# CALCULUS 2 - FALL 2026



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# Calculus 2 (Active Calculus & OpenStax)

1. Finding Antiderivatives and Evaluating Integrals
2. Using Definite Integrals
3. Differential Equations
4. Sequences and Series
5. Parametric Equations and Polar Coordinates
6. Appendices

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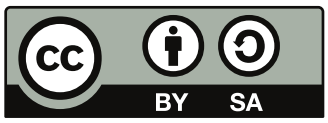
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# Students! Read this! (Active Calculus)

This book is different.



The text is available in three different formats: HTML, PDF, and print, each of which is available via links on the landing page at **activecalculus.org/**. The first two formats are free. If you are going to use the book electronically, the best mode is the **HTML**. The HTML version looks great in any browser, including on a smartphone, and the links are much easier to navigate in HTML than in PDF. Some particular direct suggestions about using the HTML follow among the next few paragraphs; alternatively, you can **watch this short video from the author**. It is also wise to download and save the PDF, since you can use the PDF offline, while the HTML version requires an internet connection. A print copy costs about $25 via **Amazon**.

This book is intended to be read sequentially and engaged with, much more than to be used as a lookup reference. For example, each section begins with a short introduction and a Preview Activity; you should read the short introduction and complete the Preview Activity prior to class. Your instructor may require you to do this. Most Preview Activities can be completed in 15-20 minutes and are intended to be accessible based on the understanding you have from preceding sections. There are not answers provided to Preview Activities, as these are designed simply to get you thinking about ideas that will be helpful in work on upcoming new material.

As you use the book, think of it as a workbook, not a worked-book. There is a great deal of scholarship that shows people learn better when they interactively engage and struggle with ideas themselves, rather than passively watch others. Thus, instead of reading worked examples or watching an instructor complete examples, you will engage with Activities that prompt you to grapple with concepts and develop deep understanding. You should expect to spend time in class working with peers on Activities and getting feedback from them and from your instructor. You can purchase a separate Activities Workbook from Amazon (**Chapters 1-4**, **Chapters 5-8**) in which to record your work on the activities, or you can find free PDFs at **the home page for the textbooks** (scroll down in the left panel and look for links to “download PDF 1-4” and “download PDF 5-8”) that have only the activities along with room to record your work. Your goal should be to do all of the activities in the relevant sections of the text and keep a careful record of your work. You can find answers to the activities **in the back matter**.

Each section concludes with a Summary. Reading the Summary after you have read the section and worked the Activities is a good way to find a short list of key ideas that are most essential to take from the section. A good study habit is to write similar summaries in your own words.

At the end of each section, you’ll find two types of Exercises. First, there are several anonymous WeBWorK exercises. These are online, interactive exercises that allow you to submit answers for immediate feedback with unlimited attempts without penalty; to submit answers, you have to be using the HTML version of the text (see **this short video** on the HTML version that includes a WeBWorK demonstration). You should use these exercises as a way to test your understanding of basic ideas in the preceding section. If your institution uses WeBWorK, you may also need to log in to a server as directed by your instructor to complete assigned WeBWorK sets as part of your course grade. The WeBWorK exercises included in this text are ungraded and not connected to any individual account. Following the WeBWorK exercises there are 3-4 additional challenging exercises that are designed to encourage you to connect ideas, investigate new situations, and write about your understanding. There are answers to most of the non-WeBWorK exercises **in the back matter**.

You can find additional support for your work in learning calculus from the **GVSU Math 201 YouTube Channel** and **GVSU Math 202 YouTube Channel** where there are several short video tutorials for each section of the text, numbered in alignment with the textbook sections. Math 201 corresponds to Chapters 1-4 and Math 202 to Chapters 5-8; there are about 90 videos for each, totaling more than 180.

The best way to be successful in mathematics generally and calculus specifically is to strive to make sense of the main ideas. We make sense of ideas by asking questions, interacting with others, attempting to solve problems, making mistakes, revising attempts, and writing and speaking about our understanding. This text has been designed to help you make sense of calculus; we wish you the very best as you undertake the large and challenging task of doing so.

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# Features of the Text (Active Calculus)

Instructors and students alike will find several consistent features in the presentation, including (click on each topic below):

**Motivating Questions**

At the start of each section, we list 2–3 *motivating questions* that provide motivation for why the following material is of interest to us. One overall goal of each section is to answer each of the motivating questions.

**Preview Activities**

Each section of the text begins with a short introduction, followed by a *preview activity*. This brief reading and the preview activity are designed to foreshadow the upcoming ideas in the remainder of the section; both the reading and preview activity are intended to be accessible to students *in advance* of class, and to be completed by students before the day on which a particular section is to be considered.

**Activities**

A typical section in the text has at least three *activities*. These are designed to engage students in an inquiry-based style that encourages them to construct solutions to key examples on their own, working individually or in small groups.

**Exercises**

There are dozens of calculus texts with (collectively) tens of thousands of exercises. Rather than repeat standard and routine exercises in this text, we recommend the use of WeBWorK with its access to the Open Problem Library and around 20,000 calculus problems. In this text, each section includes a small number of anonymous WeBWorK exercises, as well as 3–4 challenging problems per section. The WeBWorK exercises are best completed in **the .html version of the text**, as this provides students with immediate feedback without penalty. Almost every non- WeBWorK exercise has multiple parts, requires the student to connect several key ideas, and expects that the student will do at least a modest amount of writing to answer the questions and explain their findings. For instructors interested in a more conventional source of exercises, consider the freely available APEX Calculus text by Greg Hartmann et al., available from **www.apexcalculus.com**.

**Graphics**

We strive to demonstrate key fundamental ideas visually, and to encourage students to do the same. Throughout the text, we use full-color graphics to exemplify and magnify key ideas, and to use this graphical perspective alongside both numerical and algebraic representations of calculus.

**Links to interactive graphics**

Many of the ideas of calculus are best understood dynamically; java applets offer an often ideal format for investigations and demonstrations. Relying primarily on the work of David Austin of Grand Valley State University and Marc Renault of Shippensburg University, each of whom has developed a large library of applets for calculus, we frequently point the reader (through active links in the electronic versions of the text) to applets that are relevant for key ideas under consideration.

**Summary of Key Ideas**

Each section concludes with a summary of the key ideas encountered in the preceding section; this summary normally reflects responses to the motivating questions that began the section.

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# 5: ACTIVE CALCULUS - FINDING ANTIDERIVATIVES AND EVALUATING INTEGRALS


Construction Accurate Graphs of Antiderivatives

5.1: Constructing Accurate Graphs of Antiderivatives*

5.2: The Second Fundamental Theorem of Calculus

5.3: Integration by Substitution

**5.3: B - Trigonometric Substitution (OpenStax)**

5.3.1: Exercises for Section 7.3

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5.6: Numerical Integration (Simpson's Rule)*

5.7: Finding Antiderivatives and Evaluating Integrals (Exercises)



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# Construction Accurate Graphs of Antiderivatives

## Motivating Questions

- Given the graph of a function's derivative, how can we construct a completely accurate graph of the original function?
- How many antiderivatives does a given function have? What do those antiderivatives all have in common?
- Given a function $f$, how does the rule $A(x) = \int_0^x f(t)\ dt$ define a new function $A$?

A recurring theme in our discussion of differential calculus has been the question "Given information about the derivative of an unknown function $f$, how much information can we obtain about $f$ itself?" In Activity 1.8.3, the graph of $y = f'(x)$ was known (along with the value of $f$ at a single point) and we endeavored to sketch a possible graph of $f$ near the known point. In Example 3.1.2 — we investigated how the first derivative test enables us to use information about $f'$ to determine where the original function $f$ is increasing and decreasing, as well as where $f$ has relative extreme values. If we know a formula or graph of $f'$, by computing $f''$ we can find where the original function $f$ is concave up and concave down. Thus, knowing $f'$ and $f''$ enables us to understand the shape of the graph of $f$.

We returned to this question in even more detail in Section 4.1. In that setting, we knew the instantaneous velocity of a moving object and worked to determine as much as possible about the object's position function. We found connections between the net signed area under the velocity function and the corresponding change in position of the function, and the Total Change Theorem further illuminated these connections between $f'$ and $f$, showing that the total change in the value of $f$ over an interval $[a, b]$ is determined by the net signed area bounded by $f'$ and the $x$-axis on the same interval.

In what follows, we explore the situation where we possess an accurate graph of the derivative function along with a single value of the function $f$. From that information, we'd like to determine a graph of $f$ that shows where $f$ is increasing, decreasing, concave up, and concave down, and also provides an accurate function value at any point.

## Preview Activity 1

Suppose that the following information is known about a function $f$: the graph of its derivative, $y = f'(x)$, is given in Figure 1. Further, assume that $f'$ is piecewise linear (as pictured) and that for $x \leq 0$ and $x \geq 6$, $f'(x) = 0$. Finally, it is given that $f(0) = 1$.

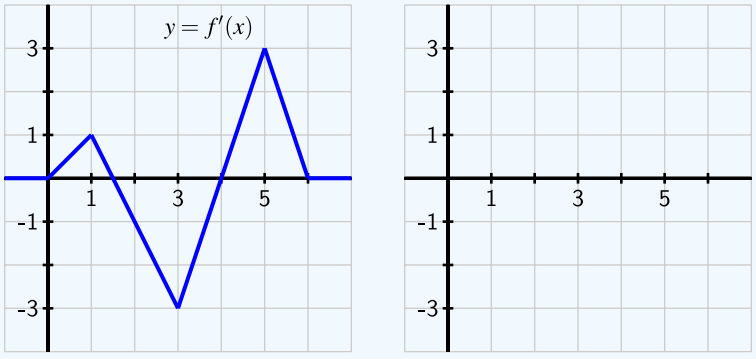



Figure 1. At left, the graph of $y = f'(x)$; at right, axes for plotting $y = f(x)$.

1. On what interval(s) is $f$ an increasing function? On what intervals is $f$ decreasing?
2. On what interval(s) is $f$ concave up? concave down?
3. At what point(s) does $f$ have a relative minimum? a relative maximum?
4. Recall that the Total Change Theorem tells us that

$$f(1) - f(0) = \int_0^1 f'(x)\,dx.$$

What is the exact value of $f(1)$?

5. Use the given information and similar reasoning to that in (d) to determine the exact value of $f(2)$, $f(3)$, $f(4)$, $f(5)$, and $f(6)$.
6. Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = f(x)$ on the axes provided, being sure to indicate the behavior of $f$ for $x < 0$ and $x > 6$.

## Constructing the graph of an antiderivative

Preview Activity 1 demonstrates that when we can find the exact area under the graph of a function on any given interval, it is possible to construct a graph of the function's antiderivative. That is, we can find a function whose derivative is given. We can now determine not only the overall shape of the antiderivative graph, but also the actual *height* of the graph at any point of interest.

This is a consequence of the Fundamental Theorem of Calculus: if we know a function $f$ and the value of the antiderivative $F$ at some starting point $a$, we can determine the value of $F(b)$ via the definite integral. Since $F(b) - F(a) = \int_a^b f(x)\ dx$, it follows that

$$F(b) = F(a) + \int_a^b f(x)\,dx. \tag{1}$$

We can also interpret the equation $F(b) - F(a) = \int_a^b f(x)\ dx$ in terms of the graphs of $f$ and $F$ as follows. On an interval $[a, b]$,

> *differences in heights on the graph of the antiderivative given by $F(b) - F(a)$ correspond to the net signed area bounded by the original function on the interval $[a, b]$, which is given by $\int_a^b f(x)\ dx$.*

### Activity 2

Suppose that the function $y = f(x)$ is given by the graph shown in Figure 2, and that the pieces of $f$ are either portions of lines or portions of circles. In addition, let $F$ be an antiderivative of $f$ and say that $F(0) = -1$. Finally, assume that for $x \leq 0$ and $x \geq 7$, $f(x) = 0$.

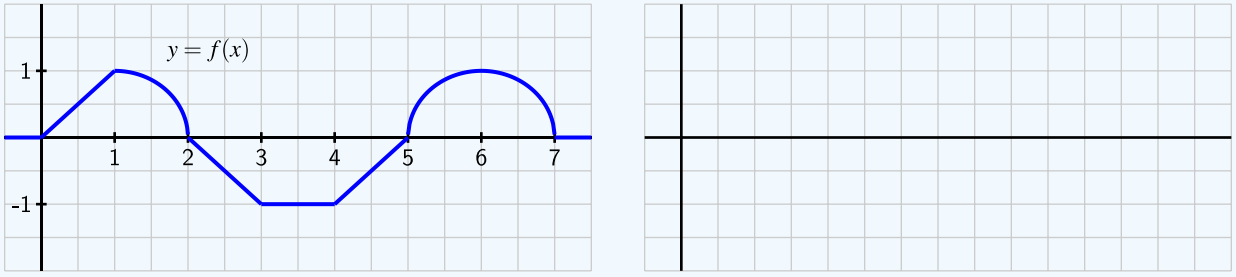



Figure 2. At left, the graph of $y = f(x)$.

1. On what interval(s) is $F$ an increasing function? On what intervals is $F$ decreasing?
2. On what interval(s) is $F$ concave up? concave down? neither?
3. At what point(s) does $F$ have a relative minimum? a relative maximum?
4. Use the given information to determine the exact value of $F(x)$ for $x = 1, 2, \ldots, 7$. In addition, what are the values of $F(-1)$ and $F(8)$?
5. Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = F(x)$ on the axes provided, being sure to indicate the behavior of $F$ for $x < 0$ and $x > 7$. Clearly indicate the scale on the vertical and horizontal axes of your graph.

6. What happens if we change one key piece of information: in particular, say that $G$ is an antiderivative of $f$ and $G(0) = 0$. How (if at all) would your answers to the preceding questions change? Sketch a graph of $G$ on the same axes as the graph of $F$ you constructed in (e).

## Multiple antiderivatives of a single function

In the final question of Activity 2, we encountered a very important idea: a function $f$ has more than one antiderivative. Each antiderivative of $f$ is determined uniquely by its value at a single point. For example, suppose that $f$ is the function given at left in Figure 3, and suppose further that $F$ is an antiderivative of $f$ that satisfies $F(0) = 1$.

