ANALYTIC GEOMETRY AND CALCULUS 1 (IVC MATH C2210 - HUBER)



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Analytic Geometry and Calculus 1 (IVC Math C2210 - Huber)

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1.1: Functions, Domains, and Limit Fundamentals

Learning Objectives

- Determine the domain of a function.
- Create new functions by combining two functions together and find the domain.
- Define the concept of a limit.
- Evaluate a limit of a explicitly or graphically defined function.
- Understand the limit laws.

Most of mathematics is the study of relationships between objects or numbers. A **relation** between two variables x and y is essentially any equation or inequality involving these two letters. A **function** is a specific type of relationship that follows the rule of being well-defined: for each input value x, there can be at most one corresponding output value y.

Question of the Day

How do we describe where a function is heading, even if it never actually gets there?

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1.1: Functions and Domains

Graphically, if x and y are related (or $f(x) = y$), we plot the ordered pair (x, y). A relationship represents a function if its graph passes the **vertical line test**, meaning no vertical line intersects the graph in more than one point.

Definition: Domain and Range

For a given function f, there are two fundamental sets of values:

i. The **domain** of a function is the set of all possible (valid) input values x.
ii. The **range** of a function is the set of all possible (valid) output values y.

To find the domain of an algebraic function, keep three key restrictions in mind:

1. No division by zero.
2. No square roots (or even roots) of negative numbers.
3. Logarithms must always take strictly positive arguments.

Example 1.1.1: Finding Domains of Functions

Find the domain of each of the following functions:

(a) $a(x) = \dfrac{\sqrt{12 - 2x}}{x + 7}$

(b) $b(x) = \log_5(4 - \sqrt{x + 1})$

(c) $c(x) = \dfrac{x^2 - 4}{x^2 - 5x + 6}$

(d) $d(x) = \sqrt{9 - x^2} + \dfrac{1}{x - 1}$

Solution:

(a) We must satisfy two conditions simultaneously: the denominator cannot be zero, and the expression under the square root must be nonnegative.

First, set the denominator equal to zero to identify restricted points:

$$x + 7 \neq 0 \implies x \neq -7$$

Next, require the radicand to be nonnegative:

$$12 - 2x \geq 0 \implies 12 \geq 2x \implies x \leq 6$$

Combining these restrictions yields the domain: $\boxed{(-\infty, -7) \cup (-7, 6]}$

(b) For $b(x)$, we must ensure the inner radicand is nonnegative and the argument of the logarithm is strictly positive.

First, for the radical:

$$x + 1 \geq 0 \implies x \geq -1$$

Second, for the logarithm:

$$4 - \sqrt{x+1} > 0 \implies \sqrt{x+1} < 4 \implies x+1 < 16 \implies x < 15$$

Combining these inequalities gives the domain: $\boxed{[-1, 15)}$

(c) For $c(x)$, the only restriction is that the denominator cannot be zero.

$$x^2 - 5x + 6 = (x-2)(x-3) \neq 0 \implies x \neq 2 \text{ and } x \neq 3$$

Domain: $\boxed{(-\infty, 2) \cup (2, 3) \cup (3, \infty)}$

(d) For $d(x)$, we need two conditions: the radicand must be nonnegative and the denominator must not be zero.

For the square root:

$$9 - x^2 \geq 0 \implies x^2 \leq 9 \implies -3 \leq x \leq 3$$

For the denominator:

$$x - 1 \neq 0 \implies x \neq 1$$

Combining these gives the domain: $\boxed{[-3, 1) \cup (1, 3]}$

Common Pitfall

When finding domains, remember to consider **all** restrictions simultaneously. A common mistake is to forget that both the radicand and denominator restrictions must be satisfied at the same time. Always combine your conditions using intersection (and).

Rule: Function Combinations and Domains

If $f(x)$ has domain D_f and $g(x)$ has domain D_g, we can form new functions whose domains are determined by the intersections of D_f and D_g:

i. *Sum*: $(f+g)(x) = f(x) + g(x)$, with domain $D_f \cap D_g$
ii. *Difference*: $(f-g)(x) = f(x) - g(x)$, with domain $D_f \cap D_g$
iii. *Product*: $(f \cdot g)(x) = f(x) \cdot g(x)$, with domain $D_f \cap D_g$
iv. *Quotient*: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, with domain $D_f \cap D_g \cap \{x : g(x) \neq 0\}$
v. *Composition*: $(f \circ g)(x) = f(g(x))$, with domain $\{x \in D_g : g(x) \in D_f\}$

For composition, it is often helpful to think: "The input x must be valid for g, and the output of $g(x)$ must be valid for f."

Example 1.1.2: Combining Functions and Finding Domains

Let $f(x) = x^2 - x$ (domain $D_f = \mathbb{R}$) and $g(x) = 1 - \frac{1}{x}$ (domain $D_g = (-\infty, 0) \cup (0, \infty)$). Find each combination and its domain:

(a) $(f+g)(x)$

(b) $\left(\frac{f}{g}\right)(x)$

(c) $(g \circ f)(x)$

(d) $(f \circ g)(x)$

Solution:

(a)

$$(f+g)(x) = x^2 - x + 1 - \frac{1}{x}$$

The domain requires $x \neq 0$, so Domain: $\boxed{(-\infty, 0) \cup (0, \infty)}$

(b)

$$\left(\frac{f}{g}\right)(x) = \frac{x^2 - x}{1 - \frac{1}{x}} = \frac{x(x-1)}{\frac{x-1}{x}} = x^2 \quad \text{for } x \neq 1$$

We must exclude $x = 0$ (from D_g) and $x = 1$ (where $g(x) = 0$). Domain: $\boxed{(-\infty, 0) \cup (0, 1) \cup (1, \infty)}$

(c)

$$(g \circ f)(x) = g(f(x)) = 1 - \frac{1}{x^2 - x}$$

This requires $f(x) \neq 0$, meaning $x(x-1) \neq 0 \implies x \neq 0$ and $x \neq 1$. Domain: $\boxed{(-\infty, 0) \cup (0, 1) \cup (1, \infty)}$

(d)

$$(f \circ g)(x) = f(g(x)) = \left(1 - \frac{1}{x}\right)^2 - \left(1 - \frac{1}{x}\right)$$

This requires $x \neq 0$ (so that $g(x)$ is defined) and $g(x)$ must be a real number (which it is for all $x \neq 0$). Domain: $\boxed{(-\infty, 0) \cup (0, \infty)}$

Example 1.1.3: More Function Combinations

Let $f(x) = \sqrt{x+3}$ and $g(x) = \dfrac{1}{x-2}$. Find each combination and its domain:

(a) $(f \cdot g)(x)$

(b) $\left(\dfrac{g}{f}\right)(x)$

(c) $(f \circ g)(x)$

Solution:

First, find the individual domains:

$$D_f : x + 3 \geq 0 \implies x \geq -3 \quad \text{so} \quad D_f = [-3, \infty)$$

$$D_g : x - 2 \neq 0 \implies x \neq 2 \quad \text{so} \quad D_g = (-\infty, 2) \cup (2, \infty)$$

(a) $(f \cdot g)(x) = \sqrt{x+3} \cdot \dfrac{1}{x-2} = \dfrac{\sqrt{x+3}}{x-2}$

Domain: $x \geq -3$ and $x \neq 2 \implies \boxed{[-3, 2) \cup (2, \infty)}$

(b) $\left(\dfrac{g}{f}\right)(x) = \dfrac{1/(x-2)}{\sqrt{x+3}} = \dfrac{1}{(x-2)\sqrt{x+3}}$

Domain: $x \geq -3$, $x \neq 2$, and $f(x) \neq 0 \implies \sqrt{x+3} \neq 0 \implies x \neq -3$

So domain: $\boxed{(-3, 2) \cup (2, \infty)}$

(c) $(f \circ g)(x) = f(g(x)) = \sqrt{\dfrac{1}{x-2} + 3} = \sqrt{\dfrac{1+3x-6}{x-2}} = \sqrt{\dfrac{3x-5}{x-2}}$

Domain requires: $x \neq 2$ (for g) and $\dfrac{3x-5}{x-2} \geq 0$

Important points: $x = \dfrac{5}{3}$ and $x = 2$. Testing intervals gives: $\boxed{\left(-\infty, \dfrac{5}{3}\right] \cup (2, \infty)}$

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1.1: Limits

Here is the updated content with the new sections inserted as requested, maintaining the same formatting and style. ---

Now let's examine the foundational concept of modern calculus: the **limit**.

Consider the rational function $f(x) = \dfrac{4}{x-1}$ near $x = -1$ and $x = 1$.

The function is defined at $x = -1$, but let's look at its behavior nearby using numerical values:

Evaluating $f(x)$ as x approaches -1 from the right ($x > -1$):

$$f(-0.9) = \frac{4}{-0.9-1} = -\frac{40}{19}, \quad f(-0.99) = \frac{4}{-0.99-1} = -\frac{400}{199}, \quad f(-0.999) = -\frac{4000}{1999}, \quad f(-0.9999) = -\frac{40000}{19999}$$

Since these function values are approaching -2, we write the right-sided limit:

$$\lim_{x \to -1^+} \frac{4}{x-1} = -2$$

Evaluating $f(x)$ as x approaches 1 from the left ($x < 1$):

$$f(-1.1) = \frac{4}{-1.1-1} = -\frac{40}{21}, \quad f(-1.01) = \frac{4}{-1.01-1} = -\frac{400}{201}, \quad f(-1.001) = \frac{4000}{-2001}, \quad f(-1.0001) = -\frac{40000}{20001}$$

Since these function values are approaching -2, we write the left-sided limit:

$$\lim_{x \to -1^-} \frac{4}{x-1} = -2$$

As x gets closer to -1, $f(x)$ gets closer to -2 from both the right and the left sides. Because it approaches a single real number from both sides, the overall limit is that number, and we can write

$$\lim_{x \to -1} \frac{4}{x-1} = -2$$

Note that although $f(-1) = -2$, this number was not used in the calculation of the limit.

Alternatively, the function is undefined at $x = 1$, but we can still analyze its behavior nearby using numerical values:

Evaluating $f(x)$ as x approaches 1 from the right ($x > 1$):

$$f(1.1) = \frac{4}{1.1-1} = 40, \quad f(1.01) = \frac{4}{1.01-1} = 400, \quad f(1.001) = 4000, \quad f(1.0001) = 40000$$

Since these function values are growing large and positive without bound, we write the right-sided limit:

$$\lim_{x \to 1^+} \frac{4}{x-1} = +\infty$$

Evaluating $f(x)$ as x approaches 1 from the left ($x < 1$):

$$f(0.9) = \frac{4}{0.9-1} = -40, \quad f(0.99) = \frac{4}{0.99-1} = -400, \quad f(0.999) = -4000, \quad f(0.9999) = -40000$$

Since these function values are growing large and negative without bound, we write the left-sided limit:

$$\lim_{x \to 1^-} \frac{4}{x-1} = -\infty$$

As x gets closer to 1, $f(x)$ grows without bound in magnitude, heading toward $+\infty$ from the right and $-\infty$ from the left. Because

it does not approach a single real number from both sides, the overall limit does not exist.

Definition: Limit Notation

We write the three following variants of the limit:

$$\lim_{x \to a^+} f(x) = L$$

means that as x gets arbitrarily close to a from the right side (when $x > a$), the value of $f(x)$ gets arbitrarily close to L.

$$\lim_{x \to a^-} f(x) = L$$

means that as x gets arbitrarily close to a from the left side (when $x < a$), the value of $f(x)$ gets arbitrarily close to L.

$$\lim_{x \to a} f(x) = L$$

to mean that as x gets arbitrarily close to a (from either side, but $x \neq a$), the value of $f(x)$ gets arbitrarily close to L.

A two-sided limit exists if and only if both one-sided limits exist and are equal:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Important Note

The value of $f(a)$ itself—whether it exists or not—**does not matter** when evaluating a limit! The limit only cares about what happens *near* a, not at a itself. This is why we can find limits even when functions are undefined at the point in question.

Example 1.1.4: Evaluating Limits of a Linear Function

Let $f(x) = 3x - 2$. Find the following limits:

(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to -2} f(x)$

Solution:

For a linear function, we can evaluate the limit by direct substitution (as we will justify later). The idea is that as x gets closer to a value, the line's output gets closer to the corresponding value.

(a) As x approaches 1, $3x - 2$ approaches $3(1) - 2 = 1$. We can check numerically: for $x = 0.9, 0.99, 0.999$, we get $0.7, 0.97, 0.997$, which approach 1. Thus:

$$\lim_{x \to 1}(3x - 2) = 1$$

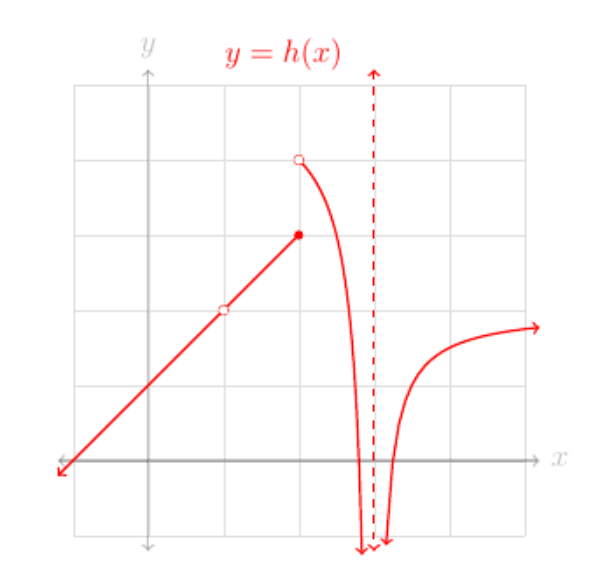
(b) As x approaches -2, $3x - 2$ approaches $3(-2) - 2 = -8$. Numerical check: for $x = -1.9, -1.99, -1.999$, we get $-7.7, -7.97, -7.997$, which approach -8. Thus:

$$\lim_{x \to -2}(3x - 2) = -8$$

Notice that in both cases, the limit equals the function value at that point. For linear functions (and more generally, continuous functions), this is always true.

Example 1.1.5: Finding Limits from a Graph

Consider the graph of a function $h(x)$ shown below.



Use the graph to find the following limits:

(a) $\lim_{x \to 1} h(x)$

(b) $\lim_{x \to 2} h(x)$

(c) $\lim_{x \to 3} h(x)$

Solution:

(a) At $x = 1$ the graph has a hole, so $h(1)$ is not defined.

As x approaches 1 from the left (values like 0.9, 0.99), the graph's y-values approach 2, so $\lim_{x \to 1^-} h(x) = 2$

From the right (values like 1.1, 1.01), the graph also approaches 2, so $\lim_{x \to 1^+} h(x) = 2$

Since both one-sided limits equal 2, we have $\boxed{\lim_{x \to 1} h(x) = 2}$

Notice that even if $h(1)$ is not defined or takes a different value, this does not effect the value of the limit.

(b) At $x = 2$, the graph has a sudden jump.

From the left, the y-values approach 3, so $\lim_{x \to 2^-} h(x) = 3$

From the right, the y-values approach 4. So $\lim_{x \to 2^+} h(x) = 4$

Since the left-hand limit (3) and the right-hand limit (4) are not equal, the two-sided limit does not exist. We write $\boxed{\lim_{x \to 2} h(x) = \text{DNE}}$

(c) Near $x = 3$, the graph has a vertical asymptote.

As x approaches 3 from the left, the y-values decrease without bound, so $\lim_{x\to 3^-} h(x) = -\infty$

As x approaches 3 from the right, the y-values also decrease without bound, so $\lim_{x\to 3^+} h(x) = -\infty$

Since both one-sided limits equal $-\infty$, we have $\boxed{\lim_{x\to 2} h(x) = -\infty}$

Rule: The Limit Laws

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ exist. Let c be a constant. Then the following algebraic laws hold:

i. *Sum Law*:

$$\lim_{x\to a}[f(x) + g(x)] = L + M$$

ii. *Difference Law*:

$$\lim_{x\to a}[f(x) - g(x)] = L - M$$

iii. *Constant Multiple Law*:

$$\lim_{x\to a}[c \cdot f(x)] = c \cdot L$$

iv. *Product Law*:

$$\lim_{x\to a}[f(x) \cdot g(x)] = L \cdot M$$

v. *Quotient Law*:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{provided } M \neq 0$$

These laws also apply identically to one-sided limits, provided the limits involved exist.

Example 1.1.6: Using Limit Laws

Suppose $\lim_{x\to 2} f(x) = 5$ and $\lim_{x\to 2} g(x) = -3$. Evaluate:

(a) $\lim_{x\to 2} [3f(x) - 2g(x)]$

(b) $\lim_{x\to 2} [f(x) \cdot g(x) + 4]$

(c) $\lim_{x\to 2} \dfrac{g(x)}{f(x) - 2}$

Solution:

(a) Using the Constant Multiple, Difference, and Sum Laws:

$$\lim_{x\to 2} [3f(x) - 2g(x)] = 3(5) - 2(-3) = 15 + 6 = 21$$

(b) Using the Product and Sum Laws:

$$\lim_{x\to 2} [f(x) \cdot g(x) + 4] = (5)(-3) + 4 = -15 + 4 = -11$$

(c) Using the Quotient and Difference Laws (note: $\lim_{x\to 2}[f(x) - 2] = 5 - 2 = 3 \neq 0$):

$$\lim_{x\to 2}\frac{g(x)}{f(x)-2}=\frac{-3}{5-2}=\frac{-3}{3}=-1$$

What Does "Arbitrarily Close" Mean Exactly?

You might wonder: what does it really mean for $f(x)$ to get "arbitrarily close" to L as x gets "arbitrarily close" to a? To make this precise, mathematicians use the **epsilon-delta (ϵ-δ) definition** of a limit.

The formal definition says:

> $\lim_{x\to a} f(x)=L$ means that for every $\epsilon>0$ (no matter how small), there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $|f(x)-L|<\epsilon$.

In plain English: You can make $f(x)$ as close as you want to L (within any tolerance $\epsilon>0$) by making x sufficiently close to a (within some distance $\delta>0$), but not equal to a. The symbol ϵ (epsilon) represents the allowable error in the output, and δ (delta) is the corresponding required closeness in the input.

Let's revisit our linear example, $f(x)=3x-2$, and $\lim_{x\to 1} f(x)=1$, to see how the epsilon-delta definition works.

Suppose we want $f(x)$ to be within $\epsilon=0.01$ of 1. That is, we want $|f(x)-1|<0.01$. Compute:

$$|(3x-2)-1|=|3x-3|=3|x-1|<0.01 \quad\Longrightarrow\quad |x-1|<\frac{0.01}{3}\approx 0.00333.$$

So if we choose $\delta=0.00333$ (or anything smaller), then whenever $0<|x-1|<\delta$, we are guaranteed that $|f(x)-1|<0.01$. For any $\epsilon>0$, we can choose $\delta=\frac{\epsilon}{3}$. This shows formally that the limit is indeed 1.

This definition is the rigorous backbone of calculus. It eliminates the vague idea of "approaching" and gives a precise, testable condition for a limit to exist.

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1.1: Practice Set

Exercise 1.1.1a

Find the domain of $f(x) = \dfrac{x^2 - 4}{2x - 8}$ in interval notation.

Answer available at the end of this section.

Exercise 1.1.1b

Find the domain of $g(x) = \dfrac{\sqrt{3x - 9}}{x - 5}$ in interval notation.

Answer available at the end of this section.

Exercise 1.1.1c

Find the domain of $h(x) = \sqrt{16 - x^2}$ in interval notation.

Answer available at the end of this section.

Exercise 1.1.1d

Find the domain of $k(x) = \log_3(5 - 2x)$ in interval notation.

Answer available at the end of this section.

Exercise 1.1.1e

Find the domain of $p(x) = \log_5(\sqrt{x + 4} - 3)$ in interval notation.

Answer available at the end of this section.

Exercise 1.1.2

Given $f(x) = \sqrt{x + 2}$ and $g(x) = \dfrac{1}{x - 3}$, state the simplified formula and find the domain of

(a) $(f + g)(x)$

(b) $\left(\dfrac{f}{g}\right)(x)$

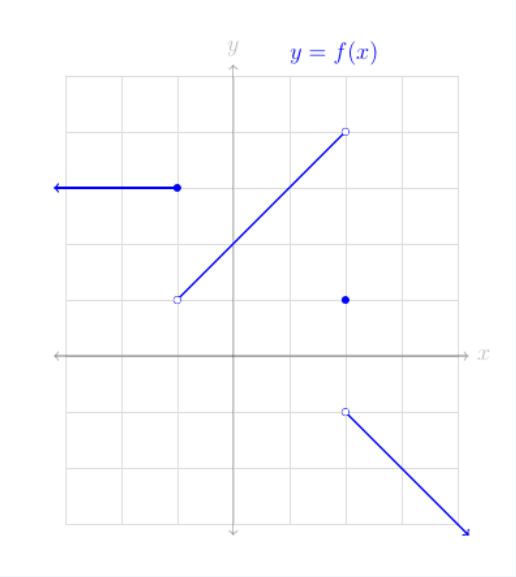
(c) $(f \circ g)(x)$

(d) $(g \circ f)(x)$

Answer available at the end of this section.

Exercise 1.1.3

Suppose $f(x)$ is defined graphically as follows:



Determine each of the following limits by checking both one-sided limits.

(a) $\lim_{x \to -1} f(x)$

(b) $\lim_{x \to 2} f(x)$

Answer available at the end of this section.

Exercise 1.1.4

Suppose that $\lim_{x \to 3} f(x) = -4$, $\lim_{x \to 3} g(x) = 2$, $\lim_{x \to 3^+} h(x) = 0$and $\lim_{x \to 3^-} h(x) = 1$.

Evaluate each of the following limits:

(a) $\lim_{x \to 3} [3f(x) - 2g(x)]$

(b) $\lim_{x \to 3} [f(x) \cdot g(x) + 5]$

(c) $\lim_{x \to 3} \dfrac{g(x) + 6}{f(x)}$

(d) $\lim_{x \to 3} [f(x) + h(x)]$

(e) $\lim_{x \to 3} [g(x) \cdot h(x)]$

(f) $\lim_{x \to 3^-} \dfrac{f(x)}{h(x)}$

(g) $\lim_{x \to 3^+} \dfrac{f(x)}{h(x)}$

Answer available at the end of this section.

Answers

Exercise 1.1.1*a*

We must exclude any value of x that makes the denominator zero:

$$2x - 8 = 0 \implies x = 4$$

Therefore, the domain is all real numbers except $x = 4$:

$$(-\infty, 4) \cup (4, \infty)$$

Exercise 1.1.1*b*

We need two conditions:

1. The radicand must be non-negative:

$$3x - 9 \geq 0 \implies x \geq 3$$

2. The denominator cannot be zero:

$$x - 5 \neq 0 \implies x \neq 5$$

Combining these gives the domain:

$$[3, 5) \cup (5, \infty)$$

Exercise 1.1.1*c*

The radicand must be non-negative:

$$16 - x^2 \geq 0 \implies x^2 \leq 16 \implies -4 \leq x \leq 4$$

Therefore the domain is:

$$[-4, 4]$$

Exercise 1.1.1*d*

The argument of a logarithm must be positive:

$$5 - 2x > 0 \implies -2x > -5 \implies x < \frac{5}{2}$$

Thus the domain is:

$$\left(-\infty, \frac{5}{2}\right)$$

Exercise 1.1.1*e*

We need two conditions:

1. The square root requires a non-negative radicand:

$$x + 4 \geq 0 \implies x \geq -4$$

2. The argument of the logarithm must be positive:

$$\sqrt{x+4} - 3 > 0 \implies \sqrt{x+4} > 3 \implies x + 4 > 9 \implies x > 5$$

Since $x > 5$ is more restrictive than $x \geq -4$, the domain is:

$$(5, \infty)$$

Exercise 1.1.2

(a) The sum is:

$$(f + g)(x) = \sqrt{x+2} + \frac{1}{x-3}$$

The domain requires:

1. $x + 2 \geq 0 \implies x \geq -2$
2. $x - 3 \neq 0 \implies x \neq 3$

Therefore the domain is:

$$[-2, 3) \cup (3, \infty)$$

(b) The quotient is:

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+2}}{1/(x-3)} = \sqrt{x+2} \cdot (x-3)$$

The domain requires:

1. $x + 2 \geq 0 \implies x \geq -2$ (domain of f)
2. $x - 3 \neq 0 \implies x \neq 3$ (domain of g)
3. $g(x) \neq 0$, which holds for all x

Therefore:

$$[-2, 3) \cup (3, \infty)$$

(c) The composition is:

$$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)+2} = \sqrt{\frac{1}{x-3}+2}$$

Simplify:

$$\sqrt{\frac{1+2(x-3)}{x-3}} = \sqrt{\frac{2x-5}{x-3}}$$

The domain requires:

1. $x-3 \neq 0 \implies x \neq 3$ (domain of g)
2. $\frac{2x-5}{x-3} \geq 0 \implies x > 3$ or $x \leq \frac{5}{2}$ ($g(x)$ inside of domain of f)

Critical points: $x = \frac{5}{2}$ and $x = 3$. Testing intervals gives:

$$(-\infty, \frac{5}{2}] \cup (3, \infty)$$

(d) The composition is:

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)-3} = \frac{1}{\sqrt{x+2}-3}$$

The domain requires:

1. $x+2 \geq 0 \implies x \geq -2$ (domain of f)
2. $\sqrt{x+2}-3 \neq 0 \implies \sqrt{x+2} \neq 3 \implies x+2 \neq 9 \implies x \neq 7$ ($f(x)$ inside of domain of g)

Therefore:

$$[-2, 7) \cup (7, \infty)$$

Exercise 1.1.3

(a) As $x \to -1^-$, the function is on the horizontal line $y = 3$, so:

$$\lim_{x \to -1^-} f(x) = 3$$

As $x \to -1^+$, the function is on the line segment from $(-1, 1)$ to $(2, 4)$, approaching the point $(-1, 1)$, so:

$$\lim_{x \to -1^+} f(x) = 1$$

Since the left-hand and right-hand limits are not equal ($3 \neq 1$), the two-sided limit does not exist:

$$\lim_{x \to -1} f(x) = \text{DNE}$$

(b) As $x \to 2^-$, the function is on the line segment from $(-1, 1)$ to $(2, 4)$, approaching the open circle at $(2, 4)$, so:

$$\lim_{x \to 2^-} f(x) = 4$$

As $x \to 2^+$, the function is on the line segment from the open circle at $(2, -1)$ to $(5, -4)$, approaching the open circle at $(2, -1)$, so:

$$\lim_{x \to 2^+} f(x) = -1$$

Since the left-hand and right-hand limits are not equal ($4 \neq -1$), the two-sided limit does not exist:

$$\lim_{x \to 2} f(x) = \text{DNE}$$

Exercise 1.1.4

(a) Using the limit laws:

$$\lim_{x \to 3}[3f(x) - 2g(x)] = 3\lim_{x \to 3} f(x) - 2\lim_{x \to 3} g(x) = 3(-4) - 2(2) = -12 - 4 = -16$$

(b) Using the limit laws:

$$\lim_{x \to 3}[f(x) \cdot g(x) + 5] = \left(\lim_{x \to 3} f(x)\right) \cdot \left(\lim_{x \to 3} g(x)\right) + 5 = (-4)(2) + 5 = -8 + 5 = -3$$

(c) Using the limit laws, and noting $\lim_{x \to 3} f(x) = -4 \neq 0$:

$$\lim_{x \to 3} \frac{g(x)+6}{f(x)} = \frac{\lim_{x \to 3} g(x) + 6}{\lim_{x \to 3} f(x)} = \frac{2+6}{-4} = \frac{8}{-4} = -2$$

(d) Using the limit laws, for each one sided limit gives:

$$\lim_{x \to 3^+}[f(x) + h(x)] = \left(\lim_{x \to 3^+} f(x)\right) + \left(\lim_{x \to 3^+} h(x)\right) = -4 + 0 = -4$$

$$\lim_{x \to 3^-}[f(x) + h(x)] = \left(\lim_{x \to 3^-} f(x)\right) + \left(\lim_{x \to 3^-} h(x)\right) = -4 + 1 = -3$$

Since these do not match, we have:

$$\lim_{x \to 3}[f(x) + h(x)] = \text{DNE}$$

(e) Using the limit laws, but noting that $\lim_{x \to 3} h(x)$ does not exist since the left and right limits differ:

$$\lim_{x \to 3^+}[g(x) \cdot h(x)] = \left(\lim_{x \to 3^+} g(x)\right) \cdot \left(\lim_{x \to 3^+} h(x)\right) = 2 \cdot 0 = 0$$

$$\lim_{x \to 3^-}[g(x) \cdot h(x)] = \left(\lim_{x \to 3^-} g(x)\right) \cdot \left(\lim_{x \to 3^-} h(x)\right) = 2 \cdot 1 = 2$$

Since these do not match, we have:

$$\lim_{x \to 3}[g(x) \cdot h(x)] = \text{DNE}$$

(f) Using the limit laws for the one-sided limit, noting $\lim_{x\to 3^-} h(x) = 1 \neq 0$:

$$\lim_{x\to 3^-} \frac{f(x)}{h(x)} = \frac{\lim\limits_{x\to 3^-} f(x)}{\lim\limits_{x\to 3^-} h(x)} = \frac{-4}{1} = -4$$

(g) As $x \to 3^+$, $h(x) \to 0$ so the limit laws do not apply. We cannot determine the limit value without more information about h.

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1.2: Polynomial and Rational Functions

Learning Objectives

- Recognize polynomial functions and take their limits.
- Recognize rational functions and take their limits.
- Determine when points outside of the domain of a rational function are holes or asymptotes.

In this section, we build on our foundational understanding of functions by reviewing fundamental function families, graph transformations, and algebraic simplification techniques essential for calculating limits.

Question of the Day

What happens when a rational function has a factor that cancels? What happens when it doesn't?

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1.2: Algebraic Functions

Definition: Linear and Power Functions

A **linear function** is a function of the form

$$f(x) = mx + b$$

where m and b are real constants. Its graph is a straight line with slope m and y-intercept $(0, b)$.

A **power function** has the form

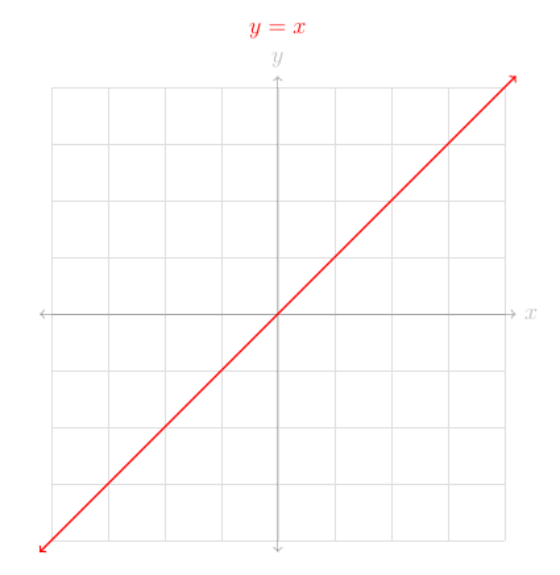
$$f(x) = x^n$$

where n is a constant.

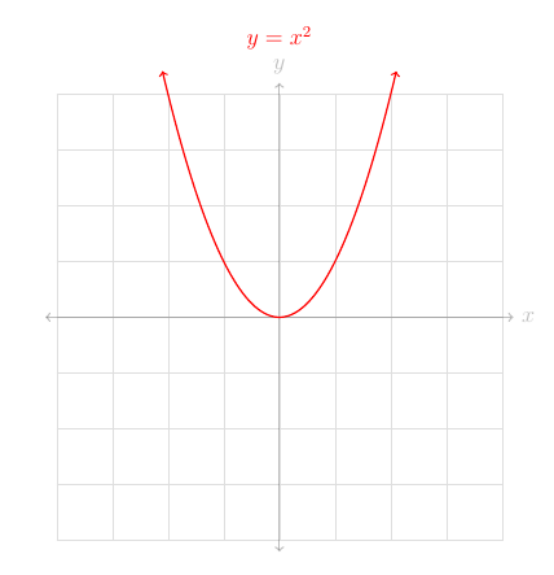
Common Parent Functions to Know

Familiarity with the graphs of basic parent functions is essential for understanding transformations and limits. The most common parent functions are:

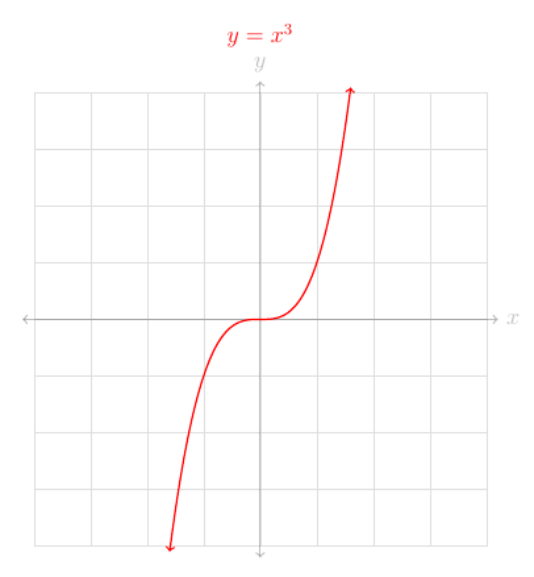
Linear:



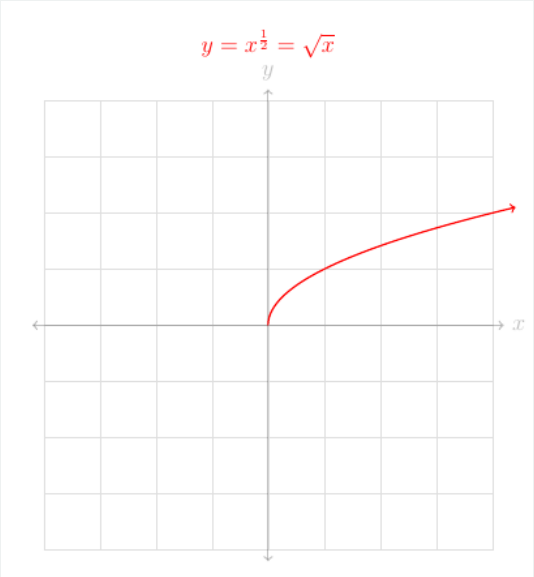
Quadratic:



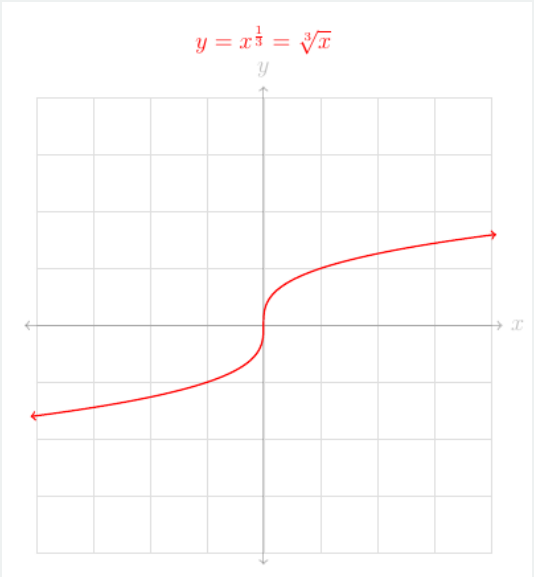
Cubic:



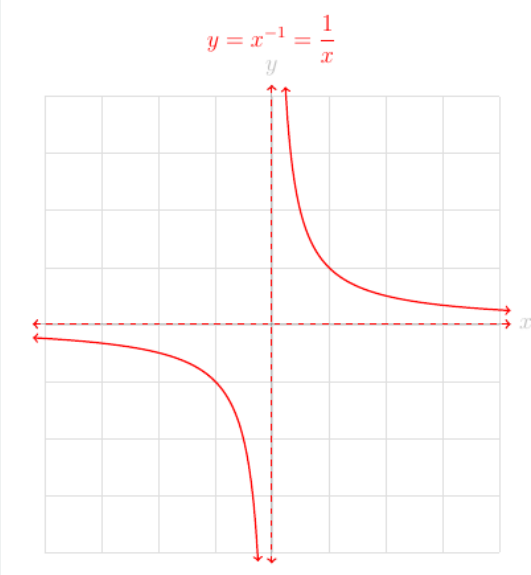
Square Root:



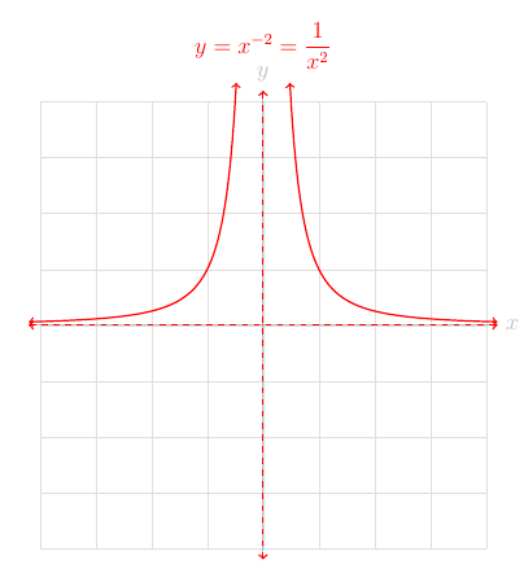
Cube Root:



Reciprocal:



Reciprocal Squared



Rule: Graph Transformations

Given a base function $y = f(x)$ and a constant $c > 0$, we can transform its graph as follows:

- $y = f(x) + c$: Vertical shift upward by c units
- $y = f(x) - c$: Vertical shift downward by c units
- $y = f(x + c)$: Horizontal shift left by c units
- $y = f(x - c)$: Horizontal shift right by c units
- $y = c \cdot f(x)$: Vertical stretch (if $c > 1$) or compression (if $0 < c < 1$) by a factor of c
- $y = f(c \cdot x)$: Horizontal compression (if $c > 1$) or stretch (if $0 < c < 1$) by a factor of $\frac{1}{c}$
- $y = -f(x)$: Reflection across the x-axis
- $y = f(-x)$: Reflection across the y-axis

Example 1.2.1: Describing Transformations

Describe the sequence of transformations needed to obtain each function from its parent function.

(a) $g(x) = -(x-2)^2 + 1$ from $f(x) = x^2$

(b) $h(x) = \dfrac{4}{x+1} - 3$ from $f(x) = \dfrac{1}{x}$

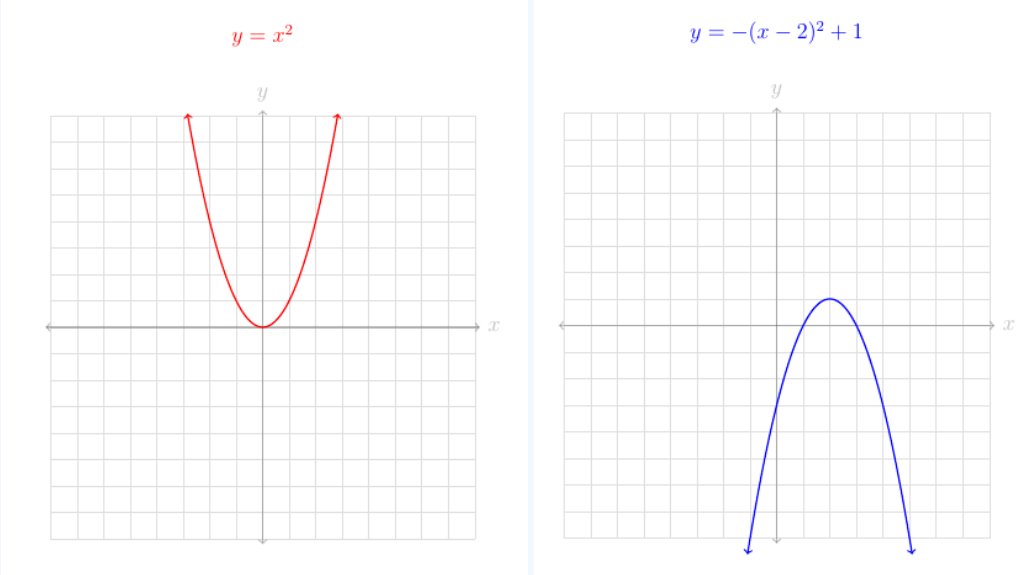
(c) $k(x) = -\sqrt{x+5}$ from $f(x) = \sqrt{x}$

Solution:

(a) Starting with $f(x) = x^2$:

1. Shift right by 2 units: $(x-2)^2$
2. Reflect across the x-axis: $-(x-2)^2$
3. Shift up by 1 unit: $3(x-2)^2 + 1$

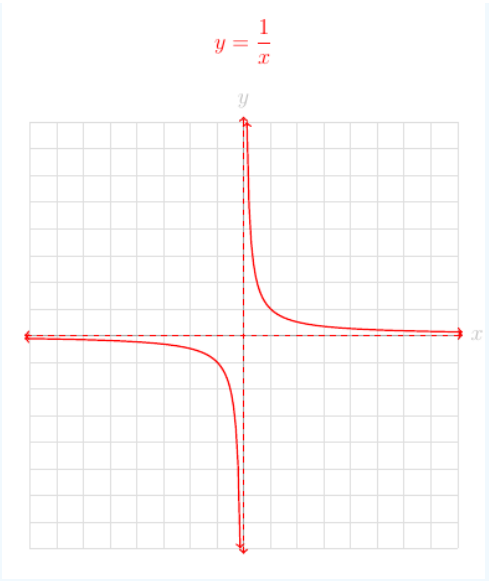
So $g(x)$ is obtained by shifting $f(x)$ **right 2 units**, **reflecting across the x-axis**, and **shifting up 1 unit**.

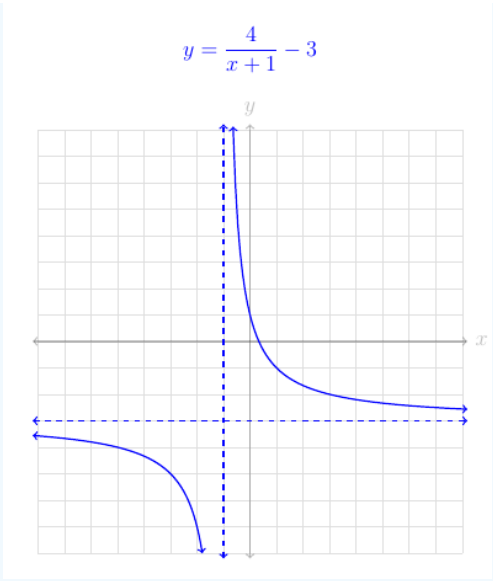


(b) Starting with $f(x) = \frac{1}{x}$:

1. Shift left by 1 unit: $\frac{1}{x+1}$
2. Vertically stretch by a factor of 4: $\frac{4}{x+1}$
3. Shift down by 3 units: $\frac{4}{x+1} - 3$

So $h(x)$ is obtained by shifting $f(x)$ **left 1 unit**, **vertically stretching by 4**, and **shifting down 3 units**.

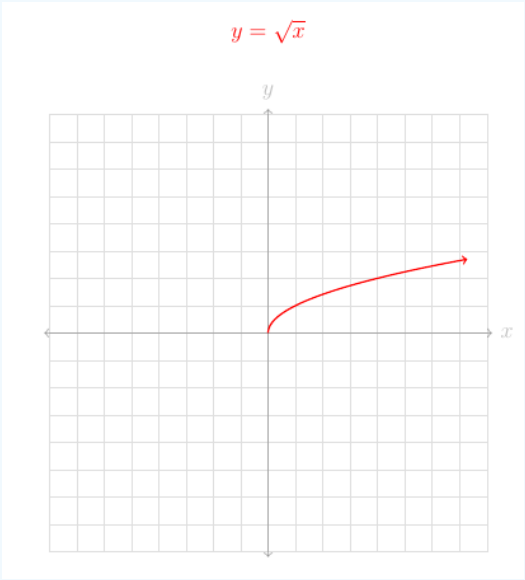


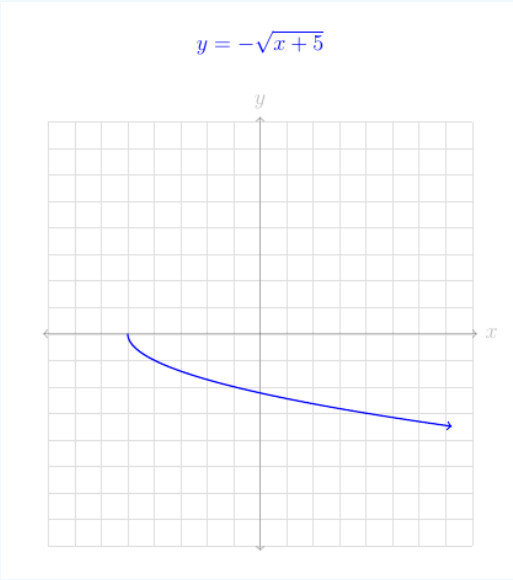


(c) Starting with $f(x) = \sqrt{x}$:

1. Shift left by 5 units: $\sqrt{x+5}$
2. Reflect across the x-axis: $-\sqrt{x+5}$

So $k(x)$ is obtained by shifting $f(x)$ **left 5 units** and **reflecting across the x-axis**.





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1.2: Holes and Asymptotes for Rational Functions

Definition: Polynomials and Rational Functions

A **polynomial** is a sum of power functions with nonnegative integer exponents:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$ and n is a nonnegative integer representing the **degree** of the polynomial.

A **rational function** is a ratio of two polynomials:

$$R(x) = \frac{p(x)}{q(x)}$$

The domain of $R(x)$ consists of all real numbers except values of x where $q(x) = 0$.

Rule: Limits of Polynomials and Rational Functions

Because limit laws hold for sums, differences, and products, polynomials are continuous everywhere. Thus, for any real number a and any polynomial $p(x)$:

$$\lim_{x \to a} p(x) = p(a)$$

Similarly, by the Quotient Law, if $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function and $q(a) \neq 0$:

$$\lim_{x \to a} R(x) = R(a) = \frac{p(a)}{q(a)}$$

When direct substitution gives the indeterminate form $\dfrac{0}{0}$, algebraic simplification (factoring and canceling) is required.

Example 1.2.2: Evaluating Limits of Polynomials and Rational Functions

Evaluate each of the following limits:

(a) $\lim\limits_{x \to 1} (x^2 + 4x)$

(b) $\lim\limits_{x \to 1} \dfrac{x^2 + 4x}{x^2 + 4x + 4}$

(c) $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x - 1}$

(d) $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$

(e) $\lim\limits_{x \to 0} \dfrac{2x - 1}{x^2}$

Solution:

(a) Since $p(x) = x^2 + 4x$ is a polynomial, evaluate directly by plugging in $x = 1$:

$$\lim_{x\to 1}(x^2+4x) = 1^2+4(1) = \boxed{5}$$

(b) Direct substitution gives a nonzero denominator $1^2+4(1)+4 = 9 \neq 0$:

$$\lim_{x\to 1}\frac{x^2+4x}{x^2+4x+4} = \frac{1^2+4(1)}{1^2+4(1)+4} = \boxed{\frac{5}{9}}$$

(c) Direct substitution yields the indeterminate form $\frac{0}{0}$. Factor the numerator to cancel the common factor $(x-1)$:

$$\lim_{x\to 1}\frac{x^2+x-2}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1}(x+2) = 1+2 = \boxed{3}$$

(d) Direct substitution gives $\frac{0}{0}$. Factor the numerator using the difference of cubes:

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 4+4+4 = \boxed{12}$$

(e) Direct substitution gives $\frac{-1}{0}$, which is undefined. We check both one-sided limits to be safe:

$$\lim_{x\to 0^+}\frac{2x-1}{x^2} = \frac{2\cdot 0-1}{(0^+)^2} = \frac{-1}{0^+} = +\infty$$

$$\lim_{x\to 0^-}\frac{2x-1}{x^2} = \frac{2\cdot 0-1}{(0^-)^2} = \frac{-1}{0^+} = +\infty$$

Since these match, we have $\lim_{x\to 0}\frac{2x-1}{x^2} = \boxed{+\infty}$

Reducing for $\frac{0}{0}$ Limits

When direct substitution yields $\frac{0}{0}$, we are always allowed to reduce a function that is inside of a limit.

This is because the reduced form of a function can only differ from its original form at a finite number of points, which cannot effect the value of a limit.

Remember: You can only cancel factors that are multiplied in the numerator and denominator, not terms that are added or subtracted!

Definition: Holes vs. Vertical Asymptotes

For a rational function $R(x) = \frac{p(x)}{q(x)}$ and a point $x = a$ where $q(a) = 0$:

- **Hole (Removable Discontinuity)**: Occurs when the factor $(x-a)$ cancels completely from the denominator. The limit $\lim_{x\to a} R(x)$ exists and is finite, but the function is undefined at $x = a$.
- **Vertical Asymptote**: Occurs when the factor $(x-a)$ remains in the denominator after reducing. The limits $\lim_{x\to a^+} R(x) = \pm\infty$ and $\lim_{x\to a^-} R(x) = \pm\infty$ so $\lim_{x\to a} R(x) = \pm\infty$ when both agree or does not exist when they disagree.

Example 1.2.3: Identifying Holes and Vertical Asymptotes

Find all holes and vertical asymptotes for each function:

(a) $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}$

(b) $g(x) = \dfrac{x^3 - x}{x^3 - 2x^2 + x}$

(c) $h(x) = \dfrac{3x^2 - 12}{x^2 - 4x + 4}$

Solution:

(a) Factor completely:

$$f(x) = \frac{x^2 - 4}{x^2 - 3x + 2} = \frac{(x-2)(x+2)}{(x-1)(x-2)}$$

The values outside of the domain are $x = 1$ and $x = 2$.

Reducing for $x \neq 2$: $f(x) = \dfrac{x+2}{x-1}$

- At $x = 2$: The factor $(x-2)$ cancels completely. There is a **hole** at $x = 2$. The y-value is $\dfrac{2+2}{2-1} = 4$, so the hole is at $\boxed{(2, 4)}$
- At $x = 1$: The factor $(x-1)$ remains in the denominator. There is a **vertical asymptote** at $\boxed{x = 1}$

(b) Factor completely:

$$g(x) = \frac{x(x^2 - 1)}{x(x^2 - 2x + 1)} = \frac{x(x-1)(x+1)}{x(x-1)^2}$$

The values outside of the domain are $x = 0$ and $x = 1$.

Reducing for $x \neq 0, 1$: $g(x) = \dfrac{x+1}{x-1}$

- At $x = 0$: The factor x cancels completely. There is a **hole** at $x = 0$. The y-value is $\dfrac{0+1}{0-1} = -1$, so the hole is at $\boxed{(0, -1)}$
- At $x = 1$: One factor of $(x-1)$ remains in the denominator. There is a **vertical asymptote** at $\boxed{x = 1}$

(c) Factor completely:

$$h(x) = \frac{3(x^2 - 4)}{(x-2)^2} = \frac{3(x-2)(x+2)}{(x-2)^2}$$

The value outside of the domain is $x = 2$.

Reducing for $x \neq 2$: $h(x) = \dfrac{3(x+2)}{x-2}$

- At $x = 2$: The factor $(x-2)$ remains in the denominator after canceling one factor. There is a **vertical asymptote** at $\boxed{x = 2}$
- No holes since the factor does not cancel completely.

Example 1.2.4: Limit at a Hole

Find all value(s) of b that will yield a finite value for $\lim\limits_{x \to 3} \dfrac{x^2 + x + b}{x^2 - 4x + 3}$ and find the value of that limit.

Solution:

We can see that $x = 3$ causes the denominator to be 0, so in order for the limit to be finite we need the numerator to also go to 0.

$$\lim_{x \to 3} \left(x^2 + x + b\right) = 3^2 + 3 + b = 12 + b$$

Setting this equal to zero gives $12 + b = 0 \implies \boxed{b = -12}$

When $b = -12$, our limit becomes:

$$\lim_{x \to 3} \frac{x^2 + x - 12}{x^2 - 4x + 3} = \lim_{x \to 3} \frac{(x-3)(x+4)}{(x-1)(x-3)} = \lim_{x \to 3} \frac{x+4}{x-1} = \frac{3+4}{3-1} = \boxed{\frac{7}{2}}$$

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1.2: Practice Set

Exercise 1.2.1

Evaluate $\lim_{x \to -3} \left(4x^2 - 3x + 7\right)$ using direct substitution and limit laws.

Answer available at the end of this section.

Exercise 1.2.2

Evaluate $\lim_{x \to \frac{1}{2}} \left(8x^3 - 4x + 1\right)$ using direct substitution and limit laws.

Answer available at the end of this section.

Exercise 1.2.3

Evaluate $\lim_{x \to 2} \left(x^4 - 3x^2 + 5\right)$ using direct substitution and limit laws.

Answer available at the end of this section.

Exercise 1.2.4

Evaluate $\lim_{x \to -2} \dfrac{x^2 - 4}{x + 2}$. If the limit does not exist, state so.

Answer available at the end of this section.

Exercise 1.2.5

Evaluate $\lim_{x \to 3} \dfrac{x^2 - 5x + 6}{x^2 - 9}$. If the limit does not exist, state so.

Answer available at the end of this section.

Exercise 1.2.6

Evaluate $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$. If the limit does not exist, state so.

Answer available at the end of this section.

Exercise 1.2.7

Evaluate $\lim_{x \to 5} \dfrac{2x^2 - 9x - 5}{x - 5}$. If the limit does not exist, state so.

Answer available at the end of this section.

Exercise 1.2.8

Evaluate $\lim_{x \to -3} \dfrac{x^2 + 6x + 9}{x^2 + 2x - 3}$. If the limit does not exist, state so.

Answer available at the end of this section.

Exercise 1.2.9

Evaluate $\lim_{x \to 4} \dfrac{x^2 - 16}{x^2 - 8x + 16}$. If the limit does not exist, state so.

Answer available at the end of this section.

Exercise 1.2.10

For $f(x) = \dfrac{3x - 6}{x^2 - 4}$, determine its domain, find the coordinates (x, y) of any holes, and state the equations of any vertical asymptotes.

Answer available at the end of this section.

Exercise 1.2.11

For $g(x) = \dfrac{x^2 - 3x - 10}{x^2 - 25}$, determine its domain, find the coordinates (x, y) of any holes, and state the equations of any vertical asymptotes.

Answer available at the end of this section.

Exercise 1.2.12

For $h(x) = \dfrac{x^3 - 4x}{x^2 - 2x}$, determine its domain, find the coordinates (x, y) of any holes, and state the equations of any vertical asymptotes.

Answer available at the end of this section.

Exercise 1.2.13

For $k(x) = \dfrac{2x^2 + 5x - 3}{6x^2 - 5x + 1}$, determine its domain, find the coordinates (x, y) of any holes, and state the equations of any vertical asymptotes.

Answer available at the end of this section.

Exercise 1.2.14

For $p(x) = \dfrac{x^3 - x^2 - 6x}{x^3 - 9x}$, determine its domain, find the coordinates (x, y) of any holes, and state the equations of any vertical asymptotes.

Answer available at the end of this section.

Exercise 1.2.15

Let $f(x) = x^{-1}$. Describe the sequence of transformations needed to obtain the graph of $g(x) = \dfrac{4}{x-2}$, and state its domain and vertical asymptote.

Answer available at the end of this section.

Exercise 1.2.16

Given $p(x) = x^2$, write the formula for the function $q(x)$ obtained by shifting the graph of $p(x)$ left by 3 units, vertically stretching it by a factor of 2, and shifting it down by 5 units.

Answer available at the end of this section.

Exercise 1.2.17

Explain why a rational function $R(x) = \dfrac{p(x)}{q(x)}$ has a hole at $x = a$ when $(x - a)$ cancels completely from $q(x)$, whereas it has a vertical asymptote at $x = a$ when $(x - a)$ remains in $q(x)$ after fully simplifying.

Answer available at the end of this section.

Answers

Exercise 1.2.1

Since this is a polynomial, we can evaluate by direct substitution:

$$\lim_{x \to -3} (4x^2 - 3x + 7) = 4(-3)^2 - 3(-3) + 7 = 4(9) + 9 + 7 = 36 + 9 + 7 = 52 \tag{1.2.1}$$

$$\boxed{52} \tag{1.2.2}$$

Exercise 1.2.2

Since this is a polynomial, we can evaluate by direct substitution:

$$\lim_{x \to \frac{1}{2}} (8x^3 - 4x + 1) = 8\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right) + 1 = 8\left(\frac{1}{8}\right) - 2 + 1 = 1 - 2 + 1 = 0 \tag{1.2.3}$$

$$\boxed{0} \tag{1.2.4}$$

Exercise 1.2.3

Since this is a polynomial, we can evaluate by direct substitution:

$$\lim_{x \to 2} (x^4 - 3x^2 + 5) = 2^4 - 3(2^2) + 5 = 16 - 3(4) + 5 = 16 - 12 + 5 = 9 \tag{1.2.5}$$

$$\boxed{9} \tag{1.2.6}$$

Exercise 1.2.4

Direct substitution gives $\frac{0}{0}$, so we factor and simplify:

$$\frac{x^2 - 4}{x + 2} = \frac{(x-2)(x+2)}{x+2} = x - 2 \quad \text{for } x \neq -2 \tag{1.2.7}$$

Therefore:

$$\lim_{x\to -2}\frac{x^2-4}{x+2}=\lim_{x\to -2}(x-2)=-2-2=-4 \tag{1.2.8}$$

$$\boxed{-4} \tag{1.2.9}$$

Exercise 1.2.5

Direct substitution gives $\frac{0}{0}$, so we factor and simplify:

$$\frac{x^2-5x+6}{x^2-9}=\frac{(x-2)(x-3)}{(x-3)(x+3)}=\frac{x-2}{x+3}\quad \text{for } x\neq 3 \tag{1.2.10}$$

Therefore:

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^2-9}=\lim_{x\to 3}\frac{x-2}{x+3}=\frac{3-2}{3+3}=\frac{1}{6} \tag{1.2.11}$$

$$\boxed{\frac{1}{6}} \tag{1.2.12}$$

Exercise 1.2.6

Direct substitution gives $\frac{0}{0}$, so we factor and simplify:

$$\frac{x^3-1}{x^2-1}=\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}=\frac{x^2+x+1}{x+1}\quad \text{for } x\neq 1 \tag{1.2.13}$$

Therefore:

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\lim_{x\to 1}\frac{x^2+x+1}{x+1}=\frac{1+1+1}{1+1}=\frac{3}{2} \tag{1.2.14}$$

$$\boxed{\frac{3}{2}} \tag{1.2.15}$$

Exercise 1.2.7

Direct substitution gives $\frac{0}{0}$, so we factor and simplify:

$$\frac{2x^2-9x-5}{x-5}=\frac{(2x+1)(x-5)}{x-5}=2x+1\quad \text{for } x\neq 5 \tag{1.2.16}$$

Therefore:

$$\lim_{x\to 5}\frac{2x^2-9x-5}{x-5}=\lim_{x\to 5}(2x+1)=2(5)+1=11 \tag{1.2.17}$$

$$\boxed{11} \tag{1.2.18}$$

Exercise 1.2.8

Direct substitution gives $\frac{0}{0}$, so we factor and simplify:

$$\frac{x^2+6x+9}{x^2+2x-3}=\frac{(x+3)^2}{(x+3)(x-1)}=\frac{x+3}{x-1}\quad \text{for } x\neq -3 \tag{1.2.19}$$

Therefore:

$$\lim_{x\to -3}\frac{x^2+6x+9}{x^2+2x-3}=\lim_{x\to -3}\frac{x+3}{x-1}=\frac{-3+3}{-3-1}=\frac{0}{-4}=0 \tag{1.2.20}$$

$$\boxed{0} \tag{1.2.21}$$

Exercise 1.2.9

Direct substitution gives $\frac{0}{0}$, so we factor and simplify:

$$\frac{x^2-16}{x^2-8x+16}=\frac{(x-4)(x+4)}{(x-4)^2}=\frac{x+4}{x-4}\quad \text{for } x\neq 4 \tag{1.2.22}$$

As $x\to 4^-$, the denominator approaches 0^-, and as $x\to 4^+$, the denominator approaches 0^+. The one-sided limits are not equal (they approach $-\infty$ and $+\infty$ respectively), so:

$$\boxed{\text{The limit does not exist}} \tag{1.2.23}$$

Exercise 1.2.10

Factor the numerator and denominator:

$$f(x)=\frac{3(x-2)}{(x-2)(x+2)}=\frac{3}{x+2}\quad \text{for } x\neq 2 \tag{1.2.24}$$

The domain is all real numbers except $x=\pm 2$: $\boxed{(-\infty,-2)\cup(-2,2)\cup(2,\infty)}$.

Since $(x-2)$ cancels completely, there is a hole at $x=2$. To find the y-coordinate, evaluate the simplified function at $x=2$:

$$y=\frac{3}{2+2}=\frac{3}{4} \tag{1.2.25}$$

Hole: $\boxed{\left(2,\frac{3}{4}\right)}$

Since $(x+2)$ remains in the denominator, there is a vertical asymptote at:

$$\boxed{x=-2} \tag{1.2.26}$$

Exercise 1.2.11

Factor the numerator and denominator:

$$g(x) = \frac{(x-5)(x+2)}{(x-5)(x+5)} = \frac{x+2}{x+5} \quad \text{for } x \neq 5 \tag{1.2.27}$$

The domain is all real numbers except $x = \pm 5$: $\boxed{(-\infty, -5) \cup (-5, 5) \cup (5, \infty)}$.

Since $(x-5)$ cancels completely, there is a hole at $x = 5$. To find the y-coordinate, evaluate the simplified function at $x = 5$:

$$y = \frac{5+2}{5+5} = \frac{7}{10} \tag{1.2.28}$$

Hole: $\boxed{\left(5, \frac{7}{10}\right)}$

Since $(x+5)$ remains in the denominator, there is a vertical asymptote at:

$$\boxed{x = -5} \tag{1.2.29}$$

Exercise 1.2.12

Factor the numerator and denominator:

$$h(x) = \frac{x(x^2-4)}{x(x-2)} = \frac{x(x-2)(x+2)}{x(x-2)} = x+2 \quad \text{for } x \neq 0, 2 \tag{1.2.30}$$

The domain is all real numbers except $x = 0, 2$: $\boxed{(-\infty, 0) \cup (0, 2) \cup (2, \infty)}$.

Both x and $(x-2)$ cancel completely, so there are holes at $x = 0$ and $x = 2$. Find the y-coordinates using the simplified function:

At $x = 0$: $y = 0 + 2 = 2$

At $x = 2$: $y = 2 + 2 = 4$

Holes: $\boxed{(0, 2)}$ and $\boxed{(2, 4)}$

No factors remain in the denominator, so there are:

$$\boxed{\text{No vertical asymptotes}} \tag{1.2.31}$$

Exercise 1.2.13

Factor the numerator and denominator:

$$2x^2 + 5x - 3 = (2x-1)(x+3) \tag{1.2.32}$$

$$6x^2 - 5x + 1 = (3x-1)(2x-1) \tag{1.2.33}$$

$$k(x) = \frac{(2x-1)(x+3)}{(3x-1)(2x-1)} = \frac{x+3}{3x-1} \quad \text{for } x \neq \frac{1}{2} \tag{1.2.34}$$

The domain is all real numbers except $x = \frac{1}{2}, \frac{1}{3}$: $\boxed{(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{2}, \infty)}$.

Since $(2x-1)$ cancels completely, there is a hole at $x = \frac{1}{2}$. Find the y-coordinate using the simplified function:

$$y = \frac{\frac{1}{2}+3}{3(\frac{1}{2})-1} = \frac{\frac{7}{2}}{\frac{3}{2}-1} = \frac{\frac{7}{2}}{\frac{1}{2}} = 7 \tag{1.2.35}$$

Hole: $\boxed{\left(\frac{1}{2}, 7\right)}$

Since $(3x-1)$ remains in the denominator, set $3x - 1 = 0 \implies x = \frac{1}{3}$. There is a vertical asymptote at:

$$\boxed{x = \frac{1}{3}} \tag{1.2.36}$$

Exercise 1.2.14

Factor the numerator and denominator:

$$p(x) = \frac{x(x^2-x-6)}{x(x^2-9)} = \frac{x(x-3)(x+2)}{x(x-3)(x+3)} = \frac{x+2}{x+3} \quad \text{for } x \neq 0, 3 \tag{1.2.37}$$

The domain is all real numbers except $x = -3, 0, 3$: $\boxed{(-\infty, -3) \cup (-3, 0) \cup (0, 3) \cup (3, \infty)}$.

Both x and $(x-3)$ cancel completely, so there are holes at $x = 0$ and $x = 3$. Find the y-coordinates using the simplified function:

At $x = 0$: $y = \frac{0+2}{0+3} = \frac{2}{3}$

At $x = 3$: $y = \frac{3+2}{3+3} = \frac{5}{6}$

Holes: $\boxed{\left(0, \frac{2}{3}\right)}$ and $\boxed{\left(3, \frac{5}{6}\right)}$

Since $(x+3)$ remains in the denominator, set $x + 3 = 0 \implies x = -3$. There is a vertical asymptote at:

$$\boxed{x = -3} \tag{1.2.38}$$

Exercise 1.2.15

Starting with $f(x) = \frac{1}{x}$:

1. Shift right by 2 units: $\frac{1}{x-2}$

2. Vertically stretch by a factor of 4: $\frac{4}{x-2}$

The domain is all real numbers except $x = 2$: $\boxed{(-\infty, 2) \cup (2, \infty)}$

The vertical asymptote is: $\boxed{x = 2}$

Exercise 1.2.16

Starting with $p(x) = x^2$:

1. Shift left by 3 units: $(x+3)^2$

2. Vertically stretch by a factor of 2: $2(x+3)^2$

3. Shift down by 5 units: $2(x+3)^2-5$

$$\boxed{q(x)=2(x+3)^2-5} \tag{1.2.39}$$

Exercise 1.2.17

When $(x-a)$ appears in both the numerator and denominator, it can be canceled algebraically. If it cancels completely from the denominator after full simplification, then the function is defined (in the simplified form) at $x=a$, but the original function was not. This creates a removable discontinuity—a hole—at $x=a$.

However, if after canceling all common factors, a factor of $(x-a)$ still remains in the denominator, then as $x \to a$, the denominator approaches zero while the numerator approaches a nonzero value. This causes the function to grow without bound ($\pm\infty$), creating a vertical asymptote at $x=a$.

In summary:

- $(x-a)$ cancels completely from denominator **hole**
- $(x-a)$ remains in denominator after simplifying **vertical asymptote**

$$\boxed{\text{Hole if factor cancels completely; vertical asymptote if factor remains in denominator}} \tag{1.2.40}$$

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1.3: Trigonometric Limits and the Squeeze Theorem

Learning Objectives

- Review the graphs and domains of the six trigonometric functions.
- Evaluate limits involving the six trigonometric functions.
- Use the Squeeze Theorem to evaluate limits that oscillate as they approach a value.

In this section, we review the fundamental trigonometric functions, analyze the behavior of oscillating functions, and introduce the Squeeze Theorem to establish essential trigonometric limits.

Question of the Day

How do we find limits of functions that involve trigonometric functions, oscillate, or are trapped between two other functions?

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1.3: Trigonometric Limits

Definition: Trigonometric Functions

On the unit circle $x^2 + y^2 = 1$, an angle θ in standard position determines a point $(x, y) = (\cos\theta, \sin\theta)$. The remaining trigonometric functions are defined in terms of sine and cosine:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

Key Trigonometric Values to Memorize

The following values appear frequently in trigonometric limits:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\cot(\theta)$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec(\theta)$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\csc(\theta)$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

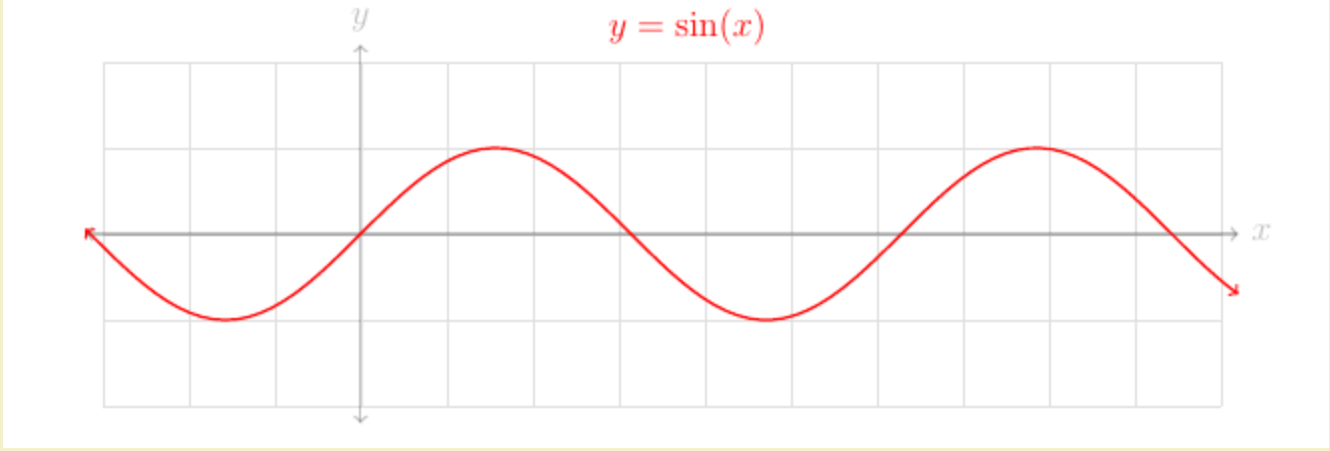
Also note: $\sin(-\theta) = -\sin(\theta)$, $\cos(-\theta) = \cos(\theta)$

Therefore, we have $\tan(-\theta) = -\tan(\theta)$, $\cot(-\theta) = -\cot(\theta)$, $\sec(-\theta) = \sec(\theta)$, and $\csc(-\theta) = -\csc(\theta)$

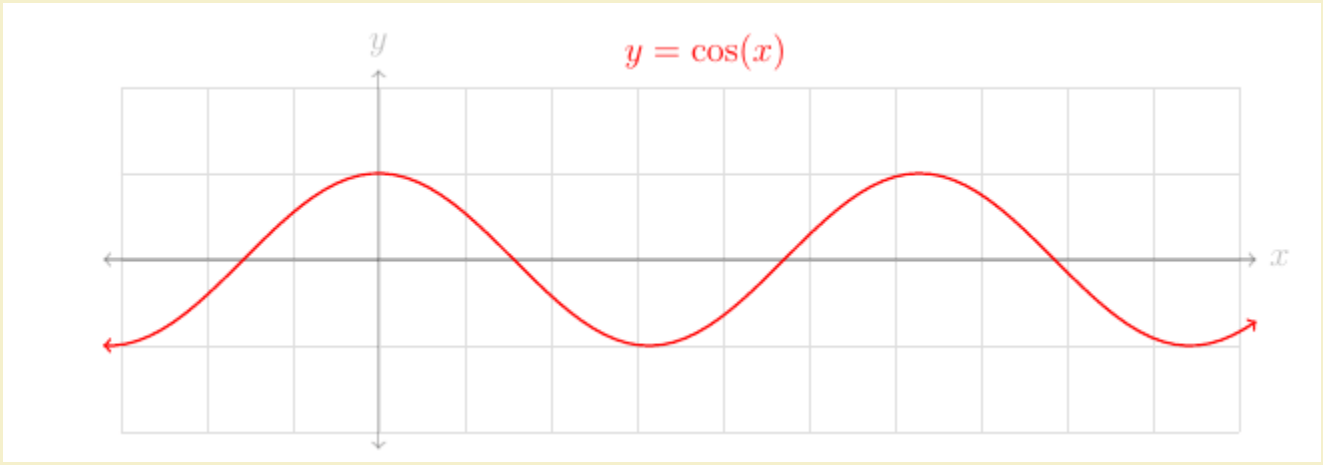
Review: The Six Trigonometric Functions

Each of these functions' domains, ranges, and graphs are provided below:

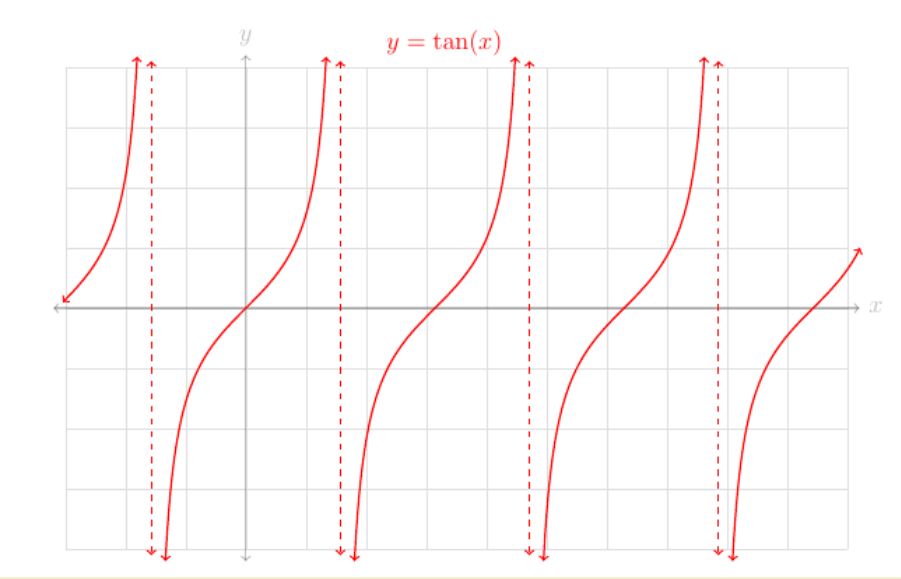
$f(x) = \sin(x)$ has domain $(-\infty, \infty)$ and range $[-1, 1]$



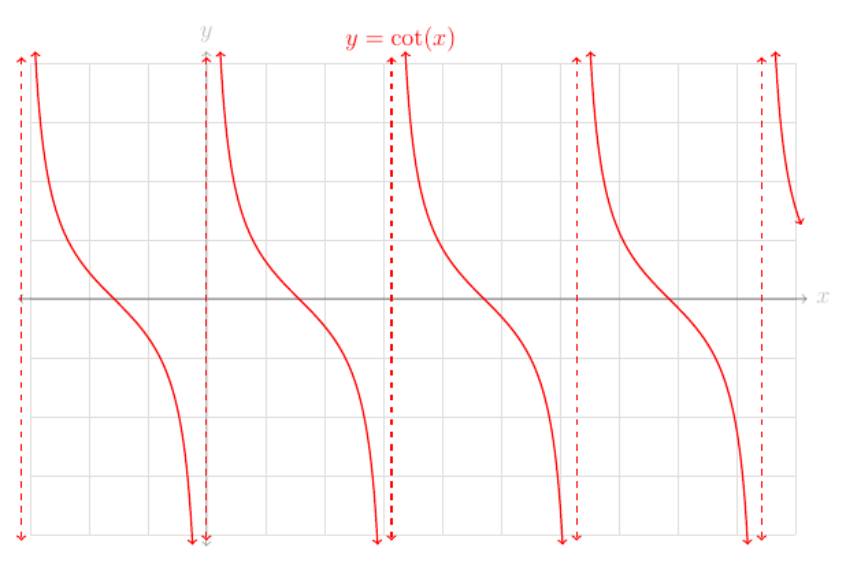
$f(x) = \cos(x)$ has domain $(-\infty, \infty)$ and range $[-1, 1]$



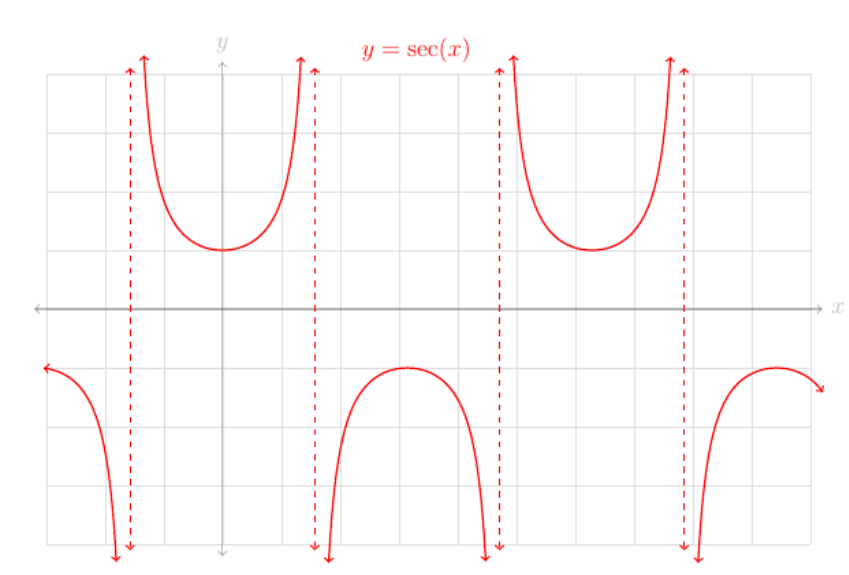
$f(x) = \tan(x)$ has domain $\left\{x \mid x \neq \frac{(2k+1)\pi}{2} \text{ for all integers } k\right\}$ and range $(-\infty, \infty)$



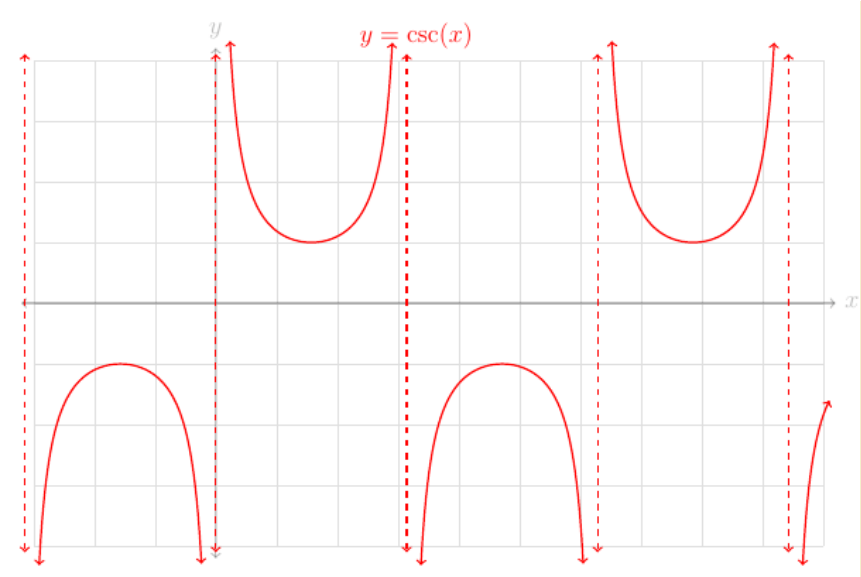
$f(x) = \cot(x)$ has domain $\{x \mid x \neq k\pi \text{ for all integers } k\}$ and range $(-\infty, \infty)$



$f(x) = \sec(x)$ has domain $\left\{x \mid x \neq \dfrac{(2k+1)\pi}{2} \text{ for all integers } k\right\}$ and range $(-\infty, -1] \cup [1, \infty)$



$f(x) = \csc(x)$ has domain $\{x \mid x \neq k\pi \text{ for all integers } k\}$ and range $(-\infty, -1] \cup [1, \infty)$



Example 1.3.1: Basic Trigonometric Limits

Evaluate each limit using direct substitution or visual inspection of basic trigonometric graphs:

(a) $\lim_{x \to -\pi} \sin(x)$

(b) $\lim_{x \to \pi} \cos(x)$

(c) $\lim_{x \to 0^+} \tan(x)$

(d) $\lim_{x \to \pi^-} \cot(x)$

(e) $\lim_{x \to \frac{\pi}{2}^+} \sec(x)$

Solution:

(a) Since $\sin(x)$ is defined everywhere, the limit matches the function value:

$$\lim_{x \to -\pi} \sin(x) = \sin(-\pi) = \boxed{0}$$

(b) Since $\cos(x)$ is defined everywhere, the limit matches the function value:

$$\lim_{x \to \pi} \cos(x) = \cos(\pi) = \boxed{-1}$$

(c) Since $\tan(x)$ is defined at zero, the limit matches the function value:

$$\lim_{x \to 0^+} \tan(x) = \tan(0) = \boxed{0}$$

(d) As $x \to \pi^-$, $\sin(x) \to 0^+$ while $\cos(x) \to -1$:

$$\lim_{x \to \pi^-} \cot(x) = \lim_{x \to \pi^-} \frac{\cos(x)}{\sin(x)} = \frac{-1}{0^+} = \boxed{-\infty}$$

(e) As $x \to \dfrac{\pi}{2}^+$, $\cos(x) \to 0^-$:

$$\lim_{x\to\frac{\pi}{2}^+}\sec(x)=\lim_{x\to\frac{\pi}{2}^+}\frac{1}{\cos(x)}=\frac{1}{0^-}=\boxed{-\infty}$$

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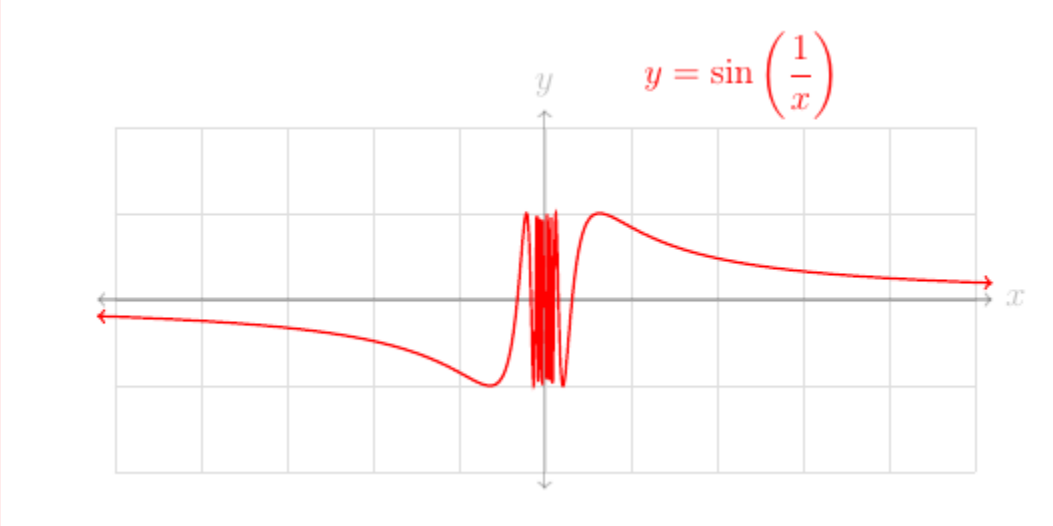
1.3: The Squeeze Theorem

Now that we can safely take the limits of any vanilla trigonometric function, we explore some tougher variants.

Warning: Not all One-Sided Limits Exist

Consider the limit $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$

The graph of this function is provided below:

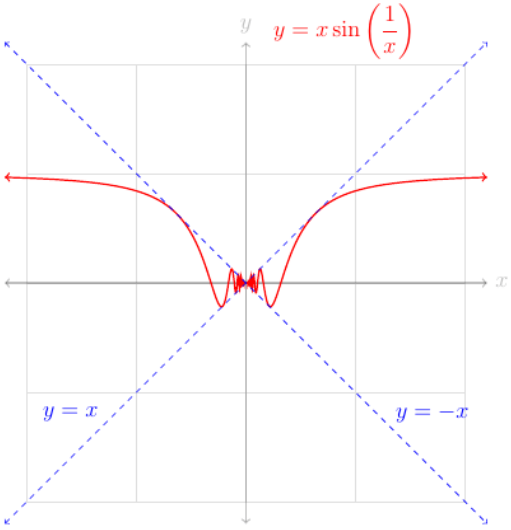


Notice that as x approaches zero from the right, the function values do not apprach a single value, but instead oscillate faster and faster between all values in the interval $[-1, 1]$. Because of this, we say that $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right) = \text{DNE}$

Since the function $f(x) = \sin\left(\frac{1}{x}\right)$ does not have a limit as $x \to 0^+$, we cannot use our limit laws when evaluating something like

$$\lim_{x\to 0^+} x \sin\left(\frac{1}{x}\right)$$

A quick look at the graph of $y = x \sin\left(\frac{1}{x}\right)$ tells us that this limit should in fact be zero:



To help us determine when this happens, we introduce a powerful theorem.

Theorem: The Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ for all x near a (except possibly at a) and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} f(x) = L$$

This theorem is especially useful when a function is multiplied by a bounded factor that oscillates but is "squeezed" to zero by a factor that tends to zero.

Illustration 1.3.1: Applying the Squeeze Theorem

Evaluate $\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right)$

Since the sine function is bounded between -1 and 1 for all real arguments, we have for all $x \neq 0$:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Multiplying the inequality by x (which is strictly positive for $x \geq 0$):

$$-x \leq x \sin\left(\frac{1}{x}\right) \leq x$$

Evaluating the limits of the bounding functions as $x \to 0$:

$$\lim_{x \to 0^=} -x = 0 \quad \text{and} \quad \lim_{x \to 0^+} x = 0$$

By the Squeeze Theorem, $\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right) = 0$

Example 1.3.2: The Squeeze Theorem with Absolute Value

Evaluate $\lim_{x\to 0}\sqrt{|x|}\cos\left(\frac{2}{x}\right)$

Solution:

Since $-1 \le \cos\left(\frac{2}{x}\right) \le 1$ for all $x \neq 0$:

$$-\sqrt{|x|} \le \sqrt{|x|}\cos\left(\frac{2}{x}\right) \le \sqrt{|x|}$$

As $x \to 0$, both $-\sqrt{|x|}$ and $\sqrt{|x|}$ approach 0. By the Squeeze Theorem:

$$\lim_{x\to 0}\sqrt{|x|}\cos\left(\frac{2}{x}\right) = \boxed{0}$$

Example 1.3.3: The Squeeze Theorem

Evaluate $\lim_{x\to 1}\dfrac{(x-1)^2\sin\left(\frac{\pi}{x-1}\right)+6}{2x^2+1}$

Solution:

Since $-1 \le \sin\left(\frac{\pi}{x-1}\right) \le 1$ for all $x \neq 1$, multiply through by $(x-1)^2 \ge 0$:

$$-(x-1)^2 \le (x-1)^2\sin\left(\frac{\pi}{x-1}\right) \le (x-1)^2$$

Add 6 to all sides:

$$6-(x-1)^2 \le (x-1)^2\sin\left(\frac{\pi}{x-1}\right)+6 \le 6+(x-1)^2$$

Divide by $2x^2+1$ (which is always positive):

$$\frac{6-(x-1)^2}{2x^2+1} \le \frac{(x-1)^2\sin\left(\frac{\pi}{x-1}\right)+6}{2x^2+1} \le \frac{6+(x-1)^2}{2x^2+1}$$

Now we take the limit of each side as $x \to 1$:

$$\lim_{x\to 1}\frac{6-(x-1)^2}{2x^2+1} = \frac{6-0}{2+1} = \frac{6}{3} = 2$$

$$\lim_{x\to 1}\frac{6+(x-1)^2}{2x^2+1} = \frac{6+0}{2+1} = \frac{6}{3} = 2$$

Since both bounds approach 2, by the Squeeze Theorem:

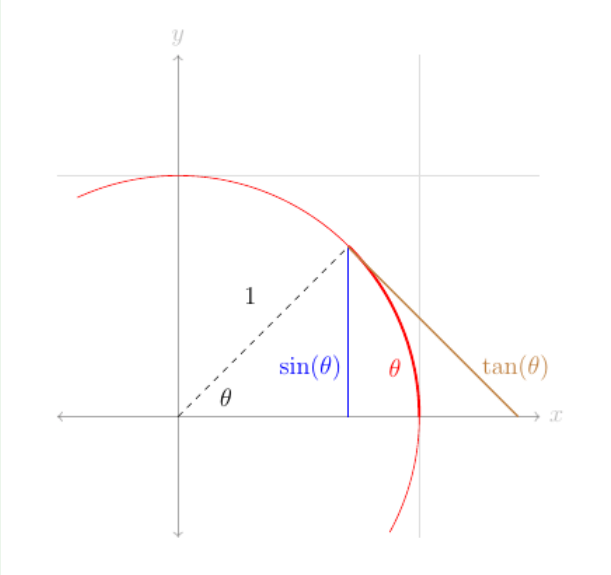
$$\lim_{x\to 1}\frac{(x-1)^2\sin\left(\frac{\pi}{x-1}\right)+6}{2x^2+1} = \boxed{2}$$

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1.3: The Fundamental Trigonometric Limit

Proof of the Fundamental Trigonometric Limit

Consider the part of the unit circle with $0 < \theta < \frac{\pi}{2}$:



Comparing the lengths of the three curves in the image above we get:

$$\sin(\theta) \leq \theta \leq \tan(\theta)$$

Dividing all sides by $\sin(\theta)$ and then taking reciprocals gives:

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$$

As $\theta \to 0^+$, $\cos(\theta) \to 1$, so by the Squeeze Theorem, $\frac{\sin(\theta)}{\theta} \to 1$.

The same argument works for $\theta \to 0^-$ since $\frac{\sin(\theta)}{\theta}$ is an even function.

Therefore, we have an incredibly useful limit:

$$\lim_{x \to 0} \frac{\sin(\theta)}{\theta} = 1$$

Example 1.3.4: Using the Fundamental Trigonometric Limit

Evaluate each of the following trigonometric limits:

(a) $\lim_{x \to 0} \frac{\sin(4x)}{2x}$

(b) $\lim_{x \to 0} \frac{\tan(6x)}{2x}$

(c) $\lim_{x\to 0}\dfrac{\sin(8x)}{\tan(2x)}$

(d) $\lim_{x\to 0}\dfrac{\sin^2(6x)}{3x^2}$

(e) $\lim_{x\to 0}\dfrac{1-\cos(x)}{x^2}$

Solution:

(a) Rewrite the denominator to match the argument $2x$:

$$\lim_{x\to 0}\frac{\sin(4x)}{2x}=\lim_{x\to 0}\frac{\sin(4x)}{4x}\cdot\frac{4x}{2x}=\left(\lim_{x\to 0}\frac{\sin(4x)}{4x}\right)\cdot\left(\lim_{x\to 0}2\right)=1\cdot 2=\boxed{2}$$

(b) Express $\tan(6x)$ as $\dfrac{\sin(6x)}{\cos(6x)}$:

$$\lim_{x\to 0}\frac{\tan(6x)}{2x}=\lim_{x\to 0}\frac{\sin(6x)}{6x}\cdot\frac{6x}{2x\cos(6x)}=\left(\lim_{x\to 0}\frac{\sin(6x)}{6x}\right)\cdot\left(\lim_{x\to 0}\frac{3}{\cos(6x)}\right)=1\cdot\frac{3}{\cos(0)}=\boxed{3}$$

(c) Rewrite the fraction in the limit to match the arguments:

$$\lim_{x\to 0}\frac{\sin(8x)}{\tan(2x)}=\lim_{x\to 0}\frac{\sin(8x)}{8x}\cdot\frac{2x}{\sin(2x)}\cdot\frac{8x\cos(2x)}{2x}=\left(\lim_{x\to 0}\frac{\sin(8x)}{8x}\right)\cdot\left(\lim_{x\to 0}\frac{2x}{\sin(2x)}\right)\cdot\left(\lim_{x\to 0}\lim_{x\to 0}4\cos(2x)\right)=1\cdot 1\cdot 4\cos(0)=\boxed{4}$$

(d) Rewrite the fraction in the limit to match the argument:

$$\lim_{x\to 0}\frac{\sin^2(6x)}{3x^2}=\lim_{x\to 0}\frac{\sin(6x)}{6x}\cdot\frac{\sin(6x)}{6x}\cdot\frac{6x\cdot 6x}{3x^2}=\left(\lim_{x\to 0}\frac{\sin(6x)}{6x}\right)^2\cdot\left(\lim_{x\to 0}12\right)=1^2\cdot 12=\boxed{12}$$

(e) Use the identity $1-\cos^2(x)=\sin^2(x)$ by multiplying the fraction by $1+\cos(x)$:

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}\cdot\frac{1+\cos(x)}{1+\cos(x)}=\lim_{x\to 0}\frac{1-\cos^2(x)}{x^2\,(1+\cos(x))}=\lim_{x\to 0}\frac{\sin^2(x)}{x^2\,(1+\cos(x))}=\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{\sin(x)}{x}\cdot\frac{1}{1+\cos(x)}$$
$$=\left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)^2\cdot\left(\lim_{x\to 0}\frac{1}{1+\cos(x)}\right)=1^2\cdot\frac{1}{1+\cos(0)}=\boxed{\frac{1}{2}}$$

Remember, this trick only works for limits of the form $\dfrac{0}{0}$.

For instance, $\lim_{x\to 7}\dfrac{\sin(x)}{x}=\dfrac{\sin(7)}{7}\neq 1$ and $\lim_{x\to \pi}\dfrac{\sin(x)}{x}=\dfrac{\sin(\pi)}{\pi}=0\neq 1$

Summary: Trigonometric Limit Strategies

When evaluating trigonometric limits:

1. **Direct substitution** if the function is defined at the point.
2. **Consider the graph** of the trigonometric function if there is a vertical asymptote at the point to determine $\pm\infty$.
3. **Use the Squeeze Theorem** for functions involving oscillating factors multiplied by a term that tends to zero.
4. **Manipulate to use** $\lim_{x\to 0}\dfrac{\sin(\theta)}{\theta}=1$ by multiplying numerator and denominator to match arguments.

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1.3: Practice Set

Exercise 1.3.1

Evaluate each of the following limits:

(a) $\lim_{x \to \frac{\pi}{3}} (\sin(x) + \cos(x))$

(b) $\lim_{x \to \frac{\pi}{4}^+} \tan(x)$

(c) $\lim_{x \to 0^+} \csc(x)$

(d) $\lim_{x \to \pi^-} \cot(x)$

(e) $\lim_{x \to \frac{\pi}{2}^-} \tan(x) \sec(x)$

(f) $\lim_{x \to \frac{\pi}{2}^-} (\cot(x) - \csc(x))$

Answer available at the end of this section.

Exercise 1.3.2

Use the Squeeze Theorem to evaluate the following limits:

(a) $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right)$

(b) $\lim_{x \to 0} \sqrt{|x|} \sin\left(\frac{1}{x^2}\right)$

(c) $\lim_{x \to 0^+} \frac{x \sin\left(\frac{1}{x}\right) + 3}{x^2 + 5}$

Answer available at the end of this section.

Exercise 1.3.3

Evaluate each of the following limits by using the Fundamental Trigonometric Limit:

(a) $\lim_{x \to 0} \frac{\sin(4x)}{\sin(2x)}$

(b) $\lim_{x \to 0} \frac{\sin(7x)}{3x}$

(c) $\lim_{x \to 0} \frac{\tan(6x)}{x \cos(3x)}$

(d) $\lim_{x \to 0} \frac{\sin(3x)}{\tan(9x)}$

(e) $\lim_{x\to 0} \dfrac{\sin^2(5x)}{x^2}$

(f) $\lim_{x\to 0} \dfrac{x\sin(3x)}{\tan(4x^2)}$

(g) $\lim_{x\to 0} \dfrac{1-\cos(2x)}{x}$

Answer available at the end of this section.

Exercise 1.3.4

Find the domain and any holes/vertical asymptotes of each function below:

(a) $f(x) = \dfrac{x}{\cos(x)-1}$

(b) $g(x) = \dfrac{\sin(x-1)}{x^2-2x+1}$

(c) $h(x) = \dfrac{\sin(\pi x)}{x^2-1}$

Answer available at the end of this section.

Exercise 1.3.5

Given that $2-x^2 \le f(x) \le 2\cos(x)$ for all $x \neq 0$, find $\lim_{x\to 0} f(x)$

Answer available at the end of this section.

Answers

Exercise 1.3.1

(a) At $x = \dfrac{\pi}{3}$, both $\sin(x)$ and $\cos(x)$ are defined, so using direct substitution:

$$\lim_{x\to\frac{\pi}{3}} (\sin(x)+\cos(x)) = \sin\left(\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}+\frac{1}{2} = \frac{\sqrt{3}+1}{2}$$

(b) At $x = \frac{\pi}{4}$, $\tan(x)$ is defined, so using direct substitution:

$$\lim_{x\to\frac{\pi}{4}^+} \tan(x) = \tan\left(\frac{\pi}{4}\right) = 1$$

(c) At $x = 0$, $\csc(x)$ is not defined.
Since $\csc(x) = \frac{1}{\sin(x)}$ and as $x \to 0^+$, $\sin(x) \to 0^+$, we have:

$$\lim_{x\to 0^+} \csc(x) = \frac{1}{0^+} = +\infty$$

(d) At $x = \pi$, $\cot(x)$ is not defined.
Since $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and as $x \to \pi^-$, $\cos(x) \to -1$ and $\sin(x) \to 0^+$, we have:

$$\lim_{x\to\pi^-} \cot(x) = \frac{-1}{0^+} = -\infty$$

(e) At $x = \frac{\pi}{2}$, both $\tan(x)$ and $\sec(x)$ are not defined. Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\sec(x) = \frac{1}{\cos(x)}$:

$$\tan(x)\sec(x) = \frac{\sin(x)}{\cos^2(x)}$$

As $x \to \frac{\pi}{2}^-$, $\sin(x) \to 1$ and $\cos(x) \to 0^+$, so $\cos^2(x) \to 0^+$. Therefore:

$$\lim_{x\to\frac{\pi}{2}^-} \tan(x)\sec(x) = \frac{1}{0^+} = +\infty$$

(f) At $x = \frac{\pi}{2}$, both $\cot(x)$ and $\csc(x)$ are not defined. Rewrite using $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and $\csc(x) = \frac{1}{\sin(x)}$:

$$\cot(x) - \csc(x) = \frac{\cos(x)}{\sin(x)} - \frac{1}{\sin(x)} = \frac{\cos(x) - 1}{\sin(x)}$$

As $x \to \frac{\pi}{2}^-$, $\cos(x) \to 0$ and $\sin(x) \to 1$. Using direct substitution:

$$\lim_{x \to \frac{\pi}{2}^-} (\cot(x) - \csc(x)) = \frac{0 - 1}{1} = -1$$

Exercise 1.3.2

(a) We know that $-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$ for all $x \neq 0$.

Multiply by $x^4 \geq 0$:

$$-x^4 \leq x^4 \cos\left(\frac{2}{x}\right) \leq x^4$$

Since $\lim_{x \to 0}(-x^4) = 0$ and $\lim_{x \to 0} x^4 = 0$, by the Squeeze Theorem:

$$\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$$

(b) We know that $-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$ for all $x \neq 0$.

Multiply by $\sqrt{|x|} \geq 0$:

$$-\sqrt{|x|} \leq \sqrt{|x|} \sin\left(\frac{1}{x^2}\right) \leq \sqrt{|x|}$$

Since $\lim_{x \to 0}(-\sqrt{|x|}) = 0$ and $\lim_{x \to 0} \sqrt{|x|} = 0$, by the Squeeze Theorem:

$$\lim_{x \to 0} \sqrt{|x|} \sin\left(\frac{1}{x^2}\right) = 0$$

(c) We know that $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, so $-x \leq x \sin\left(\frac{1}{x}\right) \leq x$.

Add 3 to all parts:

$$3 - x \leq x \sin\left(\frac{1}{x}\right) + 3 \leq 3 + x$$

Divide by $x^2 + 5 > 0$:

$$\frac{3 - x}{x^2 + 5} \leq \frac{x \sin\left(\frac{1}{x}\right) + 3}{x^2 + 5} \leq \frac{3 + x}{x^2 + 5}$$

Since $\lim_{x \to 0^+} \frac{3 - x}{x^2 + 5} = \frac{3}{5}$ and $\lim_{x \to 0^+} \frac{3 + x}{x^2 + 5} = \frac{3}{5}$, by the Squeeze Theorem:

$$\lim_{x \to 0^+} \frac{x \sin\left(\frac{1}{x}\right) + 3}{x^2 + 5} = \frac{3}{5}$$

Exercise 1.3.3

(a) We rewrite the function:

$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(2x)} = \lim_{x \to 0} \frac{\sin(4x)}{4x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{4x}{2x} = 1 \cdot 1 \cdot 2 = 2$$

(b) We rewrite the function:

$$\lim_{x \to 0} \frac{\sin(7x)}{3x} = \lim_{x \to 0} \frac{\sin(7x)}{7x} \cdot \frac{7}{3} = 1 \cdot \frac{7}{3} = \frac{7}{3}$$

(c) We rewrite the function:

$$\lim_{x \to 0} \frac{\tan(6x)}{x \cos(3x)} = \lim_{x \to 0} \frac{\sin(6x)}{x \cos(6x) \cos(3x)} = \lim_{x \to 0} \frac{\sin(6x)}{6x} \cdot \frac{6}{\cos(6x) \cos(3x)} = 1 \cdot \frac{6}{1 \cdot 1} = 6$$

(d) We rewrite the function:

$$\lim_{x \to 0} \frac{\sin(3x)}{\tan(9x)} = \lim_{x \to 0} \frac{\sin(3x)}{\frac{\sin(9x)}{\cos(9x)}} = \lim_{x \to 0} \frac{\sin(3x) \cos(9x)}{\sin(9x)}$$

$$= \lim_{x \to 0} \left(\frac{\sin(3x)}{3x} \cdot \frac{9x}{\sin(9x)} \cdot \frac{3x}{9x} \cdot \cos(9x) \right) = 1 \cdot 1 \cdot \frac{1}{3} \cdot 1 = \frac{1}{3}$$

(e) We rewrite the function:

$$\lim_{x \to 0} \frac{\sin^2(5x)}{x^2} = \lim_{x \to 0} \left(\frac{\sin(5x)}{x} \right)^2 = \lim_{x \to 0} \left(\frac{\sin(5x)}{5x} \cdot 5 \right)^2 = (1 \cdot 5)^2 = 25$$

(f) We rewrite the function:

$$\lim_{x \to 0} \frac{x \sin(3x)}{\tan(4x^2)} = \lim_{x \to 0} \frac{x \sin(3x)}{\frac{\sin(4x^2)}{\cos(4x^2)}} = \lim_{x \to 0} \frac{x \sin(3x) \cos(4x^2)}{\sin(4x^2)}$$

$$= \lim_{x \to 0} \left(\frac{\sin(3x)}{3x} \cdot \frac{4x^2}{\sin(4x^2)} \cdot \frac{x \cdot 3x}{4x^2} \cdot \cos(4x^2) \right) = 1 \cdot 1 \cdot \frac{3}{4} \cdot 1 = \frac{3}{4}$$

(g) We rewrite the function using the identity $1 - \cos(2x) = 2\sin^2(x)$:

$$\lim_{x \to 0} \frac{1 - \cos(2x)}{x} = \lim_{x \to 0} \frac{2\sin^2(x)}{x} = \lim_{x \to 0} 2 \cdot \frac{\sin(x)}{x} \cdot \sin(x) = 2 \cdot 1 \cdot 0 = 0$$

Alternatively, use $\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0$ with $h = 2x$:

$$\lim_{x\to 0}\frac{1-\cos(2x)}{x}=\lim_{x\to 0}\frac{1-\cos(2x)}{2x}\cdot 2=0\cdot 2=0$$

Exercise 1.3.4

(a) The denominator is zero when $\cos(x)-1=0 \implies \cos(x)=1 \implies x=2\pi k$ for any integer k.

Domain: $\{x\in\mathbb{R}\mid x\neq 2\pi k,\ k\in\mathbb{Z}\}$ or $\mathbb{R}\setminus\{2\pi k : k\in\mathbb{Z}\}$.

To determine holes vs. vertical asymptotes, use the identity $\cos(x)-1=-2\sin^2\left(\frac{x}{2}\right)$:

$$f(x)=\frac{x}{-2\sin^2\left(\frac{x}{2}\right)}$$

At $x=0$, the numerator is also 0. Using $\sin\left(\frac{x}{2}\right)\approx\frac{x}{2}$ for small x:

$$f(x)\approx\frac{x}{-2\left(\frac{x}{2}\right)^2}=\frac{x}{-\frac{x^2}{2}}=-\frac{2}{x}$$

So as $x\to 0$, $f(x)\to\pm\infty$. Thus there is a **vertical asymptote** at $x=0$ (and no hole).

At $x=2\pi k$ for $k\neq 0$, the numerator is $2\pi k\neq 0$ while the denominator is 0. Thus there are **vertical asymptotes** at $x=2\pi k$ for all integers k.

Therefore: No holes. Vertical asymptotes at $x=2\pi k$ for all integers k.

(b) Factor the denominator:

$$x^2-2x+1=(x-1)^2$$

Domain: $x\neq 1$.

$$g(x)=\frac{\sin(x-1)}{(x-1)^2}$$

Let $u=x-1$. As $x\to 1$, $u\to 0$, and:

$$\lim_{x\to 1}g(x)=\lim_{u\to 0}\frac{\sin(u)}{u^2}=\lim_{u\to 0}\frac{\sin(u)}{u}\cdot\frac{1}{u}$$

Since $\frac{\sin(u)}{u}\to 1$ and $\frac{1}{u}\to\pm\infty$ depending on the side, the limit does not exist (it goes to $+\infty$ on one side and $-\infty$ on the other).

Therefore: No holes. Vertical asymptote at $x=1$.

(c) Factor the denominator:

$$x^2-1=(x-1)(x+1)$$

Domain: $x\neq\pm 1$.

$$h(x)=\frac{\sin(\pi x)}{(x-1)(x+1)}$$

At $x=1$: $\sin(\pi)=0$, so both numerator and denominator are 0. Let $u=x-1$. Then $x=u+1$ and $x+1=u+2$:

$$\lim_{x\to 1}h(x)=\lim_{u\to 0}\frac{\sin(\pi(u+1))}{u(u+2)}=\lim_{u\to 0}\frac{\sin(\pi u+\pi)}{u(u+2)}=\lim_{u\to 0}\frac{-\sin(\pi u)}{u(u+2)}$$
$$=\lim_{u\to 0}\left(-\frac{\sin(\pi u)}{\pi u}\cdot\frac{\pi}{u+2}\right)=-1\cdot\frac{\pi}{2}=-\frac{\pi}{2}$$

Since the limit exists and is finite, there is a **hole** at $x=1$ with y-value $-\frac{\pi}{2}$.

At $x=-1$: $\sin(-\pi)=0$, so both numerator and denominator are 0. Let $u=x+1$. Then $x=u-1$ and $x-1=u-2$:

$$\lim_{x\to -1}h(x)=\lim_{u\to 0}\frac{\sin(\pi(u-1))}{u(u-2)}=\lim_{u\to 0}\frac{\sin(\pi u-\pi)}{u(u-2)}=\lim_{u\to 0}\frac{-\sin(\pi u)}{u(u-2)}$$
$$=\lim_{u\to 0}\left(-\frac{\sin(\pi u)}{\pi u}\cdot\frac{\pi}{u-2}\right)=-1\cdot\frac{\pi}{-2}=\frac{\pi}{2}$$

Since the limit exists and is finite, there is a **hole** at $x=-1$ with y-value $\frac{\pi}{2}$.

Therefore: Holes at $x=1$ (y = $-\frac{\pi}{2}$) and $x=-1$ (y = $\frac{\pi}{2}$). No vertical asymptotes.

Exercise 1.3.5

Since $\lim_{x\to 0}(2-x^2)=2$ and $\lim_{x\to 0}2\cos(x)=2\cos(0)=2$,

by the Squeeze Theorem:

$$\lim_{x\to 0}f(x)=2$$

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1.4: Piece-wise Functions, Continuity, and the Intermediate Value Theorem

Learning Objectives

- Review how to evaluate and sketch piecewise functions.
- Define continuity and the four types of discontinuities.
- Understand continuity at a point and on an interval.
- Use the Intermediate Value Theorem to prove the existence of solutions to equations involving continuous functions.

In this section, we study piecewise functions, formalize the concept of continuity, explore types of discontinuities, and apply the Intermediate Value Theorem.

Question of the Day

What does it mean for a function to interact well with the limit?

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1.4: Piece-wise Functions

Definition: Piecewise Functions

A **piecewise function** is a function that is defined by different formulas or expressions on different intervals of its domain. It is written as:

$$f(x) = \begin{cases} \text{formula}_1, & \text{if } x \text{ satisfies condition}_1 \\ \text{formula}_2, & \text{if } x \text{ satisfies condition}_2 \\ \vdots & \vdots \\ \text{formula}_n, & \text{if } x \text{ satisfies condition}_n \end{cases}$$

The domain of a piecewise function is the union of all intervals specified in the conditions.

