ \cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\ \cancel{(a-2b)}}$$

$$\frac{a-2b}{2(a+2b)}$$

Try It 7.2.8

Simplify: $\frac{2x^2-12xy+18y^2}{3x^2-27y^2}$.

Answer

$$\frac{2(x-3y)}{3(x+3y)}$$

Try It 7.2.9

Simplify: $\frac{5x^2-30xy+25y^2}{2x^2-50y^2}$.

Answer

$$\frac{5(x-y)}{2(x+5y)}$$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a = -1 \cdot a$.

The numerical fraction, say $\frac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\frac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.

Let's look at the expression $b-a$.	$b-a$
Rewrite.	$-a+b$
Factor out -1.	$-1(a-b)$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

OPPOSITES IN A RATIONAL EXPRESSION

The opposite of $a-b$ is $b-a$.

$$\frac{a-b}{b-a} = -1 \quad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a \neq -b$, then $\frac{a+b}{b+a} = 1$.

EXAMPLE 7.2.10

Simplify: $\frac{x^2-4x-32}{64-x^2}$

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Try It 7.2.11

Simplify: $\frac{x^2-4x-5}{25-x^2}$

Answer

$-\frac{x+1}{x+5}$

Try It 7.2.12

Simplify: $\dfrac{x^2+x-2}{1-x^2}$.

Answer

$-\dfrac{x+2}{x+1}$

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

MULTIPLICATION OF RATIONAL EXPRESSIONS

If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$, $x \neq 3$, and $x \neq 4$.

EXAMPLE 7.2.13: How to Multiply Rational Expressions

Simplify: $\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$.

Solution

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$ $\dfrac{2x}{(x-3)(x-4)} \cdot \dfrac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\dfrac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\dfrac{2 \cdot x \cdot (x-3)(x+3)}{2 \cdot 3 \cdot x \cdot x \cdot (x-3)(x-4)}$ $\dfrac{(x+3)}{3x(x-4)}$

Try It 7.2.14

Simplify: $\dfrac{5x}{x^2+5x+6} \cdot \dfrac{x^2-4}{10x}$.

Answer

$\dfrac{x-2}{2(x+3)}$

Try It 7.2.15

Simplify: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

Answer

$\frac{3(x-6)}{x+5}$

MULTIPLY RATIONAL EXPRESSIONS.

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out common factors.

EXAMPLE 7.2.16

Multiply: $\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$.

Solution

$$\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$$

Factor the numerators and denominators and then multiply.

$$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Simplify by dividing out common factors.

$$\frac{\cancel{(3a+1)}(a-3)\ \cancel{(a+5)}(a+5)}{(a-5)\ \cancel{(a+5)}\ \cancel{(3a+1)}(a-5)}$$

Simplify.

$$\frac{(a-3)(a+5)}{(a-5)(a-5)}$$

Rewrite $(a-5)(a-5)$ using an exponent.

$$\frac{(a-3)(a+5)}{(a-5)^2}$$

Try It 7.2.17

Simplify: $\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$.

Answer

$\frac{x-4}{x-5}$

Try It 7.2.18

Simplify: $\frac{4b^2+7b-2}{1-b^2} \cdot \frac{b^2-2b+1}{4b^2+15b-4}$.

Answer

$$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$$

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

DIVISION OF RATIONAL EXPRESSIONS

If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

EXAMPLE 7.2.19: How to Divide Rational Expressions

Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$ $\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\frac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Try It 7.2.20

Simplify: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Answer

$$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$$

Try It 7.2.21

Simplify: $\dfrac{2z^2}{z^2-1} \div \dfrac{z^3-z^2+z}{z^3+1}$.

Answer

$\dfrac{2z}{z-1}$

DIVISION OF RATIONAL EXPRESSIONS

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

Recall from Use the Language of Algebra that a complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

EXAMPLE 7.2.22

Divide: $\dfrac{\dfrac{6x^2-7x+2}{4x-8}}{\dfrac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\frac{\dfrac{6x^2-7x+2}{4x-8}}{\dfrac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign. $\dfrac{6x^2-7x+2}{4x-8} \div \dfrac{2x^2-7x+3}{x^2-5x+6}$

Rewrite as product of first times reciprocal of second. $\dfrac{6x^2-7x+2}{4x-8} \cdot \dfrac{x^2-5x+6}{2x^2-7x+3}$

Factor the numerators and the denominators, and then multiply. $\dfrac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$

Simplify by dividing out common factors. $\dfrac{\cancel{(2x-1)}(3x-2)\ \cancel{(x-2)}\ \cancel{(x-3)}}{4\ \cancel{(x-2)}\ \cancel{(2x-1)}\ \cancel{(x-3)}}$

Simplify. $\dfrac{3x-2}{4}$

Try It 7.2.23

Simplify: $\dfrac{\dfrac{3x^2+7x+2}{4x+24}}{\dfrac{3x^2-14x-5}{x^2+x-30}}$.

Answer

$\dfrac{x+2}{4}$

Try It 7.2.24

Simplify: $\dfrac{\dfrac{y^2-36}{2y^2+11y-6}}{\dfrac{2y^2-2y-60}{8y-4}}$.

Answer

$\dfrac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

EXAMPLE 7.2.25

Perform the indicated operations: $\dfrac{3x-6}{4x-4}\cdot\dfrac{x^2+2x-3}{x^2-3x-10}\div\dfrac{2x+12}{8x+16}$.

Solution

	$\frac{3x-6}{4x-4}\cdot\frac{x^2+2x-3}{x^2-3x-10}\div\frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4}\cdot\frac{x^2+2x-3}{x^2-3x-10}\cdot\frac{8x+16}{2x+12}$
Factor the numerators and the denominators.	$\frac{3(x-2)}{4(x-1)}\cdot\frac{(x+3)(x-1)}{(x+2)(x-5)}\cdot\frac{8(x+2)}{2(x+6)}$
Multiply the fractions. Bringing the constants to the front will help when removing common factors.	
Simplify by dividing out common factors.	$\frac{3\cdot 8(x-2)(x+3)(x-1)(x+2)}{4\cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

Try It 7.2.26

Perform the indicated operations: $\dfrac{4m+4}{3m-15}\cdot\dfrac{m^2-3m-10}{m^2-4m-32}\div\dfrac{12m-36}{6m-48}$.

Answer

$\dfrac{2(m+1)(m+2)}{3(m+4)(m-3)}$

Try It 7.2.27

Perform the indicated operations: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Answer

$$\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$$

Multiply and Divide Rational Functions

We started this section stating that a **rational expression** is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$. Similarly, we define a **rational function** as a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

RATIONAL FUNCTION

A rational function is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

The domain of a rational function is all real numbers except for those values that would cause division by zero. We must eliminate any values that make $q(x) = 0$.

DETERMINE THE DOMAIN OF A RATIONAL FUNCTION.

1. Set the denominator equal to zero.
2. Solve the equation.
3. The domain is all real numbers excluding the values found in Step 2.

EXAMPLE 7.2.28

Find the domain of $R(x) = \frac{2x^2-14x}{4x^2-16x-48}$.

Solution

The domain will be all real numbers except those values that make the denominator zero. We will set the denominator equal to zero , solve that equation, and then exclude those values from the domain.

Set the denominator to zero.	$4x^2-16x-48=0$
Factor, first factor out the GCF.	$4(x^2-4x-12)=0$
	$4(x-6)(x+2)=0$
Use the Zero Product Property.	$4 \neq 0 \quad x-6=0 \quad x+2=0$
Solve.	$x=6 \quad x=-2$
	The domain of $R(x)$ is all real numbers where $x \neq 6$ and $x \neq -2$

Try It 7.2.29

Find the domain of $R(x) = \dfrac{2x^2 - 10x}{4x^2 - 16x - 20}$.

Answer

The domain of $R(x)$ is all real numbers where $x \neq 5$ and $x \neq -1$.

Try It 7.2.30

Find the domain of $R(x) = \dfrac{4x^2 - 16x}{8x^2 - 16x - 64}$.

Answer

The domain of $R(x)$ is all real numbers where $x \neq 4$ and $x \neq -2$.

To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.

EXAMPLE 7.2.31

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{2x-6}{x^2-8x+15}$ and $g(x) = \dfrac{x^2-25}{2x+10}$.

Solution

$$R(x) = f(x) \cdot g(x)$$

$$R(x) = \frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$$

Factor each numerator and denominator. $$R(x) = \frac{2(x-3)}{(x-3)(x-5)} \cdot \frac{(x-5)(x+5)}{2(x+5)}$$

Multiply the numerators and denominators. $$R(x) = \frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$$

Remove common factors. $$R(x) = \frac{\cancel{2}\,\cancel{(x-3)}\,\cancel{(x-5)}\,\cancel{(x+5)}}{\cancel{2}\,\cancel{(x-3)}\,\cancel{(x-5)}\,\cancel{(x+5)}}$$

Simplify. $$R(x) = 1$$

Try It 7.2.32

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{3x-21}{x^2-9x+14}$ and $g(x) = \dfrac{2x^2-8}{3x+6}$.

Answer

$R(x) = 2$

Try It 7.2.33

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \dfrac{x^2 - x}{3x^2 + 27x - 30}$ and $g(x) = \dfrac{x^2 - 100}{x^2 - 10x}$.

Answer

$$R(x) = \frac{1}{3}$$

To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

EXAMPLE 7.2.34

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{3x^2}{x^2 - 4x}$ and $g(x) = \dfrac{9x^2 - 45x}{x^2 - 7x + 10}$.

Solution

$$R(x) = \frac{f(x)}{g(x)}$$

Substitute in the functions $f(x),\ g(x)$.

$$R(x) = \frac{\dfrac{3x^2}{x^2 - 4x}}{\dfrac{9x^2 - 45x}{x^2 - 7x + 10}}$$

Rewrite the division as the product of $f(x)$ and the reciprocal of $g(x)$.

$$R(x) = \frac{3x^2}{x^2 - 4x} \cdot \frac{x^2 - 7x + 10}{9x^2 - 45x}$$

Factor the numerators and denominators and then multiply.

$$R(x) = \frac{3 \cdot x \cdot x \cdot (x-5)(x-2)}{x(x-4) \cdot 3 \cdot 3 \cdot x \cdot (x-5)}$$

Simplify by dividing out common factors.

$$R(x) = \frac{\cancel{3} \cdot \cancel{x} \cdot \cancel{x}\,\cancel{(x-5)}(x-2)}{\cancel{x}(x-4) \cdot \cancel{3} \cdot 3 \cdot \cancel{x}\,\cancel{(x-5)}}$$

$$R(x) = \frac{x-2}{3(x-4)}$$

Try It 7.2.35

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{2x^2}{x^2 - 8x}$ and $g(x) = \dfrac{8x^2 + 24x}{x^2 + x - 6}$.

Answer

$$R(x) = \frac{x-2}{4(x-8)}$$

Try It 7.2.36

Find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x) = \dfrac{15x^2}{3x^2 + 33x}$ and $g(x) = \dfrac{5x - 5}{x^2 + 9x - 22}$.

Answer

$$R(x) = \frac{x(x-2)}{x-1}$$

Key Concepts

- **Determine the values for which a rational expression is undefined.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then

  $$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$
- **How to simplify a rational expression.**
  1. Factor the numerator and denominator completely.
  2. Simplify by dividing out common factors.
- **Opposites in a Rational Expression**
  The opposite of $a - b$ is $b - a$.

  $$\frac{a-b}{b-a} = -1 \qquad a \neq b$$

  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

  $$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$
- **How to multiply rational expressions.**
  1. Factor each numerator and denominator completely.
  2. Multiply the numerators and denominators.
  3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

  $$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$
- **How to divide rational expressions.**
  1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  2. Factor the numerators and denominators completely.
  3. Multiply the numerators and denominators together.
  4. Simplify by dividing out common factors.
- **How to determine the domain of a rational function.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
  3. The domain is all real numbers excluding the values found in Step 2.

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

simplified rational expression

A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

rational function

A rational function is a function of the form $R(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

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7.2E: Exercises

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

1. a. $\frac{2x^2}{z}$ b. $\frac{4p-1}{6p-5}$ c. $\frac{n-3}{n^2+2n-8}$

Answer

a. $z=0$
b. $p=\frac{5}{6}$
c. $n=-4, n=2$

2. a. $\frac{10m}{11n}$ b. $\frac{6y+13}{4y-9}$ c. $\frac{b-8}{b^2-36}$

3. a. $\frac{4x^2y}{3y}$ b. $\frac{3x-2}{2x+1}$ c. $\frac{u-1}{u^2-3u-28}$

Answer

a. $y=0$
b. $x=-\frac{1}{2}$
c. $u=-4, u=7$

4. a. $\frac{5pq^2}{9q}$ b. $\frac{7a-4}{3a+5}$ c. $\frac{1}{x^2-4}$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

5. $-\frac{44}{55}$

Answer

$-\frac{4}{5}$

6. $\frac{56}{63}$

7. $\frac{8m^3n}{12mn^2}$

Answer

$\frac{2m^2}{3n}$

8. $\frac{36v^3w^2}{27vw^3}$

9. $\dfrac{8n-96}{3n-36}$

Answer

$\dfrac{8}{3}$

10. $\dfrac{12p-240}{5p-100}$

11. $\dfrac{x^2+4x-5}{x^2-2x+1}$

Answer

$\dfrac{x+5}{x-1}$

12. $\dfrac{y^2+3y-4}{y^2-6y+5}$

13. $\dfrac{a^2-4}{a^2+6a-16}$

Answer

$\dfrac{a+2}{a+8}$

14. $\dfrac{y^2-2y-3}{y^2-9}$

15. $\dfrac{p^3+3p^2+4p+12}{p^2+p-6}$

Answer

$\dfrac{p^2+4}{p-2}$

16. $\dfrac{x^3-2x^2-25x+50}{x^2-25}$

17. $\dfrac{8b^2-32b}{2b^2-6b-80}$

Answer

$\dfrac{4b(b-4)}{(b+5)(b-8)}$

18. $\dfrac{-5c^2-10c}{-10c^2+30c+100}$

19. $\frac{3m^2+30mn+75n^2}{4m^2-100n^2}$

Answer

$\frac{3(m+5n)}{4(m-5n)}$

20. $\frac{5r^2+30rs-35s^2}{r^2-49s^2}$

21. $\frac{a-5}{5-a}$

Answer

-1

22. $\frac{5-d}{d-5}$

23. $\frac{20-5y}{y^2-16}$

Answer

$\frac{-5}{y+4}$

24. $\frac{4v-32}{64-v^2}$

25. $\frac{w^3+21}{6w^2-36}$

Answer

$\frac{w^2-6w+3}{6w-6}$

26. $\frac{v^3+125}{v^2-25}$

27. $\frac{z^2-9z+20}{16-z^2}$

Answer

$\frac{-z-5}{4+z}$

28. $\frac{a^2-5z-36}{81-a^2}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

29. $\frac{12}{16} \cdot \frac{4}{10}$

Answer

$\frac{3}{10}$

30. $\frac{3}{25} \cdot \frac{16}{24}$

31. $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

Answer

$\frac{x^3}{8y}$

32. $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

33. $\frac{5p^2}{p^2-5p-36} \cdot \frac{p^2-16}{10p}$

Answer

$\frac{p(p-4)}{2(p-9)}$

34. $\frac{3q^2}{q^2+q-6} \cdot \frac{q^2-9}{9q}$

35. $\frac{2y^2-10y}{y^2+10y+25} \cdot \frac{y+5}{6y}$

Answer

$\frac{y-5}{3(y+5)}$

36. $\frac{z^2+3z}{z^2-3z-4} \cdot \frac{z-4}{z^2}$

37. $\frac{28-4b}{3b-3} \cdot \frac{b^2+8b-9}{b^2-49}$

Answer

$\frac{-4(b+9)}{3(b+7)}$

38. $\frac{72m-12m^2}{8m+32} \cdot \frac{m^2+10m+24}{m^2-36}$

39. $\frac{c^2-10c+25}{c^2-25} \cdot \frac{c^2+10c+25}{3c^2-14c-5}$

Answer

$\frac{c+5}{3c+1}$

40. $\frac{2d^2+d-3}{d^2-16} \cdot \frac{d^2-8d+16}{2d^2-9d-18}$

41. $\frac{2m^2-3m-2}{2m2+7m+3} \cdot \frac{3m^2-14m+15}{3m^2+17m-20}$

Answer

$\frac{(m-3)(m-2)}{(m+4)(m+3)}$

42. $\frac{2n^2-3n-14}{25-n^2} \cdot \frac{n^2-10n+25}{2n^2-13n+21}$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

43. $\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

Answer

$-\frac{1}{v+5}$

44. $\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

45. $\frac{3s^2}{s^2-16} \div \frac{s^3-4s^2+16s}{s^3-64}$

Answer

$\frac{3s}{s+4}$

46. $\frac{r^2-9}{15} \div \frac{r^3-27}{5r^2+15r+45}$

47. $\frac{p^3+q^3}{3p^2+3pq+3q^2} \div \frac{p^2-q^2}{12}$

Answer

$\frac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

48. $\dfrac{v^3-8w^3}{2v^2+4vw+8w^2} \div \dfrac{v^2-4w^2}{4}$

49. $\dfrac{x^2+3x-10}{4x} \div (2x^2+20x+50)$

Answer

$$\frac{x-2}{8x(x+5)}$$

50. $\dfrac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)$

51. $\dfrac{\dfrac{2a^2-a-21}{5a+20}}{\dfrac{a^2+7a+12}{a^2+8a+16}}$

Answer

$$\frac{2a-7}{5}$$

52. $\dfrac{\dfrac{3b^2+2b-8}{12b+18}}{\dfrac{3b^2+2b-8}{2b^2-7b-15}}$

53. $\dfrac{\dfrac{12c^2-12}{2c^2-3c+1}}{\dfrac{4c+4}{6c^2-13c+5}}$

Answer

$$3(3c-5)$$

54. $\dfrac{\dfrac{4d^2+7d-2}{35d+10}}{\dfrac{d^2-4}{7d^2-12d-4}}$

For the following exercises, perform the indicated operations.

55. $\dfrac{10m^2+80m}{3m-9} \cdot \dfrac{m^2+4m-21}{m^2-9m+20} \div \dfrac{5m^2+10m}{2m-10}$

Answer

$$\frac{4(m+8)(m+7)}{3(m-4)(m+2)}$$

56. $\dfrac{4n^2+32n}{3n+2} \cdot \dfrac{3n^2-n-2}{n^2+n-30} \div \dfrac{108n^2-24n}{n+6}$

57. $\dfrac{12p^2+3p}{p+3} \div \dfrac{p^2+2p-63}{p^2-p-12} \cdot \dfrac{p-7}{9p^3-9p^2}$

Answer

$$\frac{(4p+1)(p-4)}{3p(p+9)(p-1)}$$

58. $\dfrac{6q+3}{9q^2-9q} \div \dfrac{q^2+14q+33}{q^2+4q-5} \cdot \dfrac{4q^2+12q}{12q+6}$

Multiply and Divide Rational Functions

In the following exercises, find the domain of each function.

59. $R(x) = \dfrac{x^3-2x^2-25x+50}{x^2-25}$

Answer

$x \neq 5$ and $x \neq -5$

60. $R(x) = \dfrac{x^3+3x^2-4x-12}{x^2-4}$

61. $R(x) = \dfrac{3x^2+15x}{6x^2+6x-36}$

Answer

$x \neq 2$ and $x \neq -3$

62. $R(x) = \dfrac{8x^2-32x}{2x^2-6x-80}$

For the following exercises, find $R(x) = f(x) \cdot g(x)$ where $f(x)$ and $g(x)$ are given.

63. $f(x) = \dfrac{6x^2-12x}{x^2+7x-18} \quad g(x) = \dfrac{x^2-81}{3x^2-27x}$

Answer

$R(x) = 2$

64. $f(x) = \dfrac{x^2-2x}{x^2+6x-16} \quad g(x) = \dfrac{x^2-64}{x^2-8x}$

65. $f(x) = \dfrac{4x}{x^2-3x-10} \quad g(x) = \dfrac{x^2-25}{8x^2}$

Answer

$$R(x) = \frac{x+5}{2x(x+2)}$$

66. $f(x) = \dfrac{2x^2+8x}{x^2-9x+20} \quad g(x) = \dfrac{x-5}{x^2}$

For the following exercises, find $R(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are given.

67. $f(x) = \dfrac{27x^2}{3x-21} \quad g(x) = \dfrac{3x^2+18x}{x^2+13x+42}$

Answer

$R(x) = \dfrac{3x(x+7)}{x-7}$

68. $f(x) = \dfrac{24x^2}{2x-8} \quad g(x) = \dfrac{4x^3+28x^2}{x^2+11x+28}$

69. $f(x) = \dfrac{16x^2}{4x+36} \quad g(x) = \dfrac{4x^2-24x}{x^2+4x-45}$

Answer

$R(x) = \dfrac{x(x-5)}{x-6}$

70. $f(x) = \dfrac{24x^2}{2x-4} \quad g(x) = \dfrac{12x^2+36x}{x^2-11x+18}$

Writing Exercises

71. Explain how you find the values of *x* for which the rational expression $\dfrac{x^2-x-20}{x^2-4}$ is undefined.

Answer

Answers will vary.

72. Explain all the steps you take to simplify the rational expression $\dfrac{p^2+4p-21}{9-p^2}$.

73. a. Multiply $\dfrac{7}{4} \cdot \dfrac{9}{10}$ and explain all your steps.
b. Multiply $\dfrac{n}{n-3} \cdot \dfrac{9}{n+3}$ and explain all your steps.
c. Evaluate your answer to part b. when $n = 7$. Did you get the same answer you got in part a.? Why or why not?

Answer

Answers will vary.

74. a. Divide $\dfrac{24}{5} \div 6$ and explain all your steps.
b. Divide $\dfrac{x^2-1}{x} \div (x+1)$ and explain all your steps.
c. Evaluate your answer to part b. when $x = 5$. Did you get the same answer you got in part a.? Why or why not?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
simplify rational expressions.			
multiply rational expressions.			
divide rational expressions.			
multiply and divide rational functions.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

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7.3: Add and Subtract Rational Expressions

Summary

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
   If you missed this problem, review [link].
2. Subtract: $\frac{3x}{4}-\frac{8}{9}$.
   If you missed this problem, review [link].
3. Subtract: $6(2x+1)-4(x-5)$.
   If you missed this problem, review [link].

Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

RATIONAL EXPRESSION ADDITION AND SUBTRACTION

If p, q, and r are polynomials where $r\neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of x should be excluded in the next example?

Example 7.3.1

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.

Solution

Since the denominator is $x+4$, we must exclude the value $x=-4$.

	$\frac{11x+28}{x+4}+\frac{x^2}{x+4},\ x\neq -4$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{11x+28+x^2}{x+4}$
Write the degrees in descending order.	$\frac{x^2+11x+28}{x+4}$
Factor the numerator.	$\frac{(x+4)(x+7)}{x+4}$
Simplify by removing common factors.	$\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$
Simplify.	$x+7$

The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x\neq -4$.

? Try It 7.3.2

Simplify: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

$x+2$

? Try It 7.3.3

Simplify: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

$x+3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

✓ Example 7.3.4

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\ \cancel{(x-6)}}{(x+3)\ \cancel{(x-6)}}$
	$(x-2)(x+3)$

? Try It 7.3.5

Subtract: $\frac{4x^2-11x+8}{x^2-3x+2}-\frac{3x^2+x-3}{x^2-3x+2}$.

Answer

$\frac{x-11}{x-2}$

? Try It 7.3.6

Subtract: $\frac{6x^2-x+20}{x^2-81}-\frac{5x^2+11x-7}{x^2-81}$.

Answer

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\frac{7}{d}+\frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d}+\frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d}+\frac{-5}{d}$

Simplify.	$\frac{2}{d}$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

Example 7.3.7

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify.	$\frac{m-2}{m+1}$

Try It 7.3.8

Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

Answer

$\frac{y+3}{y+2}$

Try It 7.3.9

Subtract: $\frac{2n^2+8n-1}{n^2-1}-\frac{n^2-7n-1}{1-n^2}$.

Answer

$\frac{3n-2}{n-1}$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12}+\frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}$$
$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$
$$\frac{21}{36}+\frac{10}{36}$$

$$\begin{aligned}12&=2\cdot 2\cdot 3\\18&=2\cdot\ \ 3\cdot 3\\\text{LCD}&=2\cdot 2\cdot 3\cdot 3\\\text{LCD}&=36\end{aligned}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

1. Factor each denominator completely.
2. List the factors of each denominator. Match factors vertically when possible.
3. Bring down the columns by including all factors, but do not include common factors twice.
4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of xx should we exclude in this next example?

Example 7.3.10

a. Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and b. rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

a.

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	
Factor each denominator completely, lining up common factors. Bring down the columns.	$x^2-2x-3=(x+1)(x-3)$ $x^2+4x+3=(x+1)(x+3)$ $\text{LCD}\quad=(x+1)(x-3)(x+3)$
Write the LCD as the product of the factors.	The LCD is $(x+1)(x-3)(x+3)$.

b.

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Try It 7.3.11

a. Find the LCD for the expressions $\frac{2}{x^2-x-12}$, $\frac{1}{x^2-16}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x-4)(x+3)(x+4)$

b. $\frac{2x+8}{(x-4)(x+3)(x+4)}$, $\frac{x+3}{(x-4)(x+3)(x+4)}$

Try It 7.3.12

a. Find the LCD for the expressions $\frac{3x}{x^2-3x+10}$, $\frac{5}{x^2+3x+2}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x+2)(x-5)(x+1)$

b. $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$, $\frac{5x-25}{(x+2)(x-5)(x+1)}$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

Example 7.3.13: How to Add Rational Expressions with Unlike Denominators

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$. Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$. Keep the denominators factored!	$x-3:(x-3)$ $x-2$: $(x-2)$ LCD : $(x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

Try It 7.3.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$$\frac{7x-4}{(x-2)(x+3)}$$

Try It 7.3.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$$\frac{7m+25}{(m+3)(m+4)}$$

The steps used to add rational expressions are summarized here.

ADD OR SUBTRACT RATIONAL EXPRESSIONS.

1. Determine if the expressions have a common denominator.
   - **Yes** – go to step 2.
   - **No** – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after you have combined the numerators.

Example 7.3.16

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $x^2-2x-3=(x+1)(x-3)$ $x^2+4x+3=(x+1)\quad(x+3)$ $LCD=(x+1)(x-3)(x+3)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

Try It 7.3.17

Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

Answer

$$\frac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$$

Try It 7.3.18

Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

Answer

$$\frac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Example 7.3.19

Subtract: $\frac{8y}{y^2-16}-\frac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16}-\frac{4}{y-4}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $y^2-16=(y-4)(y+4)$, $y-4=y-4$, $LCD=(y-4)(y+4)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)}-\frac{4y+16}{(y-4)(y+4)}$

Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerator.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors	$\frac{4(\cancel{y-4})}{(\cancel{y-4})(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Try It 7.3.20

Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

Try It 7.3.21

Subtract: $\frac{3}{z+3}-\frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful.

Example 7.3.22

Subtract: $\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(-1)(2-n)}$
Write $(-1)(2-n)$ as $n-2$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(n-2)}$
Simplify. Remember, $a-(-b)=a+b$.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)}{(n-2)}$
Do the rational expressions have a common denominator? No.	

Find the LCD. $n^2+n-6=(n-2)(n+3)$ $n-2=(n-2)$ $LCD=\quad(n-2)(n+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{n^2+6n+9}{(n-2)(n+3)}$
Add the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n(n+3)}{(n-2)(n+3)}$
Simplify.	$\frac{n}{(n-2)}$

Try It 7.3.23

Subtract: $\frac{3x-1}{x^2-5x-6}-\frac{2}{6-x}$.

Answer

$\frac{5x+1}{(x-6)(x+1)}$

Try It 7.3.24

Subtract: $\frac{-2y-2}{y^2+2y-8}-\frac{y-1}{2-y}$.

Answer

$\frac{y+3}{y+4}$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

Example 7.3.25

Subtract: $\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$.

Solution

	$\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$
Factor the denominators.	$\frac{4}{(a+1)(a+5)}-\frac{3}{(a+2)(a+5)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $a^2+6a+5=(a+1)(a+5)$ $a^2+7a+10=(a+5)(a+2)$ $LCD=(a+1)(a+5)(a+2)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{4(a+2)}{(a+1)(a+5)(a+2)} - \frac{3(a+1)}{(a+2)(a+5)(a+1)}$
Simplify the numerators.	$\frac{4a+8}{(a+1)(a+5)(a+2)} - \frac{3a+3}{(a+2)(a+5)(a+1)}$
Subtract the rational expressions.	$\frac{4a+8-(3a+3)}{(a+1)(a+5)(a+2)}$
Simplify the numerator.	$\frac{4a+8-3a+3}{(a+1)(a+5)(a+2)}$
	$\frac{a+5}{(a+1)(a+5)(a+2)}$
Look for common factors.	$\frac{\cancel{(a+5)}}{(a+1)\cancel{(a+5)}(a+2)}$
Simplify.	$\frac{1}{(a+1)(a+2)}$

Try It 7.3.26

Subtract: $\frac{3}{b^2-4b-5} - \frac{2}{b^2-6b+5}$.

Answer

$$\frac{1}{(b+1)(b-1)}$$

Try It 7.3.27

Subtract: $\frac{4}{x^2-4} - \frac{3}{x^2-x-2}$.

Answer

$$\frac{1}{(x+2)(x+1)}$$

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example, we will start by factoring all three denominators to find their LCD.

Example 7.3.28

Simplify: $\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$.

Solution

	$\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $u-1=(u-1)$ $u=u$ $u^2-u=u(u-1)$ $LCD=u(u-1)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u \cdot u}{(u-1)u} + \frac{1 \cdot (u-1)}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u} + \frac{u-1}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

Try It 7.3.29

Simplify: $\frac{v}{v+1} + \frac{3}{v-1} - \frac{6}{v^2-1}$.

Answer

$\frac{v+3}{v+1}$

Try It 7.3.30

Simplify: $\frac{3w}{w+2} + \frac{2}{w+7} - \frac{17w+4}{w^2+9w+14}$.

Answer

$\frac{3w}{w+7}$

Add and subtract rational functions

To add or subtract rational functions, we use the same techniques we used to add or subtract polynomial functions.

Example 7.3.31

Find $R(x) = f(x) - g(x)$ where $f(x) = \frac{x+5}{x-2}$ and $g(x) = \frac{5x+18}{x^2-4}$.

xolution

	$R(x) = f(x) - g(x)$
Substitute in the functions $f(x)$, $g(x)$.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{x^2-4}$
Factor the denominators.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{(x-2)(x+2)}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $x-2 = (x-2)$, $x^2-4 = (x-2)(x+2)$, $LCD = (x-2)(x+2)$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$R(x)=\frac{(x+5)(x+2)}{(x-2)(x+2)}-\frac{5x+18}{(x-2)(x+2)}$
Write as one rational expression.	$R(x)=\frac{(x+5)(x+2)-(5x+18)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{x^2+7x+10-5x-18}{(x-2)(x+2)}$
	$R(x)=\frac{x^2+2x-8}{(x-2)(x+2)}$
Factor the numerator, and remove common factors.	$R(x)=\frac{(x+4)(x-2)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{(x+4)}{(x+2)}$

Try It 7.3.32

Find $R(x)=f(x)-g(x)$ where $f(x)=\frac{x+1}{x+3}$ and $g(x)=\frac{x+17}{x^2-x-12}$.

Answer

$\frac{x-7}{x-4}$

Try It 7.3.33

Find $R(x)=f(x)+g(x)$ where $f(x)=\frac{x-4}{x+3}$ and $g(x)=\frac{4x+6}{x^2-9}$.

Answer

$\frac{x^2-3x+18}{(x+3)(x-3)}$

Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- Add and Subtract Rational Expressions- Unlike Denominators

Key Concepts

- **Rational Expression Addition and Subtraction**
  If p, q, and r are polynomials where $r\neq 0$, then
  $$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$
- **How to find the least common denominator of rational expressions.**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**
  1. Determine if the expressions have a common denominator.
     - Yes – go to step 2.
     - No – Rewrite each rational expression with the LCD.
       - Find the LCD.

- Rewrite each rational expression as an equivalent rational expression with the LCD.

2. Add or subtract the rational expressions.
3. Simplify, if possible.

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7.3E: Exercises

Practice Makes Perfect

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, add.

1. $\frac{2}{15}+\frac{7}{15}$

Answer

$\frac{3}{5}$

2. $\frac{7}{24}+\frac{11}{24}$

3. $\frac{3c}{4c-5}+\frac{5}{4c-5}$

Answer

$\frac{3c+5}{4c-5}$

4. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

5. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

Answer

$r+8$

6. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

7. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

Answer

$\frac{2w}{w-4}$

8. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

In the following exercises, subtract.

9. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$

Answer

$3a+7$

10. $\frac{25b^2}{5b-6} - \frac{36}{5b-6}$

11. $\frac{3m^2}{6m-30} - \frac{21m-30}{6m-30}$

Answer

$\frac{m-2}{2}$

12. $\frac{2n^2}{4n-32} - \frac{18n-16}{4n-32}$

13. $\frac{6p^2+3p+4}{p^2+4p-5} - \frac{5p^2+p+7}{p^2+4p-5}$

Answer

$\frac{p+3}{p+5}$

14. $\frac{5q^2+3q-9}{q^2+6q+8} - \frac{4q^2+9q+7}{q^2+6q+8}$

15. $\frac{5r^2+7r-33}{r^2-49} - \frac{4r^2+5r+30}{r^2-49}$

Answer

$\frac{r+9}{r+7}$

16. $\frac{7t^2-t-4}{t^2-25} - \frac{6t^2+12t-44}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add or subtract.

17. $\frac{10v}{2v-1} + \frac{2v+4}{1-2v}$

Answer

4

18. $\frac{20w}{5w-2} + \frac{5w+6}{2-5w}$

19. $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

Answer

$x+2$

20. $\frac{6y^2+2y-11}{3y-7}+\frac{3y^2-3y+17}{7-3y}$

21. $\frac{z^2+6z}{z^2-25}-\frac{3z+20}{25-z^2}$

Answer

$\frac{z+4}{z-5}$

22. $\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}$

23. $\frac{2b^2+30b-13}{b^2-49}-\frac{2b^2-5b-8}{49-b^2}$

Answer

$\frac{4b-3}{b-7}$

24. $\frac{c^2+5c-10}{c^2-16}-\frac{c^2-8c-10}{16-c^2}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, a. find the LCD for the given rational expressions b. rewrite them as equivalent rational expressions with the lowest common denominator.

25. $\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$

Answer

a. $(x+2)(x-4)(x+3)$

b. $\frac{5x+15}{(x+2)(x-4)(x+3)}, \frac{2x^2+4x}{(x+2)(x-4)(x+3)}$

26. $\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$

27. $\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$

Answer

a. $(z-2)(z+4)(z-4)$

b. $\frac{9z-36}{(z-2)(z+4)(z-4)}, \frac{4z^2-8z}{(z-2)(z+4)(z-4)}$

28. $\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$

29. $\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$

Answer

a. $(b+3)(b+3)(b-5)$

b. $\frac{4b-20}{(b+3)(b+3)(b-5)}, \frac{2b^2+6b}{(b+3)(b+3)(b-5)}$

30. $\frac{5}{c^2-4c+4}, \frac{3c}{c^2-7c+10}$

31. $\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$

Answer

a. $(d+5)(3d-1)(d-6)$

b. $\frac{2d-12}{(d+5)(3d-1)(d-6)}, \frac{5d^2+25d}{(d+5)(3d-1)(d-6)}$

32. $\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

33. $\frac{7}{10x^2y}+\frac{4}{15xy^2}$

Answer

$\frac{21y+8x}{30x^2y^2}$

34. $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

35. $\frac{3}{r+4}+\frac{2}{r-5}$

Answer

$\frac{5r-7}{(r+4)(r-5)}$

36. $\frac{4}{s-7}+\frac{5}{s+3}$

37. $\frac{5}{3w-2}+\frac{2}{w+1}$

Answer

$$\frac{11w+1}{(3w-2)(w+1)}$$

38. $\frac{4}{2x+5}+\frac{2}{x-1}$

39. $\frac{2y}{y+3}+\frac{3}{y-1}$

Answer

$$\frac{2y^2+y+9}{(y+3)(y-1)}$$

40. $\frac{3z}{z-2}+\frac{1}{z+5}$

41. $\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$

Answer

$$\frac{b(5b+10+2a^2)}{a^2(b-2)(b+2)}$$

42. $\frac{4}{cd+3c}+\frac{1}{d^2-9}$

43. $\frac{-3m}{3m-3}+\frac{5m}{m^2+3m-4}$

Answer

$$-\frac{m}{m+4}$$

44. $\frac{8}{4n+4}+\frac{6}{n^2-n-2}$

45. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

Answer

$$\frac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$$

46. $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

47. $\frac{t}{t-6}-\frac{t-2}{t+6}$

Answer

$$\frac{2(7t-6)}{(t-6)(t+6)}$$

48. $\frac{x-3}{x+6}-\frac{x}{x+3}$

49. $\frac{5a}{a+3}-\frac{a+2}{a+6}$

Answer

$\frac{4a^2+25a-6}{(a+3)(a+6)}$

50. $\frac{3b}{b-2}-\frac{b-6}{b-8}$

51. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

Answer

$\frac{-6}{m-6}$

52. $\frac{4}{n+4}-\frac{8n}{n^2-16}$

53. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

Answer

$\frac{p+2}{p+3}$

54. $\frac{-13q-8}{q^2+2q-24}-\frac{q+2}{4-q}$

55. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

Answer

$\frac{3}{r-2}$

56. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

57. $\frac{2x+7}{10x-1}+3$

Answer

$\frac{4(8x+1)}{10x-1}$

58. $\frac{8y-4}{5y+2}-6$

59. $\dfrac{3}{x^2-3x-4}-\dfrac{2}{x^2-5x+4}$

Answer

$\dfrac{x-5}{(x-4)(x+1)(x-1)}$

60. $\dfrac{4}{x^2-6x+5}-\dfrac{3}{x^2-7x+10}$

61. $\dfrac{5}{x^2+8x-9}-\dfrac{4}{x^2+10x+9}$

Answer

$\dfrac{1}{(x-1)(x+1)}$

62. $\dfrac{3}{2x^2+5x+2}-\dfrac{1}{2x^2+3x+1}$

63. $\dfrac{5a}{a-2}+\dfrac{9}{a}-\dfrac{2a+18}{a^2-2a}$

Answer

$\dfrac{5a^2+7a-36}{a(a-2)}$

64. $\dfrac{2b}{b-5}+\dfrac{3}{2b}-\dfrac{2b-15}{2b^2-10b}$

65. $\dfrac{c}{c+2}+\dfrac{5}{c-2}-\dfrac{10c}{c^2-4}$

Answer

$\dfrac{c-5}{c+2}$

66. $\dfrac{6d}{d-5}+\dfrac{1}{d+4}+\dfrac{7d-5}{d^2-d-20}$

67. $\dfrac{3d}{d+2}+\dfrac{4}{d}-\dfrac{d+8}{d^2+2d}$

Answer

$\dfrac{3(d+1)}{d+2}$

68. $\dfrac{2q}{q+5}+\dfrac{3}{q-3}-\dfrac{13q+15}{q^2+2q-15}$

Add and Subtract Rational Functions

In the following exercises, find a. $R(x)=f(x)+g(x)$ b. $R(x)=f(x)-g(x)$.

69. $f(x)=\dfrac{-5x-5}{x^2+x-6}$ and $g(x)=\dfrac{x+1}{2-x}$

Answer

a. $R(x)=-\dfrac{(x+8)(x+1)}{(x-2)(x+3)}$

b. $R(x)=\dfrac{x+1}{x+3}$

70. $f(x)=\dfrac{-4x-24}{x^2+x-30}$ and $g(x)=\dfrac{x+7}{5-x}$

71. $f(x)=\dfrac{6x}{x^2-64}$ and $g(x)=\dfrac{3}{x-8}$

Answer

a. $R(x)=\dfrac{3(3x+8)}{(x-8)(x+8)}$

b. $R(x)=\dfrac{3}{x+8}$

72. $f(x)=\dfrac{5}{x+7}$ and $g(x)=\dfrac{10x}{x^2-49}$

Writing Exercises

73. Donald thinks that $\dfrac{3}{x}+\dfrac{4}{x}$ is $\dfrac{7}{2x}$. Is Donald correct? Explain.

Answer

Answers will vary.

74. Explain how you find the Least Common Denominator of x^2+5x+4 and x^2-16.

75. Felipe thinks $\dfrac{1}{x}+\dfrac{1}{y}$ is $\dfrac{2}{x+y}$.

a. Choose numerical values for *x* and *y* and evaluate $\dfrac{1}{x}+\dfrac{1}{y}$.

b. Evaluate $\dfrac{2}{x+y}$ for the same values of *x* and *y* you used in part a..

c. Explain why Felipe is wrong.

d. Find the correct expression for $1x+1y$.

Answer

a. Answers will vary.

b. Answers will vary.

c. Answers will vary.

d. $\dfrac{x+y}{x}$

76. Simplify the expression $\dfrac{4}{n^2+6n+9}-\dfrac{1}{n^2-9}$ and explain all your steps.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			
find the least common denominator of rational expressions.			
add and subtract rational expressions with unlike denominators.			
add or subtract rational functions.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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7.4: Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Before you get started, take this readiness quiz.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.
   If you missed this problem, review [link].
2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
   If you missed this problem, review [link].
3. Solve: $\dfrac{1}{2x}+\dfrac{1}{4}=\dfrac{1}{8}$.
   If you missed this problem, review [link].

Complex Fraction

A **complex fraction** is a fraction in which the numerator and/or the denominator contains a fraction.

We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:

$$\left(\frac{6x^2-7x+2}{4x-8}\right)\div\left(\frac{2x^2-8x+3}{x^2-5x+6}\right)$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

Example 7.4.1

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$$

Solution

Rewrite the complex fraction as division.

$$\frac{6}{x-4}\div\frac{3}{x^2-16}$$

Rewrite as the product of first times the reciprocal of the second.Rewrite as the product of first times the reciprocal of the second.Rewrite as the product of first times the reciprocal of the second.

$$\frac{6}{x-4}\cdot\frac{x^2-16}{3}$$

Factor.

$$\frac{3\cdot 2}{x-4}\cdot\frac{(x-4)(x+4)}{3}$$

Multiply.

$$\frac{3\cdot 2(x-4)(x+4)}{3(x-4)}$$

Remove common factors.

$$\frac{\cancel{3}\cdot 2\ \cancel{(x-4)}(x+4)}{\cancel{3}\ \cancel{(x-4)}}$$

Simplify.

$$2(x+4)$$

Are there any value(s) of x that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

Try It 7.4.1

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$$

Answer

$$\frac{2}{3(x-1)}$$

Try It 7.4.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$$

Answer

$$\frac{1}{2(x-3)}$$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Example 7.4.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$

Solution

Simplify the numerator and denominator. Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.

$$\frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$$

Simplify the numerator and denominator.

$$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$$

Rewrite the complex rational expression as a division problem.

$$\frac{3}{6}\div\frac{1}{6}$$

Multiply the first by the reciprocal of the second.

$$\frac{3}{6} \cdot \frac{6}{1}$$

Simplify.

$$3$$

? Try It 7.4.3

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$$

Answer

$$\frac{14}{11}$$

? Try It 7.4.4

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$$

Answer

$$\frac{10}{23}$$

We follow the same procedure when the complex rational expression contains variables.

✓ Example 7.4.3: How to Simplify a Complex Rational Expression using Division

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Solution

Step 1. Simplify the numerator.

We will simplify the sum in and denominator. the numerator and difference in the denominator.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Find a common denominator and add the fractions in the numerator.

$$\frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$$

$$\frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$$

Find a common denominator and subtract the fractions in the denominator.

$$\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$$

We now have just one rational expression in the numerator and one in the denominator.

Step 2. Rewrite the complex rational expression as a division problem.

We write the numerator divided by the denominator.

$$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$$

Step 3. Divide the expressions.

Multiply the first by the reciprocal of the second.

$$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$$

Factor any expressions if possible.

$$\frac{xy(y+x)}{xy(x-y)(x+y)}$$

Remove common factors.

$$\frac{\cancel{xy}\ \cancel{(y+x)}}{\cancel{xy}(x-y)\ \cancel{(x+y)}}$$

Simplify.

$$\frac{1}{x-y}$$

? Try It 7.4.5

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$$

Answer

$$\frac{y+x}{y-x}$$

Try It 7.4.6

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$$

Answer

$$\frac{ab}{b-a}$$

We summarize the steps here.

How to simplify a complex rational expression by writing it as division.

1. Rewrite the complex rational expression as a division problem.
2. Divide the expressions.

Example 7.4.4

Simplify the complex rational expression by writing it as division:

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$$

Solution

Simplify the numerator and denominator. Find common denominators for the numerator and denominator.

$$\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$$

Simplify the numerators.

$$\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$$

Subtract the rational expressions in the numerator and add in the denominator.

$$\frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$

Simplify. (We now have one rational expression over one rational expression.)

$$\frac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$$

Rewrite as fraction division.

$$\frac{n^2+n}{n+5} \div \frac{2n}{(n+5)(n-5)}$$

Multiply the first times the reciprocal of the second.

$$\frac{n^2+n}{n+5} \cdot \frac{(n+5)(n-5)}{2n}$$

Factor any expressions if possible.

$$\frac{n(n+1)(n+5)(n-5)}{(n+5)2n}$$

Remove common factors.

$$\frac{\cancel{n}(n+1)\ \cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\ \cancel{n}}$$

Simplify.

$$\frac{(n+1)(n-5)}{2}$$

? Try It 7.4.7

Simplify the complex rational expression by writing it as division:

$$\frac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}$$

Answer

$$\frac{b(b+2)(b-5)}{3b-5}$$

? Try It 7.4.8

Simplify the complex rational expression by writing it as division:

$$\frac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}$$

Answer

$$\frac{3}{c+3}$$

Simplify a Complex Rational Expression by Using the LCD

We "cleared" the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in Example 7.4.2. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{LCD}{LCD}$ we are multiplying by 1, so the value stays the same.

Example 7.4.5

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$

Solution

The LCD of all the fractions in the whole expression is 6.

Clear the fractions by multiplying the numerator and denominator by that LCD.

$$\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$$

Distribute.

$$\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$$

Simplify.

$$\frac{2+1}{3-2}$$

$$\frac{3}{1}$$

$$3$$

Try It 7.4.9

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$$

Answer

$\frac{7}{3}$

Try It 7.4.10

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$$

Answer

$\frac{10}{3}$

We will use the same example as in Example 7.4.3. Decide which method works better for you.

Example 7.4.6: How to Simplify a Complex Rational Expressing using the LCD

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Solution

Step 1. Find the LCD of all fractions in the is complex rational expression.

The LCD of all the fractions xy.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$$

Step 2. Multiply the numerator and denominator by the LCD.

Multiply both the numerator and denominator by xy.

$$\frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$$

Step 3. Simplify the expression.

Distribute.

$$\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$$

$$\frac{y+x}{x^2-y^2}$$

Simplify.

$$\frac{\cancel{(y+x)}}{(x-y)\ \cancel{(x+y)}}$$

Remove common factors.

$$\frac{1}{x-y}$$

Try It 7.4.11

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$$

Answer

$$\frac{b+a}{a^2+b^2}$$

Try It 7.4.12

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$$

Answer

$$\frac{y-x}{xy}$$

How to simplify a complex rational expression by using the LCD.

1. Multiply the numerator and denominator by the LCD.
2. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example 7.4.7

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Solution

Find the LCD of all fractions in the complex rational expression. The LCD is:

$$x^2-36=(x+6)(x-6)$$

Multiply the numerator and denominator by the LCD.

$$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$$

Simplify the expression.

Distribute in the denominator.

$$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$$

Simplify.

$$\frac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{\cancel{(x+6)}\ \cancel{(x-6)}\left(\frac{4}{x-6}\right)-\ \cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$$

Simplify.

$$\frac{2(x-6)}{4(x+6)-4}$$

To simplify the denominator, distribute and combine like terms.

$$\frac{2(x-6)}{4x+20}$$

Factor the denominator.

$$\frac{2(x-6)}{4(x+5)}$$

Remove common factors.

$$\frac{\cancel{2}(x-6)}{\cancel{2}\cdot 2(x+5)}$$

Simplify.

$$\frac{x-6}{2(x+5)}$$

Notice that there are no more factors common to the numerator and denominator.

Try It 7.4.13

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$$

Answer

$$\frac{3(x-2)}{5x+7}$$

Try It 7.4.14

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$$

Answer

$$\frac{x+21}{6x-43}$$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

Example 7.4.8

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$

Solution

Find the LCD of all fractions in the complex rational expression.

The LCD is $(m-3)(m-4)$.

Multiply the numerator and denominator by the LCD.

$$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$$

Simplify.

$$\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\ \cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$$

Simplify.

$$\frac{4}{3(m-4)-2(m-3)}$$

Distribute.

$$\frac{4}{3m-12-2m+6}$$

Combine like terms.

$$\frac{4}{m-6}$$

Try It 7.4.15

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$$

Answer

$$\frac{3}{5x+22}$$

Try It 7.4.16

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$$

Answer

$$\frac{2\left(2y^2+13y+5\right)}{3y}$$

Example 7.4.9

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$

Solution

Find the LCD of all fractions in the complex rational expression.

The LCD is $(y+1)(y-1)$.

Multiply the numerator and denominator by the LCD.

$$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$$

Distribute in the denominator and simplify.

$$\frac{\cancel{(y+1)}(y-1)\frac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\ \cancel{(y-1)}\left(\frac{1}{\cancel{(y-1)}}\right)}$$

Simplify.

$$\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$$

Simplify the denominator and leave the numerator factored.

$$\frac{y(y-1)}{y^2-1+y+1}$$

$$\frac{y(y-1)}{y^2+y}$$

Factor the denominator and remove factors common with the numerator.

$$\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$$

Simplify.

$$\frac{y-1}{y+1}$$

? Try It 7.4.17

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$$

Answer

$$\frac{x}{x+4}$$

? Try It 7.4.18

Simplify the complex rational expression by using the LCD:

$$\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$$

Answer

$$\frac{x(x+1)}{3(x-1)}$$

Access this online resource for additional instruction and practice with complex fractions.

- Complex Fractions

Key Concepts

- **How to simplify a complex rational expression by writing it as division.**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.
  3. Simplify the expression.

Glossary

complex rational expression

A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

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7.4E: Exercises

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.

1. $\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}}$

Answer

$\dfrac{a-4}{2a}$

2. $\dfrac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}}$

3. $\dfrac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{c+7}}$

Answer

$\dfrac{1}{2(c-2)}$

4. $\dfrac{\dfrac{8}{d^2+9d+18}}{\dfrac{12}{d+6}}$

5. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

Answer

$\dfrac{12}{13}$

6. $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

7. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

Answer

$$\frac{20}{57}$$

8. $\dfrac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

9. $\dfrac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

Answer

$$\frac{n^2+m}{m-n^2}$$

10. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

11. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Answer

$$\frac{rt}{t-r}$$

12. $\dfrac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

13. $\dfrac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

Answer

$$\frac{(x+1)(x-3)}{2}$$

14. $\dfrac{y-\frac{2y}{y-4}}{\frac{2}{y-4}+\frac{2}{y+4}}$

15. $\dfrac{2-\dfrac{2}{a+3}}{\dfrac{1}{a+3}+\dfrac{a}{2}}$

Answer

$\dfrac{4}{a+1}$

16. $\dfrac{4+\dfrac{4}{b-5}}{\dfrac{1}{b-5}+\dfrac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify each complex rational expression by using the LCD.

17. $\dfrac{\dfrac{1}{3}+\dfrac{1}{8}}{\dfrac{1}{4}+\dfrac{1}{12}}$

Answer

$\dfrac{11}{8}$

18. $\dfrac{\dfrac{1}{4}+\dfrac{1}{9}}{\dfrac{1}{6}+\dfrac{1}{12}}$

19. $\dfrac{\dfrac{5}{6}+\dfrac{2}{9}}{\dfrac{7}{18}-\dfrac{1}{3}}$

Answer

19

20. $\dfrac{\dfrac{1}{6}+\dfrac{4}{15}}{\dfrac{3}{5}-\dfrac{1}{2}}$

21. $\dfrac{\dfrac{c}{d}+\dfrac{1}{d}}{\dfrac{1}{d}-\dfrac{d}{c}}$

Answer

$\dfrac{c^2+c}{c-d^2}$

22. $\dfrac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

23. $\dfrac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

Answer

$\dfrac{pq}{q-p}$

24. $\dfrac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

25. $\dfrac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$

Answer

$\dfrac{2x-10}{3x+16}$

26. $\dfrac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$

27. $\dfrac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$

Answer

$\dfrac{3z-19}{3z+8}$

28. $\dfrac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$

29. $\dfrac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$

Answer

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$$\frac{4}{3a-7}$$

30. $$\frac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$$

31. $$\frac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$$

Answer

$$\frac{2c+29}{5c}$$

32. $$\frac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$$

33. $$\frac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$$

Answer

$$\frac{2p-5}{5}$$

34. $$\frac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$$

35. $$\frac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$$

Answer

$$\frac{m(m-5)}{(4m-19)(m+5)}$$

36. $$\frac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$$

In the following exercises, simplify each complex rational expression using either method.

37. $\dfrac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

Answer

$\frac{13}{24}$

38. $\dfrac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

39. $\dfrac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

Answer

$2(a-4)$

40. $\dfrac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

41. $\dfrac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

Answer

$\frac{3mn}{n-m}$

42. $\dfrac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

43. $\dfrac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

Answer

$\frac{(x-1)(x-2)}{6}$

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44. $\dfrac{\dfrac{y}{y+3}}{2+\dfrac{1}{y-3}}$

Writing Exercises

45. In this section, you learned to simplify the complex fraction $\dfrac{\dfrac{3}{x+2}}{\dfrac{x}{x^2-4}}$ two ways: rewriting it as a division problem or multiplying the numerator and denominator by the LCD. Which method do you prefer? Why?

Answer

Answers will vary.

44. Efraim wants to start simplifying the complex fraction $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator, $\dfrac{\frac{1}{\cancel{a}}+\frac{1}{\cancel{b}}}{\frac{1}{\cancel{a}}-\frac{1}{\cancel{b}}}$. Explain what is wrong with Efraim's plan.

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7.5: Solve Rational Equations

Learning Objectives

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$
   If you missed this problem, review Example 2.2.9.
2. Solve: $n^2-5n-36=0$
   If you missed this problem, review Example 6.6.4.
3. Solve the formula $5x+2y=10$ for y.
   If you missed this problem, review Example 2.4.10.

After defining the terms ‘expression’ and ‘equation’ earlier, we have used them throughout this book. We have simplified many kinds of expressions and solved many kinds of equations. We have simplified many rational expressions so far in this chapter. Now we will solve a **rational equation**.

Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don’t end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example 7.5.1: How to Solve a Rational Equation

Solve:

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

Solution

Step 1. Note any value of the variable that would make any denominator zero.

If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$$

Step 2. Find the least common denominator of all denominators in the equation.

Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$

The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Multiply both sides of the equation by the LCD, $6x$.

$$6x \cdot \left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot \left(\frac{5}{6}\right)$$

Use the Distributive Property.

$$6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot \left(\frac{5}{6}\right)$$

Simplify - and notice, no more fractions!

$$6+2x=5x$$

Step 4. Solve the resulting equation.

Simplify.

$$6=3x$$
$$2=x$$

Step 5. Check.

If any values found in Step 1 are algebraic solutions, discard them. Check any remaining solutions in the original equation.

We did not get 0 as an algebraic solution.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

We substitute $x=2$ into the original equation.

$$\frac{1}{2}+\frac{1}{3} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{3}{6}+\frac{2}{6} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{5}{6}=\frac{5}{6}\checkmark$$

The solution is $x=2$

? Exercise 7.5.1

Solve:

$$\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$$

Answer

$$y=-\frac{15}{7}$$

? Exercise 7.5.2

Solve:

$$\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$$

Answer

$$x=\frac{15}{3}$$

The steps of this method are shown.

how to Solve equations with rational expressions.

- Step 1. Note any value of the variable that would make any denominator zero.
- Step 2. Find the least common denominator of all denominators in the equation.
- Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
- Step 4. Solve the resulting equation.
- Step 5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

✓ Example 7.5.2: How to Solve a Rational Equation using the Zero Product Property

Solve:

$$1-\frac{5}{y}=-\frac{6}{y^2}$$

Solution

Note any value of the variable that would make any denominator zero.

$$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$$

Find the least common denominator of all denominators in the equation. The LCD is y^2.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Distribute.

$$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Multiply.

$$y^2-5y=-6$$

Solve the resulting equation. First write the quadratic equation in standard form.

$$y^2-5y+6=0$$

Factor.

$$(y-2)(y-3)=0$$

Use the Zero Product Property.

$$y-2=0 \text{ or } y-3=0$$

Solve.

$$y=2 \text{ or } y=3$$

Check. We did not get 0 as an algebraic solution.

Check $y=2$ and $y=3$ in the original equation.

$$1-\frac{5}{y}=-\frac{6}{y^2} \qquad 1-\frac{5}{y}=-\frac{6}{y^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad \frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4} \qquad -\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}=-\frac{3}{2}\surd \qquad -\frac{2}{3}=-\frac{2}{3}\surd$$

The solution is $y=2, y=3$

Exercise 7.5.3

Solve:

$$1-\frac{2}{x}=\frac{15}{x^2}$$

Answer

$x=-3, x=5$

Exercise 7.5.4

Solve:

$$1-\frac{4}{y}=\frac{12}{y^2}$$

Answer

$y=-2, y=6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example 7.5.3

Solve:

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

Solution

Note any value of the variable that would make any denominator zero.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x\neq -2, x\neq 2$$

Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Distribute.

$$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Remove common factors.

$$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\ \cancel{(x-2)}\frac{4}{\cancel{x-2}}=\ \cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$$

Simplify.

$$2(x-2)+4(x+2)=x-1$$

Distribute.

$$2x-4+4x+8=x-1$$

Solve.

$$\begin{aligned}6x+4&=x-1\\5x&=-5\\x&=-1\end{aligned}$$

Check: We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$
$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$
$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$
$$\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$$
$$\frac{2}{3}=\frac{2}{3}\surd$$

The solution is $x=-1$.

Exercise 7.5.5

Solve:

$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$$

Answer

$x=\frac{2}{3}$

Exercise 7.5.6

Solve:

$$\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$$

Answer

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

Example 7.5.4

Solve:

$$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$$

Solution

Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.

$$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$$

Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$$

Distribute.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$$

Remove common factors.

$$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\ \cancel{(m-1)}\frac{3}{\cancel{m-1}}$$

Simplify.

$$m+11=5(m-1)-3(m-4)$$

Solve the resulting equation.

$$\begin{aligned} m+11 &= 5m-5-3m+12 \\ 4 &= m \end{aligned}$$

Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

? Exercise 7.5.7

Solve:

$$\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$$

Answer

There is no solution.

? Exercise 7.5.8

Solve:

$$\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$$

Answer

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

✓ Example 7.5.5

Solve:

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$$

Find the least common denominator. The LCD is $(y-6)(y+6)$

Clear the fractions.

$$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$$

Simplify.

$$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$$

Simplify.

$$y(y-6)=72+4\left(y^2-36\right)$$

Solve the resulting equation.

$$\begin{aligned} y^2-6y &= 72+4y^2-144 \\ 0 &= 3y^2+6y-72 \\ 0 &= 3\left(y^2+2y-24\right) \\ 0 &= 3(y+6)(y-4) \\ y &= -6, y=4 \end{aligned}$$

Check.

$y=-6$ is an extraneous solution. Check $y=4$ in the original equation.

$$\begin{aligned} \frac{y}{y+6} &= \frac{72}{y^2-36}+4 \\ \frac{4}{4+6} &\stackrel{?}{=} \frac{72}{4^2-36}+4 \\ \frac{4}{10} &\stackrel{?}{=} \frac{72}{-20}+4 \\ \frac{4}{10} &\stackrel{?}{=} -\frac{36}{10}+\frac{40}{10} \\ \frac{4}{10} &= \frac{4}{10}\surd \end{aligned}$$

The solution is $y=4$.

Exercise 7.5.9

Solve:

$$\frac{x}{x+4}=\frac{32}{x^2-16}+5$$

Answer

$x=3$

Exercise 7.5.10

Solve:

$$\frac{y}{y+8}=\frac{128}{y^2-64}+9$$

Answer

$y=7$

In some cases, all the algebraic solutions are extraneous.

Example 7.5.6

Solve:

$$\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$$

Solution

We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}$$

Note any value of the variable that would make any denominator zero.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}, x\neq 1, x\neq -1$$

Find the least common denominator. The LCD is $12(x-1)(x+1)$.

Clear the fractions.

$$12(x-1)(x+1)\left(\frac{x}{2(x-1)}-\frac{2}{3(x+1)}\right)=12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$$

Simplify.

$$6(x+1)\cdot x-4(x-1)\cdot 2=5x^2-2x+9$$

Simplify.

$$6x(x+1)-4\cdot 2(x-1)=5x^2-2x+9$$

Solve the resulting equation.

$$\begin{aligned} 6x^2+6x-8x+8 &= 5x^2-2x+9 \\ x^2-1 &= 0 \\ (x-1)(x+1) &= 0 \\ x &= 1 \text{ or } x=-1 \end{aligned}$$

Check.

$x=1$ and $x=-1$ are extraneous solutions.

The equation has no solution.

Exercise 7.5.11

Solve:

$$\frac{y}{5y-10}-\frac{5}{3y+6}=\frac{2y^2-19y+54}{15y^2-60}$$

Answer

There is no solution.

Exercise 7.5.12

Solve:

$$\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$$

Answer

There is no solution.

Example 7.5.7

Solve:

$$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}, x\neq -1, x\neq \frac{1}{3}, x\neq 3$$

Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.

Clear the fractions.

$$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$$

Simplify.

$$4(x+1)+3(x-3)=2(3x-1)$$

Distribute.

$$4x+4+3x-9=6x-2$$

Simplify.

$$7x-5=6x-2$$

$$x=3$$

The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Exercise 7.5.13

Solve:

$$\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}$$

Answer

There is no solution.

Exercise 7.5.14

Solve:

$$\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}$$

Answer

There is no solution.

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

Example 7.5.8

For rational function, $f(x)=\frac{2x-6}{x^2-8x+15}$:

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value.

Solution

a. The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

$$\begin{aligned} x^2-8x+15 &= 0 \\ (x-3)(x-5) &= 0 \quad \text{Factor the trinomial.} \\ x-3 &= 0 \quad \text{Use the Zero Product Property.} \\ x-5 &= 0 \quad \text{Use the Zero Product Property.} \\ x=3 \quad x &= 5 \quad \text{Solve.} \end{aligned}$$

The domain is all real numbers except $x\neq 3, x\neq 5$

b. $f(x)=1$

Substitute in the rational expression.

$$\frac{2x-6}{x^2-8x+15}=1$$

Factor the denominator.

$$\frac{2x-6}{(x-3)(x-5)}=1$$

Multiply both sides by the LCD, $(x-3)(x-5)$

$$(x-3)(x-5)\left(\frac{2x-6}{(x-3)(x-5)}\right)=(x-3)(x-5)(1)$$

Simplify.

$$2x-6=x^2-8x+15$$

Solve.

$$0=x^2-10x+21$$

Factor.

$$0 = (x-7)(x-3)$$

Use the Zero Product Property.

$$x-7=0 \quad x-3=0$$

Solve.

$$x=7 \quad x=3$$

c. The value of the function is 1 when $x=7, x=3$. So the points on the graph of this function when $f(x)=1$, will be $(7,1),(3,1)$. However, now we need to check for extraneous solutions. Since $x=3$ is not in the domain of the original function, we must remove $(3,1)$, since it would be considered extraneous.