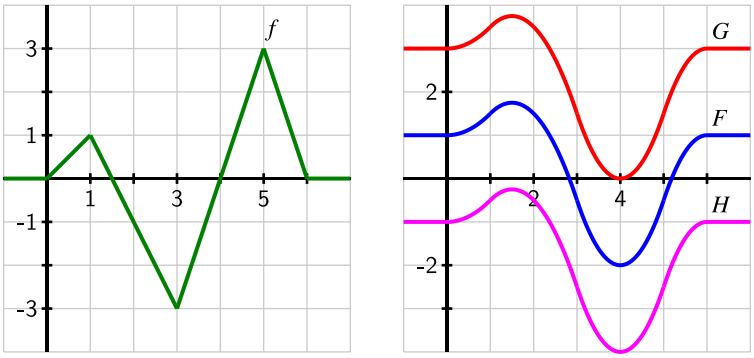



Figure 3. At left, the graph of $y = f(x)$. At right, three different antiderivatives of $f$.

Then, using Equation (1), we can compute

$$\begin{aligned} F(1) &= F(0) + \int_0^1 f(x)\,dx \\ &= 1 + 0.5 \\ &= 1.5. \end{aligned}$$

Similarly, $F(2) = 1.5$, $F(3) = -0.5$, $F(4) = -2$, $F(5) = -0.5$, and $F(6) = 1$. In addition, we can use the fact that $F' = f$ to ascertain where $F$ is increasing and decreasing, concave up and concave down, and has relative extremes and inflection points. We ultimately find that the graph of $F$ is the one given in blue in Figure 3.

If we want an antiderivative $G$ for which $G(0) = 3$, then $G$ will have the exact same shape as $F$ (since both share the derivative $f$ ), but $G$ will be shifted vertically from the graph of $F$, as pictured in red in Figure 3. Note that $G(1) - G(0) = \int_0^1 f(x)\ dx = 0.5$, just as $F(1) - F(0) = 0.5$, but since $G(0) = 3$, $G(1) = G(0) + 0.5 = 3.5$, whereas $F(1) = 1.5$. In the same way, if we assigned a different initial value to the antiderivative, say $H(0) = -1$, we would get still another antiderivative, as shown in magenta in Figure 13.

This example demonstrates an important fact that holds more generally:

**Note**

If $G$ and $H$ are both antiderivatives of a function $f$, then the function $G - H$ must be constant.

To see why this result holds, observe that if $G$ and $H$ are both antiderivatives of $f$, then $G' = f$ and $H' = f$. Hence,

$$\frac{d}{dx}[G(x) - H(x)] = G'(x) - H'(x) = f(x) - f(x) = 0.$$

Since the only way a function can have derivative zero is by being a constant function, it follows that the function $G - H$ must be constant.

We now see that if a function has at least one antiderivative, it must have infinitely many: we can add any constant of our choice to the antiderivative and get another antiderivative. For this reason, we sometimes refer to the *general antiderivative* of a function $f$.

To identify a particular antiderivative of $f$, we must know a single value of the antiderivative $F$ (this value is often called an *initial condition*). For example, if $f(x) = x^2$, its general antiderivative is $F(x) = \frac{1}{3}x^3 + C$, where we include the "$+C$" to indicate that $F$ includes *all* of the possible antiderivatives of $f$. If we know that $F(2) = 3$, we substitute 2 for $x$ in $F(x) = \frac{1}{3}x^3 + C$, and find that

$$3 = \frac{1}{3}(2)^3 + C,$$

or $C = 3 - \frac{8}{3} = \frac{1}{3}$. Therefore, the particular antiderivative in this case is $F(x) = \frac{1}{3}x^3 + \frac{1}{3}$.

**Activity 3**

For each of the following functions, sketch an accurate graph of the antiderivative that satisfies the given initial condition. In addition, sketch the graph of two additional antiderivatives of the given function, and state the corresponding initial conditions that each of them satisfy. If possible, find an algebraic formula for the antiderivative that satisfies the initial condition.

1. original function: $g(x) = |x| - 1$; initial condition: $G(-1) = 0$; interval for sketch: $[-2, 2]$
2. original function: $h(x) = \sin(x)$; initial condition: $H(0) = 1$; interval for sketch: $[0, 4\pi]$
3. original function: $p(x) = \begin{cases} x^2, & \text{if } 0 < x < 1 \\ -(x-2)^2, & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$; initial condition: $P(0) = 1$; interval for sketch: $[-1, 3]$

## Functions defined by integrals

Equation (1) allows us to compute the value of the antiderivative $F$ at a point $b$, provided that we know $F(a)$ and can evaluate the definite integral from $a$ to $b$ of $f$. That is,

$$F(b) = F(a) + \int_a^b f(x)\,dx.$$

In several situations, we have used this formula to compute $F(b)$ for several different values of $b$, and then plotted the points $(b, F(b))$ to help us draw an accurate graph of $F$. This suggests that we may want to think of $b$, the upper limit of integration, as a variable itself. To that end, we introduce the idea of an *integral function*, a function whose formula involves a definite integral.

**Definition 1**

If $f$ is a continuous function, we define the corresponding ***integral function*** $A$ according to the rule

$$A(x) = \int_a^x f(t)\,dt. \tag{12}$$

Note that because $x$ is the independent variable in the function $A$, and determines the endpoint of the interval of integration, we need to use a different variable as the variable of integration. A standard choice is $t$, but any variable other than $x$ is acceptable.

One way to think of the function $A$ is as the "net signed area from $a$ up to $x$" function, where we consider the region bounded by $y = f(t)$. For example, in Figure 4, we see a function $f$ pictured at left, and its corresponding area function (choosing $a = 0$), $A(x) = \int_0^x f(t)\ dt$ shown at right.

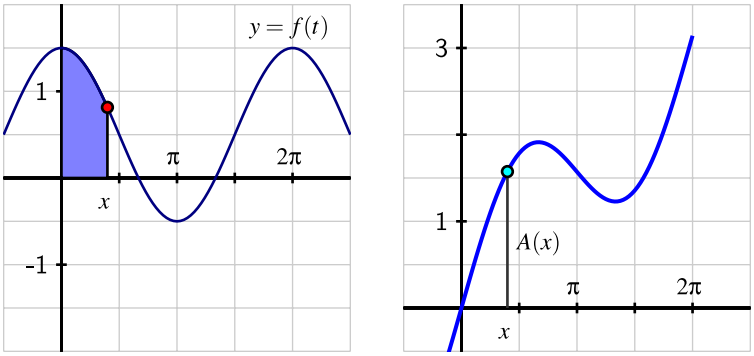



Figure 4. At left, the graph of the given function $f$. At right, the area function $A(x) = \int_0^x f(t)\ dt$.

The function $A$ measures the net signed area from $t = 0$ to $t = x$ bounded by the curve $y = f(t)$; this value is then reported as the corresponding height on the graph of $y = A(x)$. At http://gvsu.edu/s/cz, we find a java applet [1] that brings the static picture in Figure 4 to life. There, the user can move the red point on the function $f$ and see how the corresponding height changes at the light blue point on the graph of $A$.

David Austin, Grand Valley State University

The choice of $a$ is somewhat arbitrary. In the activity that follows, we explore how the value of $a$ affects the graph of the integral function.

## Activity 4

Suppose that $g$ is given by the graph at left in Figure 5 and that $A$ is the corresponding integral function defined by $A(x) = \int_1^x g(t)\ dt$.

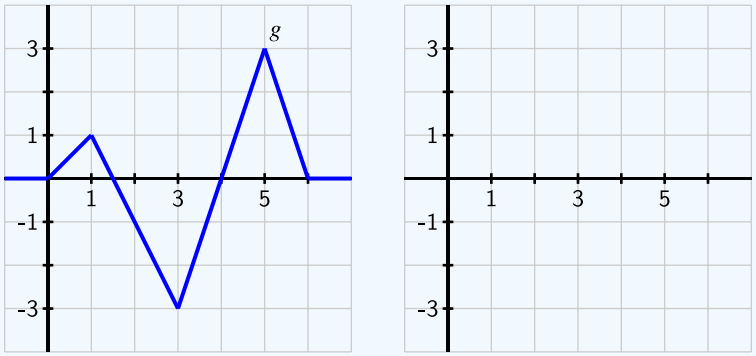



Figure 5. At left, the graph of $y = g(t)$; at right, axes for plotting $y = A(x)$, where $A$ is defined by the formula $A(x) = \int_1^x g(t)\ dt$.

1. On what interval(s) is $A$ an increasing function? On what intervals is $A$ decreasing? Why?
2. On what interval(s) do you think $A$ is concave up? concave down? Why?
3. At what point(s) does $A$ have a relative minimum? a relative maximum?
4. Use the given information to determine the exact values of $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, $A(5)$, and $A(6)$.
5. Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = A(x)$ on the axes provided, being sure to indicate the behavior of $A$ for $x < 0$ and $x > 6$.
6. How does the graph of $B$ compare to $A$ if $B$ is instead defined by $B(x) = \int_0^x g(t)\ dt$?

## Summary

- Given the graph of a function $f$, we can construct the graph of its antiderivative $F$ provided that (a) we know a starting value of $F$, say $F(a)$, and (b) we can evaluate the integral $\int_a^b f(x)\ dx$ exactly for relevant choices of $a$ and $b$. For instance, if we wish to know $F(3)$, we can compute $F(3) = F(a) + \int_a^3 f(x)\ dx$. When we combine this information about the function values of $F$ together with our understanding of how the behavior of $F' = f$ affects the overall shape of $F$, we can develop a completely accurate graph of the antiderivative $F$.
- Because the derivative of a constant is zero, if $F$ is an antiderivative of $f$, it follows that $G(x) = F(x) + C$ will also be an antiderivative of $f$. Moreover, any two antiderivatives of a function $f$ differ precisely by a constant. Thus, any function with at least one antiderivative in fact has infinitely many, and the graphs of any two antiderivatives will differ only by a vertical translation.
- Given a function $f$, the rule $A(x) = \int_a^x f(t)\ dt$ defines a new function $A$ that measures the net-signed area bounded by $f$ on the interval $[a, x]$. We call the function $A$ the integral function corresponding to $f$.

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# 5.1: Constructing Accurate Graphs of Antiderivatives*

## Motivating Questions

- Given the graph of a function's derivative, how can we construct a completely accurate graph of the original function?
- How many antiderivatives does a given function have? What do those antiderivatives all have in common?
- Given a function $f$, how does the rule $A(x) = \int_0^x f(t)\ dt$ define a new function $A$?

A recurring theme in our discussion of differential calculus has been the question "Given information about the derivative of an unknown function $f$, how much information can we obtain about $f$ itself?" In Activity 1.8.3, the graph of $y = f'(x)$ was known (along with the value of $f$ at a single point) and we endeavored to sketch a possible graph of $f$ near the known point. In Example 3.1.2 — we investigated how the first derivative test enables us to use information about $f'$ to determine where the original function $f$ is increasing and decreasing, as well as where $f$ has relative extreme values. If we know a formula or graph of $f'$, by computing $f''$ we can find where the original function $f$ is concave up and concave down. Thus, knowing $f'$ and $f''$ enables us to understand the shape of the graph of $f$.

We returned to this question in even more detail in Section 4.1. In that setting, we knew the instantaneous velocity of a moving object and worked to determine as much as possible about the object's position function. We found connections between the net signed area under the velocity function and the corresponding change in position of the function, and the Total Change Theorem further illuminated these connections between $f'$ and $f$, showing that the total change in the value of $f$ over an interval $[a, b]$ is determined by the net signed area bounded by $f'$ and the $x$-axis on the same interval.

In what follows, we explore the situation where we possess an accurate graph of the derivative function along with a single value of the function $f$. From that information, we'd like to determine a graph of $f$ that shows where $f$ is increasing, decreasing, concave up, and concave down, and also provides an accurate function value at any point.

## Preview Activity 5.1.1

Suppose that the following information is known about a function $f$: the graph of its derivative, $y = f'(x)$, is given in Figure 5.1.1. Further, assume that $f'$ is piecewise linear (as pictured) and that for $x \leq 0$ and $x \geq 6$, $f'(x) = 0$. Finally, it is given that $f(0) = 1$.

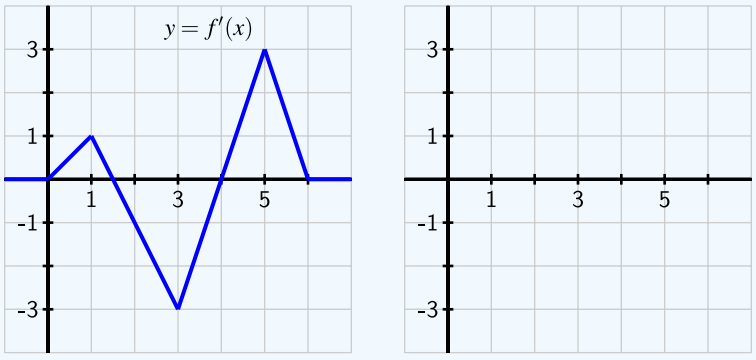



Figure 5.1.1. At left, the graph of $y = f'(x)$; at right, axes for plotting $y = f(x)$.

1. On what interval(s) is $f$ an increasing function? On what intervals is $f$ decreasing?
2. On what interval(s) is $f$ concave up? concave down?
3. At what point(s) does $f$ have a relative minimum? a relative maximum?
4. Recall that the Total Change Theorem tells us that

$$f(1) - f(0) = \int_0^1 f'(x)\,dx.$$

What is the exact value of $f(1)$?

5. Use the given information and similar reasoning to that in (d) to determine the exact value of $f(2)$, $f(3)$, $f(4)$, $f(5)$, and $f(6)$.
6. Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = f(x)$ on the axes provided, being sure to indicate the behavior of $f$ for $x < 0$ and $x > 6$.

## Constructing the graph of an antiderivative

Preview Activity 5.1.1 demonstrates that when we can find the exact area under the graph of a function on any given interval, it is possible to construct a graph of the function's antiderivative. That is, we can find a function whose derivative is given. We can now determine not only the overall shape of the antiderivative graph, but also the actual *height* of the graph at any point of interest.