Example 1.4.1: Evaluating a Piecewise Function

Consider the piecewise function:

$$f(x) = \begin{cases} x^2 + 1, & \text{if } x < 1 \\ 2x + 3, & \text{if } x > 1 \end{cases}$$

Evaluate the following:

(a) $f(0)$
(b) $f(2)$
(c) $f(-3)$
(d) $f(1)$

Solution:

For each input, we determine which condition it satisfies and use the corresponding formula.

(a) $x = 0$. Since $0 \leq 1$, we use $f(x) = x^2 + 1$:

$$f(0) = (0)^2 + 1 = \boxed{1}$$

(b) $x = 2$. Since $2 > 1$, we use $f(x) = 2x + 3$:

$$f(2) = 2(2) + 3 = \boxed{7}$$

(c) $x = -3$. Since $-3 \leq 1$, we use $f(x) = x^2 + 1$:

$$f(-3) = (-3)^2 + 1 = \boxed{10}$$

(d) $x = 1$. Since 1 does not satisfy either of $x > 1$ or $x < 1$, the function does not accept it in its domain:

$$f(1) = \boxed{\text{undefined}}$$

Important Note on Evaluation

When evaluating piecewise functions, **always check the condition** first. Pay close attention to whether the boundary value belongs to the left piece (using $\leq$ or $\geq$) or the right piece (using $>$ or $<$).

Example 1.4.2: Sketching a Piecewise Function

Sketch the graph of:

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ 2, & \text{if } 0 \leq x < 3 \\ x - 1, & \text{if } x \geq 3 \end{cases}$$

Solution:

We sketch each piece on its specified interval:

Piece 1: $f(x) = -x$ for $x < 0$

- This is a line with slope -1 passing through the origin.
- For $x = 0$, the point would be $(0, 0)$, but since $x < 0$, we use an **open circle** at $(0, 0)$.
- Plot points: $(-1, 1)$, $(-2, 2)$, etc., and draw the line extending left.

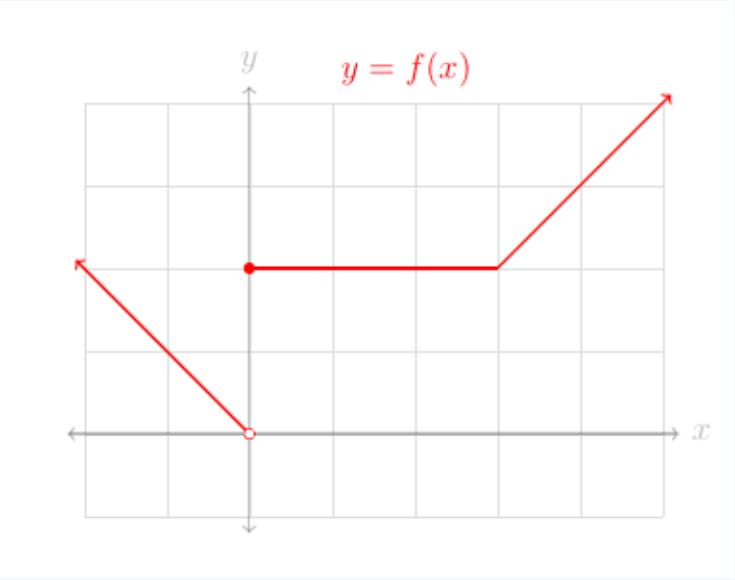
Piece 2: $f(x) = 2$ for $0 \leq x < 3$

- This is a horizontal line at $y = 2$.
- At $x = 0$, the point is $(0, 2)$ with a **closed circle** (since $0 \leq x$).
- At $x = 3$, the point would be $(3, 2)$, but since $x < 3$, we use an **open circle** at $(3, 2)$.
- Draw the horizontal segment from $x = 0$ to $x = 3$.

Piece 3: $f(x) = x - 1$ for $x \geq 3$

- This is a line with slope 1 and y-intercept -1.
- At $x = 3$, $f(3) = 2$. Since $x \geq 3$, we use a **closed circle** at $(3, 2)$.
- Plot points: $(4, 3)$, $(5, 4)$, etc., and draw the line extending right.

The resulting graph should look like three line segments with appropriate open/closed endpoints.



Strategy: Evaluating Limits of Piecewise Functions

To evaluate limits of piecewise functions at a point $x = a$:

1. **Identify which piece(s) apply** as x approaches a from the left and from the right.
2. **Evaluate the one-sided limits** using the appropriate formula(s):
   - If a lies strictly inside one interval (not at a boundary), the limit is simply the value of that piece at a (if it's continuous there).
   - If a is a **boundary point**, you must compute both one-sided limits separately.
3. **Compare the one-sided limits**:
   - If both one-sided limits exist and are equal, the two-sided limit exists and equals that common value.
   - If they are not equal, or if either one does not exist, the two-sided limit does **not** exist.

Example 1.4.3: Limits of a Piecewise Function

Consider the piecewise function from Example 2:

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ 2, & \text{if } 0 \le x < 3 \\ x - 1, & \text{if } x \ge 3 \end{cases}$$

Evaluate the following limits:

(a) $\lim_{x \to -1} f(x)$
(b) $\lim_{x \to 0} f(x)$
(c) $\lim_{x \to 2} f(x)$
(d) $\lim_{x \to 3} f(x)$

Solution:

(a) $x \to -1$: Since $-1 < 0$, both sides of $x = -1$ are on the first piece $f(x) = -x$:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} -x = -(-1) = \boxed{1}$$

(b) $x \to 0$: This is a boundary point, so each side of $x = 0$ uses a different rule:

Left-hand limit: As $x \to 0^-$, $x < 0$, so use $f(x) = -x$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = -0 = 0$$

Right-hand limit: As $x \to 0^+$, $0 \le x < 3$, so use $f(x) = 2$:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 = 2$$

Since $0 \neq 2$, the two-sided limit does **not** exist:

$$\lim_{x \to 0} f(x) = \text{DNE}$$

(c) $x \to 2$: Since 2 is strictly inside the interval $0 \le x < 3$, both sides will use $f(x) = 2$:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} 2 = \boxed{2}$$

(d) $x \to 3$: This is also a boundary point, so each side of $x = 3$ uses a different rule:

Left-hand limit: As $x \to 3^-$, $0 \le x < 3$, so use $f(x) = 2$:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 2 = 2$$

Right-hand limit: As $x \to 3^+$, $x \ge 3$, so use $f(x) = x - 1$:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 1) = 3 - 1 = 2$$

Since both one-sided limits equal 2, the two-sided limit exists:

$$\lim_{x \to 3} f(x) = \boxed{2}$$

Common Pitfalls with Piecewise Limits

When evaluating limits at boundary points, be careful:

- **Do not** assume the limit equals the function value at the boundary (it may not be continuous there).
- **Always check both sides** independently—one-sided limits can differ even if the function is defined at the point.
- If the boundary condition uses $\leq$ on one side, that side's formula **includes** the endpoint, but the limit only cares about values *near* the endpoint, not at it.

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1.4: Continuity

Definition: Formal Continuity

A function f is **continuous at** $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

Implicitly, this requires three conditions to hold simultaneously:

1. $f(a)$ is defined (meaning a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists and is finite.
3. $\lim_{x \to a} f(x) = f(a)$.

If any of these conditions fail, f is **discontinuous** at $x = a$.

Continuity Check: The Three Conditions

When testing whether a function is continuous at a point, always check the three conditions in order:

1. **Does** $f(a)$ **exist?** If not, the function is discontinuous.
2. **Does** $\lim_{x \to a} f(x)$ **exist?** If not, the function is discontinuous.
3. **Does** $\lim_{x \to a} f(x) = f(a)$**?** If not, the function is discontinuous.

Only if all three conditions are satisfied is the function continuous at $x = a$.

Example 1.4.4: Testing Continuity at a Point

Determine whether each function is continuous at the given point:

(a) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & x \neq 3 \\ 6 & x = 3 \end{cases}$ at $x = 3$

(b) $g(x) = \begin{cases} x^2 + 1 & x < 2 \\ 7 & x = 2 \\ 2x + 1 & x > 2 \end{cases}$ at $x = 2$

(c) $h(x) = \dfrac{1}{x - 1}$ at $x = 1$

Solution:

(a) Check the three conditions at $x = 3$:

1. $f(3) = 6$, so $f(3)$ exists.

2. $\lim_{x \to 3} f(x) = \lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \lim_{x \to 3} \dfrac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 3} (x + 3) = 6$, so the limit exists.

3. $\lim_{x \to 3} f(x) = 6 = f(3)$, so the limit equals the function value.

All three conditions are satisfied: $\boxed{\text{Continuous at } x = 3}$.

(b) Check the three conditions at $x = 2$:

1. $g(2) = 7$, so $g(2)$ exists.

2. Compute the one-sided limits:

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (x^2 + 1) = 5$$

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (2x + 1) = 5$$

Since both one-sided limits are equal, $\lim_{x \to 2} g(x) = 5$.

3. $\lim_{x \to 2} g(x) = 5 \neq 7 = g(2)$, so the limit does not equal the function value.

Therefore: $\boxed{\text{Not continuous at } x = 2}$ (removable discontinuity).

(c) Check the three conditions at $x = 1$:

1. $h(1) = \frac{1}{0}$, which is undefined. So $h(1)$ does not exist.

Since the first condition fails: $\boxed{\text{Not continuous at } x = 1}$ (infinite discontinuity).

Example 1.4.5: Enforcing Continuity in Piecewise Functions

Find the value of k such that the following piecewise function is continuous at $x = 2$:

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3x + 1, & \text{if } x > 2 \end{cases}$$

Solution:

For continuity at $x = 2$, we need:

$$\lim_{x \to 2} f(x) = f(2)$$

Since the limit exists if and only if the one-sided limits are equal:

Left-hand limit: As $x \to 2^-$, use $f(x) = kx^2$:

$$\lim_{x \to 2^-} f(x) = k(2)^2 = 4k$$

Right-hand limit: As $x \to 2^+$, use $f(x) = 3x + 1$:

$$\lim_{x \to 2^+} f(x) = 3(2) + 1 = 7$$

For the two-sided limit to exist, we need $4k = 7$, so:

$$k = \frac{7}{4}$$

Check: $f(2) = k(2)^2 = \frac{7}{4} \cdot 4 = 7$, which matches the limit. Therefore, $k = \frac{7}{4}$ makes the function continuous at $x = 2$.

Example 1.4.6: Enforcing Continuity in Piecewise Functions

Consider the piecewise function

$$f(x) = \begin{cases} 2x + a & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ bx^2 - 8 & \text{if } x > 3 \end{cases}$$

Find the values of a and b such that $f(x)$ is continuous at $x = 3$.

Solution:

For the two-sided limit to exist and equal $f(3) = 1$, both one-sided limits at $x = 3$ must equal 1:

Left-hand limit:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x + a) = 2 \cdot 3 + a = a + 6$$

Setting this equal to 1: $a + 6 = 1 \implies \boxed{a = -5}$

Right-hand limit:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx^2 - 8) = b \cdot 3^2 - 8 = 9b - 8$$

Setting this equal to 1: $9b - 8 = 1 \implies \boxed{b = 1}$

Definition: Types of Discontinuities

When a function f fails to be continuous at $x = a$, the discontinuity is classified as follows:

- **Removable Discontinuity (Hole)**: $\lim_{x \to a} f(x)$ exists and is finite, but $\lim_{x \to a} f(x) \neq f(a)$ (or $f(a)$ is undefined).
- **Jump Discontinuity**: Both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are finite, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.
- **Infinite Discontinuity (Vertical Asymptote)**: At least one of the one-sided limits $\lim_{x \to a^\pm} f(x)$ is $\pm\infty$.
- **Oscillating Discontinuity**: At least one of the one-sided limits $\lim_{x \to a^\pm} f(x)$ fails to exist due to wild oscillation.

Example 1.4.7: Classifying Discontinuities

Classify all discontinuities for the function

$$f(x) = \begin{cases} \dfrac{1}{2 - x} & \text{if } x \geq 1 \\ \dfrac{\sin(x)}{x} & \text{if } 0 < x < 1 \\ x^2 + 1 & \text{if } x < 0 \end{cases}$$

Solution:

We check the transition points $x = 0$ and $x = 1$, as well as internal domain restrictions (such as division by zero at $x = 2$):

At $x = 2$: $\lim_{x \to 2^+} \dfrac{1}{2 - x} = \dfrac{1}{0^-} = -\infty$ and $\lim_{x \to 2^-} \dfrac{1}{2 - x} = \dfrac{1}{0^+} = +\infty$

There is an $\boxed{\text{infinite discontinuity at } x = 2}$

At $x = 1$: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x^3} = 1$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 - x) = 0 = f(1)$.

Since the one-sided limits are finite but unequal, there is a $\boxed{\text{jump discontinuity at } x = 1}$

At $x = 2$: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 1) = 1$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to -1^+} \frac{\sin(x)}{x} = 1$

Since $\lim_{x \to 0} f(x) = 1$, but $f(0)$ is undefined, there is a $\boxed{\text{removable discontinuity at } x = 0}$

Summary: Continuity Checklist

To determine continuity of a function:

1. **Polynomials, sine, cosine, and exponential functions** are continuous everywhere on their domains.
2. **Rational functions** are continuous everywhere except where the denominator is zero.
3. **Piecewise functions** require checking each individual piece and the boundary points where they might meet.
4. **Sums, differences, products, and compositions of continuous functions** are continuous wherever they are defined.

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1.4: Intermediate Value Theorem

Definition: Continuity on an Interval

A function f is **continuous on an open interval** (a, b) if it is continuous at every point in the interval.

A function f is **continuous on a closed interval** $[a, b]$ if it is continuous on (a, b) and exhibits continuous one-sided boundaries:

- $\lim_{x \to a^+} f(x) = f(a)$ (continuous from the right at a)
- $\lim_{x \to b^-} f(x) = f(b)$ (continuous from the left at b)

Theorem: The Intermediate Value Theorem (IVT)

If f is continuous on the closed interval $[a, b]$ and k is any number strictly between $f(a)$ and $f(b)$, then there exists at least one value $c \in (a, b)$ such that

$$f(c) = k$$

In particular, if $f(a)$ and $f(b)$ have opposite signs, then there is at least one root $c \in (a, b)$ such that $f(c) = 0$.

Example 1.4.8: Applying the Intermediate Value Theorem

Show that the equation $2\sin(\pi x) + x^2 = 7$ has at least one solution in the interval $[2, 3]$.

Solution:

Define $g(x) = 2\sin(\pi x) + x^2$, which is continuous on the interval $[2, 3]$.

If we evaluate $g(x)$ at the endpoints:

$$g(2) = 2\sin(\pi \cdot 2) + 2^2 = 4 < 7$$

$$g(3) = 2\sin(\pi \cdot 3) + 3^2 = 9 > 7$$

$\boxed{\text{Since } g(2) < 7 < g(3), \text{ by the Intermediate Value Theorem, there exists at least one number } c \in (2, 3) \text{ such that } g(c) = 7}$

Example 1.4.9: The IVT with Polynomials

Show that $P(x) = 1 + 3x - x^3$ has at least positive real root.

Solution:

We know that $P(x)$ is a polynomial, so it is continuous everywhere.

If we evaluate $P(x)$ at a few points:

$$P(0) = 1 + 3 \cdot 0 - 0^3 = 1 > 0$$

$$P(1) = 1 + 3 \cdot 1 - 1^3 = 3 > 0$$

$$P(2) = 1 + 3 \cdot 2 - 2^3 = -1 < 0$$

$\boxed{\text{Since } P(2) < 0 < P(1), \text{ by the Intermediate Value Theorem, there exists at least one number } c \in (1, 2) \text{ such that } P(c) = 0}$

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1.4: Practice Set

Exercise 1.4.1

State whether each function below is continuous at $a = 3$. If not, determine the type of discontinuity.

(a) $f(x) = \dfrac{x^2 - 9}{x - 3}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & x \neq 3 \\ 6 & x = 3 \end{cases}$

(c) $h(x) = \begin{cases} x^2 + 1 & x < 3 \\ 2x + 1 & x \geq 3 \end{cases}$

Answer available at the end of this section.

Exercise 1.4.2

For each piecewise function below, find all values of x where the function is discontinuous and classify each discontinuity.

(a) $f(x) = \begin{cases} 2 & x \leq 1 \\ \dfrac{x^2 - 16}{x^2 - 5x + 4} & 1 < x < 4 \\ 5 & x \geq 4 \end{cases}$

(b) $g(x) = \begin{cases} 4 - 2x & x \geq 2 \\ \dfrac{16}{x^4} & -2 < x < 2 \\ \cos(\pi x) & x < -2 \end{cases}$

(c) $h(x) = \begin{cases} 5x & x > 3 \\ \dfrac{x^2 - 5x + 6}{x^2 - x - 2} & 0 < x < 3 \\ \sqrt[3]{x - 27} & x \leq 0 \end{cases}$

(d) $p(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 8 & x = 0 \end{cases}$

Answer available at the end of this section.

Exercise 1.4.3

Find c such that $g(x) = \begin{cases} cx + 1 & x \leq 2 \\ cx^2 - 3 & x > 2 \end{cases}$ is continuous on $(-\infty, \infty)$.

Answer available at the end of this section.

Exercise 1.4.4

Find k such that $h(x) = \begin{cases} \dfrac{6x^3 \sin(2x)}{\sin^2(3x^2)} & x \neq 0 \\ k & x = 0 \end{cases}$ is continuous at $x = 0$.

Answer available at the end of this section.

Exercise 1.4.5

Find a and b such that $f(x) = \begin{cases} x^2 + a & x < 3 \\ 10 & x = 3 \\ bx + 4 & x > 3 \end{cases}$ is continuous at $x = 3$.

Answer available at the end of this section.

Exercise 1.4.6

Explain why the function $f(x) = x^4 - \cos(\pi x)$ must have at least one real zero in the interval $[0, 1]$.

Answer available at the end of this section.

Exercise 1.4.7

Show that $\sin(\pi x^2) + x = \cos(\pi x)$ has at least one positive solution.

Answer available at the end of this section.

Exercise 1.4.8

Show that the equation $x^2 = 1 + \dfrac{8}{x - 1}$ has at least one solution.

Answer available at the end of this section.

Exercise 1.4.9

A space probe is sent from Earth to Mars in 400 days. Is there a day where the probe must be exactly equidistant between the two planets? Justify your answer.

Answer available at the end of this section.

Answers

Exercise 1.4.1

(a) $f(3) = \frac{9-9}{3-3} = \frac{0}{0}$, which is undefined. So $f(3)$ does not exist.

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3} x + 3 = 6$$

Thus, $f(x)$ has a removable discontinuity at $x = 3$.

(b) In the piecewise function, we see that $g(3)$ exists and is defined to be 6.

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3} x + 3 = 6$$

These match, so $g(x)$ is continuous at $x = 3$.

(c) In the piecewise function, we see that $h(3)$ exists and is defined by $h(3) = 2 \cdot 3 + 1 = 7$

$$\lim_{x\to 3^-} h(x) = \lim_{x\to 3^-} (x^2 + 1) = 10$$
$$\lim_{x\to 3^+} h(x) = \lim_{x\to 3^+} (2x + 1) = 7$$

Since these two limits do not match, $h(x)$ has a jump discontinuity at $x = 3$.

Exercise 1.4.2

(a) For $x > 4$ or $x < 1$, the top and bottom functions are constant, and clearly continuous. We factor the rational function in the middle to see:

\[f(x)=\dfrac{x^2-16}{x^2 - 5x + 4} = \dfrac{(x-4)(x+4)}{(x - 1)(x - 4)}=\frac{x+4}{x-1} \text{ for }1

This is also continuous (where we are using it) so we only need to check the edge cases $x = 1$ and $x = 4$.

At $x = 1$, we see that $f(1) = 2$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2 = 2$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x+4}{x-1} = \frac{5}{0^+} = +\infty$$

Therefore, $f(x)$ has an infinite discontinuity at $x = 1$.

At $x = 4$: we see that $f(4) = 5$.

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{x+4}{x-1} = \frac{8}{3}$$
$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} 5 = 5$$

Therefore, $f(x)$ has a jump discontinuity at $x = 4$.

(b) For $x > 2$ and $x < -2$, the top and bottom functions are clearly continuous. For \(-2

At $x = -2$, we see that $g(-2)$ is undefined.

$$\lim_{x\to -2^-} g(x) = \lim_{x\to -2^-} \cos(\pi x) = \cos(-2\pi) = 1$$
$$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} \frac{16}{x^4} = 1$$

Therefore, $g(x)$ has a removable discontinuity at $x = -2$.

At $x = 0$, we see that $g(0)$ is undefined.

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{16}{x^4} = \frac{16}{0^+} = +\infty$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{16}{x^4} = \frac{16}{0^+} = +\infty$$

Therefore, $g(x)$ has an infinite discontinuity at $x = 0$.

At $x = 2$, we see that $g(2) = 0$.

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} \frac{16}{x^4} = 1$$
$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (4 - 2x) = 0$$

Therefore, $g(x)$ has a jump discontinuity at $x = 2$.

(c) For $x > 3$, $h(x) = 5x$ and for $x < 0$, $h(x) = \sqrt[3]{x - 27}$. These are both clearly continuous everywhere. We factor the rational function in the middle to see:

\[h(x)=\dfrac{x^2-5x+6}{x^2-x-2} = \dfrac{(x-2)(x-3)}{(x-2)(x+1)} = \frac{x-3}{x+1} \text{ for }0

This is continuous for \(0

We will check the edge cases $x = 0$ and $x = 3$, and the point $x = 2$ inside the middle interval.

At $x = 0$, we see that $h(0) = -3$.

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} \sqrt[3]{x - 27} = -3$$
$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \frac{x-3}{x+1} = \frac{-3}{1} = -3$$

Therefore, $h(x)$ is continuous at $x = 0$.

At $x = 2$, we see that $h(2)$ is undefined.

$$\lim_{x\to 2^-} h(x) = \lim_{x\to 2^-} \frac{x-3}{x+1} = -\frac{1}{3}$$
$$\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+} \frac{x-3}{x+1} = -\frac{1}{3}$$

Therefore, $h(x)$ has a removable discontinuity at $x = 2$.

At $x = 3$, we see that $h(3) = 15$.

$$\lim_{x\to 3^-} h(x) = \lim_{x\to 3^-} \frac{x-3}{x+1} = 0$$
$$\lim_{x\to 3^+} h(x) = \lim_{x\to 3^+} 5x = 15$$

Therefore, $h(x)$ has a jump discontinuity at $x = 3$.

(d) For $x \neq 0$, $p(x) = x^2 \sin\left(\frac{1}{x}\right)$ is a product of continuous functions, so it is continuous everywhere except possibly at $x = 0$. We check the only possible discontinuity at $x = 0$.

At $x = 0$, we see that $p(0) = 8$.

Since $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist, we use the Squeeze Theorem to evaluate $\lim_{x\to 0} p(x)$.

We take $-1 \le \sin\left(\frac{1}{x}\right) \le 1$, and multiply both sides by x^2 to get:

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

We see that $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} -x^2 = 0$, so by the Squeeze Theorem:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

Therefore, $p(x)$ has a removable discontinuity at $x = 0$.

Exercise 1.4.3

For continuity on $(-\infty, \infty)$, we only need to check the boundary at $x = 2$.

We see that $g(2) = c \cdot 2 + 1 = 2c + 1$, so we need:

$$\lim_{x\to 2^-} g(x) = 2c + 1 \quad \text{and} \quad \lim_{x\to 2^+} g(x) = 2c + 1$$

First, we evaluate the left limit:

$$\lim_{x\to 2^-} (cx + 1) = 2c + 1$$

This is already equal to $g(2)$, so we move on to the right limit:

$$\lim_{x\to 2^+} (cx^2 - 3) = 4c - 3$$

Setting this equal to $g(2)$ gives:

$$2c + 1 = 4c - 3 \implies 4 = 2c \implies c = 2$$

Exercise 1.4.4

For continuity at $x = 0$, we need $\lim_{x\to 0} h(x) = h(0)$

Since $h(0) = k$, and both sides of $x = 0$ use the rule $h(x) = \dfrac{6x^3 \sin(2x)}{\sin^2(3x^2)}$, we compute the limit:

$$\lim_{x\to 0} \frac{6x^3 \sin(2x)}{\sin^2(3x^2)} = \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot \frac{3x^2}{\sin(3x^2)} \cdot \frac{3x^2}{\sin(3x^2)} \cdot \frac{12x^4}{9x^4} = 1 \cdot 1 \cdot 1 \cdot \frac{4}{3} = \frac{4}{3}$$

Setting this equal to $h(0)$, we arrive at our solution: $k = \dfrac{4}{3}$

Exercise 1.4.5

We see that $f(3) = 10$, so to ensure continuity at $x = 3$, we need:

$$\lim_{x\to 3^-} f(x) = 10 \quad \text{and} \quad \lim_{x\to 3^+} f(x) = 10$$

Left limit:

$$\lim_{x\to 3^-} (x^2 + a) = 9 + a$$

Setting this equal to 10 gives:

$$9 + a = 10 \implies a = 1$$

Right limit:

$$\lim_{x\to 3^+} (bx + 4) = 3b + 4$$

Setting this equal to 10 gives:

$$3b + 4 = 10 \implies b = 2$$

Exercise 1.4.6

f is continuous on $[0, 1]$ since it is the sum of a polynomial and a cosine function, both of which are continuous everywhere.

Evaluate at the endpoints:

$$f(0) = 0^4 - \cos(0) = 0 - 1 = -1 < 0$$
$$f(1) = 1^4 - \cos(\pi) = 1 - (-1) = 2 > 0$$

Since $f(0) < 0 < f(1)$, by the Intermediate Value Theorem, there must exist some $c \in (0, 1)$ such that $f(c) = 0$.

Exercise 1.4.7

To use the IVT, we must first rewrite as $\sin(\pi x^2) + x - \cos(\pi x) = 0$.

$F(x) = \sin(\pi x^2) + x - \cos(\pi x)$ is continuous for all x since it is a sum of continuous functions.

Evaluate at $x = 0$:

$$F(0) = \sin(0) + 0 - \cos(0) = 0 + 0 - 1 = -1 < 0$$

Evaluate at $x = 1$:

$$F(1) = \sin(\pi) + 1 - \cos(\pi) = 0 + 1 - (-1) = 2 > 0$$

Since \(F(0) < 0

Therefore, the equation $\sin(\pi x^2) + x = \cos(\pi x)$ has at least one positive solution in $(0, 1)$.

Exercise 1.4.8

Rewrite as $x^2 - \frac{8}{x-1} = 1$.

$F(x) = x^2 - \frac{8}{x-1}$ is continuous on any closed interval that does not contain $x = 1$.

Evaluating a few points leads us to:

$$F(2) = 4 - \frac{8}{1} = -4 < 1$$

$$F(5) = 25 - \frac{8}{4} = 23 > 1$$

Since $F(2) < 1 < F(5)$, by the Intermediate Value Theorem, there must exist some $c \in (2, 5)$ such that $F(c) = 1$.

Therefore, the equation $x^2 = 1 + \frac{8}{x-1}$ has at least one solution in $(2, 5)$.

Exercise 1.4.9

Let $d(t) = (\text{the distance from the probe to Earth}) - (\text{the distance from the probe to Mars})$ at time t (in days), where $t = 0$ is the start and $t = 400$ is the end.

At the start, the probe is at Earth, so the distance to Mars is clearly greater than the distance to Earth. This means $d(0) < 0$

At the end, the probe is at Mars, so the distance to Mars is clearly smaller than the distance to Earth. This means $d(400) > 0$

Equidistant between the two planets means $d(t) = 0$.

Assuming the probe cannot teleport, the distance function $d(t)$ is continuous on the interval $[0, 400]$.

Since \(d(0) < 0

Therefore, there is at least one day during the trip when the probe is exactly equidistant between Earth and Mars.

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1.5: Limits at Infinity, Transcendental Functions, and Inverses

Learning Objectives

- Take limits of functions at positive and negative infinity.
- Find the horizontal asymptotes of algebraic functions.
- Find the inverse of an invertible function.
- Define the inverses of the six trigonometric functions.
- Take limits involving inverse trigonometric functions.
- Define the exponential and logarithmic functions.
- Take limits of exponential and logarithmic functions.
- Use the properties of logarithms to simplify expressions and solve equations.

In this section, we study the end behavior of functions at infinity, formalize horizontal asymptotes, review inverse functions (including restricted inverse trigonometric functions), and examine exponential and logarithmic functions.

Questions of the Day

1. How do functions behave as the input grows without bound?
2. What happens when we switch x and y in a function?

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1.5: Limits at Infinity

Definition: Horizontal Asymptotes & Limits at Infinity

We write $\lim_{x\to\infty} f(x) = L$ to mean that as x becomes arbitrarily large (positive), $f(x)$ gets arbitrarily close to the finite number L. Similarly, $\lim_{x\to-\infty} f(x) = L$ means as x becomes arbitrarily negative, $f(x)$ gets arbitrarily close to L.

If either one-sided limit at infinity exists as a finite number, the line $\mathbf{y = L}$ is called a **horizontal asymptote** of the graph of $y = f(x)$. A function can have at most two horizontal asymptotes (one as $x \to \infty$ and one as $x \to -\infty$), and they need not be the same.

For rational functions $R(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials:

- If $\deg(p) < \deg(q)$, then $\lim_{x\to\pm\infty} R(x) = 0$. The horizontal asymptote is $y = 0$.
- If $\deg(p) = \deg(q)$, then $\lim_{x\to\pm\infty} R(x) = \dfrac{\text{leading coefficient of } p}{\text{leading coefficient of } q}$.
- If $\deg(p) > \deg(q)$, then the limit is infinite ($\pm\infty$) and **no horizontal asymptote** exists.

These rules work for both $x \to \infty$ and $x \to -\infty$. Be careful with the sign when the degree of the numerator is larger and $x \to -\infty$.

Strategy for Limits at Infinity

For rational functions, the fastest way to evaluate limits at infinity is to compare degrees:

1. **Numerator degree < Denominator degree**: Limit = 0 (HA: $y = 0$)
2. **Numerator degree = Denominator degree**: Limit = ratio of leading coefficients (HA: $y = \text{that ratio}$)
3. **Numerator degree > Denominator degree**: Limit = ∞ or $-\infty$ (no HA)

For non-rational functions, a reliable method is to **divide by the highest power of** x that appears in the denominator (or factor out the dominant term). This often works for algebraic functions involving roots as well. For trigonometric functions, remember that $\sin x$ and $\cos x$ oscillate and do not settle to a single value, so their limits at infinity do not exist (unless they are multiplied by a term that goes to 0).

A function may have a horizontal asymptote even if it is not rational. Always check the behavior as $x \to \infty$ and $x \to -\infty$ separately.

Example 1.5.1: Limits at Infinity for Rational Functions

Evaluate the following limits at infinity.

(a) $\lim_{x\to\infty} \dfrac{2x^2 + 7}{x^2 - 5x + 2}$

(b) $\lim_{x\to-\infty} \dfrac{8x^9 + 5x^4 - 3x}{2x^{10} - 1}$

(c) $\lim_{x\to\infty} \dfrac{2x^3 - 13}{9x^2 + 7}$

(d) $\lim_{x\to-\infty} \dfrac{2x^3 - 13}{9x^2 + 7}$

Solution:

(a) Divide numerator and denominator by the highest power of x on the bottom, which is x^2:

$$\lim_{x\to\infty}\frac{2x^2+7}{x^2-5x+2}=\lim_{x\to\infty}\frac{2+\frac{7}{x^2}}{1-\frac{5}{x}+\frac{2}{x^2}}=\frac{2+0}{1-0+0}=\boxed{2}$$

(b) Divide numerator and denominator by the highest power of x on the bottom, which is x^{10}::

$$\lim_{x\to-\infty}\frac{8x^9+5x^4-3x}{2x^{10}-1}=\lim_{x\to-\infty}\frac{\frac{8}{x}+\frac{5}{x^6}-\frac{3}{x^9}}{2-\frac{1}{x^{10}}}=\frac{0+0-0}{2-0}=\boxed{0}$$

(c) Divide numerator and denominator by the highest power of x on the bottom, which is x^2:

$$\lim_{x\to\infty}\frac{2x^3-13}{9x^2+7}=\lim_{x\to\infty}\frac{2x-\frac{13}{x^2}}{9+\frac{7}{x^2}}=\frac{+\infty-0}{9+0}=\boxed{+\infty}$$

(d) Divide numerator and denominator by the highest power of x on the bottom, which is x^2:

$$\lim_{x\to-\infty}\frac{2x^3-13}{9x^2+7}=\lim_{x\to\infty}\frac{2x-\frac{13}{x^2}}{9+\frac{7}{x^2}}=\frac{-\infty-0}{9+0}=\boxed{-\infty}$$

Example 1.5.2: Finding Horizontal Asymptotes for Algebraic Functions

Find the horizontal asymptotes of the functions below:

(a) $y=\dfrac{\sqrt{4x^2+1}}{3x-5}$

(b) $y=\left(\sqrt{x^2+3x}-x\right)$

Solution:

First, we evaluate $\lim\limits_{x\to\infty}\dfrac{\sqrt{4x^2+1}}{3x-5}$

Divide numerator and denominator by x (the highest power in the denominator):

$$\lim_{x\to\infty}\frac{\sqrt{4x^2+1}}{3x-5}=\lim_{x\to\infty}\frac{\frac{\sqrt{4x^2+1}}{x}}{3-\frac{5}{x}}=\lim_{x\to\infty}\frac{\sqrt{4+\frac{1}{x^2}}}{3-\frac{5}{x}}=\frac{\sqrt{4+0}}{3-0}=\frac{2}{3}$$

Now, we evaluate $\lim\limits_{x\to-\infty}\dfrac{\sqrt{4x^2+1}}{3x-5}$

For $x<0$, we have $x=-|x|=-\sqrt{x^2}$. Divide by x carefully:

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2+1}}{3x-5}=\lim_{x\to-\infty}\frac{\frac{\sqrt{4x^2+1}}{x}}{3-\frac{5}{x}}=\lim_{x\to-\infty}\frac{-\sqrt{4+\frac{1}{x^2}}}{3-\frac{5}{x}}=\frac{-\sqrt{4}}{3}=-\frac{2}{3}$$

So this function has two different horizontal asymptotes: $\boxed{y=\dfrac{2}{3}\text{ and }y=-\dfrac{2}{3}}$

(b) This is an indeterminate form $\infty-\infty$. We rationalize by multiplying by the conjugate:

$$\sqrt{x^2+3x}-x=\frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{\sqrt{x^2+3x}+x}=\frac{(x^2+3x)-x^2}{\sqrt{x^2+3x}+x}=\frac{3x}{\sqrt{x^2+3x}+x}$$

Now divide numerator and denominator by x (since $x>0$ for large positive x):

$$\lim_{x\to\infty}\frac{3x}{\sqrt{x^2+3x}+x}=\lim_{x\to\infty}\frac{3}{\sqrt{1+\frac{3}{x}}+1}=\frac{3}{1+1}=\frac{3}{2}$$

Horizontal asymptote: $y=\dfrac{3}{2}$

For $x<0$, dividing by x changes the sign inside the root. Alternatively, factor x^2 from the root:

$$\sqrt{x^2+3x}=|x|\sqrt{1+\frac{3}{x}}=-x\sqrt{1+\frac{3}{x}}\quad(\text{since } x<0)$$

Then the expression becomes

$$-x\sqrt{1+\frac{3}{x}}-x=-x\left(\sqrt{1+\frac{3}{x}}+1\right)$$

As $x\to-\infty$, $\sqrt{1+\dfrac{3}{x}}\to 1$, so $-x\cdot(1+1)\to+\infty$ because $-x$ is positive and grows without bound. Therefore,

$$\lim_{x\to-\infty}\left(\sqrt{x^2+3x}-x\right)=+\infty$$

Therefore, this function has one horizontal asymptote: $\boxed{y=\dfrac{3}{2}}$

Example 1.5.3: Trigonometric Functions

Evaluate the following limits to infinity involving trigonometric functions:

(a) $\lim\limits_{x\to\infty}\sin(x)$

(b) $\lim\limits_{x\to\infty}\dfrac{\sin(x)}{x}$

Solution:

(a) This limit does not exist because $\sin(x)$ keeps oscillating between -1 and 1 and never approaches a single finite value. There is no horizontal asymptote on its graph.

We write $\lim\limits_{x\to\infty}\sin(x)=\boxed{\text{DNE}}$

(b) The numerator $\sin x$ oscillates between -1 and 1, while the denominator grows without bound. By the squeeze theorem,

$$-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$$

Since $\lim\limits_{x\to\infty}-\dfrac{1}{x}=0$ and $\lim\limits_{x\to\infty}\dfrac{1}{x}=0$, we have

$$\lim_{x\to\infty}\frac{\sin x}{x}=\boxed{0}$$

Note

As with $\sin(x)$ in part (a) of the Example above, we have $\lim_{x\to\infty}\cos(x)$ does not exist, and $\lim_{x\to\infty}\tan(x)$ also does not exist. The same can be said of any of the standard periodic trigonometric functions' limits at positive and negative infinity.

However, if a trig function is multiplied by a term that decays to 0, the product may have a limit of 0, as seen above in part (b).

Common Pitfalls

- **Assuming a function has only one horizontal asymptote.** Some functions have two different horizontal asymptotes (one for each direction). Always check both ∞ and $-\infty$.
- **Forgetting absolute values when dealing with square roots.** For even roots, $\sqrt{x^2} = |x|$, which is x for $x \to \infty$ and $-x$ for $x \to -\infty$. This affects the sign of the limit.
- **Confusing "no horizontal asymptote" with "limit does not exist."** A limit can be infinite (e.g., ∞) and still be a valid statement; it just means no horizontal asymptote. A limit DNE means the function oscillates or behaves erratically without approaching any value (finite or infinite).
- **Applying rational function rules to non-rational functions.** The degree comparison rule only works for rational functions. For others, use algebraic manipulation or the squeeze theorem.

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1.5: Inverse Functions

Definition: Inverse Functions

A function f is **one-to-one** (or **injective**) if every horizontal line intersects the graph of f at most once. Equivalently, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

If a function f is one-to-one on its domain, then it has an **inverse function**, denoted f^{-1}, that "undoes" the action of f. That is, for every x in the domain of f:

$$f^{-1}(f(x)) = x$$

And for every x in the range of f:

$$f\left(f^{-1}(x)\right) = x$$

The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

Finding an inverse function:

1. Replace $f(x)$ with y.
2. Swap the variables x and y.
3. Solve for y in terms of x.
4. Replace y with $f^{-1}(x)$.

Important Note

The notation f^{-1} does **not** mean $\dfrac{1}{f(x)}$. It denotes the inverse function, not the reciprocal.

Example 1.5.4: Finding Inverse Functions

Find the explicit formula for the inverse of each function and state its domain and range:

(a) $f(x) = 2x^5 - 3$

(b) $g(x) = \dfrac{4x - 3}{2x + 1}$

(c) $h(y) = \dfrac{1}{1 - y^2}$ with domain restricted to $0 < y < 1$

Solution:

(a) Set $y = 2x^5 - 3$, swap x and y: $x = 2y^5 - 3$.

Solve for y: $2y^5 = x + 3 \implies y = \sqrt[5]{\dfrac{x+3}{2}}$

$$\boxed{f^{-1}(x) = \sqrt[5]{\frac{x+3}{2}}}$$

Domain of f^{-1}: $(-\infty, \infty)$
Range of f^{-1}: $(-\infty, \infty)$

(b) Set $y = \dfrac{4x-3}{2x+1}$ and swap x and y:

$$x = \frac{4y-3}{2y+1}$$

Solve explicitly for y:

$$\begin{aligned} x(2y+1) &= 4y-3 \\ 2xy+x &= 4y-3 \\ 2xy-4y &= -x-3 \\ y(2x-4) &= -(x+3) \\ y &= \frac{-x-3}{2x-4} = \frac{x+3}{4-2x} \end{aligned}$$

$$\boxed{g^{-1}(x) = \frac{x+3}{4-2x}}$$

Domain of g^{-1}: $4-2x \neq 0 \implies x \neq 2$, so $(-\infty, 2) \cup (2, \infty)$

Range of g^{-1}: Same as the domain of g, where $2x+1 \neq 0 \implies x \neq -\dfrac{1}{2}$, so $\left(-\infty, -\dfrac{1}{2}\right) \cup \left(-\dfrac{1}{2}, \infty\right)$

(c) Given $x = \dfrac{1}{1-y^2}$, solve for y:

$$\begin{aligned} x(1-y^2) &= 1 \\ x - xy^2 &= 1 \\ xy^2 &= x-1 \\ y^2 &= \frac{x-1}{x} \end{aligned}$$

Since $0 < y < 1$, take the positive root. Also $x > 1$ for the restriction to hold.

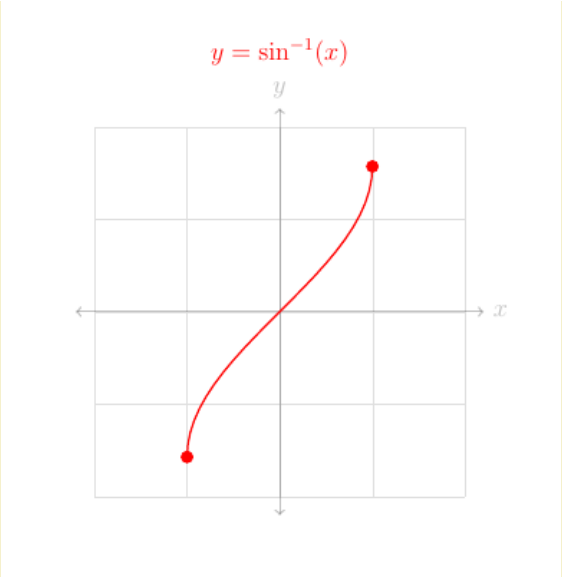
$$\boxed{h^{-1}(x) = \sqrt{\frac{x-1}{x}}}$$

Domain of h^{-1}: $(1, \infty)$
Range of h^{-1}: $(0, 1)$

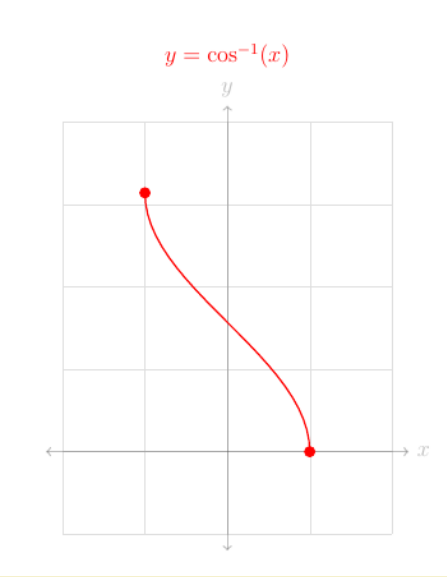
Review: Inverse Trigonometric Functions

Trigonometric functions are periodic and not one-to-one on their entire domains; therefore, their domains must be restricted to define inverse functions:

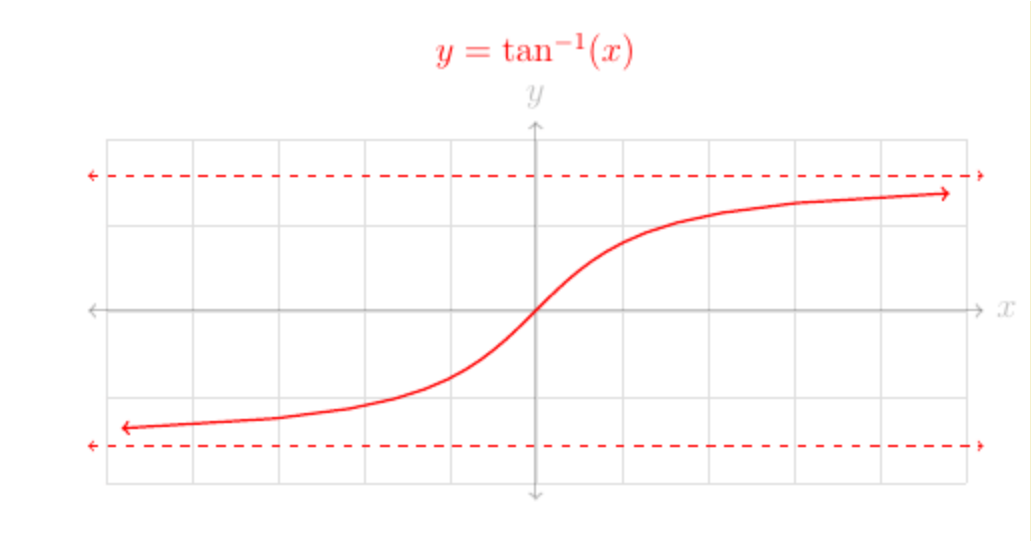
$y = \sin^{-1}(x)$ has domain $[-1, 1]$ and range $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$



$y = \cos^{-1}(x)$ has domain $[-1, 1]$ and range $[0, \pi]$

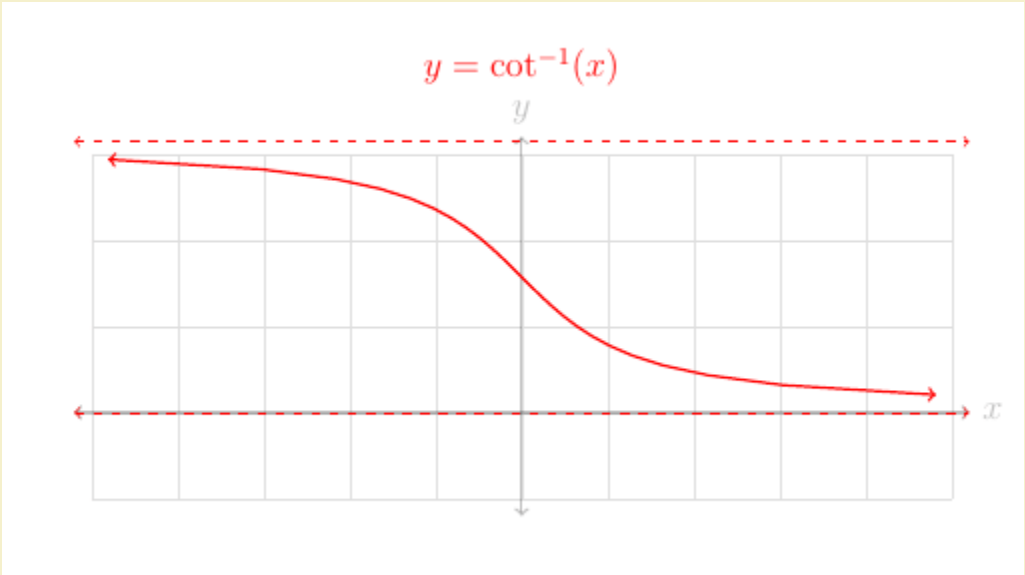


$y = \tan^{-1}(x)$ has domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



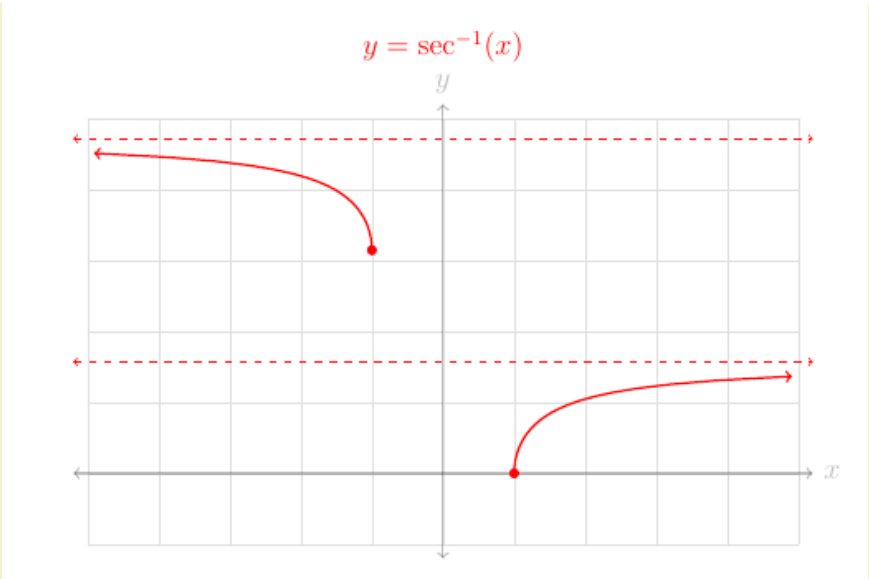
Horizontal Asymptotes: $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$

$y = \cot^{-1}(x)$ has domain $(-\infty, \infty)$ and range $(0, \pi)$



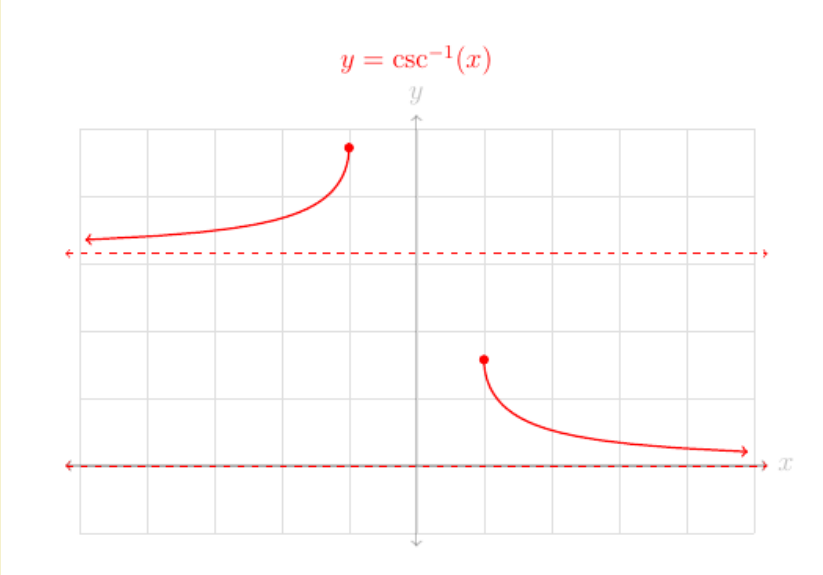
Horizontal Asymptotes: $\lim_{x\to\infty} \cot^{-1}(x) = 0$ and $\lim_{x\to-\infty} \cot^{-1}(x) = \pi$

$y = \sec^{-1}(x)$ has domain $(-\infty, -1] \cup [1, \infty)$ and range $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$



Horizontal Asymptotes: $\lim_{x\to\infty} \sec^{-1}(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \sec^{-1}(x) = \frac{3\pi}{2}$

$y = \csc^{-1}(x)$ has domain $(-\infty, -1] \cup [1, \infty)$ and range $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$



Horizontal Asymptotes: $\lim_{x\to\infty} \csc^{-1}(x) = 0$ and $\lim_{x\to-\infty} \csc^{-1}(x) = \pi$

Example 1.5.5: Inverse Trigonometric Evaluations

Evaluate the following inverse trigonometric limits:

(a) $\lim_{x\to-\frac{1}{2}} \sin^{-1}(x)$

(b) $\lim_{x\to-\frac{1}{2}} \cos^{-1}(x)$

(c) $\lim_{x\to -1^+} \sin^{-1}(x)$

(d) $\lim_{x\to -1^-} \tan^{-1}(x)$

(e) $\lim_{x\to -\infty} \tan^{-1}(x) + \cot^{-1}(x)$

(f) $\lim_{x\to 2} \dfrac{\sec^{-1}(x)}{x}$

Solution:

(a) Since $\sin^{-1}(x)$ is continuous on its domain $[-1,1]$, we can evaluate by direct substitution:

$$\lim_{x\to -\frac{1}{2}} \sin^{-1}(x) = \sin^{-1}\left(-\frac{1}{2}\right) = \boxed{-\frac{\pi}{6}}$$

(b) Since $\cos^{-1}(x)$ is continuous on its domain $[-1,1]$, we can evaluate by direct substitution:

$$\lim_{x\to -\frac{1}{2}} \cos^{-1}(x) = \cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$$

(c) As $x \to -1^+$, $\sin^{-1}(x)$ approaches the left endpoint of its range:

$$\lim_{x\to -1^+} \sin^{-1}(x) = \boxed{-\frac{\pi}{2}}$$

(d) As $x \to -1^-$, $\tan^{-1}(x)$ approaches $\tan^{-1}(-1)$ from the left. Since $\tan^{-1}(x)$ is continuous on $(-\infty,\infty)$:

$$\lim_{x\to -1^-} \tan^{-1}(x) = \tan^{-1}(-1) = \boxed{-\frac{\pi}{4}}$$

(e) As $x \to -\infty$, $\tan^{-1}(x) \to -\dfrac{\pi}{2}$ and $\cot^{-1}(x) \to \pi$. Therefore:

$$\lim_{x\to -\infty} \left[\tan^{-1}(x) + \cot^{-1}(x)\right] = -\frac{\pi}{2} + \pi = \boxed{\frac{\pi}{2}}$$

(f) Since $\sec^{-1}(x)$ is continuous on its domain and $x = 2$ is in the domain, we can evaluate by direct substitution:

$$\lim_{x\to 2} \frac{\sec^{-1}(x)}{x} = \frac{\sec^{-1}(2)}{2} = \frac{\frac{\pi}{3}}{2} = \boxed{\frac{\pi}{6}}$$

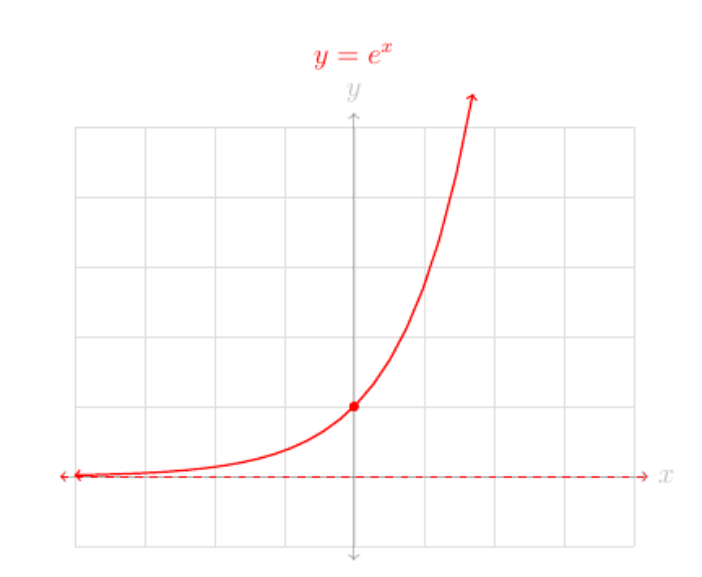
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1.5: Transcendental Functions

Definition: Exponential & Logarithmic Functions

An **exponential function** has the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$. It has domain $(-\infty, \infty)$ and range $(0, \infty)$.

The natural exponential function is $f(x) = e^x$ where $e \approx 2.718\ldots$.



Horizontal Asymptotes: $\lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0$

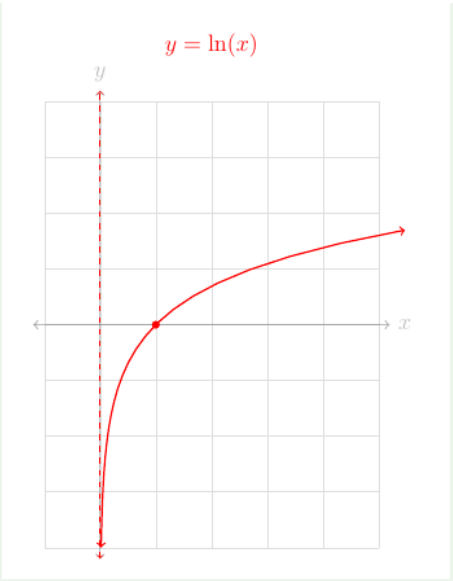
Since exponential functions are continuous and pass the horizontal line test, they are invertible.

The inverse of the exponential function is the **logarithm function**:

$$y = \log_a(x) \iff a^y = x \quad \text{for } a > 0, a \neq 1, \text{ and } x > 0$$

The logarithm function has domain $(0, \infty)$ and range $(-\infty, \infty)$.

The natural logarithm function is $f(x) = \ln(x) = \log_e(x)$.



Key Limits: $\lim_{x\to\infty} \ln(x) = \infty$ and $\lim_{x\to 0^+} \ln(x) = -\infty$

Example 1.5.6: Exponential and Logarithmic Limits

Evaluate each of the following limits:

(a) $\lim_{x\to\infty} e^{-x}$

(b) $\lim_{x\to-\infty} e^{-x}$

(c) $\lim_{x\to\infty} \dfrac{3e^x + 2}{5e^x - 1}$

(d) $\lim_{x\to 0} \ln(x^2)$

(e) $\lim_{x\to 8} \log_2(x)$

Solution:

(a) Since the graph of $y = e^{-x}$ is a horizontal reflection of $y = e^x$, we see:

$$\lim_{x\to\infty} e^{-x} = \lim_{x\to-\infty} e^x = \boxed{0}$$

(b) Since $e^x \to 0^+$ as $x \to -\infty$, we have:

$$\lim_{x\to-\infty} e^{-x} = \lim_{x\to-\infty} \frac{1}{e^x} = \frac{1}{0^+} = \boxed{+\infty}$$

(c) Divide numerator and denominator by e^x:

$$\lim_{x\to\infty} \frac{3e^x + 2}{5e^x - 1} = \lim_{x\to\infty} \frac{3 + 2e^{-x}}{5 - e^{-x}} = \frac{3+0}{5-0} = \boxed{\frac{3}{5}}$$

(d) Since $y = x^2$ approaces 0 and is positive as $x \to 0$ we have:

$$\lim_{x\to 0} \ln(x^2) = \lim_{y\to 0^+} \ln(y) = \boxed{-\infty}$$

(e) Since $y = \log_b(x)$ is continuous on its domain for all b we can plug in $x = 8$ to get:

$$\lim_{x \to 8} \log_2(x) = \log_2(8) = \log_2(2^3) = \boxed{3}$$

Properties of Logarithms

For $a > 0$, $a \neq 1$, and $x, y > 0$, the following properties hold:

- **Product Property:** $\log_a(xy) = \log_a(x) + \log_a(y)$
- **Quotient Property:** $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- **Power Property:** $\log_a(x^r) = r \log_a(x)$
- **Change of Base:** $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$
- **Inverse Properties:** $\log_a(a^x) = x$ and $a^{\log_a(x)} = x$

Example 1.5.7: Simplifying Logarithmic Expressions

Simplify each expression:

(a) $\log_2(8x^3) - \log_2(2x)$ for $x > 0$

(b) $\ln(e^{3x}) + \ln(e^2)$

(c) $\log_4(16x^2) - \log_4(2x)$ for $x > 0$

Solution:

(a) Use the quotient property:

$$\log_2(8x^3) - \log_2(2x) = \log_2\left(\frac{8x^3}{2x}\right) = \log_2(4x^2) = \log_2(4) + \log_2(x^2) = \boxed{2 + 2\log_2(x)}$$

(b) $\ln(e^{3x}) = 3x$ and $\ln(e^2) = 2$, so:

$$\ln(e^{3x}) + \ln(e^2) = \boxed{3x + 2}$$

(c) Use the quotient property:

$$\log_4(16x^2) - \log_4(2x) = \log_4\left(\frac{16x^2}{2x}\right) = \log_4(8x) = \log_4(8) + \log_4(x)$$

Since $8 = 4^{\frac{3}{2}}$, we have $\log_4(8) = \frac{3}{2}$. Therefore:

$$\log_4(16x^2) - \log_4(2x) = \boxed{\frac{3}{2} + \log_4(x)}$$

Example 1.5.8: Solving Logarithmic Equations

Solve each equation for x:

(a) $\log_3(2x - 1) + \log_3(x - 2) = 3$

(b) $\log_5(2x + 1) - \log_5(x - 1) = 1$

(c) $\log_2(x+3)+\log_2(x-1)=3$

Solution:

(a) Use the product property:

$$\log_3((2x-1)(x-2))=3$$

Convert to exponential form:

$$\begin{aligned}(2x-1)(x-2)&=3^3=27\\ 2x^2-5x+2&=27\\ 2x^2-5x-25&=0\\ (2x+5)(x-5)&=0\\ x=-\frac{5}{2}&\text{ or } x=5\end{aligned}$$

Check domain: $x>2$, so $x=-\frac{5}{2}$ is extraneous. $\boxed{x=5}$

(b) Use the quotient property:

$$\log_5\left(\frac{2x+1}{x-1}\right)=1$$

Convert to exponential form:

$$\begin{aligned}\frac{2x+1}{x-1}&=5^1=5\\ 2x+1&=5x-5\\ 6&=3x\\ x&=2\end{aligned}$$

Check domain: $x>1$, so $x=2$ is valid. $\boxed{x=2}$

(c) Use the product property:

$$\log_2((x+3)(x-1))=3$$

Convert to exponential form:

$$\begin{aligned}(x+3)(x-1)&=2^3=8\\ x^2+2x-3&=8\\ x^2+2x-11&=0\end{aligned}$$

Use the quadratic formula:

$$\begin{aligned}x&=\frac{-2\pm\sqrt{4+44}}{2}\\ &=\frac{-2\pm\sqrt{48}}{2}\\ &=\frac{-2\pm4\sqrt{3}}{2}\\ &=-1\pm2\sqrt{3}\end{aligned}$$

Check domain: $x>1$ (since $x-1>0$), so $x=-1-2\sqrt{3}$ is extraneous. $\boxed{x=-1+2\sqrt{3}}$

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1.5: Practice Set

Exercise 1.5.1

Evaluate each limit:

(a) $\lim_{x\to\infty} \dfrac{7x^4 - 3x^2 + 2}{5x^4 + 8x - 1}$

(b) $\lim_{x\to-\infty} \dfrac{3x^5 - 2x^3 + x}{4x^5 + 7x^2 - 5}$

(c) $\lim_{x\to\infty} \dfrac{9x^2 + 4}{2x^3 - 5x + 1}$

(d) $\lim_{x\to-\infty} \dfrac{6x^4 - 3x}{8x^6 + 2x - 7}$

(e) $\lim_{x\to\infty} \dfrac{2x^5 - 3x^3 + x}{6x^3 + 4x - 9}$

Answer available at the end of this section.

Exercise 1.5.2

Find all horizontal asymptotes of each function:

(a) $f(x) = \dfrac{6x^2 - 5x + 2}{3x^2 + x - 7}$

(b) $g(x) = \dfrac{4x^3 - 2x + 1}{8x^4 + 3x^2 - 5}$

(c) $h(x) = \dfrac{5x^7 - 4x^2 + 9}{2x^5 + 7x - 3}$

(d) $p(x) = \dfrac{\sqrt{4x^2 + 1} - x}{x + 2}$

(e) $q(x) = \dfrac{\sqrt{9x^2 - 2x + 1} + 3x}{2x + 9}$

Answer available at the end of this section.

Exercise 1.5.3

Find the inverse function of each function and state its domain and range:

(a) $f(x) = 3x - 4$

(b) $g(x) = \dfrac{5x - 2}{3x + 1}$

(c) $h(x) = x^3 + 7$

(d) $p(x) = \sqrt[5]{x-2} + 1$

(e) $q(x) = \log_2(2x+1) - 1$

(f) $r(x) = \dfrac{1}{1-e^{-x^2}}$ with domain restricted to $x < 0$

Answer available at the end of this section.

Exercise 1.5.4

Evaluate each limit:

(a) $\lim\limits_{x \to \frac{1}{2}} \sin^{-1}(x)$

(b) $\lim\limits_{x \to -\frac{\sqrt{2}}{2}} \cos^{-1}(x)$

(c) $\lim\limits_{x \to -1^+} x \sin^{-1}(x)$

(d) $\lim\limits_{x \to 0^-} \cos^{-1}(e^x)$

(e) $\lim\limits_{x \to -\infty} \left[\tan^{-1}(x) + \sec^{-1}(x)\right]$

(f) $\lim\limits_{x \to \sqrt{3}} \dfrac{\cot^{-1}(x)}{x^2+1}$

Answer available at the end of this section.

Exercise 1.5.5

Evaluate each limit:

(a) $\lim\limits_{x \to \infty} e^{-2x}$

(b) $\lim\limits_{x \to 3} \log_8\left(\dfrac{1}{x+1}\right)$

(c) $\lim\limits_{x \to \infty} \dfrac{4e^x + 3}{2e^x - 5}$

(d) $\lim\limits_{x \to 1^+} \ln(x^2 - 1)$

Answer available at the end of this section.

Exercise 1.5.6

Solve each equation for x:

(a) $\log_2(x+1) + \log_2(x-1) = 3$

(b) $\log_4(3x-2) - \log_4(x) = 1$

Answer available at the end of this section.

Answers

Exercise 1.5.1

(a) Divide numerator and denominator by the highest power of x in the denominator, x^4:

$$\lim_{x\to\infty}\frac{7x^4-3x^2+2}{5x^4+8x-1}=\lim_{x\to\infty}\frac{7-\frac{3}{x^2}+\frac{2}{x^4}}{5+\frac{8}{x^3}-\frac{1}{x^4}}=\frac{7-0+0}{5+0-0}=\frac{7}{5}$$

(b) Divide numerator and denominator by x^5:

$$\lim_{x\to-\infty}\frac{3x^5-2x^3+x}{4x^5+7x^2-5}=\lim_{x\to-\infty}\frac{3-\frac{2}{x^2}+\frac{1}{x^4}}{4+\frac{7}{x^3}-\frac{5}{x^5}}=\frac{3-0+0}{4+0-0}=\frac{3}{4}$$

(c) Divide numerator and denominator by x^3 :

$$\lim_{x\to\infty}\frac{9x^2+4}{2x^3-5x+1}=\lim_{x\to\infty}\frac{\frac{9}{x}+\frac{4}{x^3}}{2-\frac{5}{x^2}+\frac{1}{x^3}}=\frac{0+0}{2-0+0}=0$$

(d) Divide numerator and denominator by x^6:

$$\lim_{x\to-\infty}\frac{6x^4-3x}{8x^6+2x-7}=\lim_{x\to-\infty}\frac{\frac{6}{x^2}-\frac{3}{x^5}}{8+\frac{2}{x^5}-\frac{7}{x^6}}=\frac{0-0}{8+0-0}=0$$

(e) Divide numerator and denominator by x^3:

$$\lim_{x\to\infty}\frac{2x^5-3x^3+x}{6x^3+4x-9}=\lim_{x\to\infty}\frac{2x^2-3+\frac{1}{x^2}}{6+\frac{4}{x^2}-\frac{9}{x^3}}=\infty$$

The numerator grows without bound while the denominator approaches 6, so the limit is ∞.

Exercise 1.5.2

(a) We take the limits to both positive and negative infinity:

$$\lim_{x\to\infty}\frac{6x^2-5x+2}{3x^2+x-7}=\lim_{x\to\infty}\frac{6-\frac{5}{x}+\frac{2}{x^2}}{3+\frac{1}{x}-\frac{7}{x^2}}=\frac{6}{3}=2$$

$$\lim_{x\to-\infty}\frac{6x^2-5x+2}{3x^2+x-7}=\lim_{x\to-\infty}\frac{6-\frac{5}{x}+\frac{2}{x^2}}{3+\frac{1}{x}-\frac{7}{x^2}}=\frac{6}{3}=2$$

HA: $y=2$

(b) We take the limits to both positive and negative infinity:

$$\lim_{x\to\infty}\frac{4x^3-2x+1}{8x^4+3x^2-5}=\lim_{x\to\infty}\frac{\frac{4}{x}-\frac{2}{x^3}+\frac{1}{x^4}}{8+\frac{3}{x^2}-\frac{5}{x^4}}=\frac{0}{8}=0$$

$$\lim_{x\to-\infty}\frac{4x^3-2x+1}{8x^4+3x^2-5}=\lim_{x\to-\infty}\frac{\frac{4}{x}-\frac{2}{x^3}+\frac{1}{x^4}}{8+\frac{3}{x^2}-\frac{5}{x^4}}=\frac{0}{8}=0$$

HA: $y=0$

(c) We take the limits to both positive and negative infinity:

$$\lim_{x\to\infty}\frac{5x^7-4x^2+9}{2x^5+7x-3}=\lim_{x\to\infty}\frac{5x^2-\frac{4}{x^3}+\frac{9}{x^5}}{2+\frac{7}{x^4}-\frac{3}{x^5}}=+\infty$$

$$\lim_{x\to-\infty}\frac{5x^7-4x^2+9}{2x^5+7x-3}=\lim_{x\to-\infty}\frac{5x^2-\frac{4}{x^3}+\frac{9}{x^5}}{2+\frac{7}{x^4}-\frac{3}{x^5}}=+\infty$$

No horizontal asymptote

(d) We take the (careful) limits to both positive and negative infinity:

$$\lim_{x\to\infty}\frac{\sqrt{4x^2+1}-x}{x+2}=\lim_{x\to\infty}\frac{\sqrt{4+\frac{1}{x^2}}-1}{1+\frac{2}{x}}=\frac{2-1}{1+0}=1$$

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2+1}-x}{x+2}=\lim_{x\to-\infty}\frac{-\sqrt{4+\frac{1}{x^2}}-1}{1+\frac{2}{x}}=\frac{-2-1}{1+0}=-3$$

HA: $y=1$ and $y=-3$

(e) We take the (careful) limits to both positive and negative infinity:

$$\lim_{x\to\infty}\frac{\sqrt{9x^2-2x+1}+3x}{2x+9}=\lim_{x\to\infty}\frac{\sqrt{9-\frac{2}{x}+\frac{1}{x^2}}+3}{2+\frac{9}{x}}=\frac{3+3}{2+0}=3$$

$$\lim_{x\to-\infty}\frac{\sqrt{9x^2-2x+1}+3x}{2x+9}=\lim_{x\to-\infty}\frac{-\sqrt{9-\frac{2}{x}+\frac{1}{x^2}}+3}{2+\frac{9}{x}}=\frac{-3+3}{2+0}=0$$

HA: $y=3$ and $y=0$

Exercise 1.5.3

(a) Replace $f(x)$ with y: $y=3x-4$. Swap x and y: $x=3y-4$

Solve for y:

$$\begin{aligned}x&=3y-4\\x+4&=3y\\y&=\frac{x+4}{3}\\f^{-1}(x)&=\frac{x+4}{3}\end{aligned}$$

Domain: $(-\infty,\infty)$ Range: $(-\infty,\infty)$

(b) $y=\frac{5x-2}{3x+1}$. Swap x and y: $x=\frac{5y-2}{3y+1}$

Solve for y:

$$\begin{aligned}x(3y+1)&=5y-2\\3xy+x&=5y-2\\3xy-5y&=-x-2\\y(3x-5)&=-x-2\\y&=\frac{-x-2}{3x-5}=\frac{x+2}{5-3x}\\g^{-1}(x)&=\frac{x+2}{5-3x}\end{aligned}$$

Domain: $5-3x\neq 0 \implies x\neq\frac{5}{3}$, so $\left(-\infty,\frac{5}{3}\right)\cup\left(\frac{5}{3},\infty\right)$

Range: Same as domain of g, where $3x+1\neq 0 \implies x\neq-\frac{1}{3}$, so $\left(-\infty,-\frac{1}{3}\right)\cup\left(-\frac{1}{3},\infty\right)$

(c) $y=x^3+7$. Swap x and y: $x=y^3+7$

Solve for y:

$$\begin{aligned}x&=y^3+7\\x-7&=y^3\\y&=\sqrt[3]{x-7}\\h^{-1}(x)&=\sqrt[3]{x-7}\end{aligned}$$

Domain: $(-\infty,\infty)$ Range: $(-\infty,\infty)$

(d) $y=\sqrt[5]{x-2}+1$. Swap x and y: $x=\sqrt[5]{y-2}+1$

Solve for y:

$$\begin{aligned}x&=\sqrt[5]{y-2}+1\\x-1&=\sqrt[5]{y-2}\\(x-1)^5&=y-2\\y&=(x-1)^5+2\\p^{-1}(x)&=(x-1)^5+2\end{aligned}$$

Domain: $(-\infty,\infty)$ Range: $(-\infty,\infty)$

(e) $y=\log_2(2x+1)-1$. Swap x and y: $x=\log_2(2y+1)-1$

Solve for y:

$$\begin{aligned}x&=\log_2(2y+1)-1\\x+1&=\log_2(2y+1)\\2^{x+1}&=2y+1\\2^{x+1}-1&=2y\\y&=\frac{2^{x+1}-1}{2}\\q^{-1}(x)&=\frac{2^{x+1}-1}{2}\end{aligned}$$

Domain: $(-\infty,\infty)$ Range: $(-\infty,\infty)$

(f) $y=\frac{1}{1-e^{-x^2}}$ with domain restricted to $x<0$. Swap x and y: $x=\frac{1}{1-e^{-y^2}}$

Solve for y:

$$\begin{aligned} x &= \frac{1}{1-e^{-y^2}} \\ x(1-e^{-y^2}) &= 1 \\ 1-e^{-y^2} &= \frac{1}{x} \\ e^{-y^2} &= 1-\frac{1}{x} \\ e^{-y^2} &= \frac{x-1}{x} \\ -y^2 &= \ln\left(\frac{x-1}{x}\right) \\ y^2 &= -\ln\left(\frac{x-1}{x}\right) = \ln\left(\frac{x}{x-1}\right) \end{aligned}$$

Since the original domain is $x < 0$, we take the negative root for y to maintain the restriction:

$$r^{-1}(x) = -\sqrt{\ln\left(\frac{x}{x-1}\right)}$$

Domain: $(0,1)$ Range: $(-\infty, 0)$

Exercise 1.5.4

(a) Since $\sin^{-1}(x)$ is continuous on its domain $[-1,1]$, we can evaluate by direct substitution:

$$\lim_{x\to\frac{1}{2}} \sin^{-1}(x) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

(b) Since $\cos^{-1}(x)$ is continuous on its domain $[-1,1]$, we can evaluate by direct substitution:

$$\lim_{x\to-\frac{\sqrt{2}}{2}} \cos^{-1}(x) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

(c) As $x \to -1^+$, $\sin^{-1}(x)$ approaches the left endpoint of its range $\left(-1, -\frac{\pi}{2}\right)$:

$$\lim_{x\to-1^+} x\sin^{-1}(x) = -1\cdot\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

(d) As $x \to 0^-$, $y = e^x$ approaches 1 from below, so:

$$\lim_{x\to0^-} \cos^{-1}\left(e^x\right) = \lim_{y\to1^-} \cos^{-1}(y) = \pi$$

(e) As $x \to -\infty$, $\tan^{-1}(x) \to -\frac{\pi}{2}$ and $\sec^{-1}(x) \to \frac{3\pi}{2}$. Therefore:

$$\lim_{x\to-\infty} \left[\tan^{-1}(x) + \sec^{-1}(x)\right] = -\frac{\pi}{2} + \frac{3\pi}{2} = \pi$$

(f) Since $\cot^{-1}(x)$ is continuous on its domain and $x = \sqrt{3}$ is in the domain, we can evaluate by direct substitution:

$$\lim_{x\to\sqrt{3}} \frac{\cot^{-1}(x)}{x^2+1} = \frac{\cot^{-1}\left(\sqrt{3}\right)}{4} = \frac{\frac{\pi}{6}}{4} = \frac{\pi}{24}$$

Exercise 1.5.5

(a) $\lim_{x\to\infty} e^{-2x} = \lim_{x\to\infty} \frac{1}{e^{2x}} = 0$

(b) Since $x = 3$ is in the domain of $y = \log_8\left(\frac{1}{x+1}\right)$, we use direct substitution and the change of base formula:

$$\lim_{x\to3} \log_8\left(\frac{1}{x+1}\right) = \log_8\left(\frac{1}{4}\right) = \frac{\log_2\left(\frac{1}{4}\right)}{\log_2(8)} = \frac{-2}{3}$$

(c) Divide numerator and denominator by e^x:

$$\lim_{x\to\infty} \frac{4e^x+3}{2e^x-5} = \lim_{x\to\infty} \frac{4+\frac{3}{e^x}}{2-\frac{5}{e^x}} = \frac{4+0}{2-0} = 2$$

(d) As $x \to 1^+$, $y = x^2 - 1 \to 0$ from above, so:

$$\lim_{x\to1^+} \ln(x^2-1) = \lim_{y\to0^+} \ln(y) = -\infty$$

Exercise 1.5.6

(a) Use the product property:

$$\log_2((x+1)(x-1)) = 3$$

Convert to exponential form:

$$\begin{aligned} (x+1)(x-1) &= 2^3 = 8 \\ x^2-1 &= 8 \\ x^2 &= 9 \\ x &= \pm3 \end{aligned}$$

Check domain: $x+1>0 \implies x>-1$ and $x-1>0 \implies x>1$. So $x>1$

Thus $x=-3$ is extraneous. $x=3$ is valid.

$$x = 3$$

(b) Use the quotient property:

$$\log_4\left(\frac{3x-2}{x}\right)=1$$

Convert to exponential form:

$$\frac{3x-2}{x}=4^1=4$$
$$3x-2=4x$$
$$-2=x$$

Check domain: $3x-2>0 \implies x>\frac{2}{3}$ and $x>0$. So $x>\frac{2}{3}$

Thus $x=-2$ is extraneous. No solution.

$$\text{No solution}$$

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1.6: The Derivative and Tangent Lines

Learning Objectives

- Explore the different between average and instantaneous velocity.
- Define the derivative of a function at a point as a limit.
- Find the tangent line to a function by using the derivative.

In this section, we introduce the concept of the derivative at a point, explore its geometric interpretation as the slope of a tangent line, and compute tangent line equations for fundamental algebraic functions.

Question of the Day

How do we find the slope of a tangent line (or instantaneous speed) at a single point?

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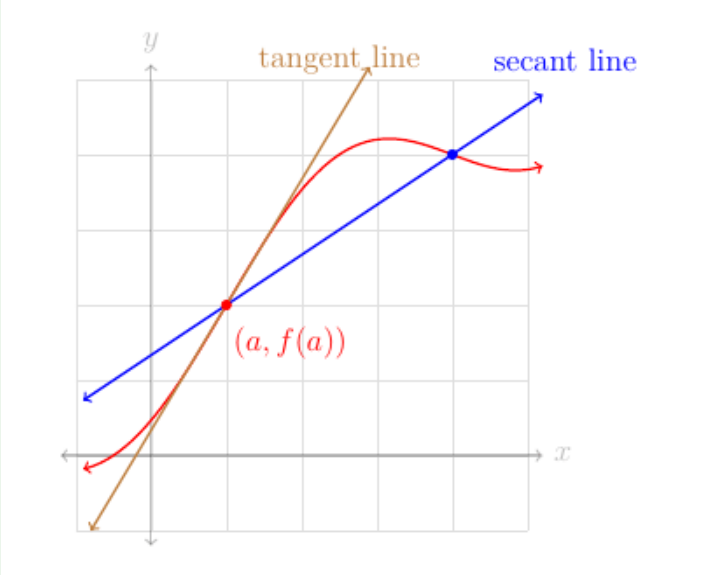
1.6: Tangent Lines

Definition: Average Velocity vs. Instantaneous Velocity

Suppose an object moves along a straight line and its position at time t is given by a function $s(t)$. The **average velocity** of the object over the time interval $[a, a+h]$ is the change in position divided by the change in time:

$$v_{\text{avg}} = \frac{s(a+h) - s(a)}{h}$$

Geometrically, this is the slope of the **secant line** connecting the points $(a, s(a))$ and $(a+h, s(a+h))$.



As we make the time interval smaller and smaller (letting $h \to 0$), the secant line approaches the **tangent line** at the point $(a, s(a))$. The slope of this tangent line is the **instantaneous velocity** at time $t = a$:

$$v_{\text{inst}} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Illustration 1.6.1: Average vs. Instantaneous Rate of Change

An object is dropped from a height of 70 feet. Its height above ground in feet after t seconds is given by $h(t) = 70 - 16t^2$. The object hits the ground at $t = 2$ seconds. What is the instantaneous velocity of the object at the exact moment it hits at $t = 2$ seconds?

The average velocity over the interval $[x, 2]$ is the slope of the secant line:

$$\begin{aligned} v_{\text{avg}} &= \frac{h(2)-h(x)}{2-x} \\ &= \frac{(70-16(2)^2)-(70-16x^2)}{2-x} \\ &= \frac{6-(70-16x^2)}{2-x} \\ &= \frac{16x^2-64}{2-x} \\ &= \frac{16(x^2-4)}{2-x} \\ &= \frac{16(x-2)(x+2)}{-(x-2)} \\ &= -16(x+2)\ \text{ft/s} \end{aligned}$$

The average velocity between $t=x$ and $t=2$ is $-16(x+2)$ ft/s. The instantaneous velocity at $t=2$ is the limit of the average velocity as $x \to 2$:

$$\lim_{x\to 2} \frac{h(x)-h(2)}{x-2} = \lim_{x\to 2} -16(x+2) = -16(2+2) = -64\ \text{ft/s}$$

Thus, the object is moving at an instantaneous velocity of -64 ft/s when it hits the ground.

Definition: The Derivative at a Point

The **derivative** of a function $f(x)$ at $x=a$, denoted by $f'(a)$, represents the instantaneous rate of change or the slope of the tangent line at $(a, f(a))$:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Equivalently, letting $h = x-a$, we have the **alternate form of the derivative**:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

The **equation of the tangent line** to the curve $y=f(x)$ at the point $(a, f(a))$ is:

$$y - f(a) = f'(a)(x-a)$$

Interpreting the Derivative

The derivative $f'(a)$ can be interpreted in several ways:

- **Geometrically**: The slope of the tangent line to the graph of f at the point $(a, f(a))$.
- **Physically**: The instantaneous rate of change of f with respect to x at $x=a$.
- **Algebraically**: The limit of the difference quotient $\frac{f(a+h)-f(a)}{h}$ as $h \to 0$.

Example 1.6.1: Evaluating Derivatives Using the Limit Definition

Use the limit definition of the derivative to find the line tangent to each function at the given value of a:

(a) $f(x) = 7-2x$ at $a=4$

(b) $f(x) = x^2 - 10$ at $a = 5$

(c) $f(x) = 4x^2 - x$ at $a = -1$

(d) $f(x) = x^3 + 2$ at $a = 1$

Solution:

(a) Using the $h \to 0$ form:

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h\to 0} \frac{[7 - 2(4+h)] - [7 - 2(4)]}{h} \\ &= \lim_{h\to 0} \frac{(7 - 8 - 2h) - (7 - 8)}{h} \\ &= \lim_{h\to 0} \frac{(-1 - 2h) - (-1)}{h} \\ &= \lim_{h\to 0} \frac{-2h}{h} \\ &= \lim_{h\to 0} (-2) = -2 \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(4) &= \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} \\ &= \lim_{x\to 4} \frac{(7 - 2x) - (-1)}{x - 4} \\ &= \lim_{x\to 4} \frac{8 - 2x}{x - 4} \\ &= \lim_{x\to 4} \frac{-2(x - 4)}{x - 4} \\ &= \lim_{x\to 4} (-2) = -2 \end{aligned}$$

Thus $f'(4) = -2$. The tangent line at $(4, f(4)) = (4, -1)$ is:

$$\boxed{y + 1 = -2(x - 4)}$$

(b) Using the $h \to 0$ form:

$$\begin{aligned} f'(5) &= \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} \\ &= \lim_{h\to 0} \frac{[(5+h)^2 - 10] - [25 - 10]}{h} \\ &= \lim_{h\to 0} \frac{(25 + 10h + h^2 - 10) - 15}{h} \\ &= \lim_{h\to 0} \frac{10h + h^2}{h} \\ &= \lim_{h\to 0} (10 + h) = 10 \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(5) &= \lim_{x\to 5}\frac{f(x)-f(5)}{x-5} \\ &= \lim_{x\to 5}\frac{(x^2-10)-15}{x-5} \\ &= \lim_{x\to 5}\frac{x^2-25}{x-5} \\ &= \lim_{x\to 5}\frac{(x-5)(x+5)}{x-5} \\ &= \lim_{x\to 5}(x+5) = 10 \end{aligned}$$

Thus $f'(5) = 10$. The tangent line at $(5, f(5)) = (5, 15)$ is:

$$\boxed{y - 15 = 10(x-5)}$$

(c) Using the $h \to 0$ form:

$$\begin{aligned} f'(-1) &= \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} \\ &= \lim_{h\to 0}\frac{[4(-1+h)^2-(-1+h)]-[4(1)-(-1)]}{h} \\ &= \lim_{h\to 0}\frac{[4(1-2h+h^2)+1-h]-[4+1]}{h} \\ &= \lim_{h\to 0}\frac{(4-8h+4h^2+1-h)-5}{h} \\ &= \lim_{h\to 0}\frac{-9h+4h^2}{h} \\ &= \lim_{h\to 0}(-9+4h) = -9 \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(-1) &= \lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)} \\ &= \lim_{x\to -1}\frac{(4x^2-x)-5}{x+1} \\ &= \lim_{x\to -1}\frac{4x^2-x-5}{x+1} \\ &= \lim_{x\to -1}\frac{(x+1)(4x-5)}{x+1} \\ &= \lim_{x\to -1}(4x-5) = -4-5 = -9 \end{aligned}$$

Thus $f'(-1) = -9$. The tangent line at $(-1, f(-1)) = (-1, 5)$ is:

$$\boxed{y - 5 = -9(x+1)}$$

(d) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0}\frac{[(1+h)^3+2]-[1+2]}{h} \\
&= \lim_{h\to 0}\frac{(1+3h+3h^2+h^3+2)-3}{h} \\
&= \lim_{h\to 0}\frac{3h+3h^2+h^3}{h} \\
&= \lim_{h\to 0}(3+3h+h^2) = 3
\end{aligned}
$$

Using the $x \to a$ form:

$$
\begin{aligned}
f'(1) &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \\
&= \lim_{x\to 1}\frac{(x^3+2)-3}{x-1} \\
&= \lim_{x\to 1}\frac{x^3-1}{x-1} \\
&= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1} \\
&= \lim_{x\to 1}(x^2+x+1) = 1+1+1 = 3
\end{aligned}
$$

Thus $f'(1) = 3$. The tangent line at $(1, f(1)) = (1, 3)$ is:

$$\boxed{y-3 = 3(x-1)}$$

Example 1.6.2: More Challenging Derivatives

Find the equation of the line tangent to each curve below at the given point:

(a) $y = \dfrac{2}{x+1}$ at $(1, 1)$

(b) $y = \dfrac{2x}{x^2-3}$ at $(-2, -4)$

(c) $y = \sqrt{x}$ at $(100, 10)$

(d) $y = \sqrt{x^2+5}$ at $(2, 3)$

Solution:

(a) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0}\frac{\dfrac{2}{1+h+1}-1}{h} \\
&= \lim_{h\to 0}\frac{\dfrac{2}{h+2}-1}{h} \\
&= \lim_{h\to 0}\frac{\dfrac{2-(h+2)}{h+2}}{h} \\
&= \lim_{h\to 0}\frac{\dfrac{-h}{h+2}}{h} \\
&= \lim_{h\to 0}\frac{-1}{h+2} = -\frac{1}{2}
\end{aligned}
$$

Using the $x \to a$ form:

$$
\begin{aligned}
f'(1) &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \\
&= \lim_{x\to 1}\frac{\dfrac{2}{x+1}-1}{x-1} \\
&= \lim_{x\to 1}\frac{\dfrac{2-(x+1)}{x+1}}{x-1} \\
&= \lim_{x\to 1}\frac{\dfrac{1-x}{x+1}}{x-1} \\
&= \lim_{x\to 1}\frac{-(x-1)}{(x+1)(x-1)} \\
&= \lim_{x\to 1}\frac{-1}{x+1} = -\frac{1}{2}
\end{aligned}
$$

Thus $f'(1) = -\dfrac{1}{2}$. The tangent line at $(1,1)$ is:

$$\boxed{y-1 = -\frac{1}{2}(x-1)}$$

(b) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(-2) &= \lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{2(-2+h)}{(-2+h)^2-3}-(-4)}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{-4+2h}{4-4h+h^2-3}+4}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{-4+2h}{1-4h+h^2}+4}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{-4+2h+4(1-4h+h^2)}{1-4h+h^2}}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{-4+2h+4-16h+4h^2}{1-4h+h^2}}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{-14h+4h^2}{1-4h+h^2}}{h}\\
&= \lim_{h\to 0}\frac{-14+4h}{1-4h+h^2} = \frac{-14}{1} = -14
\end{aligned}
$$

Using the $x \to a$ form:

$$
\begin{aligned}
f'(-2) &= \lim_{x\to -2}\frac{f(x)-f(-2)}{x-(-2)}\\
&= \lim_{x\to -2}\frac{\dfrac{2x}{x^2-3}-(-4)}{x+2}\\
&= \lim_{x\to -2}\frac{\dfrac{2x}{x^2-3}+4}{x+2}\\
&= \lim_{x\to -2}\frac{\dfrac{2x+4(x^2-3)}{x^2-3}}{x+2}\\
&= \lim_{x\to -2}\frac{\dfrac{4x^2+2x-12}{x^2-3}}{x+2}\\
&= \lim_{x\to -2}\frac{2(2x^2+x-6)}{(x^2-3)(x+2)}\\
&= \lim_{x\to -2}\frac{2(2x-3)(x+2)}{(x^2-3)(x+2)}\\
&= \lim_{x\to -2}\frac{2(2x-3)}{x^2-3} = \frac{2(-4-3)}{4-3} = \frac{-14}{1} = -14
\end{aligned}
$$

Thus $f'(-2) = -14$. The tangent line at $(-2,-4)$ is:

$$\boxed{y+4 = -14(x+2)}$$

(c) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(100) &= \lim_{h\to 0}\frac{f(100+h)-f(100)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{100+h}-10}{h}\\
&= \lim_{h\to 0}\frac{(\sqrt{100+h}-10)(\sqrt{100+h}+10)}{h(\sqrt{100+h}+10)}\\
&= \lim_{h\to 0}\frac{(100+h)-100}{h(\sqrt{100+h}+10)}\\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{100+h}+10)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{100+h}+10} = \frac{1}{10+10} = \frac{1}{20}
\end{aligned}
$$

Using the $x \to a$ form:

$$
\begin{aligned}
f'(100) &= \lim_{x\to 100}\frac{f(x)-f(100)}{x-100}\\
&= \lim_{x\to 100}\frac{\sqrt{x}-10}{x-100}\\
&= \lim_{x\to 100}\frac{\sqrt{x}-10}{(\sqrt{x}-10)(\sqrt{x}+10)}\\
&= \lim_{x\to 100}\frac{1}{\sqrt{x}+10} = \frac{1}{10+10} = \frac{1}{20}
\end{aligned}
$$

Thus $f'(100) = \dfrac{1}{20}$. The tangent line at $(100, 10)$ is:

$$\boxed{y - 10 = \frac{1}{20}(x - 100)}$$

(d) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{(2+h)^2+5}-3}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{4+4h+h^2+5}-3}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{h^2+4h+9}-3}{h}\\
&= \lim_{h\to 0}\frac{(\sqrt{h^2+4h+9}-3)(\sqrt{h^2+4h+9}+3)}{h(\sqrt{h^2+4h+9}+3)}\\
&= \lim_{h\to 0}\frac{(h^2+4h+9)-9}{h(\sqrt{h^2+4h+9}+3)}\\
&= \lim_{h\to 0}\frac{h^2+4h}{h(\sqrt{h^2+4h+9}+3)}\\
&= \lim_{h\to 0}\frac{h+4}{\sqrt{h^2+4h+9}+3} = \frac{4}{\sqrt{9}+3} = \frac{4}{3+3} = \frac{2}{3}
\end{aligned}
$$

Using the $x \to a$ form:

$$
\begin{aligned}
f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\
&= \lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x-2} \\
&= \lim_{x\to 2} \frac{(\sqrt{x^2+5}-3)(\sqrt{x^2+5}+3)}{(x-2)(\sqrt{x^2+5}+3)} \\
&= \lim_{x\to 2} \frac{(x^2+5)-9}{(x-2)(\sqrt{x^2+5}+3)} \\
&= \lim_{x\to 2} \frac{x^2-4}{(x-2)(\sqrt{x^2+5}+3)} \\
&= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+5}+3)} \\
&= \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+5}+3} = \frac{4}{\sqrt{4+5}+3} = \frac{4}{3+3} = \frac{2}{3}
\end{aligned}
$$

Thus $f'(2) = \frac{2}{3}$. The tangent line at $(2,3)$ is:

$$\boxed{y-3 = \frac{2}{3}(x-2)}$$

Summary: Derivative at a Point

To find $f'(a)$ using the limit definition:

1. **Write the difference quotient**: $\frac{f(a+h)-f(a)}{h}$ (or $\frac{f(x)-f(a)}{x-a}$).
2. **Simplify algebraically**: Factor, rationalize, combine fractions, or use trigonometric identities.
3. **Take the limit**: As $h \to 0$ (or $x \to a$), evaluate the simplified expression.

Remember: The derivative exists only if this limit exists and is finite!

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1.6: Practice Set

Exercise 1.6.1

Find the equation of the tangent line to each function at the given point using the limit definition of the derivative. Show both the $h \to 0$ and $x \to a$ methods.

(a) $f(x) = 4x - 7$ at $(2, 1)$

(b) $f(x) = x^2 + 3x$ at $(1, 4)$

(c) $f(x) = 2x^2 - 5x + 1$ at $(-1, 8)$

(d) $f(x) = \sqrt{2x + 1}$ at $(4, 3)$

(e) $f(x) = \dfrac{3}{x - 2}$ at $(5, 1)$

(f) $f(x) = \dfrac{2x + 1}{x - 3}$ at $(4, 9)$

Answer available at the end of this section.

Exercise 1.6.2

Use the limit definition of the derivative to find $f'(a)$ at the given value of a. Show both the $h \to 0$ and $x \to a$ methods.

(a) $f(x) = 3x^2 - 2x + 5$ at $a = 2$

(b) $f(x) = \sqrt{4x - 3}$ at $a = 3$

(c) $f(x) = \sqrt{x^2 + 9}$ at $a = 4$

(d) $f(x) = \dfrac{5}{x^2 + 1}$ at $a = 2$

(e) $f(x) = \dfrac{1}{\sqrt{x}}$ at $a = 4$

Answer available at the end of this section.

Exercise 1.6.3

A particle moves along a straight line with position function $s(t) = 4t^2 - 3t + 2$, where s is measured in meters and t in seconds.

(a) Find the average velocity over the interval $[1, 3]$

(b) Find the average velocity over the interval $[1, 1.5]$

(c) Find the instantaneous velocity at $t = 1$

Answer available at the end of this section.

Exercise 1.6.4

A particle moves along a straight line with position function $s(t) = \sqrt{3t+1}$, where s is measured in feet and t in minutes.

(a) Find the average velocity over the interval $[1, 5]$

(b) Find the average velocity over the interval $[1, 3]$

(c) Find the instantaneous velocity at $t = 1$

Answer available at the end of this section.

Answers

Exercise 1.6.1

(a) Using the $h \to 0$ form:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{[4(2+h) - 7] - [4(2) - 7]}{h} \\ &= \lim_{h\to 0} \frac{(8 + 4h - 7) - (8 - 7)}{h} \\ &= \lim_{h\to 0} \frac{(1 + 4h) - 1}{h} \\ &= \lim_{h\to 0} \frac{4h}{h} \\ &= \lim_{h\to 0} 4 = 4 \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{(4x - 7) - 1}{x - 2} \\ &= \lim_{x\to 2} \frac{4x - 8}{x - 2} \\ &= \lim_{x\to 2} \frac{4(x - 2)}{x - 2} \\ &= \lim_{x\to 2} 4 = 4 \end{aligned}$$

Thus $f'(2) = 4$. The tangent line at $(2, 1)$ is:

$$y - 1 = 4(x - 2)$$

(b) Using the $h \to 0$ form:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{[(1+h)^2 + 3(1+h)] - [1+3]}{h} \\ &= \lim_{h\to 0} \frac{(1 + 2h + h^2 + 3 + 3h) - 4}{h} \\ &= \lim_{h\to 0} \frac{5h + h^2}{h} \\ &= \lim_{h\to 0} (5 + h) = 5 \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{(x^2 + 3x) - 4}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 + 3x - 4}{x - 1} \\ &= \lim_{x\to 1} \frac{(x - 1)(x + 4)}{x - 1} \\ &= \lim_{x\to 1} (x + 4) = 5 \end{aligned}$$

Thus $f'(1) = 5$. The tangent line at $(1, 4)$ is:

$$y - 4 = 5(x - 1)$$

(c) Using the $h \to 0$ form:

$$\begin{aligned} f'(-1) &= \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} \\ &= \lim_{h\to 0} \frac{[2(-1+h)^2 - 5(-1+h) + 1] - [2(1) - 5(-1) + 1]}{h} \\ &= \lim_{h\to 0} \frac{[2(1 - 2h + h^2) + 5 - 5h + 1] - [2 + 5 + 1]}{h} \\ &= \lim_{h\to 0} \frac{(2 - 4h + 2h^2 + 5 - 5h + 1) - 8}{h} \\ &= \lim_{h\to 0} \frac{-9h + 2h^2}{h} \\ &= \lim_{h\to 0} (-9 + 2h) = -9 \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(-1) &= \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} \\ &= \lim_{x\to -1} \frac{(2x^2 - 5x + 1) - 8}{x + 1} \\ &= \lim_{x\to -1} \frac{2x^2 - 5x - 7}{x + 1} \\ &= \lim_{x\to -1} \frac{(x + 1)(2x - 7)}{x + 1} \\ &= \lim_{x\to -1} (2x - 7) = -2 - 7 = -9 \end{aligned}$$

Thus $f'(-1) = -9$. The tangent line at $(-1, 8)$ is:

$$y - 8 = -9(x + 1)$$

(d) Using the $h \to 0$ form:

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2(4+h)+1} - 3}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9 + 2h} - 3}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{9+2h} - 3)(\sqrt{9+2h} + 3)}{h(\sqrt{9+2h} + 3)} \\ &= \lim_{h\to 0} \frac{(9 + 2h) - 9}{h(\sqrt{9+2h} + 3)} \\ &= \lim_{h\to 0} \frac{2h}{h(\sqrt{9+2h} + 3)} \\ &= \lim_{h\to 0} \frac{2}{\sqrt{9+2h} + 3} = \frac{2}{3+3} = \frac{1}{3} \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(4) &= \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} \\ &= \lim_{x\to 4} \frac{\sqrt{2x+1} - 3}{x - 4} \\ &= \lim_{x\to 4} \frac{(\sqrt{2x+1} - 3)(\sqrt{2x+1} + 3)}{(x - 4)(\sqrt{2x+1} + 3)} \\ &= \lim_{x\to 4} \frac{(2x + 1) - 9}{(x - 4)(\sqrt{2x+1} + 3)} \\ &= \lim_{x\to 4} \frac{2x - 8}{(x - 4)(\sqrt{2x+1} + 3)} \\ &= \lim_{x\to 4} \frac{2(x - 4)}{(x - 4)(\sqrt{2x+1} + 3)} \\ &= \lim_{x\to 4} \frac{2}{\sqrt{2x+1} + 3} = \frac{2}{3+3} = \frac{1}{3} \end{aligned}$$

Thus $f'(4) = \frac{1}{3}$. The tangent line at $(4, 3)$ is:

$$y - 3 = \frac{1}{3}(x - 4)$$

(e) Using the $h \to 0$ form:

$$\begin{aligned} f'(5) &= \lim_{h\to 0} \frac{f(5+h)-f(5)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{3}{5+h-2}-1}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{3}{h+3}-1}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{3-(h+3)}{h+3}}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{-h}{h+3}}{h} \\ &= \lim_{h\to 0} \frac{-1}{h+3} = -\frac{1}{3} \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(5) &= \lim_{x\to 5} \frac{f(x)-f(5)}{x-5} \\ &= \lim_{x\to 5} \frac{\dfrac{3}{x-2}-1}{x-5} \\ &= \lim_{x\to 5} \frac{\dfrac{3-(x-2)}{x-2}}{x-5} \\ &= \lim_{x\to 5} \frac{\dfrac{5-x}{x-2}}{x-5} \\ &= \lim_{x\to 5} \frac{-(x-5)}{(x-2)(x-5)} \\ &= \lim_{x\to 5} \frac{-1}{x-2} = -\frac{1}{3} \end{aligned}$$

Thus $f'(5) = -\frac{1}{3}$. The tangent line at $(5,1)$ is:

$$y - 1 = -\frac{1}{3}(x-5)$$

(f) Using the $h \to 0$ form:

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{2(4+h)+1}{4+h-3}-9}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{8+2h+1}{h+1}-9}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{9+2h}{h+1}-9}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{9+2h-9(h+1)}{h+1}}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{9+2h-9h-9}{h+1}}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{-7h}{h+1}}{h} \\ &= \lim_{h\to 0} \frac{-7}{h+1} = -7 \end{aligned}$$

Using the $x \to a$ form:

$$
\begin{aligned}
f'(4) &= \lim_{x\to 4} \frac{f(x)-f(4)}{x-4} \\
&= \lim_{x\to 4} \frac{\dfrac{2x+1}{x-3} - 9}{x-4} \\
&= \lim_{x\to 4} \frac{\dfrac{2x+1-9(x-3)}{x-3}}{x-4} \\
&= \lim_{x\to 4} \frac{\dfrac{2x+1-9x+27}{x-3}}{x-4} \\
&= \lim_{x\to 4} \frac{\dfrac{-7x+28}{x-3}}{x-4} \\
&= \lim_{x\to 4} \frac{\dfrac{-7(x-4)}{x-3}}{x-4} \\
&= \lim_{x\to 4} \frac{-7}{x-3} = \frac{-7}{1} = -7
\end{aligned}
$$

Thus $f'(4) = -7$. The tangent line at $(4, 9)$ is:

$$y - 9 = -7(x-4)$$

Exercise 1.6.2

(a) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{[3(2+h)^2 - 2(2+h)+5] - [3(4)-4+5]}{h} \\
&= \lim_{h\to 0} \frac{[3(4+4h+h^2) - 4 - 2h + 5] - 13}{h} \\
&= \lim_{h\to 0} \frac{(12+12h+3h^2 - 4 - 2h + 5) - 13}{h} \\
&= \lim_{h\to 0} \frac{(13+10h+3h^2) - 13}{h} \\
&= \lim_{h\to 0} \frac{10h+3h^2}{h} \\
&= \lim_{h\to 0} (10+3h) = 10
\end{aligned}
$$

Using the $x \to a$ form:

$$
\begin{aligned}
f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\
&= \lim_{x\to 2} \frac{(3x^2-2x+5) - 13}{x-2} \\
&= \lim_{x\to 2} \frac{3x^2-2x-8}{x-2} \\
&= \lim_{x\to 2} \frac{(x-2)(3x+4)}{x-2} \\
&= \lim_{x\to 2} (3x+4) = 6+4 = 10
\end{aligned}
$$

(b) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4(3+h)-3} - 3}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9+4h} - 3}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{9+4h}-3)(\sqrt{9+4h}+3)}{h(\sqrt{9+4h}+3)} \\
&= \lim_{h\to 0} \frac{(9+4h)-9}{h(\sqrt{9+4h}+3)} \\
&= \lim_{h\to 0} \frac{4h}{h(\sqrt{9+4h}+3)} \\
&= \lim_{h\to 0} \frac{4}{\sqrt{9+4h}+3} = \frac{4}{3+3} = \frac{2}{3}
\end{aligned}
$$

Using the $x \to a$ form:

$$\begin{aligned} f'(3) &= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} \\ &= \lim_{x\to 3}\frac{\sqrt{4x-3}-3}{x-3} \\ &= \lim_{x\to 3}\frac{(\sqrt{4x-3}-3)(\sqrt{4x-3}+3)}{(x-3)(\sqrt{4x-3}+3)} \\ &= \lim_{x\to 3}\frac{(4x-3)-9}{(x-3)(\sqrt{4x-3}+3)} \\ &= \lim_{x\to 3}\frac{4x-12}{(x-3)(\sqrt{4x-3}+3)} \\ &= \lim_{x\to 3}\frac{4(x-3)}{(x-3)(\sqrt{4x-3}+3)} \\ &= \lim_{x\to 3}\frac{4}{\sqrt{4x-3}+3} = \frac{4}{3+3} = \frac{2}{3} \end{aligned}$$

(c) Using the $h \to 0$ form:

$$\begin{aligned} f'(4) &= \lim_{h\to 0}\frac{f(4+h)-f(4)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{(4+h)^2+9}-5}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{16+8h+h^2+9}-5}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{h^2+8h+25}-5}{h} \\ &= \lim_{h\to 0}\frac{(\sqrt{h^2+8h+25}-5)(\sqrt{h^2+8h+25}+5)}{h(\sqrt{h^2+8h+25}+5)} \\ &= \lim_{h\to 0}\frac{(h^2+8h+25)-25}{h(\sqrt{h^2+8h+25}+5)} \\ &= \lim_{h\to 0}\frac{h^2+8h}{h(\sqrt{h^2+8h+25}+5)} \\ &= \lim_{h\to 0}\frac{h+8}{\sqrt{h^2+8h+25}+5} = \frac{8}{5+5} = \frac{4}{5} \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} f'(4) &= \lim_{x\to 4}\frac{f(x)-f(4)}{x-4} \\ &= \lim_{x\to 4}\frac{\sqrt{x^2+9}-5}{x-4} \\ &= \lim_{x\to 4}\frac{(\sqrt{x^2+9}-5)(\sqrt{x^2+9}+5)}{(x-4)(\sqrt{x^2+9}+5)} \\ &= \lim_{x\to 4}\frac{(x^2+9)-25}{(x-4)(\sqrt{x^2+9}+5)} \\ &= \lim_{x\to 4}\frac{x^2-16}{(x-4)(\sqrt{x^2+9}+5)} \\ &= \lim_{x\to 4}\frac{(x-4)(x+4)}{(x-4)(\sqrt{x^2+9}+5)} \\ &= \lim_{x\to 4}\frac{x+4}{\sqrt{x^2+9}+5} = \frac{8}{5+5} = \frac{4}{5} \end{aligned}$$

(d) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{5}{(2+h)^2+1}-1}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{5}{h^2+4h+5}-1}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{5-(h^2+4h+5)}{h^2+4h+5}}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{-h^2-4h}{h^2+4h+5}}{h}\\
&= \lim_{h\to 0}\frac{-h-4}{h^2+4h+5} = \frac{-4}{5} = -\frac{4}{5}
\end{aligned}
$$

Using the $x\to a$ form:

$$
\begin{aligned}
f'(2) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\\
&= \lim_{x\to 2}\frac{\dfrac{5}{x^2+1}-1}{x-2}\\
&= \lim_{x\to 2}\frac{\dfrac{5-(x^2+1)}{x^2+1}}{x-2}\\
&= \lim_{x\to 2}\frac{\dfrac{4-x^2}{x^2+1}}{x-2}\\
&= \lim_{x\to 2}\frac{\dfrac{-(x-2)(x+2)}{x^2+1}}{x-2}\\
&= \lim_{x\to 2}\frac{-(x+2)}{x^2+1} = \frac{-4}{5} = -\frac{4}{5}
\end{aligned}
$$

(e) Using the $h\to 0$ form:

$$
\begin{aligned}
f'(4) &= \lim_{h\to 0}\frac{f(4+h)-f(4)}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{1}{\sqrt{4+h}}-\dfrac{1}{2}}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{2-\sqrt{4+h}}{2\sqrt{4+h}}}{h}\\
&= \lim_{h\to 0}\frac{2-\sqrt{4+h}}{2h\sqrt{4+h}}\\
&= \lim_{h\to 0}\frac{(2-\sqrt{4+h})(2+\sqrt{4+h})}{2h\sqrt{4+h}(2+\sqrt{4+h})}\\
&= \lim_{h\to 0}\frac{4-(4+h)}{2h\sqrt{4+h}(2+\sqrt{4+h})}\\
&= \lim_{h\to 0}\frac{-h}{2h\sqrt{4+h}(2+\sqrt{4+h})}\\
&= \lim_{h\to 0}\frac{-1}{2\sqrt{4+h}(2+\sqrt{4+h})} = \frac{-1}{2(2)(2+2)} = -\frac{1}{16}
\end{aligned}
$$

Using the $x\to a$ form:

$$
\begin{aligned}
f'(4) &= \lim_{x\to 4}\frac{f(x)-f(4)}{x-4} \\
&= \lim_{x\to 4}\frac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4} \\
&= \lim_{x\to 4}\frac{2-\sqrt{x}}{2\sqrt{x}(x-4)} \\
&= \lim_{x\to 4}\frac{(2-\sqrt{x})(2+\sqrt{x})}{2\sqrt{x}(x-4)(2+\sqrt{x})} \\
&= \lim_{x\to 4}\frac{4-x}{2\sqrt{x}(x-4)(2+\sqrt{x})} \\
&= \lim_{x\to 4}\frac{-(x-4)}{2\sqrt{x}(x-4)(2+\sqrt{x})} \\
&= \lim_{x\to 4}\frac{-1}{2\sqrt{x}(2+\sqrt{x})} = \frac{-1}{2(2)(2+2)} = -\frac{1}{16}
\end{aligned}
$$

Exercise 1.6.3

(a) The average velocity over $[1,3]$ is:

$$
\begin{aligned}
v_{\text{avg}} &= \frac{s(3)-s(1)}{3-1} \\
&= \frac{[4(9)-3(3)+2]-[4(1)-3(1)+2]}{2} \\
&= \frac{(36-9+2)-(4-3+2)}{2} \\
&= \frac{29-3}{2} = \frac{26}{2} = 13 \text{ m/s}
\end{aligned}
$$

(b) The average velocity over $[1,1.5]$ is:

$$
\begin{aligned}
v_{\text{avg}} &= \frac{s(1.5)-s(1)}{1.5-1} \\
&= \frac{[4(2.25)-3(1.5)+2]-3}{0.5} \\
&= \frac{(9-4.5+2)-3}{0.5} \\
&= \frac{6.5-3}{0.5} = \frac{3.5}{0.5} = 7 \text{ m/s}
\end{aligned}
$$

(c) The instantaneous velocity at $t=1$ is the derivative $s'(1)$.