? Exercise 7.5.15

For rational function, $f(x)=\dfrac{8-x}{x^2-7x+12}$

a. Find the domain of the function.
b. Solve $f(x)=3$.
c. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 3$ and $x \neq 4$
b. $x=2, x=\dfrac{14}{3}$
c. $(2,3), \left(\dfrac{14}{3},3\right)$

? Exercise 7.5.16

For rational function, $f(x)=\dfrac{x-1}{x^2-6x+5}$

a. Solve $f(x)=4$.
b. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 1$ and $x \neq 5$
b. $x=\dfrac{21}{4}$
c. $\left(\dfrac{21}{4},4\right)$

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$$
\begin{aligned}
m &= \frac{y-y_1}{x-x_1} \\
m\,(x-x_1) &= \left(\frac{y-y_1}{x-x_1}\right)(x-x_1) \quad \text{Multiply both sides of the equation by } x-x_1. \\
m\,(x-x_1) &= y-y_1 \quad \text{Simplify.} \\
y-y_1 &= m\,(x-x_1) \quad \text{Rewrite the equation with the y terms on the left.}
\end{aligned}
$$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2,3)$. We will add one more step to solve for y.

Example 7.5.9

Solve: $m = \dfrac{y-2}{x-3}$ for y.

Solution

$$m = \frac{y-2}{x-3}$$

Note any value of the variable that would make any denominator zero.

$$m = \frac{y-2}{x-3}, x \neq 3$$

Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.

$$(x-3)m = (x-3)\left(\frac{y-2}{x-3}\right)$$

Simplify.

$$xm-3m = y-2$$

Isolate the term with y.

$$xm-3m+2 = y$$

Exercise 7.5.17

Solve: $m = \dfrac{y-5}{x-4}$ for y.

Answer

$y = mx-4m+5$

Exercise 7.5.18

Solve: $m = \dfrac{y-1}{x+5}$ for y.

Answer

$y = mx+5m+1$

Remember to multiply both sides by the LCD in the next example.

Example 7.5.10

Solve: $\frac{1}{c}+\frac{1}{m}=1$ for c

Solution

$$\frac{1}{c}+\frac{1}{m}=1 \text{ for } c$$

Note any value of the variable that would make any denominator zero.

$$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$$

Clear the fractions by multiplying both sides of the equations by the LCD, cm.

$$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$$

Distribute.

$$cm\left(\frac{1}{c}\right)+cm\frac{1}{m}=cm(1)$$

Simplify.

$$m+c=cm$$

Collect the terms with c to the right.

$$m=cm-c$$

Factor the expression on the right.

$$m=c(m-1)$$

To isolate c, divide both sides by $m-1$.

$$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$$

Simplify by removing common factors.

$$\frac{m}{m-1}=c$$

Notice that even though we excluded $c=0$ and $m=0$ from the original equation, we must also now state that $m\neq 1$.

Exercise 7.5.19

Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$$a=\frac{b}{cb-1}$$

Exercise 7.5.20

Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y

Answer

$$y = \frac{3x}{x+6}$$

Media Access Additional Online Resources

Access this online resource for additional instruction and practice with equations with rational expressions.

- Equations with Rational Expressions

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7.5E: Exercises

Solve Rational Equations

In the following exercises, solve each rational equation.

1. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

Answer

$a=10$

2. $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

3. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

Answer

$v=\frac{40}{21}$

4. $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

5. $1-\frac{2}{m}=\frac{8}{m^2}$

Answer

$m=-2,\ m=4$

6. $1+\frac{4}{n}=\frac{21}{n^2}$

7. $1+\frac{9}{p}=\frac{-20}{p^2}$

Answer

$p=-5,\ p=-4$

8. $1-\frac{7}{q}=\frac{-6}{q^2}$

9. $\frac{5}{3v-2}=\frac{7}{4v}$

Answer

$v=14$

10. $\frac{8}{2w+1}=\frac{3}{w}$

11. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

Answer

$x=-\frac{4}{5}$

12. $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

13. $\frac{8}{z-10}-\frac{7}{z+10}=\frac{5}{z^2-100}$

Answer

$z=-145$

14. $\frac{9}{a+11}-\frac{6}{a-11}=\frac{6}{a^2-121}$

15. $\frac{-10}{q-2}-\frac{7}{q+4}=1$

Answer

$q=-18,\ q=-1$

16. $\frac{2}{s+7}-\frac{3}{s-3}=1$

17. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

Answer

no solution

18. $\frac{w+8}{w^2-11w+28}=\frac{5}{w-7}+\frac{2}{w-4}$

19. $\frac{x-10}{x^2+8x+12}=\frac{3}{x+2}+\frac{4}{x+6}$

Answer

no solution

20. $\frac{y-5}{y^2-4y-5}=\frac{1}{y+1}+\frac{1}{y-5}$

21. $\frac{b+3}{3b}+\frac{b}{24}=\frac{1}{b}$

Answer

$b=-8$

22. $\frac{c+3}{12c}+\frac{c}{36}=\frac{1}{4c}$

23. $\frac{d}{d+3}=\frac{18}{d^2-9}+4$

Answer

$d=2$

24. $\frac{m}{m+5}=\frac{50}{m^2-25}+6$

25. $\frac{n}{n+2}-3=\frac{8}{n^2-4}$

Answer

$m=1$

26. $\frac{p}{p+7}-8=\frac{98}{p^2-49}$

27. $\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$

Answer

no solution

28. $\frac{r}{3r-15}-\frac{1}{4r+20}=\frac{3r^2+17r+40}{12r^2-300}$

29. $\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$

Answer

$s=\frac{5}{4}$

30. $\frac{t}{6t-12}-\frac{5}{2t+10}=\frac{t^2-23t+70}{12t^2+36t-120}$

31. $\frac{2}{x^2+2x-8}-\frac{1}{x^2+9x+20}=\frac{4}{x^2+3x-10}$

Answer

$x=-\frac{4}{3}$

32. $\frac{5}{x^2+4x+3}+\frac{2}{x^2+x-6}=\frac{3}{x^2-x-2}$

33. $\frac{3}{x^2-5x-6}+\frac{3}{x^2-7x+6}=\frac{6}{x^2-1}$

Answer

no solution

34. $\dfrac{2}{x^2+2x-3}+\dfrac{3}{x^2+4x+3}=\dfrac{6}{x^2-1}$

Solve Rational Equations that Involve Functions

35. For rational function, $f(x)=\dfrac{x-2}{x^2+6x+8}$:

a. Find the domain of the function
b. Solve $f(x)=5$
c. Find the points on the graph at this function value

Answer

a. The domain is all real numbers except $x\neq -2$ and $x\neq -4$
b. $x=-3, x=-\dfrac{14}{5}$
c. $(-3,5), \left(-\dfrac{14}{5},5\right)$

36. For rational function, $f(x)=\dfrac{x+1}{x^2-2x-3}$:

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value

37. For rational function, $f(x)=\dfrac{2-x}{x^2-7x+10}$:

a. Find the domain of the function
b. Solve $f(x)=2$
c. Find the points on the graph at this function value

Answer

a. The domain is all real numbers except $x\neq 2$ and $x\neq 5$
b. $x=\dfrac{9}{2}$
c. $\left(\dfrac{9}{2},2\right)$

38. For rational function, $f(x)=\dfrac{5-x}{x^2+5x+6}$:

a. Find the domain of the function
b. Solve $f(x)=3$
c. Find the points on the graph at this function value

Solve a Rational Equation for a Specific Variable

In the following exercises, solve:

39. $\dfrac{c}{r}=2\pi$ for r

Answer

$$r = \frac{C}{2\pi}$$

40. $\frac{I}{r} = P$ for r

41. $\frac{v+3}{w-1} = \frac{1}{2}$ for w

Answer

$$w = 2v+7$$

42. $\frac{x+5}{2-y} = \frac{4}{3}$ for y

43. $a = \frac{b+3}{c-2}$ for c

Answer

$$c = \frac{b+3+2a}{a}$$

44. $m = \frac{n}{2-n}$ for n

45. $\frac{1}{p} + \frac{2}{q} = 4$ for p

Answer

$$p = \frac{q}{4q-2}$$

46. $\frac{3}{s} + \frac{1}{t} = 2$ for s

47. $\frac{2}{v} + \frac{1}{5} = \frac{3}{w}$ for w

Answer

$$w = \frac{15v}{10+v}$$

48. $\frac{6}{x} + \frac{2}{3} = \frac{1}{y}$ for y

49. $\frac{m+3}{n-2} = \frac{4}{5}$ for n

Answer

$$n = \frac{5m+23}{4}$$

50. $r = \dfrac{s}{3-t}$ for t

51. $\dfrac{E}{c} = m^2$ for c

Answer

$$c = \frac{E}{m^2}$$

52. $\dfrac{R}{T} = W$ for T

53. $\dfrac{3}{x} - \dfrac{5}{y} = \dfrac{1}{4}$ for y

Answer

$$y = \frac{20x}{12-x}$$

54. $c = \dfrac{2}{a} + \dfrac{b}{5}$ for a

Writing Exercises

55. Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.

Answer

Answers will vary.

56. Alek thinks the equation $\dfrac{y}{y+6} = \dfrac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Alek is wrong.

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7.6: Solve Uniform Motion and Work Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve uniform motion applications
- Solve work applications

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. An express train and a local bus leave Chicago to travel to Champaign. The express bus can make the trip in 2 hours and the local bus takes 5 hours for the trip. The speed of the express bus is 42 miles per hour faster than the speed of the local bus. Find the speed of the local bus.
   If you missed this problem, review [link].
2. Solve $\frac{1}{3}x+\frac{1}{4}x=\frac{5}{6}$.
   If you missed this problem, review [link].
3. Solve: $18t^2-30=-33t$.
   If you missed this problem, review [link].

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula D=rt in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

The formula D=rt assumes we know *r* and *t* and use them to find D. If we know D and *r* and need to find *t*, we would solve the equation for *t* and get the formula $t=\frac{D}{r}$

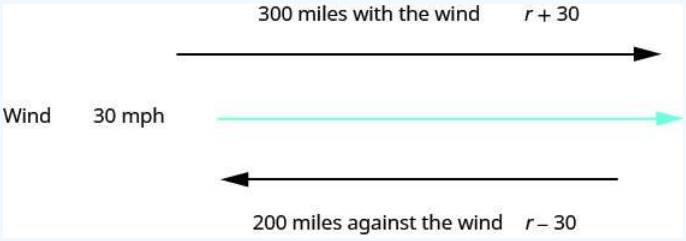
We have also explained how flying with or against a current affects the speed of a vehicle. We will revisit that idea in the next example.

Example 7.6.1

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane.	Let r= the speed of the airplane.

When the plane flies with the wind, the wind increases its speed and the rate is r+30.	
When the plane flies against the wind, the wind decreases its speed and the rate is r−30.	
Write in the rates. Write in the distances. Since D=r·t , we solve for t and get $\frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate • Time = Distance</th></tr><tr><td>Headwind</td><td>$r-30$ $\quad$ $\frac{200}{r-30}$ $\quad$ 200</td></tr><tr><td>Tailwind</td><td>$r+30$ $\quad$ $\frac{300}{r+30}$ $\quad$ 300</td></tr></table>
We know the times are equal and so we write our equation.	$\frac{200}{r-30}=\frac{300}{r+30}$
We multiply both sides by the LCD. 200(r+30)=300(r−30)	
Simplify.	(r+30)(200)=(r−30)(300)
	200r+6000=300r−9000
Solve.	15000=100r 150=r
Check.	
Is 150 mph a reasonable speed for an airplane? Yes. If the plane is traveling 150 mph and the wind is 30 mph:	
Tailwind 150+30=180 mph $\frac{300}{180}=\frac{5}{3}$ hours	
Headwind 150−30=120 mph $\frac{200}{120}=\frac{5}{3}$ hours	
The times are equal, so it checks.	The plane was traveling 150 mph.

? Try It 7.6.2

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

Answer

15 mph

? Try It 7.6.3

Judy can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Judy's boat without a wind?

Answer

17 mph

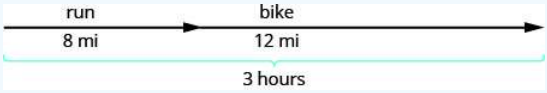
In the next example, we will know the total time resulting from travelling different distances at different speeds.

Example 7.6.4

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Jazmine's running speed.	Let r= Jazmine's running sp
Her biking speed is 4 miles faster than her running speed.	r+4= her biking speed
The distances are given, enter them into the chart.	
Since D=r·t , we solve for t and get $t = \frac{D}{r}$ We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate • Time = Distance</th></tr><tr><td>Run</td><td>r $\frac{8}{r}$ 8</td></tr><tr><td>Bike</td><td>$r+4$ $\frac{24}{r+4}$ 24</td></tr><tr><td></td><td>3</td></tr></table>
Write a word sentence.	Her time plus the time biking
Translate the sentence to get the equation.	$\frac{8}{r} + \frac{24}{r+4} = 3$
Solve.	$r(r+4)(\frac{8}{r} + \frac{24}{r+4}) = 3r(r+4)$ $8(r+4) + 24r = 3r^2 + 12r$ $8r + 32 + 24r = 3r^2 + 12r$ $32 + 32r = 3r^2 + 12r$ $0 = 3r^2 - 20r - 32$ $0 = (3r+4)(r-8)$
	(3r+4)=0, (r−8)=0
	$r = -\frac{4}{3}$, r=8
Check. r=8	
A negative speed does not make sense in this problem, so r=8 is the solution.	
Is 8 mph a reasonable running speed? Yes.	
Run 8 mph, $\frac{8miles}{8mph} = 1hour$ Bike 12 mph, $\frac{24miles}{12mph} = 2hours$ Total 3 hours. Jazmine's running speed is 8 mph.	

Try It 7.6.5

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

Answer

10 mph

Try It 7.6.6

Tony drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

Answer

50 mph

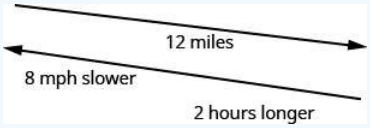
Once again, we will use the uniform motion formula solved for the variable *t*.

Example 7.6.7

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed.	Let r= Hamilton's downhill speed.
His uphill speed is 8 miles per hour slower. Enter the rates into the chart.	h−8= Hamilton's uphill speed
The distance is the same in both directions, 12 miles. Since D=r·t , we solve for *t* and get $t = \frac{D}{r}$ We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate • Time = Distance</th></tr><tr><td>Downhill</td><td>h	$\frac{12}{h}$	12</td></tr><tr><td>Uphill</td><td>$h-8$	$\frac{12}{h-8}$	12</td></tr></table>
Write a word sentence about the time.	He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.
Translate the sentence to get the equation. Solve.	$\frac{12}{h-8} = \frac{12}{h} + 2$ $h(h-8)(\frac{12}{h-8}) = h(h-8)(\frac{12}{h} + 2)$ 12h=12(h−8)+2h(h−8) $12h = 12h - 96 + 2h^2 - 16h$ $0 = 2h^2 - 16h - 96$ $0 = 2(h^2 - 8h - 48)$ 0=2(h−12)(h+4) h−12=0, h+4=0 h=12, h=−4
Check. Is 12 mph a reasonable speed for biking downhill? Yes.	
Downhill 12 mph, $\frac{12miles}{12mph} = 1hour$	

Uphill 12−8=4 mph $\frac{12miles}{4mph} = 3hours$	
	The uphill time is 2 hours more than the downhill time. Hamilton's downhill speed is 12 mph.

Try It 7.6.8

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

Answer

15 mph

Try It 7.6.9

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 18 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Answer

6 mph

Solve Work Applications

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

This is a typical 'work' application. There are three quantities involved here – the time it would take each of the two people to do the job alone and the time it would take for them to do the job together.

Let's get back to Pete and Alicia painting the room. We will let t be the number of hours it would take them to paint the room together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take them if they worked together, we solve for t.

	Pete's part + Alicia's part = part of total $\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$
	$\frac{1}{10}+\frac{1}{8}=\frac{1}{t}$
Multiply by the LCD, 40t	$40t\left(\frac{1}{10}+\frac{1}{8}\right)=40t\left(\frac{1}{t}\right)$
Distribute.	$40t\cdot\frac{1}{10}+40t\cdot\frac{1}{8}=40t\left(\frac{1}{t}\right)$
Simplify and solve.	$4t+5t=40$

	$9t = 40$
	$t = \frac{40}{9}$
We'll write as a mixed number so that we can convert it to hours and minutes.	$t = 4\frac{4}{9}$ hours
Remember, 1 hour = 60 minutes.	$t = 4$ hours $+ \frac{4}{9}(60$ minutes$)$
Multiply, and then round to the nearest minute.	$t = 4$ hours $+ 27$ minutes
	It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

Keep in mind, it should take less time for two people to complete a job working together than for either person to do it alone.

Example 7.6.10

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

Solution

This is a work problem. A chart will help us organize the information.

Let t= the number of hours needed to complete the job together.	
Enter the hours per job for Press #1, Press #2 and when they work together. If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed. Similarly find the part of the job completed/hours for Press #2 and when they both work together.	<table><tr><th></th><th>Number of hours to complete the job</th><th>Part of job completed/ hour</th></tr><tr><td>Press #1</td><td>6</td><td>$\frac{1}{6}$</td></tr><tr><td>Press #2</td><td>12</td><td>$\frac{1}{12}$</td></tr><tr><td>Together</td><td>t</td><td>$\frac{1}{t}$</td></tr></table>
Write a word sentence.	
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.	
Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Solve.	$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Multiply by the LCD, 12t.	$12t\left(\frac{1}{6} + \frac{1}{12}\right) = 12t\left(\frac{1}{t}\right)$
Simplify.	$2t + t = 12$
	$3t = 12$
	$t = 4$
	When both presses are running it takes 4 hours to do the job.

Try It 7.6.11

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

Answer

2 hours and 24 minutes

Try It 7.6.12

Carrie can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

Answer

2 hours and 6 minutes

Example 7.6.13

Corey can shovel all the snow from the sidewalk and driveway in 4 hours. If he and his twin Casey work together, they can finish shoveling the snow in 2 hours. How many hours would it take Casey to do the job by himself?

Solution

This is a work application. A chart will help us organize the information.	
We are looking for how many hours it would take Casey to complete the job by himself.	
Let t= the number of hours needed for Casey to complete.	
Enter the hours per job for Corey, Casey, and when they work together. If Corey takes 4 hours, then in 1 hour $\frac{1}{4}$ of the job is completed. Similarly find the part of the job completed/hours for Casey and when they both work together.	<table><tr><th></th><th>Number of hours needed to complete the job</th><th>Part of job completed/ hour</th></tr><tr><td>Corey</td><td>4</td><td>$\frac{1}{4}$</td></tr><tr><td>Casey</td><td>t</td><td>$\frac{1}{t}$</td></tr><tr><td>Together</td><td>2</td><td>$\frac{1}{2}$</td></tr></table>
Write a word sentence.	
The part completed by Corey plus the part completed by Casey equals the amount completed together.	
Translate to an equation:	$\overbrace{\text{Corey} + \text{Casey}}^{} = \text{Together}$ $\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$
Solve.	$\frac{1}{4}+\frac{1}{t}=\frac{1}{2}$
Multiply by the LCD, 4t.	$4t\left(\frac{1}{4}+\frac{1}{t}\right)=4t\left(\frac{1}{2}\right)$
Simplify.	$t+4=2t$
	$4=t$
	It would take Casey 4 hours to do the job alone.

Try It 7.6.14

Two hoses can fill a swimming pool in 10 hours. It would take one hose 26 hours to fill the pool by itself. How long would it take for the other hose, working alone, to fill the pool?

Answer

16.25 hours

Try It 7.6.15

Cara and Cindy, working together, can rake the yard in 4 hours. Working alone, it takes Cindy 6 hours to rake the yard. How long would it take Cara to rake the yard alone?

Answer

12 hours

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7.6E: Exercises

Practice Makes Perfect

Solve Uniform Motion Applications

In the following exercises, solve uniform motion applications

Exercise 1

Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35 mph headwind in the same amount of time it can travel 702 miles with a 35 mph tailwind. Find the speed of the helicopter.

Answer

160 mph

Exercise 2

A private jet can fly 1210 miles against a 25 mph headwind in the same amount of time it can fly 1694 miles with a 25 mph tailwind. Find the speed of the jet.

Exercise 3

A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

Answer

29 mph

Exercise 4

Darrin can skateboard 2 miles against a 4 mph wind in the same amount of time he skateboards 6 miles with a 4 mph wind. Find the speed Darrin skateboards with no wind.

Exercise 5

Jane spent 2 hours exploring a mountain with a dirt bike. When she rode the 40 miles uphill, she went 5 mph slower than when she reached the peak and rode for 12 miles along the summit. What was her rate along the summit?

Answer

30 mph

Exercise 6

Jill wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

Exercise 7

Bill wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

Answer

20 mph

Exercise 8

Nancy took a 3 hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?

Exercise 9

Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, l, what was his uphill rate?

Answer

4 mph

Exercise 10

Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

Exercise 11

Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

Answer

60 mph

Exercise 12

Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel?

Exercise 13

John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

Answer

650 mph

Exercise 14

Jim's speedboat can travel 20 miles upstream against a 3 mph current in the same amount of time it travels 22 miles downstream with a 3 mph current speed. Find the speed of the Jim's boat.

Exercise 15

Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

Answer

50 mph

Exercise 16

Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

Exercise 17

When driving the 9 hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

Answer

50 mph

Exercise 18

Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

Solve Work Applications

In the following exercises, solve work applications.

Exercise 19

Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

Answer

2 hours

Exercise 20

It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

Exercise 21

Mary can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

Answer

2 hours and 44 minutes

Exercise 22

Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

Exercise 23

Leeson can proofread a newspaper copy in 4 hours. If Ryan helps, they can do the job in 3 hours. How long would it take for Ryan to do his job alone?

Answer

12 hours

Exercise 24

Paul can clean a classroom floor in 3 hours. When his assistant helps him, the job takes 2 hours. How long would it take the assistant to do it alone?

Exercise 25

Josephine can correct her students' test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

Answer

7 hours and 30 minutes

Exercise 26

Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take the Levi's dad to wash the car by himself?

Exercise 27

Jackson can remove the shingles off of a house in 7 hours, while Martin can remove the shingles in 5 hours. How long will it take them to remove the shingles if they work together?

Answer

2 hours and 55 minutes

Exercise 28

At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

Exercise 29

Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Answer

4 hours

Exercise 30

It takes Tina 3 hours to frost her holiday cookies, but if Candy helps her it takes 2 hours. How long would it take Candy to frost the holiday cookies by herself?

Everyday Math

Exercise 31

Dana enjoys taking her dog for a walk, but sometimes her dog gets away and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

Answer

3 mph

Exercise 32

Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Writing Exercises

Exercise 33

In Example, the solution h=−4 is crossed out. Explain why.

Exercise 34

Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3}+\frac{1}{4}=\frac{1}{t}$ can be used to find *t*, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve uniform motion applications.			
solve work applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Rational Equation Number Applications

In the following exercises, solve number applications

Exercise 35

The sum of the reciprocals of two consecutive odd integers is $-\frac{16}{63}$. Find the two numbers.

Answer

−9, −7

Exercise 36

The sum of the reciprocals of two consecutive odd integers is $\frac{28}{195}$. Find the two numbers.

Exercise 37

The sum of the reciprocals of two consecutive even integers is $\frac{5}{12}$. Find the two numbers.

Answer

4, 6

Exercise 38

The sum of the reciprocals of two consecutive even integers is $\frac{11}{60}$. Find the two numbers

Exercise 39

The sum of the reciprocals of two consecutive integers is $-\frac{19}{90}$. Find the two numbers.

Answer

−10, −9

Exercise 40

The sum of the reciprocals of two consecutive integers is $\frac{19}{90}$. Find the two numbers.

Exercise 41

The sum of a number and its reciprocal is $\frac{5}{2}$. Find the number(s).

Answer

2, $\frac{1}{2}$

Exercise 42

The sum of a number and its reciprocal is $\frac{41}{20}$. Find the number(s).

Exercise 43

The sum of a number and twice its reciprocal is $\frac{9}{2}$. Find the number(s).

Answer

$\frac{1}{2}$, 4

Exercise 44

The sum of a number and twice its reciprocal is $\frac{17}{6}$. Find the number(s).

Exercise 45

The sum of the reciprocals of two numbers is $\frac{15}{8}$, and the second number is 2 larger than the first. Find the two numbers.

Answer

$\{\frac{2}{3}, \frac{8}{3}\}$ and $\{-\frac{8}{5}, \frac{2}{5}\}$

Exercise 46

The sum of the reciprocals of two numbers is $\frac{16}{15}$, and the second number is 1 larger than the first. Find the two numbers.

Exercise 47

When the same number is added to both the numerator and denominator of the fraction $\frac{3}{7}$, the result is $\frac{2}{3}$. What is the number?

Answer

The number added is 5.

Exercise 48

When the same number is added to both the numerator and denominator of the fraction $\frac{5}{8}$, the result is $\frac{3}{4}$. What is the number?

Exercise 49

When the same number is added to both the numerator and denominator of the fraction $\frac{3}{8}$, the result is $\frac{1}{6}$. What is the number?

Answer

The number added is -2.

Exercise 50

When the same number is added to both the numerator and denominator of the fraction $\frac{7}{9}$, the result is $\frac{2}{3}$. What is the number?

Exercise 51

When the same number is subtracted to both the numerator and denominator of the fraction $\frac{1}{10}$, the result is $\frac{2}{3}$. What is the number?

Answer

The number subtracted is -17.

Exercise 52

When the same number is subtracted to both the numerator and denominator of the fraction $\frac{3}{4}$, the result is $\frac{5}{6}$. What is the number?

Exercise 53

When the same number is subtracted from both the numerator and denominator of the fraction $\frac{7}{12}$, the result is $\frac{1}{2}$. What is the number?

Answer

2

Exercise 54

When the same number is added to both the numerator and denominator of the fraction $\frac{13}{15}$, the result is $\frac{8}{9}$. What is the number?

Exercise 55

One-third of a number added to the reciprocal of number yields $\frac{13}{6}$. What is the number?

Answer

$x = \frac{1}{2}, 6$

Exercise 56

Four-fifths of a number added to the reciprocal of number yields $\frac{81}{10}$. What is the number?

Exercise 57

One-half of a number added to twice the reciprocal of the number yields 2. What is the number?

Answer

2

Exercise 58

One-fourth of a number added to four times the reciprocal of the number yields $\frac{-10}{3}$. What is the number?

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Chapter 7 Review Exercises

Chapter Review Exercises

Simplify Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

Exercise 1

$\frac{2a+1}{3a-2}$

Answer

$a \neq \frac{2}{3}$

Exercise 2

$\frac{b-3}{b^2-16}$

Exercise 3

$\frac{3xy^2}{5y}$

Answer

$y \neq 0$

Exercise 4

$\frac{u-3}{u^2-u-30}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expressions for the given values.

Exercise 5

$\frac{4p-1}{p^2+5}$ when $p=-1$

Answer

$-\frac{5}{6}$

Exercise 6

$\frac{q^2-5}{q+3}$ when $q=7$

Exercise 7

$\frac{y^2-8}{y^2-y-2}$ when $y=1$

Answer

$\frac{7}{2}$

Example 8

$\frac{z^2+2}{4z-z^2}$ when $z=3$

Simplify Rational Expressions

In the following exercises, simplify.

Exercise 9

$\frac{10}{24}$

Answer

$\frac{5}{12}$

Exercise 10

$\frac{8m^4}{16mn^3}$

Exercise 11

$\frac{14a-14}{a-1}$

Answer

14

Exercise 12

$\frac{b^2+7b+12}{b^2+8b+16}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify.

Exercise 13

$\frac{c^2-c-2}{4-c^2}$

Answer

$$-\frac{c+1}{c+2}$$

Exercise 14

$$\frac{d-16}{16-d}$$

Exercise 15

$$\frac{7v-35}{25-v^2}$$

Answer

$$-\frac{7}{5+v}$$

Exercise 16

$$\frac{w^2-3w-28}{49-w^2}$$

Multiply and Divide Rational Expressions

Multiply Rational Expressions

In the following exercises, multiply.

Exercise 17

$$\frac{3}{8}\cdot\frac{2}{15}$$

Answer

$$\frac{1}{20}$$

Exercise 18

$$\frac{2xy^2}{8y^3}\cdot\frac{16y}{24x}$$

Exercise 19

$$\frac{3a^2+21a}{a^2+6a-7}\cdot\frac{a-1}{ab}$$

Answer

$$\frac{3}{b}$$

Exercise 20

$$\frac{5z^2}{5z^2+40z+35}\cdot\frac{z^2-1}{3z}$$

Divide Rational Expressions

In the following exercises, divide.

Exercise 21

$$\frac{t^2-4t-12}{t^2+8t+12}\div\frac{t^2-36}{6t}$$

Answer

$$\frac{6t}{(t+6)^2}$$

Exercise 22

$$\frac{r^2-16}{4}\div\frac{r^3-64}{2r^2-8r+32}$$

Exercise 23

$$\frac{11+w}{w-9}\div\frac{121-w^2}{9-w}$$

Answer

$$\frac{1}{11+w}$$

Exercise 24

$$\frac{3y^2-12y-63}{4y+3}\div(6y^2-42y)$$

Exercise 25

$$\frac{\frac{c^2-64}{3c^2+26c+16}}{\frac{c^2-4c-32}{15c+10}}$$

Answer

$$5c+4$$

Exercise 26

$$\frac{8m^2-8m}{m-4}\cdot\frac{m^2+2m-24}{m^2+7m+10}\div\frac{2m^2-6m}{m+5}$$

Add and Subtract Rational Expressions with a Common Denominator

Add Rational Expressions with a Common Denominator

In the following exercises, add.

Exercise 27

$\frac{3}{5}+\frac{2}{5}$

Answer

1

Exercise 28

$\frac{4a^2}{2a-1}-\frac{1}{2a-1}$

Exercise 29

$\frac{p^2+10p}{p+5}+\frac{25}{p+5}$

Answer

$p+5$

Exercise 30

$\frac{3x}{x-1}+\frac{2}{x-1}$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

Exercise 31

$\frac{d^2}{d+4}-\frac{3d+28}{d+4}$

Answer

$d-7$

Exercise 32

$\frac{z^2}{z+10}-\frac{100}{z+10}$

Exercise 33

$$\frac{4q^2-q+3}{q^2+6q+5}-\frac{3q^2+q+6}{q^2+6q+5}$$

Answer

$$\frac{q-3}{q+5}$$

Exercise 34

$$\frac{5t+4t+3}{t^2-25}-\frac{4t^2-8t-32}{t^2-25}$$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add and subtract.

Exercise 35

$$\frac{18w}{6w-1}+\frac{3w-2}{1-6w}$$

Answer

$$\frac{15w+2}{6w-1}$$

Exercise 36

$$\frac{a^2+3a}{a^2-4}-\frac{3a-8}{4-a^2}$$

Exercise 37

$$\frac{2b^2+3b-15}{b^2-49}-\frac{b^2+16b-1}{49-b^2}$$

Answer

$$\frac{3b-2}{b+7}$$

Exercise 38

$$\frac{8y^2-10y+7}{2y-5}+\frac{2y^2+7y+2}{5-2y}$$

Add and Subtract Rational Expressions With Unlike Denominators

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

Exercise 38

$\frac{4}{m^2-3m-10}, \quad \frac{2m}{m^2-m-20}$

Answer

$(m+2)(m-5)(m+4)$

Exercise 39

$\frac{6}{n^2-4}, \quad \frac{2n}{n^2-4n+4}$

Exercise 40

$\frac{5}{3p^2+17p-6}, \quad \frac{2m}{3p^2-23p-8}$

Answer

$(3p+1)(p+6)(p+8)$

Find Equivalent Rational Expressions

In the following exercises, rewrite as equivalent rational expressions with the given denominator.

Exercise 41

Rewrite as equivalent rational expressions with denominator $(m+2)(m-5)(m+4)$

$\frac{4}{m^2-3m-10}, \quad \frac{2m}{m^2-m-20}$.

Exercise 42

Rewrite as equivalent rational expressions with denominator $(n-2)(n-2)(n+2)$

$\frac{6}{n^2-4n+4}, \quad \frac{2n}{n^2-4}$.

Answer

$\frac{6n+12}{(n-2)(n-2)(n+2)}, \quad \frac{2n^2-4n}{(n-2)(n-2)(n+2)}$

Exercise 43

Rewrite as equivalent rational expressions with denominator $(3p+1)(p+6)(p+8)$

$\frac{5}{3p^2+19p+6}, \quad \frac{7p}{3p^2+25p+8}$

Add Rational Expressions with Different Denominators

In the following exercises, add.

Exercise 44

$\frac{2}{3}+\frac{3}{5}$

Answer

$\frac{19}{15}$

Exercise 45

$\frac{7}{5a}+\frac{3}{2b}$

Exercise 46

$\frac{2}{c-2}+\frac{9}{c+3}$

Answer

$\frac{11c-12}{(c-2)(c+3)}$

Exercise 47

$\frac{3d}{d^2-9}+\frac{5}{d^2+6d+9}$

Exercise 48

$\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

Answer

$\frac{5x^2+26x}{(x+4)(x+4)(x+6)}$

Exercise 49

$\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

Subtract Rational Expressions with Different Denominators

In the following exercises, subtract and add.

Exercise 50

$\frac{3v}{v+2}-\frac{v+2}{v+8}$

Answer

$\frac{2(v^2+10v-2)}{(v+2)(v+8)}$

Exercise 51

$$\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$$

Exercise 52

$$\frac{7m+3}{m+2}-5$$

Answer

$$\frac{2m-7}{m+2}$$

Exercise 53

$$\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$$

Exercise 54

$$\frac{8d}{d^2-64}-\frac{4}{d+8}$$

Answer

$$4d-8$$

Exercise 55

$$\frac{5}{12x^2y}+\frac{7}{20xy^3}$$

Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify.

Exercise 56

$$\frac{\frac{5a}{a+2}}{\frac{10a^2}{a^2-4}}$$

Answer

$$\frac{a-2}{2a}$$

Exercise 57

$$\frac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$$

Exercise 58

$$\frac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$$

Answer

$$\frac{(x-8)(x-5)}{2}$$

Exercise 59

$$\frac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

Exercise 60

$$\frac{6+\frac{2}{q-4}}{\frac{5}{q}+4}$$

Answer

$$\frac{(q-2)(q+4)}{5(q-4)}$$

Exercise 61

$$\frac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$$

Exercise 62

$$\frac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$$

Answer

$$\frac{z-5}{21z+21}$$

Exercise 63

$$\frac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$$

Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

Exercise 64

$$\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$$

Answer

$$\frac{6}{7}$$

Exercise 65

$$1-\frac{2}{m}=\frac{8}{m^2}$$

Exercise 66

$$\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$$

Answer

$$\frac{3}{2}$$

Exercise 67

$$\frac{3}{q+8}-\frac{2}{q-2}=1$$

Exercise 68

$$\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$$

Answer

no solution

Exercise 69

$$\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

Exercise 70

$\frac{V}{l} = hw$ for l

Answer

$l = \frac{V}{hw}$

Exercise 71

$\frac{1}{x} - \frac{2}{y} = 5$ for y

Exercise 72

$x = \frac{y+5}{z-7}$ for z

Answer

$z = \frac{y+5+7x}{x}$

Exercise 73

$P = \frac{k}{V}$ for V

Solve Proportion and Similar Figure Applications Similarity

Solve Proportions

In the following exercises, solve.

Exercise 74

$\frac{x}{4} = \frac{3}{5}$

Answer

$\frac{12}{5}$

Exercise 75

$\frac{3}{y} = \frac{9}{5}$

Exercise 76

$$\frac{s}{s+20}=\frac{3}{7}$$

Answer

15

Exercise 77

$$\frac{t-3}{5}=\frac{t+2}{9}$$

In the following exercises, solve using proportions.

Exercise 78

Rachael had a 21 ounce strawberry shake that has 739 calories. How many calories are there in a 32 ounce shake?

Answer

1161 calories

Exercise 79

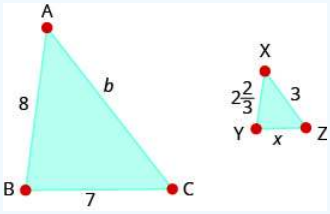
Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.

Exercise 80

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.



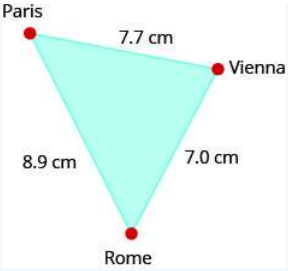
Answer

$b=9;\ x=2\frac{1}{3}$

Exercise 81

On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

1. a. Paris to Rome
2. b. Paris to Vienna



Exercise 82

Tony is 5.75 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

Answer

23 feet

Exercise 83

The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5 foot tall Natalie cast a 1.1 foot shadow How long would the shadow of the lighthouse be?

Solve Uniform Motion and Work Applications Problems

Solve Uniform Motion Applications

In the following exercises, solve.

Exercise 84

When making the 5-hour drive home from visiting her parents, Lisa ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles in the bad weather. How fast did she drive when the weather was bad?

Answer

45 mph

Exercise 85

Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

Exercise 86

John can ride his bicycle 8 mph faster than Luke can ride his bike. It takes Luke 3 hours longer than John to ride 48 miles. How fast can John ride his bike?

Answer

16 mph

Exercise 87

Mark was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

Exercise 88

Jerry can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

Answer

$\frac{4}{5}$ hour

Exercise 89

Lisa takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

Exercise 90

Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

Answer

12 days

Exercise 91

Sue and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

Use Direct and Inverse Variation

Solve Direct Variation Problems

In the following exercises, solve.

Exercise 92

If y varies directly as x, when $y = 9$ and $x = 3$, find x when $y = 21$.

Answer

7

Exercise 93

If y varies directly as x, when $y = 20$ and $x = 2$, find y when $x = 4$.

Exercise 94

If m varies inversely with the square of n, when $m = 4$ and $n = 6$, find m when $n = 2$.

Answer

36

Exercise 95

Vanessa is traveling to see her fiancé. The distance, d, varies directly with the speed, v, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

Exercise 96

If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $\$12$, how much would a 6” diameter pizza cost?

Answer

$\$9$

Exercise 97

The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Solve Inverse Variation Problems

In the following exercises, solve.

Exercise 98

The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $\$6$, how many tickets can Madelyn afford to buy if the price increased to $\$8$?

Answer

97 tickets

Exercise 99

On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12-inch string have?

Practice Test

In the following exercises, simplify.

Exercise 1

$\dfrac{3a^2b}{6ab^2}$

Answer

$\dfrac{a}{2b}$

Exercise 2

$$\frac{5b-25}{b^2-25}$$

In the following exercises, perform the indicated operation and simplify.

Exercise 3

$$\frac{4x}{x+2}\cdot\frac{x^2+5x+6}{12x^2}$$

Answer

$$\frac{x+3}{3x}$$

Exercise 4

$$\frac{5y}{4y-8}\cdot\frac{y^2-4}{10}$$

Exercise 5

$$\frac{4p}{q}+\frac{5}{p}$$

Answer

$$\frac{4+5q}{pq}$$

Exercise 6

$$\frac{1}{z-9}-\frac{3}{z+9}$$

Exercise 7

$$\frac{\frac{2}{3}+\frac{3}{5}}{\frac{2}{5}}$$

Answer

$$\frac{19}{16}$$

Exercise 8

$$\frac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$$

In the following exercises, solve each equation.

Exercise 9

$$\frac{1}{2}+\frac{2}{7}=\frac{1}{x}$$

Answer

$$x=\frac{14}{11}$$

Exercise 10

$$\frac{5}{y-6}=\frac{3}{y+6}$$

Exercise 11

$$\frac{1}{z-5}+\frac{1}{z+5}=\frac{1}{z^2-25}$$

Answer

$$z=\frac{1}{2}$$

Exercise 12

$$\frac{t}{4}=\frac{3}{5}$$

Exercise 13

$$\frac{2}{r-2}=\frac{3}{r-1}$$

Answer

$$r=4$$

In the following exercises, solve.

Exercise 14

If y varies directly with x, and $x=5$ when $y=30$, find x when $y=42$.

Exercise 15

If y varies inversely with x and $x=6$ when $y=20$, find y when $x=2$.

Answer

$$y=60$$

Exercise 16

If y varies inversely with the square of x and $x=3$ when $y=9$, find y when $x=4$.

Exercise 17

The recommended erythromycin dosage for dogs, is 5 mg for every pound the dog weighs. If Daisy weighs 25 pounds, how many milligrams of erythromycin should her veterinarian prescribe?

Answer

125 mg

Exercise 18

Julia spent 4 hours Sunday afternoon exercising at the gym. She ran on the treadmill for 10 miles and then biked for 20 miles. Her biking speed was 5 mph faster than her running speed on the treadmill. What was her running speed?

Exercise 19

Kurt can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Kurt's speed on his bike?

Answer

14 mph

Exercise 20

Amanda jogs to the park 8 miles using one route and then returns via a 14-mile route. The return trip takes her 1 hour longer than her jog to the park. Find her jogging rate.

Exercise 21

An experienced window washer can wash all the windows in Mike's house in 2 hours, while a new trainee can wash all the windows in 7 hours. How long would it take them working together?

Answer

$1\frac{5}{9}$ hour

Exercise 22

Josh can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Josh's dad working alone to split the logs?

Exercise 23

The price that Tyler pays for gas varies directly with the number of gallons he buys. If 24 gallons cost him $59.76, what would 30 gallons cost?

Answer

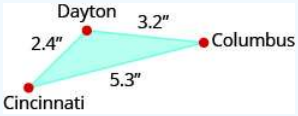
$74.70

Exercise 24

The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

Exercise 25

The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio, as shown on the figure below, that gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?

Answer

64 miles

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CHAPTER OVERVIEW

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Chapter 8 Review Exercises

Thumbnail: The mathematical expression "The (principal) square root of x". (GPL, David Vignoni (original icon); Flamurai (SVG convertion); bayo (color)).

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8.0: Prelude to Roots and Radicals

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

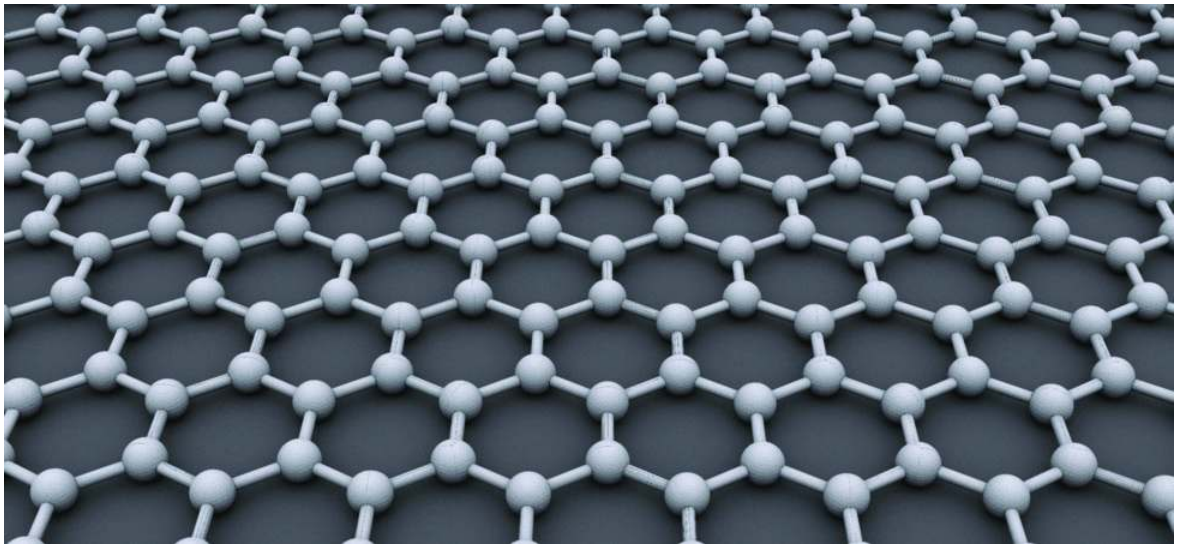


Figure 8.0.1 : Graphene is an incredible strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: "AlexanderAIUS" / Wikimedia Commons)

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8.1: Simplify Expressions with Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Before you get started, take this readiness quiz.

1. Simplify: a. $(-9)^2$ b. -9^2 c. $(-9)^3$
   If you missed this problem, review Example 2.21.
2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review Example 1.34.
3. Simplify: a. $x^3 \cdot x^3$ b. $y^2 \cdot y^2 \cdot y^2$ c. $z^3 \cdot z^3 \cdot z^3 \cdot z^3$
   If you missed this problem, review Example 5.12.

Simplify Expressions with Roots

In Foundations, we briefly looked at square roots. Remember that when a real number n is multiplied by itself, we write n^2 and read it 'n^2 squared'. This number is called the **square** of n, and n is called the **square root**. For example,

13^2 is read "13 squared"

169 is called the square of 13, since $13^2 = 169$

13 is a square root of 169

Definition 8.1.1: Square and Square Root of a Number

Square

If $n^2 = m$, then m is the **square** of n.

Square Root

If $n^2 = m$, then n is a **square root** of m.

Notice $(-13)^2 = 169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of m. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Definition 8.1.2: Square Root Notation

$\sqrt{m}$ is read "the square root of m."

If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

radical sign $\longrightarrow \sqrt{m} \longleftarrow$ radicand

Figure 8.1.1

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

Example 8.1.1

Simplify:

a. $\sqrt{144}$
b. $-\sqrt{289}$

Solution:

a.

$$\sqrt{144}$$

Since $12^2 = 144$.

$$12$$

b.

$$-\sqrt{289}$$

Since $17^2 = 289$ and the negative is in front of the radical sign.

$$-17$$

Exercise 8.1.1

Simplify:

a. $-\sqrt{64}$
b. $\sqrt{225}$

Answer

a. -8
b. 15

Exercise 8.1.2

Simplify:

a. $\sqrt{100}$
b. $-\sqrt{121}$

Answer

a. 10
b. -11

Can we simplify $-\sqrt{49}$? Is there a number whose square is -49?

$$(\_\_\_)^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

Example 8.1.2

Simplify:

a. $\sqrt{-196}$
b. $-\sqrt{64}$

Solution:

a.

$$\sqrt{-196}$$

There is no real number whose square is -196.

$\sqrt{-196}$ is not a real number.

b.

$$-\sqrt{64}$$

The negative is in front of the radical.

$$-8$$

Exercise 8.1.3

Simplify:

a. $\sqrt{-169}$
b. $-\sqrt{81}$

Answer

a. not a real number
b. -9

Exercise 8.1.4

Simplify:

a. $-\sqrt{49}$
b. $\sqrt{-121}$

Answer

a. -7
b. not a real number

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fifth power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5 to 5. See *Figure 8.1.2*

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Figure 8.1.2

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of -2 to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Even power **Positive** result Odd power **Negative** result

Figure 8.1.3

We will now extend the square root definition to higher roots.

Definition 8.1.3: Nth Root of a Number

If $b^n = a$, then b is an nth root of a.

The principal n^{th} root of a is written $\sqrt[n]{a}$.

The n is called the **index** of the radical.

Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We can refer to *Figure 8.1.2* to help find higher roots.

$$4^3 = 64 \qquad \sqrt[3]{64} = 4$$
$$3^4 = 81 \qquad \sqrt[4]{81} = 3$$
$$(-2)^5 = -32 \qquad \sqrt[5]{-32} = -2$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$

When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all the values of a.

We will apply these properties in the next two examples.

Example 8.1.3

Simplify:

a. $\sqrt[3]{64}$
b. $\sqrt[4]{81}$
c. $\sqrt[5]{32}$

Solution:

a.

$$\sqrt[3]{64}$$

Since $4^3 = 64$.

$$4$$

b.

$$\sqrt[4]{81}$$

Since $(3)^4 = 81$.

$$3$$

c.

$$\sqrt[5]{32}$$

Since $(2)^5 = 32$.

$$2$$

Exercise 8.1.5

Simplify:

a. $\sqrt[3]{27}$
b. $\sqrt[4]{256}$
c. $\sqrt[5]{243}$

Answer

a. 3
b. 4
c. 3

Exercise 8.1.6

Simplify:

a. $\sqrt[3]{1000}$
b. $\sqrt[4]{16}$
c. $\sqrt[5]{243}$

Answer

a. 10
b. 2
c. 3

In this example be alert for the negative signs as well as even and odd powers.

Example 8.1.4

Simplify:

a. $\sqrt[3]{-125}$
b. $\sqrt[4]{16}$
c. $\sqrt[5]{-243}$

Solution:

a.

$$\sqrt[3]{-125}$$

Since $(-5)^3 = -125$.

$$-5$$

b.

$$\sqrt[4]{16}$$

Think, $(?)^4 = -16$. No real number raised to the fourth power is negative.

Not a real number.

c.

$$\sqrt[5]{-243}$$

Since $(-3)^5 = -243$.

$$-3$$

Exercise 8.1.7

Simplify:

a. $\sqrt[3]{-27}$
b. $\sqrt[4]{-256}$
c. $\sqrt[5]{-32}$

Answer

a. -3
b. not real
c. -2

Exercise 8.1.8

Simplify:

a. $\sqrt[3]{-216}$
b. $\sqrt[4]{-81}$
c. $\sqrt[5]{-1024}$

Answer

a. -6
b. not real
c. -4

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4} = 2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for **perfect square** numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

Number	Square Root
4	2
9	3
16	4
25	5

11 $\sqrt{11}$

$9 < 11 < 16$

$3 < \sqrt{11} < 4$

Number	Cube Root
8	2
27	3
64	4
125	5

91 $\sqrt[3]{91}$

$64 < 91 < 125$

$4 < \sqrt[3]{91} < 5$

Figure 8.1.4

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

Example 8.1.5

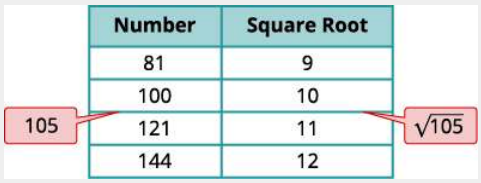
Estimate each root between two consecutive whole numbers:

a. $\sqrt{105}$
b. $\sqrt[3]{43}$

Solution:

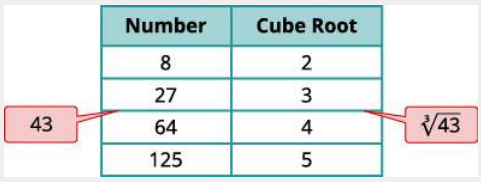
a. Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

Table 8.1.1

	$\sqrt{105}$
	
Locate 105 between two consecutive perfect squares.	$100 < 105 < 121$
$\sqrt{105}$ is between their square roots.	$10 < \sqrt{105} < 11$

b. Similarly we locate 43 between two perfect cube numbers.

Table 8.1.2

	$\sqrt[3]{43}$
	
Locate 43 between two consecutive perfect cubes.	$27 < 43 < 64$
$\sqrt[3]{43}$ is between their cube roots.	$3 < \sqrt[3]{43} < 4$

Exercise 8.1.9

Estimate each root between two consecutive whole numbers:

a. $\sqrt{38}$
b. $\sqrt[3]{93}$

Answer

a. $6 < \sqrt{38} < 7$
b. $4 < \sqrt[3]{93} < 5$

Exercise 8.1.10

Estimate each root between two consecutive whole numbers:

a. $\sqrt{84}$
b. $\sqrt[3]{152}$

Answer

a. $9 < \sqrt{84} < 10$
b. $5 < \sqrt[3]{152} < 6$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$(2.236067978)^2 = 5.000000002 \qquad (3.105422799)^4 = 92.999999991$$
$$(2.24)^2 = 5.0176 \qquad (3.11)^4 = 93.54951841$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

Example 8.1.6

Round to two decimal places:

a. $\sqrt{17}$
b. $\sqrt[3]{49}$
c. $\sqrt[4]{51}$

Solution:

a.

$$\sqrt{17}$$

Use the calculator square root key.

$$4.123105626\ldots$$

Round to two decimal places.

$$4.12$$

$$\sqrt{17} \approx 4.12$$

b.

$$\sqrt[3]{49}$$

Use the calculator $\sqrt[y]{x}$ key.

$$3.659305710\ldots$$

Round to two decimal places.

$$3.66$$

$$\sqrt[3]{49} \approx 3.66$$

c.

$$\sqrt[4]{51}$$

Use the calculator $\sqrt[y]{x}$ key.

$$2.6723451177\ldots$$

Round to two decimal places.

$$2.67$$

$$\sqrt[4]{51} \approx 2.67$$

Exercise 8.1.11

Round to two decimal places:

1. $\sqrt{11}$
2. $\sqrt[3]{71}$
3. $\sqrt[4]{127}$

Answer

a. ≈ 3.32
b. ≈ 4.14
c. ≈ 3.36

Exercise 8.1.12

Round to two decimal places:

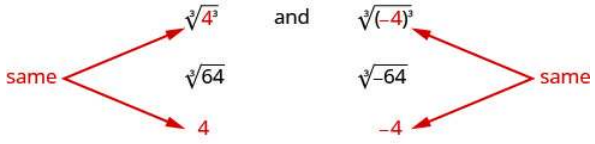
1. $\sqrt{13}$
2. $\sqrt[3]{84}$
3. $\sqrt[4]{98}$

Answer

a. ≈ 3.61
b. ≈ 4.38
c. ≈ 3.15

Simplify Variable Expressions with Roots

The odd root of a number can be either positive or negative. For example,

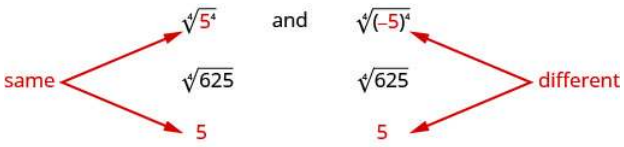


In either case, when n is odd, $\sqrt[n]{a^n} = a$.

Figure 8.1.13

But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,



Here we see that sometimes when n is even, $\sqrt[n]{a^n} \neq a$.

Figure 8.1.14

How can we make sure the fourth root of -5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

Definition 8.1.4: Simplifying Odd and Even Roots

For any integer $n \geq 2$,

$$\text{when the index } n \text{ is odd } \sqrt[n]{a^n} = a$$

$$\text{when the index } n \text{ is even } \sqrt[n]{a^n} = |a|$$

We must use the **absolute value** signs when we take an even root of an expression with a variable in the radical.

Example 8.1.7

Simplify:

a. $\sqrt{x^2}$
b. $\sqrt[3]{n^3}$
c. $\sqrt[4]{p^4}$
d. $\sqrt[5]{y^5}$

Solution:

a. We use the absolute value to be sure to get the positive root.

$$\sqrt{x^2}$$

Since the index n is even, $\sqrt[n]{a^n} = |a|$.

b. This is an odd indexed root so there is no need for an absolute value sign.

$$\sqrt[3]{m^3}$$

Since the index is n is odd, $\sqrt[n]{a^n} = a$.

$$m$$

c.

$$\sqrt[4]{p^4}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$|p|$$

d.

$$\sqrt[5]{y^5}$$

Since the index n is odd, $\sqrt[n]{a^n} = a$.

$$y$$

Exercise 8.1.13

Simplify:

a. $\sqrt{b^2}$
b. $\sqrt[3]{w^3}$
c. $\sqrt[4]{m^4}$
d. $\sqrt[5]{q^5}$

Answer

a. $|b|$
b. w
c. $|m|$
d. q

Exercise 8.1.14

Simplify:

a. $\sqrt{y^2}$
b. $\sqrt[3]{p^3}$
c. $\sqrt[4]{z^4}$
d. $\sqrt[5]{q^5}$

Answer

a. $|y|$
b. p
c. $|z|$
d. q

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root.

$$\sqrt{a^{2m}}$$

Since $(a^m)^2 = a^{2m}$.

$$\sqrt{(a^m)^2}$$

Since n is even $\sqrt[n]{a^n} = |a|$.

$$|a^m|$$

So $\sqrt{a^{2m}} = |a^m|$.

We apply this concept in the next example.

Example 8.1.8

Simplify:

a. $\sqrt{x^6}$
b. $\sqrt{y^{16}}$

Solution:

a.

$$\sqrt{x^6}$$

Since $\left(x^3\right)^2 = x^6$.

$$\sqrt{(x^3)^2}$$

Since the index n is even $\sqrt{a^n} = |a|$.

$$|x^3|$$

b.

$$\sqrt{y^{16}}$$

Since $\left(y^8\right)^2 = y^{16}$.

$$\sqrt{(y^8)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$y^8$$

In this case the absolute value sign is not needed as y^8 is positive.

Exercise 8.1.15

Simplify:

a. $\sqrt{y^{18}}$
b. $\sqrt{z^{12}}$

Answer

a. $|y^9|$
b. z^6

Exercise 8.1.16

Simplify:

a. $\sqrt{m^4}$
b. $\sqrt{b^{10}}$

Answer

a. m^2
b. $|b^5|$

The next example uses the same idea for higher roots.

Example 8.1.9

Simplify:

a. $\sqrt[3]{y^{18}}$
b. $\sqrt[4]{z^8}$

Solution:

a.

$$\sqrt[3]{y^{18}}$$

Since $\left(y^6\right)^3 = y^{18}$.

$$\sqrt[3]{\left(y^6\right)^3}$$

Since n is odd, $\sqrt[n]{a^n} = a$.

$$y^6$$

b.

$$\sqrt[4]{z^8}$$

Since $\left(z^2\right)^4 = z^8$.

$$\sqrt[4]{\left(z^2\right)^4}$$

Since z^2 is positive, we do not need an absolute value sign.

$$z^2$$

Exercise 8.1.17

Simplify:

a. $\sqrt[4]{u^{12}}$
b. $\sqrt[3]{v^{15}}$

Answer

a. $|u^3|$
b. v^5

Exercise 8.1.18

Simplify:

a. $\sqrt[5]{c^{20}}$
b. $\sqrt[6]{d^{24}}$

Answer

a. c^4
b. d^4

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = |a^m|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always positive.

Example 8.1.10

Simplify:

a. $\sqrt{16n^2}$
b. $-\sqrt{81c^2}$

Solution:

a.

$$\sqrt{16n^2}$$

Since $(4n)^2 = 16n^2$.

$$\sqrt{(4n)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$4|n|$$

b.

$$-\sqrt{81c^2}$$

Since $(9c)^2 = 81c^2$.

$$-\sqrt{(9c)^2}$$

Since the index n is even $\sqrt[n]{a^n} = |a|$.

$$-9|c|$$

Exercise 8.1.19

Simplify:

a. $\sqrt{64x^2}$
b. $-\sqrt{100p^2}$

Answer

a. $8|x|$
b. $-10|p|$

Exercise 8.1.20

Simplify:

a. $\sqrt{169y^2}$
b. $-\sqrt{121y^2}$

Answer

a. $13|y|$
b. $-11|y|$

This example just takes the idea farther as it has roots of higher index.

Example 8.1.11

Simplify:

a. $\sqrt[3]{64p^6}$
b. $\sqrt[4]{16q^{12}}$

Solution:

a.

$$\sqrt[3]{64p^6}$$

Rewrite $64p^6$ as $\left(4p^2\right)^3$.

$$\sqrt[3]{\left(4p^2\right)^3}$$

Take the cube root.

$$4p^2$$

b.

$$\sqrt[4]{16q^{12}}$$

Rewrite the radicand as a fourth power.

$$\sqrt[4]{\left(2q^3\right)^4}$$

Take the fourth root.

$$2|q^3|$$

Exercise 8.1.21

Simplify:

a. $\sqrt[3]{27x^{27}}$
b. $\sqrt[4]{81q^{28}}$

Answer

a. $3x^9$
b. $3|q^7|$

Exercise 8.1.22

Simplify:

a. $\sqrt[3]{125q^9}$
b. $\sqrt[5]{243q^{25}}$

Answer

a. $5p^3$
b. $3q^5$

The next examples have two variables.

Example 8.1.12

Simplify:

a. $\sqrt{36x^2y^2}$
b. $\sqrt{121a^6b^8}$
c. $\sqrt[3]{64p^{63}q^9}$

Solution:

a.

$$\sqrt{36x^2y^2}$$

Since $(6xy)^2 = 36x^2y^2$

$$\sqrt{(6xy)^2}$$

Take the square root.

$$6|xy|$$

b.

$$\sqrt{121a^6b^8}$$

Since $\left(11a^3b^4\right)^2 = 121a^6b^8$

$$\sqrt{\left(11a^3b^4\right)^2}$$

Take the square root.

$$11\left|a^3\right|b^4$$

c.

$$\sqrt[3]{64p^{63}q^9}$$

Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$

$$\sqrt[3]{(4p^{21}q^3)^3}$$

Take the cube root.

$$4p^{21}q^3$$

? Exercise 8.1.23

Simplify:

a. $\sqrt{100a^2b^2}$
b. $\sqrt{144p^{12}q^{20}}$
c. $\sqrt[3]{8x^{30}y^{12}}$

Answer

a. $10|ab|$
b. $12p^6q^{10}$
c. $2x^{10}y^4$

? Exercise 8.1.24

Simplify:

a. $\sqrt{225m^2n^2}$
b. $\sqrt{169x^{10}y^{14}}$
c. $\sqrt[3]{27w^{36}z^{15}}$

Answer

a. $15|mn|$
b. $13\,|x^5y^7|$
c. $3w^{12}z^5$

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- Simplifying Variables Exponents with Roots using Absolute Values

Key Concepts

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of m'
  - If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

    Figure 8.1.1
  - The square root of m, $\sqrt{m}$, is a positive number whose square is m.
- **n^{th} Root of a Number**
  - If $b^n = a$, then b is an nth root of a.
  - The principal n^{th} root of a is written $\sqrt[n]{a}$.
  - n is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
  - When n is an even number and
    - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
    - $a < 0$, then $\sqrt[n]{a}$ is not a real number
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Simplifying Odd and Even Roots**

- For any integer $n \geq 2$,
  - when n is odd $\sqrt[n]{a^n} = a$
  - when n is even $\sqrt[n]{a^n} = |a|$
- We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Glossary

square of a number

If $n^2 = m$, then m is the square of n.

square root of a number

If $n^2 = m$, then n is a square root of m.

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8.1E: Exercises

Simplifying Expressions with Roots

In the following exercises, simplify.

1. a. $\sqrt{64}$ b. $-\sqrt{81}$

Answer

a. 8 b. -9

2. a. $\sqrt{169}$ b. $-\sqrt{100}$

3. a. $\sqrt{196}$ b. $-\sqrt{1}$

Answer

a. 14 b. -1

4. a. $\sqrt{144}$ b. $-\sqrt{121}$

5. a. $\sqrt{\frac{4}{9}}$ b. $-\sqrt{0.01}$

Answer

a. $\frac{2}{3}$ b. -0.1

6. a. $\sqrt{\frac{64}{121}}$ b. $-\sqrt{0.16}$

7. a. $\sqrt{-121}$ b. $-\sqrt{289}$

Answer

a. not a real number b. -17

8. a. $-\sqrt{400}$ b. $\sqrt{-36}$

9. a. $-\sqrt{225}$ b. $\sqrt{-9}$

Answer

a. -15 b. not a real number

10. a. $\sqrt{-49}$ b. $-\sqrt{256}$

11. a. $\sqrt[3]{216}$ b. $\sqrt[4]{256}$

Answer

a. 6 b. 4

12. a. $\sqrt[3]{27}$ b. $\sqrt[4]{16}$ c. $\sqrt[5]{243}$

13. a. $\sqrt[3]{512}$ b. $\sqrt[4]{81}$ c. $\sqrt[5]{1}$

Answer

a. 8 b. 3 b. 1

14. a. $\sqrt[3]{125}$ b. $\sqrt[4]{1296}$ c. $\sqrt[5]{1024}$

15. a. $\sqrt[3]{-8}$ b. $\sqrt[4]{-81}$ c. $\sqrt[5]{-32}$

Answer

a. -2 b. not a real number c. -2

16. a. $\sqrt[3]{-64}$ b. $\sqrt[4]{-16}$ c. $\sqrt[5]{-243}$

17. a. $\sqrt[3]{-125}$ b. $\sqrt[4]{-1296}$ c. $\sqrt[5]{-1024}$

Answer

a. -5 b. not a real number c. -4

18. a. $\sqrt[3]{-512}$ b. $\sqrt[4]{-81}$ c. $\sqrt[5]{-1}$

In the following exercises, estimate each root by giving the interval of two consecutive whole numbers in which the root lies.