This is a consequence of the Fundamental Theorem of Calculus: if we know a function $f$ and the value of the antiderivative $F$ at some starting point $a$, we can determine the value of $F(b)$ via the definite integral. Since $F(b) - F(a) = \int_a^b f(x)\ dx$, it follows that

$$F(b) = F(a) + \int_a^b f(x)\,dx. \tag{5.1.1}$$

We can also interpret the equation $F(b) - F(a) = \int_a^b f(x)\ dx$ in terms of the graphs of $f$ and $F$ as follows. On an interval $[a, b]$,

> *differences in heights on the graph of the antiderivative given by $F(b) - F(a)$ correspond to the net signed area bounded by the original function on the interval $[a, b]$, which is given by $\int_a^b f(x)\ dx$.*

### Activity 5.1.2

Suppose that the function $y = f(x)$ is given by the graph shown in Figure 5.1.2, and that the pieces of $f$ are either portions of lines or portions of circles. In addition, let $F$ be an antiderivative of $f$ and say that $F(0) = -1$. Finally, assume that for $x \le 0$ and $x \ge 7$, $f(x) = 0$.

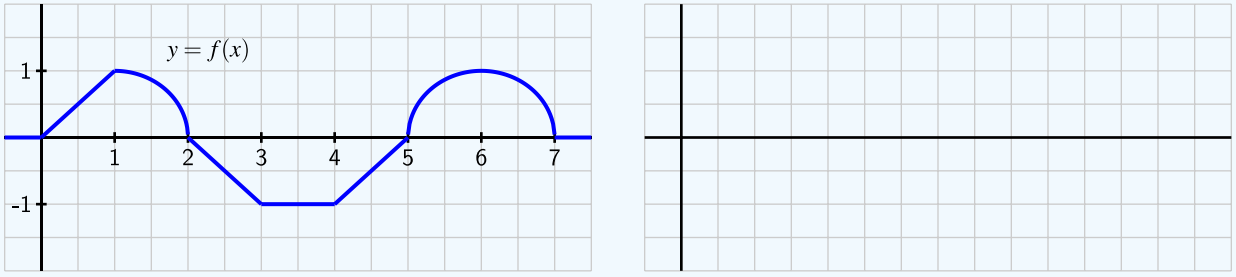



Figure 5.1.2. At left, the graph of $y = f(x)$.

1. On what interval(s) is $F$ an increasing function? On what intervals is $F$ decreasing?
2. On what interval(s) is $F$ concave up? concave down? neither?
3. At what point(s) does $F$ have a relative minimum? a relative maximum?
4. Use the given information to determine the exact value of $F(x)$ for $x = 1, 2, \ldots, 7$. In addition, what are the values of $F(-1)$ and $F(8)$?
5. Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = F(x)$ on the axes provided, being sure to indicate the behavior of $F$ for $x < 0$ and $x > 7$. Clearly indicate the scale on the vertical and horizontal axes of your graph.

6. What happens if we change one key piece of information: in particular, say that $G$ is an antiderivative of $f$ and $G(0) = 0$. How (if at all) would your answers to the preceding questions change? Sketch a graph of $G$ on the same axes as the graph of $F$ you constructed in (e).

## Multiple antiderivatives of a single function

In the final question of Activity 5.1.2, we encountered a very important idea: a function $f$ has more than one antiderivative. Each antiderivative of $f$ is determined uniquely by its value at a single point. For example, suppose that $f$ is the function given at left in Figure 5.1.3, and suppose further that $F$ is an antiderivative of $f$ that satisfies $F(0) = 1$.

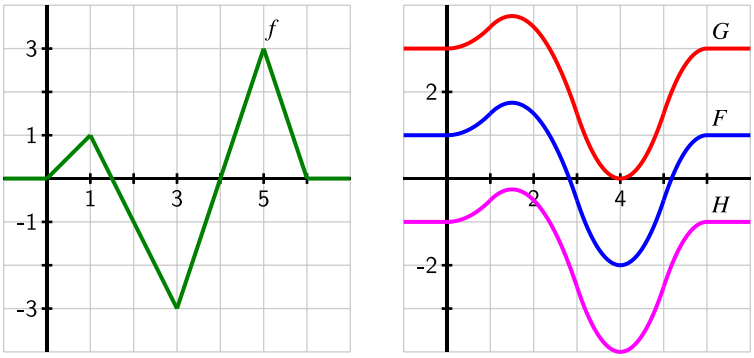



Figure 5.1.3. At left, the graph of $y = f(x)$. At right, three different antiderivatives of $f$.

Then, using Equation (5.1.1), we can compute

$$\begin{aligned} F(1) &= F(0) + \int_0^1 f(x)\,dx \\ &= 1 + 0.5 \\ &= 1.5. \end{aligned}$$

Similarly, $F(2) = 1.5$, $F(3) = -0.5$, $F(4) = -2$, $F(5) = -0.5$, and $F(6) = 1$. In addition, we can use the fact that $F' = f$ to ascertain where $F$ is increasing and decreasing, concave up and concave down, and has relative extremes and inflection points. We ultimately find that the graph of $F$ is the one given in blue in Figure 5.1.3.

If we want an antiderivative $G$ for which $G(0) = 3$, then $G$ will have the exact same shape as $F$ (since both share the derivative $f$), but $G$ will be shifted vertically from the graph of $F$, as pictured in red in Figure 5.1.3. Note that $G(1) - G(0) = \int_0^1 f(x)\ dx = 0.5$, just as $F(1) - F(0) = 0.5$, but since $G(0) = 3$, $G(1) = G(0) + 0.5 = 3.5$, whereas $F(1) = 1.5$. In the same way, if we assigned a different initial value to the antiderivative, say $H(0) = -1$, we would get still another antiderivative, as shown in magenta in Figure 5.1.13.

This example demonstrates an important fact that holds more generally:

**Note**

If $G$ and $H$ are both antiderivatives of a function $f$, then the function $G - H$ must be constant.

To see why this result holds, observe that if $G$ and $H$ are both antiderivatives of $f$, then $G' = f$ and $H' = f$. Hence,

$$\frac{d}{dx}[G(x) - H(x)] = G'(x) - H'(x) = f(x) - f(x) = 0.$$

Since the only way a function can have derivative zero is by being a constant function, it follows that the function $G - H$ must be constant.

We now see that if a function has at least one antiderivative, it must have infinitely many: we can add any constant of our choice to the antiderivative and get another antiderivative. For this reason, we sometimes refer to the *general antiderivative* of a function $f$.

To identify a particular antiderivative of $f$, we must know a single value of the antiderivative $F$ (this value is often called an *initial condition*). For example, if $f(x) = x^2$, its general antiderivative is $F(x) = \frac{1}{3}x^3 + C$, where we include the "$+C$" to indicate that $F$ includes *all* of the possible antiderivatives of $f$. If we know that $F(2) = 3$, we substitute 2 for $x$ in $F(x) = \frac{1}{3}x^3 + C$, and find that

$$3 = \frac{1}{3}(2)^3 + C,$$

or $C = 3 - \frac{8}{3} = \frac{1}{3}$. Therefore, the particular antiderivative in this case is $F(x) = \frac{1}{3}x^3 + \frac{1}{3}$.

**Activity 5.1.3**

For each of the following functions, sketch an accurate graph of the antiderivative that satisfies the given initial condition. In addition, sketch the graph of two additional antiderivatives of the given function, and state the corresponding initial conditions that each of them satisfy. If possible, find an algebraic formula for the antiderivative that satisfies the initial condition.

1. original function: $g(x) = |x| - 1$; initial condition: $G(-1) = 0$; interval for sketch: $[-2, 2]$
2. original function: $h(x) = \sin(x)$; initial condition: $H(0) = 1$; interval for sketch: $[0, 4\pi]$
3. original function: $p(x) = \begin{cases} x^2, & \text{if } 0 < x < 1 \\ -(x-2)^2, & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$; initial condition: $P(0) = 1$; interval for sketch: $[-1, 3]$

## Functions defined by integrals

Equation (5.1.1) allows us to compute the value of the antiderivative $F$ at a point $b$, provided that we know $F(a)$ and can evaluate the definite integral from $a$ to $b$ of $f$. That is,

$$F(b) = F(a) + \int_a^b f(x)\,dx.$$

In several situations, we have used this formula to compute $F(b)$ for several different values of $b$, and then plotted the points $(b, F(b))$ to help us draw an accurate graph of $F$. This suggests that we may want to think of $b$, the upper limit of integration, as a variable itself. To that end, we introduce the idea of an *integral function*, a function whose formula involves a definite integral.

**Definition 5.1.1**

If $f$ is a continuous function, we define the corresponding ***integral function*** $A$ according to the rule

$$A(x) = \int_a^x f(t)\,dt. \tag{5.1.12}$$

Note that because $x$ is the independent variable in the function $A$, and determines the endpoint of the interval of integration, we need to use a different variable as the variable of integration. A standard choice is $t$, but any variable other than $x$ is acceptable.

One way to think of the function $A$ is as the "net signed area from $a$ up to $x$" function, where we consider the region bounded by $y = f(t)$. For example, in Figure 5.1.4, we see a function $f$ pictured at left, and its corresponding area function (choosing $a = 0$), $A(x) = \int_0^x f(t)\ dt$ shown at right.

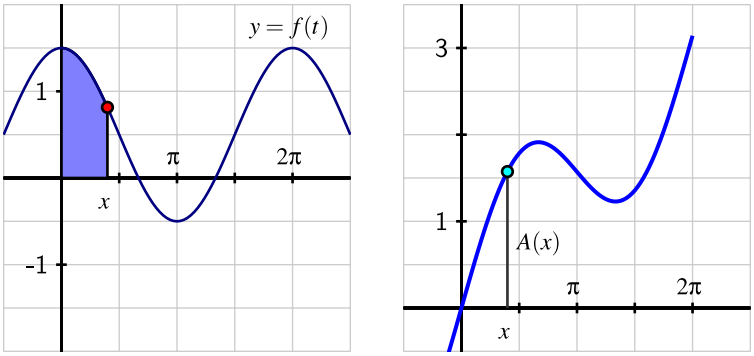



Figure 5.1.4. At left, the graph of the given function $f$. At right, the area function $A(x) = \int_0^x f(t)\ dt$.

The function $A$ measures the net signed area from $t = 0$ to $t = x$ bounded by the curve $y = f(t)$; this value is then reported as the corresponding height on the graph of $y = A(x)$. At http://gvsu.edu/s/cz, we find a java applet[1] that brings the static picture in Figure 5.1.4 to life. There, the user can move the red point on the function $f$ and see how the corresponding height changes at the light blue point on the graph of $A$.

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The choice of $a$ is somewhat arbitrary. In the activity that follows, we explore how the value of $a$ affects the graph of the integral function.

## Activity 5.1.4

Suppose that $g$ is given by the graph at left in Figure 5.1.5 and that $A$ is the corresponding integral function defined by $A(x) = \int_1^x g(t)\ dt$.

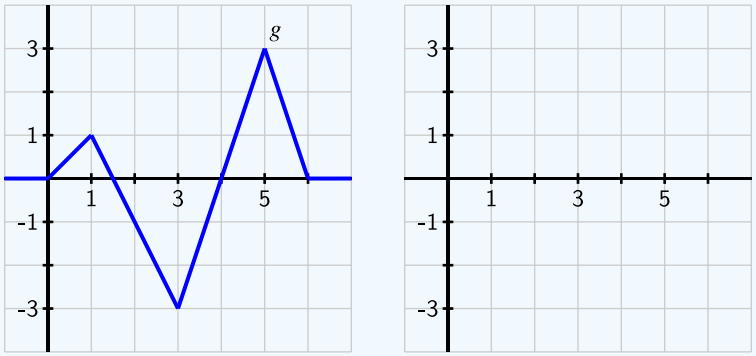



Figure 5.1.5. At left, the graph of $y = g(t)$; at right, axes for plotting $y = A(x)$, where $A$ is defined by the formula $A(x) = \int_1^x g(t)\ dt$.

1. On what interval(s) is $A$ an increasing function? On what intervals is $A$ decreasing? Why?
2. On what interval(s) do you think $A$ is concave up? concave down? Why?
3. At what point(s) does $A$ have a relative minimum? a relative maximum?
4. Use the given information to determine the exact values of $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, $A(5)$, and $A(6)$.
5. Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = A(x)$ on the axes provided, being sure to indicate the behavior of $A$ for $x < 0$ and $x > 6$.
6. How does the graph of $B$ compare to $A$ if $B$ is instead defined by $B(x) = \int_0^x g(t)\ dt$?

## Summary

- Given the graph of a function $f$, we can construct the graph of its antiderivative $F$ provided that (a) we know a starting value of $F$, say $F(a)$, and (b) we can evaluate the integral $\int_a^b f(x)\ dx$ exactly for relevant choices of $a$ and $b$. For instance, if we wish to know $F(3)$, we can compute $F(3) = F(a) + \int_a^3 f(x)\ dx$. When we combine this information about the function values of $F$ together with our understanding of how the behavior of $F' = f$ affects the overall shape of $F$, we can develop a completely accurate graph of the antiderivative $F$.
- Because the derivative of a constant is zero, if $F$ is an antiderivative of $f$, it follows that $G(x) = F(x) + C$ will also be an antiderivative of $f$. Moreover, any two antiderivatives of a function $f$ differ precisely by a constant. Thus, any function with at least one antiderivative in fact has infinitely many, and the graphs of any two antiderivatives will differ only by a vertical translation.
- Given a function $f$, the rule $A(x) = \int_a^x f(t)\ dt$ defines a new function $A$ that measures the net-signed area bounded by $f$ on the interval $[a, x]$. We call the function $A$ the integral function corresponding to $f$.

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# 5.2: The Second Fundamental Theorem of Calculus

## Motivating Questions

- How does the integral function $A(x) = \int_1^x f(t)\ dt$ define an antiderivative of $f$?
- What is the statement of the Second Fundamental Theorem of Calculus?
- How do the First and Second Fundamental Theorems of Calculus enable us to formally see how differentiation and integration are almost inverse processes?