Using the $h\to 0$ form:

$$
\begin{aligned}
s'(1) &= \lim_{h\to 0}\frac{s(1+h)-s(1)}{h} \\
&= \lim_{h\to 0}\frac{[4(1+h)^2-3(1+h)+2]-3}{h} \\
&= \lim_{h\to 0}\frac{[4(1+2h+h^2)-3-3h+2]-3}{h} \\
&= \lim_{h\to 0}\frac{(4+8h+4h^2-3-3h+2)-3}{h} \\
&= \lim_{h\to 0}\frac{(3+5h+4h^2)-3}{h} \\
&= \lim_{h\to 0}\frac{5h+4h^2}{h} \\
&= \lim_{h\to 0}(5+4h) = 5 \text{ m/s}
\end{aligned}
$$

Using the $x\to a$ form:

$$
\begin{aligned}
s'(1) &= \lim_{t\to 1}\frac{s(t)-s(1)}{t-1} \\
&= \lim_{t\to 1}\frac{(4t^2-3t+2)-3}{t-1} \\
&= \lim_{t\to 1}\frac{4t^2-3t-1}{t-1} \\
&= \lim_{t\to 1}\frac{(t-1)(4t+1)}{t-1} \\
&= \lim_{t\to 1}(4t+1) = 5 \text{ m/s}
\end{aligned}
$$

Exercise 1.6.4

(a) The average velocity over $[1,5]$ is:

$$\begin{aligned} v_{\text{avg}} &= \frac{s(5)-s(1)}{5-1} \\ &= \frac{\sqrt{3(5)+1}-\sqrt{3(1)+1}}{4} \\ &= \frac{\sqrt{16}-\sqrt{4}}{4} \\ &= \frac{4-2}{4} = \frac{2}{4} = \frac{1}{2} \text{ ft/min} \end{aligned}$$

(b) The average velocity over $[1,3]$ is:

$$\begin{aligned} v_{\text{avg}} &= \frac{s(3)-s(1)}{3-1} \\ &= \frac{\sqrt{3(3)+1}-\sqrt{3(1)+1}}{2} \\ &= \frac{\sqrt{10}-2}{2} \text{ ft/min} \end{aligned}$$

(c) The instantaneous velocity at $t=1$ is the derivative $s'(1)$.
Using the $h \to 0$ form:

$$\begin{aligned} s'(1) &= \lim_{h\to 0} \frac{s(1+h)-s(1)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3(1+h)+1}-2}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3h+4}-2}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{3h+4}-2)(\sqrt{3h+4}+2)}{h(\sqrt{3h+4}+2)} \\ &= \lim_{h\to 0} \frac{(3h+4)-4}{h(\sqrt{3h+4}+2)} \\ &= \lim_{h\to 0} \frac{3h}{h(\sqrt{3h+4}+2)} \\ &= \lim_{h\to 0} \frac{3}{\sqrt{3h+4}+2} = \frac{3}{2+2} = \frac{3}{4} \text{ ft/min} \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} s'(1) &= \lim_{t\to 1} \frac{s(t)-s(1)}{t-1} \\ &= \lim_{t\to 1} \frac{\sqrt{3t+1}-2}{t-1} \\ &= \lim_{t\to 1} \frac{(\sqrt{3t+1}-2)(\sqrt{3t+1}+2)}{(t-1)(\sqrt{3t+1}+2)} \\ &= \lim_{t\to 1} \frac{(3t+1)-4}{(t-1)(\sqrt{3t+1}+2)} \\ &= \lim_{t\to 1} \frac{3t-3}{(t-1)(\sqrt{3t+1}+2)} \\ &= \lim_{t\to 1} \frac{3(t-1)}{(t-1)(\sqrt{3t+1}+2)} \\ &= \lim_{t\to 1} \frac{3}{\sqrt{3t+1}+2} = \frac{3}{2+2} = \frac{3}{4} \text{ ft/min} \end{aligned}$$

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1.7: Differentiability and Higher-Order Derivatives

Learning Objectives

- Define the derivative as a function.
- Determine whether a function is differentiable at a given point based on its graph.
- Define second, third, and other higher order derivatives of a function.

In this section, we transition from finding the derivative at a single point to treating the derivative as a function itself, examine the relationship between the graphs of $f(x)$ and $f'(x)$, establish the conditions for differentiability, and introduce higher-order derivatives.

Question of the Day

How does the graph of $f'(x)$ relate to the graph of $f(x)$?

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1.7: The Derivative Function

Definition: The Derivative as a Function

Instead of computing a separate limit for each specific point a, we can treat x as an independent variable to define the **derivative function** $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Equivalently, using the $x \to a$ form with a replaced by x:

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

The domain of f' consists of all values of x in the domain of f for which this limit exists.

Derivative Notation

The derivative of a function can be written in several equivalent ways:

- $f'(x)$ — Lagrange notation
- y' — Prime notation (when $y = f(x)$)
- $\frac{df}{dx}$ or $\frac{dy}{dx}$ — Leibniz notation
- $\frac{d}{dx}[f(x)]$ — Operator notation

All of these represent the same concept: the instantaneous rate of change of f with respect to x.

Example 1.7.1: Computing Derivative Functions

Use the limit definition to find the derivative function for each:

(a) $f(x) = x^2 - 2x + 3$

(b) $g(x) = \frac{1}{x^2}$

(c) $h(x) = 3x^2 + 5x - 7$

(d) $k(x) = \frac{1}{x+1}$

Solution:

(a) Using the $h \to 0$ form:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{[(x+h)^2 - 2(x+h) + 3] - [x^2 - 2x + 3]}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h + 3 - x^2 + 2x - 3}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} \\
&= \lim_{h\to 0} (2x + h - 2) = 2x - 2
\end{aligned}
$$

Using the $x \to a$ form (with a replaced by x):

$$
\begin{aligned}
f'(x) &= \lim_{t\to x} \frac{f(t) - f(x)}{t - x} \\
&= \lim_{t\to x} \frac{(t^2 - 2t + 3) - (x^2 - 2x + 3)}{t - x} \\
&= \lim_{t\to x} \frac{t^2 - x^2 - 2t + 2x}{t - x} \\
&= \lim_{t\to x} \frac{(t - x)(t + x) - 2(t - x)}{t - x} \\
&= \lim_{t\to x} (t + x - 2) = 2x - 2
\end{aligned}
$$

$$
\boxed{f'(x) = 2x - 2}
$$

(b) Using the $h \to 0$ form:

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{x^2 - (x+h)^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{h \cdot x^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2xh - h^2}{h \cdot x^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}
\end{aligned}
$$

Using the $x \to a$ form:

$$
\begin{aligned}
g'(x) &= \lim_{t\to x} \frac{\dfrac{1}{t^2} - \dfrac{1}{x^2}}{t - x} \\
&= \lim_{t\to x} \frac{\dfrac{x^2 - t^2}{x^2 t^2}}{t - x} \\
&= \lim_{t\to x} \frac{-(t - x)(t + x)}{x^2 t^2 (t - x)} \\
&= \lim_{t\to x} \frac{-(t + x)}{x^2 t^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}
\end{aligned}
$$

$$
\boxed{g'(x) = -\frac{2}{x^3}}
$$

(c) Using the $h \to 0$ form:

$$\begin{aligned} h'(x) &= \lim_{h\to 0} \frac{[3(x+h)^2 + 5(x+h) - 7] - [3x^2 + 5x - 7]}{h} \\ &= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 5x + 5h - 7 - 3x^2 - 5x + 7}{h} \\ &= \lim_{h\to 0} \frac{6xh + 3h^2 + 5h}{h} \\ &= \lim_{h\to 0} (6x + 3h + 5) = 6x + 5 \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} h'(x) &= \lim_{t\to x} \frac{(3t^2 + 5t - 7) - (3x^2 + 5x - 7)}{t - x} \\ &= \lim_{t\to x} \frac{3(t^2 - x^2) + 5(t - x)}{t - x} \\ &= \lim_{t\to x} \frac{(t - x)(3t + 3x + 5)}{t - x} \\ &= \lim_{t\to x} (3t + 3x + 5) = 6x + 5 \end{aligned}$$

$$\boxed{h'(x) = 6x + 5}$$

(d) Using the $h \to 0$ form:

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{\dfrac{1}{x+h+1} - \dfrac{1}{x+1}}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{(x+1) - (x+h+1)}{(x+h+1)(x+1)}}{h} \\ &= \lim_{h\to 0} \frac{-h}{h(x+h+1)(x+1)} \\ &= \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} = -\frac{1}{(x+1)^2} \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} k'(x) &= \lim_{t\to x} \frac{\dfrac{1}{t+1} - \dfrac{1}{x+1}}{t - x} \\ &= \lim_{t\to x} \frac{\dfrac{(x+1) - (t+1)}{(t+1)(x+1)}}{t - x} \\ &= \lim_{t\to x} \frac{-(t - x)}{(t - x)(t+1)(x+1)} \\ &= \lim_{t\to x} \frac{-1}{(t+1)(x+1)} = -\frac{1}{(x+1)^2} \end{aligned}$$

$$\boxed{k'(x) = -\frac{1}{(x+1)^2}}$$

Theorem: Differentiability and Continuity

A function f is **differentiable at** $x = a$ if $f'(a)$ exists and is finite.

Differentiability Implies Continuity: If f is differentiable at $x = a$, then f is continuous at $x = a$.

Note: The converse is false. Continuity does **not** guarantee differentiability. A continuous function fails to be differentiable at $x = a$ if it has:

- A **corner** or **cusp** (sharp turns where left-hand and right-hand derivative limits differ)
- A **vertical tangent line** (where the derivative limit evaluates to $\pm\infty$)
- A **discontinuity** (jump, removable, or infinite)

Illustration 1.7.1: Non-Differentiability at a Corner

The function $f(x) = |x|$ is not differentiable at $x = 0$, despite being continuous there.

First, we verify continuity at $x = 0$:

$$\lim_{x\to 0} |x| = 0 = f(0)$$

So f is continuous at $x = 0$.

Now check differentiability using the $h \to 0$ form:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

Evaluate the one-sided limits:

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

Using the $x \to a$ form:

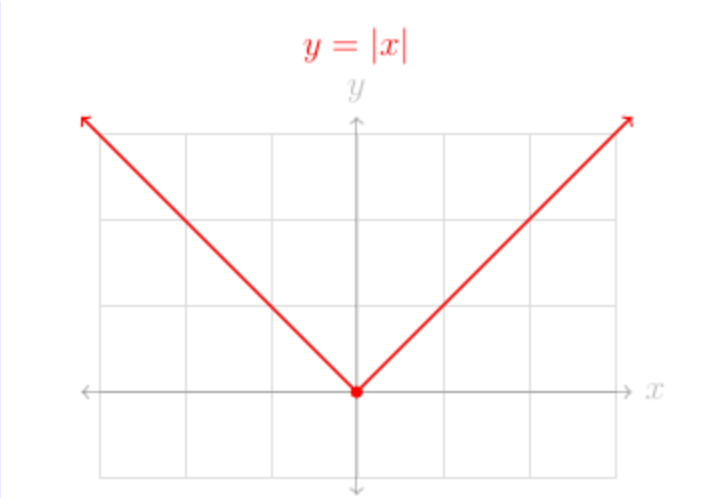
$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}$$

Again, the one-sided limits differ:

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1, \qquad \lim_{x\to 0^-} \frac{|x|}{x} = -1$$

Since the left-hand and right-hand derivative limits are not equal, the derivative at $x = 0$ does not exist.

The graph of $y = |x|$ has a sharp corner at the origin, which is why it is not differentiable there even though it is continuous.



Example 1.7.2: Parallel Tangent Lines

Is there a line tangent to $y = x^2$ which is parallel to $y = 6x + 5$?

Solution:

A line parallel to $y = 6x + 5$ must have a slope of $m = 6$. We need to find an x-value where the derivative (slope of the tangent line) equals 6.

Using the definition of the derivative at a point x:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x \end{aligned}$$

Setting the slope equal to 6:

$$2x = 6 \implies x = 3$$

At $x = 3$, $y = 3^2 = 9$. The point of tangency is $(3, 9)$, and the equation of the tangent line is:

$$y - 9 = 6(x - 3)$$

$$\boxed{y = 6x - 9}$$

Example 1.7.3: Finding Constants for Differentiability

What values of a and b will make the function differentiable at $x = 1$?

$$f(x) = \begin{cases} \dfrac{2x}{x-2} & \text{if } x \leq 1 \\ ax^2 + b & \text{if } x > 1 \end{cases}$$

Solution:

1. Enforce Continuity at $x = 1$:

For the function to be continuous, the left-hand and right-hand limits must equal $f(1)$.

$$f(1) = \frac{2(1)}{1-2} = -2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{2x}{x-2} = -2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(ax^2 + b\right) = a(1)^2 + b = a + b$$

Setting these equal gives: $a + b = -2$.

2. Enforce Equal One-Sided Derivatives at $x = 1$:

Using the $h \to 0$ form for the left side of the derivative limit gives:

$$\begin{aligned} f'(1^-) &= \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0^-} \frac{\dfrac{2(1+h)}{(1+h)-2} - (-2)}{h} \\ &= \lim_{h \to 0^-} \frac{\dfrac{2+2h}{h-1} + 2}{h} \\ &= \lim_{h \to 0^-} \frac{\dfrac{2+2h+2h-2}{h-1}}{h} \\ &= \lim_{h \to 0^-} \frac{\dfrac{4h}{h-1}}{h} \\ &= \lim_{h \to 0^-} \frac{4}{h-1} = -4 \end{aligned}$$

So $f'(1^-) = -\dfrac{4}{(1-2)^2} = -4$.

Using the $h \to 0$ form for the right side of the derivative limit gives:

$$\begin{aligned} f'(1^+) &= \lim_{h \to 0^+} \frac{a(1+h)^2 + b - (a+b)}{h} \\ &= \lim_{h \to 0^+} \frac{a(1+2h+h^2) + b - a - b}{h} \\ &= \lim_{h \to 0^+} \frac{2ah + ah^2}{h} = \lim_{h \to 0^+} (2a + ah) = 2a \end{aligned}$$

Setting the one-sided derivatives equal:

$$2a = -4 \implies a = -2$$

Substituting $a = -2$ into $a + b = -2$ gives $b = 0$ so our final answers are:

$$\boxed{a = -2,\ b = 0}$$

Example 1.7.4: Finding Constants for Differentiability (Another Case)

Find the values of m and b that make $g(x) = \begin{cases} x^2 & x \leq 2 \\ mx + b & x > 2 \end{cases}$ differentiable at $x = 2$.

Solution:

1. Enforce Continuity at $x = 2$:

For the function to be continuous, the left-hand and right-hand limits must equal $g(2)$.

$$g(2) = 2^2 = 4$$

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} x^2 = 4$$

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (mx + b) = m(2) + b = 2m + b$$

Setting these equal gives: $2m + b = 4$.

2. Enforce Equal One-Sided Derivatives at $x = 2$:

Using the $h \to 0$ form for the left side of the derivative limit gives:

$$\begin{aligned} g'(2^-) &= \lim_{h \to 0^-} \frac{g(2+h) - g(2)}{h} \\ &= \lim_{h \to 0^-} \frac{(2+h)^2 - 4}{h} \\ &= \lim_{h \to 0^-} \frac{4 + 4h + h^2 - 4}{h} \\ &= \lim_{h \to 0^-} \frac{4h + h^2}{h} \\ &= \lim_{h \to 0^-} (4 + h) = 4 \end{aligned}$$

Using the $h \to 0$ form for the right side of the derivative limit gives:

$$\begin{aligned} g'(2^+) &= \lim_{h \to 0^+} \frac{g(2+h) - g(2)}{h} \\ &= \lim_{h \to 0^+} \frac{[m(2+h) + b] - 4}{h} \\ &= \lim_{h \to 0^+} \frac{2m + mh + b - 4}{h} \end{aligned}$$

For this limit to exist, the constant term must be zero (since $h \to 0$, a nonzero constant over h would diverge). From the continuity equation $2m + b = 4$, we have $2m + b - 4 = 0$, so the constant term is already zero.

Thus:

$$g'(2^+) = \lim_{h \to 0^+} \frac{mh}{h} = \lim_{h \to 0^+} m = m$$

Setting the one-sided derivatives equal:

$$m = 4$$

Substituting $m = 4$ into $2m + b = 4$:

$$2(4) + b = 4 \implies 8 + b = 4 \implies b = -4$$

Thus our final answers are:

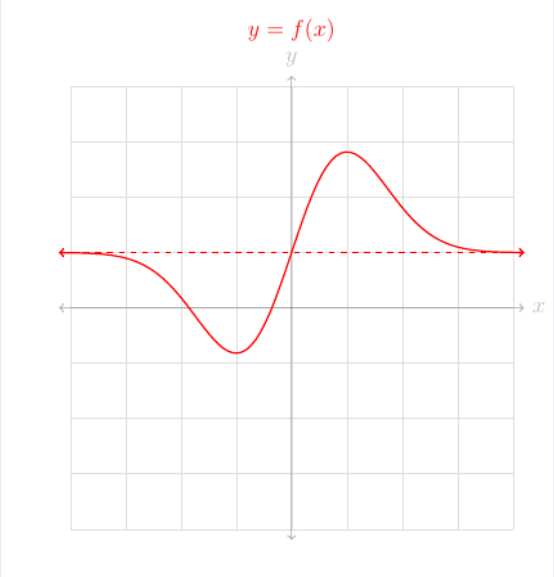
$$\boxed{m = 4,\ b = -4}$$

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1.7: Higher Order Derivatives

Illustration \(\PageIndex{2}\): Sketching the Derivative from a Graph

Given the graph of \(y = f(x)\) below, sketch the graph of its derivative \(y = f'(x)\).



To sketch the derivative, we analyze the slope of the tangent line at various points:

- Where \(f\) is increasing, \(f'(x) > 0\) (the derivative is positive)
- Where \(f\) is decreasing, \(f'(x) < 0\) (the derivative is negative)
- Where \(f\) has a horizontal tangent (local max/min or stationary point), \(f'(x) = 0\)

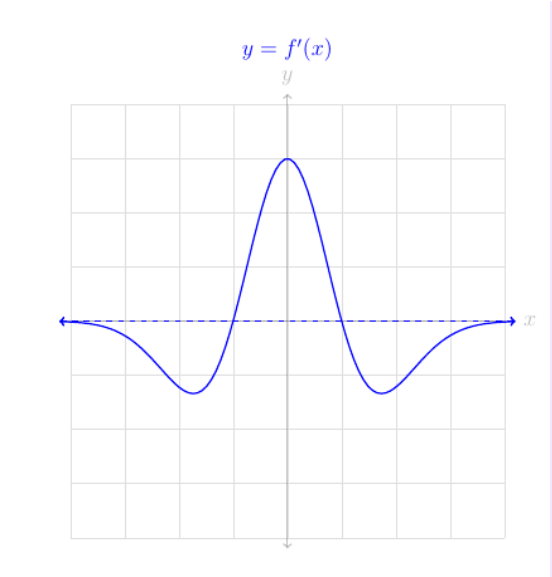
The easiest points to graph for a derivative are places where the function has horizontal tangent lines (slope equal to zero). This occurs twice on our graph of \(y=f(x)\), once at \(x=-1\) and once at \(x=1\). This means that we have \(f'(-1)=0\) and \(f'(1)=0\).

For \(x<-1\), the function starts with a very small negative slope and decreases until it reaches \(x=-1\). The slope of the tangent line is steepest at around \(x=-1.7\), with an approximate slope of \(m=-1.3\). Therefore, we draw the graph of \(y=f'(x)\) to start barely negative on the far left, and dip down to \(f'(-1.7)=-1.3\), then return up toward \(f'(-1)=0\). (your estimates may vary, but should be in the same ballpark).

For \(-1

For \(x>1\), the function is decreasing again. The slopes of the tangent lines begin with \(f'(1)=0\), then become negative, reaching their steepest point around \(x=1.7\), with an approximate slope of \(m=-1.3\). After this point, slopes remain negative, but are less and less steep, tending toward zero. Therefore, we draw the graph of \(y=f'(x)\) to start at zero, dip down to \(f'(1.7)=-1.3\), and then return up toward zero on the far right end.

We use a horizontal asymptote to denote that fact that the slopes tend toward zero on the far edges of the graph of \(y=f'(x)\), as pictured below:



Definition: Higher-Order Derivatives

If \(f'\) is itself a differentiable function, its derivative is called the **second derivative**:

\[f''(x) = \dfrac{d}{dx}[f'(x)] = \lim_{h \to 0} \dfrac{f'(x + h) - f'(x)}{h} \nonumber \]

Notation: \(f''(x)\), \(y''\), or \(\dfrac{d^2f}{dx^2}\).

In general, the \(n^{\text{th}}\) derivative is obtained by differentiating iteratively \(n\) times:

\[f^{(n)}(x) = \dfrac{d^n f}{dx^n} = y^{(n)} \nonumber \]

Higher-order derivatives have important interpretations in physics (acceleration is the second derivative of position) and geometry (curvature).

Example \(\PageIndex{5}\): Higher-Order Derivatives

(a) For \(f(x) = x^3 - 3x^2 + 2x - 1\), find \(f'''(x)\)

(b) For \(g(x) = \sqrt{x}\), find \(g''(100)\)

Solution:

(a) Compute successive derivatives:

\[f'(x) = 3x^2 - 6x + 2 \nonumber \]

\[f''(x) = 6x - 6 \nonumber \]

\[f'''(x) = 6 \nonumber \]

\[\boxed{f'''(x) = 6} \nonumber \]

(b) Compute the first and second derivatives of \(g(x) = x^{1/2}\):

\[g'(x) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}} \nonumber \]

$$g''(x) = -\dfrac{1}{4}x^{-3/2} = -\dfrac{1}{4\sqrt{x^3}} \nonumber$$

Evaluate at $x = 100$:

$$g''(100) = -\dfrac{1}{4\sqrt{100^3}} = -\dfrac{1}{4(1000)} = -\dfrac{1}{4000} \nonumber$$

$$\boxed{g''(100) = -\dfrac{1}{4000}} \nonumber$$

Example $\PageIndex{6}$: Differentiability Analysis

Consider the function $f(x) = \begin{cases} x^{4/3} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Is $f(x)$ continuous at $x = 0$?

(b) Is $f(x)$ differentiable at $x = 0$?

(c) Is $f''(0)$ defined?

Solution:

(a) Check continuity at $x = 0$:

$$\lim_{x \to 0} x^{4/3} = 0 = f(0) \nonumber$$

$$\boxed{\text{Yes, } f \text{ is continuous at } x = 0.} \nonumber$$

(b) Evaluate the derivative limit at $x = 0$ using the $h \to 0$ form:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{h^{4/3} - 0}{h} \\[4pt] &= \lim_{h \to 0} h^{1/3} = 0 \end{aligned} \nonumber$$

Using the $x \to a$ form:

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} \\[4pt] &= \lim_{x \to 0} \dfrac{x^{4/3}}{x} \\[4pt] &= \lim_{x \to 0} x^{1/3} = 0 \end{aligned} \nonumber$$

$$\boxed{\text{Yes, } f \text{ is differentiable at } x = 0,\ f'(0) = 0.} \nonumber$$

(c) For $x \neq 0$, compute the derivative:

$$f'(x) = \dfrac{4}{3}x^{1/3} \nonumber$$

Now check if $f''(0)$ exists:

$$\begin{aligned} f''(0) &= \lim_{h \to 0} \dfrac{f'(h) - f'(0)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{\dfrac{4}{3}h^{1/3} - 0}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{4}{3} h^{-2/3} = \infty \end{aligned} \nonumber$$

Using the $x \to a$ form:

$$\begin{aligned} f''(0) &= \lim_{x \to 0} \dfrac{f'(x) - f'(0)}{x - 0} \\[4pt] &= \lim_{x \to 0} \dfrac{\dfrac{4}{3}x^{1/3}}{x} \\[4pt] &= \lim_{x \to 0} \dfrac{4}{3} x^{-2/3} = \infty \end{aligned} \nonumber$$

Since the limit evaluates to ∞, the second derivative does not exist as a finite value.

$$\boxed{f''(0) \text{ does not exist (infinite).}} \nonumber$$

Example \(\PageIndex{7}\): Higher-Order Differentiability

What values of \(a, b, c \in \mathbb{R}\) make \(h''(-1)\) exist for

$$h(x) = \begin{cases} 2x^3 + 1 & \text{if } x \ge -1 \\ ax^2 + bx + c & \text{if } x < -1 \end{cases} \nonumber$$

Solution:

For \(h''(-1)\) to exist, \(h(x)\), \(h'(x)\), and \(h''(x)\) must all be continuous at \(x = -1\).

1. **Continuity of \(h(x)\) at \(x = -1\)**:

$$h(-1) = 2(-1)^3 + 1 = -1 \nonumber$$

$$\lim_{x \to -1^-} h(x) = a(-1)^2 + b(-1) + c = a - b + c \nonumber$$

Equation 1: \(a - b + c = -1\)

2. **Continuity of \(h'(x)\) at \(x = -1\)**:

For \(x \ge -1\), \(h'(x) = 6x^2 \implies h'(-1) = 6\).

For \(x < -1\), \(h'(x) = 2ax + b \implies \lim_{x \to -1^-} h'(x) = -2a + b\).

Equation 2: \(-2a + b = 6\)

3. **Continuity of \(h''(x)\) at \(x = -1\)**:

For \(x \ge -1\), \(h''(x) = 12x \implies h''(-1) = -12\).

For \(x < -1\), \(h''(x) = 2a \implies \lim_{x \to -1^-} h''(x) = 2a\).

Setting these equal: \(2a = -12 \implies a = -6\).

Substitute \(a = -6\) into Equation 2:

$$-2(-6) + b = 6 \implies 12 + b = 6 \implies b = -6 \nonumber$$

Substitute \(a = -6\) and \(b = -6\) into Equation 1:

$$(-6) - (-6) + c = -1 \implies c = -1 \nonumber$$

$$\boxed{a = -6,\ b = -6,\ c = -1} \nonumber$$

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1.7: Practice Set

Exercise \(\PageIndex{1}\)

Use the limit definition to find the derivative function for each. Show both the \(h \to 0\) and \(x \to a\) methods.

(a) \(f(x) = 5x - 3\)

(b) \(g(x) = 2x^2 - 4x + 1\)

(c) \(h(x) = \sqrt{2x + 5}\)

(d) \(p(x) = \dfrac{3x - 1}{x + 2}\)

(e) \(q(x) = \dfrac{\sqrt{x}}{x + 1}\)

Answer available at the end of this section.

Exercise \(\PageIndex{2}\)

Find the second derivative of each function.

(a) \(f(x) = 3x^2 - 2x + 7\)

(b) \(g(x) = 2x^3 - 5x^2 + 4x - 1\)

(c) \(h(x) = \dfrac{1}{x - 3}\)

(d) \(p(x) = \dfrac{2x + 1}{x - 4}\)

Answer available at the end of this section.

Exercise \(\PageIndex{3}\)

For each piecewise function, find all values of \(x\) where the function is not differentiable. Justify your answer.

(a) \(f(x) = \begin{cases} x^2 + 1 & x \le 2 \\ 5x - 3 & x > 2 \end{cases}\)

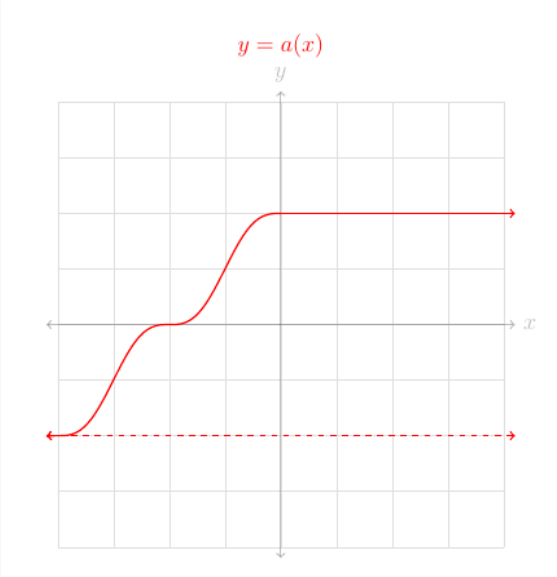
(b) \(g(x) = \begin{cases} 2x + 3 & x \le 1 \\ x^2 + 4 & x > 1 \end{cases}\)

(c) \(h(x) = \begin{cases} x^2 & x \le 0 \\ 2x & 0 < x \le 2 \\ -x + 6 & x > 2 \end{cases}\)

Answer available at the end of this section.

Exercise \(\PageIndex{4a}\)

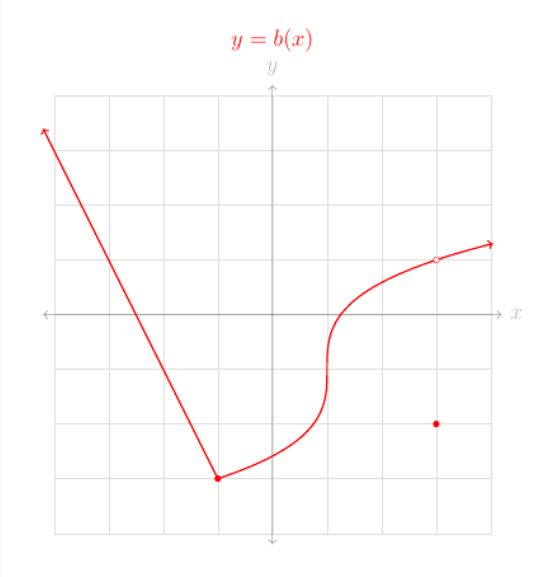
Given the graph of the function below, sketch the graph of its derivative.



Answer available at the end of this section.

Exercise \(\PageIndex{4b}\)

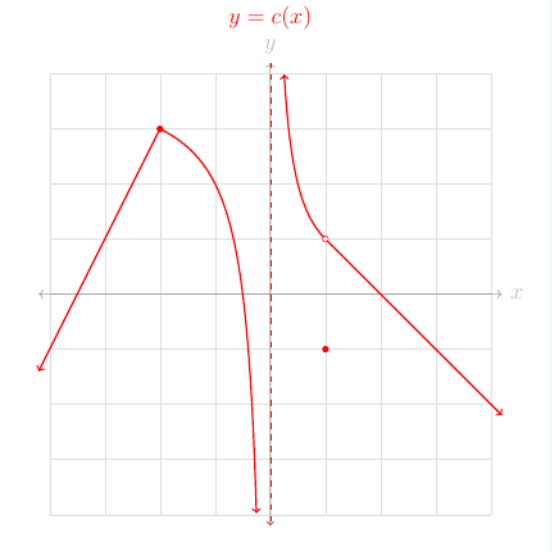
Given the graph of the function below, sketch the graph of its derivative.



Answer available at the end of this section.

Exercise \(\PageIndex{4c}\)

Given the graph of the below, sketch the graph of its derivative.



Answer available at the end of this section.

Exercise \(\PageIndex{5}\): Finding Constants for Differentiability

Find the values of \(a\), \(b\), and \(c\) that make \(h''(2)\) exist for:

$$h(x) = \begin{cases} x^3 - 4 & \text{if } x \ge 2 \\ ax^2 + bx + c & \text{if } x < 2 \end{cases}$$

Answer available at the end of this section.

Answers

Exercise \(\PageIndex{1}\)

(a) Using the \(h \to 0\) form:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[5(x+h) - 3] - [5x - 3]}{h} = \lim_{h \to 0} \frac{5x + 5h - 3 - 5x + 3}{h} = \lim_{h \to 0} \frac{5h}{h} = \lim_{h \to 0} 5 = 5$$

Using the \(x \to a\) form:

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{(5t - 3) - (5x - 3)}{t - x} = \lim_{t \to x} \frac{5t - 5x}{t - x} = \lim_{t \to x} \frac{5(t - x)}{t - x} = \lim_{t \to x} 5 = 5$$

Thus \(f'(x) = 5\).

(b) Using the \(h \to 0\) form:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{[2(x+h)^2 - 4(x+h) + 1] - [2x^2 - 4x + 1]}{h} = \lim_{h \to 0} \frac{[2(x^2 + 2xh + h^2) - 4x - 4h + 1] - [2x^2 - 4x + 1]}{h} = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 4x - 4h + 1 - 2x^2 + 4x - 1}{h} = \lim_{h \to 0} \frac{4xh + 2h^2 - 4h}{h} = \lim_{h \to 0} (4x + 2h - 4) = 4x - 4$$

Using the \(x \to a\) form:

$$g'(x) = \lim_{t \to x} \frac{g(t) - g(x)}{t - x} = \lim_{t \to x} \frac{(2t^2 - 4t + 1) - (2x^2 - 4x + 1)}{t - x} = \lim_{t \to x} \frac{2(t^2 - x^2) - 4(t - x)}{t - x} = \lim_{t \to x} \frac{2(t - x)(t + x) - 4(t - x)}{t - x} = \lim_{t \to x} \frac{(t - x)(2t + 2x - 4)}{t - x} = \lim_{t \to x} (2t + 2x - 4) = 2x + 2x - 4 = 4x - 4$$

Thus \(g'(x) = 4x - 4\).

(c) Using the \(h \to 0\) form:

$$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h} = \lim_{h \to 0} \frac{\sqrt{2(x+h) + 5} - \sqrt{2x + 5}}{h} = \lim_{h \to 0} \frac{\sqrt{2x + 2h + 5} - \sqrt{2x + 5}}{h} = \lim_{h \to 0} \frac{(\sqrt{2x + 2h + 5} - \sqrt{2x + 5})(\sqrt{2x + 2h + 5} + \sqrt{2x + 5})}{h(\sqrt{2x + 2h + 5} + \sqrt{2x + 5})} = \lim_{h \to 0} \frac{(2x + 2h + 5) - (2x + 5)}{h(\sqrt{2x + 2h + 5} + \sqrt{2x + 5})} = \lim_{h \to 0} \frac{2h}{h(\sqrt{2x + 2h + 5} + \sqrt{2x + 5})} = \lim_{h \to 0} \frac{2}{\sqrt{2x + 2h + 5} + \sqrt{2x + 5}} = \frac{2}{\sqrt{2x + 5} + \sqrt{2x + 5}} = \frac{2}{2\sqrt{2x + 5}} = \frac{1}{\sqrt{2x + 5}}$$

Using the \(x \to a\) form:

$$h'(x) = \lim_{t \to x} \frac{h(t) - h(x)}{t - x} = \lim_{t \to x} \frac{\sqrt{2t + 5} - \sqrt{2x + 5}}{t - x} = \lim_{t \to x} \frac{(\sqrt{2t + 5} - \sqrt{2x + 5})(\sqrt{2t + 5} + \sqrt{2x + 5})}{(t - x)(\sqrt{2t + 5} + \sqrt{2x + 5})} = \lim_{t \to x} \frac{(2t + 5) - (2x + 5)}{(t - x)(\sqrt{2t + 5} + \sqrt{2x + 5})} = \lim_{t \to x} \frac{2(t - x)}{(t - x)(\sqrt{2t + 5} + \sqrt{2x + 5})} = \lim_{t \to x} \frac{2}{\sqrt{2t + 5} + \sqrt{2x + 5}} = \frac{2}{\sqrt{2x + 5} + \sqrt{2x + 5}} = \frac{2}{2\sqrt{2x + 5}} = \frac{1}{\sqrt{2x + 5}}$$

Thus $h'(x) = \dfrac{1}{\sqrt{2x + 5}}$.

(d) Using the $h \to 0$ form:

$$\begin{aligned} p'(x) &= \lim_{h \to 0} \dfrac{p(x+h) - p(x)}{h} \\ &= \lim_{h \to 0} \dfrac{\dfrac{3(x+h) - 1}{(x+h) + 2} - \dfrac{3x - 1}{x + 2}}{h} \\ &= \lim_{h \to 0} \dfrac{\dfrac{3x + 3h - 1}{x + h + 2} - \dfrac{3x - 1}{x + 2}}{h} \\ &= \lim_{h \to 0} \dfrac{\dfrac{(3x + 3h - 1)(x + 2) - (3x - 1)(x + h + 2)}{(x + h + 2)(x + 2)}}{h} \\ &= \lim_{h \to 0} \dfrac{(3x + 3h - 1)(x + 2) - (3x - 1)(x + h + 2)}{h(x + h + 2)(x + 2)} \end{aligned}$$

Expand the numerator:

$$\begin{aligned} &(3x + 3h - 1)(x + 2) - (3x - 1)(x + h + 2) \\ &= (3x^2 + 6x + 3xh + 6h - x - 2) - (3x^2 + 3xh + 6x - x - h - 2) \\ &= (3x^2 + 5x + 3xh + 6h - 2) - (3x^2 + 5x + 3xh - h - 2) \\ &= 3x^2 + 5x + 3xh + 6h - 2 - 3x^2 - 5x - 3xh + h + 2 \\ &= 7h \end{aligned}$$

Thus:

$$p'(x) = \lim_{h \to 0} \dfrac{7h}{h(x + h + 2)(x + 2)} = \lim_{h \to 0} \dfrac{7}{(x + h + 2)(x + 2)} = \dfrac{7}{(x + 2)^2}$$

Using the $x \to a$ form:

$$\begin{aligned} p'(x) &= \lim_{t \to x} \dfrac{p(t) - p(x)}{t - x} \\ &= \lim_{t \to x} \dfrac{\dfrac{3t - 1}{t + 2} - \dfrac{3x - 1}{x + 2}}{t - x} \\ &= \lim_{t \to x} \dfrac{(3t - 1)(x + 2) - (3x - 1)(t + 2)}{(t - x)(t + 2)(x + 2)} \\ &= \lim_{t \to x} \dfrac{(3tx + 6t - x - 2) - (3xt + 6x - t - 2)}{(t - x)(t + 2)(x + 2)} \\ &= \lim_{t \to x} \dfrac{3tx + 6t - x - 2 - 3xt - 6x + t + 2}{(t - x)(t + 2)(x + 2)} \\ &= \lim_{t \to x} \dfrac{7t - 7x}{(t - x)(t + 2)(x + 2)} \\ &= \lim_{t \to x} \dfrac{7(t - x)}{(t - x)(t + 2)(x + 2)} \\ &= \lim_{t \to x} \dfrac{7}{(t + 2)(x + 2)} = \dfrac{7}{(x + 2)(x + 2)} = \dfrac{7}{(x + 2)^2} \end{aligned}$$

Thus $p'(x) = \dfrac{7}{(x + 2)^2}$.

(e) Using the $h \to 0$ form:

$$\begin{aligned} q'(x) &= \lim_{h \to 0} \dfrac{q(x+h) - q(x)}{h} \\ &= \lim_{h \to 0} \dfrac{\dfrac{\sqrt{x + h}}{x + h + 1} - \dfrac{\sqrt{x}}{x + 1}}{h} \\ &= \lim_{h \to 0} \dfrac{\sqrt{x + h}(x + 1) - \sqrt{x}(x + h + 1)}{h(x + h + 1)(x + 1)} \end{aligned}$$

Rewrite the numerator:

$$\begin{aligned} &\sqrt{x + h}(x + 1) - \sqrt{x}(x + h + 1) \\ &= (x + 1)(\sqrt{x + h} - \sqrt{x}) - h\sqrt{x} \end{aligned}$$

Thus:

$$q'(x) = \lim_{h \to 0} \dfrac{(x + 1)(\sqrt{x + h} - \sqrt{x}) - h\sqrt{x}}{h(x + h + 1)(x + 1)}$$

Rationalize $\sqrt{x + h} - \sqrt{x}$:

$$\sqrt{x + h} - \sqrt{x} = \dfrac{(x + h) - x}{\sqrt{x + h} + \sqrt{x}} = \dfrac{h}{\sqrt{x + h} + \sqrt{x}}$$

Substitute:

$$\begin{aligned} q'(x) &= \lim_{h \to 0} \dfrac{(x + 1)\left(\dfrac{h}{\sqrt{x + h} + \sqrt{x}}\right) - h\sqrt{x}}{h(x + h + 1)(x + 1)} \\ &= \lim_{h \to 0} \dfrac{h\left(\dfrac{x + 1}{\sqrt{x + h} + \sqrt{x}} - \sqrt{x}\right)}{h(x + h + 1)(x + 1)} \\ &= \lim_{h \to 0} \dfrac{\dfrac{x + 1}{\sqrt{x + h} + \sqrt{x}} - \sqrt{x}}{(x + h + 1)(x + 1)} \\ &= \dfrac{\dfrac{x + 1}{2\sqrt{x}} - \sqrt{x}}{(x + 1)(x + 1)} \\ &= \dfrac{\dfrac{x + 1 - 2x}{2\sqrt{x}}}{(x + 1)^2} \\ &= \dfrac{1 - x}{2\sqrt{x}(x + 1)^2} \end{aligned}$$

Using the $x \to a$ form:

$$\begin{aligned} q'(x) &= \lim_{t \to x} \dfrac{q(t) - q(x)}{t - x} \\ &= \lim_{t \to x} \dfrac{\dfrac{\sqrt{t}}{t + 1} - \dfrac{\sqrt{x}}{x + 1}}{t - x} \\ &= \lim_{t \to x} \dfrac{\sqrt{t}(x + 1) - \sqrt{x}(t + 1)}{(t - x)(t + 1)(x + 1)} \\ &= \lim_{t \to x} \dfrac{x\sqrt{t} + \sqrt{t} - t\sqrt{x} - \sqrt{x}}{(t - x)(t + 1)(x + 1)} \end{aligned}$$

Rewrite the numerator:

$$x\sqrt{t} + \sqrt{t} - t\sqrt{x} - \sqrt{x} = (x + 1)(\sqrt{t} - \sqrt{x}) - \sqrt{x}(t - x)$$

Thus:

$$q'(x) = \lim_{t \to x} \dfrac{(x + 1)(\sqrt{t} - \sqrt{x}) - \sqrt{x}(t - x)}{(t - x)(t + 1)(x + 1)}$$

Using $\sqrt{t} - \sqrt{x} = \dfrac{t - x}{\sqrt{t} + \sqrt{x}}$:

$$\begin{aligned} q'(x) &= \lim_{t \to x} \dfrac{(x + 1)\left(\dfrac{t - x}{\sqrt{t} + \sqrt{x}}\right) - \sqrt{x}(t - x)}{(t - x)(t + 1)(x + 1)} \\ &= \lim_{t \to x} \dfrac{(t - x)\left(\dfrac{x + 1}{\sqrt{t} + \sqrt{x}} - \sqrt{x}\right)}{(t - x)(t + 1)(x + 1)} \\ &= \lim_{t \to x} \dfrac{\dfrac{x + 1}{\sqrt{t} + \sqrt{x}} - \sqrt{x}}{(t + 1)(x + 1)} \\ &= \dfrac{\dfrac{x + 1}{2\sqrt{x}} - \sqrt{x}}{(x + 1)(x + 1)} \\ &= \dfrac{\dfrac{x + 1 - 2x}{2\sqrt{x}}}{(x + 1)^2} \\ &= \dfrac{1 - x}{2\sqrt{x}(x + 1)^2} \end{aligned}$$

Thus $q'(x) = \dfrac{1 - x}{2\sqrt{x}(x + 1)^2}$.

Exercise $\PageIndex{2}$

(a) Using the $h \to 0$ form for the first derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \dfrac{[3(x+h)^2 - 2(x+h) + 7] - [3x^2 - 2x + 7]}{h} \\ &= \lim_{h \to 0} \dfrac{[3x^2 + 6xh + 3h^2 - 2x - 2h + 7] - [3x^2 - 2x + 7]}{h} \\ &= \lim_{h \to 0} \dfrac{6xh + 3h^2 - 2h}{h} \\ &= \lim_{h \to 0} (6x + 3h - 2) = 6x - 2 \end{aligned}$$

Using the $x \to a$ form for the first derivative:

$$\begin{aligned} f'(x) &= \lim_{t \to x} \dfrac{f(t) - f(x)}{t - x} \\ &= \lim_{t \to x} \dfrac{(3t^2 - 2t + 7) - (3x^2 - 2x + 7)}{t - x} \\ &= \lim_{t \to x} \dfrac{3(t^2 - x^2) - 2(t - x)}{t - x} \\ &= \lim_{t \to x} \dfrac{(t - x)(3t + 3x - 2)}{t - x} \\ &= \lim_{t \to x} (3t + 3x - 2) = 6x - 2 \end{aligned}$$

Now find $f''(x)$ using the $h \to 0$ form:

$$\begin{aligned} f''(x) &= \lim_{h \to 0} \dfrac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \to 0} \dfrac{[6(x+h) - 2] - [6x - 2]}{h} \\ &= \lim_{h \to 0} \dfrac{6x + 6h - 2 - 6x + 2}{h} \\ &= \lim_{h \to 0} \dfrac{6h}{h} = 6 \end{aligned}$$

Using the $x \to a$ form for the second derivative:

$$\begin{aligned} f''(x) &= \lim_{t \to x} \dfrac{f'(t) - f'(x)}{t - x} \\ &= \lim_{t \to x} \dfrac{(6t - 2) - (6x - 2)}{t - x} \\ &= \lim_{t \to x} \dfrac{6t - 6x}{t - x} \\ &= \lim_{t \to x} \dfrac{6(t - x)}{t - x} = 6 \end{aligned}$$

Thus $f''(x) = 6$.

(b) Using the $h \to 0$ form for the first derivative:

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \dfrac{[2(x+h)^3 - 5(x+h)^2 + 4(x+h) - 1] - [2x^3 - 5x^2 + 4x - 1]}{h} \\ &= \lim_{h \to 0} \dfrac{[2x^3 + 6x^2h + 6xh^2 + 2h^3 - 5x^2 - 10xh - 5h^2 + 4x + 4h - 1] - [2x^3 - 5x^2 + 4x - 1]}{h} \\ &= \lim_{h \to 0} \dfrac{6x^2h + 6xh^2 + 2h^3 - 10xh - 5h^2 + 4h}{h} \\ &= \lim_{h \to 0} (6x^2 + 6xh + 2h^2 - 10x - 5h + 4) = 6x^2 - 10x + 4 \end{aligned}$$

Using the $x \to a$ form for the first derivative:

$$\begin{aligned} g'(x) &= \lim_{t \to x} \dfrac{g(t) - g(x)}{t - x} \\ &= \lim_{t \to x} \dfrac{(2t^3 - 5t^2 + 4t - 1) - (2x^3 - 5x^2 + 4x - 1)}{t - x} \\ &= \lim_{t \to x} \dfrac{2(t^3 - x^3) - 5(t^2 - x^2) + 4(t - x)}{t - x} \\ &= \lim_{t \to x} \dfrac{(t - x)(2t^2 + 2tx + 2x^2 - 5t - 5x + 4)}{t - x} \\ &= \lim_{t \to x} (2t^2 +$$

$$2tx + 2x^2 - 5t - 5x + 4) \\[4pt] &= 2x^2 + 2x^2 + 2x^2 - 5x - 5x + 4 = 6x^2 - 10x + 4 \end{align*} \nonumber \]$$

Now find $g''(x)$ using the $h \to 0$ form:

$$\[\begin{align*} g''(x) &= \lim_{h \to 0} \dfrac{g'(x+h) - g'(x)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{[6(x+h)^2 - 10(x+h) + 4] - [6x^2 - 10x + 4]}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{[6x^2 + 12xh + 6h^2 - 10x - 10h + 4] - [6x^2 - 10x + 4]}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{12xh + 6h^2 - 10h}{h} \\[4pt] &= \lim_{h \to 0} (12x + 6h - 10) = 12x - 10 \end{align*} \nonumber \]$$

Using the $x \to a$ form for the second derivative:

$$\[\begin{align*} g''(x) &= \lim_{t \to x} \dfrac{g'(t) - g'(x)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{(6t^2 - 10t + 4) - (6x^2 - 10x + 4)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{6(t^2 - x^2) - 10(t - x)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{(t - x)(6t + 6x - 10)}{t - x} \\[4pt] &= \lim_{t \to x} (6t + 6x - 10) = 12x - 10 \end{align*} \nonumber \]$$

Thus $g''(x) = 12x - 10$.

(c) Using the $h \to 0$ form for the first derivative:

$$\[\begin{align*} h'(x) &= \lim_{h \to 0} \dfrac{h(x+h) - h(x)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{\dfrac{1}{x+h-3} - \dfrac{1}{x-3}}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{\dfrac{(x-3) - (x+h-3)}{(x+h-3)(x-3)}}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{-h}{h(x+h-3)(x-3)} \\[4pt] &= \lim_{h \to 0} \dfrac{-1}{(x+h-3)(x-3)} = -\dfrac{1}{(x-3)^2} \end{align*} \nonumber \]$$

Using the $x \to a$ form for the first derivative:

$$\[\begin{align*} h'(x) &= \lim_{t \to x} \dfrac{h(t) - h(x)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{\dfrac{1}{t-3} - \dfrac{1}{x-3}}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{\dfrac{(x-3) - (t-3)}{(t-3)(x-3)}}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{-(t - x)}{(t - x)(t-3)(x-3)} \\[4pt] &= \lim_{t \to x} \dfrac{-1}{(t-3)(x-3)} = -\dfrac{1}{(x-3)^2} \end{align*} \nonumber \]$$

Now find $h''(x)$ using the $h \to 0$ form:

$$\[\begin{align*} h''(x) &= \lim_{h \to 0} \dfrac{h'(x+h) - h'(x)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{-\dfrac{1}{(x+h-3)^2} + \dfrac{1}{(x-3)^2}}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{\dfrac{1}{(x-3)^2} - \dfrac{1}{(x+h-3)^2}}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{\dfrac{(x+h-3)^2 - (x-3)^2}{(x-3)^2(x+h-3)^2}}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{(x+h-3 - x + 3)(x+h-3 + x - 3)}{h(x-3)^2(x+h-3)^2} \\[4pt] &= \lim_{h \to 0} \dfrac{h(2x + h - 6)}{h(x-3)^2(x+h-3)^2} \\[4pt] &= \lim_{h \to 0} \dfrac{2x + h - 6}{(x-3)^2(x+h-3)^2} = \dfrac{2x - 6}{(x-3)^4} = \dfrac{2}{(x-3)^3} \end{align*} \nonumber \]$$

Using the $x \to a$ form for the second derivative:

$$\[\begin{align*} h''(x) &= \lim_{t \to x} \dfrac{h'(t) - h'(x)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{-\dfrac{1}{(t-3)^2} + \dfrac{1}{(x-3)^2}}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{\dfrac{1}{(x-3)^2} - \dfrac{1}{(t-3)^2}}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{\dfrac{(t-3)^2 - (x-3)^2}{(x-3)^2(t-3)^2}}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{(t-3 - x + 3)(t-3 + x - 3)}{(t - x)(x-3)^2(t-3)^2} \\[4pt] &= \lim_{t \to x} \dfrac{(t - x)(t + x - 6)}{(t - x)(x-3)^2(t-3)^2} \\[4pt] &= \lim_{t \to x} \dfrac{t + x - 6}{(x-3)^2(t-3)^2} = \dfrac{2x - 6}{(x-3)^4} = \dfrac{2}{(x-3)^3} \end{align*} \nonumber \]$$

Thus $h''(x) = \dfrac{2}{(x-3)^3}$.

(d) Using the $h \to 0$ form for the first derivative:

$$\[\begin{align*} p'(x) &= \lim_{h \to 0} \dfrac{p(x+h) - p(x)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{\dfrac{2(x+h) + 1}{x+h-4} - \dfrac{2x + 1}{x-4}}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{(2x + 2h + 1)(x - 4) - (2x + 1)(x + h - 4)}{h(x+h-4)(x-4)} \end{align*} \nonumber \]$$

Expand the numerator:

$$\[\begin{align*} &(2x + 2h + 1)(x - 4) - (2x + 1)(x + h - 4) \\[4pt] &= (2x^2 - 8x + 2xh - 8h + x - 4) - (2x^2 + 2xh - 8x + x + h - 4) \\[4pt] &= (2x^2 - 7x + 2xh - 8h - 4) - (2x^2 - 7x + 2xh + h - 4) \\[4pt] &= -9h \end{align*} \nonumber \]$$

Thus:

$$\[p'(x) = \lim_{h \to 0} \dfrac{-9h}{h(x+h-4)(x-4)} = \lim_{h \to 0} \dfrac{-9}{(x+h-4)(x-4)} = -\dfrac{9}{(x-4)^2} \nonumber \]$$

Using the $x \to a$ form for the first derivative:

$$\[\begin{align*} p'(x) &= \lim_{t \to x} \dfrac{p(t) - p(x)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{\dfrac{2t + 1}{t - 4} - \dfrac{2x + 1}{x - 4}}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{(2t + 1)(x - 4) - (2x + 1)(t - 4)}{(t - x)(t - 4)(x - 4)} \\[4pt] &= \lim_{t \to x} \dfrac{(2tx - 8t + x - 4) - (2xt - 8x + t - 4)}{(t - x)(t - 4)(x - 4)} \\[4pt] &= \lim_{t \to x} \dfrac{-9t + 9x}{(t - x)(t - 4)(x - 4)} \\[4pt] &= \lim_{t \to x} \dfrac{-9(t - x)}{(t - x)(t - 4)(x - 4)} \\[4pt] &= \lim_{t \to x} \dfrac{-9}{(t - 4)(x - 4)} = -\dfrac{9}{(x - 4)^2} \end{align*} \nonumber \]$$

Now find $p''(x)$ using the $h \to 0$ form:

$$\[\begin{align*} p''(x) &= \lim_{h \to 0} \dfrac{p'(x+h) - p'(x)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{-\dfrac{9}{(x+h-4)^2} + \dfrac{9}{(x-4)^2}}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{9\left(\dfrac{1}{(x-4)^2} - \dfrac{1}{(x+h-4)^2}\right)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{9\left(\dfrac{(x+h-4)^2 - (x-4)^2}{(x-4)^2(x+h-4)^2}\right)}{h} \\[4pt] &= \lim_{h \to 0} \dfrac{9h(2x + h - 8)}{h(x-4)^2(x+h-4)^2} \\[4pt] &= \lim_{h \to 0} \dfrac{9(2x + h - 8)}{(x-4)^2(x+h-4)^2} = \dfrac{9(2x - 8)}{(x-4)^4} = \dfrac{18}{(x-4)^3} \end{align*} \nonumber \]$$

Using the $x \to a$ form for the second derivative:

$$\[\begin{align*} p''(x) &= \lim_{t \to x} \dfrac{p'(t) - p'(x)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{-\dfrac{9}{(t-4)^2} + \dfrac{9}{(x-4)^2}}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{9\left(\dfrac{1}{(x-4)^2} - \dfrac{1}{(t-4)^2}\right)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{9\left(\dfrac{(t-4)^2 - (x-4)^2}{(x-4)^2(t-4)^2}\right)}{t - x} \\[4pt] &= \lim_{t \to x} \dfrac{9(t - x)(t + x - 8)}{(t - x)(x-4)^2(t-4)^2} \\[4pt] &= \lim_{t \to x} \dfrac{9(t + x - 8)}{(x-4)^2(t-4)^2} = \dfrac{9(2x - 8)}{(x-4)^4} = \dfrac{18}{(x-4)^3} \end{align*} \nonumber \]$$

Thus $p''(x) = \dfrac{18}{(x-4)^3}$.

Exercise $\PageIndex{3}$

(a) Check differentiability at the boundary $x = 2$.

First, check continuity from both sides at $x = 2$:

$$\[f(2) = 2^2 + 1 = 5 \nonumber \]$$

$$\[\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x - 3) = 10 - 3 = 7 \nonumber \]$$

Since $f(2) \ne \lim_{x \to 2^+} f(x)$, the function is not continuous at $x = 2$. Therefore, it is not differentiable at $x = 2$.

Check differentiability from both sides:

Using the $h \to 0$ form for the left derivative:

$$\[\begin{align*} f'(2^-) &= \lim_{h \to 0^-} \dfrac{f(2+h) - f(2)}{h} \\[4pt] &= \lim_{h \to 0^-} \dfrac{[(2+h)^2 + 1] - 5}{h} \\[4pt] &= \lim_{h \to 0^-} \dfrac{4 + 4h + h^2 + 1 - 5}{h} \\[4pt] &= \lim_{h \to 0^-} \dfrac{4h + h^2}{h} \\[4pt] &= \lim_{h \to 0^-} (4 + h) = 4 \end{align*} \nonumber \]$$

Using the $h \to 0$ form for the right derivative:

$$\[\begin{align*} f'(2^+) &= \lim_{h \to 0^+} \dfrac{f(2+h) - f(2)}{h} \\[4pt] &= \lim_{h \to 0^+} \dfrac{[5(2+h) - 3] - 5}{h} \\[4pt] &= \lim_{h \to 0^+} \dfrac{10 + 5h - 3 - 5}{h} \\[4pt] &= \lim_{h \to 0^+} \dfrac{2 + 5h}{h} \end{align*} \nonumber \]$$

As $h \to 0^+$, the numerator approaches 2 while the denominator approaches 0^+, so the limit is $+\infty$.

Thus f is not differentiable at $x = 2$.

(b) Check differentiability at the boundary $x = 1$.

First, check continuity from both sides at $x = 1$:

$$g(1) = 2(1) + 3 = 5$$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (x^2 + 4) = 1 + 4 = 5$$

So g is continuous at $x = 1$.

Now check differentiability from both sides:

Using the $h \to 0$ form for the left derivative:

$$\begin{aligned} g'(1^-) &= \lim_{h \to 0^-} \dfrac{g(1+h) - g(1)}{h} \\ &= \lim_{h \to 0^-} \dfrac{[2(1+h) + 3] - 5}{h} \\ &= \lim_{h \to 0^-} \dfrac{2 + 2h + 3 - 5}{h} \\ &= \lim_{h \to 0^-} \dfrac{2h}{h} = 2 \end{aligned}$$

Using the $h \to 0$ form for the right derivative:

$$\begin{aligned} g'(1^+) &= \lim_{h \to 0^+} \dfrac{g(1+h) - g(1)}{h} \\ &= \lim_{h \to 0^+} \dfrac{[(1+h)^2 + 4] - 5}{h} \\ &= \lim_{h \to 0^+} \dfrac{1 + 2h + h^2 + 4 - 5}{h} \\ &= \lim_{h \to 0^+} \dfrac{2h + h^2}{h} \\ &= \lim_{h \to 0^+} (2 + h) = 2 \end{aligned}$$

Since $g'(1^-) = g'(1^+) = 2$, the function is differentiable at $x = 1$.

Thus g is differentiable everywhere.

(c) Check differentiability at the boundaries $x = 0$ and $x = 2$.

At $x = 0$:

Continuity from both sides:

$$h(0) = 0^2 = 0$$

$$\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} 2x = 0$$

So h is continuous at $x = 0$.

Derivatives from both sides using the $h \to 0$ form:

Left derivative:

$$\begin{aligned} h'(0^-) &= \lim_{h \to 0^-} \dfrac{h(0+h) - h(0)}{h} \\ &= \lim_{h \to 0^-} \dfrac{h^2 - 0}{h} \\ &= \lim_{h \to 0^-} h = 0 \end{aligned}$$

Right derivative:

$$\begin{aligned} h'(0^+) &= \lim_{h \to 0^+} \dfrac{h(0+h) - h(0)}{h} \\ &= \lim_{h \to 0^+} \dfrac{2h - 0}{h} \\ &= \lim_{h \to 0^+} 2 = 2 \end{aligned}$$

Since $h'(0^-) \ne h'(0^+)$, h is not differentiable at $x = 0$.

At $x = 2$:

Continuity from both sides:

$$h(2) = 2(2) = 4$$

$$\lim_{x \to 2^+} h(x) = \lim_{x \to 2^+} (-x + 6) = -2 + 6 = 4$$

So h is continuous at $x = 2$.

Derivatives from both sides using the $h \to 0$ form:

Left derivative:

$$\begin{aligned} h'(2^-) &= \lim_{h \to 0^-} \dfrac{h(2+h) - h(2)}{h} \\ &= \lim_{h \to 0^-} \dfrac{2(2+h) - 4}{h} \\ &= \lim_{h \to 0^-} \dfrac{4 + 2h - 4}{h} \\ &= \lim_{h \to 0^-} \dfrac{2h}{h} = 2 \end{aligned}$$

Right derivative:

$$\begin{aligned} h'(2^+) &= \lim_{h \to 0^+} \dfrac{h(2+h) - h(2)}{h} \\ &= \lim_{h \to 0^+} \dfrac{[-(2+h) + 6] - 4}{h} \\ &= \lim_{h \to 0^+} \dfrac{(-2 - h + 6) - 4}{h} \\ &= \lim_{h \to 0^+} \dfrac{4 - h - 4}{h} \\ &= \lim_{h \to 0^+} \dfrac{-h}{h} = -1 \end{aligned}$$

Since $h'(2^-) \ne h'(2^+)$, h is not differentiable at $x = 2$.

Thus h is not differentiable at $x = 0$ and $x = 2$.

Exercise $\PageIndex{4a}$

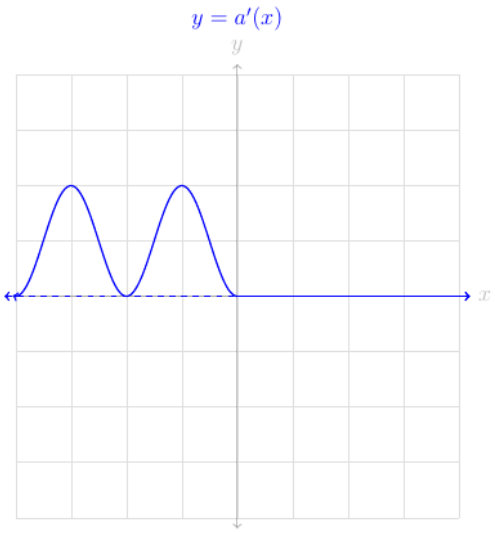
For $x>0$, the function $a(x)$ is constant, so $a'(x)=0$ for all such x.

For $x=0,-2,-4$, the function has flat spots, so $a'(-4)=a'(-2)=a'(0)=0$.

For -4

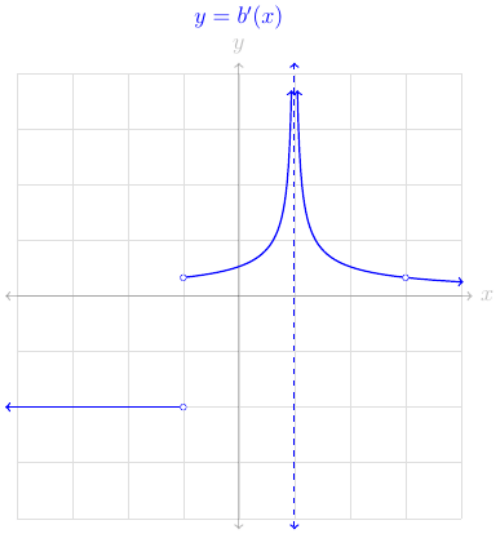
For -2

The resulting derivative function is graphed below:



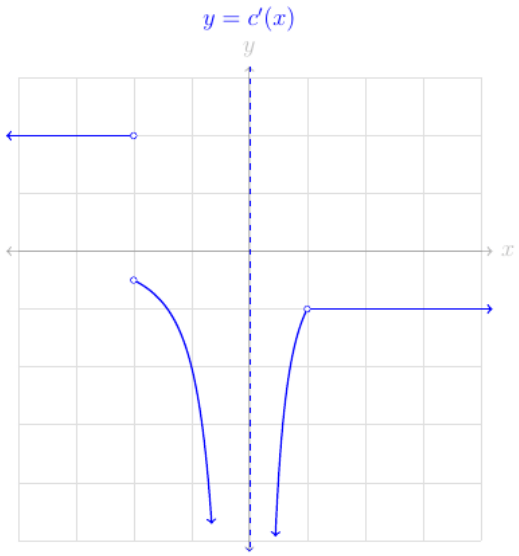
Exercise \(\PageIndex{4b}\)

For \(x<-1\), the function \(b(x)\) is a line with slope \(-2\), so \(b'(x)=-2\) for all such \(x\).
At \(x=-1\), the function has a sharp turn, so \(b'(-1)\) does not exist.
For \(-1
At \(x=1\), the tangent line is vertical, so \(b'(1)\) does not exist. (There is a vertical asymptote on the graph of \(y=b'(x)\))
For \(1
At \(x=3\), there is a removable discontinuity on \(b(x)\), so \(b'(3)\) does not exist.
For \(x>3\), the graph is increasing at at a slower and slower rate, starting at \(m=\dfrac{1}{2}\) and tending toward zero.
The resulting derivative function is graphed below:



Exercise \(\PageIndex{4c}\)

For \(x<-2\), the function \(c(x)\) is a line with slope \(2\), so \(c'(x)=2\) for all such \(x\).
At \(x=-2\), the function has a sharp turn, so \(c'(-2)\) does not exist.
For \(-2
At \(x=0\), the graph has a vertical asymptote, so \(c'(0)\) does not exist. (There is a vertical asymptote on the graph of \(y=c'(x)\))
For \(0
At \(x=1\), there is a removable discontinuity on \(c(x)\), so \(c'(1)\) does not exist.
For \(x>1\), the graph is a line with slope \(-1\), so \(c'(x)=-1\) for all such \(x\).
The resulting derivative function is graphed below:



Exercise \(\PageIndex{5}\): Finding Constants for Differentiability

For \(h''(2)\) to exist, \(h(x)\), \(h'(x)\), and \(h''(x)\) must all be continuous at \(x = 2\).

1. Continuity of \(h(x)\) at \(x = 2\):

From the left:

$$h(2) = 2^3 - 4 = 8 - 4 = 4$$

From the right:

$$\lim_{x \to 2^-} h(x) = a(2)^2 + b(2) + c = 4a + 2b + c$$

Equation 1: \(4a + 2b + c = 4\)

2. Continuity of \(h'(x)\) at \(x = 2\):

Using the \(h \to 0\) form for the left derivative (using \(x \ge 2\) piece):

$$\begin{aligned} h'(2^-) &= \lim_{h \to 0^-} \dfrac{h(2+h) - h(2)}{h} \\ &= \lim_{h \to 0^-} \dfrac{[(2+h)^3 - 4] - 4}{h} \\ &= \lim_{h \to 0^-} \dfrac{8 + 12h + 6h^2 + h^3 - 4 - 4}{h} \\ &= \lim_{h \to 0^-} \dfrac{12h + 6h^2 + h^3}{h} \\ &= \lim_{h \to 0^-} (12 + 6h + h^2) = 12 \end{aligned}$$

Using the \(h \to 0\) form for the right derivative (using \(x < 2\) piece):

$$\begin{aligned} h'(2^+) &= \lim_{h \to 0^+} \dfrac{h(2+h) - h(2)}{h} \\ &= \lim_{h \to 0^+} \dfrac{[a(2+h)^2 + b(2+h) + c] - 4}{h} \\ &= \lim_{h \to 0^+} \dfrac{[4a + 4ah + ah^2 + 2b + bh + c] - 4}{h} \\ &= \lim_{h \to 0^+} \dfrac{(4a + 2b + c - 4) + (4a + b)h + ah^2}{h} \end{aligned}$$

For this limit to exist, the constant term must be zero (since \(h \to 0\), a nonzero constant over \(h\) would diverge). From Equation 1, \(4a + 2b + c - 4 = 0\), so the constant term is already zero.

Thus:

$$h'(2^+) = \lim_{h \to 0^+} [(4a + b) + ah] = 4a + b$$

For differentiability, \(h'(2^-) = h'(2^+)\):

$$12 = 4a + b$$

Equation 2: \(4a + b = 12\)

3. Continuity of \(h''(x)\) at \(x = 2\):

Using the \(h \to 0\) form for the left second derivative:

$$\begin{aligned} h''(2^-) &= \lim_{h \to 0^-} \dfrac{h'(2+h) - h'(2)}{h} \\ &= \lim_{h \to 0^-} \dfrac{[3(2+h)^2] - 12}{h} \\ &= \lim_{h \to 0^-} \dfrac{3(4 + 4h + h^2) - 12}{h} \\ &= \lim_{h \to 0^-} \dfrac{12 + 12h + 3h^2 - 12}{h} \\ &= \lim_{h \to 0^-} \dfrac{12h + 3h^2}{h} \\ &= \lim_{h \to 0^-} (12 + 3h) = 12 \end{aligned}$$

Using the \(h \to 0\) form for the right second derivative:

$$\begin{aligned} h''(2^+) &= \lim_{h \to 0^+} \dfrac{h'(2+h) - h'(2)}{h} \\ &= \lim_{h \to 0^+} \dfrac{[2a(2+h) + b] - 12}{h} \\ &= \lim_{h \to 0^+} \dfrac{4a + 2ah + b - 12}{h} \end{aligned}$$

For this limit to exist, the constant term must be zero. From Equation 2, \(4a + b = 12\), so this is satisfied.

Thus:

$$h''(2^+) = \lim_{h \to 0^+} \dfrac{2ah}{h} = \lim_{h \to 0^+} 2a = 2a$$

For the second derivative to exist, \(h''(2^-) = h''(2^+)\):

$$12 = 2a \implies a = 6$$

Substitute \(a = 6\) into Equation 2:

$$4(6) + b = 12 \implies 24 + b = 12 \implies b = -12$$

Substitute \(a = 6\) and \(b = -12\) into Equation 1:

$$4(6) + 2(-12) + c = 4 \implies 24 - 24 + c = 4 \implies c = 4$$

Thus \(a = 6,\ b = -12,\ c = 4\).

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2: MASTERING THE DERIVATIVE

2.1: Derivative Shortcuts

2.1: Basic Derivative Rules

2.1: Sum, Difference, and Product Rules

2.1: Practice Set

2.2: Derivative Shortcuts for Division

2.2: Quotient Rule

2.2: Remaining Trig Derivatives

2.2: Practice Set

2.3: Derivatives Shortcuts for Composition

2.3: The Chain Rule

2.3: Implicit Differentiation

2.3: Practice Set

2.4: Derivatives of Inverse and Hyperbolic Functions

2.4: Derivatives of Inverse Functions

2.4: Logarithmic Differentiation

2.4: Hyperbolic Trig Functions

2.4: Practice Set

2.5: Linearization and the Mean Value Theorem

2.5: Linearization for Approximation

2.5: The Mean Value Theorem

2.5: Practice Set

2.6: Related Rates

2.6: Related Rates

2.6: Practice Set

2.7: L'Hopital's Rule

2.7: Type 1 Indeterminant Forms

2.7: Type 2 and 3 Indeterminant Forms

2.7: Practice Set

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2.1: Derivative Shortcuts

Learning Objectives

- Use the power rule to take the derivative of power functions without a limit.
- Discover the formula for the derivative of a general and natural exponential function.
- Use the sum, difference, and constant multiple rules to determine the derivative of a polynomial function.
- Use the product rule to determine the derivative of functions that are the product of two or more functions.
- Use the derivative shortcuts to solve tangent line problems without setting up limits.

Calculating derivatives using the limit definition can quickly become tedious for complicated functions. In this section, we establish shortcuts and algebraic patterns derived directly from limit laws, allowing us to compute derivatives quickly and efficiently without evaluating limits every time.

Question of the Day

Are there patterns for derivatives so we don't have to take a limit every time?

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2.1: Basic Derivative Rules

Applying the limit definition to polynomial terms of the form $f(x) = x^n$ yields a consistent pattern: the exponent drops to become a constant coefficient, and the new exponent is reduced by one.

Rule: The Power Rule for Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

This rule holds for any real number exponent n, including negative numbers, fractions, and irrational values.

Note: Special Cases of the Power Rule

- **Constant Functions ($n = 0$):** If $f(x) = c = c \cdot x^0$, the derivative is zero because horizontal lines have a slope of zero:
$$\frac{d}{dx}(c) = 0$$
- **Linear Functions ($n = 1$):** If $f(x) = x$, the derivative is constant because a line of the form $y = x$ has a constant slope of 1:
$$\frac{d}{dx}(x) = 1$$

Example 2.1.1: Applying the Power Rule

Find the derivative of each function:

(a) $f(x) = x^5$

(b) $g(x) = x^{1234}$

(c) $h(x) = \sqrt[3]{x}$

(d) $p(x) = \dfrac{1}{x^4}$

Solution:

(a) $f'(x) = 5x^{5-1} = 5x^4$

$$\boxed{f'(x) = 5x^4}$$

(b) $g'(x) = 1234x^{1234-1} = 1234x^{1233}$

$$\boxed{g'(x) = 1234x^{1233}}$$

(c) Rewrite as a power: $h(x) = x^{1/3}$.

$$h'(x) = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3}$$

$$\boxed{h'(x) = \frac{1}{3\sqrt[3]{x^2}}}$$

(d) Rewrite as a power: $p(x) = x^{-4}$.

$$p'(x) = -4x^{-4-1} = -4x^{-5}$$

$$\boxed{p'(x) = -\frac{4}{x^5}}$$

Common Mistake Alert

When rewriting radicals or reciprocals as powers, remember:

- $\sqrt[n]{x} = x^{\frac{1}{n}}$
- $\frac{1}{x^n} = x^{-n}$
- $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

Always rewrite before applying the Power Rule!

Example 2.1.2: More Power Rule Practice

Find the derivative of each function:

(a) $f(x) = x^{\frac{2}{3}}$

(b) $g(x) = \dfrac{1}{\sqrt{x}}$

(c) $h(x) = x^{2\pi}$

Solution:

(a) $f'(x) = \dfrac{2}{3}x^{\frac{2}{3}-1} = \boxed{\dfrac{2}{3}x^{-\frac{1}{3}}}$

(b) Rewrite: $g(x) = x^{-\frac{1}{2}}$, so $g'(x) = -\dfrac{1}{2}x^{-\frac{1}{2}-1} = \boxed{-\dfrac{1}{2}x^{-\frac{3}{2}}}$

(c) $h'(x) = \boxed{2\pi x^{2\pi-1}}$

For a general exponential function $f(x) = b^x$ (where base $b > 0$), computing the derivative via the limit definition gives:

$$f'(x) = \lim_{h\to 0}\frac{b^{x+h} - b^x}{h} = \lim_{h\to 0}\frac{b^x(b^h - 1)}{h} = b^x \cdot \left(\lim_{h\to 0}\frac{b^h - 1}{h}\right)$$

Letting $L(b) = \lim_{h\to 0}\dfrac{b^h - 1}{h}$, numerical approximations demonstrate how this factor behaves for different bases:

- For base 2: $L(2) \approx \dfrac{2^{0.0001} - 1}{0.0001} \approx 0.693$
- For base 3: $L(3) \approx \dfrac{3^{0.0001} - 1}{0.0001} \approx 1.099$

Since $L(b)$ passes continuously from less than 1 to greater than 1 as b increases from 2 to 3, there exists a unique fundamental number $e \approx 2.71828\ldots$ such that $L(e) = 1$.

Rule: The Derivative of an Exponential Function

$$\frac{d}{dx}\left(e^x\right) = e^x$$

In general, for any base $b > 0$, the limit factor $L(b) = \ln(b)$, giving:

$$\frac{d}{dx}\left(b^x\right) = b^x \ln(b)$$

Notice that the derivative of e^x is the special case of the general rule where $\ln(e) = 1$.

Example 2.1.3: Differentiating Exponential Functions

Find the derivative of each function:

(a) $f(x) = 5^x$

(b) $g(t) = e^t$

(c) $h(x) = 10^x$

Solution:

(a) Using the general exponential derivative formula:

$$\boxed{f'(x) = 5^x \ln(5)}$$

(b) Using the natural exponential derivative formula:

$$\boxed{g'(t) = e^t}$$

(c) Using the general exponential derivative formula:

$$\boxed{h'(x) = 10^x \ln(10)}$$

Example 2.1.4: Tangent Line to a Power Function

Find the equation of the tangent line to $f(x) = x^3$ at the point $(2, 8)$.

Solution:

First find the derivative using the Power Rule:

$$f'(x) = 3x^2$$

Evaluate the derivative at $x = 2$ to find the slope of the tangent line:

$$f'(2) = 3(2)^2 = 12$$

The tangent line passes through $(2, 8)$ with slope $m = 12$.

Using point-slope form:

$$\boxed{y - 8 = 12(x - 2)}$$

Example 2.1.5: Tangent Line to an Exponential Function

Find the equation of the tangent line to $f(x) = e^x$ at the point $(0, 1)$.

Solution:

First find the derivative:

$$f'(x) = e^x$$

Evaluate the derivative at $x = 0$ to find the slope of the tangent line:

$$f'(0) = e^0 = 1$$

The tangent line passes through $(0, 1)$ with slope $m = 1$.

Using point-slope form:

$$y - 1 = 1(x - 0)$$

$$\boxed{y = x + 1}$$

Example 2.1.6: Tangent Line Parallel to a Given Line

Find the point on $f(x) = x^2$ where the tangent line is parallel to the line $y = -8x + 3$.

Solution:

A line parallel to $y = -8x + 3$ must have slope $m = -8$

First find the derivative:

$$f'(x) = 2x$$

Set the derivative equal to the desired slope:

$$2x = -8 \implies x = -4$$

Find the corresponding y-value:

$$f(-4) = (-4)^2 = 16$$

Thus the point of tangency is $(-4, 16)$.

$$\boxed{(-4, 16)}$$

Example 2.1.7: Tangent Line Parallel to a Given Line

Find the equation of the tangent line to $f(x) = \sqrt{x}$ that is perpendicular to $y = -8x + 3$

Solution:

A line perpendicular to $y = -8x + 3$ must have slope $m = \dfrac{1}{8}$

Find the derivative using the Power Rule, since $f(x) = x^{\frac{1}{2}}$:

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Set the derivative equal to the desired slope:

$$\frac{1}{2\sqrt{x}} = \frac{1}{8} \implies \sqrt{x} = 4 \implies x = 16$$

Since $f(16) = \sqrt{16} = 4$, the tangent line passes through $(16, 4)$ with slope $m = \frac{1}{8}$.

Using point-slope form:

$$\boxed{y - 4 = \frac{1}{8}(x - 16)}$$

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2.1: Sum, Difference, and Product Rules

Constant Multiple and Sum/Difference Rules

Because derivatives are built directly on limits, they inherit linear properties from limit operations.

- **Constant Multiple Rule:** Multiplicative constants factor out of derivatives:

$$\frac{d}{dx}\left[k \cdot f(x)\right] = k \cdot f'(x) \quad \text{where } k \in \mathbb{R}$$

- **Sum and Difference Rule:** The derivative of a sum or difference is simply the sum or difference of the individual derivatives:

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

Together with the Power Rule, these properties allow us to differentiate any polynomial or algebraic sum term-by-term.

Example 2.1.8: Differentiating Polynomials

Find the derivative of each polynomial:

(a) $p(x) = x^8 + x^5 + 3x^2 - 5$

(b) $q(x) = 10x^{10} - 20x^5 + 100x$

(c) $f(x) = \frac{3}{4}x^4 - 2x^3 + \frac{1}{2}x^2 - 7x + 10$

Solution:

(a) Differentiate term-by-term:

$$p'(x) = 8x^7 + 5x^4 + 3(2x) - 0 = \boxed{8x^7 + 5x^4 + 6x}$$

(b) Differentiate term-by-term:

$$q'(x) = 10(10x^9) - 20(5x^4) + 100(1) = \boxed{100x^9 - 100x^4 + 100}$$

(c) Differentiate term-by-term:

$$\begin{aligned} f'(x) &= \frac{3}{4}(4x^3) - 2(3x^2) + \frac{1}{2}(2x) - 7(1) + 0 \\ &= \boxed{3x^3 - 6x^2 + x - 7} \end{aligned}$$

The Product Rule

The derivative of a product of two functions is **not** simply the product of their derivatives. Instead, rate-of-change interactions require using the Product Rule.

The Product Rule Formula:

If $h(x) = f(x) \cdot g(x)$, where both f and g are differentiable, then:

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

In words: "The derivative of the first times the second, plus the first times the derivative of the second."

Common Mistake Alert

Many students incorrectly assume that $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$. **This is false!** Always use the Product Rule: $\frac{d}{dx}[fg] = f'g + fg'$.

For example, if $f(x) = x$ and $g(x) = x$, then $f'g' = 1 \cdot 1 = 1$, but the correct derivative is $\frac{d}{dx}(x^2) = 2x$.

Illustration 2.1.1: Verifying the Product Rule

We will verify the Product Rule for $h(x) = (x^2 + 4)(2x^3 - 7)$ by expanding first, then taking the derivative, and comparing it with the Product Rule formula.

Expansion Method:

$$h(x) = (x^2 + 4)(2x^3 - 7) = 2x^5 - 7x^2 + 8x^3 - 28$$

$$h'(x) = 10x^4 - 14x + 24x^2$$

Product Rule Method:

Let $f = x^2 + 4$ and $g = 2x^3 - 7$.

$f' = 2x$, $g' = 6x^2$.

$$\begin{aligned} h'(x) &= (2x)(2x^3 - 7) + (x^2 + 4)(6x^2) \\ &= 4x^4 - 14x + 6x^4 + 24x^2 \\ &= 10x^4 + 24x^2 - 14x \end{aligned}$$

Both methods give the same result!

Example 2.1.9: Product Rule

Differentiate each function using the Product Rule:

(a) $a(x) = x^7 \cdot 3^x$

(b) $b(x) = \sqrt{x}e^x$

(c) $c(x) = (x^5 + 3x^2 - 4x - 1)(9x - 9^x)$

Solution:

(a) Let $f = x^7$ and $g = 3^x$.

Then, we know $f' = 7x^6$ and $g' = 3^x \ln(3)$.

$$\begin{aligned} a'(x) &= (7x^6)(3^x) + (x^7)(3^x \ln(3)) \\ &= x^6 \cdot 3^x(7 + x\ln(3)) \end{aligned}$$

$$\boxed{b'(x) = x^6 \cdot 3^x(7 + x\ln(3))}$$

(b) Let $f = \sqrt{x} = x^{\frac{1}{2}}$ and $g = e^x$.

Then, we know $f' = \frac{1}{2}x^{-\frac{1}{2}}$ and $g' = e^x$.

$$b'(x) = \left(\frac{1}{2}x^{-1/2}\right)(e^x) + (x^{1/2})(e^x)$$
$$= \boxed{e^x\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)}$$

(c) Let $f = x^5 + 3x^2 - 4x - 1$ and $g = 9x - 9^x$.

Then, we know $f' = 5x^4 + 6x - 4$ and $g' = 9 - 9^x\ln(9)$.

Using the Product Rule:

$$\boxed{c'(x) = (5x^4 + 6x - 4)(9x - 9^x) + (x^5 + 3x^2 - 4x - 1)(9 - 9^x\ln(9))}$$

Example 2.1.10: Finding Tangent Lines

Find the equation of the tangent line to $y = 3x^3 - 4x^2 - 7x + 2$ at the point $(1, -6)$.

Solution:

First, find the derivative:

$$y' = 9x^2 - 8x - 7$$

Evaluate at $x = 1$:

$$y'(1) = 9(1)^2 - 8(1) - 7 = 9 - 8 - 7 = -6$$

So the slope of the tangent line is $m = -6$.

Using point-slope form at $(1, -6)$:

$$y - (-6) = -6(x - 1) \implies y + 6 = -6x + 6$$

Therefore, the tangent line simplifies to:

$$\boxed{y = -6x}$$

Example 2.1.11: Finding Horizontal Tangents

Find all points (x, y) where the tangent line to $y = x^2e^x$ is horizontal.

Solution:

A horizontal tangent line has slope 0, so we need $y' = 0$.

Using the Product Rule with $f = x^2$ and $g = e^x$:

$$y' = (2x)(e^x) + (x^2)(e^x)$$
$$= e^x(2x + x^2) = e^x x(x + 2)$$

Set $y' = 0$. Since $e^x \neq 0$ for all x:

$$x(x+2) = 0 \implies x = 0 \text{ or } x = -2$$

Find the corresponding y-values:

At $x = 0$: $y = 0^2 e^0 = 0$. Point: $(0, 0)$.

At $x = -2$: $y = (-2)^2 e^{-2} = 4e^{-2}$. Point: $(-2, 4e^{-2})$.

$$\boxed{(0,0) \text{ and } (-2, 4e^{-2})}$$

Extension: The Triple Product Rule

When differentiating three multiplied functions $f(x) = u(x) \cdot v(x) \cdot w(x)$, grouping two functions and applying the Product Rule iteratively yields the general expansion:

$$f'(x) = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)$$

Each term differentiates exactly one factor while holding all other factors constant.

Example 2.1.12: Triple Product Rule

Let $f(x)$ be a function which has tangent line $y = 3x - 1$ at the point $(1, 2)$. Find the equation of the line tangent to $y = x^2 e^x f(x)$ when $x = 1$.

Solution:

Let $g(x) = x^2 e^x f(x)$. This is a product of three functions: $u = x^2$, $v = e^x$, and $w = f(x)$.

Then $u' = 2x$, $v' = e^x$, and $w' = f'(x)$.

Using the Triple Product Rule $g'(x) = u'vw + uv'w + uvw'$:

$$\begin{aligned} g'(x) &= (2x)(e^x)(f(x)) + (x^2)(e^x)(f(x)) + (x^2)(e^x)(f'(x)) \\ &= xe^x\left(2f(x) + xf(x) + xf'(x)\right) \end{aligned}$$

We are given that the tangent line to $f(x)$ at $(1, 2)$ is $y = 3x - 1$. This means $f(1) = 2$ and $f'(1) = 3$.

Evaluate $g'(1)$:

$$\begin{aligned} g'(1) &= 1e^1\left(2(2) + 1(2) + 1(3)\right) \\ &= e(4 + 2 + 3) = 9e \end{aligned}$$

Find the point of tangency: $g(1) = 1^2 e^1 f(1) = e(2) = 2e$.

The tangent line at $(1, 2e)$ with slope $m = 9e$:

$$\boxed{y - 2e = 9e(x - 1)}$$

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2.1: Practice Set

Exercise 2.1.1

Use the power rule or exponential derivative rules to find the derivative of each function.

(a) $f(x) = x^9$

(b) $g(x) = x^{-3}$

(c) $h(x) = \sqrt[4]{x^3}$

(d) $p(x) = \dfrac{1}{x^6}$

(e) $q(x) = e^x$

(f) $r(x) = 5^x$

(g) $s(x) = 2e^x$

(h) $t(x) = 4 \cdot 7^x$

Answer available at the end of this section.

Exercise 2.1.2

Use the sum/difference and product rules to find the derivative of each function.

(a) $f(x) = 4x^6 - 3x^4 + 8x^2 - 5$

(b) $g(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$

(c) $h(x) = 3e^x - 2x^5 + \dfrac{4}{x^3}$

(d) $p(x) = (3x^4 - 2)(x^2 + 4)$

(e) $q(x) = (2x^3 + 5x)e^x$

(f) $r(x) = (4x - 1)(x^3 + 2x^2)$

(g) $s(x) = (x^3 + 2x)(3e^x)$

(h) $t(x) = (x^2 - 4)(x^3 + x)$

Answer available at the end of this section.

Exercise 2.1.3

Find the derivative of each function using appropriate rules.

(a) $f(x) = (3x^4 - 5x^2)(e^x)$

(b) $g(x) = \dfrac{6x^5 - 4x^3 + 8x - 2}{2}$

(c) $h(x) = 6^x - 2x^5 + 3x^3 - 7$

(d) $p(x) = (e^x + 3x)(5x - 2)$

(e) $q(x) = \sqrt[3]{x}(x^4 - 3)$

(f) $r(x) = (4x^3 - 2)(3^x)$

Answer available at the end of this section.

Exercise 2.1.4

Find the equation of the tangent line to each function at the given point. Use the derivative shortcuts to find the slope.

(a) $f(x) = 3x^2 - 4x + 1$ at $x = 2$

(b) $g(x) = x^4 - 2x^2 + 5$ at $x = -1$

(c) $h(x) = 2e^x + 3x$ at $x = 0$

(d) $p(x) = (x^2 + 1)(x - 3)$ at $x = 1$

(e) $q(x) = 4^x + x^3 - 2x$ at $x = 1$

(f) $r(x) = (2x - 1)\sqrt{x}$ at $x = 4$

Answer available at the end of this section.

Answers

Exercise 2.1.1

(a) Apply the power rule:

$$f'(x) = 9x^8$$

(b) Apply the power rule:

$$g'(x) = -3x^{-4} = -\frac{3}{x^4}$$

(c) Rewrite as $h(x) = x^{\frac{3}{4}}$, then apply the power rule:

$$h'(x) = \frac{3}{4}x^{-\frac{1}{4}} = \frac{3}{4\sqrt[4]{x}}$$

(d) Rewrite as $p(x) = x^{-6}$, then apply the power rule:

$$p'(x) = -6x^{-7} = -\frac{6}{x^7}$$

(e) Recall that $\frac{d}{dx}[e^x] = e^x$:

$$q'(x) = e^x$$

(f) Recall that $\frac{d}{dx}[a^x] = a^x \ln(a)$:

$$r'(x) = 5^x \ln(5)$$

(g) Since 2 is a constant:

$$s'(x) = 2e^x$$

(h) Since 4 is a constant:

$$t'(x) = 4 \cdot 7^x \ln(7)$$

Exercise 2.1.2

(a) Apply the power rule term by term:

$$f'(x) = 24x^5 - 12x^3 + 16x$$

(b) Rewrite as $g(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$:

$$g'(x) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}$$

(c) Rewrite as $h(x) = 3e^x - 2x^5 + 4x^{-3}$:

$$h'(x) = 3e^x - 10x^4 - 12x^{-4} = 3e^x - 10x^4 - \frac{12}{x^4}$$

(d) Let $u = 3x^4 - 2$ and $v = x^2 + 4$. Then $u' = 12x^3$ and $v' = 2x$.
Using the product rule $f'(x) = u'v + uv'$:

$$\begin{aligned} p'(x) &= (12x^3)(x^2+4) + (3x^4-2)(2x) \\ &= 12x^5 + 48x^3 + 6x^5 - 4x \\ &= 18x^5 + 48x^3 - 4x \end{aligned}$$

(e) Let $u = e^x$ and $v = 2x^3 + 5x$. Then $u' = e^x$ and $v' = 6x^2 + 5$.
Using the product rule:

$$\begin{aligned} q'(x) &= (e^x)(2x^3+5x) + (e^x)(6x^2+5) \\ &= e^x(2x^3 + 5x + 6x^2 + 5) \\ &= e^x(2x^3 + 6x^2 + 5x + 5) \end{aligned}$$

(f) Let $u = 4x - 1$ and $v = x^3 + 2x^2$. Then $u' = 4$ and $v' = 3x^2 + 4x$.
Using the product rule:

$$\begin{aligned} r'(x) &= (4)(x^3+2x^2) + (4x-1)(3x^2+4x) \\ &= 4x^3 + 8x^2 + (12x^3 + 16x^2 - 3x^2 - 4x) \\ &= 4x^3 + 8x^2 + 12x^3 + 13x^2 - 4x \\ &= 16x^3 + 21x^2 - 4x \end{aligned}$$

(g) Let $u = x^3 + 2x$ and $v = 3e^x$. Then $u' = 3x^2 + 2$ and $v' = 3e^x$.
Using the product rule:

$$\begin{aligned} s'(x) &= (3x^2+2)(3e^x) + (x^3+2x)(3e^x) \\ &= 3e^x(3x^2 + 2 + x^3 + 2x) \\ &= 3e^x(x^3 + 3x^2 + 2x + 2) \end{aligned}$$

(h) Let $u = x^2 - 4$ and $v = x^3 + x$. Then $u' = 2x$ and $v' = 3x^2 + 1$.
Using the product rule:

$$\begin{aligned} t'(x) &= (2x)(x^3+x) + (x^2-4)(3x^2+1) \\ &= 2x^4 + 2x^2 + 3x^4 + x^2 - 12x^2 - 4 \\ &= 5x^4 - 9x^2 - 4 \end{aligned}$$

Exercise 2.1.3

(a) Use the product rule with $u = 3x^4 - 5x^2$ and $v = e^x$. Then $u' = 12x^3 - 10x$ and $v' = e^x$.

$$\begin{aligned} f'(x) &= (12x^3 - 10x)(e^x) + (3x^4 - 5x^2)(e^x) \\ &= e^x(12x^3 - 10x + 3x^4 - 5x^2) \\ &= e^x(3x^4 + 12x^3 - 5x^2 - 10x) \end{aligned}$$

Thus $f'(x) = e^x(3x^4 + 12x^3 - 5x^2 - 10x)$.

(b) Rewrite as $g(x) = \frac{1}{2}(6x^5 - 4x^3 + 8x - 2) = 3x^5 - 2x^3 + 4x - 1$.

$$g'(x) = 15x^4 - 6x^2 + 4$$

(c) Apply the sum/difference rules:

$$h'(x) = 6^x \ln(6) - 10x^4 + 9x^2$$

(d) Use the product rule with $u = e^x + 3x$ and $v = 5x - 2$. Then $u' = e^x + 3$ and $v' = 5$.

$$\begin{aligned} p'(x) &= (e^x+3)(5x-2) + (e^x+3x)(5) \\ &= (5x-2)e^x + 15x - 6 + 5e^x + 15x \\ &= e^x(5x+3) + 30x - 6 \end{aligned}$$

(e) Rewrite as $q(x) = x^{1/3}(x^4 - 3)$. Use the product rule with $u = x^{1/3}$ and $v = x^4 - 3$. Then $u' = \frac{1}{3}x^{-2/3}$ and $v' = 4x^3$.

$$\begin{aligned} q'(x) &= \left(\frac{1}{3}x^{-2/3}\right)(x^4-3) + (x^{1/3})(4x^3) \\ &= \frac{x^4-3}{3\sqrt[3]{x^2}} + 4x^{10/3} \\ &= \frac{x^4-3}{3\sqrt[3]{x^2}} + \frac{12x^4}{3\sqrt[3]{x^2}} \\ &= \frac{13x^4-3}{3\sqrt[3]{x^2}} \end{aligned}$$

(f) Use the product rule with $u = 4x^3 - 2$ and $v = 3^x$. Then $u' = 12x^2$ and $v' = 3^x \ln(3)$.

$$\begin{aligned} r'(x) &= (12x^2)(3^x) + (4x^3 - 2)(3^x \ln(3)) \\ &= 3^x(12x^2 + (4x^3 - 2)\ln(3)) \end{aligned}$$

Exercise 2.1.4

(a) First find the derivative:

$$f'(x) = 6x - 4$$

Evaluate at $x = 2$: $f'(2) = 6(2) - 4 = 12 - 4 = 8$.
Find the point: $f(2) = 3(4) - 4(2) + 1 = 12 - 8 + 1 = 5$.
The tangent line at $(2, 5)$ with slope $m = 8$:

$$y - 5 = 8(x - 2)$$

(b) First find the derivative:

$$g'(x) = 4x^3 - 4x$$

Evaluate at $x = -1$: $g'(-1) = 4(-1)^3 - 4(-1) = -4 + 4 = 0$.
Find the point: $g(-1) = (-1)^4 - 2(-1)^2 + 5 = 1 - 2 + 5 = 4$.
The tangent line at $(-1, 4)$ with slope $m = 0$:

$$y - 4 = 0(x + 1)$$

(c) First find the derivative:

$$h'(x) = 2e^x + 3$$

Evaluate at $x = 0$: $h'(0) = 2e^0 + 3 = 2 + 3 = 5$.
Find the point: $h(0) = 2e^0 + 3(0) = 2 + 0 = 2$.
The tangent line at $(0, 2)$ with slope $m = 5$:

$$y - 2 = 5(x - 0)$$

(d) First expand or use the product rule. Using the product rule with $u = x^2 + 1$ and $v = x - 3$: $u' = 2x$, $v' = 1$.

$$\begin{aligned} p'(x) &= (2x)(x - 3) + (x^2 + 1)(1) \\ &= 2x^2 - 6x + x^2 + 1 = 3x^2 - 6x + 1 \end{aligned}$$

Evaluate at $x = 1$: $p'(1) = 3(1) - 6(1) + 1 = 3 - 6 + 1 = -2$.
Find the point: $p(1) = (1^2 + 1)(1 - 3) = (2)(-2) = -4$.
The tangent line at $(1, -4)$ with slope $m = -2$:

$$y - (-4) = -2(x - 1)$$

(e) First find the derivative:

$$q'(x) = 4^x \ln(4) + 3x^2 - 2$$

Evaluate at $x = 1$: $q'(1) = 4^1 \ln(4) + 3(1)^2 - 2 = 4\ln(4) + 1$.
Find the point: $q(1) = 4^1 + 1^3 - 2(1) = 4 + 1 - 2 = 3$.
The tangent line at $(1, 3)$ with slope $m = 4\ln(4) + 1$:

$$y - 3 = (4\ln(4) + 1)(x - 1)$$

(f) Rewrite as $r(x) = x^{1/2}(2x - 1)$. Use the product rule with $u = x^{1/2}$ and $v = 2x - 1$: $u' = \frac{1}{2}x^{-1/2}$, $v' = 2$.

$$\begin{aligned} r'(x) &= \left(\frac{1}{2}x^{-1/2}\right)(2x - 1) + (x^{1/2})(2) \\ &= \frac{2x - 1}{2\sqrt{x}} + 2\sqrt{x} \\ &= \frac{2x - 1}{2\sqrt{x}} + \frac{4x}{2\sqrt{x}} = \frac{6x - 1}{2\sqrt{x}} \end{aligned}$$

Evaluate at $x = 4$: $r'(4) = \dfrac{6(4) - 1}{2\sqrt{4}} = \dfrac{24 - 1}{2(2)} = \dfrac{23}{4}$.
Find the point: $r(4) = \sqrt{4}(2(4) - 1) = 2(8 - 1) = 2(7) = 14$.
The tangent line at $(4, 14)$ with slope $m = \dfrac{23}{4}$:

$$y - 14 = \frac{23}{4}(x - 4)$$

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2.2: Derivative Shortcuts for Division

Learning Objectives

- Use the quotient rule to determine the derivative of functions that are the result of division of two functions.
- Determine the derivative of the six trigonometric functions without using limits.
- Find the equation of tangent lines to functions we know how to differentiate with our known shortcuts.

While the Product Rule handles functions multiplied together, evaluating the derivative of a ratio of functions using the limit definition quickly becomes algebraically cumbersome. In this section, we establish the Quotient Rule and combine our derivative toolkit to find the rates of change for all six trigonometric functions.

Question of the Day

How do we differentiate a ratio of two functions without using the limit definition every time?

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2.2: Quotient Rule

The Quotient Rule

If two functions $f(x)$ and $g(x)$ are differentiable and $g(x) \neq 0$, the derivative of their quotient is given by taking the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

The Quotient Rule Formula:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Mnemonic tip: Many students remember this rule as "*Low d-High minus High d-Low, over the square of Low.*"

Simplification Strategy: Before applying the Quotient Rule blindly, check if the denominator contains a single monomial term. If so, it is often much faster to split the fraction and use the Power Rule instead.

Common Mistake Alert

Many students incorrectly assume that $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'}{g'}$. **This is false!** Always use the Quotient Rule: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$.

For example, if $f(x) = x^2$ and $g(x) = x$, then $f'/g' = \frac{2x}{1} = 2x$, but the correct derivative is $\frac{d}{dx}(x) = 1$.

Illustration 2.2.1: Verifying the Quotient Rule

We will verify the Quotient Rule for $h(x) = \frac{x^8}{x^3}$ by computing $h'(x)$ in two different ways.

Method 1: Simplify first

$$h(x) = \frac{x^8}{x^3} = x^5$$

$$h'(x) = 5x^4$$

Method 2: Quotient Rule

Let $f(x) = x^8$ and $g(x) = x^3$.

$f'(x) = 8x^7$, $g'(x) = 3x^2$.

$$h'(x) = \frac{(8x^7)(x^3) - (x^8)(3x^2)}{(x^3)^2}$$

$$= \frac{8x^{10} - 3x^{10}}{x^6}$$

$$= \frac{5x^{10}}{x^6} = 5x^4$$

Both methods give the same result:

$$\boxed{h'(x) = 5x^4}$$

Example 2.2.1: Applying the Quotient Rule

Differentiate each function using the Quotient Rule:

(a) $h(x) = \dfrac{x}{x^2+1}$

(b) $g(x) = \dfrac{2x+3}{4x-1}$

(c) $f(x) = \dfrac{e^x}{x^3+2}$

Solution:

(a) Let $f(x) = x$ and $g(x) = x^2 + 1$.

Then, we know $f'(x) = 1$ and $g'(x) = 2x$.

$$\begin{aligned} h'(x) &= \frac{(1)(x^2+1) - (x)(2x)}{(x^2+1)^2} \\ &= \frac{x^2+1-2x^2}{(x^2+1)^2} \\ &= \boxed{\frac{1-x^2}{(x^2+1)^2}} \end{aligned}$$

(b) Let $f(x) = 2x + 3$ and $g(x) = 4x - 1$.

Then, we know $f'(x) = 2$ and $g'(x) = 4$.

$$\begin{aligned} g'(x) &= \frac{(2)(4x-1) - (2x+3)(4)}{(4x-1)^2} \\ &= \frac{8x-2-8x-12}{(4x-1)^2} \\ &= \boxed{-\frac{14}{(4x-1)^2}} \end{aligned}$$

(c) Let $f(x) = e^x$ and $g(x) = x^3 + 2$.

Then, we know $f'(x) = e^x$ and $g'(x) = 3x^2$.

$$\begin{aligned} f'(x) &= \frac{e^x(x^3+2) - e^x(3x^2)}{(x^3+2)^2} \\ &= \boxed{\frac{e^x(x^3-3x^2+2)}{(x^3+2)^2}} \end{aligned}$$

Example 2.2.2: Quotient Rule with Exponential Functions

Find the line tangent to $y = \dfrac{5^x}{x^2+1}$ at $x = 2$

Solution:

For $h(x) = \dfrac{5^x}{x^2+1}$, we see that $h(2) = \dfrac{25}{5} = 5$.

To find $h'(2)$, we must differentiate by using the Quotient Rule.

Let $f(x) = 5^x$ and $g(x) = x^2 + 1$.

Then, we know $f'(x) = 5^x \ln(5)$ and $g'(x) = 2x$.

$$\begin{aligned} h'(x) &= \frac{(5^x \ln(5))(x^2+1) - (5^x)(2x)}{(x^2+1)^2} \\ &= \frac{5^x[(x^2+1)\ln(5) - 2x]}{(x^2+1)^2} \end{aligned}$$

Therefore, $h'(2) = \dfrac{25\,[5\ln(5) - 10]}{25} = 5\ln(5) - 10$

The tangent line must then be:

$$\boxed{y - 5 = (5\ln(5) - 10)\,(x - 2)}$$

Simplification Before Differentiation

When the denominator is a single monomial term, splitting the fraction into separate terms often simplifies the differentiation process significantly.

Illustration 2.2.2: Simplifying Before Differentiating

We will differentiate $h(x) = \dfrac{3x^4 + 6x^3 - 9x^2}{\sqrt[3]{x}}$ first by dividing term-by-term, and then via the Quotient Rule.

Dividing Term-by-Term:

Rewrite $\sqrt[3]{x} = x^{\frac{1}{3}}$. Divide each term:

$$h(x) = \frac{3x^4}{x^{\frac{1}{3}}} + \frac{6x^3}{x^{\frac{1}{3}}} - \frac{9x^2}{x^{\frac{1}{3}}} = 3x^{\frac{11}{3}} + 6x^{\frac{8}{3}} - 9x^{\frac{5}{3}}$$

Now differentiate term-by-term:

$$\begin{aligned} h'(x) &= 3\left(\frac{11}{3}x^{\frac{8}{3}}\right) + 6\left(\frac{8}{3}x^{\frac{5}{3}}\right) - 9\left(\frac{5}{3}x^{\frac{2}{3}}\right) \\ &= 11x^{\frac{8}{3}} + 16x^{\frac{5}{3}} - 15x^{\frac{2}{3}} \end{aligned}$$

Using the Quotient Rule:

We let $f(x) = 3x^4 + 6x^3 - 9x^2$ and $g(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$.

Then, we know $f'(x) = 12x^3 + 18x^2 - 18x$ and $g'(x) = \dfrac{1}{3}x^{-\frac{2}{3}}$.

Applying the Quotient Rule, we see that

$$h'(x) = \frac{(12x^3 + 18x^2 - 18x)\left(x^{\frac{1}{3}}\right) - (3x^4 + 6x^3 - 9x^2)\left(\frac{1}{3}x^{-\frac{2}{3}}\right)}{\left(x^{\frac{1}{3}}\right)^2}$$

This is surprisingly equivalent to the answer we got above, but takes significantly more effort to reach!

This is why, when possible, we always prefer to reduce and simplify before differentiating instead of blinding using the Quotient Rule.

Example 2.2.3: More Simplification Before Differentiating

Differentiate $y = \dfrac{7x^5 - 2x^2 + 4}{x^2}$ by dividing term-by-term first.

Solution:

Divide each term by x^2:

$$y = \frac{7x^5}{x^2} - \frac{2x^2}{x^2} + \frac{4}{x^2} = 7x^3 - 2 + 4x^{-2}$$

Now differentiate term-by-term:

$$y' = 7(3x^2) - 0 + 4(-2x^{-3}) = \boxed{21x^2 - 8x^{-3}}$$

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2.2: Remaining Trig Derivatives

Derivatives of Sine and Cosine

Using the limit definition of the derivative, we can derive the derivatives of $\sin(x)$ and $\cos(x)$.

For $f(x) = \sin(x)$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \left[\sin(x)\cdot \frac{\cos(h)-1}{h} + \cos(x)\cdot \frac{\sin(h)}{h}\right] \\ &= \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x) \end{aligned}$$

Using the standard limits $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$.

For $f(x) = \cos(x)$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\ &= \lim_{h\to 0} \left[\cos(x)\cdot \frac{\cos(h)-1}{h} - \sin(x)\cdot \frac{\sin(h)}{h}\right] \\ &= \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x) \end{aligned}$$

Thus we have:

$$\frac{d}{dx}[\sin(x)] = \cos(x) \quad \text{and} \quad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Derivatives of Trigonometric Functions

By writing $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$ in terms of $\sin(x)$ and $\cos(x)$ and applying the Quotient Rule, we derive the complete table of trigonometric derivatives:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$	$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$	$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$	$\csc(x)$	$-\csc(x)\cot(x)$

Notice that every "co-" trigonometric function ($\cos$, $\cot$, $\csc$) has a negative sign in its derivative!