19. a. $\sqrt{70}$ b. $\sqrt[3]{71}$

Answer

a. $8 < \sqrt{70} < 9$ b. $4 < \sqrt[3]{71} < 5$

20. a. $\sqrt{55}$ b. $\sqrt[3]{119}$

21. a. $\sqrt{200}$ b. $\sqrt[3]{137}$

Answer

a. $14 < \sqrt{200} < 15$ b. $5 < \sqrt[3]{137} < 6$

22. a. $\sqrt{172}$ b. $\sqrt[3]{200}$

In the following exercises, approximate each root and round to two decimal places.

23. a. $\sqrt{19}$ b. $\sqrt[3]{89}$ c. $\sqrt[4]{97}$

Answer

a. ≈ 4.36 b. ≈ 4.46 c. ≈ 3.14

24. a. $\sqrt{21}$ b. $\sqrt[3]{93}$ c. $\sqrt[4]{101}$

25. a. $\sqrt{53}$ b. $\sqrt[3]{147}$ c. $\sqrt[4]{452}$

Answer

a. ≈ 7.28 b. ≈ 5.28 c. ≈ 4.61

26. a. $\sqrt{47}$ b. $\sqrt[3]{163}$ c. $\sqrt[4]{527}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

27. a. $\sqrt[5]{u^5}$ b. $\sqrt[8]{v^8}$

Answer

a. u b. $|v|$

28. a. $\sqrt[3]{a^3}$ b. $\sqrt[9]{b^9}$

29. a. $\sqrt[4]{y^4}$ b. $\sqrt[7]{m^7}$

Answer

a. $|y|$ b. m

30. a. $\sqrt[8]{k^8}$ b. $\sqrt[6]{p^6}$

31. a. $\sqrt{x^6}$ b. $\sqrt{y^{16}}$

Answer

a. $|x^3|$ b. y^8

32. a. $\sqrt{a^{14}}$ b. $\sqrt{w^{24}}$

33. a. $\sqrt{x^{24}}$ b. $\sqrt{y^{22}}$

Answer

a. x^{12} b. $|y^{11}|$

34. a. $\sqrt{a^{12}}$ b. $\sqrt{b^{26}}$

35. a. $\sqrt[3]{x^9}$ b. $\sqrt[4]{y^{12}}$

Answer

a. x^3 b. $|y^3|$

36. a. $\sqrt[5]{a^{10}}$ b. $\sqrt[3]{b^{27}}$

37. a. $\sqrt[4]{m^8}$ b. $\sqrt[5]{n^{20}}$

Answer

a. m^2 b. n^4

38. a. $\sqrt[6]{r^{12}}$ b. $\sqrt[3]{s^{30}}$

39. a. $\sqrt{49x^2}$ b. $-\sqrt{81x^{18}}$

Answer

a. $7|x|$ b. $-9|x^9|$

40. a. $\sqrt{100y^2}$ b. $-\sqrt{100m^{32}}$

41. a. $\sqrt{121m^{20}}$ b. $-\sqrt{64a^2}$

Answer

a. $11m^{10}$ b. $-8|a|$

42. a. $\sqrt{81x^{36}}$ b. $-\sqrt{25x^2}$

43. a. $\sqrt[4]{16x^8}$ b. $\sqrt[6]{64y^{12}}$

Answer

a. $2x^2$ b. $2y^2$

44. a. $\sqrt[3]{-8c^9}$ b. $\sqrt[3]{125d^{15}}$

45. a. $\sqrt[3]{216a^6}$ b. $\sqrt[5]{32b^{20}}$

Answer

a. $6a^2$ b. $2b^4$

46. a. $\sqrt[7]{128r^{14}}$ b. $\sqrt[4]{81s^{24}}$

47. a. $\sqrt{144x^2y^2}$ b. $\sqrt{169w^8y^{10}}$ c. $\sqrt[3]{8a^{51}b^6}$

Answer

a. $12|xy|$ b. $13w^4\,|y^5|$ c. $2a^{17}b^2$

48. a. $\sqrt{196a^2b^2}$ b. $\sqrt{81p^{24}q^6}$ c. $\sqrt[3]{27p^{45}q^9}$

49. a. $\sqrt{121a^2b^2}$ b. $\sqrt{9c^8d^{12}}$ c. $\sqrt[3]{64x^{15}y^{66}}$

Answer

a. $11|ab|$ b. $3c^4d^6$ c. $4x^5y^{22}$

50. a. $\sqrt{225x^2y^2z^2}$ b. $\sqrt{36r^6s^{20}}$ c. $\sqrt[3]{125y^{18}z^{27}}$

Writing Exercises

51. Why is there no real number equal to $\sqrt{-64}$?

Answer

Since the square of any real number is positive, it's not possible for a real number to square to -64.

52. What is the difference between 9^2 and $\sqrt{9}$?

53. Explain what is meant by the nth root of a number.

Answer

If you raise this root to the nth power, it will give you back the original number (under the radical).

54. Explain the difference of finding the nth root of a number when the index is even compared to when the index is odd.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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8.2: Simplify Rational Exponents

8.4: Simplify Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
   If you missed this problem, review Example 1.28.
2. Simplify: $(4x^2y^5)^3$.
   If you missed this problem, review Example 5.18.
3. Simplify: 5^{-3}.
   If you missed this problem, review Example 5.14.

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n=a^{m\cdot n}$ when m and n are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number p such that $(8^p)^3=8$. We will use the Power Property of Exponents to find the value of p.

$$(8^p)^3=8$$

Multiple the exponents on the left.

$$8^{3p}=8$$

Write the exponent 1 on the right.

$$8^{3p}=8^1$$

Since the bases are the same, the exponents must be equal.

$$3p=1$$

Solve for p.

$$p=\frac{1}{3}$$

So $\left(8^{\frac{1}{3}}\right)^3=8$. But we know also $(\sqrt[3]{8})^3=8$. Then it must be that $8^{\frac{1}{3}}=\sqrt[3]{8}$.

This same logic can be used for any positive integer exponent n to show that $a^{\frac{1}{n}}=\sqrt[n]{a}$.

Definition 8.2.1: Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n\geq 2$, then

$$a^{\frac{1}{n}}=\sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Example 8.2.1

Write as a radical expression:

a. $x^{\frac{1}{2}}$
b. $y^{\frac{1}{3}}$
c. $z^{\frac{1}{4}}$

Solution:

We want to write each expression in the form $\sqrt[n]{a}$.

a.

$$x^{\frac{1}{2}}$$

The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.

$$\sqrt{x}$$

b.

$$y^{\frac{1}{3}}$$

The denominator of the exponent is 3, so the index is 3.

$$\sqrt[3]{y}$$

c.

$$z^{\frac{1}{4}}$$

The denominator of the exponent is \4, so the index is 4.

$$\sqrt[4]{z}$$

Exercise 8.2.1

Write as a radical expression:

a. $t^{\frac{1}{2}}$
b. $m^{\frac{1}{3}}$
c. $r^{\frac{1}{4}}$

Answer

a. $\sqrt{t}$
b. $\sqrt[3]{m}$
c. $\sqrt[4]{r}$

Exercise 8.2.2

Write as a radical expression:

a. $b^{\frac{1}{6}}$
b. $z^{\frac{1}{5}}$
c. $p^{\frac{1}{4}}$

Answer

a. $\sqrt[6]{b}$
b. $\sqrt[5]{z}$
c. $\sqrt[4]{p}$

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

Example 8.2.2

Write with a rational exponent:

a. $\sqrt{5y}$
b. $\sqrt[3]{4x}$
c. $3\sqrt[4]{5z}$

Solution:

We want to write each radical in the form $a^{\frac{1}{n}}$

a.

$$\sqrt{5y}$$

No index is shown, so it is 2.

The denominator of the exponent will be 2.

Put parentheses around the entire expression $5y$.

$$(5y)^{\frac{1}{2}}$$

b.

$$\sqrt[3]{4x}$$

The index is 3, so the denominator of the exponent is 3. Include parentheses $(4x)$.

$$(4x)^{\frac{1}{3}}$$

c.

$$3\sqrt[4]{5z}$$

The index is 4, so the denominator of the exponent is 4. Put parentheses only around the $5z$ since 3 is not under the radical sign.

$$3(5z)^{\frac{1}{4}}$$

Exercise 8.2.3

Write with a rational exponent:

a. $\sqrt{10m}$
b. $\sqrt[5]{3n}$
c. $3\sqrt[4]{6y}$

Answer

a. $(10m)^{\frac{1}{2}}$
b. $(3n)^{\frac{1}{5}}$
c. $3(6y)^{\frac{1}{4}}$

Exercise 8.2.4

Write with a rational exponent:

a. $\sqrt[7]{3k}$
b. $\sqrt[4]{5j}$
c. $8\sqrt[3]{2a}$

Answer

a. $(3k)^{\frac{1}{7}}$
b. $(5j)^{\frac{1}{4}}$
c. $8(2a)^{\frac{1}{3}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example 8.2.3

Simplify:

a. $25^{\frac{1}{2}}$
b. $64^{\frac{1}{3}}$
c. $256^{\frac{1}{4}}$

Solution:

a.

$$25^{\frac{1}{2}}$$

Rewrite as a square root.

$$\sqrt{25}$$

Simplify.

$$5$$

b.

$$64^{\frac{1}{3}}$$

Rewrite as a cube root.

$$\sqrt[3]{64}$$

Recognize 64 is a perfect cube.

$$\sqrt[3]{4^3}$$

Simplify.

$$4$$

c.

$$256^{\frac{1}{4}}$$

Rewrite as a fourth root.

$$\sqrt[4]{256}$$

Recognize 256 is a perfect fourth power.

$$\sqrt[4]{4^4}$$

Simplify.

4

Exercise 8.2.5

Simplify:

a. $36^{\frac{1}{2}}$
b. $8^{\frac{1}{3}}$
c. $16^{\frac{1}{4}}$

Answer

a. 6
b. 2
c. 2

Exercise 8.2.6

Simplify:

a. $100^{\frac{1}{2}}$
b. $27^{\frac{1}{3}}$
c. $81^{\frac{1}{4}}$

Answer

a. 10
b. 3
c. 3

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

Example 8.2.4

Simplify:

a. $(-16)^{\frac{1}{4}}$
b. $-16^{\frac{1}{4}}$
c. $(16)^{-\frac{1}{4}}$

Solution:

a.

$$(-16)^{\frac{1}{4}}$$

Rewrite as a fourth root.

$$\sqrt[4]{-16}$$

$$\sqrt[4]{(-2)^4}$$

Simplify.

No real solution

b.

$$-16^{\frac{1}{4}}$$

The exponent only applies to the 16. Rewrite as a fourth root.

$$-\sqrt[4]{16}$$

Rewrite 16 as 2^4

$$-\sqrt[4]{2^4}$$

Simplify.

$$-2$$

c.

$$(16)^{-\frac{1}{4}}$$

Rewrite using the property $a^{-n}=\frac{1}{a^n}$.

$$\frac{1}{(16)^{\frac{1}{4}}}$$

Rewrite as a fourth root.

$$\frac{1}{\sqrt[4]{16}}$$

Rewrite 16 as 2^4.

$$\frac{1}{\sqrt[4]{2^4}}$$

Simplify.

$$\frac{1}{2}$$

Exercise 8.2.7

Simplify:

a. $(-64)^{-\frac{1}{2}}$
b. $-64^{\frac{1}{2}}$
c. $(64)^{-\frac{1}{2}}$

Answer

a. No real solution
b. -8
c. $\frac{1}{8}$

Exercise 8.2.8

Simplify:

a. $(-256)^{\frac{1}{4}}$
b. $-256^{\frac{1}{4}}$
c. $(256)^{-\frac{1}{4}}$

Answer

a. No real solution
b. -4
c. $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and $(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following defintion.

Definition 8.2.2: Rational Exponent $a^{\frac{m}{n}}$

For any positive integers m and n,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

Example 8.2.5

Write with a rational exponent:

a. $\sqrt{y^3}$
b. $(\sqrt[3]{2x})^4$
c. $\sqrt{\left(\frac{3a}{4b}\right)^3}$

Solution:

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$

a.

Let

$$\sqrt[2]{y^3} \tag{8.2.1}$$

The numerator of the exponent is the exponent 3.
The denominator of the exponent is the index of the radical, 2.

$$y^{\frac{3}{2}} \tag{8.2.2}$$

b.

Let

$$(\sqrt[3]{2x})^4 \tag{8.2.3}$$

The numerator of the exponent is the exponent 4.
The denominator of the exponent is the index of the radical, 3.

$$(2x)^{\frac{4}{3}} \tag{8.2.4}$$

c.

Let

$$\sqrt{\left(\frac{3a}{4b}\right)^3} \tag{8.2.5}$$

The numerator of the exponent is the exponent 3.
The denominator of the exponent is the index of the radical, 2.

$$\left(\frac{3a}{4b}\right)^{\frac{3}{2}} \tag{8.2.6}$$

? Exercise 8.2.9

Write with a rational exponent:

a. $\sqrt{x^5}$
b. $(\sqrt[4]{3y})^3$
c. $\sqrt{\left(\frac{2m}{3n}\right)^5}$

Answer

a. $x^{\frac{5}{2}}$
b. $(3y)^{\frac{3}{4}}$
c. $\left(\frac{2m}{3n}\right)^{\frac{5}{2}}$

? Exercise 8.2.10

Write with a rational exponent:

a. $\sqrt[5]{a^2}$
b. $(\sqrt[3]{5ab})^5$
c. $\sqrt{\left(\frac{7xy}{z}\right)^3}$

Answer

a. $a^{\frac{2}{5}}$
b. $(5ab)^{\frac{5}{3}}$
c. $\left(\frac{7xy}{z}\right)^{\frac{3}{2}}$

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

✓ Example 8.2.6

Simplify:

a. $125^{\frac{2}{3}}$
b. $16^{-\frac{3}{2}}$
c. $32^{-\frac{2}{5}}$

Solution:

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

a.

$$125^{\frac{2}{3}}$$

The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.

$$(\sqrt[3]{125})^2$$

Simplify.

$$(5)^2$$

$$25$$

b. We will rewrite each expression first using $a^{-n} = \frac{1}{a^n}$ and then change to radical form.

$$16^{-\frac{3}{2}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{16^{\frac{3}{2}}}$$

Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.

$$\frac{1}{(\sqrt{16})^3}$$

Simplify.

$$\frac{1}{4^3}$$

$$\frac{1}{64}$$

c.

$$32^{-\frac{2}{5}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{32^{\frac{2}{5}}}$$

Change to radical form.

$$\frac{1}{(\sqrt[5]{32})^2}$$

Rewrite the radicand as a power.

$$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$$

Simplify.

$$\frac{1}{2^2}$$

$$\frac{1}{4}$$

? Exercise 8.2.11

Simplify:

a. $27^{\frac{2}{3}}$
b. $81^{-\frac{3}{2}}$
c. $16^{-\frac{3}{4}}$

Answer

a. 9
b. $\frac{1}{729}$
c. $\frac{1}{8}$

Exercise 8.2.12

Simplify:

a. $4^{\frac{3}{2}}$

b. $27^{-\frac{2}{3}}$

c. $625^{-\frac{3}{4}}$

Answer

a. 8

b. $\frac{1}{9}$

c. $\frac{1}{125}$

Example 8.2.7

Simplify:

a. $-25^{\frac{3}{2}}$

b. $-25^{-\frac{3}{2}}$

c. $(-25)^{\frac{3}{2}}$

Solution:

a.

$$-25^{\frac{3}{2}}$$

Rewrite in radical form.

$$-(\sqrt{25})^3$$

Simplify the radical.

$$-(5)^3$$

Simplify.

$$-125$$

b.

$$-25^{-\frac{3}{2}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$.

$$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$$

Rewrite in radical form.

$$-\left(\frac{1}{(\sqrt{25})^3}\right)$$

Simplify the radical.

$$-\left(\frac{1}{(5)^3}\right)$$

Simplify.

$$-\frac{1}{125}$$

c.

$$(-25)^{\frac{3}{2}}$$

Rewrite in radical form.

$$(\sqrt{-25})^3$$

There is no real number whose square root is -25.

Not a real number.

Exercise 8.2.13

Simplify:

a. $-16^{\frac{3}{2}}$
b. $-16^{-\frac{3}{2}}$
c. $(-16)^{-\frac{3}{2}}$

Answer

a. -64
b. $-\frac{1}{64}$
c. Not a real number

Exercise 8.2.14

Simplify:

a. $-81^{\frac{3}{2}}$
b. $-81^{-\frac{3}{2}}$
c. $(-81)^{-\frac{3}{2}}$

Answer

a. -729
b. $-\frac{1}{729}$
c. Not a real number

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponents here to have them for reference as we simplify expressions.

Properties of Exponents

If a and b are real numbers and m and n are rational numbers, then

Product Property

$$a^m \cdot a^n = a^{m+n}$$

Power Property

$$(a^m)^n = a^{m \cdot n}$$

Product to a Power

$$(ab)^m = a^m b^m$$

Quotient Property

$$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

Zero Exponent Definition

$$a^0 = 1, a \neq 0$$

Quotient to a Power Property

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Negative Exponent Property

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

We will apply these properties in the next example.

Example 8.2.8

Simplify:

a. $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$

b. $\left(z^9\right)^{\frac{2}{3}}$

c. $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

Solution

a. The Product Property tells us that when we multiple the same base, we add the exponents.

$$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$$

The bases are the same, so we add the exponents.

$$x^{\frac{1}{2}+\frac{5}{6}}$$

Add the fractions.

$$x^{\frac{8}{6}}$$

Simplify the exponent.

$$x^{\frac{4}{3}}$$

b. The Power Property tells us that when we raise a power to a power, we multiple the exponents.

$$\left(z^9\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$z^{9 \cdot \frac{2}{3}}$$

Simplify.

$$z^6$$

c. The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

$$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$$

To divide with the same base, we subtract the exponents.

$$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$$

Simplify.

$$\frac{1}{x^{\frac{4}{3}}}$$

Exercise 8.2.15

Simplify:

a. $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$

b. $\left(x^{6}\right)^{\frac{4}{3}}$

c. $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$

Answer

a. $x^{\frac{3}{2}}$

b. x^{8}

c. $\frac{1}{x}$

Exercise 8.2.16

Simplify:

a. $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$

b. $\left(m^{9}\right)^{\frac{2}{9}}$

c. $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$

Answer

a. $y^{\frac{11}{8}}$

b. m^{2}

c. $\frac{1}{d}$

Sometimes we need to use more than one property. In the next example, we will use both the **Product to a Power Property** and then the **Power Property**.

Example 8.2.9

Simplify:

a. $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$

b. $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$

Solution:

a.

$$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

First we use the Product to a Power Property.

$$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

Rewrite 27 as a power of 3.

$$\left(3^{3}\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$$

Simplify.

$$9u^{\frac{1}{3}}$$

b.

$$\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

First we use the Product to a Power Property.

$$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

To raise a power to a power, we multiply the exponents.

$$mn^{\frac{3}{4}}$$

? Exercise 8.2.17

Simplify:

a. $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$

b. $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$

Answer

a. $8x^{\frac{1}{5}}$

b. $x^{\frac{1}{2}}y^{\frac{1}{3}}$

? Exercise 8.2.18

Simplify:

a. $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$

b. $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$

Answer

a. $729n^{\frac{3}{5}}$

b. $a^2b^{\frac{2}{3}}$

We will use both the **Product Property** and the **Quotient Property** in the next example.

✓ Example 8.2.10

Simplify:

a. $\frac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$

b. $\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$

Solution:

a.

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

Use the Quotient Property, subtract the exponents.

$$x^{\frac{8}{4}}$$

Simplify.

$$x^2$$

b.

$$\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$$

Use the Quotient Property, subtract the exponents.

$$\left(\frac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$$

Simplify.

$$\left(\frac{16x^2}{y}\right)^{\frac{1}{2}}$$

Use the Product to a Power Property, multiply the exponents.

$$\frac{4x}{y^{\frac{1}{2}}}$$

Exercise 8.2.19

Simplify:

a. $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$

b. $\left(\frac{25m^{\frac{1}{6}}n^{\frac{11}{6}}}{m^{\frac{2}{3}}n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$

Answer

a. m^2

b. $\frac{5n}{m^{\frac{1}{4}}}$

Exercise 8.2.20

Simplify:

a. $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$

b. $\left(\frac{27x^{\frac{4}{5}}y^{\frac{1}{6}}}{x^{\frac{1}{5}}y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$

Answer

a. u^3

b. $3x^{\frac{1}{5}}y^{\frac{1}{3}}$

Access these online resources for additional instruction and practice with simplifying rational exponents.

- Review-Rational Exponents
- Using Laws of Exponents on Radicals: Properties of Rational Exponents

Key Concepts

- **Rational Exponent $a^{\frac{1}{n}}$**
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **Rational Exponent $a^{\frac{m}{n}}$**
  - For any positive integers m and n,
    $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If a, b are real numbers and m, n are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $(a^m)^n = a^{m \cdot n}$
    - **Product to a Power** $(ab)^m = a^m b^m$
    - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
    - **Zero Exponent Definition** $a^0 = 1, a \neq 0$
    - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
    - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, a \neq 0$

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8.2E: Exercises

Practice Makes Perfect

Exercise SET A: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

1. a. $x^{\frac{1}{2}}$ b. $y^{\frac{1}{3}}$ c. $z^{\frac{1}{4}}$
2. a. $r^{\frac{1}{2}}$ b. $s^{\frac{1}{3}}$ c. $t^{\frac{1}{4}}$
3. a. $u^{\frac{1}{5}}$ b. $v^{\frac{1}{9}}$ c. $w^{\frac{1}{20}}$
4. a. $g^{\frac{1}{7}}$ b. $h^{\frac{1}{5}}$ c. $j^{\frac{1}{25}}$

Answer

1. a. $\sqrt{x}$ b. $\sqrt[3]{y}$ c. $\sqrt[4]{z}$
3. a. $\sqrt[5]{u}$ b. $\sqrt[9]{v}$ c. $\sqrt[20]{w}$

Exercise Set B: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt[7]{x}$ b. $\sqrt[9]{y}$ c. $\sqrt[5]{f}$
2. a. $\sqrt[8]{4}$ b. $\sqrt[10]{s}$ c. $\sqrt[4]{t}$
3. a. $\sqrt[3]{7c}$ b. $\sqrt[7]{12d}$ c. $2\sqrt[4]{6b}$
4. a. $\sqrt[4]{5x}$ b. $\sqrt[8]{9y}$ c. $7\sqrt[5]{3z}$
5. a. $\sqrt{21p}$ b. $\sqrt[4]{8q}$ c. $4\sqrt[6]{36r}$
6. a. $\sqrt[3]{25a}$ b. $\sqrt{3b}$ c. $\sqrt[8]{40c}$

Answer

1. a. $x^{\frac{1}{7}}$ b. $y^{\frac{1}{9}}$ c. $f^{\frac{1}{5}}$
3. a. $(7c)^{\frac{1}{4}}$ b. $(12d)^{\frac{1}{7}}$ c. $2(6b)^{\frac{1}{4}}$
5. a. $(21p)^{\frac{1}{2}}$ b. $(8q)^{\frac{1}{4}}$ c. $4(36r)^{\frac{1}{6}}$

Exercise SET C: simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, simplify.

1. a. $81^{\frac{1}{2}}$ b. $125^{\frac{1}{3}}$ c. $64^{\frac{1}{2}}$
2. a. $625^{\frac{1}{4}}$ b. $243^{\frac{1}{5}}$ c. $32^{\frac{1}{5}}$
3. a. $16^{\frac{1}{4}}$ b. $16^{\frac{1}{2}}$ c. $625^{\frac{1}{4}}$
4. a. $64^{\frac{1}{3}}$ b. $32^{\frac{1}{5}}$ c. $81^{\frac{1}{4}}$
5. a. $(-216)^{\frac{1}{3}}$ b. $-216^{\frac{1}{3}}$ c. $(216)^{-\frac{1}{3}}$
6. a. $(-1000)^{\frac{1}{3}}$ b. $-1000^{\frac{1}{3}}$ c. $(1000)^{-\frac{1}{3}}$
7. a. $(-81)^{\frac{1}{4}}$ b. $-81^{\frac{1}{4}}$ c. $(81)^{-\frac{1}{4}}$
8. a. $(-49)^{\frac{1}{2}}$ b. $-49^{\frac{1}{2}}$ c. $(49)^{-\frac{1}{2}}$
9. a. $(-36)^{\frac{1}{2}}$ b. $-36^{\frac{1}{2}}$ c. $(36)^{-\frac{1}{2}}$
10. a. $(-16)^{\frac{1}{4}}$ b. $-16^{\frac{1}{4}}$ c. $16^{-\frac{1}{4}}$
11. a. $(-100)^{\frac{1}{2}}$ b. $-100^{\frac{1}{2}}$ c. $(100)^{-\frac{1}{2}}$

12. a. $(-32)^{\frac{1}{5}}$ b. $(243)^{-\frac{1}{5}}$ c. $-125^{\frac{1}{3}}$

Answer

1. a. 9 b. 5 c. 8
3. a. 2 b. 4 c. 5
5. a. -6 b. -6 c. $\frac{1}{6}$
7. a. not real b. -3 c. $\frac{1}{3}$
9. a. not real b. -6 c. $\frac{1}{6}$
11. a. not real b. -10 c. $\frac{1}{10}$

Exercise SET D: simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt{m^5}$ b. $(\sqrt[3]{3y})^7$ c. $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$
2. a. $\sqrt[4]{r^7}$ b. $(\sqrt[5]{2pq})^3$ c. $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$
3. a. $\sqrt[5]{u^2}$ b. $(\sqrt[3]{6x})^5$ c. $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$
4. a. $\sqrt[3]{a}$ b. $(\sqrt[4]{21v})^3$ c. $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$

Answer

1. a. $m^{\frac{5}{2}}$ b. $(3y)^{\frac{7}{3}}$ c. $\left(\frac{4x}{5y}\right)^{\frac{3}{5}}$
3. a. $u^{\frac{2}{5}}$ b. $(6x)^{\frac{5}{3}}$ c. $\left(\frac{18a}{5b}\right)^{\frac{7}{4}}$

Exercise SET E: simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, simplify.

1. a. $64^{\frac{5}{2}}$ b. $81^{\frac{-3}{2}}$ c. $(-27)^{\frac{2}{3}}$
2. a. $25^{\frac{3}{2}}$ b. $9^{-\frac{3}{2}}$ c. $(-64)^{\frac{2}{3}}$
3. a. $32^{\frac{2}{5}}$ b. $27^{-\frac{2}{3}}$ c. $(-25)^{\frac{1}{2}}$
4. a. $100^{\frac{3}{2}}$ b. $49^{-\frac{5}{2}}$ c. $(-100)^{\frac{3}{2}}$
5. a. $-9^{\frac{3}{2}}$ b. $-9^{-\frac{3}{2}}$ c. $(-9)^{\frac{3}{2}}$
6. a. $-64^{\frac{3}{2}}$ b. $-64^{-\frac{3}{2}}$ c. $(-64)^{\frac{3}{2}}$

Answer

1. a. $32,768$b. $\frac{1}{729}$ c. 9
3. a. 4 b. $\frac{1}{9}$ c. not real
5. a. -27 b. $-\frac{1}{27}$ c. not real

Exercise SET F: use the laws of exponents to simplify expressions with rational exponents

In the following exercises, simplify. Assume all variables are positive.

1. a. $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$ b. $\left(p^{12}\right)^{\frac{3}{4}}$ c. $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$
2. a. $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$ b. $\left(b^{15}\right)^{\frac{3}{5}}$ c. $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$
3. a. $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$ b. $\left(x^{12}\right)^{\frac{2}{3}}$ c. $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$
4. a. $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$ b. $\left(h^{6}\right)^{\frac{4}{3}}$ c. $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$
5. a. $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$ b. $\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)^{\frac{3}{2}}$
6. a. $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$ b. $\left(m^{\frac{4}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{4}}$
7. a. $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$ b. $\left(4p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$
8. a. $\left(625n^{\frac{8}{3}}\right)^{\frac{3}{4}}$ b. $\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$
9. a. $\dfrac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$ b. $\left(\dfrac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$
10. a. $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$ b. $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$
11. a. $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$ b. $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$
12. a. $\dfrac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$ b. $\left(\dfrac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

Answer

1. a. $c^{\frac{7}{8}}$ b. p^9 c. $\frac{1}{r}$
3. a. $y^{\frac{5}{4}}$ b. x^8 c. $\dfrac{1}{m}$
5. a. $81q^2$ b. $a^{\frac{1}{2}}b$
7. a. $8u^{\frac{1}{4}}$ b. $8p^{\frac{1}{2}}q^{\frac{3}{4}}$
9. a. $r^{\frac{7}{2}}$ b. $\dfrac{6s}{t}$
11. a. c^2 b. $\dfrac{2x}{3y}$

Exercise SET G: writing exercises

1. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.
2. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

Figure 8.3.4

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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8.3: Simplify Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Before you get started, take this readiness quiz.

1. Simplify: $\frac{x^9}{x^4}$.
   If you missed this problem, review Example 5.13.
2. Simplify: $\frac{y^3}{y^{11}}$.
   If you missed this problem, review Example 5.13.
3. Simplify: $\left(n^2\right)^6$.
   If you missed this problem, review Example 5.17.

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A **radical expression**, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Definition 8.3.1: Simplified Radical Expression

For real numbers a and m, and $n \geq 2$,

$\sqrt[n]{a}$ is considered simplified if a has no factors of m^n

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

Similarly, $\sqrt[3]{4}$ is simplified because there are no perfect cube factors in 4. But $\sqrt[3]{24}$ is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that

$$(ab)^n = a^n b^n. \tag{8.3.1}$$

The corresponding of **Product Property of Roots** says that

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}. \tag{8.3.2}$$

Definition 8.3.2: Product Property of nth Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

Example 8.3.1: Simplify square roots using the product property of roots

Simplify: $\sqrt{98}$.

Solution:

Step 1: Find the largest factor in the radicand that is a perfect power of the index.	We see that 49 is the largest factor of 98 that has a power of 2.	$\sqrt{98}$
Rewrite the radicand as a product of two factors, using that factor.	In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{49 \cdot 2}$
Step 2: Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Step 3: Simplify the root of the perfect power.		$7\sqrt{2}$

Try It 8.3.1

Simplify: $\sqrt{48}$

Answer

$4\sqrt{3}$

Try It 8.3.2

Simplify: $\sqrt{45}$.

Answer

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

Simplify a Radical Expression Using the Product Property

1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

Example 8.3.2

Simplify:

a. $\sqrt{500}$
b. $\sqrt[3]{16}$

c. $\sqrt[4]{243}$

Solution:

a.

$$\sqrt{500}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{100 \cdot 5}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{100} \cdot \sqrt{5}$$

Simplify.

$$10\sqrt{5}$$

b.

$$\sqrt[3]{16}$$

Rewrite the radicand as a product using the greatest perfect cube factor. $2^3 = 8$

$$\sqrt[3]{8 \cdot 2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8} \cdot \sqrt[3]{2}$$

Simplify.

$$2\sqrt[3]{2}$$

c.

$$\sqrt[4]{243}$$

Rewrite the radicand as a product using the greatest perfect fourth power factor. $3^4 = 81$

$$\sqrt[4]{81 \cdot 3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{81} \cdot \sqrt[4]{3}$$

Simplify.

$$3\sqrt[4]{3}$$

? Try It 8.3.3

Simplify: a. $\sqrt{288}$ b. $\sqrt[3]{81}$ c. $\sqrt[4]{64}$

Answer

a. $12\sqrt{2}$ b. $3\sqrt[3]{3}$ c. $2\sqrt[4]{4}$

? Try It 8.3.4

Simplify: a. $\sqrt{432}$ b. $\sqrt[3]{625}$ c. $\sqrt[4]{729}$

Answer

a. $12\sqrt{3}$ b. $5\sqrt[3]{5}$ c. $3\sqrt[4]{9}$

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Example 8.3.3

Simplify:

a. $\sqrt{x^3}$
b. $\sqrt[3]{x^4}$
c. $\sqrt[4]{x^7}$

Solution:

a.

$$\sqrt{x^3}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{x^2 \cdot x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{x^2} \cdot \sqrt{x}$$

Simplify.

$$|x|\sqrt{x}$$

b.

$$\sqrt[3]{x^4}$$

Rewrite the radicand as a product using the largest perfect cube factor.

$$\sqrt[3]{x^3 \cdot x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$$

Simplify.

$$x\sqrt[3]{x}$$

c.

$$\sqrt[4]{x^7}$$

Rewrite the radicand as a product using the greatest perfect fourth power factor.

$$\sqrt[4]{x^4 \cdot x^3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$$

Simplify.

$$|x|\sqrt[4]{x^3}$$

Try It 8.3.5

Simplify: a. $\sqrt{b^5}$ b. $\sqrt[4]{y^6}$ c. $\sqrt[3]{z^5}$

Answer

a. $b^2\sqrt{b}$ b. $|y|\sqrt[4]{y^2}$ c. $z\sqrt[3]{z^2}$

Try It 8.3.6

Simplify: a. $\sqrt{p^9}$ b. $\sqrt[5]{y^8}$ c. $\sqrt[6]{q^{13}}$

Answer

a. $p^4\sqrt{p}$ b. $p\sqrt[5]{p^3}$ c. $q^2\sqrt[6]{q}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

Example 8.3.4

Simplify:

a. $\sqrt{72n^7}$
b. $\sqrt[3]{24x^7}$
c. $\sqrt[4]{80y^{14}}$

Solution:

a.

$$\sqrt{72n^7}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{36n^6 \cdot 2n}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{36n^6} \cdot \sqrt{2n}$$

Simplify.

$$6\left|n^3\right|\sqrt{2n}$$

b.

$$\sqrt[3]{24x^7}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{8x^6 \cdot 3x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$$

Rewrite the first radicand as $\left(2x^2\right)^3$.

$$\sqrt[3]{\left(2x^2\right)^3} \cdot \sqrt[3]{3x}$$

Simplify.

$$2x^2\sqrt[3]{3x}$$

c.

$$\sqrt[4]{80y^{14}}$$

Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16y^{12} \cdot 5y^2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{16y^{12}} \cdot \sqrt[4]{5y^2}$$

Rewrite the first radicand as $\left(2y^3\right)^4$.

$$\sqrt[4]{(2y^3)^4}\cdot\sqrt[4]{5y^2}$$

Simplify.

$$2\left|y^3\right|\sqrt[4]{5y^2}$$

Try It 8.3.7

Simplify: a. $\sqrt{32y^5}$ b. $\sqrt[3]{54p^{10}}$ c. $\sqrt[4]{64q^{10}}$

Answer

a. $4y^2\sqrt{2y}$ b. $3p^3\sqrt[3]{2p}$ c. $2q^2\sqrt[4]{4q^2}$

Try It 8.3.8

Simplify: a. $\sqrt{75a^9}$ b. $\sqrt[3]{128m^{11}}$ c. $\sqrt[4]{162n^7}$

Answer

a. $5a^4\sqrt{3a}$ b. $4m^3\sqrt[3]{2m^2}$ c. $3|n|\sqrt[4]{2n^3}$

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

Example 8.3.5

Simplify:

a. $\sqrt{63u^3v^5}$
b. $\sqrt[3]{40x^4y^5}$
c. $\sqrt[4]{48x^4y^7}$

Solution:

a.

$$\sqrt{63u^3v^5}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9u^2v^4\cdot 7uv}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{9u^2v^4}\cdot\sqrt{7uv}$$

Rewrite the first radicand as $\left(3uv^2\right)^2$.

$$\sqrt{(3uv^2)^2}\cdot\sqrt{7uv}$$

Simplify.

$$3|u|v^2\sqrt{7uv}$$

b.

$$\sqrt[3]{40x^4y^5}$$

Rewrite the radicand as a product using the largest perfect cube factor.

$$\sqrt[3]{8x^3y^3\cdot 5xy^2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{5xy^2}$$

Rewrite the first radicand as $(2xy)^3$.

$$\sqrt[3]{(2xy)^3} \cdot \sqrt[3]{5xy^2}$$

Simplify.

$$2xy\sqrt[3]{5xy^2}$$

c.

$$\sqrt[4]{48x^4y^7}$$

Rewrite the radicand as a product using the largest perfect fourth power factor.

$$\sqrt[4]{16x^4y^4 \cdot 3y^3}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{16x^4y^4} \cdot \sqrt[4]{3y^3}$$

Rewrite the first radicand as $(2xy)^4$.

$$\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$$

Simplify.

$$2|xy|\sqrt[4]{3y^3}$$

Try It 8.3.9

Simplify:

a. $\sqrt{98a^7b^5}$
b. $\sqrt[3]{56x^5y^4}$
c. $\sqrt[4]{32x^5y^8}$

Answer

a. $7|a^3|b^2\sqrt{2ab}$
b. $2xy\sqrt[3]{7x^2y}$
c. $2|x|y^2\sqrt[4]{2x}$

Try It 8.3.10

Simplify:

a. $\sqrt{180m^9n^{11}}$
b. $\sqrt[3]{72x^6y^5}$
c. $\sqrt[4]{80x^7y^4}$

Answer

a. $6m^4|n^5|\sqrt{5mn}$
b. $2x^2y\sqrt[3]{9y^2}$
c. $2|xy|\sqrt[4]{5x^3}$

Example 8.3.6

Simplify:

a. $\sqrt[3]{-27}$
b. $\sqrt[4]{-16}$

Solution:

a.

$$\sqrt[3]{-27}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3}$$

Take the cube root.

$$-3$$

b.

$$\sqrt[4]{-16}$$

There is no real number n where $n^4 = -16$.

Not a real number

Try It 8.3.11

Simplify:

a. $\sqrt[3]{-64}$
b. $\sqrt[4]{-81}$

Answer

a. -4
b. no real number

Try It 8.3.12

Simplify:

a. $\sqrt[3]{-625}$
b. $\sqrt[4]{-324}$

Answer

a. $-5\sqrt[3]{5}$
b. no real number

We have seen how to use the **order of operations** to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

Example 8.3.7

Simplify:

a. $3+\sqrt{32}$
b. $\dfrac{4-\sqrt{48}}{2}$

Solution:

a.

$$3+\sqrt{32}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$3+\sqrt{16\cdot 2}$$

Rewrite the radical as the product of two radicals.

$$3+\sqrt{16}\cdot\sqrt{2}$$

Simplify.

$$3+4\sqrt{2}$$

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

b.

$$\frac{4-\sqrt{48}}{2}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\frac{4-\sqrt{16\cdot 3}}{2}$$

Rewrite the radical as the product of two radicals.

$$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$$

Simplify.

$$\frac{4-4\sqrt{3}}{2}$$

Factor the common factor from the numerator.

$$\frac{4(1-\sqrt{3})}{2}$$

Remove the common factor, 2, from the numerator and denominator.

$$\frac{\cancel{2}\cdot 2(1-\sqrt{3})}{\cancel{2}}$$

Simplify.

$$2(1-\sqrt{3})$$

? Try It 8.3.13

Simplify:

a. $5+\sqrt{75}$

b. $\dfrac{10-\sqrt{75}}{5}$

Answer

a. $5+5\sqrt{3}$

b. $2-\sqrt{3}$

? Try It 8.3.14

Simplify:

a. $2+\sqrt{98}$

b. $\dfrac{6-\sqrt{45}}{3}$

Answer

a. $2+7\sqrt{2}$
b. $2-\sqrt{5}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

Example 8.3.8

Simplify:

a. $\sqrt{\dfrac{45}{80}}$
b. $\sqrt[3]{\dfrac{16}{54}}$
c. $\sqrt[4]{\dfrac{5}{80}}$

Solution:

a.

$$\sqrt{\frac{45}{80}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt{\frac{5\cdot 9}{5\cdot 16}}$$

Simplify the fraction by removing common factors.

$$\sqrt{\frac{9}{16}}$$

Simplify. Note $\left(\dfrac{3}{4}\right)^2=\dfrac{9}{16}$.

$$\frac{3}{4}$$

b.

$$\sqrt[3]{\frac{16}{54}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt[3]{\frac{2\cdot 8}{2\cdot 27}}$$

Simplify the fraction by removing common factors.

$$\sqrt[3]{\frac{8}{27}}$$

Simplify. Note $\left(\dfrac{2}{3}\right)^3=\dfrac{8}{27}$.

$$\frac{2}{3}$$

c.

$$\sqrt[4]{\frac{5}{80}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt[4]{\frac{5 \cdot 1}{5 \cdot 16}}$$

Simplify the fraction by removing common factors.

$$\sqrt[4]{\frac{1}{16}}$$

Simplify. Note $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

$$\frac{1}{2}$$

? Try It 8.3.15

Simplify:

a. $\sqrt{\frac{75}{48}}$

b. $\sqrt[3]{\frac{54}{250}}$

c. $\sqrt[4]{\frac{32}{162}}$

Answer

a. $\frac{5}{4}$

b. $\frac{3}{5}$

c. $\frac{2}{3}$

? Try It 8.3.16

Simplify:

a. $\sqrt{\frac{98}{162}}$

b. $\sqrt[3]{\frac{24}{375}}$

c. $\sqrt[4]{\frac{4}{324}}$

Answer

a. $\frac{7}{9}$

b. $\frac{2}{5}$

c. $\frac{1}{3}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the **Quotient Property** to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Example 8.3.9

Simplify:

a. $\sqrt{\frac{m^6}{m^4}}$

b. $\sqrt[3]{\frac{a^8}{a^5}}$

c. $\sqrt[4]{\frac{a^{10}}{a^2}}$

Solution:

a.

$$\sqrt{\frac{m^6}{m^4}}$$

Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.

$$\sqrt{m^2}$$

Simplify.

$$|m|$$

b.

$$\sqrt[3]{\frac{a^8}{a^5}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical first.

$$\sqrt[3]{a^3}$$

Simplify.

$$a$$

c.

$$\sqrt[4]{\frac{a^{10}}{a^2}}$$

Use the Quotient Property of exponents to simplify the fraction under the radical first.

$$\sqrt[4]{a^8}$$

Rewrite the radicand using perfect fourth power factors.

$$\sqrt[4]{(a^2)^4}$$

Simplify.

$$a^2$$

Try It 8.3.17

Simplify:

a. $\sqrt{\frac{a^8}{a^6}}$
b. $\sqrt[4]{\frac{x^7}{x^3}}$
c. $\sqrt[4]{\frac{y^{17}}{y^5}}$

Answer

a. $|a|$
b. $|x|$
c. y^3

Try It 8.3.18

Simplify:

a. $\sqrt{\frac{x^{14}}{x^{10}}}$
b. $\sqrt[3]{\frac{m^{13}}{m^7}}$
c. $\sqrt[5]{\frac{n^{12}}{n^2}}$

Answer

a. x^2
b. m^2
c. n^2

Remember the **Quotient to a Power Property**? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Definition 8.3.3

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example 8.3.10 how to simplify the quotient of radical expressions

Simplify: $\sqrt{\frac{27m^3}{196}}$

Solution:

Step 1: Simplify the fraction in the radicand, if possible.

$\frac{27m^3}{196}$ cannot be simplified.

$$\sqrt{\frac{27m^3}{196}}$$

Step 2: Use the Quotient Property to rewrite the radical as the quotient of two radicals.

We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.

$$\frac{\sqrt{27m^3}}{\sqrt{196}}$$

Step 3: Simplify the radicals in the numerator and the denominator.

$9m^2$ and 196 are perfect squares.

$$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$$

$$\frac{3m\sqrt{3m}}{14}$$

? Try It 8.3.19

Simplify: $\sqrt{\frac{24p^3}{49}}$.

Answer

$$\frac{2|p|\sqrt{6p}}{7}$$

? Try It 8.3.20

Simplify: $\sqrt{\frac{48x^5}{100}}$.

Answer

$$\frac{2x^2\sqrt{3x}}{5}$$

Simplify a Square Root Using the Quotient Property

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

✓ Example 8.3.11

Simplify:

a. $\sqrt{\frac{45x^5}{y^4}}$

b. $\sqrt[3]{\frac{24x^7}{y^3}}$

c. $\sqrt[4]{\frac{48x^{10}}{y^8}}$

Solution:

a.

$$\sqrt{\frac{45x^5}{y^4}}$$

We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.

$$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$$

Simplify.

$$\frac{3x^2\sqrt{5x}}{y^2}$$

b.

$$\sqrt[3]{\frac{24x^7}{y^3}}$$

The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6\cdot 3x}}{\sqrt[3]{y^3}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3}\cdot\sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

Simplify.

$$\frac{2x^2\sqrt[3]{3x}}{y}$$

c.

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified.

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8\cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4}\cdot\sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

Try It 8.3.21

Simplify:

a. $\sqrt{\frac{80m^3}{n^6}}$

b. $\sqrt[3]{\frac{108c^{10}}{d^6}}$

c. $\sqrt[4]{\frac{80x^{10}}{y^4}}$

Answer

a. $\frac{4|m|\sqrt{5m}}{|n^3|}$

b. $\frac{3c^3\sqrt[3]{4c}}{d^2}$

c. $\frac{2x^2\sqrt[4]{5x^2}}{|y|}$

Try It 8.3.22

Simplify:

a. $\sqrt{\frac{54u^7}{v^8}}$

b. $\sqrt[3]{\frac{40r^3}{s^6}}$

c. $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$

Answer

a. $\frac{3u^3\sqrt{6u}}{v^4}$

b. $\frac{2r\sqrt[3]{5}}{s^2}$

c. $\frac{3\left|m^3\right|\sqrt[4]{2m^2}}{|n^3|}$

Be sure to simplify the fraction in the radicand first, if possible.

Example 8.3.12

Simplify:

a. $\sqrt{\frac{18p^5q^7}{32pq^2}}$

b. $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$

c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

Solution:

a.

$$\sqrt{\frac{18p^5q^7}{32pq^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt{\frac{9p^4q^5}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$$

Simplify.

$$\frac{3p^2q^2\sqrt{q}}{4}$$

b.

$$\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt[3]{\frac{8x^3y^5}{27}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y^2}}{\sqrt[3]{27}}$$

Simplify.

$$\frac{2xy\sqrt[3]{y^2}}{3}$$

c.

$$\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt[4]{\frac{a^5b^4}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$$

Simplify.

$$\frac{|ab|\sqrt[4]{a}}{2}$$

? Try It 8.3.23

Simplify:

a. $\sqrt{\frac{50x^5y^3}{72x^4y}}$

b. $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$

c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

Answer

a. $\frac{5|y|\sqrt{x}}{6}$

b. $\frac{2xy\sqrt[3]{y^2}}{3}$

c. $\frac{|ab|\sqrt[4]{a}}{2}$

? Try It 8.3.24

Simplify:

a. $\sqrt{\frac{48m^7n^2}{100m^5n^8}}$

b. $\sqrt[3]{\frac{54x^7y^5}{250x^2y^2}}$

c. $\sqrt[4]{\frac{32a^9b^7}{162a^3b^3}}$

Answer

a. $\frac{2|m|\sqrt{3}}{5\,|n^3|}$

b. $\frac{3xy\sqrt[3]{x^2}}{5}$

c. $\frac{2|ab|\sqrt[4]{a^2}}{3}$

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the **Quotient Property** again, to combine them into one radical. We will then look to see if we can simplify the expression.

Example 8.3.13

Simplify:

a. $\dfrac{\sqrt{48a^7}}{\sqrt{3a}}$

b. $\dfrac{\sqrt[3]{-108}}{\sqrt[3]{2}}$

c. $\dfrac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$

Solution:

a.

$$\frac{\sqrt{48a^7}}{\sqrt{3a}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt{\frac{48a^7}{3a}}$$

Simplify the fraction under the radical.

$$\sqrt{16a^6}$$

Simplify.

$$4\left|a^3\right|$$

b.

$$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt[3]{\frac{-108}{2}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{-54}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3\cdot 2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{(-3)^3}\cdot\sqrt[3]{2}$$

Simplify.

$$-3\sqrt[3]{2}$$

c.

$$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt[4]{\frac{96x^7}{3x^2}}$$

Simplify the fraction under the radical.

$$\sqrt[4]{32x^5}$$

Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$$

Simplify.

$$2|x|\sqrt[4]{2x}$$

? Try It 8.3.25

Simplify:

a. $\dfrac{\sqrt{98z^5}}{\sqrt{2z}}$

b. $\dfrac{\sqrt[3]{-500}}{\sqrt[3]{2}}$

c. $\dfrac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$

Answer

a. $7z^2$

b. $-5\sqrt[3]{2}$

c. $3|m|\sqrt[4]{2m^2}$

? Try It 8.3.26

Simplify:

a. $\dfrac{\sqrt{128m^9}}{\sqrt{2m}}$

b. $\dfrac{\sqrt[3]{-192}}{\sqrt[3]{3}}$

c. $\dfrac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$

Answer

a. $8m^4$

b. -4

c. $3|n|\sqrt[4]{2}$

Access these online resources for additional instruction and practice with simplifying radical expressions.

- Simplifying Square Root and Cube Root with Variables
- Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents
- Simplifying Cube Roots

Key Concepts

- **Simplified Radical Expression**
  - For real numbers a, m and $n \geq 2$
    $\sqrt[n]{a}$ is considered simplified if a has no factors of m^n
- **Product Property of nth Roots**

- For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
  $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

- **How to simplify a radical expression using the Product Property**
  1. Find the largest factor in the radicand that is a perfect power of the index.
     Rewrite the radicand as a product of two factors, using that factor.
  2. Use the product rule to rewrite the radical as the product of two radicals.
  3. Simplify the root of the perfect power.

- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

- **How to simplify a radical expression using the Quotient Property.**
  1. Simplify the fraction in the radicand, if possible.
  2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
  3. Simplify the radicals in the numerator and the denominator.

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8.3E: Exercises

Practice Makes Perfect

Exercise SET A: use the product property to simplify radical expressions

In the following exercises, use the Product Property to simplify radical expressions.

1. $\sqrt{27}$
2. $\sqrt{80}$
3. $\sqrt{125}$
4. $\sqrt{96}$
5. $\sqrt{147}$
6. $\sqrt{450}$
7. $\sqrt{800}$
8. $\sqrt{675}$
9. a. $\sqrt[4]{32}$
   b. $\sqrt[5]{64}$
10. a. $\sqrt[3]{625}$
    b. $\sqrt[6]{128}$
11. a. $\sqrt[5]{64}$
    b. $\sqrt[3]{256}$
12. a. $\sqrt[4]{3125}$
    b. $\sqrt[3]{81}$

Answer

1. $3\sqrt{3}$
3. $5\sqrt{5}$
5. $7\sqrt{3}$
7. $20\sqrt{2}$
9. a. $2\sqrt[4]{2}$
   b. $2\sqrt[5]{2}$
11. a. $2\sqrt[5]{2}$
    b. $4\sqrt[3]{4}$

Exercise SET B: use the product property to simplify radical expressions

In the following exercises, simplify using absolute value signs as needed.

1. a. $\sqrt{y^{11}}$
   b. $\sqrt[3]{r^5}$
   c. $\sqrt[4]{s^{10}}$
2. a. $\sqrt{m^{13}}$
   b. $\sqrt[5]{u^7}$
   c. $\sqrt[6]{v^{11}}$
3. a. $\sqrt{n^{21}}$
   b. $\sqrt[3]{q^8}$

c. $\sqrt[8]{n^{10}}$

4. a. $\sqrt{r^{25}}$
   b. $\sqrt[5]{p^8}$
   c. $\sqrt[4]{m^5}$
5. a. $\sqrt{125r^{13}}$
   b. $\sqrt[3]{108x^5}$
   c. $\sqrt[4]{48y^6}$
6. a. $\sqrt{80s^{15}}$
   b. $\sqrt[5]{96a^7}$
   c. $\sqrt[6]{128b^7}$
7. a. $\sqrt{242m^{23}}$
   b. $\sqrt[4]{405m10}$
   c. $\sqrt[5]{160n^8}$
8. a. $\sqrt{175n^{13}}$
   b. $\sqrt[5]{512p^5}$
   c. $\sqrt[4]{324q^7}$
9. a. $\sqrt{147m^7n^{11}}$
   b. $\sqrt[3]{48x^6y^7}$
   c. $\sqrt[4]{32x^5y^4}$
10. a. $\sqrt{96r^3s^3}$
    b. $\sqrt[3]{80x^7y^6}$
    c. $\sqrt[4]{80x^8y^9}$
11. a. $\sqrt{192q^3r^7}$
    b. $\sqrt[3]{54m^9n^{10}}$
    c. $\sqrt[4]{81a^9b^8}$
12. a. $\sqrt{150m^9n^3}$
    b. $\sqrt[3]{81p^7q^8}$
    c. $\sqrt[4]{162c^{11}d^{12}}$
13. a. $\sqrt[3]{-864}$
    b. $\sqrt[4]{-256}$
14. a. $\sqrt[5]{-486}$
    b. $\sqrt[6]{-64}$
15. a. $\sqrt[5]{-32}$
    b. $\sqrt[8]{-1}$
16. a. $\sqrt[3]{-8}$
    b. $\sqrt[4]{-16}$
17. a. $5+\sqrt{12}$
    b. $\dfrac{10-\sqrt{24}}{2}$
18. a. $8+\sqrt{96}$
    b. $\dfrac{8-\sqrt{80}}{4}$
19. a. $1+\sqrt{45}$
    b. $\dfrac{3+\sqrt{90}}{3}$
20. a. $3+\sqrt{125}$
    b. $\dfrac{15+\sqrt{75}}{5}$

Answer

1.

a. $|y^5|\sqrt{y}$
b. $r\sqrt[3]{r^2}$
c. $s^2\sqrt[4]{s^2}$

3.

a. $n^{10}\sqrt{n}$
b. $q^2\sqrt[3]{q^2}$
c. $|n|\sqrt[8]{n^2}$

5.

a. $5r^6\sqrt{5r}$
b. $3x\sqrt[3]{4x^2}$
c. $2|y|\sqrt[4]{3y^2}$

7.

a. $11\,|m^{11}|\sqrt{2m}$
b. $3m^2\sqrt[4]{5m^2}$
c. $2n\sqrt[5]{5n^3}$

9.

a. $7\,|m^3n^5|\sqrt{3mn}$
b. $2x^2y^2\sqrt[3]{6y}$
c. $2|xy|\sqrt[4]{2x}$

11.

a. $8\,|qr^3|\sqrt{3qr}$
b. $3m^3n^3\sqrt[3]{2n}$
c. $3a^2b^2\sqrt[4]{a}$

13.

a. $-6\sqrt[3]{4}$
b. not real

15.

a. -2
b. not real

17.

a. $5+2\sqrt{3}$
b. $5-\sqrt{6}$

19.

a. $1+3\sqrt{5}$
b. $1+\sqrt{10}$

Exercise Set C: use the quotient property to simplify radical expressions

In the following exercises, use the Quotient Property to simplify square roots.

1. a. $\sqrt{\frac{45}{80}}$

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b. $\sqrt{\frac{8}{27}}$

c. $\sqrt[4]{\frac{1}{81}}$

2. a. $\sqrt{\frac{72}{98}}$

b. $\sqrt{\frac{24}{81}}$

c. $\sqrt[4]{\frac{6}{96}}$

3. a. $\sqrt{\frac{100}{36}}$

b. $\sqrt{\frac{81}{375}}$

c. $\sqrt[4]{\frac{1}{256}}$

4. a. $\sqrt{\frac{121}{16}}$

b. $\sqrt{\frac{16}{250}}$

c. $\sqrt[4]{\frac{32}{162}}$

5. a. $\sqrt{\frac{x^{10}}{x^6}}$

b. $\sqrt{\frac{p^{11}}{p^2}}$

c. $\sqrt[4]{\frac{q^{17}}{q^{13}}}$

6. a. $\sqrt{\frac{p^{20}}{p^{10}}}$

b. $\sqrt{\frac{d^{12}}{d^7}}$

c. $\sqrt{\frac{m^{12}}{m^4}}$

7. a. $\sqrt{\frac{y^4}{y^8}}$

b. $\sqrt{\frac{u^{21}}{u^{11}}}$

c. $\sqrt{\frac{v^{30}}{v^{12}}}$

8. a. $\sqrt{\frac{q^8}{q^{14}}}$

b. $\sqrt{\frac{r^{14}}{r^5}}$

c. $\sqrt{\frac{c^{21}}{c^9}}$

9. $\sqrt{\frac{96x^7}{121}}$

10. $\sqrt{\frac{108y^4}{49}}$
11. $\sqrt{\frac{300m^5}{64}}$
12. $\sqrt{\frac{125n^7}{169}}$
13. $\sqrt{\frac{98r^5}{100}}$
14. $\sqrt{\frac{180s^{10}}{144}}$
15. $\sqrt{\frac{28q^6}{225}}$
16. $\sqrt{\frac{150r^3}{256}}$
17. a. $\sqrt{\frac{75r^9}{s^8}}$
    b. $\sqrt[3]{\frac{54a^8}{b^3}}$
    c. $\sqrt[4]{\frac{64c^5}{d^4}}$
18. a. $\sqrt{\frac{72x^5}{y^6}}$
    b. $\sqrt[5]{\frac{96r^{11}}{s^5}}$
    c. $\sqrt[6]{\frac{128u^7}{v^{12}}}$
19. a. $\sqrt{\frac{28p^7}{q^2}}$
    b. $\sqrt[3]{\frac{81s^8}{t^3}}$
    c. $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$
20. a. $\sqrt{\frac{45r^3}{s^{10}}}$
    b. $\sqrt[3]{\frac{625u^{10}}{v^3}}$
    c. $\sqrt[4]{\frac{729c^{21}}{d^8}}$
21. a. $\sqrt{\frac{32x^5y^3}{18x^3y}}$
    b. $\sqrt[3]{\frac{5x^6y^9}{40x^5y^3}}$
    c. $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$
22. a. $\sqrt{\frac{75r^6s^8}{48rs^4}}$
    b. $\sqrt[3]{\frac{24x^8y^4}{81x^2y}}$

c. $\sqrt[4]{\frac{32m^9n^2}{162mn^2}}$

23. a. $\sqrt{\frac{27p^2q}{108p^4q^3}}$

b. $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$

c. $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

24. a. $\sqrt{\frac{50r^5s^2}{128r^2s^6}}$

b. $\sqrt[3]{\frac{24m^9n^7}{375m^4n}}$

c. $\sqrt[4]{\frac{81m^2n^8}{256m^1n^2}}$

25. a. $\frac{\sqrt{45p^9}}{\sqrt{5q^2}}$

b. $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$

c. $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

26. a. $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

b. $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

c. $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

27. a. $\frac{\sqrt{50m^7}}{\sqrt{2m}}$

b. $\sqrt[3]{\frac{1250}{2}}$

c. $\sqrt[4]{\frac{486y^9}{2y^3}}$

28. a. $\frac{\sqrt{72n^{11}}}{\sqrt{2n}}$

b. $\sqrt[3]{\frac{162}{6}}$

c. $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

Answer

1.

a. $\frac{3}{4}$

b. $\frac{2}{3}$

c. $\frac{1}{3}$

3.

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a. $\frac{5}{3}$
b. $\frac{3}{5}$
c. $\frac{1}{4}$

5.

a. x^2
b. p^3
c. $|q|$

7.

a. $\frac{1}{y^2}$
b. u^2
c. $|v^3|$

9. $\frac{4\left|x^3\right|\sqrt{6x}}{11}$

11. $\frac{10m^2\sqrt{3m}}{8}$

13. $\frac{7r^2\sqrt{2r}}{10}$

15. $\frac{2\left|q^3\right|\sqrt{7}}{15}$

17.

a. $\frac{5r^4\sqrt{3r}}{s^4}$
b. $\frac{3a^2\sqrt[3]{2a^2}}{|b|}$
c. $\frac{2|c|\sqrt[4]{4c}}{|d|}$

19.

a. $\frac{2\left|p^3\right|\sqrt{7p}}{|q|}$
b. $\frac{3s^2\sqrt[3]{3s^2}}{t}$
c. $\frac{2\left|p^3\right|\sqrt[4]{4p^3}}{|q^3|}$

21.

a. $\frac{4|xy|}{3}$
b. $\frac{y^2\sqrt[3]{x}}{2}$
c. $\frac{|ab|\sqrt[4]{a}}{4}$

23.

a. $\frac{1}{2|pq|}$

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b. $\frac{2cd\sqrt[5]{2d^2}}{5}$

c. $\frac{|mn|\sqrt[6]{2}}{2}$

25.

a. $\frac{3p^4\sqrt{p}}{|q|}$

b. $2\sqrt[4]{2}$

c. $2x\sqrt[5]{2x}$

27.

a. $5\left|m^3\right|$

b. $5\sqrt[3]{5}$

c. $3|y|\sqrt[4]{3y^2}$

Exercise SET D: writing exercises

1. Explain why $\sqrt{x^4}=x^2$. Then explain why $\sqrt{x^{16}}=x^8$.
2. Explain why $7+\sqrt{9}$ is not equal to $\sqrt{7+9}$.
3. Explain how you know that $\sqrt[5]{x^{10}}=x^2$.
4. Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Answer

1. Answers may vary

3. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

Figure 8.2.1

b. After reviewing this checklist, what will you do to become confident for all objectives?

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8.4: Add, Subtract, and Multiply Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Before you get started, take this readiness quiz.