In Section 4.4, we learned the Fundamental Theorem of Calculus (FTC), which from here forward will be referred to as the *First* Fundamental Theorem of Calculus, as in this section we develop a corresponding result that follows it. Recall that the First FTC tells us that if $f$ is a continuous function on $[a, b]$ and $F$ is any antiderivative of $f$ (that is, $F' = f$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We have used this result in two settings:

1. If we have a graph of $f$ and we can compute the exact area bounded by $f$ on an interval $[a, b]$, we can compute the change in an antiderivative $F$ over the interval.
2. If we can find an algebraic formula for an antiderivative of $f$, we can evaluate the integral to find the net signed area bounded by the function on the interval.

For the former, see Preview Activity 5.1.1 or Activity 5.1.2. For the latter, we can easily evaluate exactly integrals such as

$$\int_1^4 x^2\,dx,$$

since we know that the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$. Thus,

$$\begin{aligned}\int_1^4 x^2\,dx &= \frac{1}{3}x^3\Big|_1^4 \\ &= \frac{1}{3}(4)^3 - \frac{1}{3}(1)^3 \\ &= 21.\end{aligned}$$

Thus, the First FTC can used in two ways. First, to find the difference $F(b) - F(a)$ for an antiderivative $F$ of the integrand $f$, even if we may not have a formula for $F$ itself. To do this, we must know the value of the integral $\int_a^b f(x)\ dx$ exactly, perhaps through known geometric formulas for area. In addition, the First FTC provides a way to find the exact value of a definite integral, and hence a certain net signed area exactly, by finding an antiderivative of the integrand and evaluating its total change over the interval. In this case, we need to know a formula for the antiderivative $F$. Both of these perspectives are reflected in Figure 5.2.1.

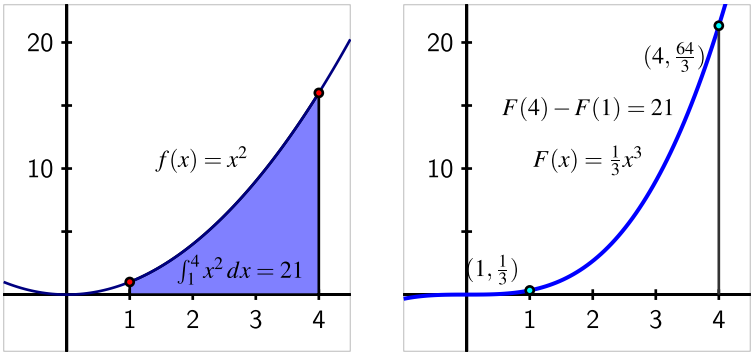



Figure 5.2.1. At left, the graph of $f(x) = x^2$ on the interval $[1, 4]$ and the area it bounds. At right, the antiderivative function $F(x) = \frac{1}{3}x^3$, whose total change on $[1, 4]$ is the value of the definite integral at left.

The value of a definite integral may have additional meaning depending on context: as the change in position when the integrand is a velocity function, the total amount of pollutant leaked from a tank when the integrand is the rate at which pollution is leaking, or other total changes if the integrand is a rate function. Also, the value of the definite integral is connected to the average value of a continuous function on a given interval: $f_{\text{AVG}[a,b]} = \frac{1}{b-a}\int_a^b f(x)\ dx$.

In the last part of Section 5.1, we studied integral functions of the form $A(x) = \int_c^x f(t)\ dt$. Figure 5.1.5 is a particularly important image to keep in mind as we work with integral functions, and the corresponding java applet at gvsu.edu/s/cz can help us understand the function $A$. In what follows, we use the First FTC to gain additional understanding of the function $A(x) = \int_c^x f(t)\ dt$, where the integrand $f$ is given (either through a graph or a formula), and $c$ is a constant.

### Preview Activity 5.2.1

Consider the function $A$ defined by the rule

$$A(x) = \int_1^x f(t)\,dt,$$

where $f(t) = 4 - 2t$.

1. Compute $A(1)$ and $A(2)$ exactly.
2. Use the First Fundamental Theorem of Calculus to find a formula for $A(x)$ that does not involve integrals. That is, use the first FTC to evaluate $\int_1^x (4 - 2t)\ dt$.
3. Observe that $f$ is a linear function; what kind of function is $A$?
4. Using the formula you found in (b) that does not involve integrals, compute $A'(x)$.
5. While we have defined $f$ by the rule $f(t) = 4 - 2t$, it is equivalent to say that $f$ is given by the rule $f(x) = 4 - 2x$. What do you observe about the relationship between $A$ and $f$?

## The Second Fundamental Theorem of Calculus

The result of Preview Activity 5.2.1 is not particular to the function $f(t) = 4 - 2t$, nor to the choice of "1" as the lower bound in the integral that defines the function $A$. For instance, if we let $f(t) = \cos(t) - t$ and set $A(x) = \int_2^x f(t)\ dt$, we can determine a formula for $A$ by the First FTC. Specifically,

$$\begin{aligned} A(x) &= \int_2^x (\cos(t) - t)\,dt \\ &= \sin(t) - \frac{1}{2}t^2\Big|_2^x \\ &= \sin(x) - \frac{1}{2}x^2 - (\sin(2) - 2)\,. \end{aligned}$$

Differentiating $A(x)$, since $(\sin(2) - 2)$ is constant, it follows that

$$A'(x) = \cos(x) - x,$$

and thus we see that $A'(x) = f(x)$, so $A$ is an antiderivative of $f$. And since $A(2) = \int_2^2 f(t)\ dt = 0$, $A$ is the only antiderivative of $f$ for which $A(2) = 0$.

In general, if $f$ is any continuous function, and we define the function $A$ by the rule

$$A(x) = \int_c^x f(t)\,dt,$$

where $c$ is an arbitrary constant, then we can show that $A$ is an antiderivative of $f$. To see why, let's demonstrate that $A'(x) = f(x)$ by using the limit definition of the derivative. Doing so, we observe that

$$\begin{aligned} A'(x) &= \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \\ &= \lim_{h\to 0} \frac{\int_c^{x+h} f(t)\,dt - \int_c^x f(t)\,dt}{h} \\ &= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}, \end{aligned} \tag{5.2.1}$$

where Equation (5.2.1) follows from the fact that $\int_c^x f(t)\ dt + \int_x^{x+h} f(t)\ dt = \int_c^{x+h} f(t)\ dt$. Now, observe that for small values of $h$,

$$\int_x^{x+h} f(t)\,dt \approx f(x) \cdot h,$$

by a simple left-hand approximation of the integral. Thus, as we take the limit in Equation (5.2.1), it follows that

$$A'(x) = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0} \frac{f(x)\cdot h}{h} = f(x).$$

Hence, $A$ is indeed an antiderivative of $f$. In addition, $A(c) = \int_c^c f(t)\ dt = 0$. The preceding argument demonstrates the truth of the Second Fundamental Theorem of Calculus, which we state as follows.

### The Second Fundamental Theorem of Calculus

If $f$ is a continuous function and $c$ is any constant, then $f$ has a unique antiderivative $A$ that satisfies $A(c) = 0$, and that antiderivative is given by the rule $A(x) = \int_c^x f(t)\ dt$.

### Activity 5.2.2

Suppose that $f$ is the function given in Figure 5.2.2 and that $f$ is a piecewise function whose parts are either portions of lines or portions of circles, as pictured.

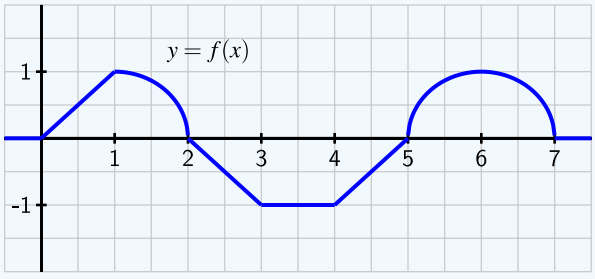





Figure 5.2.2. At left, the graph of $y = f(x)$. At right, axes for sketching $y = A(x)$.

In addition, let $A$ be the function defined by the rule $A(x) = \int_2^x f(t)\ dt$.

1. What does the Second FTC tell us about the relationship between $A$ and $f$?
2. Compute $A(1)$ and $A(3)$ exactly.
3. Sketch a precise graph of $y = A(x)$ on the axes at right that accurately reflects where $A$ is increasing and decreasing, where $A$ is concave up and concave down, and the exact values of $A$ at $x = 0, 1, \ldots, 7$.
4. How is $A$ similar to, but different from, the function $F$ that you found in Activity 5.1.2?
5. With as little additional work as possible, sketch precise graphs of the functions $B(x) = \int_3^x f(t)\ dt$ and $C(x) = \int_1^x f(t)\ dt$. Justify your results with at least one sentence of explanation.

## Understanding Integral Functions

The Second FTC provides us with a way to construct an antiderivative of any continuous function. In particular, if we are given a continuous function $g$ and wish to find an antiderivative $G$, we can now say that

$$G(x) = \int_c^x g(t)\,dt$$

provides the rule for such an antiderivative, and moreover that $G(c) = 0$. Note especially that we know that $G'(x) = g(x)$, or

$$\frac{d}{dx}\left[\int_c^x g(t)\,dt\right] = g(x). \tag{5.2.2}$$

This result is useful for understanding the graph of $G$.

### Example 5.2.1

Investigate the behavior of the integral function

$$E(x) = \int_0^x e^{-t^2}\,dt.$$

*Answer available at the end of this section.*

### Activity 5.2.3

Suppose that $f(t) = \frac{t}{1+t^2}$ and $F(x) = \int_0^x f(t)\ dt$.

1. On the axes at left in Figure 5.2.4, plot a graph of $f(t) = \frac{t}{1+t^2}$ on the interval $-10 \le t \le 10$. Clearly label the vertical axes with appropriate scale.
2. What is the key relationship between $F$ and $f$, according to the Second FTC?
3. Use the first derivative test to determine the intervals on which $F$ is increasing and decreasing.
4. Use the second derivative test to determine the intervals on which $F$ is concave up and concave down. Note that $f'(t)$ can be simplified to be written in the form $f'(t) = \frac{1-t^2}{(1+t^2)^2}$.
5. Using technology appropriately, estimate the values of $F(5)$ and $F(10)$ through appropriate Riemann sums.
6. Sketch an accurate graph of $y = F(x)$ on the righthand axes provided, and clearly label the vertical axes with appropriate scale.



Figure 5.2.4. Axes for plotting $f$ and $F$.

## Differentiating an Integral Function

We have seen that the Second FTC enables us to construct an antiderivative $F$ for any continuous function $f$ as the integral function $F(x) = \int_c^x f(t)\ dt$. If we have a function of the form $F(x) = \int_c^x f(t)\ dt$, then we know that $F'(x) = \frac{d}{dx}\left[\int_c^x f(t)\,dt\right] = f(x)$. This shows that integral functions, while perhaps having the most complicated formulas of any functions we have encountered, are nonetheless particularly simple to differentiate. For instance, if

$$F(x) = \int_\pi^x \sin(t^2)\,dt,$$

then by the Second FTC, we know immediately that

$$F'(x) = \sin(x^2).$$

In general, we know by the Second FTC that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

This equation says that "the derivative of the integral function whose integrand is $f$, is $f$." We see that if we first integrate the function $f$ from $t = a$ to $t = x$, and then differentiate with respect to $x$, these two processes "undo" each other.

What happens if we differentiate a function $f(t)$ and then integrate the result from $t = a$ to $t = x$? That is, what can we say about the quantity

$$\int_a^x \frac{d}{dt}\left[f(t)\right]\,dt?$$

We note that $f(t)$ is an antiderivative of $\frac{d}{dt}\left[f(t)\right]$ and apply the First FTC. We see that

$$\begin{aligned}\int_a^x \frac{d}{dt}\left[f(t)\right]\,dt &= f(t)\Big|_a^x \\ &= f(x) - f(a).\end{aligned}$$

Thus, we see that if we first differentiate $f$ and then integrate the result from $a$ to $x$, we return to the function $f$, minus the constant value $f(a)$. So the two processes almost undo each other, up to the constant $f(a)$.

The observations made in the preceding two paragraphs demonstrate that differentiating and integrating (where we integrate from a constant up to a variable) are almost inverse processes. This should not be surprising: integrating involves antidifferentiating, which reverses the process of differentiating. On the other hand, we see that there is some subtlety involved, because integrating the derivative of a function does not quite produce the function itself. This is because every function has an entire family of antiderivatives, and any two of those antiderivatives differ only by a constant.

**Activity 5.2.4**

Evaluate each of the following derivatives and definite integrals. Clearly cite whether you use the First or Second FTC in so doing.

1. $\frac{d}{dx}\left[\int_4^x e^{t^2}\,dt\right]$
2. $\int_{-2}^{x} \frac{d}{dt}\left[\frac{t^4}{1+t^4}\right]\,dt$
3. $\frac{d}{dx}\left[\int_x^1 \cos(t^3)\,dt\right]$
4. $\int_3^x \frac{d}{dt}\left[\ln(1+t^2)\right]\,dt$
5. $\frac{d}{dx}\left[\int_4^{x^3} \sin(t^2)\,dt\right]$.

## Summary

- For a continuous function $f$, the integral function $A(x) = \int_1^x f(t)\,dt$ defines an antiderivative of $f$.
- The Second Fundamental Theorem of Calculus is the formal, more general statement of the preceding fact: if $f$ is a continuous function and $c$ is any constant, then $A(x) = \int_c^x f(t)\,dt$ is the unique antiderivative of $f$ that satisfies $A(c) = 0$.
- Together, the First and Second FTC enable us to formally see how differentiation and integration are almost inverse processes through the observations that

$$\int_c^x \frac{d}{dt}\left[f(t)\right]\,dt = f(x) - f(c)$$

and

$$\frac{d}{dx}\left[\int_c^x f(t)\,dt\right] = f(x).$$

## Answers

**Example** 5.2.1

$E$ is closely related to the well known *error function* [1] in probability and statistics. It turns out that the function $e^{-t^2}$ does not have an elementary antiderivative. The error function is defined by the rule $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ and has the key property that $0 \le \operatorname{erf}(x) < 1$ for all $x \ge 0$ and moreover that $\lim_{x\to\infty} \operatorname{erf}(x) = 1$.