Illustration 2.2.3: Deriving $\frac{d}{dx}[\tan(x)] = \sec^2(x)$

Use $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and the Quotient Rule to show that $\dfrac{d}{dx}(\tan(x)) = \sec^2(x)$.

Solution:

Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$.

$f'(x) = \cos(x)$, $g'(x) = -\sin(x)$.

$$\begin{aligned}\frac{d}{dx}[\tan(x)] &= \frac{f'g - fg'}{g^2} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} = \sec^2(x)\end{aligned}$$

$$\boxed{\frac{d}{dx}[\tan(x)] = \sec^2(x)}$$

Example 2.2.4: Combining Trigonometric Rules

Differentiate each function:

(a) $f(x) = \tan(x) + \cot(x)$

(b) $h(x) = x^2\cot(x)$

(c) $g(x) = \dfrac{\csc(x)}{x^2 + 1}$

Solution:

(a)

$$f'(x) = \sec^2(x) - \csc^2(x)$$

$$\boxed{f'(x) = \sec^2(x) - \csc^2(x)}$$

(b) Use the Product Rule with $u = x^2$ and $v = \cot(x)$.

$u' = 2x$, $v' = -\csc^2(x)$.

$$h'(x) = (2x)(\cot(x)) + (x^2)(-\csc^2(x))$$

$$\boxed{h'(x) = 2x\cot(x) - x^2\csc^2(x)}$$

(c) Use the Quotient Rule with $f(x) = \csc(x)$ and $g(x) = x^2 + 1$.

$f'(x) = -\csc(x)\cot(x)$, $g'(x) = 2x$.

$$g'(x) = \frac{(-\csc(x)\cot(x))(x^2+1) - (\csc(x))(2x)}{(x^2+1)^2}$$

$$= \frac{-\csc(x)[(x^2+1)\cot(x) + 2x]}{(x^2+1)^2}$$

$$\boxed{g'(x) = \frac{-\csc(x)[(x^2+1)\cot(x) + 2x]}{(x^2+1)^2}}$$

Illustration 2.2.4: Simplifying Trigonometric Expressions First

We can differentiate $j(x) = \dfrac{x\tan(x)}{\sec(x)}$ two ways.

The Product and Quotient Rule:

We use the Quotient Rule with $f(x) = x\tan(x)$ and $g(x) = \sec(x)$

The Product Rule gives us $f'(x) = 1 \cdot \tan(x) + x \cdot \sec^2(x) = \tan(x) + x\sec^2(x)$ and $g'(x) = \sec(x)\tan(x)$.

Therefore, we have:

$$j'(x) = \frac{\left(\tan(x) + x\sec^2(x)\right)\left(\sec(x)\right) - \left(x\tan(x)\right)\left(\sec(x)\tan(x)\right)}{\left(\sec(x)\right)^2}$$

Simplifying First:

Simplify first: $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and $\sec(x) = \dfrac{1}{\cos(x)}$.

$$j(x) = \frac{x \cdot \dfrac{\sin(x)}{\cos(x)}}{\dfrac{1}{\cos(x)}} = x\sin(x)$$

Now differentiate using the Product Rule:

$$j'(x) = (1)(\sin(x)) + (x)(\cos(x)) = \sin(x) + x\cos(x)$$

Example 2.2.5: Tangent Line with Quotient Rule

Find the equation of the line tangent to $y = \dfrac{\cos(x)}{x}$ at $x = \pi$.

Solution:

Find the y-coordinate at $x = \pi$:

$$y(\pi) = \frac{\cos(\pi)}{\pi} = -\frac{1}{\pi}$$

Find the derivative using the Quotient Rule:

$$y' = \frac{(-\sin(x))(x) - (\cos(x))(1)}{x^2} = \frac{-x\sin(x) - \cos(x)}{x^2}$$

Evaluate at $x = \pi$:

$$y'(\pi) = \frac{-\pi\sin(\pi) - \cos(\pi)}{\pi^2} = \frac{-\pi(0) - (-1)}{\pi^2} = \frac{1}{\pi^2}$$

Using point-slope form at $\left(\pi, -\frac{1}{\pi}\right)$:

$$y - \left(-\frac{1}{\pi}\right) = \frac{1}{\pi^2}(x - \pi)$$

$$y + \frac{1}{\pi} = \frac{x}{\pi^2} - \frac{1}{\pi}$$

$$\boxed{y = \frac{x}{\pi^2} - \frac{2}{\pi}}$$

Example 2.2.6: Horizontal Tangents with Trigonometric Functions

Find all points (x, y) where $y = e^x \sin(x)$ has a horizontal tangent line on the interval $[0, 2\pi]$.

Solution:

A horizontal tangent line has slope 0, so we need $y' = 0$.

Using the Product Rule:

$$y' = e^x \sin(x) + e^x \cos(x) = e^x(\sin(x) + \cos(x))$$

Set $y' = 0$. Since $e^x \neq 0$ for all x:

$$\sin(x) + \cos(x) = 0 \implies \sin(x) = -\cos(x) \implies \tan(x) = -1$$

On $[0, 2\pi]$, $\tan(x) = -1$ at $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

Find the corresponding y-values:

At $x = \frac{3\pi}{4}$: $y = e^{3\pi/4} \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} e^{3\pi/4}$

At $x = \frac{7\pi}{4}$: $y = e^{7\pi/4} \sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2} e^{7\pi/4}$

$$\boxed{\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2} e^{3\pi/4}\right) \text{ and } \left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2} e^{7\pi/4}\right)}$$

Example 2.2.7: Tangent Line with Triple Product

Find the equation of the line tangent to $y = xe^x \sec(x)$ at $x = \pi$.

Solution:

Find the y-coordinate at $x = \pi$:

$$y(\pi) = \pi e^\pi \sec(\pi) = \pi e^\pi(-1) = -\pi e^\pi$$

Find the derivative using the Triple Product Rule:

Let $u = x$, $v = e^x$, $w = \sec(x)$.

$u' = 1$, $v' = e^x$, $w' = \sec(x)\tan(x)$.

$$y' = u'vw + uv'w + uvw' = e^x \sec(x) + xe^x \sec(x) + xe^x \sec(x)\tan(x)$$

Evaluate at $x = \pi$:

$\sec(\pi) = -1$, $\tan(\pi) = 0$.

$$y'(\pi) = e^\pi(-1) + \pi e^\pi(-1) + \pi e^\pi(-1)(0) = -e^\pi - \pi e^\pi = -e^\pi(1 + \pi)$$

Using point-slope form at $(\pi, -\pi e^\pi)$:

$$\boxed{y - (-\pi e^\pi) = -e^\pi(1 + \pi)(x - \pi)}$$

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2.2: Practice Set

Exercise 2.2.1

Use the quotient rule to find the derivative of each function.

(a) $f(x) = \dfrac{2x+5}{3x-1}$

(b) $g(x) = \dfrac{x^2-4}{x+3}$

(c) $h(x) = \dfrac{5x^3-2x}{x^2+1}$

(d) $p(x) = \dfrac{3x^4+2x}{x^3-5}$

(e) $q(x) = \dfrac{e^x}{x^2+1}$

(f) $r(x) = \dfrac{4^x}{2x-3}$

Answer available at the end of this section.

Exercise 2.2.2

Simplify each expression using a trig identity, then find the derivative.

(a) $f(x) = \sin(x)\csc(x)$

(b) $g(x) = \tan(x)\cos(x)$

(c) $h(x) = \sec(x)\cot(x)$

(d) $p(x) = \dfrac{\sin(x)}{\cos(x)} \cdot \cos^2(x)$

(e) $q(x) = \dfrac{\tan(x)}{\sec(x)}$

(f) $r(x) = \sin(x)\cot(x)$

Answer available at the end of this section.

Exercise 2.2.3

Find the equation of the tangent line to each function at the given point.

(a) $f(x) = \tan(x)$ at $\left(\dfrac{\pi}{4}, 1\right)$

(b) $g(x) = \sec(x)$ at $(0, 1)$

(c) $h(x) = \dfrac{x^2 - 1}{x + 2}$ at $x = 1$

(d) $p(x) = \csc(x)$ at $\left(\dfrac{\pi}{2}, 1\right)$

(e) $q(x) = \dfrac{e^x}{x + 1}$ at $x = 0$

(f) $r(x) = \dfrac{x^3 - 8}{x - 2}$ at $x = 3$

Answer available at the end of this section.

Exercise 2.2.4

For each function, find all x-values where the tangent line is parallel to the line $y = 2x + 5$.

(a) $f(x) = \dfrac{x - 1}{x + 1}$

(b) $g(x) = \tan(x)$ on the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(c) $h(x) = \sec(x)\cos(x)$ (simplify first)

Answer available at the end of this section.

Answers

Exercise 2.2.1

(a) Let $u = 2x + 5$ and $v = 3x - 1$. Then $u' = 2$ and $v' = 3$.

Using the quotient rule $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$:

$$\begin{aligned} f'(x) &= \frac{(2)(3x-1) - (2x+5)(3)}{(3x-1)^2} \\ &= \frac{6x - 2 - 6x - 15}{(3x-1)^2} \\ &= \frac{-17}{(3x-1)^2} \end{aligned}$$

Thus $f'(x) = \dfrac{-17}{(3x-1)^2}$.

(b) Let $u = x^2 - 4$ and $v = x + 3$. Then $u' = 2x$ and $v' = 1$.

$$\begin{aligned} g'(x) &= \frac{(2x)(x+3) - (x^2-4)(1)}{(x+3)^2} \\ &= \frac{2x^2 + 6x - x^2 + 4}{(x+3)^2} \\ &= \frac{x^2 + 6x + 4}{(x+3)^2} \end{aligned}$$

Thus $g'(x) = \dfrac{x^2 + 6x + 4}{(x+3)^2}$.

(c) Let $u = 5x^3 - 2x$ and $v = x^2 + 1$. Then $u' = 15x^2 - 2$ and $v' = 2x$.

$$\begin{aligned} h'(x) &= \frac{(15x^2-2)(x^2+1) - (5x^3-2x)(2x)}{(x^2+1)^2} \\ &= \frac{15x^4 + 15x^2 - 2x^2 - 2 - (10x^4 - 4x^2)}{(x^2+1)^2} \\ &= \frac{15x^4 + 13x^2 - 2 - 10x^4 + 4x^2}{(x^2+1)^2} \\ &= \frac{5x^4 + 17x^2 - 2}{(x^2+1)^2} \end{aligned}$$

Thus $h'(x) = \dfrac{5x^4 + 17x^2 - 2}{(x^2+1)^2}$.

(d) Let $u = 3x^4 + 2x$ and $v = x^3 - 5$. Then $u' = 12x^3 + 2$ and $v' = 3x^2$.

$$\begin{aligned} p'(x) &= \frac{(12x^3+2)(x^3-5) - (3x^4+2x)(3x^2)}{(x^3-5)^2} \\ &= \frac{12x^6 - 60x^3 + 2x^3 - 10 - (9x^6 + 6x^3)}{(x^3-5)^2} \\ &= \frac{12x^6 - 58x^3 - 10 - 9x^6 - 6x^3}{(x^3-5)^2} \\ &= \frac{3x^6 - 64x^3 - 10}{(x^3-5)^2} \end{aligned}$$

Thus $p'(x) = \dfrac{3x^6 - 64x^3 - 10}{(x^3-5)^2}$.

(e) Let $u = e^x$ and $v = x^2 + 1$. Then $u' = e^x$ and $v' = 2x$.

$$q'(x) = \frac{(e^x)(x^2+1) - (e^x)(2x)}{(x^2+1)^2} = \frac{e^x(x^2 - 2x + 1)}{(x^2+1)^2} = \frac{e^x(x-1)^2}{(x^2+1)^2}$$

Thus $q'(x) = \dfrac{e^x(x-1)^2}{(x^2+1)^2}$.

(f) Let $u = 4^x$ and $v = 2x - 3$. Then $u' = 4^x \ln(4)$ and $v' = 2$.

$$r'(x) = \frac{(4^x \ln(4))(2x-3) - (4^x)(2)}{(2x-3)^2} = \frac{4^x((2x-3)\ln(4) - 2)}{(2x-3)^2}$$

Thus $r'(x) = \dfrac{4^x((2x-3)\ln(4) - 2)}{(2x-3)^2}$.

Exercise 2.2.2

(a) Simplify using $\csc(x) = \dfrac{1}{\sin(x)}$:

$$f(x) = \sin(x) \cdot \frac{1}{\sin(x)} = 1$$

Therefore $f'(x) = 0$.

(b) Simplify using $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$:

$$g(x) = \frac{\sin(x)}{\cos(x)} \cdot \cos(x) = \sin(x)$$

Therefore $g'(x) = \cos(x)$.

(c) Simplify using $\sec(x) = \dfrac{1}{\cos(x)}$ and $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$:

$$h(x) = \frac{1}{\cos(x)} \cdot \frac{\cos(x)}{\sin(x)} = \frac{1}{\sin(x)} = \csc(x)$$

Therefore $h'(x) = -\csc(x)\cot(x)$.

(d) Simplify using $\dfrac{\sin(x)}{\cos(x)} = \tan(x)$:

$$p(x) = \tan(x) \cdot \cos^2(x) = \frac{\sin(x)}{\cos(x)} \cdot \cos^2(x) = \sin(x)\cos(x)$$

Using the product rule or the double-angle identity, $\dfrac{d}{dx}[\sin(x)\cos(x)] = \cos^2(x) - \sin^2(x) = \cos(2x)$.

Therefore $p'(x) = \cos^2(x) - \sin^2(x) = \cos(2x)$.

(e) Simplify using $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and $\sec(x) = \dfrac{1}{\cos(x)}$:

$$q(x) = \frac{\dfrac{\sin(x)}{\cos(x)}}{\dfrac{1}{\cos(x)}} = \sin(x)$$

Therefore $q'(x) = \cos(x)$.

(f) Simplify using $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$:

$$r(x) = \sin(x) \cdot \frac{\cos(x)}{\sin(x)} = \cos(x)$$

Therefore $r'(x) = -\sin(x)$.

Exercise 2.2.3

(a) Recall that $\dfrac{d}{dx}[\tan(x)] = \sec^2(x)$.

Evaluate at $x = \dfrac{\pi}{4}$:

$$f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = 2$$

The tangent line at $\left(\frac{\pi}{4}, 1\right)$ with slope $m = 2$:

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

Thus $y - 1 = 2\left(x - \frac{\pi}{4}\right)$.

(b) Recall that $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$.

Evaluate at $x = 0$:

$$g'(0) = \sec(0)\tan(0) = (1)(0) = 0$$

The tangent line at $(0, 1)$ with slope $m = 0$:

$$y - 1 = 0(x - 0)$$

Thus $y = 1$.

(c) Use the quotient rule with $u = x^2 - 1$ and $v = x + 2$. Then $u' = 2x$ and $v' = 1$.

$$h'(x) = \frac{(2x)(x+2) - (x^2-1)(1)}{(x+2)^2} = \frac{2x^2 + 4x - x^2 + 1}{(x+2)^2} = \frac{x^2 + 4x + 1}{(x+2)^2}$$

Evaluate at $x = 1$: $h'(1) = \frac{1 + 4 + 1}{(1+2)^2} = \frac{6}{9} = \frac{2}{3}$.

Find the point: $h(1) = \frac{1-1}{1+2} = 0$.

The tangent line at $(1, 0)$ with slope $m = \frac{2}{3}$:

$$y - 0 = \frac{2}{3}(x - 1)$$

Thus $y = \frac{2}{3}(x - 1)$.

(d) Recall that $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$.

Evaluate at $x = \frac{\pi}{2}$:

$$p'\left(\frac{\pi}{2}\right) = -\csc\left(\frac{\pi}{2}\right)\cot\left(\frac{\pi}{2}\right) = -(1)(0) = 0$$

The tangent line at $\left(\frac{\pi}{2}, 1\right)$ with slope $m = 0$:

$$y - 1 = 0\left(x - \frac{\pi}{2}\right)$$

Thus $y = 1$.

(e) Use the quotient rule with $u = e^x$ and $v = x + 1$. Then $u' = e^x$ and $v' = 1$.

$$q'(x) = \frac{(e^x)(x+1) - (e^x)(1)}{(x+1)^2} = \frac{e^x(x + 1 - 1)}{(x+1)^2} = \frac{xe^x}{(x+1)^2}$$

Evaluate at $x = 0$: $q'(0) = \frac{0 \cdot e^0}{(1)^2} = 0$.

Find the point: $q(0) = \frac{e^0}{1} = 1$.

The tangent line at $(0, 1)$ with slope $m = 0$:

$$y - 1 = 0(x - 0)$$

Thus $y = 1$.

(f) Notice that $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$.

For $x \neq 2$, $r(x) = x^2 + 2x + 4$.

Therefore $r'(x) = 2x + 2$.

Evaluate at $x = 3$: $r'(3) = 2(3) + 2 = 8$.

Find the point: $r(3) = 3^2 + 2(3) + 4 = 9 + 6 + 4 = 19$.

The tangent line at $(3, 19)$ with slope $m = 8$:

$$y - 19 = 8(x - 3)$$

Thus $y - 19 = 8(x - 3)$.

Exercise 2.2.4

A line parallel to $y = 2x + 5$ must have slope $m = 2$. We need to find where the derivative equals 2.

(a) Use the quotient rule with $u = x - 1$ and $v = x + 1$. Then $u' = 1$ and $v' = 1$.

$$f'(x) = \frac{(1)(x+1) - (x-1)(1)}{(x+1)^2} = \frac{x + 1 - x + 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Set $f'(x) = 2$:

$$\frac{2}{(x+1)^2} = 2 \implies \frac{1}{(x+1)^2} = 1 \implies (x+1)^2 = 1 \implies x + 1 = \pm 1$$

Thus $x = 0$ or $x = -2$.

Therefore the tangent line to $f(x) = \frac{x-1}{x+1}$ is parallel to $y = 2x + 5$ at $x = 0$ and $x = -2$.

(b) Recall that $\frac{d}{dx}[\tan(x)] = \sec^2(x)$.

Set $g'(x) = 2$:

$$\sec^2(x) = 2 \implies \frac{1}{\cos^2(x)} = 2 \implies \cos^2(x) = \frac{1}{2}$$

$$\cos(x) = \pm\frac{\sqrt{2}}{2}$$

On the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos(x) > 0$, so $\cos(x) = \frac{\sqrt{2}}{2}$.

Thus $x = \frac{\pi}{4}$ or $x = -\frac{\pi}{4}$.

Therefore the tangent line to $g(x) = \tan(x)$ is parallel to $y = 2x + 5$ at $x = \frac{\pi}{4}$ and $x = -\frac{\pi}{4}$.

(c) Simplify first using $\sec(x) = \frac{1}{\cos(x)}$:

$$h(x) = \sec(x)\cos(x) = \frac{1}{\cos(x)} \cdot \cos(x) = 1$$

Therefore $h'(x) = 0$ for all x.

Since $h'(x) = 0 \neq 2$ for all x, there are no x-values where the tangent line is parallel to $y = 2x + 5$.

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2.3: Derivatives Shortcuts for Composition

Learning Objectives

- Objective 1
- Objective 2

Intro Here.

Question of the Day

How do we differentiate a function that is composed inside another function?

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2.3: The Chain Rule

So far, we have learned how to differentiate sums, products, and quotients of basic functions. However, many mathematical models involve composite functions—where one function is evaluated inside another. In this section, we introduce the Chain Rule to differentiate composite functions, and we extend this technique to implicit equations where y cannot be isolated explicitly.

The Chain Rule

When a function $h(x)$ is formed by composing an outer function f with an inner function g, written as $h(x) = f(g(x))$, its rate of change is the derivative of the outer function evaluated at the inner function, multiplied by the derivative of the inner function.

The Chain Rule Formula:

$$h'(x) = f'(g(x)) \cdot g'(x) \tag{2.3.1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{2.3.2}$$

Common Composite Patterns:

- **General Power Rule:** $\frac{d}{dx}\left[(g(x))^n\right] = n(g(x))^{n-1} \cdot g'(x)$
- **Composite Exponential:** $\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \cdot g'(x)$
- **Composite Trigonometric:** $\frac{d}{dx}\left[\sin(g(x))\right] = \cos(g(x)) \cdot g'(x)$

Common Mistake Alert

Many students forget to multiply by the derivative of the inner function when using the Chain Rule. Remember: **always multiply by** $g'(x)$!

For example, $\frac{d}{dx}[\sin(x^2)]$ is **not** $\cos(x^2)$. It is $\cos(x^2) \cdot 2x$.

Example 2.3.1: Applying the Chain Rule

Differentiate each function using the Chain Rule:

(a) $a(x) = \sqrt{x^4 + 4}$

(b) $b(x) = e^{3x^2 - 5}$

(c) $c(x) = \sin(4x^3 - 2x)$

Solution:

(a) Rewrite as a power: $a(x) = (x^4 + 4)^{1/2}$.

Outer function: $u^{1/2}$, inner function: $u = x^4 + 4$.

$$a'(x) = \frac{1}{2}(x^4 + 4)^{-1/2} \cdot (4x^3) = \frac{2x^3}{\sqrt{x^4 + 4}} \tag{2.3.3}$$

$$\boxed{a'(x) = \frac{2x^3}{\sqrt{x^4+4}}} \tag{2.3.4}$$

(b) Outer function: e^u, inner function: $u = 3x^2 - 5$.

$$b'(x) = e^{3x^2-5} \cdot (6x) = \boxed{6xe^{3x^2-5}} \tag{2.3.5}$$

(c) Outer function: $\sin(u)$, inner function: $u = 4x^3 - 2x$.

$$c'(x) = \cos(4x^3 - 2x) \cdot (12x^2 - 2) \tag{2.3.6}$$

$$\boxed{c'(x) = (12x^2 - 2)\cos(4x^3 - 2x)} \tag{2.3.7}$$

Example 2.3.2: Chain Rule with Combined Rules

Differentiate each function:

(a) $f(x) = x^2e^{3x}$

(b) $g(x) = (2x^3 - x^2 + 4)^5$

(c) $h(x) = \cos^3(2x)$

Solution:

(a) Use the Product Rule with $u = x^2$ and $v = e^{3x}$.

$u' = 2x$, $v' = e^{3x} \cdot 3 = 3e^{3x}$.

$$f'(x) = (2x)(e^{3x}) + (x^2)(3e^{3x}) = e^{3x}(2x + 3x^2) \tag{2.3.8}$$

$$\boxed{f'(x) = e^{3x}(3x^2 + 2x)} \tag{2.3.9}$$

(b) Outer function: u^5, inner function: $u = 2x^3 - x^2 + 4$.

$$g'(x) = 5(2x^3 - x^2 + 4)^4 \cdot (6x^2 - 2x) \tag{2.3.10}$$

$$\boxed{g'(x) = (30x^2 - 10x)(2x^3 - x^2 + 4)^4} \tag{2.3.11}$$

(c) Rewrite as $h(x) = [\cos(2x)]^3$.

Outer function: u^3, inner function: $u = \cos(2x)$.

$$h'(x) = 3[\cos(2x)]^2 \cdot (-\sin(2x) \cdot 2) \tag{2.3.12}$$

$$= -6\cos^2(2x)\sin(2x) \tag{2.3.13}$$

$$\boxed{h'(x) = -6\cos^2(2x)\sin(2x)} \tag{2.3.14}$$

Question of the Day

What is the difference between $\sin^3(x)$ and $\sin(x^3)$?

Answer In $\sin^3(x) = (\sin(x))^3$, the sine function is inside the power of 3, so the derivative uses the Chain Rule with outer function u^3: $3\sin^2(x)\cos(x)$. In $\sin(x^3)$, the power is inside the sine, so the derivative uses the Chain Rule with outer function $\sin(u)$: $\cos(x^3) \cdot 3x^2$.

Example 2.3.3: Distinguishing Composite Structures

Differentiate $c(x) = \tan^3(x)$ and $d(x) = \tan(x^3)$.

Solution:

For $c(x) = (\tan(x))^3$, the outer function is u^3 and the inner function is $\tan(x)$.

$$c'(x) = 3\tan^2(x) \cdot \sec^2(x) \tag{2.3.15}$$

$$\boxed{c'(x) = 3\tan^2(x)\sec^2(x)} \tag{2.3.16}$$

For $d(x) = \tan(x^3)$, the outer function is $\tan(u)$ and the inner function is $u = x^3$.

$$d'(x) = \sec^2(x^3) \cdot 3x^2 \tag{2.3.17}$$

$$\boxed{d'(x) = 3x^2\sec^2(x^3)} \tag{2.3.18}$$

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2.3: Implicit Differentiation

Implicit Differentiation

Not every curve can be expressed in explicit form $y = f(x)$. Curves defined implicitly by equations involving both x and y (such as circles, ellipses, or complex algebraic curves) still possess tangent lines.

To find $\frac{dy}{dx}$ implicitly:

1. Differentiate both sides of the equation with respect to x.
2. Apply the Chain Rule whenever differentiating terms containing y, treating y as an implicit function of x (so $\frac{d}{dx}[y] = \frac{dy}{dx}$ or y').
3. Collect all terms containing $\frac{dy}{dx}$ on one side of the equation and move all other terms to the opposite side.
4. Factor out $\frac{dy}{dx}$ and solve algebraically.

Key Insight for Implicit Differentiation

When differentiating terms that contain y, always multiply by $\frac{dy}{dx}$. For example:

- $\frac{d}{dx}[y^2] = 2y \cdot \frac{dy}{dx}$
- $\frac{d}{dx}[\sin(y)] = \cos(y) \cdot \frac{dy}{dx}$
- $\frac{d}{dx}[e^y] = e^y \cdot \frac{dy}{dx}$
- $\frac{d}{dx}[xy] = (1)(y) + (x)\left(\frac{dy}{dx}\right) = y + x\frac{dy}{dx}$ (Product Rule!)

Example 2.3.4: Basic Implicit Differentiation

Find $\frac{dy}{dx}$ for the circle equation $x^2 + y^2 = 25$.

Solution:

Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25) \tag{2.3.1}$$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \tag{2.3.2}$$

Solve for $\frac{dy}{dx}$:

$$2y \cdot \frac{dy}{dx} = -2x \tag{2.3.3}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}} \tag{2.3.4}$$

This matches what we would get by solving for $y = \pm\sqrt{25 - x^2}$ and differentiating, but works without isolating y.

Example 2.3.5: Implicit Differentiation with Product Rule

Find $\frac{dy}{dx}$ for $x^2y + 3xy^2 = 10$.

Solution:

Differentiate both sides with respect to x, using the Product Rule where needed:

For x^2y: differentiate as product: $(2x)(y) + (x^2)\left(\frac{dy}{dx}\right)$

For $3xy^2$: differentiate as product: $(3)(y^2) + (3x)(2y)\left(\frac{dy}{dx}\right)$

$$\frac{d}{dx}[x^2y] + \frac{d}{dx}[3xy^2] = \frac{d}{dx}[10] \tag{2.3.5}$$

$$\left(2xy + x^2\frac{dy}{dx}\right) + \left(3y^2 + 6xy\frac{dy}{dx}\right) = 0 \tag{2.3.6}$$

Collect terms with $\frac{dy}{dx}$:

$$x^2\frac{dy}{dx} + 6xy\frac{dy}{dx} = -2xy - 3y^2 \tag{2.3.7}$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x^2 + 6xy) = -2xy - 3y^2 \tag{2.3.8}$$

$$\boxed{\frac{dy}{dx} = \frac{-2xy - 3y^2}{x^2 + 6xy}} \tag{2.3.9}$$

Example 2.3.6: Implicit Differentiation with Trigonometric Functions

Find $\frac{dy}{dx}$ for $\sin(xy) + x^2 = y^3$.

Solution:

Differentiate both sides with respect to x:

For $\sin(xy)$: use the Chain Rule: $\cos(xy) \cdot \frac{d}{dx}[xy]$

$$\frac{d}{dx}[xy] = y + x\frac{dy}{dx} \tag{2.3.10}$$

So $\frac{d}{dx}[\sin(xy)] = \cos(xy)(y + x\frac{dy}{dx})$

$$\frac{d}{dx}[\sin(xy)] + \frac{d}{dx}[x^2] = \frac{d}{dx}[y^3] \tag{2.3.11}$$

$$\cos(xy)\left(y + x\frac{dy}{dx}\right) + 2x = 3y^2\frac{dy}{dx} \tag{2.3.12}$$

Distribute the $\cos(xy)$:

$$y\cos(xy) + x\cos(xy)\frac{dy}{dx} + 2x = 3y^2\frac{dy}{dx} \tag{2.3.13}$$

Collect terms with $\frac{dy}{dx}$:

$$x\cos(xy)\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -y\cos(xy) - 2x \tag{2.3.14}$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x\cos(xy) - 3y^2) = -y\cos(xy) - 2x \tag{2.3.15}$$

$$\boxed{\frac{dy}{dx} = \frac{-y\cos(xy) - 2x}{x\cos(xy) - 3y^2}} \tag{2.3.16}$$

Tangent Lines with Implicit Differentiation

Once we find $\frac{dy}{dx}$ implicitly, we can evaluate it at a given point to find the slope of the tangent line.

Example 2.3.7: Tangent Line from Implicit Equation

Find the equation of the tangent line to $x^2 + y^2 = 25$ at the point $(3, 4)$.

Solution:

From Example 4, $\frac{dy}{dx} = -\frac{x}{y}$.

At $(3, 4)$: $\frac{dy}{dx} = -\frac{3}{4}$.

Using point-slope form at $(3, 4)$:

$$y - 4 = -\frac{3}{4}(x - 3) \tag{2.3.17}$$

$$y = -\frac{3}{4}x + \frac{9}{4} + 4 = -\frac{3}{4}x + \frac{25}{4} \tag{2.3.18}$$

$$\boxed{y = -\frac{3}{4}x + \frac{25}{4}} \tag{2.3.19}$$

Example 2.3.8: Tangent Line with Trigonometric Implicit Equation

Find the equation of the tangent line to $\sin(xy) + x^2 = y^3$ at the point $(0, 1)$.

Solution:

From Example 6, $\frac{dy}{dx} = \frac{-y\cos(xy)-2x}{x\cos(xy)-3y^2}$.

At $(0, 1)$:

$\cos(0 \cdot 1) = \cos(0) = 1$.

$$\frac{dy}{dx} = \frac{-(1)(1) - 2(0)}{(0)(1) - 3(1)^2} = \frac{-1}{-3} = \frac{1}{3} \tag{2.3.20}$$

Using point-slope form at $(0, 1)$:

$$y - 1 = \frac{1}{3}(x - 0) \tag{2.3.21}$$

$$\boxed{y = \frac{1}{3}x + 1} \tag{2.3.22}$$

Higher-Order Derivatives with Implicit Differentiation

We can find second derivatives implicitly by differentiating $\frac{dy}{dx}$ again with respect to x, remembering that $\frac{dy}{dx}$ may contain both x and y.

Example 2.3.9: Second Derivative Implicitly

Find $\frac{d^2y}{dx^2}$ for $x^2 + y^2 = 25$.

Solution:

We know $\frac{dy}{dx} = -\frac{x}{y}$.

Differentiate $\frac{dy}{dx}$ with respect to x using the Quotient Rule or treat as $-xy^{-1}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{x}{y}\right] \tag{2.3.23}$$

Using the Quotient Rule with $f(x) = -x$ and $g(x) = y$:

$$\frac{d^2y}{dx^2} = \frac{(-1)(y) - (-x)\left(\frac{dy}{dx}\right)}{y^2} = \frac{-y + x\frac{dy}{dx}}{y^2} \tag{2.3.24}$$

Substitute $\frac{dy}{dx} = -\frac{x}{y}$:

$$\frac{d^2y}{dx^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y - \frac{x^2}{y}}{y^2} = \frac{-\frac{y^2+x^2}{y}}{y^2} \tag{2.3.25}$$

Since $x^2 + y^2 = 25$:

$$\frac{d^2y}{dx^2} = \frac{-\frac{25}{y}}{y^2} = \boxed{-\frac{25}{y^3}} \tag{2.3.26}$$

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2.3: Practice Set

Exercise 2.3.1

Find the derivative of each function:

(a) $f(x) = (3x^2 + 2x - 5)^4$

(b) $g(x) = \sqrt{5x^3 - 2x + 1}$

(c) $h(x) = \sin(4x^2 - 3x)$

(d) $p(x) = \tan^3(2x)$

(e) $q(x) = \sec(e^x + x^2)$

(f) $r(x) = (x^2 + 1)^3(2x - 5)^2$

(g) $s(x) = \dfrac{\cos(3x)}{(2x + 1)^2}$

Answer available at the end of this section.

Exercise 2.3.2

Find the equation of the tangent line to each function at the given point using the chain rule.

(a) $f(x) = \sqrt{4x^2 + 9}$ at $x = 2$

(b) $g(x) = \sin(3x - \pi)$ at $x = \dfrac{\pi}{3}$

(c) $h(x) = (x^3 - 2x)^4$ at $x = 1$

(d) $p(x) = \sec(2x)$ at $x = \dfrac{\pi}{6}$

(e) $q(x) = \tan(x^2 + 1)$ at $x = 0$

(f) $r(x) = x^2 \cos(2x)$ at $x = \dfrac{\pi}{4}$

Answer available at the end of this section.

Exercise 2.3.3

Use implicit differentiation to find $\dfrac{dy}{dx}$ and the equation of the tangent line at the given point.

(a) $x^2 + y^2 = 25$ at $(3, 4)$

(b) $x^3 + y^3 = 9xy$ at $(2, 4)$

(c) $\sin(x) + \cos(y) = 1$ at $(0, 0)$

(d) $x^2 + xy + y^2 = 7$ at $(1, 2)$

(e) $x^2y + y^2x = 6$ at $(1, 2)$

(f) $\cos(xy) = \frac{1}{2}$ at $\left(\frac{\pi}{3}, 1\right)$

Answer available at the end of this section.

Exercise 2.3.4

Find all horizontal and vertical tangent lines to each implicit curve.

(a) $x^2 + y^2 = 9$

(b) $x^2 + xy + y^2 = 7$

(c) $x^2y + y^2x = 6$

(d) $x^3 + y^3 = 9xy$ (the folium of Descartes)

Answer available at the end of this section.

Answers

Exercise 2.3.1

(a) Let $u = 3x^2 + 2x - 5$. Then $u' = 6x + 2$.

Using the chain rule $\frac{d}{dx}[u^n] = nu^{n-1}u'$:

$$f'(x) = 4(3x^2 + 2x - 5)^3(6x + 2)$$

Thus $f'(x) = 4(3x^2 + 2x - 5)^3(6x + 2)$.

(b) Rewrite as $g(x) = (5x^3 - 2x + 1)^{1/2}$. Let $u = 5x^3 - 2x + 1$. Then $u' = 15x^2 - 2$.

$$g'(x) = \frac{1}{2}(5x^3 - 2x + 1)^{-1/2}(15x^2 - 2) = \frac{15x^2 - 2}{2\sqrt{5x^3 - 2x + 1}}$$

Thus $g'(x) = \frac{15x^2 - 2}{2\sqrt{5x^3 - 2x + 1}}$.

(c) Let $u = 4x^2 - 3x$. Then $u' = 8x - 3$.

Using the chain rule $\frac{d}{dx}[\sin(u)] = \cos(u)u'$:

$$h'(x) = \cos(4x^2 - 3x)(8x - 3)$$

Thus $h'(x) = (8x - 3)\cos(4x^2 - 3x)$.

(d) Rewrite as $p(x) = [\tan(2x)]^3$. Let $u = \tan(2x)$. Then $u' = \sec^2(2x) \cdot 2 = 2\sec^2(2x)$.

$$p'(x) = 3[\tan(2x)]^2 \cdot 2\sec^2(2x) = 6\tan^2(2x)\sec^2(2x)$$

Thus $p'(x) = 6\tan^2(2x)\sec^2(2x)$.

(e) Let $u = e^x + x^2$. Then $u' = e^x + 2x$.

Using the chain rule $\frac{d}{dx}[\sec(u)] = \sec(u)\tan(u)u'$:

$$q'(x) = \sec(e^x + x^2)\tan(e^x + x^2)(e^x + 2x)$$

Thus $q'(x) = (e^x + 2x)\sec(e^x + x^2)\tan(e^x + x^2)$.

(f) Let $u = (x^2 + 1)^3$ and $v = (2x - 5)^2$.

Using the chain rule for each: $u' = 3(x^2 + 1)^2(2x) = 6x(x^2 + 1)^2$ and $v' = 2(2x - 5)(2) = 4(2x - 5)$.

Using the product rule $r'(x) = u'v + uv'$:

$$\begin{aligned} r'(x) &= [6x(x^2 + 1)^2](2x - 5)^2 + (x^2 + 1)^3[4(2x - 5)] \\ &= 2(x^2 + 1)^2(2x - 5)\left[3x(2x - 5) + 2(x^2 + 1)\right] \\ &= 2(x^2 + 1)^2(2x - 5)(6x^2 - 15x + 2x^2 + 2) \\ &= 2(x^2 + 1)^2(2x - 5)(8x^2 - 15x + 2) \end{aligned}$$

Thus $r'(x) = 2(x^2 + 1)^2(2x - 5)(8x^2 - 15x + 2)$.

(g) Let $u = \cos(3x)$ and $v = (2x + 1)^2$.

Using the chain rule: $u' = -\sin(3x)(3) = -3\sin(3x)$ and $v' = 2(2x + 1)(2) = 4(2x + 1)$.

Using the quotient rule:

$$\begin{aligned} s'(x) &= \frac{u'v - uv'}{v^2} \\ &= \frac{[-3\sin(3x)](2x+1)^2 - [\cos(3x)][4(2x+1)]}{(2x+1)^4} \\ &= \frac{(2x+1)[-3(2x+1)\sin(3x) - 4\cos(3x)]}{(2x+1)^4} \\ &= \frac{-3(2x+1)\sin(3x) - 4\cos(3x)}{(2x+1)^3} \end{aligned}$$

Thus $s'(x) = \dfrac{-3(2x+1)\sin(3x) - 4\cos(3x)}{(2x+1)^3}$.

Exercise 2.3.2

(a) Rewrite as $f(x) = (4x^2+9)^{1/2}$. Let $u = 4x^2+9$. Then $u' = 8x$.

$$f'(x) = \frac{1}{2}(4x^2+9)^{-1/2}(8x) = \frac{4x}{\sqrt{4x^2+9}}$$

Evaluate at $x = 2$: $f'(2) = \dfrac{4(2)}{\sqrt{4(4)+9}} = \dfrac{8}{\sqrt{16+9}} = \dfrac{8}{5}$.

Find the point: $f(2) = \sqrt{4(4)+9} = \sqrt{25} = 5$.

The tangent line at $(2,5)$ with slope $m = \dfrac{8}{5}$:

$$y - 5 = \frac{8}{5}(x-2)$$

Thus $y - 5 = \dfrac{8}{5}(x-2)$.

(b) Let $u = 3x - \pi$. Then $u' = 3$.

$$g'(x) = \cos(3x-\pi)(3) = 3\cos(3x-\pi)$$

Evaluate at $x = \dfrac{\pi}{3}$: $g'\left(\dfrac{\pi}{3}\right) = 3\cos(\pi-\pi) = 3\cos(0) = 3$.

Find the point: $g\left(\dfrac{\pi}{3}\right) = \sin(3\cdot\dfrac{\pi}{3} - \pi) = \sin(\pi-\pi) = \sin(0) = 0$.

The tangent line at $\left(\dfrac{\pi}{3}, 0\right)$ with slope $m = 3$:

$$y - 0 = 3\left(x - \frac{\pi}{3}\right)$$

Thus $y = 3\left(x - \dfrac{\pi}{3}\right)$.

(c) Let $u = x^3 - 2x$. Then $u' = 3x^2 - 2$.

$$h'(x) = 4(x^3-2x)^3(3x^2-2)$$

Evaluate at $x = 1$: $h'(1) = 4(1-2)^3(3-2) = 4(-1)^3(1) = -4$.

Find the point: $h(1) = (1-2)^4 = (-1)^4 = 1$.

The tangent line at $(1,1)$ with slope $m = -4$:

$$y - 1 = -4(x-1)$$

Thus $y - 1 = -4(x-1)$.

(d) Let $u = 2x$. Then $u' = 2$.

$$p'(x) = \sec(2x)\tan(2x)(2) = 2\sec(2x)\tan(2x)$$

Evaluate at $x = \dfrac{\pi}{6}$:

$$p'\left(\frac{\pi}{6}\right) = 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 2(2)(\sqrt{3}) = 4\sqrt{3}$$

Find the point: $p\left(\dfrac{\pi}{6}\right) = \sec\left(\dfrac{\pi}{3}\right) = 2$.

The tangent line at $\left(\dfrac{\pi}{6}, 2\right)$ with slope $m = 4\sqrt{3}$:

$$y - 2 = 4\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

Thus $y - 2 = 4\sqrt{3}\left(x - \dfrac{\pi}{6}\right)$.

(e) Let $u = x^2 + 1$. Then $u' = 2x$.

$$q'(x) = \sec^2(x^2+1)(2x) = 2x\sec^2(x^2+1)$$

Evaluate at $x = 0$: $q'(0) = 0$ since the factor of $2x$ is 0.

Find the point: $q(0) = \tan(1)$.

The tangent line at $(0, \tan(1))$ with slope $m = 0$:

$$y - \tan(1) = 0(x-0)$$

Thus $y = \tan(1)$.

(f) Use the product rule with $u = x^2$ and $v = \cos(2x)$.

$u' = 2x$ and $v' = -\sin(2x)(2) = -2\sin(2x)$.

$$r'(x) = (2x)\cos(2x) + (x^2)(-2\sin(2x)) = 2x\cos(2x) - 2x^2\sin(2x)$$

Evaluate at $x = \frac{\pi}{4}$:

$$r'\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi^2}{16}\right)\sin\left(\frac{\pi}{2}\right) = 0 - \frac{\pi^2}{8}(1) = -\frac{\pi^2}{8}$$

Find the point: $r\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2\cos\left(\frac{\pi}{2}\right) = \frac{\pi^2}{16}(0) = 0$.

The tangent line at $\left(\frac{\pi}{4}, 0\right)$ with slope $m = -\frac{\pi^2}{8}$:

$$y - 0 = -\frac{\pi^2}{8}\left(x - \frac{\pi}{4}\right)$$

Thus $y = -\frac{\pi^2}{8}\left(x - \frac{\pi}{4}\right)$.

Exercise 2.3.3

(a) Differentiate both sides with respect to x:

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[25] \implies 2x + 2y\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = -\frac{x}{y}$$

Evaluate at $(3, 4)$: $\frac{dy}{dx} = -\frac{3}{4}$.

The tangent line at $(3, 4)$ with slope $m = -\frac{3}{4}$:

$$y - 4 = -\frac{3}{4}(x - 3)$$

Thus $y - 4 = -\frac{3}{4}(x - 3)$.

(b) Differentiate both sides with respect to x:

$$3x^2 + 3y^2\frac{dy}{dx} = 9\left(y + x\frac{dy}{dx}\right)$$
$$3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}$$

Collect terms with $\frac{dy}{dx}$:

$$3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$$
$$\frac{dy}{dx}(3y^2 - 9x) = 9y - 3x^2$$
$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3(3y - x^2)}{3(y^2 - 3x)} = \frac{3y - x^2}{y^2 - 3x}$$

Evaluate at $(2, 4)$: $\frac{dy}{dx} = \frac{3(4) - 4}{16 - 3(2)} = \frac{12 - 4}{16 - 6} = \frac{8}{10} = \frac{4}{5}$.

The tangent line at $(2, 4)$ with slope $m = \frac{4}{5}$:

$$y - 4 = \frac{4}{5}(x - 2)$$

Thus $y - 4 = \frac{4}{5}(x - 2)$.

(c) Differentiate both sides with respect to x:

$$\cos(x) - \sin(y)\frac{dy}{dx} = 0$$
$$-\sin(y)\frac{dy}{dx} = -\cos(x) \implies \frac{dy}{dx} = \frac{\cos(x)}{\sin(y)}$$

Evaluate at $(0, 0)$: $\frac{dy}{dx} = \frac{\cos(0)}{\sin(0)} = \frac{1}{0}$, which is undefined.

Therefore the tangent line is vertical at $x = 0$.

Thus the tangent line is $x = 0$.

(d) Differentiate both sides with respect to x:

$$2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$
$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Collect terms with $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x + 2y) = -2x - y$$
$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

Evaluate at $(1,2)$: $\frac{dy}{dx} = \frac{-2(1)-2}{1+2(2)} = \frac{-2-2}{1+4} = -\frac{4}{5}$.

The tangent line at $(1,2)$ with slope $m = -\frac{4}{5}$:

$$y - 2 = -\frac{4}{5}(x-1)$$

Thus $y - 2 = -\frac{4}{5}(x-1)$.

(e) Differentiate both sides with respect to x:

$$\frac{d}{dx}[x^2 y] + \frac{d}{dx}[y^2 x] = 0$$

$$(2xy + x^2\frac{dy}{dx}) + (2xy\frac{dy}{dx} + y^2) = 0$$

$$2xy + x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} + y^2 = 0$$

Collect terms with $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

Evaluate at $(1,2)$: $\frac{dy}{dx} = \frac{-2(1)(2)-4}{1+2(1)(2)} = \frac{-4-4}{1+4} = -\frac{8}{5}$.

The tangent line at $(1,2)$ with slope $m = -\frac{8}{5}$:

$$y - 2 = -\frac{8}{5}(x-1)$$

Thus $y - 2 = -\frac{8}{5}(x-1)$.

(f) Differentiate both sides with respect to x:

$$-\sin(xy)\cdot\left(y + x\frac{dy}{dx}\right) = 0$$

Since $\cos(xy) = \frac{1}{2}$, at the given point $\sin(xy) \neq 0$, so:

$$y + x\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{y}{x}$$

Evaluate at $\left(\frac{\pi}{3}, 1\right)$: $\frac{dy}{dx} = -\frac{1}{\pi/3} = -\frac{3}{\pi}$.

The tangent line at $\left(\frac{\pi}{3}, 1\right)$ with slope $m = -\frac{3}{\pi}$:

$$y - 1 = -\frac{3}{\pi}\left(x - \frac{\pi}{3}\right)$$

Thus $y - 1 = -\frac{3}{\pi}\left(x - \frac{\pi}{3}\right)$.

Exercise 2.3.4

(a) From Exercise 3(a), $\frac{dy}{dx} = -\frac{x}{y}$.

Horizontal tangents: Set $\frac{dy}{dx} = 0$:

$$-\frac{x}{y} = 0 \implies x = 0$$

Substitute into $x^2 + y^2 = 9$: $0 + y^2 = 9 \implies y = \pm 3$.

Horizontal tangent lines: $y = 3$ and $y = -3$ at points $(0,3)$ and $(0,-3)$.

Vertical tangents: Set denominator of $\frac{dy}{dx}$ equal to 0:

$$y = 0$$

Substitute into $x^2 + y^2 = 9$: $x^2 + 0 = 9 \implies x = \pm 3$.

Vertical tangent lines: $x = 3$ and $x = -3$ at points $(3,0)$ and $(-3,0)$.

(b) From Exercise 3(d), $\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$.

Horizontal tangents: Set numerator equal to 0:

$$-2x - y = 0 \implies y = -2x$$

Substitute into $x^2 + xy + y^2 = 7$:

$$x^2 + x(-2x) + (-2x)^2 = 7 \implies x^2 - 2x^2 + 4x^2 = 7 \implies 3x^2 = 7 \implies x^2 = \frac{7}{3}$$

$$x = \pm\sqrt{\frac{7}{3}} \implies y = \mp 2\sqrt{\frac{7}{3}}$$

Horizontal tangent lines: $y = -2x$ at points $\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right)$ and $\left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right)$.

Vertical tangents: Set denominator equal to 0:

$$x + 2y = 0 \implies x = -2y$$

Substitute into $x^2 + xy + y^2 = 7$:

$$(-2y)^2 + (-2y)y + y^2 = 7 \implies 4y^2 - 2y^2 + y^2 = 7 \implies 3y^2 = 7 \implies y^2 = \frac{7}{3}$$

$$y = \pm\sqrt{\frac{7}{3}} \implies x = \mp 2\sqrt{\frac{7}{3}}$$

Vertical tangent lines: $x = -2y$ at points $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$ and $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$.

(c) From Exercise 3(e), $\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy} = \frac{-y(2x + y)}{x(x + 2y)}$.

Horizontal tangents: Set numerator equal to 0:

$$-y(2x + y) = 0 \implies y = 0 \text{ or } y = -2x$$

If $y = 0$, substitute into $x^2y + y^2x = 6$: $0 = 6$, impossible.

If $y = -2x$, substitute into $x^2y + y^2x = 6$:

$$x^2(-2x) + (-2x)^2x = 6 \implies -2x^3 + 4x^3 = 6 \implies 2x^3 = 6 \implies x^3 = 3 \implies x = \sqrt[3]{3}$$

$$y = -2\sqrt[3]{3}$$

Horizontal tangent line: $y = -2\sqrt[3]{3}$ at point $\left(\sqrt[3]{3}, -2\sqrt[3]{3}\right)$.

Vertical tangents: Set denominator equal to 0:

$$x(x + 2y) = 0 \implies x = 0 \text{ or } x = -2y$$

If $x = 0$, substitute into $x^2y + y^2x = 6$: $0 = 6$, impossible.

If $x = -2y$, substitute into $x^2y + y^2x = 6$:

$$(-2y)^2y + y^2(-2y) = 6 \implies 4y^3 - 2y^3 = 6 \implies 2y^3 = 6 \implies y^3 = 3 \implies y = \sqrt[3]{3}$$

$$x = -2\sqrt[3]{3}$$

Vertical tangent line: $x = -2\sqrt[3]{3}$ at point $\left(-2\sqrt[3]{3}, \sqrt[3]{3}\right)$.

(d) From Exercise 3(b), $\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$.

Horizontal tangents: Set numerator equal to 0:

$$3y - x^2 = 0 \implies y = \frac{x^2}{3}$$

Substitute into $x^3 + y^3 = 9xy$:

$$x^3 + \left(\frac{x^2}{3}\right)^3 = 9x\left(\frac{x^2}{3}\right) \implies x^3 + \frac{x^6}{27} = 3x^3$$

$$\frac{x^6}{27} = 2x^3 \implies x^6 = 54x^3 \implies x^3(x^3 - 54) = 0$$

If $x = 0$, then $y = 0$, but $(0, 0)$ is not on the curve (or is a singular point).

If $x^3 = 54$, then $x = \sqrt[3]{54} = 3\sqrt[3]{2}$, and $y = \frac{x^2}{3} = \frac{(3\sqrt[3]{2})^2}{3} = \frac{9\sqrt[3]{4}}{3} = 3\sqrt[3]{4}$.

Horizontal tangent line: $y = 3\sqrt[3]{4}$ at point $\left(3\sqrt[3]{2}, 3\sqrt[3]{4}\right)$.

Vertical tangents: Set denominator equal to 0:

$$y^2 - 3x = 0 \implies x = \frac{y^2}{3}$$

Substitute into $x^3 + y^3 = 9xy$:

$$\left(\frac{y^2}{3}\right)^3 + y^3 = 9\left(\frac{y^2}{3}\right)y \implies \frac{y^6}{27} + y^3 = 3y^3$$

$$\frac{y^6}{27} = 2y^3 \implies y^6 = 54y^3 \implies y^3(y^3 - 54) = 0$$

If $y = 0$, then $x = 0$, but $(0, 0)$ is not on the curve (or is a singular point).

If $y^3 = 54$, then $y = \sqrt[3]{54} = 3\sqrt[3]{2}$, and $x = \frac{y^2}{3} = \frac{(3\sqrt[3]{2})^2}{3} = \frac{9\sqrt[3]{4}}{3} = 3\sqrt[3]{4}$.

Vertical tangent line: $x = 3\sqrt[3]{4}$ at point $\left(3\sqrt[3]{4}, 3\sqrt[3]{2}\right)$.

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2.4: Derivatives of Inverse and Hyperbolic Functions

Learning Objectives

- Objective 1
- Objective 2

Intro Here.

Question of the Day

How do we differentiate inverse functions without explicitly solving for them?

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2.4: Derivatives of Inverse Functions

Inverse functions reverse the roles of inputs and outputs. In this section, we develop methods to differentiate inverse functions—even when we cannot solve for them explicitly—and use implicit differentiation to establish derivative rules for logarithmic, inverse trigonometric, and hyperbolic functions. Finally, we introduce logarithmic differentiation to handle complex products, quotients, and variable exponents.

Derivative of an Inverse Function

If a function f is one-to-one, differentiable, and invertible, the derivative of its inverse function f^{-1} can be computed using the chain rule on the identity $f(f^{-1}(x)) = x$:

General Formula:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \tag{2.4.1}$$

To evaluate $(f^{-1})'(a)$ at a specific value $x = a$:

1. Set $f(b) = a$ and solve for $b = f^{-1}(a)$.
2. Compute $f'(x)$ and evaluate it at $x = b$.
3. Take the reciprocal: $(f^{-1})'(a) = \frac{1}{f'(b)}$.

Example 2.4.1: Derivative of an Inverse Function

Find $(f^{-1})'(3)$ for $f(x) = x^3 + 2x$.

Solution:

First, solve $x^3 + 2x = 3$ to find b. By inspection, $b = 1$ works since $1^3 + 2(1) = 3$.

Then $f'(x) = 3x^2 + 2$, so $f'(1) = 3(1)^2 + 2 = 5$.

Therefore, $(f^{-1})'(3) = \frac{1}{f'(1)} = \boxed{\frac{1}{5}}$.

Example 2.4.2: Inverse Derivative at a Point

Find $(f^{-1})'(4)$ for $f(x) = 2x^3 + 1$.

Solution:

Solve $2x^3 + 1 = 4$: $2x^3 = 3 \implies x^3 = \frac{3}{2} \implies b = \sqrt[3]{\frac{3}{2}}$.

Then $f'(x) = 6x^2$, so $f'(b) = 6\left(\sqrt[3]{\frac{3}{2}}\right)^2 = 6\left(\frac{3}{2}\right)^{2/3}$.

$$(f^{-1})'(4) = \frac{1}{6\left(\frac{3}{2}\right)^{2/3}} \tag{2.4.2}$$

$$\boxed{(f^{-1})'(4) = \frac{1}{6\left(\frac{3}{2}\right)^{2/3}}} \tag{2.4.3}$$

Geometric Interpretation

The formula $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$ has a nice geometric interpretation: the slopes of inverse functions are reciprocals at corresponding points. If the point (a, b) lies on f, then (b, a) lies on f^{-1}, and the tangent slopes are reciprocals.

Derivatives of Logarithmic Functions

By writing $y = \log_b(x)$ in exponential form $b^y = x$ and taking implicit derivatives, we obtain the derivative formula for logarithmic functions:

$$\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)} \qquad \text{and specifically} \qquad \frac{d}{dx}(\ln(x)) = \frac{1}{x} \tag{2.4.4}$$

With the chain rule applied, $\frac{d}{dx}(\ln(g(x))) = \frac{g'(x)}{g(x)}$.

Example 2.4.3: Derivatives of Logarithmic Functions

Differentiate each function:

(a) $f(x) = \ln(3x^2 + 5)$

(b) $g(x) = \log_5(x^3 + 2x)$

(c) $h(x) = \ln(\sin(x))$

Solution:

(a) Using $\frac{d}{dx}[\ln(g(x))] = \frac{g'(x)}{g(x)}$:

$$f'(x) = \frac{6x}{3x^2 + 5} \tag{2.4.5}$$

$$\boxed{f'(x) = \frac{6x}{3x^2 + 5}} \tag{2.4.6}$$

(b) Using $\frac{d}{dx}[\log_b(g(x))] = \frac{g'(x)}{g(x)\ln(b)}$:

$$g'(x) = \frac{3x^2 + 2}{(x^3 + 2x)\ln(5)} \tag{2.4.7}$$

$$\boxed{g'(x) = \frac{3x^2 + 2}{(x^3 + 2x)\ln(5)}} \tag{2.4.8}$$

(c) Using the chain rule:

$$h'(x) = \frac{\cos(x)}{\sin(x)} = \cot(x) \tag{2.4.9}$$

$$\boxed{h'(x) = \cot(x)} \tag{2.4.10}$$

Question of the Day

Why does $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ but $\frac{d}{dx}[\log_b(x)] = \frac{1}{x\ln(b)}$?

Answer The natural logarithm $\ln(x)$ is the inverse of e^x, which has the special property that its derivative is itself. The factor $\ln(b)$ appears because the derivative of b^x includes $\ln(b)$, and the inverse relationship produces a reciprocal factor.

Derivatives of Inverse Trigonometric Functions

Applying implicit differentiation to restricted trigonometric equations (e.g., setting $y = \sin^{-1}(x) \implies \sin(y) = x$) and utilizing right-triangle geometry yields the formulas for inverse trigonometric functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$	$\cot^{-1}(x)$	$-\frac{1}{1+x^2}$
$\sec^{-1}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\csc^{-1}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

Example 2.4.4: Deriving $\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$

Use implicit differentiation to find the derivative of $\sin^{-1}(x)$.

Solution:

Let $y = \sin^{-1}(x)$. Then $\sin(y) = x$.

Differentiate both sides with respect to x:

$$\cos(y) \cdot \frac{dy}{dx} = 1 \tag{2.4.11}$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} \tag{2.4.12}$$

Since $\sin(y) = x$, we know $\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$ (positive because $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$).

$$\boxed{\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}} \tag{2.4.13}$$

Example 2.4.5: Derivatives of Inverse Trigonometric Functions

Differentiate each function:

(a) $f(x) = \sin^{-1}(3x)$

(b) $g(x) = \tan^{-1}(x^2)$

(c) $h(x) = \cos^{-1}(e^x)$

Solution:

(a) Using the Chain Rule:

$$f'(x) = \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 = \frac{3}{\sqrt{1-9x^2}} \tag{2.4.14}$$

$$\boxed{f'(x) = \frac{3}{\sqrt{1-9x^2}}} \tag{2.4.15}$$

(b) Using the Chain Rule:

$$g'(x) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4} \tag{2.4.16}$$

$$\boxed{g'(x) = \frac{2x}{1+x^4}} \tag{2.4.17}$$

(c) Using the Chain Rule:

$$h'(x) = -\frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = -\frac{e^x}{\sqrt{1-e^{2x}}} \tag{2.4.18}$$

$$\boxed{h'(x) = -\frac{e^x}{\sqrt{1-e^{2x}}}} \tag{2.4.19}$$

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2.4: Logarithmic Differentiation

Logarithmic Differentiation

For complicated algebraic expressions involving multiple products, quotients, or powers—or functions with variable bases and exponents of the form $y = [f(x)]^{g(x)}$—we use **logarithmic differentiation**:

1. Take the natural logarithm $\ln$ of both sides: $\ln(y) = \ln([f(x)]^{g(x)})$.
2. Use logarithm properties to expand and simplify the right side.
3. Differentiate implicitly with respect to x: $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\text{expanded side}]$.
4. Multiply by y to solve for $\frac{dy}{dx}$ and substitute the original expression for y.

When to Use Logarithmic Differentiation

Logarithmic differentiation is particularly useful for:

- Functions with variable exponents: $y = x^{\sin(x)}$, $y = (\ln(x))^x$
- Functions with multiple products and quotients: $y = \frac{x^2(x-1)^3}{(x+2)^4\sqrt{x+5}}$
- Functions where the Power Rule alone doesn't apply because the exponent is not constant

Example 2.4.7: Logarithmic Differentiation with Variable Exponent

Use logarithmic differentiation to find $\frac{dy}{dx}$ for $y = x^{\sin(x)}$.

Solution:

Step 1: Take $\ln$ of both sides:

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x)\ln(x) \tag{2.4.1}$$

Step 2: Differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x) + \sin(x)\left(\frac{1}{x}\right) \tag{2.4.2}$$

Step 3: Multiply by y:

$$\frac{dy}{dx} = y\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right) \tag{2.4.3}$$

Step 4: Substitute $y = x^{\sin(x)}$:

$$\boxed{\frac{dy}{dx} = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)} \tag{2.4.4}$$

Example 2.4.8: Logarithmic Differentiation with Products and Quotients

Use logarithmic differentiation to find $\frac{dy}{dx}$ for $y = \frac{x^2(x-1)^3}{(x+2)^4\sqrt{x+5}}$.

Solution:

Step 1: Take $\ln$ of both sides:

$$\ln(y) = \ln\left(\frac{x^2(x-1)^3}{(x+2)^4\sqrt{x+5}}\right) \tag{2.4.5}$$

Step 2: Use logarithm properties to expand:

$$\ln(y) = \ln(x^2) + \ln((x-1)^3) - \ln((x+2)^4) - \ln(\sqrt{x+5}) \tag{2.4.6}$$

$$\ln(y) = 2\ln(x) + 3\ln(x-1) - 4\ln(x+2) - \frac{1}{2}\ln(x+5) \tag{2.4.7}$$

Step 3: Differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{3}{x-1} - \frac{4}{x+2} - \frac{1}{2(x+5)} \tag{2.4.8}$$

Step 4: Multiply by y and substitute:

$$\boxed{\frac{dy}{dx} = \frac{x^2(x-1)^3}{(x+2)^4\sqrt{x+5}}\left(\frac{2}{x} + \frac{3}{x-1} - \frac{4}{x+2} - \frac{1}{2(x+5)}\right)} \tag{2.4.9}$$

Example 2.4.9: Logarithmic Differentiation with Exponential Base

Use logarithmic differentiation to find $\frac{dy}{dx}$ for $y = (\ln(x))^x$.

Solution:

Step 1: Take $\ln$ of both sides:

$$\ln(y) = \ln((\ln(x))^x) = x\ln(\ln(x)) \tag{2.4.10}$$

Step 2: Differentiate implicitly (using the Product Rule):

$$\frac{1}{y}\frac{dy}{dx} = \ln(\ln(x)) + x \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x} = \ln(\ln(x)) + \frac{1}{\ln(x)} \tag{2.4.11}$$

Step 3: Multiply by y:

$$\frac{dy}{dx} = y\left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right) \tag{2.4.12}$$

Step 4: Substitute $y = (\ln(x))^x$:

$$\boxed{\frac{dy}{dx} = (\ln(x))^x\left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right)} \tag{2.4.13}$$

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2.4: Hyperbolic Trig Functions

Derivatives of Hyperbolic Functions

Unlike circular trigonometric functions which satisfy $\cos^2(t) + \sin^2(t) = 1$, hyperbolic functions satisfy the identity $\cosh^2(t) - \sinh^2(t) = 1$. They are defined exponentially:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \tag{2.4.1}$$

Their primary derivatives are:

$$\frac{d}{dx}(\sinh(x)) = \cosh(x) \qquad \text{and} \qquad \frac{d}{dx}(\cosh(x)) = \sinh(x) \tag{2.4.2}$$

The remaining hyperbolic functions are defined analogously to trigonometric functions:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)}, \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)}, \quad \operatorname{csch}(x) = \frac{1}{\sinh(x)} \tag{2.4.3}$$

Their derivatives are:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sinh(x)$	$\cosh(x)$	$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\operatorname{sech}^2(x)$	$\coth(x)$	$-\operatorname{csch}^2(x)$
$\operatorname{sech}(x)$	$-\operatorname{sech}(x)\tanh(x)$	$\operatorname{csch}(x)$	$-\operatorname{csch}(x)\coth(x)$

Example 2.4.6: Derivatives of Hyperbolic Functions

Differentiate each function:

(a) $f(x) = \sinh(2x)$

(b) $g(x) = \cosh(x^2 + 1)$

(c) $h(x) = \tanh(\ln(x))$

Solution:

(a) Using the Chain Rule:

$$f'(x) = \cosh(2x) \cdot 2 = 2\cosh(2x) \tag{2.4.4}$$

$$\boxed{f'(x) = 2\cosh(2x)} \tag{2.4.5}$$

(b) Using the Chain Rule:

$$g'(x) = \sinh(x^2 + 1) \cdot 2x = 2x\sinh(x^2 + 1) \tag{2.4.6}$$

$$\boxed{g'(x) = 2x\sinh(x^2 + 1)} \tag{2.4.7}$$

(c) Using the Chain Rule with $\frac{d}{dx}[\tanh(u)] = \operatorname{sech}^2(u) \cdot u'$:

$$h'(x) = \operatorname{sech}^2(\ln(x)) \cdot \frac{1}{x} \tag{2.4.8}$$

$$\boxed{h'(x) = \frac{\operatorname{sech}^2(\ln(x))}{x}} \tag{2.4.9}$$

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2.4: Practice Set

Exercise 2.4.1

Find the derivative of each function.

(a) $f(x) = \sin^{-1}(3x)$

(b) $g(x) = \cos^{-1}(x^2)$

(c) $h(x) = \tan^{-1}(e^x)$

(d) $p(x) = \sec^{-1}(2x + 1)$

(e) $q(x) = \ln(5x^3 - 2x + 1)$

(f) $r(x) = \log_3(x^2 + 4)$

(g) $s(x) = \ln(\sin(x))$

(h) $t(x) = \ln(x^2 + 1) \cdot e^x$ (product rule)

Answer available at the end of this section.

Exercise 2.4.2

Use logarithmic differentiation to find y'.

(a) $y = x^x$

(b) $y = (2x + 1)^{x^2}$

(c) $y = x^{\sin(x)}$

(d) $y = \dfrac{(x^2 + 1)^3}{(3x - 2)^4}$

(e) $y = \sqrt{\dfrac{x - 1}{x + 2}}$

(f) $y = \dfrac{x^2 \cdot e^x}{(x - 5)^3}$

Answer available at the end of this section.

Exercise 2.4.3

Find the derivative of each hyperbolic function.

(a) $f(x) = \sinh(3x)$

(b) $g(x) = \cosh(x^2)$

(c) $h(x) = \tanh(e^x)$

(d) $p(x) = \sinh^{-1}(2x)$

(e) $q(x) = \cosh^{-1}(x^2 + 1)$

(f) $r(x) = \tanh^{-1}\left(\frac{x}{2}\right)$

Answer available at the end of this section.

Exercise 2.4.4

Find the equation of the tangent line to each function at the given point.

(a) $f(x) = \ln(x^2 + 1)$ at $x = 1$

(b) $g(x) = \tan^{-1}(2x)$ at $x = \frac{1}{2}$

(c) $h(x) = \sinh(x)$ at $x = 0$

(d) $p(x) = \cosh^{-1}(x)$ at $x = 2$

(e) $q(x) = x^x$ at $x = 1$

(f) $r(x) = \sin^{-1}(x^2)$ at $x = \frac{1}{2}$

Answer available at the end of this section.

Exercise 2.4.5

Find the derivative of each function, which may require multiple rules (chain, product, quotient, inverse trig, logs, etc.).

(a) $f(x) = \ln\left(\frac{x^2 + 1}{x - 3}\right)$

(b) $g(x) = \sin^{-1}(e^x) \cdot \cos(x)$

(c) $h(x) = \frac{\tan^{-1}(x)}{\ln(x)}$

(d) $p(x) = \ln(\sec(x) + \tan(x))$

(e) $q(x) = \cosh^{-1}(x^2 + 2x)$

(f) $r(x) = x^{\ln(x)}$

Answer available at the end of this section.

Exercise 2.4.6

Find the equation of the tangent line to each implicit function at the given point.

(a) $\ln(x) + \ln(y) = 1$ at $(e, 1)$

(b) $\tan^{-1}(x) + \tan^{-1}(y) = \frac{\pi}{4}$ at $(0, 1)$

(c) $\sinh(x)+\cosh(y)=1$ at $(0,0)$

(d) $\ln(x^2+y^2)=2$ at $(1,\sqrt{e^2-1})$

Answer available at the end of this section.

Answers

Exercise 2.4.1

(a) Recall that $\frac{d}{dx}[\sin^{-1}(u)]=\frac{u'}{\sqrt{1-u^2}}$.

Let $u=3x$. Then $u'=3$.

$$f'(x)=\frac{3}{\sqrt{1-(3x)^2}}=\frac{3}{\sqrt{1-9x^2}}$$

Thus $f'(x)=\frac{3}{\sqrt{1-9x^2}}$.

(b) Recall that $\frac{d}{dx}[\cos^{-1}(u)]=-\frac{u'}{\sqrt{1-u^2}}$.

Let $u=x^2$. Then $u'=2x$.

$$g'(x)=-\frac{2x}{\sqrt{1-(x^2)^2}}=-\frac{2x}{\sqrt{1-x^4}}$$

Thus $g'(x)=-\frac{2x}{\sqrt{1-x^4}}$.

(c) Recall that $\frac{d}{dx}[\tan^{-1}(u)]=\frac{u'}{1+u^2}$.

Let $u=e^x$. Then $u'=e^x$.

$$h'(x)=\frac{e^x}{1+(e^x)^2}=\frac{e^x}{1+e^{2x}}$$

Thus $h'(x)=\frac{e^x}{1+e^{2x}}$.

(d) Recall that $\frac{d}{dx}[\sec^{-1}(u)]=\frac{u'}{|u|\sqrt{u^2-1}}$.

Let $u=2x+1$. Then $u'=2$.

$$p'(x)=\frac{2}{|2x+1|\sqrt{(2x+1)^2-1}}$$

Thus $p'(x)=\frac{2}{|2x+1|\sqrt{(2x+1)^2-1}}$.

(e) Recall that $\frac{d}{dx}[\ln(u)]=\frac{u'}{u}$.

Let $u=5x^3-2x+1$. Then $u'=15x^2-2$.

$$q'(x)=\frac{15x^2-2}{5x^3-2x+1}$$

Thus $q'(x)=\frac{15x^2-2}{5x^3-2x+1}$.

(f) Recall that $\frac{d}{dx}[\log_a(u)]=\frac{u'}{u\ln(a)}$.

Let $u=x^2+4$. Then $u'=2x$.

$$r'(x)=\frac{2x}{(x^2+4)\ln(3)}$$

Thus $r'(x)=\frac{2x}{(x^2+4)\ln(3)}$.

(g) Let $u=\sin(x)$. Then $u'=\cos(x)$.

$$s'(x)=\frac{\cos(x)}{\sin(x)}=\cot(x)$$

Thus $s'(x)=\cot(x)$.

(h) Use the product rule with $u=\ln(x^2+1)$ and $v=e^x$.

$u'=\frac{2x}{x^2+1}$ and $v'=e^x$.

$$t'(x)=\left(\frac{2x}{x^2+1}\right)e^x+\ln(x^2+1)e^x=e^x\left(\frac{2x}{x^2+1}+\ln(x^2+1)\right)$$

Thus $t'(x)=e^x\left(\frac{2x}{x^2+1}+\ln(x^2+1)\right)$.

Exercise 2.4.2

(a) Take the natural log of both sides:

$$\ln(y) = \ln(x^x) = x\ln(x)$$

Differentiate both sides with respect to x:

$$\frac{y'}{y} = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

Multiply by $y = x^x$:

$$y' = x^x(\ln(x) + 1)$$

Thus $y' = x^x(\ln(x) + 1)$.

(b) Take the natural log of both sides:

$$\ln(y) = \ln((2x+1)^{x^2}) = x^2\ln(2x+1)$$

Differentiate both sides with respect to x:

$$\frac{y'}{y} = 2x\ln(2x+1) + x^2 \cdot \frac{2}{2x+1}$$

Multiply by $y = (2x+1)^{x^2}$:

$$y' = (2x+1)^{x^2}\left(2x\ln(2x+1) + \frac{2x^2}{2x+1}\right)$$

Thus $y' = (2x+1)^{x^2}\left(2x\ln(2x+1) + \frac{2x^2}{2x+1}\right)$.

(c) Take the natural log of both sides:

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x)\ln(x)$$

Differentiate both sides with respect to x:

$$\frac{y'}{y} = \cos(x)\ln(x) + \sin(x) \cdot \frac{1}{x}$$

Multiply by $y = x^{\sin(x)}$:

$$y' = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$$

Thus $y' = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$.

(d) Take the natural log of both sides:

$$\ln(y) = \ln\left(\frac{(x^2+1)^3}{(3x-2)^4}\right) = 3\ln(x^2+1) - 4\ln(3x-2)$$

Differentiate both sides with respect to x:

$$\frac{y'}{y} = 3 \cdot \frac{2x}{x^2+1} - 4 \cdot \frac{3}{3x-2} = \frac{6x}{x^2+1} - \frac{12}{3x-2}$$

Multiply by $y = \frac{(x^2+1)^3}{(3x-2)^4}$:

$$y' = \frac{(x^2+1)^3}{(3x-2)^4}\left(\frac{6x}{x^2+1} - \frac{12}{3x-2}\right)$$

Thus $y' = \frac{(x^2+1)^3}{(3x-2)^4}\left(\frac{6x}{x^2+1} - \frac{12}{3x-2}\right)$.

(e) Rewrite as $y = \left(\frac{x-1}{x+2}\right)^{1/2}$.

Take the natural log of both sides:

$$\ln(y) = \frac{1}{2}\ln\left(\frac{x-1}{x+2}\right) = \frac{1}{2}\left[\ln(x-1) - \ln(x+2)\right]$$

Differentiate both sides with respect to x:

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+2}\right) = \frac{1}{2}\left(\frac{(x+2)-(x-1)}{(x-1)(x+2)}\right) = \frac{3}{2(x-1)(x+2)}$$

Multiply by $y = \sqrt{\frac{x-1}{x+2}}$:

$$y' = \sqrt{\frac{x-1}{x+2}} \cdot \frac{3}{2(x-1)(x+2)}$$

Thus $y' = \frac{3}{2(x-1)(x+2)}\sqrt{\frac{x-1}{x+2}}$.

(f) Take the natural log of both sides:

$$\ln(y) = \ln\left(\frac{x^2e^x}{(x-5)^3}\right) = 2\ln(x) + x - 3\ln(x-5)$$

Differentiate both sides with respect to x:

$$\frac{y'}{y} = \frac{2}{x} + 1 - \frac{3}{x-5}$$

Multiply by $y = \dfrac{x^2e^x}{(x-5)^3}$:

$$y' = \frac{x^2e^x}{(x-5)^3}\left(\frac{2}{x} + 1 - \frac{3}{x-5}\right)$$

Thus $y' = \dfrac{x^2e^x}{(x-5)^3}\left(\dfrac{2}{x} + 1 - \dfrac{3}{x-5}\right)$.

Exercise 2.4.3

(a) Recall that $\dfrac{d}{dx}[\sinh(u)] = \cosh(u)u'$.

Let $u = 3x$. Then $u' = 3$.

$$f'(x) = 3\cosh(3x)$$

Thus $f'(x) = 3\cosh(3x)$.

(b) Recall that $\dfrac{d}{dx}[\cosh(u)] = \sinh(u)u'$.

Let $u = x^2$. Then $u' = 2x$.

$$g'(x) = 2x\sinh(x^2)$$

Thus $g'(x) = 2x\sinh(x^2)$.

(c) Recall that $\dfrac{d}{dx}[\tanh(u)] = \operatorname{sech}^2(u)u'$.

Let $u = e^x$. Then $u' = e^x$.

$$h'(x) = e^x\operatorname{sech}^2(e^x)$$

Thus $h'(x) = e^x\operatorname{sech}^2(e^x)$.

(d) Recall that $\dfrac{d}{dx}[\sinh^{-1}(u)] = \dfrac{u'}{\sqrt{1+u^2}}$.

Let $u = 2x$. Then $u' = 2$.

$$p'(x) = \frac{2}{\sqrt{1+(2x)^2}} = \frac{2}{\sqrt{1+4x^2}}$$

Thus $p'(x) = \dfrac{2}{\sqrt{1+4x^2}}$.

(e) Recall that $\dfrac{d}{dx}[\cosh^{-1}(u)] = \dfrac{u'}{\sqrt{u^2-1}}$, for $u > 1$.

Let $u = x^2 + 1$. Then $u' = 2x$.

$$q'(x) = \frac{2x}{\sqrt{(x^2+1)^2-1}} = \frac{2x}{\sqrt{x^4+2x^2+1-1}} = \frac{2x}{\sqrt{x^4+2x^2}} = \frac{2x}{|x|\sqrt{x^2+2}}$$

For $x > 0$, $q'(x) = \dfrac{2}{\sqrt{x^2+2}}$. For $x < 0$, $q'(x) = -\dfrac{2}{\sqrt{x^2+2}}$.

Thus $q'(x) = \dfrac{2x}{|x|\sqrt{x^2+2}}$.

(f) Recall that $\dfrac{d}{dx}[\tanh^{-1}(u)] = \dfrac{u'}{1-u^2}$, for $|u| < 1$.

Let $u = \dfrac{x}{2}$. Then $u' = \dfrac{1}{2}$.

$$r'(x) = \frac{\frac{1}{2}}{1-\left(\frac{x}{2}\right)^2} = \frac{\frac{1}{2}}{1-\frac{x^2}{4}} = \frac{2}{4-x^2}$$

Thus $r'(x) = \dfrac{2}{4-x^2}$.

Exercise 2.4.4

(a) $f'(x) = \dfrac{2x}{x^2+1}$.

Evaluate at $x = 1$: $f'(1) = \dfrac{2(1)}{1+1} = 1$.

Find the point: $f(1) = \ln(1+1) = \ln(2)$.

The tangent line at $(1, \ln(2))$ with slope $m = 1$:

$$y - \ln(2) = 1(x-1)$$

Thus $y - \ln(2) = x - 1$.

(b) $g'(x) = \dfrac{2}{1+(2x)^2} = \dfrac{2}{1+4x^2}$.

Evaluate at $x = \dfrac{1}{2}$: $g'\left(\dfrac{1}{2}\right) = \dfrac{2}{1+4\left(\frac{1}{4}\right)} = \dfrac{2}{1+1} = 1$.

Find the point: $g\left(\dfrac{1}{2}\right) = \tan^{-1}(1) = \dfrac{\pi}{4}$.

The tangent line at $\left(\dfrac{1}{2}, \dfrac{\pi}{4}\right)$ with slope $m = 1$:

$$y - \frac{\pi}{4} = 1\left(x - \frac{1}{2}\right)$$

Thus $y - \frac{\pi}{4} = x - \frac{1}{2}$.

(c) $h'(x) = \cosh(x)$.

Evaluate at $x = 0$: $h'(0) = \cosh(0) = 1$.

Find the point: $h(0) = \sinh(0) = 0$.

The tangent line at $(0, 0)$ with slope $m = 1$:

$$y - 0 = 1(x - 0)$$

Thus $y = x$.

(d) Recall that $\frac{d}{dx}[\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2 - 1}}$.

Evaluate at $x = 2$: $p'(2) = \frac{1}{\sqrt{4 - 1}} = \frac{1}{\sqrt{3}}$.

Find the point: $p(2) = \cosh^{-1}(2) = \ln(2 + \sqrt{3})$.

The tangent line at $(2, \ln(2 + \sqrt{3}))$ with slope $m = \frac{1}{\sqrt{3}}$:

$$y - \ln(2 + \sqrt{3}) = \frac{1}{\sqrt{3}}(x - 2)$$

Thus $y - \ln(2 + \sqrt{3}) = \frac{1}{\sqrt{3}}(x - 2)$.

(e) From Exercise 2(a), $q'(x) = x^x(\ln(x) + 1)$.

Evaluate at $x = 1$: $q'(1) = 1^1(\ln(1) + 1) = 1(0 + 1) = 1$.

Find the point: $q(1) = 1^1 = 1$.

The tangent line at $(1, 1)$ with slope $m = 1$:

$$y - 1 = 1(x - 1)$$

Thus $y - 1 = x - 1$ or $y = x$.

(f) $r'(x) = \frac{2x}{\sqrt{1 - (x^2)^2}} = \frac{2x}{\sqrt{1 - x^4}}$.

Evaluate at $x = \frac{1}{2}$: $r'\left(\frac{1}{2}\right) = \frac{2\left(\frac{1}{2}\right)}{\sqrt{1 - \left(\frac{1}{16}\right)}} = \frac{1}{\sqrt{\frac{15}{16}}} = \frac{4}{\sqrt{15}}$.

Find the point: $r\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{1}{4}\right)$.

The tangent line at $\left(\frac{1}{2}, \sin^{-1}\left(\frac{1}{4}\right)\right)$ with slope $m = \frac{4}{\sqrt{15}}$:

$$y - \sin^{-1}\left(\frac{1}{4}\right) = \frac{4}{\sqrt{15}}\left(x - \frac{1}{2}\right)$$

Thus $y - \sin^{-1}\left(\frac{1}{4}\right) = \frac{4}{\sqrt{15}}\left(x - \frac{1}{2}\right)$.

Exercise 2.4.5

(a) Simplify using log properties:

$$f(x) = \ln(x^2 + 1) - \ln(x - 3)$$

$$f'(x) = \frac{2x}{x^2 + 1} - \frac{1}{x - 3}$$

Thus $f'(x) = \frac{2x}{x^2 + 1} - \frac{1}{x - 3}$.

(b) Use the product rule with $u = \sin^{-1}(e^x)$ and $v = \cos(x)$.

$u' = \frac{e^x}{\sqrt{1 - e^{2x}}}$ and $v' = -\sin(x)$.

$$g'(x) = \frac{e^x}{\sqrt{1 - e^{2x}}}\cos(x) + \sin^{-1}(e^x)(-\sin(x))$$

Thus $g'(x) = \frac{e^x \cos(x)}{\sqrt{1 - e^{2x}}} - \sin(x)\sin^{-1}(e^x)$.

(c) Use the quotient rule with $u = \tan^{-1}(x)$ and $v = \ln(x)$.

$u' = \frac{1}{1 + x^2}$ and $v' = \frac{1}{x}$.

$$h'(x) = \frac{\left(\frac{1}{1 + x^2}\right)\ln(x) - \tan^{-1}(x)\left(\frac{1}{x}\right)}{[\ln(x)]^2}$$

Thus $h'(x) = \frac{\frac{\ln(x)}{1 + x^2} - \frac{\tan^{-1}(x)}{x}}{[\ln(x)]^2}$.

(d) Let $u = \sec(x) + \tan(x)$. Then $u' = \sec(x)\tan(x) + \sec^2(x)$.

$$p'(x) = \frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x) + \tan(x)}$$

Factor the numerator: $\sec(x)(\tan(x) + \sec(x))$.

$$p'(x) = \frac{\sec(x)(\sec(x) + \tan(x))}{\sec(x) + \tan(x)} = \sec(x)$$

Thus $p'(x) = \sec(x)$.

(e) Let $u = x^2 + 2x$. Then $u' = 2x + 2$.

$$q'(x) = \frac{2x+2}{\sqrt{(x^2+2x)^2 - 1}} = \frac{2(x+1)}{\sqrt{x^4 + 4x^3 + 4x^2 - 1}}$$

Thus $q'(x) = \dfrac{2(x+1)}{\sqrt{x^4 + 4x^3 + 4x^2 - 1}}$.

(f) Take the natural log of both sides:

$$\ln(r) = \ln(x^{\ln(x)}) = \ln(x) \cdot \ln(x) = [\ln(x)]^2$$

Differentiate both sides:

$$\frac{r'}{r} = 2\ln(x) \cdot \frac{1}{x} = \frac{2\ln(x)}{x}$$

Multiply by $r = x^{\ln(x)}$:

$$r'(x) = x^{\ln(x)} \cdot \frac{2\ln(x)}{x}$$

Thus $r'(x) = 2x^{\ln(x)-1}\ln(x)$.

Exercise 2.4.6

(a) Differentiate both sides with respect to x:

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} \implies \frac{dy}{dx} = -\frac{y}{x}$$

Evaluate at $(e, 1)$: $\dfrac{dy}{dx} = -\dfrac{1}{e}$.

The tangent line at $(e, 1)$ with slope $m = -\dfrac{1}{e}$:

$$y - 1 = -\frac{1}{e}(x - e)$$

Thus $y - 1 = -\dfrac{1}{e}(x - e)$.

(b) Differentiate both sides with respect to x:

$$\frac{1}{1+x^2} + \frac{1}{1+y^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{1+y^2}{1+x^2}$$

Evaluate at $(0, 1)$: $\dfrac{dy}{dx} = -\dfrac{1+1}{1+0} = -2$.

The tangent line at $(0, 1)$ with slope $m = -2$:

$$y - 1 = -2(x - 0)$$

Thus $y - 1 = -2x$.

(c) Differentiate both sides with respect to x:

$$\cosh(x) + \sinh(y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\cosh(x)}{\sinh(y)}$$

Evaluate at $(0, 0)$: $\dfrac{dy}{dx} = -\dfrac{\cosh(0)}{\sinh(0)} = -\dfrac{1}{0}$, which is undefined.

Therefore the tangent line is vertical at $x = 0$.

Thus the tangent line is $x = 0$.

(d) Differentiate both sides with respect to x:

$$\frac{1}{x^2+y^2}(2x + 2y\frac{dy}{dx}) = 0$$

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$$

Evaluate at $(1, \sqrt{e^2-1})$: $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{e^2-1}}$.

The tangent line at $(1, \sqrt{e^2-1})$ with slope $m = -\dfrac{1}{\sqrt{e^2-1}}$:

$$y - \sqrt{e^2-1} = -\frac{1}{\sqrt{e^2-1}}(x - 1)$$

Thus $y - \sqrt{e^2 - 1} = -\dfrac{1}{\sqrt{e^2 - 1}}(x - 1)$.

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2.5: Linearization and the Mean Value Theorem

Learning Objectives

- Describe the linear approximation to a function at a point.
- Write the linearization of a given function.
- Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials Introduction to Power Series and Functions.

Question of the Day

How can we use tangent lines to approximate function values near a known point?

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2.5: Linearization for Approximation

Tangent lines provide the best linear approximation of a smooth function near a specific point. In this section, we construct linearizations to estimate function values without complex computations, evaluate propagated errors, and explore the Mean Value Theorem—a fundamental theorem connecting local derivatives to average rates of change.

Linearization (Tangent Line Approximation)

When x is close to a known point a, a differentiable function $f(x)$ can be approximated by its tangent line at $x = a$. This tangent line function is called the **linearization** of f at a.

Linearization Formula:

$$L(x) = f(a) + f'(a)(x - a) \tag{2.5.1}$$

For values of x near a, we have $f(x) \approx L(x)$.

Geometric Interpretation

The linearization $L(x)$ is simply the equation of the tangent line to $f(x)$ at $x = a$. The closer x is to a, the better the approximation. This is why linearization is often called the **tangent line approximation**.

Example 2.5.1: Linearization of a Square Root Function

Find the linearization of $f(x) = \sqrt{x}$ at $a = 100$, and use it to estimate $\sqrt{102}$.

Solution:

$$f(100) = \sqrt{100} = 10 \tag{2.5.2}$$

$$f'(x) = \frac{1}{2\sqrt{x}} \implies f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05 \tag{2.5.3}$$

$$L(x) = f(100) + f'(100)(x - 100) = 10 + 0.05(x - 100) \tag{2.5.4}$$

$$\boxed{L(x) = 10 + \frac{x - 100}{20}} \tag{2.5.5}$$

To estimate $\sqrt{102}$:

$$f(102) \approx L(102) = 10 + \frac{102 - 100}{20} = 10 + \frac{2}{20} = 10.1 \tag{2.5.6}$$

$$\boxed{\sqrt{102} \approx 10.1} \tag{2.5.7}$$

(Actual value: $\sqrt{102} \approx 10.0995$, so the approximation is very accurate.)

Example 2.5.2: Linearization of a Trigonometric Function

Find the linearization of $f(x) = \sin(x)$ at $a = 0$, and use it to estimate $\sin(0.1)$.

Solution:

$$f(0) = \sin(0) = 0 \tag{2.5.8}$$

$$f'(x) = \cos(x) \implies f'(0) = \cos(0) = 1 \tag{2.5.9}$$

$$L(x) = f(0) + f'(0)(x - 0) = 0 + 1(x) = x \tag{2.5.10}$$

$$\boxed{L(x) = x} \tag{2.5.11}$$

To estimate $\sin(0.1)$:

$$f(0.1) \approx L(0.1) = 0.1 \tag{2.5.12}$$

$$\boxed{\sin(0.1) \approx 0.1} \tag{2.5.13}$$

(Actual value: $\sin(0.1) \approx 0.09983$, which is very close to 0.1.)

Example 2.5.3: Linearization of a General Function

Find the linearization of $f(x) = e^x$ at $a = 0$, and use it to approximate $e^{0.2}$.

Solution:

$$f(0) = e^0 = 1 \tag{2.5.14}$$

$$f'(x) = e^x \implies f'(0) = e^0 = 1 \tag{2.5.15}$$

$$L(x) = f(0) + f'(0)(x - 0) = 1 + 1(x) = 1 + x \tag{2.5.16}$$

$$\boxed{L(x) = 1 + x} \tag{2.5.17}$$

To estimate $e^{0.2}$:

$$e^{0.2} \approx L(0.2) = 1 + 0.2 = 1.2 \tag{2.5.18}$$

$$\boxed{e^{0.2} \approx 1.2} \tag{2.5.19}$$

(Actual value: $e^{0.2} \approx 1.2214$.)

Error Estimation and Propagation

Linearization can estimate how small measurement errors propagate into calculated quantities. For a function $f(x)$, if x has a small error Δx, then the approximate error in $f(x)$ is:

$$\Delta f \approx f'(a)\Delta x \tag{2.5.20}$$

The **relative error** is approximately $\frac{\Delta f}{f(a)} \approx \frac{f'(a)}{f(a)}\Delta x$.

Example 2.5.4: Propagated Error in Volume

The volume of a cube with side length s is $V(s) = s^3$. If the side length is measured as $s = 10$ cm with a maximum error of ± 0.1 cm, estimate the maximum possible error in the volume.

Solution:

$$V'(s) = 3s^2 \tag{2.5.21}$$

At $s = 10$: $V'(10) = 3(10)^2 = 300$.

$$\Delta V \approx V'(10)\Delta s = 300(0.1) = 30 \text{ cm}^3 \tag{2.5.22}$$

$$\boxed{\text{The volume error is approximately } \pm 30 \text{ cm}^3} \tag{2.5.23}$$

The relative error is:

$$\frac{\Delta V}{V} \approx \frac{30}{1000} = 0.03 = 3\% \tag{2.5.24}$$

Example 2.5.5: Propagated Error in Surface Area

The surface area of a sphere is $A(r) = 4\pi r^2$. If the radius is measured as $r = 5$ meters with an error of ± 0.02 meters, estimate the maximum possible error in the surface area.

Solution:

$$A'(r) = 8\pi r \tag{2.5.25}$$

At $r = 5$: $A'(5) = 8\pi(5) = 40\pi$.

$$\Delta A \approx A'(5)\Delta r = 40\pi(0.02) = 0.8\pi \approx 2.513 \text{ m}^2 \tag{2.5.26}$$

$$\boxed{\text{The surface area error is approximately } \pm 0.8\pi \text{ m}^2 \approx \pm 2.51 \text{ m}^2} \tag{2.5.27}$$

Question of the Day

Why does linearization work for small changes?

Answer The derivative $f'(a)$ represents the instantaneous rate of change at a. For small changes Δx, the function behaves almost linearly, so the tangent line provides an excellent approximation. The error is proportional to $(\Delta x)^2$, so the smaller the change, the better the approximation.

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2.5: The Mean Value Theorem

The Mean Value Theorem (MVT)

The Mean Value Theorem bridges the gap between the average rate of change over an interval and the instantaneous rate of change at a specific point.

The Mean Value Theorem:

If $f(x)$ is **continuous** on the closed interval $[a, b]$ and **differentiable** on the open interval (a, b), then there exists at least one number $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{2.5.1}$$

In geometric terms, there is at least one point c where the tangent line to the curve is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Rolle's Theorem: A Special Case of MVT

Rolle's Theorem is a special case of the Mean Value Theorem where $f(a) = f(b)$. In this case, there exists $c \in (a, b)$ such that $f'(c) = 0$.

This is particularly useful for proving that a function has at most one root on an interval.

Example 2.5.6: Applying the Mean Value Theorem

Verify that the Mean Value Theorem applies to $f(x) = x^3 - x$ on $[0, 2]$, and find all values of c that satisfy the theorem.

Solution:

$f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the MVT applies.

$$f(0) = 0^3 - 0 = 0 \tag{2.5.2}$$

$$f(2) = 2^3 - 2 = 8 - 2 = 6 \tag{2.5.3}$$

$$\frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2} = 3 \tag{2.5.4}$$

Set $f'(c) = 3$:

$$f'(x) = 3x^2 - 1 \implies 3c^2 - 1 = 3 \tag{2.5.5}$$

$$3c^2 = 4 \implies c^2 = \frac{4}{3} \implies c = \pm\frac{2}{\sqrt{3}} \tag{2.5.6}$$

Only $c = \frac{2}{\sqrt{3}} \approx 1.155$ is in $(0, 2)$.

$$\boxed{c = \frac{2}{\sqrt{3}}} \tag{2.5.7}$$

Example 2.5.7: Mean Value Theorem with a Rational Function

Verify that the Mean Value Theorem applies to $f(x) = \frac{1}{x}$ on $[1, 4]$, and find all values of c that satisfy the theorem.

Solution:

$f(x)$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$, so the MVT applies.

$$f(1) = 1 \tag{2.5.8}$$

$$f(4) = \frac{1}{4} \tag{2.5.9}$$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\frac{1}{4} - 1}{3} = \frac{-\frac{3}{4}}{3} = -\frac{1}{4} \tag{2.5.10}$$

Set $f'(c) = -\frac{1}{4}$:

$$f'(x) = -\frac{1}{x^2} \implies -\frac{1}{c^2} = -\frac{1}{4} \tag{2.5.11}$$

$$c^2 = 4 \implies c = \pm 2 \tag{2.5.12}$$

Only $c = 2$ is in $(1, 4)$.

$$\boxed{c = 2} \tag{2.5.13}$$

Example 2.5.8: When MVT Does Not Apply

Explain why the Mean Value Theorem cannot be applied to $f(x) = |x|$ on $[-1, 2]$.

Solution:

The function $f(x) = |x|$ is continuous on $[-1, 2]$, but it is **not differentiable** at $x = 0$ (there is a sharp corner). Since the function is not differentiable on the entire open interval $(-1, 2)$, the MVT does not apply.

$$\boxed{\text{The MVT does not apply because } f \text{ is not differentiable at } x = 0} \tag{2.5.14}$$

Applications of the Mean Value Theorem: Proving Uniqueness

The Mean Value Theorem (along with Rolle's Theorem) is frequently used to prove that an equation has *at most one* solution by demonstrating that assuming two roots leads to a contradiction.

Example 2.5.9: Proving at Most One Solution

Show that the equation $x^3 - 3x + 1 = 0$ has at most one solution in the interval $(0, 1)$.

Solution:

Suppose, for contradiction, that there are two solutions a and b in $(0, 1)$ with $a < b$. Then $f(a) = f(b) = 0$ where $f(x) = x^3 - 3x + 1$.

By Rolle's Theorem, there must exist $c \in (a, b)$ such that $f'(c) = 0$.

But $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$.

For $x \in (0,1)$, $x^2 < 1$, so $x^2 - 1 < 0$, meaning $f'(x) < 0$ for all $x \in (0,1)$.

This contradicts the existence of $c \in (a,b) \subset (0,1)$ where $f'(c) = 0$.

Therefore, the equation has **at most one** solution in $(0,1)$.

$$\boxed{\text{At most one solution in } (0,1)} \tag{2.5.15}$$

Example 2.5.10: Proving a Unique Root Exists

Show that the equation $x^3 - 3x + 1 = 0$ has exactly one solution in $(0,1)$.

Solution:

We already showed at most one solution exists in Example 9.

Now we show at least one solution exists using the Intermediate Value Theorem:

$$f(0) = 1 > 0 \tag{2.5.16}$$

$$f(1) = 1 - 3 + 1 = -1 < 0 \tag{2.5.17}$$

Since f is continuous and $f(0) > 0 > f(1)$, by the IVT, there is at least one root in $(0,1)$.

Since we have at least one and at most one, there is **exactly one** solution in $(0,1)$.

$$\boxed{\text{Exactly one solution in } (0,1)} \tag{2.5.18}$$

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2.5: Practice Set

Exercise 2.5.1

Find the linearization function, $L(x)$, of each function at the given point a:

(a) $f(x) = \sqrt{x}$ at $a = 9$

(b) $g(x) = \sin(x)$ at $a = 0$

(c) $h(x) = e^x$ at $a = 0$

(d) $p(x) = \ln(x)$ at $a = 1$

(e) $q(x) = \tan(x)$ at $a = \frac{\pi}{4}$

(f) $r(x) = x^3 - 2x + 1$ at $a = 2$

Answer available at the end of this section.

Exercise 2.5.2

Use the linearization from Exercise 1 to approximate each value:

(a) $\sqrt{10}$

(b) $\sin(0.1)$

(c) $e^{0.2}$

(d) $\ln(1.1)$

(e) $\tan\left(\frac{\pi}{4} + 0.1\right)$

(f) $(2.1)^3 - 2(2.1) + 1$

Answer available at the end of this section.

Exercise 2.5.3

Use the Mean Value Theorem to find all values c in the given interval that satisfy the conclusion of the theorem:

(a) $f(x) = x^2$ on $[0, 4]$

(b) $g(x) = x^3 - 3x + 1$ on $[-2, 2]$

(c) $h(x) = \sqrt{x}$ on $[1, 4]$

(d) $p(x) = \sin(x)$ on $[0, \pi]$

(e) $q(x) = \ln(x)$ on $[1, e]$

(f) $r(x) = e^x$ on $[0, 1]$

Answer available at the end of this section.

Exercise 2.5.4

For each function and interval, verify that the Mean Value Theorem applies and find all values c that satisfy the conclusion:

(a) $f(x) = x^2 - 4x + 3$ on $[1, 5]$

(b) $g(x) = \cos(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) $h(x) = \dfrac{1}{x-1}$ on $[2, 4]$

(d) $p(x) = \tan^{-1}(x)$ on $[0, 1]$

(e) $q(x) = x^{\frac{3}{2}}$ on $[0, 4]$

Answer available at the end of this section.

Exercise 2.5.5

Use the Mean Value Theorem to show that each equation has exactly one solution in the given interval:

(a) $x^3 + x - 1 = 0$

(b) $\cos(x) = 2x$

(c) $e^x = 2 - x$

(d) $\ln(x) = 2 - x$

Answer available at the end of this section.

Exercise 2.5.6

Solve each application problem:

(a) A car's position is given by $s(t) = t^3 - 6t^2 + 9t + 2$ for $0 \le t \le 4$. Find the time(s) when the instantaneous velocity equals the average velocity over the interval $[0, 4]$.

(b) A runner completes a 10 km race in 40 minutes. Show that there must be a moment during the race when the runner's instantaneous speed equals 15 km/hr.

Answer available at the end of this section.

Answers

Exercise 2.5.1

(a) $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

At $a = 9$: $f(9) = 3$, $f'(9) = \frac{1}{2(3)} = \frac{1}{6}$

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Thus, $L(x) = 3 + \frac{1}{6}(x - 9)$

(b) $g(x) = \sin(x)$, $g'(x) = \cos(x)$
At $a = 0$: $g(0) = 0$, $g'(0) = 1$

$$L(x) = 0 + 1(x - 0) = x$$

Thus, $L(x) = x$
(c) $h(x) = e^x$, $h'(x) = e^x$
At $a = 0$: $h(0) = 1$, $h'(0) = 1$

$$L(x) = 1 + 1(x - 0) = 1 + x$$

Thus, $L(x) = 1 + x$
(d) $p(x) = \ln(x)$, $p'(x) = \dfrac{1}{x}$
At $a = 1$: $p(1) = 0$, $p'(1) = 1$

$$L(x) = 0 + 1(x - 1) = x - 1$$

Thus, $L(x) = x - 1$
(e) $q(x) = \tan(x)$, $q'(x) = \sec^2(x)$
At $a = \dfrac{\pi}{4}$: $q\left(\dfrac{\pi}{4}\right) = 1$, $q'\left(\dfrac{\pi}{4}\right) = \sec^2\left(\dfrac{\pi}{4}\right) = 2$.

$$L(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$$

Thus, $L(x) = 1 + 2\left(x - \dfrac{\pi}{4}\right)$
(f) $r(x) = x^3 - 2x + 1$, $r'(x) = 3x^2 - 2$
At $a = 2$: $r(2) = 8 - 4 + 1 = 5$, $r'(2) = 12 - 2 = 10$

$$L(x) = 5 + 10(x - 2)$$

Thus, $L(x) = 5 + 10(x - 2)$

Exercise 2.5.2

(a) Using $L(x) = 3 + \dfrac{1}{6}(x - 9)$:

$$\sqrt{10} \approx L(10) = 3 + \frac{1}{6}(1) = 3 + \frac{1}{6} = \frac{19}{6} \approx 3.1667$$

Actual value: $\sqrt{10} \approx 3.1623$
(b) Using $L(x) = x$:

$$\sin(0.1) \approx L(0.1) = 0.1$$

Actual value: $\sin(0.1) \approx 0.0998$
(c) Using $L(x) = 1 + x$:

$$e^{0.2} \approx L(0.2) = 1 + 0.2 = 1.2$$

Actual value: $e^{0.2} \approx 1.2214$
(d) Using $L(x) = x - 1$:

$$\ln(1.1) \approx L(1.1) = 1.1 - 1 = 0.1$$

Actual value: $\ln(1.1) \approx 0.0953$
(e) Using $L(x) = 1 + 2\left(x - \dfrac{\pi}{4}\right)$:
Let $x = \dfrac{\pi}{4} + 0.1$. Then $x - \dfrac{\pi}{4} = 0.1$.

$$\tan\left(\frac{\pi}{4} + 0.1\right) \approx L\left(\frac{\pi}{4} + 0.1\right) = 1 + 2(0.1) = 1.2$$

Actual value: $\tan\left(\dfrac{\pi}{4} + 0.1\right) = \tan(0.8854) \approx 1.2230$
(f) Using $L(x) = 5 + 10(x - 2)$:

$$(2.1)^3 - 2(2.1) + 1 \approx L(2.1) = 5 + 10(0.1) = 6$$

Actual value: $2.1^3 - 2(2.1) + 1 = 9.261 - 4.2 + 1 = 6.061$.

Exercise 2.5.3

(a) $f(x) = x^2$, $f'(x) = 2x$

$$\frac{f(4) - f(0)}{4 - 0} = \frac{16 - 0}{4} = 4$$

Set $2c = 4 \implies c = 2$
Thus, $c = 2$
(b) $g(x) = x^3 - 3x + 1$, $g'(x) = 3x^2 - 3$

$$\frac{g(2) - g(-2)}{2 - (-2)} = \frac{(8 - 6 + 1) - (-8 + 6 + 1)}{4} = \frac{3 - (-1)}{4} = \frac{4}{4} = 1$$

Set $3c^2 - 3 = 1 \implies 3c^2 = 4 \implies c^2 = \dfrac{4}{3} \implies c = \pm\dfrac{2}{\sqrt{3}}$
Both values lie in $(-2, 2)$, so $c = \dfrac{2}{\sqrt{3}}$ and $c = -\dfrac{2}{\sqrt{3}}$

(c) $h(x) = \sqrt{x}$, $h'(x) = \dfrac{1}{2\sqrt{x}}$

$$\frac{h(4) - h(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}$$

Set $\dfrac{1}{2\sqrt{c}} = \dfrac{1}{3} \implies 2\sqrt{c} = 3 \implies \sqrt{c} = \dfrac{3}{2} \implies c = \dfrac{9}{4}$

Thus, $c = \dfrac{9}{4}$

(d) $p(x) = \sin(x)$, $p'(x) = \cos(x)$

$$\frac{p(\pi) - p(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

Set $\cos(c) = 0 \implies c = \dfrac{\pi}{2}$ on $(0, \pi)$

Thus, $c = \dfrac{\pi}{2}$

(e) $q(x) = \ln(x)$, $q'(x) = \dfrac{1}{x}$

$$\frac{q(e) - q(1)}{e - 1} = \frac{1 - 0}{e - 1} = \frac{1}{e - 1}$$

Set $\dfrac{1}{c} = \dfrac{1}{e - 1} \implies c = e - 1$

Thus, $c = e - 1$

(f) $r(x) = e^x$, $r'(x) = e^x$

$$\frac{r(1) - r(0)}{1 - 0} = \frac{e - 1}{1} = e - 1$$

Set $e^c = e - 1 \implies c = \ln(e - 1)$

Thus, $c = \ln(e - 1)$

Exercise 2.5.4

(a) $f(x) = x^2 - 4x + 3$ is continuous on $[1, 5]$ and differentiable on $(1, 5)$ with $f'(x) = 2x - 4$

$$\frac{f(5) - f(1)}{5 - 1} = \frac{(25 - 20 + 3) - (1 - 4 + 3)}{4} = \frac{8 - 0}{4} = 2$$

Set $2c - 4 = 2 \implies 2c = 6 \implies c = 3$

Thus, $c = 3$

(b) $g(x) = \cos(x)$ is continuous on $[-\pi/2, \pi/2]$ and differentiable on $\left(-\dfrac{pi}{2}, \dfrac{\pi}{2}\right)$ with $g'(x) = -\sin(x)$

$$\frac{g\left(\frac{\pi}{2}\right) - g\left(-\frac{\pi}{2}\right)}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)} = \frac{0 - 0}{\pi} = 0$$

Set $-\sin(c) = 0 \implies \sin(c) = 0 \implies c = 0$ on $\left(-\dfrac{pi}{2}, \dfrac{\pi}{2}\right)$

Thus, $c = 0$

(c) $h(x) = \dfrac{1}{x - 1}$ is continuous on $[2, 4]$ and differentiable on $(2, 4)$ with $h'(x) = -\dfrac{1}{(x - 1)^2}$

$$\frac{h(4) - h(2)}{4 - 2} = \frac{\frac{1}{3} - 1}{2} = \frac{-\frac{2}{3}}{2} = -\frac{1}{3}$$

Set $-\dfrac{1}{(c - 1)^2} = -\dfrac{1}{3} \implies (c - 1)^2 = 3 \implies c - 1 = \pm\sqrt{3}$.

Since $c \in (2, 4)$, $c - 1 > 1$, so $c - 1 = \sqrt{3} \implies c = 1 + \sqrt{3}$

Thus, $c = 1 + \sqrt{3}$

(d) $p(x) = \tan^{-1}(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with $p'(x) = \dfrac{1}{1 + x^2}$

$$\frac{p(1) - p(0)}{1 - 0} = \frac{\frac{\pi}{4} - 0}{1} = \frac{\pi}{4}$$

Set $\dfrac{1}{1 + c^2} = \dfrac{\pi}{4} \implies 1 + c^2 = \dfrac{4}{\pi} \implies c^2 = \dfrac{4}{\pi} - 1$

Since $c \in (0, 1)$, $c = \sqrt{\dfrac{4}{\pi} - 1}$

Thus, $c = \sqrt{\dfrac{4}{\pi} - 1}$

(e) \(q(x) = x^{\frac{3}{2}}\) is continuous on $[0, 4]$ and differentiable on $(0, 4)$ with $q'(x) = \dfrac{3}{2}x^{\frac{1}{2}} = \dfrac{3\sqrt{x}}{2}$

$$\frac{q(4) - q(0)}{4 - 0} = \frac{8 - 0}{4} = 2$$

Set $\dfrac{3\sqrt{c}}{2} = 2 \implies 3\sqrt{c} = 4 \implies \sqrt{c} = \dfrac{4}{3} \implies c = \dfrac{16}{9}$

Thus, $c = \dfrac{16}{9}$

Exercise 2.5.5

(a) Let $f(x) = x^3 + x - 1$

Existence: $f(0) = -1 < 0$ and $f(1) = 1 > 0$. By the Intermediate Value Theorem, there is at least one root in $(0, 1)$.

Uniqueness: $f'(x) = 3x^2 + 1 > 0$ for all x. Thus f is strictly increasing and can have at most one root.

Therefore $x^3 + x - 1 = 0$ has exactly one solution.

(b) Let $f(x) = \cos(x) - 2x$

Existence: $f(0) = 1 > 0$ and $f(1) = \cos(1) - 2 < 0$. By the IVT, there is at least one root in $(0, 1)$.

Uniqueness: $f'(x) = -\sin(x) - 2 < 0$ for all x . Thus f is strictly decreasing and can have at most one root.

Therefore $\cos(x) = 2x$ has exactly one solution.