1. Add: $3x^2+9x-5-(x^2-2x+3)$.
   If you missed this problem, review Example 5.5.
2. Simplify: $(2+a)(4-a)$.
   If you missed this problem, review Example 5.28.
3. Simplify: $(9-5y)^2$.
   If you missed this problem, review Example 5.31.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Definition 8.4.1: Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x+8x$ is $11x$.Similarly we add $3\sqrt{x}+8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

Example 8.4.1

Simplify:

a. $2\sqrt{2}-7\sqrt{2}$
b. $5\sqrt[3]{y}+4\sqrt[3]{y}$
c. $7\sqrt[4]{x}-2\sqrt[4]{y}$

Solution:

a.

$$2\sqrt{2}-7\sqrt{2}$$

Since the radicals are like, we subtract the coefficients.

$$-5\sqrt{2}$$

b.

$$5\sqrt[3]{y}+4\sqrt[3]{y}$$

Since the radicals are like, we add the coefficients.

$$9\sqrt[3]{y}$$

c.

$$7\sqrt[4]{x}-2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

Exercise 8.4.1

Simplify:

a. $8\sqrt{2}-9\sqrt{2}$
b. $4\sqrt[3]{x}+7\sqrt[3]{x}$
c. $3\sqrt[4]{x}-5\sqrt[4]{y}$

Answer

a. $-\sqrt{2}$
b. $11\sqrt[3]{x}$
c. $3\sqrt[4]{x}-5\sqrt[4]{y}$

Exercise 8.4.2

Simplify:

a. $5\sqrt{3}-9\sqrt{3}$
b. $5\sqrt[3]{y}+3\sqrt[3]{y}$
c. $5\sqrt[4]{m}-2\sqrt[3]{m}$

Answer

a. $-4\sqrt{3}$
b. $8\sqrt[3]{y}$
c. $5\sqrt[4]{m}-2\sqrt[3]{m}$

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

Example 8.4.2

Simplify:

a. $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$
b. $\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$

Solution:

a.

$$2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$$

Since the radicals are like, we combine them.

$$0\sqrt{5n}$$

Simplify.

$$0$$

b.

$$\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$$

Since the radicals are like, we combine them.

$$2\sqrt[4]{3xy}$$

Exercise 8.4.3

Simplify:

a. $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$
b. $4\sqrt[4]{5xy}+2\sqrt[4]{5xy}-7\sqrt[4]{5xy}$

Answer

a. $-2\sqrt{7x}$
b. $-\sqrt[4]{5xy}$

Exercise 8.4.4

Simplify:

a. $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$
b. $6\sqrt[3]{7mn}+\sqrt[3]{7mn}-4\sqrt[3]{7mn}$

Answer

a. $-\sqrt{3y}$
b. $3\sqrt[3]{7mn}$

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

Example 8.4.3

Simplify:

a. $\sqrt{20}+3\sqrt{5}$
b. $\sqrt[3]{24}-\sqrt[3]{375}$
c. $\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$

Solution:

a.

$$\sqrt{20}+3\sqrt{5}$$

Simplify the radicals, when possible.

$$\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$$
$$2\sqrt{5}+3\sqrt{5}$$

Combine the like radicals.

$$5\sqrt{5}$$

b.

$$\sqrt[3]{24}-\sqrt[3]{375}$$

Simplify the radicals.

$$\sqrt[3]{8}\cdot\sqrt[3]{3}-\sqrt[3]{125}\cdot\sqrt[3]{3}$$
$$2\sqrt[3]{3}-5\sqrt[3]{3}$$

Combine the like radicals.

$$-3\sqrt[3]{3}$$

c.

$$\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$$

Simplify the radicals.

$$\frac{1}{2}\sqrt[4]{16}\cdot\sqrt[4]{3}-\frac{2}{3}\sqrt[4]{81}\cdot\sqrt[4]{3}$$
$$\frac{1}{2}\cdot 2\cdot\sqrt[4]{3}-\frac{2}{3}\cdot 3\cdot\sqrt[4]{3}$$
$$\sqrt[4]{3}-2\sqrt[4]{3}$$

Combine the like radicals.

$$-\sqrt[4]{3}$$

Exercise 8.4.5

Simplify:

a. $\sqrt{18}+6\sqrt{2}$
b. $6\sqrt[3]{16}-2\sqrt[3]{250}$
c. $\frac{2}{3}\sqrt[3]{81}-\frac{1}{2}\sqrt[3]{24}$

Answer

a. $9\sqrt{2}$
b. $2\sqrt[3]{2}$
c. $\sqrt[3]{3}$

Exercise 8.4.6

Simplify:

a. $\sqrt{27}+4\sqrt{3}$
b. $4\sqrt[3]{5}-7\sqrt[3]{40}$
c. $\frac{1}{2}\sqrt[3]{128}-\frac{5}{3}\sqrt[3]{54}$

Answer

a. $7\sqrt{3}$
b. $-10\sqrt[3]{5}$
c. $-3\sqrt[3]{2}$

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

Example 8.4.4

Simplify:

a. $9\sqrt{50m^2}-6\sqrt{48m^2}$
b. $\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$

Solution:

a.

$$9\sqrt{50m^2}-6\sqrt{48m^2}$$

Simplify the radicals.

$$9\sqrt{25m^2}\cdot\sqrt{2}-6\sqrt{16m^2}\cdot\sqrt{3}$$

$$9 \cdot 5m \cdot \sqrt{2} - 6 \cdot 4m \cdot \sqrt{3}$$

$$45m\sqrt{2} - 24m\sqrt{3}$$

The radicals are not like and so cannot be combined.

b.

$$\sqrt[3]{54n^5} - \sqrt[3]{16n^5}$$

Simplify the radicals.

$$\sqrt[3]{27n^3} \cdot \sqrt[3]{2n^2} - \sqrt[3]{8n^3} \cdot \sqrt[3]{2n^2}$$

$$3n\sqrt[3]{2n^2} - 2n\sqrt[3]{2n^2}$$

Combine the like radicals.

$$n\sqrt[3]{2n^2}$$

? Exercise 8.4.7

Simplify:

a. $\sqrt{32m^7} - \sqrt{50m^7}$
b. $\sqrt[3]{135x^7} - \sqrt[3]{40x^7}$

Answer

a. $-m^3\sqrt{2m}$
b. $x^2\sqrt[3]{5x}$

? Exercise 8.4.8

Simplify:

a. $\sqrt{27p^3} - \sqrt{48p^3}$
b. $\sqrt[3]{256y^5} - \sqrt[3]{32n^5}$

Answer

a. $-p\sqrt{3p}$
b. $4y\sqrt[3]{4y^2} - 2n\sqrt[3]{4n^2}$

Multiply Radical Expressions

We have used the **Product Property of Roots** to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots 'in reverse' to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Definition 8.4.2: Product Property of Roots

For any real numbers, $\sqrt[n]{a}$ and $\sqrt[b]{n}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is $12xy$. Keep this in mind as you do these examples.

Example 8.4.5

Simplify:

a. $(6\sqrt{2})(3\sqrt{10})$
b. $(-5\sqrt[3]{4})(-4\sqrt[3]{6})$

Solution:

a.

$$(6\sqrt{2})(3\sqrt{10})$$

Multiply using the Product Property.

$$18\sqrt{20}$$

Simplify the radical.

$$18\sqrt{4}\cdot\sqrt{5}$$

Simplify.

$$18\cdot 2\cdot\sqrt{5}$$
$$36\sqrt{5}$$

b.

$$(-5\sqrt[3]{4})(-4\sqrt[3]{6})$$

Multiply using the Product Property.

$$20\sqrt[3]{24}$$

Simplify the radical.

$$20\sqrt[3]{8}\cdot\sqrt[3]{3}$$

Simplify.

$$20\cdot 2\cdot\sqrt[3]{3}$$
$$40\sqrt[3]{3}$$

Exercise 8.4.9

Simplify:

a. $(3\sqrt{2})(2\sqrt{30})$
b. $(2\sqrt[3]{18})(-3\sqrt[3]{6})$

Answer

a. $12\sqrt{15}$
b. $-18\sqrt[3]{2}$

Exercise 8.4.10

Simplify:

a. $(3\sqrt{3})(3\sqrt{6})$
b. $(-4\sqrt[3]{9})(3\sqrt[3]{6})$

Answer

a. $27\sqrt{2}$
b. $-36\sqrt[3]{2}$

We follow the same procedures when there are variables in the radicands.

Example 8.4.6

Simplify:

a. $\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$
b. $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$

Solution:

a.

$$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$$

Multiply.

$$40\sqrt{18p^4}$$

Simplify the radical.

$$40\sqrt{9p^4}\cdot\sqrt{2}$$

Simplify.

$$40\cdot 3p^2\cdot\sqrt{3}$$
$$120p^2\sqrt{3}$$

b. When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

$$\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$$

Multiply.

$$6\sqrt[4]{4\cdot 5\cdot 4\cdot 7y^5}$$

Simplify the radical.

$$6\sqrt[4]{16y^4}\cdot\sqrt[4]{35y}$$

Simplify.

$$6\cdot 2y\sqrt[4]{35y}$$

Multiply.

$$12y\sqrt[4]{35y}$$

Exercise 8.4.11

Simplify:

a. $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$
b. $\left(-4\sqrt[4]{12y^3}\right)\left(-\sqrt[4]{8y^3}\right)$

Answer

a. $36x^3\sqrt{5}$
b. $8y\sqrt[4]{3y^2}$

Exercise 8.4.12

Simplify:

a. $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$
b. $\left(-4\sqrt[4]{9a^3}\right)\left(3\sqrt[4]{27a^2}\right)$

Answer

a. $144y^2\sqrt{5y}$
b. $-36\sqrt[4]{3a}$

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the **Distributive Property** to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

Example 8.4.7

Simplify:

a. $\sqrt{6}(\sqrt{2}+\sqrt{18})$
b. $\sqrt[3]{9}(5-\sqrt[3]{18})$

Solution:

a.

$$\sqrt{6}(\sqrt{2}+\sqrt{18})$$

Multiply.

$$\sqrt{12}+\sqrt{108}$$

Simplify.

$$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$$

Simplify.

$$2\sqrt{3}+6\sqrt{3}$$

Combine like radicals.

$$8\sqrt{3}$$

b.

$$\sqrt[3]{9}(5-\sqrt[3]{18})$$

Distribute.

$$5\sqrt[3]{9}-\sqrt[3]{162}$$

Simplify.

$$5\sqrt[3]{9}-\sqrt[3]{27}\cdot\sqrt[3]{6}$$

Simplify.

$$5\sqrt[3]{9}-3\sqrt[3]{6}$$

Exercise 8.4.13

Simplify:

a. $\sqrt{6}(1+3\sqrt{6})$
b. $\sqrt[3]{4}(-2-\sqrt[3]{6})$

Answer

a. $18+\sqrt{6}$
b. $-2\sqrt[3]{4}-2\sqrt[3]{3}$

Exercise 8.4.14

Simplify:

a. $\sqrt{8}(2-5\sqrt{8})$
b. $\sqrt[3]{3}(-\sqrt[3]{9}-\sqrt[3]{6})$

Answer

a. $-40+4\sqrt{2}$
b. $-3-\sqrt[3]{18}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example 8.4.8

Simplify:

a. $(3-2\sqrt{7})(4-2\sqrt{7})$
b. $(\sqrt[3]{x}-2)(\sqrt[3]{x}+4)$

Solution:

a.

$$(3-2\sqrt{7})(4-2\sqrt{7})$$

Multiply.

$$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$$

Simplify.

$$12-6\sqrt{7}-8\sqrt{7}+28$$

Combine like terms.

$$40-14\sqrt{7}$$

b.

$$(\sqrt[3]{x}-2)(\sqrt[3]{x}+4)$$

Multiply.

$$\sqrt[3]{x^2}+4\sqrt[3]{x}-2\sqrt[3]{x}-8$$

Combine like terms.

$$\sqrt[3]{x^2}+2\sqrt[3]{x}-8$$

Exercise 8.4.15

Simplify:

a. $(6-3\sqrt{7})(3+4\sqrt{7})$
b. $(\sqrt[3]{x}-2)(\sqrt[3]{x}-3)$

Answer

a. $-66+15\sqrt{7}$
b. $\sqrt[3]{x^2}-5\sqrt[3]{x}+6$

Exercise 8.4.16

Simplify:

a. $(2-3\sqrt{11})(4-\sqrt{11})$
b. $(\sqrt[3]{x}+1)(\sqrt[3]{x}+3)$

Answer

a. $41-14\sqrt{11}$
b. $\sqrt[3]{x^2}+4\sqrt[3]{x}+3$

Example 8.4.9

Simplify: $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$

Solution:

$$(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$$

Multiply.

$$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$$

Simplify.

$$6+12\sqrt{10}-\sqrt{10}-20$$

Combine like terms.

$$-14+11\sqrt{10}$$

Exercise 8.4.17

Simplify: $(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})$

Answer

$1+9\sqrt{21}$

Exercise 8.4.18

Simplify: $(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8})$

Answer

$-12-20\sqrt{3}$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products

Binomial Squares

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$

Product of Conjugates

$$(a+b)(a-b)=a^2-b^2$$

We will use the special product formulas in the next few examples. We will start with the **Product of Binomial Squares Pattern**.

Example 8.4.10

Simplify:

a. $(2+\sqrt{3})^2$
b. $(4-2\sqrt{5})^2$

Solution:

a.

Table 8.4.1

	$(a+b)^2$ $(2+\sqrt{3})^2$
Multiply using the Product of Binomial Squares Pattern.	$a^2+2ab+b^2$ $2^2+2\cdot 2\cdot\sqrt{3}+(\sqrt{3})^2$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

b.

Table 8.4.2

	$(a-b)^2$ $(4-2\sqrt{5})^2$
Multiple, using the Product of Binomial Squares Pattern.	$a^2-2ab+b^2$ $4^2-2\cdot 4\cdot 2\sqrt{5}+(2\sqrt{5})^2$
Simplify.	$16-16\sqrt{5}+4\cdot 5$
	$16-16\sqrt{5}+20$
Combine like terms.	$36-16\sqrt{5}$

Exercise 8.4.19

Simplify:

a. $(10+\sqrt{2})^2$
b. $(1+3\sqrt{6})^2$

Answer

a. $102+20\sqrt{2}$
b. $55+6\sqrt{6}$

Exercise 8.4.20

Simplify:

a. $(6-\sqrt{5})^2$
b. $(9-2\sqrt{10})^2$

Answer

a. $41-12\sqrt{5}$
b. $121-36\sqrt{10}$

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

Example 8.4.11

Simplify: $(5-2\sqrt{3})(5+2\sqrt{3})$

Solution:

Table 8.4.3

	$(a-b)(a+b)$ $(5-2\sqrt{3})(5+2\sqrt{3})$
Multiply using the Product of Conjugates Pattern.	a^2-b^2 $5^2-(2\sqrt{3})^2$
Simplify.	$25-4\cdot 3$
	13

Exercise 8.4.21

Simplify: $(3-2\sqrt{5})(3+2\sqrt{5})$

Answer

-11

Exercise 8.4.22

Simplify: $(4+5\sqrt{7})(4-5\sqrt{7})$

Answer

-159

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- Multiplying Adding Subtracting Radicals
- Multiplying Special Products: Square Binomials Containing Square Roots
- Multiplying Conjugates

Key Concepts

- **Product Property of Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$ $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ and $\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$
- **Special Products**

Binomial Squares	Product of Conjugates
$(a+b)^2=a^2+2ab+b^2$	$(a+b)(a-b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	

Glossary

like radicals

Like radicals are radical expressions with the same index and the same radicand.

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8.4E: Exercises

Practice Makes Perfect

Exercise A: add and subtract radical expressions

In the following exercises, simplify. Assume all variables are greater than or equal to zero so that absolute values are not needed.

1. a. $8\sqrt{2}-5\sqrt{2}$ b. $5\sqrt[3]{m}+2\sqrt[3]{m}$ c. $8\sqrt[4]{m}-2\sqrt[4]{n}$
2. a. $7\sqrt{2}-3\sqrt{2}$ b. $7\sqrt[3]{p}+2\sqrt[3]{p}$ c. $5\sqrt[3]{x}-3\sqrt[3]{x}$
3. a. $3\sqrt{5}+6\sqrt{5}$ b. $9\sqrt[3]{a}+3\sqrt[3]{a}$ c. $5\sqrt[4]{2z}+\sqrt[4]{2z}$
4. a. $4\sqrt{5}+8\sqrt{5}$ b. $\sqrt[3]{m}-4\sqrt[3]{m}$ c. $\sqrt{n}+3\sqrt{n}$
5. a. $3\sqrt{2a}-4\sqrt{2a}+5\sqrt{2a}$ b. $5\sqrt[4]{3ab}-3\sqrt[4]{3ab}-2\sqrt[4]{3ab}$
6. a. $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$ b. $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$
7. a. $8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$ b. $2\sqrt[3]{4pq}-5\sqrt[3]{4pq}+4\sqrt[3]{4pq}$
8. a. $3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$ b. $11\sqrt[3]{2rs}-9\sqrt[3]{2rs}+3\sqrt[3]{2rs}$
9. a. $\sqrt{27}-\sqrt{75}$ b. $\sqrt[3]{40}-\sqrt[3]{320}$ c. $\frac{1}{2}\sqrt[4]{32}+\frac{2}{3}\sqrt[4]{162}$
10. a. $\sqrt{72}-\sqrt{98}$ b. $\sqrt[3]{24}+\sqrt[3]{81}$ c. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$
11. a. $\sqrt{48}+\sqrt{27}$ b. $\sqrt[3]{54}+\sqrt[3]{128}$ c. $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$
12. a. $\sqrt{45}+\sqrt{80}$ b. $\sqrt[3]{81}-\sqrt[3]{192}$ c. $\frac{5}{2}\sqrt[4]{80}+\frac{7}{3}\sqrt[4]{405}$
13. a. $\sqrt{72a^5}-\sqrt{50a^5}$ b. $9\sqrt[4]{80p^4}-6\sqrt[4]{405p^4}$
14. a. $\sqrt{48b^5}-\sqrt{75b^5}$ b. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$
15. a. $\sqrt{80c^7}-\sqrt{20c^7}$ b. $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$
16. a. $\sqrt{96d^9}-\sqrt{24d^9}$ b. $5\sqrt[4]{243s^6}+2\sqrt[4]{3s^6}$
17. $3\sqrt{128y^2}+4y\sqrt{162}-8\sqrt{98y^2}$
18. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Answer

1. a. $3\sqrt{2}$ b. $7\sqrt[3]{m}$ c. $6\sqrt[4]{m}$
3. a. $9\sqrt{5}$ b. $12\sqrt[3]{a}$ c. $6\sqrt[4]{2z}$
5. a. $4\sqrt{2a}$ b. 0
7. a. $\sqrt{3c}$ b. $\sqrt[3]{4pq}$
9. a. $-2\sqrt{3}$ b. $-2\sqrt[3]{5}$ c. $3\sqrt[4]{2}$

11. a. $7\sqrt{3}$ b. $7\sqrt[3]{2}$ c. $3\sqrt[4]{5}$

13. a. $a^2\sqrt{2a}$ b. 0

15. a. $2c^3\sqrt{5c}$ b. $14r^2\sqrt[4]{2r^2}$

17. $4y\sqrt{2}$

Exercise B: multiply radical expressions

In the following exercises, simplify.

1. a. $(-2\sqrt{3})(3\sqrt{18})$

   b. $(8\sqrt[3]{4})(-4\sqrt[3]{18})$

   c.

2. a. $(-4\sqrt{5})(5\sqrt{10})$

   b. $(-2\sqrt[3]{9})(7\sqrt[3]{9})$

   c.

3. a. $(5\sqrt{6})(-\sqrt{12})$

   b. $(-2\sqrt[4]{18})(-\sqrt[4]{9})$

   c.

4. a. $(-2\sqrt{7})(-2\sqrt{14})$

   b. $(-3\sqrt[4]{8})(-5\sqrt[4]{6})$

   c.

5. a. $\left(4\sqrt{12z^3}\right)\left(3\sqrt{9z}\right)$

   b. $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

c.

6. a. $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$

   b. $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

   c.

7. a. $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$

   b. $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

   c.

8. a. $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$

   b. $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

   c.

Answer

1.

   a. $-18\sqrt{6}$

   b. $-64\sqrt[3]{9}$

   c.

3.

   a. $-30\sqrt{2}$

   b. $6\sqrt[4]{2}$

c.

5.

a. $72z^2\sqrt{3}$

b. $45x^2\sqrt[3]{2}$

c.

7.

a. $-42z^5\sqrt{2z}$

b. $-8y\sqrt[4]{6y}$

Exercise C: use polynomial multiplication to multiply radical expressions

In the following exercises, multiply.

1. a. $\sqrt{7}(5+2\sqrt{7})$

   b. $\sqrt[3]{6}(4+\sqrt[3]{18})$

   c.

2. a. $\sqrt{11}(8+4\sqrt{11})$

   b. $\sqrt[3]{3}(\sqrt[3]{9}+\sqrt[3]{18})$

   c.

3. a. $\sqrt{11}(-3+4\sqrt{11})$

   b. $\sqrt[4]{3}(\sqrt[4]{54}+\sqrt[4]{18})$

   c.

4. a. $\sqrt{2}(-5+9\sqrt{2})$

   b. $\sqrt[4]{2}(\sqrt[4]{12}+\sqrt[4]{24})$

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c.

5. $(7+\sqrt{3})(9-\sqrt{3})$

6. $(8-\sqrt{2})(3+\sqrt{2})$

7. a. $(9-3\sqrt{2})(6+4\sqrt{2})$

   b. $(\sqrt[3]{x}-3)(\sqrt[3]{x}+1)$

   c.

8. a. $(3-2\sqrt{7})(5-4\sqrt{7})$

   b. $(\sqrt[3]{x}-5)(\sqrt[3]{x}-3)$

   c.

9. a. $(1+3\sqrt{10})(5-2\sqrt{10})$

   b. $(2\sqrt[3]{x}+6)(\sqrt[3]{x}+1)$

   c.

10. a. $(7-2\sqrt{5})(4+9\sqrt{5})$

    b. $(3\sqrt[3]{x}+2)(\sqrt[3]{x}-2)$

    c.

11. $(\sqrt{3}+\sqrt{10})(\sqrt{3}+2\sqrt{10})$

12. $(\sqrt{11}+\sqrt{5})(\sqrt{11}+6\sqrt{5})$

13. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$

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14. $(4\sqrt{6}+7\sqrt{13})(8\sqrt{6}-3\sqrt{13})$

15. a. $(3+\sqrt{5})^2$

    b. $(2-5\sqrt{3})^2$

    c.

16. a. $(4+\sqrt{11})^2$

    b. $(3-2\sqrt{5})^2$

    c.

17. a. $(9-\sqrt{6})^2$

    b. $(10+3\sqrt{7})^2$

    c.

18. a. $(5-\sqrt{10})^2$

    b. $(8+3\sqrt{2})^2$

    c.

19. $(4+\sqrt{2})(4-\sqrt{2})$

20. $(7+\sqrt{10})(7-\sqrt{10})$

21. $(4+9\sqrt{3})(4-9\sqrt{3})$

22. $(1+8\sqrt{2})(1-8\sqrt{2})$

23. $(12-5\sqrt{5})(12+5\sqrt{5})$

24. $(9-4\sqrt{3})(9+4\sqrt{3})$

25. $(\sqrt[3]{3x}+2)(\sqrt[3]{3x}-2)$

26. $(\sqrt[3]{4x}+3)(\sqrt[3]{4x}-3)$

Answer

1.

a. $14+5\sqrt{7}$

b. $4\sqrt[3]{6}+3\sqrt[3]{4}$

c.

3.

a. $44-3\sqrt{11}$

b. $3\sqrt[4]{2}+\sqrt[4]{54}$

c.

5. $60+2\sqrt{3}$

7.

a. $30+18\sqrt{2}$

b. $\sqrt[3]{x^2}-2\sqrt[3]{x}-3$

c.

9.

a. $-54+13\sqrt{10}$

b. $2\sqrt[3]{x^2}+8\sqrt[3]{x}+6$

c.

11. $23+3\sqrt{30}$

13. $-439-2\sqrt{77}$

15.

a. $14+6\sqrt{5}$

b. $79-20\sqrt{3}$

17.

a. $87-18\sqrt{6}$

b. $163+60\sqrt{7}$

19. 14

21. -227

23. 19

25. $\sqrt[3]{9x^2}-4$

Exercise D: mixed practice

1. $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$
2. $\sqrt{175k^4}-\sqrt{63k^4}$
3. $\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$
4. $\sqrt[3]{24}+\sqrt[3]{81}$
5. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$
6. $8\sqrt[4]{13}-4\sqrt[4]{13}-3\sqrt[4]{13}$
7. $5\sqrt{12c^4}-3\sqrt{27c^6}$
8. $\sqrt{80a^5}-\sqrt{45a^5}$
9. $\frac{3}{5}\sqrt{75}-\frac{1}{4}\sqrt{48}$
10. $21\sqrt[3]{9}-2\sqrt[3]{9}$
11. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$
12. $11\sqrt{11}-10\sqrt{11}$
13. $\sqrt{3}\cdot\sqrt{21}$
14. $(4\sqrt{6})(-\sqrt{18})$
15. $(7\sqrt[3]{4})(-3\sqrt[3]{18})$
16. $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$
17. $(\sqrt{29})^2$
18. $(-4\sqrt{17})(-3\sqrt{17})$
19. $(-4+\sqrt{17})(-3+\sqrt{17})$
20. $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

21. $(6-3\sqrt{2})^2$

22. $\sqrt{3}(4-3\sqrt{3})$

23. $\sqrt[3]{3}(2\sqrt[3]{9}+\sqrt[3]{18})$

24. $(\sqrt{6}+\sqrt{3})(\sqrt{6}+6\sqrt{3})$

Answer

1. $5\sqrt{3}$
3. $9\sqrt{2}$
5. $-\sqrt[4]{5}$
7. $10c^2\sqrt{3}-9c^3\sqrt{3}$
9. $2\sqrt{3}$
11. $17q^2$
13. $3\sqrt{7}$
15. $-42\sqrt[3]{9}$
17. 29
19. $29-7\sqrt{17}$
21. $72-36\sqrt{2}$
23. $6+3\sqrt[3]{2}$

Exercise E: writing exercises

1. Explain when a radical expression is in simplest form.
2. Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.
3. a. Explain why $(-\sqrt{n})^2$ is always non-negative, for $n \geq 0$.
   b. Explain why $-(\sqrt{n})^2$ is always non-positive, for $n \geq 0$.
4. Use the binomial square pattern to simplify $(3+\sqrt{2})^2$. Explain all your steps.

Answer

1. Answers will vary
3. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

Figure 8.4.14

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b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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8.5: Divide Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Before you get started, take this readiness quiz.

1. Simplify: $\frac{30}{48}$.
   If you missed this problem, review Example 1.24.
2. Simplify: $x^2 \cdot x^4$.
   If you missed this problem, review Example 5.12.
3. Multiply: $(7+3x)(7-3x)$.
   If you missed this problem, review Example 5.32.

Divide Radical Expressions

We have used the **Quotient Property of Radical Expressions** to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals. We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Definition 8.5.1: Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

Example 8.5.1

Simplify:

a. $\frac{\sqrt{72x^3}}{\sqrt{162x}}$

b. $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$

Solution:

a.

$$\frac{\sqrt{72x^3}}{\sqrt{162x}}$$

Rewrite using the quotient property,

$$\sqrt{\frac{72x^3}{162x}}$$

Remove common factors.

$$\sqrt{\frac{\cancel{18}\cdot 4\cdot x^2\cdot \cancel{x}}{\cancel{18}\cdot 9\cdot \cancel{x}}}$$

Simplify.

$$\sqrt{\frac{4x^2}{9}}$$

Simplify the radical.

$$\frac{2x}{3}$$

b.

$$\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$$

Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$.

$$\sqrt[3]{\frac{32x^2}{4x^5}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{\frac{8}{x^3}}$$

Simplify the radical.

$$\frac{2}{x}$$

Exercise 8.5.1

Simplify:

a. $\frac{\sqrt{50s^3}}{\sqrt{128s}}$

b. $\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$

Answer

a. $\frac{5s}{8}$

b. $\frac{2}{a}$

Exercise 8.5.2

Simplify:

a. $\frac{\sqrt{75q^5}}{\sqrt{108q}}$

b. $\frac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$

Answer

a. $\frac{5q^2}{6}$

b. $\frac{2}{b}$

Example 8.5.2

Simplify:

a. $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$

b. $\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$

Solution:

a.

$$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$$

Rewrite using the quotient property.

$$\sqrt{\frac{147ab^8}{3a^3b^4}}$$

Remove common factors in the fraction.

$$\sqrt{\frac{49b^4}{a^2}}$$

Simplify the radical.

$$\frac{7b^2}{a}$$

b.

$$\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$$

Rewrite using the quotient property.

$$\sqrt[3]{\frac{-250mn^{-2}}{2m^{-2}n^4}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{\frac{-125m^3}{n^6}}$$

Simplify the radical.

$$-\frac{5m}{n^2}$$

Exercise 8.5.3

Simplify:

a. $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$

b. $\frac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$

Answer

a. $\frac{9x^2}{y^2}$

b. $\frac{-4x}{y}$

Exercise 8.5.4

Simplify:

a. $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$

b. $\frac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$

Answer

a. $\frac{10n^3}{m}$

b. $\frac{-3p}{q^2}$

Example 8.5.3

Simplify: $\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$

Solution:

$$\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$$

Rewrite using the quotient property.

$$\sqrt{\frac{54x^5y^3}{3x^2y}}$$

Remove common factors in the fraction.

$$\sqrt{18x^3y^2}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9x^2y^2 \cdot 2x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{9x^2y^2} \cdot \sqrt{2x}$$

Simplify.

$$3xy\sqrt{2x}$$

Exercise 8.5.5

Simplify: $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$

Answer

$4xy\sqrt{2x}$

Exercise 8.5.6

Simplify: $\frac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$

Answer

$4ab\sqrt{3b}$

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator. This process is still used today, and is useful in other areas of mathematics, too.

Definition 8.5.2: Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a **radical expression** is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example 8.5.4

Simplify:

a. $\frac{4}{\sqrt{3}}$

b. $\sqrt{\frac{3}{20}}$

c. $\frac{3}{\sqrt{6x}}$

Solution:

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.1

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.2

	$\sqrt{\frac{3}{20}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{3}}{\sqrt{20}}$
Simplify the denominator.	$\frac{\sqrt{3}}{2\sqrt{5}}$
Multiply the numerator and denominator by $\sqrt{5}$.	$\frac{\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$
Simplify.	$\frac{\sqrt{15}}{2 \cdot 5}$
Simplify.	$\frac{\sqrt{15}}{10}$

c.

Table 8.5.3

	$\frac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$\frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$
Simplify.	$\frac{3\sqrt{6x}}{6x}$
Simplify.	$\frac{\sqrt{6x}}{2x}$

? Exercise 8.5.7

Simplify:

a. $\frac{5}{\sqrt{3}}$

b. $\sqrt{\frac{3}{32}}$

c. $\frac{2}{\sqrt{2x}}$

Answer

a. $\frac{5\sqrt{3}}{3}$

b. $\frac{\sqrt{6}}{8}$

c. $\dfrac{\sqrt{2x}}{x}$

Exercise 8.5.8

Simplify:

a. $\dfrac{6}{\sqrt{5}}$

b. $\sqrt{\dfrac{7}{18}}$

c. $\dfrac{5}{\sqrt{5x}}$

Answer

a. $\dfrac{6\sqrt{5}}{5}$

b. $\dfrac{\sqrt{14}}{6}$

c. $\dfrac{\sqrt{5x}}{x}$

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,

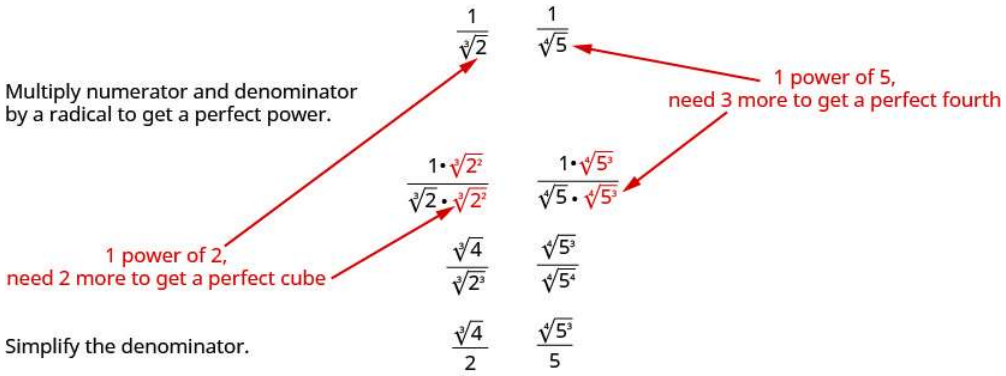


Figure 8.5.14

We will use this technique in the next examples.

Example 8.5.5

Simplify:

a. $\dfrac{1}{\sqrt[3]{6}}$

b. $\sqrt[3]{\dfrac{7}{24}}$

c. $\dfrac{3}{\sqrt[3]{4x}}$

Solution:

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.4

	$\frac{1}{\sqrt[3]{6}}$
The radical in the denominator has one factor of 6. Multiply both the numerator and denominator by $\sqrt[3]{6^2}$, which gives us 2 more factors of 6.	$\frac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$
Multiply. Notice the radicand in the denominator has 3 powers of 6.	$\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$
Simplify the cube root in the denominator.	$\frac{\sqrt[3]{36}}{6}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.5

	$\sqrt[3]{\frac{7}{24}}$
The fraction is not a perfect cube, so rewrite using the Quotient Property.	$\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$
Simplify the denominator.	$\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$
Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3.	$\frac{\sqrt[3]{7} \cdot \sqrt[3]{3^2}}{2\sqrt[3]{3} \cdot \sqrt[3]{3^2}}$
Simplify.	$\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$
Remember, $\sqrt[3]{3^3} = 3$.	$\frac{\sqrt[3]{63}}{2 \cdot 3}$
Simplify.	$\frac{\sqrt[3]{63}}{6}$

c.

Table 8.5.6

	$\frac{3}{\sqrt[3]{4x}}$
Rewrite the radicand to show the factors.	$\frac{3}{\sqrt[3]{2^2 \cdot x}}$
Multiple the numerator and denominator by $\sqrt[3]{2 \cdot x^2}$. This will get us 3 factors of 2 and 3 factors of x.	$\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$
Simplify.	$\frac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$
Simplify the radical in the denominator.	$\frac{3\sqrt[3]{2x^2}}{2x}$

Exercise 8.5.9

Simplify:

a. $\frac{1}{\sqrt[3]{7}}$

b. $\sqrt[3]{\frac{5}{12}}$

c. $\frac{5}{\sqrt[3]{9y}}$

Answer

a. $\frac{\sqrt[3]{49}}{7}$

b. $\frac{\sqrt[3]{90}}{6}$

c. $\frac{5\sqrt[3]{3y^2}}{3y}$

Exercise 8.5.10

Simplify:

a. $\frac{1}{\sqrt[3]{2}}$

b. $\sqrt[3]{\frac{3}{20}}$

c. $\frac{2}{\sqrt[3]{25n}}$

Answer

a. $\frac{\sqrt[3]{4}}{2}$

b. $\frac{\sqrt[3]{150}}{10}$

c. $\frac{2\sqrt[3]{5n^2}}{5n}$

Example 8.5.6

Simplify:

a. $\frac{1}{\sqrt[4]{2}}$

b. $\sqrt[4]{\frac{5}{64}}$

c. $\frac{2}{\sqrt[4]{8x}}$

Solution:

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

Table 8.5.7

	$\frac{1}{\sqrt[4]{2}}$
The radical in the denominator has one factor of 2. Multiply both the numerator and denominator by $\sqrt[4]{2^3}$, which gives us 3 more factors of 2.	$\frac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$
Multiply. Notice the radicand in the denominator has 4 powers of 2.	$\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$
Simplify the fourth root in the denominator.	$\frac{\sqrt[4]{8}}{2}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

Table 8.5.8

	$\sqrt[4]{\frac{5}{64}}$
The fraction is not a perfect fourth power, so rewrite using the Quotient Property.	$\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$
Rewrite the radicand in the denominator to show the factors.	$\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$
Simplify the denominator.	$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$
Multiply the numerator and denominator by $\sqrt[4]{2^2}$. This will give us 4 factors of 2.	$\frac{\sqrt[4]{5} \cdot \sqrt[4]{2^2}}{2\sqrt[4]{2^2} \cdot \sqrt[4]{2^2}}$
Simplify.	$\frac{\sqrt[4]{5} \cdot \sqrt[4]{4}}{2\sqrt[4]{2^4}}$
Remember, $\sqrt[4]{2^4} = 2$.	$\frac{\sqrt[4]{20}}{2 \cdot 2}$
Simplify.	$\frac{\sqrt[4]{20}}{4}$

c.

Table 8.5.9

	$\frac{2}{\sqrt[4]{8x}}$
Rewrite the radicand to show the factors.	$\frac{2}{\sqrt[4]{2^3 \cdot x}}$
Multiply the numerator and denominator by $\sqrt[4]{2 \cdot x^3}$. This will get us 4 factors of 2 and 4 factors of x.	$\frac{2 \cdot \sqrt[4]{2 \cdot x^3}}{\sqrt[4]{2^3 x} \cdot \sqrt[4]{2 \cdot x^3}}$
Simplify.	$\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4 x^4}}$
Simplify the radical in the denominator.	$\frac{2\sqrt[4]{2x^3}}{2x}$
Simplify the fraction.	$\frac{\sqrt[4]{2x^3}}{x}$

Exercise 8.5.11

Simplify:

a. $\frac{1}{\sqrt[4]{3}}$

b. $\sqrt[4]{\frac{3}{64}}$

c. $\frac{3}{\sqrt[4]{125x}}$

Answer

a. $\frac{\sqrt[4]{27}}{3}$

b. $\frac{\sqrt[4]{12}}{4}$

c. $\frac{3\sqrt[4]{5x^3}}{5x}$

Exercise 8.5.12

Simplify:

a. $\frac{1}{\sqrt[4]{5}}$

b. $\sqrt[4]{\frac{7}{128}}$

c. $\frac{4}{\sqrt[4]{4x}}$

Answer

a. $\frac{\sqrt[4]{125}}{5}$

b. $\frac{\sqrt[4]{224}}{8}$

c. $\frac{\sqrt[4]{64x^3}}{x}$

Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the **Product of Conjugates Pattern** to **rationalize the denominator**.

$$(a-b)(a+b) \qquad (2-\sqrt{5})(2+\sqrt{5})$$
$$a^2-b^2 \qquad 2^2-(\sqrt{5})^2$$
$$4-5$$
$$-1$$

When we multiple a binomial that includes a square root by its conjugate, the product has no square roots.

Example 8.5.7

Simplify: $\frac{5}{2-\sqrt{3}}$

Solution:

Table 8.5.10

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

? Exercise 8.5.13

Simplify: $\frac{3}{1-\sqrt{5}}$.

Answer

$-\frac{3(1+\sqrt{5})}{4}$

? Exercise 8.5.14

Simplify: $\frac{2}{4-\sqrt{6}}$.

Answer

$\frac{4+\sqrt{6}}{5}$

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

✔ Example 8.5.8

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution:

Table 8.5.11

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$
Simplify the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

Exercise 8.5.15

Simplify: $\dfrac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Answer

$$\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$$

Exercise 8.5.16

Simplify: $\dfrac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$

Answer

$$\frac{\sqrt{10}(\sqrt{y}+\sqrt{3})}{y-3}$$

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

Example 8.5.9

Simplify: $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution:

Table 8.5.12

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

Exercise 8.5.17

Simplify: $\dfrac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Answer

$$\frac{(\sqrt{p}+\sqrt{2})^2}{p-2}$$

? Exercise 8.5.18

Simplify: $\dfrac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$

Answer

$$\frac{(\sqrt{q}-\sqrt{10})^2}{q-10}$$

Key Concepts

- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$
- **Simplified Radical Expressions**
  - A radical expression is considered simplified if there are:
    - no factors in the radicand that have perfect powers of the index
    - no fractions in the radicand
    - no radicals in the denominator of a fraction

Glossary

rationalizing the denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

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8.5E: Exercises

Practice Makes Perfect

Exercise SET A: divide square roots

In the following exercises, simplify.

1. a. $\frac{\sqrt{128}}{\sqrt{72}}$ b. $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$
2. a. $\frac{\sqrt{48}}{\sqrt{75}}$ b. $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$
3. a. $\frac{\sqrt{200m^5}}{\sqrt{98m}}$ b. $\frac{\sqrt[3]{54y^2}}{\sqrt[3]{2y^5}}$
4. a. $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$ b. $\frac{\sqrt[3]{54y}}{\sqrt[3]{16y^4}}$
5. a. $\frac{\sqrt{75r^3}}{\sqrt{108r^7}}$ b. $\frac{\sqrt[3]{24x^7}}{\sqrt[3]{81x^4}}$
6. a. $\frac{\sqrt{196q}}{\sqrt{484q^5}}$ b. $\frac{\sqrt[3]{16m^4}}{\sqrt[3]{54m}}$
7. a. $\frac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$ b. $\frac{\sqrt[3]{-16a^4b^{-2}}}{\sqrt[3]{2a^{-2}b}}$
8. a. $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$ b. $\frac{\sqrt[3]{-375y^4z^2}}{\sqrt[3]{3y^{-2}z^4}}$
9. a. $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$ b. $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$
10. a. $\frac{\sqrt{810c^{-3}d^7}}{\sqrt{1000cd}}$ b. $\frac{\sqrt[3]{24a^7b^{-1}}}{\sqrt[3]{-81a^{-2}b^2}}$
11. $\frac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$
12. $\frac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$
13. $\frac{\sqrt[3]{48a^3b^6}}{\sqrt[3]{3a^{-1}b^3}}$
14. $\frac{\sqrt[3]{162x^{-3}y^6}}{\sqrt[3]{2x^3y^{-2}}}$

Answer

1. a. $\frac{4}{3}$ b. $\frac{4}{3}$
3. a. $\frac{10m^2}{7}$ b. $\frac{3}{y}$
5. a. $\frac{5}{6r^2}$ b. $\frac{2x}{3}$

7. a. $\frac{6p}{q^2}$ b. $-\frac{2a^2}{b}$

9. a. $\frac{8m^4}{3n^4}$ b. $-\frac{2x^2}{3y^2}$

11. $4x^4\sqrt{7y}$

13. $2ab\sqrt[3]{2a}$

Exercise SET B: Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

15. a. $\frac{10}{\sqrt{6}}$ b. $\sqrt{\frac{4}{27}}$ c. $\frac{10}{\sqrt{5x}}$

16. a. $\frac{8}{\sqrt{3}}$ b. $\sqrt{\frac{7}{40}}$ c. $\frac{8}{\sqrt{2y}}$

17. a. $\frac{6}{\sqrt{7}}$ b. $\sqrt{\frac{8}{45}}$ c. $\frac{12}{\sqrt{3p}}$

18. a. $\frac{4}{\sqrt{5}}$ b. $\sqrt{\frac{27}{80}}$ c. $\frac{18}{\sqrt{6q}}$

19. a. $\frac{1}{\sqrt[3]{5}}$ b. $\sqrt[3]{\frac{5}{24}}$ c. $\frac{4}{\sqrt[3]{36a}}$

20. a. $\frac{1}{\sqrt[3]{3}}$ b. $\sqrt[3]{\frac{5}{32}}$ c. $\frac{7}{\sqrt[3]{49b}}$

21. a. $\frac{1}{\sqrt[3]{11}}$ b. $\sqrt[3]{\frac{7}{54}}$ c. $\frac{3}{\sqrt[3]{3x^2}}$

22. a. $\frac{1}{\sqrt[3]{13}}$ b. $\sqrt[3]{\frac{3}{128}}$ c. $\frac{3}{\sqrt[3]{6y^2}}$

23. a. $\frac{1}{\sqrt[4]{7}}$ b. $\sqrt[4]{\frac{5}{32}}$ c. $\frac{4}{\sqrt[4]{4x^2}}$

24. a. $\frac{1}{\sqrt[4]{4}}$ b. $\sqrt[4]{\frac{9}{32}}$ c. $\frac{6}{\sqrt[4]{9x^3}}$

25. a. $\frac{1}{\sqrt[4]{9}}$ b. $\sqrt[4]{\frac{25}{128}}$ c. $\frac{6}{\sqrt[4]{27a}}$

26. a. $\frac{1}{\sqrt[4]{8}}$ b. $\sqrt[4]{\frac{27}{128}}$ c. $\frac{16}{\sqrt[4]{64b^2}}$

Answer

15. a. $\frac{5\sqrt{6}}{3}$ b. $\frac{2\sqrt{3}}{9}$ c. $\frac{2\sqrt{5x}}{x}$

17. a. $\frac{6\sqrt{7}}{7}$ b. $\frac{2\sqrt{10}}{15}$ c. $\frac{4\sqrt{3p}}{p}$

19. a. $\frac{\sqrt[3]{25}}{5}$ b. $\frac{\sqrt[3]{45}}{6}$ c. $\frac{2\sqrt[3]{6a^2}}{3a}$

21. a. $\frac{\sqrt[3]{121}}{11}$ b. $\frac{\sqrt[3]{28}}{6}$ c. $\frac{\sqrt[3]{9x}}{x}$

23. a. $\frac{\sqrt[4]{343}}{7}$ b. $\frac{\sqrt[4]{40}}{4}$ c. $\frac{2\sqrt[4]{4x^2}}{x}$

25. a. $\frac{\sqrt[4]{9}}{3}$ b. $\frac{\sqrt[4]{50}}{4}$ c. $\frac{2\sqrt[4]{3a^2}}{a}$

Exercise SET C: Rationalize a Two Term Denominator

In the following exercises, simplify.

27. $\frac{8}{1-\sqrt{5}}$

28. $\frac{7}{2-\sqrt{6}}$

29. $\frac{6}{3-\sqrt{7}}$

30. $\frac{5}{4-\sqrt{11}}$

31. $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

32. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

33. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

34. $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

35. $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

36. $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

37. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

38. $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Answer

27. $-2(1+\sqrt{5})$

29. $3(3+\sqrt{7})$

31. $\frac{\sqrt{3}(\sqrt{m}+\sqrt{5})}{m-5}$

33. $\frac{\sqrt{2}(\sqrt{x}+\sqrt{6})}{x-6}$

35. $\frac{(\sqrt{r}+\sqrt{5})^2}{r-5}$

37. $\frac{(\sqrt{x}+2\sqrt{2})^2}{x-8}$

Exercise SET D: writing exercises

1. a. Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

   b. Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

   c. Why are the two methods of simplifying square roots different?

2. Explain what is meant by the word rationalize in the phrase, "rationalize a denominator."
3. Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an epxression with no radicals.
4. Explain why multiplying $\frac{7}{\sqrt[3]{x}}$ by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ does not rationalize the denominator.

Answer

1. Answers will vary
3. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

Figure 8.5.63

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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8.6: Solve Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review Example 5.31.
2. Solve: $2x-5=0$.
   If you missed this problem, review Example 2.2.
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review Example 6.45.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Definition: Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the nth power. This will eliminate the radical.

$$\text{For } a \geq 0, (\sqrt[n]{a})^n = a \text{ .}$$

Example 8.6.1 how to solve a radical equation

Solve: $\sqrt{5n-4}-9=0$.

Solution:

Table 8.6.1

Step 1: Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2: Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$(\sqrt{5n-4})^2=(9)^2$
Step 3: Solve the new equation.	Remember, $(\sqrt{a})^2=a$.	$5n-4=81$ $5n=85$ $n=17$

Step 4: Check the answer in the original equation.		Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9=0$ $0=0$ The solution is $n=17$.

Exercise 8.6.1

Solve: $\sqrt{3m+2}-5=0$.

Answer

$m=\frac{23}{3}$

Exercise 8.6.2

Solve: $\sqrt{10z+1}-2=0$.

Answer

$z=\frac{3}{10}$

Solve a Radical Equation With One Radical

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

Example 8.6.2

Solve: $\sqrt{9k-2}+1=0$.

Solution:

Table 8.6.2

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

Exercise 8.6.3

Solve: $\sqrt{2r-3}+5=0$.

Answer

no solution

Exercise 8.6.4

Solve: $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Definition: Binomial Squares

$(a+b)^2=a^2+2ab+b^2$
$(a-b)^2=a^2-2ab+b^2$

Don't forget the middle term!

Example 8.6.3

Solve: $\sqrt{p-1}+1=p$.

Solution:

Table 8.6.3

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right, Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the Zero Product Property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	
$p=1$ $\sqrt{p-1}+1=p$ $\sqrt{1-1}+1\stackrel{?}{=}1$ $\sqrt{0}+1\stackrel{?}{=}1$ $1=1$ ✓ $p=2$ $\sqrt{p-1}+1=p$ $\sqrt{2-1}+1\stackrel{?}{=}2$ $\sqrt{1}+1\stackrel{?}{=}2$ $2=2$ ✓	

The solutions are $p=1, p=2$.

Exercise 8.6.5

Solve: $\sqrt{x-2}+2=x$.

Answer

$x = 2, x = 3$

Exercise 8.6.6

Solve: $\sqrt{y-5}+5=y$.

Answer

$y = 5, y = 6$

When the index of the radical is 3, we cube both sides to remove the radical.

$$(\sqrt[3]{a})^3 = a$$

Example 8.6.4

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution:

Table 8.6.4

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$(\sqrt[3]{5x+1})^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	
$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$ $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ $\sqrt[3]{-64}+8\stackrel{?}{=}4$ $-4+8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=-13$.

Exercise 8.6.7

Solve: $\sqrt[3]{4x-3}+8=5$

Answer

$x = -6$

Exercise 8.6.8

Solve: $\sqrt[3]{6x-10}+1=-3$

Answer

$x = -9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n = a^{m \cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2 = x, \left(x^{\frac{1}{3}}\right)^3 = x$$

Remember, $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Example 8.6.5

Solve: $(3x-2)^{\frac{1}{4}} + 3 = 5$.

Solution:

Table 8.6.5

	$(3x-2)^{\frac{1}{4}} + 3 = 5$
To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}} = 2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4 = (2)^4$
Simplify.	$3x - 2 = 16$
Solve the equation.	$3x = 18$
	$x = 6$
Check the answer.	
$x = 6 \quad (3x-2)^{\frac{1}{4}} + 3 = 5$ $(3 \cdot 6 - 2)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $(16)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $2 + 3 \stackrel{?}{=} 5$ $5 = 5$ ✓	
	The solution is $x = 6$.

Exercise 8.6.9

Solve: $(9x+9)^{\frac{1}{4}} - 2 = 1$

Answer

$x = 8$

Exercise 8.6.10

Solve: $(4x-8)^{\frac{1}{4}} + 5 = 7$

Answer

$x = 6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an **extraneous solution**!

Example 8.6.6

Solve: $\sqrt{r+4}-r+2=0$.

Solution:

Table 8.6.6

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Simplify and then solve the equation.	$r+4=r^2-4r+4$
If it is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$
Check your answer.	
$r=0$, $\sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\stackrel{?}{=}0$ $\sqrt{4}+2\stackrel{?}{=}0$ $4\neq 0$ $r=5$, $\sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2\stackrel{?}{=}0$ $\sqrt{9}-3\stackrel{?}{=}0$ $0=0\checkmark$	The solution is $r=5$.
	$r=0$ is an extreneous solution.

Exercise 8.6.11

Solve: $\sqrt{m+9}-m+3=0$

Answer

$m=7$

Exercise 8.6.12

Solve: $\sqrt{n+1}-n+1=0$.

Answer

$n=3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

Example 8.6.7

Solve: $3\sqrt{3x-5}-8=4$.

Solution:

Table 8.6.7

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$

Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	
$x=7$ $3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4$ ✓	
	The solution is $x=7$.

Exercise 8.6.13

Solve: $2\sqrt{4a+4}-16=16$.

Answer

$a=63$

Exercise 8.6.14

Solve: $3\sqrt{2b+3}-25=50$

Answer

$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Example 8.6.8

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

Solution:

The radical terms are isolated.

$$\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$$

Since the index is 3, cube both sides of the equation.

$$(\sqrt[3]{4x-3})^3=(\sqrt[3]{3x+2})^3$$

Simplify, then solve the new equation.

$$\begin{aligned} 4x-3 &= 3x+2 \\ x-3 &= 2 \\ x &= 5 \end{aligned}$$

The solution is $x=5$.

Check the answer.

We leave it to you to show that 5 checks!

? Exercise 8.6.15

Solve: $\sqrt[3]{5x-4} = \sqrt[3]{2x+5}$.

Answer

$x=3$

? Exercise 8.6.16

Solve: $\sqrt[3]{7x+1} = \sqrt[3]{2x-5}$.

Answer

$x=-\frac{6}{5}$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

✓ Example 8.6.9 how to solve a radical equation

Solve: $\sqrt{m}+1 = \sqrt{m+9}$.

Solution:

Table 8.6.8

Step 1: Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2: Raise both sides of the equation to the power of the index.	We square both sides. Simplify--be very careful as you multiply!	$(\sqrt{m}+1)^2=(\sqrt{m+9})^2$
Step 3: Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$\begin{aligned} m+2\sqrt{m}+1 &= m+9 \\ 2\sqrt{m} &= 8 \\ \sqrt{m} &= 4 \\ (\sqrt{m})^2 &= (4)^2 \\ m &= 16 \end{aligned}$
Step 4: Check the answer in the original equation.		$\begin{aligned} \sqrt{m}+1 &= \sqrt{m+9} \\ \sqrt{16}+1 &\stackrel{?}{=} \sqrt{16+9} \\ 4+1 &\stackrel{?}{=} 5 \\ 5 &= 5 \end{aligned}$ The solution is $m=16$.

Exercise 8.6.17

Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Answer

$x=4$

Exercise 8.6.18

Solve: $\sqrt{x}+2=\sqrt{x+16}$.

Answer

$x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 8.6.10

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution:

Table 8.6.9

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$\left(6\sqrt{q-2}\right)^2=\left(3q-6\right)^2$ $a^2 - 2ab + b^2$ $6^2\left(\sqrt{q-2}\right)^2=(3q)^2-2\cdot 3q\cdot 6+6^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9(q^2-8q+12)$ $0=9(q-6)(q-2)$

Use the Zero Product Property.	$q-6=0 \quad q-2=0$ $q=6 \quad q=2$
The checks are left to you.	The solutions are $q=6$ and $q=2$.

Exercise 8.6.19

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer

$x=5$

Exercise 8.6.20

Solve: $\sqrt{x}+2=\sqrt{3x+4}$

Answer

$x=0x=4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a Problem Solving Strategy for Applications With Formulas

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of **gravity** on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Definition: Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t=\frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h=64$ into the formula.

Table 8.6.10

	$t=\frac{\sqrt{h}}{4}$
	$t=\frac{\sqrt{64}}{4}$

Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 8.6.11

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:

Table 8.6.11

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking.	The time it takes for the sunglasses to reach the river.
Step 3: **Name** what we are looking.	Let (t=\) time.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5: **Solve the equation**.	$t=\frac{20}{4}$
	$t=5$
Step 6: **Check** the answer in the problem and make sure it makes sense.	$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$ $5\stackrel{?}{=}\frac{20}{4}$ $5=5\checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7: **Answer** the equation.	It will take 5 seconds for the sunglasses to reach the river.

Exercise 8.6.21

A helicopter dropped a rescue package from a height of $1,296$ feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Exercise 8.6.22

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the **speed**, in miles per hour, a car was going before applying the brakes.

Definition: Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

Example 8.6.12

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

Table 8.6.12

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	The speed of a car.
Step 3: **Name** what we are looking for.	Let $s =$ the speed.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5: **Solve the equation**.	$s = \sqrt{4{,}560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \stackrel{?}{\approx} \sqrt{24(190)}$ $67.5 \stackrel{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277...$ ✓
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Exercise 8.6.23

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7mph

Exercise 8.6.24

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1mph

Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

Key Concepts

- **Binomial Squares**
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**
  1. Isolate one of the radical terms on one side of the equation.
  2. Raise both sides of the equation to the power of the index.
  3. Are there any more radicals?
     If yes, repeat Step 1 and Step 2 again.
     If no, solve the new equation.
  4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  2. Identify what we are looking for.
  3. Name what we are looking for by choosing a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

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8.6E: Exercises

Practice Makes Perfect

Exercise SET A: Solve Radical Equations

In the following exercises, solve.

1. $\sqrt{5x-6}=8$
2. $\sqrt{4x-3}=7$
3. $\sqrt{5x+1}=-3$
4. $\sqrt{3y-4}=-2$
5. $\sqrt[3]{2x}=-2$
6. $\sqrt[3]{4x-1}=3$
7. $\sqrt{2m-3}-5=0$
8. $\sqrt{2n-1}-3=0$
9. $\sqrt{6v-2}-10=0$
10. $\sqrt{12u+1}-11=0$
11. $\sqrt{4m+2}+2=6$
12. $\sqrt{6n+1}+4=8$
13. $\sqrt{2u-3}+2=0$
14. $\sqrt{5v-2}+5=0$
15. $\sqrt{u-3}+3=u$
16. $\sqrt{v-10}+10=v$
17. $\sqrt{r-1}=r-1$
18. $\sqrt{s-8}=s-8$
19. $\sqrt[3]{6x+4}=4$
20. $\sqrt[3]{11x+4}=5$
21. $\sqrt[3]{4x+5}-2=-5$
22. $\sqrt[3]{9x-1}-1=-5$
23. $(6x+1)^{\frac{1}{2}}-3=4$
24. $(3x-2)^{\frac{1}{2}}+1=6$
25. $(8x+5)^{\frac{1}{3}}+2=-1$
26. $(12x-5)^{\frac{1}{3}}+8=3$
27. $(12x-3)^{\frac{1}{4}}-5=-2$
28. $(5x-4)^{\frac{1}{4}}+7=9$
29. $\sqrt{x+1}-x+1=0$
30. $\sqrt{y+4}-y+2=0$
31. $\sqrt{z+100}-z=-10$
32. $\sqrt{w+25}-w=-5$

33. $3\sqrt{2x-3}-20=7$

34. $2\sqrt{5x+1}-8=0$

35. $2\sqrt{8r+1}-8=2$

36. $3\sqrt{7y+1}-10=8$

Answer

1. $m=14$
3. no solution
5. $x=-4$
7. $m=14$
9. $v=17$
11. $m=\frac{7}{2}$
13. no solution
15. $u=3, u=4$
17. $r=1, r=2$
19. $x=10$
21. $x=-8$
23. $x=8$
25. $x=-4$
27. $x=7$
29. $x=3$
31. $z=21$
33. $x=42$
35. $r=3$

Exercise SET B: Solve Radical Equations with Two Radicals

In the following exercises, solve.

37. $\sqrt{3u+7}=\sqrt{5u+1}$

38. $\sqrt{4v+1}=\sqrt{3v+3}$

39. $\sqrt{8+2r}=\sqrt{3r+10}$

40. $\sqrt{10+2c}=\sqrt{4c+16}$

41. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$

42. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$

43. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

44. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$

45. $\sqrt{a}+2=\sqrt{a+4}$

46. $\sqrt{r}+6=\sqrt{r+8}$

47. $\sqrt{u}+1=\sqrt{u+4}$

48. $\sqrt{x}+1=\sqrt{x+2}$

49. $\sqrt{a+5}-\sqrt{a}=1$
50. $-2=\sqrt{d-20}-\sqrt{d}$
51. $\sqrt{2x+1}=1+\sqrt{x}$
52. $\sqrt{3x+1}=1+\sqrt{2x-1}$
53. $\sqrt{2x-1}-\sqrt{x-1}=1$
54. $\sqrt{x+1}-\sqrt{x-2}=1$
55. $\sqrt{x+7}-\sqrt{x-5}=2$
56. $\sqrt{x+5}-\sqrt{x-3}=2$

Answer

37. $u=3$
39. $r=-2$
41. $x=1$
43. $x=-8, x=2$
45. $a=0$
47. $u=\frac{9}{4}$
49. $a=4$
51. $x=0\ x=4$
53. $x=1\ x=5$
55. $x=9$

Exercise SET C: Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

57. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.
58. **Landscaping** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.
59. **Gravity** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
60. **Gravity** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.
61. **Accident investigation** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.
62. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

57. 8.7 feet
59. 4.7 seconds
61. 72 feet

Exercise SET D: Writing Exercises

63. Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.

64. a. Solve the equations $\sqrt{r+4}-r+2=0$.
    b. Explain why one of the "solutions" that was found was not actually a solution to the equation.

Answer

63. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

Figure 8.6.42

b. After reviewing this checklist, what will you do to become confident for all objectives?

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8.7: Use Radicals in Functions

Learning Objectives

By the end of this section, you will be able to:

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Before you get started, take this readiness quiz.

1. Solve: $1-2x \geq 0$.
   If you missed this problem, review Example 2.50.
2. For $f(x)=3x-4$, evaluate $f(2), f(-1), f(0)$
   If you missed this problem, review Example 3.48.
3. Graph $f(x)=\sqrt{x}$. State the domain and range of the function in interval notation.
   If you missed this problem, review Example 3.56.

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

- The square root function is $f(x)=\sqrt{x}$.
- The cube root function is $f(x)=\sqrt[3]{x}$.

Definition 8.7.1: radical function

A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of $f(x)$ for a given value of x just as we did in our previous work with functions.

Example 8.7.1

For the function $f(x)=\sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solution:

a.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(5)$, substitute 5 for x.

$$f(5)=\sqrt{2\cdot 5-1}$$

Simplify.

$$f(5)=\sqrt{9}$$

Take the square root.

$$f(5)=3$$

b.

$$f(x)=\sqrt{2x-1}$$

To evaluate $f(-2)$, substitute -2 for x.

$$f(-2)=\sqrt{2(-2)-1}$$

Simplify.

$$f(-2)=\sqrt{-5}$$

Since the square root of a negative number is not a real number, the function does not have a value at $x=-2$.

Exercise 8.7.1

For the function $f(x)=\sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answer

a. $f(6)=4$
b. no value at $x=0$

Exercise 8.7.2

For the function $g(x)=\sqrt{5x+5}$, find

a. $g(4)$
b. $g(-3)$

Answer

a. $g(4)=5$
b. no value at $f(-3)$

We follow the same procedure to evaluate cube roots.

Example 8.7.2

For the function $g(x)=\sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solution:

a.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(14)$, substitute 14 for x.

$$g(14)=\sqrt[3]{14-6}$$

Simplify.

$$g(14)=\sqrt[3]{8}$$

Take the cube root.

$$g(14)=2$$

b.

$$g(x)=\sqrt[3]{x-6}$$

To evaluate $g(-2)$, substitute -2 for x.

$$g(-2)=\sqrt[3]{-2-6}$$

Simplify.

$$g(-2) = \sqrt[3]{-8}$$

Take the cube root.

$$g(-2) = -2$$

Exercise 8.7.3

For the function $g(x) = \sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answer

a. $g(4) = 2$
b. $g(1) = -1$

Exercise 8.7.4

For the function $h(x) = \sqrt[3]{5x-2}$, find

a. $h(2)$
b. $h(-5)$

Answer

a. $h(2) = 2$
b. $h(-5) = -3$

The next example has fourth roots.

Example 8.7.3

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solution:

a.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(4)$, substitute 4 for x.

$$f(4) = \sqrt[4]{5 \cdot 4 - 4}$$

Simplify.

$$f(4) = \sqrt[4]{16}$$

Take the fourth root.

$$f(4) = 2$$

b.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(-12)$, substitute -12 for x.

$$f(-12) = \sqrt[4]{5(-12)-4}$$

Simplify.

$$f(-12) = \sqrt[4]{-64}$$

Since the fourth root of a negative number is not a real number, the function does not have a value at $x = -12$.

Exercise 8.7.5

For the function $f(x) = \sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answer

a. $f(4) = 2$
b. $f(-1) = 1$

Exercise 8.7.6

For the function $g(x) = \sqrt[4]{5x+1}$, find

a. $g(16)$
b. $g(3)$

Answer

a. $g(16) = 3$
b. $g(3) = 2$

Find the Domain of a Radical Function

To find the **domain** and **range** of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Properties of $\sqrt[n]{a}$

When n is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Domain of a Radical Function

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.

When the **index** of the radical is **odd**, the radicand can be any real number.

Example 8.7.4

Find the domain of the function, $f(x) = \sqrt{3x-4}$. Write the domain in interval notation.

Solution:

Since the function, $f(x) = \sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

Solve.

$$\begin{aligned} 3x-4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \frac{4}{3} \end{aligned}$$

The domain of $f(x)=\sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Exercise 8.7.7

Find the domain of the function, $f(x)=\sqrt{6x-5}$. Write the domain in interval notation.

Answer

$\left[\frac{5}{6}, \infty\right)$

Exercise 8.7.8

Find the domain of the function, $f(x)=\sqrt{4-5x}$. Write the domain in interval notation.

Answer

$\left(-\infty, \frac{4}{5}\right]$

Example 8.7.5

Find the domain of the function, $g(x)=\sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution:

Solve the function, $g(x)=\sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x-1>0$ and solve.

$$x-1>0$$

Solve.

$$x>1$$

Also, since the radicand is a fraction, we must realize that the denominator cannot be zero.

We solve $x-1=0$ to find the value that must be eliminated from the domain.

$$x-1=0$$

Solve.

$$x=1 \text{ so } x/neq1 \text{ in the domain.}$$

Putting this together we get the domain is $x>1$ and we write it as $(1, \infty)$.

Exercise 8.7.9

Find the domain of the function, $f(x)=\sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Answer

$(-3,\infty)$

Exercise 8.7.10

Find the domain of the function, $h(x)=\sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

Answer

$(5,\infty)$

The next example involves a cube root and so will require different thinking.

Example 8.7.6

Find the domain of the function, $f(x)=\sqrt[3]{2x^2+3}$. Write the domain in interval notation.

Solution:

Since the function, $f(x)=\sqrt[3]{2x^2+3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $(-\infty,\infty)$.

The domain of $f(x)=\sqrt[3]{2x^2+3}$ is all real numbers and we write it in interval notation as $(-\infty,\infty)$.

Exercise 8.7.11

Find the domain of the function, $f(x)=\sqrt[3]{3x^2-1}$. Write the domain in interval notation.

Answer

$(-\infty,\infty)$

Exercise 8.7.12

Find the domain of the function, $g(x)=\sqrt[3]{5x-4}$. Write the domain in interval notation.

Answer

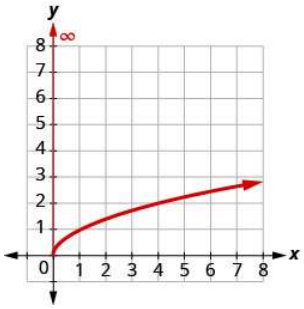
$(-\infty,\infty)$

Graph Radical Functions

Before we graph any radical function, we first find the **domain** of the function. For the function, $f(x)=\sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x\geq 0$ and we write this in interval notation as $[0,\infty)$.