While we cannot evaluate $E$ exactly for any value other than $x = 0$, we still can gain a tremendous amount of information about the function $E$. By applying the rule in Equation (5.2.2) to $E$, it follows that

$$E'(x) = \frac{d}{dx}\left[\int_0^x e^{-t^2}\,dt\right] = e^{-x^2},$$

so we know a formula for the derivative of $E$, and we know that $E(0) = 0$. This information is precisely the type we were given in Activity 3.1.2, where we were given information about the derivative of a function, but lacked a formula for the function itself.

Using the first and second derivatives of $E$, along with the fact that $E(0) = 0$, we can determine more information about the behavior of $E$. First, we note that for all real numbers $x$, $e^{-x^2} > 0$, and thus $E'(x) > 0$ for all $x$. Thus $E$ is an always increasing function. Further, as $x \to \infty$, $E'(x) = e^{-x^2} \to 0$, so the slope of the function $E$ tends to zero as $x \to \infty$ (and similarly as $x \to -\infty$). Indeed, it turns out that $E$ has horizontal asymptotes as $x$ increases or decreases without bound.

In addition, we can observe that $E''(x) = -2xe^{-x^2}$, and that $E''(0) = 0$, while $E''(x) < 0$ for $x > 0$ and $E''(x) > 0$ for $x < 0$. This information tells us that $E$ is concave up for $x < 0$ and concave down for $x > 0$ with a point of inflection at $x = 0$.

The only thing we lack at this point is a sense of how big $E$ can get as $x$ increases. If we use a midpoint Riemann sum with 10 subintervals to estimate $E(2)$, we see that $E(2) \approx 0.8822$; a similar calculation to estimate $E(3)$ shows little change ($E(3) \approx 0.8862$), so it appears that as $x$ increases without bound, $E$ approaches a value just larger than $0.886$, which aligns with the fact that $E$ has horizontal asymptotes. Putting all of this information together (and using the symmetry of $f(t) = e^{-t^2}$), we see the results shown in Figure 5.2.3.

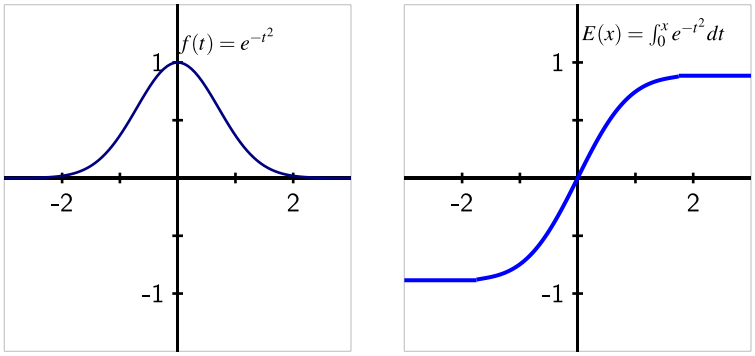



Figure 5.2.3. At left, the graph of $f(t) = e^{-t^2}$. At right, the integral function $E(x) = \int_0^x e^{-t^2}\, dt$, which is the unique antiderivative of $f$ that satisfies $E(0) = 0$.

Because $E$ is the antiderivative of $f(t) = e^{-t^2}$ that satisfies $E(0) = 0$, values on the graph of $y = E(x)$ represent the net signed area of the region bounded by $f(t) = e^{-t^2}$ from 0 up to $x$. We see that the value of $E$ increases rapidly near zero but then levels off as $x$ increases, since there is less and less additional accumulated area bounded by $f(t) = e^{-t^2}$ as $x$ increases.

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# 5.3: Integration by Substitution

## Motivating Questions

- How can we begin to find algebraic formulas for antiderivatives of more complicated algebraic functions?
- What is an indefinite integral and how is its notation used in discussing antiderivatives?
- How does the technique of $u$-substitution work to help us evaluate certain indefinite integrals, and how does this process rely on identifying function-derivative pairs?

In Section 4.4, we learned the key role that antiderivatives play in the process of evaluating definite integrals exactly. The Fundamental Theorem of Calculus tells us that if $F$ is any antiderivative of $f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Furthermore, we realized that each elementary derivative rule developed in Chapter 2 leads to a corresponding elementary antiderivative, as summarized in Table 4.4.5. Thus, if we wish to evaluate an integral such as

$$\int_0^1 \left(x^3 - \sqrt{x} + 5^x\right)\,dx,$$

it is straightforward to do so, since we can easily antidifferentiate $f(x) = x^3 - \sqrt{x} + 5^x$. Because one antiderivative of $f$ is $F(x) = \frac{1}{4}x^4 - \frac{2}{3}x^{3/2} + \frac{1}{\ln(5)}5^x$, the Fundamental Theorem of Calculus tells us that

$$\begin{aligned}\int_0^1 \left(x^3 - \sqrt{x} + 5^x\right)\,dx &= \frac{1}{4}x^4 - \frac{2}{3}x^{3/2} + \frac{1}{\ln(5)}5^x\Big|_0^1 \\ &= \left(\frac{1}{4}(1)^4 - \frac{2}{3}(1)^{3/2} + \frac{1}{\ln(5)}5^1\right) - \left(0 - 0 + \frac{1}{\ln(5)}5^0\right) \\ &= -\frac{5}{12} + \frac{4}{\ln(5)}.\end{aligned}$$

We see that we have a natural interest in being able to find such algebraic antiderivatives. We emphasize *algebraic* antiderivatives, as opposed to any antiderivative, since we know by the Second Fundamental Theorem of Calculus that $G(x) = \int_a^x f(t)\,dt$ is indeed an antiderivative of the given function $f$, but one that still involves a definite integral. Our goal in this section is to "undo" the process of differentiation to find an algebraic antiderivative for a given function.

## Preview Activity 5.3.1

In Section 2.5, we learned the Chain Rule and how it can be applied to find the derivative of a composite function. In particular, if $u$ is a differentiable function of $x$, and $f$ is a differentiable function of $u(x)$, then

$$\frac{d}{dx}\left[f(u(x))\right] = f'(u(x)) \cdot u'(x).$$

In words, we say that the derivative of a composite function $c(x) = f(u(x))$, where $f$ is considered the "outer" function and $u$ the "inner" function, is "the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function."

1. For each of the following functions, use the Chain Rule to find the function's derivative. Be sure to label each derivative by name (e.g., the derivative of $g(x)$ should be labeled $g'(x)$).
   1. $g(x) = e^{3x}$
   2. $h(x) = \sin(5x + 1)$
   3. $p(x) = \arctan(2x)$

4. $q(x) = (2-7x)^4$
5. $r(x) = 3^{4-11x}$

2. For each of the following functions, use your work in (a) to help you determine the general antiderivative [1] of the function. Label each antiderivative by name (e.g., the antiderivative of $m$ should be called $M$). In addition, check your work by computing the derivative of each proposed antiderivative.
   1. $m(x) = e^{3x}$
   2. $n(x) = \cos(5x+1)$
   3. $s(x) = \dfrac{1}{1+4x^2}$
   4. $v(x) = (2-7x)^3$
   5. $w(x) = 3^{4-11x}$
3. Based on your experience in parts (a) and (b), conjecture an antiderivative for each of the following functions. Test your conjectures by computing the derivative of each proposed antiderivative.
   1. $a(x) = \cos(\pi x)$
   2. $b(x) = (4x+7)^{11}$
   3. $c(x) = xe^{x^2}$

Recall that the general antiderivative of a function includes "$+C$" to reflect the entire family of functions that share the same derivative.

## Reversing the Chain Rule: First Steps

Whenever $f$ is a familiar function whose antiderivative is known and $u(x)$ is a linear function, it is straightforward to antidifferentiate a function of the form

$$h(x) = f(u(x)).$$

**Example 5.3.1**

Determine the general antiderivative of

$$h(x) = (5x-3)^6.$$

Check the result by differentiating.

For this composite function, the outer function $f$ is $f(u) = u^6$, while the inner function is $u(x) = 5x-3$. Since the antiderivative of $f$ is $F(u) = \frac{1}{7}u^7 + C$, we see that the antiderivative of $h$ is

$$H(x) = \frac{1}{7}(5x-3)^7 \cdot \frac{1}{5} + C = \frac{1}{35}(5x-3)^7 + C.$$

The inclusion of the constant $\frac{1}{5}$ is essential precisely because the derivative of the inner function is $u'(x) = 5$. Indeed, if we now compute $H'(x)$, we find by the Chain Rule (and Constant Multiple Rule) that

$$H'(x) = \frac{1}{35} \cdot 7(5x-3)^6 \cdot 5 = (5x-3)^6 = h(x),$$

and thus $H$ is indeed the general antiderivative of $h$.

Hence, in the special case where the outer function is familiar and the inner function is linear, we can antidifferentiate composite functions according to the following rule.

**Note**

If $h(x) = f(ax+b)$ and $F$ is a known algebraic antiderivative of $f$, then the general antiderivative of $h$ is given by

$$H(x) = \frac{1}{a}F(ax+b) + C.$$

It is useful to have shorthand notation that indicates the instruction to find an antiderivative. Thus, in a similar way to how the notation

$$\frac{d}{dx}\left[f(x)\right]$$

represents the derivative of $f(x)$ with respect to $x$, we use the notation of the *indefinite integral*,

$$\int f(x)\,dx$$

to represent the general antiderivative of $f$ with respect to $x$. Returning to the earlier example with $h(x) = (5x-3)^6$, we can rephrase the relationship between $h$ and its antiderivative $H$ through the notation

$$\int (5x-3)^6\,dx = \frac{1}{35}(5x-6)^7 + C.$$

When we find an antiderivative, we will often say that we *evaluate an indefinite integral*. Just as the notation $\frac{d}{dx}[\square]$ means "find the derivative with respect to $x$ of $\square$," the notation $\int \square\ dx$ means "find a function of $x$ whose derivative is $\square$."

**Activity 5.3.2**

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

1. $\displaystyle\int \sin(8-3x)\ dx$
2. $\displaystyle\int \sec^2(4x)\ dx$
3. $\displaystyle\int \frac{1}{11x-9}\ dx$
4. $\displaystyle\int \csc(2x+1)\cot(2x+1)\ dx$
5. $\displaystyle\int \frac{1}{\sqrt{1-16x^2}}\ dx$
6. $\displaystyle\int 5^{-x}\ dx$

## Reversing the Chain Rule: $u$-substitution

A natural question arises from our recent work: what happens when the inner function is not linear? For example, can we find antiderivatives of such functions as

$$g(x) = xe^{x^2} \text{ and } h(x) = e^{x^2}?$$

It is important to remember that differentiation and antidifferentiation are almost inverse processes (that they are not is due to the $+C$ that arises when antidifferentiating). This almost-inverse relationship enables us to take any known derivative rule and rewrite it as a corresponding rule for an indefinite integral. For example, since

$$\frac{d}{dx}\left[x^5\right] = 5x^4,$$

we can equivalently write

$$\int 5x^4\,dx = x^5 + C.$$

Recall that the Chain Rule states that

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x).$$

Restating this relationship in terms of an indefinite integral,

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C. \tag{5.3.1}$$

Equation (5.3.1) tells us that if we can view a given function as $f'(g(x))g'(x)$ for some appropriate choices of $f$ and $g$, then we can antidifferentiate the function by reversing the Chain Rule. Note that both $g(x)$ and $g'(x)$ appear in the form of $f'(g(x))g'(x)$; we will sometimes say that we seek to *identify a function-derivative pair* ($g(x)$ and $g'(x)$) when trying to apply the rule in Equation (5.3.1).

If we can identify a function-derivative pair, we will introduce a new variable $u$ to represent the function $g(x)$. With $u = g(x)$, it follows in Leibniz notation that $\frac{du}{dx} = g'(x)$, so that in terms of differentials[2], $du = g'(x)\ dx$. Now converting the indefinite integral to a new one in terms of $u$, we have

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du.$$

If we recall from the definition of the derivative that $\frac{du}{dx} \approx \frac{\Delta u}{\Delta x}$ and use the fact that $\frac{du}{dx} = g'(x)$, then we see that $g'(x) \approx \frac{\Delta u}{\Delta x}$. Solving for $\Delta u$, $\Delta u \approx g'(x)\Delta x$. It is this last relationship that, when expressed in "differential" notation enables us to write $du = g'(x)\ dx$ in the change of variable formula.

Provided that $f'$ is an elementary function whose antiderivative is known, we can easily evaluate the indefinite integral in $u$, and then go on to determine the desired overall antiderivative of $f'(g(x))g'(x)$. We call this process *$u$-substitution*, and summarize the rule as follows:

**Note**

With the substitution $u = g(x)$,

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + C = f(g(x)) + C.$$

To see $u$-substitution at work, we consider the following example.

**Example 5.3.2**

Evaluate the indefinite integral

$$\int x^3 \cdot \sin(7x^4 + 3)\,dx$$

and check the result by differentiating.

*Answer available at the end of this section.*

The $u$-substitution worked because the function multiplying $\sin(7x^4 + 3)$ was $x^3$. If instead that function was $x^2$ or $x^4$, the substitution process would not have worked. This is one of the primary challenges of antidifferentiation: slight changes in the integrand make tremendous differences. For instance, we can use $u$-substitution with $u = x^2$ and $du = 2xdx$ to find that

$$\begin{aligned}\int xe^{x^2}\,dx &= \int e^u \cdot \frac{1}{2}\,du \\ &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C.\end{aligned}$$

However, for the similar indefinite integral

$$\int e^{x^2}\,dx,$$

the $u$-substitution $u = x^2$ is no longer possible because the factor of $x$ is missing. Hence, part of the lesson of $u$-substitution is just how specialized the process is: it only applies to situations where, up to a missing constant, the integrand is the result of applying the Chain Rule to a different, related function.