(c) Let $f(x) = e^x + x - 2$

Existence: $f(0) = 1 - 2 = -1 < 0$ and $f(1) = e + 1 - 2 = e - 1 > 0$. By the IVT, there is at least one root in $(0, 1)$.

Uniqueness: $f'(x) = e^x + 1 > 0$ for all x. Thus f is strictly increasing and can have at most one root.

Therefore $e^x = 2 - x$ has exactly one solution.

(d) Let $f(x) = \ln(x) + x - 2$

Existence: $f(1) = 0 + 1 - 2 = -1 < 0$ and $f(2) = \ln(2) + 2 - 2 = \ln(2) > 0$. By the IVT, there is at least one root in $(1, 2)$.

Uniqueness: $f'(x) = \dfrac{1}{x} + 1 > 0$ for all $x > 0$. Thus f is strictly increasing and can have at most one root.

Therefore $\ln(x) = 2 - x$ has exactly one solution.

Exercise 2.5.6

(a) The average velocity over $[0, 4]$ is:

$$v_{\text{avg}} = \frac{s(4) - s(0)}{4 - 0} = \frac{(64 - 96 + 36 + 2) - 2}{4} = \frac{6 - 2}{4} = 1$$

The instantaneous velocity is $s'(t) = 3t^2 - 12t + 9$.

Set $s'(t) = 1$: $3t^2 - 12t + 9 = 1 \implies 3t^2 - 12t + 8 = 0$.

Using the quadratic formula:

$$t = \frac{12 \pm \sqrt{144 - 96}}{6} = \frac{12 \pm \sqrt{48}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$$

Both values lie in $[0, 4]$, so $t = 2 \pm \dfrac{2\sqrt{3}}{3}$.

(b) The runner's average speed is 10 km in $\dfrac{40}{60} = \dfrac{2}{3}$ hours:

$$v_{\text{avg}} = \frac{10}{2/3} = 15 \text{ km/hr}$$

Let $s(t)$ be the distance traveled as a function of time, with $s(0) = 0$ and $s(2/3) = 10$. By the Mean Value Theorem, there exists $c \in (0, 2/3)$ such that $s'(c) = \dfrac{s(2/3) - s(0)}{2/3 - 0} = 15$.

Since $s'(c)$ is the instantaneous velocity at time c, there must be a moment when the runner's speed is exactly 15 km/hr.

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2.6: Related Rates

Learning Objectives

- Express changing quantities in terms of derivatives.
- Find relationships among the derivatives in a given problem.
- Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

Question of the Day

How do we find the rate at which one quantity changes when another rate is known?

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2.6: Related Rates

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 2.6.1: Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$ (Figure 2.6.1). How fast is the radius increasing when the radius is 3 cm?

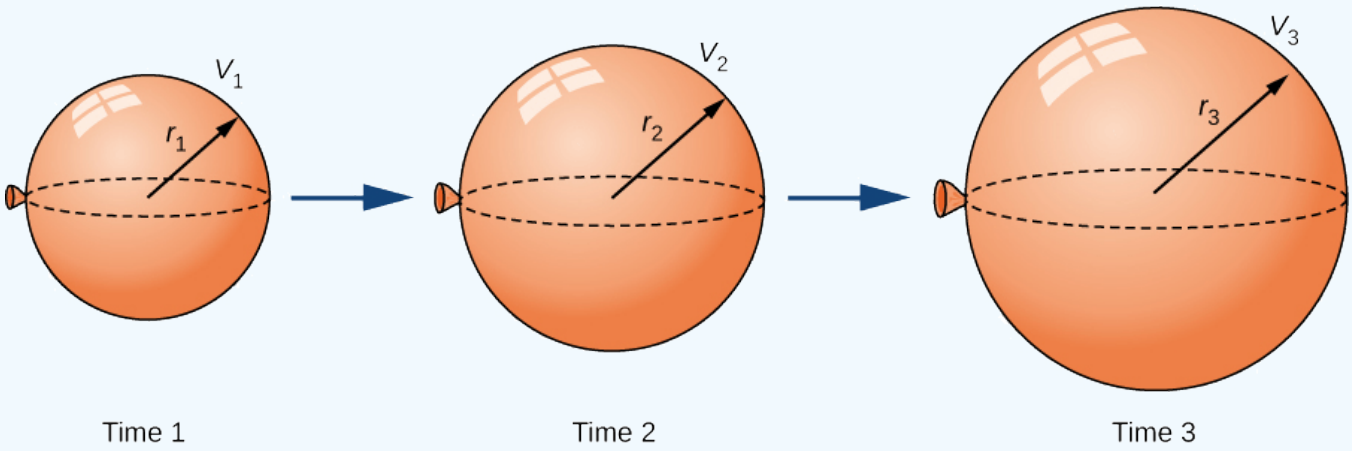


Figure 2.6.1: As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution

The volume of a sphere of radius r centimeters is

$$V = \frac{4}{3}\pi r^3\ \text{cm}^3$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \frac{4}{3}\pi\big[r(t)\big]^3\text{cm}^3$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi\big[r(t)\big]^2 r'(t)$$

The balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$, so $V'(t) = 2\ \text{cm}^3/\text{sec}$. Therefore,

$$2\ \text{cm}^3/\text{sec} = \Big(4\pi\big[r(t)\big]^2\ \text{cm}^2\Big)\cdot\Big(r'(t)\ \text{cm/s}\Big)$$

which implies

$$r'(t) = \frac{1}{2\pi\left[r(t)\right]^2} \text{ cm/sec}$$

When the radius $r = 3$ cm,

$$\boxed{r'(t) = \frac{1}{18\pi} \text{ cm/sec}}$$

Exercise 2.6.1

What is the instantaneous rate of change of the radius when $r = 6$ cm?

Hint available at the end of this section.

Answer available at the end of this section.

Before looking at other examples, let's outline the problem-solving strategy we will be using to solve related-rates problems.

Problem-Solving Strategy: Solving a Related-Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change

Note that when solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4. We examine this potential error in the following example.

Examples of the Process

Let's now implement the strategy just described to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the plane and the rate at which the distance between the plane and a person on the ground is changing.

Example 2.6.2: An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Solution

Step 1. Draw a picture, introducing variables to represent the different quantities involved.

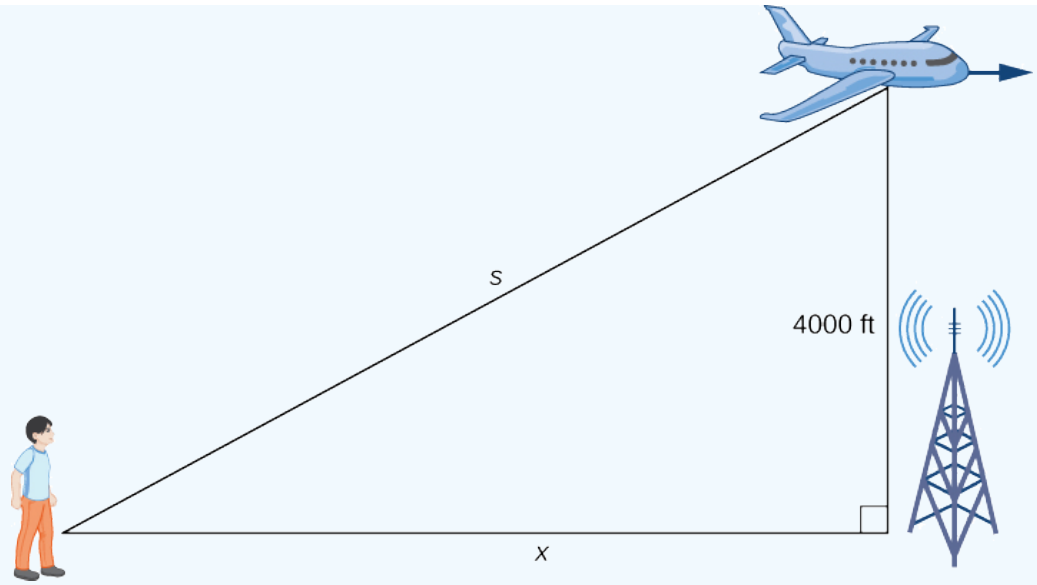


Figure 2.6.2: An airplane is flying at a constant height of 4000 ft. The distance between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables s and x, respectively.

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the height of the plane because it remains at a constant elevation of 4000 ft. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Step 2. Since x denotes the horizontal distance between the man and the point on the ground below the plane, $\frac{dx}{dt}$ represents the speed of the plane. We are told the speed of the plane is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find $\frac{ds}{dt}$ when $x = 3000$ ft.

Step 3. From the figure, we can use the **Pythagorean theorem** to write an equation relating x and s:

$$[x(t)]^2 + 4000^2 = [s(t)]^2$$

Step 4. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

$$2x \cdot \frac{dx}{dt} + 0 = 2s \cdot \frac{ds}{dt}$$

Step 5. Find the rate at which the distance between the man and the plane is increasing when the plane is directly over the radio tower. That is, find $\frac{ds}{dt}$ when $x = 3000$ ft. Since the speed of the plane is 600 ft/sec, we know that $\frac{dx}{dt} = 600$ ft/sec. We are not given an explicit value for s; however, since we are trying to find $\frac{ds}{dt}$ when $x = 3000$ ft, we can use the Pythagorean theorem to determine the distance s when $x = 3000$ ft and the height is 4000 ft. Solving the equation

$$3000^2 + 4000^2 = s^2$$

for s, we have $s = 5000$ ft at the time of interest. Using these values, we conclude that $\frac{ds}{dt}$ is a solution of the equation

$$2 \cdot 3000 \cdot 600 + 0 = 2 \cdot 5000 \cdot \frac{ds}{dt}$$

Therefore,

$$\boxed{\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360\,\text{ft/sec}}$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in step 3, we related the variable quantities $x(t)$ and $s(t)$ by the equation

$$[x(t)]^2 + 4000^2 = [s(t)]^2$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

$$3000^2 + 4000^2 = [s(t)]^2$$

After differentiating, our equation would become

$$0 = s(t)\frac{ds}{dt}$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0$.

Exercise 2.6.2

What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

Hint available at the end of this section.

Answer available at the end of this section.

Example 2.6.3: Chapter Opener - A Rocket Launch

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec.



Figure 2.6.3: (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

Solution

Step 1. Draw a picture introducing the variables.

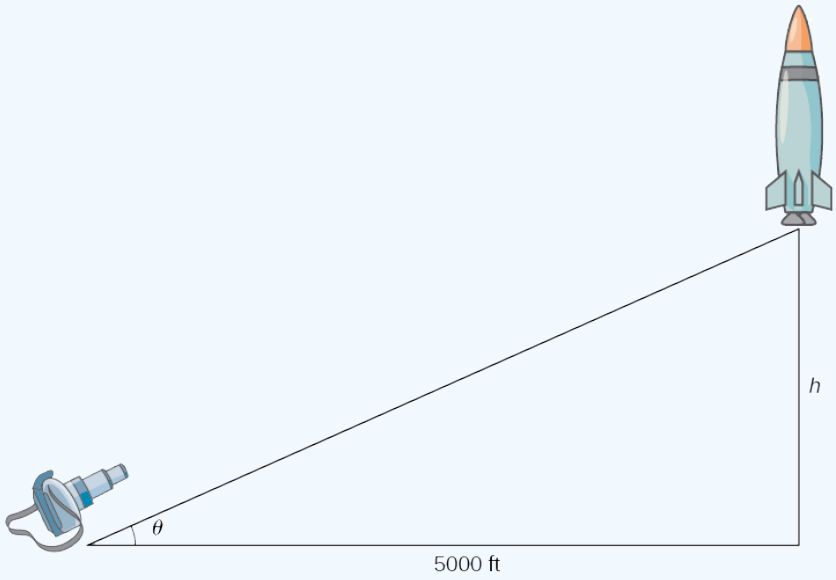


Figure 2.6.4: A camera is positioned 5000 ft from the launch pad of the rocket. The height of the rocket and the angle of the camera are changing with respect to time. We denote those quantities with the variables h and θ,respectively.

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Step 2. We are trying to find the rate of change in the angle of the camera with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600$ ft/sec.

Step 3. Now we need to find an equation relating the two quantities that are changing with respect to time: h and θ. How can we create such an equation? Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan\theta$ is the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

$$\tan(\theta) = \frac{h}{5000}$$

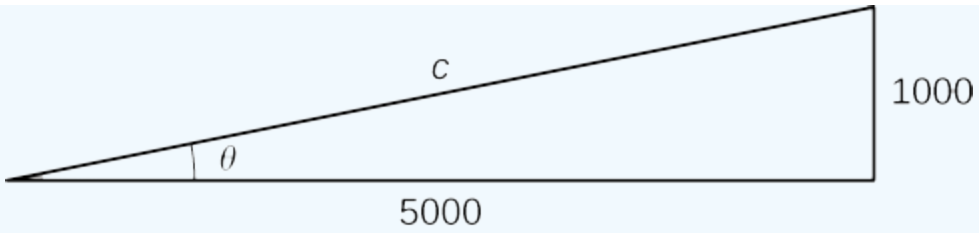
This gives us the equation

$$h = 5000\tan(\theta)$$

Step 4. Differentiating this equation with respect to time t, we obtain

$$\frac{dh}{dt} = 5000\sec^2(\theta)\cdot\frac{d\theta}{dt}$$

Step 5. We want to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At this time, we know that $\frac{dh}{dt} = 600$ ft/sec. We need to determine $\sec^2(\theta)$. Recall that $\sec(\theta)$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is 5000 ft. To determine the length of the hypotenuse, we use the Pythagorean theorem, where the length of one leg is 5000 ft, the length of the other leg is $h = 1000$ ft, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

$$c = 1000\sqrt{26} \text{ ft}$$

Therefore, when $h = 1000$, we have

$$\sec^2(\theta) = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}$$

Recall from step 4 that the equation relating $\frac{d\theta}{dt}$ to our known values is

$$\frac{dh}{dt} = 5000\sec^2(\theta) \cdot \frac{d\theta}{dt}$$

When $h = 1000$ ft, we know that $\frac{dh}{dt} = 600$ ft/sec and $\sec^2(\theta) = \frac{26}{25}$. Substituting these values into the previous equation, we arrive at the equation

$$600 = 5000 \cdot \frac{26}{25} \cdot \frac{d\theta}{dt}$$

Therefore, we solve for $\frac{d\theta}{dt}$ to get our final answer $\boxed{\frac{d\theta}{dt} = \frac{3}{26} \text{ rad/sec}}$

Exercise 2.6.3

What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Hint available at the end of this section.

Answer available at the end of this section.

In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example 2.6.4: Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03 \text{ ft}^3/\text{sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}$ ft?

Solution

Step 1: Draw a picture introducing the variables.

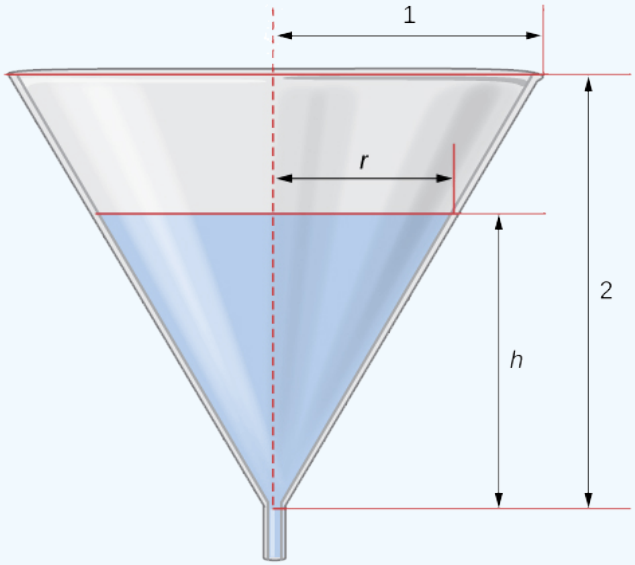


Figure 2.6.5: Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03$ ft/sec.

Step 3: The volume of water in the cone is

$$V = \frac{1}{3}\pi r^2 h$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is *leaving* at the rate of $0.03\ \text{ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03\ \text{ft}^3/\text{sec}$. Therefore,

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2 \frac{dh}{dt}$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}$$

It follows that

$$\boxed{\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\,\text{ft/sec}}$$

Exercise 2.6.4

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 2.6.1

$\frac{dr}{dt} = \frac{1}{2\pi r^2}$

Exercise 2.6.1

$\frac{1}{72\pi}$ cm/sec, or approximately 0.0044 cm/sec

Exercise 2.6.2

$\frac{ds}{dt} = 300$ ft/sec

Exercise 2.6.2

500 ft/sec

Exercise 2.6.3

Find $\frac{d\theta}{dt}$ when $h = 2000$ ft. At that time, $\frac{dh}{dt} = 500$ ft/sec.

Exercise 2.6.3

$\frac{1}{10}$ rad/sec

Exercise 2.6.4

We need to find $\frac{dh}{dt}$ when $h = \frac{1}{4}$.

Exercise 2.6.4

-0.61 ft/sec

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2.6: Practice Set

Exercise 2.6.1

A spherical balloon is being inflated at a rate of $8\ \mathrm{cm}^3/\mathrm{s}$. How fast is the radius increasing when the radius is $4\ \mathrm{cm}$?

Answer available at the end of this section.

Exercise 2.6.2

A ladder 10 feet long is leaning against a wall. The bottom of the ladder is sliding away from the wall at a rate of $2\ \mathrm{ft}/\mathrm{s}$. How fast is the top of the ladder sliding down the wall when the bottom is 6 feet from the wall?

Answer available at the end of this section.

Exercise 2.6.3

A conical tank (with vertex down) has a height of 12 feet and a radius of 4 feet. Water is flowing into the tank at a rate of $5\ \mathrm{ft}^3/\mathrm{min}$. How fast is the water level rising when the water is 6 feet deep?

Answer available at the end of this section.

Exercise 2.6.4

A car is traveling north at $60\ \mathrm{mph}$ and a truck is traveling east at $40\ \mathrm{mph}$. The car is 3 miles south of an intersection and the truck is 4 miles west of the same intersection. How fast is the distance between them changing?

Answer available at the end of this section.

Exercise 2.6.5

A cylindrical tank with radius 3 feet is being filled with water at a rate of $10\ \mathrm{ft}^3/\mathrm{min}$. How fast is the water level rising?

Answer available at the end of this section.

Exercise 2.6.6

A particle is moving along the curve $y = x^2 + 1$. At the point $(2, 5)$, the x-coordinate is changing at $3\ \mathrm{units}/\mathrm{s}$. How fast is the y-coordinate changing?

Answer available at the end of this section.

Exercise 2.6.7

A square is expanding. The side length is increasing at $2\ \mathrm{cm}/\mathrm{s}$. How fast is the area increasing when the side length is $5\ \mathrm{cm}$?

Answer available at the end of this section.

Exercise 2.6.8

A circle's radius is increasing at $3\ \mathrm{cm/s}$. How fast is the circumference increasing when the radius is $6\ \mathrm{cm}$?

Answer available at the end of this section.

Exercise 2.6.9

A rectangle's length is increasing at $4\ \mathrm{cm/s}$ and its width is decreasing at $2\ \mathrm{cm/s}$. At the instant when the length is $10\ \mathrm{cm}$ and the width is $3\ \mathrm{cm}$, how fast is the area changing?

Answer available at the end of this section.

Exercise 2.6.10

A kite is flying 100 feet above the ground. The string is being let out at a rate of $3\ \mathrm{ft/s}$. How fast is the horizontal distance from the person to the kite changing when the string is 125 feet long?

Answer available at the end of this section.

Exercise 2.6.11

A 6-foot tall man is walking away from a 20-foot tall streetlight at a rate of $4\ \mathrm{ft/s}$. How fast is the length of his shadow increasing?

Answer available at the end of this section.

Exercise 2.6.12

An oil spill forms a circular slick. The radius is increasing at $2\ \mathrm{ft/min}$. How fast is the area of the slick increasing when the radius is $10\ \mathrm{ft}$?

Answer available at the end of this section.

Exercise 2.6.13

An inverted cone (vertex down) is being drained at a rate of $2\ \mathrm{ft^3/min}$. The cone has a radius of 3 feet and height of 6 feet. How fast is the water level dropping when the water is 4 feet deep?

Answer available at the end of this section.

Exercise 2.6.14

Two cars leave an intersection at the same time. One travels north at $50\ \mathrm{mph}$ and the other travels east at $30\ \mathrm{mph}$. How fast is the distance between them increasing after 2 hours?

Answer available at the end of this section.

Exercise 2.6.15

A water trough has a triangular cross-section with height 4 feet and width 6 feet, and the length of the trough is a constant 10 feet. Water is flowing into the trough at $8\ \text{ft}^3/\text{min}$. How fast is the water level rising when the water is 2 feet deep?

Answer available at the end of this section.

Answers

Exercise 2.6.1

Volume of a sphere: $V = \frac{4}{3}\pi r^3$. Differentiate with respect to t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Given $\frac{dV}{dt} = 8$ and $r = 4$:

$$8 = 4\pi(4)^2 \frac{dr}{dt} = 64\pi \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{8}{64\pi} = \frac{1}{8\pi}\ \text{cm/s}$$

Thus, $\frac{dr}{dt} = \frac{1}{8\pi}\ \text{cm/s}$

Exercise 2.6.2

Let x be the distance from the wall to the bottom of the ladder and y be the height of the top of the ladder. By the Pythagorean theorem:

$$x^2 + y^2 = 100$$

Differentiate with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Given $\frac{dx}{dt} = 2$ and $x = 6$:

$$y = \sqrt{100 - 36} = \sqrt{64} = 8$$

$$6(2) + 8\frac{dy}{dt} = 0 \implies 12 + 8\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{12}{8} = -\frac{3}{2}\ \text{ft/s}$$

The top of the ladder is sliding down at $\frac{3}{2}\ \text{ft/s}$

Exercise 2.6.3

Volume of a cone: $V = \frac{1}{3}\pi r^2 h$. By similar triangles, $\frac{r}{h} = \frac{4}{12} = \frac{1}{3}$, so $r = \frac{h}{3}$.

$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{3}\pi \cdot \frac{h^2}{9} \cdot h = \frac{\pi}{27}h^3$$

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

Given $\frac{dV}{dt} = 5$ and $h = 6$:

$$5 = \frac{\pi}{9}(36)\frac{dh}{dt} = 4\pi\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{5}{4\pi}\ \text{ft/min}$$

Thus $\frac{dh}{dt} = \frac{5}{4\pi}\ \text{ft/min}$

Exercise 2.6.4

Let x be the truck's distance west of the intersection and y be the car's distance north of the intersection. The distance between them is $z = \sqrt{x^2 + y^2}$.

Given $\frac{dx}{dt} = -40$ (moving toward the intersection) and $\frac{dy}{dt} = 60$ (moving away).

Differentiate $z^2 = x^2 + y^2$:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \implies z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

At the given instant, $x = 4$, $y = 3$, so $z = \sqrt{16 + 9} = 5$.

$$5\frac{dz}{dt} = 4(-40) + 3(60) = -160 + 180 = 20 \implies \frac{dz}{dt} = 4\ \text{mph}$$

The distance between them is increasing at $4\ \text{mph}$

Exercise 2.6.5

Volume of a cylinder: $V = \pi r^2 h$. Since r is constant ($r = 3$):

$$V = 9\pi h$$

Differentiate with respect to t:

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

Given $\frac{dV}{dt} = 10$:

$$10 = 9\pi \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{10}{9\pi} \text{ ft/min}$$

Thus, $\frac{dh}{dt} = \frac{10}{9\pi}$ ft/min

Exercise 2.6.6

Given $y = x^2 + 1$. Differentiate with respect to t:

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

At $x = 2$ and $\frac{dx}{dt} = 3$:

$$\frac{dy}{dt} = 2(2)(3) = 12 \text{ units/s}$$

Thus, $\frac{dy}{dt} = 12$ units/s

Exercise 2.6.7

Area of a square: $A = s^2$. Differentiate with respect to t:

$$\frac{dA}{dt} = 2s \frac{ds}{dt}$$

Given $s = 5$ and $\frac{ds}{dt} = 2$:

$$\frac{dA}{dt} = 2(5)(2) = 20 \text{ cm}^2/\text{s}$$

Thus, $\frac{dA}{dt} = 20$ cm^2/s

Exercise 2.6.8

Circumference of a circle: $C = 2\pi r$. Differentiate with respect to t:

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

Given $\frac{dr}{dt} = 3$:

$$\frac{dC}{dt} = 2\pi(3) = 6\pi \text{ cm/s}$$

Thus, $\frac{dC}{dt} = 6\pi$ cm/s

Exercise 2.6.9

Area of a rectangle: $A = lw$. Differentiate with respect to t:

$$\frac{dA}{dt} = \frac{dl}{dt} w + l \frac{dw}{dt}$$

Given $\frac{dl}{dt} = 4$, $\frac{dw}{dt} = -2$, $l = 10$, and $w = 3$:

$$\frac{dA}{dt} = 4(3) + 10(-2) = 12 - 20 = -8 \text{ cm}^2/\text{s}$$

Thus, $\frac{dA}{dt} = -8$ cm^2/s

Exercise 2.6.10

Let x be the horizontal distance, $y = 100$ be the height (constant), and s be the length of the string.

$$s^2 = x^2 + y^2 = x^2 + 10000$$

Differentiate with respect to t:

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} \implies s \frac{ds}{dt} = x \frac{dx}{dt}$$

Given $s = 125$ and $\frac{ds}{dt} = 3$. Then $x = \sqrt{125^2 - 10000} = \sqrt{15625 - 10000} = \sqrt{5625} = 75$.

$$125(3) = 75 \frac{dx}{dt} \implies \frac{dx}{dt} = \frac{375}{75} = 5 \text{ ft/s}$$

Thus, $\frac{dx}{dt} = 5$ ft/s

Exercise 2.6.11

Let x be the distance from the man to the streetlight and s be the length of his shadow. By similar triangles:

$$\frac{20}{x+s} = \frac{6}{s} \implies 20s = 6x + 6s \implies 14s = 6x \implies s = \frac{3}{7}x$$

Differentiate with respect to t:

$$\frac{ds}{dt} = \frac{3}{7} \frac{dx}{dt}$$

Given $\frac{dx}{dt} = 4$:

$$\frac{ds}{dt} = \frac{3}{7}(4) = \frac{12}{7} \text{ ft/s}$$

Thus, $\frac{ds}{dt} = \frac{12}{7}$ ft/s

Exercise 2.6.12

Area of a circle: $A = \pi r^2$. Differentiate with respect to t:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Given $r = 10$ and $\frac{dr}{dt} = 2$:

$$\frac{dA}{dt} = 2\pi(10)(2) = 40\pi \text{ ft}^2/\text{min}$$

Thus, $\frac{dA}{dt} = 40\pi$ ft^2/min

Exercise 2.6.13

By similar triangles, $\frac{r}{h} = \frac{3}{6} = \frac{1}{2}$, so $r = \frac{h}{2}$.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$

Given $\frac{dV}{dt} = -2$ (draining) and $h = 4$:

$$-2 = \frac{\pi}{4}(16)\frac{dh}{dt} = 4\pi \frac{dh}{dt} \implies \frac{dh}{dt} = -\frac{1}{2\pi} \text{ ft/min}$$

The water level is dropping at $\frac{1}{2\pi}$ ft/min

Exercise 2.6.14

Let x be the eastward distance and y be the northward distance. The distance between them is $z = \sqrt{x^2 + y^2}$.

Given $\frac{dx}{dt} = 30$ and $\frac{dy}{dt} = 50$.

Differentiate $z^2 = x^2 + y^2$:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \implies z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

After 2 hours: $x = 30(2) = 60$, $y = 50(2) = 100$, so $z = \sqrt{3600 + 10000} = \sqrt{13600} = 20\sqrt{34}$.

$$20\sqrt{34}\frac{dz}{dt} = 60(30) + 100(50) = 1800 + 5000 = 6800$$

$$\frac{dz}{dt} = \frac{6800}{20\sqrt{34}} = \frac{340}{\sqrt{34}} \approx 58.31 \text{ mph}$$

Thus, $\frac{dz}{dt} = \frac{340}{\sqrt{34}}$ mph

Exercise 2.6.15

By similar triangles, the width of the water at depth h is $w = \frac{6}{4}h = \frac{3}{2}h$.

The cross-sectional area of the water is $A = \frac{1}{2}wh = \frac{1}{2}\left(\frac{3}{2}h\right)h = \frac{3}{4}h^2$.

Since the trough has length 10, the volume is $V = A \cdot 10 = \frac{15}{2}h^2$.

Differentiate with respect to t:

$$\frac{dV}{dt} = 15h\frac{dh}{dt}$$

Given $\frac{dV}{dt} = 8$ and $h = 2$:

$$8 = 15 \cdot 2 \cdot \frac{dh}{dt} = 30\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{4}{15} \text{ ft/min}$$

Thus, $\frac{dh}{dt} = \frac{4}{15}$ ft/min

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2.7: L'Hopital's Rule

Learning Objectives

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule,** uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Question of the Day

How can we evaluate limits that give indeterminate forms like 0/0 or infinity/infinity?

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- 4.3: Indeterminate Forms and l'Hospital's Rule by Leonard Lamp, Kenn Huber, and Mohamad Youssef is licensed CC BY-NC-SA 4.0.

2.7: Type 1 Indeterminant Forms

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}$$

However, what happens if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin(x)}{x}$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4$$

For $\lim_{x\to 0}\frac{\sin(x)}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more importantly, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and such that $g'(a) \neq 0$ For x near a,we can write

$$f(x) \approx f(a) + f'(a)(x-a)$$

and

$$g(x) \approx g(a) + g'(a)(x-a)$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}$$

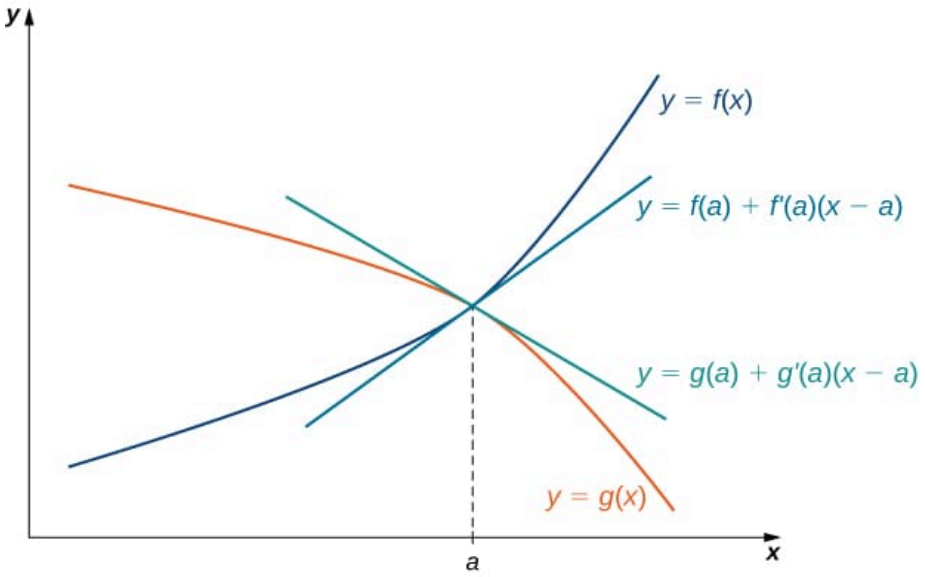


Figure 2.7.1: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x\to a} f(x) = 0$. Similarly, $g(a) = \lim_{x\to a} g(x) = 0$. If we also assume that $f\,'$ and $g\,'$ are continuous at $x = a$, then $f\,'(a) = \lim_{x\to a} f\,'(x)$ and $g\,'(a) = \lim_{x\to a} g\,'(x)$. Using these ideas, we conclude that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Note that the assumption that $f\,'$ and $g\,'$ are continuous at a and $g\,'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

Theorem 2.7.1: L'Hôpital's Rule ($\frac{0}{0}$ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof

We provide a proof of this theorem in the special case when $f, g, f\,'$, and $g\,'$ are all continuous over an open interval containing a. In that case, since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{Since } f(a)=0=g(a)\\
&= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} && \text{Multiply numerator and denominator by } \frac{1}{x-a}\\
&= \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} && \text{The limit of a quotient is the quotient of the limits.}\\
&= \frac{f'(a)}{g'(a)} && \text{By the definition of the derivative}\\
&= \frac{\lim\limits_{x\to a}f'(x)}{\lim\limits_{x\to a}g'(x)} && \text{By the continuity of } f' \text{ and } g'\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)} && \text{The limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 2.7.1: Applying L'Hôpital's Rule ($\frac{0}{0}$ Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

(a) $\lim\limits_{x\to 0}\dfrac{1-\cos(x)}{x}$

(b) $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{\ln(x)}$

(c) $\lim\limits_{x\to\infty}\dfrac{e^{\frac{1}{x}}-1}{\frac{1}{x}}$

(d) $\lim\limits_{x\to 0}\dfrac{\sin(x)-x}{x^2}$

Solution

(a) Since the numerator $1-\cos(x)\to 0$ and the denominator $x\to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

$$\lim_{x\to 0}\frac{1-\cos(x)}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}\big(1-\cos(x)\big)}{\frac{d}{dx}\big(x\big)}=\lim_{x\to 0}\frac{\sin(x)}{1}=\frac{\lim\limits_{x\to 0}\sin(x)}{\lim\limits_{x\to 0}1}=\frac{0}{1}=\boxed{0}$$

(b) As $x\to 1$, the numerator $\sin(\pi x)\to 0$ and the denominator $\ln(x)\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 1}\frac{\sin(\pi x)}{\ln(x)} = \lim_{x\to 1}\frac{\pi\cos(\pi x)}{\frac{1}{x}}$$
$$= \lim_{x\to 1}\pi x\cos(\pi x)$$
$$= (\pi\cdot 1)(-1) = \boxed{-\pi}$$

(c) As $x\to\infty$, the numerator $e^{\frac{1}{x}}-1\to 0$ and the denominator $\frac{1}{x}\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}} = \lim_{x\to\infty}\frac{e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty}e^{\frac{1}{x}} = e^0 = \boxed{1}$$

(d) As $x\to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 0}\frac{\sin(x)-x}{x^2} = \lim_{x\to 0}\frac{\cos(x)-1}{2x}$$

Since the numerator and denominator of this new quotient both approach zero as $x\to 0$, we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x\to 0}\frac{\cos(x)-1}{2x} = \lim_{x\to 0}\frac{-\sin(x)}{2} = 0$$

Therefore, we conclude that

$$\lim_{x\to 0}\frac{\sin x - x}{x^2} = \boxed{0}$$

Exercise 2.7.1

Evaluate $\lim_{x\to 0}\frac{x}{\tan(x)}$

Hint available at the end of this section.

Answer available at the end of this section.

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$. Limits of this form are classified as indeterminate forms of type $\frac{\infty}{\infty}$. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, $\frac{\infty}{\infty}$ is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

Theorem 2.7.2: L'Hôpital's Rule ($\frac{\infty}{\infty}$ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a}f(x)=\infty$ (or $-\infty$) and $\lim_{x\to a}g(x)=\infty$ (or $-\infty$). Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a=\infty$ or $-\infty$, or the limit is one-sided.

Example 2.7.2: Applying L'Hôpital's Rule ($\frac{\infty}{\infty}$) Case

Evaluate each of the following limits by applying L'Hôpital's rule.

(a) $\lim\limits_{x\to\infty}\dfrac{3x+5}{2x+1}$

(b) $\lim\limits_{x\to 0^+}\dfrac{\ln(x)}{\cot(x)}$

Solution

(a) Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim\limits_{x\to\infty}(3x+5)=\infty$ and $\lim\limits_{x\to\infty}(2x+1)=\infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\boxed{\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3}{2}=\frac{3}{2}}$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+\frac{5}{x}}{2+\frac{1}{x}}=\frac{3}{2}$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

(b) Here, $\lim\limits_{x\to 0^+}\ln(x)=-\infty$ and $\lim\limits_{x\to 0^+}\cot(x)=\infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+}\frac{\ln(x)}{\cot(x)}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc^2(x)}=\lim_{x\to 0^+}\frac{1}{-x\csc^2(x)}$$

Now as $x\to 0^+,\csc^2(x)\to\infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc(x)$ to write

$$\lim_{x\to 0^+}\frac{1}{-x\csc^2(x)}=\lim_{x\to 0^+}\frac{\sin^2(x)}{-x}$$

Now $\lim\limits_{x\to 0^+}\sin^2(x)=0$ and $\lim\limits_{x\to 0^+}-x=0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x\to 0^+}\frac{\sin^2(x)}{-x}=\lim_{x\to 0^+}\frac{2\sin(x)\cos(x)}{-1}=\frac{0}{-1}=0$$

We conclude that

$$\boxed{\lim_{x\to 0^+}\frac{\ln(x)}{\cot(x)}=0}$$

Exercise 2.7.2

Evaluate $\lim\limits_{x\to\infty}\dfrac{\ln(x)}{5x}$

Answer available at the end of this section.

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Consider the following example.

Illustration 2.7.1: When L'Hôpital's Rule Does Not Apply

Consider $\lim\limits_{x\to 1}\dfrac{x^2+5}{3x+4}$

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5)=2x$$

and

$$\frac{d}{dx}(3x+4)=3$$

At which point we would conclude erroneously that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\lim_{x\to 1}\frac{2x}{3}=\frac{2}{3}$$

However, since $\lim\limits_{x\to 1}(x^2+5)=6$ and $\lim\limits_{x\to 1}(3x+4)=7$, we actually have

$$\boxed{\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}}$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\frac{d}{dx}(x^2+5)}{\frac{d}{dx}(3x+4)}$$

Exercise 2.7.3

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim\limits_{x\to 0^+}\dfrac{\cos(x)}{x}$. Evaluate $\lim\limits_{x\to 0^+}\dfrac{\cos(x)}{x}$ by other means.

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 2.7.1

$\frac{d}{dx}\big(\tan(x)\big)=\sec^2(x)$

Exercise 2.7.1

1

Exercise 2.7.2

0

Exercise 2.7.3

Determine the limits of the numerator and denominator separately.

Exercise 2.7.3

$\lim_{x \to 0^+} \cos(x) = 1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞.

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2.7: Type 2 and 3 Indeterminant Forms

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0 \cdot \infty, \infty - \infty, 1^{\infty}, \infty^0$, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim_{x \to a} f(x) \cdot g(x)$, where $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form that arises in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x \to \infty$.

For example, let n be a positive integer and consider $f(x) = \dfrac{1}{x^n + 1}$ and $g(x) = 3x^2$:

As $x \to \infty, f(x) \to 0$ and $g(x) \to \infty$. However, the limit as $x \to \infty$ of $f(x)g(x) = \dfrac{3x^2}{(x^n + 1)}$ varies, depending on n.

If $n = 1$, then $\lim_{x \to \infty} f(x)g(x) = \infty$
If $n = 2$, then $\lim_{x \to \infty} f(x)g(x) = 3$
If $n = 3$, then $\lim_{x \to \infty} f(x)g(x) = 0$

Here we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 2.7.3: Indeterminate Form of Type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln(x)$

Solution

First, rewrite the function $x \ln(x)$ as a quotient to apply L'Hôpital's rule. If we write

$$x \ln(x) = \frac{\ln(x)}{\frac{1}{x}}$$

we see that $\ln(x) \to -\infty$ as $x \to 0^+$ and $\dfrac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{d}{dx}\left(\ln(x)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} (-x) = 0$$

We conclude that

$$\lim_{x\to 0^+} x\ln(x) = 0$$

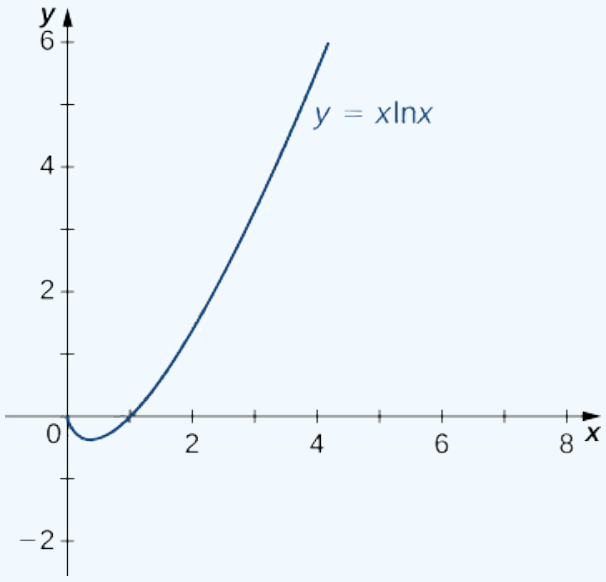


Figure 2.7.2: Finding the limit at $x = 0$ of the function $f(x) = x\ln(x)$

Exercise 2.7.4

Evaluate $\lim_{x\to 0} x\cot(x)$

Hint available at the end of this section.

Answer available at the end of this section.

Indeterminate Form of Type ∞-∞

Another type of indeterminate form is $\infty - \infty$. Consider the following example.

Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty, f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim_{x\to\infty}(f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit.

For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x\to\infty}(f(x) - g(x)) = \lim_{x\to\infty}(3x^3 - 3x^2 - 5) = \infty$$

On the other hand, if $n = 2$, then

$$\lim_{x\to\infty}(f(x) - g(x)) = \lim_{x\to\infty}(3x^2 - 3x^2 - 5) = -5$$

However, if $n = 1$, then

$$\lim_{x\to\infty}(f(x) - g(x)) = \lim_{x\to\infty}(3x - 3x^2 - 5) = -\infty$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 2.7.4: Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan(x)} \right)$

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan(x)} = \frac{\tan(x) - x^2}{x^2 \tan(x)}$$

As $x \to 0^+$, the numerator $\tan(x) - x^2 \to 0$ and the denominator $x^2 \tan(x) \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{\tan(x) - x^2}{x^2 \tan(x)} = \lim_{x \to 0^+} \frac{\sec^2(x) - 2x}{x^2 \sec^2(x) + 2x \tan(x)}$$

As $x \to 0^+$, $(\sec^2(x)) - 2x \to 1$ and $x^2 \sec^2(x) + 2x \tan(x) \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{\sec^2(x) - 2x}{x^2 \sec^2(x) + 2x \tan(x)} = \infty$$

Therefore,

$$\boxed{\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan(x)} \right) = \infty}$$

Exercise 2.7.5

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right)$

Hint available at the end of this section.

Answer available at the end of this section.

Indeterminate Forms Involving Exponents

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions $0^0, \infty^0$, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}$$

Then,

$$\ln y = \ln \left(f(x)^{g(x)} \right) = g(x) \ln \left(f(x) \right)$$

Therefore,

$$\lim_{x\to a} \ln(y) = \lim_{x\to a} g(x)\ln\left(f(x)\right)$$

Since $\lim_{x\to a} f(x) = \infty$, we know that $\lim_{x\to a} \ln\left(f(x)\right) = \infty$. Therefore, $\lim_{x\to a} g(x)\ln\left(f(x)\right)$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x\to a} g(x)\ln\left(f(x)\right) = L$, where L may be ∞ or $-\infty$. Then

$$\lim_{x\to a} \ln(y) = L$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to a} y\right) = L$$

which gives us

$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^L$$

Example 2.7.5: Indeterminate Form of Type ∞^0

Evaluate $\lim_{x\to\infty} x^{\frac{1}{x}}$

Solution

Let $y = x^{\frac{1}{x}}$. Then,

$$\ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x}\ln(x) = \frac{\ln(x)}{x}$$

We need to evaluate $\lim_{x\to\infty} \dfrac{\ln(x)}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Therefore, $\lim_{x\to\infty} \ln(y) = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty} y\right) = 0$$

which leads to

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = e^{\ln\left(\lim\limits_{x\to\infty} y\right)} = e^0 = 1$$

Hence,

$$\boxed{\lim_{x\to\infty} x^{\frac{1}{x}} = 1}$$

Exercise 2.7.6

Evaluate $\lim_{x\to\infty} x^{\frac{1}{\ln(x)}}$

Hint available at the end of this section.

Answer available at the end of this section.

Example 2.7.6: Indeterminate Form of Type 0^0

Evaluate $\lim\limits_{x\to 0^+} x^{\sin(x)}$

Solution

Let

$$y = x^{\sin(x)}$$

Therefore,

$$\ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\ln(x)$$

We now evaluate $\lim\limits_{x\to 0^+} \sin(x)\ln(x)$. Since $\lim\limits_{x\to 0^+} \sin(x) = 0$ and $\lim\limits_{x\to 0^+} \ln(x) = -\infty$, we have the indeterminate form $0\cdot\infty$. To apply L'Hôpital's rule, we need to rewrite $\sin(x)\ln(x)$ as a fraction. We could write

$$\sin(x)\ln(x) = \frac{\sin(x)}{\frac{1}{\ln(x)}} \text{ or } \sin(x)\ln(x) = \frac{\ln(x)}{\frac{1}{\sin(x)}} = \frac{\ln(x)}{\csc(x)}$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\sin(x)}{\frac{1}{\ln(x)}} = \lim_{x\to 0^+} \frac{\cos(x)}{-\frac{1}{x}(\ln(x))^2} = \lim_{x\to 0^+} \left(-x\left(\ln(x)\right)^2\cos(x)\right)$$

Unfortunately, we not only have another expression involving the indeterminate form $0\cdot\infty$, but the new limit is even more complicated to evaluate than the one with which we started.

Instead, we try the second option. By applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc(x)\cot(x)} = \lim_{x\to 0^+} \frac{-1}{x\csc(x)\cot(x)}$$

Using the fact that $\csc(x) = \dfrac{1}{\sin(x)}$ and $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, we can rewrite the expression on the right-hand side as

$$\lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+} \left[\frac{\sin(x)}{x}\right](-\tan(x)) = \left(\lim_{x\to 0^+} \frac{\sin(x)}{x}\right)\cdot\left(\lim_{x\to 0^+} (-\tan(x))\right) = 1\cdot 0 = 0$$

We conclude that $\lim\limits_{x\to 0^+} \ln(y) = 0$. Therefore, $\ln\left(\lim\limits_{x\to 0^+} y\right) = 0$ and we have

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sin(x)} = e^0 = 1$$

Hence,

$$\boxed{\lim_{x\to 0^+} x^{\sin(x)} = 1}$$

Exercise 2.7.7

Evaluate $\lim\limits_{x\to 0^+} x^x$

Hint available at the end of this section.

Answer available at the end of this section.

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as Table 2.7.1 shows, the values of x^3 are growing much faster than the values of x^2.

Table 2.7.1: Comparing the Growth Rates of x^2 and x^3

x	10	100	1000	10000
$f(x) = x^2$	100	10000	1000000	100000000
$g(x) = x^3$	1000	1000000	1000000000	1000000000000

In fact,

$$\lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$$

or, equivalently

$$\lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in Table 2.7.2. In fact,

$$\lim_{x\to\infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}$$

Table 2.7.2: Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

x	10	100	1000	10000
$f(x) = x^2$	100	10000	1000000	100000000
$g(x) = 3x^2 + 4x + 1$	341	30401	3004001	300040001

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty \quad \text{or equivalently,} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 2.7.7: Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

(a) $f(x)=x^2$ and $g(x)=e^x$

(b) $f(x)=\ln(x)$ and $g(x)=x^2$

Solution

(a) Since $\lim_{x\to\infty}x^2=\infty$ and $\lim_{x\to\infty}e^x=\infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty}\left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}$$

Since $\lim_{x\to\infty}2x=\infty$ and $\lim_{x\to\infty}e^x=\infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$$

we conclude that

$$\lim_{x\to\infty}\frac{x^2}{e^x}=0$$

Therefore, $\boxed{e^x \text{ grows more rapidly than } x^2}$ as $x\to\infty$ (See Figure 2.7.3 and Table 2.7.3)

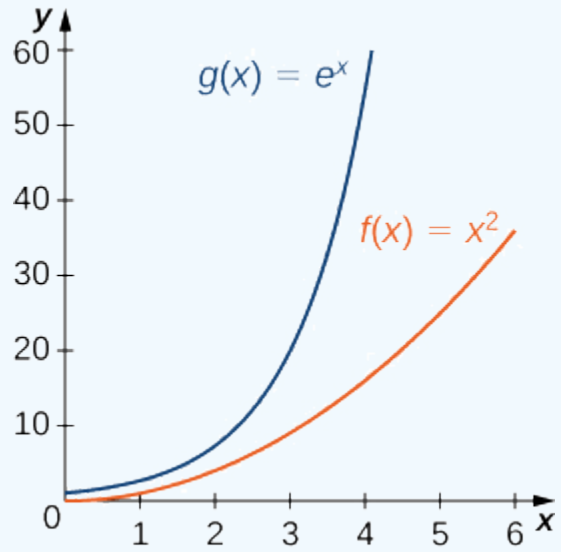


Figure 2.7.3: An exponential function grows at a faster rate than a power function.

Table 2.7.3: Growth rates of a power function and an exponential function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22026	3269017	485165195

(b) Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$. We obtain

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

Thus, $\boxed{x^2 \text{ grows more rapidly than } \ln(x)}$ as $x \to \infty$ (see Figure 2.7.4 and Table 2.7.4).

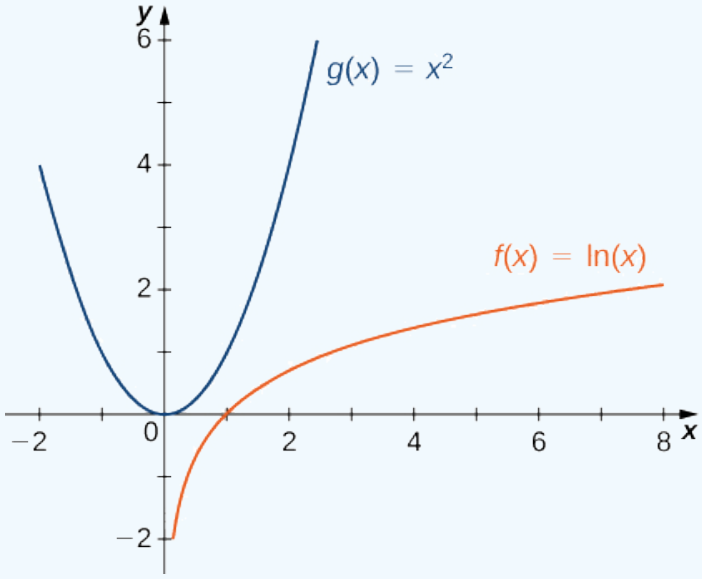


Figure 2.7.4: A power function grows at a faster rate than a logarithmic function.

Table 2.7.4: Growth rates of a power function and a logarithmic function

x	10	100	1000	10000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10000	1000000	100000000

Exercise 2.7.8

Compare the growth rates of x^{100} and 2^x

Hint available at the end of this section.

Answer available at the end of this section.

Using the same ideas as in Example 2.7.8*a*. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure 2.7.5 and Table 2.7.5, we compare e^x with x^3 and x^4 as $x \to \infty$.

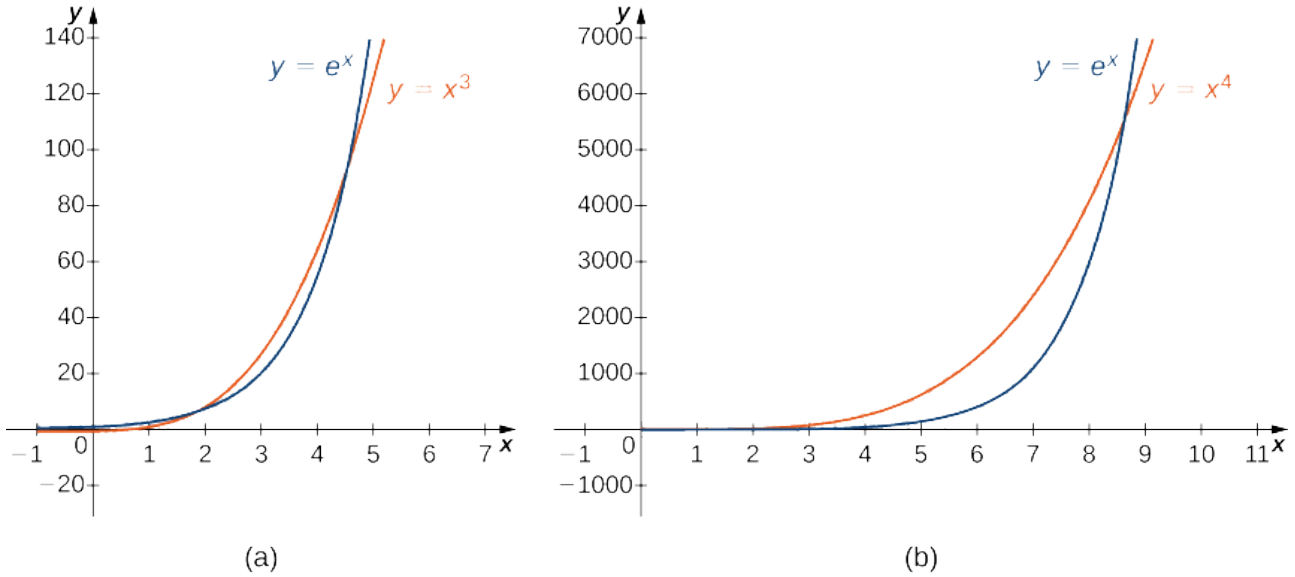


Figure 2.7.5: The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

Table 2.7.5: An exponential function grows at a faster rate than any power function

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10000	50625	160000
e^x	148	22026	3269017	485165195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln(x)$ for any $p > 0$. In Figure 2.7.6 and Table 2.7.6, we compare $\ln(x)$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

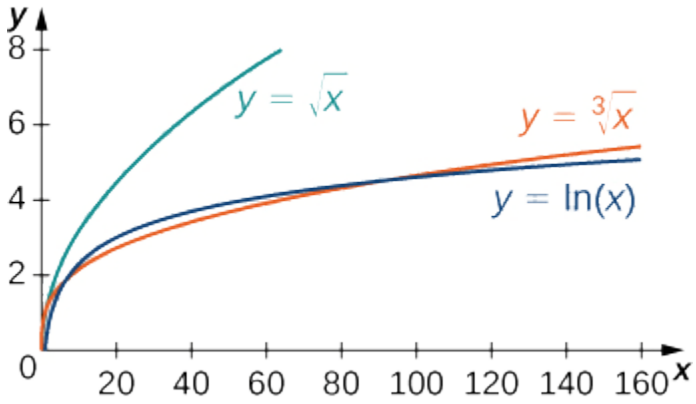


Figure 2.7.6: The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

Table 2.7.6: A logarithmic function grows at a slower rate than any root function

x	10	100	1000	10000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Answers

Exercise 2.7.4

Write $x\cot(x) = \dfrac{x\cos(x)}{\sin(x)}$

Exercise 2.7.4

1

Exercise 2.7.5

Rewrite the difference of fractions as a single fraction.

Exercise 2.7.5

0

Exercise 2.7.6

Let $y = x^{\frac{1}{\ln(x)}}$ and apply the natural logarithm to both sides of the equation.

Exercise 2.7.6

e

Exercise 2.7.7

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Exercise 2.7.7

1

Exercise 2.7.8

Apply L'Hôpital's rule to $x^{100}/2^x$.

Exercise 2.7.8

The function 2^x grows faster than x^{100}

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2.7: Practice Set

Exercise 2.7.1

Evaluate each limit using L'Hôpital's Rule. Some may require multiple applications.

(a) $\lim_{x\to 0} \dfrac{\sin(3x)}{x}$

(b) $\lim_{x\to 0} \dfrac{1-\cos(x)}{x^2}$

(c) $\lim_{x\to \infty} \dfrac{3x^2+2x-1}{5x^2-4x+7}$

(d) $\lim_{x\to 0} \dfrac{e^x-1-x}{x^2}$

(e) $\lim_{x\to 0} \dfrac{\tan(x)-x}{x^3}$

(f) $\lim_{x\to \infty} \dfrac{e^x}{x^3}$

(g) $\lim_{x\to 0} \dfrac{\sin(x)-x}{x^3}$

(h) $\lim_{x\to \infty} \dfrac{\ln(x)}{x^2}$

Answer available at the end of this section.

Exercise 2.7.2

Evaluate each limit of the form $\infty - \infty$.

(a) $\lim_{x\to 0^+} \left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$

(b) $\lim_{x\to 1} \left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)$

(c) $\lim_{x\to 0^+} \left(\dfrac{1}{\sin(x)}-\dfrac{1}{x}\right)$

(d) $\lim_{x\to \infty} \left(\sqrt{x^2+x}-x\right)$

(e) $\lim_{x\to 0^+} (\csc(x)-\cot(x))$

Answer available at the end of this section.

Exercise 2.7.3

Evaluate each limit of the form $0\cdot\infty$.

(a) $\lim_{x\to 0^+} x\ln(x)$

(b) $\lim_{x\to\infty} xe^{-x}$

(c) $\lim_{x\to 0^+} \sin(x)\ln(x)$

(d) $\lim_{x\to\infty} \dfrac{\ln(x)}{\csc(x)}$ (hint: rewrite as $\ln(x)\cdot\sin(x)$)

(e) $\lim_{x\to 0^+} x\csc(x)$

Answer available at the end of this section.

Exercise 2.7.4

Evaluate each limit of the form 1^∞ or ∞^0.

(a) $\lim_{x\to 0} (1+x)^{1/x}$

(b) $\lim_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$

(c) $\lim_{x\to 0^+} x^x$

(d) $\lim_{x\to\infty} x^{1/x}$

(e) $\lim_{x\to 0} (\cos(x))^{1/x^2}$

(f) $\lim_{x\to 0^+} (\sin(x))^x$

Answer available at the end of this section.

Answers

Exercise 2.7.1

(a) This is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\sin(3x)}{x}=\lim_{x\to 0}\frac{3\cos(3x)}{1}=3\cos(0)=3$$

Thus $\lim_{x\to 0}\frac{\sin(3x)}{x}=3$.

(b) This is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\lim_{x\to 0}\frac{\sin(x)}{2x}$$

This is still $\frac{0}{0}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{\sin(x)}{2x}=\lim_{x\to 0}\frac{\cos(x)}{2}=\frac{1}{2}$$

Thus $\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\frac{1}{2}$.

(c) This is of the form $\frac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{3x^2+2x-1}{5x^2-4x+7}=\lim_{x\to\infty}\frac{6x+2}{10x-4}$$

This is still $\frac{\infty}{\infty}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to\infty}\frac{6x+2}{10x-4}=\lim_{x\to\infty}\frac{6}{10}=\frac{3}{5}$$

Thus $\lim_{x\to\infty}\dfrac{3x^2+2x-1}{5x^2-4x+7}=\dfrac{3}{5}$.

(d) This is of the form $\dfrac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\frac{e^x-1}{2x}$$

This is still $\dfrac{0}{0}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}$$

Thus $\lim_{x\to 0}\dfrac{e^x-1-x}{x^2}=\dfrac{1}{2}$.

(e) This is of the form $\dfrac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\tan(x)-x}{x^3}=\lim_{x\to 0}\frac{\sec^2(x)-1}{3x^2}$$

This is still $\dfrac{0}{0}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{\sec^2(x)-1}{3x^2}=\lim_{x\to 0}\frac{2\sec^2(x)\tan(x)}{6x}$$

This is still $\dfrac{0}{0}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{2\sec^2(x)\tan(x)}{6x}=\lim_{x\to 0}\frac{2(\sec^2(x)\sec^2(x)+\tan(x)\cdot 2\sec^2(x)\tan(x))}{6}$$
$$=\lim_{x\to 0}\frac{2(\sec^4(x)+2\sec^2(x)\tan^2(x))}{6}=\frac{2(1+0)}{6}=\frac{1}{3}$$

Thus $\lim_{x\to 0}\dfrac{\tan(x)-x}{x^3}=\dfrac{1}{3}$.

(f) This is of the form $\dfrac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{e^x}{x^3}=\lim_{x\to\infty}\frac{e^x}{3x^2}$$

This is still $\dfrac{\infty}{\infty}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to\infty}\frac{e^x}{3x^2}=\lim_{x\to\infty}\frac{e^x}{6x}$$

This is still $\dfrac{\infty}{\infty}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to\infty}\frac{e^x}{6x}=\lim_{x\to\infty}\frac{e^x}{6}=\infty$$

Thus $\lim_{x\to\infty}\dfrac{e^x}{x^3}=\infty$.

(g) This is of the form $\dfrac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\sin(x)-x}{x^3}=\lim_{x\to 0}\frac{\cos(x)-1}{3x^2}$$

This is still $\dfrac{0}{0}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{\cos(x)-1}{3x^2}=\lim_{x\to 0}\frac{-\sin(x)}{6x}$$

This is still $\dfrac{0}{0}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{-\sin(x)}{6x}=\lim_{x\to 0}\frac{-\cos(x)}{6}=-\frac{1}{6}$$

Thus $\lim_{x\to 0}\dfrac{\sin(x)-x}{x^3}=-\dfrac{1}{6}$.

(h) This is of the form $\dfrac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln(x)}{x^2}=\lim_{x\to\infty}\frac{\frac{1}{x}}{2x}=\lim_{x\to\infty}\frac{1}{2x^2}=0$$

Thus $\lim_{x\to\infty}\dfrac{\ln(x)}{x^2}=0$.

Exercise 2.7.2

(a) Combine into a single fraction:

$$\frac{1}{x}-\frac{1}{x^2}=\frac{x-1}{x^2}$$

This is of the form $\dfrac{0}{0}$ as $x\to 0^+$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0^+}\frac{x-1}{x^2}=\lim_{x\to 0^+}\frac{1}{2x}=-\infty$$

Thus $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=-\infty$.

(b) Combine into a single fraction:

$$\frac{1}{x-1}-\frac{2}{x^2-1}=\frac{1}{x-1}-\frac{2}{(x-1)(x+1)}=\frac{(x+1)-2}{(x-1)(x+1)}=\frac{x-1}{(x-1)(x+1)}=\frac{1}{x+1}$$

For $x\neq 1$, the expression simplifies to $\frac{1}{x+1}$.

$$\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$$

Thus $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)=\frac{1}{2}$.

(c) Combine into a single fraction:

$$\frac{1}{\sin(x)}-\frac{1}{x}=\frac{x-\sin(x)}{x\sin(x)}$$

This is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{x-\sin(x)}{x\sin(x)}=\lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)+x\cos(x)}$$

This is still $\frac{0}{0}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)+x\cos(x)}=\lim_{x\to 0}\frac{\sin(x)}{\cos(x)+\cos(x)-x\sin(x)}=\lim_{x\to 0}\frac{\sin(x)}{2\cos(x)-x\sin(x)}=\frac{0}{2}=0$$

Thus $\lim_{x\to 0^+}\left(\frac{1}{\sin(x)}-\frac{1}{x}\right)=0$.

(d) Rewrite using rationalization:

$$\sqrt{x^2+x}-x=\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}=\frac{x}{\sqrt{x^2+x}+x}$$

This is of the form $\frac{\infty}{\infty}$. Divide numerator and denominator by x:

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{1+1}=\frac{1}{2}$$

Thus $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\frac{1}{2}$.

(e) Rewrite in terms of sine and cosine:

$$\csc(x)-\cot(x)=\frac{1}{\sin(x)}-\frac{\cos(x)}{\sin(x)}=\frac{1-\cos(x)}{\sin(x)}$$

This is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)}=\lim_{x\to 0}\frac{\sin(x)}{\cos(x)}=\frac{0}{1}=0$$

Thus $\lim_{x\to 0^+}(\csc(x)-\cot(x))=0$.

Exercise 2.7.3

(a) Rewrite as $\frac{\ln(x)}{1/x}$, which is of the form $\frac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0^+}x\ln(x)=\lim_{x\to 0^+}\frac{\ln(x)}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0$$

Thus $\lim_{x\to 0^+}x\ln(x)=0$.

(b) Rewrite as $\frac{x}{e^x}$, which is of the form $\frac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0$$

Thus $\lim_{x\to\infty}xe^{-x}=0$.

(c) Rewrite as $\frac{\ln(x)}{1/\sin(x)}=\frac{\ln(x)}{\csc(x)}$, which is of the form $\frac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\lim_{x\to 0^+}\frac{\ln(x)}{\csc(x)}=\lim_{x\to 0^+}\frac{1/x}{-\csc(x)\cot(x)}=\lim_{x\to 0^+}\frac{-\sin(x)\tan(x)}{x}$$

This is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule again:

$$\lim_{x\to 0^+}\frac{-\sin(x)\tan(x)}{x}=\lim_{x\to 0^+}\frac{-(\cos(x)\tan(x)+\sin(x)\sec^2(x))}{1}=0$$

Thus $\lim_{x\to 0^+}\sin(x)\ln(x)=0$.

(d) Rewrite as $\ln(x)\cdot\sin(x)$, which is of the form $0\cdot\infty$ as $x\to\infty$ ($\ln(x)\to\infty$ but $\sin(x)$ oscillates, so this limit does not exist in the usual sense).

Actually, rewrite as $\frac{\ln(x)}{\csc(x)}$, which is $\frac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln(x)}{\csc(x)}=\lim_{x\to\infty}\frac{1/x}{-\csc(x)\cot(x)}=\lim_{x\to\infty}\frac{-\sin(x)\tan(x)}{x}$$

Since $\sin(x)\tan(x)$ is bounded (between -1 and 1 in magnitude when $\tan(x)$ is defined), the limit is 0.

Thus $\lim_{x\to\infty}\frac{\ln(x)}{\csc(x)}=0$.

(e) Rewrite as $\frac{x}{\sin(x)}$, which is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule:

$$\lim_{x\to0^+}\frac{x}{\sin(x)}=\lim_{x\to0^+}\frac{1}{\cos(x)}=1$$

Thus $\lim_{x\to0^+}x\csc(x)=1$.

Exercise 2.7.4

(a) This is of the form 1^∞. Let $y=\lim_{x\to0}(1+x)^{1/x}$.

Take the natural log:

$$\ln(y)=\lim_{x\to0}\frac{\ln(1+x)}{x}$$

This is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule:

$$\ln(y)=\lim_{x\to0}\frac{1/(1+x)}{1}=\lim_{x\to0}\frac{1}{1+x}=1$$

Thus $y=e^1=e$.

So $\lim_{x\to0}(1+x)^{1/x}=e$.

(b) This is of the form 1^∞. Let $y=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Take the natural log:

$$\ln(y)=\lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

This is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule:

$$\ln(y)=\lim_{x\to\infty}\frac{\frac{1}{1+1/x}\cdot(-1/x^2)}{-1/x^2}=\lim_{x\to\infty}\frac{1}{1+1/x}=1$$

Thus $y=e^1=e$.

So $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.

(c) This is of the form 0^0. Let $y=\lim_{x\to0^+}x^x$.

Take the natural log:

$$\ln(y)=\lim_{x\to0^+}x\ln(x)$$

From Exercise 3(a), $\lim_{x\to0^+}x\ln(x)=0$.

Thus $y=e^0=1$.

So $\lim_{x\to0^+}x^x=1$.

(d) This is of the form ∞^0. Let $y=\lim_{x\to\infty}x^{1/x}$.

Take the natural log:

$$\ln(y)=\lim_{x\to\infty}\frac{\ln(x)}{x}$$

This is of the form $\frac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\ln(y)=\lim_{x\to\infty}\frac{1/x}{1}=0$$

Thus $y=e^0=1$.

So $\lim_{x\to\infty}x^{1/x}=1$.

(e) This is of the form 1^∞. Let $y=\lim_{x\to0}(\cos(x))^{1/x^2}$.

Take the natural log:

$$\ln(y)=\lim_{x\to0}\frac{\ln(\cos(x))}{x^2}$$

This is of the form $\frac{0}{0}$. Apply L'Hôpital's Rule:

$$\ln(y)=\lim_{x\to0}\frac{-\tan(x)}{2x}$$

This is still $\frac{0}{0}$. Apply L'Hôpital's Rule again:

$$\ln(y)=\lim_{x\to0}\frac{-\sec^2(x)}{2}=-\frac{1}{2}$$

Thus $y = e^{-1/2} = \dfrac{1}{\sqrt{e}}$.

So $\lim\limits_{x\to 0} (\cos(x))^{1/x^2} = \dfrac{1}{\sqrt{e}}$.

(f) This is of the form 0^0. Let $y = \lim_{x\to 0^+} (\sin(x))^x$.

Take the natural log:

$$\ln(y) = \lim_{x\to 0^+} x\ln(\sin(x)) = \lim_{x\to 0^+} \frac{\ln(\sin(x))}{1/x}$$

This is of the form $\dfrac{\infty}{\infty}$. Apply L'Hôpital's Rule:

$$\ln(y) = \lim_{x\to 0^+} \frac{\cot(x)}{-1/x^2} = \lim_{x\to 0^+} -x^2\cot(x) = \lim_{x\to 0^+} -\frac{x^2\cos(x)}{\sin(x)}$$

This is of the form $\dfrac{0}{0}$. Apply L'Hôpital's Rule:

$$\ln(y) = \lim_{x\to 0^+} -\frac{2x\cos(x) - x^2\sin(x)}{\cos(x)} = -\frac{0}{1} = 0$$

Thus $y = e^0 = 1$.

So $\lim\limits_{x\to 0^+} (\sin(x))^x = 1$.

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3: THE ANATOMY OF A CURVE

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3.1: Maximums and Minimums

Learning Objectives

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance, because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Question of the Day

How do derivatives tell us where a function reaches its highest and lowest values?

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3.1: Local Extrema

Local Extrema and Critical Points

Consider the function f shown in Figure $3.1.3$. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.

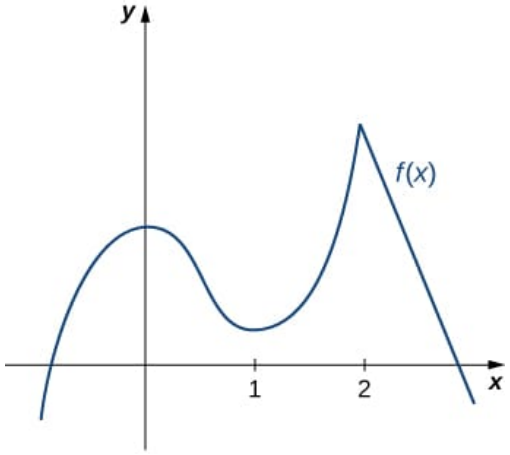


Figure $3.1.3$: This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition: Local Extrema

A function f has a *local maximum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$.

A function f has a *local minimum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$.

A function f has a *local extremum* at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure $3.1.3$ again. The local extrema occur at $x = 0, x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition: Critical Points

Let c be an interior point in the domain of f. We say that c is a *critical point* of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as **Fermat's theorem.**

Theorem 3.1.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c, but $x > c$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$. We conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In Figure 3.1.4, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

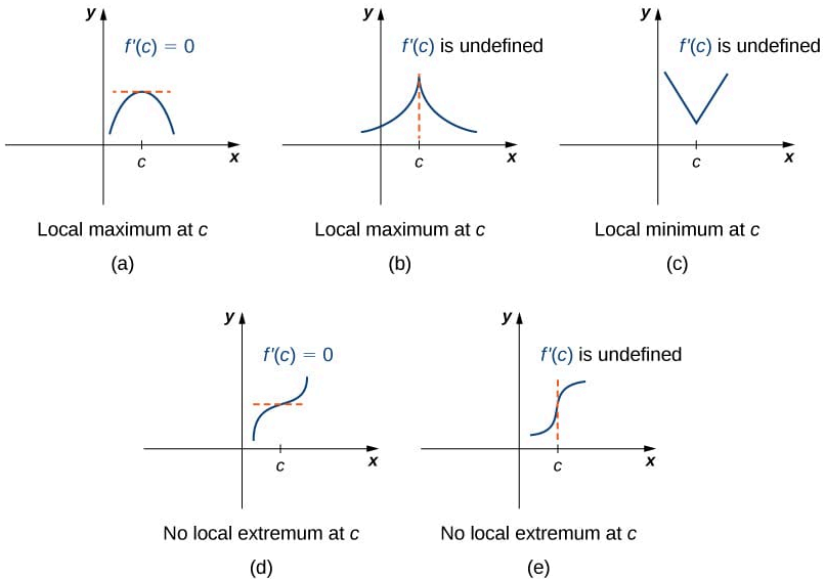


Figure 3.1.4: (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 3.1.1: Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

(a) $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

(b) $f(x) = (x^2 - 1)^3$

(c) $f(x) = \frac{4x}{1 + x^2}$

Solution

(a) The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in Figure 3.1.5, we see that f has:

$$\boxed{\text{a local maximum at } x = 1 \text{ and a local minimum at } x = 4}$$

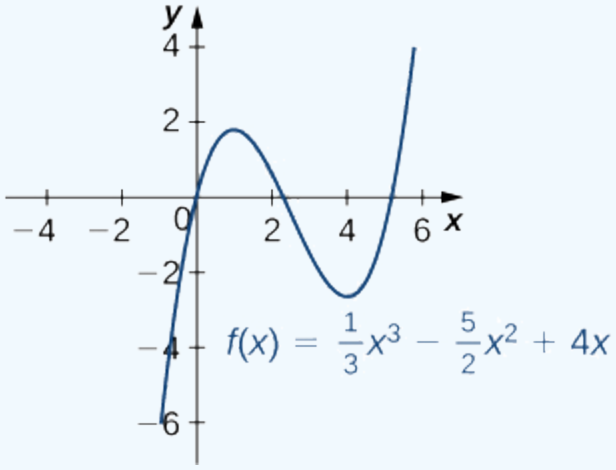


Figure 3.1.5: This function has a local maximum and a local minimum.

(b) Using the chain rule, we see the derivative is

$$f'(x) = 3(x^2-1)^2(2x) = 6x(x^2-1)^2$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in Figure 3.1.6, we see that f has:

$$\boxed{\text{a local (and absolute) minimum at } x = 0, \text{ but does not have a local extremum at } x = 1 \text{ or } x = -1}$$

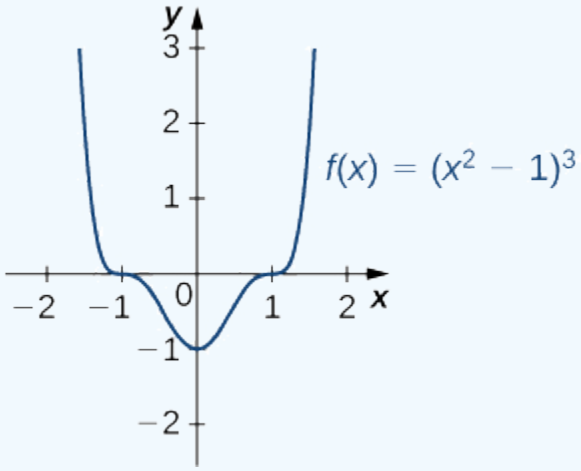


Figure 3.1.6: This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

(c) By the quotient rule, we see that the derivative is

$$f'(x) = \frac{4(1+x^2) - 4x(2x)}{(1+x^2)^2} = \frac{4-4x^2}{(1+x^2)^2}$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in Figure 3.1.7, we see that f has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has:

$$\boxed{\text{a local maximum at } x = 1 \text{ and a local minimum at } x = -1}$$

(Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

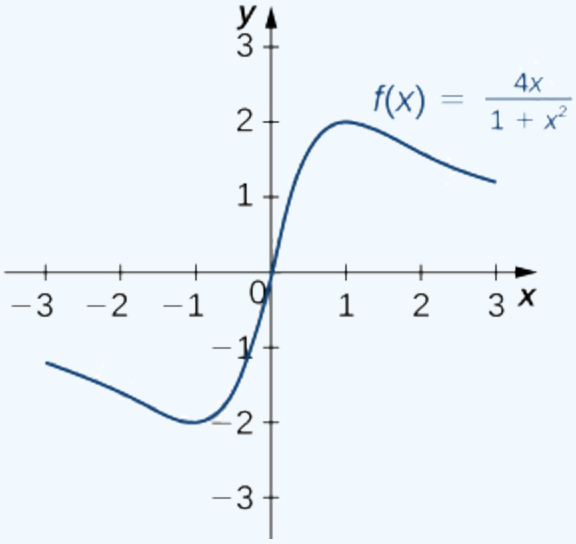


Figure 3.1.7: This function has an absolute maximum and an absolute minimum.

Exercise 3.1.1

Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 3.1.1

Calculate $f'(x)$.

Exercise 3.1.1

$x = \frac{-2}{3}, x = 1$

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3.1: The Extreme Value Theorem

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty, f(x) \to \infty$.Therefore, the function does not have a largest value.

However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$.

We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$.
We say that $f(x) = x^2 + 1$ does not have an absolute maximum (Figure $3.1.1$).

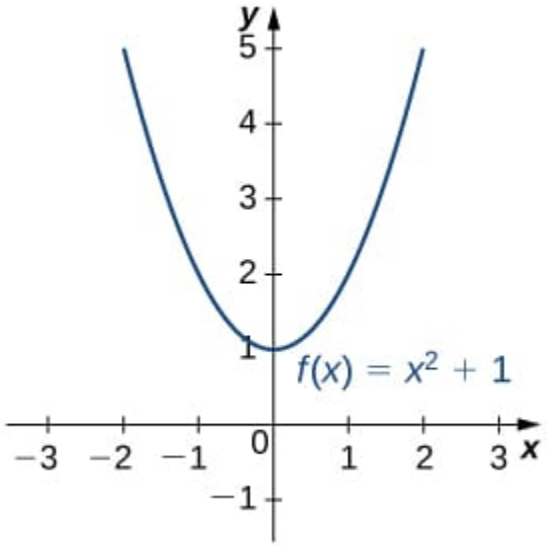


Figure $3.1.1$: The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition: Absolute Extrema

Let f be a function defined over an interval I and let $c \in I$. We say f has an *absolute maximum* on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an *absolute minimum* on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has *an absolute extremum* on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = \dfrac{1}{x^2 + 1}$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$, as shown in Figure $3.1.2$(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure $3.1.2$ shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

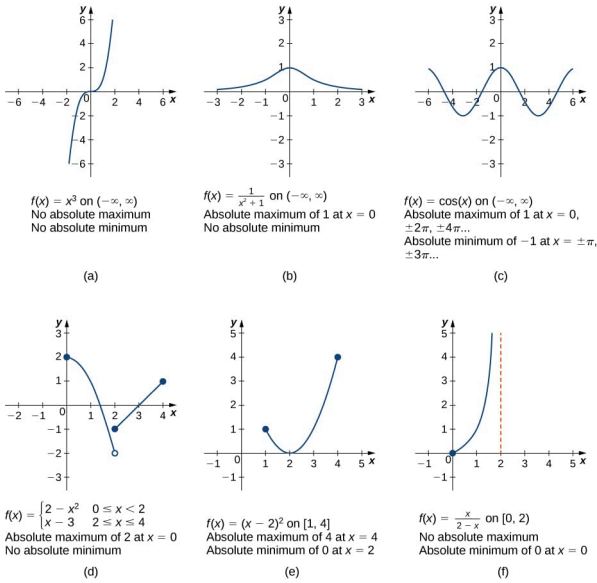


Figure 3.1.2: Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem 3.1.1: Extreme Value Theorem

If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in Figure 3.1.2 (d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in Figure 3.1.2, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by Fermat's Theorem, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 3.1.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From "Location of Absolute Extrema," the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Illustration 3.1.1: Locating Absolute Extrema

For each of the following functions, we find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

(a) $f(x) = -x^2 + 3x - 2$ over $[1, 3]$

Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$:

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, we know f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$, and $\frac{3}{2}$ is in the interval $[1, 3]$, we see $f\left(\frac{3}{2}\right)$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f\left(\frac{3}{2}\right)$ and find

$$f\left(\frac{3}{2}\right) = \frac{1}{4}$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$.

The absolute minimum of f over the interval $[1, 3]$ is -2, and it occurs at $x = 3$, as shown in Figure 3.1.8.

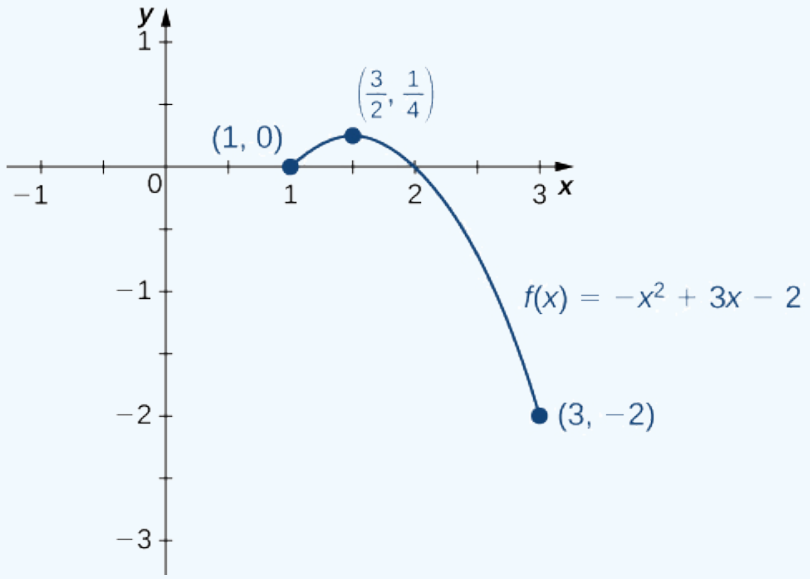


Figure 3.1.8: This function has both an absolute maximum and an absolute minimum.

(b) $f(x) = x^2 - 3x^{\frac{2}{3}}$ over $[0, 2]$

Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3 \cdot 2^{\frac{2}{3}} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - 2x^{-\frac{1}{3}} = \frac{2x^{\frac{4}{3}} - 2}{x^{\frac{1}{3}}}$$

for $x \neq 0$. The derivative is zero when $2x^{\frac{4}{3}} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval $[0, 2]$ is zero, and it occurs at $x = 0$.

The absolute minimum is -2, and it occurs at $x = 1$, as shown in Figure 3.1.9.

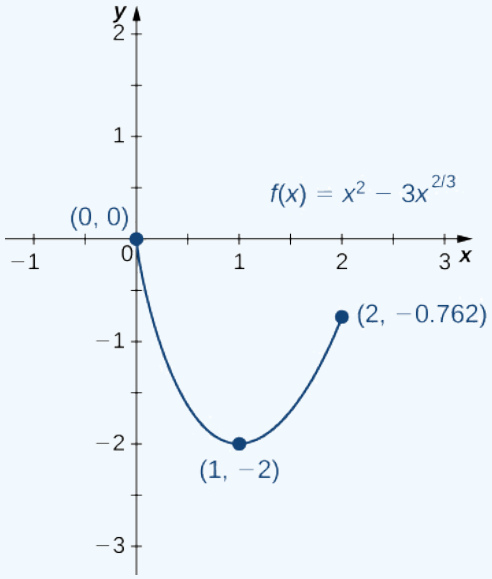


Figure 3.1.9: This function has an absolute maximum at an endpoint of the interval.

Exercise 3.1.2

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$

Hint available at the end of this section.

Answer available at the end of this section.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Answers

Exercise 3.1.2

Look for critical points. Evaluate f at all critical points and at the endpoints.

Exercise 3.1.2

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

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3.1: Practice Set

Exercise 3.1.1

Find the critical numbers of each function.

(a) $f(x) = x^3 - 12x + 5$

(b) $g(x) = 2x^3 - 3x^2 - 36x + 7$

(c) $h(x) = x^{4/3} - 4x^{1/3}$

(d) $p(x) = \dfrac{x^2}{x - 2}$

(e) $q(x) = \sin(x)\cos(x)$ on $[0, 2\pi]$

(f) $r(x) = xe^{-x}$

Answer available at the end of this section.

Exercise 3.1.2

Find the absolute maximum and minimum values of each function on the given interval.

(a) $f(x) = x^3 - 3x^2 + 1$ on $[-1, 4]$

(b) $g(x) = x^4 - 2x^2 + 3$ on $[-2, 2]$

(c) $h(x) = \sqrt{x} - x$ on $[0, 4]$

(d) $p(x) = \sin(x) + \cos(x)$ on $[0, \pi]$

(e) $q(x) = \dfrac{x}{x^2 + 1}$ on $[-2, 2]$

(f) $r(x) = xe^{-x/2}$ on $[0, 4]$

Answer available at the end of this section.

Exercise 3.1.3

Find all local extrema of each function.

(a) $f(x) = x^3 - 6x^2 + 9x + 2$

(b) $g(x) = x^4 - 8x^2 + 16$

(c) $h(x) = x^{2/3}$

(d) $p(x) = \dfrac{x^2}{x - 1}$

(e) $q(x) = \sin(x) + \cos(x)$ on $[0, 2\pi]$

(f) $r(x) = \ln(x^2 + 1)$

Answer available at the end of this section.

Exercise 3.1.4

Find all absolute extrema of each function on the given interval.

(a) $f(x) = x^3 - 3x + 1$ on $[-2, 2]$

(b) $g(x) = \dfrac{x}{x^2 + 4}$ on $[-3, 3]$

(c) $h(x) = \sin(2x)$ on $[0, \pi]$

(d) $p(x) = \ln(x^2 + 1)$ on $[-2, 2]$

(e) $q(x) = x^2 e^{-x}$ on $[-1, 3]$

Answer available at the end of this section.

Exercise 3.1.5

For each function, find all critical numbers and classify each as a local maximum, local minimum, or neither.

(a) $f(x) = 2x^3 - 9x^2 + 12x - 3$

(b) $g(x) = x^4 - 4x^3 + 4x^2$

(c) $h(x) = \dfrac{x^2 - 4}{x - 2}$

(d) $p(x) = \cos(2x)$ on $[0, 2\pi]$

(e) $q(x) = x^{5/3} - 5x^{2/3}$

(f) $r(x) = \dfrac{\ln(x)}{x}$ for $x > 0$

Answer available at the end of this section.

Exercise 3.1.6

Find the absolute maximum and minimum values of each function on the given interval.

(a) $f(x) = x^3 - 3x^2 - 9x + 5$ on $[-2, 4]$

(b) $g(x) = \dfrac{x^2}{x^2 + 1}$ on $[-2, 2]$

(c) $h(x) = \sin(x) - \cos(x)$ on $[0, \pi]$

(d) $p(x) = x\sqrt{4 - x^2}$ on $[-2, 2]$

(e) $q(x) = xe^{-x^2}$ on $[-2, 2]$

Answer available at the end of this section.

Answers

Exercise 3.1.1

(a) Find $f'(x) = 3x^2 - 12$.

Set $f'(x) = 0$: $3x^2 - 12 = 0 \implies x^2 = 4 \implies x = \pm 2$.

Thus the critical numbers are $x = -2$ and $x = 2$.

(b) Find $g'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x - 3)(x + 2)$.

Set $g'(x) = 0$: $x = 3$ or $x = -2$.

Thus the critical numbers are $x = -2$ and $x = 3$.

(c) Find $h'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x-1)}{3x^{2/3}}$.

Set $h'(x) = 0$: $x = 1$.

$h'(x)$ is undefined when $x = 0$.

Thus the critical numbers are $x = 0$ and $x = 1$.

(d) Find $p'(x) = \frac{(2x)(x-2) - x^2(1)}{(x-2)^2} = \frac{2x^2 - 4x - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$.

Set $p'(x) = 0$: $x = 0$ or $x = 4$.

$p'(x)$ is undefined when $x = 2$ (but $x = 2$ is not in the domain of p).

Thus the critical numbers are $x = 0$ and $x = 4$.

(e) Find $q'(x) = \cos^2(x) - \sin^2(x) = \cos(2x)$.

Set $q'(x) = 0$: $\cos(2x) = 0 \implies 2x = \frac{\pi}{2} + k\pi \implies x = \frac{\pi}{4} + \frac{k\pi}{2}$.

On $[0, 2\pi]$: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Thus the critical numbers are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

(f) Find $r'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$.

Set $r'(x) = 0$: $e^{-x}(1 - x) = 0 \implies x = 1$.

Thus the critical number is $x = 1$.

Exercise 3.1.2

(a) $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Critical numbers: $x = 0, 2$.

Evaluate at endpoints and critical numbers:

$f(-1) = -1 - 3 + 1 = -3$

$f(0) = 1$

$f(2) = 8 - 12 + 1 = -3$

$f(4) = 64 - 48 + 1 = 17$

Absolute maximum: 17 at $x = 4$.

Absolute minimum: -3 at $x = -1$ and $x = 2$.

(b) $g'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1)$. Critical numbers: $x = -1, 0, 1$.

Evaluate at endpoints and critical numbers:

$g(-2) = 16 - 8 + 3 = 11$

$g(-1) = 1 - 2 + 3 = 2$

$g(0) = 3$

$g(1) = 1 - 2 + 3 = 2$

$g(2) = 16 - 8 + 3 = 11$

Absolute maximum: 11 at $x = \pm 2$.

Absolute minimum: 2 at $x = \pm 1$.

(c) $h'(x) = \frac{1}{2\sqrt{x}} - 1$. Critical number: $x = \frac{1}{4}$.

Evaluate at endpoints and critical numbers:

$h(0) = 0$

$h\left(\frac{1}{4}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$h(4) = 2 - 4 = -2$

Absolute maximum: $\frac{1}{4}$ at $x = \frac{1}{4}$.

Absolute minimum: -2 at $x = 4$.

(d) $p'(x) = \cos(x) - \sin(x)$. Critical number: $\tan(x) = 1 \implies x = \frac{\pi}{4}$.

Evaluate at endpoints and critical numbers:

$p(0) = 1$

$p\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

$p(\pi) = -1$

Absolute maximum: $\sqrt{2}$ at $x = \frac{\pi}{4}$.

Absolute minimum: -1 at $x = \pi$.

(e) $q'(x) = \dfrac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$. Critical numbers: $x = \pm 1$.

Evaluate at endpoints and critical numbers:

$q(-2) = -\dfrac{2}{5}$

$q(-1) = -\dfrac{1}{2}$

$q(1) = \dfrac{1}{2}$

$q(2) = \dfrac{2}{5}$

Absolute maximum: $\dfrac{1}{2}$ at $x = 1$.

Absolute minimum: $-\dfrac{1}{2}$ at $x = -1$.

(f) $r'(x) = e^{-x/2} - \dfrac{x}{2}e^{-x/2} = e^{-x/2}\left(1 - \dfrac{x}{2}\right)$. Critical number: $x = 2$.

Evaluate at endpoints and critical numbers:

$r(0) = 0$

$r(2) = 2e^{-1} = \dfrac{2}{e}$

$r(4) = 4e^{-2} = \dfrac{4}{e^2}$

Absolute maximum: $\dfrac{2}{e}$ at $x = 2$.

Absolute minimum: 0 at $x = 0$.

Exercise 3.1.3

(a) $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$.

Critical numbers: $x = 1, 3$.

Use the first derivative test:

For $x < 1$, $f'(x) > 0$ (e.g., $x = 0$: $3(-1)(-3) = 9 > 0$).

For $1 < x < 3$, $f'(x) < 0$ (e.g., $x = 2$: $3(1)(-1) = -3 < 0$).

For $x > 3$, $f'(x) > 0$ (e.g., $x = 4$: $3(3)(1) = 9 > 0$).

Local maximum at $x = 1$: $f(1) = 1 - 6 + 9 + 2 = 6$.

Local minimum at $x = 3$: $f(3) = 27 - 54 + 27 + 2 = 2$.

(b) $g'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$.

Critical numbers: $x = -2, 0, 2$.

Use the first derivative test:

For $x < -2$, $g'(x) < 0$ (e.g., $x = -3$: negative).

For $-2 < x < 0$, $g'(x) > 0$ (e.g., $x = -1$: $4(-1)(-3)(1) = 12 > 0$).

For $0 < x < 2$, $g'(x) < 0$ (e.g., $x = 1$: $4(1)(-1)(3) = -12 < 0$).

For $x > 2$, $g'(x) > 0$ (e.g., $x = 3$: positive).

Local minimum at $x = -2$: $g(-2) = 16 - 32 + 16 = 0$.

Local maximum at $x = 0$: $g(0) = 16$.

Local minimum at $x = 2$: $g(2) = 16 - 32 + 16 = 0$.

(c) $h'(x) = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}$.

$h'(x)$ is undefined at $x = 0$.

For $x < 0$, $h'(x) < 0$.

For $x > 0$, $h'(x) > 0$.

Local minimum at $x = 0$: $h(0) = 0$.

(d) $p'(x) = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$.

Critical numbers: $x = 0, 2$. Note: $x = 1$ is not in the domain.

For $x < 0$, $p'(x) > 0$ (e.g., $x = -1$: positive).

For $0 < x < 1$, $p'(x) < 0$ (e.g., $x = 0.5$: negative).

For $1 < x < 2$, $p'(x) < 0$ (e.g., $x = 1.5$: negative).

For $x > 2$, $p'(x) > 0$ (e.g., $x = 3$: positive).

Local maximum at $x = 0$: $p(0) = 0$.

Local minimum at $x = 2$: $p(2) = \dfrac{4}{1} = 4$.

(e) $q'(x) = \cos(x) - \sin(x)$. Critical numbers: $\tan(x) = 1 \implies x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$.

Use the first derivative test:

For x near 0, $q'(x) > 0$ (e.g., $x = 0$: $1 > 0$).

For $\dfrac{\pi}{4} < x < \dfrac{5\pi}{4}$, $q'(x) < 0$ (e.g., $x = \pi$: $-1 < 0$).

For $x > \dfrac{5\pi}{4}$, $q'(x) > 0$ (e.g., $x = 3\pi/2$: $1 > 0$).

Local maximum at $x = \frac{\pi}{4}$: $q\left(\frac{\pi}{4}\right) = \sqrt{2}$.

Local minimum at $x = \frac{5\pi}{4}$: $q\left(\frac{5\pi}{4}\right) = -\sqrt{2}$.

(f) $r'(x) = \frac{2x}{x^2+1}$. Critical number: $x = 0$.

For $x < 0$, $r'(x) < 0$.

For $x > 0$, $r'(x) > 0$.

Local minimum at $x = 0$: $r(0) = 0$.

Exercise 3.1.4

(a) $f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$. Critical numbers: $x = \pm 1$.

Evaluate:

$f(-2) = -8 + 6 + 1 = -1$

$f(-1) = -1 + 3 + 1 = 3$

$f(1) = 1 - 3 + 1 = -1$

$f(2) = 8 - 6 + 1 = 3$

Absolute maximum: 3 at $x = -1$ and $x = 2$.

Absolute minimum: -1 at $x = -2$ and $x = 1$.

(b) $g'(x) = \frac{(1)(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$. Critical numbers: $x = \pm 2$.

Evaluate:

$g(-3) = \frac{-3}{13}$

$g(-2) = \frac{-2}{8} = -\frac{1}{4}$

$g(2) = \frac{2}{8} = \frac{1}{4}$

$g(3) = \frac{3}{13}$

Absolute maximum: $\frac{1}{4}$ at $x = 2$.

Absolute minimum: $-\frac{1}{4}$ at $x = -2$.

(c) $h'(x) = 2\cos(2x)$. Critical numbers: $\cos(2x) = 0 \implies 2x = \frac{\pi}{2} + k\pi \implies x = \frac{\pi}{4} + \frac{k\pi}{2}$.

On $[0, \pi]$: $x = \frac{\pi}{4}, \frac{3\pi}{4}$.

Evaluate:

$h(0) = 0$

$h\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$h\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$

$h(\pi) = 0$

Absolute maximum: 1 at $x = \frac{\pi}{4}$.

Absolute minimum: -1 at $x = \frac{3\pi}{4}$.

(d) $p'(x) = \frac{2x}{x^2+1}$. Critical number: $x = 0$.

Evaluate:

$p(-2) = \ln(5)$

$p(0) = \ln(1) = 0$

$p(2) = \ln(5)$

Absolute maximum: $\ln(5)$ at $x = \pm 2$.

Absolute minimum: 0 at $x = 0$.

(e) $q'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$. Critical numbers: $x = 0, 2$.

Evaluate:

$q(-1) = 1 \cdot e^1 = e$

$q(0) = 0$

$q(2) = 4e^{-2} = \frac{4}{e^2}$

$q(3) = 9e^{-3} = \frac{9}{e^3}$

Absolute maximum: e at $x = -1$.

Absolute minimum: 0 at $x = 0$.

Exercise 3.1.5

(a) $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$. Critical numbers: $x = 1, 2$.

For $x < 1$, $f'(x) > 0$.

For $1 < x < 2$, $f'(x) < 0$.

For $x > 2$, $f'(x) > 0$.

Local maximum at $x = 1$: $f(1) = 2 - 9 + 12 - 3 = 2$.
Local minimum at $x = 2$: $f(2) = 16 - 36 + 24 - 3 = 1$.
(b) $g'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x - 1)(x - 2)$. Critical numbers: $x = 0, 1, 2$.
For $x < 0$, $g'(x) < 0$.
For $0 < x < 1$, $g'(x) > 0$.
For $1 < x < 2$, $g'(x) < 0$.
For $x > 2$, $g'(x) > 0$.
Local minimum at $x = 0$: $g(0) = 0$.
Local maximum at $x = 1$: $g(1) = 1 - 4 + 4 = 1$.
Local minimum at $x = 2$: $g(2) = 16 - 32 + 16 = 0$.
(c) Note that $h(x) = \dfrac{x^2 - 4}{x - 2} = \dfrac{(x - 2)(x + 2)}{x - 2} = x + 2$ for $x \neq 2$.
$h'(x) = 1$ for all $x \neq 2$.
There are no critical numbers (the function has a removable discontinuity at $x = 2$, but no local extrema).
(d) $p'(x) = -2\sin(2x)$. Critical numbers: $\sin(2x) = 0 \implies 2x = k\pi \implies x = \dfrac{k\pi}{2}$.
On $[0, 2\pi]$: $x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$.
For $x \in (0, \pi/2)$, $p'(x) < 0$.
For $x \in (\pi/2, \pi)$, $p'(x) > 0$.
For $x \in (\pi, 3\pi/2)$, $p'(x) < 0$.
For $x \in (3\pi/2, 2\pi)$, $p'(x) > 0$.
Local minimum at $x = \dfrac{\pi}{2}$: $p\left(\dfrac{\pi}{2}\right) = \cos(\pi) = -1$.
Local maximum at $x = \pi$: $p(\pi) = \cos(2\pi) = 1$.
Local minimum at $x = \dfrac{3\pi}{2}$: $p\left(\dfrac{3\pi}{2}\right) = \cos(3\pi) = -1$.
(e) $q'(x) = \dfrac{5}{3}x^{2/3} - \dfrac{10}{3}x^{-1/3} = \dfrac{5}{3}x^{-1/3}(x - 2) = \dfrac{5(x - 2)}{3x^{1/3}}$. Critical numbers: $x = 0, 2$.
For $x < 0$, $q'(x) > 0$.
For $0 < x < 2$, $q'(x) < 0$.
For $x > 2$, $q'(x) > 0$.
Local maximum at $x = 0$: $q(0) = 0$.
Local minimum at $x = 2$: $q(2) = 2^{5/3} - 5(2^{2/3}) = 2^{2/3}(2 - 5) = -3 \cdot 2^{2/3}$.
(f) $r'(x) = \dfrac{(1/x)(x) - \ln(x)(1)}{x^2} = \dfrac{1 - \ln(x)}{x^2}$. Critical number: $\ln(x) = 1 \implies x = e$.
For $0 < x < e$, $r'(x) > 0$.
For $x > e$, $r'(x) < 0$.
Local maximum at $x = e$: $r(e) = \dfrac{1}{e}$.
No local minimum.

Exercise 3.1.6

(a) $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$. Critical numbers: $x = -1, 3$.
Evaluate:
$f(-2) = -8 - 12 + 18 + 5 = 3$
$f(-1) = -1 - 3 + 9 + 5 = 10$
$f(3) = 27 - 27 - 27 + 5 = -22$
$f(4) = 64 - 48 - 36 + 5 = -15$
Absolute maximum: 10 at $x = -1$.
Absolute minimum: -22 at $x = 3$.
(b) $g'(x) = \dfrac{2x(x^2 + 1) - x^2(2x)}{(x^2 + 1)^2} = \dfrac{2x}{(x^2 + 1)^2}$. Critical number: $x = 0$.
Evaluate:
$g(-2) = \dfrac{4}{5}$
$g(0) = 0$
$g(2) = \dfrac{4}{5}$
Absolute maximum: $\dfrac{4}{5}$ at $x = \pm 2$.
Absolute minimum: 0 at $x = 0$.
(c) $h'(x) = \cos(x) + \sin(x)$. Critical number: $\tan(x) = -1 \implies x = \dfrac{3\pi}{4}$.
Evaluate:
$h(0) = -1$
$h\left(\dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2} - \left(-\dfrac{\sqrt{2}}{2}\right) = \sqrt{2}$
$h(\pi) = 0 - (-1) = 1$

Absolute maximum: $\sqrt{2}$ at $x = \dfrac{3\pi}{4}$.

Absolute minimum: -1 at $x = 0$.

(d) $p(x) = x(4 - x^2)^{1/2}$. Domain: $[-2, 2]$.

$p'(x) = (4 - x^2)^{1/2} + x \cdot \dfrac{1}{2}(4 - x^2)^{-1/2}(-2x) = \dfrac{4 - x^2}{\sqrt{4 - x^2}} - \dfrac{x^2}{\sqrt{4 - x^2}} = \dfrac{4 - 2x^2}{\sqrt{4 - x^2}}$.

Critical number: $4 - 2x^2 = 0 \implies x^2 = 2 \implies x = \pm\sqrt{2}$.

Evaluate:

$p(-2) = 0$

$p(-\sqrt{2}) = -\sqrt{2}\sqrt{2} = -2$

$p(\sqrt{2}) = \sqrt{2}\sqrt{2} = 2$

$p(2) = 0$

Absolute maximum: 2 at $x = \sqrt{2}$.

Absolute minimum: -2 at $x = -\sqrt{2}$.

(e) $q'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2)$. Critical numbers: $x = \pm\dfrac{1}{\sqrt{2}}$.

Evaluate:

$q(-2) = -2e^{-4}$

$q\left(-\dfrac{1}{\sqrt{2}}\right) = -\dfrac{1}{\sqrt{2}}e^{-1/2}$

$q\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{\sqrt{2}}e^{-1/2}$

$q(2) = 2e^{-4}$

Absolute maximum: $\dfrac{1}{\sqrt{2e}}$ at $x = \dfrac{1}{\sqrt{2}}$.

Absolute minimum: $-\dfrac{1}{\sqrt{2e}}$ at $x = -\dfrac{1}{\sqrt{2}}$.

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3.2: Curve Sketching

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.
- Recognize a horizontal asymptote on the graph of a function.
- Estimate the end behavior of a function as x increases or decreases without bound.
- Recognize an oblique asymptote on the graph of a function.
- Analyze a function and its derivatives to draw its graph.

Earlier in this chapter we stated that if a function f has a local extremum at a point c, then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

Question of the Day

How do we use all our derivative information to sketch a complete graph of a function?

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3.2: Curve Sketching Basics

The First Derivative Test

Corollary \(3\) of the Mean Value Theorem showed that if the derivative of a function is positive over an interval \(I\) then the function is increasing over \(I\). On the other hand, if the derivative of the function is negative over an interval \(I\), then the function is decreasing over \(I\) as shown in the following figure.

This figure is broken into four figures labeled a, b, c, and d. Figure a shows a function increasing convexly from (a, f(a)) to (b, f(b)). At two points the derivative is taken and it is noted that at both f' /
0. In other words, f is increasing. Figure b shows a function increasing concavely from (a, f(a)) to (b, f(b)). At two points the derivative is taken and it is noted that at both f' > 0. In other words, f is increasing. Figure c shows a function decreasing concavely from (a, f(a)) to (b, f(b)). At two points the derivative is taken and it is noted that at both f' < 0. In other words, f is decreasing. Figure d shows a function decreasing convexly from (a, f(a)) to (b, f(b)). At two points the derivative is taken and it is noted that at both f' < 0. In other words, f is decreasing." src="/@api/deki/files/2428/CNX_Calc_Figure_04_05_001.jpeg">

Figure \(\PageIndex{1}\): Both functions are increasing over the interval \((a,b)\). At each point \(x\), the derivative \(f'(x)>0\). Both functions are decreasing over the interval \((a,b)\). At each point \(x\), the derivative \(f'(x)<0.\)

A continuous function \(f\) has a local maximum at point \(c\) if and only if \(f\) switches from increasing to decreasing at point \(c\). Similarly, \(f\) has a local minimum at \(c\) if and only if \(f\) switches from decreasing to increasing at \(c\). If \(f\) is a continuous function over an interval \(I\) containing \(c\) and differentiable over \(I\), except possibly at \(c\), the only way \(f\) can switch from increasing to decreasing (or vice versa) at point \(c\) is if \(f'\) changes sign as \(x\) increases through \(c\). If \(f\) is differentiable at \(c\), the only way that \(f'\) can change sign as \(x\) increases through \(c\) is if \(f'(c)=0\). Therefore, for a function \(f\) that is continuous over an interval \(I\) containing \(c\) and differentiable over \(I\), except possibly at \(c\), the only way \(f\) can switch from increasing to decreasing (or vice versa) is if \(f'(c)=0\) or \(f'(c)\) is undefined. Consequently, to locate local extrema for a function \(f\), we look for points \(c\) in the domain of \(f\) such that \(f'(c)=0\) or \(f'(c)\) is undefined. Recall that such points are called critical points of \(f\).

Note that \(f\) need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure \(\PageIndex{2}\), we show that if a continuous function \(f\) has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if \(f\) has a local extremum at a critical point, then the sign of \(f'\) switches as \(x\) increases through that point.

A function f(x) is graphed. It starts in the second quadrant and increases to x = a, which is too sharp and hence f'(a) is undefined. In this section f' /
0. Then, f decreases from x = a to x = b (so f' < 0 here), before increasing at x = b. It is noted that f'(b) = 0. While increasing from x = b to x = c, f' > 0. The function has an inversion point at c, and it is marked f'(c) = 0. The function increases some more to d (so f' > 0), which is the global maximum. It is marked that f'(d) = 0. Then the function decreases and it is marked that f' > 0." style="width: 867px; height: 429px;" width="867px" height="429px" src="/@api/deki/files/34649/CNX_Calc_Figure_04_05_002b.jpg">

Figure \(\PageIndex{2}\): The function \(f\) has four critical points: \(a,b,c\),and \(d\). The function \(f\) has local maxima at \(a\) and \(d\), and a local minimum at \(b\). The function \(f\) does not have a local extremum at \(c\). The sign of \(f'\) changes at all local extrema.

Using Figure \(\PageIndex{2}\), we summarize the main results regarding local extrema.

- If a continuous function \(f\) has a local extremum, it must occur at a critical point \(c\).
- The function has a local extremum at the critical point \(c\) if and only if the derivative \(f'\) switches sign as \(x\) increases through \(c\).
- Therefore, to test whether a function has a local extremum at a critical point \(c\), we must determine the sign of \(f'(x)\) to the left and right of \(c\).

This result is known as the **first derivative test.**

Theorem \(\PageIndex{1}\): First Derivative Test

Suppose that \(f\) is a continuous function over an interval \(I\) containing a critical point \(c\). If \(f\) is differentiable over \(I\), except possibly at point \(c\), then \(f(c)\) satisfies one of the following descriptions:

i. If \(f'\) changes sign from positive when \(xc\), then \(f(c)\) is a local maximum of \(f\).
ii. If \(f'\) changes sign from negative when \(xc\), then \(f(c)\) is a local minimum of \(f\).
iii. If \(f'\) has the same sign for \(xc\), then \(f(c)\) is neither a local maximum nor a local minimum of \(f\).

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example \(\PageIndex{1}\): Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for \(f(x)=x^3−3x^2−9x−1\)

Solution

Step 1. The derivative is \(f'(x)=3x^2−6x−9.\) To find the critical points, we need to find where \(f'(x)=0.\) Factoring the polynomial, we conclude that the critical points must satisfy

\[3(x^2−2x−3)=3(x−3)(x+1)=0 \nonumber \]

Therefore, the critical points are \(x=3,−1.\) Now divide the interval \((−∞,∞)\) into the smaller intervals \((−∞,−1),(−1,3)\) and \((3,∞).\)

Step 2. Since \(f'\) is a continuous function, to determine the sign of \(f'(x)\) over each subinterval, it suffices to choose a point over each of the intervals \((−∞,−1),(−1,3)\) and \((3,∞)\) and determine the sign of \(f'\) at each of these points. For example, let's choose \(x=−2\), \(x=0\), and \(x=4\) as test points.