Previously we used point plotting to graph the function, $f(x)=\sqrt{x}$. We chose x-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Figure 8.7.1

Once we see the graph, we can find the range of the function. The y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

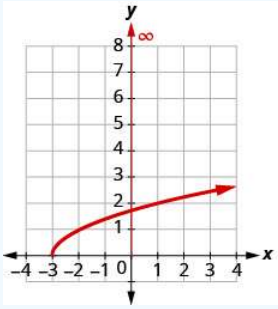
Example 8.7.7

For the function $f(x)=\sqrt{x+3}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x+3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$.
b. To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x)=\sqrt{x+3}$	$(x, f(x))$
–3	0	(–3, 0)
–2	1	(–2, 1)
1	2	(1, 2)
6	3	(6, 3)

Figure 8.7.2

c. Looking at the graph, we see the y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

Exercise 8.7.13

For the function $f(x)=\sqrt{x+2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[-2, \infty)$

b.

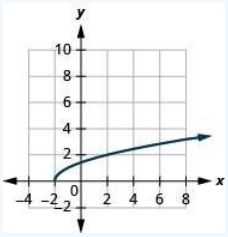


Figure 8.7.3

c. range: $[0, \infty)$

? Exercise 8.7.14

For the function $f(x) = \sqrt{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[2, \infty)$
b.

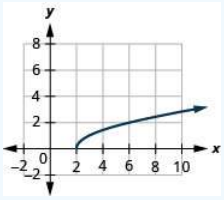


Figure 8.7.4

c. range: $[0, \infty)$

In our previous work graphing functions, we graphed $f(x) = x^3$ but we did not graph the function $f(x) = \sqrt[3]{x}$. We will do this now in the next example.

✓ Example 8.7.8

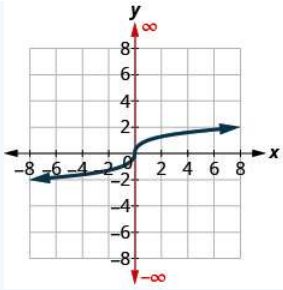
For the function, $f(x) = \sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty, \infty)$

b. To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x)=\sqrt[3]{x}$	$(x, f(x))$
–8	–2	(–8, –2)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

Figure 8.7.5

c. Looking at the graph, we see the y-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

Exercise 8.7.15

For the function $f(x)=-\sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

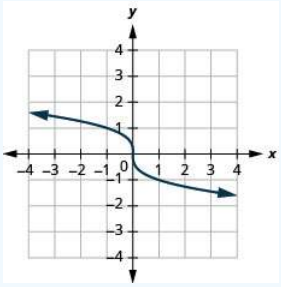


Figure 8.7.6

c. range: $(-\infty, \infty)$

Exercise 8.7.16

For the function $f(x)=\sqrt[3]{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

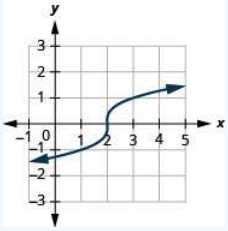


Figure 8.7.7

c. range: $(-\infty, \infty)$

Access these online resources for additional instruction and practice with radical functions.

- Domain of a Radical Function
- Domain of a Radical Function 2
- Finding Domain of a Radical Function

Key Concepts

- **Properties of $\sqrt[n]{a}$**
  - When n is an **even** number and:
    $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
    $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

Glossary

radical function

A radical function is a function that is defined by a radical expression.

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8.7E: Exercises

Practice Makes Perfect

Exercise $8.7E.17$ Evaluate a Radical Function

In the following exercises, evaluate each function.

1. $f(x)=\sqrt{4x-4}$, find
   a. $f(5)$
   b. $f(0)$
2. $f(x)=\sqrt{6x-5}$, find
   a. $f(5)$
   b. $f(-1)$
3. $g(x)=\sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
4. $g(x)=\sqrt{3x+1}$, find
   a. $g(8)$
   b. $g(5)$
5. $F(x)=\sqrt{3-2x}$, find
   a. $F(1)$
   b. $F(-11)$
6. $F(x)=\sqrt{8-4x}$, find
   a. $F(1)$
   b. $F(-2)$
7. $G(x)=\sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
8. $G(x)=\sqrt{4x+1}$, find
   a. $G(11)$
   b. $G(2)$
9. $g(x)=\sqrt[3]{2x-4}$, find
   a. $g(6)$
   b. $g(-2)$
10. $g(x)=\sqrt[3]{7x-1}$, find
    a. $g(4)$
    b. $g(-1)$
11. $h(x)=\sqrt[3]{x^2-4}$, find
    a. $h(-2)$
    b. $h(6)$
12. $h(x)=\sqrt[3]{x^2+4}$, find
    a. $h(-2)$
    b. $h(6)$
13. For the function $f(x)=\sqrt[4]{2x^3}$, find
    a. $f(0)$
    b. $f(2)$
14. For the function $f(x)=\sqrt[4]{3x^3}$, find

a. $f(0)$
b. $f(3)$

15. For the function $g(x)=\sqrt[4]{4-4x}$, find
    a. $g(1)$
    b. $g(-3)$
16. For the function $g(x)=\sqrt[4]{8-4x}$, find
    a. $g(-6)$
    b. $g(2)$

Answer

1.
    a. $f(5)=4$
    b. no value at $x=0$
3.
    a. $g(4)=5$
    b. $g(8)=7$
5.
    a. $F(1)=1$
    b. $F(-11)=5$
7.
    a. $G(5)=2\sqrt{6}$
    b. $G(2)=3$
9.
    a. $g(6)=2$
    b. $g(-2)=-2$
11.
    a. $h(-2)=0$
    b. $h(6)=2\sqrt[3]{4}$
13.
    a. $f(0)=0$
    b. $f(2)=2$
15.
    a. $g(1)=0$
    b. $g(-3)=2$

Exercise $8.7E.18$ Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

1. $f(x)=\sqrt{3x-1}$
2. $f(x)=\sqrt{4x-2}$
3. $g(x)=\sqrt{2-3x}$
4. $g(x)=\sqrt{8-x}$
5. $h(x)=\sqrt{\frac{5}{x-2}}$
6. $h(x)=\sqrt{\frac{6}{x+3}}$

7. $f(x)=\sqrt{\frac{x+3}{x-2}}$
8. $f(x)=\sqrt{\frac{x-1}{x+4}}$
9. $g(x)=\sqrt[3]{8x-1}$
10. $g(x)=\sqrt[3]{6x+5}$
11. $f(x)=\sqrt[3]{4x^2-16}$
12. $f(x)=\sqrt[3]{6x^2-25}$
13. $F(x)=\sqrt[4]{8x+3}$
14. $F(x)=\sqrt[4]{10-7x}$
15. $G(x)=\sqrt[5]{2x-1}$
16. $G(x)=\sqrt[5]{6x-3}$

Answer

1. $\left[\frac{1}{3},\infty\right)$
3. $\left(-\infty,\frac{2}{3}\right]$
5. $(2,\infty)$
7. $(-\infty,-3]\cup(2,\infty)$
9. $(-\infty,\infty)$
11. $(-\infty,\infty)$
13. $\left[-\frac{3}{8},\infty\right)$
15. $(-\infty,\infty)$

? Exercise $8.7E.19$ graph radical functions

In the following exercises,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $f(x)=\sqrt{x+1}$
2. $f(x)=\sqrt{x-1}$
3. $g(x)=\sqrt{x+4}$
4. $g(x)=\sqrt{x-4}$
5. $f(x)=\sqrt{x}+2$
6. $f(x)=\sqrt{x}-2$
7. $g(x)=2\sqrt{x}$
8. $g(x)=3\sqrt{x}$
9. $f(x)=\sqrt{3-x}$
10. $f(x)=\sqrt{4-x}$
11. $g(x)=-\sqrt{x}$
12. $g(x)=-\sqrt{x}+1$
13. $f(x)=\sqrt[3]{x+1}$
14. $f(x)=\sqrt[3]{x-1}$
15. $g(x)=\sqrt[3]{x+2}$
16. $g(x)=\sqrt[3]{x-2}$
17. $f(x)=\sqrt[3]{x}+3$
18. $f(x)=\sqrt[3]{x}-3$
19. $g(x)=\sqrt[3]{x}$
20. $g(x)=-\sqrt[3]{x}$

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21. $f(x) = 2\sqrt[3]{x}$
22. $f(x) = -2\sqrt[3]{x}$

Answer

1.

a. domain: $[-1, \infty)$

b.

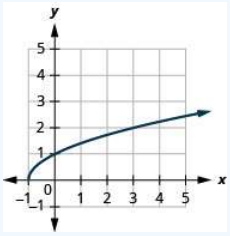


Figure 8.7.8

c. $[0, \infty)$

3.

a. domain: $[-4, \infty)$

b.

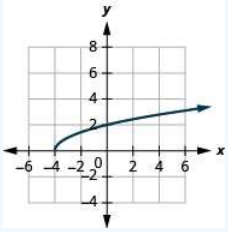


Figure 8.7.9

c. $[0, \infty)$

5.

a. domain: $[0, \infty)$

b.

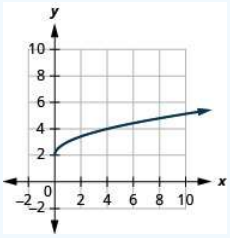


Figure 8.7.10

c. $[2, \infty)$

7.

a. domain: $[0, \infty)$

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b.

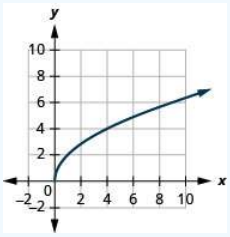


Figure 8.7.11

c. $[0, \infty)$

9.

a. domain: $(-\infty, 3]$

b.

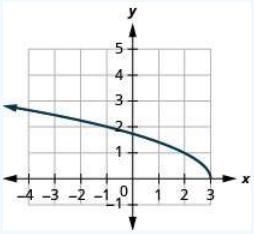


Figure 8.7.12

c. $[0, \infty)$

11.

a. domain: $[0, \infty)$

b.

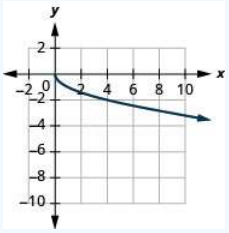


Figure 8.7.13

c. $(-\infty, 0]$

13.

a. domain: $(-\infty, \infty)$

b.

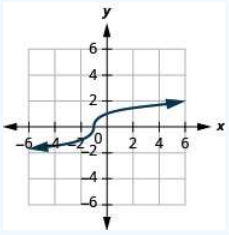


Figure 8.7.14

c. $(-\infty, \infty)$

15.

a. domain: $(-\infty, \infty)$

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b.

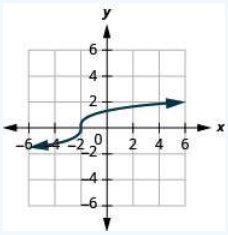


Figure 8.7.15

c. $(-\infty, \infty)$

17.

a. domain: $(-\infty, \infty)$

b.

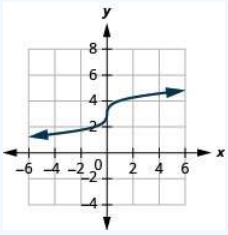


Figure 8.7.16

c. $(-\infty, \infty)$

19.

a. domain: $(-\infty, \infty)$

b.

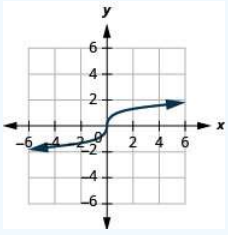


Figure 8.7.17

c. $(-\infty, \infty)$

21.

a. domain: $(-\infty, \infty)$

b.

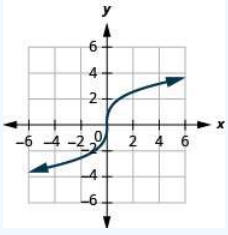


Figure 8.7.18

c. $(-\infty, \infty)$

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Exercise $8.7E.20$ writing exercises

1. Explain how to find the domain of a fourth root function.
2. Explain how to find the domain of a fifth root function.
3. Explain why $y = \sqrt[3]{x}$ is a function.
4. Explain why the process of finding the domain of a radical function with an even index is different from the process when the index is odd.

Answer

1. Answers may vary

3. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate a radical function.			
find the domain of a radical function.			
graph a radical function.			

Figure 8.7.19

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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8.8: Use the Complex Number System

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Before you get started, take this readiness quiz.

1. Given the numbers $-4, -\sqrt{7}, 0.\overline{5}, \frac{7}{3}, 3, \sqrt{81}$, list the
   a. rational numbers
   b. irrational numbers
   c. real numbers

   If you missed this problem, review Example 1.42.
2. Multiply: $(x-3)(2x+5)$.
   If you missed this problem, review Example 5.28.
3. Rationalize the denominator: $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$
   If you missed this problem, review Example 5.32.

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1}$, we are looking for a real number x so that $x^2 = -1$. Since all real numbers squared are positive numbers, there is no real number that equals -1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is -1.

Definition 8.8.1

The **imaginary unit** i is the number whose square is -1.

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

Definition 8.8.2

Square Root of a Negative Number

If b is a positive real number, then

$$\sqrt{-b} = \sqrt{b}i$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b} = i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b} = \sqrt{b}i$ is considered standard form.

Example 8.8.1

Write each expression in terms of i and simplify is possible:

a. $\sqrt{-25}$
b. $\sqrt{-7}$
c. $\sqrt{-12}$

Solution:

a.

$$\sqrt{-25}$$

Use the definition of the square root of negative numbers.

$$\sqrt{25}i$$

Simplify.

$$5i$$

b.

$$\sqrt{-7}$$

Use the definition of the square root of negative numbers.

$$\sqrt{7}i$$

Simplify.

Be careful that it is clear that i is not under the radical sign.

c.

$$\sqrt{-12}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i$$

Simplify $\sqrt{12}$.

$$2\sqrt{3}i$$

Exercise 8.8.1

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-81}$
b. $\sqrt{-5}$
c. $\sqrt{-18}$

Answer

a. $9i$
b. $\sqrt{5}i$
c. $3\sqrt{2}i$

Exercise 8.8.2

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-36}$
b. $\sqrt{-3}$
c. $\sqrt{-27}$

Answer

a. $6i$
b. $\sqrt{3}i$
c. $3\sqrt{3}i$

Now that we are familiar with the imaginary number i, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a+bi$, where a, b are real numbers. We call a the real part and b the imaginary part.

Definition 8.8.3

A **complex number** is of the form $a+bi$, where a and b are real numbers.

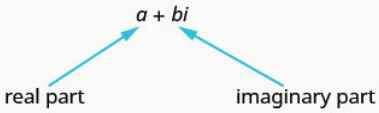


Figure 8.8.1

A complex number is in standard form when written as $a+bi$, where a and b are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq 0$, then $a+bi$ is an imaginary number.

If $a=0$, then $a+bi$ becomes $0+bi=bi$, and is called a pure imaginary number.

We summarize this here.

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$R	$0+bi$ bi	Pure imaginary numbe4

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.

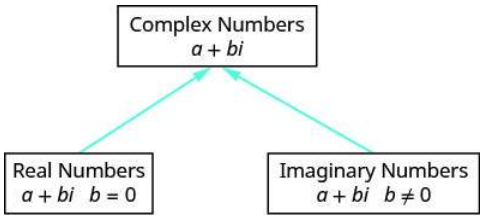


Figure 8.8.2

Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Example 8.8.2

Add: $\sqrt{-12}+\sqrt{-27}$.

Solution:

$$\sqrt{-12}+\sqrt{-27}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i+\sqrt{27}i$$

Simplify the square roots.

$$2\sqrt{3}i+3\sqrt{3}i$$

Add.

$$5\sqrt{3}i$$

Exercise 8.8.3

Add: $\sqrt{-8}+\sqrt{-32}$.

Answer

$6\sqrt{2}i$

Exercise 8.8.4

Add: $\sqrt{-27}+\sqrt{-48}$

Answer

$7\sqrt{3}i$

Remember to add both the real parts and the imaginary parts in this next example.

Example 8.8.3

Simplify:

a. $(4-3i)+(5+6i)$
b. $(2-5i)-(5-2i)$

Solution:

a.

$$(4-3i)+(5+6i)$$

Use the Associative Property to put the real parts and the imaginary parts together.

$$(4+5)+(-3i+6i)$$

Simplify.

$$9+3i$$

b.

$$(2-5i)-(5-2i)$$

Distribute.

$$2-5i-5+2i$$

Use the Associative Property to put the real parts and the imaginary parts together.

$$2-5-5i+2i$$

Simplify.

$$-3-3i$$

Exercise 8.8.5

Simplify:

a. $(2+7i)+(4-2i)$
b. $(8-4i)-(2-i)$

Answer

a. $6+5i$
b. $6-3i$

Exercise 8.8.6

Simplify:

a. $(3-2i)+(-5-4i)$
b. $(4+3i)-(2-6i)$

Answer

a. $-2-6i$
b. $2+9i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

Example 8.8.4

Multiply: $2i(7-5i)$

Solution:

$$2i(7-5i)$$

Distribute.

$$14i-10i^2$$

Simplify i^2.

$$14i-10(-1)$$

Multiply.

$$14i+10$$

Write in standard form.

$$10+14i$$

Exercise 8.8.7

Multiply: $4i(5-3i)$.

Answer

$12+20i$

Exercise 8.8.8

Multiply: $-3i(2+4i)$.

Answer

$12-6i$

In the next example, we multiply the binomials using the **Distributive Property** or **FOIL**.

Example 8.8.5

Multiply: $(3+2i)(4-3i)$.

Solution:

$$(3+2i)(4-3i)$$

Use FOIL.

$$12-9i+8i-6i^2$$

Simplify i^2 and combine like terms.

$$12-i-6(-1)$$

Multiply.

$$12-i+6$$

Combine the real parts.

$$18-i$$

Exercise 8.8.9

Multiple: $(5-3i)(-1-2i)$.

Answer

$-11-7i$

Exercise 8.8.10

Multiple: $(-4-3i)(2+i)$.

Answer

$-5-10i$

In the next example, we could use FOIL or the **Product of Binomial Squares Pattern**.

Example 8.8.6

Multiply: $(3+2i)^2$

Solution:

Table 8.8.2

	$(a+b)^2$ $(3+2i)^2$
Use the Product of Binomial Squares Pattern, $(a+b)^2=a^2+2ab+b^2$.	$a^2+2ab+b^2$ $3^2+2\cdot3\cdot2i+(2i)^2$
Simplify.	$9+12i+4i^2$
Simplify i^2.	$9+12i+4(-1)$
Simplify.	$5+12i$

Exercise 8.8.11

Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

Answer

$-21+20i$

Exercise 8.8.12

Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Answer

$9-40i$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b}=\sqrt{b}i$.This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

Example 8.8.7

Multiply: $\sqrt{-36}\cdot\sqrt{-4}$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

$$\sqrt{-36}\cdot\sqrt{-4}$$

Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.

$$\sqrt{36}i\cdot\sqrt{4}i$$

Simplify.

$$6i\cdot2i$$

Multiply.

$$12i^2$$

Simplify i^2 and multiply.

$$-12$$

Exercise 8.8.13

Multiply: $\sqrt{-49} \cdot \sqrt{-4}$.

Answer

-14

Exercise 8.8.14

Multiply: $\sqrt{-36} \cdot \sqrt{-81}$.

Answer

-54

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

Example 8.8.8

Multiply: $(3-\sqrt{-12})(5+\sqrt{-27})$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

$$(3-\sqrt{-12})(5+\sqrt{-27})$$

Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.

$$(3-2\sqrt{3}i)(5+3\sqrt{3}i)$$

Use FOIL.

$$15+9\sqrt{3}i-10\sqrt{3}i-6\cdot 3i^2$$

Combine like terms and simplify i^2.

$$15-\sqrt{3}i-6\cdot(-3)$$

Multiply and combine like terms.

$$33-\sqrt{3}i$$

Exercise 8.8.15

Multiply: $(4-\sqrt{-12})(3-\sqrt{-48})$.

Answer

$-12-22\sqrt{3}i$

Exercise 8.8.16

Multiply: $(-2+\sqrt{-8})(3-\sqrt{-18})$.

Answer

$6+12\sqrt{2}i$

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Definition 8.8.4

A **complex conjugate pair** is of the form $a+bi, a-bi$.

We will multiply a complex conjugate pair in the next example.

Example 8.8.9

Multiply: $(3-2i)(3+2i)$.

Solution:

$$(3-2i)(3+2i)$$

Use FOIL

$$9+6i-6i-4i^2$$

Combine like terms and simplify i^2.

$$9-4(-1)$$

Multiply and combine like terms.

$$13$$

Exercise 8.8.17

Multiply: $(4-3i)\cdot(4+3i)$.

Answer

25

Exercise 8.8.18

Multiply: $(-2+5i)\cdot(-2-5i)$.

Answer

29

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$ .The result is called a **difference of squares**. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the **Product of Conjugates Pattern**.

$$(a-b)(a+b)$$

$$(3-2i)(3+2i)$$

$$a^2-b^2$$

$$(3)^2-(2i)^2$$

$$9-4i^2$$

$$9-4(-1)$$

$$13$$

Figure 8.8.8

Notice this is the same result we found in Example 8.8.9.

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a^2+b^2$

Definition 8.8.5

Product of Complex Conjugates

If a and b are real numbers, then

$$(a-bi)(a+bi)=a^2+b^2$$

Example 8.8.10

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

Solution:

Table 8.8.3

	$(a-b)(a+b)$ $(8-2i)(8+2i)$
Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$.	a^2+b^2 8^2+2^2
Simplify the squares.	$64+4$
Add.	68

Exercise 8.8.19

Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

Answer

109

Exercise 8.8.20

Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

Answer

41

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

Example 8.8.11 how to divide complex numbers

Divide: $\frac{4+3i}{3-4i}$.

Solution:

Table 8.8.4

Step 1: Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$
Step 3: Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator. Combine like terms. Simplify. Write the result in standard form.	$\frac{12+16i+9i+12i^2}{9+16}$ $\frac{12+25i-12}{25}$ $\frac{25i}{25}$ i

Exercise 8.8.21

Divide: $\frac{2+5i}{5-2i}$.

Answer

i

Exercise 8.8.22

Divide: $\frac{1+6i}{6-i}$.

Answer

i

We summarize the steps here.

How to Divide Complex Numbers

1. Write both the numerator and denominator in standard form.
2. Multiply both the numerator and denominator by the complex conjugate of the denominator.
3. Simplify and write the result in standard form.

Example 8.8.12

Divide, writing the answers in standard form: $\frac{-3}{5+2i}$.

Solution:

$$\frac{-3}{5+2i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{-3(5-2i)}{(5+2i)(5-2i)}$$

Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator.

$$\frac{-15+6i}{5^2+2^2}$$

Simplify.

$$\frac{-15+6i}{29}$$

Write in standard form.

$$-\frac{15}{29}+\frac{6}{29}i$$

Exercise 8.8.23

Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

Answer

$\frac{4}{17}+\frac{16}{17}i$

Exercise 8.8.24

Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Answer

$\frac{2}{5}+\frac{4}{5}i$

Be careful as you find the conjugate of the denominator.

Example 8.8.13

Divide: $\frac{5+3i}{4i}$.

Solution:

$$\frac{5+3i}{4i}$$

Write the denominator in standard form.

$$\frac{5+3i}{0+4i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$$

Simplify.

$$\frac{(5+3i)(-4i)}{(4i)(-4i)}$$

Multiply.

$$\frac{-20i-12i^2}{-16i^2}.$$

Simplify the i^2.

$$\frac{-20i+12}{16}$$

Rewrite in standard form.

$$\frac{12}{16}-\frac{20}{16}i$$

Simplify the fractions.

$$\frac{3}{4}-\frac{5}{4}i$$

Exercise 8.8.25

Divide: $\frac{3+3i}{2i}$.

Answer

$\frac{3}{2}-\frac{3}{2}i$

Exercise 8.8.26

Divide: $\frac{2+4i}{5i}$.

Answer

$\frac{4}{5}-\frac{2}{5}i$

Simplify Powers of i

The powers of i make an interesting pattern that will help us simplify higher powers of i. Let's evaluate the powers of i to see the pattern.

$$\begin{array}{cccc} i^1 & i^2 & i^3 & i^4 \\ i & -1 & i^2\cdot i & i^2\cdot i^2 \\ & & -1\cdot i & (-1)(-1) \\ & & -i & 1 \end{array}$$

$$\begin{array}{cccc} i^5 & i^6 & i^7 & i^8 \\ i^4\cdot i & i^4\cdot i^2 & i^4\cdot i^3 & i^4\cdot i^4 \\ 1\cdot i & 1\cdot i^2 & 1\cdot i^3 & 1\cdot 1 \\ i & i^2 & i^3 & 1 \\ & -1 & -i & \end{array}$$

We summarize this now.

$$\begin{array}{ll} i^1=i & i^5=i \\ i^2=-1 & i^6=-1 \\ i^3=-i & i^7=-i \\ i^4=1 & i^8=1 \end{array}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of i. Since $i^4=1$, we rewrite each power, i^n, as a product using i^4 to a power and another power of i.

We rewrite it in the form $i^n=\left(i^4\right)^q\cdot i^r$, where the exponent, q, is the quotient of n divided by 4 and the exponent, r, is the remainder from this division. For example, to simplify i^{57}, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57=4\cdot 14+1$. So we write $i^{57}=\left(1^4\right)^{14}\cdot i^1$ and then simplify from there.

$$\begin{array}{r|l} 14 & i^{57} \\ 4\overline{)57} & \left(1^4\right)^{14}\cdot i^1 \\ \underline{4} & \\ 17 & 1\cdot i \\ \underline{16} & \\ 1 & i \end{array}$$

Figure 8.8.13

Example 8.8.14

Simplify: i^{86}.

Solution:

$$i^{86}$$

Divide 86 by 4 and rewrite i^{86} in the $i^n=\left(i^4\right)^q\cdot i^r$ form.

$$\left(1^4\right)^{21}\cdot i^2$$

$$\begin{array}{rl} 21 & \text{(divide the exponent by the number of repeats in the imaginary number i)} \\ 4\overline{)86} & \\ \underline{8} & \text{(four goes into eight twice with zero remainders)} \\ 06 & \\ \underline{4} & \text{(four goes into 6 one time with 2 remainders)} \\ 2 & \text{(remainder is 2 so calculate the imaginary number with an exponent of two)} \end{array} \tag{8.8.1}$$

Simplify.

$$(1)^{21} \cdot (-1)$$

Simplify.

$$-1$$

? Exercise 8.8.27

Simplify: i^{74}.

Answer

-1

? Exercise 8.8.28

Simplify: i^{92}.

Answer

1

Access these online resources for additional instruction and practice with the complex number system.

- Expressing Square Roots of Negative Numbers with i
- Subtract and Multiply Complex Numbers
- Dividing Complex Numbers
- Rewriting Powers of i

Key Concepts

- **Square Root of a Negative Number**
  - If b is a positive real number, then \(\sqrt{-b}=\sqrt{b} i\

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bi$ bi	Pure imaginary number

- A complex number is in **standard form** when written as $a + bi$, where a, b are real numbers.

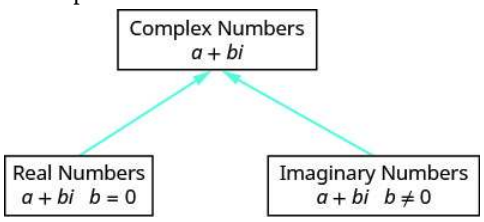


Figure 8.8.2

- **Product of Complex Conjugates**
  - If a, b are real numbers, then $(a-bi)(a+bi) = a^2 + b^2$
- **How to Divide Complex Numbers**
  1. Write both the numerator and denominator in standard form.
  2. Multiply the numerator and denominator by the complex conjugate of the denominator.
  3. Simplify and write the result in standard form.

Glossary

complex conjugate pair

A complex conjugate pair is of the form $a+bi, a-bi$.

complex number

A complex number is of the form $a+bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system

The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit

The imaginary unit i is the number whose square is -1. $i^2 = -1$ or $i = \sqrt{-1}$.

standard form

A complex number is in standard form when written as $a+bi$, where a, b are real numbers.

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8.8E: Exercises

Practice Makes Perfect

Exercises 1 - 4: Evaluate a Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

1. a. $\sqrt{-16}$ b. $\sqrt{-11}$ c. $\sqrt{-8}$
2. a. $\sqrt{-121}$ b. $\sqrt{-1}$ c. $\sqrt{-20}$
3. a. $\sqrt{-100}$ b. $\sqrt{-13}$ c. $\sqrt{-45}$
4. a. $\sqrt{-49}$ b. $\sqrt{-15}$ c. $\sqrt{-75}$

Answer

1. a. $4i$ b. $i\sqrt{11}$ c. $2i\sqrt{2}$
3. a. $10i$ b. $i\sqrt{13}$ c. $3i\sqrt{5}$

ExerciseS 5 - 21: Add or Subtract Complex Numbers

In the following exercises, add or subtract, putting the answer in $a+bi$ form.

5. $\sqrt{-75}+\sqrt{-48}$
6. $\sqrt{-12}+\sqrt{-75}$
7. $\sqrt{-50}+\sqrt{-18}$
8. $\sqrt{-72}+\sqrt{-8}$
9. $(1+3i)+(7+4i)$
10. $(6+2i)+(3-4i)$
11. $(8-i)+(6+3i)$
12. $(7-4i)+(-2-6i)$
13. $(1-4i)-(3-6i)$
14. $(8-4i)-(3+7i)$
15. $(6+i)-(-2-4i)$
16. $(-2+5i)-(-5+6i)$
17. $(5-\sqrt{-36})+(2-\sqrt{-49})$
18. $(-3+\sqrt{-64})+(5-\sqrt{-16})$
19. $(-7-\sqrt{-50})-(-32-\sqrt{-18})$
20. $(-5+\sqrt{-27})-(-4-\sqrt{-48})$

Answer

5. $0+\left(9\sqrt{3}\right)i$
7. $0+\left(8\sqrt{2}\right)i$
9. $8+7i$
11. $14+2i$
13. $-2+2i$
15. $8+5i$

17. $7-13i$

19. $25-\left(2\sqrt{2}\right)i$

? ExerciseS 21 - 28: Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

21. $4i(5-3i)$

22. $2i(-3+4i)$

23. $-6i(-3-2i)$

24. $-i(6+5i)$

25. $(4+3i)(-5+6i)$

26. $(-2-5i)(-4+3i)$

27. $(-3+3i)(-2-7i)$

28. $(-6-2i)(-3-5i)$

Answer

21. $12+20i$

23. $-12+18i$

25. $-38+9i$

27. $27+15i$

? ExerciseS 29 - 32: Multiply Complex Numbers

In the following exercises, multiply using the Product of Binomial Squares Pattern, putting the answer in $a+bi$ form.

29. $(3+4i)^2$

30. $(-1+5i)^2$

31. $(-2-3i)^2$

32. $(-6-5i)^2$

Answer

29. $-7+24i$

31. $-5-12i$

? Exercises 33 - 46: Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

33. $\sqrt{-25}\cdot\sqrt{-36}$

34. $\sqrt{-4}\cdot\sqrt{-16}$

35. $\sqrt{-9}\cdot\sqrt{-100}$

36. $\sqrt{-64}\cdot\sqrt{-9}$

37. $(-2-\sqrt{-27})(4-\sqrt{-48})$

38. $(5-\sqrt{-12})(-3+\sqrt{-75})$

39. $(2+\sqrt{-8})(-4+\sqrt{-18})$

40. $(5+\sqrt{-18})(-2-\sqrt{-50})$

41. $(2-i)(2+i)$

42. $(4-5i)(4+5i)$

43. $(7-2i)(7+2i)$

44. $(-3-8i)(-3+8i)$

Answer

33. $30i = 0+30i$

35. $-30 = -30+0i$

37. $-44+\left(4\sqrt{3}\right)i$

39. $-20-\left(2\sqrt{2}\right)i$

41. $5 = 5+0i$

43. $53 = 53+0i$

? ExerciseS 45 - 49: Multiply Complex Numbers

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

45. $(7-i)(7+i)$

46. $(6-5i)(6+5i)$

47. $(9-2i)(9+2i)$

48. $(-3-4i)(-3+4i)$

Answer

45. 50

47. 85

? ExerciseS 49 - 60: Divide Complex Numbers

In the following exercises, divide, putting the answer in $a+bi$ form.

49. $\dfrac{3+4i}{4-3i}$

50. $\dfrac{5-2i}{2+5i}$

51. $\dfrac{2+i}{3-4i}$

52. $\dfrac{3-2i}{6+i}$

53. $\dfrac{3}{2-3i}$

54. $\dfrac{2}{4-5i}$

55. $\dfrac{-4}{3-2i}$

56. $\dfrac{-1}{3+2i}$

57. $\dfrac{1+4i}{3i}$

58. $\dfrac{4+3i}{7i}$

59. $\dfrac{-2-3i}{4i}$

60. $\dfrac{-3-5i}{2i}$

Answer

49. $i=0+i$

51. $\frac{2}{25}+\frac{11}{25}i$

53. $\frac{6}{13}+\frac{9}{13}i$

55. $-\frac{12}{13}-\frac{8}{13}i$

57. $\frac{4}{3}-\frac{1}{3}i$

59. $-\frac{3}{4}+\frac{1}{2}i$

? ExerciseS 61 - 68: Simplify Powers of i

In the following exercises, simplify.

61. i^{41}

62. i^{39}

63. i^{66}

64. i^{48}

65. i^{128}

66. i^{162}

67. i^{137}

68. i^{255}

Answer

61. $i^{41}=i^{40}\cdot i=\left(i^4\right)^{10}\cdot i=i$

63. $i^{66}=i^{64}\cdot i^2=\left(i^4\right)^{16}\cdot(-1)=-1$

65. $i^{128}=\left(i^4\right)^{32}=1$

67. $i^{137}=i^{136}\cdot i=\left(i^4\right)^{34}\cdot i=1\cdot i=i$

? ExerciseS 69 - 72: Writing Exercises

69. Explain the relationship between real numbers and complex numbers.

70. Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$\sqrt{-7}\cdot\sqrt{-7}$

$\sqrt{49}$

7

71. Why is $\sqrt{-64} = 8i$ but $\sqrt[3]{-64} = -4$.

72. Explain how dividing complex numbers is similar to rationalizing a denominator.

Answer

69. Answers may vary

71. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate the square root of a negative number.			
add or subtract complex numbers.			
multiply complex numbers.			
divide complex numbers.			
simplify powers of *i*.			

Figure 8.8.15

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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Chapter 8 Review Exercises

Chapter Review Exercises

Simplify Expressions with Roots

Exercise 1 Simplify Expressions with Roots

In the following exercises, simplify.

1. a. $\sqrt{225}$
   b. $-\sqrt{16}$
2. a. $-\sqrt{169}$
   b. $\sqrt{-8}$
3. a. $\sqrt[3]{8}$
   b. $\sqrt[4]{81}$
   c. $\sqrt[5]{243}$
4. a. $\sqrt[3]{-512}$
   b. $\sqrt[4]{-81}$
   c. $\sqrt[5]{-1}$

Answer

1.
   a. 15
   b. -4
3.
   a. 2
   b. 3
   c. 3

Exercise 2 Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

1. a. $\sqrt{68}$
   b. $\sqrt[3]{84}$

Answer

1.
   a. $8 < \sqrt{68} < 9$
   b. $4 < \sqrt[3]{84} < 5$

Exercise 3 Estimate and Approximate Roots

In the following exercises, approximate each root and round to two decimal places.

1. a. $\sqrt{37}$
   b. $\sqrt[3]{84}$
   c. $\sqrt[4]{125}$

Answer

1. Solve for yourself

Exercise 4 Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

1. a. $\sqrt[3]{a^3}$
   b. $\sqrt[7]{b^7}$
2. a. $\sqrt{a^{14}}$
   b. $\sqrt{w^{24}}$
3. a. $\sqrt[4]{m^8}$
   b. $\sqrt[5]{n^{20}}$
4. a. $\sqrt{121m^{20}}$
   b. $-\sqrt{64a^2}$
5. a. $\sqrt[3]{216a^6}$
   b. $\sqrt[5]{32b^{20}}$
6. a. $\sqrt{144x^2y^2}$
   b. $\sqrt{169w^8y^{10}}$
   c. $\sqrt[3]{8a^{51}b^6}$

Answer

1.
   a. a
   b. $|b|$
3.
   a. m^2
   b. n^4
5.
   a. $6a^2$
   b. $2b^4$

Simplify Radical Expressions

Exercise 5 Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

1. $\sqrt{125}$
2. $\sqrt{675}$
3. a. $\sqrt[3]{625}$
   b. $\sqrt[6]{128}$

Answer

1. $5\sqrt{5}$
3.
   a. $5\sqrt[3]{5}$
   b. $2\sqrt[6]{2}$

Exercise 6 Use the Product Property to Simplify Radical Expressions

In the following exercises, simplify using absolute value signs as needed.

1. a. $\sqrt{a^{23}}$
   b. $\sqrt[3]{b^8}$
   c. $\sqrt[8]{c^{13}}$
2. a. $\sqrt{80s^{15}}$
   b. $\sqrt[5]{96a^7}$
   c. $\sqrt[6]{128b^7}$
3. a. $\sqrt{96r^3s^3}$
   b. $\sqrt[3]{80x^7y^6}$
   c. $\sqrt[4]{80x^8y^9}$
4. a. $\sqrt[5]{-32}$
   b. $\sqrt[8]{-1}$
5. a. $8+\sqrt{96}$
   b. $\frac{2+\sqrt{40}}{2}$

Answer

2.

a. $4\left|s^7\right|\sqrt{5s}$
b. $2a\sqrt[5]{3a^2}$
c. $2|b|\sqrt[6]{2b}$

4.

a. -2
b. not real

Exercise 7 Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

1. a. $\sqrt{\frac{72}{98}}$
   b. $\sqrt[3]{\frac{24}{81}}$
   c. $\sqrt[4]{\frac{6}{96}}$
2. a. $\sqrt{\frac{y^4}{y^8}}$
   b. $\sqrt[5]{\frac{u^{21}}{u^{11}}}$
   c. $\sqrt[6]{\frac{v^{30}}{v^{12}}}$
3. $\sqrt{\frac{300m^5}{64}}$
4. a. $\sqrt{\frac{28p^7}{q^2}}$
   b. $\sqrt[3]{\frac{81s^8}{t^3}}$
   c. $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$
5. a. $\sqrt{\frac{27p^2q}{108p^4q^3}}$

b. $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$

c. $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

6. a. $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

b. $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

c. $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

Answer

1.

a. $\frac{6}{7}$

b. $\frac{2}{3}$

c. $\frac{1}{2}$

3. $\frac{10m^2\sqrt{3m}}{8}$

5.

a. $\frac{1}{2|pq|}$

b. $\frac{2cd\sqrt[5]{2d^2}}{5}$

c. $\frac{|mn|\sqrt[6]{2}}{2}$

Simplify Rational Exponents

Exercise 8 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

1. a. $r^{\frac{1}{2}}$

b. $s^{\frac{1}{3}}$

c. $t^{\frac{1}{4}}$

Answer

1.

a. $\sqrt{r}$

b. $\sqrt[3]{s}$

c. $\sqrt[4]{t}$

Exercise 9 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt{21p}$

b. $\sqrt[4]{8q}$

c. $4\sqrt[6]{36r}$

Answer

1. Solve for yourself

Exercise 10 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, simplify.

1. a. $625^{\frac{1}{4}}$
   b. $243^{\frac{1}{5}}$
   c. $32^{\frac{1}{5}}$
2. a. $(-1,000)^{\frac{1}{3}}$
   b. $-1,000^{\frac{1}{3}}$
   c. $(1,000)^{-\frac{1}{3}}$
3. a. $(-32)^{\frac{1}{5}}$
   b. $(243)^{-\frac{1}{5}}$
   c. $-125^{\frac{1}{3}}$

Answer

1.
   a. 5
   b. 3
   c. 2
3.
   a. -2
   b. $\frac{1}{3}$
   c. -5

Exercise 11 Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt[4]{r^7}$
   b. $(\sqrt[5]{2pq})^3$
   c. $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

Answer

1. Solve for yourself

Exercise 12 Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, simplify.

1. a. $25^{\frac{3}{2}}$
   b. $9^{-\frac{3}{2}}$
   c. $(-64)^{\frac{2}{3}}$
2. a. $-64^{\frac{3}{2}}$
   b. $-64^{-\frac{3}{2}}$
   c. $(-64)^{\frac{3}{2}}$

Answer

1.

a. 125
b. $\frac{1}{27}$
c. 16

Exercise 13 Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

1. a. $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
   b. $\left(b^{15}\right)^{\frac{3}{5}}$
   c. $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$
2. a. $\frac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$
   b. $\left(\frac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

Answer

1.
   a. 6^3
   b. b^9
   c. $\frac{1}{w}$

Add, Subtract and Multiply Radical Expressions

Exercise 14 add and Subtract Radical Expressions

In the following exercises, simplify.

1. a. $7\sqrt{2}-3\sqrt{2}$
   b. $7\sqrt[3]{p}+2\sqrt[3]{p}$
   c. $5\sqrt[3]{x}-3\sqrt[3]{x}$
2. a. $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$
   b. $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$
3. a. $\sqrt{48}+\sqrt{27}$
   b. $\sqrt[3]{54}+\sqrt[3]{128}$
   c. $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$
4. a. $\sqrt{80c^7}-\sqrt{20c^7}$
   b. $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$
5. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Answer

1.
   a. $4\sqrt{2}$
   b. $9\sqrt[3]{p}$
   c. $2\sqrt[3]{x}$
3.
   a. $7\sqrt{3}$
   b. $7\sqrt[3]{2}$

c. $3\sqrt[4]{5}$

5. $37y\sqrt{3}$

Exercise 15 Multiply Radical Expressions

In the following exercises, simplify.

1. a. $(5\sqrt{6})(-\sqrt{12})$
   b. $(-2\sqrt[4]{18})(-\sqrt[4]{9})$
2. a. $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$
   b. $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

Answer

2.

a. $126x^2\sqrt{2}$
b. $48a\sqrt[3]{a^2}$

Exercise 16 Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

1. a. $\sqrt{11}(8+4\sqrt{11})$
   b. $\sqrt[3]{3}(\sqrt[3]{9}+\sqrt[3]{18})$
2. a. $(3-2\sqrt{7})(5-4\sqrt{7})$
   b. $(\sqrt[3]{x}-5)(\sqrt[3]{x}-3)$
3. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$
4. a. $(4+\sqrt{11})^2$
   b. $(3-2\sqrt{5})^2$
5. $(7+\sqrt{10})(7-\sqrt{10})$
6. $(\sqrt[3]{3x}+2)(\sqrt[3]{3x}-2)$

Answer

2.

a. $71-22\sqrt{7}$
b. $\sqrt[3]{x^2}-8\sqrt[3]{x}+15$

4.

a. $27+8\sqrt{11}$
b. $29-12\sqrt{5}$

6. $\sqrt[3]{9x^2}-4$

Divide Radical Expressions

Exercise 17 Divide Square Roots

In the following exercises, simplify.

1. a. $\frac{\sqrt{48}}{\sqrt{75}}$
   b. $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$
2. a. $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$

b. $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

Answer

2.

a. $\frac{8m^4}{3n^4}$

b. $-\frac{x^2}{2y^2}$

Exercise 18 rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

1. a. $\frac{8}{\sqrt{3}}$
   b. $\sqrt{\frac{7}{40}}$
   c. $\frac{8}{\sqrt{2y}}$
2. a. $\frac{1}{\sqrt[3]{11}}$
   b. $\sqrt[3]{\frac{7}{54}}$
   c. $\frac{3}{\sqrt[3]{3x^2}}$
3. a. $\frac{1}{\sqrt[4]{4}}$
   b. $\sqrt[4]{\frac{9}{32}}$
   c. $\frac{6}{\sqrt[4]{9x^3}}$

Answer

2.

a. $\frac{\sqrt[3]{121}}{11}$

b. $\frac{\sqrt[3]{28}}{6}$

c. $\frac{\sqrt[3]{9x}}{x}$

Exercise 19 Rationalize a Two Term Denominator

In the following exercises, simplify.

1. $\frac{7}{2-\sqrt{6}}$
2. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$
3. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

Answer

1. $-\frac{7(2+\sqrt{6})}{2}$
3. $\frac{(\sqrt{x}+2\sqrt{2})^2}{x-8}$

Solve Radical Equations

Exercise 20 Solve Radical Equations

In the following exercises, solve.

1. $\sqrt{4x-3}=7$
2. $\sqrt{5x+1}=-3$
3. $\sqrt[3]{4x-1}=3$
4. $\sqrt{u-3}+3=u$
5. $\sqrt[3]{4x+5}-2=-5$
6. $(8x+5)^{\frac{1}{3}}+2=-1$
7. $\sqrt{y+4}-y+2=0$
8. $2\sqrt{8r+1}-8=2$

Answer

2. no solution

4. $u=3, u=4$

6. $x=-4$

8. $r=3$

Exercise 21 Solve Radical Equations with Two Radicals

In the following exercises, solve.

1. $\sqrt{10+2c}=\sqrt{4c+16}$
2. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$
3. $\sqrt{r}+6=\sqrt{r+8}$
4. $\sqrt{x+1}-\sqrt{x-2}=1$

Answer

2. $x=-8, x=2$

4. $x=3$

Exercise 22 Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

1. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answers to th nearest tenth of a foot.
2. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

2. 64.8feet

Use Radicals in Functions

Exercise 23 Evaluate a Radical Function

In the following exercises, evaluate each function.

1. $g(x)=\sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
2. $G(x)=\sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
3. $h(x)=\sqrt[3]{x^2-4}$, find
   a. $h(-2)$
   b. $h(6)$
4. For the function $g(x)=\sqrt[4]{4-4x}$, find
   a. $g(1)$
   b. $g(-3)$

Answer

2.

a. $G(5)=2\sqrt{6}$
b. $G(2)=3$

4.

a. $g(1)=0$
b. $g(-3)=2$

Exercise 24 Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

1. $g(x)=\sqrt{2-3x}$
2. $F(x)=\sqrt{\frac{x+3}{x-2}}$
3. $f(x)=\sqrt[3]{4x^2-16}$
4. $F(x)=\sqrt[4]{10-7x}$

Answer

2. $(2,\infty)$

4. $\left[\frac{7}{10},\infty\right)$

Exercise 25 graph Radical Functions

In the following exercises,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $g(x)=\sqrt{x+4}$
2. $g(x)=2\sqrt{x}$
3. $f(x)=\sqrt[3]{x-1}$
4. $f(x)=\sqrt[3]{x}+3$

Answer

2.

a. domain: $[0, \infty)$

b.

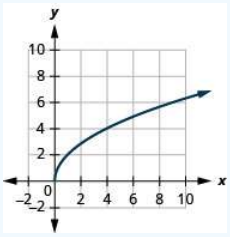


Figure 8.E.1

c. range: $[0, \infty)$

4.

a. domain: $(-\infty, \infty)$

b.

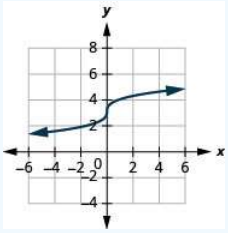


Figure 8.E.2

c. range: $(-\infty, \infty)$

Use the Complex Number System

Exercise 26 evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

1. a. $\sqrt{-100}$
   b. $\sqrt{-13}$
   c. $\sqrt{-45}$

Answer

Solve for yourself

Exercise 27 Add or Subtract Complex Numbers

In the following exercises, add or subtract.

1. $\sqrt{-50} + \sqrt{-18}$
2. $(8 - i) + (6 + 3i)$
3. $(6 + i) - (-2 - 4i)$
4. $(-7 - \sqrt{-50}) - (-32 - \sqrt{-18})$

Answer

1. $8\sqrt{2}i$
3. $8 + 5i$

Exercise 28 Multiply Complex Numbers

In the following exercises, multiply.

1. $(-2-5i)(-4+3i)$
2. $-6i(-3-2i)$
3. $\sqrt{-4}\cdot\sqrt{-16}$
4. $(5-\sqrt{-12})(-3+\sqrt{-75})$

Answer

1. $23+14i$
3. -6

Exercise 29 Multiply Complex Numbers

In the following exercises, multiply using the Product of Binomial Squares Pattern.

1. $(-2-3i)^2$

Answer

1. $-5-12i$

Exercise 30 Multiply Complex Numbers

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

1. $(9-2i)(9+2i)$

Answer

Solve for yourself

Exercise 31 divide Complex Numbers

In the following exercises, divide.

1. $\frac{2+i}{3-4i}$
2. $\frac{-4}{3-2i}$

Answer

1. $\frac{2}{25}+\frac{11}{25}i$

Exercise 32 Simplify Powers of i

In the following exercises, simplify.

1. i^{48}
2. i^{255}

Answer

1. 1

Practice Test

Exercise 33

In the following exercises, simplify using absolute values as necessary.

1. $\sqrt[3]{125x^9}$
2. $\sqrt{169x^8y^6}$
3. $\sqrt[3]{72x^8y^4}$
4. $\sqrt{\frac{45x^3y^4}{180x^5y^2}}$

Answer

1. $5x^3$
3. $2x^2y\sqrt[3]{9x^2y}$

Exercise 34

In the following exercises, simplify. Assume all variables are positive.

1. a. $216^{-\frac{1}{4}}$
   b. $-49^{\frac{3}{2}}$
2. $\sqrt{-45}$
3. $\frac{x^{-\frac{1}{4}} \cdot x^{\frac{5}{4}}}{x^{-\frac{3}{4}}}$
4. $\left(\frac{8x^{\frac{2}{3}}y^{-\frac{5}{2}}}{x^{-\frac{7}{3}}y^{\frac{1}{2}}}\right)^{\frac{1}{3}}$
5. $\sqrt{48x^5} - \sqrt{75x^5}$
6. $\sqrt{27x^2} - 4x\sqrt{12} + \sqrt{108x^2}$
7. $2\sqrt{12x^5} \cdot 3\sqrt{6x^3}$
8. $\sqrt[3]{4}(\sqrt[3]{16} - \sqrt[3]{6})$
9. $(4 - 3\sqrt{3})(5 + 2\sqrt{3})$
10. $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$
11. $\frac{\sqrt{245xy^{-4}}}{\sqrt{45x^4y^3}}$
12. $\frac{1}{\sqrt[3]{5}}$
13. $\frac{3}{2+\sqrt{3}}$
14. $\sqrt{-4} \cdot \sqrt{-9}$
15. $-4i(-2 - 3i)$
16. $\frac{4+i}{3-2i}$
17. i^{172}

Answer

1.
   a. $\frac{1}{4}$
   b. -343
3. $x^{\frac{7}{4}}$
5. $-x^2\sqrt{3x}$
7. $36x^4\sqrt{2}$

9. $2-7\sqrt{3}$

11. $\frac{7x^5}{3y^7}$

13. $3(2-\sqrt{3})$

15. $-12+8i$

17. $-i$

Exercise 35

In the following exercises, solve.

1. $\sqrt{2x+5}+8=6$
2. $\sqrt{x+5}+1=x$
3. $\sqrt[3]{2x^2-6x-23}=\sqrt[3]{x^2-3x+5}$

Answer

2. $x=4$

Exercise 36

In the following exercise,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $g(x)=\sqrt{x+2}$

Answer

1.

a. domain: $[-2,\infty)$
b.

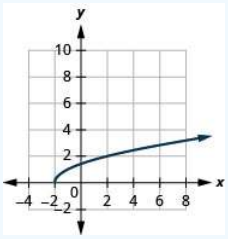


Figure 8.E.3

c. range: $[0,\infty)$

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CHAPTER OVERVIEW

9: Quadratic Equations and Functions

9.0: Prelude to Quadratic Equations and Functions
9.1: Solve Quadratic Equations Using the Square Root Property
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Chapter 9 Review Exercises

Thumbnail: Plot of the quadratic function. (Public Domain; N.Mori).

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9.0: Prelude to Quadratic Equations and Functions

Blink your eyes. You've taken a photo. That's what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

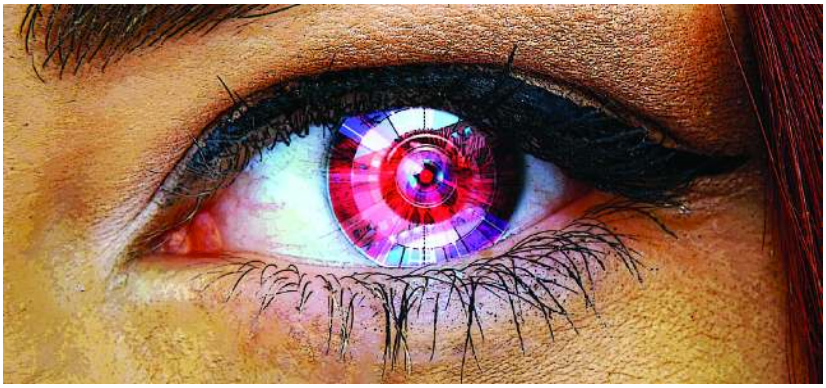


Figure 9.0.1: Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: "intographics"/Pixabay)

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9.1: Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
   If you missed this problem, review Example 8.13.
2. Simplify: $\sqrt{\frac{32}{5}}$.
   If you missed this problem, review Example 8.50.
3. Factor: $9x^2 - 12x + 4$.
   If you missed this problem, review Example 6.23.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the Form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

$$x^2 = 9$$

Put the equation in standard form.

$$x^2 - 9 = 0$$

Factor the difference of squares.

$$(x-3)(x+3) = 0$$

Use the Zero Produce Property.

$$x - 3 = 0 \quad x - 3 = 0$$

Solve each equation.

$$x = 3 \quad x = -3$$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4, x = -4$ and $x = 5, x = -5$

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If $n^2 = m$, then n is a square root of m.

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Definition 9.1.1

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

$$\begin{aligned} & & x^2 &= 9 \\ &\text{Use the Square Root Property.} & x &= \pm\sqrt{9} \\ & & x &= \pm 3 \end{aligned}$$

So $x = 3$ or $x = -3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

$$x^2 = 7$$

Use the Square Root Property. $x = \sqrt{7}, \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

Example 9.1.1 How to Solve a Quadratic Equation of the form $ax^2 - k$ Using the Square Root Property

Solve: $x^2 - 50 = 0$.

Solution:

Table 9.1.1

Step 1: Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2: Use the Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$
Step 3: Simplify the radical.	Rewrite to show two solutions.	$x = \pm\sqrt{25}\cdot\sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2},\ x = -5\sqrt{2}$
Step 4: Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25\cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25\cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$

Exercise 9.1.1

Solve: $x^2 - 48 = 0$.

Answer

$x = 4\sqrt{3}, x = -4\sqrt{3}$

Exercise 9.1.2

Solve: $y^2 - 27 = 0$.

Answer

$y = 3\sqrt{3}, y = -3\sqrt{3}$

The steps to take to use the **Square Root Property** to solve a quadratic equation are listed here.

Solve a Quadratic Equation Using the Square Root Property

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

Example 9.1.2

Solve: $3z^2 = 108$.

Solution:

Table 9.1.2

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6, \quad z = -6$
Check the solutions: $3z^2 = 108$ $\quad$ $3z^2 = 108$ $3(6)^2 \stackrel{?}{=} 108$ $\quad$ $3(-6)^2 \stackrel{?}{=} 108$ $3(36) \stackrel{?}{=} 108$ $\quad$ $3(36) \stackrel{?}{=} 108$ $108 = 108$ ✓ $\quad$ $108 = 108$ ✓ Figure 9.1.1	

Exercise 9.1.3

Solve: $2x^2 = 98$.

Answer

$x = 7, x = -7$

Exercise 9.1.4

Solve: $5m^2 = 80$.

Answer

$m = 4, m = -4$

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

Example 9.1.3

Solve: $x^2 + 72 = 0$.

Solution:

Table 9.1.3

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}i$
Simplify the radical.	$x = \pm 6\sqrt{2}i$
Rewrite to show two solutions	$x = 6\sqrt{2}i, x = -6\sqrt{2}i$
Check the solutions: $x^2 + 72 = 0$ $(6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$ $(6)^2 \cdot (\sqrt{2})^2 \cdot i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0$ ✓	$x^2 + 72 = 0$ $(-6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$ $(-6)^2 \cdot (\sqrt{2})^2 \cdot i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0$ ✓

Exercise 9.1.5

Solve: $c^2 + 12 = 0$.

Answer

$c = 2\sqrt{3}i, \quad c = -2\sqrt{3}i$

Exercise 9.1.6

Solve: $q^2 + 24 = 0$.

Answer

$c = 2\sqrt{6}i, \quad c = -2\sqrt{6}i$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

Example 9.1.4

Solve: $\frac{2}{3}u^2 + 5 = 17$.

Solution:

Table 9.1.4

	$\frac{2}{3}u^2+5=17$
Isolate the quadratic term.	$\frac{2}{3}u^2=12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2}\cdot\frac{2}{3}u^2=\frac{3}{2}\cdot 12$
Simplify.	$u^2=18$
Use the Square Root Property.	$u=\pm\sqrt{18}$
Simplify the radical.	$u=\pm\sqrt{9\cdot 2}$
Simplify.	$u=\pm 3\sqrt{2}$
Rewrite to show two solutions.	$u=3\sqrt{2},\quad u=-3\sqrt{2}$
Check: $\frac{2}{3}u^2+5=17$ $\frac{2}{3}(3\sqrt{2})^2+5\stackrel{?}{=}17$ $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$ $12+5\stackrel{?}{=}17$ $17=17\checkmark$	$\frac{2}{3}u^2+5=17$ $\frac{2}{3}(-3\sqrt{2})^2+5\stackrel{?}{=}17$ $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$ $12+5\stackrel{?}{=}17$ $17=17\checkmark$

? Exercise 9.1.7

Solve: $\frac{1}{2}x^2+4=24$.

Answer

$x=2\sqrt{10}, x=-2\sqrt{10}$

? Exercise 9.1.8

Solve: $\frac{3}{4}y^2-3=18$.

Answer

$y=2\sqrt{7}, y=-2\sqrt{7}$

The solutions to some equations may have fractions inside the radicals. When this happens, we must **rationalize the denominator**.

✓ Example 9.1.5

Solve: $2x^2-8=41$.

Solution:

Table 9.1.5

	$2x^2-8=41$
Isolate the quadratic term.	$2x^2=49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2}=\frac{49}{2}$
Simplify.	$x^2=\frac{49}{2}$

Use the Square Root Property.	$x=\pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x=\pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x=\pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$
Simplify.	$x=\pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x=\frac{7\sqrt{2}}{2},\quad x=-\frac{7\sqrt{2}}{2}$
Check: We leave te check for you.	

? Exercise 9.1.9

Solve: $5r^2-2=34$.

Answer

$r=\frac{6\sqrt{5}}{5},\quad r=-\frac{6\sqrt{5}}{5}$

? Exercise 9.1.10

Solve: $3t^2+6=70$.

Answer

$t=\frac{8\sqrt{3}}{3},\quad t=-\frac{8\sqrt{3}}{3}$

Solve Quadratic Equation of the Form $a(x-h)^2=k$ Using the Square Root Property

We can use the **Square Root Property** to solve an equation of the form $a(x-h)^2=k$ as well. Notice that the quadratic term, x, in the original form $ax^2=k$ is replaced with $(x-h)$.

$ax^2=k$ $\quad$ $a(x-h)^2=k$

Figure 9.1.20

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x-h)^2$.

✓ Example 9.1.6

Solve: $4(y-7)^2=48$.

Solution:

Table 9.1.6

	$4(y-7)^2=48$
Divide both sides by the coefficient 4.	$(y-7)^2=12$
Use the Square Root Property on the binomial.	$y-7=\pm\sqrt{12}$
Simplify the radical.	$y-7=\pm2\sqrt{3}$
Solve for y.	$y=7\pm2\sqrt{3}$

Rewrite to show two solutions.	$y = 7 + 2\sqrt{3}$ $y = 7 - 2\sqrt{3}$

Check:

$4(y-7)^2 = 48$ | $4(y-7)^2 = 48$

$4(7+2\sqrt{3}-7)^2 \stackrel{?}{=} 48$ | $4(7-2\sqrt{3}-7)^2 \stackrel{?}{=} 48$

$4(2\sqrt{3})^2 \stackrel{?}{=} 48$ | $4(-2\sqrt{3})^2 \stackrel{?}{=} 48$

$4(12) \stackrel{?}{=} 48$ | $4(12) \stackrel{?}{=} 48$

$48 = 48$ ✓ | $48 = 48$ ✓

Figure 9.1.21

? Exercise 9.1.11

Solve: $3(a-3)^2 = 54$.

Answer

$a = 3 + 3\sqrt{2}, \quad a = 3 - 3\sqrt{2}$

? Exercise 9.1.12

Solve: $2(b+2)^2 = 80$.

Answer

$b = -2 + 2\sqrt{10}, \quad b = -2 - 2\sqrt{10}$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

✓ Example 9.1.7

Solve: $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$.

Solution:

$$\left(x - \tfrac{1}{3}\right)^2 = \tfrac{5}{9}$$

Use the Square Root Property.

$$x - \tfrac{1}{3} = \pm\sqrt{\tfrac{5}{9}}$$

Rewrite the radical as a fraction of square roots.

$$x - \tfrac{1}{3} = \pm\tfrac{\sqrt{5}}{\sqrt{9}}$$

Simplify the radical.

$$x - \tfrac{1}{3} = \pm\tfrac{\sqrt{5}}{3}$$

Solve for x.

$$x = \tfrac{1}{3} \pm \tfrac{\sqrt{5}}{3}$$

Rewrite to show two solutions.

$$x = \tfrac{1}{3} + \tfrac{\sqrt{5}}{3}, x = \tfrac{1}{3} - \tfrac{\sqrt{5}}{3}$$

Check:

We leave the check for you.

Exercise 9.1.13

Solve: $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

Answer

$x=\frac{1}{2}+\frac{\sqrt{5}}{2}, x=\frac{1}{2}-\frac{\sqrt{5}}{2}$

Exercise 9.1.14

Solve: $\left(y+\frac{3}{4}\right)^2=\frac{7}{16}$.

Answer

$y=-\frac{3}{4}+\frac{\sqrt{7}}{4}, y=-\frac{3}{4}-\frac{\sqrt{7}}{4}$

We will start the solution to the next example by isolating the binomial term.

Example 9.1.8

Solve: $2(x-2)^2+3=57$.

Solution:

$$2(x-2)^2+3=57$$

Subtract 3 from both sides to isolate the binomial term.

$$2(x-2)^2=54$$

Divide both sides by 2.

$$(x-2)^2=27$$

Use the Square Root Property.

$$x-2=\pm\sqrt{27}$$

Simplify the radical.

$$x-2=\pm 3\sqrt{3}$$

Solve for x.

$$x=2\pm 3\sqrt{3}$$

Rewrite to show two solutions.

$$x=2+3\sqrt{3}, x=2-3\sqrt{3}$$

Check:

We leave the check for you.

Exercise 9.1.15

Solve: $5(a-5)^2+4=104$.

Answer

$a=5+2\sqrt{5}, a=5-2\sqrt{5}$

Exercise 9.1.16

Solve: $3(b+3)^2-8=88$.

Answer

$b=-3+4\sqrt{2}, \quad b=-3-4\sqrt{2}$

Sometimes the solutions are complex numbers.

Example 9.1.9

Solve: $(2x-3)^2=-12$.

Solution:

$$(2x-3)^2=-12$$

Use the Square Root Property.

$$2x-3=\pm\sqrt{-12}$$

Simplify the radical.

$$2x-3=\pm 2\sqrt{3}i$$

Add 3 to both sides.

$$2x=3\pm 2\sqrt{3}i$$

Divide both sides by 2.

$$x=\frac{3\pm 2\sqrt{3}i}{2}$$

Rewrite in standard form.

$$x=\frac{3}{2}\pm\frac{2\sqrt{3}i}{2}$$

Simplify.

$$x=\frac{3}{2}\pm\sqrt{3}i$$

Rewrite to show two solutions.

$$x=\frac{3}{2}+\sqrt{3}i, x=\frac{3}{2}-\sqrt{3}i$$

Check:

We leave the check for you.

Exercise 9.1.17

Solve: $(3r+4)^2=-8$.