**Activity 5.3.3**

Evaluate each of the following indefinite integrals by using these steps:

- Find two functions within the integrand that form (up to a possible missing constant) a function-derivative pair;
- Make a substitution and convert the integral to one involving $u$ and $du$;
- Evaluate the new integral in $u$;
- Convert the resulting function of $u$ back to a function of $x$ by using your earlier substitution;
- Check your work by differentiating the function of $x$. You should come up with the integrand originally given.

1. $\displaystyle\int \frac{x^2}{5x^3+1}\,dx$
2. $\displaystyle\int e^x \sin(e^x)\,dx$
3. $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

## Evaluating Definite Integrals via $u$-substitution

We have introduced $u$-substitution as a means to evaluate indefinite integrals of functions that can be written, up to a constant multiple, in the form $f(g(x))g'(x)$. This same technique can be used to evaluate definite integrals involving such functions, though we need to be careful with the corresponding limits of integration. Consider, for instance, the definite integral

$$\int_2^5 xe^{x^2}\,dx.$$

Whenever we write a definite integral, it is implicit that the limits of integration correspond to the variable of integration. To be more explicit, observe that

$$\int_2^5 xe^{x^2}\,dx = \int_{x=2}^{x=5} xe^{x^2}\,dx.$$

When we execute a $u$-substitution, we change the *variable* of integration; it is essential to note that this also changes the *limits* of integration. For instance, with the substitution $u = x^2$ and $du = 2x\ dx$, it also follows that when $x = 2$, $u = 2^2 = 4$, and when $x = 5$, $u = 5^2 = 25$. Thus, under the change of variables of $u$-substitution, we now have

$$\begin{aligned}\int_{x=2}^{x=5} xe^{x^2}\,dx &= \int_{u=4}^{u=25} e^u \cdot \frac{1}{2}\,du \\ &= \frac{1}{2}e^u\Big|_{u=4}^{u=25} \\ &= \frac{1}{2}e^{25} - \frac{1}{2}e^4.\end{aligned}$$

Alternatively, we could consider the related indefinite integral $\int xe^{x^2}\,dx$, find the antiderivative $\frac{1}{2}e^{x^2}$ through $u$-substitution, and then evaluate the original definite integral. With that method, we'd have

$$\begin{aligned}\int_2^5 xe^{x^2}\,dx &= \frac{1}{2}e^{x^2}\Big|_2^5 \\ &= \frac{1}{2}e^{25} - \frac{1}{2}e^4,\end{aligned}$$

which is, of course, the same result.

**Activity 5.3.4**

Evaluate each of the following definite integrals exactly through an appropriate $u$-substitution.

1. $\displaystyle\int_1^2 \frac{x}{1+4x^2}\,dx$
2. $\displaystyle\int_0^1 e^{-x}(2e^{-x}+3)^9\,dx$
3. $\displaystyle\int_{2/\pi}^{4/\pi} \frac{\cos\left(\frac{1}{x}\right)}{x^2}\,dx$

## Summary

- To find algebraic formulas for antiderivatives of more complicated algebraic functions, we need to think carefully about how we can reverse known differentiation rules. To that end, it is essential that we understand and recall known derivatives of basic functions, as well as the standard derivative rules.
- The indefinite integral provides notation for antiderivatives. When we write "$\int f(x)\,dx$," we mean "the general antiderivative of $f$." In particular, if we have functions $f$ and $F$ such that $F' = f$, the following two statements say the exact thing:

  $$\frac{d}{dx}[F(x)] = f(x) \text{ and } \int f(x)\,dx = F(x) + C.$$

  That is, $f$ is the derivative of $F$, and $F$ is an antiderivative of $f$.

- The technique of $u$-substitution helps us to evaluate indefinite integrals of the form $\int f(g(x))g'(x)\,dx$ through the substitutions $u = g(x)$ and $du = g'(x)\,dx$, so that

  $$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

  A key part of choosing the expression in $x$ to be represented by $u$ is the identification of a function-derivative pair. To do so, we often look for an "inner" function $g(x)$ that is part of a composite function, while investigating whether $g'(x)$ (or a constant multiple of $g'(x)$) is present as a multiplying factor of the integrand.

## Answers

**Example** 5.3.2

We can make two algebraic observations regarding the integrand, $x^3 \cdot \sin(7x^4+3)$. First, $\sin(7x^4+3)$ is a composite function; as such, we know we'll need a more sophisticated approach to antidifferentiating. Second, $x^3$ is almost the derivative of $(7x^4+3)$; the only issue is a missing constant. Thus, $x^3$ and $(7x^4+3)$ are nearly a function-derivative pair. Furthermore, we know the antiderivative of $f(u) = \sin(u)$. The combination of these observations suggests that we can evaluate the given indefinite integral by reversing the chain rule through $u$-substitution.

Letting $u$ represent the inner function of the composite function $\sin(7x^4+3)$, we have $u = 7x^4+3$, and thus $\frac{du}{dx} = 28x^3$. In differential notation, it follows that $du = 28x^3\ dx$, and thus $x^3\ dx = \frac{1}{28}\ du$. The original indefinite integral may be slightly rewritten as

$$\int \sin(7x^4+3)\cdot x^3\,dx,$$

and so by substituting $u$ for $7x^4+3$ and $\frac{1}{28}\ du$ for $x^3\ dx$, it follows that

$$\int \sin(7x^4+3)\cdot x^3\,dx = \int \sin(u)\cdot\frac{1}{28}\,du.$$

Now we may evaluate the easier integral in $u$, and then replace $u$ by the expression $7x^4+3$. Doing so, we find

$$\begin{aligned}\int \sin(7x^4+3)\cdot x^3\,dx &= \int \sin(u)\cdot\frac{1}{28}\,du\\ &= \frac{1}{28}\int \sin(u)\,du\\ &= \frac{1}{28}(-\cos(u)) + C\\ &= -\frac{1}{28}\cos(7x^4+3) + C.\end{aligned}$$

To check our work, we observe by the Chain Rule that

$$\frac{d}{dx}\left[-\frac{1}{28}\cos(7x^4+3)\right] = -\frac{1}{28}\cdot(-1)\sin(7x^4+3)\cdot 28x^3 = \sin(7x^4+3)\cdot x^3,$$

which is indeed the original integrand.

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# 5.3: B - TRIGONOMETRIC SUBSTITUTION (OPENSTAX)

## 5.3.1: Exercises for Section 7.3

**Learning Objectives**

- Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$, where the values of $a$ are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of *trigonometric substitution* comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

## Integrals Involving $\sqrt{a^2-x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2-x^2}$, consider the integral $\int \sqrt{9-x^2}dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x=3\sin\theta$, we have $dx=3\cos\theta\ d\theta$. After substituting into the integral, we have

$$\int \sqrt{9-x^2}\,dx = \int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta.$$

After simplifying, we have

$$\int \sqrt{9-x^2}\,dx = \int 9\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta.$$

Letting $1-\sin^2\theta=\cos^2\theta$, we now have

$$\int \sqrt{9-x^2}\,dx = \int 9\sqrt{\cos^2\theta}\cos\theta\,d\theta.$$

Assuming that $\cos\theta\geq 0$, we have

$$\int \sqrt{9-x^2}\,dx = \int 9\cos^2\theta\,d\theta.$$

At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2-x^2}$, we make the substitution $x=a\sin\theta$ and $dx=a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2-x^2}$ is $[-a,a]$. Thus,

$$-a\leq x\leq a.$$

Consequently,

$$-1\leq\frac{x}{a}\leq 1.$$

Since the range of $\sin x$ over $[-(\pi/2),\pi/2]$ is $[-1,1]$, there is a unique angle $\theta$ satisfying $-(\pi/2)\leq\theta\leq\pi/2$ so that $\sin\theta=x/a$, or equivalently, so that $x=a\sin\theta$. If we substitute $x=a\sin\theta$ into $\sqrt{a^2-x^2}$, we get

$$\begin{aligned}
\sqrt{a^2-x^2} &= \sqrt{a^2-(a\sin\theta)^2} && \text{Let } x = a\sin\theta \text{ where } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.\\
& && \text{Simplify.}\\
&= \sqrt{a^2-a^2\sin^2\theta} && \text{Factor out } a^2.\\
&= \sqrt{a^2(1-\sin^2\theta)} && \text{Substitute } 1-\sin^2 x = \cos^2 x.\\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.}\\
&= |a\cos\theta|\\
&= a\cos\theta
\end{aligned}$$

Since $\cos x \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $a > 0, |a\cos\theta| = a\cos\theta$. We can see, from this discussion, that by making the substitution $x = a\sin\theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving $x$. To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin\theta = \frac{x}{a}$, we can draw the reference triangle in Figure 5.3.1 to assist in expressing the values of $\cos\theta,\ \tan\theta$, and the remaining trigonometric functions in terms of $x$. It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at $\theta$ for all $\theta$ satisfying $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2-x^2}$ actually appears as the length of one side of the triangle. Last, should $\theta$ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

This figure is a right triangle. It has an angle labeled theta. This angle is opposite the vertical side. The hypotenuse is labeled a, the vertical leg is labeled x, and the horizontal leg is labeled as the square root of (a^2 – x^2). To the left of the triangle is the equation sin(theta) = x/a.

Figure 5.3.1: A reference triangle can help express the trigonometric functions evaluated at $\theta$ in terms of $x$.

The essential part of this discussion is summarized in the following problem-solving strategy.

### Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2-x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2-x^2}}dx$, $\int \frac{x}{\sqrt{a^2-x^2}}dx$, and $\int x\sqrt{a^2-x^2}\ dx$, they can each be integrated directly either by formula or by a simple $u$-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta\ d\theta$. Note: This substitution yields $\sqrt{a^2-x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 1 to rewrite the result in terms of $x$. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

### Example 5.3.1: Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate

$$\int \sqrt{9-x^2}dx.$$

**Solution**

Begin by making the substitutions $x = 3\sin\theta$ and $dx = 3\cos\theta\ d\theta$. Since $\sin\theta = \frac{x}{3}$, we can construct the reference triangle shown in Figure 2.

This figure is a right triangle. It has an angle labeled theta. This angle is opposite the vertical side. The hypotenuse is labeled 3, the vertical leg is labeled x, and the horizontal leg is labeled as the square root of (9 – x^2). To the left of the triangle is the equation sin(theta) = x/3.

Figure 5.3.2: A reference triangle can be constructed for Example 5.3.1.

Thus,

$$\int \sqrt{9-x^2}\ dx = \int \sqrt{9-(3\sin\theta)^2}3\cos\theta\ d\theta \text{ Substitute } x = 3\sin\theta \text{ and } dx = 3\cos\theta\ d\theta.$$

$$= \int \sqrt{9(1-\sin^2\theta)}\cdot 3\cos\theta\ d\theta \text{ Simplify.}$$

$$= \int \sqrt{9\cos^2\theta}\cdot 3\cos\theta\ d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta.$$

$$= \int 3|\cos\theta|3\cos\theta\ d\theta \text{ Take the square root.}$$

$$= \int 9\cos^2\theta\ d\theta \text{ Simplify. Since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \cos\theta \ge 0 \text{ and } |\cos\theta| = \cos\theta.$$

$$= \int 9\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)\ d\theta \text{ Use the strategy for integrating an even power of } \cos\theta.$$

$$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \text{ Evaluate the integral.}$$

$$= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C \text{ Substitute } \sin(2\theta) = 2\sin\theta\cos\theta.$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C \text{ Substitute } \sin^{-1}\left(\frac{x}{3}\right) = \theta \text{ and } \sin\theta = \tfrac{x}{3}.$$

Use the reference triangle to see that $\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$ and make this substitution. Simplify.

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C. \text{ Simplify.}$$

## Example 5.3.2: Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate

$$\int \frac{\sqrt{4-x^2}}{x}dx.$$

**Solution**

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta\ d\theta$. Since $\sin\theta = \dfrac{x}{2}$, we can construct the reference triangle shown in Figure 5.3.3.

This figure is a right triangle. It has an angle labeled theta. This angle is opposite the vertical side. The vertical leg is labeled x, and the horizontal leg is labeled as the square root of (4 – x^2). To the left of the triangle is the equation sin(theta) = x/2.

Figure 5.3.3: A reference triangle can be constructed for Example 5.3.2.

Thus,

$$\int \frac{\sqrt{4-x^2}}{x}dx = \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta}2\cos\theta\ d\theta \text{ Substitute } x = 2\sin\theta \text{ and } dx = 2\cos\theta\ d\theta.$$

$$= \int \frac{2\cos^2\theta}{\sin\theta}\ d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta \text{ and simplify.}$$

$= \int \frac{2(1-\sin^2\theta)}{\sin\theta}\ d\theta$ Substitute $\cos^2\theta = 1-\sin^2\theta$.

$= \int (2\csc\theta - 2\sin\theta)\ d\theta$ Separate the numerator, simplify, and use $\csc\theta = \frac{1}{\sin\theta}$.

$= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C$ Evaluate the integral.

$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C$. Use the reference triangle to rewrite the expression in terms of $x$ and simplify.

In the next example, we see that we sometimes have a choice of methods.

## Example 5.3.3: Integrating an Expression Involving $\sqrt{a^2-x^2}$ Two Ways

Evaluate $\int x^3\sqrt{1-x^2}dx$ two ways: first by using the substitution $u = 1-x^2$ and then by using a trigonometric substitution.

**Method 1**

Let $u = 1-x^2$ and hence $x^2 = 1-u$. Thus, $du = -2x\ dx$. In this case, the integral becomes

$\int x^3\sqrt{1-x^2}\ dx = -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\ dx)$ Make the substitution.

$= -\frac{1}{2}\int (1-u)\sqrt{u}\ du$ Expand the expression.

$= -\frac{1}{2}\int (u^{1/2} - u^{3/2})\ du$ Evaluate the integral.

$= -\frac{1}{2}(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}) + C$ Rewrite in terms of x.

$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C.$

**Method 2**

Let $x = \sin\theta$. In this case, $dx = \cos\theta\ d\theta$. Using this substitution, we have

$\int x^3\sqrt{1-x^2}dx = \int \sin^3\theta\cos^2\theta\ d\theta$

$= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\ d\theta$ Let $u = \cos\theta$. Thus,$du = -\sin\theta\ d\theta$.