Table: \(\PageIndex{1}\): First Derivative Test for \(f(x)=x^3−3x^2−9x−1\)

Interval	Test Point	Sign of \(f'(x)\)	Conclusion
\((−∞,−1)\)	\(x=−2\)	(+)(−)(−)=+	\(f\) increasing
\((−1,3)\)	\(x=0\)	(+)(−)(+)=-	\(f\) decreasing
\((3,∞)\)	\(x=4\)	(+)(+)(+)=+	\(f\) increasing

Step 3. Since \(f'\) switches sign from positive to negative as \(x\) increases through \(-1\), \(\boxed{f(x) \text{ has a local maximum at }x=−1}\). Since \(f'\) switches sign from negative to positive as \(x\) increases through \(3\), \(\boxed{f(x) \text{ has a local minimum at }x=3}\). These analytical results agree with the following graph.

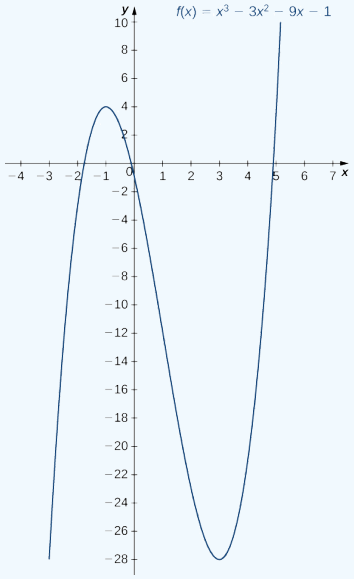


Figure \(\PageIndex{3}\): The function \(f\) has a maximum at \(x=−1\) and a minimum at \(x=3\)

Exercise \(\PageIndex{1}\)

Use the first derivative test to locate all local extrema for \(f(x)=−x^3+\dfrac{3}{2}x^2+18x\)

Hint available at the end of this section.

Answer available at the end of this section.

Example \(\PageIndex{2}\): Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for \(f(x)=5x^{\frac{1}{3}}−x^{\frac{5}{3}}\)

Solution

Step 1. The derivative is

\[f'(x)=\frac{5}{3}x^{−\frac{2}{3}}−\frac{5}{3}x^{\frac{2}{3}}=\frac{5}{3x^{\frac{2}{3}}}−\frac{5x^{\frac{2}{3}}}{3}=\frac{5−5x^{\frac
\]

The derivative \(f'(x)=0\) when \(1−x^{\frac{4}{3}}=0.\) Therefore, \(f'(x)=0\) at \(x=±1\). The derivative \(f'(x)\) is undefined at \(x=0.\) Therefore, we have three critical points: \(x=0\), \(x=1\), and \(x=−1\). Consequently, divide the interval \((−∞,∞)\) into the smaller intervals \((−∞,−1),\,(−1,0),\,(0,1)\), and \((1,∞)\).

Step 2: Since \(f'\) is continuous over each subinterval, it suffices to choose a test point \(x\) in each of the intervals from step 1 and determine the sign of \(f'\) at each of these points. The points \(x=−2,\,x=−\dfrac{1}{2},\,x=\dfrac{1}{2}\), and \(x=2\) are test points for these intervals.

Table: \(\PageIndex{2}\): First Derivative Test for \(f(x)=5x^{\frac{1}{3}}−x^{\frac{5}{3}}\)

Interval	Test Point	Sign of \(f'(x)\)	Conclusion
\((−∞,−1)\)	\(x=−2\)	\(\dfrac{(+)(−)}{+}=−\)	\(f\) decreasing
\((−1,0)\)	\(x=−\dfrac{1}{2}\)	\(\dfrac{(+)(+)}{+}=+\)	\(f\) increasing
\((0,1)\)	\(x=\dfrac{1}{2}\)	\(\dfrac{(+)(+)}{+}=+\)	\(f\) increasing

Interval	Test Point	Sign of $f'(x)$	Conclusion
$(1,\infty)$	$x=2$	$\dfrac{(+)(-)}{+}=-$	f decreasing

Step 3: Since f is decreasing over the interval $(-\infty,-1)$ and increasing over the interval $(-1,0)$, $\boxed{f(x)\text{ has a local minimum at }x=-1}$. Since f is increasing over the interval $(-1,0)$ and the interval $(0,1)$, $\boxed{f(x)\text{ does not have a local extremum at }x=0}$. Since f is increasing over the interval $(0,1)$ and decreasing over the interval $(1,\infty)$, $\boxed{f(x)\text{ has a local maximum at }x=1}$. The analytical results agree with the following graph.

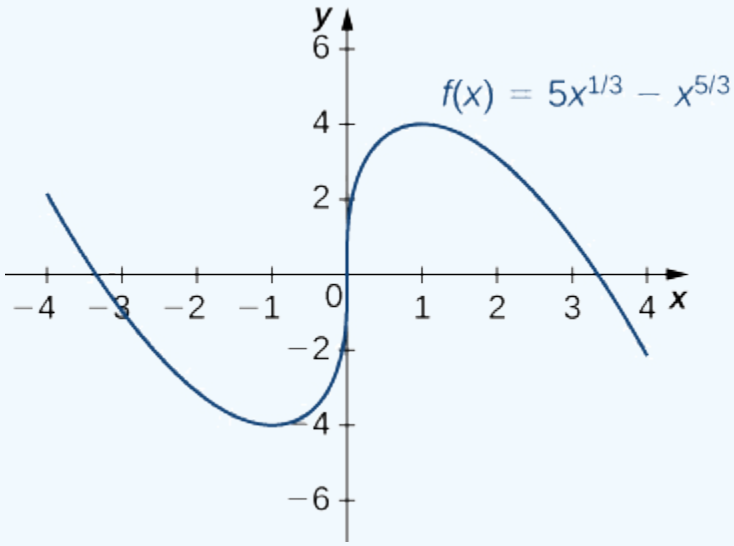


Figure $\PageIndex{4}$: The function f has a local minimum at $x=-1$ and a local maximum at $x=1$

Exercise $\PageIndex{2}$

Use the first derivative test to find all local extrema for $f(x)=\dfrac{3}{x-1}$

Hint available at the end of this section.

Answer available at the end of this section.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure $\PageIndex{5a}$ shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. Figure $\PageIndex{5b}$ shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition: Concavity Test

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

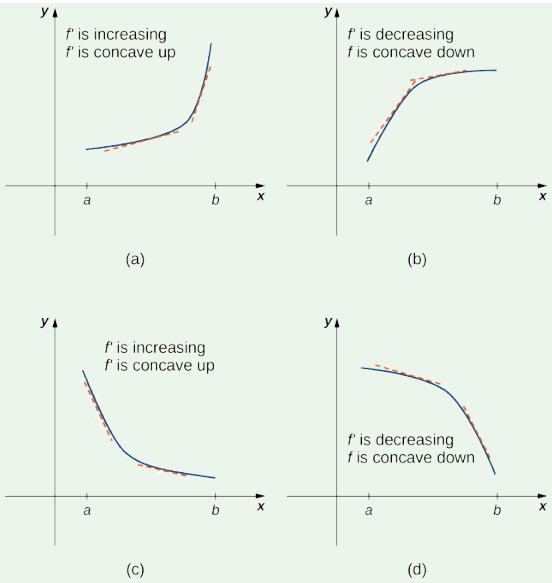


Figure \(\PageIndex{5}\): (a), (c) Since \(f'\) is increasing over the interval \((a,b)\), we say \(f\) is concave up over \((a,b)\). (b), (d)\) Since \(f'\) is decreasing over the interval \((a,b)\), we say \(f\) is concave down over \((a,b).\)

In general, without having the graph of a function \(f\), how can we determine its concavity? By definition, a function \(f\) is concave up if \(f'\) is increasing. From Corollary \(3\), we know that if \(f'\) is a differentiable function, then \(f'\) is increasing if its derivative \(f''(x)>0\). Therefore, a function \(f\) that is twice differentiable is concave up when \(f''(x)>0\). Similarly, a function \(f\) is concave down if \(f'\) is decreasing. We know that a differentiable function \(f'\) is decreasing if its derivative \(f''(x)<0\). Therefore, a twice-differentiable function \(f\) is concave down when \(f''(x)<0\). Applying this logic is known as the *concavity test*.

Test for Concavity

Let \(f\) be a function that is twice differentiable over an interval \(I\).

i. If \(f''(x)>0\) for all \(x\in I\), then \(f\) is concave up over \(I\).
ii. If \(f''(x)<0\) for all \(x\in I,\) then \(f\) is concave down over \(I\).

We conclude that we can determine the concavity of a function \(f\) by looking at the second derivative of \(f\). In addition, we observe that a function \(f\) can switch concavity (Figure \(\PageIndex{6}\)). However, a continuous function can switch concavity only at a point \(x\) if \(f''(x)=0\) or \(f''(x)\) is undefined. Consequently, to determine the intervals where a function \(f\) is concave up and concave down, we look for those values of \(x\) where \(f''(x)=0\) or \(f''(x)\) is undefined. When we have determined these points, we divide the domain of \(f\) into smaller intervals and determine the sign of \(f''\) over each of these smaller intervals. If \(f''\) changes sign as we pass through a point \(x\), then \(f\) changes concavity. It is important to remember that a function \(f\) may not change concavity at a point \(x\) even if \(f''(x)=0\) or \(f''(x)\) is undefined. If, however, \(f\) does change concavity at a point \(a\) and \(f\) is continuous at \(a\), we say the point \((a,f(a))\) is an **inflection point** of \(f\).

Definition: Inflection Point

If \(f\) is continuous at \(a\) and \(f\) changes concavity at \(a\), the point \((a, \,f(a))\) is an *inflection point* of \(f\).

A sinusoidal function is shown that has been shifted into the first quadrant. The function starts decreasing, so f' < 0 and f'' / 0. The function reaches the local minimum and starts increasing, so f' > 0 and f'' > 0. It is noted that the slope is increasing for these two intervals. The function then reaches an inflection point (a, f(a)) and from here the slop is decreasing even though the function continues to increase, so f' > 0 and f'' < 0. The function reaches the maximum and then starts decreasing, so f' < 0 and f'' < 0." style="width: 731px; height: 401px;" width="731px" height="401px" src="/@api/deki/files/2433/CNX_Calc_Figure_04_05_004.jpeg">

Figure \(\PageIndex{6}\): Since \(f''(x)>0\) for \(xa\), the function \(f\) is concave down over the interval \((a,∞)\). The point \((a,f(a))\) is an inflection point of \(f\).

Example \(\PageIndex{3}\): Testing for Concavity

For the function \(f(x)=x^3−6x^2+9x+30,\) determine all intervals where \(f\) is concave up and all intervals where \(f\) is concave down. List all inflection points for \(f\).

Solution

To determine concavity, we need to find the second derivative \(f''(x).\) The first derivative is \(f'(x)=3x^2−12x+9,\) so the second derivative is \(f''(x)=6x−12.\) If the function changes concavity, it occurs either when \(f''(x)=0\) or \(f''(x)\) is undefined. Since \(f''\) is defined for all real numbers \(x\), we need only find where \(f''(x)=0\). Solving the equation \(6x−12=0\), we see that \(x=2\) is the only place where \(f\) could change concavity. We now test points over the intervals \((−∞,2)\) and \((2,∞)\) to determine the concavity of \(f\). The points \(x=0\) and \(x=3\) are test points for these intervals.

Table: \(\PageIndex{3}\): Test for Concavity for \(f(x)=x^3−6x^2+9x+30\)

Interval	Test Point	Sign of \(f''(x)=6x−12\) at Test Point	Conclusion
\((−∞,2)\)	\(x=0\)	−	\(f\) concave down
\((2,∞)\)	\(x=3\)	+	\(f\) concave up

We conclude that \(f\) is **concave down over the interval \((−∞,2)\) and concave up over the interval \((2,∞)\)**.

Since \(f\) changes concavity at \(x=2\), the point \(\boxed{\left(2,f(2)\right)=(2,32) \text{ is an inflection point}}\). Figure \(\PageIndex{7}\) confirms the analytical results.

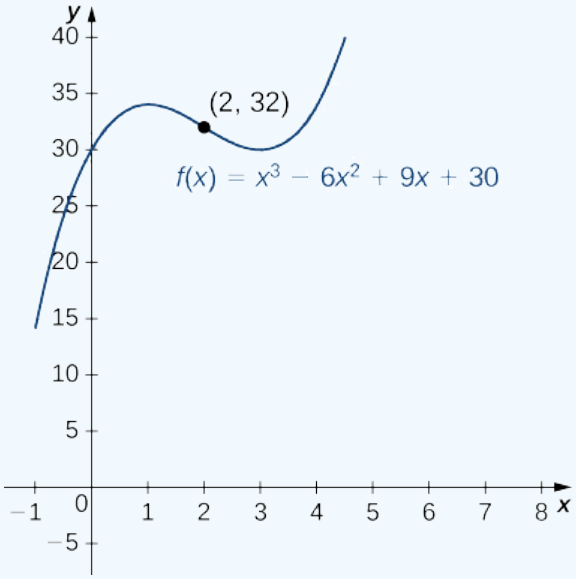


Figure \(\PageIndex{7}\): The given function has a point of inflection at \((2,32)\) where the graph changes concavity.

Exercise \(\PageIndex{3}\)

For \(f(x)=−x^3+\dfrac{3}{2}x^2+18x\), find all intervals where \(f\) is concave up and all intervals where \(f\) is concave down.

Hint available at the end of this section.

Answer available at the end of this section.

We now summarize, in Table \(\PageIndex{4}\), the information that the first and second derivatives of a function \(f\) provide about the graph of \(f\), and illustrate this information in Figure \(\PageIndex{8}\).

Table: \(\PageIndex{4}\): What Derivatives Tell Us about Graphs

Sign of \(f'\)	Sign of \(f''\)	Is \(f\) increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

A function is graphed in the first quadrant. It is broken up into four sections, with the breaks coming at the local minimum, inflection point, and local maximum, respectively. The first section is decreasing and concave up; here, f’ < 0 and f’’ / 0. The second section is increasing and concave up; here, f’ > 0 and f’’ > 0. The third section is increasing and concave down; here, f’ > 0 and f’’ < 0. The fourth section is decreasing and concave down; here, f’ < 0 and f’’ < 0." style="width: 937px; height: 475px;" width="937px" height="475px" src="/@api/deki/files/78800/OpenStaxCalcFig4-5-8.jpeg">

Figure \(\PageIndex{8}\): Consider a twice-differentiable function \(f\) over an open interval \(I\). If \(f'(x)>0\) for all \(x∈I\), the function is increasing over \(I\). If \(f'(x)<0\) for all \(x∈I\), the function is decreasing over \(I\). If \(f''(x)>0\) for all \(x∈I\), the function is concave up. If \(f''(x)<0\) for all \(x∈I\), the function is concave down on \(I\).

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extremum, it must occur at a critical point. However, a function need not have a local extremum at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let \(f\) be a twice-differentiable function such that \(f'(a)=0\) and \(f''\) is continuous over an open interval \(I\) containing \(a\). Suppose \(f''(a)<0\). Since \(f''\) is continuous over \(I, f''(x)<0\) for all \(x∈I\) (Figure \(\PageIndex{9}\)). Then, by Corollary \(3\), \(f'\) is a decreasing function over \(I\). Since \(f'(a)=0\), we conclude that for all \(x∈I, \,f'(x)>0\) if \(xa\). Therefore, by the first derivative test, \(f\) has a local maximum at \(x=a\).

On the other hand, suppose there exists a point \(b\) such that \(f'(b)=0\) but \(f''(b)>0\). Since \(f''\) is continuous over an open interval \(I\) containing \(b\), then \(f''(x)>0\) for all \(x∈I\) (Figure \(\PageIndex{9}\)). Then, by Corollary \(3\), \(f'\) is an increasing function over \(I\). Since \(f'(b)=0\), we conclude that for all \(x∈I\), \(f'(x)<0\) if \(x0\) if \(x>b\). Therefore, by the first derivative test, \(f\) has a local minimum at \(x=b.\)

A function f(x) is graphed in the first quadrant with a and b marked on the x-axis. The function is vaguely sinusoidal, increasing first to x = a, then decreasing to x = b, and increasing again. At (a, f(a)), the tangent is marked, and it is noted that f’(a) = 0 and f’’(a) < 0. At (b, f(b)), the tangent is marked, and it is noted f’(b) = 0 and f’’(b) / 0." style="width: 487px; height: 272px;" width="487px" height="272px" src="/@api/deki/files/2436/CNX_Calc_Figure_04_05_006.jpeg">

Figure \(\PageIndex{9}\): Consider a twice-differentiable function \(f\) such that \(f''\) is continuous. Since \(f'(a)=0\) and \(f''(a)<0\), there is an interval \(I\) containing \(a\) such that for all \(x\) in \(I\), \(f\) is increasing if \(xa\). As a result, \(f\) has a local maximum at \(x=a\). Since \(f'(b)=0\) and \(f''(b)>0\), there is an interval \(I\) containing \(b\) such that for all \(x\) in \(I\), \(f\) is decreasing if \(xb\). As a result, \(f\) has a local minimum at \(x=b\).

Theorem \(\PageIndex{2}\): Second Derivative Test

Suppose \(f'(c)=0\) and \(f"\) is continuous over an interval containing \(c\).

i. If \(f"(c)>0\), then \(f\) has a local minimum at \(c\).
ii. If \(f"(c)<0\), then \(f\) has a local maximum at \(c\).
iii. If \(f"(c)=0,\) then the test is inconclusive.

Note that for case iii. when \(f"(c)=0\), then \(f\) may have a local maximum, local minimum, or neither at \(c\). For example, the functions \(f(x)=x^3, \; f(x)=x^4,\) and \(f(x)=–x^4\) all have critical points at \(x=0\). In each case, the second derivative is zero at \(x=0\). However, the function \(f(x)=x^4\) has a local minimum at \(x=0\) whereas the function \(f(x)=–x^4\) has a local maximum at \(x=0\), and the function \(f(x)=x^3\) does not have a local extremum at \(x=0\).

Let's now look at how to use the second derivative test to determine whether \(f\) has a local maximum or local minimum at a critical point \(c\) where \(f'(c)=0.\)

Example \(\PageIndex{4}\): Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for \(f(x)=x^5–5x^3\)

Solution

To apply the second derivative test, we first need to find critical points \(c\) where \(f'(c)=0\). The derivative is \(f'(x)=5x^4–15x^2\).

Therefore, \(f'(x)=5x^4–15x^2=5x^2(x^2–3)=0\) when \(x=0,\,±\sqrt{3}\).

To determine whether \(f\) has a local extremum at any of these points, we need to evaluate the sign of \(f"\) at these points. The second derivative is

$$f"(x)=20x^3–30x=10x(2x^2–3)$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether \(f\) has a local maximum or local minimum at any of these points.

Table: \(\PageIndex{5}\): Second Derivative Test for \(f(x)=x^5–5x^3\)

\(x\)	\(f"(x)\)	Conclusion
\(–\sqrt{3}\)	\(–30\sqrt{3}\)	Local maximum
\(0\)	\(0\)	Second derivative test is inconclusive
\(\sqrt{3}\)	\(30\sqrt{3}\)	Local minimum

By the second derivative test, we conclude that \(\boxed{f(x) \text{ has a local maximum at }x=–\sqrt{3}\text{ and a local minimum at }x=\sqrt{3}}\). The second derivative test is inconclusive at \(x=0\). To determine whether \(f\) has a local extrema at \(x=0,\) we apply the first derivative test. To evaluate the sign of \(f'(x)=5x^2(x^2–3)\) for \(x∈\left(–\sqrt{3},0\right)\) and \(x∈\left(0,\sqrt{3}\right)\), let \(x=–1\) and \(x=1\) be the two test points. Since \(f'(–1)<0\) and \(f'(1)<0\), we conclude that \(f\) is decreasing on both intervals and, therefore, \(\boxed{f(x)\text{ does not have a local extrema at }x=0}\) as shown in the following graph.

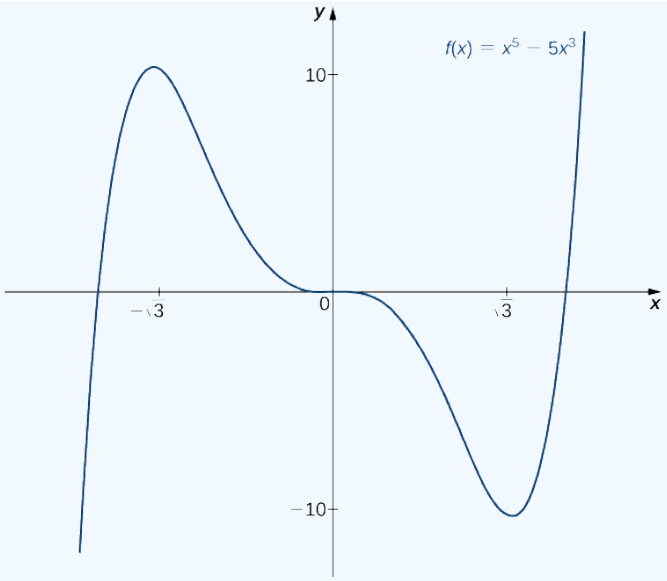


Figure \(\PageIndex{10}\):The function \(f\) has a local maximum at \(x=−\sqrt{3}\) and a local minimum at \(x=\sqrt{3}\)

Exercise \(\PageIndex{4}\)

Consider the function \(f(x)=x^3−\dfrac{3}{2}x^2−18x\). The points \(c=3,\,−2\) satisfy \(f'(c)=0\). Use the second derivative test to determine whether \(f\) has a local maximum or local minimum at those points.

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise \(\PageIndex{1}\)

Find all critical points of \(f\) and determine the signs of \(f'(x)\) over particular intervals determined by the critical points.

Exercise \(\PageIndex{1}\)

\(f\) has a local minimum at \(−2\) and a local maximum at \(3\)

Exercise \(\PageIndex{2}\)

The only critical point of \(f\) is \(x=1.\)

Exercise \(\PageIndex{2}\)

\(f\) has no local extrema because \(f'\) does not change sign at \(x=1\)

Exercise \(\PageIndex{3}\)

Find where \(f''(x)=0\)

Exercise \(\PageIndex{3}\)

\(f\) is concave up over the interval \(\left(−∞,\dfrac{1}{2}\right)\) and concave down over the interval \(\left(\dfrac{1}{2},∞\right)\)

Exercise \(\PageIndex{4}\)

\(f''(x)=6x−3\)

Exercise \(\PageIndex{4}\)

\(f\) has a local maximum at \(−2\) and a local minimum at \(3\)

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3.2: Curve Sketching with Asymptotes

We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$. In this section, we see how limits at infinity affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function f.

End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where n is a positive integer. From Figure 3.2.1 and Figure 3.2.2, we see that

$$\lim_{x\to\infty} x^n = \infty \text{ for all } n = 1, 2, 3, \ldots \quad \text{and} \quad \lim_{x\to-\infty} x^n = \begin{cases} \infty & \text{for } n = 2, 4, 6, \ldots \\ -\infty & \text{for } n = 1, 3, 5, \ldots \end{cases}$$

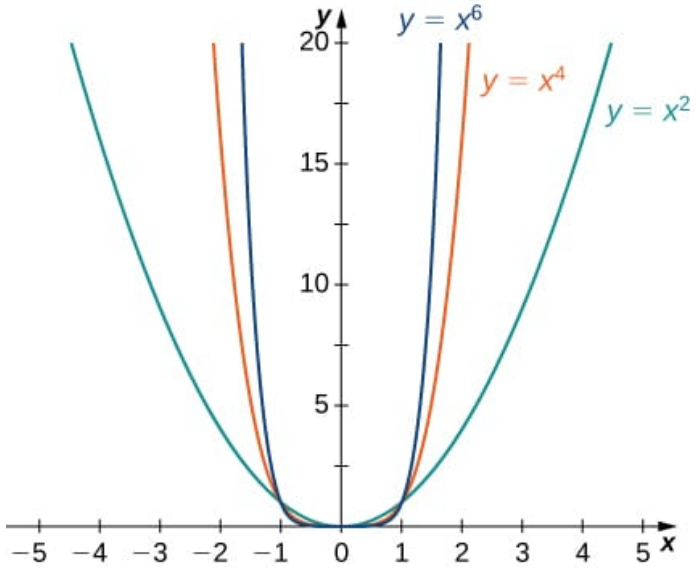


Figure 3.2.1: For power functions with an even power n, $\lim_{x\to\infty} x^n = \infty = \lim_{x\to-\infty} x^n$

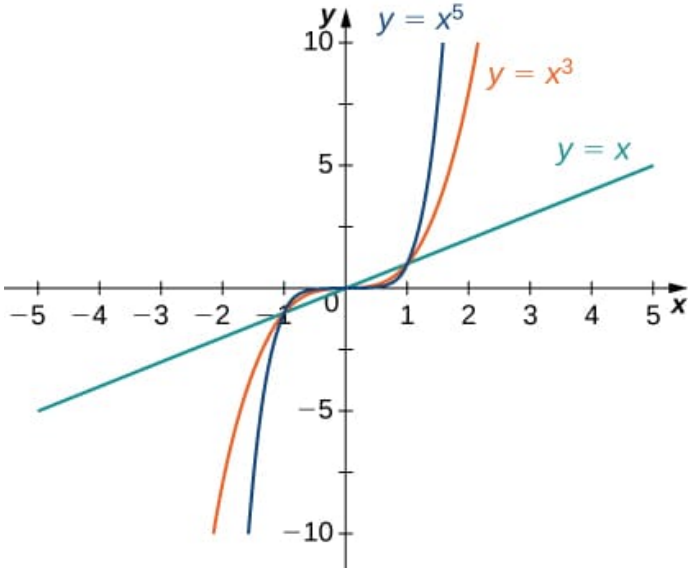


Figure 3.2.2: For power functions with an odd power n, $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to-\infty} x^n = -\infty$

Using these facts, it is not difficult to evaluate $\lim\limits_{x\to\infty} cx^n$ and $\lim\limits_{x\to-\infty} cx^n$, where c is any constant and n is a positive integer. If $c > 0$, the graph of $y = cx^n$is a vertical stretch or compression of $y = x^n$, and therefore

$$\lim_{x\to\infty} cx^n = \lim_{x\to\infty} x^n \text{ and } \lim_{x\to-\infty} cx^n = \lim_{x\to-\infty} x^n \text{ if } c > 0$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the x-axis, and therefore

$$\lim_{x\to\infty} cx^n = -\lim_{x\to\infty} x^n \text{ and } \lim_{x\to-\infty} cx^n = -\lim_{x\to-\infty} x^n \text{ if } c < 0$$

If $c = 0, y = cx^n = 0$, in which case $\lim\limits_{x\to\infty} cx^n = 0 = \lim\limits_{x\to-\infty} cx^n$.

Example 3.2.1: Limits at Infinity for Power Functions

For each function f, evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$

(a) $f(x) = -5x^3$

(b) $f(x) = 2x^4$

Solution

(a) Since the coefficient of x^3 is -5, the graph of $f(x) = -5x^3$ involves a vertical stretch and reflection of the graph of $y = x^3$ about the x-axis.

Therefore, $\boxed{\lim\limits_{x\to\infty}(-5x^3) = -\infty \text{ and } \lim\limits_{x\to-\infty}(-5x^3) = \infty}$

(b) Since the coefficient of x^4 is 2, the graph of $f(x) = 2x^4$ is a vertical stretch of the graph of $y = x^4$.

Therefore, $\boxed{\lim\limits_{x\to\infty} 2x^4 = \infty \text{ and } \lim\limits_{x\to-\infty} 2x^4 = \infty}$

Exercise 3.2.1

Let $f(x) = -3x^4$. Find $\lim_{x\to\infty} f(x)$

Hint available at the end of this section.

Answer available at the end of this section.

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x\to\pm\infty} f(x)$ for any polynomial function f. Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

of degree $n \geq 1$ so that $a_n \neq 0$. Factoring, we see that

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \ldots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right)$$

As $x \to \pm\infty$, all the terms inside the parentheses approach zero except the first term. We conclude that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$ as shown in Figure 3.2.3 and Table 3.2.1.

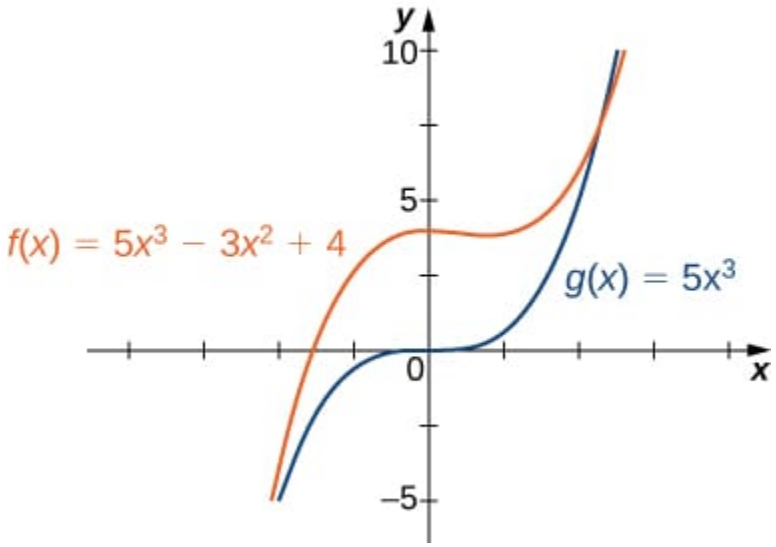


Figure 3.2.3: The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

Table 3.2.1: A polynomial's end behavior is determined by the term with the largest exponent

x	10	100	1000
$f(x) = 5x^3 - 3x^2 + 4$	4704	4970004	4997000004
$g(x) = 5x^3$	5000	5000000	5000000000
x	−10	−100	−1000
$f(x) = 5x^3 - 3x^2 + 4$	−5296	−5029996	−5002999996
$g(x) = 5x^3$	−5000	−5000000	−5000000000

End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In Example 3.2.2, we show that the limits at infinity of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ depend on the relationship between the degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x.

Example 3.2.2: Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \to \infty$ and $x \to -\infty$. Then, use this information to describe the end behavior of the function.

(a) $f(x) = \dfrac{3x-1}{2x+5}$

(b) $f(x) = \dfrac{3x^2+2x}{4x^3-5x+7}$

(c) $f(x) = \dfrac{3x^2+4x}{x+2}$

Solution

(a) The highest power of the numerator and denominator are both x, so they are the same.

Dividing the numerator and denominator by x and applying the algebraic limit laws, we see that:

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{3x-1}{2x+5} &= \lim_{x\to\pm\infty} \frac{3-\frac{1}{x}}{2+\frac{5}{x}} \\ &= \frac{\lim\limits_{x\to\pm\infty} 3-\dfrac{1}{x}}{\lim\limits_{x\to\pm\infty} 2+\dfrac{5}{x}} \\ &= \frac{3-0}{2+0} = \frac{3}{2}\end{aligned}$$

Since $\boxed{\lim_{x\to\pm\infty} f(x) = \frac{3}{2}}$, we know that $\boxed{y = \frac{3}{2} \text{ is a horizontal asymptote}}$ for this function as shown in the following graph.

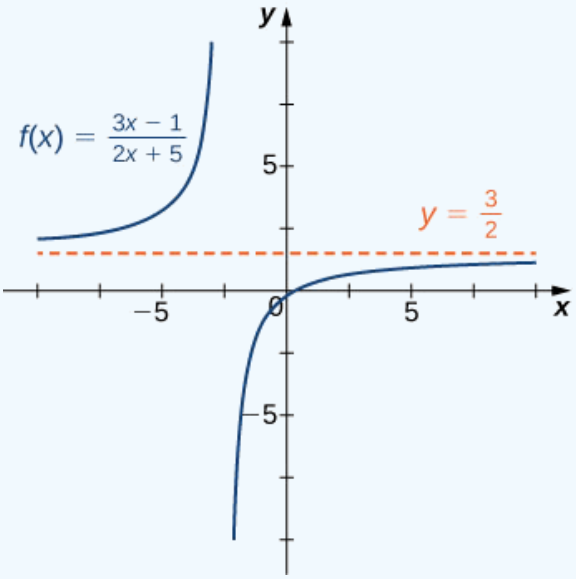


Figure 3.2.4: The graph of this rational function approaches the horizontal asymptote $y = \frac{3}{2}$ as $x \to \pm\infty$

(b) The degree of numerator is less than the degree of the denominator. Since the largest power of x appearing in the denominator is x^3, divide the numerator and denominator by x^3. After doing so and applying algebraic limit laws, we obtain

$$\lim_{x\to\pm\infty} \frac{3x^2+2x}{4x^3-5x+7} = \lim_{x\to\pm\infty} \frac{\frac{3}{x}+\frac{2}{x^2}}{4-\frac{5}{x^2}+\frac{7}{x^3}} = \frac{3\cdot 0+2\cdot 0}{4-5\cdot 0+7\cdot 0} = \frac{0}{4} = 0$$

Since $\boxed{\lim_{x\to\pm\infty} f(x) = 0}$, we know $\boxed{f(x) \text{ has a horizontal asymptote at } y = 0}$ as shown in the following graph.

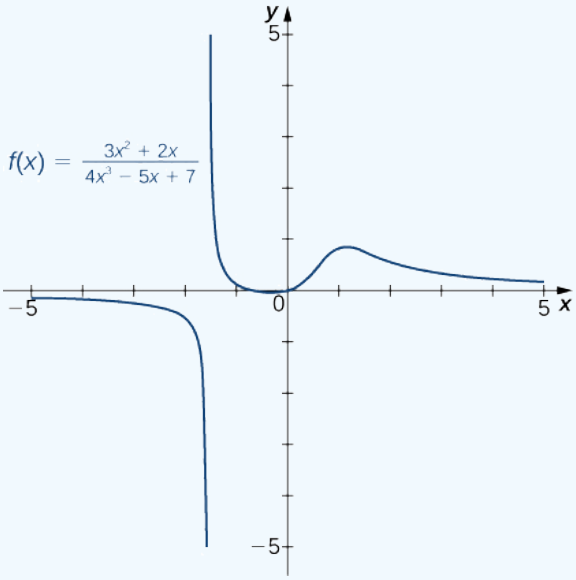


Figure 3.2.5: The graph of this rational function approaches the horizontal asymptote $y = 0$ as $x \to \pm\infty$.

(c) The degree of the numerator is greater than the degree of the denominator. Since the largest power of x appearing in the denominator is x, we divide the numerator and denominator by x:

$$\lim_{x\to\pm\infty} \frac{3x^2+4x}{x+2} = \lim_{x\to\pm\infty} \frac{3x+4}{1+\frac{2}{x}} = \frac{\pm\infty}{1+0}$$

As $x \to \infty$, the numerator approaches $+\infty$, so $\boxed{\lim_{x\to\infty} f(x) = \infty}$

As $x \to -\infty$, the numerator approaches $-\infty$, so $\boxed{\lim_{x\to-\infty} f(x) = -\infty}$

This means the graph has $\boxed{\text{no horizontal asymptotes}}$, as shown in the following figure.

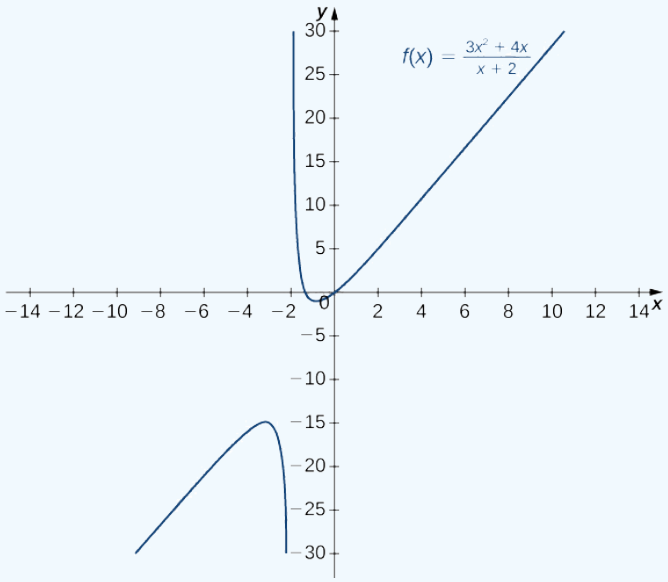


Figure 3.2.6: As $x \to \infty$, the values $f(x) \to \infty$, and as $x \to -\infty$, the values $f(x) \to -\infty$

Exercise 3.2.2

Evaluate $\lim_{x\to\pm\infty} \dfrac{3x^2+2x-1}{5x^2-4x+7}$ and use these limits to determine the end behavior of $f(x) = \dfrac{3x^2+2x-1}{5x^2-4x+7}$

Hint available at the end of this section.

Answer available at the end of this section.

Before proceeding, consider the graph of $f(x) = \dfrac{3x^2+4x}{x+2}$ shown in Figure 3.2.16. As $x \to \infty$ and $x \to -\infty$, the graph of f appears almost linear. Although f is certainly not a linear function, we now investigate why the graph of f seems to be approaching a linear function. First, using long division of polynomials, we can write

$$f(x) = \frac{3x^2+4x}{x+2} = 3x - 2 + \frac{4}{x+2}$$

Since $\dfrac{4}{x+2} \to 0$ as $x \to \pm\infty$, we conclude that

$$\lim_{x\to\pm\infty} (f(x) - (3x-2)) = \lim_{x\to\pm\infty} \frac{4}{x+2} = 0$$

Therefore, the graph of f approaches the line $y = 3x - 2$ as $x \to \pm\infty$. This line is known as an **oblique asymptote** for f (Figure 3.2.7).

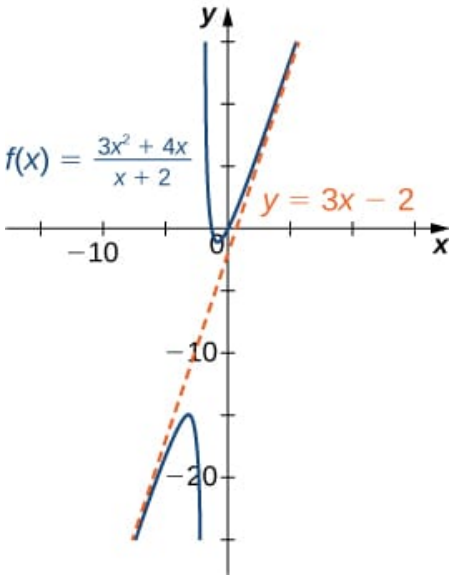


Figure 3.2.7: The graph of the rational function $f(x) = \dfrac{3x^2 + 4x}{x + 2}$ approaches the oblique asymptote $y = 3x - 2$ as $x \to \pm\infty$

We can summarize the results of Example 3.2.2 to make the following conclusion regarding end behavior for rational functions.

Summary of End Behavior for Rational Functions

Consider a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} \quad \text{where} \quad a_n \neq 0 \text{ and } b_m \neq 0$$

1. If the degree of the numerator is the same as the degree of the denominator $(n = m)$, then f has a horizontal asymptote of $y = \dfrac{a_n}{b_m}$ as $x \to \pm\infty$.
2. If the degree of the numerator is less than the degree of the denominator \((n
3. If the degree of the numerator is greater than the degree of the denominator $(n > m)$, then f does not have a horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)}$$

where the degree of $r(x)$ is less than the degree of $q(x)$. As a result, $\lim\limits_{x \to \pm\infty} \dfrac{r(x)}{q(x)} = 0$. Therefore, the values of $[f(x) - g(x)]$ approach zero as $x \to \pm\infty$. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ (i.e., $n = m + 1$), the function $g(x)$ is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

Example 3.2.3: Determining End Behavior for a Function Involving a Radical

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3x - 2}{\sqrt{4x^2 + 5}}$ and describe the end behavior of f.

Solution

Let's use the same strategy as we did for rational functions: divide the numerator and denominator by a power of x. To determine the appropriate power of x, consider the expression $\sqrt{4x^2 + 5}$ in the denominator. Since

$$\sqrt{4x^2+5} \approx \sqrt{4x^2} = 2|x|$$

for large values of x in effect x appears just to the first power in the denominator. Therefore, we divide the numerator and denominator by $|x|$. Then, using the fact that $|x| = x$ for $x > 0$, $|x| = -x$ for $x < 0$, and $|x| = \sqrt{x^2}$ for all x, we calculate the limits as follows:

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to\infty} \frac{\frac{1}{|x|}\cdot(3x-2)}{\frac{1}{|x|}\cdot\sqrt{4x^2+5}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}\cdot(3x-2)}{\sqrt{\frac{1}{x^2}\cdot(4x^2+5)}} \\
&= \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{4+\frac{5}{x^2}}} \\
&= \frac{3}{\sqrt{4}} = \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to-\infty} \frac{\frac{1}{|x|}\cdot(3x-2)}{\frac{1}{|x|}\cdot\sqrt{4x^2+5}} \\
&= \lim_{x\to-\infty} \frac{\frac{1}{-x}\cdot(3x-2)}{\sqrt{\frac{1}{x^2}\cdot(4x^2+5)}} \\
&= \lim_{x\to-\infty} \frac{-3+\frac{2}{x}}{\sqrt{4+\frac{5}{x^2}}} \\
&= \frac{-3}{\sqrt{4}} = -\frac{3}{2}
\end{aligned}$$

Therefore, $f(x)$ approaches the $\boxed{\text{horizontal asymptote } y = \frac{3}{2} \text{ as } x \to \infty}$ and the $\boxed{\text{horizontal asymptote } y = -\frac{3}{2} \text{ as } x \to -\infty}$ as shown in the following graph.

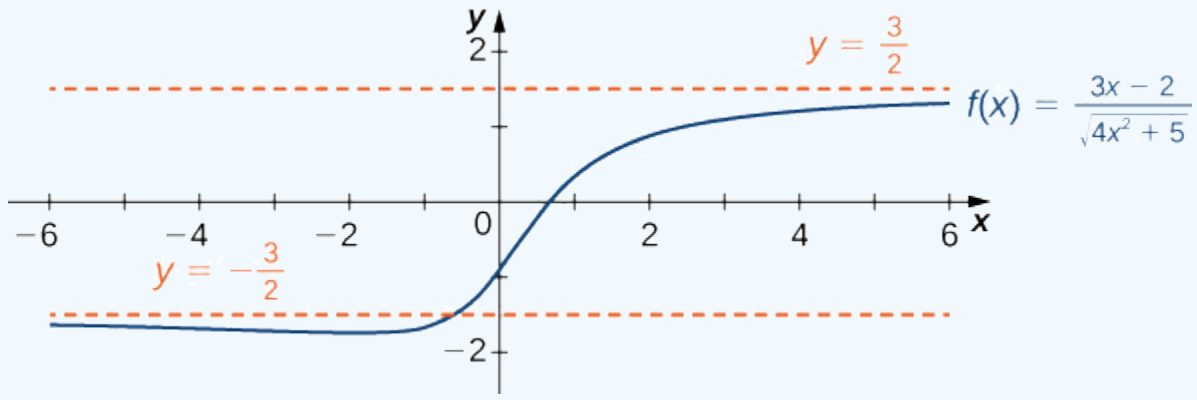


Figure 3.2.8: This function has two horizontal asymptotes and it crosses one of the asymptotes.

Exercise 3.2.3

Evaluate $\lim_{x\to\infty} \frac{\sqrt{3x^2+4}}{x+6}$

Hint available at the end of this section.

Answer available at the end of this section.

Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as $x \to \pm\infty$. For example, $\sin(x)$ oscillates between 1 and –1 (Figure $3.2.9$). The tangent function x has an infinite number of vertical asymptotes as $x \to \pm\infty$; therefore, it does not approach a finite limit nor does it approach $\pm\infty$ as $x \to \pm\infty$ as shown in Figure $3.2.10$.

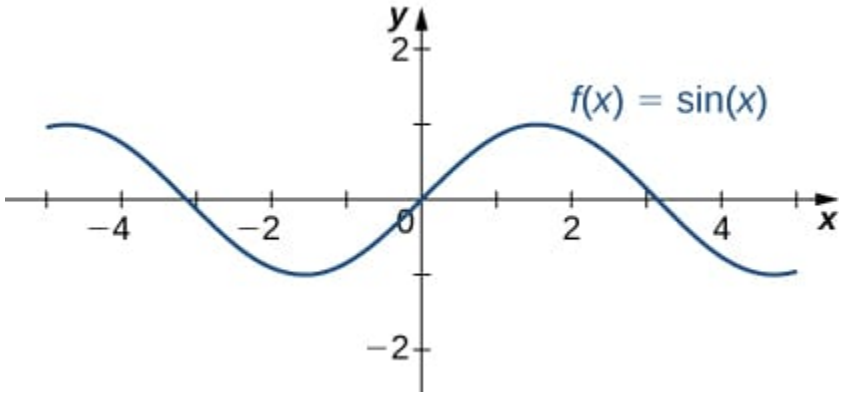


Figure $3.2.9$: The function $f(x) = \sin(x)$ oscillates between 1 and -1 as $x \to \pm\infty$

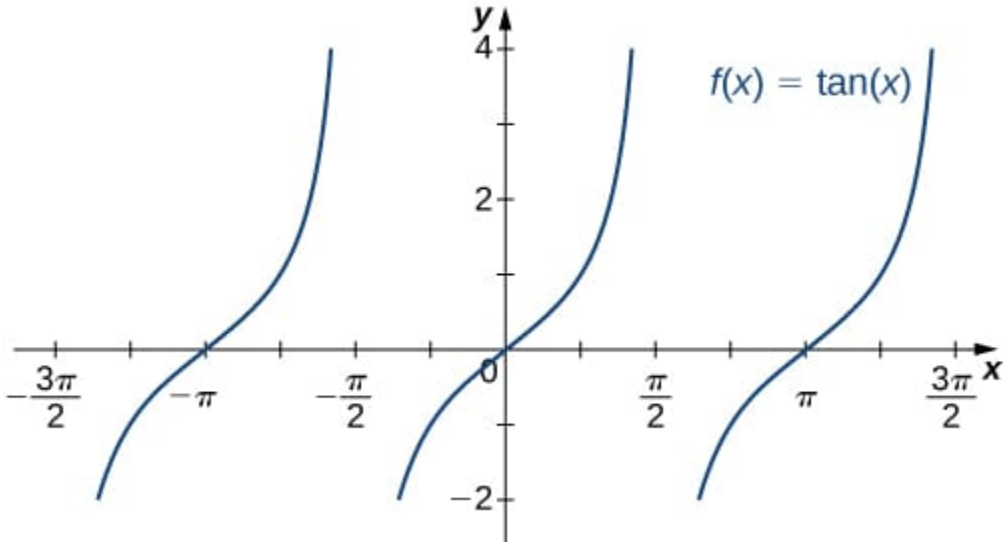


Figure $3.2.10$: The function $f(x) = \tan(x)$ does not approach a limit and does not approach $\pm\infty$ as $x \to \pm\infty$

Recall that for any base $b > 0, \ \ b \neq 1$, the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1, \ \ y = b^x$ is increasing over $(-\infty, \infty)$. If \(01\). Therefore, $f(x) = e^x$ is increasing on $(-\infty, \infty)$ and the range is $(0, \infty)$. The exponential function $f(x) = e^x$ approaches ∞ as $x \to \infty$ and approaches 0 as $x \to -\infty$ as shown in Table $3.2.2$ and Figure $3.2.11$.

Table $3.2.2$: End behavior of the natural exponential function

x	–5	–2	0	2	5
e^x	0.00674	0.135	1	7.389	148.413

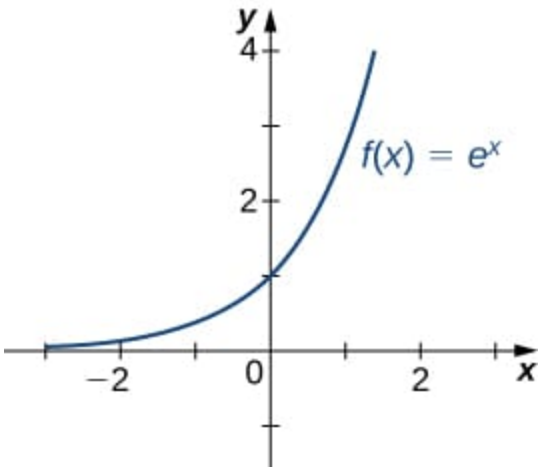


Figure 3.2.11: The exponential function approaches zero as $x \to -\infty$ and approaches ∞ as $x \to \infty$

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \to -\infty$ as $x \to 0^+$ and $\ln(x) \to \infty$ as $x \to \infty$ as shown in Figure 3.2.12 and Table 3.2.3.

Table 3.2.3: End behavior of the natural logarithm function

x	0.01	0.1	1	10	100
$\ln(x)$	−4.605	−2.303	0	2.303	4.605

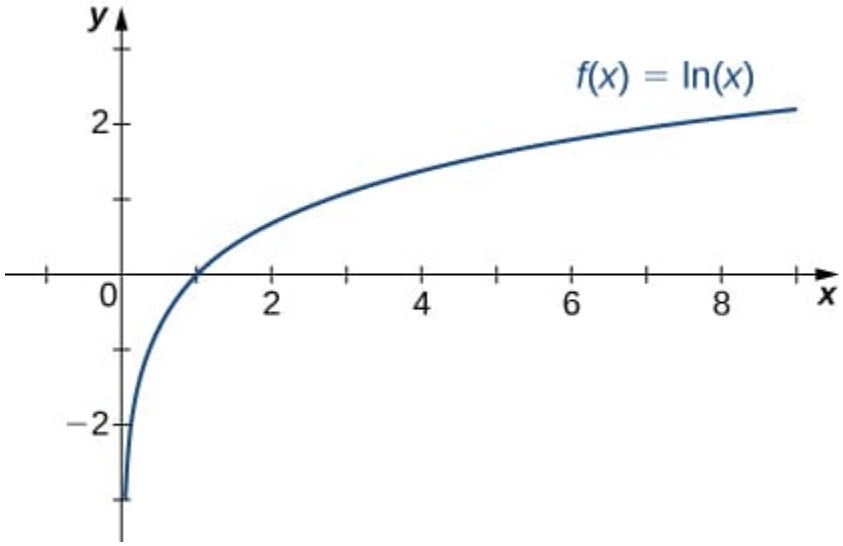


Figure 3.2.12: The natural logarithm function approaches ∞ as $x \to \infty$

Example 3.2.4: Determining End Behavior for a Transcendental Function

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{2 + 3e^x}{7 - 5e^x}$ and describe the end behavior of f.

Solution

To find the limit as $x \to \infty$, divide the numerator and denominator by e^x:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2 + 3e^x}{7 - 5e^x} = \lim_{x \to \infty} \frac{2e^{-x} + 3}{7e^{-x} - 5} = \frac{\lim\limits_{x \to \infty} 2e^{-x} + 3}{\lim\limits_{x \to \infty} 7e^{-x} - 5}$$

As shown in Figure 3.2.11, $e^x \to 0$ as $x \to -\infty$. Equivalently $e^{-x} \to 0$ as \(x\to \infty). Therefore,

$$\lim_{x\to\infty} \frac{2e^{-x}+3}{7e^{-x}-5} = \frac{0+3}{0-5} = -\frac{3}{5}$$

We conclude that $\boxed{\lim_{x\to\infty} f(x) = -\frac{3}{5}}$, and the graph of f approaches the $\boxed{\text{horizontal asymptote } y = -\frac{3}{5} \text{ as } x \to \infty}$

To find the limit as $x \to -\infty$, use the fact that $e^x \to 0$ as $x \to -\infty$ and the original form of $f(x)$, so:

$$\lim_{x\to-\infty} \frac{2+3e^x}{7-5e^x} = \frac{2+0}{7-0} = \frac{2}{7}$$

Thus, $\boxed{\lim_{x\to-\infty} f(x) = \frac{2}{7}}$, and the graph of f approaches the $\boxed{\text{horizontal asymptote } y = \frac{2}{7} \text{ as } x \to -\infty}$

Exercise 3.2.4

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3e^x - 4}{5e^x + 2}$

Hint available at the end of this section.

Answer available at the end of this section.

Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

Problem-Solving Strategy: Drawing the Graph of a Function

Given a function f, use the following steps to sketch a graph of f:

1. Determine the domain of the function.
2. Locate the x- and y-intercepts.
3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number L, then $y = L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x) = \dfrac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as
$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)}$$
where the degree of $r(x)$ is less than the degree of $q(x)$. The values of $f(x)$ approach the values of $g(x)$ as $x \to \pm\infty$. If $g(x)$ is a linear function, it is known as an oblique asymptote.
4. Determine whether f has any vertical asymptotes.
5. Calculate f'. Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
6. Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Illustration 3.2.1: Sketching a Graph of a Polynomial

We demonstrate these steps by sketching a graph of $f(x) = (x-1)^2(x+2)$

Step 1: Since f is a polynomial, the domain is the set of all real numbers.

Step 2: When $x = 0, \quad f(x) = 2$. Therefore, the y-intercept is $(0, 2)$. To find the x-intercepts, we need to solve the equation $(x-1)^2(x+2) = 0$, which gives us the x-intercepts $(1, 0)$ and $(-2, 0)$

Step 3: We need to evaluate the end behavior of f. As $x \to \infty, \quad (x-1)^2 \to \infty$ and $(x+2) \to \infty$. Therefore, $\lim\limits_{x\to\infty} f(x) = \infty$

As $x \to -\infty, \quad (x-1)^2 \to \infty$ and $(x+2) \to -\infty$. Therefore, $\lim\limits_{x\to-\infty} f(x) = -\infty$

To get even more information about the end behavior of f, we can multiply the factors of f. When doing so, we see that

$$f(x) = (x-1)^2(x+2) = x^3 - 3x + 2$$

Since the leading term of f is x^3, we conclude that f behaves like $y = x^3$ as $x \to \pm\infty$.

Step 4: Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5: The first derivative of f is

$$f'(x) = 3x^2 - 3$$

Therefore, f has two critical points: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1), \quad (-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2, x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f\,'(x)$ at each of these test points, as shown in the following table.

Interval	Test point	Sign of Derivative $f'(x)$	Conclusion
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f increasing
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f decreasing
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f increasing

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6: The second derivative of f is

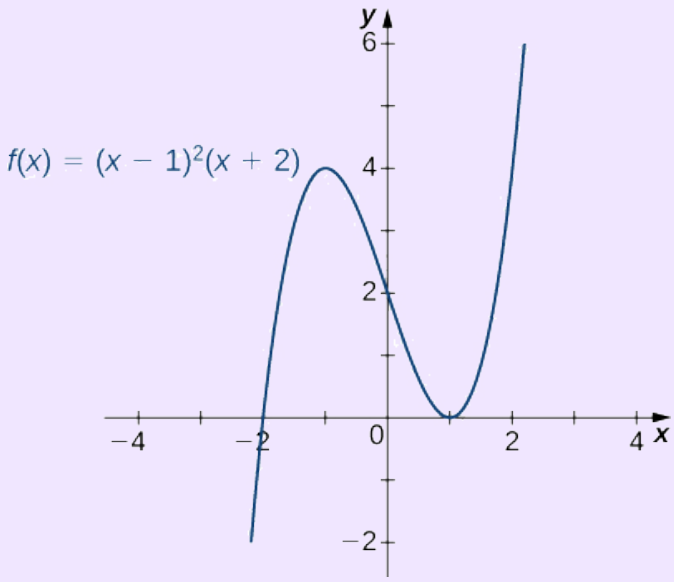
$$f''(x) = 6x$$

The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	Test Point	Sign of $f''(x)$	Conclusion
$(-\infty, 0)$	$x = -1$	$-$	f concave down
$(0, \infty)$	$x = 1$	$+$	f concave up

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f\,'(x) = 0$ at those points—combined with the fact that f'' changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f.

Combining this information, we arrive at the graph of $f(x) = (x-1)^2(x+2)$ shown in the following graph:



Exercise 3.2.5

Sketch a graph of $f(x) = (x-1)^3(x+2)$

Hint available at the end of this section.

Illustration 3.2.2: Sketching a Rational Function

Now we sketch a rational function: $f(x) = \dfrac{x^2}{1-x^2}$

Step 1: The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2: Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\dfrac{x^2}{1-x^2} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since $f$is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2. We obtain

$$\lim_{x\to\pm\infty} \frac{x^2}{1-x^2} = \lim_{x\to\pm\infty} \frac{1}{\frac{1}{x^2}-1} = -1$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \to \infty$ and $x \to -\infty$.

Step 4: To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f, evaluate $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to -1} f(x)$. By looking at each one-sided limit as $x \to 1$, we see that

$$\lim_{x\to 1^+} \frac{x^2}{1-x^2} = -\infty \text{ and } \lim_{x\to 1^-} \frac{x^2}{1-x^2} = \infty$$

In addition, by looking at each one-sided limit as $x \to -1$, we find that

$$\lim_{x\to -1^+}\frac{x^2}{1-x^2}=\infty \text{ and } \lim_{x\to -1^-}\frac{x^2}{1-x^2}=-\infty$$

Step 5: Calculate the first derivative:

$$f\,'(x)=\frac{(1-x^2)(2x)-x^2(-2x)}{(1-x^2)^2}=\frac{2x}{(1-x^2)^2}$$

Critical points occur at points x where $f\,'(x)=0$ or $f\,'(x)$ is undefined. We see that $f\,'(x)=0$ when $x=0$. The derivative $f\,'$ is not undefined at any point in the domain of f. However, $x=\pm 1$ are not in the domain of f. Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty,\infty)$ into four smaller intervals: $(-\infty,-1),(-1,0),(0,1)$, and $(1,\infty)$, and choose a test point in each interval to determine the sign of $f\,'(x)$ in each of these intervals. The values $x=-2$, $x=-\frac{1}{2}$, $x=\frac{1}{2}$, and $x=2$ are good choices for test points as shown in the following table.

Interval	Test point	Sign of $f\,'(x)$	Conclusion
$(-\infty,-1)$	$x=-2$	$\frac{-}{+}=-$	f decreasing
$(-1,0)$	$x=-\frac{1}{2}$	$\frac{-}{+}=-$	f decreasing
$(0,1)$	$x=\frac{1}{2}$	$\frac{+}{+}=+$	f increasing
$(1,\infty)$	$x=2$	$\frac{+}{+}=+$	f increasing

From this analysis, we conclude that f has a local minimum at $x=0$ but no local maximum.

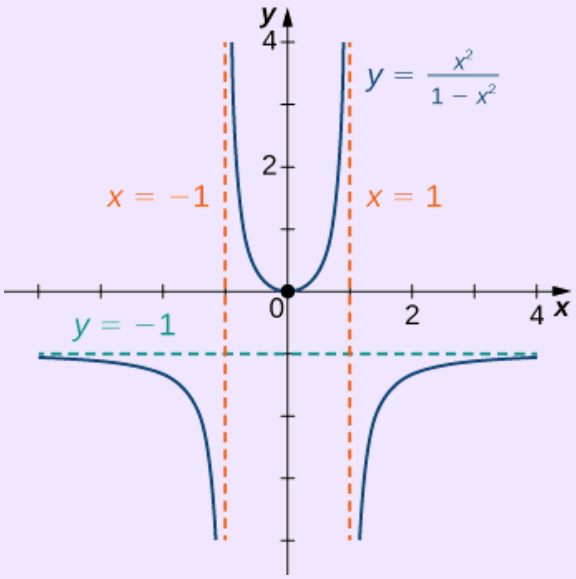
Step 6: Calculate the second derivative:

$$\begin{aligned} f''(x)&=\frac{(1-x^2)^2(2)-2x(2(1-x^2)(-2x))}{(1-x^2)^4}\\ &=\frac{(1-x^2)\left[2\left(1-x^2\right)+8x^2\right]}{(1-x^2)^4}\\ &=\frac{2\left(1-x^2\right)+8x^2}{(1-x^2)^3}\\ &=\frac{6x^2+2}{(1-x^2)^3}\end{aligned}$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x)=0$ or $f''(x)$ is undefined. Since the numerator $6x^2+2\neq 0$ for any x, $f''(x)$ is never zero. Furthermore, f'' is not undefined for any x in the domain of f. However, as discussed earlier, $x=\pm 1$ are not in the domain of f. Therefore, to determine the concavity of f, we divide the interval $(-\infty,\infty)$ into the three smaller intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. The values $x=-2$, $x=0$, and $x=2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x)$	Conclusion
$(-\infty,-1)$	$x=-2$	$\frac{+}{-}=-$	f concave down
$(-1,1)$	$x=0$	$\frac{+}{+}=+$	f concave up
$(1,\infty)$	$x=2$	$\frac{+}{-}=-$	f concave down

Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x=-1$ and $x=1$, there are no inflection points at either of these places because f is not continuous at $x=-1$ or $x=1$.



Exercise 3.2.6

Sketch a graph of $f(x) = \dfrac{3x+5}{8+4x}$

Hint available at the end of this section.

Answer available at the end of this section.

Illustration 3.2.3: Sketching a Rational Function with an Oblique Asymptote

Consider the graph of $f(x) = \dfrac{x^2}{x-1}$

Step 1: The domain of f is the set of all real numbers x except $x = 1$.

Step 2: Find the intercepts. We can see that when $x = 0,\ \ f(x) = 0$, so $(0, 0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

Since $\dfrac{1}{x-1} \to 0$ as $x \to \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \to \pm\infty$. The line $y = x + 1$ is an oblique asymptote for f.

Step 4: To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \to 1$, we find

$$\lim_{x\to 1^+} \frac{x^2}{x-1} = \infty \text{ and } \lim_{x\to 1^-} \frac{x^2}{x-1} = -\infty$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5: Calculate the first derivative:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

We have $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical points. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0), (0, 1), (1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. For example, let $x = -1, x = \frac{1}{2}, x = \frac{3}{2}$, and $x = 3$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x)$	Conclusion
$(-\infty, 0)$	$x = -1$	$\frac{(-)(-)}{+} = +$	f increasing
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(-)}{+} = -$	f decreasing
$(1, 2)$	$x = \frac{3}{2}$	$\frac{(+)(-)}{+} = -$	f decreasing
$(2, \infty)$	$x = 3$	$\frac{(+)(+)}{+} = +$	f increasing

From this table, we see that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. The value of f at the local maximum is $f(0) = 0$ and the value of f at the local minimum is $f(2) = 4$. Therefore, $(0, 0)$ and $(2, 4)$ are important points on the graph.

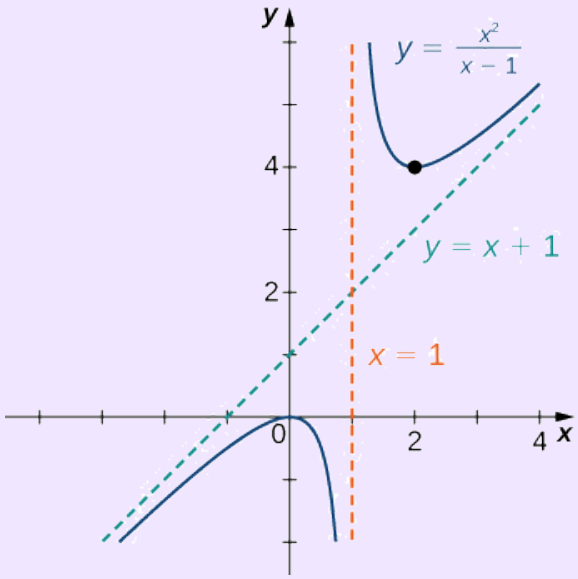
Step 6. Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(x-1)^2(2x-2) - 2(x-1)(x^2-2x)}{(x-1)^4} \\ &= \frac{2(x-1)[(x-1)^2 - (x^2-2x)]}{(x-1)^4} \\ &= \frac{2[x^2 - 2x + 1 - x^2 + 2x]}{(x-1)^3} \\ &= \frac{2}{(x-1)^3} \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x = 1$, to check concavity we just divide the interval $(-\infty, \infty)$ into the two smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x = 0$ and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x)$	Conclusion
$(-\infty, 1)$	$x = 0$	$\frac{+}{-} = -$	f concave down
$(1, \infty)$	$x = 2$	$\frac{+}{+} = +$	f concave up

From the information gathered, we arrive at the following graph for f:



Exercise 3.2.7

Find the oblique asymptote for $f(x) = \dfrac{3x^3 - 2x + 1}{2x^2 - 4}$

Hint available at the end of this section.

Answer available at the end of this section.

Illustration 3.2.4: Sketching the Graph of a Function with a Cusp

Lastly, we sketch a graph of $f(x) = (x-1)^{\frac{2}{3}}$

Step 1: Since the cube-root function is defined for all real numbers x and $(x-1)^{\frac{2}{3}} = \left(\sqrt[3]{x-1}\right)^2$, the domain of f is all real numbers.

Step 2: To find the y-intercept, evaluate $f(0)$. Since $f(0) = 1$, the y-intercept is $(0, 1)$. To find the x-intercept, solve $(x-1)^{\frac{2}{3}} = 0$. The solution of this equation is $x = 1$, so the x-intercept is $(1, 0)$.

Step 3: Since $\lim\limits_{x \to \pm\infty} (x-1)^{\frac{2}{3}} = \infty$, the function continues to grow without bound as $x \to \infty$ and $x \to -\infty$.

Step 4: The function has no vertical asymptotes.

Step 5: To determine where f is increasing or decreasing, calculate $f\,'$. We find

$$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3(x-1)^{\frac{1}{3}}}$$

This function is not zero anywhere, but it is undefined when $x = 1$. Therefore, the only critical point is $x = 1$. Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points in each of these intervals to determine the sign of $f\,'(x)$ in each of these smaller intervals. Let $x = 0$ and $x = 2$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f\,'(x)$	Conclusion
$(-\infty, 1)$	$x = 0$	$\frac{+}{-} = -$	f decreasing

Interval	Test Point	Sign of $f'(x)$	Conclusion
$(1,\infty)$	$x=2$	$\frac{+}{+}=+$	f increasing

We conclude that f has a local minimum at $x=1$. Evaluating f at $x=1$, we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim_{x\to 1} f'(x)$. Looking at the one-sided limits, we have

$$\lim_{x\to 1^+} \frac{2}{3(x-1)^{\frac{1}{3}}} = \infty \text{ and } \lim_{x\to 1^-} \frac{2}{3(x-1)^{\frac{1}{3}}} = -\infty$$

Therefore, f has a cusp at $x=1$.

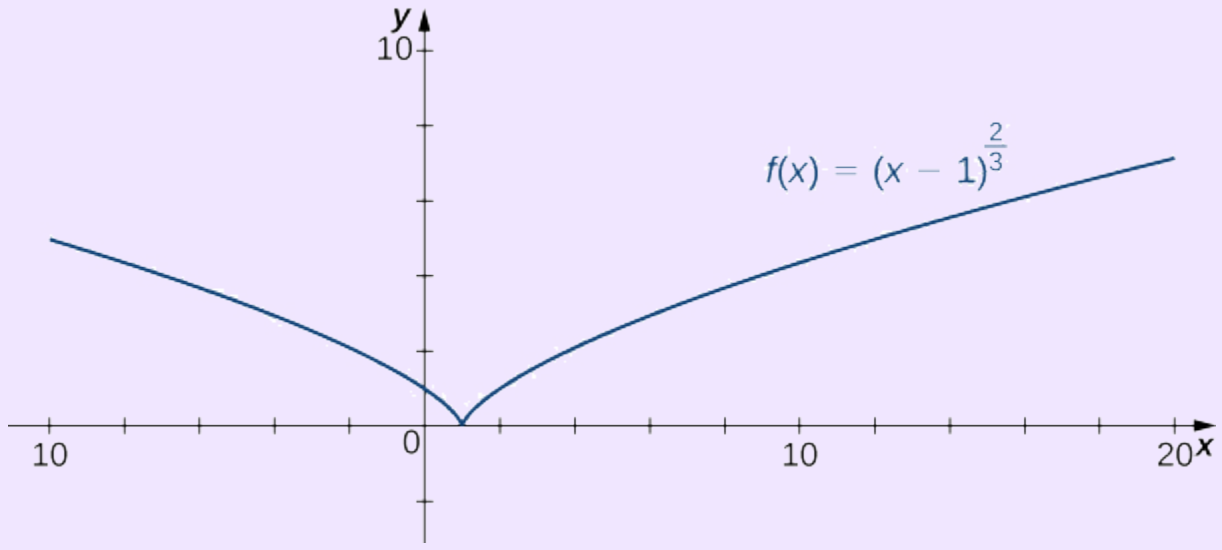
Step 6: To determine concavity, we calculate the second derivative of f :

$$f''(x) = -\frac{2}{9}(x-1)^{-\frac{4}{3}} = \frac{-2}{9(x-1)^{\frac{4}{3}}}$$

We find that $f''(x)$ is defined for all x, but is undefined when $x=1$. Therefore, divide the interval $(-\infty,\infty)$ into the smaller intervals $(-\infty,1)$ and $(1,\infty)$, and choose test points to evaluate the sign of $f''(x)$ in each of these intervals. As we did earlier, let $x=0$ and $x=2$ be test points as shown in the following table.

Interval	Test Point	Sign of $f''(x)$	Conclusion
$(-\infty,1)$	$x=0$	$\frac{-}{+}=-$	f concave down
$(1,\infty)$	$x=2$	$\frac{-}{+}=-$	f concave down

From this table, we conclude that f is concave down everywhere. Combining all of this information, we arrive at the following graph for f:



Exercise 3.2.8

Consider the function $f(x)=5-x^{\frac{2}{3}}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f.

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 3.2.1

The coefficient -3 is negative.

Exercise 3.2.1

$-\infty$

Exercise 3.2.2

Divide the numerator and denominator by x^2.

Exercise 3.2.2

$\frac{3}{5}$

Exercise 3.2.3

Divide the numerator and denominator by x.

Exercise 3.2.3

$\sqrt{3}$

Exercise 3.2.4

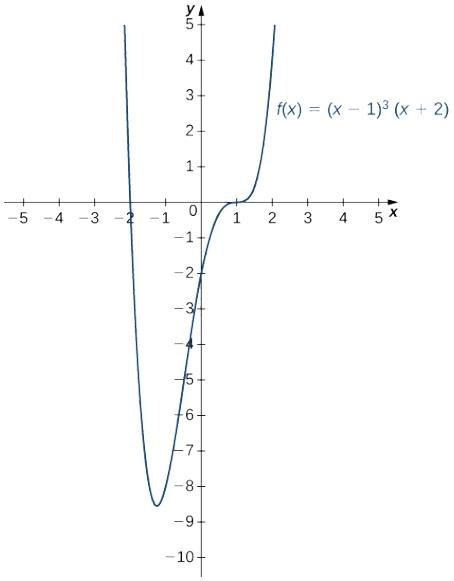
$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

Exercise 3.2.4

$\lim_{x\to\infty} f(x) = \frac{3}{5}, \quad \lim_{x\to-\infty} f(x) = -2$

Exercise 3.2.5

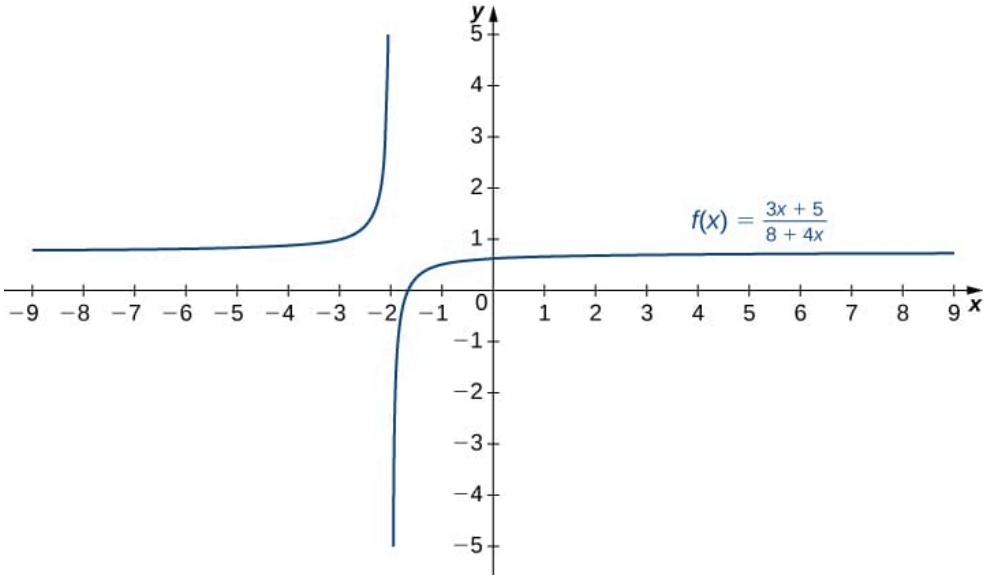
f is a fourth-degree polynomial.



Exercise 3.2.6

A line $y = L$ is a horizontal asymptote of f if the limit as $x \to \infty$ or the limit as $x \to -\infty$ of $f(x)$ is L. A line $x = a$ is a vertical asymptote if at least one of the one-sided limits of f as $x \to a$ is ∞ or $-\infty$.

Exercise 3.2.6



Exercise 3.2.7

Use long division of polynomials.

Exercise 3.2.7

$y = \frac{3}{2}x$

Exercise 3.2.8

A function f has a cusp at a point a if $f(a)$ exists, $f'(a)$ is undefined, one of the one-sided limits as $x \to a$ of $f'(x)$ is $+\infty$, and the other one-sided limit is $-\infty$.

Exercise 3.2.8

The function f has a cusp at $(0, 5)$, since $\lim_{x \to 0^-} f'(x) = \infty$ and $\lim_{x \to 0^+} f'(x) = -\infty$. For end behavior, $\lim_{x \to \pm\infty} f(x) = -\infty$.

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3.2: Practice Set

Exercise 3.2.1

For each function, find the domain, intercepts, critical points, inflection points, intervals of increase/decrease, intervals of concavity, and any asymptotes. Then sketch the curve.

(a) $f(x) = x^3 - 3x^2 - 9x + 5$

(b) $g(x) = x^4 - 8x^2 + 16$

(c) $h(x) = x^3 - 6x^2 + 9x + 2$

(d) $p(x) = x^4 - 4x^3 + 4x^2$

Answer available at the end of this section.

Exercise 3.2.2

For each function, find the domain, intercepts, critical points, inflection points, intervals of increase/decrease, intervals of concavity, and any asymptotes. Then sketch the curve.

(a) $f(x) = \dfrac{x}{x^2 + 1}$

(b) $g(x) = \dfrac{x^2}{x^2 + 1}$

(c) $h(x) = \dfrac{x^2}{x^2 - 1}$

(d) $p(x) = \dfrac{x^2 - 1}{x}$

Answer available at the end of this section.

Exercise 3.2.3

For each function, find the domain, intercepts, critical points, inflection points, intervals of increase/decrease, intervals of concavity, and any asymptotes. Then sketch the curve.

(a) $f(x) = e^{-x^2}$

(b) $g(x) = xe^{-x}$

(c) $h(x) = \ln(x^2 + 1)$

(d) $p(x) = \sin(x)$ on $[0, 2\pi]$

Answer available at the end of this section.

Exercise 3.2.4

For each function, find the domain, intercepts, critical points, inflection points, intervals of increase/decrease, intervals of concavity, and any asymptotes. Then sketch the curve.

(a) $f(x) = x^3 - 9x^2 + 24x - 16$

(b) $g(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

(c) $h(x) = \dfrac{x}{x-2}$

(d) $p(x) = \dfrac{x^2}{x-1}$

Answer available at the end of this section.

Answers

Exercise 3.2.1

(a) $f(x) = x^3 - 3x^2 - 9x + 5$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $f(0) = 5$. x-intercepts: solve $x^3 - 3x^2 - 9x + 5 = 0$. $x = -1$ is a root: $(-1)^3 - 3(1) - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10 \neq 0$. Try $x = 5$: $125 - 75 - 45 + 5 = 10$. Try $x = 1$: $1 - 3 - 9 + 5 = -6$. The roots are not nice; we leave as approximate or factored form.

Critical points: $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$. Critical numbers: $x = -1, 3$.

$f(-1) = -1 - 3 + 9 + 5 = 10$. $f(3) = 27 - 27 - 27 + 5 = -22$.

Intervals of increase/decrease:

For $x < -1$, $f'(x) > 0$ (increasing).

For $-1 < x < 3$, $f'(x) < 0$ (decreasing).

For $x > 3$, $f'(x) > 0$ (increasing).

Local maximum at $(-1, 10)$. Local minimum at $(3, -22)$.

Inflection points: $f''(x) = 6x - 6 = 6(x-1)$. Inflection point at $x = 1$.

$f(1) = 1 - 3 - 9 + 5 = -6$.

Intervals of concavity:

For $x < 1$, $f''(x) < 0$ (concave down).

For $x > 1$, $f''(x) > 0$ (concave up).

Asymptotes: None.

(b) $g(x) = x^4 - 8x^2 + 16 = (x^2 - 4)^2 = (x-2)^2(x+2)^2$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $g(0) = 16$. x-intercepts: $x = \pm 2$.

Critical points: $g'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$. Critical numbers: $x = -2, 0, 2$.

$g(-2) = 0$, $g(0) = 16$, $g(2) = 0$.

Intervals of increase/decrease:

For $x < -2$, $g'(x) < 0$ (decreasing).

For $-2 < x < 0$, $g'(x) > 0$ (increasing).

For $0 < x < 2$, $g'(x) < 0$ (decreasing).

For $x > 2$, $g'(x) > 0$ (increasing).

Local minima at $(-2, 0)$ and $(2, 0)$. Local maximum at $(0, 16)$.

Inflection points: $g''(x) = 12x^2 - 16 = 4(3x^2 - 4)$. Inflection points at $x = \pm\dfrac{2}{\sqrt{3}}$.

$g\left(\pm\dfrac{2}{\sqrt{3}}\right) = \left(\dfrac{16}{9}\right)^2 - 8\left(\dfrac{4}{3}\right) + 16 = \dfrac{256}{81} - \dfrac{32}{3} + 16 = \dfrac{256 - 864 + 1296}{81} = \dfrac{688}{81}$.

Intervals of concavity:

For $x < -\dfrac{2}{\sqrt{3}}$, $g''(x) > 0$ (concave up).

For $-\dfrac{2}{\sqrt{3}} < x < \dfrac{2}{\sqrt{3}}$, $g''(x) < 0$ (concave down).

For $x > \dfrac{2}{\sqrt{3}}$, $g''(x) > 0$ (concave up).

Asymptotes: None.

(c) $h(x) = x^3 - 6x^2 + 9x + 2$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $h(0) = 2$. x-intercepts: solve $x^3 - 6x^2 + 9x + 2 = 0$. Try $x = 2$: $8 - 24 + 18 + 2 = 4$. Try $x = -1$: $-1 - 6 - 9 + 2 = -14$. Try $x = 2 + \sqrt{3} \approx 3.732$: approximate root. (Roots are not nice.)

Critical points: $h'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$. Critical numbers: $x = 1, 3$.

$h(1) = 1 - 6 + 9 + 2 = 6$. $h(3) = 27 - 54 + 27 + 2 = 2$.

Intervals of increase/decrease:

For $x < 1$, $h'(x) > 0$ (increasing).

For $1 < x < 3$, $h'(x) < 0$ (decreasing).

For $x > 3$, $h'(x) > 0$ (increasing).

Local maximum at $(1, 6)$. Local minimum at $(3, 2)$.

Inflection points: $h''(x) = 6x - 12 = 6(x - 2)$. Inflection point at $x = 2$.

$h(2) = 8 - 24 + 18 + 2 = 4$.

Intervals of concavity:

For $x < 2$, $h''(x) < 0$ (concave down).

For $x > 2$, $h''(x) > 0$ (concave up).

Asymptotes: None.

(d) $p(x) = x^4 - 4x^3 + 4x^2 = x^2(x^2 - 4x + 4) = x^2(x - 2)^2$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $p(0) = 0$. x-intercepts: $x = 0, 2$.

Critical points: $p'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x - 1)(x - 2)$. Critical numbers: $x = 0, 1, 2$.

$p(0) = 0$, $p(1) = 1 - 4 + 4 = 1$, $p(2) = 16 - 32 + 16 = 0$.

Intervals of increase/decrease:

For $x < 0$, $p'(x) < 0$ (decreasing).

For $0 < x < 1$, $p'(x) > 0$ (increasing).

For $1 < x < 2$, $p'(x) < 0$ (decreasing).

For $x > 2$, $p'(x) > 0$ (increasing).

Local minimum at $(0, 0)$. Local maximum at $(1, 1)$. Local minimum at $(2, 0)$.

Inflection points: $p''(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2)$.

Solve $3x^2 - 6x + 2 = 0$: $x = \dfrac{6 \pm \sqrt{36 - 24}}{6} = \dfrac{6 \pm \sqrt{12}}{6} = \dfrac{6 \pm 2\sqrt{3}}{6} = 1 \pm \dfrac{\sqrt{3}}{3}$.

Inflection points at $x = 1 \pm \dfrac{\sqrt{3}}{3}$.

Intervals of concavity:

For $x < 1 - \dfrac{\sqrt{3}}{3}$, $p''(x) > 0$ (concave up).

For $1 - \dfrac{\sqrt{3}}{3} < x < 1 + \dfrac{\sqrt{3}}{3}$, $p''(x) < 0$ (concave down).

For $x > 1 + \dfrac{\sqrt{3}}{3}$, $p''(x) > 0$ (concave up).

Asymptotes: None.

Exercise 3.2.2

(a) $f(x) = \dfrac{x}{x^2 + 1}$

Domain: $(-\infty, \infty)$ (since $x^2 + 1 > 0$ for all x).

Intercepts: y-intercept: $f(0) = 0$. x-intercept: $x = 0$.

Critical points: $f'(x) = \dfrac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \dfrac{1 - x^2}{(x^2 + 1)^2}$. Critical numbers: $x = \pm 1$.

$f(1) = \dfrac{1}{2}$. $f(-1) = -\dfrac{1}{2}$.

Intervals of increase/decrease:

For $x < -1$, $f'(x) < 0$ (decreasing).

For $-1 < x < 1$, $f'(x) > 0$ (increasing).

For $x > 1$, $f'(x) < 0$ (decreasing).

Local minimum at $\left(-1, -\dfrac{1}{2}\right)$. Local maximum at $\left(1, \dfrac{1}{2}\right)$.

Inflection points: $f''(x) = \dfrac{2x(x^2 - 3)}{(x^2 + 1)^3}$. Inflection points at $x = 0, \pm\sqrt{3}$.

$f(0) = 0$. $f(\sqrt{3}) = \dfrac{\sqrt{3}}{4}$. $f(-\sqrt{3}) = -\dfrac{\sqrt{3}}{4}$.

Intervals of concavity:

For $x < -\sqrt{3}$, $f''(x) < 0$ (concave down).

For $-\sqrt{3} < x < 0$, $f''(x) > 0$ (concave up).

For $0 < x < \sqrt{3}$, $f''(x) < 0$ (concave down).

For $x > \sqrt{3}$, $f''(x) > 0$ (concave up).

Asymptotes: Horizontal asymptote: $y = 0$ as $x \to \pm\infty$. No vertical asymptotes.

(b) $g(x) = \dfrac{x^2}{x^2 + 1}$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $g(0) = 0$. x-intercept: $x = 0$.

Critical points: $g'(x) = \dfrac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \dfrac{2x}{(x^2+1)^2}$. Critical number: $x = 0$.

$g(0) = 0$.

Intervals of increase/decrease:

For $x < 0$, $g'(x) < 0$ (decreasing).

For $x > 0$, $g'(x) > 0$ (increasing).

Local minimum at $(0, 0)$.

Inflection points: $g''(x) = \dfrac{2(1-3x^2)}{(x^2+1)^3}$. Inflection points at $x = \pm\dfrac{1}{\sqrt{3}}$.

$g\left(\pm\dfrac{1}{\sqrt{3}}\right) = \dfrac{1/3}{1/3+1} = \dfrac{1/3}{4/3} = \dfrac{1}{4}$.

Intervals of concavity:

For $x < -\dfrac{1}{\sqrt{3}}$, $g''(x) < 0$ (concave down).

For $-\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$, $g''(x) > 0$ (concave up).

For $x > \dfrac{1}{\sqrt{3}}$, $g''(x) < 0$ (concave down).

Asymptotes: Horizontal asymptote: $y = 1$ as $x \to \pm\infty$. No vertical asymptotes.

(c) $h(x) = \dfrac{x^2}{x^2-1}$

Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Intercepts: y-intercept: $h(0) = 0$. x-intercept: $x = 0$.

Critical points: $h'(x) = \dfrac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$. Critical number: $x = 0$.

$h(0) = 0$.

Intervals of increase/decrease:

For $x < -1$, $h'(x) > 0$ (increasing).

For $-1 < x < 0$, $h'(x) > 0$ (increasing).

For $0 < x < 1$, $h'(x) < 0$ (decreasing).

For $x > 1$, $h'(x) < 0$ (decreasing).

Local maximum at $(0, 0)$.

Inflection points: $h''(x) = \dfrac{2(3x^2+1)}{(x^2-1)^3}$. No inflection points (the numerator is always positive, so concavity only changes at asymptotes).

Intervals of concavity:

For $x < -1$, $h''(x) < 0$ (concave down).

For $-1 < x < 1$, $h''(x) > 0$ (concave up).

For $x > 1$, $h''(x) < 0$ (concave down).

Asymptotes: Vertical asymptotes: $x = \pm 1$. Horizontal asymptote: $y = 1$ as $x \to \pm\infty$.

(d) $p(x) = \dfrac{x^2-1}{x} = x - \dfrac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Intercepts: y-intercept: none (undefined at $x = 0$). x-intercepts: $x = \pm 1$.

Critical points: $p'(x) = 1 + \dfrac{1}{x^2} = \dfrac{x^2+1}{x^2}$. No critical numbers (derivative is never zero).

Intervals of increase/decrease: $p'(x) > 0$ for all $x \neq 0$. The function is increasing on $(-\infty, 0)$ and $(0, \infty)$. No local extrema.

Inflection points: $p''(x) = -\dfrac{2}{x^3}$. No inflection points (concavity changes at the asymptote).

Intervals of concavity:

For $x < 0$, $p''(x) > 0$ (concave up).

For $x > 0$, $p''(x) < 0$ (concave down).

Asymptotes: Vertical asymptote: $x = 0$. Oblique asymptote: $y = x$ (since $p(x) = x - \dfrac{1}{x}$).

Exercise 3.2.3

(a) $f(x) = e^{-x^2}$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $f(0) = 1$. x-intercepts: none (always positive).

Critical points: $f'(x) = -2xe^{-x^2}$. Critical number: $x = 0$.

$f(0) = 1$.

Intervals of increase/decrease:

For $x < 0$, $f'(x) > 0$ (increasing).

For $x > 0$, $f'(x) < 0$ (decreasing).

Local maximum at $(0, 1)$.

Inflection points: $f''(x) = e^{-x^2}(4x^2 - 2) = 2e^{-x^2}(2x^2 - 1)$. Inflection points at $x = \pm\frac{1}{\sqrt{2}}$.

$f\left(\pm\frac{1}{\sqrt{2}}\right) = e^{-1/2} = \frac{1}{\sqrt{e}}$.

Intervals of concavity:

For $x < -\frac{1}{\sqrt{2}}$, $f''(x) > 0$ (concave up).

For $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, $f''(x) < 0$ (concave down).

For $x > \frac{1}{\sqrt{2}}$, $f''(x) > 0$ (concave up).

Asymptotes: Horizontal asymptote: $y = 0$ as $x \to \pm\infty$.

(b) $g(x) = xe^{-x}$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $g(0) = 0$. x-intercept: $x = 0$.

Critical points: $g'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$. Critical number: $x = 1$.

$g(1) = e^{-1} = \frac{1}{e}$.

Intervals of increase/decrease:

For $x < 1$, $g'(x) > 0$ (increasing).

For $x > 1$, $g'(x) < 0$ (decreasing).

Local maximum at $\left(1, \frac{1}{e}\right)$.

Inflection points: $g''(x) = e^{-x}(x - 2)$. Inflection point at $x = 2$.

$g(2) = 2e^{-2} = \frac{2}{e^2}$.

Intervals of concavity:

For $x < 2$, $g''(x) < 0$ (concave down).

For $x > 2$, $g''(x) > 0$ (concave up).

Asymptotes: Horizontal asymptote: $y = 0$ as $x \to \infty$. As $x \to -\infty$, $g(x) \to -\infty$.

(c) $h(x) = \ln(x^2 + 1)$

Domain: $(-\infty, \infty)$ (since $x^2 + 1 > 0$ for all x).

Intercepts: y-intercept: $h(0) = 0$. x-intercept: $x = 0$.

Critical points: $h'(x) = \frac{2x}{x^2 + 1}$. Critical number: $x = 0$.

$h(0) = 0$.

Intervals of increase/decrease:

For $x < 0$, $h'(x) < 0$ (decreasing).

For $x > 0$, $h'(x) > 0$ (increasing).

Local minimum at $(0, 0)$.

Inflection points: $h''(x) = \frac{2(1 - x^2)}{(x^2 + 1)^2}$. Inflection points at $x = \pm 1$.

$h(1) = h(-1) = \ln(2)$.

Intervals of concavity:

For $x < -1$, $h''(x) < 0$ (concave down).

For $-1 < x < 1$, $h''(x) > 0$ (concave up).

For $x > 1$, $h''(x) < 0$ (concave down).

Asymptotes: None (but grows like $2\ln(x)$ as $x \to \infty$).

(d) $p(x) = \sin(x)$ on $[0, 2\pi]$

Domain: $[0, 2\pi]$

Intercepts: y-intercept: $p(0) = 0$. x-intercepts: $x = 0, \pi, 2\pi$.

Critical points: $p'(x) = \cos(x)$. Critical numbers: $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

$p\left(\frac{\pi}{2}\right) = 1$. $p\left(\frac{3\pi}{2}\right) = -1$.

Intervals of increase/decrease:

For $0 < x < \frac{\pi}{2}$, $p'(x) > 0$ (increasing).

For $\frac{\pi}{2} < x < \frac{3\pi}{2}$, $p'(x) < 0$ (decreasing).

For $\frac{3\pi}{2} < x < 2\pi$, $p'(x) > 0$ (increasing).

Local maximum at $\left(\frac{\pi}{2}, 1\right)$. Local minimum at $\left(\frac{3\pi}{2}, -1\right)$.

Inflection points: $p''(x) = -\sin(x)$. Inflection points at $x = 0, \pi, 2\pi$ (endpoints and middle).

On the interval, inflection point at $(\pi, 0)$.

Intervals of concavity:

For $0 < x < \pi$, $p''(x) < 0$ (concave down).

For $\pi < x < 2\pi$, $p''(x) > 0$ (concave up).

Asymptotes: None.

Exercise 3.2.4

(a) $f(x) = x^3 - 9x^2 + 24x - 16$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $f(0) = -16$. x-intercepts: solve $x^3 - 9x^2 + 24x - 16 = 0$. Try $x = 1$: $1 - 9 + 24 - 16 = 0$. So $(x-1)(x^2 - 8x + 16) = (x-1)(x-4)^2 = 0$. Roots: $x = 1, 4$ (double).

Critical points: $f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$. Critical numbers: $x = 2, 4$.

$f(2) = 8 - 36 + 48 - 16 = 4$. $f(4) = 64 - 144 + 96 - 16 = 0$.

Intervals of increase/decrease:

For $x < 2$, $f'(x) > 0$ (increasing).

For $2 < x < 4$, $f'(x) < 0$ (decreasing).

For $x > 4$, $f'(x) > 0$ (increasing).

Local maximum at $(2, 4)$. Local minimum at $(4, 0)$.

Inflection points: $f''(x) = 6x - 18 = 6(x-3)$. Inflection point at $x = 3$.

$f(3) = 27 - 81 + 72 - 16 = 2$.

Intervals of concavity:

For $x < 3$, $f''(x) < 0$ (concave down).

For $x > 3$, $f''(x) > 0$ (concave up).

Asymptotes: None.

(b) $g(x) = x^4 - 4x^3 + 6x^2 - 4x + 1 = (x-1)^4$

Domain: $(-\infty, \infty)$

Intercepts: y-intercept: $g(0) = 1$. x-intercept: $x = 1$.

Critical points: $g'(x) = 4(x-1)^3$. Critical number: $x = 1$.

$g(1) = 0$.

Intervals of increase/decrease:

For $x < 1$, $g'(x) < 0$ (decreasing).

For $x > 1$, $g'(x) > 0$ (increasing).

Local minimum at $(1, 0)$.

Inflection points: $g''(x) = 12(x-1)^2$. No inflection points ($g''(x) = 0$ at $x = 1$ but does not change sign).

Intervals of concavity:

For all $x \neq 1$, $g''(x) > 0$ (concave up).

Asymptotes: None.

(c) $h(x) = \dfrac{x}{x-2}$

Domain: $(-\infty, 2) \cup (2, \infty)$

Intercepts: y-intercept: $h(0) = 0$. x-intercept: $x = 0$.

Critical points: $h'(x) = \dfrac{(1)(x-2) - x(1)}{(x-2)^2} = \dfrac{-2}{(x-2)^2}$. No critical numbers (derivative never zero).

Intervals of increase/decrease: $h'(x) < 0$ for all $x \neq 2$. Function is decreasing on $(-\infty, 2)$ and $(2, \infty)$. No local extrema.

Inflection points: $h''(x) = \dfrac{4}{(x-2)^3}$. No inflection points.

Intervals of concavity:

For $x < 2$, $h''(x) < 0$ (concave down).

For $x > 2$, $h''(x) > 0$ (concave up).

Asymptotes: Vertical asymptote: $x = 2$. Horizontal asymptote: $y = 1$ as $x \to \pm\infty$.

(d) $p(x) = \dfrac{x^2}{x-1}$

Domain: $(-\infty, 1) \cup (1, \infty)$

Intercepts: y-intercept: $p(0) = 0$. x-intercept: $x = 0$.

Critical points: $p'(x) = \dfrac{2x(x-1) - x^2(1)}{(x-1)^2} = \dfrac{2x^2 - 2x - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$. Critical numbers: $x = 0, 2$.

$p(0) = 0$. $p(2) = \dfrac{4}{1} = 4$.

Intervals of increase/decrease:

For $x < 0$, $p'(x) > 0$ (increasing).

For $0 < x < 1$, $p'(x) < 0$ (decreasing).

For $1 < x < 2$, $p'(x) < 0$ (decreasing).

For $x > 2$, $p'(x) > 0$ (increasing).

Local maximum at $(0, 0)$. Local minimum at $(2, 4)$.

Inflection points: $p''(x) = \dfrac{2}{(x-1)^3}$. No inflection points.

Intervals of concavity:

For $x < 1$, $p''(x) < 0$ (concave down).

For $x > 1$, $p''(x) > 0$ (concave up).

Asymptotes: Vertical asymptote: $x = 1$. Oblique asymptote: $y = x + 1$ (since $\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$).

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3.3: Optimization

Learning Objectives

- Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Question of the Day

How do we use derivatives to find the best possible outcome in real-world situations?

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3.3: Optimization

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 3.3.1, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example 3.3.1: Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 3.3.1). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

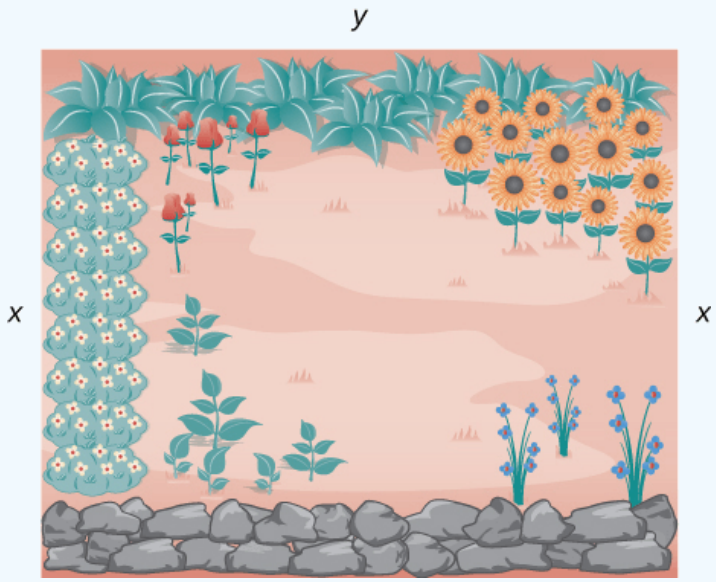


Figure 3.3.1: We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From Figure 3.3.1, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden.

Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x$$

Therefore, the only critical point is $x = 25$ (Figure 3.3.2). We conclude that the maximum area must occur when $x = 25$.

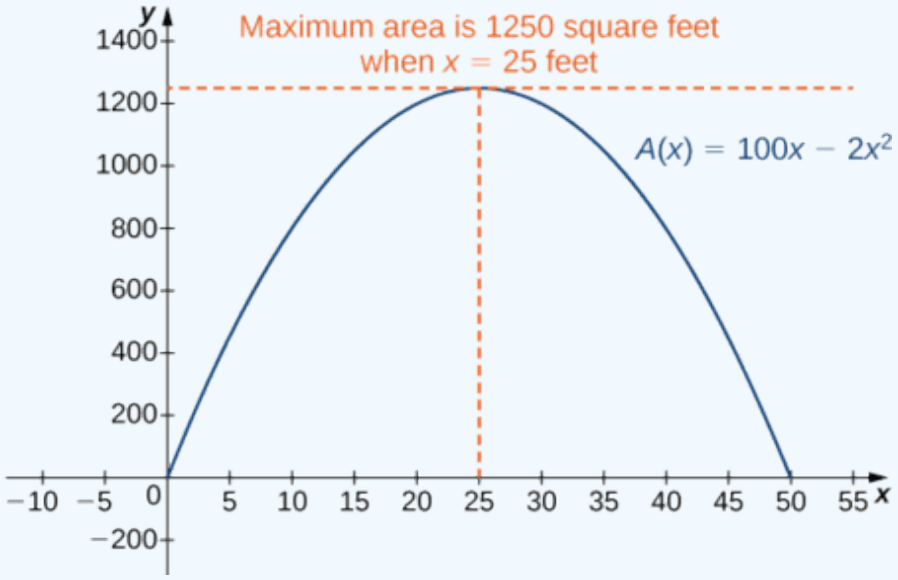


Figure 3.3.2: To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$

Then we have $y = 100 - 2x = 100 - 2(25) = 50$.

To maximize the area of the garden, let $\boxed{x = 25 \text{ ft and } y = 50 \text{ ft}}$. The area of this garden is $\boxed{1250 \text{ ft}^2}$

Exercise 3.3.1

Determine the maximum area if we want to make the same rectangular garden as in Figure 3.3.2, but we have 200 ft of fencing.

Hint available at the end of this section.

Answer available at the end of this section.

Now let's look at a general strategy for solving optimization problems similar to Example 3.3.1.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example $3.3.2$: Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (Figure $3.3.3$). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

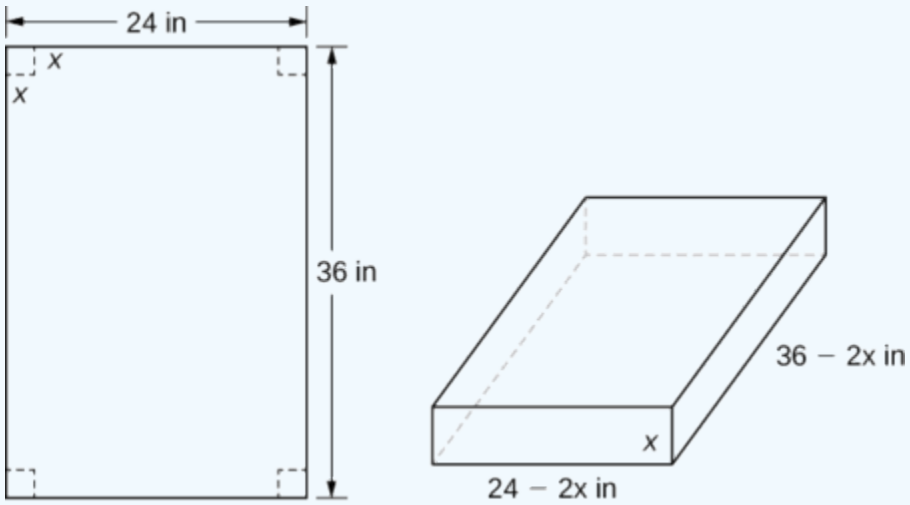


Figure $3.3.3$: A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is

$$V = L \cdot W \cdot H$$

where $L,\ W$,and H are the length, width, and height, respectively.

Step 4: From Figure $3.3.3$, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x$$

Step 5: To determine the domain of consideration, let's examine Figure 3.3.3. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for \(0

$$V\,'(x) = 12x^2 - 240x + 864$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0$$

Using the quadratic formula, we find that the critical points are

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} = \frac{20 \pm \sqrt{112}}{2} = \frac{20 \pm 4\sqrt{7}}{2} = 10 \pm 2\sqrt{7}$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$.

Therefore, $\boxed{\text{the volume is maximized if we let } x = 10 - 2\sqrt{7} \text{ in.}}$

The maximum volume is $V\left(10 - 2\sqrt{7}\right) = 640 + 448\sqrt{7} \approx 1825 \text{ in}^3$ as shown in the following graph.

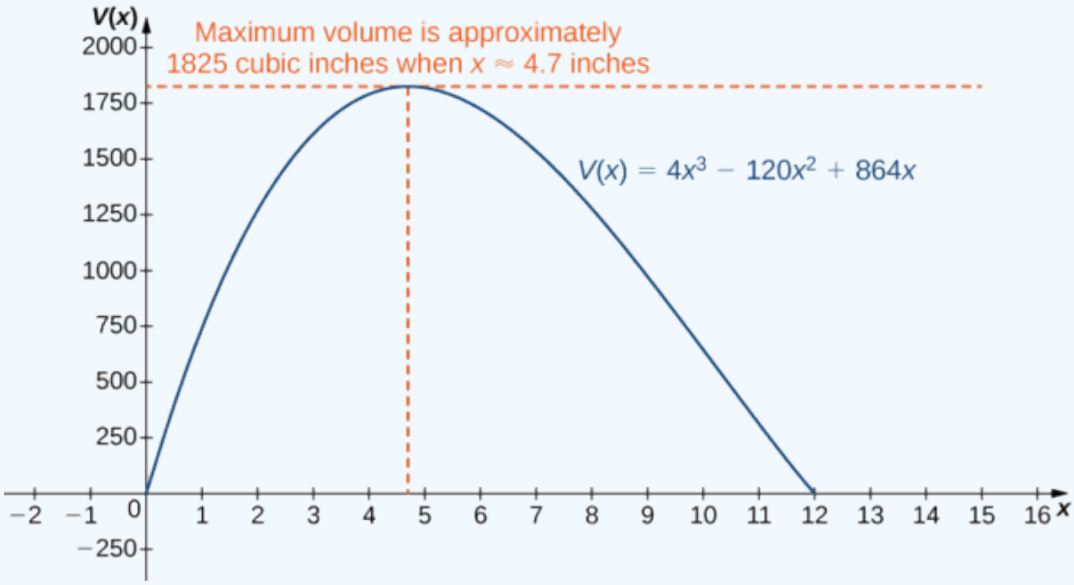


Figure 3.3.4: Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Exercise 3.3.2

Suppose the dimensions of the cardboard in Example 3.3.2 are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.3.3: Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (Figure $3.3.5$). Let T be the time it takes to get from the cabin to the island.

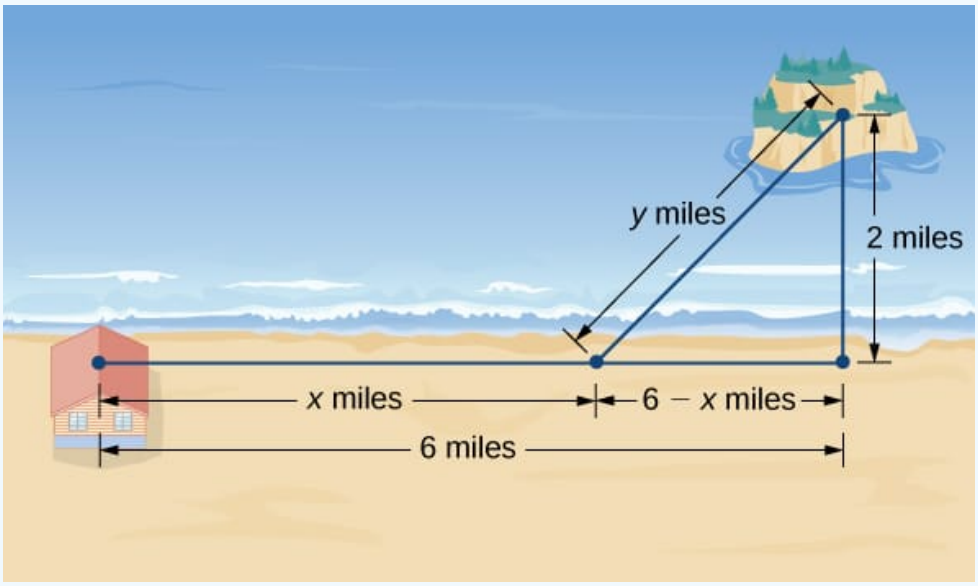


Figure $3.3.5$: How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time $(D = R \times T)$, the time spent running is

$$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8}$$

and the time spent swimming is

$$T_{swimming} = \frac{D_{swimming}}{R_{swimming}} = \frac{y}{3}$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}$$

Step 4: From Figure $3.3.5$, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem, $2^2 + (6 - x)^2 = y^2$, and we obtain $y = \sqrt{(6 - x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6-x)^2+4}}{3}$$

Step 5: From Figure 3.3.5, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$\begin{aligned} T'(x) &= \frac{1}{8} - \frac{1}{2}\frac{[(6-x)^2+4]^{-1/2}}{3} \cdot 2(6-x) \\ &= \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2+4}} \end{aligned}$$

If $T\,'(x) = 0$,, then

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2+4}}$$

Therefore,

$$3\sqrt{(6-x)^2+4} = 8(6-x)$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9[(6-x)^2+4] = 64(6-x)^2$$

which implies

$$55(6-x)^2 = 36$$

We conclude that if x is a critical point, then x satisfies

$$(x-6)^2 = \frac{36}{55}$$

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}$$

Since $x = 6 + \frac{6}{\sqrt{55}}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - \frac{6}{\sqrt{55}}$ is in the domain. Since we squared both sides of an equation to arrive at the possible critical points, it remains to verify that $x = 6 - \frac{6}{\sqrt{55}}$ is not an extraneous solution.

Since $x = 6 - \frac{6}{\sqrt{55}}$ does satisfy that equation, we conclude that $x = 6 - \frac{6}{\sqrt{55}}$ is a critical point, and it is the only one.

To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - \frac{6}{\sqrt{55}}$.

We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas

$$T\left(6 - \frac{6}{\sqrt{55}}\right) \approx 1.368\,\text{h}$$

Therefore, we conclude that T has a local minimum if the distance ran is $\boxed{x \approx 5.19 \text{ mi.}}$

Exercise 3.3.3

Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Hint available at the end of this section.

Answer available at the end of this section.

In business, companies are interested in **maximizing revenue.** In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

Example 3.3.4: Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge $\$50$ per day or less, they will rent all their cars. If they charge $\$200$ per day or more, they will not rent any cars. Assuming the owners plan to charge customers between $\$50$ per day and $\$200$ per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \cdot p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$R(p) = n \cdot p = (1000 - 5p)p = -5p^2 + 1000p$$

Step 5: Since the owners plan to charge between $\$50$ per car per day and $\$200$ per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$.

Therefore, the critical point is $p = 100$. When $p = 100, R(100) = \$50,000$. When $p = 50, R(p) = \$37,500$. When $p = 200, R(p) = \$0$.

Therefore, the absolute maximum occurs at $p = \$100$. $\boxed{\text{The car rental company should charge } \$100 \text{ per day per car}}$ to maximize revenue as shown in the following figure.