Answer

$r=-\frac{4}{3}+\frac{2\sqrt{2}i}{3}, r=-\frac{4}{3}-\frac{2\sqrt{2}i}{3}$

Exercise 9.1.18

Solve: $(2t-8)^2=-10$.

Answer

$t=4+\frac{\sqrt{10}i}{2}, t=4-\frac{\sqrt{10}i}{2}$

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example 9.1.10

Solve: $4n^2+4n+1=16$.

Solution:

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

Table 9.1.7

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1=\pm\sqrt{16}$
Simplify the radical.	$2n+1=\pm 4$
Solve for n.	$2n=-1\pm 4$
Divide each side by 2.	$\frac{2n}{2}=\frac{-1\pm 4}{2}$ $n=\frac{-1\pm 4}{2}$
Rewrite to show two solutions.	$n=\frac{-1+4}{2}, n=\frac{-1-4}{2}$
Simplify each equation.	$n=\frac{3}{2}, \quad n=-\frac{5}{2}$
Check: $4n^2+4n+1=16$ $4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $9+6+1\stackrel{?}{=}16$ $16=16$ ✓ $4n^2+4n+1=16$ $4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $25-10+1\stackrel{?}{=}16$ $16=16$ ✓	

Figure 9.1.22

Exercise 9.1.19

Solve: $9m^2-12m+4=25$.

Answer

$m=\frac{7}{3}, \quad m=-1$

Exercise 9.1.20

Solve: $16n^2+40n+25=4$.

Answer

$n=-\frac{3}{4}, \quad n=-\frac{7}{4}$

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- Solving Quadratic Equations: The Square Root Property

- Using the Square Root Property to Solve Quadratic Equations

Key Concepts

- Square Root Property
  - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$or $x = \pm\sqrt{k}$

  How to solve a quadratic equation using the square root property.

  1. Isolate the quadratic term and make its coefficient one.
  2. Use Square Root Property.
  3. Simplify the radical.
  4. Check the solutions.

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9.1E: Exercises

Practice Makes Perfect

? ExerciseS 1 - 22: Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

1. $a^2 = 49$
2. $b^2 = 144$
3. $r^2 - 24 = 0$
4. $t^2 - 75 = 0$
5. $u^2 - 300 = 0$
6. $v^2 - 80 = 0$
7. $4m^2 = 36$
8. $3n^2 = 48$
9. $\frac{4}{3}x^2 = 48$
10. $\frac{5}{3}y^2 = 60$
11. $x^2 + 25 = 0$
12. $y^2 + 64 = 0$
13. $x^2 + 63 = 0$
14. $y^2 + 45 = 0$
15. $\frac{4}{3}x^2 + 2 = 110$
16. $\frac{2}{3}y^2 - 8 = -2$
17. $\frac{2}{5}a^2 + 3 = 11$
18. $\frac{3}{2}b^2 - 7 = 41$
19. $7p^2 + 10 = 26$
20. $2q^2 + 5 = 30$
21. $5y^2 - 7 = 25$
22. $3x^2 - 8 = 46$

Answer

1. $a = \pm 7$
3. $r = \pm 2\sqrt{6}$
5. $u = \pm 10\sqrt{3}$
7. $m = \pm 3$
9. $x = \pm 6$
11. $x = \pm 5i$
13. $x = \pm 3\sqrt{7}i$
15. $x = \pm 9$
17. $a = \pm 2\sqrt{5}$

19. $p = \pm\frac{4\sqrt{7}}{7}$

21. $y = \pm\frac{4\sqrt{10}}{5}$

Exercises 23 - 46: Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

23. $(u-6)^2 = 64$

24. $(v+10)^2 = 121$

25. $(m-6)^2 = 20$

26. $(n+5)^2 = 32$

27. $\left(r-\frac{1}{2}\right)^2 = \frac{3}{4}$

28. $\left(x+\frac{1}{5}\right)^2 = \frac{7}{25}$

29. $\left(y+\frac{2}{3}\right)^2 = \frac{8}{81}$

30. $\left(t-\frac{5}{6}\right)^2 = \frac{11}{25}$

31. $(a-7)^2+5 = 55$

32. $(b-1)^2-9 = 39$

33. $4(x+3)^2-5 = 27$

34. $5(x+3)^2-7 = 68$

35. $(5c+1)^2 = -27$

36. $(8d-6)^2 = -24$

37. $(4x-3)^2+11 = -17$

38. $(2y+1)^2-5 = -23$

39. $m^2-4m+4 = 8$

40. $n^2+8n+16 = 27$

41. $x^2-6x+9 = 12$

42. $y^2+12y+36 = 32$

43. $25x^2-30x+9 = 36$

44. $9y^2+12y+4 = 9$

45. $36x^2-24x+4 = 81$

46. $64x^2+144x+81 = 25$

Answer

23. $u = 14, u = -2$

25. $m = 6 \pm 2\sqrt{5}$

27. $r = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

29. $y = -\frac{2}{3} \pm \frac{2\sqrt{2}}{9}$

31. $a = 7 \pm 5\sqrt{2}$

33. $x = -3 \pm 2\sqrt{2}$

35. $c = -\frac{1}{5} \pm \frac{3\sqrt{3}}{5}i$

37. $x = \frac{3}{4} \pm \frac{\sqrt{7}}{2}i$

39. $m = 2 \pm 2\sqrt{2}$

41. $x = 3 + 2\sqrt{3}, x = 3 - 2\sqrt{3}$

43. $x = -\frac{3}{5}, x = \frac{9}{5}$

45. $x = -\frac{7}{6}, x = \frac{11}{6}$

ExerciseS 47 - 68: Mixed Practice

In the following exercises, solve using the Square Root Property.

47. $2r^2 = 32$

48. $4t^2 = 16$

49. $(a-4)^2 = 28$

50. $(b+7)^2 = 8$

51. $9w^2 - 24w + 16 = 1$

52. $4z^2 + 4z + 1 = 49$

53. $a^2 - 18 = 0$

54. $b^2 - 108 = 0$

55. $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$

56. $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$

57. $m^2 + 12 = 0$

58. $n^2 + 48 = 0$

59. $u^2 - 14u + 49 = 72$

60. $v^2 + 18v + 81 = 50$

61. $(m-4)^2 + 3 = 15$

62. $(n-7)^2 - 8 = 64$

63. $(x+5)^2 = 4$

64. $(y-4)^2 = 64$

65. $6c^2 + 4 = 29$

66. $2d^2 - 4 = 77$

67. $(x-6)^2 + 7 = 3$

68. $(y-4)^2 + 10 = 9$

Answer

47. $r = \pm 4$

49. $a = 4 \pm 2\sqrt{7}$

51. $w = 1, w = \frac{5}{3}$

53. $a = \pm 3\sqrt{2}$

https://math.libretexts.org/@go/page/30908

55. $p = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

57. $m = \pm 2\sqrt{2i}$

59. $u = 7 \pm 6\sqrt{2}$

61. $m = 4 \pm 2\sqrt{3}$

63. $x = -3, x = -7$

65. $c = \pm \frac{5\sqrt{6}}{6}$

67. $x = 6 \pm 2i$

? ExerciseS 69 - 70: Writing exercises

69. In your own words, explain the Square Root Property.

70. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2 = 16$.

Answer

69. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x-h)^2 = k$ using the square root property.			

Figure 9.1.23

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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9.2: Solve Quadratic Equations by Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2+bx+c=0$ by completing the square
- Solve quadratic equations of the form $ax^2+bx+c=0$ by completing the square

Before you get started, take this readiness quiz.

1. Expand: $(x+9)^2$.
   If you missed this problem, review Example 5.32.
2. Factor $y^2-14y+49$.
   If you missed this problem, review Example 6.9.
3. Factor $5n^2+40n+80$.
   If you missed this problem, review Example 6.14.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y-7)^2=12$ because the left side was a perfect square.

$$\begin{aligned}(y-7)^2&=12\\ y-7&=\pm\sqrt{12}\\ y-7&=\pm 2\sqrt{3}\\ y&=7\pm 2\sqrt{3}\end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)^2$ in order to use the Square Root Property.

$$\begin{aligned}x^2-10x+25&=18\\ (x-5)^2&=18\end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$$\begin{array}{cc}(x+9)^2 & (y-7)^2\\ (x+9)(x+9) & (y-7)(y-7)\\ x^2+9x+9x+81 & y^2-7y-7y+49\\ x^2+18x+81 & y^2-14y+49\end{array}$$

We restate the patterns here for reference.

Definition 9.2.1: Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2=a^2+2ab+b^2 \qquad \underbrace{(a+b)^2}_{\text{(binomial)}^2}=\underbrace{a^2}_{\text{(first term)}^2}+\underbrace{2ab}_{2\cdot\text{(product of terms)}}+\underbrace{b^2}_{\text{(second term)}^2}$$

$$(a-b)^2=a^2-2ab+b^2 \qquad \underbrace{(a-b)^2}_{\text{(binomial)}^2}=\underbrace{a^2}_{\text{(first term)}^2}-\underbrace{2ab}_{2\cdot\text{(product of terms)}}+\underbrace{b^2}_{\text{(second term)}^2}$$

We can use this pattern to “make” a perfect square.

We will start with the expression x^2+6x. Since there is a plus sign between the two terms, we will use the $(a+b)^2$ pattern, $a^2+2ab+b^2=(a+b)^2$.

$$\underset{x^2+6x+\_}{\overset{a^2+2ab+b^2}{}}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of x^2+6x is a square, x^2. This tells us that $a=x$.

$$\underset{x^2+2\cdot x\cdot b+b^2}{\overset{a^2+\quad 2ab\quad +b^2}{}}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b=3$.

$$\underset{x^2+2\cdot 3\cdot x+\_}{\overset{a^2+\quad 2ab\quad +b^2}{}}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2=9$.

$$\underset{x^2+6x+9}{\overset{a^2+2ab+b^2}{}}$$

We can now factor.

$$\underset{(x+3)^2}{\overset{(a+b)^2}{}}$$

So we found that adding 9 to x^2+6x ‘completes the square’, and we write it as $(x+3)^2$.

Howto: Complete a Square of x^2+bx

1. Identify b, the coefficient of x.
2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx.
4. Factor the perfect square trinomial, writing it as a binomial squared.

Example 9.2.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. x^2-26x
b. y^2-9y
c. $n^2+\frac{1}{2}n$

Solution:

a.

	$\underset{x^2-26x}{\overset{x^2-\quad bx}{}}$
The coefficient of x is -26.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-26)\right)^2$ $(13)^2$ 169	
Add 169 to the binomial to complete the square.	$x^2-26x+169$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-13)^2$

b.

	$x^2 - bx$ $y^2 - 9y$
The coefficient of y is -9.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2 - 9y + \frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(y - \frac{9}{2}\right)^2$

c.

	$x^2 + bx$ $n^2 + \frac{1}{2}n$
The coefficient of n is $\frac{1}{2}$.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2 + \frac{1}{2}n + \frac{1}{16}$
Rewrite as a binomial square.	$\left(n + \frac{1}{4}\right)^2$

Exercise 9.2.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. $a^2 - 20a$
b. $m^2 - 5m$
c. $p^2 + \frac{1}{4}p$

Answer

a. $(a-10)^2$
b. $\left(b - \frac{5}{2}\right)^2$
c. $\left(p + \frac{1}{8}\right)^2$

Exercise 9.2.2

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. $b^2 - 4b$
b. $n^2 + 13n$
c. $q^2 - \frac{2}{3}q$

Answer

a. $(b-2)^2$

b. $\left(n+\frac{13}{2}\right)^2$

c. $\left(q-\frac{1}{3}\right)^2$

Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a **quadratic equation** by **completing the square** too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2+6x=40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2+6x=40$
	$x^2+6x+\_\_=40+\_\_$
	$x^2+6x+9=40+9$
Add 9 to both sides to complete the square.	$(x+3)^2=49$

Now the equation is in the form to solve using the **Square Root Property**! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

Example 9.2.2 How to Solve a Quadratic Equation of the Form $x^2+bx+x=0$ by Completing the Square

Solve by completing the square: $x^2+8x=48$.

Solution:

Step 1: Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$x^2+bx \quad c$ $x^2+8x=48$
Step 2: Find $\left(\frac{1}{2}\cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2=16$ Add 16 to BOTH sides of the equation.	$x^2+8x+\underline{\quad\quad}=48$ $\left(\frac{1}{2}\cdot 8\right)^2$ $x^2+8x+16=48+16$
Step 3: Factor the perfect square trinomial as a binomial square.	$x^2+8x+16=(x+4)^2$ Add the terms on the right.	$(x+4)^2=64$
Step 4: Use the Square Root Property.		$x+4=\pm\sqrt{64}$
Step 5: Simplify the radical and then solve the two resulting equations.		$x+4=\pm 8$ $x+4=8 \quad x+4=-8$ $x=4 \quad\quad x=-12$
Step 6: Check the solutions.	Put each answer in the original equation to check. Substitute $x=4$ and $x=-12$.	$x^2+8x=48$ $(4)^2+8(4)\stackrel{?}{=}48$ $16+32\stackrel{?}{=}48$ $48=48$ $x^2+8x=48$ $(-12)^2+8(-12)\stackrel{?}{=}48$ $144-96\stackrel{?}{=}48$ $48=48$

Exercise 9.2.3

Solve by completing the square: $x^2+4x=5$.

Answer

$x=-5, x=-1$

Exercise 9.2.4

Solve by completing the square: $y^2-10y=-9$.

Answer

$y=1, y=9$

The steps to solve a quadratic equation by completing the square are listed here.

Solve a Quadratic Equation of the Form $x^2+bx+c=0$ by Completing the Square

1. Isolate the variable terms on one side and the constant terms on the other.
2. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
4. Use the Square Root Property.
5. Simplify the radical and then solve the two resulting equations.
6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

Example 9.2.3

Solve by completing the square: $x^2+4x=-21$.

Solution:

	$x^2+4x=-21$ (x^2: b, x: c)
The variable terms are on the left side. Take half of 4 and square it.	$x^2+4x+\underset{\left(\frac{1}{2}\cdot 4\right)^2}{\_\_\_\_}=-21$
$\left(\frac{1}{2}(4)\right)^2=4$	
Add 4 to both sides.	$x^2+4x+4=-21+4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+2)^2=-17$
Use the Square Root Property.	$x+2=\pm\sqrt{-17}$
Simplifying using complex numbers.	$x+2=\pm\sqrt{17}\,i$
Subtract 2 from each side.	$x=-2\pm\sqrt{17}\,i$
Rewrite to show two solutions.	$x=-2+\sqrt{17}\,i, \quad x=-2-\sqrt{17}\,i$
We leave the check to you.	

Exercise 9.2.5

Solve by completing the square: $y^2 - 10y = -35$.

Answer

$y = 5 + \sqrt{15}i, y = 5 - \sqrt{15}i$

Exercise 9.2.6

Solve by completing the square: $z^2 + 8z = -19$.

Answer

$z = -4 + \sqrt{3}i, z = -4 - \sqrt{3}i$

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

Example 9.2.4

Solve by completing the square: $y^2 - 18y = -6$.

Solution:

	$\underset{x^2}{y^2} - \underset{bx}{18y} = \underset{c}{-6}$
The variable terms are on the left side. Take half of -18 and square it.	
$\left(\frac{1}{2}(-18)\right)^2 = 81$	$y^2 - 18y + \underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\_\_\_\_\_} = -6$
Add 81 to both sides.	$y^2 - 18y + 81 = -6 + 81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y-9)^2 = 75$
Use the Square Root Property.	$y - 9 = \pm\sqrt{75}$
Simplify the radical.	$y - 9 = \pm 5\sqrt{3}$
Solve for y.	$y = 9 \pm 5\sqrt{3}$
Check. $y^2 - 18y = -6$ $(9+5\sqrt{3})^2 - 18(9+5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓ $y^2 - 18y = -6$ $(9-5\sqrt{3})^2 - 18(9-5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓	$y^2 - 18y = -6$ $(9+5\sqrt{3})^2 - 18(9+5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓ $y^2 - 18y = -6$ $(9-5\sqrt{3})^2 - 18(9-5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6$ ✓

Figure 9.2.35

Another way to check this would be to use a calculator. Evaluate $y^2 - 18y$ for both of the solutions. The answer should be -6.

Exercise 9.2.7

Solve by completing the square: $x^2 - 16x = -16$.

Answer

$x = 8 + 4\sqrt{3}, x = 8 - 4\sqrt{3}$

Exercise 9.2.8

Solve by completing the square: $y^2+8y=11$.

Answer

$y=-4+3\sqrt{3}, y=-4-3\sqrt{3}$

We will start the next example by isolating the variable terms on the left side of the equation.

Example 9.2.5

Solve by completing the square: $x^2+10x+4=15$.

Solution:

	$x^2+10x+4=15$
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2+10x=11$
Take half of 10 and square it.	
$\left(\frac{1}{2}(10)\right)^2=25$	$x^2-10x+\underset{\left(\frac{1}{2}\cdot(10)\right)^2}{\_\_\_\_\_}=11$
Add 25 to both sides.	$x^2+10x+25=11+25$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+5)^2=36$
Use the Square Root Property.	$x+5=\pm\sqrt{36}$
Simplify the radical.	$x+5=\pm 6$
Solve for x.	$x=-5\pm 6$
Rewrite to show two solutions.	$x=-5+6, \quad x=-5-6$
Solve the equations.	$x=1, \quad x=-11$
Check: $x^2+10x+4=15$ $(1)^2+10(1)+4\stackrel{?}{=}15$ $1+10+4\stackrel{?}{=}15$ $15=15$ ✓ $x^2+10x+4=15$ $(-11)^2+10(-11)+4\stackrel{?}{=}15$ $121+110+4\stackrel{?}{=}15$ $15=15$ ✓	

Figure 9.2.47

Exercise 9.2.9

Solve by completing the square: $a^2+4a+9=30$.

Answer

$a=-7, a=3$

Exercise 9.2.10

Solve by completing the square: $b^2+8b-4=16$.

Answer

$b=-10, b=2$

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

Example 9.2.6

Solve by completing the square: $n^2=3n+11$.

Solution:

	$n^2=3n+11$
Subtract $3n$ to get the variable terms on the left side.	$n^2-3n=11$
Take half of -3 and square it.	
$\left(\frac{1}{2}(-3)\right)^2=\frac{9}{4}$	$n^2-3n+\underset{\left(\frac{1}{2}\cdot(-3)\right)^2}{\_\_\_\_}=11$
Add $\frac{9}{4}$ to both sides.	$n^2-3n+\frac{9}{4}=11+\frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n-\frac{3}{2}\right)^2=\frac{44}{4}+\frac{9}{4}$
Add the fractions on the right side.	$\left(n-\frac{3}{2}\right)^2=\frac{53}{4}$
Use the Square Root Property.	$n-\frac{3}{2}=\pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n-\frac{3}{2}=\pm\frac{\sqrt{53}}{2}$
Solve for n.	$n=\frac{3}{2}\pm\frac{\sqrt{53}}{2}$
Rewrite to show two solutions.	$n=\frac{3}{2}+\frac{\sqrt{53}}{2}, \quad n=\frac{3}{2}-\frac{\sqrt{53}}{2}$
Check: We leave the check for you!	

Exercise 9.2.11

Solve by completing the square: $p^2=5p+9$.

Answer

$p=\frac{5}{2}+\frac{\sqrt{61}}{2}, p=\frac{5}{2}-\frac{\sqrt{61}}{2}$

Exercise 9.2.12

Solve by completing the square: $q^2=7q-3$.

Answer

$q = \frac{7}{2} + \frac{\sqrt{37}}{2}, q = \frac{7}{2} - \frac{\sqrt{37}}{2}$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the **Zero Product Property** since it says "If $a \cdot b = 0$, then $a = 0$ or $b = 0$." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

Example 9.2.7

Solve by completing the square: $(x-3)(x+5) = 9$.

Solution:

	$(x-3)(x+5) = 9$
We multiple the binomials on the left.	$x^2 + 2x - 15 = 9$
Add 15 to isolate the constant terms on the right.	$x^2 + 2x \quad = 24$
Take half of 2 and square it.	
$\left(\frac{1}{2} \cdot (2)\right)^2 = 1$	$x^2 + 2x + \underset{\left(\frac{1}{2} \cdot (2)\right)^2}{\_\_\_\_} = 24$
Add 1 to both sides.	$x^2 + 2x + 1 = 24 + 1$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+1)^2 = 25$
Use the Square Root Property.	$x + 1 = \pm\sqrt{25}$
Solve for x.	$x = -1 \pm 5$
Rewrite to show two solutions.	$x = -1 + 5, x = -1 - 5$
Simplify.	$x = 4, \quad x = -6$
Check: We leave the check for you!	

Exercise 9.2.13

Solve by completing the square: $(c-2)(c+8) = 11$.

Answer

$c = -9, c = 3$

Exercise 9.2.14

Solve by completing the square: $(d-7)(d+3) = 56$.

Answer

$d = 11, d = -7$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of **completing the square** works best when the coefficient of x^2 is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Example 9.2.8

Solve by completing the square: $3x^2 - 12x - 15 = 0$.

Solution:

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$
Simplify.	$x^2 - 4x - 5 = 0$
Add 5 to get the constant terms on the right side.	$x^2 - 4x = 5$
Take half of 4 and square it.	
$\left(\frac{1}{2}(-4)\right)^2 = 4$	$x^2 - 4x + \underset{\left(\frac{1}{2}\cdot(4)\right)^2}{\_\_\_\_} = 5$
Add 4 to both sides.	$x^2 - 4x + 4 = 5 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-2)^2 = 9$
Use the Square Root Property.	$x - 2 = \pm\sqrt{9}$
Solve for x.	$x - 2 = \pm 3$
Rewrite to show two solutions.	$x = 2 + 3,\ x = 2 - 3$
Simplify.	$x = 5,\quad x = -1$
Check: $x=5$: $3x^2-12x-15=0$; $3(5)^2-12(5)-15 \stackrel{?}{=} 0$; $75-60-15 \stackrel{?}{=} 0$; $0=0$ ✓ — $x=-1$: $3x^2-12x-15=0$; $3(-1)^2-12(-1)-15 \stackrel{?}{=} 0$; $3+12-15 \stackrel{?}{=} 0$; $0=0$ ✓	

Figure 9.2.80

Exercise 9.2.15

Solve by completing the square: $2m^2 + 16m + 14 = 0$.

Answer

$m = -7, m = -1$

Exercise 9.2.16

Solve by completing the square: $4n^2 - 24n - 56 = 8$.

Answer

$n = -2, n = 8$

To complete the square, the coefficient of the x^2 must be 1. When the **leading coefficient** is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

Example 9.2.9

Solve by completing the square: $2x^2 - 3x = 20$.

Solution:

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it.	
$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underset{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2}{\_\_\_\_} = 10$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for x.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show two solutions.	$x = \frac{3}{4} + \frac{13}{4}, \quad x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4, \qquad x = -\frac{5}{2}$
Check: We leave the check for you!	

Exercise 9.2.17

Solve by completing the square: $3r^2 - 2r = 21$.

Answer

$r = -\frac{7}{3}, r = 3$

Exercise 9.2.18

Solve by completing the square: $4t^2 + 2t = 20$.

Answer

$t = -\frac{5}{2}, t = 2$

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a **quadratic equation** by completing the square to include equations of the form $ax^2 + bx + c = 0$.

Howto: Solve a Quadratic Equation of the Form $ax^2 + bx + c = 0$ by Completing the Square

1. Divide by aa to make the coefficient of x^2 term 1.
2. Isolate the variable terms on one side and the constant terms on the other.
3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
5. Use the Square Root Property.
6. Simplify the radical and then solve the two resulting equations.
7. Check the solutions.

Example 9.2.10

Solve by completing the square: $3x^2 + 2x = 4$.

Solution:

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2 + 2x = 4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2 + 2x}{3} = \frac{4}{3}$
Simplify.	$x^2 + \frac{2}{3}x = \frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2 = \frac{1}{9}$	$x^2 + \frac{2}{3}x + \underset{\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2}{\_\_\_\_} = \frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x + \frac{1}{3}\right)^2 = \frac{12}{9} + \frac{1}{9}$
Use the Square Root Property.	$x + \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x + \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$
Solve for x.	$x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$
Rewrite to show two solutions.	$x = -\frac{1}{3} + \frac{\sqrt{13}}{3}, \quad x = -\frac{1}{3} - \frac{\sqrt{13}}{3}$
Check: We leave the check for you!	

? Exercise 9.2.19

Solve by completing the square: $4x^2+3x=2$.

Answer

$x=-\frac{3}{8}+\frac{\sqrt{41}}{8}, x=-\frac{3}{8}-\frac{\sqrt{41}}{8}$

? Exercise 9.2.20

Solve by completing the square: $3y^2-10y=-5$.

Answer

$y=\frac{5}{3}+\frac{\sqrt{10}}{3}, y=\frac{5}{3}-\frac{\sqrt{10}}{3}$

Access these online resources for additional instruction and practice with completing the square.

- Completing Perfect Square Trinomials
- Completing the Square 1
- Completing the Square to Solve Quadratic Equations
- Completing the Square to Solve Quadratic Equations: More Examples
- Completing the Square 4

Key Concepts

- Binomial Squares Pattern
  If a and b are real numbers,

$(a+b)^2=a^2+2ab+b^2$

$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$

$(a-b)^2=a^2-2ab+b^2$

$\underbrace{(a-b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} - \underbrace{2ab}_{2\cdot\text{(product of terms)}} + \underbrace{b^2}_{\text{(second term)}^2}$

- How to Complete a Square
  1. Identify b, the coefficient of x.
  2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
  3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx
  4. Rewrite the trinomial as a binomial square
- How to solve a quadratic equation of the form $ax^2+bx+c=0$ by completing the square.
  1. Divide by a to make the coefficient of x^2 term 1.
  2. Isolate the variable terms on one side and the constant terms on the other.
  3. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
  4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.
  5. Use the Square Root Property.
  6. Simplify the radical and then solve the two resulting equations.
  7. Check the solutions.

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9.2E: Exercises

Practice Makes Perfect

? ExerciseS 1 - 4: Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. a. m^2-24m
   b. x^2-11x
   c. $p^2-\frac{1}{3}p$
2. a. n^2-16n
   b. y^2+15y
   c. $q^2+\frac{3}{4}q$
3. a. p^2-22p
   b. y^2+5y
   c. $m^2+\frac{2}{5}m$
4. a. q^2-6q
   b. x^2-7x
   c. $n^2-\frac{2}{3}n$

Answer

1. a. $(m-12)^2$ b. $\left(x-\frac{11}{2}\right)^2$ c. $\left(p-\frac{1}{6}\right)^2$

3. a. $(p-11)^2$ b. $\left(y+\frac{5}{2}\right)^2$ c. $\left(m+\frac{1}{5}\right)^2$

? ExerciseS 5 - 28: Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In the following exercises, solve by completing the square.

5. $u^2+2u=3$
6. $z^2+12z=-11$
7. $x^2-20x=21$
8. $y^2-2y=8$
9. $m^2+4m=-44$
10. $n^2-2n=-3$
11. $r^2+6r=-11$
12. $t^2-14t=-50$
13. $a^2-10a=-5$
14. $b^2+6b=41$
15. $x^2+5x=2$
16. $y^2-3y=2$
17. $u^2-14u+12=-1$
18. $z^2+2z-5=2$
19. $r^2-4r-3=9$
20. $t^2-10t-6=5$

21. $v^2 = 9v + 2$

22. $w^2 = 5w - 1$

23. $x^2 - 5 = 10x$

24. $y^2 - 14 = 6y$

25. $(x + 6)(x - 2) = 9$

26. $(y + 9)(y + 7) = 80$

27. $(x + 2)(x + 4) = 3$

28. $(x - 2)(x - 6) = 5$

Answer

5. $u = -3, u = 1$

7. $x = -1, x = 21$

9. $m = -2 \pm 2\sqrt{10}i$

11. $r = -3 \pm \sqrt{2}i$

13. $a = 5 \pm 2\sqrt{5}$

15. $x = -\frac{5}{2} \pm \frac{\sqrt{33}}{2}$

17. $u = 1, u = 13$

19. $r = -2, r = 6$

21. $v = \frac{9}{2} \pm \frac{\sqrt{89}}{2}$

23. $x = 5 \pm \sqrt{30}$

25. $x = -7, x = 3$

27. $x = -5, x = -1$

? ExerciseS 29 - 40: Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

29. $3m^2 + 30m - 27 = 6$

30. $2x^2 - 14x + 12 = 0$

31. $2n^2 + 4n = 26$

32. $5x^2 + 20x = 15$

33. $2c^2 + c = 6$

34. $3d^2 - 4d = 15$

35. $2x^2 + 7x - 15 = 0$

36. $3x^2 - 14x + 8 = 0$

37. $2p^2 + 7p = 14$

38. $3q^2 - 5q = 9$

39. $5x^2 - 3x = -10$

40. $7x^2 + 4x = -3$

Answer

29. $m = -11, m = 1$

31. $n = 1 \pm \sqrt{14}$

33. $c = -2, c = \frac{3}{2}$

35. $x = -5, x = \frac{3}{2}$

37. $p = -\frac{7}{4} \pm \frac{\sqrt{161}}{4}$

39. $x = \frac{3}{10} \pm \frac{\sqrt{191}}{10} i$

? ExerciseS 41 - 42: writing exercises

41. Solve the equation $x^2 + 10x = -25$

a. by using the Square Root Property
b. by Completing the Square
c. Which method do you prefer? Why?

42. Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Answer

41. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

Figure 9.2.103

b. After reviewing this checklist, what will you do to become confident for all objectives?

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9.3: Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Before you get started, take this readiness quiz.

1. Evaluate b^2-4ab when $a=3$ and $b=-2$.
2. Simplify $\sqrt{108}$.
3. Simplify $\sqrt{50}$.

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for x.

We start with the standard form of a quadratic equation and solve it for x by completing the square.

	$ax^2+bx+c=0, \quad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx \quad =-c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x \quad =-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x \quad =-\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The final equation is called the "Quadratic Formula."

Definition 9.3.1: Quadratic Formula

The solutions to a **quadratic equation** of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{9.3.1}$$

To use the **Quadratic Formula**, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the Quadratic Formula (Equation 9.3.1) is an equation. Make sure you use both sides of the equation.

Example 9.3.1 How to Solve a Quadratic Equation Using the Quadratic Formula

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution:

Step 1: Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$
Step 2: Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a=2, b=9, c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$
Step 3: Simplify the fraction, and solve for x.		$x=\frac{-9\pm\sqrt{81-(-40)}}{4}$ $x=\frac{-9\pm\sqrt{121}}{4}$ $x=\frac{-9\pm 11}{4}$ $x=\frac{-9+11}{4}$ $\quad x=\frac{-9-11}{4}$ $x=\frac{2}{4}$ $\quad x=\frac{-20}{4}$ $x=\frac{1}{2}$ $\quad x=-5$

Step 4: Check the solutions.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$ and $x=-5$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+0\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\stackrel{?}{=}0$ $\frac{10}{2}-5\stackrel{?}{=}0$ $5-5\stackrel{?}{=}0$ $0=0$ $2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\stackrel{?}{=}0$ $2\cdot25-45-5\stackrel{?}{=}0$ $50-45-5\stackrel{?}{=}0$ $0=0$

Exercise 9.3.1

Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

Answer

$y=1, y=\frac{2}{3}$

Exercise 9.3.2

Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

Answer

$z=1, z=-\frac{3}{2}$

HowTo: Solve a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x=$".

Example 9.3.2

Solve by using the Quadratic Formula: $x^2-6x=-5$.

Solution:

	$x^2-6x=-5$

Write the equation in standard form by adding 5 to each side.	$x^2-6x+5=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2-6x+5=0$
Identify the values of a, b, c.	$a=1, b=-6, c=5$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(-6)\pm\sqrt{(-6)^2-4\cdot 1\cdot(5)}}{2\cdot 1}$
Simplify.	$x=\frac{6\pm\sqrt{36-20}}{2}$ $x=\frac{6\pm\sqrt{16}}{2}$ $x=\frac{6\pm 4}{2}$
Rewrite to show two solutions.	$x=\frac{6+4}{2}, \quad x=\frac{6-4}{2}$
Simplify.	$x=\frac{10}{2}, \quad x=\frac{2}{2}$
	$x=5, \quad x=1$
Check: $x^2-6x+5=0$ $\quad$ $x^2-6x+5=0$ $5^2-6\cdot 5+5\stackrel{?}{=}0$ $\quad$ $1^2-6\cdot 1+5\stackrel{?}{=}0$ $25-30+5\stackrel{?}{=}0$ $\quad$ $1-6+5\stackrel{?}{=}0$ $0=0$ ✓ $\quad$ $0=0$ ✓	

? Exercise 9.3.3

Solve by using the Quadratic Formula: $a^2-2a=15$.

Answer

$a=-3, a=5$

? Exercise 9.3.4

Solve by using the Quadratic Formula: $b^2+24=-10b$.

Answer

$b=-6, b=-4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the **Quadratic Formula**. If we get a **radical** as a solution, the final answer must have the radical in its simplified form.

✓ Example 9.3.3

Solve by using the Quadratic Formula: $2x^2+10x+11=0$.

Solution:

	$2x^2+10x+11=0$
This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+10x+11=0$

Identify the values of a, b and c.	$a = 2,\ b = 10,\ c = 11$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \dfrac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \dfrac{-10 \pm \sqrt{100 - 88}}{4}$
	$x = \dfrac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \dfrac{-10 \pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \dfrac{2(-5 \pm \sqrt{3})}{4}$ $x = \dfrac{2(-5 \pm \sqrt{3})}{4}$
Remove the common factors.	$x = \dfrac{-5 \pm \sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \dfrac{-5 + \sqrt{3}}{2}, \quad x = \dfrac{-5 - \sqrt{3}}{2}$
Check: We leave the check for you!	

? Exercise 9.3.5

Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

Answer

$$m = \frac{-6 + \sqrt{15}}{3}, m = \frac{-6 - \sqrt{15}}{3}$$

? Exercise 9.3.6

Solve by using the Quadratic Formula: $5n^2 + 4n - 4 = 0$.

Answer

$$n = \frac{-2 + 2\sqrt{6}}{5}, n = \frac{-2 - 2\sqrt{6}}{5}$$

When we substitute a, b, and c into the Quadratic Formula and the **radicand** is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

✓ Example 9.3.4

Solve by using the Quadratic Formula: $3p^2 + 2p + 9 = 0$.

Solution:

Table 9.3.5

	$3p^2 + 2p + 9 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $3p^2 + 2p + 9 = 0$

Identify the values of a, b, c.	$a=3,\ b=2,\ c=9$
Write the Quadratic Formula.	$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$p=\frac{-(2)\pm\sqrt{(2)^2-4\cdot 3\cdot(9)}}{2\cdot 3}$
Simplify.	$p=\frac{-2\pm\sqrt{4-108}}{6}$
	$p=\frac{-2\pm\sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2\pm\sqrt{104}\,i}{6}$
Simplify the radical.	$p=\frac{-2\pm 2\sqrt{26}\,i}{6}$
Factor the common factor in the numerator.	$p=\frac{2\left(-1\pm\sqrt{26}\,i\right)}{6}$
Remove the common factors.	$p=\frac{-1\pm\sqrt{26}\,i}{3}$
Rewrite in standard $a+bi$ form.	$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$
Write as two solutions.	$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$

? Exercise 9.3.7

Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

Answer

$$a=\frac{1}{4}+\frac{\sqrt{31}}{4}i,\quad a=\frac{1}{4}-\frac{\sqrt{31}}{4}i$$

? Exercise 9.3.8

Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Answer

$$b=-\frac{1}{5}+\frac{\sqrt{19}}{5}i,\quad b=-\frac{1}{5}-\frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

✓ Example 9.3.5

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Solution:

Our first step is to get the equation in standard form.

Table 9.3.6

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$

This equation is now in standard form.	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$
Factor the common factor in the numerator.	$x=\frac{2(-3\pm 2\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm 2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\ x=-3-2\sqrt{5}$
Check: We leave the check for you!	

Exercise 9.3.9

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

Answer

$x=-1+\sqrt{6}, x=-1-\sqrt{6}$

Exercise 9.3.10

Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

Answer

$y=1+\sqrt{2}, y=1-\sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

Example 9.3.6

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

Table 9.3.7

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$

Multiply.	$3u^2 + 4u = 2$
Subtract 2 to get the equation in standard form.	$ax^2 + bx + c = 0$ $3u^2 + 4u - 2 = 0$
Identify the values of a, b, and c.	$a = 3,\ b = 4,\ c = -2$
Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$
	$u = \frac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \frac{-4 \pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u = \frac{2(-2 \pm \sqrt{10})}{6}$
Remove the common factors.	$u = \frac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \frac{-2 + \sqrt{10}}{3}, \quad u = \frac{-2 - \sqrt{10}}{3}$
Check: We leave the check for you!	

? Exercise 9.3.11

Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

Answer

$c = \frac{2+\sqrt{7}}{3}, \quad c = \frac{2-\sqrt{7}}{3}$

? Exercise 9.3.12

Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Answer

$d = \frac{9+\sqrt{33}}{4}, d = \frac{9-\sqrt{33}}{4}$

Think about the equation $(x-3)^2 = 0$. We know from the **Zero Product Property** that this equation has only one solution, $x = 3$.

We will see in the next example how using the **Quadratic Formula** to solve an equation whose standard form is a perfect square **trinomial** equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

Example 9.3.7

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution:

Table 9.3.8

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$ax^2 + bx + c = 0$ $4x^2 - 20x + 25 = 0$
Identify the values of a, b, and c.	$a = 4,\ b = -20,\ c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Exercise 9.3.13

Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

Answer

$r = -5$

Exercise 9.3.14

Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Answer

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Definition 9.3.2

Discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

Figure 9.3.85

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Table 9.3.9

Quadratic Equation (in standard form)	Discriminat $b^2 - 4ac$	Value of the Discriminant	Number and Type of Solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	$+$	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	$-$	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x = \frac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x = \frac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x = \frac{-b \pm \sqrt{-}}{2a}$

Figure 9.3.86

Using the Discriminant $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example 9.3.8

Determine the number of solutions to each quadratic equation.

a. $3x^2 + 7x - 9 = 0$
b. $5n^2 + n + 4 = 0$
c. $9y^2 - 6y + 1 = 0$

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

$$3x^2 + 7x - 9 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 3, \quad b = 7, \quad c = -9$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(7)^2-4\cdot 3\cdot(-9)$$

Simplify.

$$\begin{array}{c} 49+108 \\ 157 \end{array}$$

Since the discriminant is positive, there are 2 real solutions to the equation.

b.

$$5n^2+n+4=0$$

The equation is in standard form, identify a, b, and c.

$$a=5,\quad b=1,\quad c=4$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(1)^2-4\cdot 5\cdot 4$$

Simplify.

$$\begin{array}{c} 1-80 \\ -79 \end{array}$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

c.

$$9y^2-6y+1=0$$

The equation is in standard form, identify a, b, and c.

$$a=9,\quad b=-6,\quad c=1$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(-6)^2-4\cdot 9\cdot 1$$

Simplify.

$$\begin{array}{c} 36-36 \\ 0 \end{array}$$

Since the discriminant is 0, there is 1 real solution to the equation.

Exercise 9.3.15

Determine the number and type of solutions to each quadratic equation.

a. $8m^2-3m+6=0$
b. $5z^2+6z-2=0$
c. $9w^2+24w+16=0$

Answer

a. 2 complex solutions
b. 2 real solutions

c. 1 real solution

Exercise 9.3.16

Determine the number and type of solutions to each quadratic equation.

a. $b^2+7b-13=0$
b. $5a^2-6a+10=0$
c. $4r^2-20r+25=0$

Answer

a. 2 real solutions
b. 2 complex solutions
c. 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Identify the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example 9.3.9

Identify the most appropriate method to use to solve each quadratic equation.

a. $5z^2=17$
b. $4x^2-12x+9=0$
c. $8u^2+6u=11$

Solution:

a.

$$5z^2=17$$

Since the equation is in the $ax^2=k$, the most appropriate method is to use the Square Root Property.

b.

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

c.

$$8u^2 + 6u = 11$$

Put the equation in standard form.

$$8u^2 + 6u - 11 = 0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Exercise 9.3.17

Identify the most appropriate method to use to solve each quadratic equation.

a. $x^2 + 6x + 8 = 0$
b. $(n-3)^2 = 16$
c. $5p^2 - 6p = 9$

Answer

a. Factoring
b. Square Root Property
c. Quadratic Formula

Exercise 9.3.18

Identify the most appropriate method to use to solve each quadratic equation.

a. $8a^2 + 3a - 9 = 0$
b. $4b^2 + 4b + 1 = 0$
c. $5c^2 = 125$

Answer

a. Quadratic Formula
b. Factoring or Square Root Property
c. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, c.
  2. Write the Quadratic Formula. Then substitute in the values of a, b, c.
  3. Simplify.

4. Check the solutions.

- Using the Discriminant, b^2-4ac, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2+bx+c=0, a\neq 0$,
    - If $b^2-4ac>0$, the equation has 2 real solutions.
    - If $b^2-4ac=0$, the equation has 1 real solution.
    - If $b^2-4ac<0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.
  1. Try Factoring first. If the quadratic factors easily, this method is very quick.
  2. Try the **Square Root Property** next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property.
  3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

Glossary

discriminant

In the Quadratic Formula, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, the quantity b^2-4ac is called the discriminant.

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9.3E: Exercises

Practice Makes Perfect

? ExerciseS 1 - 32: Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4m^2+m-3=0$
2. $4n^2-9n+5=0$
3. $2p^2-7p+3=0$
4. $3q^2+8q-3=0$
5. $p^2+7p+12=0$
6. $q^2+3q-18=0$
7. $r^2-8r=33$
8. $t^2+13t=-40$
9. $3u^2+7u-2=0$
10. $2p^2+8p+5=0$
11. $2a^2-6a+3=0$
12. $5b^2+2b-4=0$
13. $x^2+8x-4=0$
14. $y^2+4y-4=0$
15. $3y^2+5y-2=0$
16. $6x^2+2x-20=0$
17. $2x^2+3x+3=0$
18. $2x^2-x+1=0$
19. $8x^2-6x+2=0$
20. $8x^2-4x+1=0$
21. $(v+1)(v-5)-4=0$
22. $(x+1)(x-3)=2$
23. $(y+4)(y-7)=18$
24. $(x+2)(x+6)=21$
25. $\frac{1}{4}m^2+\frac{1}{12}m=\frac{1}{3}$
26. $\frac{1}{3}n^2+n=-\frac{1}{2}$
27. $\frac{3}{4}b^2+\frac{1}{2}b=\frac{3}{8}$
28. $\frac{1}{9}c^2+\frac{2}{3}c=3$
29. $16c^2+24c+9=0$
30. $25d^2-60d+36=0$
31. $25q^2+30q+9=0$

32. $16y^2+8y+1=0$

Answer

1. $m=-1, m=\frac{3}{4}$
3. $p=\frac{1}{3}, p=2$
5. $p=-4, p=-3$
7. $r=-3, r=11$
9. $u=\frac{-7\pm\sqrt{73}}{6}$
11. $a=\frac{3\pm\sqrt{3}}{2}$
13. $x=-4\pm2\sqrt{5}$
15. $y=-\frac{2}{3}, y=-1$
17. $x=-\frac{3}{4}\pm\frac{\sqrt{15}}{4}i$
19. $x=\frac{3}{8}\pm\frac{\sqrt{7}}{8}i$
21. $v=2\pm2\sqrt{2}$
23. $y=-4, y=7$
25. $m=1, m=\frac{-4}{3}$
27. $b=\frac{-2\pm\sqrt{22}}{6}$
29. $c=-\frac{3}{4}$
31. $q=-\frac{3}{5}$

? ExerciseS 33 - 36 Use the Discriminant to Predict the Number of Real Solutions of a Quadratic Equation

In the following exercises, determine the number of real solutions for each quadratic equation.

33. a. $4x^2-5x+16=0$
    b. $36y^2+36y+9=0$
    c. $6m^2+3m-5=0$
34. a. $9v^2-15v+25=0$
    b. $100w^2+60w+9=0$
    c. $5c^2+7c-10=0$
35. a. $r^2+12r+36=0$
    b. $8t^2-11t+5=0$
    c. $3v^2-5v-1=0$
36. a. $25p^2+10p+1=0$
    b. $7q^2-3q-6=0$
    c. $7y^2+2y+8=0$

Answer

33. a. no real solutions b. 1 c. 2

35. a. 1 b. no real solutions c. 2

ExerciseS 37 - 40: Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

37. a. $x^2-5x-24=0$
    b. $(y+5)^2=12$
    c. $14m^2+3m=11$
38. a. $(8v+3)^2=81$
    b. $w^2-9w-22=0$
    c. $4n^2-10=6$
39. a. $6a^2+14=20$
    b. $\left(x-\frac{1}{4}\right)^2=\frac{5}{16}$
    c. $y^2-2y=8$
40. a. $8b^2+15b=4$
    b. $\frac{5}{9}v^2-\frac{2}{3}v=1$
    c. $\left(w+\frac{4}{3}\right)^2=\frac{2}{9}$

Answer

37. a. Factor b. Square Root c. Quadratic Formula

39. a. Quadratic Formula b. Square Root c. Factor

ExerciseS 41 - 42: Writing Exercises

41. Solve the equation $x^2+10x=120$
    a. by completing the square
    b. using the Quadratic Formula
    c. Which method do you prefer? Why?
42. Solve the equation $12y^2+23y=24$
    a. by completing the square
    b. using the Quadratic Formula
    c. Which method do you prefer? Why?

Answer

41. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

Figure 9.3.87

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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9.4: Solve Quadratic Equations in Quadratic Form

Learning Objectives

By the end of this section, you will be able to:

- Solve equations in quadratic form

Before you get started

... take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20$.
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15$.
3. Simplify
   a. $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$
   b. $\left(x^{\frac{1}{3}}\right)^2$
   c. $\left(x^{-1}\right)^2$

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

	$x^4 - 4x^2 - 5$
	$\left(x^2\right)^2 - 4\left(x^2\right) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$\left(x^2+1\right)\left(x^2-5\right)$

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

Example 9.4.1 How to Solve Equations in Quadratic Form

Solve: $6x^4 - 7x^2 + 2 = 0$

Solution:

Step 1: Identify a substitution that will put the equation in quadratic form.	Since $\left(x^2\right)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$

Step 2: Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6\left(x^2\right)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3: Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u-1)(3u-2) = 0$ $2u - 1 = 0, 3u - 2 = 0$ $2u = 1, 3u = 2$ $u = \frac{1}{2} u = \frac{2}{3}$
Step 4: Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2} \quad x^2 = \frac{2}{3}$
Step 5: Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}} \quad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2} \quad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2} \quad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2} \quad x = -\frac{\sqrt{6}}{3}$
Step 6: Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0$ We leave the other checks to you!

? Exercise 9.4.1

Solve: $x^4 - 6x^2 + 8 = 0$.

Answer

$x = \sqrt{2}, x = -\sqrt{2}, x = 2, x = -2$

? Exercise 9.4.2

Solve: $x^4 - 11x^2 + 28 = 0$.

Answer

$x = \sqrt{7}, x = -\sqrt{7}, x = 2, x = -2$

We summarize the steps to solve an equation in quadratic form.

Solve Equations in Quadratic Form

1. Identify a substitution that will put the equation in quadratic form.
2. Rewrite the equation with the substitution to put it in quadratic form.
3. Solve the quadratic equation for u.
4. Substitute the original variable back into the results, using the substitution.

5. Solve for the original variable.
6. Check the solutions.

In the next example, the binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

Example 9.4.2

Solve: $(x-2)^2+7(x-2)+12=0$.

Solution:

	$(x-2)^2+7(x-2)+12=0$
Prepare for the substitution.	$(x-2)^2+7(x-2)+12=0$
Let $u=x-2$ and substitute.	$u^2+7u+12=0$
Solve by factoring.	$(u+3)(u+4)=0$ $u+3=0, \quad u+4=0$ $u=-3, \quad u=-4$
Replace u with $x-2$.	$x-2=-3, \quad x-2=-4$
Solve for x.	$x=-1, \quad x=-2$
Check: $x=-1$ $(x-2)^2+7(x-2)+12=0$ $(-1-2)^2+7(-1-2)+12\stackrel{?}{=}0$ $(-3)^2+7(-3)+12\stackrel{?}{=}0$ $9-21+12\stackrel{?}{=}0$ $0=0$ ✓ $x=-2$ $(x-2)^2+7(x-2)+12=0$ $(-2-2)^2+7(-2-2)+12\stackrel{?}{=}0$ $(-4)^2+7(-4)+12\stackrel{?}{=}0$ $16-28+12\stackrel{?}{=}0$ $0=0$ ✓ Figure 9.4.14	

Exercise 9.4.3

Solve: $(x-5)^2+6(x-5)+8=0$.

Answer

$x=3, x=1$

Exercise 9.4.4

Solve: $(y-4)^2+8(y-4)+15=0$.

Answer

$y=-1, y=1$

In the next example, we notice that $(\sqrt{x})^2=x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

Example 9.4.3

Solve: $x-3\sqrt{x}+2=0$.

Solution:

The $\sqrt{x}$ in the middle term, is squared in the first term $(\sqrt{x})^2=x$. If we let $u=\sqrt{x}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$x-3\sqrt{x}+2=0$
Rewrite the trinomial to prepare for the substitution.	$(\sqrt{x})^2-3\sqrt{x}+2=0$
Let $u=\sqrt{x}$ and substitute.	$u^2-3u+2=0$
Solve by factoring.	$(u-2)(u-1)=0$ $u-2=0, \quad u-1=0$ $u=2, \quad u=1$ \)
Replace u with $\sqrt{x}$.	$\sqrt{x}=2, \quad \sqrt{x}=1$
Solve for x, by squaring both sides.	$x=4, \quad x=1$
Check: $x=4$: $x-3\sqrt{x}+2=0$; $4-3\sqrt{4}+2\stackrel{?}{=}0$; $4-6+2\stackrel{?}{=}0$; $0=0$ ✓ $x=1$: $x-3\sqrt{x}+2=0$; $1-3\sqrt{1}+2\stackrel{?}{=}0$; $1-3+2\stackrel{?}{=}0$; $0=0$ ✓	

Exercise 9.4.5

Solve: $x-7\sqrt{x}+12=0$.

Answer

$x=9, x=16$

Exercise 9.4.6

Solve: $x-6\sqrt{x}+8=0$.

Answer

$x=4, x=16$

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

Example 9.4.4

Solve: $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$.

Solution:

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2=x^{\frac{2}{3}}$. If we let $u=x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$

Rewrite the trinomial to prepare for the substitution.	$\left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 24 = 0$
Let $u = x^{\frac{1}{3}}$	$u^2 - 2u - 24 = 0$
Solve by factoring.	$(u-6)(u+4) = 0$ $u-6=0, \quad u+4=0$ $u=6, \quad u=-4$
Replace u with $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}} = 6, \quad x^{\frac{1}{3}} = -4$
Solve for x by cubing both sides.	$\left(x^{\frac{1}{3}}\right)^3 = (6)^3, \quad \left(x^{\frac{1}{3}}\right)^3 = (-4)^3$ $x = 216, \quad x = -64$
Check: $x = 216$ $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$ $(216)^{\frac{2}{3}} - 2(216)^{\frac{1}{3}} - 24 \stackrel{?}{=} 0$ $36 - 12 - 24 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x = -64$ $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$ $(-64)^{\frac{2}{3}} - 2(-64)^{\frac{1}{3}} - 24 \stackrel{?}{=} 0$ $16 + 8 - 24 \stackrel{?}{=} 0$ $0 = 0$ ✓ Figure 9.4.33	

? Exercise 9.4.7

Solve: $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 14 = 0$.

Answer

$x = -8, x = 343$

? Exercise 9.4.8

Solve: $x^{\frac{1}{2}} + 8x^{\frac{1}{4}} + 15 = 0$.

Answer

$x = 81, x = 625$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

✓ Example 9.4.5

Solve: $3x^{-2} - 7x^{-1} + 2 = 0$.

Solution:

The x^{-1} in the middle term is squared in the first term $\left(x^{-1}\right)^2 = x^{-2}$. If we let $u = x^{-1}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$3x^{-2} - 7x^{-1} + 2 = 0$
Rewrite the trinomial to prepare for the substitution.	$3(x^{-1})^2 - 7(x^{-1}) + 2 = 0$
Let $u = x^{-1}$ and substitute.	$3u^2 - 7u + 2 = 0$
Solve by factoring.	$(3u-1)(u-2) = 0$
	$3u-1=0, \quad u-2=0$

	$u=\frac{1}{3}, \quad u=2$
Replace u with x^{-1}.	$x^{-1}=\frac{1}{3}, \quad x^{-1}=2$
Solve for x by taking the reciprocal since $x^{-1}=\frac{1}{x}$.	$x=3, \quad x=\frac{1}{2}$
Check: $x=3$ $3x^{-2}-7x^{-1}+2=0$ $3(3)^{-2}-7(3)^{-1}+2\stackrel{?}{=}0$ $3\left(\frac{1}{9}\right)-7\left(\frac{1}{3}\right)+2\stackrel{?}{=}0$ $\frac{1}{3}-\left(\frac{7}{3}\right)+\frac{6}{3}\stackrel{?}{=}0$ $0=0$ ✓ $x=\frac{1}{2}$ $3x^{-2}-7x^{-1}+2=0$ $3\left(\frac{1}{2}\right)^{-2}-7\left(\frac{1}{2}\right)^{-1}+2\stackrel{?}{=}0$ $3(4)-7(2)+2\stackrel{?}{=}0$ $12-14+2\stackrel{?}{=}0$ $0=0$ ✓ Figure 9.4.42	

Exercise 9.4.9

Solve: $8x^{-2}-10x^{-1}+3=0$.

Answer

$x=\frac{4}{3}, x=2$

Exercise 9.4.10

Solve: $6x^{-2}-23x^{-1}+20=0$.

Answer

$x=\frac{2}{5}, x=\frac{3}{4}$

Access this online video for additional instruction and practice with solving quadratic equations: https://www.youtube.com/watch?v=7X-CZMbpxuw

Key Concepts

- How to solve equations in quadratic form.
  1. Identify a substitution that will put the equation in quadratic form.
  2. Rewrite the equation with the substitution to put it in quadratic form.
  3. Solve the quadratic equation for u.
  4. Substitute the original variable back into the results, using the substitution.
  5. Solve for the original variable.
  6. Check the solutions.

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9.4E: Exercises

Practice Makes Perfect

? Exercise $9.4E.11$ Solve equations in quadratic form

In the following exercises, solve.

1. $x^4-7x^2+12=0$
2. $x^4-9x^2+18=0$
3. $x^4-13x^2-30=0$
4. $x^4+5x^2-36=0$
5. $2x^4-5x^2+3=0$
6. $4x^4-5x^2+1=0$
7. $2x^4-7x^2+3=0$
8. $3x^4-14x^2+8=0$
9. $(x-3)^2-5(x-3)-36=0$
10. $(x+2)^2-3(x+2)-54=0$
11. $(3y+2)^2+(3y+2)-6=0$
12. $(5y-1)^2+3(5y-1)-28=0$
13. $\left(x^2+1\right)^2-5\left(x^2+1\right)+4=0$
14. $\left(x^2-4\right)^2-4\left(x^2-4\right)+3=0$
15. $2\left(x^2-5\right)^2-5\left(x^2-5\right)+2=0$
16. $2\left(x^2-5\right)^2-7\left(x^2-5\right)+6=0$
17. $x-\sqrt{x}-20=0$
18. $x-8\sqrt{x}+15=0$
19. $x+6\sqrt{x}-16=0$
20. $x+4\sqrt{x}-21=0$
21. $6x+\sqrt{x}-2=0$
22. $6x+\sqrt{x}-1=0$
23. $10x-17\sqrt{x}+3=0$
24. $12x+5\sqrt{x}-3=0$
25. $x^{\frac{2}{3}}+9x^{\frac{1}{3}}+8=0$
26. $x^{\frac{2}{3}}-3x^{\frac{1}{3}}=28$
27. $x^{\frac{2}{3}}+4x^{\frac{1}{3}}=12$
28. $x^{\frac{2}{3}}-11x^{\frac{1}{3}}+30=0$
29. $6x^{\frac{2}{3}}-x^{\frac{1}{3}}=12$
30. $3x^{\frac{2}{3}}-10x^{\frac{1}{3}}=8$
31. $8x^{\frac{2}{3}}-43x^{\frac{1}{3}}+15=0$
32. $20x^{\frac{2}{3}}-23x^{\frac{1}{3}}+6=0$
33. $x-8x^{\frac{1}{2}}+7=0$
34. $2x-7x^{\frac{1}{2}}=15$
35. $6x^{-2}+13x^{-1}+5=0$
36. $15x^{-2}-26x^{-1}+8=0$
37. $8x^{-2}-2x^{-1}-3=0$
38. $15x^{-2}-4x^{-1}-4=0$

Answer

1. $x=\pm\sqrt{3}, x=\pm 2$
3. $x=\pm\sqrt{15}, x=\pm\sqrt{2}i$

5. $x = \pm 1, x = \frac{\pm\sqrt{6}}{2}$

7. $x = \pm\sqrt{3}, x = \pm\frac{\sqrt{2}}{2}$

9. $x = -1, x = 12$

11. $x = -\frac{5}{3}, x = 0$

13. $x = 0, x = \pm\sqrt{3}$

15. $x = \pm\frac{11}{2}, x = \pm\frac{\sqrt{22}}{2}$

17. $x = 25$

19. $x = 4$

21. $x = \frac{1}{4}$

23. $x = \frac{1}{25}, x = \frac{9}{4}$

25. $x = -1, x = -512$

27. $x = 8, x = -216$

29. $x = \frac{27}{8}, x = -\frac{64}{27}$

31. $x = 27, x = 64,000$

33. $x = 1, x = 49$

35. $x = -2, x = -\frac{3}{5}$

37. $x = -2, x = \frac{4}{3}$

Exercise $9.4E.12$ writing exercises

1. Explain how to recognize an equation in quadratic form.
2. Explain the procedure for solving an equation in quadratic form.

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

Figure 9.4.43

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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9.5: Solve Applications of Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is -100. Find the numbers.
   If you missed this problem, review Example 2.18.
2. Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.
   If you missed this problem, review Example 7.35.
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review Example 2.34.

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

Use a Problem-Solving Strategy

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We have solved number applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

	Consecutive even integers
	$64, 66, 68$
n	1st even integer
$n+2$	2nd consecutive even integer
$n+4$	3rd consecutive even integer

	Consecutive odd integers
	$77, 79, 81$
n	1st odd integer
$n+2$	2nd consecutive odd integer
$n+4$	3rd consecutive odd integer

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example 9.5.1

The product of two consecutive odd integers is 195. Find the integers.

Solution:

Step 1: **Read** the problem

Step 2: **Identify** what we are looking for.

We are looking for two consecutive odd integers.

Step 3: **Name** what we are looking for.

Let $n =$ the first odd integer.

$n+2 =$ the next odd integer.

Step 4: **Translate** into an equation. State the problem in one sentence.

"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.

Translate into an equation.

$$n(n+2) = 195$$

Step 5: **Solve** the equation. Distribute.

$$n^2 + 2n = 195$$

Write the equation in standard form.

$$n^2 + 2n - 195 = 0$$

Factor.

$$(n+15)(n-13) = 0$$

Use the Zero Product Property.

$$n+15 = 0 \quad n-13 = 0$$

Solve each equation.

$$n = -15, \quad n = 13$$

There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.

First odd integer $n = 13$	First odd integer $n = -15$
next odd integer $n+2$	next odd integer $n+2$
$13+2$	$-15+2$
15	-13

Step 6: **Check** the answer.

Do these pairs work? Are they consecutive odd integers?

$13, 15$ yes
$-13, -15$ yes

Is their product 195?

$13 \cdot 15 = 195$ yes

$-13(-15) = 195$ yes

Step 7: **Answer** the question.

Two consecutive odd integers whose product is 195 are $13, 15$ and $-13, -15$.

Exercise 9.5.1

The product of two consecutive odd integers is 99. Find the integers.

Answer

The two consecutive odd integers whose product is 99 are $9, 11$, and $-9, -11$.

Exercise 9.5.2

The product of two consecutive even integers is 168. Find the integers.

Answer

The two consecutive even integers whose product is 128 are $12, 14$ and $-12, -14$.

We will use the formula for the area of a triangle to solve the next example.

Definition 9.5.1

Area of a Triangle

For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

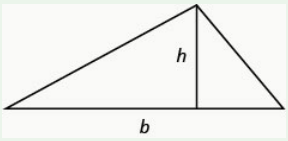


Figure 9.5.1

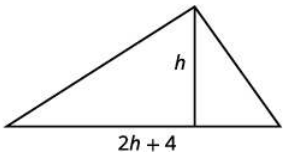
Recall that when we solve geometric applications, it is helpful to draw the figure.

Example 9.5.2

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Solution:

Table 9.5.1

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the base and height.
Step 3: **Name** what we are looking for.	Let $h =$ the height of the triangle. $2h + 4 =$ the base of the triangle.

Step 4: **Translate** into an equation. We know the area. Write the formula for the area of a triangle.	$A=\frac{1}{2}bh$
Step 5: **Solve** the equation. Substitute in the values.	$120=\frac{1}{2}(2h+4)h$
Distribute.	$120=h^2+2h$
This is a quadratic equation, rewrite it in standard form.	$h^2+2h-120=0$
Factor.	$(h-10)(h+12)=0$
Use the Zero Product Property.	$h-10=0 \quad h+12=0$
Simplify.	$h=10, \quad h=\cancel{-12}$
Since h is the height of a window, a value of $h=-12$ does not make sense.	
The height of the triangle $h=10$.	
The base of the triangle $2h+4$. $2\cdot 10+4$ 24	
Step 6: **Check** the answer. Does a triangle with height 10 and base 24 have area 120? Yes.	
Step 7: **Answer** the question.	The height of the triangular window is 10 feet and the base is 24 feet.

? Exercise 9.5.3

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

Answer

The height of the triangle is 12 inches and the base is 76 inches.

? Exercise 9.5.4

If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

Answer

The height of the triangle is 11 feet and the base is 20 feet.

In the two preceding examples, the number in the radical in the **Quadratic Formula** was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

Definition 9.5.2

Area of a Rectangle

For a rectangle with length, L, and width, W, the area, A, is given by the formula $A=LW$.

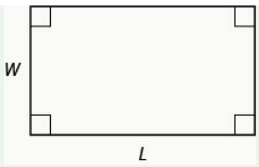


Figure 9.5.3

Example 9.5.3

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution:

Table 9.5.2

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the length and width.
Step 3: **Name** what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4: **Translate** into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5: **Solve** the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$
This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula.	$ax^2 + bx + c = 0$ $3w^2 - w - 150 = 0$
Identify the a, b, c values.	$a = 3,\ b = -1,\ c = -150$
Write the Quadratic Formula.	$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$w = \frac{1 \pm \sqrt{1 + 1800}}{6}$ $w = \frac{1 \pm \sqrt{1801}}{6}$ Figure 9.5.13
Rewrite to show two solutions.	$w = \frac{1 + \sqrt{1801}}{6}, \quad w = \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2, \quad$ ~~$w \approx -6.9$~~ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Step 6: **Check** the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.	
Step 7: **Answer** the question.	The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet.

Exercise 9.5.5

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

Answer

The length of the garden is approximately 18 feet and the width 11 feet.

Exercise 9.5.6

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot?

Answer

The length of the tablecloth is approximately 11.8 feet and the width 6.8 feet.

The **Pythagorean Theorem** gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Definition 9.5.3

Pythagorean Theorem

In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.

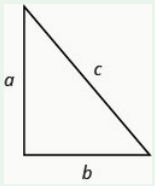


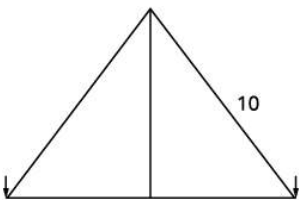
Figure 9.5.16

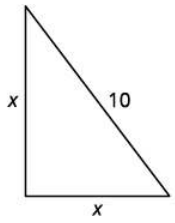
Example 9.5.4

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution:

Table 9.5.3

Step 1: **Read** the problem. Draw a picture.	
Step 2: **Identify** what we are looking for.	We are looking for the height of the pole.