$= \int (u^4 - u^2)\ du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$ Substitute $\cos\theta = u$.

$= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C$ Use a reference triangle to see that $\cos\theta = \sqrt{1-x^2}$.

$= \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C.$

## Exercise 5.3.1

Rewrite the integral $\int \frac{x^3}{\sqrt{25-x^2}}\ dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Integrating Expressions Involving $\sqrt{a^2+x^2}$

For integrals containing $\sqrt{a^2+x^2}$,let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of $x$, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a\tan\theta$ or $x = a\cot\theta$. Either of these substitutions would actually work, but the standard substitution is $x = a\tan\theta$ or, equivalently, $\tan\theta = x/a$. With this substitution, we make the assumption that $-(\pi/2) < \theta < \pi/2$, so that we also have $\theta = \tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

### Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta\ d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2sec^2\theta} = |a\sec\theta| = a\sec\theta$. (Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 5.3.4 to rewrite the result in terms of $x$. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

This figure is a right triangle. It has an angle labeled theta. This angle is opposite the vertical side. The hypotenuse is labeled the square root of (a^2+x^2), the vertical leg is labeled x, and the horizontal leg is labeled a. To the left of the triangle is the equation tan(theta) = x/a.

Figure 5.3.4: A reference triangle can be constructed to express the trigonometric functions evaluated at $\theta$ in terms of $x$.

### Example 5.3.4: Integrating an Expression Involving $\sqrt{a^2+x^2}$

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

**Solution**

Begin with the substitution $x = \tan\theta$ and $dx = sec^2\theta\ d\theta$. Since $\tan\theta = x$, draw the reference triangle in Figure 5.3.5.

This figure is a right triangle. It has an angle labeled theta. This angle is opposite the vertical side. The hypotenuse is labeled the square root of (1+x^2), the vertical leg is labeled x, and the horizontal leg is labeled 1. To the left of the triangle is the equation tan(theta) = x/1.

Figure 5.3.5: The reference triangle for Example 5.3.4.

Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta}d\theta \qquad \text{Substitute } x=\tan\theta \text{ and } dx=\sec^2\theta\,d\theta.$$

$$\text{This substitution makes } \sqrt{1+x^2}=\sec\theta. \text{ Simplify.}$$

$$= \int \sec\theta\,d\theta \qquad \text{Evaluate the integral.}$$

$$= \ln|\sec\theta+\tan\theta|+C \qquad \text{Use the reference triangle to express the result in terms of } x.$$

$$= \ln|\sqrt{1+x^2}+x|+C$$

To check the solution, differentiate:

$$\frac{d}{dx}\left(\ln|\sqrt{1+x^2}+x|\right) = \frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right) = \frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

Since $\sqrt{1+x^2}+x>0$ for all values of $x$, we could rewrite $\ln|\sqrt{1+x^2}+x|+C=\ln(\sqrt{1+x^2}+x)+C$, if desired.

## Example 5.3.5: Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x=\sinh\theta$ to evaluate $\int \frac{dx}{\sqrt{1+x^2}}$.

**Solution**

Because $\sinh\theta$ has a range of all real numbers, and $1+\sinh^2\theta=\cosh^2\theta$, we may also use the substitution $x=\sinh\theta$ to evaluate this integral. In this case, $dx=\cosh\theta\ d\theta$. Consequently,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}d\theta \qquad \text{Substitute } x=\sinh\theta \text{ and } dx=\cosh\theta\,d\theta.$$

$$\text{Substitute } 1+\sinh^2\theta=\cosh^2\theta.$$

$$= \int \frac{\cosh\theta}{\sqrt{\cosh^2\theta}}d\theta \qquad \text{Since } \sqrt{\cosh^2\theta}=|\cosh\theta|$$

$$= \int \frac{\cosh\theta}{|\cosh\theta|}d\theta \qquad |\cosh\theta|=\cosh\theta \text{ since } \cosh\theta>0 \text{ for all } \theta.$$

$$= \int \frac{\cosh\theta}{\cosh\theta}d\theta \qquad \text{Simplify.}$$

$$= \int 1\,d\theta \qquad \text{Evaluate the integral.}$$

$$= \theta+C \qquad \text{Since } x=\sinh\theta, \text{ we know } \theta=\sinh^{-1}x.$$

$$= \sinh^{-1}x+C.$$

**Analysis**

This answer looks quite different from the answer obtained using the substitution $x=\tan\theta$. To see that the solutions are the same, set $y=\sinh^{-1}x$. Thus, $\sinh y=x$. From this equation we obtain:

$$\frac{e^y-e^{-y}}{2}=x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y}-2xe^y-1=0.$$

Use the quadratic equation to solve for $e^y$:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}.$$

Simplifying, we have:

$$e^y = x \pm \sqrt{x^2+1}.$$

Since $x - \sqrt{x^2+1} < 0$, it must be the case that $e^y = x + \sqrt{x^2+1}$. Thus,

$$y = \ln(x + \sqrt{x^2+1}).$$

Last, we obtain

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$

After we make the final observation that, since $x + \sqrt{x^2+1} > 0$,

$$\ln(x + \sqrt{x^2+1}) = \ln | \sqrt{1+x^2} + x |,$$

we see that the two different methods produced equivalent solutions.

## Example 5.3.6: Finding an Arc Length

Find the length of the curve $y = x^2$ over the interval $[0, \frac{1}{2}]$.

**Solution**

Because $\frac{dy}{dx} = 2x$, the arc length is given by

$$\int_0^{1/2} \sqrt{1+(2x)^2}dx = \int_0^{1/2} \sqrt{1+4x^2}dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta \ d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

$\int_0^{1/2} \sqrt{1+4x^2}dx = \int_0^{\pi/4} \sqrt{1+\tan^2\theta} \cdot \frac{1}{2}\sec^2\theta \ d\theta$ After substitution, $\sqrt{1+4x^2} = \sec\theta$. (Substitute $1+\tan^2\theta = \sec^2\theta$ and simplify.)

$= \frac{1}{2}\int_0^{\pi/4} \sec^3\theta \ d\theta$ We derived this integral in the previous section.

$= \frac{1}{2}(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|) \ |_0^{\pi/4}$ Evaluate and simplify.

$= \frac{1}{4}(\sqrt{2} + \ln(\sqrt{2}+1)).$

## Exercise 5.3.2

Rewrite $\int x^3\sqrt{x^2+4}dx$ by using a substitution involving $\tan\theta$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

# Integrating Expressions Involving $\sqrt{x^2 - a^2}$

The domain of the expression $\sqrt{x^2 - a^2}$ is $(-\infty, -a] \cup [a, +\infty)$. Thus, either $x \leq -a$ or $x \geq a$. Hence, $\frac{x}{a} \leq -1$ or $\frac{x}{a} \geq 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, it makes sense to use the substitution $\sec\theta = \frac{x}{a}$ or, equivalently, $x = a\sec\theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$. The corresponding substitution for $dx$ is $dx = a\sec\theta\tan\theta\ d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

## Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\ d\theta$. This substitution yields
$$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|.$$
For $x \geq a, |a\tan\theta| = a\tan\theta$ and for $x \leq -a, |a\tan\theta| = -a\tan\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from Figure $5.3.6$ to rewrite the result in terms of $x$.
6. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

This figure has two right triangles. The first triangle is in the first quadrant of the xy coordinate system and has an angle labeled theta. This angle is opposite the vertical side. The hypotenuse is labeled x, the vertical leg is labeled the square root of (x^2-a^2), and the horizontal leg is labeled a. The horizontal leg is on the x-axis. To the left of the triangle is the equation sec(theta) = x/a, x
a. There are also the equations sin(theta)= the square root of (x^2-a^2)/x, cos(theta) = a/x, and tan(theta) = the square root of (x^2-a^2)/a. The second triangle is in the second quadrant, with the hypotenuse labeled –x. The horizontal leg is labeled –a and is on the negative x-axis. The vertical leg is labeled the square root of (x^2-a^2). To the right of the triangle is the equation sec(theta) = x/a, x<-a. There are also the equations sin(theta)= the negative square root of (x^2-a^2)/x, cos(theta) = a/x, and tan(theta) = the negative square root of (x^2-a^2)/a." class="internal" loading="lazy" src="https://math.libretexts.orghttps://math.libretexts.org/@api/deki/files/12447/7.3.4.png?revision=1">

Figure $5.3.6$: Use the appropriate reference triangle to express the trigonometric functions evaluated at $\theta$ in terms of $x$.

## Example $5.3.7$: Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.

**Solution**

First, sketch a rough graph of the region described in the problem, as shown in the following figure.

This figure is the graph of the function f(x) = the square root of (x^2-9). It is an increasing curve that starts on the x-axis at 3 and is in the first quadrant. Under the curve above the x-axis is a shaded region bounded to the right at x = 5.

Figure $5.3.7$: Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2 - 9}dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\ d\theta$ . We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}(\frac{5}{3})$. After making these substitutions and simplifying, we have

**Area** $= \int_3^5 \sqrt{x^2 - 9}dx$

$= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta\ d\theta$ Use $\tan^2\theta = \sec^2\theta - 1$.

$= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta - 1)\sec\theta \; d\theta$ Expand.

$= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta - \sec\theta) \; d\theta$ Evaluate the integral.

$= (\frac{9}{2}\ln|\sec\theta + \tan\theta| + \frac{9}{2}\sec\theta\tan\theta) - 9\ln|\sec\theta + \tan\theta| \; |_0^{\sec^{-1}(5/3)}$ Simplify.

$= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| \; |_0^{\sec^{-1}(5/3)}$ Evaluate. Use $\sec(\sec^{-1}\frac{5}{3}) = \frac{5}{3}$ and $\tan(\sec^{-1}\frac{5}{3}) = \frac{4}{3}$.

$= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln|\frac{5}{3} + \frac{4}{3}| - (\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1+0|)$

$= 10 - \frac{9}{2}\ln 3$

**Exercise 5.3.3**

Evaluate

$$\int \frac{dx}{\sqrt{x^2-4}}.$$

Assume that $x > 2$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Key Concepts

- For integrals involving $\sqrt{a^2-x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta \; d\theta$.
- For integrals involving $\sqrt{a^2+x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta \; d\theta$.
- For integrals involving $\sqrt{x^2-a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta \; d\theta$.

## Glossary

**trigonometric substitution**
an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ into a trigonometric integral

## Answers

**Exercise** 5.3.1
Substitute $x = 5\sin\theta$ and $dx = 5\cos\theta \; d\theta$.

**Exercise** 5.3.1
$\int 125\sin^3\theta \; d\theta$

**Exercise** 5.3.2
Use $x = 2\tan\theta$ and $dx = 2\sec^2\theta \; d\theta$.

**Exercise** 5.3.2
$$\int 32\tan^3\theta\sec^3\theta \, d\theta$$

**Exercise** 5.3.3
Substitute $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta \; d\theta$.

**Exercise** 5.3.3
$$\ln|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}| + C$$

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# 5.3.1: Exercises for Section 7.3

**Simplify the expressions in exercises 1 - 5 by writing each one using a single trigonometric function.**

1) $4 - 4\sin^2\theta$

2) $9\sec^2\theta - 9$

**Answer**

$$9\sec^2\theta - 9 \quad = \quad 9\tan^2\theta$$

3) $a^2 + a^2\tan^2\theta$

4) $a^2 + a^2\sinh^2\theta$

**Answer**

$$a^2 + a^2\sinh^2\theta \quad = \quad a^2\cosh^2\theta$$

5) $16\cosh^2\theta - 16$

**Use the technique of completing the square to express each trinomial in exercises 6 - 8 as the square of a binomial.**

6) $4x^2 - 4x + 1$

**Answer**

$4(x - \frac{1}{2})^2$

7) $2x^2 - 8x + 3$

8) $-x^2 - 2x + 4$

**Answer**

$-(x+1)^2 + 5$

**In exercises 9 - 28, integrate using the method of trigonometric substitution. Express the final answer in terms of the original variable.**

9) $\int \frac{dx}{\sqrt{4 - x^2}}$

10) $\int \frac{dx}{\sqrt{x^2 - a^2}}$

**Answer**

$$\int \frac{dx}{\sqrt{x^2 - a^2}} \quad = \quad \ln\left|\, x + \sqrt{-a^2 + x^2}\,\right| + C$$

11) $\int \sqrt{4 - x^2}\; dx$

12) $\int \frac{dx}{\sqrt{1 + 9x^2}}$

**Answer**

$$\int \frac{dx}{\sqrt{1 + 9x^2}} \quad = \quad \tfrac{1}{3}\ln\left|\, \sqrt{9x^2 + 1} + 3x\,\right| + C$$

13) $\int \frac{x^2\, dx}{\sqrt{1 - x^2}}$

14) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

**Answer**

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} + C$$

15) $\int \frac{dx}{(1+x^2)^2}$

16) $\int \sqrt{x^2+9}\ dx$

**Answer**

$$\int \sqrt{x^2+9}\ dx = 9\left[\frac{x\sqrt{x^2+9}}{18} + \tfrac{1}{2}\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right|\right] + C$$

17) $\int \frac{\sqrt{x^2-25}}{x}\ dx$

18) $\int \frac{\theta^3}{\sqrt{9-\theta^2}}\ d\theta$

**Answer**

$$\int \frac{\theta^3 d\theta}{\sqrt{9-\theta^2}}\ d\theta = -\tfrac{1}{3}\sqrt{9-\theta^2}(18+\theta^2) + C$$

19) $\int \frac{dx}{\sqrt{x^6-x^2}}$

20) $\int \sqrt{x^6-x^8}\ dx$

**Answer**

$$\int \sqrt{x^6-x^8}\ dx = \frac{(-1+x^2)(2+3x^2)\sqrt{x^6-x^8}}{15x^3} + C$$

21) $\int \frac{dx}{(1+x^2)^{3/2}}$

22) $\int \frac{dx}{(x^2-9)^{3/2}}$

**Answer**

$$\int \frac{dx}{(x^2-9)^{3/2}} = -\frac{x}{9\sqrt{x^2-9}} + C$$

23) $\int \frac{\sqrt{1+x^2}}{x}\ dx$

24) $\int \frac{x^2}{\sqrt{x^2-1}}\ dx$

**Answer**

$$\int \frac{x^2}{\sqrt{x^2-1}}\ dx = \tfrac{1}{2}(\ln\mid x+\sqrt{x^2-1}\mid + x\sqrt{x^2-1}) + C$$