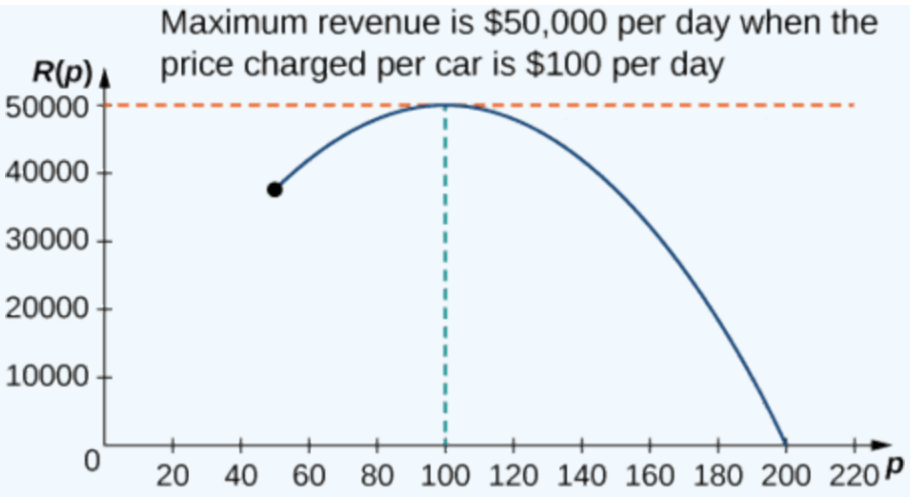


Figure 3.3.6: To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

Exercise 3.3.4

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.3.5: Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1$$

What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

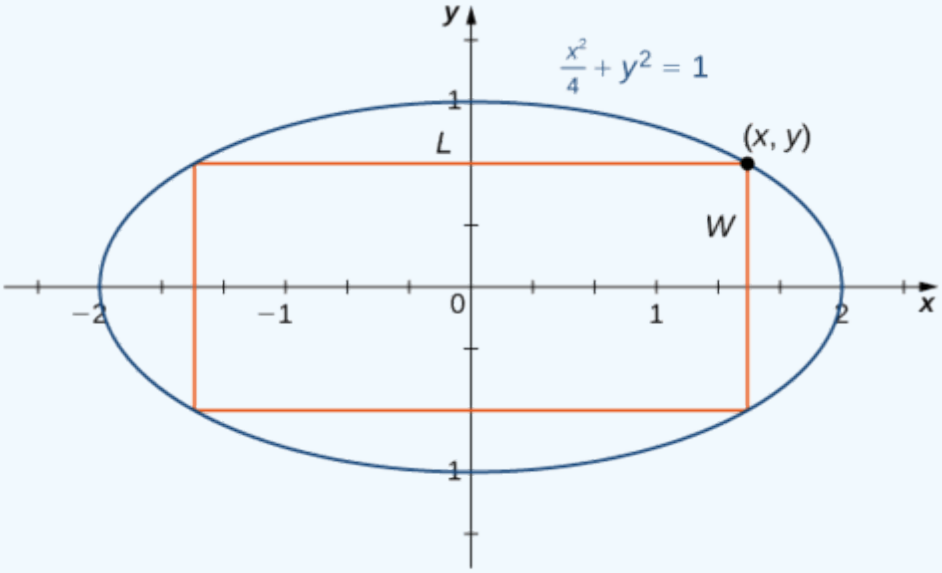


Figure 3.3.7: We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in Figure 3.3.7. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

Step 5: From Figure 3.3.7, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy \(0

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For \(00\).

Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4 - x^2} + 2x \cdot \frac{1}{2\sqrt{4 - x^2}}(-2x) \\ &= 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} \\ &= \frac{8 - 4x^2}{\sqrt{4 - x^2}} \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

$$\frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0$$

then x must satisfy

$$8 - 4x^2 = 0$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are two possible critical points. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not.

Therefore, we check whether $\sqrt{2}$ is a solution of the original equation. Since $x = \sqrt{2}$ is a legitimate solution, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$.

Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

Plugging these into our original function gives $A = LW = (2\sqrt{2})(\sqrt{2}) = 4$. Thus, the $\boxed{\text{maximum area is } 4}$

Exercise 3.3.5

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Hint available at the end of this section.

Answer available at the end of this section.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example 3.3.6: Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of $216\ \text{in}^3$ is to be constructed. What is the minimum surface area of the box?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

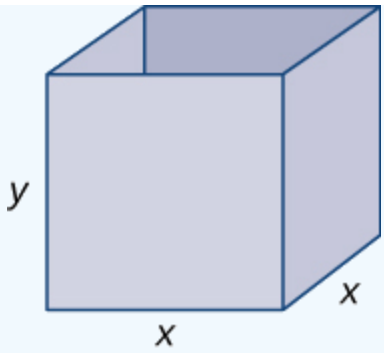


Figure 3.3.8: We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2$$

Step 4: Since the volume of this box is x^2y and the volume is given as 216 in^3, the constraint equation is

$$x^2y = 216$$

Solving the constraint equation for y, we have $y = \dfrac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2$$

Therefore, $S(x) = \dfrac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+,\ S(x) \to \infty$. Also, as $x \to \infty,\ S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x$$

Therefore, $S'(x) = 0$ when $2x = \dfrac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see Figure 3.3.9).

When $x = 6\sqrt[3]{2}$, $y = \dfrac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

$$S\left(6\sqrt[3]{2}\right) = \frac{864}{6\sqrt[3]{2}} + (6\sqrt[3]{2})^2 = \boxed{108\sqrt[3]{4} \text{ in}^2}$$

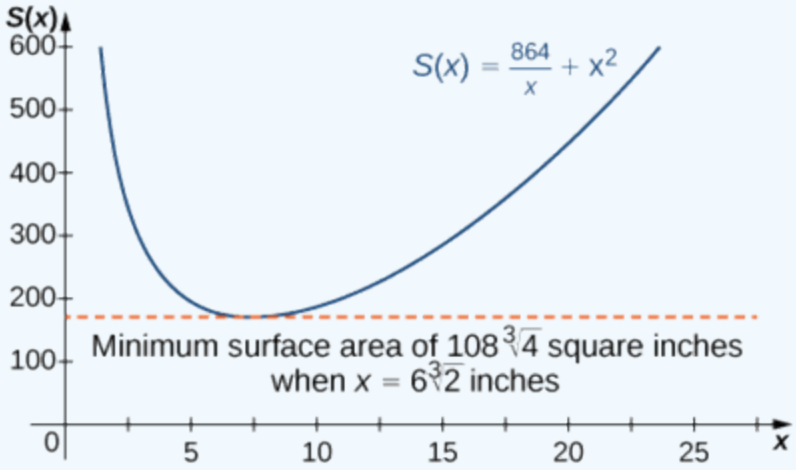


Figure 3.3.9: We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

Exercise 3.3.6

Consider the same open-top box, which is to have volume $216\ \text{in}^3$. Suppose the cost of the material for the base is $20¢/\text{in}^2$ and the cost of the material for the sides is $30¢/\text{in}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 3.3.1

We need to maximize the function $A(x) = 200x - 2x^2$ over the interval $[0, 100]$.

Exercise 3.3.1

The maximum area is $5000\ \text{ft}^2$

Exercise 3.3.2

The volume of the box is $L \cdot W \cdot H$.

Exercise 3.3.2

$V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

Exercise 3.3.3

The time $T = T_{running} + T_{swimming}$.

Exercise 3.3.3

$$T(x) = \frac{x}{6} + \frac{\sqrt{(15-x)^2 + 1}}{2.5}$$

Exercise 3.3.4

$R(p) = n \cdot p$, where n is the number of cars rented and p is the price charged per car.

Exercise 3.3.4

The company should charge $\$75$ per car per day.

Exercise 3.3.5

If (x, y) is the vertex of the rectangle that lies in the first quadrant, then the area of the square is $A = (2x)(2y) = 4xy$.

Exercise 3.3.5

$A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

Exercise 3.3.6

If the cost of one of the sides is $30¢/\text{in}^2$, the cost of that side is $0.30xy$ dollars.

Exercise 3.3.6

$c(x) = \frac{259.2}{x} + 0.2x^2$ dollars

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3.3: Practice Set

Exercise 3.3.1

Find two positive numbers whose sum is 20 and whose product is maximized.

Answer available at the end of this section.

Exercise 3.3.2

Find two positive numbers whose product is 100 and whose sum is minimized.

Answer available at the end of this section.

Exercise 3.3.3

A farmer has 120 meters of fencing to enclose a rectangular field. One side of the field is along a river and does not need fencing. What dimensions maximize the enclosed area?

Answer available at the end of this section.

Exercise 3.3.4

A rectangular box with a square base and an open top is to be constructed from 108 square inches of material. What dimensions maximize the volume?

Answer available at the end of this section.

Exercise 3.3.5

A cylindrical can is to hold 1000π cubic centimeters of liquid. What dimensions minimize the amount of material used (surface area)?

Answer available at the end of this section.

Exercise 3.3.6

Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(4, 0)$.

Answer available at the end of this section.

Exercise 3.3.7

A rectangular garden is to be built against a wall. The garden has a length of x feet and a width of y feet. If 120 feet of fencing are available for the other three sides, find the dimensions that maximize the area of the garden.

Answer available at the end of this section.

Exercise 3.3.8

A rectangular box with a square base and no top is to be made from 150 square inches of material. Find the dimensions that maximize the volume.

Answer available at the end of this section.

Exercise 3.3.9

A closed cylindrical can is to have a volume of 16π cubic inches. What dimensions minimize the surface area?

Answer available at the end of this section.

Exercise 3.3.10

Find the dimensions of the rectangle with the largest area that can be inscribed in a semicircle of radius 3 (with the base on the diameter).

Answer available at the end of this section.

Exercise 3.3.11

A rectangular sheet of cardboard 12 inches by 16 inches is used to make an open box by cutting equal squares from each corner and folding up the sides. What size squares should be cut to maximize the volume?

Answer available at the end of this section.

Exercise 3.3.12

A rectangle is inscribed in a circle of radius 5. Find the dimensions of the rectangle with maximum area.

Answer available at the end of this section.

Exercise 3.3.13

A company wants to construct a cylindrical container with a volume of 1000π cubic centimeters. The material for the top and bottom costs 2 cents per square centimeter, while the material for the side costs 1 cent per square centimeter. What dimensions minimize the cost?

Answer available at the end of this section.

Exercise 3.3.14

Find two positive numbers whose sum is 15 and the sum of their squares is minimized.

Answer available at the end of this section.

Exercise 3.3.15

A rectangular pen is to be built with one side against a barn. The other three sides require fencing. If 60 meters of fencing are available, what dimensions maximize the area of the pen?

Answer available at the end of this section.

Answers

Exercise 3.3.1

Let the two numbers be x and y. We are given $x + y = 20$, so $y = 20 - x$.

The product is $P = xy = x(20 - x) = 20x - x^2$.

$P'(x) = 20 - 2x$. Set $P'(x) = 0$: $20 - 2x = 0 \implies x = 10$.

Then $y = 20 - 10 = 10$.

Thus the two numbers are 10 and 10.

Exercise 3.3.2

Let the two numbers be x and y. We are given $xy = 100$, so $y = \dfrac{100}{x}$.

The sum is $S = x + y = x + \dfrac{100}{x}$.

$S'(x) = 1 - \dfrac{100}{x^2}$. Set $S'(x) = 0$: $1 - \dfrac{100}{x^2} = 0 \implies x^2 = 100 \implies x = 10$.

Then $y = \dfrac{100}{10} = 10$.

Thus the two numbers are 10 and 10.

Exercise 3.3.3

Let x be the length of the side parallel to the river and y be the length of each of the other two sides.

The fencing used is $x + 2y = 120$, so $x = 120 - 2y$.

The area is $A = xy = (120 - 2y)y = 120y - 2y^2$.

$A'(y) = 120 - 4y$. Set $A'(y) = 0$: $120 - 4y = 0 \implies y = 30$.

Then $x = 120 - 2(30) = 60$.

Thus the dimensions are 60 m by 30 m.

Exercise 3.3.4

Let x be the side length of the square base and y be the height of the box.

Surface area (open top): $x^2 + 4xy = 108$, so $y = \dfrac{108 - x^2}{4x}$.

The volume is $V = x^2 y = x^2 \cdot \dfrac{108 - x^2}{4x} = \dfrac{108x - x^3}{4} = 27x - \dfrac{1}{4}x^3$.

$V'(x) = 27 - \dfrac{3}{4}x^2$. Set $V'(x) = 0$: $27 - \dfrac{3}{4}x^2 = 0 \implies x^2 = 36 \implies x = 6$.

Then $y = \dfrac{108 - 36}{4(6)} = \dfrac{72}{24} = 3$.

Thus the dimensions are $6 \times 6 \times 3$ inches.

Exercise 3.3.5

Let r be the radius and h be the height of the cylinder.

Volume: $V = \pi r^2 h = 1000\pi$, so $h = \dfrac{1000}{r^2}$.

Surface area (including top and bottom): $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \dfrac{1000}{r^2} = 2\pi r^2 + \dfrac{2000\pi}{r}$.

$S'(r) = 4\pi r - \dfrac{2000\pi}{r^2}$. Set $S'(r) = 0$: $4\pi r - \dfrac{2000\pi}{r^2} = 0 \implies 4r^3 = 2000 \implies r^3 = 500 \implies r = \sqrt[3]{500} = 5\sqrt[3]{4}$.

Then $h = \dfrac{1000}{r^2} = \dfrac{1000}{(5\sqrt[3]{4})^2} = \dfrac{1000}{25\sqrt[3]{16}} = \dfrac{40}{\sqrt[3]{16}} = 10\sqrt[3]{4}$.

Thus $r = 5\sqrt[3]{4}$ cm and $h = 10\sqrt[3]{4}$ cm.

Exercise 3.3.6

Let $(x, \sqrt{x})$ be a point on the curve. The distance squared to $(4, 0)$ is:

$$D^2 = (x - 4)^2 + (\sqrt{x} - 0)^2 = (x - 4)^2 + x$$

$$D^2 = x^2 - 8x + 16 + x = x^2 - 7x + 16$$

Minimize $f(x) = x^2 - 7x + 16$. $f'(x) = 2x - 7$. Set $f'(x) = 0$: $2x - 7 = 0 \implies x = \dfrac{7}{2}$.

Then $y = \sqrt{\dfrac{7}{2}}$.

Thus the closest point is $\left(\dfrac{7}{2}, \sqrt{\dfrac{7}{2}}\right)$.

Exercise 3.3.7

Let x be the length parallel to the wall and y be the width.

Fencing: $x + 2y = 120$, so $x = 120 - 2y$.

Area: $A = xy = (120 - 2y)y = 120y - 2y^2$.

$A'(y) = 120 - 4y$. Set $A'(y) = 0$: $120 - 4y = 0 \implies y = 30$.

Then $x = 120 - 2(30) = 60$.

Thus the dimensions are 60 ft by 30 ft.

Exercise 3.3.8

Let x be the side length of the square base and y be the height.

Surface area: $x^2 + 4xy = 150$, so $y = \dfrac{150 - x^2}{4x}$.

Volume: $V = x^2 y = x^2 \cdot \dfrac{150 - x^2}{4x} = \dfrac{150x - x^3}{4} = 37.5x - \dfrac{1}{4}x^3$.

$V'(x) = 37.5 - \dfrac{3}{4}x^2$. Set $V'(x) = 0$: $37.5 - \dfrac{3}{4}x^2 = 0 \implies x^2 = 50 \implies x = 5\sqrt{2}$.

Then $y = \dfrac{150 - 50}{4(5\sqrt{2})} = \dfrac{100}{20\sqrt{2}} = \dfrac{5}{\sqrt{2}} = \dfrac{5\sqrt{2}}{2}$.

Thus the dimensions are $5\sqrt{2} \times 5\sqrt{2} \times \dfrac{5\sqrt{2}}{2}$.

Exercise 3.3.9

Let r be the radius and h be the height.

Volume: $V = \pi r^2 h = 16\pi$, so $h = \dfrac{16}{r^2}$.

Surface area: $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \cdot \dfrac{16}{r^2} = 2\pi r^2 + \dfrac{32\pi}{r}$.

$S'(r) = 4\pi r - \dfrac{32\pi}{r^2}$. Set $S'(r) = 0$: $4\pi r - \dfrac{32\pi}{r^2} = 0 \implies 4r^3 = 32 \implies r^3 = 8 \implies r = 2$.

Then $h = \dfrac{16}{4} = 4$.

Thus the dimensions are radius 2 inches and height 4 inches.

Exercise 3.3.10

Place the semicircle with center at the origin. The equation is $x^2 + y^2 = 9$ with $y \geq 0$.

Let the rectangle have vertices $(\pm x, 0)$ and $(\pm x, y)$. Then $x^2 + y^2 = 9$, so $y = \sqrt{9 - x^2}$.

Area: $A = 2xy = 2x\sqrt{9 - x^2}$.

$A'(x) = 2\sqrt{9 - x^2} - \dfrac{2x^2}{\sqrt{9 - x^2}} = \dfrac{18 - 4x^2}{\sqrt{9 - x^2}}$.

Set $A'(x) = 0$: $18 - 4x^2 = 0 \implies x^2 = \dfrac{9}{2} \implies x = \dfrac{3}{\sqrt{2}}$.

Then $y = \sqrt{9 - \dfrac{9}{2}} = \sqrt{\dfrac{9}{2}} = \dfrac{3}{\sqrt{2}}$.

Thus the rectangle has width $2x = 3\sqrt{2}$ and height $\dfrac{3}{\sqrt{2}}$.

Exercise 3.3.11

Let x be the side length of each square cut from the corners.

The resulting box has dimensions: length $16 - 2x$, width $12 - 2x$, height x.

Volume: $V = x(16 - 2x)(12 - 2x) = x(192 - 56x + 4x^2) = 192x - 56x^2 + 4x^3$.

$V'(x) = 192 - 112x + 12x^2 = 4(3x^2 - 28x + 48)$.

Set $V'(x) = 0$: $3x^2 - 28x + 48 = 0$.

$$x = \frac{28 \pm \sqrt{784 - 576}}{6} = \frac{28 \pm \sqrt{208}}{6} = \frac{28 \pm 4\sqrt{13}}{6} = \frac{14 \pm 2\sqrt{13}}{3}$$

Approximately: $x \approx \dfrac{14 - 7.21}{3} \approx 2.26$ or $x \approx \dfrac{14 + 7.21}{3} \approx 7.07$ (invalid since $12 - 2(7.07) < 0$).

Thus $x = \dfrac{14 - 2\sqrt{13}}{3}$ inches.

Exercise 3.3.12

Let the rectangle have dimensions $2x$ by $2y$ with center at the origin. Then $x^2 + y^2 = 25$.

Area: $A = 4xy = 4x\sqrt{25 - x^2}$.

$A'(x) = 4\sqrt{25 - x^2} - \dfrac{4x^2}{\sqrt{25 - x^2}} = \dfrac{100 - 8x^2}{\sqrt{25 - x^2}}$.

Set $A'(x) = 0$: $100 - 8x^2 = 0 \implies x^2 = \dfrac{25}{2} \implies x = \dfrac{5}{\sqrt{2}}$.

Then $y = \sqrt{25 - \dfrac{25}{2}} = \dfrac{5}{\sqrt{2}}$.

Thus the rectangle is a square with side length $2x = 2y = 5\sqrt{2}$.

Exercise 3.3.13

Let r be the radius and h be the height.

Volume: $\pi r^2 h = 1000\pi$, so $h = \dfrac{1000}{r^2}$.

Cost: $C = 2(\text{top and bottom area}) + 1(\text{side area}) = 2(2\pi r^2) + 1(2\pi r h) = 4\pi r^2 + 2\pi r \cdot \dfrac{1000}{r^2} = 4\pi r^2 + \dfrac{2000\pi}{r}$.

$C'(r) = 8\pi r - \dfrac{2000\pi}{r^2}$. Set $C'(r) = 0$: $8\pi r - \dfrac{2000\pi}{r^2} = 0 \implies 8r^3 = 2000 \implies r^3 = 250 \implies r = \sqrt[3]{250} = 5\sqrt[3]{2}$.

Then $h = \dfrac{1000}{r^2} = \dfrac{1000}{25\sqrt[3]{4}} = \dfrac{40}{\sqrt[3]{4}} = 10\sqrt[3]{2}$.

Thus $r = 5\sqrt[3]{2}$ cm and $h = 10\sqrt[3]{2}$ cm.

Exercise 3.3.14

Let the numbers be x and $15 - x$.

Sum of squares: $S = x^2 + (15 - x)^2 = x^2 + 225 - 30x + x^2 = 2x^2 - 30x + 225$.

$S'(x) = 4x - 30$. Set $S'(x) = 0$: $4x - 30 = 0 \implies x = \dfrac{15}{2}$.

Then $15 - x = \dfrac{15}{2}$.

Thus the two numbers are $\dfrac{15}{2}$ and $\dfrac{15}{2}$.

Exercise 3.3.15

Let x be the length parallel to the barn and y be the width.

Fencing: $x + 2y = 60$, so $x = 60 - 2y$.

Area: $A = xy = (60 - 2y)y = 60y - 2y^2$.

$A'(y) = 60 - 4y$. Set $A'(y) = 0$: $60 - 4y = 0 \implies y = 15$.

Then $x = 60 - 2(15) = 30$.

Thus the dimensions are 30 m by 15 m.

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3.4: Distances and Areas

Learning Objectives

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Question of the Day

How can we approximate the area of a region with an irregular shape?

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3.4: Estimating Areas

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20$$

We could probably skip writing a couple of terms and write

$$1+2+3+4+\cdots+19+20$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the i is called the $index$. Each term is evaluated, then we sum all the values, beginning with the value when $i=1$ and ending with the value when $i=n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2+s_3+s_4+s_5+s_6+s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use $i,\ j,\ k,\ m$, and n for indices.

Let's try a couple of examples of using sigma notation.

Example 3.4.1: Using Sigma Notation

(a) Write in sigma notation and evaluate the sum of terms 3^i for $i=1,2,3,4,5$.

(b) Write the sum in sigma notation:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$$

Solution

(a) Write

$$\sum_{i=1}^{5} 3^i = 3+3^2+3^3+3^4+3^5 = \boxed{363}$$

(b) The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\boxed{\sum_{i=1}^{5} \frac{1}{i^2}}$

Exercise 3.4.1

Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$

Hint available at the end of this section.

Answer available at the end of this section.

The properties associated with the summation process are given in the following rule.

Theorem 3.4.1: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \leq m \leq n$.

i. $\sum_{i=1}^{n} c = nc$

ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

iii. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

iv. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof

We prove properties (ii.) and (iii.) here, and leave proof of the other properties to the Exercises.

(ii.) We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i$$

(iii.) We have

$$\begin{aligned} \sum_{i=1}^{n} (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) \\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \end{aligned}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Example 3.4.2: Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

(a) The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$

(b) The sum of the terms $(i^3 - i^2)$ for $i = 1, 2, 3, 4, 5, 6$

Solution

(a) Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}
\sum_{i=1}^{200} (i-3)^2 &= \sum_{i=1}^{200} (i^2 - 6i + 9) \\
&= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \\
&= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\
&= \frac{200(200+1)(400+1)}{6} - 6\left[\frac{200(200+1)}{2}\right] + 9(200) \\
&= 2686700 - 120600 + 1800 \\
&= \boxed{2567900}
\end{aligned}$$

(b) Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}
\sum_{i=1}^{6} (i^3 - i^2) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\
&= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\
&= \frac{1764}{4} - \frac{546}{6} \\
&= \boxed{350}
\end{aligned}$$

Exercise 3.4.2

Find the sum of the values of $4+3i$ for $i=1,2,\ldots,100$

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.4.3: Finding the Sum of the Function Values

Find the sum of the values of $f(x)=x^3$ over the integers $1,2,3,\ldots,10$

Solution

Using the sum of cubes formula above, we have

$$\sum_{i=0}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100(121)}{4} = \boxed{3025}$$

Exercise 3.4.3

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$

Hint available at the end of this section.

Answer available at the end of this section.

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a,b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x=a$ on the left, and the line $x=b$ on the right (Figure 3.4.1).

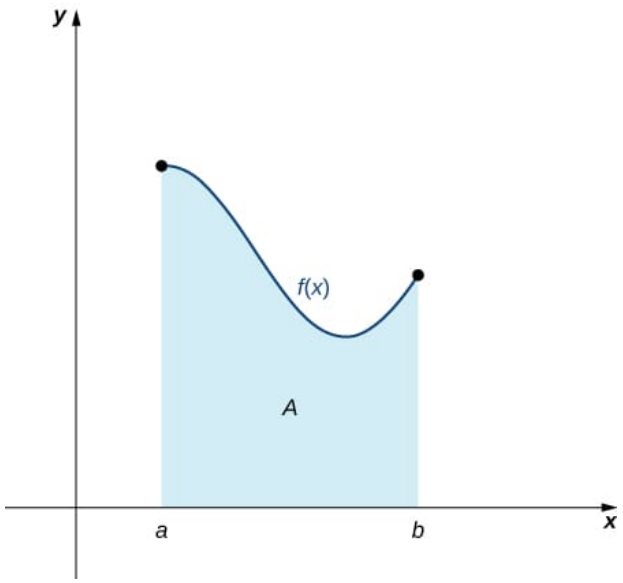


Figure 3.4.1: An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n} \text{ for } i = 1, 2, 3, \ldots, n$$

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x \text{ for } i = 1, 2, 3, \ldots, n$$

This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

Definition: Partitions

A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Definition: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 3.4.2). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

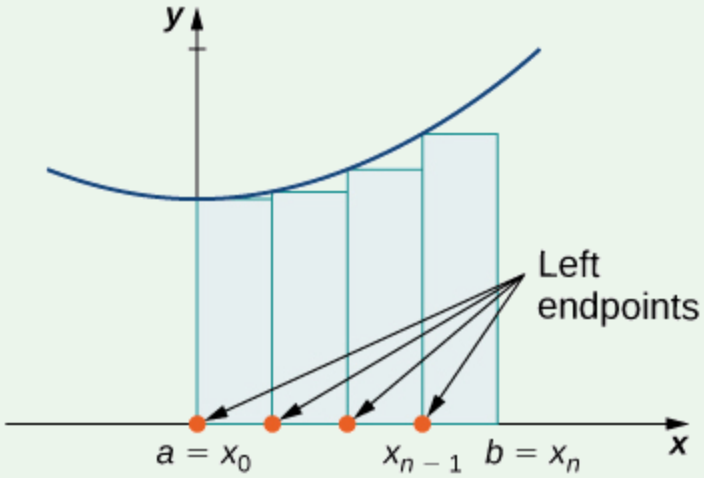


Figure 3.4.2: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Definition: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\ \Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 3.4.3).

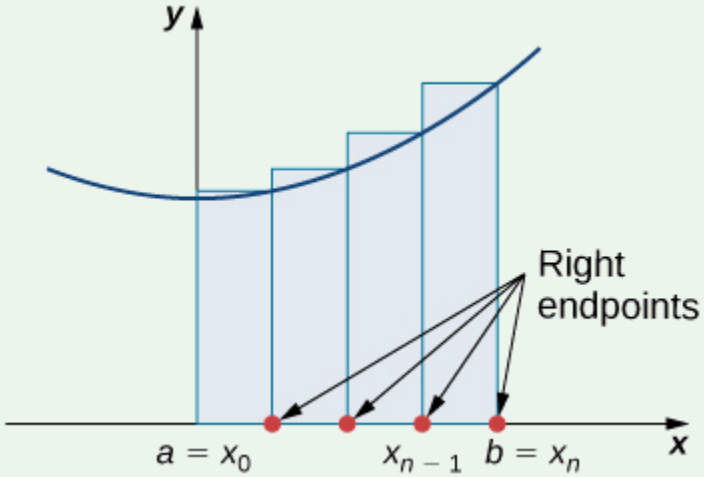


Figure 3.4.3: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure 3.4.2.

The graphs in Figure 3.4.4 represent the curve $f(x) = \dfrac{x^2}{2}$. In Figure 3.4.4*b* we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width $.5$. Thus, $\Delta x = .5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1} , the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, .5], [.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width.

Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)(.5) + f(.5)(.5) + f(1)(.5) + f(1.5)(.5) + f(2)(.5) + f(2.5)(.5) \\ &= (0)(.5) + (.125)(.5) + (.5)(.5) + (1.125)(.5) + (2)(.5) + (3.125)(.5) \\ &= 0 + .0625 + .25 + .5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

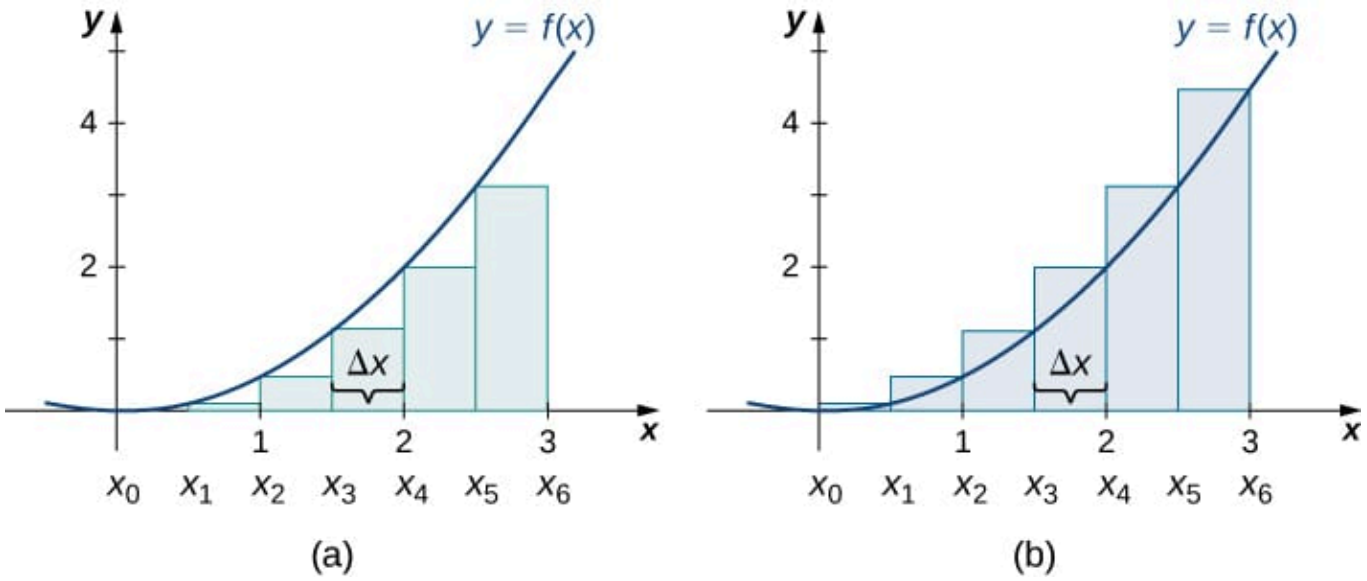


Figure 3.4.4: Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In Figure 3.4.4*b*, we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(.5)(.5) + f(1)(.5) + f(1.5)(.5) + f(2)(.5) + f(2.5)(.5) + f(3)(.5) \\ &= (.125)(.5) + (.5)(.5) + (1.125)(.5) + (2)(.5) + (3.125)(.5) + (4.5)(.5) \\ &= .0625 + .25 + .5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2 \end{aligned}$$

Example 3.4.4: Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \dfrac{(2-0)}{4} = .5$. This is the width of each rectangle. The intervals $[0, .5], [.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure 3.4.5. Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(.5) = .25$, $f(1) = 1$, and $f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(.5) + .25(.5) + 1(.5) + 2.25(.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

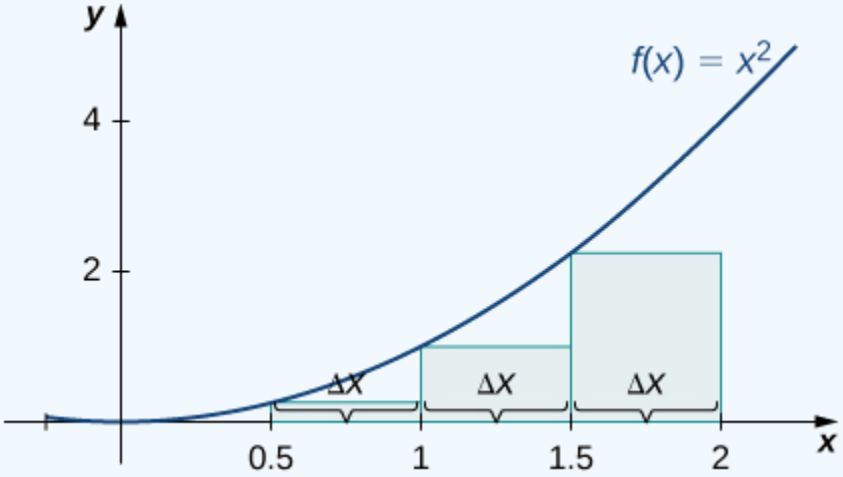


Figure 3.4.5: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 3.4.6. The intervals are the same, $\Delta x = .5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= .25(.5) + 1(.5) + 2.25(.5) + 4(.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$

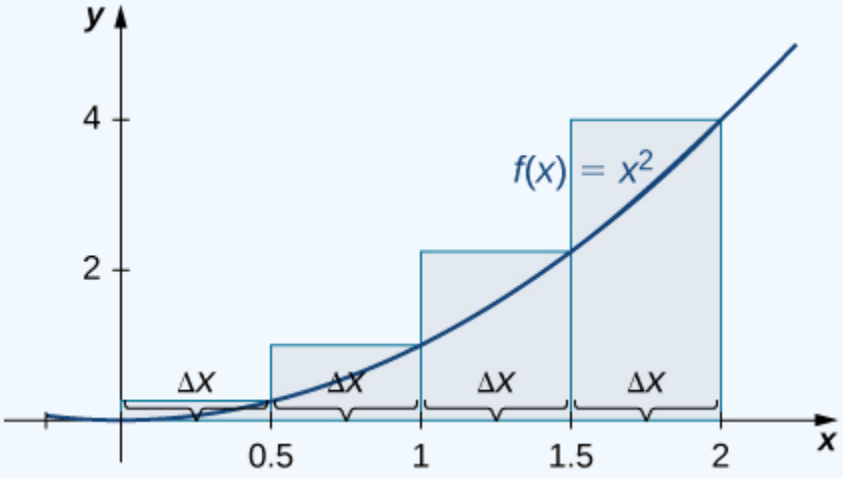


Figure 3.4.6: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is $\boxed{1.75 \text{ units}^2}$ and the right-endpoint approximation is $\boxed{3.75 \text{ units}^2}$

Exercise 3.4.4

Sketch left-endpoint and right-endpoint approximations for $f(x) = \dfrac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Hint available at the end of this section.

Answer available at the end of this section.

Looking at Figure 3.4.4 and the graphs in Example 3.4.4, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. Figure 3.4.7 shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

The area is approximated by the summed areas of the rectangles, or

$$L_4 = f(0)(.5) + f(.5)(.5) + f(1)(.5) + f(1.5).5 = 7.5 \text{ units}^2$$

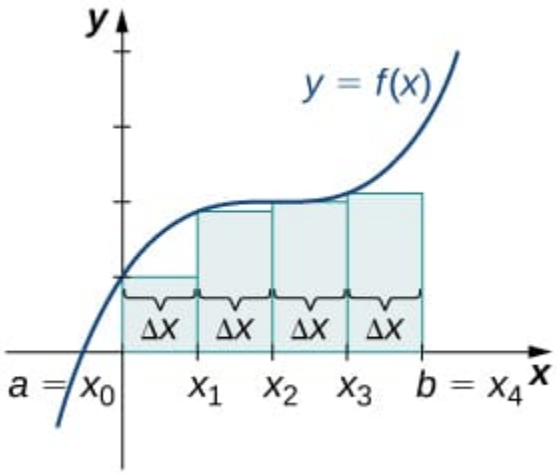


Figure 3.4.7: With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure 3.4.8 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure 3.4.7 with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$L_8 = f(0)(.25) + f(.25)(.25) + f(.5)(.25) + f(.75)(.25) + f(1)(.25) + f(1.25)(.25) + f(1.5)(.25) + f(1.75)(.25) = 7.75 \text{ units}^2$$

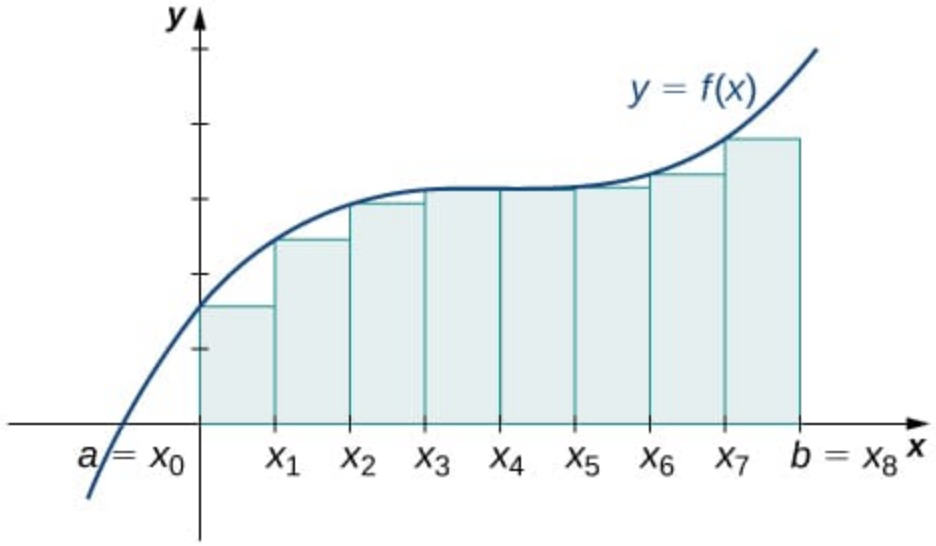


Figure 3.4.8: The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in Figure 3.4.9 shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(.0625) + f(.0625)(.0625) + f(.125)(.0625) + \cdots + f(1.9375)(.0625) = 7.9375 \text{ units}^2$$

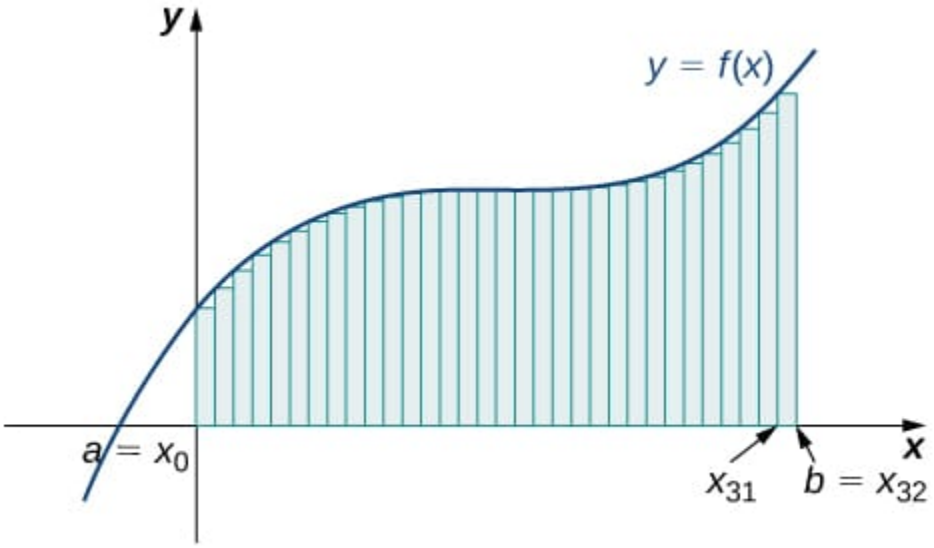


Figure 3.4.9: Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure 3.4.10), yields an area

$$R_4 = f(.5)(.5) + f(1)(.5) + f(1.5)(.5) + f(2)(.5) = 8.5 \text{ units}^2$$

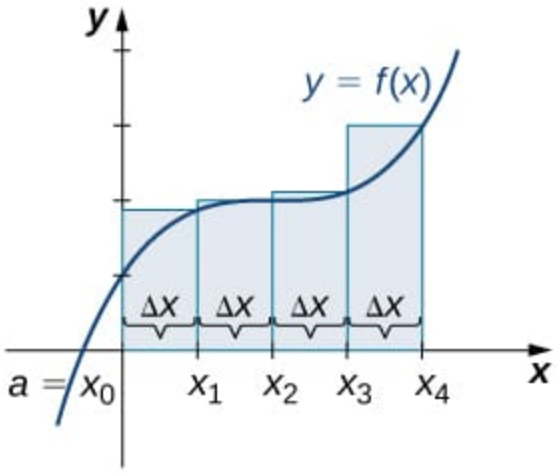


Figure 3.4.10: Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \dfrac{2-0}{8} = 0.25$. The graph is shown in Figure 3.4.11. The area is

$$R_8 = f(.25)(.25) + f(.5)(.25) + f(.75)(.25) + f(1)(.25) + f(1.25)(.25) + f(1.5)(.25) + f(1.75)(.25) + f(2)(.25) = 8.25 \text{ units}^2$$

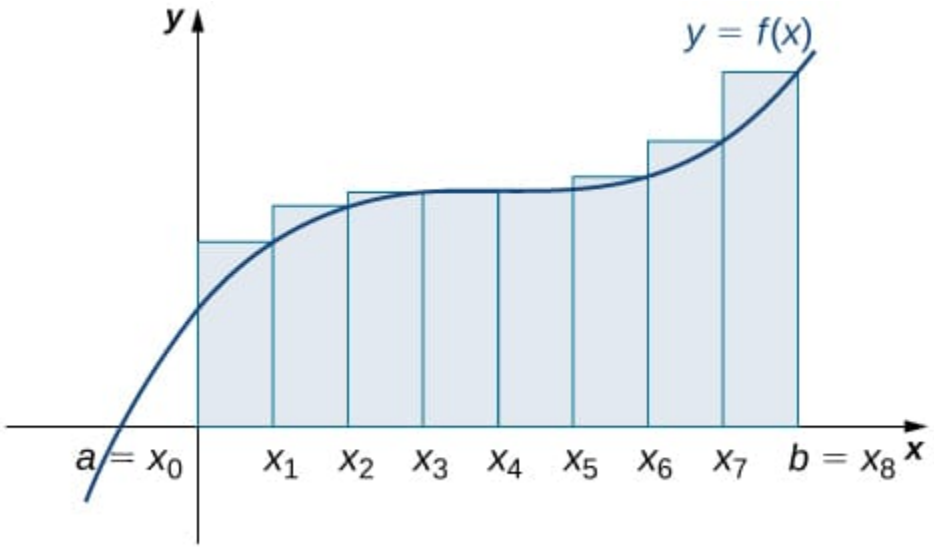


Figure 3.4.11: Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 3.4.12). The area is approximately

$$R_{32} = f(.0625)(.0625) + f(.125)(.0625) + f(.1875)(.0625) + \cdots + f(2)(.0625) = 8.0625 \text{ units}^2$$

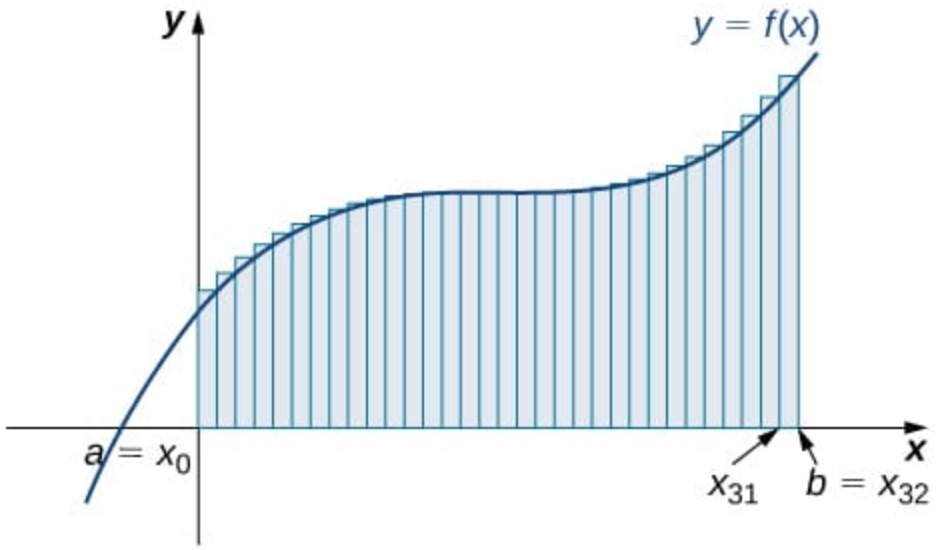


Figure 3.4.12: The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table 3.4.15 shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Table 3.4.15: Converging Values of Left- and Right-Endpoint Approximations as n Increases

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25
$n = 32$	7.94	8.06

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1},\ x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \dfrac{b-a}{n}$ for any natural number of intervals n.

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\displaystyle\sum_{i=1}^{n} f(x_i^*)\ \Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on Sequences and Series; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of x_i^*. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of x_i^*).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for x_i^*. Although any choice for x_i^* gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for x_i^* to guarantee one result or the other.

If we want an overestimate, for example, we can choose x_i^* such that for $i = 1, 2, 3, \ldots, n,\ f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose x_i^* so that for $i = 1, 2, 3, \ldots, n,\ f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select x_i^* in this way, then the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose x_i^* so that for $i = 1, 2, 3, \ldots, n,\ f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example 3.4.5: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$ with $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2],\ \Delta x = \dfrac{1}{4}$. We can list the intervals as $[1, 1.25],\ [1.25, 1.5],\ [1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure shows that a lower sum is obtained by using the right endpoints.

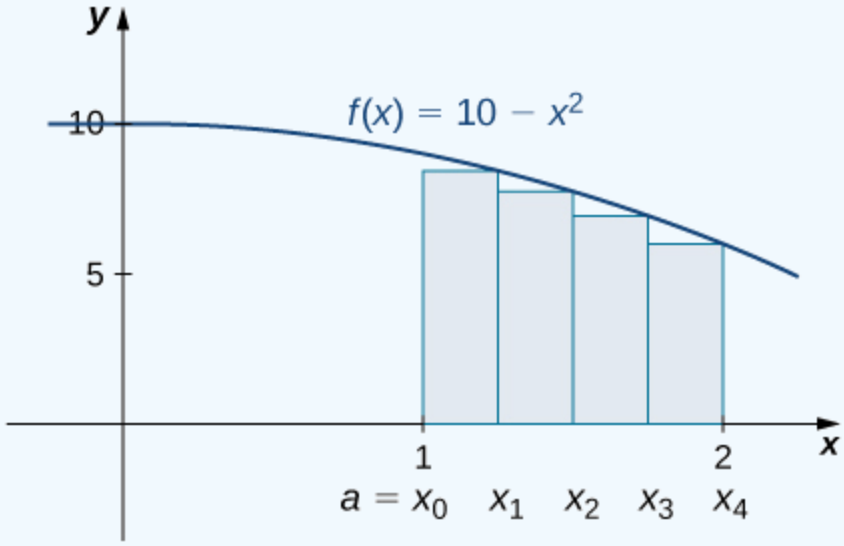


Figure 3.4.13: The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x_k^2)(0.25) &= 0.25\Big[\left(10 - (1.25)^2\right) + \left(10 - (1.5)^2\right) + \left(10 - (1.75)^2\right) + \left(10 - (2)^2\right)\Big] \\ &= 0.25\Big[8.4375 + 7.75 + 6.9375 + 6\Big] \\ &= 7.28\,\text{units}^2\end{aligned}$$

We have clearly found an **underestimate** for the area: $\boxed{7.28\ \text{units}^2}$

Exercise 3.4.5

Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$ with $n = 4$ and sketch the approximation.

Hint available at the end of this section.

Example 3.4.6: Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin(x)$ over the interval $[a, b] = \left[0, \frac{\pi}{2}\right]$ with $n = 6$ subintervals.

Solution

Let's first look at the graph in Figure 3.4.14 to get a better idea of the area of interest.

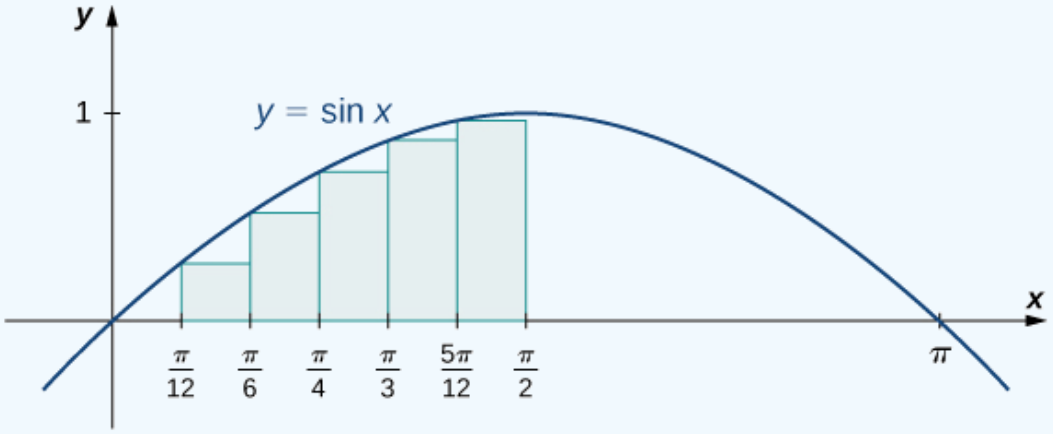


Figure 3.4.14: The graph of $y = \sin(x)$ is divided into six regions with $\Delta x = \frac{\frac{\pi}{2}}{6} = \frac{\pi}{12}$

The intervals are $\left[0, \frac{\pi}{12}\right]$, $\left[\frac{\pi}{12}, \frac{\pi}{6}\right]$, $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, $\left[\frac{\pi}{3}, \frac{5\pi}{12}\right]$, and $\left[\frac{5\pi}{12}, \frac{\pi}{2}\right]$. Note that $f(x) = \sin(x)$ is increasing on the interval $\left[0, \frac{\pi}{2}\right]$, so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin(x_i)\left(\frac{\pi}{12}\right)$. We have

$$A \approx \sin(0) \cdot \frac{\pi}{12} + \sin\left(\frac{\pi}{12}\right) \cdot \frac{\pi}{12} + \sin\left(\frac{\pi}{6}\right) \cdot \frac{\pi}{12} + \sin\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{12} + \sin\left(\frac{\pi}{3}\right) \cdot \frac{\pi}{12} + \sin\left(\frac{5\pi}{12}\right) \cdot \frac{\pi}{12} \approx \boxed{0.863 \text{ units}^2}$$

Exercise 3.4.6

Using the function $f(x) = \sin(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$, find an upper sum with $n = 6$ subintervals.

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 3.4.1

Use the solving steps in Example 3.4.1 as a guide.

Exercise 3.4.1

$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$

Exercise 3.4.2

Use the properties of sigma notation to solve the problem.

Exercise 3.4.2

15550

Exercise 3.4.3

Use the formula for sums of powers of integers above.

Exercise 3.4.3

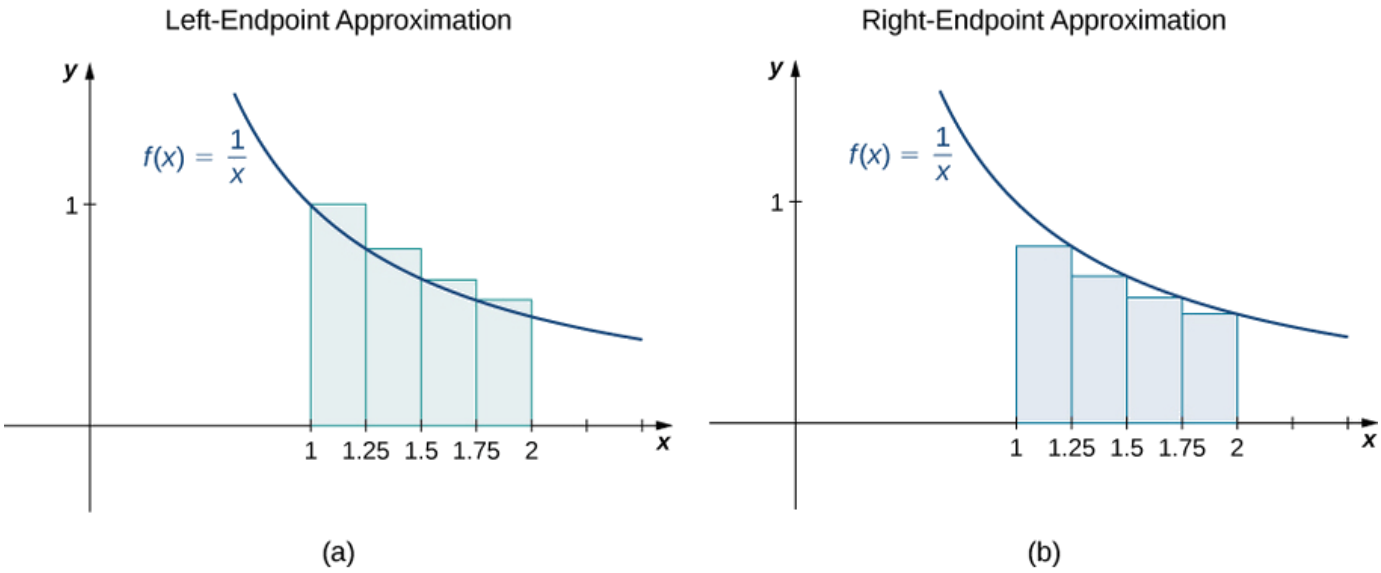
440

Exercise 3.4.4

Follow the solving strategy in Example 3.4.4 step-by-step.

Exercise 3.4.4

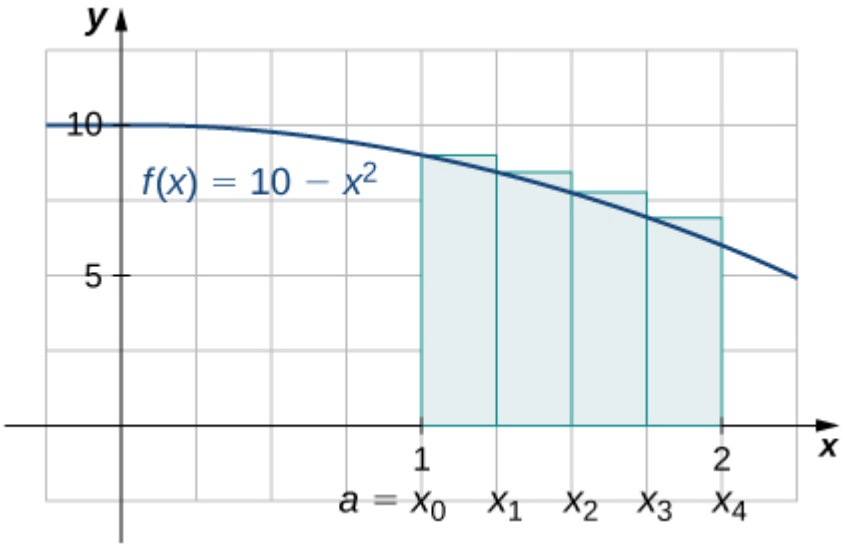
The left-endpoint approximation is 0.7595 units^2. The right-endpoint approximation is 0.6345 units^2.



Exercise 3.4.5

$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Upper sum=$8.0313\ \text{units}^2$



Exercise 3.4.6

Follow the steps from Example 3.4.6.

Exercise 3.4.6

$A \approx 1.125\ \text{units}^2$

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3.4: Practice Set

Exercise 3.4.1

Evaluate each summation.

(a) $\sum_{k=1}^{5} k$

(b) $\sum_{k=1}^{6} (2k+1)$

(c) $\sum_{k=1}^{4} k^2$

(d) $\sum_{k=1}^{5} (3k^2 - 2k + 1)$

(e) $\sum_{k=1}^{3} 2^k$

(f) $\sum_{k=1}^{4} \frac{1}{k}$

Answer available at the end of this section.

Exercise 3.4.2

Use summation formulas to evaluate each sum.

(a) $\sum_{k=1}^{20} k$

(b) $\sum_{k=1}^{15} k^2$

(c) $\sum_{k=1}^{12} (3k+2)$

(d) $\sum_{k=1}^{10} (k^2 - 3k)$

(e) $\sum_{k=1}^{8} (2k^2 - k + 1)$

(f) $\sum_{k=1}^{25} k^3$

Answer available at the end of this section.

Exercise 3.4.3

Approximate the area under each curve on the given interval using the specified number of rectangles and the indicated method. Use the right endpoints.

(a) $f(x) = x^2$ on $[0, 2]$ with 4 rectangles

(b) $f(x) = 3x + 1$ on $[0, 4]$ with 4 rectangles

(c) $f(x) = \sqrt{x}$ on $[0, 4]$ with 4 rectangles

(d) $f(x) = 2x + 1$ on $[0, 2]$ with 4 rectangles

Answer available at the end of this section.

Exercise 3.4.4

Approximate the area under each curve on the given interval using the specified number of rectangles and the indicated method. Use the left endpoints.

(a) $f(x) = x^2 + 1$ on $[0, 2]$ with 4 rectangles

(b) $f(x) = x^3$ on $[0, 2]$ with 4 rectangles

(c) $f(x) = 2x$ on $[0, 4]$ with 4 rectangles

(d) $f(x) = \sqrt{x + 1}$ on $[0, 3]$ with 3 rectangles

Answer available at the end of this section.

Exercise 3.4.5

Approximate the area under each curve on the given interval using the specified number of rectangles and the indicated method. Use the midpoints.

(a) $f(x) = x^2$ on $[0, 2]$ with 4 rectangles

(b) $f(x) = \sin(x)$ on $[0, \pi]$ with 4 rectangles

(c) $f(x) = 4 - x^2$ on $[0, 2]$ with 4 rectangles

(d) $f(x) = e^x$ on $[0, 2]$ with 4 rectangles

Answer available at the end of this section.

Exercise 3.4.6

For each function and interval, compare the left, right, and midpoint approximations using 4 rectangles. Determine which is closest to the actual area (if known).

(a) $f(x) = x^2$ on $[0, 2]$

(b) $f(x) = 2x + 1$ on $[0, 2]$

(c) $f(x) = \sqrt{x}$ on $[0, 4]$

Answer available at the end of this section.

Exercise 3.4.7

Write each sum in summation notation.

(a) $1 + 2 + 3 + 4 + 5 + 6$

(b) $2 + 4 + 6 + 8 + 10$

(c) $1 + 4 + 9 + 16 + 25$

(d) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

(e) $3 + 7 + 11 + 15 + 19$

(f) $1 - 2 + 3 - 4 + 5 - 6$

Answer available at the end of this section.

Exercise 3.4.8

Approximate the area under $f(x) = x^2 + 1$ on $[0, 2]$ using $n = 4$ rectangles for each method. Then use $n = 8$ rectangles for the right endpoint method.

(a) Left endpoints with $n = 4$

(b) Right endpoints with $n = 4$

(c) Midpoints with $n = 4$

(d) Right endpoints with $n = 8$

Answer available at the end of this section.

Exercise 3.4.9

A car's velocity in ft/s is recorded every 2 seconds. Use the data to estimate the total distance traveled using left and right endpoint approximations.

t (seconds): $0, 2, 4, 6, 8$

$v(t)$ (ft/s): $0, 8, 12, 15, 10$

Answer available at the end of this section.

Exercise 3.4.10

Approximate the area under $f(x) = \cos(x)$ on $[0, \pi]$ using $n = 4$ rectangles and the right endpoint method.

Answer available at the end of this section.

Answers

Exercise 3.4.1

(a) $\sum_{k=1}^{5} k = 1+2+3+4+5 = 15$

(b) $\sum_{k=1}^{6}(2k+1) = (2(1)+1)+(2(2)+1)+(2(3)+1)+(2(4)+1)+(2(5)+1)+(2(6)+1)$

$$= 3+5+7+9+11+13 = 48$$

(c) $\sum_{k=1}^{4} k^2 = 1^2+2^2+3^2+4^2 = 1+4+9+16 = 30$

(d) $\sum_{k=1}^{5}(3k^2-2k+1)$

$$= (3-2+1)+(12-4+1)+(27-6+1)+(48-8+1)+(75-10+1)$$
$$= 2+9+22+41+66 = 140$$

(e) $\sum_{k=1}^{3} 2^k = 2^1+2^2+2^3 = 2+4+8 = 14$

(f) $\sum_{k=1}^{4} \frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{12}{12}+\frac{6}{12}+\frac{4}{12}+\frac{3}{12} = \frac{25}{12}$

Exercise 3.4.2

Recall the summation formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(a) $\sum_{k=1}^{20} k = \frac{20(21)}{2} = 210$

(b) $\sum_{k=1}^{15} k^2 = \frac{15(16)(31)}{6} = \frac{7440}{6} = 1240$

(c) $\sum_{k=1}^{12}(3k+2) = 3\sum_{k=1}^{12} k + \sum_{k=1}^{12} 2 = 3\left(\frac{12(13)}{2}\right) + 2(12)$

$$= 3(78)+24 = 234+24 = 258$$

(d) $\sum_{k=1}^{10}(k^2-3k) = \sum_{k=1}^{10} k^2 - 3\sum_{k=1}^{10} k$

$$= \frac{10(11)(21)}{6} - 3\left(\frac{10(11)}{2}\right) = \frac{2310}{6} - 3(55) = 385-165 = 220$$

(e) $\sum_{k=1}^{8}(2k^2-k+1) = 2\sum_{k=1}^{8} k^2 - \sum_{k=1}^{8} k + \sum_{k=1}^{8} 1$

$$= 2\left(\frac{8(9)(17)}{6}\right) - \left(\frac{8(9)}{2}\right) + 8 = 2(204) - 36 + 8 = 408 - 36 + 8 = 380$$

(f) $\sum_{k=1}^{25} k^3 = \left[\frac{25(26)}{2}\right]^2 = [325]^2 = 105625$

Exercise 3.4.3

Recall: $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$, and Area $\approx \sum_{k=1}^{n} f(x_k)\Delta x$.

(a) $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. Right endpoints: $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$.

$$A \approx [f(0.5)+f(1)+f(1.5)+f(2)]\left(\frac{1}{2}\right) = [0.25+1+2.25+4]\left(\frac{1}{2}\right) = 7.5\left(\frac{1}{2}\right) = 3.75$$

(b) $\Delta x = \frac{4-0}{4} = 1$. Right endpoints: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$.

$$A \approx [f(1)+f(2)+f(3)+f(4)](1) = [4+7+10+13] = 34$$

(c) $\Delta x = \frac{4-0}{4} = 1$. Right endpoints: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$.

$$A \approx [f(1)+f(2)+f(3)+f(4)](1) = [1+\sqrt{2}+\sqrt{3}+2] = 3+\sqrt{2}+\sqrt{3} \approx 6.146$$

(d) $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. Right endpoints: $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$.

$$A \approx [f(0.5)+f(1)+f(1.5)+f(2)]\left(\frac{1}{2}\right) = [2+3+4+5]\left(\frac{1}{2}\right) = 14\left(\frac{1}{2}\right) = 7$$

Exercise 3.4.4

Recall: $\Delta x = \frac{b-a}{n}$, $x_k = a + (k-1)\Delta x$, and Area $\approx \sum_{k=1}^{n} f(x_k)\Delta x$.

(a) $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. Left endpoints: $x_1 = 0$, $x_2 = 0.5$, $x_3 = 1$, $x_4 = 1.5$.

$$A \approx [f(0)+f(0.5)+f(1)+f(1.5)]\left(\frac{1}{2}\right) = [1+1.25+2+3.25]\left(\frac{1}{2}\right) = 7.5\left(\frac{1}{2}\right) = 3.75$$

(b) $\Delta x = \dfrac{2-0}{4} = \dfrac{1}{2}$. Left endpoints: $x_1 = 0$, $x_2 = 0.5$, $x_3 = 1$, $x_4 = 1.5$.

$$A \approx [f(0)+f(0.5)+f(1)+f(1.5)]\left(\frac{1}{2}\right) = [0+0.125+1+3.375]\left(\frac{1}{2}\right) = 4.5\left(\frac{1}{2}\right) = 2.25$$

(c) $\Delta x = \dfrac{4-0}{4} = 1$. Left endpoints: $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$.

$$A \approx [f(0)+f(1)+f(2)+f(3)](1) = [0+2+4+6] = 12$$

(d) $\Delta x = \dfrac{3-0}{3} = 1$. Left endpoints: $x_1 = 0$, $x_2 = 1$, $x_3 = 2$.

$$A \approx [f(0)+f(1)+f(2)](1) = [1+\sqrt{2}+\sqrt{3}] \approx 4.146$$

Exercise 3.4.5

Recall: $\Delta x = \dfrac{b-a}{n}$, $x_k = a + \left(k - \dfrac{1}{2}\right)\Delta x$, and Area $\approx \sum_{k=1}^{n} f(x_k)\Delta x$.

(a) $\Delta x = \dfrac{2-0}{4} = \dfrac{1}{2}$. Midpoints: $x_1 = 0.25$, $x_2 = 0.75$, $x_3 = 1.25$, $x_4 = 1.75$.

$$A \approx [f(0.25)+f(0.75)+f(1.25)+f(1.75)]\left(\frac{1}{2}\right)$$

$$= [0.0625+0.5625+1.5625+3.0625]\left(\frac{1}{2}\right) = 5.25\left(\frac{1}{2}\right) = 2.625$$

(b) $\Delta x = \dfrac{\pi-0}{4} = \dfrac{\pi}{4}$. Midpoints: $x_1 = \dfrac{\pi}{8}$, $x_2 = \dfrac{3\pi}{8}$, $x_3 = \dfrac{5\pi}{8}$, $x_4 = \dfrac{7\pi}{8}$.

$$A \approx \left[\sin\left(\frac{\pi}{8}\right)+\sin\left(\frac{3\pi}{8}\right)+\sin\left(\frac{5\pi}{8}\right)+\sin\left(\frac{7\pi}{8}\right)\right]\left(\frac{\pi}{4}\right)$$

$$= \left[\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2+\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}\right]\left(\frac{\pi}{4}\right)$$

$$= \left[\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}\right]\left(\frac{\pi}{4}\right) \approx [0.765+1.848]\left(\frac{\pi}{4}\right) = 2.613\left(\frac{\pi}{4}\right) \approx 2.052$$

(c) $\Delta x = \dfrac{2-0}{4} = \dfrac{1}{2}$. Midpoints: $x_1 = 0.25$, $x_2 = 0.75$, $x_3 = 1.25$, $x_4 = 1.75$.

$$A \approx [f(0.25)+f(0.75)+f(1.25)+f(1.75)]\left(\frac{1}{2}\right)$$

$$= [3.9375+3.4375+2.4375+0.9375]\left(\frac{1}{2}\right) = 10.75\left(\frac{1}{2}\right) = 5.375$$

(d) $\Delta x = \dfrac{2-0}{4} = \dfrac{1}{2}$. Midpoints: $x_1 = 0.25$, $x_2 = 0.75$, $x_3 = 1.25$, $x_4 = 1.75$.

$$A \approx [e^{0.25}+e^{0.75}+e^{1.25}+e^{1.75}]\left(\frac{1}{2}\right)$$

$$\approx [1.284+2.117+3.490+5.755]\left(\frac{1}{2}\right) = 12.646\left(\frac{1}{2}\right) = 6.323$$

Exercise 3.4.6

(a) $f(x) = x^2$ on $[0,2]$, $n = 4$, $\Delta x = \dfrac{1}{2}$.

Left: $x = 0, 0.5, 1, 1.5$

$$L_4 = [0+0.25+1+2.25]\left(\frac{1}{2}\right) = 3.5\left(\frac{1}{2}\right) = 1.75$$

Right: $x = 0.5, 1, 1.5, 2$

$$R_4 = [0.25+1+2.25+4]\left(\frac{1}{2}\right) = 7.5\left(\frac{1}{2}\right) = 3.75$$

Midpoint: $x = 0.25, 0.75, 1.25, 1.75$

$$M_4 = [0.0625+0.5625+1.5625+3.0625]\left(\frac{1}{2}\right) = 5.25\left(\frac{1}{2}\right) = 2.625$$

Actual area: $\int_0^2 x^2 dx = \dfrac{8}{3} \approx 2.667$. Midpoint is closest.

(b) $f(x) = 2x+1$ on $[0,2]$, $n = 4$, $\Delta x = \dfrac{1}{2}$.

Left: $x = 0, 0.5, 1, 1.5$

$$L_4 = [1+2+3+4]\left(\frac{1}{2}\right) = 10\left(\frac{1}{2}\right) = 5$$

Right: $x = 0.5, 1, 1.5, 2$

$$R_4 = [2+3+4+5]\left(\frac{1}{2}\right) = 14\left(\frac{1}{2}\right) = 7$$

Midpoint: $x = 0.25, 0.75, 1.25, 1.75$

$$M_4 = [1.5+2.5+3.5+4.5]\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right) = 6$$

Actual area: trapezoid area $= \dfrac{1+5}{2}(2) = 6$. All three give the same since f is linear.

(c) $f(x) = \sqrt{x}$ on $[0, 4]$, $n = 4$, $\Delta x = 1$.
Left: $x = 0, 1, 2, 3$

$$L_4 = [0 + 1 + \sqrt{2} + \sqrt{3}](1) = 1 + \sqrt{2} + \sqrt{3} \approx 4.146$$

Right: $x = 1, 2, 3, 4$

$$R_4 = [1 + \sqrt{2} + \sqrt{3} + 2](1) = 3 + \sqrt{2} + \sqrt{3} \approx 6.146$$

Midpoint: $x = 0.5, 1.5, 2.5, 3.5$

$$M_4 = [\sqrt{0.5} + \sqrt{1.5} + \sqrt{2.5} + \sqrt{3.5}](1) \approx 0.707 + 1.225 + 1.581 + 1.871 = 5.384$$

Actual area: $\int_0^4 \sqrt{x}dx = \frac{16}{3} \approx 5.333$. Midpoint is closest.

Exercise 3.4.7

(a) $\sum_{k=1}^{6} k$

(b) $\sum_{k=1}^{5} 2k$

(c) $\sum_{k=1}^{5} k^2$

(d) $\sum_{k=1}^{5} \frac{1}{k}$

(e) $\sum_{k=1}^{5} (4k - 1)$

(f) $\sum_{k=1}^{6} (-1)^{k+1} k$

Exercise 3.4.8

$f(x) = x^2 + 1$ on $[0, 2]$, actual area $= \int_0^2 (x^2 + 1)dx = \frac{8}{3} + 2 = \frac{14}{3} \approx 4.667$.

(a) $n = 4$, $\Delta x = \frac{1}{2}$. Left: $x = 0, 0.5, 1, 1.5$

$$L_4 = [f(0) + f(0.5) + f(1) + f(1.5)]\left(\frac{1}{2}\right) = [1 + 1.25 + 2 + 3.25]\left(\frac{1}{2}\right) = 7.5\left(\frac{1}{2}\right) = 3.75$$

(b) $n = 4$, $\Delta x = \frac{1}{2}$. Right: $x = 0.5, 1, 1.5, 2$

$$R_4 = [f(0.5) + f(1) + f(1.5) + f(2)]\left(\frac{1}{2}\right) = [1.25 + 2 + 3.25 + 5]\left(\frac{1}{2}\right) = 11.5\left(\frac{1}{2}\right) = 5.75$$

(c) $n = 4$, $\Delta x = \frac{1}{2}$. Midpoints: $x = 0.25, 0.75, 1.25, 1.75$

$$M_4 = [f(0.25) + f(0.75) + f(1.25) + f(1.75)]\left(\frac{1}{2}\right)$$

$$= [1.0625 + 1.5625 + 2.5625 + 4.0625]\left(\frac{1}{2}\right) = 9.25\left(\frac{1}{2}\right) = 4.625$$

(d) $n = 8$, $\Delta x = \frac{1}{4}$. Right endpoints: $x_k = \frac{k}{4}$ for $k = 1, 2, \ldots, 8$.

$$R_8 = \sum_{k=1}^{8} f\left(\frac{k}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{4}\sum_{k=1}^{8}\left[\left(\frac{k}{4}\right)^2 + 1\right] = \frac{1}{4}\sum_{k=1}^{8}\left(\frac{k^2}{16} + 1\right)$$

$$= \frac{1}{4}\left[\frac{1}{16}\sum_{k=1}^{8} k^2 + 8\right] = \frac{1}{4}\left[\frac{1}{16}\left(\frac{8(9)(17)}{6}\right) + 8\right] = \frac{1}{4}\left[\frac{1}{16}(204) + 8\right] = \frac{1}{4}[12.75 + 8] = \frac{20.75}{4} = 5.1875$$

Exercise 3.4.9

Total distance traveled is the area under the velocity curve. $\Delta t = 2$.
Left endpoints: use $t = 0, 2, 4, 6$.

$$D_L = [v(0) + v(2) + v(4) + v(6)](2) = [0 + 8 + 12 + 15](2) = 35(2) = 70 \text{ ft}$$

Right endpoints: use $t = 2, 4, 6, 8$.

$$D_R = [v(2) + v(4) + v(6) + v(8)](2) = [8 + 12 + 15 + 10](2) = 45(2) = 90 \text{ ft}$$

The actual distance is between 70 and 90 feet. A better estimate would be the average:

$$\frac{70 + 90}{2} = 80 \text{ ft}$$

Exercise 3.4.10

$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$. Right endpoints: $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$.

$$A \approx \left[\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{4}\right) + \cos(\pi)\right]\left(\frac{\pi}{4}\right)$$

$$= \left[\frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1\right]\left(\frac{\pi}{4}\right) = [-1]\left(\frac{\pi}{4}\right) = -\frac{\pi}{4} \approx -0.785$$

Note: The negative approximation indicates the area under the x-axis for part of the interval. If calculating "area" as a positive quantity, you'd take absolute values.

The actual signed area is $\int_0^{\pi} \cos(x)dx = 0$.

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3.5: Definite Integrals

Learning Objectives

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use geometry and the properties of definite integrals to evaluate them.
- Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Question of the Day

How do we go from an approximation (Riemann sum) to an exact area?

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3.5: The Definite Integral

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition: Definite Integral

If $f(x)$ is a function defined on an interval $[a, b]$, the ***definite integral*** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on Applications of Derivatives, where we used the indefinite integral symbol (without the a and b above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm **Leibniz**, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol $\int$ is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers a and b are x-values and are called the **limits of integration**; specifically, a is the lower limit and b is the upper limit. To clarify, we are using the word limit in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \to \infty$. Second, the boundaries of the region are called the *limits of integration*.

We call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a **dummy variable**, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a, b]$, then the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Theorem 3.5.1: Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities or removable discontinuities on a closed interval are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes

the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

Illustration 3.5.1: Evaluating an Integral Using the Definition

We will use the definition of the definite integral to evaluate $\int_0^2 x^2 \, dx$ with a right-endpoint approximation to generate the Riemann sum.

We first want to set up a Riemann sum. Based on the limits of integration, we have $a = 0$ and $b = 2$.

For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[0, 2]$. Then

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each i, we need to calculate the function value at the right endpoint of the interval $[x_{i-1}, x_i]$. The right endpoint of the interval is x_i, and since P is a regular partition,

$$x_i = x_0 + i\Delta x = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i) = x_i^2 = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}$$

Then the Riemann sum takes the form

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{4i^2}{n^2}\right)\frac{2}{n} = \sum_{i=1}^{n} \frac{8i^2}{n^3} = \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

Using the summation formula for $\sum_{i=1}^{n} i^2$, we have

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] \\
&= \frac{16n^3+24n^2+n}{6n^3} \\
&= \frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2}
\end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n \to \infty$. We get

$$\begin{aligned}
\int_0^2 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2}\right) \\
&= \frac{8}{3} + 0 + 0 = \frac{8}{3}
\end{aligned}$$

Exercise 3.5.1

Use the definition of the definite integral to evaluate $\int_0^3 (2x-1)\ dx$

Use a right-endpoint approximation to generate the Riemann sum.

Hint available at the end of this section.

Answer available at the end of this section.

Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x-axis.

Example 3.5.1: Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\ dx$

Solution

The function describes a semicircle with radius 3. To find

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx$$

we want to find the area under the curve over the interval $[3,6]$. The formula for the area of a circle is $A=\pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A=\frac{1}{2}\pi r^2$. The shaded area in Figure 3.5.1 covers one-half of the semicircle, or $A=\frac{1}{4}\pi r^2$. Thus,

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx = \frac{1}{4}\pi(3)^2 = \boxed{\frac{9}{4}\pi}$$

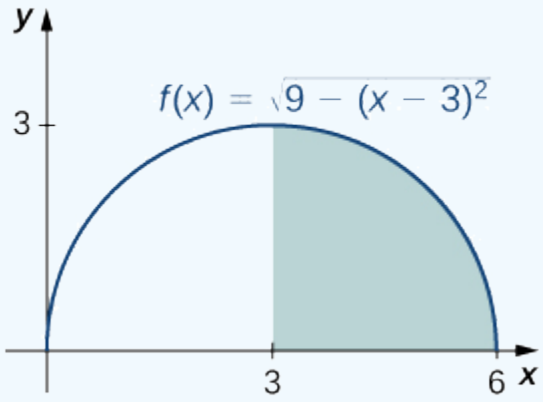


Figure 3.5.1: The value of the integral of the function $f(x)$ over the interval $[3,6]$ is the area of the shaded region.

Exercise 3.5.2

Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)\ dx$

Hint available at the end of this section.

Answer available at the end of this section.

Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in Figure 3.5.2) on the interval $[0, 2]$. Use $n = 8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the negative of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

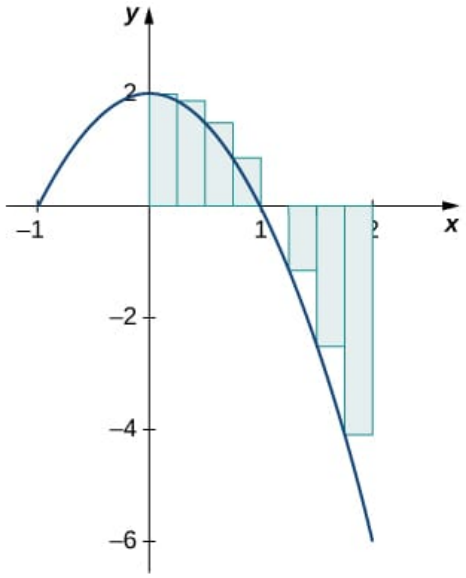


Figure 3.5.2: For a function that is partly negative, the Riemann sum is the area of the rectangles above the x-axis less the area of the rectangles below the x-axis.

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the x-axis and the x-axis, less the area between the curve below the x-axis and the x-axis, as shown in Figure 3.5.3. Then,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = A_1 - A_2$$

The quantity $A_1 - A_2$ is called the **net signed area.**

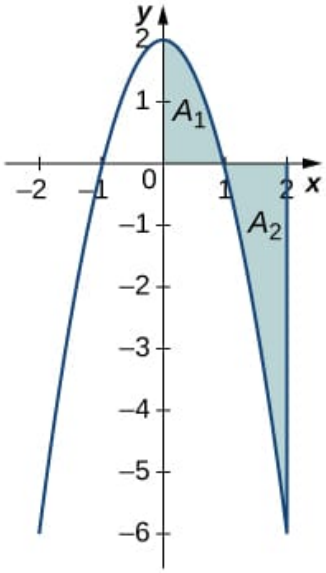


Figure 3.5.3: In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the x-axis is larger, the net signed area is positive. If the area below the x-axis is larger, the net signed area is negative. If the areas above and below the x-axis are equal, the net signed area is zero.

Example 3.5.2: Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ (Figure 3.5.4). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9$$

where 3 is the base and $2(3) = 6$ is the height. The area of triangle A_2, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9$$

where 3 is the base and 6 is the height. Thus, the net area is

$$\int_{-3}^{3} 2x \ dx = A_1 - A_2 = 9 - 9 = \boxed{0}$$

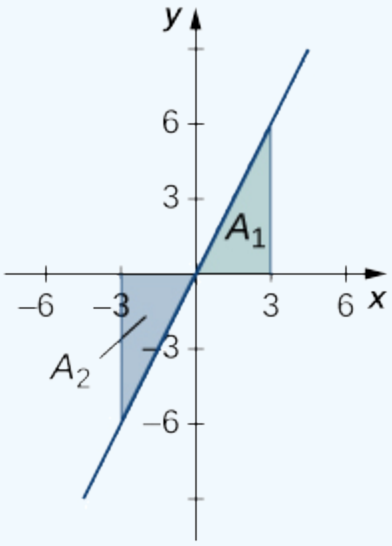


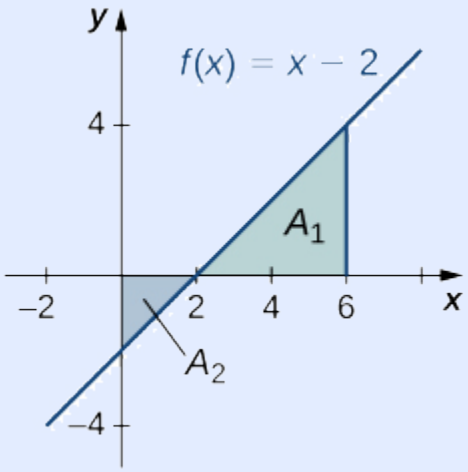
Figure 3.5.4: The area above the curve and below the x-axis equals the area below the curve and above the x-axis.

Analysis

If A_1 is the area above the x-axis and A_2 is the area below the x-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

Exercise 3.5.3

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.



Hint available at the end of this section.

Answer available at the end of this section.

Total Area

One application of the definite integral is finding **displacement** when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 70 mph for 2 hours, then it is 140 miles away from its original position (Figure $3.5.5$). Using integral notation, we have

$$\int_0^2 70\,dt = 140 \text{ miles}$$

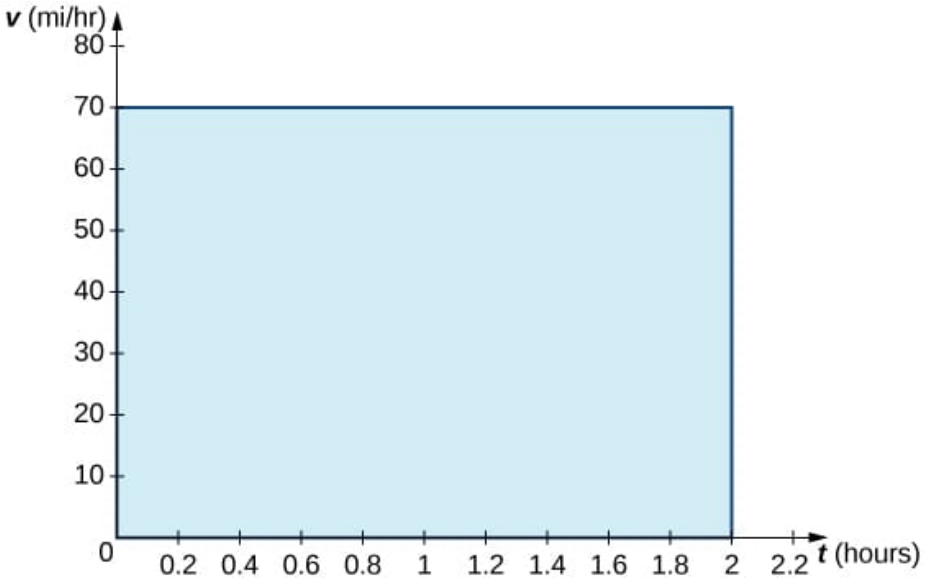


Figure $3.5.5$: The area under the curve $v(t) = 70$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 miles north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at its starting position (Figure $3.5.6$). Again, using integral notation, we have

$$\int_0^2 60\,dt + \int_2^5 -40\,dt = 120 - 120 = 0$$

In this case the displacement is zero.

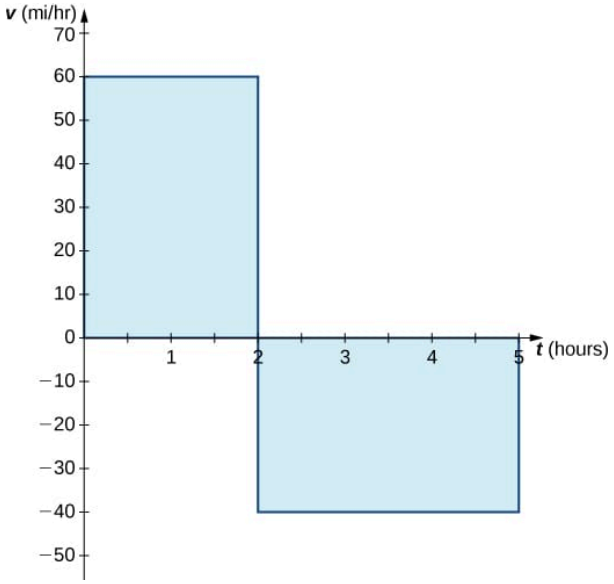


Figure $3.5.6$: The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between

the curve and the t-axis, regardless of whether that area is above or below the axis. This is called the **total area.**

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\int_0^2 |60|\,dt + \int_2^5 |-40|\,dt = \int_0^2 60\,dt + \int_2^5 40\,dt = 120 + 120 = 240$$

Bringing these ideas together formally, we state the following definitions.

Definition: Net Signed Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies above the axis and let A_2 represent the area between $f(x)$ and the x-axis that lies below the axis. Then, the *net signed area* between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2$$

The total area between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|\,dx = A_1 + A_2$$

Example 3.5.3: Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

Solution

Calculate the x-intercept as $(2, 0)$ (set $y = 0$, solve for x). To find the total area, take the area below the x-axis over the subinterval $[0, 2]$ and add it to the area above the x-axis on the subinterval $[2, 6]$ (Figure 3.5.7).

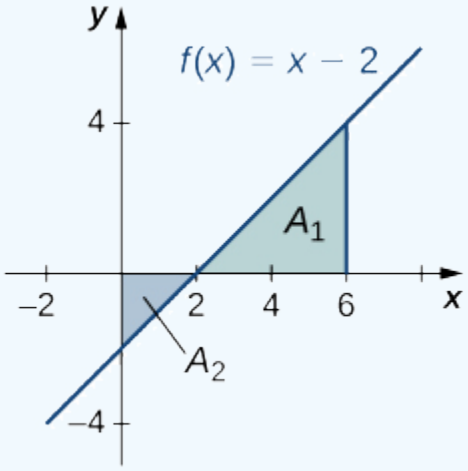


Figure 3.5.7: The total area between the line and the x-axis over $[0, 6]$ is A_2 plus A_1.

We have

$$\int_0^6 |x - 2|\,dx = A_2 + A_1$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2 = 2$$

$$A_1 = \frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot 4 = 8$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = \boxed{10 \text{ units}^2}$$

Exercise 3.5.4

Find the total area between the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 3.5.1

Use the solving strategy from Example 3.5.1.

Exercise 3.5.1

6

Exercise 3.5.2

Graph the function $f(x)$ and calculate the area under the function on the interval $[2, 4]$.

Exercise 3.5.2

18 square units

Exercise 3.5.3

Use the solving method described in Example 3.5.3.

Exercise 3.5.3

6

Exercise 3.5.4

Review the solving strategy in Example 3.5.4.

Exercise 3.5.4

18 units^2

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3.5: Properties of the Definite Integral

Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

Rule: Properties of the Definite Integral

1.

$$\int_a^a f(x)\,dx = 0$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b \big[f(x) + g(x)\big]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b \big[f(x) - g(x)\big]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

The integral of a difference is the difference of the integrals.

5.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Example 3.5.4: Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

Solution

Using integral notation, we have $\int_{-2}^{1} (-3x^3 + 2x + 2)\ dx$. We apply properties 3. and 5. to get

$$\int_{-2}^{1} (-3x^3 + 2x + 2)\, dx = \int_{-2}^{1} -3x^3\, dx + \int_{-2}^{1} 2x\, dx + \int_{-2}^{1} 2\, dx$$

$$= \boxed{-3\int_{-2}^{1} x^3\, dx + 2\int_{-2}^{1} x\, dx + \int_{-2}^{1} 2\, dx}$$

Exercise 3.5.5

Use the properties of the definite integral to express the definite integral of $f(x) = 6x^3 - 4x^2 + 2x - 3$ over the interval $[1, 3]$ as the sum of four definite integrals.

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.5.5: Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)\ dx = 10$ and $\int_0^5 f(x)\ dx = 5$, find the value of $\int_5^8 f(x)\ dx$

Solution

By property 6,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Thus,

$$\int_0^8 f(x)\, dx = \int_0^5 f(x)\, dx + \int_5^8 f(x)\, dx$$

$$10 = 5 + \int_5^8 f(x)\, dx$$

$$\boxed{\int_5^8 f(x)\, dx = 5}$$

Exercise 3.5.6

If it is known that $\int_1^5 f(x)\ dx = -3$ and $\int_2^5 f(x)\ dx = 4$, find the value of $\int_1^2 f(x)\ dx$

Hint available at the end of this section.

Answer available at the end of this section.

Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the x-axis is greater than the area between $g(x)$ and the x-axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b,\ \ a = b$, or="" $a = ''\ '' > b$. The following properties, however, concern only the case $a \leq b$, and are used when we want to compare the sizes of integrals.</b,\,a=b\),>

Theorem 3.5.2: Comparison Theorem

i. If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq 0$$

ii. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

iii. If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Illustration 3.5.2: Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0,1]$

Solution

Graphing these functions is necessary to understand how they compare over the interval $[0,1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, on the interval $[0,1]$, the graphs appear to be on top of each other. We need to zoom in to see that, on the interval $[0,1],\ \ g(x)$ is above $f(x)$. The two functions intersect at $x = 0$ and $x = 1$ (Figure 3.5.8).

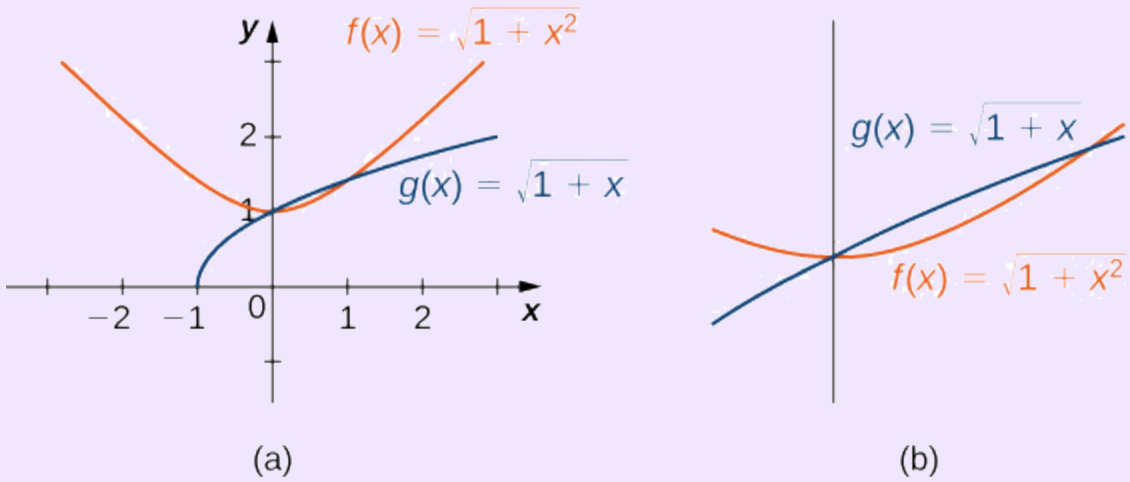


Figure 3.5.8: (a) The function $f(x)$ appears above the function $g(x)$ except over the interval $[0, 1]$ (b) Viewing the same graph with a greater zoom shows this more clearly.

We can see from the graph that over the interval $[0, 1]$, $g(x) \geq f(x)$.

Comparing the integrals over the specified interval $[0, 1]$, we also see that $\boxed{\int_0^1 g(x)\,dx \geq \int_0^1 f(x)\,dx}$ (Figure 3.5.9).

The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0, 1]$.

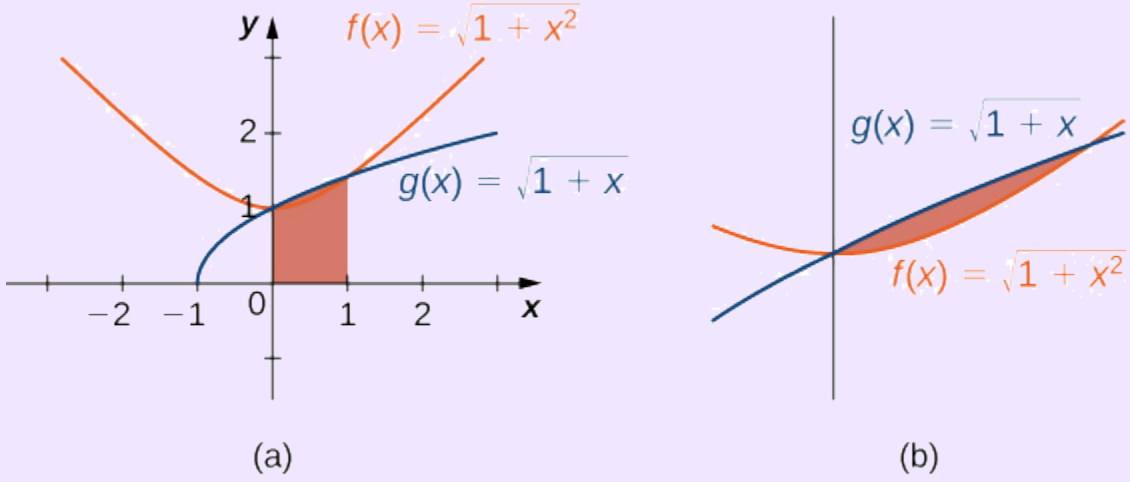


Figure 3.5.9: (a) The graph shows that over the interval $[0, 1], g(x) \geq f(x)$, where equality holds only at the endpoints of the interval. (b) Viewing the same graph with a greater zoom shows this more clearly.

Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

$$\frac{89 + 90 + 56 + 78 + 100 + 69}{6} = \frac{482}{6} \approx 80.33$$

Therefore, your average test grade is approximately 80.33, which translates to a B– at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time t, and we want to find the object's

average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a, b]$ and let $[a, b]$ be divided into n subintervals of width $\Delta x = (b - a)/n$. Choose a representative x_i^* in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2, \ldots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \dfrac{b-a}{n}$, so $n = \dfrac{b-a}{\Delta x}$, and we get

$$f_{ave} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\left(\dfrac{b-a}{\Delta x}\right)}$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^{n} f(x_i^*)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

$$\frac{\displaystyle\sum_{i=1}^{n} f(x_i^*)}{\left(\dfrac{b-a}{\Delta x}\right)} = \left(\frac{\Delta x}{b-a}\right)\sum_{i=1}^{n} f(x_i^*) = \left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)\Delta x$$

This is a Riemann sum. Then, to get the exact average value, take the limit as n goes to infinity. Thus, the average value of a function is given by

$$\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx$$

Definition: Average Value of a Function

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the *average value of the function* $f(x)$ (or f_{ave}) on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 3.5.6: Finding the Average Value of a Linear Function

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$

Solution

First, graph the function on the stated interval, as shown in Figure 3.5.10.

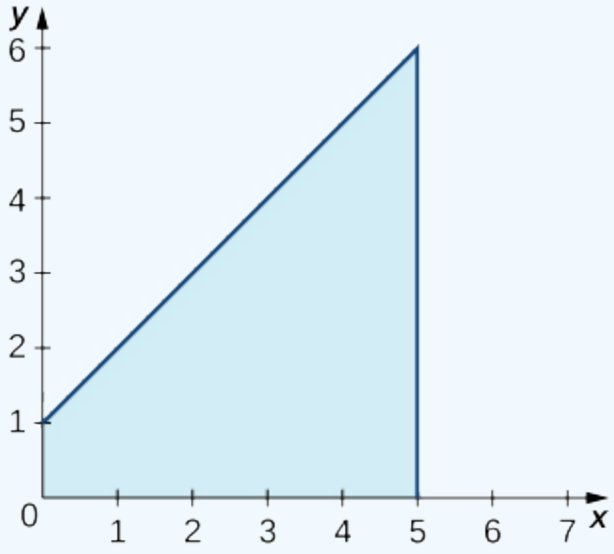


Figure 3.5.10:The graph shows the area under the function $(x) = x + 1$ over $[0, 5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a + b)$, where h represents height, and a and b represent the two parallel sides. Then,

$$\int_0^5 (x + 1)\ dx = \frac{1}{2}h(a + b) = \frac{1}{2} \cdot 5 \cdot (1 + 6) = \frac{35}{2}$$

Thus the average value of the function is

$$\frac{1}{5 - 0}\int_0^5 (x + 1)\, dx = \frac{1}{5} \cdot \frac{35}{2} = \boxed{\frac{7}{2}}$$

Exercise 3.5.7

Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 3.5.5

Use the solving strategy from Example 3.5.5 and the properties of definite integrals.

Exercise 3.5.5

$6\int_1^3 x^3\ dx - 4\int_1^3 x^2\ dx + 2\int_1^3 x\ dx - \int_1^3 3\ dx$

Exercise 3.5.6

Use the solving strategy from Example 3.5.6 and the rule on properties of definite integrals.

Exercise 3.5.6

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Exercise 3.5.7

Use the average value formula, and use geometry to evaluate the integral.

Exercise 3.5.7

3

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3.5: Practice Set

Exercise 3.5.1

Express each limit as a definite integral.

(a) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3x_k^2+2\right)\Delta x$ on $[0,2]$

(b) $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{x_k}\Delta x$ on $[1,4]$

(c) $\lim_{n\to\infty}\sum_{k=1}^{n}\sin(x_k)\Delta x$ on $[0,\pi]$

(d) $\lim_{n\to\infty}\sum_{k=1}^{n}e^{x_k}\Delta x$ on $[0,1]$

(e) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{x_k}\Delta x$ on $[1,3]$

Answer available at the end of this section.

Exercise 3.5.2

Use the limit definition of the definite integral to evaluate each integral.

(a) $\int_0^2 4x\,dx$

(b) $\int_0^3 (2x+1)\,dx$

(c) $\int_1^4 (3x-2)\,dx$

(d) $\int_0^1 x^2\,dx$

(e) $\int_0^2 (x^2+1)\,dx$

Answer available at the end of this section.

Exercise 3.5.3

Use the properties of definite integrals to evaluate each expression. Assume $\int_0^2 f(x)dx=5$, $\int_2^4 f(x)dx=3$, $\int_0^4 g(x)dx=7$.

(a) $\int_0^4 f(x)dx$

(b) $\int_4^2 f(x)dx$

(c) $\int_0^2 3f(x)dx$

(d) $\int_0^4 [f(x)+g(x)]dx$

(e) $\int_0^2 [2f(x)-g(x)]dx$ (assuming $\int_0^2 g(x)dx = 4$)

(f) $\int_0^4 [3f(x)-2g(x)]dx$

Answer available at the end of this section.

Exercise 3.5.4

Use the limit definition of the definite integral to evaluate each integral.

(a) $\int_0^1 (2x+3)dx$

(b) $\int_0^2 3x^2 dx$

(c) $\int_1^3 x dx$

(d) $\int_0^1 (x^2+2x)dx$

Answer available at the end of this section.

Exercise 3.5.5

Evaluate each integral using the properties of definite integrals. Assume $\int_0^3 f(x)dx = 6$, $\int_3^5 f(x)dx = 2$, $\int_0^5 g(x)dx = 10$, and $\int_0^3 g(x)dx = 4$.

(a) $\int_0^5 f(x)dx$

(b) $\int_5^0 f(x)dx$

(c) $\int_0^5 [3f(x)+2g(x)]dx$

(d) $\int_0^3 [f(x)-2g(x)]dx$

(e) $\int_3^5 g(x)dx$

(f) $\int_3^5 [4f(x) - g(x)]dx$

Answer available at the end of this section.

Exercise 3.5.6

Use the limit definition of the definite integral to evaluate each integral.

(a) $\int_0^1 (3x^2 + 2x)dx$

(b) $\int_0^2 (x^2 - 1)dx$

(c) $\int_1^4 (2x)dx$

(d) $\int_0^2 (x^3)dx$

Answer available at the end of this section.

Exercise 3.5.7

Evaluate each integral using the properties of definite integrals.

(a) $\int_0^2 5x^2 dx$

(b) $\int_0^2 (x^2 + 4x)dx$

(c) $\int_0^1 (3x^2 - 2x + 1)dx$

(d) $\int_0^1 x^4 dx$

Answer available at the end of this section.

Exercise 3.5.8

Write each definite integral as a limit of a Riemann sum (using right endpoints).

(a) $\int_0^3 x^2 dx$

(b) $\int_1^4 (2x + 1)dx$

(c) $\int_0^\pi \sin(x)dx$

(d) $\int_0^2 e^x dx$

Answer available at the end of this section.

Answers

Exercise 3.5.1

Recall the definition: $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$.

(a) $\int_0^2 (3x^2+2)dx$

(b) $\int_1^4 \sqrt{x}dx$

(c) $\int_0^\pi \sin(x)dx$

(d) $\int_0^1 e^x dx$

(e) $\int_1^3 \frac{1}{x}dx$

Exercise 3.5.2

Recall: $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$. Use $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.

(a) $\int_0^2 4xdx = \lim_{n\to\infty}\sum_{k=1}^{n} 4x_k\Delta x$, where $\Delta x = \frac{2}{n}$, $x_k = \frac{2k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n} 4\left(\frac{2k}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{16k}{n^2} = \lim_{n\to\infty}\frac{16}{n^2}\cdot\frac{n(n+1)}{2} = \lim_{n\to\infty} 8\left(1+\frac{1}{n}\right) = 8$$

(b) $\int_0^3 (2x+1)dx = \lim_{n\to\infty}\sum_{k=1}^{n}(2x_k+1)\Delta x$, where $\Delta x = \frac{3}{n}$, $x_k = \frac{3k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(2\left(\frac{3k}{n}\right)+1\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{18k}{n^2}+\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{18}{n^2}\cdot\frac{n(n+1)}{2}+\frac{3}{n}\cdot n\right] = \lim_{n\to\infty}\left[9\left(1+\frac{1}{n}\right)+3\right] = 9+3 = 12$$

(c) $\int_1^4 (3x-2)dx = \lim_{n\to\infty}\sum_{k=1}^{n}(3x_k-2)\Delta x$, where $\Delta x = \frac{3}{n}$, $x_k = 1+\frac{3k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(3\left(1+\frac{3k}{n}\right)-2\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{9k}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{3}{n}\cdot n+\frac{27}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[3+\frac{27}{2}\left(1+\frac{1}{n}\right)\right] = 3+\frac{27}{2} = \frac{33}{2}$$

(d) $\int_0^1 x^2dx = \lim_{n\to\infty}\sum_{k=1}^{n} x_k^2\Delta x$, where $\Delta x = \frac{1}{n}$, $x_k = \frac{k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2 = \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \frac{2}{6} = \frac{1}{3}$$

(e) $\int_0^2 (x^2+1)dx = \lim_{n\to\infty}\sum_{k=1}^{n}(x_k^2+1)\Delta x$, where $\Delta x = \frac{2}{n}$, $x_k = \frac{2k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{2k}{n}\right)^2+1\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}+1\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{8}{n^3}\sum_{k=1}^{n}k^2+\frac{2}{n}\cdot n\right] = \lim_{n\to\infty}\left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+2\right]$$

$$= \lim_{n\to\infty}\left[\frac{8(n+1)(2n+1)}{6n^2}+2\right] = \frac{8(2)}{6}+2 = \frac{8}{3}+2 = \frac{14}{3}$$

Exercise 3.5.3

Recall the properties of definite integrals:

- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
- $\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(a) $\int_0^4 f(x)dx = \int_0^2 f(x)dx + \int_2^4 f(x)dx = 5+3 = 8$

(b) $\int_4^2 f(x)dx = -\int_2^4 f(x)dx = -3$

(c) $\int_0^2 3f(x)dx = 3\int_0^2 f(x)dx = 3(5) = 15$

(d) $\int_0^4 [f(x)+g(x)]dx = \int_0^4 f(x)dx + \int_0^4 g(x)dx = 8+7 = 15$

(e) $\int_0^2 [2f(x)-g(x)]dx = 2\int_0^2 f(x)dx - \int_0^2 g(x)dx = 2(5)-4 = 10-4 = 6$

(f) $\int_0^4 [3f(x)-2g(x)]dx = 3\int_0^4 f(x)dx - 2\int_0^4 g(x)dx = 3(8)-2(7) = 24-14 = 10$

Exercise 3.5.4

(a) $\int_0^1 (2x+3)dx$, where $\Delta x = \frac{1}{n}$, $x_k = \frac{k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(2\left(\frac{k}{n}\right)+3\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2k}{n^2}+\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{2}{n^2}\cdot\frac{n(n+1)}{2}+\frac{3}{n}\cdot n\right] = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)+3\right] = 1+3 = 4$$

(b) $\int_0^2 3x^2dx$, where $\Delta x = \frac{2}{n}$, $x_k = \frac{2k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}3\left(\frac{2k}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{24k^2}{n^3} = \lim_{n\to\infty}\frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty}4\cdot\frac{(n+1)(2n+1)}{n^2} = 4(2) = 8$$

(c) $\int_1^3 xdx$, where $\Delta x = \frac{2}{n}$, $x_k = 1+\frac{2k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{2k}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left[\frac{2}{n}\cdot n+\frac{4}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[2+2\left(1+\frac{1}{n}\right)\right] = 2+2 = 4$$

(d) $\int_0^1 (x^2+2x)dx$, where $\Delta x = \frac{1}{n}$, $x_k = \frac{k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{k}{n}\right)^2+2\left(\frac{k}{n}\right)\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty}\left[\frac{1}{n^3}\sum_{k=1}^{n}k^2+\frac{2}{n^2}\sum_{k=1}^{n}k\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{2}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{6n^2}+\left(1+\frac{1}{n}\right)\right] = \frac{2}{6}+1 = \frac{1}{3}+1 = \frac{4}{3}$$

Exercise 3.5.5

(a) $\int_0^5 f(x)dx = \int_0^3 f(x)dx + \int_3^5 f(x)dx = 6+2 = 8$

(b) $\int_5^0 f(x)dx = -\int_0^5 f(x)dx = -8$

(c) $\int_0^5 [3f(x)+2g(x)]dx = 3\int_0^5 f(x)dx + 2\int_0^5 g(x)dx = 3(8)+2(10) = 24+20 = 44$

(d) $\int_0^3 [f(x)-2g(x)]dx = \int_0^3 f(x)dx - 2\int_0^3 g(x)dx = 6-2(4) = 6-8 = -2$

(e) $\int_3^5 g(x)dx = \int_0^5 g(x)dx - \int_0^3 g(x)dx = 10-4 = 6$

(f) $\int_3^5 [4f(x)-g(x)]dx = 4\int_3^5 f(x)dx - \int_3^5 g(x)dx = 4(2)-6 = 8-6 = 2$

Exercise 3.5.6

Recall: $\sum_{k=1}^{n}k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

(a) $\int_0^1 (3x^2+2x)dx$, where $\Delta x = \frac{1}{n}$, $x_k = \frac{k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(3\left(\frac{k}{n}\right)^2+2\left(\frac{k}{n}\right)\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty}\left[\frac{3}{n^3}\sum_{k=1}^{n}k^2+\frac{2}{n^2}\sum_{k=1}^{n}k\right]$$

$$= \lim_{n\to\infty}\left[\frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{2}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{2n^2}+\left(1+\frac{1}{n}\right)\right] = \frac{2}{2}+1 = 1+1 = 2$$

(b) $\int_0^2 (x^2-1)dx$, where $\Delta x = \frac{2}{n}$, $x_k = \frac{2k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{2k}{n}\right)^2-1\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left[\frac{8}{n^3}\sum_{k=1}^{n}k^2-\frac{2}{n}\cdot n\right]$$

$$= \lim_{n\to\infty}\left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - 2\right] = \frac{8(2)}{6} - 2 = \frac{8}{3} - 2 = \frac{2}{3}$$

(c) $\int_1^4 2x\,dx$, where $\Delta x = \frac{3}{n}$, $x_k = 1 + \frac{3k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n} 2\left(1+\frac{3k}{n}\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty}\left[\frac{6}{n}\cdot n + \frac{18}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[6 + 9\left(1+\frac{1}{n}\right)\right] = 6 + 9 = 15$$

(d) $\int_0^2 x^3\,dx$, where $\Delta x = \frac{2}{n}$, $x_k = \frac{2k}{n}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2k}{n}\right)^3\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{16}{n^4}\sum_{k=1}^{n}k^3 = \lim_{n\to\infty}\frac{16}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

$$= \lim_{n\to\infty} 16\cdot\frac{(n+1)^2}{4n^2} = 4\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 4(1) = 4$$

Exercise 3.5.7

Use the result $\int_0^1 x^n dx = \frac{1}{n+1}$ and $\int_0^a x^n dx = \frac{a^{n+1}}{n+1}$.