Step 3: **Name** what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from pole to stake Each side is a right triangle. We draw a picture of one of them.  Figure 9.5.18
Step 4: **Translate** into an equation. We can use the Pythagorean Theorem to solve for x. Write the Pythagorean Theorem.	$a^2+b^2=c^2$
Step 5: **Solve** the equation. Substitute.	$x^2+x^2=10^2$
Simplify.	$2x^2=100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2}=\frac{100}{2}$
Simplify.	$x^2=50$
Use the Square Root Property.	$x=\pm\sqrt{50}$
Simplify the radical.	$x=\pm 5\sqrt{2}$
Rewrite to show two solutions.	$x=5\sqrt{2}, \quad x=-5\sqrt{2}$
	If we approximate this number to the nearest tenth with a calculator, we find $x \approx 7.1$.
Step 6: **Check** the answer. Check on your own in the Pythagorean Theorem.	
Step 7: **Answer** the question.	The pole should be about 7.1 feet tall.

? Exercise 9.5.7

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

Answer

The length of the flag pole's shadow is approximately 6.3 feet and the height of the flag pole is 18.9 feet.

? Exercise 9.5.8

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

Answer

The distance between the opposite corners is approximately 7.2 feet.

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Definition 9.5.4

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula

$$h = -16t^2 + v_0 t$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

Example 9.5.5

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution:

Table 9.5.4

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	We are looking for the number of seconds, which is time.
Step 3: **Name** what we are looking for.	Let t = the number of seconds.
Step 4: **Translate** into an equation. Use the formula.	$h = -16t^2 + v_0 t$
Step 5: **Solve** the equation. We know the velocity v_0 is 130 feet per second. The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula.	$ax^2 + bx + c = 0$ $16t^2 - 130t + 260 = 0$
Identify the values of a, b, c.	$a = 16,\ b = -130,\ c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$ Figure 9.5.26
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32}, \quad t = \frac{130 - \sqrt{260}}{32}$
Approximate the answer with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$ seconds
Step 6: **Check** the answer. The check is left to you.	
Step 7: **Answer** the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

Exercise 9.5.9

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

Answer

The arrow will reach 180 feet on its way up after 3 seconds and again on its way down after approximately 3.8 seconds.

Exercise 9.5.10

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

Answer

The ball will reach 48 feet on its way up after approximately $.6$ second and again on its way down after approximately 5.4 seconds.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate	• Time	= Distance

Figure 9.5.29

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

Some uniform motion problems are also modeled by quadratic equations.

Example 9.5.6

Professor Smith just returned from a conference that was $2,000$ miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

Solution:

This is a uniform motion situation. A diagram will help us visualize the situation.

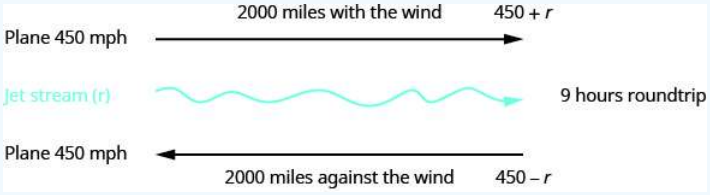


Figure 9.5.30

We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is $450 + r$.

When the plane flies against the wind, the wind decreases its speed and the rate is $450 - r$.

Table 9.5.5

Write in the rates. Write in the distances. Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th>Type</th><th>Rate</th><th>·</th><th>Time</th><th>=</th><th>Distance</th></tr><tr><td>Headwind</td><td>$450 - r$</td><td></td><td>$\frac{2000}{450-r}$</td><td></td><td>2000</td></tr><tr><td>Tailwind</td><td>$450 + r$</td><td></td><td>$\frac{2000}{450+r}$</td><td></td><td>2000</td></tr><tr><td></td><td></td><td></td><td>9</td><td></td><td></td></tr></table>
We know the times add to 9 and so we write our equation.	$\frac{2000}{450-r} + \frac{2000}{450+r} = 9$
We multiply both sides by the LCD.	$(450-r)(450+r)\left(\frac{2000}{450-r} + \frac{2000}{450+r}\right) = 9(450-r)(450+r)$
Simplify.	$2000(450+r) + 2000(450-r) = 9(450-r)(450+r)$
Factor the 2, 000.	$2000(450+r+450-r) = 9\left(450^2 - r^2\right)$
Solve.	$2000(900) = 9\left(450^2 - r^2\right)$
Divide by 9.	$2000(100) = 450^2 - r^2$
Simplify.	$200000 = 202500 - r^2$ $-2500 = -r^2$ \ $50 = r$ The speed of the jet stream is 50 mph.
Check: Is 50 mph a reasonable speed for the jet stream? Yes. If the plane is traveling 450 mph and the wind is 50 mph, Tailwind $450 + 50 = 500\text{mph}$ $\frac{2000}{500} = 4$ hours Headwind $450 - 50 = 400\text{mph}$ $\frac{2000}{400} = 5$ hours The times add to 9 hours, so it checks.	

The speed of the jet stream was 50 mph.

Exercise 9.5.11

MaryAnne just returned from a visit with her grandchildren back east. The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 100 mph.

Exercise 9.5.12

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 50 mph.

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

Example 9.5.7

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

Solution:

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Table 9.5.6

Let $x =$ the number of hours for Press #2 to complete the job. Enter the hours per job for Press #1, Press #2, and when they work together.	<table><tr><th></th><th>Number of hours needed to complete the job.</th><th>Part of job completed/hour</th></tr><tr><td>Press #1</td><td>$x+12$</td><td>$\frac{1}{x+12}$</td></tr><tr><td>Press #2</td><td>x</td><td>$\frac{1}{x}$</td></tr><tr><td>Together</td><td>8</td><td>$\frac{1}{8}$</td></tr></table>
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together. Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$
Solve.	$\frac{1}{x+12}+\frac{1}{x}=\frac{1}{8}$
Multiply by the LCD, $8x(x+12)$.	$8x(x+12)\left(\frac{1}{x+12}+\frac{1}{x}\right)=\left(\frac{1}{8}\right)8x(x+12)$
Simplify.	$8x+8(x+12)=x(x+12)$ $8x+8x+96=x^2+12x$ Figure 9.5.37 $0=x^2-4x-96$ Figure 9.5.38
Solve.	$0=(x-12)(x+8)$ $x-12=0, x+8=0$ Figure 9.5.40 $x=12, x=-8$ hours Figure 9.5.41
Since the idea of negative hours does not make sense, we use the values $x=12$.	$12+12$ 24 hours 12 12 hours Figure 9.5.43
Write our sentence answer.	Press #1 would take 24 hours and Press #2 would take 12 hours to do the job alone.

Exercise 9.5.13

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

Answer

Press #1 would take 12 hours, and Press #2 would take 6 hours to do the job alone.

Exercise 9.5.14

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

Answer

The red hose take 6 hours and the green hose take 3 hours alone.

Access these online resources for additional instruction and practice with solving applications modeled by quadratic equations.

- Word Problems Involving Quadratic Equations
- Quadratic Equation Word Problems
- Applying the Quadratic Formula

Key Concepts

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- Area of a Triangle
  - For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

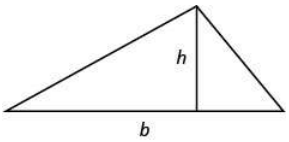


Figure 9.5.1

- Area of a Rectangle
  - For a rectangle with length,L, and width, W, the area, A, is given by the formula $A = LW$.

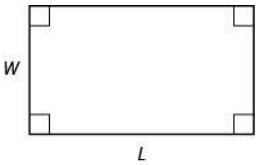


Figure 9.5.3

- Pythagorean Theorem
  - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.

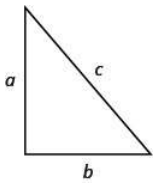


Figure 9.5.16

- Projectile motion
  - The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula $h = -16t^2 + v_0 t$.

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9.5E: Exercises

Practice Makes Perfect

Exercise $9.5E.15$ Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method.

1. The product of two consecutive odd numbers is 255. Find the numbers.
2. The product of two consecutive even numbers is 360. Find the numbers.
3. The product of two consecutive even numbers is 624. Find the numbers.
4. The product of two consecutive odd numbers is $1,023$. Find the numbers.
5. The product of two consecutive odd numbers is 483. Find the numbers.
6. The product of two consecutive even numbers is 528. Find the numbers.

Answer

1. Two consecutive odd numbers whose product is 255 are 15 and 17, and -15 and -17.

3. The first and second consecutive odd numbers are 24 and 26, and -26 and -24.

5. Two consecutive odd numbers whose product is 483 are 21 and 23, and -21 and -23.

Exercise $9.5E.16$ Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method. Round your answers to the nearest tenth, if needed.

1. A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.
2. The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.
3. The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?
4. A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?
5. The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.
6. A rectangular lawn has area 140 square yards. Its length is six yards less than twice its width. What are the length and width of the lawn?
7. A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.
8. The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?
9. The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.
10. The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.
11. A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.

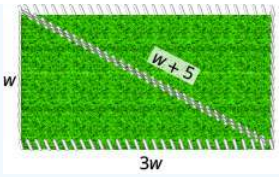


Figure 9.5.44

12. Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.

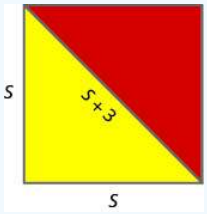


Figure 9.5.45

13. Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

14. John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

15. An arrow is shot vertically upward at a rate of $v_0 = 220$ feet per second. Use the projectile formula $h = -16t^2 + v_0 t$, to determine when the height of the arrow will be 400 feet.

16. A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

17. A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0 t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

18. A stone is dropped from a 196-foot platform. Use the formula $h = -16t^2 + v_0 t + 196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0 = 0$.)

19. The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

20. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

21. Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

22. Rick paddled up the river, spent the night camping, and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

23. Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

24. Two gardeners can do the weekly yard maintenance in 8 minutes if they work together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintenance individually?

25. It takes two hours for two machines to manufacture $10,000$ parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture $10,000$ parts alone?

26. Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the pool?

Answer

1. The width of the triangle is 5 inches and the height is 18 inches.
3. The base is 24 feet and the height of the triangle is 10 feet.
5. The length of the driveway is 15.0 feet and the width is 3.3 feet.
7. The length of table is 8 feet and the width is 3 feet.
9. The length of the legs of the right triangle are 3.2 and 9.6 cm.
11. The length of the diagonal fencing is 7.3 yards.
13. The ladder will reach 24.5 feet on the side of the house.
15. The arrow will reach 400 feet on its way up in 2.2 seconds and on the way down in 11.6 seconds.
17. The bullet will take 70 seconds to hit the ground.
19. The speed of the wind was 49 mph.
21. The speed of the current was 4.3 mph.
23. The less experienced painter takes 6 hours and the experienced painter takes 3 hours to do the job alone.
25. Machine #1 takes 3.6 hours and Machine #2 takes 4.6 hours to do the job alone.

? Exercise $9.5E.17$ writing exercises

1. Make up a problem involving the product of two consecutive odd integers.
   a. Start by choosing two consecutive odd integers. What are your integers?
   b. What is the product of your integers?
   c. Solve the equation $n(n+2)=p$, where p is the product you found in part (b).
   d. Did you get the numbers you started with?
2. Make up a problem involving the product of two consecutive even integers.
   a. Start by choosing two consecutive even integers. What are your integers?
   b. What is the product of your integers?
   c. Solve the equation $n(n+2)=p$, where p is the product you found in part (b).
   d. Did you get the numbers you started with?

Answer

1. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

Figure 9.5.46

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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9.6: Graph Quadratic Functions Using Properties

Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

Before you get started, take this readiness quiz.

1. Graph the function $f(x) = x^2$ by plotting points.
   If you missed this problem, review Example 3.54.
2. Solve: $2x^2 + 3x - 2 = 0$.
   If you missed this problem, review Example 6.45.
3. Evaluate $-\frac{b}{2a}$ when $a = 3$ and $b = -6$.
   If you missed this problem, review Example 1.21.

Recognize the Graph of a Quadratic Function

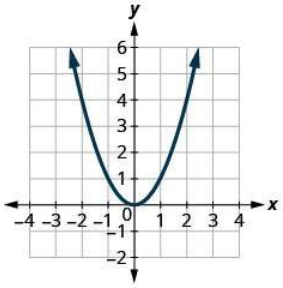
Previously we very briefly looked at the function $f(x) = x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x) = ax^2 + bx + c$ if $a \neq 0$. We call this kind of function a quadratic function.

Definition 9.6.1

A **quadratic function**, where a, b, and c are real numbers and $a \neq 0$, is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function $f(x) = x^2$ by plotting points.



x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Figure 9.6.1

Every quadratic function has a graph that looks like this. We call this figure a **parabola**. Let's practice graphing a parabola by plotting a few points.

Example 9.6.1

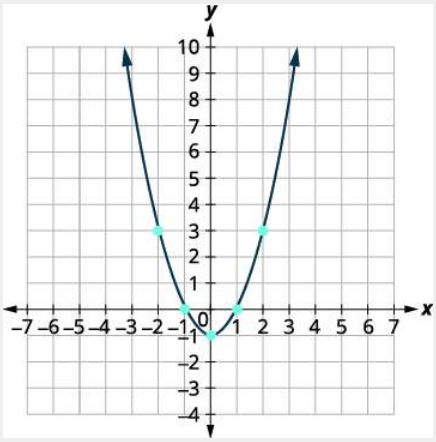
Graph: $f(x) = x^2 - 1$.

Solution:

We will graph the function by plotting points.

Choose integer values for x, substitute them into the equation and simplify to find $f(x)$. Record the values of the ordered pairs in the chart.	$f(x) = x^2 - 1$	
	x	$f(x)$
	0	–1
	1	0
	–1	0
	2	3
	–2	3

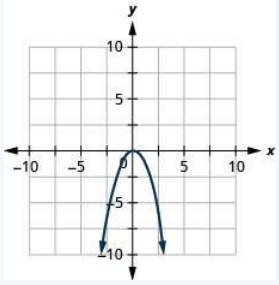
Plot the points, and then connect them with a smooth curve. The result will be the graph of the function $f(x) = x^2 - 1$.



Exercise 9.6.1

Graph $f(x) = -x^2$.

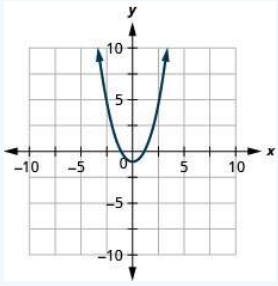
Answer



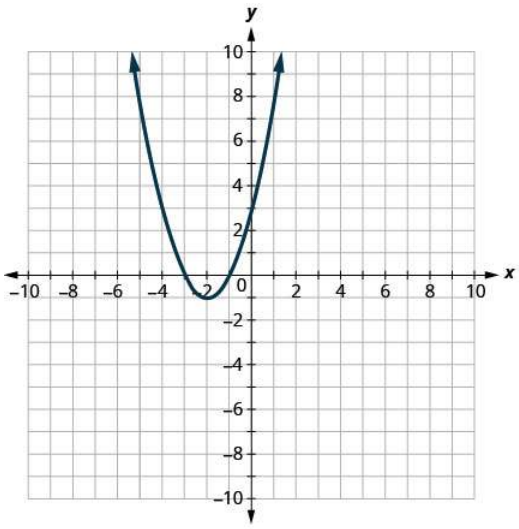
Exercise 9.6.2

Graph $f(x) = x^2 - 1$.

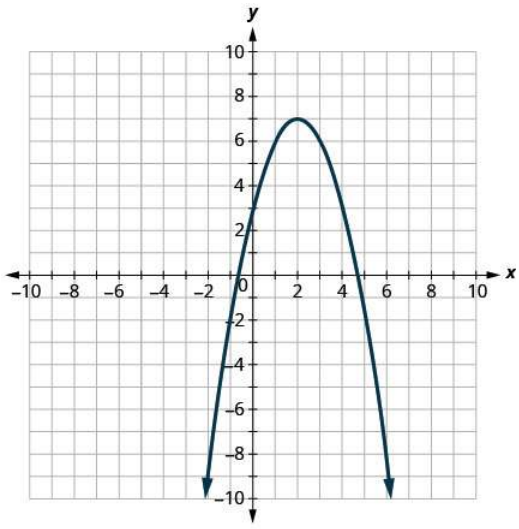
Answer



All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See *Figure 9.6.6*



$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$
$a > 0$
opens upward



$f(x) = ax^2 + bx + c$
$f(x) = -x^2 + 4x + 3$
$a < 0$
opens downward

Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in *Figure 9.6.6*). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Definition 9.6.2

Parabola Orientation

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

Example 9.6.2

Determine whether each parabola opens upward or downward:

a. $f(x) = -3x^2 + 2x - 4$
b. $f(x) = 6x^2 + 7x - 9$

Solution:

a. Find the value of a.

$f(x) = ax^2 + bx + c$
$f(x) = -3x^2 + 2x - 4$
$a = -3$

Since the a is negative, the parabola will open downward.

b. Find the value of a.

$f(x) = ax^2 + bx + c$
$f(x) = 6x^2 + 7x - 9$
$a = 6$

Since the a is positive, the parabola will open upward.

Exercise 9.6.3

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = 2x^2 + 5x - 2$
b. $f(x) = -3x^2 - 4x + 7$

Answer

a. up
b. down

Exercise 9.6.4

Determine whether the graph of each function is a parabola that opens upward or downward:

a. $f(x) = -2x^2 - 2x - 3$
b. $f(x) = 5x^2 - 2x - 1$

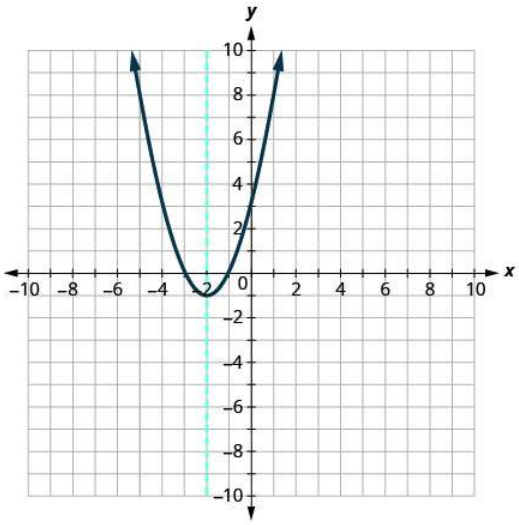
Answer

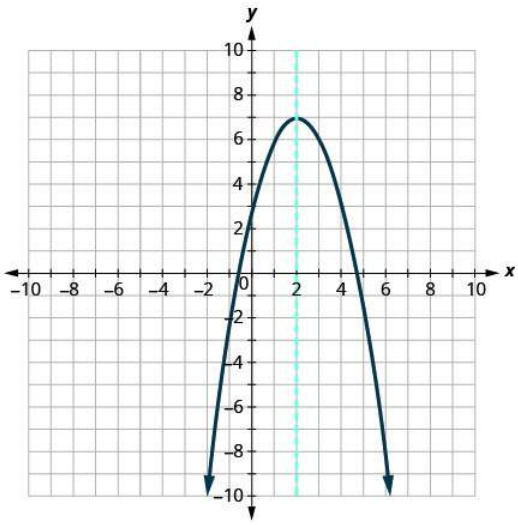
a. down
b. up

Find the Axis of Symmetry and Vertex of a Parabola

Look again at *Figure 9.6.10*. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry.





The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$	$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic function.

$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry is $x = -2$	axis of symmetry is $x = 2$
vertex is $(-2, \_\_\_)$	vertex is $(2, \_\_\_)$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
$f(x) = (-2)^2 + 4(-2) + 3$	$f(x) = -(2)^2 + 4(2) + 3$
$f(x) = -1$	$f(x) = 7$
vertex is $(-2, -1)$	vertex is $(2, 7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$
- the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

Example 9.6.3

For the graph of $f(x) = 3x^2 - 6x + 2$ find:

a. the axis of symmetry
b. the vertex

Solution:

a.

	$f(x) = ax^2 + bx + c$ $f(x) = 3x^2 - 6x + 2$
The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.	
Substitute the values a, b into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$
	The axis of symmetry is the line $x = 1$.

b.

	$f(x) = 3x^2 - 6x + 2$
The vertex is a point on the line of symmetry, so its x-coordinate will be $x = 1$. Find $f(1)$.	$f(1) = 3(1)^2 - 6(1) + 2$
Simplify.	$f(1) = 3 \cdot 1 - 6 + 2$

The result is the y-coordinate.	$f(1) = -1$
	The vertex is $(1, -1)$.

Exercise 9.6.5

For the graph of $f(x) = 2x^2 - 8x + 1$ find:

a. the axis of symmetry
b. the vertex

Answer

a. $x = 2$
b. $(2, -7)$

Exercise 9.6.6

For the graph of $f(x) = 2x^2 - 4x - 3$ find:

a. the axis of symmetry
b. the vertex

Answer

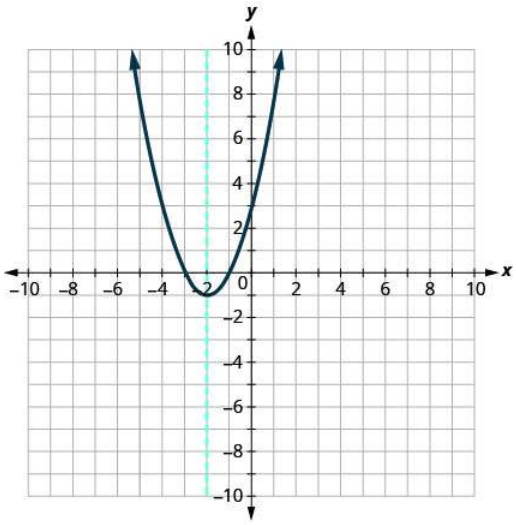
a. $x = 1$
b. $(1, -5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the x- and y-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the y-intercept the value of x is zero. So to find the y-intercept, we substitute $x = 0$ into the function.

Let's find the y-intercepts of the two parabolas shown in *Figure 9.6.20*.

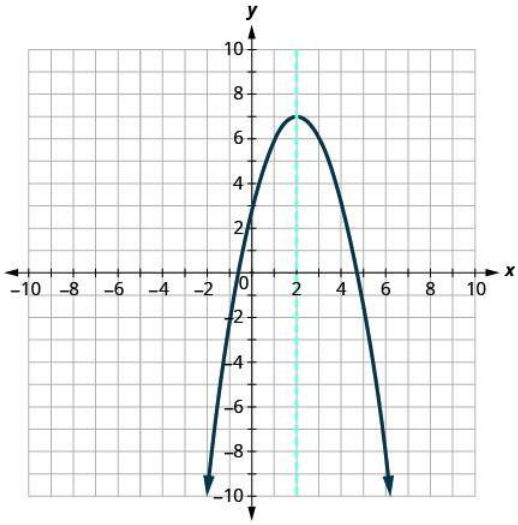


$f(x) = x^2 + 4x + 3$

$x = 0 \quad f(0) = 0^2 + 4 \cdot 0 + 3$

$f(0) = 3$

y-intercept $(0, 3)$



$f(x) = -x^2 + 4x + 3$

$x = 0 \quad f(0) = -0^2 + 4 \cdot 0 + 3$

$f(0) = 3$

y-intercept $(0, 3)$

An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x)=0$. In other words, we will need to solve the equation $0=ax^2+bx+c$ for x.

$$f(x)=ax^2+bx+c$$
$$0=ax^2+bx+c$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose function is $f(x)=x^2+4x+3$.

	$f(x)=x^2+4x+3$
Let $f(x)=0$.	$0=x^2+4x+3$
Factor.	$0=(x+1)(x+3)$
Use the Zero Product Property.	$x+1=0 \quad x+3=0$
Solve.	$x=-1 \quad x=-3$
	The x-intercepts are $(-1,0)$ and $(-3,0)$.

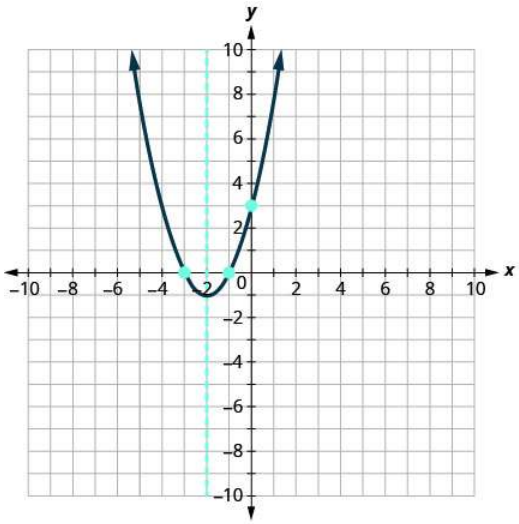
Now we will find the x-intercepts of the parabola whose function is $f(x)=-x^2+4x+3$.

	$f(x)=-x^2+4x+3$
Let $f(x)=0$.	$0=-x^2+4x+3$
This quadratic does not factor, so we use the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$a=-1, b=4, c=3$	$x=\frac{-4\pm\sqrt{4^2-4(-1)(3)}}{2(-1)}$
Simplify.	$x=\frac{-4\pm\sqrt{28}}{-2}$
	$x=\frac{-4\pm2\sqrt{7}}{-2}$
	$x=\frac{-2(2\pm\sqrt{7})}{-2}$
	$x=2\pm\sqrt{7}$
	The x-intercepts are $(2+\sqrt{7},0)$ and $(2-\sqrt{7},0)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph,

$$(2+\sqrt{7},0)\approx(4.6,0) \quad (2-\sqrt{7},0)\approx(-0.6,0)$$

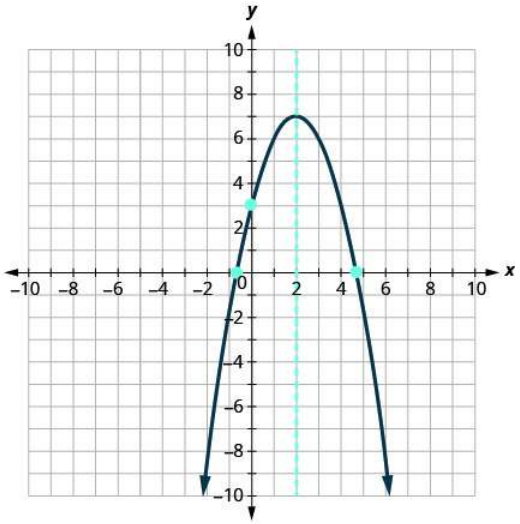
Do these results agree with our graphs? See *Figure 9.6.34*



$f(x) = x^2 + 4x + 3$

y-intercept $(0, 3)$

x-intercepts $(-1, 0)$ and $(-3, 0)$



$f(x) = -x^2 + 4x + 3$

y-intercept $(0, 3)$

x-intercepts $(2 + \sqrt{7}, 0) \approx (4.6, 0)$

$(2 - \sqrt{7}, 0) \approx (-0.6, 0)$

Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

y-intercept

Let $x = 0$ and solve for $f(x)$.

x-intercepts

Let $f(x) = 0$ and solve for x

Example 9.6.4

Find the intercepts of the parabola whose function is $f(x) = x^2 - 2x - 8$.

Solution:

To find the y-intercept, let $x = 0$ and solve for $f(x)$.	$f(x) = x^2 - 2x - 8$
	$f(0) = 0^2 - 2 \cdot 0 - 8$
	$f(0) = -8$
	When $x = 0$, then $f(0) = -8$. The y-intercept is the point $(0, -8)$.
To find the x-intercept, let $f(x) = 0$ and solve for x.	$f(x) = x^2 - 2x - 8$
	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$
	$4 = x \quad -2 = x$
	When $f(x) = 0$, then $x = 4$ or $x = -2$. The x-intercepts are the points $(4, 0)$ and $(-2, 0)$.

Exercise 9.6.7

Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

Answer

y -intercept: $(0, -8)$ x -intercepts $(-4, 0), (2, 0)$

Exercise 9.6.8

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

Answer

y -intercept: $(0, -12)$ x -intercepts $(-2, 0), (6, 0)$

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The x-intercepts of the parabolas occur where $f(x) = 0$.

For example:

Quadratic equation

$$\begin{aligned} x^2 - 2x - 15 &= 0 \quad \text{Let } f(x) = 0 \\ (x-5)(x+3) &= 0 \\ x - 5 = 0 \quad x + 3 &= 0 \\ x = 5 \quad x &= -3 \end{aligned}$$

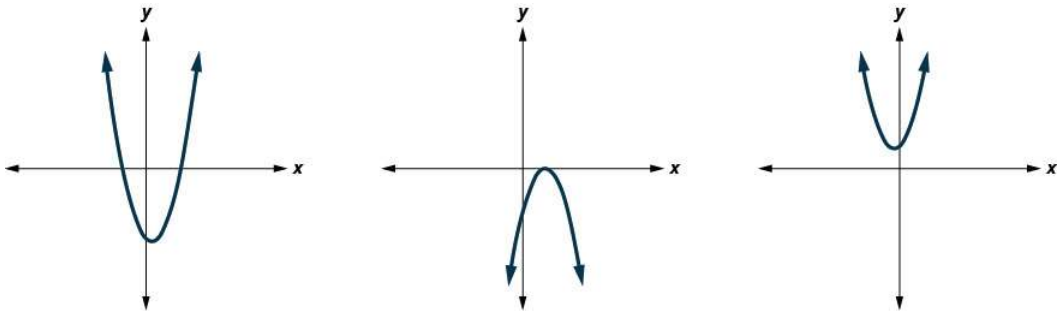
Quadratic function

$$\begin{aligned} f(x) &= x^2 - 2x - 15 \\ 0 &= x^2 - 2x - 15 \\ 0 &= (x-5)(x+3) \\ x - 5 = 0 \quad & x + 3 = 0 \\ x = 5 \quad & x = -3 \\ (5, 0) \text{ and } &(-3, 0) \\ x \text{ -intercepts} & \end{aligned}$$

The solutions of the quadratic function are the x values of the x-intercepts.

Earlier, we saw that quadratic equations have $2, 1$, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the x-intercepts of the graphs, the number of x-intercepts is the same as the number of solutions.

Previously, we used the **discriminant** to determine the number of solutions of a quadratic function of the form $ax^2 + bx + c = 0$. Now we can use the discriminant to tell us how many x-intercepts there are on the graph.



$b^2 - 4ac > 0$
Two solutions
Two x-intercepts

$b^2 - 4ac = 0$
One solution
One x-intercept

$b^2 - 4ac < 0$
No real solution
No x-intercept

Before you to find the values of the x-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example 9.6.5

Find the intercepts of the parabola for the function $f(x) = 5x^2 + x + 4$.

Solution:

	$f(x) = 5x^2 + x + 4$
To find the y-intercept, let $x = 0$ and solve for $f(x)$.	$f(0) = 5 \cdot 0^2 + 0 + 4$
	$f(0) = 4$
	When $x = 0$, then $f(0) = 4$. The y-intercept is the point $(0, 4)$.
To find the x-intercept, let $f(x) = 0$ and solve for x.	$f(x) = 5x^2 + x + 4$
	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions which is also the number of x-intercepts.	
$b^2 - 4ac$ $1^2 - 4 \cdot 5 \cdot 4$ $1 - 80$ -79	
	Since the value of the discriminant is negative, there is no real solution to the equation. There are no x-intercepts.

Exercise 9.6.9

Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.

Answer

y-intercept: $(0, 4)$ no x-intercept

Exercise 9.6.10

Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 5$

Answer

y-intercept: $(0,-5)$ x-intercepts $(-1,0),(5,0)$

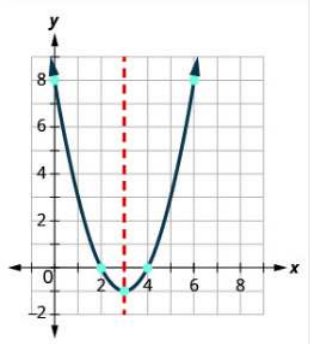
Graph Quadratic Functions Using Properties

Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

Example 9.6.6 How to Graph a Quadratic Function Using Properties

Graph $f(x)=x^2-6x+8$ by using its properties.

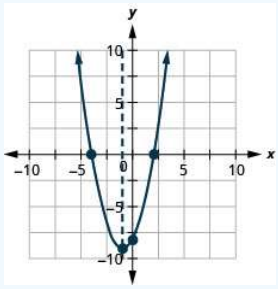
Solution:

Step 1: Determine whether the parabola opens upward or downward.	Look a in the equation $f(x)=x^2-6x+8$ Since a is positive, the parabola opens upward.	$f(x)=x^2-6x+8$ $a=1, b=-6, c=8$ **The parabola opens upward.**
Step 2: Find the axis of symmetry.	$f(x)=x^2-6x+8$ The axis of symmetry is the line $x=-\frac{b}{2a}$.	Axis of Symmetry $x=-\frac{b}{2a}$ $x=-\frac{(-6)}{2\cdot 1}$ $x=3$ **The axis of symmetry is the line $x=3$.**
Step 3: Find the vertex.	The vertex is on the axis of symmetry. Substitute $x=3$ into the function.	Vertex $f(x)=x^2-6x+8$ $f(3)=(3)^2-6(3)+8$ $f(3)=-1$ **The vertex is $(3,-1)$.**
Step 4: Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find $f(0)$. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x=3$. A point 3 units to the right of the axis of symmetry has $x=6$.	y-intercept $f(x)=x^2-6x+8$ $f(0)=(0)^2-6(0)+8$ $f(0)=8$ **The y-intercept is $(0,8)$.** Point symmetric to y-intercept: **The point is $(6,8)$.**
Step 5: Find the x-intercepts. Find additional points if needed.	We solve $f(x)=0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x)=x^2-6x+8$ $0=x^2-6x+8$ $0=(x-2)(x-4)$ $x=2 or x=4$ **The x-intercepts are $(2,0)$ and $(4,0)$.**
Step 6: Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Exercise 9.6.11

Graph $f(x) = x^2 + 2x - 8$ by using its properties.

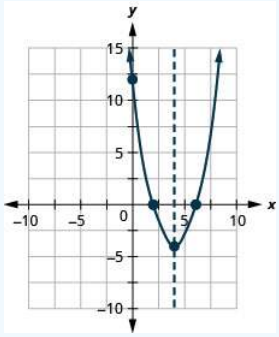
Answer



Exercise 9.6.12

Graph $f(x) = x^2 - 8x + 12$ by using its properties.

Answer



We list the steps to take in order to graph a quadratic function here.

To Graph a Quadratic Function Using Properties

1. Determine whether the parabola opens upward or downward.
2. Find the equation of the axis of symmetry.
3. Find the vertex.
4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts. Find additional points if needed.
6. Graph the parabola.

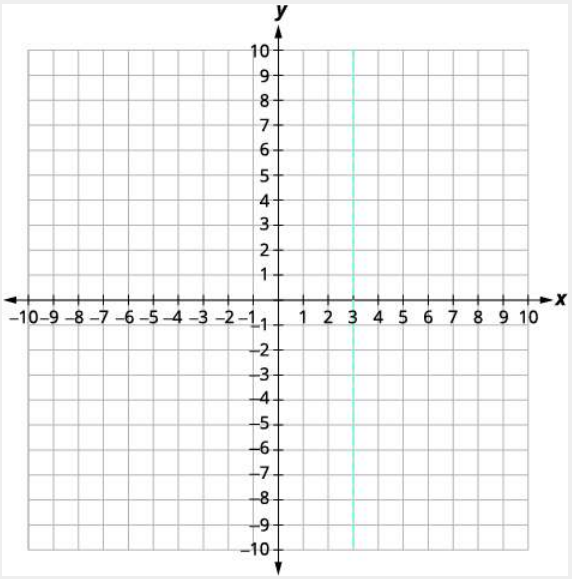
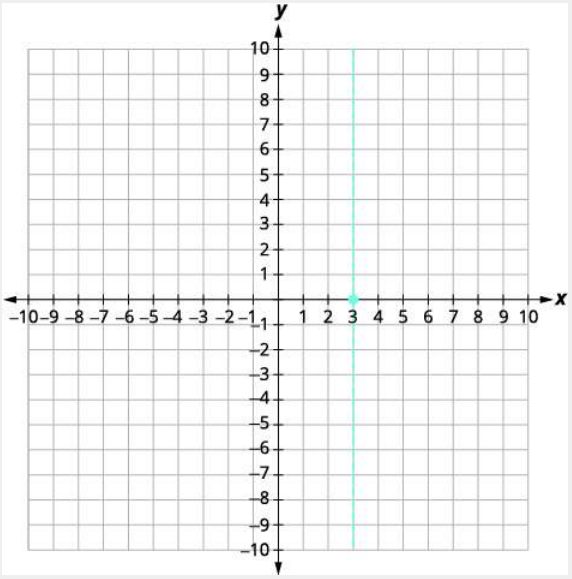
We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

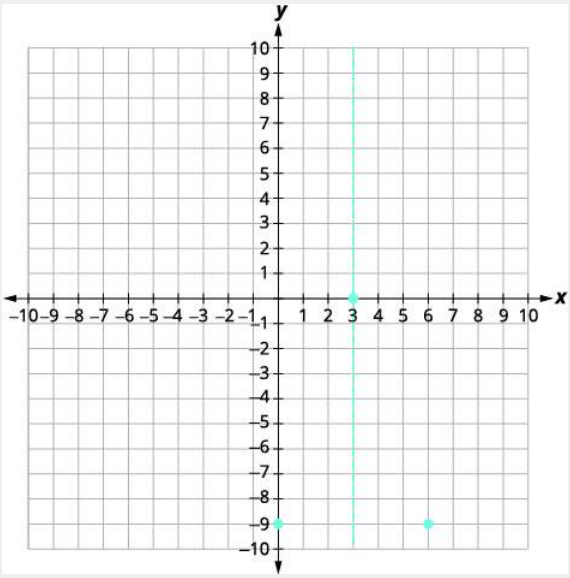
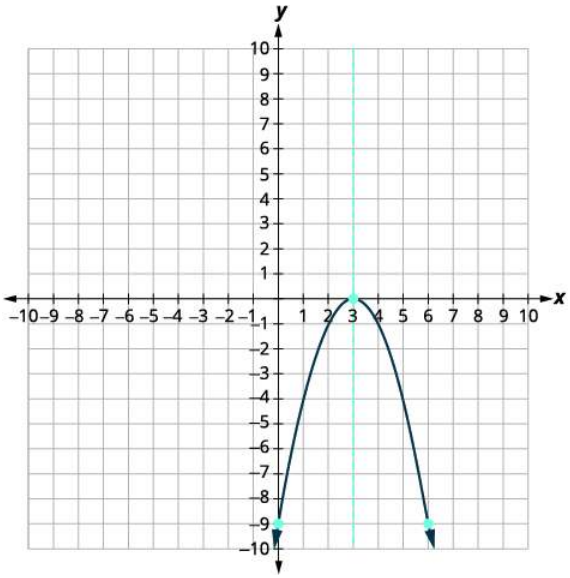
Example 9.6.7

Graph $f(x) = -x^2 + 6x - 9$ by using its properties.

Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	

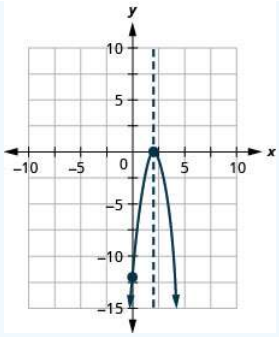
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{6}{2(-1)}$
	$x = 3$
	The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$.
	
Find $f(3)$.	$f(x) = -x^2 + 6x - 9$
	$f(3) = -3^2 + 6 \cdot 3 - 9$
	$f(3) = -9 + 18 - 9$
	$f(3) = 0$
	The vertex is $(3, 0)$.
	
The y-intercept occurs when $x = 0$. Find $f(0)$.	$f(x) = -x^2 + 6x - 9$
Substitute $x = 0$.	$f(0) = -0^2 + 6 \cdot 0 - 9$

Simplify.	$f(0) = -9$
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the y-intercept is $(6, -9)$
The x-intercept occurs when $f(x) = 0$.	$f(x) = -x^2 + 6x - 9$
Find $f(x) = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for x.	$0 = 3$
Connect the points to graph the parabola.	

? Exercise 9.6.13

Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

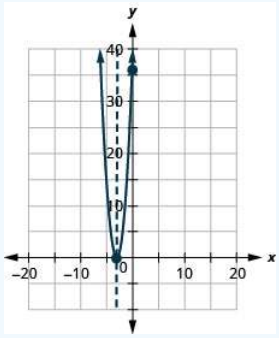
Answer



Exercise 9.6.14

Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.

Answer



For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

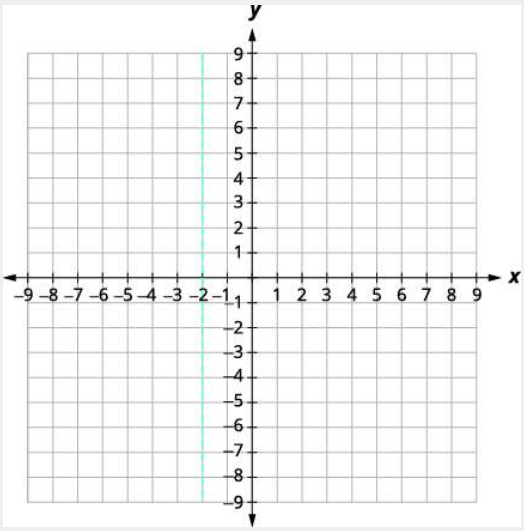
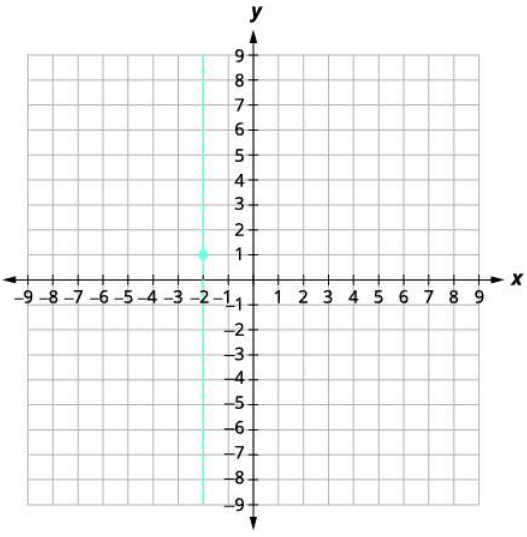
How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

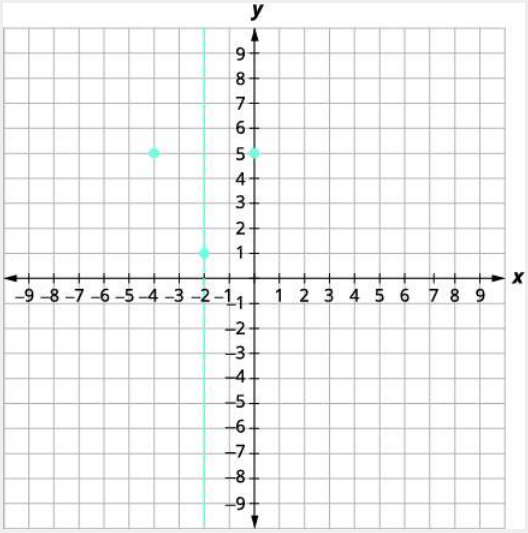
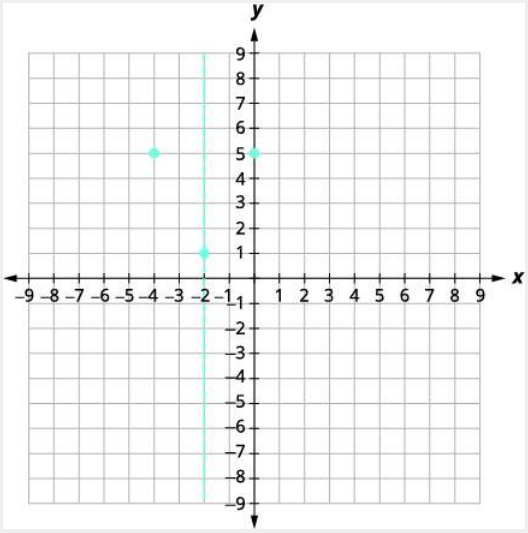
Example 9.6.8

Graph $f(x) = x^2 + 4x + 5$ by using its properties.

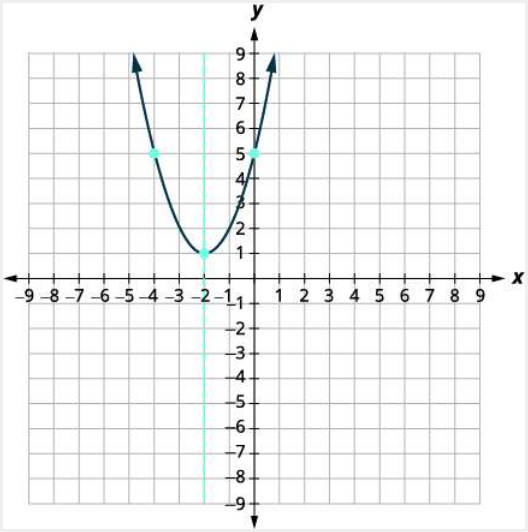
Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{4}{(2)1}$
	$x = -2$
	The equation of the axis of symmetry is \(x=-2).

	
The vertex is on the line $x = -2$.	
Find $f(x)$ when $x = -2$.	$f(x) = x^2 + 4x + 5$
	$f(-2) = (-2)^2 + 4(-2) + 5$
	$f(-2) = 4 - 8 + 5$
	$f(-2) = 1$
	The vertex is $(-2, 1)$.
	
The y-intercept occurs when $x = 0$.	$f(x) = x^2 + 4x - 5$
Find $f(0)$.	$f(0) = 5$
Simplify.	$f(0) = 5$
	The y-intercept is $(0, 5)$.

The point $(-4, 5)$ is two units to the left of the line of symmetry. The point to units to the right of the line of symmetry is \((0,5)\.	
	Point symmetric to the y-intercept is $(-4, 5)$.
The x-intercept occurs when $f(x) = 0$.	
Find $f(x) = 0$.	$0 = x^2 + 4x + 5$
Test the discriminant.	
	$b^2 - 4ac$
	$4^2 - 4 \cdot 1 \cdot 5$
	$16 - 20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no x-intercept.	

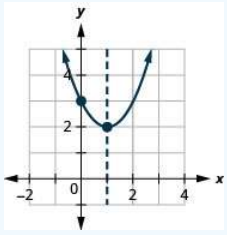
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.



Exercise 9.6.15

Graph $f(x) = x^2 - 2x + 3$ by using its properties.

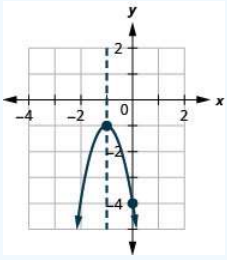
Answer



Exercise 9.6.16

Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

Answer



Finding the y-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the **Quadratic Formula** to find the x-intercepts.

Example 9.6.9

Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

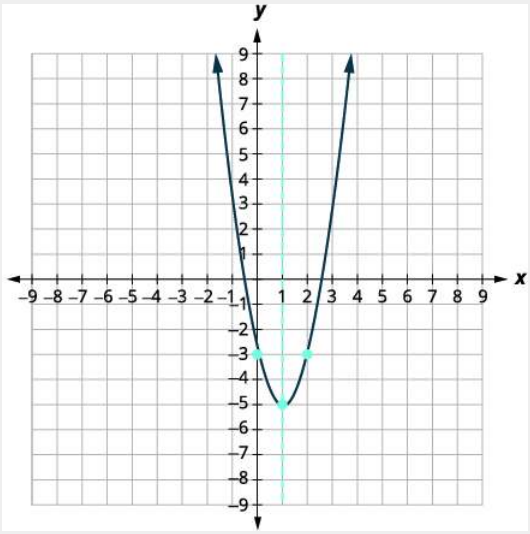
Solution:

$f(x) = ax^2 + bx + c$

$f(x) = 2x^2 - 4x - 3$

Since a is 2, the parabola opens upward.	
To find the equation of the axis of symmetry, use $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$
	$x=-\frac{-4}{2\cdot 2}$
	$x=1$
	The equation of the axis of symmetry is $x=1$.
The vertex is on the line $x=1$.	$f(x)=2x^2-4x-3$
Find $f(1)$.	$f(x)=2(1)^2-4(1)-3$
	$f(1)=2-4-3$
	\(\f(1)=-5)
	The vertex is $(1,-5)$.
The y-intercept occurs when $x=0$.	$f(x)=2x^2-4x-3$
Find $f(0)$.	$f(0)=2(0)^2-4(0)-3$
Simplify.	$f(0)=-3$
	The y-intercept is $(0,-3)$.
The point $(0,-3)$ is one unit to the left of the line of symmetry.	Point symmetric to the y-intercept is $(2,-3)$
The point one unit to the right of the line of symmetry is $(2,3)$.	
The x-intercept occurs when $y=0$.	$f(x)=2x^2-4x-3$
Find $f(x)=0$.	$0=2x^2-4x-3$
Use the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Substitute in the values of a, b and c.	$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(3)}}{2(2)}$
Simplify.	$x=\frac{-4\pm\sqrt{16+24}}{4}$
Simplify inside the radical.	$x=\frac{4\pm\sqrt{40}}{4}$
Simplify the radical.	$x=\frac{4\pm2\sqrt{10}}{4}$
Factor the GCF.	$x=\frac{2(2\pm\sqrt{10})}{4}$
Remove common factors.	$x=\frac{2\pm\sqrt{10}}{2}$
Write as two equations.	$x=\frac{2+\sqrt{10}}{2}, \quad x=\frac{2-\sqrt{10}}{2}$
Approximate the values.	$x\approx 2.5, \quad x\approx -0.6$
	The approximate values of the x-intercepts are $(2.5,0)$ and $(-0.6,0)$.

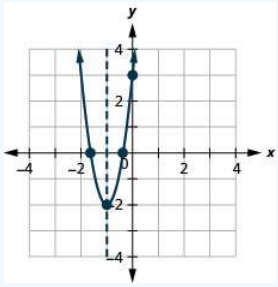
Graph the parabola using the points found.



Exercise 9.6.17

Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

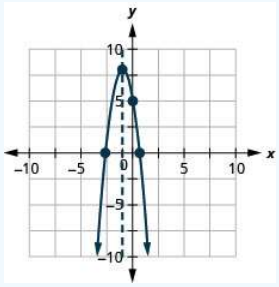
Answer



Exercise 9.6.18

Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.

Answer



Solve Maximum and Minimum Applications

Knowing that the **vertex** of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the **minimum** value of a parabola that opens upward. It is the **maximum** value of a parabola that opens downward. See *Figure 9.6.124*.

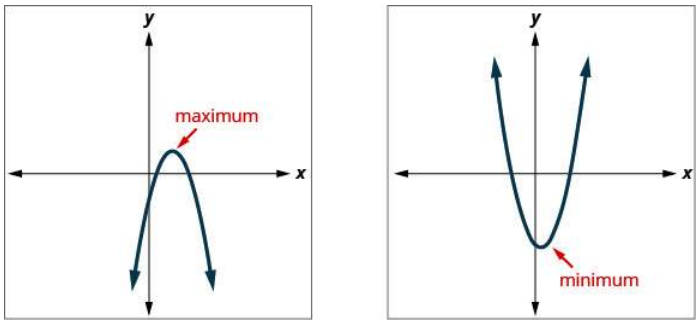


Figure 9.6.124

Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

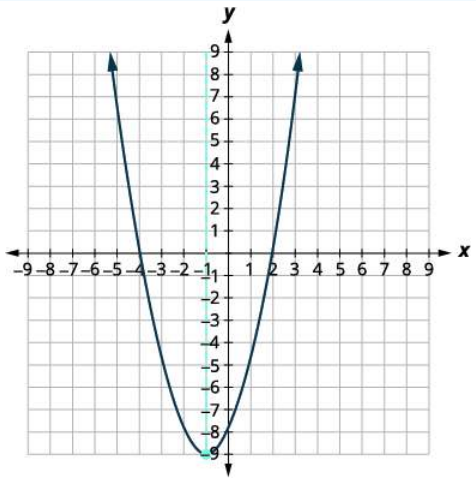
- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

Example 9.6.10

Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

Solution:

	$f(x) = x^2 + 2x - 8$
Since a is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x = -\frac{b}{2a}$
	$x = -\frac{2}{2\times 1}$
	$x = -1$
	The equation of the axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$f(x) = x^2 + 2x - 8$
Find $f(-1)$.	$f(-1) = (-1)^2 + 2(-1) - 8$
	$f(-1) = 1 - 2 - 8$
	$f(-1) = -9$
	The vertex is $(-1, -9)$.
Since the parabola has a minimum, the y-coordinate of the vertex is the minimum y-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x = -1$.	



Show the graph to verify the result.

? Exercise 9.6.19

Find the maximum or minimum value of the quadratic function $f(x) = x^2 - 8x + 12$.

Answer

The minimum value of the quadratic function is -4 and it occurs when $x = 4$.

? Exercise 9.6.20

Find the maximum or minimum value of the quadratic function $f(x) = -4x^2 + 16x - 11$.

Answer

The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

We have used the formula

$$h(t) = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

✓ Example 9.6.11

The quadratic equation $h(t) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

a. How many seconds will it take the volleyball to reach its maximum height?
b. Find the maximum height of the volleyball.

Solution:

$$h(t) = -16t^2 + 176t + 4$$

Since a is negative, the parabola opens downward. The quadratic function has a maximum.

a. Find the equation of the axis of symmetry.

$$t = -\frac{b}{2a}$$
$$t = -\frac{176}{2(-16)}$$
$$t = 5.5$$

The equation of the axis of symmetry is $t = 5.5$.

The vertex is on the line $t = 5.5$.

The maximum occurs when $t = 5.5$ seconds.

b. Find $h(5.5)$.

$$h(t) = -16t^2 + 176t + 4$$
$$h(t) = -16(5.5)^2 + 176(5.5) + 4$$

Use a calculator to simplify.

$$h(t) = 488$$

The vertex is $(5.5, 488)$.

Since the parabola has a maximum, the h-coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

? Exercise 9.6.21

Solve, rounding answers to the nearest tenth.

The quadratic function $h(t) = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

Answer

It will take 4 seconds for the stone to reach its maximum height of 288 feet.

? Exercise 9.6.22

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of $h(t) = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height?

Answer

It will take 6.5 seconds for the rocket to reach its maximum height of 676 feet.

Key Concepts

- Parabola Orientation
  - For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if
    - $a > 0$, the parabola opens upward.
    - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:
  - the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
  - the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
  - the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

- **y-intercept**
  - Let $x = 0$ and solve for $f(x)$.
- **x-intercepts**
  - Let \(f(x)=0\) and solve for x.

- How to graph a quadratic function using properties.
  1. Determine whether the parabola opens upward or downward.
  2. Find the equation of the axis of symmetry.
  3. Find the vertex.
  4. Find the y-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  5. Find the x-intercepts. Find additional points if needed.
  6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Equation
  - The y-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

Glossary

quadratic function

A quadratic function, where a, b, and c are real numbers and $a \neq 0$, is a function of the form $f(x) = ax^2 + bx + c$.

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9.6E: Graph Quadratic Functions Using Properties (Exercises)

Practice Makes Perfect

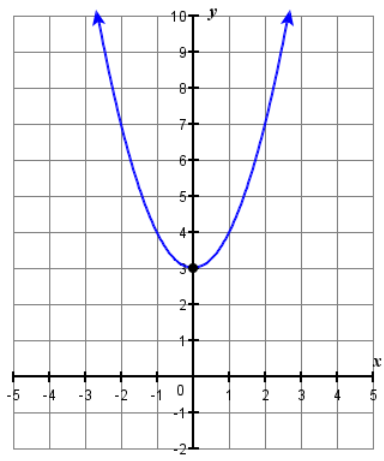
Exercises 1 - 4: Recognize the Graph of a Quadratic Function

In the following exercises, graph the functions by plotting points.

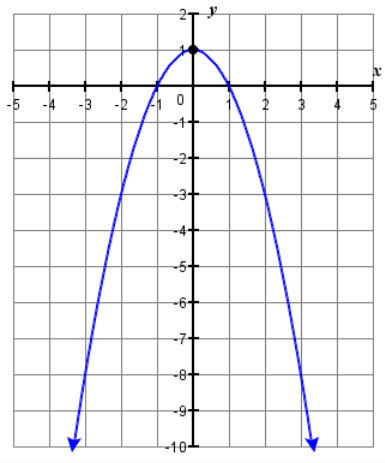
1. $f(x) = x^2 + 3$
2. $f(x) = x^2 - 3$
3. $y = -x^2 + 1$
4. $f(x) = -x^2 - 1$

Answer

1.



3.



Exercises 5 - 8: Recognize the Graph of a Quadratic Function

For each of the following exercises, determine if the parabola opens up or down.

5. a. $f(x) = -2x^2 - 6x - 7$ b. $f(x) = 6x^2 + 2x + 3$
6. a. $f(x) = 4x^2 + x - 4$ b. $f(x) = -9x^2 - 24x - 16$

7. a. $f(x) = -3x^2 + 5x - 1$ b. $f(x) = 2x^2 - 4x + 5$

8. a. $f(x) = x^2 + 3x - 4$ b. $f(x) = -4x^2 - 12x - 9$

Answer

5. a. down b. up

7. a. down b. up

Exercises 9 - 12: Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find

a. The equation of the axis of symmetry
b. The vertex of its graph

9. $f(x) = x^2 + 8x - 1$

10. $f(x) = x^2 + 10x + 25$

11. $f(x) = -x^2 + 2x + 5$

12. $f(x) = -2x^2 - 8x - 3$

Answer

9. a. Axis of symmetry: $x = -4$ b. Vertex: $(-4, -17)$

11. a. Axis of symmetry: $x = 1$ b. Vertex: $(1, 2)$

Exercises 13 - 24: Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.

13. $f(x) = x^2 + 7x + 6$

14. $f(x) = x^2 + 10x - 11$

15. $f(x) = x^2 + 8x + 12$

16. $f(x) = x^2 + 5x + 6$

17. $f(x) = -x^2 + 8x - 19$

18. $f(x) = -3x^2 + x - 1$

19. $f(x) = x^2 + 6x + 13$

20. $f(x) = x^2 + 8x + 12$

21. $f(x) = 4x^2 - 20x + 25$

22. $f(x) = -x^2 - 14x - 49$

23. $f(x) = -x^2 - 6x - 9$

24. $f(x) = 4x^2 + 4x + 1$

Answer

13. y-intercept: $(0, 6)$; x-intercept(s): $(-1, 0), (-6, 0)$

15. y-intercept: $(0, 12)$; x-intercept(s): $(-2, 0), (-6, 0)$

17. y-intercept: $(0, -19)$; x-intercept(s): none

19. y-intercept: $(0, 13)$; x-intercept(s): none

21. y-intercept: $(0, -16)$; x-intercept(s): $(\frac{5}{2}, 0)$

23. y-intercept: $(0, 9)$; x-intercept(s): $(-3, 0)$

Exercises 25 - 42: Graph Quadratic Functions Using Properties

In the following exercises, graph the function by using its properties.

25. $f(x) = x^2 + 6x + 5$
26. $f(x) = x^2 + 4x - 12$
27. $f(x) = x^2 + 4x + 3$
28. $f(x) = x^2 - 6x + 8$
29. $f(x) = 9x^2 + 12x + 4$
30. $f(x) = -x^2 + 8x - 16$
31. $f(x) = -x^2 + 2x - 7$
32. $f(x) = 5x^2 + 2$
33. $f(x) = 2x^2 - 4x + 1$
34. $f(x) = 3x^2 - 6x - 1$
35. $f(x) = 2x^2 - 4x + 2$
36. $f(x) = -4x^2 - 6x - 2$
37. $f(x) = -x^2 - 4x + 2$
38. $f(x) = x^2 + 6x + 8$
39. $f(x) = 5x^2 - 10x + 8$
40. $f(x) = -16x^2 + 24x - 9$
41. $f(x) = 3x^2 + 18x + 20$
42. $f(x) = -2x^2 + 8x - 10$

Answer

25.

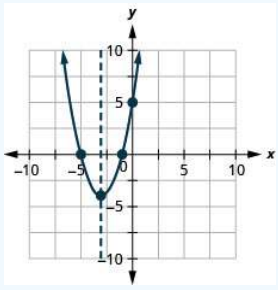


Figure 9.6.136

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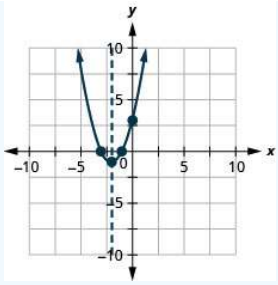


Figure 9.6.137

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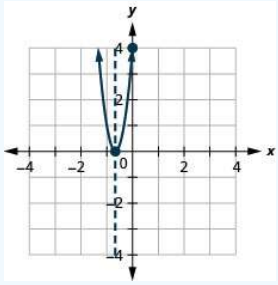


Figure 9.6.138

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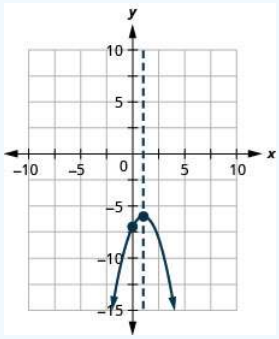


Figure 9.6.139

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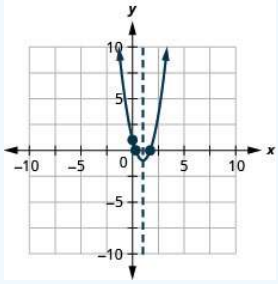


Figure 9.6.140

35.

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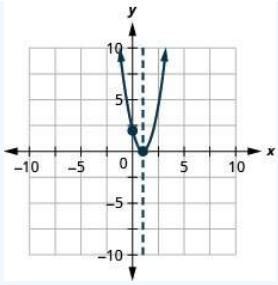


Figure 9.6.141

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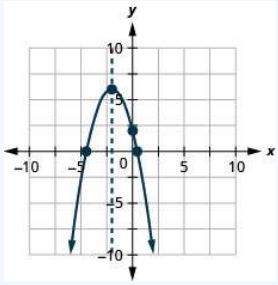


Figure 9.6.142

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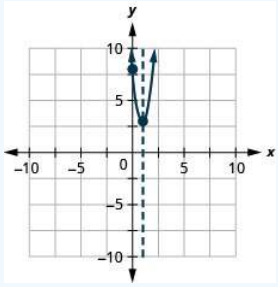


Figure 9.6.143

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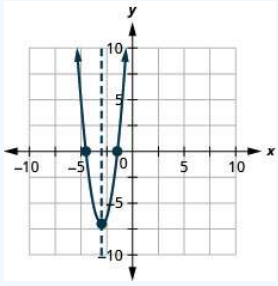


Figure 9.6.144

Exercises 43 - 48: Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

43. $f(x) = 2x^2 + x - 1$

44. $y = -4x^2 + 12x - 5$

45. $y = x^2 - 6x + 15$

46. $y = -x^2 + 4x - 5$

47. $y = -9x^2 + 16$

48. $y = 4x^2 - 49$

Answer

43. The minimum value is $-\frac{9}{8}$ when $x = -\frac{1}{4}$.

45. The maximum value is 6 when $x = 3$.

47. The maximum value is 16 when $x = 0$.

Exercises 49 - 60: Solve Maximum and Minimum Applications

In the following exercises, solve. Round answers to the nearest tenth.

49. An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

50. A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

51. A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

52. A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

53. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

54. A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

55. A retailer who sells fashion boots estimates that by selling them for x dollars each, he will be able to sell $70 - x$ boots a week. Use the quadratic function $R(x) = -x^2 + 70x$ to find the revenue received when the average selling price of a pair of fashion boots is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue per day.

56. A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x^2 + 8x$ to find the revenue received per day when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue per day, and then find the amount of the maximum revenue.

57. A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A(x) = x(240 - 2x)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

58. A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic function $A(x) = x(100 - 2x)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

59. A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

60. A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x) = x(300 - 2x)$ to determine the maximum area of the fenced in yard.

Answer

49. In 5.3 sec the arrow will reach maximum height of 486 ft.

51. In 3.4 seconds the ball will reach its maximum height of 185.6 feet.

53. 20 computers will give the maximum of $\$400$ in receipts.

55. He will be able to sell 35 pairs of boots at the maximum revenue of $\$1,225$

57. The length of the side along the river of the corral is 120 feet and the maximum area is $7,200$ square feet.

59. The maximum area of the patio is 800 feet.

? Exercises 61 - 64: Writing Exercises

61. How do the graphs of the functions $f(x) = x^2$ and $f(x) = x^2 - 1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

62. Explain the process of finding the vertex of a parabola.

63. Explain how to find the intercepts of a parabola.

64. How can you use the discriminant when you are graphing a quadratic function?

Answer

1. Answers will vary.

3. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

Figure 9.6.145

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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9.7: Systems of Nonlinear Equations and Inequalities - Two Variables

Learning Objectives

- Solve a system of nonlinear equations using substitution.
- Solve a system of nonlinear equations using elimination.
- Graph a nonlinear inequality.
- Graph a system of nonlinear inequalities.

Halley's Comet (Figure $9.7.1$) orbits the sun about once every 75 years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.



Figure $9.7.1$: Halley's Comet (credit: "NASA Blueshift"/Flickr)

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving systems of nonlinear equations are similar to those for linear equations.

Solving a System of Nonlinear Equations Using Substitution

A system of nonlinear equations is a system of two or more equations in two or more variables containing at least one equation that is not linear. Recall that a linear equation can take the form $Ax+By+C=0$. Any equation that cannot be written in this form in nonlinear. The substitution method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

Intersection of a Parabola and a Line

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line.

Possible Types of Solutions For Points of Intersection of a Parabola and a Line

Figure $9.7.2$ illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution - The line will never intersect the parabola.
- One solution - The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions - The line crosses on the inside of the parabola and intersects the parabola at two points.

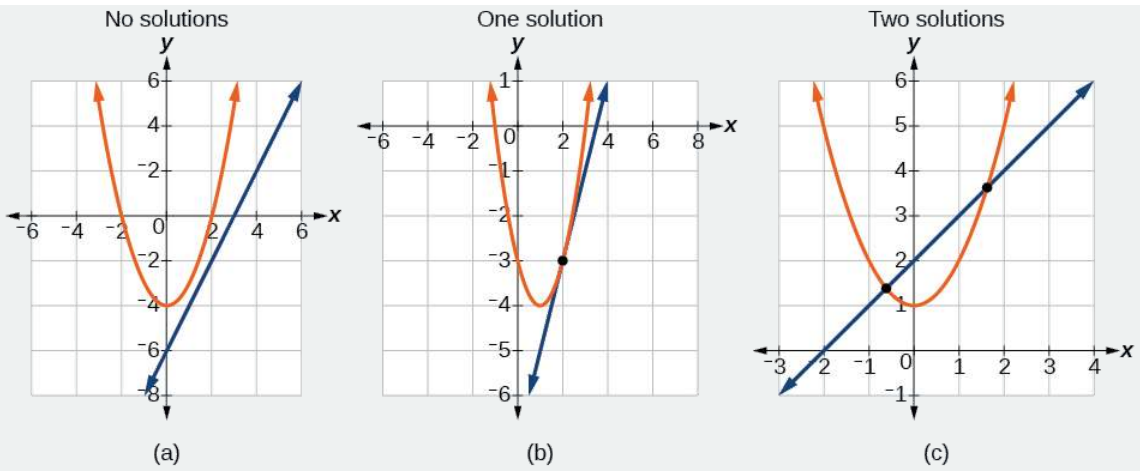


Figure 9.7.2

How to: Given a system of equations containing a line and a parabola, find the solution

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example 9.7.1: Solving a System of Nonlinear Equations Representing a Parabola and a Line

Solve the system of equations.

$$\begin{aligned} x-y &= -1 \\ y &= x^2+1 \end{aligned}$$

Solution

Solve the first equation for x and then substitute the resulting expression into the second equation.

$$\begin{aligned} x-y &= -1 \\ x &= y-1 \qquad \text{Solve for } x. \end{aligned}$$

$$\begin{aligned} y &= x^2+1 \\ y &= (y-1)^2+1 \qquad \text{Substitute expression for } x. \end{aligned}$$

Expand the equation and set it equal to zero.

$$\begin{aligned} y &= (y-1)^2+1 \\ &= (y^2-2y+1)+1 \\ &= y^2-2y+2 \\ 0 &= y^2-3y+2 \\ &= (y-2)(y-1) \end{aligned}$$

Solving for y gives $y=2$ and $y=1$. Next, substitute each value for y into the first equation to solve for x. Always substitute the value into the linear equation to check for extraneous solutions.

$$\begin{aligned} x-y &= -1 \\ x-(2) &= -1 \\ x &= 1 \\ x-(1) &= -1 \\ x &= 0 \end{aligned}$$

The solutions are $(1, 2)$ and $(0, 1)$,which can be verified by substituting these (x, y) values into both of the original equations (Figure $9.7.3$).

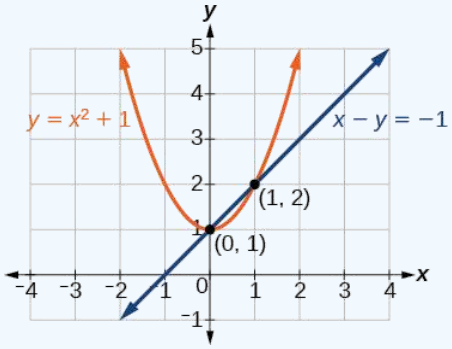


Figure 9.7.3

Q&A: Could we have substituted values for y into the second equation to solve for x in last example?

Yes, but because x is squared in the second equation this could give us extraneous solutions for x.

For $y = 1$

$$\begin{aligned} y &= x^2 + 1 \\ y &= x^2 + 1 \\ x^2 &= 0 \\ x &= \pm\sqrt{0} = 0 \end{aligned}$$

This gives us the same value as in the solution.

For $y = 2$

$$\begin{aligned} y &= x^2 + 1 \\ 2 &= x^2 + 1 \\ x^2 &= 1 \\ x &= \pm\sqrt{1} = \pm 1 \end{aligned}$$

Notice that -1 is an extraneous solution.

Exercise 9.7.1

Solve the given system of equations by substitution.

$$\begin{aligned} 3x - y &= -2 \\ 2x^2 - y &= 0 \end{aligned}$$

Answer

$\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(2, 8)$

Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

POSSIBLE TYPES OF SOLUTIONS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND A LINE

Figure $9.7.4$ illustrates possible solution sets for a system of equations involving a circle and a line.

- No solution - The line does not intersect the circle.

- One solution - The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions - The line crosses the circle and intersects it at two points.

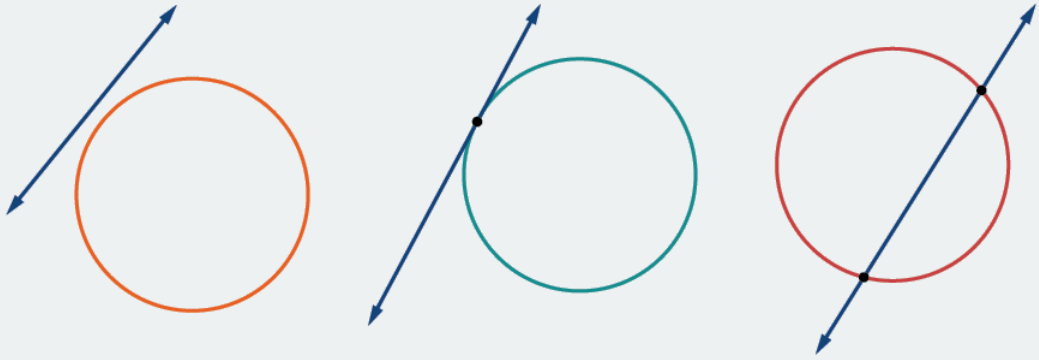


Figure 9.7.4

How to: Given a system of equations containing a line and a circle, find the solution

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example 9.7.2: Finding the Intersection of a Circle and a Line by Substitution

Find the intersection of the given circle and the given line by substitution.

$$\begin{aligned} x^2+y^2 &= 5 \\ y &= 3x-5 \end{aligned}$$

Solution

One of the equations has already been solved for y. We will substitute $y=3x-5$ into the equation for the circle.

$$\begin{aligned} x^2+(3x-5)^2 &= 5 \\ x^2+9x^2-30x+25 &= 5 \\ 10x^2-30x+20 &= 0 \end{aligned}$$

Now, we factor and solve for x.

$$\begin{aligned} 10(x2-3x+2) &= 0 \\ 10(x-2)(x-1) &= 0 \\ x &= 2 \\ x &= 1 \end{aligned}$$

Substitute the two x-values into the original linear equation to solve for y.

$$\begin{aligned} y &= 3(2)-5 \\ &= 1 \\ y &= 3(1)-5 \\ &= -2 \end{aligned}$$

The line intersects the circle at $(2,1)$ and $(1,-2)$,which can be verified by substituting these (x,y) values into both of the original equations (Figure 9.7.5).

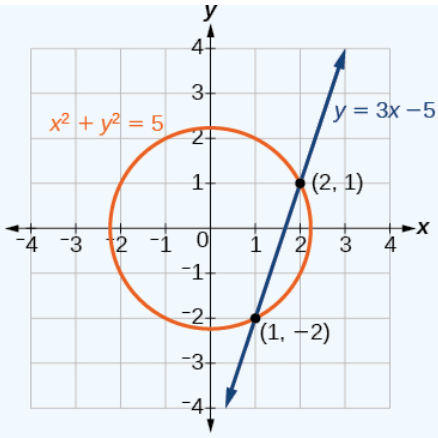


Figure 9.7.5

? Exercise 9.7.2

Solve the system of nonlinear equations.

$$\begin{aligned} x^2 + y^2 &= 10 \\ x - 3y &= -10 \end{aligned}$$

Answer

$(-1, 3)$

Solving a System of Nonlinear Equations Using Elimination

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-by-two system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

POSSIBLE TYPES OF SOLUTIONS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND AN ELLIPSE

Figure 9.7.6 illustrates possible solution sets for a system of equations involving a circle and an ellipse.

- No solution - The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution - The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions - The circle and the ellipse intersect at two points.
- Three solutions - The circle and the ellipse intersect at three points.
- Four solutions - The circle and the ellipse intersect at four points.

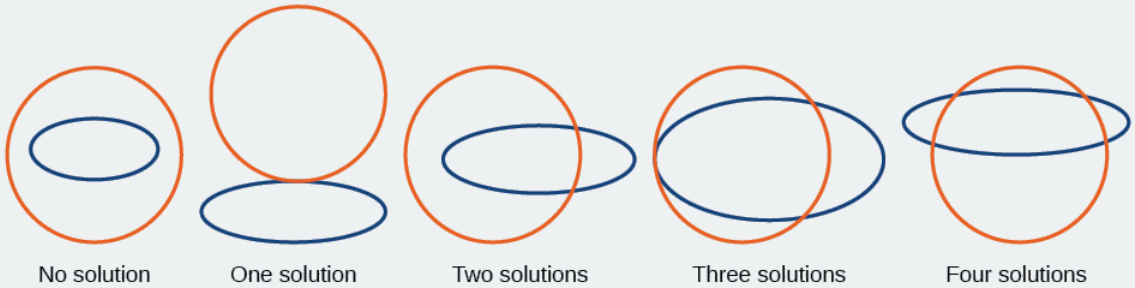


Figure 9.7.6

Example 9.7.3: Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Solve the system of nonlinear equations.

$$\begin{aligned} x^2+y^2 &= 26 \quad (1)\\ 3x^2+25y^2 &= 100 \quad (2)\end{aligned}$$

Solution

Let's begin by multiplying equation (1) by -3,and adding it to equation (2).

$$\begin{aligned} (-3)(x^2+y^2) &= (-3)(26)\\ -3x^2-3y^2 &= -78\\ 3x^2+25y^2 &= 100\\ \hline 22y^2 &= 22\end{aligned}$$

After we add the two equations together, we solve for y.

$$\begin{aligned} y^2 &= 1\\ y &= \pm\sqrt{1} = \pm 1\end{aligned}$$

Substitute $y=\pm 1$ into one of the equations and solve for x.

$$\begin{aligned} x^2+(1)^2 &= 26\\ x^2+1 &= 26\\ x^2 &= 25\\ x &= \pm\sqrt{25} = \pm 5\\ x^2+(-1)^2 &= 26\\ x^2+1 &= 26\\ x^2 &= \pm\sqrt{25} = \pm 5\end{aligned}$$

There are four solutions: $(5,1)$, $(-5,1)$, $(5,-1)$,and $(-5,-1)$. See Figure 9.7.7.

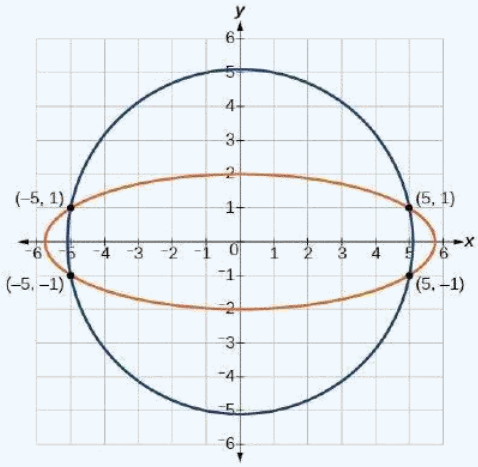


Figure 9.7.7

Exercise 9.7.3

Find the solution set for the given system of nonlinear equations.

$$\begin{aligned} 4x^2+y^2 &= 13\\ x^2+y^2 &= 10\end{aligned}$$

Answer

$(1,3),(1,-3),(-1,3),(-1,-3)$

Graphing a Nonlinear Inequality

All of the equations in the systems that we have encountered so far have involved equalities, but we may also encounter systems that involve inequalities. We have already learned to graph linear inequalities by graphing the corresponding equation, and then shading the region represented by the inequality symbol. Now, we will follow similar steps to graph a nonlinear **inequality** so that we can learn to solve systems of nonlinear inequalities. A **nonlinear inequality** is an inequality containing a nonlinear expression. Graphing a nonlinear inequality is much like graphing a linear inequality.

Recall that when the inequality is greater than, $y > a$,or less than, $y < a$,the graph is drawn with a dashed line. When the inequality is greater than or equal to, $y \geq a$,or less than or equal to, $y \leq a$,the graph is drawn with a solid line. The graphs will create regions in the plane, and we will test each region for a solution. If one point in the region works, the whole region works. That is the region we shade (Figure $9.7.8$).

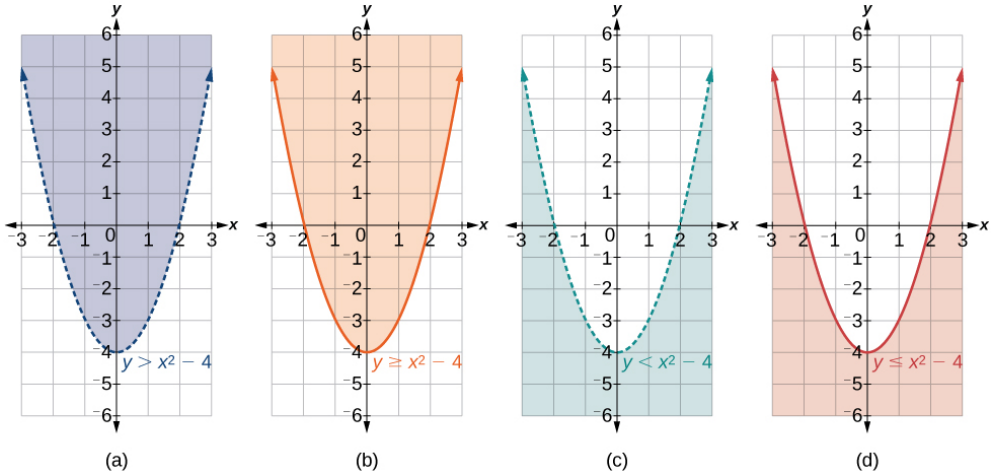


Figure $9.7.8$: (a) an example of $y > a$; (b) an example of $y \geq a$; (c) an example of $y < a$; (d) an example of $y \leq a$

How to: Given an inequality bounded by a parabola, sketch a graph

1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is $\leq$ or $\geq$), the parabola is graphed as a solid line.
3. If the boundary is not included in the region (the operator is $<$ or $>$), the parabola is graphed as a dashed line.
4. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
5. Shade the region representing the solution set.

Example $9.7.4$: Graphing an Inequality for a Parabola

Graph the inequality $y > x^2 + 1$.

Solution

First, graph the corresponding equation $y = x^2 + 1$. Since $y > x^2 + 1$ has a greater than symbol, we draw the graph with a dashed line. Then we choose points to test both inside and outside the parabola. Let's test the points

$(0, 2)$ and $(2, 0)$. One point is clearly inside the parabola and the other point is clearly outside.

$$y > x^2 + 1$$
$$2 > (0)^2 + 1$$
$$2 > 1 \qquad \text{True}$$

$$0 > (2)^2 + 1$$
$$0 > 5 \qquad \text{False}$$

The graph is shown in Figure 9.7.9. We can see that the solution set consists of all points inside the parabola, but not on the graph itself.

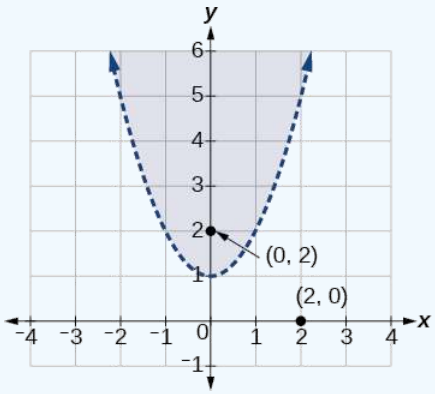


Figure 9.7.9

Graphing a System of Nonlinear Inequalities

Now that we have learned to graph nonlinear inequalities, we can learn how to graph systems of nonlinear inequalities. A **system of nonlinear inequalities** is a system of two or more inequalities in two or more variables containing at least one inequality that is not linear. Graphing a system of nonlinear inequalities is similar to graphing a system of linear inequalities. The difference is that our graph may result in more shaded regions that represent a solution than we find in a system of linear inequalities. The solution to a nonlinear system of inequalities is the region of the graph where the shaded regions of the graph of each inequality overlap, or where the regions intersect, called the **feasible region**.

How to: Given a system of nonlinear inequalities, sketch a graph

1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality.

Example 9.7.5: Graphing a System of Inequalities

Graph the given system of inequalities.

$$x^2 - y \leq 0$$
$$2x^2 + y \leq 12$$

Solution

These two equations are clearly parabolas. We can find the points of intersection by the elimination process: Add both equations and the variable y will be eliminated. Then we solve for x.

$$\begin{aligned} x^2 - y &= 0 \\ 2x^2 + y &= 12 \\ \hline 3x^2 &= 12 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Substitute the x-values into one of the equations and solve for y.

$$\begin{aligned} x^2 - y &= 0 \\ (2)^2 - y &= 0 \\ 4 - y &= 0 \\ y &= 4 \end{aligned}$$

$$\begin{aligned} (-2)^2 - y &= 0 \\ 4 - y &= 0 \\ y &= 4 \end{aligned}$$

The two points of intersection are $(2, 4)$ and $(-2, 4)$. Notice that the equations can be rewritten as follows.

$$\begin{aligned} x^2 - y &\leq 0 \\ x^2 &\leq y \\ y &\geq x^2 \end{aligned}$$

$$\begin{aligned} 2x^2 + y &\leq 12 \\ y &\leq -2x^2 + 12 \end{aligned}$$

Graph each inequality. See Figure 9.7.10. The feasible region is the region between the two equations bounded by $2x^2 + y \leq 12$ on the top and $x^2 - y \leq 0$ on the bottom.

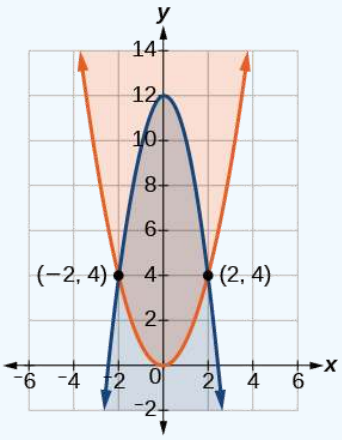


Figure 9.7.10

Exercise 9.7.5

Graph the given system of inequalities.

$$\begin{aligned} y &\geq x^2 - 1 \\ x - y &\geq -1 \end{aligned}$$

Answer

Shade the area bounded by the two curves, above the quadratic and below the line.

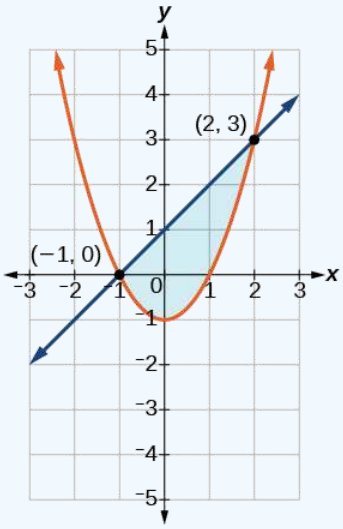


Figure 9.7.11

Media

Access these online resources for additional instruction and practice with nonlinear equations.

- Solve a System of Nonlinear Equations Using Substitution
- Solve a System of Nonlinear Equations Using Elimination

Key Concepts

- There are three possible types of solutions to a system of equations representing a line and a parabola: (1) no solution, the line does not intersect the parabola; (2) one solution, the line is tangent to the parabola; and (3) two solutions, the line intersects the parabola in two points. See Example 9.7.1.
- There are three possible types of solutions to a system of equations representing a circle and a line: (1) no solution, the line does not intersect the circle; (2) one solution, the line is tangent to the parabola; (3) two solutions, the line intersects the circle in two points. See Example 9.7.2.
- There are five possible types of solutions to the system of nonlinear equations representing an ellipse and a circle: (1) no solution, the circle and the ellipse do not intersect; (2) one solution, the circle and the ellipse are tangent to each other; (3) two solutions, the circle and the ellipse intersect in two points; (4) three solutions, the circle and ellipse intersect in three places; (5) four solutions, the circle and the ellipse intersect in four points. See Example 9.7.3.
- An inequality is graphed in much the same way as an equation, except for > or <, we draw a dashed line and shade the region containing the solution set. See Example 9.7.4.
- Inequalities are solved the same way as equalities, but solutions to systems of inequalities must satisfy both inequalities. See Example 9.7.5.

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9.7E: Exercises

Verbal

1) Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three? If not, explain why not. If so, give an example of such a system, in graph form, and explain why your choice gives two or three answers.

Answer

A nonlinear system could be representative of two circles that overlap and intersect in two locations, hence two solutions. A nonlinear system could be representative of a parabola and a circle, where the vertex of the parabola meets the circle and the branches also intersect the circle, hence three solutions.

2) When graphing an inequality, explain why we only need to test one point to determine whether an entire region is the solution?

3) When you graph a system of inequalities, will there always be a feasible region? If so, explain why. If not, give an example of a graph of inequalities that does not have a feasible region. Why does it not have a feasible region?

Answer

No. There does not need to be a feasible region. Consider a system that is bounded by two parallel lines. One inequality represents the region above the upper line; the other represents the region below the lower line. In this case, no points in the plane are located in both regions; hence there is no feasible region.

4) If you graph a revenue and cost function, explain how to determine in what regions there is profit.

5) If you perform your break-even analysis and there is more than one solution, explain how you would determine which x-values are profit and which are not.

Answer

Choose any number between each solution and plug into $C(x)$ and $R(x)$. If $C(x) < R(x)$, then there is profit.

Algebraic

For the exercises 6-10, solve the system of nonlinear equations using substitution.

6) $$\begin{aligned} x+y &= 4 \\ x^2+y^2 &= 9 \end{aligned}$$

7) $$\begin{aligned} y &= x-3 \\ x^2+y^2 &= 9 \end{aligned}$$

Answer

$(0,-3), (3,0)$

8) $$\begin{aligned} y &= x \\ x^2+y^2 &= 9 \end{aligned}$$

9) $$\begin{aligned} y &= -x \\ x^2+y^2 &= 9 \end{aligned}$$

Answer

$\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

10) $$\begin{aligned} x &= 2 \\ x^2-y^2 &= 9 \end{aligned}$$

For the exercises 11-15, solve the system of nonlinear equations using elimination.

11) $\begin{aligned} 4x^2-9y^2 &= 36 \\ 4x^2+9y^2 &= 36 \end{aligned}$

Answer

$(-3,0),(3,0)$

12) $\begin{aligned} x^2+y^2 &= 25 \\ x^2-y^2 &= 1 \end{aligned}$

13) $\begin{aligned} 2x^2+4y^2 &= 4 \\ 2x^2-4y^2 &= 25x-10 \end{aligned}$

Answer

$\left(\frac{1}{4},-\frac{\sqrt{62}}{8}\right),\left(\frac{1}{4},\frac{\sqrt{62}}{8}\right)$

14) $\begin{aligned} y^2-x^2 &= 9 \\ 3x^2+2y^2 &= 8 \end{aligned}$

15) $\begin{aligned} x^2+y^2+\frac{1}{16} &= 2500 \\ y &= 2x^2 \end{aligned}$

Answer

$\left(-\frac{\sqrt{398}}{4},\frac{199}{4}\right),\left(\frac{\sqrt{398}}{4},\frac{199}{4}\right)$

For the exercises 16-23, use any method to solve the system of nonlinear equations.

16) $\begin{aligned} -2x^2+y &= -5 \\ 6x-y &= 9 \end{aligned}$

17) $\begin{aligned} -x^2+y &= 2 \\ -x+y &= 2 \end{aligned}$

Answer

$(0,2),(1,3)$

18) $\begin{aligned} x^2+y^2 &= 1 \\ y &= 20x^2-1 \end{aligned}$

19) $\begin{aligned} x^2+y^2 &= 1 \\ y &= -x^2 \end{aligned}$

Answer

$\left(-\sqrt{\frac{1}{2}(\sqrt{5}-1)},\frac{1}{2}(1-\sqrt{5})\right),\left(\sqrt{\frac{1}{2}(\sqrt{5}-1)},\frac{1}{2}(1-\sqrt{5})\right)$

20) $\begin{aligned} 2x^3-x^2 &= y \\ y &= \frac{1}{2}-x \end{aligned}$

21) $\begin{aligned} 9x^2+25y^2 &= 225 \\ (x-6)^2+y^2 &= 1 \end{aligned}$

Answer

$(5,0)$

22) $\begin{aligned} x^4 - x^2 &= y \\ x^2 + y &= 0 \end{aligned}$

23) $\begin{aligned} 2x^3 - x^2 &= y \\ x^2 + y &= 0 \end{aligned}$

Answer

$(0, 0)$

For the exercises 24-38, use any method to solve the nonlinear system.

24) $\begin{aligned} x^2 + y^2 &= 9 \\ y &= 3 - x^2 \end{aligned}$

25) $\begin{aligned} x^2 - y^2 &= 9 \\ x &= 3 \end{aligned}$

Answer

$(3, 0)$

26) $\begin{aligned} x^2 - y^2 &= 9 \\ y &= 3 \end{aligned}$

27) $\begin{aligned} x^2 - y^2 &= 9 \\ x - y &= 0 \end{aligned}$

Answer

No Solutions Exist

28) $\begin{aligned} -x^2 + y &= 2 \\ -4x + y &= -1 \end{aligned}$

29) $\begin{aligned} -x^2 + y &= 2 \\ 2y &= -x \end{aligned}$

Answer

No Solutions Exist

30) $\begin{aligned} x^2 + y^2 &= 25 \\ x^2 - y^2 &= 36 \end{aligned}$

31) $\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= x^2 \end{aligned}$

Answer

$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

32) $\begin{aligned} 16x^2 - 9y^2 + 144 &= 0 \\ y^2 + x^2 &= 16 \end{aligned}$

33) $\begin{aligned} 3x^2 - y^2 &= 12 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$

Answer

$(2, 0)$

34) $\begin{aligned} 3x^2-y^2 &= 12 \\ (x-1)^2+y^2 &= 4 \end{aligned}$

35) $\begin{aligned} 3x^2-y^2 &= 12 \\ x^2+y^2 &= 16 \end{aligned}$

Answer

$(-\sqrt{7},-3),(-\sqrt{7},3),(\sqrt{7},-3),(\sqrt{7},3)$

36) $\begin{aligned} x^2-y^2-6x-4y-11 &= 0 \\ -x^2+y^2 &= 5 \end{aligned}$

37) $\begin{aligned} x^2+y^2-6y &= 7 \\ x^2+y &= 1 \end{aligned}$

Answer

$\left(-\sqrt{\frac{1}{2}(\sqrt{73}-5)},\frac{1}{2}(7-\sqrt{73})\right),\left(\sqrt{\frac{1}{2}(\sqrt{73}-5)},\frac{1}{2}(7-\sqrt{73})\right)$

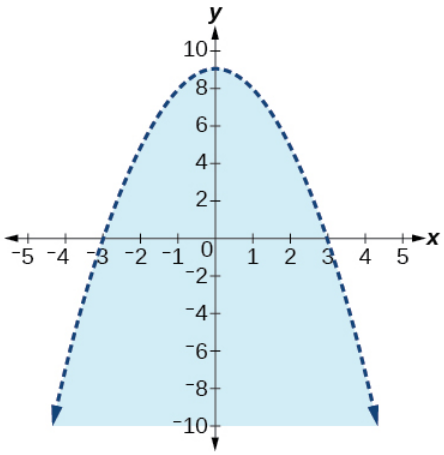
38) $\begin{aligned} x^2+y^2 &= 6 \\ xy &= 1 \end{aligned}$

Graphical

For the exercises 39-40, graph the inequality.

39) $x^2+y<9$

Answer

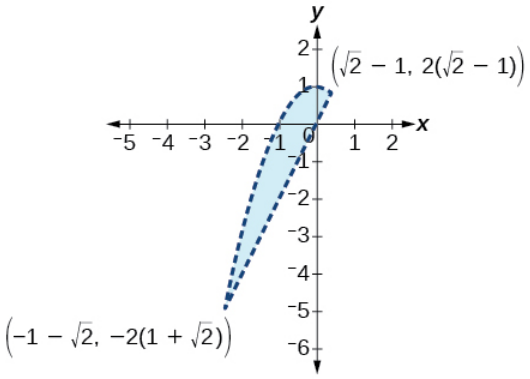


40) $x^2+y^2<4$

For the exercises 41-45, graph the system of inequalities. Label all points of intersection.

41) $\begin{aligned} x^2+y &< 1 \\ y &> 2x \end{aligned}$

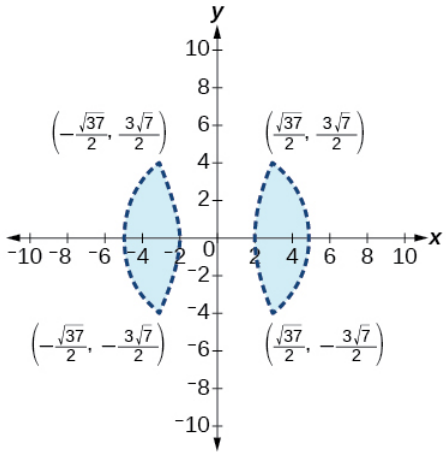
Answer



42) $\begin{aligned} x^2+y &< -5 \\ y &> 5x+10 \end{aligned}$

43) $\begin{aligned} x^2+y^2 &< 25 \\ 3x^2-y^2 &> 12 \end{aligned}$

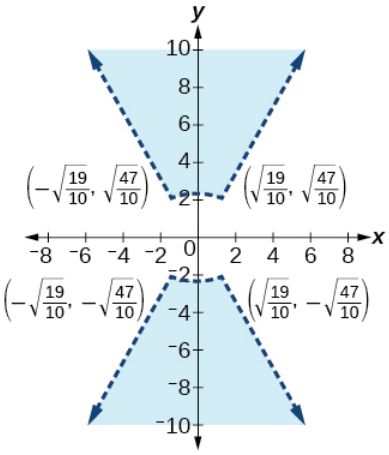
Answer



44) $\begin{aligned} x^2-y^2 &> -4 \\ x^2+y^2 &< 12 \end{aligned}$

45) $\begin{aligned} x^2+3y^2 &> 16 \\ 3x^2-y^2 &< 1 \end{aligned}$

Answer



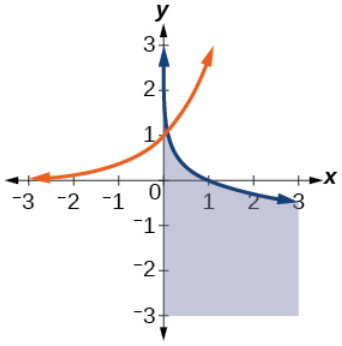
Extensions

For the exercises 46-47, graph the inequality.

46) $y \geq e^x$
$y \leq \ln(x) + 5$

47) $y \leq -\log(x)$
$y \leq e^x$

Answer



For the exercises 48-52, find the solutions to the nonlinear equations with two variables.

48) $\frac{4}{x^2} + \frac{1}{y^2} = 24$
$\frac{5}{x^2} - \frac{2}{y^2} + 4 = 0$

49) $\frac{6}{x^2} - \frac{1}{y^2} = 8$
$\frac{1}{x^2} - \frac{6}{y^2} = \frac{1}{8}$

Answer

$\left(-2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right), \left(-2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right), \left(2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right), \left(2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right)$

50) $x^2 - xy + y^2 - 2 = 0$
$x + 3y = 4$

51) $\begin{aligned} x^2 - xy - 2y^2 - 6 &= 0 \\ x^2 + y^2 &= 1 \end{aligned}$

Answer

No Solution Exists

52) $\begin{aligned} x^2 + 4xy - 2y^2 - 6 &= 0 \\ x &= y + 2 \end{aligned}$

Technology

For the exercises 53-54, solve the system of inequalities. Use a calculator to graph the system to confirm the answer.

53) $\begin{aligned} xy &< 1 \\ y &> \sqrt{x} \end{aligned}$

Answer

$x = 0$, $y > 0$ and $0 < x < 1$, $\sqrt{x} < y < \dfrac{1}{x}$

54) $\begin{aligned} x^2 + y &< 3 \\ y &> 2x \end{aligned}$

Real-World Applications

For the exercises 55-, construct a system of nonlinear equations to describe the given behavior, then solve for the requested solutions.

55) Two numbers add up to 300. One number is twice the square of the other number. What are the numbers?

Answer

$12, 288$

56) The squares of two numbers add to 360. The second number is half the value of the first number squared. What are the numbers?

57) A laptop company has discovered their cost and revenue functions for each day: $C(x) = 3x^2 - 10x + 200$ and $R(x) = -2x^2 + 100x + 50$. If they want to make a profit, what is the range of laptops per day that they should produce? Round to the nearest number which would generate profit.

Answer

2 - 20 computers

58) A cell phone company has the following cost and revenue functions: $C(x) = 8x^2 - 600x + 21,500$ and $R(x) = -3x^2 + 480x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

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Chapter 9 Review Exercises

Chapter Review Exercises

Solve Quadratic Equations Using the Square Root Property

Exercise 1 Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $y^2 = 144$
2. $n^2 - 80 = 0$
3. $4a^2 = 100$
4. $2b^2 = 72$
5. $r^2 + 32 = 0$
6. $t^2 + 18 = 0$
7. $\frac{2}{3}w^2 - 20 = 30$
8. $5c^2 + 3 = 19$

Answer

1. $y = \pm 12$
3. $a = \pm 5$
5. $r = \pm 4\sqrt{2}i$
7. $w = \pm 5\sqrt{3}$

Exercise 2 Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $(p-5)^2 + 3 = 19$
2. $(u+1)^2 = 45$
3. $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$
4. $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$
5. $(n-4)^2 - 50 = 150$
6. $(4c-1)^2 = -18$
7. $n^2 + 10n + 25 = 12$
8. $64a^2 + 48a + 9 = 81$

Answer

1. $p = -1, 9$
3. $x = \frac{1}{4} \pm \frac{\sqrt{3}}{4}$
5. $n = 4 \pm 10\sqrt{2}$
7. $n = -5 \pm 2\sqrt{3}$

Solve Quadratic Equations by Completing the Square

Exercise 3 Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. $x^2 + 22x$

2. $m^2 - 8m$
3. $a^2 - 3a$
4. $b^2 + 13b$

Answer

1. $(x+11)^2$
3. $\left(a - \frac{3}{2}\right)^2$

Exercise 4 Solve Quadratic Equations Using Completing the Square

In the following exercises, solve by completing the square.

1. $d^2 + 14d = -13$
2. $y^2 - 6y = 36$
3. $m^2 + 6m = -109$
4. $t^2 - 12t = -40$
5. $v^2 - 14v = -31$
6. $w^2 - 20w = 100$
7. $m^2 + 10m - 4 = -13$
8. $n^2 - 6n + 11 = 34$
9. $a^2 = 3a + 8$
10. $b^2 = 11b - 5$
11. $(u+8)(u+4) = 14$
12. $(z-10)(z+2) = 28$

Answer

1. $d = -13, -1$
3. $m = -3 \pm 10i$
5. $v = 7 \pm 3\sqrt{2}$
7. $m = -9, -1$
9. $a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$
11. $u = -6 \pm 2\sqrt{2}$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

Exercise 5 Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

1. $3p^2 - 18p + 15 = 15$
2. $5q^2 + 70q + 20 = 0$
3. $4y^2 - 6y = 4$
4. $2x^2 + 2x = 4$
5. $3c^2 + 2c = 9$
6. $4d^2 - 2d = 8$
7. $2x^2 + 6x = -5$
8. $2x^2 + 4x = -5$

Answer

1. $p = 0, 6$

3. $y = -\frac{1}{2}, 2$

5. $c = -\frac{1}{3} \pm \frac{2\sqrt{7}}{3}$

7. $x = \frac{3}{2} \pm \frac{1}{2}i$

Exercise 6 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4x^2 - 5x + 1 = 0$
2. $7y^2 + 4y - 3 = 0$
3. $r^2 - r - 42 = 0$
4. $t^2 + 13t + 22 = 0$
5. $4v^2 + v - 5 = 0$
6. $2w^2 + 9w + 2 = 0$
7. $3m^2 + 8m + 2 = 0$
8. $5n^2 + 2n - 1 = 0$
9. $6a^2 - 5a + 2 = 0$
10. $4b^2 - b + 8 = 0$
11. $u(u - 10) + 3 = 0$
12. $5z(z - 2) = 3$
13. $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$
14. $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$
15. $4c^2 + 4c + 1 = 0$
16. $9d^2 - 12d = -4$

Answer

1. $x = \frac{1}{4}, 1$

3. $r = -6, 7$

5. $v = \frac{-1 \pm \sqrt{21}}{8}$

7. $m = \frac{-4 \pm \sqrt{10}}{3}$

9. $a = \frac{5}{12} \pm \frac{\sqrt{23}}{12}i$

11. $u = 5 \pm \sqrt{21}$

13. $p = \frac{4 \pm \sqrt{5}}{5}$

15. $c = -\frac{1}{2}$

Exercise 7 Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

1. a. $9x^2 - 6x + 1 = 0$
   b. $3y^2 - 8y + 1 = 0$
   c. $7m^2 + 12m + 4 = 0$
   d. $5n^2 - n + 1 = 0$
2. a. $5x^2 - 7x - 8 = 0$
   b. $7x^2 - 10x + 5 = 0$
   c. $25x^2 - 90x + 81 = 0$
   d. $15x^2 - 8x + 4 = 0$

Answer

1.

a. 1
b. 2
c. 2
d. 2

Exercise 8 Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

1. a. $16r^2-8r+1=0$
   b. $5t^2-8t+3=9$
   c. $3(c+2)^2=15$
2. a. $4d^2+10d-5=21$
   b. $25x^2-60x+36=0$
   c. $6(5v-7)^2=150$

Answer

1.

a. Factor
b. Quadratic Formula
c. Square Root

Solve Equations in Quadratic Form

Exercise 9 Solve Equations in Quadratic Form

In the following exercises, solve.

1. $x^4-14x^2+24=0$
2. $x^4+4x^2-32=0$
3. $4x^4-5x^2+1=0$
4. $(2y+3)^2+3(2y+3)-28=0$
5. $x+3\sqrt{x}-28=0$
6. $6x+5\sqrt{x}-6=0$
7. $x^{\frac{2}{3}}-10x^{\frac{1}{3}}+24=0$
8. $x+7x^{\frac{1}{2}}+6=0$
9. $8x^{-2}-2x^{-1}-3=0$

Answer

1. $x=\pm\sqrt{2}, x=\pm 2\sqrt{3}$
3. $x=\pm 1, x=\pm\frac{1}{2}$
5. $x=16$
7. $x=64, x=216$
9. $x=-2, x=\frac{4}{3}$

Exercise 10 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

1. Find two consecutive odd numbers whose product is 323.
2. Find two consecutive even numbers whose product is 624.
3. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.
4. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.
5. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.

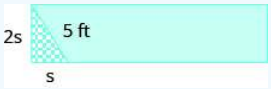


Figure 9.E.1

6. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

7. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

8. For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

9. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

10. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

11. Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

12. Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

Answer

2. Two consecutive even numbers whose product is 624 are 24 and 26, and -24 and -26.

4. The height is 14 inches and the width is 10 inches.

6. The length of the diagonal is 3.6 feet.

8. The width of the serving table is 4.7 feet and the length is 16.1 feet.



Figure 9.E.2

10. The speed of the wind was 30 mph.

12. One man takes 3 hours and the other man 6 hours to finish the repair alone.

Graph Quadratic Functions Using Properties

Exercise 11 Recognize the Graph of a Quadratic Function

In the following exercises, graph by plotting point.

1. Graph $y = x^2 - 2$
2. Graph $y = -x^2 + 3$

Answer

2.

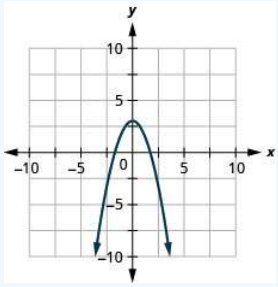


Figure 9.E.3

Exercise 12 Recognize the Graph of a Quadratic Function

In the following exercises, determine if the following parabolas open up or down.

1. a. $y = -3x^2 + 3x - 1$
   b. $y = 5x^2 + 6x + 3$
2. a. $y = x^2 + 8x - 1$
   b. $y = -4x^2 - 7x + 1$

Answer

2.

a. Up
b. Down

Exercise 13 Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find

a. The equation of the axis of symmetry
b. The vertex

1. $y = -x^2 + 6x + 8$
2. $y = 2x^2 - 8x + 1$

Answer

2. $x = 2$; $(2, -7)$

Exercise 14 Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

1. $y = x^2 - 4x + 5$

https://math.libretexts.org/@go/page/30920

2. $y = x^2 - 8x + 15$
3. $y = x^2 - 4x + 10$
4. $y = -5x^2 - 30x - 46$
5. $y = 16x^2 - 8x + 1$
6. $y = x^2 + 16x + 64$

Answer

2. $y : (0, 15)$
   $x : (3, 0), (5, 0)$
4. $y : (0, -46)$
   $x :$ none
6. $y : (0, -64)$
   $x : (-8, 0)$

Graph Quadratic Functions Using Properties

Exercise 15 Graph Quadratic Functions Using Properties

In the following exercises, graph by using its properties.

1. $y = x^2 + 8x + 15$
2. $y = x^2 - 2x - 3$
3. $y = -x^2 + 8x - 16$
4. $y = 4x^2 - 4x + 1$
5. $y = x^2 + 6x + 13$
6. $y = -2x^2 - 8x - 12$

Answer

2.

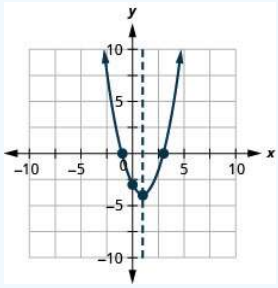


Figure 9.E.4

4.

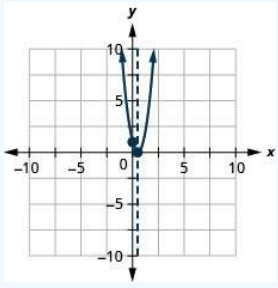


Figure 9.E.5

6.

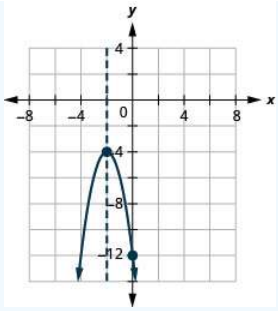


Figure 9.E.6

Exercise 16 Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

1. $y = 7x^2 + 14x + 6$
2. $y = -3x^2 + 12x - 10$

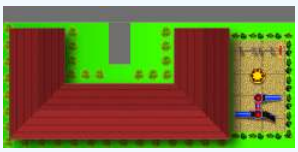
Answer

2. The maximum value is 2 when $x = 2$.

Exercise 17 Solve Maximum and Minimum Applications

In the following exercises, solve. Rounding answers to the nearest tenth.

1. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.
2. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.


Figure 9.E.7

Answer

2. The length adjacent to the building is 90 feet giving a maximum area of $4,050$ square feet.

Graph Quadratic Functions Using Transformations

Exercise 18 Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using a vertical shift.

1. $g(x) = x^2 + 4$
2. $h(x) = x^2 - 3$

Answer

2.

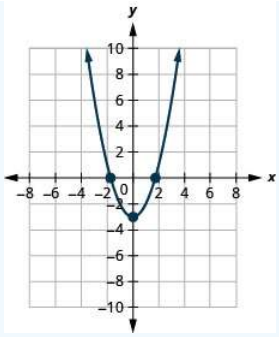


Figure 9.E.8

Exercise 19 Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using a horizontal shift.

1. $f(x) = (x+1)^2$
2. $g(x) = (x-3)^2$

Answer

2.

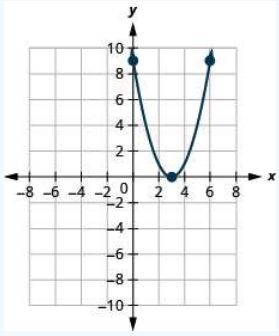


Figure 9.E.9

Exercise 20 Graph Quadratic Functions of the Form $f(x) = x^2 + k$

In the following exercises, graph each function using transformations.

1. $f(x) = (x+2)^2 + 3$
2. $f(x) = (x+3)^2 - 2$
3. $f(x) = (x-1)^2 + 4$
4. $f(x) = (x-4)^2 - 3$

Answer

2.

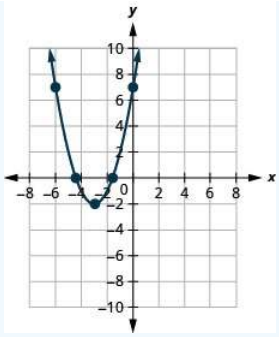


Figure 9.E.10

4.

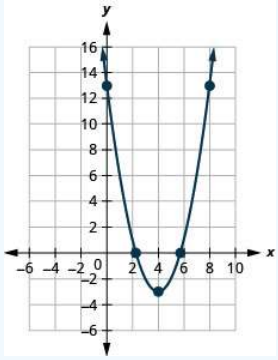


Figure 9.E.11

Exercise 21 Graph Quadratic Functions of the Form $f(x) = ax^2$

In the following exercises, graph each function.

1. $f(x) = 2x^2$
2. $f(x) = -x^2$
3. $f(x) = \frac{1}{2}x^2$

Answer

2.

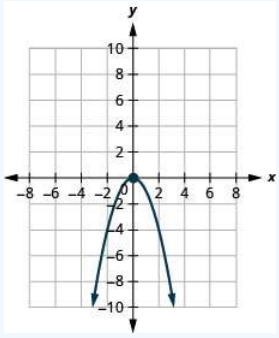


Figure 9.E.12

Exercise 22 Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x-h)^2 + k$ form by completing the square.

1. $f(x) = 2x^2 - 4x - 4$

2. $f(x) = 3x^2 + 12x + 8$

Answer

1. $f(x) = 2(x-1)^2 - 6$

Exercise 23 Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x) = a(x-h)^2 + k$ form
b. Graph it by using transformations

1. $f(x) = 3x^2 - 6x - 1$
2. $f(x) = -2x^2 - 12x - 5$
3. $f(x) = 2x^2 + 4x + 6$
4. $f(x) = 3x^2 - 12x + 7$

Answer

1.

a. $f(x) = 3(x-1)^2 - 4$
b.

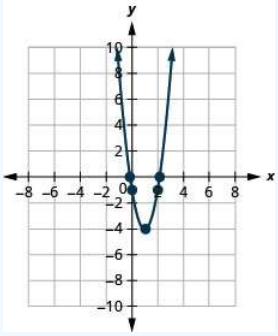


Figure 9.E.13

3.

a. $f(x) = 2(x+1)^2 + 4$
b.

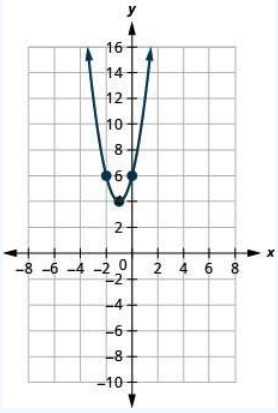


Figure 9.E.14

Exercise 24 Graph Quadratic Functions Using Transformations

In the following exercises,

a. Rewrite each function in $f(x)=a(x-h)^2+k$ form
b. Graph it using properties

1. $f(x)=-3x^2-12x-5$
2. $f(x)=2x^2-12x+7$

Answer

1.

a. $f(x)=-3(x+2)^2+7$
b.

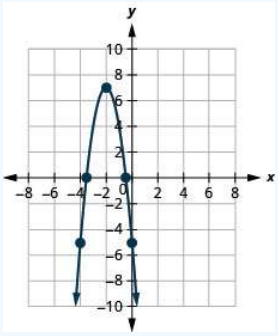


Figure 9.E.15

Exercise 25 Find a Quadratic Function From its Graph

In the following exercises, write the quadratic function in $f(x)=a(x-h)^2+k$ form.

1.

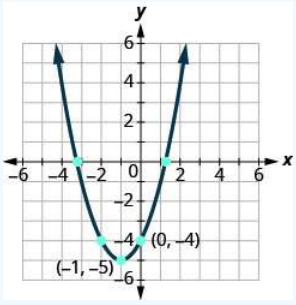


Figure 9.E.16

2.

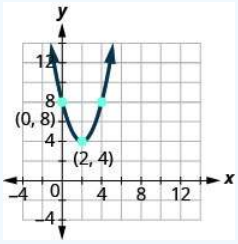


Figure 9.E.17

Answer

1. $f(x)=(x+1)^2-5$

Exercise 26 Solve Quadratic Inequalities Graphically

In the following exercises, solve graphically and write the solution in interval notation.

1. $x^2 - x - 6 > 0$
2. $x^2 + 4x + 3 \leq 0$
3. $-x^2 - x + 2 \geq 0$
4. $-x^2 + 2x + 3 < 0$

Answer

1.

a.

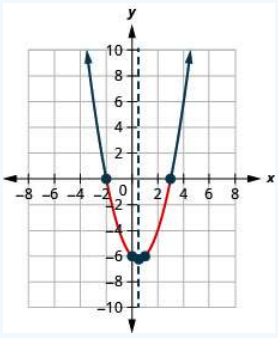


Figure 9.E.18

b. $(-\infty, -2) \cup (3, \infty)$

3.

a.

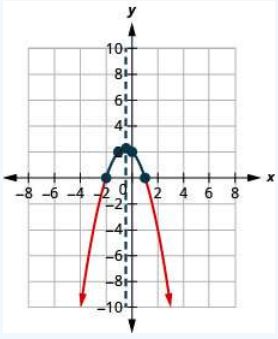


Figure 9.E.19

b. $[-2, 1]$

Exercise 27 Solve Quadratic Inequalities Graphically

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 - 6x + 8 < 0$
2. $x^2 + x > 12$
3. $x^2 - 6x + 4 \leq 0$
4. $2x^2 + 7x - 4 > 0$
5. $-x^2 + x - 6 > 0$
6. $x^2 - 2x + 4 \geq 0$

Answer

1. $(2, 4)$
3. $[3-\sqrt{5}, 3+\sqrt{5}]$
5. no solution

Practice Test

Exercise 28

1. Use the Square Root Property to solve the quadratic equation $3(w+5)^2 = 27$.
2. Use Completing the Square to solve the quadratic equation $a^2 - 8a + 7 = 23$.
3. Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

Answer

1. $w = -2, w = -8$
3. $m = 1, m = \frac{3}{2}$

Exercise 29

Solve the following quadratic equations. Use any method.

1. $2x(3x-2) - 1 = 0$
2. $\frac{9}{4}y^2 - 3y + 1 = 0$

Answer

2. $y = \frac{2}{3}$

Exercise 30

Use the discriminant to determine the number and type of solutions of each quadratic equation.

1. $6p^2 - 13p + 7 = 0$
2. $3q^2 - 10q + 12 = 0$

Answer

2. 2 complex

Exercise 31

Solve each equation.

1. $4x^4 - 17x^2 + 4 = 0$
2. $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

Answer

2. $y = 1, y = -27$

Exercise 32

For each parabola, find

a. Which direction it opens
b. The equation of the axis of symmetry

c. The vertex
d. The x-and y-intercepts
e. The maximum or minimum value

1. $y = 3x^2 + 6x + 8$
2. $y = -x^2 - 8x + 16$

Answer

2.

a. down
b. $x = -4$
c. $(-4, 0)$
d. $y : (0, 16); x : (-4, 0)$
e. minimum value of -4 when $x = 0$

Exercise 33

Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.

1. $f(x) = x^2 + 6x + 9$
2. $f(x) = -2x^2 + 8x + 4$

Answer

2.

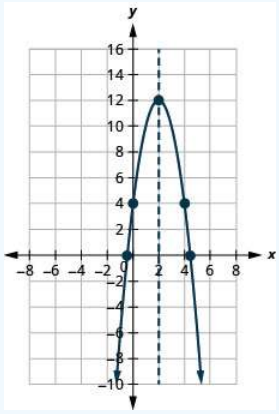


Figure 9.E.20

Exercise 34

In the following exercises, graph each function using transformations.

1. $f(x) = (x + 3)^2 + 2$
2. $f(x) = x^2 - 4x - 1$

Answer

2.

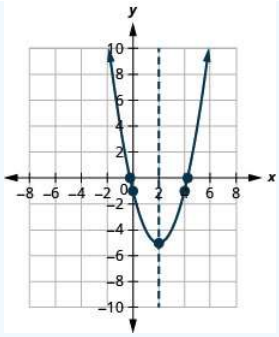


Figure 9.E.21

Exercise 35

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

1. $x^2 - 6x - 8 \leq 0$
2. $2x^2 + x - 10 > 0$

Answer

2. $\left(-\infty, -\frac{5}{2}\right) \cup (2, \infty)$

Exercise 36

Model the situation with a quadratic equation and solve by any method.

1. Find two consecutive even numbers whose product is 360.
2. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

Answer

2. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.

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CHAPTER OVERVIEW

10: Right Triangle Trigonometry

10.1: Right Triangle Trigonometry

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10.1: Right Triangle Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°(\frac{\pi}{6}), 45°(\frac{\pi}{4})$, and $60°(\frac{\pi}{3})$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure $10.1.1$ shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

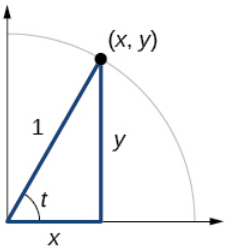


Figure 10.1.1

We know

$$\cos t = \frac{x}{1} = x \tag{10.1.1}$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \tag{10.1.2}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $10.1.2$

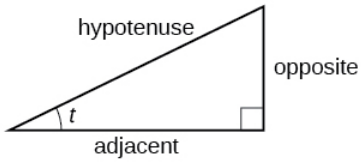


Figure 10.1.2: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{10.1.3}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{10.1.4}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{10.1.5}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 10.1.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 10.1.3, find the value of $\cos\alpha$.

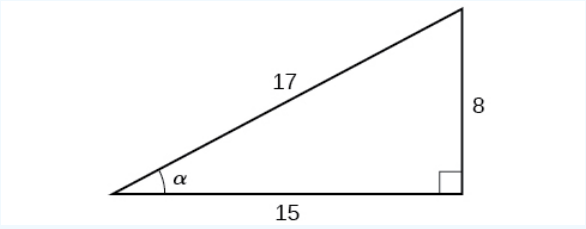


Figure 10.1.3

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 10.1.4:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{15}{17}$$

Exercise 10.1.1

Given the triangle shown in Figure 10.1.4, find the value of $\sin t$.

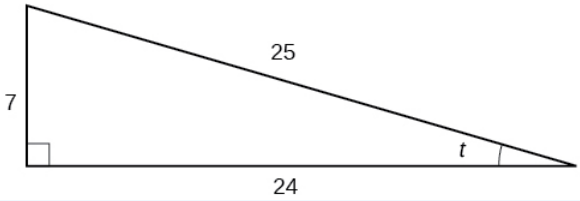


Figure 10.1.4

Answer

$\frac{7}{25}$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 10.1.5. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

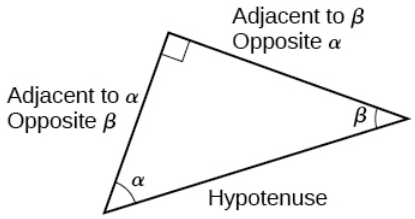


Figure 10.1.5: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 10.1.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 10.1.6, evaluate $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

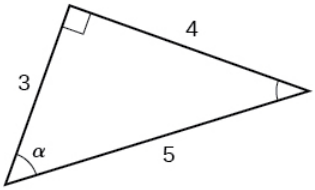


Figure 10.1.6

Solution

$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Exercise 10.1.2

Using the triangle shown in Figure 10.1.7, evaluate $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

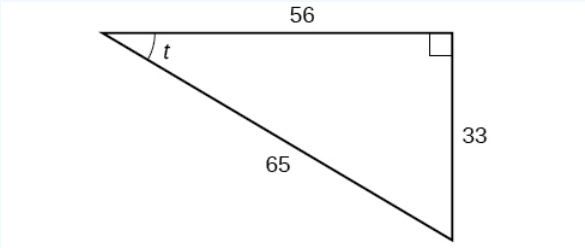


Figure 10.1.7

Answer

$$\sin t = \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56},$$

$$\sec t = \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30^\circ, 60^\circ$, and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a $30^\circ, 60^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45^\circ, 45^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 10.1.8

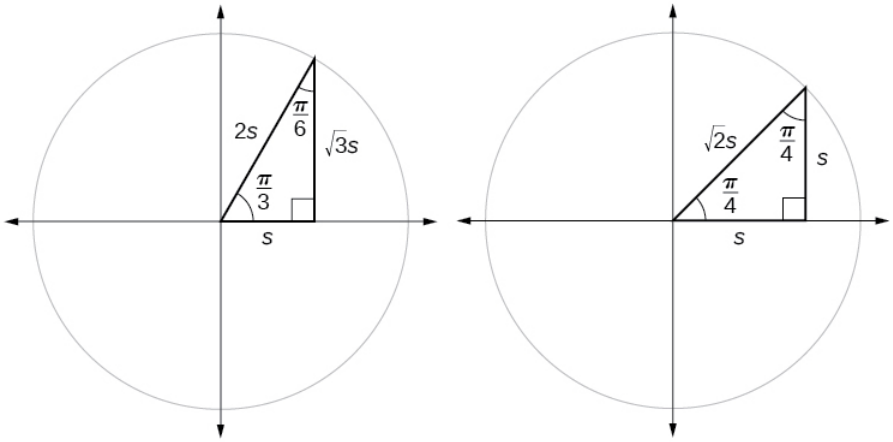


Figure 10.1.8: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 10.1.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 10.1.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 10.1.3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{\pi}{4}) = 1,$
$\sec(\frac{\pi}{4}) = \sqrt{2}, \csc(\frac{\pi}{4}) = \sqrt{2}, \cot(\frac{\pi}{4}) = 1$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the

sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 10.1.9).

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

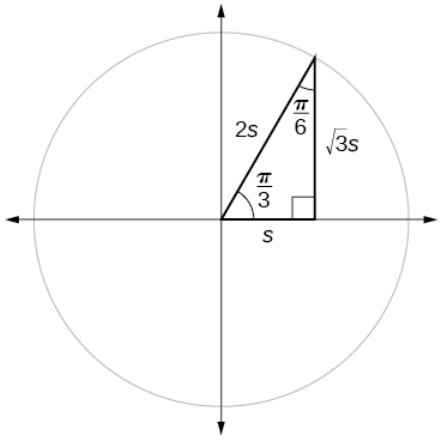
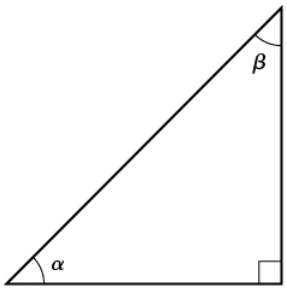


Figure 10.1.9: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure 10.1.9, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 10.1.10.



$\sin\alpha = \cos\beta$
$\sin\beta = \cos\alpha$

Figure 10.1.10: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table 10.1.1.

Table 10.1.1

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$
$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$

$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 10.1.4: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos\frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Exercise 10.1.4

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 10.1.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 10.1.11.

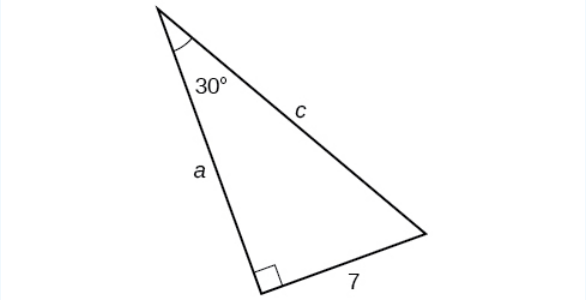


Figure 10.1.11

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30^\circ)} \tag{10.1.6}$$

$$= 12.1 \tag{10.1.7}$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30^\circ)} = 14$$

Exercise 10.1.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer

$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3}$; missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 10.1.12

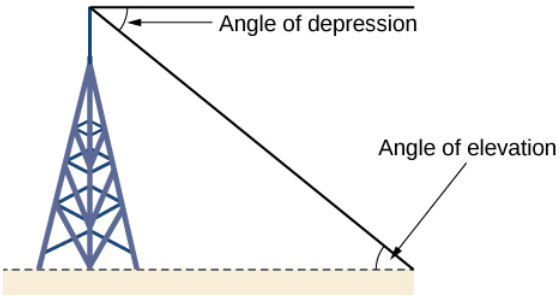


Figure 10.1.12

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.

4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 10.1.6: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure 10.1.13 Find the height of the tree.

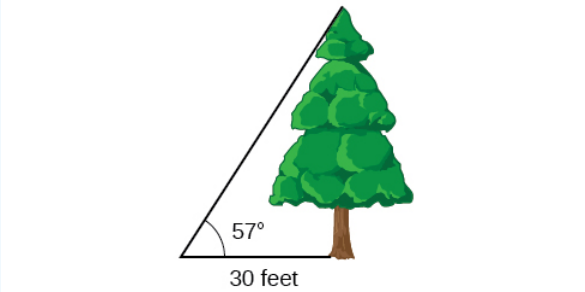


Figure 10.1.13

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{10.1.8}$$

The tree is approximately 46 feet tall.

Exercise 10.1.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer

About 52 ft

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities

$$\cos t = \sin\left(\frac{\pi}{2} - t\right)$$
$$\sin t = \cos\left(\frac{\pi}{2} - t\right)$$
$$\tan t = \cot\left(\frac{\pi}{2} - t\right)$$
$$\cot t = \tan\left(\frac{\pi}{2} - t\right)$$
$$\sec t = \csc\left(\frac{\pi}{2} - t\right)$$
$$\csc t = \sec\left(\frac{\pi}{2} - t\right)$$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See Example.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See Example.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See Example.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See Example.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See Example.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See Example.

Glossary

adjacent side
in a right triangle, the side between a given angle and the right angle

angle of depression
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
in a right triangle, the side most distant from a given angle

hypotenuse
the side of a right triangle opposite the right angle

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