(a) $\int_0^2 5x^2dx = 5\int_0^2 x^2dx = 5\left(\frac{2^3}{3}\right) = 5\left(\frac{8}{3}\right) = \frac{40}{3}$

(b) $\int_0^2 (x^2+4x)dx = \int_0^2 x^2dx + 4\int_0^2 xdx = \frac{8}{3} + 4\left(\frac{2^2}{2}\right) = \frac{8}{3} + 4(2) = \frac{8}{3} + 8 = \frac{32}{3}$

(c) $\int_0^1 (3x^2 - 2x + 1)dx = 3\int_0^1 x^2dx - 2\int_0^1 xdx + \int_0^1 1dx$

$$= 3\left(\frac{1}{3}\right) - 2\left(\frac{1}{2}\right) + 1 = 1 - 1 + 1 = 1$$

(d) $\int_0^1 x^4dx = \frac{1}{5}$

Exercise 3.5.8

Recall: $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$ for right endpoints.

(a) $\int_0^3 x^2dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3k}{n}\right)^2\left(\frac{3}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{27k^2}{n^3}$

(b) $\int_1^4 (2x+1)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(2\left(1+\frac{3k}{n}\right)+1\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{6k}{n}\right)\left(\frac{3}{n}\right)$

(c) $\int_0^\pi \sin(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\sin\left(\frac{\pi k}{n}\right)\left(\frac{\pi}{n}\right)$

(d) $\int_0^2 e^x dx = \lim_{n\to\infty}\sum_{k=1}^{n} e^{2k/n}\left(\frac{2}{n}\right)$

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3.6: The Fundamental Theorem of Calculus

Learning Objectives

- Describe the meaning of the Mean Value Theorem for Integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.
- Define the antiderivative of a function and methods for calculating it.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

Question of the Day

How are derivatives and integrals connected?

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3.6: The Fundamental Theorem of Calculus

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at some point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a, b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a, b]$. We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

Theorem 3.6.1: The Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

This formula can also be stated as

$$\int_a^b f(x)\, dx = f(c)(b-a)$$

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a, b]$. Then, for all x in $[a, b]$, we have $m \leq f(x) \leq M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$$

Proof

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a, b]$. Then, for all x in $[a, b]$, we have $m \leq f(x) \leq M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$$

Dividing by $b-a$ gives us

$$m \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq M$$

Since $\frac{1}{b-a} \int_a^b f(x)\ dx$ is a number between m and M, and since $f(x)$ is continuous and assumes the values m and M over $[a, b]$, by the Intermediate Value Theorem, there is a number c over $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and the proof is complete.

□

Example 3.6.1: Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0, 4]$ and find c such that $f(c)$ equals the average value of the function over $[0, 4]$.

Solution

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0}\int_0^4 (8-2x)\,dx$$

We can see in Figure 3.6.1 that the function represents a straight line and forms a right triangle bounded by the x- and y -axes.

The area of the triangle is $A = \frac{1}{2} \cdot \text{base} \cdot \text{height}$:

$$A = \frac{1}{2}(4)(8) = 16$$

Thus, the average value of the function is

$$\frac{1}{4-0}\int_0^4 (8-2x)\,dx = \frac{1}{4}(16) = 4$$

Set the average value equal to $f(c)$ and solve for c:

$$\begin{aligned} 8 - 2c &= 4 \\ c &= 2 \end{aligned}$$

Thus, we have shown $f(2) = \dfrac{1}{4-0}\displaystyle\int_0^4 f(x)\ dx$, so $\boxed{c = 2}$ satisfies the Mean Value Theorem for Integrals.

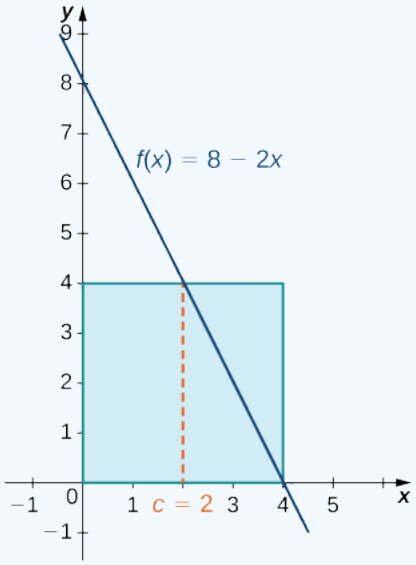


Figure 3.6.1: By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at c at least once over a closed interval.

Exercise 3.6.1

Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.6.2: Finding the Point Where a Function Takes on Its Average Value

Given $\int_0^3 x^2 \, dx = 9$, find c such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$

Solution

We are looking for the value of c such that

$$f(c) = \frac{1}{3-0} \int_0^3 x^2 \, dx = \frac{1}{3}(9) = 3$$

Replacing $f(c)$ with c^2, we have

$$c^2 = 3$$
$$c = \pm\sqrt{3}$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $\boxed{c = \sqrt{3}}$ (Figure 3.6.2).

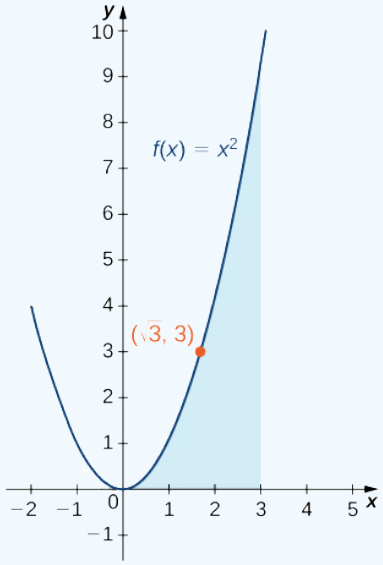


Figure 3.6.2: Over the interval $[0, 3]$, the function $f(x) = x^2$ takes on its average value at $c = \sqrt{3}$.

Exercise 3.6.2

Given $\int_1^4 (2x^2 - 1)\ dx = 39$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[1, 4]$.

Hint available at the end of this section.

Answer available at the end of this section.

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1,** is stated here. Part 1 establishes the relationship between differentiation and integration.

Theorem 3.6.2: The Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)\, dt$$

then $F'(x) = f(x)$ over $[a, b]$.

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point a to the point x. At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of x, the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of x.

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental Theorem of Calculus*. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Proof: Fundamental Theorem of Calculus, Part 1

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h}\int_x^{x+h} f(t)\ dt$ is just the average value of the function $f(x)$ over the interval $[x, x+h]$. Therefore, by the Mean Value Theorem for Integrals, there is some number c in $[x, x+h]$ such that

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c)$$

In addition, since c is between x and h, c approaches x as h approaches zero. Also, since $f(x)$ is continuous, we have

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$$

Putting all these pieces together, we have completed the proof:

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x)$$

□

Example 3.6.3: Finding a Derivative with the Fundamental Theorem of Calculus

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3+1}\,dt$$

Solution

According to the Fundamental Theorem of Calculus, the derivative is given by

$$\boxed{g'(x) = \frac{1}{x^3+1}}$$

Exercise 3.6.3

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2+4}\ dx$

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.6.4: Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Find $F'(x)$ for $F(x) = \int_1^{\sqrt{x}} \sin(t)\ dt$

Solution

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin(t)\ dt$

Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$F'(x) = \sin(u(x))\frac{du}{dx} = \sin(u(x)) \cdot \left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \boxed{\frac{\sin(\sqrt{x})}{2\sqrt{x}}}$$

Exercise 3.6.4

Find $F'(x)$ for $F(x) = \int_1^{x^3} \cos(t)\ dt$

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.6.5: Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Find $F'(x)$ for $F(x) = \int_x^{2x} t^3\ dt$

Solution

Both limits of integration in $F(x) = \int_x^{2x} t^3\ dt$ are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3\,dt \\ &= \int_x^0 t^3\,dt + \int_0^{2x} t^3\,dt \\ &= -\int_0^x t^3\,dt + \int_0^{2x} t^3\,dt \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3\,dt\right] = -x^3$$

Differentiating the second term, we first let $u(x) = 2x$. Then,

$$\begin{aligned} \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3\,dt\right] \\ &= (u(x))^3 \cdot \frac{du}{dx} \\ &= (2x)^3 \cdot 2 = 16x^3 \end{aligned}$$

Thus,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3\,dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] \\ &= -x^3 + 16x^3 \\ &= \boxed{15x^3} \end{aligned}$$

Exercise 3.6.5

Find $F'(x)$ for $F(x) = \int_x^{x^2} \cos(t)\ dt$

Hint available at the end of this section.

Answer available at the end of this section.

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Theorem 3.6.3: The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We often see the notation $F(x)\Big|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Proof

Let $P = x_i, i = 0, 1, \ldots, n$ be a regular partition of $[a, b]$. Then, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \end{aligned}$$

Now, we know F is an antiderivative of f over $[a, b]$, so by the Mean Value Theorem for derivatives for each $i = 0, 1, \ldots, n$ we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx$$

□

Illustration 3.6.1: Evaluating an Integral with the Fundamental Theorem of Calculus

We demonstrate how to use the Fundamental Theorem of Calculus, Part 2 to evaluate $\int_{-2}^{2} (t^2 - 4)\ dt$

Recall the power rule for Antiderivatives:

If $y = x^n$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

$$\begin{aligned} \int_{-2}^{2} (t^2 - 4)dt &= \left(\frac{t^3}{3} - 4t\right)\Big|_{-2}^{2} \\ &= \left[\frac{(2)^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right] \\ &= -\frac{32}{3} \end{aligned}$$

Analysis

Notice that we did not include the "$+C$" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2, any antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in Figure 3.6.3. Note that the region between the curve and the x-axis is all below the x-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

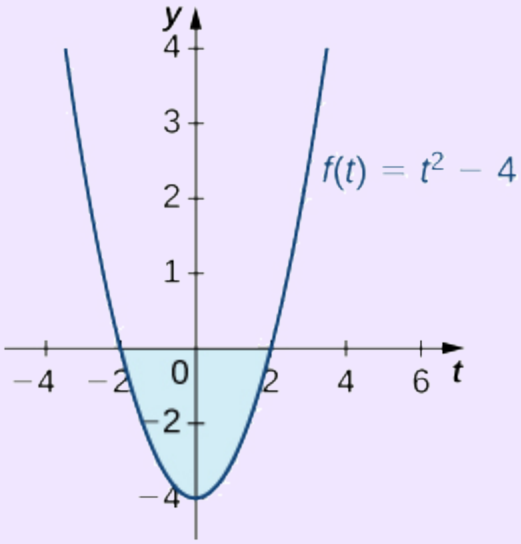


Figure 3.6.3: The evaluation of a definite integral can produce a negative value, even though area is always positive.

Example 3.6.6: Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2:

$$\int_1^9 \frac{x-1}{\sqrt{x}}dx$$

Solution

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

$$\int_1^9 \frac{x-1}{x^{\frac{1}{2}}}\,dx = \int_1^9 \left(\frac{x}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}}\right)\,dx = \int_1^9 \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\,dx$$

Now, integrate using the power rule:

$$\begin{aligned}\int_1^9 \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\,dx &= \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)\Big|_1^9 \\ &= \left[\frac{9^{\frac{3}{2}}}{\frac{3}{2}} - \frac{9^{\frac{1}{2}}}{\frac{1}{2}}\right] - \left[\frac{1^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^{\frac{1}{2}}}{\frac{1}{2}}\right] \\ &= \left[\frac{2}{3}(27) - 2(3)\right] - \left[\frac{2}{3}(1) - 2(1)\right] \\ &= \boxed{\frac{40}{3}}\end{aligned}$$

See Figure 3.6.4.

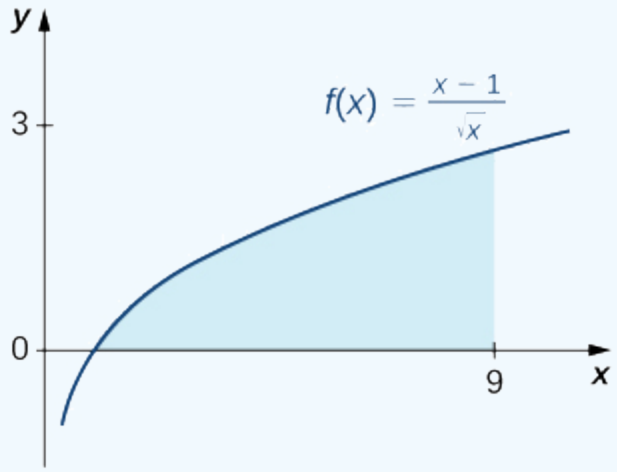


Figure 3.6.4: The area under the curve from $x = 1$ to $x = 9$ can be calculated by evaluating a definite integral.

Exercise 3.6.6

Evaluate $\int_1^2 x^{-4}\, dx$

Hint available at the end of this section.

Answer available at the end of this section.

Applications of Definite Integrals

Example 3.6.7: A Roller-Skating Race

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 10 + \cos(t)$ ft/sec, who is going to win the race?

Solution

We need to integrate both functions over the interval $[0, 5]$ and see which value is bigger. For James, we want to calculate

$$\int_0^5 (5 + 2t)\, dt$$

Using the power rule, we have

$$\begin{aligned}\int_0^5 (5 + 2t)\, dt &= \left(5t + t^2\right)\Big|_0^5 \\ &= (25 + 25) \\ &= 50\end{aligned}$$

Thus, James has skated 50 ft after 5 sec. Turning now to Kathy, we want to calculate

$$\begin{aligned}\int_0^5 (10+\cos(t))\,dt &= (10t+\sin(t))\Big|_0^5 \\ &= (50+\sin(5))-(0-\sin(0)) \\ &\approx 49.04\end{aligned}$$

Kathy has skated approximately 49.04 ft after 5 sec. **James wins**, but not by much!

Exercise 3.6.7

Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

Hint available at the end of this section.

Answer available at the end of this section.

Explore: A Parachutist in Free Fall

Julie is an avid skydiver with more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).



Figure 3.6.5: Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$.

She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower “belly down” position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?
4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground. On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec. Answer these questions based on this velocity:
5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear "**wingsuits**" (Figure 3.6.6). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.



Figure 3.6.6: The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air

Answers

Exercise 3.6.1

Use the procedures from Example 3.6.1 to solve the problem

Exercise 3.6.1

The average value is 1.5 and $c = 3$

Exercise 3.6.2

Use the procedures from Example 3.6.2 to solve the problem.

Exercise 3.6.2

$c = \sqrt{7}$

Exercise 3.6.3

Follow the procedures from Example 3.6.3 to solve the problem.

Exercise 3.6.3

$g\,'(r) = \sqrt{r^2 + 4}$

Exercise 3.6.4

Use the chain rule to solve the problem.

Exercise 3.6.4

$F\,'(x) = 3x^2 \cos\left(x^3\right)$

Exercise 3.6.5

Use the procedures from Example 3.6.5 to solve the problem

Exercise 3.6.5

$F\,'(x) = 2x \cos\left(x^2\right) - \cos(x)$

Exercise 3.6.6

Use the power rule.

Exercise 3.6.6

$\frac{7}{24}$

Exercise 3.6.7

Change the limits of integration from those in Example 3.6.7.

Exercise 3.6.7

Kathy wins: James skates 24 ft in 3 sec, but Kathy skates 30.14 ft in 3 sec.

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3.6: Antiderivatives

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in Derivatives of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s\,'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$ —that is, $a(t) = v\,'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v\,'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s\,'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (Introduction to Techniques of Integration).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F derivative f, we call F an antiderivative of f.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F\,'(x) = 2x$. Are there any other antiderivatives of f? Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives. Are there any others that are not of the form $x^2 + C$ for some constant C? The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F\,'(x) = G\,'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

Theorem 3.6.1: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;
II. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 3.6.1: Finding Antiderivatives

For each of the following functions, find all antiderivatives.

(a) $f(x) = 3x^2$

(b) $f(x) = \dfrac{1}{x}$

(c) $f(x) = \cos(x)$

(d) $f(x) = e^x$

Solution

(a) Because

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $\boxed{x^3 + C}$ is an antiderivative of $3x^2$.

(b) Let $f(x) = \ln|x|$. For $x > 0, f(x) = \ln(x)$ and

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$$

For $x < 0, f(x) = \ln(-x)$ and

$$\frac{d}{dx}\left(\ln(-x)\right) = -\frac{1}{-x} = \frac{1}{x}$$

Therefore,

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\dfrac{1}{x}$. Therefore, every antiderivative of $\dfrac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\boxed{\ln|x| + C}$ is an antiderivative of $\dfrac{1}{x}$.

(c) We have

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x)$$

so $F(x) = \sin(x)$ is an antiderivative of $\cos(x)$. Therefore, every antiderivative of $\cos(x)$ is of the form $\sin(x) + C$ for some constant C and every function of the form $\boxed{\sin(x) + C}$ is an antiderivative of $\cos(x)$.

(d) Since

$$\frac{d}{dx}\left(e^x\right) = e^x$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $\boxed{e^x + C}$ is an antiderivative of e^x.

Exercise 3.6.1

Find all antiderivatives of $f(x) = \sin(x)$

Answer available at the end of this section.

Antiderivative Shortcuts

We can see that all of our previous knowledge about derivatives has an equivalent statement about antiderivatives.

Table : Antiderivative Shortcut Formulas

$f(x) = F'(x)$	$F(x)$
0	C
k (a constant function)	$kx + C$
x^n for $n \neq -1$	$\frac{1}{n+1}x^{n+1} + C$
$\frac{1}{x}$	$\ln \lvert x \rvert + C$
e^x	$e^x + C$
b^x	$\frac{1}{\ln(b)}b^x + C$
$\cos(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\csc^2(x)$	$-\cot(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
$\csc(x)\cot(x)$	$-\csc(x) + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x) + C$
$\frac{1}{1+x^2}$	$\tan^{-1}(x) + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}(x) + C$

Similarly, we know our rules about differentiation can be turned into rules about anti-differentiation. Consider that when taking the derivative of $h(x) = x^2 + 3e^x$, we have

$$h'(x) = \frac{d}{dx}\left(x^2 + 3e^x\right) = \frac{d}{dx}\left(x^2\right) + 3\frac{d}{dx}\left(e^x\right) = 2x + 3 \cdot e^x = 2x + 3e^x$$

We know that the derivative distributes across addition (and subtraction), so we can differentiate each term in the sum separately. This means when taking antiderivatives, we can focus on taking the antiderivative of each term in a sum (or difference) separately and add (or subtract) the results to find the total antiderivative.

$$h(x) = f(x) + g(x) \implies h'(x) = f'(x) + g'(x) \quad \text{therefore,} \quad h'(x) = f'(x) + g'(x) \implies H(x) = F(x) + G(x)$$

Similarly, we know that multiplication by a constant is simple to handle for differentiation, and can be treated in a similar fashion for antiderivatives:

$$h(x) = k \cdot f(x) \implies h'(x) = k \cdot f'(x) \quad \text{therefore,} \quad h'(x) = k \cdot f'(x) \implies H(x) = k \cdot F(x)$$

We have seen that the rules for differentiating multiplication, division, and composition are significantly more complicated. These will require a special focus (in a later chapter) to find the equivalent antiderivative rules for. In this subchapter, we will avoid trying to reverse the product or quotient, or chain rule.

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A differential equation is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x)$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of equation are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation.

For example, the solutions of $\frac{dy}{dx} = 6x^2$ are given by $y(x) = 2x^3 + C$, since when we differentiate y we see $y'(x) = 6x^2$.

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$.

Definition: Initial-Value Problem

Finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition

$$y(x_0) = y_0$$

is an example of an **initial-value problem**. The condition $y(x_0) = y_0$ is known as an *initial condition*.

For example, looking for a function y that satisfies the differential equation $\frac{dy}{dx} = 6x^2$ with$y(1) = 5$ is an example of an initial-value problem.

Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

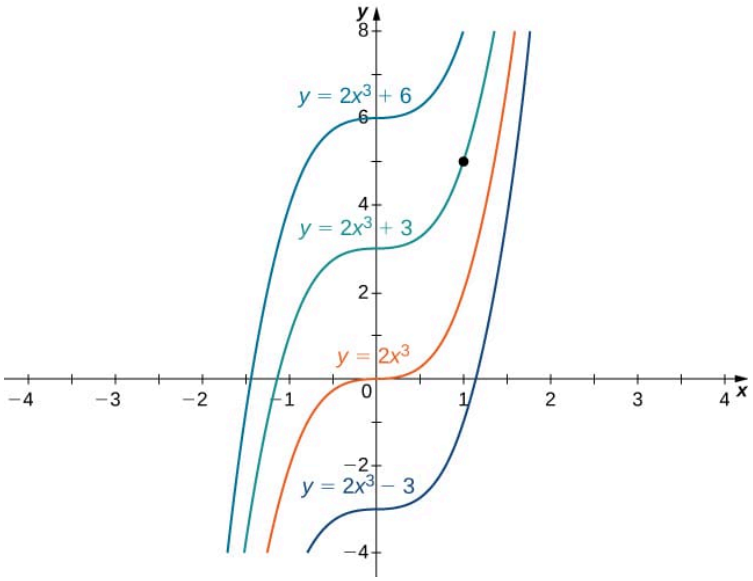


Figure $3.6.2$***:*** *Some of the solution curves of the differential equation* $\frac{dy}{dx} = 6x^2$ *are displayed. The function* $y = 2x^3 + 3$ *satisfies the differential equation and the initial condition* $y(1) = 5$.

Example $3.6.2$: Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin(x), \quad \text{with } y(0) = 5$$

Solution

First, we need to solve the differential equation. If $\frac{dy}{dx} = \sin(x)$, then $y = -\cos(x) + C$.

Next ,we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that

$$y(0) = -\cos(0) + C = 5 \implies -1 + C = 5 \implies C = 6$$

Thus, the solution of the initial-value problem is $\boxed{y = -\cos(x) + 6}$

Exercise $3.6.2$

Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}$ with $y(1) = 2$

Answer available at the end of this section.

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 3.6.3

A car is traveling at the rate of 60 ft/sec when the brakes are applied. The car begins decelerating at a constant rate of 10 ft/sec^2.

(a) How many seconds elapse before the car stops?

(b) How far does the car travel during that time?

Solution

(a) First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car is traveling at a rate of 60 ft/sec. Therefore, the initial velocity is $v(0) = 60$ ft/sec. Since the car is decelerating, the acceleration is $a(t) = -10$ ft/s^2\). Since the acceleration is the derivative of the velocity, we have $v\,'(t) = -10$.

Therefore, we have an initial-value problem to solve:

$$v\,'(t) = -10, \;\; v(0) = 60$$

This constant has simple antiderivative:

$$v(t) = -10t + C$$

Since $v(0) = 60$, we can see that $v(0) = -10 \cdot 0 + C \implies C = 60$. Thus, the velocity function is

$$v(t) = -10t + 60$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero.

Solving $-10t + 60 = 0$, we obtain $\boxed{t = 6 \text{ seconds}}$.

(b) To find how far the car travels during this time, we need to find the position of the car after 6 sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

$$s\,'(t) = -10t + 60, \;\; s(0) = 0$$

We use D to represents the (possibliy different) constant when taking this antiderivative:

$$s(t) = -10 \cdot \frac{1}{2}t^2 + 60 \cdot t + D = -5t^2 + 60t + D$$

Since $s(0) = 0$, we see that $s(0) = -5 \cdot 0^2 + 60 \cdot 0 + D \implies D = 0$. Therefore, the position function is

$$s(t) = -5t^2 + 60t$$

Plugging in $t = 6$, we have $s(6) = 180$. Therefore, the car stops after a distance of $\boxed{180 \text{ feet}}$.

Exercise 3.6.3

Suppose the car is decelerating at the rate of 6 ft/sec^2. How long does it take for the car to stop? How far will the car travel?

Answer available at the end of this section.

Answers

Exercise 3.6.1

$$-\cos(x)+C$$

Exercise 3.6.2

$$y=-\frac{3}{x}+5$$

Exercise 3.6.3

The car will stop in 10 seconds, traveling 300 feet.

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3.6: Practice Set

Exercise 3.6.1

Find the general antiderivative of each function.

(a) $f(x) = 3x^2$

(b) $f(x) = 4x^3 - 2x + 5$

(c) $f(x) = \dfrac{2}{x^3}$

(d) $f(x) = \sqrt{x}$

(e) $f(x) = e^x$

(f) $f(x) = \dfrac{1}{x}$

Answer available at the end of this section.

Exercise 3.6.2

Evaluate each definite integral using the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_0^2 3x^2 dx$

(b) $\displaystyle\int_1^3 (2x + 1) dx$

(c) $\displaystyle\int_0^1 (3x^2 - 2x + 1) dx$

(d) $\displaystyle\int_1^4 \sqrt{x} dx$

(e) $\displaystyle\int_0^2 e^x dx$

(f) $\displaystyle\int_1^e \frac{1}{x} dx$

Answer available at the end of this section.

Exercise 3.6.3

Find the particular antiderivative that satisfies the given condition.

(a) $f'(x) = 4x$, $f(1) = 3$

(b) $f'(x) = 3x^2 - 2$, $f(0) = 5$

(c) $f'(x) = \sqrt{x}$, $f(4) = 6$

(d) $f'(x) = e^x + 1$, $f(0) = 2$

(e) $f'(x) = \dfrac{1}{x}$, $f(e) = 3$

(f) $f'(x) = 2x + 3$, $f(1) = 4$

Answer available at the end of this section.

Exercise 3.6.4

Use the Fundamental Theorem of Calculus to evaluate each definite integral.

(a) $\int_{-1}^{2} (3x^2 - 2x + 1)dx$

(b) $\int_{0}^{1} (x^4 - 3x^2 + 2)dx$

(c) $\int_{1}^{4} \left(x^2 + \dfrac{1}{x^2}\right) dx$

(d) $\int_{0}^{2} (x - 1)(x + 3)dx$

(e) $\int_{0}^{\pi} \cos(x)dx$

(f) $\int_{0}^{\pi/2} \sin(x)dx$

Answer available at the end of this section.

Exercise 3.6.5

Find the derivative of each function using the Fundamental Theorem of Calculus.

(a) $F(x) = \int_{0}^{x} (t^2 + 1)dt$

(b) $F(x) = \int_{1}^{x} \sqrt{t}dt$

(c) $F(x) = \int_{0}^{x} e^t dt$

(d) $F(x) = \int_{0}^{x} \cos(t)dt$

(e) $F(x) = \int_{x}^{2} (3t^2 - 1)dt$

(f) $F(x) = \int_{0}^{x^2} \sqrt{t}dt$

Answer available at the end of this section.

Exercise 3.6.6

Find the area of the region bounded by the curve and the x-axis on the given interval.

(a) $f(x) = 2x + 1$ on $[0, 3]$

(b) $f(x) = x^2$ on $[0, 2]$

(c) $f(x) = 4 - x^2$ on $[0, 2]$

(d) $f(x) = \sin(x)$ on $[0, \pi]$

(e) $f(x) = e^x$ on $[0, 1]$

Answer available at the end of this section.

Exercise 3.6.7

Find the general antiderivative of each function.

(a) $f(x) = 5x^4 - 3x^2 + 2$

(b) $f(x) = \dfrac{3}{\sqrt{x}}$

(c) $f(x) = \dfrac{4}{x^2}$

(d) $f(x) = \sin(x) + \cos(x)$

(e) $f(x) = e^x + \dfrac{1}{x}$

(f) $f(x) = 2x^3 - \dfrac{1}{x^2} + 3e^x$

Answer available at the end of this section.

Exercise 3.6.8

Evaluate each definite integral using the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_1^2 (3x^2 - 4x + 1)dx$

(b) $\displaystyle\int_0^1 x(2x^2 - 1)dx$

(c) $\displaystyle\int_1^4 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

(d) $\displaystyle\int_0^{\pi/2} (\sin(x) + \cos(x))dx$

(e) $\displaystyle\int_0^1 (e^x - e^{-x})dx$

(f) $\int_1^4 \left(\frac{3}{x^2}+\frac{1}{x}\right)dx$

Answer available at the end of this section.

Exercise 3.6.9

Find the derivative of each function using the Fundamental Theorem of Calculus.

(a) $F(x)=\int_0^x \sin(t)dt$

(b) $F(x)=\int_2^x \frac{1}{t}dt$

(c) $F(x)=\int_x^1 (2t-1)dt$

(d) $F(x)=\int_0^{3x} e^t dt$

(e) $F(x)=\int_1^{x^2} (t^3+1)dt$

(f) $F(x)=\int_{x^2}^x \sqrt{t}dt$

Answer available at the end of this section.

Exercise 3.6.10

Find the area of the region bounded by the curve and the x-axis on the given interval. If the curve goes below the x-axis, compute the signed area and the total area.

(a) $f(x)=x^2-1$ on $[0,2]$

(b) $f(x)=\cos(x)$ on $[0,\pi]$

(c) $f(x)=x^3-2x$ on $[-1,2]$

Answer available at the end of this section.

Answers

Exercise 3.6.1

Recall: $\int x^n dx=\frac{x^{n+1}}{n+1}+C$ for $n\neq -1$, $\int \frac{1}{x}dx=\ln|x|+C$, $\int e^x dx=e^x+C$.

(a) $\int 3x^2dx=3\cdot\frac{x^3}{3}+C=x^3+C$

(b) $\int (4x^3-2x+5)dx=4\cdot\frac{x^4}{4}-2\cdot\frac{x^2}{2}+5x+C=x^4-x^2+5x+C$

(c) $\int \frac{2}{x^3}dx=\int 2x^{-3}dx=2\cdot\frac{x^{-2}}{-2}+C=-\frac{1}{x^2}+C$

(d) $\int \sqrt{x}dx=\int x^{1/2}dx=\frac{x^{3/2}}{3/2}+C=\frac{2}{3}x^{3/2}+C$

(e) $\int e^x dx=e^x+C$

(f) $\int \frac{1}{x}dx=\ln|x|+C$

Exercise 3.6.2

Recall: $\int_a^b f(x)dx = F(b) - F(a)$, where $F'(x) = f(x)$.

(a) $F(x) = x^3$. $\int_0^2 3x^2 dx = F(2) - F(0) = 8 - 0 = 8$

(b) $F(x) = x^2 + x$. $\int_1^3 (2x+1)dx = F(3) - F(1) = (9+3) - (1+1) = 12 - 2 = 10$

(c) $F(x) = x^3 - x^2 + x$. $\int_0^1 (3x^2 - 2x + 1)dx = F(1) - F(0) = (1 - 1 + 1) - 0 = 1$

(d) $F(x) = \frac{2}{3}x^{3/2}$. $\int_1^4 \sqrt{x}dx = F(4) - F(1) = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$

(e) $F(x) = e^x$. $\int_0^2 e^x dx = e^2 - e^0 = e^2 - 1$

(f) $F(x) = \ln|x|$. $\int_1^e \frac{1}{x}dx = \ln(e) - \ln(1) = 1 - 0 = 1$

Exercise 3.6.3

(a) $f(x) = \int 4x dx = 2x^2 + C$.

$f(1) = 2(1)^2 + C = 3 \implies 2 + C = 3 \implies C = 1$.

Thus $f(x) = 2x^2 + 1$.

(b) $f(x) = \int (3x^2 - 2)dx = x^3 - 2x + C$.

$f(0) = 0 - 0 + C = 5 \implies C = 5$.

Thus $f(x) = x^3 - 2x + 5$.

(c) $f(x) = \int \sqrt{x}dx = \frac{2}{3}x^{3/2} + C$.

$f(4) = \frac{2}{3}(4)^{3/2} + C = \frac{2}{3}(8) + C = \frac{16}{3} + C = 6 \implies C = \frac{2}{3}$.

Thus $f(x) = \frac{2}{3}x^{3/2} + \frac{2}{3} = \frac{2}{3}(x^{3/2} + 1)$.

(d) $f(x) = \int (e^x + 1)dx = e^x + x + C$.

$f(0) = e^0 + 0 + C = 1 + C = 2 \implies C = 1$.

Thus $f(x) = e^x + x + 1$.

(e) $f(x) = \int \frac{1}{x}dx = \ln|x| + C$. For $x > 0$, $f(x) = \ln(x) + C$.

$f(e) = \ln(e) + C = 1 + C = 3 \implies C = 2$.

Thus $f(x) = \ln(x) + 2$.

(f) $f(x) = \int (2x + 3)dx = x^2 + 3x + C$.

$f(1) = 1 + 3 + C = 4 \implies C = 0$.

Thus $f(x) = x^2 + 3x$.

Exercise 3.6.4

(a) $F(x) = x^3 - x^2 + x$.

$\int_{-1}^2 (3x^2 - 2x + 1)dx = F(2) - F(-1) = (8 - 4 + 2) - (-1 - 1 - 1) = 6 - (-3) = 9$

(b) $F(x) = \frac{x^5}{5} - x^3 + 2x$.

$\int_0^1 (x^4 - 3x^2 + 2)dx = F(1) - F(0) = \left(\frac{1}{5} - 1 + 2\right) - 0 = \frac{1}{5} + 1 = \frac{6}{5}$

(c) $F(x) = \frac{x^3}{3} - \frac{1}{x}$.

$\int_1^4 \left(x^2 + \frac{1}{x^2}\right)dx = F(4) - F(1) = \left(\frac{64}{3} - \frac{1}{4}\right) - \left(\frac{1}{3} - 1\right) = \frac{64}{3} - \frac{1}{4} - \frac{1}{3} + 1 = 21 - \frac{1}{4} + 1 = \frac{87}{4}$

(d) First expand: $(x-1)(x+3) = x^2 + 2x - 3$.

$F(x) = \frac{x^3}{3} + x^2 - 3x$.

$\int_0^2 (x^2 + 2x - 3)dx = F(2) - F(0) = \left(\frac{8}{3} + 4 - 6\right) - 0 = \frac{8}{3} - 2 = \frac{2}{3}$

(e) $F(x) = \sin(x)$.

$\int_0^\pi \cos(x)dx = \sin(\pi) - \sin(0) = 0 - 0 = 0$

(f) $F(x) = -\cos(x)$.

$\int_0^{\pi/2} \sin(x)dx = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 + 1 = 1$

Exercise 3.6.5

Recall: $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ and $\frac{d}{dx}\left[\int_a^{g(x)} f(t)dt\right] = f(g(x))g'(x)$.

(a) $F'(x) = x^2 + 1$

(b) $F'(x) = \sqrt{x}$

(c) $F'(x) = e^x$

(d) $F'(x) = \cos(x)$

(e) $F(x) = \int_x^2 (3t^2 - 1)dt = -\int_2^x (3t^2 - 1)dt$. Thus $F'(x) = -(3x^2 - 1) = 1 - 3x^2$

(f) Let $u = x^2$. Then $F(x) = \int_0^u \sqrt{t}dt$. $F'(x) = \sqrt{u} \cdot u' = \sqrt{x^2} \cdot 2x = |x| \cdot 2x = 2x|x|$. For $x \geq 0$, $F'(x) = 2x^2$. Thus $F'(x) = 2x|x|$.

Exercise 3.6.6

The area is given by $\int_a^b f(x)dx$ when $f(x) \geq 0$ on the interval.

(a) $\int_0^3 (2x + 1)dx = [x^2 + x]_0^3 = (9 + 3) - 0 = 12$

(b) $\int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$

(c) $\int_0^2 (4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \left(8 - \frac{8}{3}\right) - 0 = \frac{16}{3}$

(d) $\int_0^\pi \sin(x)dx = [-\cos(x)]_0^\pi = -\cos(\pi) - (-\cos(0)) = 1 + 1 = 2$

(e) $\int_0^1 e^x dx = [e^x]_0^1 = e - 1$

Exercise 3.6.7

(a) $\int (5x^4 - 3x^2 + 2)dx = 5 \cdot \frac{x^5}{5} - 3 \cdot \frac{x^3}{3} + 2x + C = x^5 - x^3 + 2x + C$

(b) $\int \frac{3}{\sqrt{x}}dx = \int 3x^{-1/2}dx = 3 \cdot \frac{x^{1/2}}{1/2} + C = 6\sqrt{x} + C$

(c) $\int \frac{4}{x^2}dx = \int 4x^{-2}dx = 4 \cdot \frac{x^{-1}}{-1} + C = -\frac{4}{x} + C$

(d) $\int (\sin(x) + \cos(x))dx = -\cos(x) + \sin(x) + C$

(e) $\int \left(e^x + \frac{1}{x}\right)dx = e^x + \ln|x| + C$

(f) $\int \left(2x^3 - \frac{1}{x^2} + 3e^x\right)dx = \frac{2x^4}{4} - \left(-\frac{1}{x}\right) + 3e^x + C = \frac{x^4}{2} + \frac{1}{x} + 3e^x + C$

Exercise 3.6.8

(a) $F(x) = x^3 - 2x^2 + x$.

$\int_1^2 (3x^2 - 4x + 1)dx = F(2) - F(1) = (8 - 8 + 2) - (1 - 2 + 1) = 2 - 0 = 2$

(b) Expand: $x(2x^2 - 1) = 2x^3 - x$. $F(x) = \frac{x^4}{2} - \frac{x^2}{2}$.

$\int_0^1 (2x^3 - x)dx = F(1) - F(0) = \left(\frac{1}{2} - \frac{1}{2}\right) - 0 = 0$

(c) $F(x) = \frac{2}{3}x^{3/2} - 2x^{1/2}$.

$\int_1^4 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = F(4) - F(1) = \left(\frac{2}{3}(8) - 2(2)\right) - \left(\frac{2}{3} - 2\right) = \left(\frac{16}{3} - 4\right) - \left(-\frac{4}{3}\right) = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

(d) $F(x) = -\cos(x) + \sin(x)$.

$\int_0^{\pi/2} (\sin(x) + \cos(x))dx = F\left(\frac{\pi}{2}\right) - F(0) = (0 + 1) - (-1 + 0) = 1 + 1 = 2$

(e) $F(x) = e^x + e^{-x}$.

$\int_0^1 (e^x - e^{-x})dx = F(1) - F(0) = (e + e^{-1}) - (1 + 1) = e + \frac{1}{e} - 2$

(f) $F(x) = -\frac{3}{x} + \ln|x|$.

$\int_1^4 \left(\frac{3}{x^2} + \frac{1}{x}\right)dx = F(4) - F(1) = \left(-\frac{3}{4} + \ln(4)\right) - (-3 + 0) = -\frac{3}{4} + \ln(4) + 3 = \frac{9}{4} + \ln(4)$

Exercise 3.6.9

(a) $F'(x) = \sin(x)$

(b) $F'(x) = \frac{1}{x}$

(c) $F(x) = \int_x^1 (2t - 1)dt = -\int_1^x (2t - 1)dt$. Thus $F'(x) = -(2x - 1) = 1 - 2x$

(d) Let $u = 3x$. $F(x) = \int_0^u e^t dt$. $F'(x) = e^u \cdot u' = e^{3x} \cdot 3 = 3e^{3x}$

(e) Let $u = x^2$. $F(x) = \int_1^u (t^3 + 1)dt$. $F'(x) = (u^3 + 1) \cdot u' = ((x^2)^3 + 1)(2x) = (x^6 + 1)(2x) = 2x^7 + 2x$

(f) $F(x) = \int_{x^2}^x \sqrt{t}dt = \int_{x^2}^0 \sqrt{t}dt + \int_0^x \sqrt{t}dt = -\int_0^{x^2} \sqrt{t}dt + \int_0^x \sqrt{t}dt$.

$F'(x) = -\sqrt{x^2} \cdot 2x + \sqrt{x} \cdot 1 = -|x| \cdot 2x + \sqrt{x}$.

For $x \geq 0$, $F'(x) = -2x^2 + \sqrt{x}$. For $x < 0$, $F'(x) = 2x^2 + \sqrt{x}$.

Exercise 3.6.10

(a) $f(x) = x^2 - 1$. On $[0, 1]$, $f(x) \leq 0$; on $[1, 2]$, $f(x) \geq 0$.

Signed area: $\int_0^2 (x^2 - 1)dx = \left[\frac{x^3}{3} - x\right]_0^2 = \left(\frac{8}{3} - 2\right) - 0 = \frac{2}{3}$

Total area: $\int_0^1 (1 - x^2)dx + \int_1^2 (x^2 - 1)dx = \left[x - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - x\right]_1^2$

$$= \left(1 - \frac{1}{3}\right) + \left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right] = \frac{2}{3} + \left(\frac{2}{3} + \frac{2}{3}\right) = \frac{2}{3} + \frac{4}{3} = 2$$

(b) $f(x) = \cos(x)$. On $[0, \pi/2]$, $f(x) \geq 0$; on $[\pi/2, \pi]$, $f(x) \leq 0$.

Signed area: $\int_0^{\pi} \cos(x)dx = [\sin(x)]_0^{\pi} = 0 - 0 = 0$

Total area: $\int_0^{\pi/2} \cos(x)dx + \int_{\pi/2}^{\pi} -\cos(x)dx = [\sin(x)]_0^{\pi/2} - [\sin(x)]_{\pi/2}^{\pi} = (1 - 0) - (0 - 1) = 1 + 1 = 2$

(c) $f(x) = x^3 - 2x = x(x^2 - 2)$. Roots: $x = 0, \pm\sqrt{2}$. On $[-1, 0]$, $f(x) \geq 0$; on $[0, \sqrt{2}]$, $f(x) \leq 0$; on $[\sqrt{2}, 2]$, $f(x) \geq 0$.

Signed area: $\int_{-1}^{2} (x^3 - 2x)dx = \left[\frac{x^4}{4} - x^2\right]_{-1}^{2} = (4 - 4) - \left(\frac{1}{4} - 1\right) = 0 - \left(-\frac{3}{4}\right) = \frac{3}{4}$

Total area: $\int_{-1}^{0} (x^3 - 2x)dx + \int_0^{\sqrt{2}} (2x - x^3)dx + \int_{\sqrt{2}}^{2} (x^3 - 2x)dx$

$$= \left[\frac{x^4}{4} - x^2\right]_{-1}^{0} + \left[x^2 - \frac{x^4}{4}\right]_0^{\sqrt{2}} + \left[\frac{x^4}{4} - x^2\right]_{\sqrt{2}}^{2}$$

$$= 0 - \left(\frac{1}{4} - 1\right) + (2 - 1) - 0 + (4 - 4) - (1 - 1) = \frac{3}{4} + 1 + 0 = \frac{7}{4}$$

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3.7: Applications of Definite and Indefinite Integrals

Learning Objectives

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

Intro here

Question of the Day

How do we interpret the numerical values of definite integrals?

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3.7: Applications for Definite Integrals

The Net Change Theorem

The **net change theorem** considers the integral of a **rate of change**. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Theorem 3.7.1: Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)\,dx \quad \text{or equivalently,} \quad \int_a^b F'(x)\,dx = F(b) - F(a)$$

Subtracting $F(a)$ from both sides of the top equation yields the bottom one. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a **velocity** function in which the result is **displacement.**

We looked at a simple example of this in The Definite Integral section. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 3.7.1.

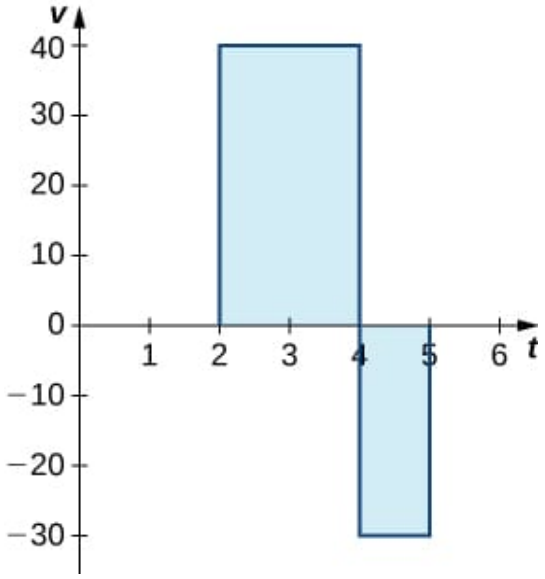


Figure 3.7.1: The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The **net displacement** is given by

$$\int_2^5 v(t)\,dt = \int_2^4 40\,dt + \int_4^5 -30\,dt = 80 - 30 = 50$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The **total distance** traveled is given by

$$\int_2^5 |v(t)|\,dt = \int_2^4 40\,dt + \int_4^5 30\,dt = 80 + 30 = 110$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example 3.7.1: Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution

Applying the net change theorem, we have

$$\int_0^3 (3t-5)\,dt = \left(\frac{3t^2}{2} - 5t\right)\Big|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0 = \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}$$

The net displacement is $\boxed{-\frac{3}{2} \text{ meters}}$

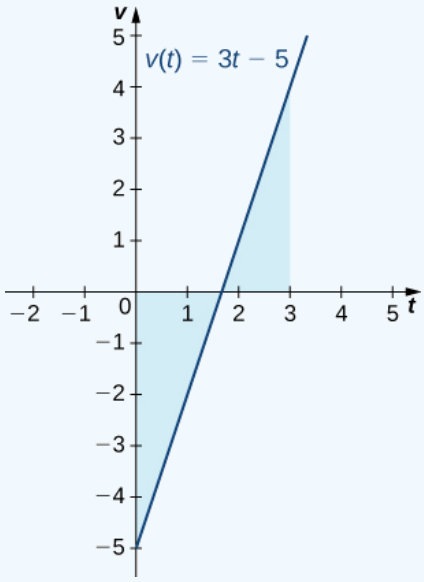


Figure 3.7.2: The graph shows velocity versus time for a particle moving with a linear velocity function.

Example 3.7.2: Finding the Total Distance Traveled

Use Example 3.7.2 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t-intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for t. Thus,

$$\begin{aligned} 3t-5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3} \end{aligned}$$

The two subintervals are $\left[0, \frac{5}{3}\right]$ and $\left[\frac{5}{3}, 3\right]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $\left[0, \frac{5}{3}\right]$, we have $|v(t)| = -v(t)$ over that interval. Over $\left[\frac{5}{3}, 3\right]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

$$\begin{aligned} \int_0^3 |v(t)|\,dt &= \int_0^{\frac{5}{3}} -v(t)\,dt + \int_{\frac{5}{3}}^3 v(t)\,dt \\ &= \int_0^{\frac{5}{3}} 5-3t\,dt + \int_{\frac{5}{3}}^3 3t-5\,dt \\ &= \left(5t - \frac{3t^2}{2}\right)\Big|_0^{\frac{5}{3}} + \left(\frac{3t^2}{2} - 5t\right)\Big|_{\frac{5}{3}}^3 \\ &= \left[5\left(\frac{5}{3}\right) - \frac{3\left(\frac{5}{3}\right)^2}{2}\right] - 0 + \left[\frac{27}{2} - 15\right] - \left[\frac{3\left(\frac{5}{3}\right)^2}{2} - \frac{25}{3}\right] \\ &= \frac{25}{3} - \frac{25}{6} + \frac{27}{2} - 15 - \frac{25}{6} + \frac{25}{3} \\ &= \frac{41}{6} \end{aligned}$$

So, the total distance traveled is $\boxed{\frac{41}{6}\text{meters}}$

Exercise 3.7.1

Find the net displacement and total distance traveled in meters given the velocity function $f(t) = \frac{1}{2}e^t - 2$ over the interval $[0, 2]$.

Hint available at the end of this section.

Answer available at the end of this section.

Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown below.

Example 3.7.3: How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at the rate of $5 - t^3$ gal/hr, how much gasoline is used in the first 2 hours?

Solution

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval [0,2]. We have

$$\begin{aligned}\int_0^2 (5-t^3)\,dt &= \left(5t-\frac{t^4}{4}\right)\Big|_0^2 \\ &= \left[5(2)-\frac{2^4}{4}\right]-\left[5(0)-\frac{0^4}{4}\right] \\ &= 6\end{aligned}$$

Thus, the motorboat uses $\boxed{6 \text{ gallons}}$ of gas in 2 hours.

Example 3.7.4: Chapter Opener: Iceboats

As we saw at the beginning of the chapter, top **iceboat** racers can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed.



Figure 3.7.3: (credit: modification of work by Carter Brown, Flickr)

Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t)=20t+5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t)=\begin{cases}20t+5 & \text{for } 0\le t\le \dfrac{1}{2} \\ 15 & \text{for } \dfrac{1}{2}\le t\le 1\end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\int_0^1 2v(t)\,dt$$

Substituting the expressions we were given for $v(t)$, we get

$$\begin{aligned}\text{Distance} &= \int_0^1 2v(t)\,dt \\ &= \int_0^{\frac{1}{2}} 2v(t)\,dt + \int_{\frac{1}{2}}^1 2v(t)\,dt \\ &= \int_0^{\frac{1}{2}} 2(20t+5)\,dt + \int_{\frac{1}{2}}^1 2(15)\,dt \\ &= \int_0^{\frac{1}{2}} (40t+10)\,dt + \int_{\frac{1}{2}}^1 30\,dt \\ &= \left(20t^2+10t\right)\Big|_0^{\frac{1}{2}} + \left(30t\right)\Big|_{\frac{1}{2}}^1 \\ &= \left(\frac{20}{4}+5\right) - 0 + (30-15) \\ &= 25\end{aligned}$$

Andrew is $\boxed{25 \text{ miles}}$ from his starting point after 1 hour.

Exercise 3.7.2

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t+5 & \text{for } 0 \le t \le \dfrac{1}{2} \\ -10t+15 & \text{for } \dfrac{1}{2} \le t \le 1 \end{cases}$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Hint available at the end of this section.

Answer available at the end of this section.

Integrating Even and Odd Functions

We saw in Functions and Graphs that an **even function** is a function in which $f(-x) = f(x)$ for all x in the domain—that is, the graph of the curve is unchanged when x is replaced with $-x$. The graphs of even functions are symmetric about the y-axis. An **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to a, involve two equal areas, because they are symmetric about the y-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the x-axis are equal.

Shortcut: Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Example 3.7.5: Integrating an Even Function

Integrate the even function $\int_{-2}^{2} \left(3x^8 - 2\right)\ dx$ and verify that the integration formula for even functions holds.

Solution

The symmetry appears in the graphs in Figure 3.7.4. Graph (a) shows the region below the curve and above the x-axis. We have to zoom in to this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the x-axis. The signed area of this region is negative. Both views illustrate the symmetry about the y-axis of an even function. We have

$$\begin{aligned}\int_{-2}^{2} \left(3x^8 - 2\right)\ dx &= \left(\frac{x^9}{3} - 2x\right)\Big|_{-2}^{2} \\ &= \left[\frac{2^9}{3} - 2(2)\right] - \left[\frac{(-2)^9}{3} - 2(-2)\right] \\ &= \left(\frac{512}{3} - 4\right) - \left(-\frac{512}{3} + 4\right) \\ &= \boxed{\frac{1000}{3}}\end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

$$\int_{0}^{2} \left(3x^8 - 2\right)\ dx = \left(\frac{x^9}{3} - 2x\right)\Big|_{0}^{2} = \frac{512}{3} - 4 = \frac{500}{3}$$

Since $2 \cdot \frac{500}{3} = \frac{1000}{3}$, we have verified the formula for even functions in this particular example.

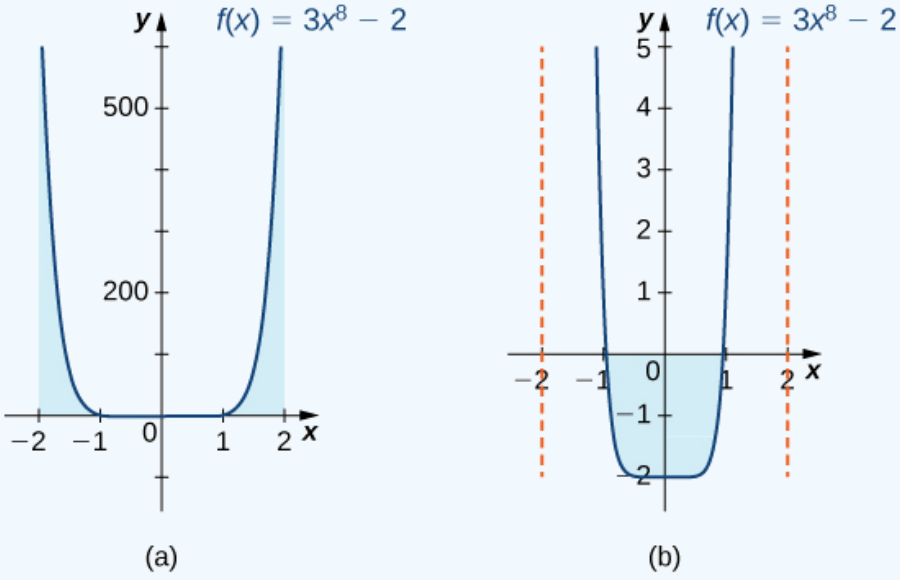


Figure 3.7.4: Graph (a) shows the positive area between the curve and the x-axis, whereas graph (b) shows the negative area between the curve and the x-axis. Both views show the symmetry about the y-axis.

Example 3.7.6: Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin(x)$ over the interval $[-\pi, \pi]$

Solution

The graph is shown in Figure 3.7.5. We can see the symmetry about the origin by the positive area above the x-axis over $[-\pi, 0]$, and the negative area below the x-axis over $[0, \pi]$. we have

$$\int_{-\pi}^{\pi} -5\sin(x)\,dx = -5(-\cos(x))\Big|_{-\pi}^{\pi} = 5\cos(x)\Big|_{-\pi}^{\pi} = 5\cos(\pi) - 5\cos(-\pi) = -5 - (-5) = \boxed{0}$$

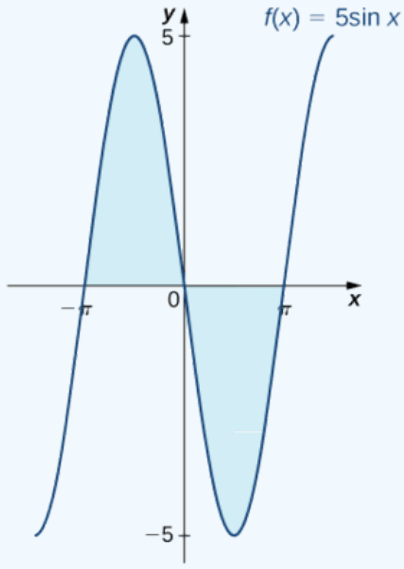


Figure 3.7.5:The graph shows areas between a curve and the x-axis for an odd function.

Exercise 3.7.3

Evaluate the integral $\int_{-2}^{2} x^4\ dx$

Hint available at the end of this section.

Answer available at the end of this section.

Answers

Exercise 3.7.1

Note that $f(t) \le 0$ for $t \le \ln(4)$ and $f(t) \ge 0$ for $t \ge \ln(4)$.

Exercise 3.7.1

Net displacement: $\frac{e^2 - 9}{2} \approx -0.8055$ m; total distance traveled: $4\ln 4 - 7.5 + \frac{e^2}{2} \approx 1.740$ m.

Exercise 3.7.2

Don't forget that Andrew's iceboat moves twice as fast as the wind.

Exercise 3.7.2

17.5 mi

Exercise 3.7.3

This is an even function.

Exercise 3.7.3

$\frac{64}{5}$

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3.7: Indefinite Integrals

Indefinite Integrals

We now look at the formal notation used to represent anti-derivatives and examine some of their properties. These properties allow us to find anti-derivatives of more complicated functions. Given a function f, we use the notation $f\,'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for anti-derivatives. If F is an anti-derivative of f, we say that $F(x)+C$ is the most general anti-derivative of f and write

$$\int f(x)\,dx = F(x)+C$$

The symbol $\int$ is called an integral sign, and $\int f(x)\ dx$ is called the indefinite integral of f.

Definition: Indefinite Integral

Given a function f, the indefinite integral of f, denoted

$$\int f(x)\,dx$$

is the most general anti-derivative of f. If F is an anti-derivative of f, then

$$\int f(x)\,dx = F(x)+C$$

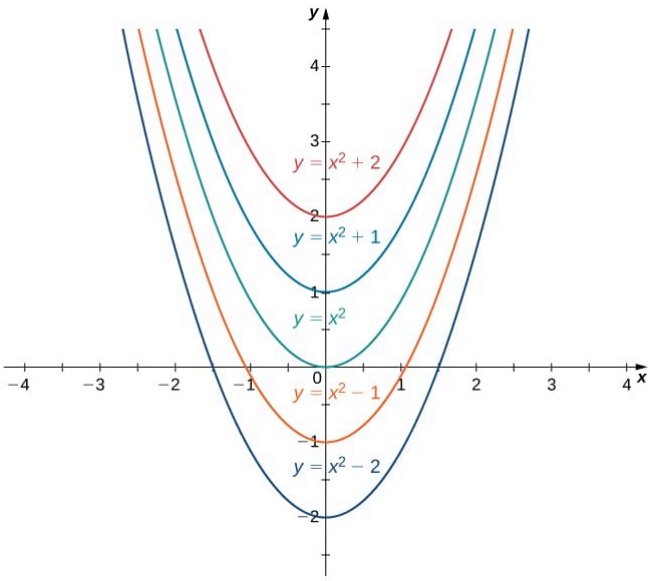
The expression $f(x)$ is called the integrand and the variable x is the variable of integration.

Given the terminology introduced in this definition, the act of finding the anti-derivatives of a function f is usually referred to as integrating f.

For a function f and an anti-derivative F, the functions $F(x)+C$, where C is any real number, is often referred to as the family of anti-derivatives of f. For example, since x^2 is an anti-derivative of $2x$ and any anti-derivative of $2x$ is of the form x^2+C, we write

$$\int 2x\,dx = x^2+C$$

The collection of all functions of the form x^2+C, where C is any real number, is known as the family of anti-derivatives of $2x$. The figure below shows a graph of this family of anti-derivatives.



Evaluating indefinite integrals for any function is the same as finding this family of anti-derivatives. This notation is used since we calculate anti-derivatives while solving definite integrals, the only difference is that we do not get numbers to plug in once we are done.

When claiming that

$$\int f(x)\,dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 3.7.7

Each of the following statements is of the form $\int f(x)\ dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$

(a) $\int (x + e^x)\ dx = \dfrac{x^2}{2} + e^x + C$

(b) $\int xe^x\ dx = xe^x - e^x + C$

Solution

(a) Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x$$

the statement

$$\int (x + e^x)\ dx = \frac{x^2}{2} + e^x + C$$

is correct.

(b) Using the product rule, we see that

$$\frac{d}{dx}\left(xe^x - e^x + C\right) = e^x + xe^x - e^x = xe^x$$

Therefore, the statement

$$\int xe^x\,dx = xe^x - e^x + C$$

is correct.

Note that we just verified an indefinite integral for a product. In general, the product of anti-derivatives is not an anti-derivative of a product.

Exercise 3.7.4

Verify that

$$\int x\cos(x)\,dx = x\sin(x) + \cos(x) + C$$

Answer available at the end of this section.

We have already discussed the properties of the anti-derivative that are inherited from differentiation rules. These properties are summarized below:

Theorem 3.7.2: Properties of Indefinite Integrals

Let F and G be anti-derivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int \big(f(x) \pm g(x)\big)\,dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int \big(kf(x)\big)\,dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with anti-derivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated (as we saw in Example 3.7.8b above). We look at and address integrals involving these more complicated functions in Introduction to Integration. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 3.7.8

Evaluate each of the following indefinite integrals:

(a) $\displaystyle\int (5x^3 - 7x^2 + 3x + 4)\,dx$

(b) $\displaystyle\int \frac{x^2 + 4\sqrt[3]{x}}{x}\,dx$

(c) $\displaystyle\int \frac{4}{1+x^2}\,dx$

(d) $\int \tan(x)\cos(x)\ dx$

Solution

(a) We can integrate each of the four terms in the integrand separately. We obtain

$$\int (5x^3 - 7x^2 + 3x + 4)\,dx = \int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx$$

Each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx = 5\int x^3\,dx - 7\int x^2\,dx + 3\int x\,dx + 4\int 1\,dx$$

Using the power rule for integrals, we conclude that

$$\int (5x^3 - 7x^2 + 3x + 4)\,dx = \boxed{\frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C}$$

(b) Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = x + 4x^{-\frac{2}{3}}$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\int \frac{x^2 + 4\sqrt[3]{x}}{x}\,dx = \int x\,dx + 4\int x^{-\frac{2}{3}}dx = \boxed{\frac{1}{2}x^2 + 12x^{\frac{1}{3}} + C}$$

(c) Write the integral as

$$\int \frac{4}{1+x^2}\,dx = 4\int \frac{1}{1+x^2}\,dx = \boxed{4\tan^{-1}(x) + C}$$

(d) Rewrite the integrand as

$$\tan(x)\cos(x) = \frac{\sin(x)}{\cos(x)}\cos(x) = \sin(x)$$

Therefore,

$$\int \tan(x)\cos(x)\,dx = \int \sin(x)\,dx = \boxed{-\cos(x) + C}$$

Answers

Exercise 3.7.4

$$\frac{d}{dx}(x\sin(x) + \cos(x) + C) = \sin(x) + x\cos(x) - \sin(x) = x\cos(x)$$

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3.7: Practice Set

Exercise 3.7.1

Find the general indefinite integral of each function.

(a) $\int (6x^2 - 4x + 3)dx$

(b) $\int \left(\frac{2}{x^3} + \frac{3}{x}\right) dx$

(c) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

(d) $\int (\sin(x) - 2\cos(x))dx$

(e) $\int (e^x - e^{-x})dx$

(f) $\int \left(3x^2 + \frac{2}{x^2} - 5e^x\right) dx$

Answer available at the end of this section.

Exercise 3.7.2

A particle moves along a line with velocity function $v(t) = 3t^2 - 2t + 1$ m/s. Find the displacement and total distance traveled over the interval $t \in [0, 3]$.

Answer available at the end of this section.

Exercise 3.7.3

A particle moves along a line with velocity function $v(t) = t^2 - 4t + 3$ m/s. Find the displacement and total distance traveled over the interval $t \in [0, 4]$.

Answer available at the end of this section.

Exercise 3.7.4

A particle moves along a line with velocity function $v(t) = 3t^2 - 12t + 9$ m/s. Find the displacement and total distance traveled over the interval $t \in [0, 4]$.

Answer available at the end of this section.

Exercise 3.7.5

A particle moves along a line with acceleration function $a(t) = 6t - 2$ m/s^2. Given $v(0) = 4$ m/s and $s(0) = 1$ m, find the position function $s(t)$.

Answer available at the end of this section.

Exercise 3.7.6

A particle moves along a line with acceleration function $a(t) = 12t$ m/s^2. Given $v(1) = 7$ m/s and $s(1) = 4$ m, find the position function $s(t)$.

Answer available at the end of this section.

Exercise 3.7.7

A particle moves along a line with velocity function $v(t) = \cos(t)$ m/s. Find the displacement and total distance traveled over the interval $t \in [0, \pi]$.

Answer available at the end of this section.

Exercise 3.7.8

A particle moves along a line with acceleration function $a(t) = 2e^t$ m/s^2. Given $v(0) = 3$ m/s and $s(0) = 2$ m, find the position function $s(t)$.

Answer available at the end of this section.

Exercise 3.7.9

A particle moves along a line with velocity function $v(t) = t^3 - 6t^2 + 9t$ m/s. Find the displacement and total distance traveled over the interval $t \in [0, 4]$.

Answer available at the end of this section.

Exercise 3.7.10

A particle moves along a line with velocity function $v(t) = 4t - t^2$ m/s. Find the displacement and total distance traveled over the interval $t \in [0, 5]$.

Answer available at the end of this section.

Exercise 3.7.11

A particle moves along a line with acceleration function $a(t) = \sin(t) + \cos(t)$ m/s^2. Given $v(0) = 1$ m/s and $s(0) = 0$ m, find the position function $s(t)$.

Answer available at the end of this section.

Exercise 3.7.12

A particle moves along a line with velocity function $v(t) = 2t - 4$ m/s. Find the displacement and total distance traveled over the interval $t \in [0, 4]$.

Answer available at the end of this section.

Exercise 3.7.13

A particle moves along a line with velocity function $v(t) = 3\sin(t)$ m/s. Find the displacement and total distance traveled over the interval $t \in [0, 2\pi]$.

Answer available at the end of this section.

Exercise 3.7.14

A particle moves along a line with acceleration function $a(t) = 6t + 4$ m/s^2. Given $v(0) = -2$ m/s and $s(1) = 3$ m, find the position function $s(t)$.

Answer available at the end of this section.

Exercise 3.7.15

A particle moves along a line with acceleration function $a(t) = \dfrac{6}{t^3}$ m/s^2. Given $v(1) = 4$ m/s and $s(1) = 2$ m, find the position function $s(t)$ for $t > 0$.

Answer available at the end of this section.

Answers

Exercise 3.7.1

Recall: $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ for $n \neq -1$, $\int \dfrac{1}{x} dx = \ln|x| + C$, $\int e^x dx = e^x + C$, $\int \sin(x) dx = -\cos(x) + C$, $\int \cos(x) dx = \sin(x) + C$.

(a) $\int (6x^2 - 4x + 3) dx = 6 \cdot \dfrac{x^3}{3} - 4 \cdot \dfrac{x^2}{2} + 3x + C = 2x^3 - 2x^2 + 3x + C$

(b) $\int \left(\dfrac{2}{x^3} + \dfrac{3}{x}\right) dx = \int (2x^{-3} + \dfrac{3}{x}) dx = 2 \cdot \dfrac{x^{-2}}{-2} + 3\ln|x| + C = -\dfrac{1}{x^2} + 3\ln|x| + C$

(c) $\int \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right) dx = \int (x^{1/2} + x^{-1/2}) dx = \dfrac{x^{3/2}}{3/2} + \dfrac{x^{1/2}}{1/2} + C = \dfrac{2}{3}x^{3/2} + 2\sqrt{x} + C$

(d) $\int (\sin(x) - 2\cos(x)) dx = -\cos(x) - 2\sin(x) + C$

(e) $\int (e^x - e^{-x}) dx = e^x + e^{-x} + C$

(f) $\int \left(3x^2 + \dfrac{2}{x^2} - 5e^x\right) dx = \int (3x^2 + 2x^{-2} - 5e^x) dx = x^3 - \dfrac{2}{x} - 5e^x + C$

Exercise 3.7.2

Recall: Displacement $= \int_a^b v(t) dt$. Total distance $= \int_a^b |v(t)| dt$.

Displacement: $\int_0^3 (3t^2 - 2t + 1) dt = \left[t^3 - t^2 + t\right]_0^3 = (27 - 9 + 3) - 0 = 21$ m.

For total distance, check if $v(t)$ changes sign on $[0, 3]$.

$v(t) = 3t^2 - 2t + 1$. The discriminant is $4 - 12 = -8 < 0$, so $v(t) > 0$ for all t.

Thus total distance equals displacement: 21 m.

Exercise 3.7.3

Displacement: $\int_0^4 (t^2 - 4t + 3) dt = \left[\dfrac{t^3}{3} - 2t^2 + 3t\right]_0^4 = \left(\dfrac{64}{3} - 32 + 12\right) - 0 = \dfrac{64}{3} - 20 = \dfrac{4}{3}$ m.

For total distance, find where $v(t) = 0$: $t^2 - 4t + 3 = (t-1)(t-3) = 0 \implies t = 1, 3$.

On $[0, 1]$, $v(t) \geq 0$. On $[1, 3]$, $v(t) \leq 0$. On $[3, 4]$, $v(t) \geq 0$.

Total distance:

$$\int_0^1 (t^2 - 4t + 3) dt + \int_1^3 -(t^2 - 4t + 3) dt + \int_3^4 (t^2 - 4t + 3) dt$$

$$= \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^1 - \left[\frac{t^3}{3} - 2t^2 + 3t\right]_1^3 + \left[\frac{t^3}{3} - 2t^2 + 3t\right]_3^4$$

Let $F(t) = \frac{t^3}{3} - 2t^2 + 3t$.

$F(0) = 0$, $F(1) = \frac{1}{3} - 2 + 3 = \frac{4}{3}$, $F(3) = 9 - 18 + 9 = 0$, $F(4) = \frac{64}{3} - 32 + 12 = \frac{4}{3}$.

Total distance: $(F(1) - F(0)) - (F(3) - F(1)) + (F(4) - F(3)) = \frac{4}{3} - \left(0 - \frac{4}{3}\right) + \frac{4}{3} = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ m.

Exercise 3.7.4

Displacement: $\int_0^4 (3t^2 - 12t + 9)dt = \left[t^3 - 6t^2 + 9t\right]_0^4 = (64 - 96 + 36) - 0 = 4$ m.

Find where $v(t) = 0$: $3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-1)(t-3) = 0 \implies t = 1, 3$.

On $[0, 1]$, $v(t) \geq 0$. On $[1, 3]$, $v(t) \leq 0$. On $[3, 4]$, $v(t) \geq 0$.

Let $F(t) = t^3 - 6t^2 + 9t$.

$F(0) = 0$, $F(1) = 1 - 6 + 9 = 4$, $F(3) = 27 - 54 + 27 = 0$, $F(4) = 64 - 96 + 36 = 4$.

Total distance: $(F(1) - F(0)) - (F(3) - F(1)) + (F(4) - F(3)) = 4 - (0 - 4) + 4 = 4 + 4 + 4 = 12$ m.

Exercise 3.7.5

$v(t) = \int a(t)dt = \int (6t - 2)dt = 3t^2 - 2t + C$.

Using $v(0) = 4$: $4 = 3(0)^2 - 2(0) + C \implies C = 4$.

Thus $v(t) = 3t^2 - 2t + 4$.

$s(t) = \int v(t)dt = \int (3t^2 - 2t + 4)dt = t^3 - t^2 + 4t + C$.

Using $s(0) = 1$: $1 = 0 - 0 + 0 + C \implies C = 1$.

Thus $s(t) = t^3 - t^2 + 4t + 1$.

Exercise 3.7.6

$v(t) = \int 12t dt = 6t^2 + C$.

Using $v(1) = 7$: $7 = 6(1)^2 + C \implies C = 1$.

Thus $v(t) = 6t^2 + 1$.

$s(t) = \int (6t^2 + 1)dt = 2t^3 + t + C$.

Using $s(1) = 4$: $4 = 2(1)^3 + 1 + C \implies C = 1$.

Thus $s(t) = 2t^3 + t + 1$.

Exercise 3.7.7

Displacement: $\int_0^\pi \cos(t)dt = [\sin(t)]_0^\pi = 0 - 0 = 0$ m.

For total distance: $v(t) = \cos(t)$ changes sign at $t = \frac{\pi}{2}$.

On $[0, \pi/2]$, $v(t) \geq 0$. On $[\pi/2, \pi]$, $v(t) \leq 0$.

Total distance: $\int_0^{\pi/2} \cos(t)dt + \int_{\pi/2}^{\pi} -\cos(t)dt = [\sin(t)]_0^{\pi/2} - [\sin(t)]_{\pi/2}^{\pi} = (1 - 0) - (0 - 1) = 1 + 1 = 2$ m.

Exercise 3.7.8

$v(t) = \int 2e^t dt = 2e^t + C$.

Using $v(0) = 3$: $3 = 2e^0 + C \implies C = 1$.

Thus $v(t) = 2e^t + 1$.

$s(t) = \int (2e^t + 1)dt = 2e^t + t + C$.

Using $s(0) = 2$: $2 = 2e^0 + 0 + C \implies C = 0$.

Thus $s(t) = 2e^t + t$.

Exercise 3.7.9

Displacement: $\int_0^4 (t^3 - 6t^2 + 9t)dt = \left[\frac{t^4}{4} - 2t^3 + \frac{9t^2}{2}\right]_0^4 = (64 - 128 + 72) - 0 = 8$ m.

Find where $v(t) = 0$: $t^3 - 6t^2 + 9t = t(t^2 - 6t + 9) = t(t-3)^2 = 0 \implies t = 0, 3$.

On $[0, 3]$, $v(t) \geq 0$. On $[3, 4]$, $v(t) \geq 0$ (since $t \geq 0$ and $(t-3)^2 \geq 0$).

Thus total distance equals displacement: 8 m.

Exercise 3.7.10

Displacement: $\int_0^5 (4t - t^2)dt = \left[2t^2 - \frac{t^3}{3}\right]_0^5 = \left(50 - \frac{125}{3}\right) - 0 = \frac{150 - 125}{3} = \frac{25}{3}$ m.

Find where $v(t) = 0$: $4t - t^2 = t(4 - t) = 0 \implies t = 0, 4$.

On $[0, 4]$, $v(t) \geq 0$. On $[4, 5]$, $v(t) \leq 0$.

Let $F(t) = 2t^2 - \frac{t^3}{3}$.

$F(0) = 0$, $F(4) = 32 - \frac{64}{3} = \frac{96 - 64}{3} = \frac{32}{3}$, $F(5) = 50 - \frac{125}{3} = \frac{150 - 125}{3} = \frac{25}{3}$.

Total distance: $(F(4) - F(0)) - (F(5) - F(4)) = \frac{32}{3} - \left(\frac{25}{3} - \frac{32}{3}\right) = \frac{32}{3} + \frac{7}{3} = \frac{39}{3} = 13$ m.

Exercise 3.7.11

$v(t) = \int (\sin(t) + \cos(t))dt = -\cos(t) + \sin(t) + C$.

Using $v(0) = 1$: $1 = -1 + 0 + C \implies C = 2$.

Thus $v(t) = -\cos(t) + \sin(t) + 2$.

$s(t) = \int(-\cos(t) + \sin(t) + 2)dt = -\sin(t) - \cos(t) + 2t + C$.

Using $s(0) = 0$: $0 = 0 - 1 + 0 + C \implies C = 1$.

Thus $s(t) = -\sin(t) - \cos(t) + 2t + 1$.

Exercise 3.7.12

Displacement: $\int_0^4 (2t-4)dt = \left[t^2 - 4t\right]_0^4 = (16-16) - 0 = 0$ m.

Find where $v(t) = 0$: $2t - 4 = 0 \implies t = 2$.

On $[0, 2]$, $v(t) \leq 0$. On $[2, 4]$, $v(t) \geq 0$.

Let $F(t) = t^2 - 4t$.

$F(0) = 0$, $F(2) = 4 - 8 = -4$, $F(4) = 16 - 16 = 0$.

Total distance: $-(F(2) - F(0)) + (F(4) - F(2)) = -(-4 - 0) + (0 - (-4)) = 4 + 4 = 8$ m.

Exercise 3.7.13

Displacement: $\int_0^{2\pi} 3\sin(t)dt = [-3\cos(t)]_0^{2\pi} = -3(1) - (-3(1)) = -3 + 3 = 0$ m.

For total distance: $v(t) = 3\sin(t)$ changes sign at $t = \pi$.

On $[0, \pi]$, $v(t) \geq 0$. On $[\pi, 2\pi]$, $v(t) \leq 0$.

Total distance: $\int_0^{\pi} 3\sin(t)dt + \int_{\pi}^{2\pi} -3\sin(t)dt = [-3\cos(t)]_0^{\pi} + [3\cos(t)]_{\pi}^{2\pi}$

$$= (-3\cos(\pi) + 3\cos(0)) + (3\cos(2\pi) - 3\cos(\pi)) = (3+3) + (3-(-3)) = 6 + 6 = 12 \text{ m}$$

Exercise 3.7.14

$v(t) = \int(6t + 4)dt = 3t^2 + 4t + C$.

Using $v(0) = -2$: $-2 = 0 + 0 + C \implies C = -2$.

Thus $v(t) = 3t^2 + 4t - 2$.

$s(t) = \int(3t^2 + 4t - 2)dt = t^3 + 2t^2 - 2t + C$.

Using $s(1) = 3$: $3 = 1 + 2 - 2 + C \implies C = 2$.

Thus $s(t) = t^3 + 2t^2 - 2t + 2$.

Exercise 3.7.15

$a(t) = 6t^{-3}$. $v(t) = \int 6t^{-3}dt = 6 \cdot \dfrac{t^{-2}}{-2} + C = -\dfrac{3}{t^2} + C$.

Using $v(1) = 4$: $4 = -3 + C \implies C = 7$.

Thus $v(t) = -\dfrac{3}{t^2} + 7$.

$s(t) = \int\left(-\dfrac{3}{t^2} + 7\right)dt = \int(-3t^{-2} + 7)dt = \dfrac{3}{t} + 7t + C$.

Using $s(1) = 2$: $2 = 3 + 7 + C \implies C = -8$.

Thus $s(t) = \dfrac{3}{t} + 7t - 8$.

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3.8: Substitution

Learning Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called *integration by substitution*, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

Question of the Day

How do we integrate a function that is the result of the chain rule?

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3.8: Substitution in Indefinite Integrals

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f\left(g(x)\right)g'(x)\ dx$.

For example, in the integral

$$\int \left(x^2-3\right)^3 \cdot 2x\,dx$$

we have $f(x)=x^3$ and $g(x)=x^2-3$. We know $g'(x)=2x$, so

$$f\left(g(x)\right)g'(x)=\left(x^2-3\right)^3 \cdot 2x$$

and we see that our integrand is in the correct form. The method is called substitution because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as *change of variables* because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Theorem 3.8.1: Substitution with Indefinite Integrals

Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\int f\left(g(x)\right)g'(x)\,dx=\int f(u)\,du=F(u)+C=F\left(g(x)\right)+C$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\left[F(g(x))\right]=F'(g(x))\,g'(x)=f\left(g(x)\right)g'(x)$$

Integrating both sides with respect to x, we see that

$$\int f\left(g(x)\right)g'(x)\,dx=F\left(g(x)\right)+C$$

If we now substitute $u=g(x)$, and $du=g'(x)\ dx$, we get

$$\int f\left(g(x)\right)g'(x)\,dx=\int f(u)\,du=F(u)+C=F\left(g(x)\right)+C$$

□

Returning to the problem we looked at originally, we let $u=x^2-3$ and then $du=2x\ dx$.

Rewrite the original integral in terms of u:

$$\int \left(x^2-3\right)^3 \cdot (2x\,dx)=\int u^3\,du$$

Using the power rule for integrals, we have

$$\int u^3\,du = \frac{u^4}{4} + C$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2-3)^4}{4} + C$$

It is a straightforward application of the chain rule to check this is the correct anti-derivative. We generalize the procedure in the following Problem-Solving Strategy.

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$. such that $g\,'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g\,'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Example 3.8.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x\left(3x^2+4\right)^4\ dx$

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\ dx$ and we already have du in the integrand.

Write the integral in terms of u:

$$\int 6x\left(3x^2+4\right)^4\,dx = \int \left(3x^2+4\right)^4 \cdot (6x\,dx) = \int u^4\,du$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C = \boxed{\frac{1}{5}\left(3x^2+4\right)^5 + C}$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand.
Picking a value for C of 1, we let $y = \frac{1}{5}\left(3x^2+4\right)^5 + 1$. We have

$$y' = \frac{1}{5}\cdot 5\left(3x^2+4\right)^4 \cdot 6x = 6x\left(3x^2+4\right)^4$$

This is exactly the expression we started with inside the integrand.

Exercise 3.8.1

Use substitution to find the antiderivative of $\int 3x^2\left(x^3-3\right)^2\ dx$

Hint available at the end of this section.

Answer available at the end of this section.

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

Example 3.8.2: Using Substitution with Alteration

Use substitution to evaluate $\int z\sqrt{z^2-5}\ dz$

Solution

Rewrite the integral as $\int z(z^2-5)^{\frac{1}{2}}\ dz$. Let $u=z^2-5$ and $du=2z\ dz$. Now we have a problem because $du=2z\ dz$ and the original expression has only $z\ dz$. We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u=z^2-5 \quad \text{yields } du=2z\,dz \leftrightarrow \frac{du}{2}=z\,dz$$

Write the integral in terms of u:

$$\int (z^2-5)^{\frac{1}{2}}\cdot(z\,dz)=\int u^{\frac{1}{2}}\,\frac{du}{2}$$

We simplify and then integrate the expression in u:

$$\begin{aligned}\int u^{\frac{1}{2}}\,\frac{du}{2} &= \frac{1}{2}\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C \\ &= \frac{1}{3}u^{\frac{3}{2}}+C \\ &= \boxed{\frac{1}{3}\left(z^2-5\right)^{\frac{3}{2}}+C}\end{aligned}$$

Exercise 3.8.2

Use substitution to evaluate $\int x^2\left(x^3+5\right)^9\ dx$

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.8.3: Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate $\int \frac{\sin(t)}{\cos^3(t)}\ dt$

Solution

We know the derivative of $\cos(t)$ is $-\sin(t)$, so we set $u=\cos(t)$. Then $du=-\sin(t)\ dt \leftrightarrow -du=\sin(t)\ dt$

Substituting into the integral, we have

$$\int \frac{\sin(t)}{\cos^3(t)}\,dt = \int \frac{-du}{u^3} = -\int u^{-3}\,du$$

Evaluating the integral, we get

$$-\int u^{-3}\,du = -\left(-\frac{u^{-2}}{2}\right) + C = \frac{1}{2}u^{-2} + C$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin(t)}{\cos^3(t)}\,dt = \boxed{\frac{1}{2\cos^2(t)} + C}$$

Exercise 3.8.3

Use substitution to evaluate the integral $\int \frac{\cos(t)}{\sin^4(t)}\,dt$

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 3.8.4

Use substitution to evaluate the indefinite integral $\int \cos^3(t)\sin(t)\,dt$

Hint available at the end of this section.

Answer available at the end of this section.

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, u should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of u. This technique should become clear in the next example.

Example 3.8.4: Finding an Antiderivative Using u-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\,dx$$

Solution

If we let $u = x - 1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We need to express x in terms of u. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u :

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)\,du = \int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)\,du$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}\int (u^{\frac{1}{2}} + u^{-\frac{1}{2}})\, du &= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C \\ &= \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C \\ &= (x-1)^{\frac{1}{2}}\left[\frac{2}{3}(x-1) + 2\right] + C \\ &= (x-1)^{\frac{1}{2}}\left(\frac{2}{3}x + \frac{4}{3}\right) \\ &= \boxed{\frac{2}{3}(x-1)^{\frac{1}{2}}(x+2) + C}\end{aligned}$$

Answers

Exercise 3.8.1

Let $u = x^3 - 3$.

Exercise 3.8.1

$$\int 3x^2 \left(x^3 - 3\right)^2 \, dx = \frac{1}{3}\left(x^3 - 3\right)^3 + C$$

Exercise 3.8.2

Use $u = x^3 + 5$ and $\frac{du}{3} = x^2 \, dx$

Exercise 3.8.2

$$\int x^2 \left(x^3 + 5\right)^9 \, dx = \frac{1}{30}\left(x^3 + 5\right)^{10} + C$$

Exercise 3.8.3

Use the process from Example 3.8.3 to solve the problem.

Exercise 3.8.3

$$\int \frac{\cos(t)}{\sin^4(t)} \, dt = -\frac{1}{3\sin^3(t)} + C$$

Exercise 3.8.4

Use the process from Example 3.8.3 to solve the problem.

Exercise 3.8.4

$$\int \cos^3(t)\sin(t) \, dt = -\frac{\cos^4(t)}{4} + C$$

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3.8: Substution in Definite Integrals

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Theorem 3.8.2: Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f\left(g(x)\right) g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f\left(g(x)\right) g'(x)\, dx = F\left(g(x)\right) + C$$

Then

$$\begin{aligned}\int_a^b f\left(g(x)\right) g'(x)\, dx &= F\left(g(x)\right)\Big|_{x=a}^{x=b} \\ &= F\left(g(b)\right) - F\left(g(a)\right) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\, du\end{aligned}$$

and we have the desired result.

Example 3.8.5: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2 \left(1 + 2x^3\right)^5\, dx$$

Solution

Let $u = 1 + 2x^3$, so $du = 6x^2\ dx \leftrightarrow \dfrac{du}{6} = x^2\ dx$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1$, and when $x = 1,\ \ u = 1 + 2(1) = 3$.

Then

$$\int_0^1 x^2 \left(1 + 2x^3\right)^5\, dx = \int_0^1 \left(1 + 2x^3\right)^5 \left(x^2\, dx\right) = \int_0^1 u^5 \cdot \left(\frac{du}{6}\right) = \frac{1}{6}\int_1^3 u^5\, du$$

Evaluating this expression, we get

$$\frac{1}{6}\int_1^3 u^5\,du = \frac{1}{6}\cdot\frac{u^6}{6}\Big|_1^3$$
$$= \frac{1}{36}\left[3^6 - 1^6\right]$$
$$= \boxed{\frac{182}{9}}$$

Exercise 3.8.5

Use substitution to evaluate $\int_{-1}^{0} y\left(2y^2-3\right)^5\ dy$

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 3.8.6

Use substitution to evaluate $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\ dx$

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.8.6: Using Substitution with an Exponential Function

Use substitution to evaluate

$$\int_0^1 xe^{4x^2+3}\,dx$$

Solution

Let $u = 4x^3 + 3$. Then, $du = 8x\ dx$. To adjust the limits of integration, we note that when $x = 0,\ u = 3$, and when $x = 1,\ u = 7$. So our substitution gives

$$\int_0^1 xe^{4x^2+3}\,dx = \int_3^7 e^u\cdot\left(\frac{du}{8}\right)$$
$$= \frac{1}{8}\int_3^7 e^u\,du$$
$$= \frac{1}{8}e^u\Big|_3^7$$
$$= \boxed{\frac{e^7-e^3}{8}}$$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for u after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in Example 3.8.7.

Example 3.8.7: Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate

$$\int_0^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta$$

Solution

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$ allows us to rewrite the integral as

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta &= \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right) d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta + \int_0^{\frac{\pi}{2}} \cos(2\theta)\, d\theta\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral.

Let $u = 2\theta$. Then, $du = 2\ d\theta$, or $\dfrac{1}{2}\ du = d\theta$. Also, when $\theta = 0,\ u = 0$, and when $\theta = \pi/2,\ u = \pi$. Expressing the second integral in terms of u, we have

$$\begin{aligned}\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\, d\theta &= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\pi} \cos u \frac{du}{2} \\ &= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} + \frac{1}{4}\sin(u)\Big|_{u=0}^{u=\pi} \\ &= \left(\frac{\pi}{4}-0\right)+(0-0) = \boxed{\frac{\pi}{4}}\end{aligned}$$

Answers

Exercise 3.8.5

Use the steps from Example 3.8.5 to solve the problem.

Exercise 3.8.5

$\displaystyle\int_{-1}^{0} y\left(2y^2-3\right)^5\ dy = \frac{91}{3}$

Exercise 3.8.6

Use the process from Example 3.8.5 to solve the problem.

Exercise 3.8.6

$\displaystyle\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right)\ dx = \frac{2}{3\pi}$

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3.8: Practice Set

Exercise 3.8.1

Evaluate each indefinite integral using substitution.

(a) $\int 2x(x^2+1)^4 dx$

(b) $\int x^2\sqrt{x^3+1}dx$

(c) $\int e^{3x}dx$

(d) $\int xe^{x^2}dx$

(e) $\int \frac{2x}{x^2+1}dx$

(f) $\int \sin(2x)dx$

Answer available at the end of this section.

Exercise 3.8.2

Evaluate each indefinite integral using substitution.

(a) $\int \cos(3x)dx$

(b) $\int \frac{1}{2x+1}dx$

(c) $\int x^3\cos(x^4)dx$

(d) $\int \frac{x}{\sqrt{1-x^2}}dx$

(e) $\int \sin^3(x)\cos(x)dx$

(f) $\int \frac{\ln(x)}{x}dx$

Answer available at the end of this section.

Exercise 3.8.3

Evaluate each definite integral using substitution.

(a) $\int_0^1 2x(x^2+1)^3dx$

(b) $\int_0^2 x^2\sqrt{x^3+1}dx$

(c) $\int_0^1 e^{2x}dx$

(d) $\int_0^{\pi/2} \sin(x)\cos(x)dx$

(e) $\int_1^e \frac{\ln(x)}{x}dx$

(f) $\int_0^2 \frac{x}{x^2+1}dx$

Answer available at the end of this section.

Exercise 3.8.4

Evaluate each indefinite integral using substitution. Some may require more creative substitutions.

(a) $\int \tan(x)dx$

(b) $\int x^2e^{x^3}dx$

(c) $\int \frac{x}{\sqrt{4-x^2}}dx$

(d) $\int \sin^4(x)\cos(x)dx$

(e) $\int \frac{1}{x\ln(x)}dx$

(f) $\int \frac{x}{(x^2+1)^3}dx$

Answer available at the end of this section.

Exercise 3.8.5

Evaluate each definite integral using substitution.

(a) $\int_{-1}^1 xe^{x^2}dx$

(b) $\int_0^1 \frac{2x}{(x^2+1)^2}dx$

(c) $\int_0^{\pi/2} \sin(x)\cos^2(x)dx$

(d) $\int_1^4 \frac{1}{x\sqrt{\ln(x)}}dx$ (hint: let $u=\ln(x)$)

(e) $\int_0^{\pi/2} \cos(x)e^{\sin(x)}dx$

(f) $\int_0^1 x^3\sqrt{x^4+1}dx$

Answer available at the end of this section.

Exercise 3.8.6

Evaluate each indefinite integral using substitution. These may require more advanced substitutions.

(a) $\int x^3\sin(x^4)dx$

(b) $\int \frac{e^x}{e^x+1}dx$

(c) $\int x\sqrt{2x^2-3}dx$

(d) $\int \sec^2(x)\tan(x)dx$

(e) $\int \frac{x^2}{x^3-1}dx$

(f) $\int x^2(x^3+2)^5dx$

Answer available at the end of this section.

Exercise 3.8.7

Evaluate each definite integral using substitution.

(a) $\int_1^2 \frac{x}{x^2+1}dx$

(b) $\int_0^2 x\sqrt{x^2+1}dx$

(c) $\int_0^{\pi/4} \sec^2(x)\tan(x)dx$

(d) $\int_0^1 \frac{e^x}{e^x+1}dx$

(e) $\int_1^3 \frac{x^2}{x^3-1}dx$

(f) $\int_0^1 x^3(x^4+1)^3dx$

Answer available at the end of this section.

Exercise 3.8.8

Find the area under each curve on the given interval using substitution.

(a) $f(x) = x\sqrt{x^2+1}$ on $[0,2]$

(b) $f(x) = \dfrac{x}{x^2+1}$ on $[1,3]$

(c) $f(x) = \sin(x)\cos^3(x)$ on $[0,\pi/2]$

(d) $f(x) = e^{2x}$ on $[0,1]$

Answer available at the end of this section.

Exercise 3.8.9

Find the particular antiderivative $F(x)$ that satisfies the given condition using substitution.

(a) $f(x) = 2xe^{x^2}$, $F(0) = 3$

(b) $f(x) = \dfrac{x}{x^2+1}$, $F(0) = 1$

(c) $f(x) = \sin(x)\cos(x)$, $F(\pi/2) = 2$

(d) $f(x) = \dfrac{e^x}{e^x+1}$, $F(0) = 0$

Answer available at the end of this section.

Exercise 3.8.10

Use substitution to find the displacement and total distance traveled for each velocity function on the given interval.

(a) $v(t) = 2te^{t^2}$ on $[0,1]$

(b) $v(t) = \dfrac{t}{t^2+1}$ on $[0,2]$

(c) $v(t) = \sin(t)\cos(t)$ on $[0,\pi/2]$

(d) $v(t) = \dfrac{e^t}{e^t+1}$ on $[0,1]$

Answer available at the end of this section.

Answers

Exercise 3.8.1

(a) Let $u = x^2+1$. Then $du = 2xdx$.

$$\int 2x(x^2+1)^4 dx = \int u^4 du = \frac{u^5}{5} + C = \frac{(x^2+1)^5}{5} + C$$

(b) Let $u = x^3+1$. Then $du = 3x^2dx$, so $x^2dx = \frac{1}{3}du$.

$$\int x^2\sqrt{x^3+1}dx = \int \sqrt{u}\cdot\frac{1}{3}du = \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} + C = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C$$

(c) Let $u = 3x$. Then $du = 3dx$, so $dx = \frac{1}{3}du$.

$$\int e^{3x}dx = \int e^u \cdot \frac{1}{3}du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C$$

(d) Let $u = x^2$. Then $du = 2xdx$, so $xdx = \frac{1}{2}du$.

$$\int xe^{x^2}dx = \int e^u \cdot \frac{1}{2}du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

(e) Let $u = x^2 + 1$. Then $du = 2xdx$.

$$\int \frac{2x}{x^2+1}dx = \int \frac{1}{u}du = \ln|u| + C = \ln(x^2+1) + C$$

(f) Let $u = 2x$. Then $du = 2dx$, so $dx = \frac{1}{2}du$.

$$\int \sin(2x)dx = \int \sin(u) \cdot \frac{1}{2}du = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(2x) + C$$

Exercise 3.8.2

(a) Let $u = 3x$. Then $du = 3dx$, so $dx = \frac{1}{3}du$.

$$\int \cos(3x)dx = \int \cos(u) \cdot \frac{1}{3}du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(3x) + C$$

(b) Let $u = 2x + 1$. Then $du = 2dx$, so $dx = \frac{1}{2}du$.

$$\int \frac{1}{2x+1}dx = \int \frac{1}{u} \cdot \frac{1}{2}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x+1| + C$$

(c) Let $u = x^4$. Then $du = 4x^3dx$, so $x^3dx = \frac{1}{4}du$.

$$\int x^3\cos(x^4)dx = \int \cos(u) \cdot \frac{1}{4}du = \frac{1}{4}\sin(u) + C = \frac{1}{4}\sin(x^4) + C$$

(d) Let $u = 1 - x^2$. Then $du = -2xdx$, so $xdx = -\frac{1}{2}du$.

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{2}\right)du = -\frac{1}{2}\int u^{-1/2}du = -\frac{1}{2} \cdot 2u^{1/2} + C = -\sqrt{1-x^2} + C$$

(e) Let $u = \sin(x)$. Then $du = \cos(x)dx$.

$$\int \sin^3(x)\cos(x)dx = \int u^3du = \frac{u^4}{4} + C = \frac{\sin^4(x)}{4} + C$$

(f) Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$.

$$\int \frac{\ln(x)}{x}dx = \int udu = \frac{u^2}{2} + C = \frac{(\ln(x))^2}{2} + C$$

Exercise 3.8.3

(a) Let $u = x^2 + 1$. Then $du = 2xdx$. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

$$\int_0^1 2x(x^2+1)^3dx = \int_1^2 u^3du = \left[\frac{u^4}{4}\right]_1^2 = \frac{16}{4} - \frac{1}{4} = 4 - \frac{1}{4} = \frac{15}{4}$$

(b) Let $u = x^3 + 1$. Then $du = 3x^2dx$, so $x^2dx = \frac{1}{3}du$. When $x = 0$, $u = 1$. When $x = 2$, $u = 9$.

$$\int_0^2 x^2\sqrt{x^3+1}dx = \int_1^9 \sqrt{u} \cdot \frac{1}{3}du = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{9}(27-1) = \frac{2}{9}(26) = \frac{52}{9}$$

(c) Let $u = 2x$. Then $du = 2dx$, so $dx = \frac{1}{2}du$. When $x = 0$, $u = 0$. When $x = 1$, $u = 2$.

$$\int_0^1 e^{2x}dx = \int_0^2 e^u \cdot \frac{1}{2}du = \frac{1}{2}[e^u]_0^2 = \frac{1}{2}(e^2 - 1)$$

(d) Let $u = \sin(x)$. Then $du = \cos(x)dx$. When $x = 0$, $u = 0$. When $x = \frac{\pi}{2}$, $u = 1$.

$$\int_0^{\pi/2} \sin(x)\cos(x)dx = \int_0^1 udu = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

(e) Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$. When $x = 1$, $u = 0$. When $x = e$, $u = 1$.

$$\int_1^e \frac{\ln(x)}{x}dx = \int_0^1 udu = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

(f) Let $u = x^2 + 1$. Then $du = 2xdx$, so $xdx = \frac{1}{2}du$. When $x = 0$, $u = 1$. When $x = 2$, $u = 5$.

$$\int_0^2 \frac{x}{x^2+1}dx = \int_1^5 \frac{1}{u} \cdot \frac{1}{2}du = \frac{1}{2}[\ln|u|]_1^5 = \frac{1}{2}\ln(5)$$

Exercise 3.8.4

(a) $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Let $u = \cos(x)$. Then $du = -\sin(x)dx$, so $\sin(x)dx = -du$.

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = \int -\frac{1}{u}du = -\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

(b) Let $u = x^3$. Then $du = 3x^2dx$, so $x^2dx = \frac{1}{3}du$.

$$\int x^2 e^{x^3}dx = \int e^u \cdot \frac{1}{3}du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

(c) Let $u = 4 - x^2$. Then $du = -2xdx$, so $xdx = -\frac{1}{2}du$.

$$\int \frac{x}{\sqrt{4-x^2}}dx = \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{2}\right)du = -\frac{1}{2}\int u^{-1/2}du = -\frac{1}{2} \cdot 2u^{1/2} + C = -\sqrt{4-x^2} + C$$

(d) Let $u = \sin(x)$. Then $du = \cos(x)dx$.

$$\int \sin^4(x)\cos(x)dx = \int u^4 du = \frac{u^5}{5} + C = \frac{\sin^5(x)}{5} + C$$

(e) Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$.

$$\int \frac{1}{x\ln(x)}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln(x)| + C$$

(f) Let $u = x^2 + 1$. Then $du = 2xdx$, so $xdx = \frac{1}{2}du$.

$$\int \frac{x}{(x^2+1)^3}dx = \int \frac{1}{u^3} \cdot \frac{1}{2}du = \frac{1}{2}\int u^{-3}du = \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{4u^2} + C = -\frac{1}{4(x^2+1)^2} + C$$

Exercise 3.8.5

(a) Let $u = x^2$. Then $du = 2xdx$, so $xdx = \frac{1}{2}du$. When $x = -1$, $u = 1$. When $x = 1$, $u = 1$.

Since the limits are the same, the integral is 0. (The function xe^{x^2} is odd on $[-1, 1]$.)

$$\int_{-1}^{1} xe^{x^2}dx = 0$$

(b) Let $u = x^2 + 1$. Then $du = 2xdx$. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

$$\int_0^1 \frac{2x}{(x^2+1)^2}dx = \int_1^2 \frac{1}{u^2}du = \left[-\frac{1}{u}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

(c) Let $u = \cos(x)$. Then $du = -\sin(x)dx$, so $\sin(x)dx = -du$. When $x = 0$, $u = 1$. When $x = \frac{\pi}{2}$, $u = 0$.

$$\int_0^{\pi/2} \sin(x)\cos^2(x)dx = \int_1^0 -u^2 du = \int_0^1 u^2 du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

(d) Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$. When $x = 1$, $u = 0$. When $x = 4$, $u = \ln(4)$.

$$\int_1^4 \frac{1}{x\sqrt{\ln(x)}}dx = \int_0^{\ln(4)} \frac{1}{\sqrt{u}}du = \int_0^{\ln(4)} u^{-1/2}du = \left[2\sqrt{u}\right]_0^{\ln(4)} = 2\sqrt{\ln(4)}$$

(e) Let $u = \sin(x)$. Then $du = \cos(x)dx$. When $x = 0$, $u = 0$. When $x = \frac{\pi}{2}$, $u = 1$.

$$\int_0^{\pi/2} \cos(x)e^{\sin(x)}dx = \int_0^1 e^u du = [e^u]_0^1 = e - 1$$

(f) Let $u = x^4 + 1$. Then $du = 4x^3dx$, so $x^3dx = \frac{1}{4}du$. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

$$\int_0^1 x^3\sqrt{x^4+1}dx = \int_1^2 \sqrt{u} \cdot \frac{1}{4}du = \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{1}{6}(2^{3/2} - 1) = \frac{1}{6}(2\sqrt{2} - 1)$$

Exercise 3.8.6

(a) Let $u = x^4$. Then $du = 4x^3dx$, so $x^3dx = \frac{1}{4}du$.

$$\int x^3\sin(x^4)dx = \int \sin(u) \cdot \frac{1}{4}du = -\frac{1}{4}\cos(u) + C = -\frac{1}{4}\cos(x^4) + C$$

(b) Let $u = e^x + 1$. Then $du = e^x dx$.

$$\int \frac{e^x}{e^x+1}dx = \int \frac{1}{u}du = \ln|u| + C = \ln(e^x + 1) + C$$

(c) Let $u = 2x^2 - 3$. Then $du = 4xdx$, so $xdx = \frac{1}{4}du$.

$$\int x\sqrt{2x^2-3}dx = \int \sqrt{u} \cdot \frac{1}{4}du = \frac{1}{4} \cdot \frac{2}{3}u^{3/2} + C = \frac{1}{6}(2x^2-3)^{3/2} + C$$

(d) Let $u = \tan(x)$. Then $du = \sec^2(x)dx$.

$$\int \sec^2(x)\tan(x)dx = \int udu = \frac{u^2}{2} + C = \frac{\tan^2(x)}{2} + C$$

(e) Let $u = x^3 - 1$. Then $du = 3x^2dx$, so $x^2dx = \frac{1}{3}du$.

$$\int \frac{x^2}{x^3-1}dx = \int \frac{1}{u} \cdot \frac{1}{3}du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3-1| + C$$

(f) Let $u = x^3 + 2$. Then $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3} du$.

$$\int x^2 (x^3+2)^5 dx = \int u^5 \cdot \frac{1}{3} du = \frac{1}{3} \cdot \frac{u^6}{6} + C = \frac{1}{18}(x^3+2)^6 + C$$

Exercise 3.8.7

(a) Let $u = x^2 + 1$. Then $du = 2x dx$, so $x dx = \frac{1}{2} du$. When $x = 1$, $u = 2$. When $x = 2$, $u = 5$.

$$\int_1^2 \frac{x}{x^2+1} dx = \int_2^5 \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} [\ln|u|]_2^5 = \frac{1}{2} \ln\left(\frac{5}{2}\right)$$

(b) Let $u = x^2 + 1$. Then $du = 2x dx$, so $x dx = \frac{1}{2} du$. When $x = 0$, $u = 1$. When $x = 2$, $u = 5$.

$$\int_0^2 x\sqrt{x^2+1} dx = \int_1^5 \sqrt{u} \cdot \frac{1}{2} du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^5 = \frac{1}{3}(5^{3/2} - 1)$$

(c) Let $u = \tan(x)$. Then $du = \sec^2(x) dx$. When $x = 0$, $u = 0$. When $x = \frac{\pi}{4}$, $u = 1$.

$$\int_0^{\pi/4} \sec^2(x) \tan(x) dx = \int_0^1 u du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

(d) Let $u = e^x + 1$. Then $du = e^x dx$. When $x = 0$, $u = 2$. When $x = 1$, $u = e + 1$.

$$\int_0^1 \frac{e^x}{e^x+1} dx = \int_2^{e+1} \frac{1}{u} du = [\ln|u|]_2^{e+1} = \ln(e+1) - \ln(2) = \ln\left(\frac{e+1}{2}\right)$$

(e) Let $u = x^3 - 1$. Then $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3} du$. When $x = 1$, $u = 0$. When $x = 3$, $u = 26$.

$$\int_1^3 \frac{x^2}{x^3-1} dx = \int_0^{26} \frac{1}{u} \cdot \frac{1}{3} du = \frac{1}{3}[\ln|u|]_0^{26}$$

Since $\ln(0)$ is undefined, the integral is improper. The correct approach is to note the singularity at $x = 1$ and evaluate as a limit.

$$\int_1^3 \frac{x^2}{x^3-1} dx = \lim_{b \to 1^+} \int_b^3 \frac{x^2}{x^3-1} dx = \lim_{b \to 1^+} \frac{1}{3}[\ln|u|]_{b^3-1}^{26} = \infty$$

Thus the integral diverges.

(f) Let $u = x^4 + 1$. Then $du = 4x^3 dx$, so $x^3 dx = \frac{1}{4} du$. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

$$\int_0^1 x^3 (x^4+1)^3 dx = \int_1^2 u^3 \cdot \frac{1}{4} du = \frac{1}{4}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{16}(16 - 1) = \frac{15}{16}$$

Exercise 3.8.8

(a) From Exercise 7(b), $\int_0^2 x\sqrt{x^2+1} dx = \frac{1}{3}(5^{3/2} - 1)$

(b) From Exercise 7(a) with different limits: Let $u = x^2 + 1$, $du = 2x dx$. When $x = 1$, $u = 2$. When $x = 3$, $u = 10$.

$$\int_1^3 \frac{x}{x^2+1} dx = \int_2^{10} \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2}[\ln|u|]_2^{10} = \frac{1}{2}\ln(5)$$

(c) From Exercise 5(c), $\int_0^{\pi/2} \sin(x)\cos^3(x) dx = \frac{1}{4}$

(d) Let $u = 2x$. Then $du = 2dx$, so $dx = \frac{1}{2} du$. When $x = 0$, $u = 0$. When $x = 1$, $u = 2$.

$$\int_0^1 e^{2x} dx = \int_0^2 e^u \cdot \frac{1}{2} du = \frac{1}{2}[e^u]_0^2 = \frac{1}{2}(e^2 - 1)$$

Exercise 3.8.9

(a) Let $u = x^2$, $du = 2x dx$.

$$F(x) = \int 2x e^{x^2} dx = \int e^u du = e^u + C = e^{x^2} + C$$

Using $F(0) = 3$: $3 = e^0 + C = 1 + C \implies C = 2$.

Thus $F(x) = e^{x^2} + 2$.

(b) Let $u = x^2 + 1$, $du = 2x dx$.

$$F(x) = \int \frac{x}{x^2+1} dx = \int \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$$

Using $F(0) = 1$: $1 = \frac{1}{2}\ln(1) + C = 0 + C \implies C = 1$.

Thus $F(x) = \frac{1}{2}\ln(x^2+1) + 1$.

(c) Let $u = \sin(x)$, $du = \cos(x) dx$.

$$F(x) = \int \sin(x)\cos(x) dx = \int u du = \frac{u^2}{2} + C = \frac{\sin^2(x)}{2} + C$$

Using $F(\pi/2) = 2$: $2 = \frac{1}{2} + C \implies C = \frac{3}{2}$.

Thus $F(x) = \frac{\sin^2(x)}{2} + \frac{3}{2}$.

(d) Let $u = e^x + 1$, $du = e^x dx$.

$$F(x) = \int \frac{e^x}{e^x+1}dx = \int \frac{1}{u}du = \ln|u| + C = \ln(e^x+1) + C$$

Using $F(0) = 0$: $0 = \ln(2) + C \implies C = -\ln(2)$.

Thus $F(x) = \ln(e^x+1) - \ln(2) = \ln\left(\frac{e^x+1}{2}\right)$.

Exercise 3.8.10

(a) $v(t) = 2te^{t^2} > 0$ on $[0,1]$. Displacement = total distance.

Let $u = t^2$, $du = 2tdt$.

$$\int_0^1 2te^{t^2}dt = \int_0^1 e^u du = [e^u]_0^1 = e - 1$$

Thus displacement and total distance are both $e-1$.

(b) $v(t) = \frac{t}{t^2+1} \geq 0$ on $[0,2]$. Displacement = total distance.

Let $u = t^2+1$, $du = 2tdt$.

$$\int_0^2 \frac{t}{t^2+1}dt = \int_1^5 \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}[\ln|u|]_1^5 = \frac{1}{2}\ln(5)$$

Thus displacement and total distance are both $\frac{1}{2}\ln(5)$.

(c) $v(t) = \sin(t)\cos(t) \geq 0$ on $[0,\pi/2]$. Displacement = total distance.

Let $u = \sin(t)$, $du = \cos(t)dt$.

$$\int_0^{\pi/2} \sin(t)\cos(t)dt = \int_0^1 udu = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

Thus displacement and total distance are both $\frac{1}{2}$.

(d) $v(t) = \frac{e^t}{e^t+1} > 0$ on $[0,1]$. Displacement = total distance.

Let $u = e^t+1$, $du = e^t dt$.

$$\int_0^1 \frac{e^t}{e^t+1}dt = \int_2^{e+1} \frac{1}{u}du = [\ln|u|]_2^{e+1} = \ln(e+1) - \ln(2) = \ln\left(\frac{e+1}{2}\right)$$

Thus displacement and total distance are both $\ln\left(\frac{e+1}{2}\right)$.

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