25) $\int \frac{x^2}{x^2+4}\ dx$

26) $\int \frac{dx}{x^2\sqrt{x^2+1}}$

**Answer**

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = -\frac{\sqrt{1+x^2}}{x} + C$$

27) $\int \frac{x^2}{\sqrt{1+x^2}}\ dx$

28) $\int_{-1}^{1} (1-x^2)^{3/2}\ dx$

**Answer**

$$\int_{-1}^{1} (1-x^2)^{3/2}\ dx = \tfrac{1}{8}\left(x(5-2x^2)\sqrt{1-x^2} + 3\arcsin x\right) + C$$

**In exercises 29 - 34, use the substitutions $x = \sinh\theta$, $\cosh\theta$, or $\tanh\theta$. Express the final answers in terms of the variable $x$.**

29) $\int \frac{dx}{\sqrt{x^2-1}}$

30) $\int \frac{dx}{x\sqrt{1-x^2}}$

**Answer**

$$\int \frac{dx}{x\sqrt{1-x^2}} = \ln x - \ln|\,1+\sqrt{1-x^2}\,| + C$$

31) $\int \sqrt{x^2-1}\ dx$

32) $\int \frac{\sqrt{x^2-1}}{x^2}\ dx$

**Answer**

$$\int \frac{\sqrt{x^2-1}}{x^2}\ dx = -\frac{\sqrt{-1+x^2}}{x} + \ln\left|x+\sqrt{-1+x^2}\right| + C$$

33) $\int \frac{dx}{1-x^2}$

34) $\int \frac{\sqrt{1+x^2}}{x^2}\ dx$

**Answer**

$$\int \frac{\sqrt{1+x^2}}{x^2}\ dx = -\frac{\sqrt{1+x^2}}{x} + \operatorname{arcsinh}\ x + C$$

**Use the technique of completing the square to evaluate the integrals in exercises 35 - 39.**

35) $\int \frac{1}{x^2-6x}\ dx$

36) $\int \frac{1}{x^2+2x+1}\ dx$

**Answer**

$$\int \frac{1}{x^2+2x+1}\ dx = -\frac{1}{1+x} + C$$

37) $\int \frac{1}{\sqrt{-x^2+2x+8}}\ dx$

38) $\int \frac{1}{\sqrt{-x^2+10x}}\ dx$

**Answer**

$\int \frac{1}{\sqrt{-x^2+10x}}\ dx \quad = \quad \arcsin\left(\frac{x-5}{5}\right)+C$

39) $\int \frac{1}{\sqrt{x^2+4x-12}}\ dx$

40) Evaluate the integral without using calculus: $\int_{-3}^{3} \sqrt{9-x^2}\ dx.$

**Answer**

$\int_{-3}^{3} \sqrt{9-x^2}\ dx \quad = \quad \frac{9\pi}{2}$; area of a semicircle with radius 3

41) Find the area enclosed by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1.$

42) Evaluate the integral $\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x=\cos\theta$ and evaluate using trigonometric substitution. Second, let $x=\sin\theta$ and use trigonometric substitution. Are the answers the same?

**Answer**

$\int \frac{dx}{\sqrt{1-x^2}} \quad = \quad \arcsin(x)+C$ is the common answer.

43) Evaluate the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x=\sec\theta$. Next, evaluate the same integral using the substitution $x=\csc\theta$. Show that the results are equivalent.

44) Evaluate the integral $\int \frac{x}{x^2+1}\ dx$ using the form $\int \frac{1}{u}\ du$. Next, evaluate the same integral using $x=\tan\theta$. Are the results the same?

**Answer**

$\int \frac{x}{x^2+1}\ dx \quad = \quad \frac{1}{2}\ln(1+x^2)+C$ is the result using either method.

45) State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2+1}\ dx$. Why did you choose this method?

46) State the method of integration you would use to evaluate the integral $\int x^2\sqrt{x^2-1}\ dx$. Why did you choose this method?

**Answer**

Use trigonometric substitution. Let $x=\sec(\theta)$.

47) Evaluate $\int_{-1}^{1} \frac{x}{x^2+1}\ dx$

48) Find the length of the arc of the curve over the specified interval: $y=\ln x, \quad [1,5]$. Round the answer to three decimal places.

**Answer**

$s=4.367$ units

49) Find the surface area of the solid generated by revolving the region bounded by the graphs of $y=x^2,\ y=0,\ x=0$, and $x=\sqrt{2}$ about the $x$-axis. (Round the answer to three decimal places).

50) The region bounded by the graph of $f(x)=\frac{1}{1+x^2}$ and the $x$-axis between $x=0$ and $x=1$ is revolved about the $x$-axis. Find the volume of the solid that is generated.

**Answer**

$$V = \left(\frac{\pi^2}{8} + \frac{\pi}{4}\right) \text{ units}^3$$

**In exercises 51 - 52, solve the initial-value problem for $y$ as a function of $x$.**

51) $(x^2 + 36)\dfrac{dy}{dx} = 1, \quad y(6) = 0$

52) $(64 - x^2)\dfrac{dy}{dx} = 1, \quad y(0) = 3$

**Answer**

$$y = \frac{1}{16}\ln\left|\frac{x+8}{x-8}\right| + 3$$

53) Find the area bounded by $y = \dfrac{2}{\sqrt{64 - 4x^2}}$, $x = 0$, $y = 0$, and $x = 2$.

54) An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \dfrac{16}{\sqrt{64 + x^2}}$, $x = 0$, $y = 0$, $x = 2$ about the $x$-axis. Find the volume of the tank (in cubic meters).

**Answer**

$$V = 24.6 \text{ m}^3$$

55) During each cycle, the velocity $v$ (in feet per second) of a robotic welding device is given by $v = 2t - \dfrac{14}{4 + t^2}$, where $t$ is time in seconds. Find the expression for the displacement $s$ (in feet) as a function of $t$ if $s = 0$ when $t = 0$.

56) Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2$.

**Answer**

$$s = \frac{2\pi}{3} \text{ units}$$

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# 5.4: Integration by Parts

## Motivating Questions

- How do we evaluate indefinite integrals that involve products of basic functions such as $\int x\sin(x)\ dx$ and $\int xe^x\ dx$?
- What is the method of integration by parts and how can we consistently apply it to integrate products of basic functions?
- How does the algebraic structure of functions guide us in identifying $u$ and $dv$ in using integration by parts?

In Section 5.3, we learned the technique of $u$-substitution for evaluating indefinite integrals. For example, the indefinite integral $\int x^3\sin(x^4)\ dx$ is perfectly suited to $u$-substitution, because one factor is a composite function and the other factor is the derivative (up to a constant) of the inner function. Recognizing the algebraic structure of a function can help us to find its antiderivative.

Next we consider integrands with a different elementary algebraic structure: a product of basic functions. For instance, suppose we are interested in evaluating the indefinite integral

$$\int x\sin(x)\,dx.$$

The integrand is the product of the basic functions $f(x)=x$ and $g(x)=\sin(x)$. We know that it is relatively complicated to compute the derivative of the product of two functions, so we should expect that antidifferentiating a product should be similarly involved. Intuitively, we expect that evaluating $\int x\sin(x)\ dx$ will involve somehow reversing the Product Rule.

To that end, in Preview Activity 5.4.1 we refresh our understanding of the Product Rule and then investigate some indefinite integrals that involve products of basic functions.

## Preview Activity 5.4.1

In Section 2.3, we developed the Product Rule and studied how it is employed to differentiate a product of two functions. In particular, recall that if $f$ and $g$ are differentiable functions of $x$, then

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\cdot g'(x)+g(x)\cdot f'(x).$$

1. For each of the following functions, use the Product Rule to find the function's derivative. Be sure to label each derivative by name (e.g., the derivative of $g(x)$ should be labeled $g'(x)$).
   1. $g(x)=x\sin(x)$
   2. $h(x)=xe^x$
   3. $p(x)=x\ln(x)$
   4. $q(x)=x^2\cos(x)$
   5. $r(x)=e^x\sin(x)$
2. Use your work in (a) to help you evaluate the following indefinite integrals. Use differentiation to check your work.
   1. $\int xe^x+e^x\ dx$
   2. $\int e^x(\sin(x)+\cos(x))\ dx$
   3. $\int 2x\cos(x)-x^2\sin(x)\ dx$
   4. $\int x\cos(x)+\sin(x)\ dx$
   5. $\int 1+\ln(x)\ dx$

3. Observe that the examples in (b) work nicely because of the derivatives you were asked to calculate in (a). Each integrand in (b) is precisely the result of differentiating one of the products of basic functions found in (a). To see what happens when an integrand is still a product but not necessarily the result of differentiating an elementary product, we consider how to evaluate

$$\int x\cos(x)\,dx.$$

   1. First, observe that

$$\frac{d}{dx}\left[x\sin(x)\right] = x\cos(x) + \sin(x).$$

      Integrating both sides indefinitely and using the fact that the integral of a sum is the sum of the integrals, we find that

$$\int \left(\frac{d}{dx}\left[x\sin(x)\right]\right)\,dx = \int x\cos(x)\,dx + \int \sin(x)\,dx.$$

      In this last equation, evaluate the indefinite integral on the left side as well as the rightmost indefinite integral on the right.

   2. In the most recent equation from (i.), solve the equation for the expression $\int x\cos(x)\ dx$.
   3. For which product of basic functions have you now found the antiderivative?

## Reversing the Product Rule: Integration by Parts

Problem (c) in Preview Activity $5.4.1$ provides a clue to the general technique known as Integration by Parts, which comes from reversing the Product Rule. Recall that the Product Rule states that

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x).$$

Integrating both sides of this equation indefinitely with respect to $x$, we find

$$\int \frac{d}{dx}\left[f(x)g(x)\right]\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx. \tag{5.4.1}$$

On the left side of Equation $(5.4.1)$, we have the indefinite integral of the derivative of a function. Temporarily omitting the constant that may arise, we have

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx. \tag{5.4.2}$$

We solve for the first indefinite integral on the left to generate the rule

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx. \tag{5.4.3}$$

Often we express Equation $(5.4.3)$ in terms of the variables $u$ and $v$, where $u = f(x)$ and $v = g(x)$. In differential notation, $du = f'(x)\ dx$ and $dv = g'(x)\ dx$, so we can state the rule for Integration by Parts in its most common form as follows:

**Note**

$$\int u\,dv = uv - \int v\,du.$$

To apply integration by parts, we look for a product of basic functions that we can identify as $u$ and $dv$. If we can antidifferentiate $dv$ to find $v$, and evaluating $\int v\ du$ is not more difficult than evaluating $\int u\ dv$, then this substitution usually proves to be fruitful. To demonstrate, we consider the following example.

**Example 5.4.1**

Evaluate the indefinite integral

$$\int x\cos(x)\,dx$$

using integration by parts.

*Answer available at the end of this section.*

The general technique of integration by parts involves trading the problem of integrating the product of two functions for the problem of integrating the product of two related functions. That is, we convert the problem of evaluating $\int u\ dv$ to that of evaluating $\int v\ du$. This clearly shapes our choice of $u$ and $v$. In Example 5.4.1, the original integral to evaluate was $\int x\cos(x)\ dx$, and through the substitution provided by integration by parts, we were instead able to evaluate $\int \sin(x)\cdot 1\ dx$. Note that the original function $x$ was replaced by its derivative, while $\cos(x)$ was replaced by its antiderivative.

**Activity 5.4.2**

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

1. $\int te^{-t}\ dt$
2. $\int 4x\sin(3x)\ dx$
3. $\int z\sec^2(z)\ dz$
4. $\int x\ln(x)\ dx$

## Some Subtleties with Integration by Parts

Sometimes integration by parts is not an obvious choice, but the technique is appropriate nonetheless. Integration by parts allows us to replace one function in a product with its derivative while replacing the other with its antiderivative. For instance, consider evaluating

$$\int \arctan(x)\,dx.$$

Initially, this problem seems ill-suited to integration by parts, since there does not appear to be a product of functions present. But if we note that $\arctan(x) = \arctan(x)\cdot 1$, and realize that we know the derivative of $\arctan(x)$ as well as the antiderivative of $1$, we see the possibility for the substitution $u = \arctan(x)$ and $dv = 1\ dx$. We explore this substitution further in Activity 5.4.3.

In a related problem, consider $\int t^3\sin(t^2)\ dt$. Observe that there is a composite function present in $\sin(t^2)$, but there is not an obvious function-derivative pair, as we have $t^3$ (rather than simply $t$) multiplying $\sin(t^2)$. In this problem we use both $u$-substitution and integration by parts. First we write $t^3 = t\cdot t^2$ and consider the indefinite integral

$$\int t\cdot t^2\cdot\sin(t^2)\,dt.$$

We let $z = t^2$ so that $dz = 2t\ dt$, and thus $t\ dt = \frac{1}{2}\ dz$. (We are using the variable $z$ to perform a "$z$-substitution" first so that we may then apply integration by parts.) Under this $z$-substitution, we now have

$$\int t\cdot t^2\cdot\sin(t^2)\,dt = \int z\cdot\sin(z)\cdot\frac{1}{2}\,dz.$$

The resulting integral can be evaluated by parts. This, too, is explored further in Activity 5.4.3.

These problems show that we sometimes must think creatively in choosing the variables for substitution in integration by parts, and that we may need to use substitution for an additional change of variables.

### Activity 5.4.3

Evaluate each of the following indefinite integrals, using the provided hints.

1. Evaluate $\int \arctan(x)\ dx$ by using Integration by Parts with the substitution $u = \arctan(x)$ and $dv = 1\ dx$.
2. Evaluate $\int \ln(z)\ dz$. Consider a similar substitution to the one in (a).
3. Use the substitution $z = t^2$ to transform the integral $\int t^3 \sin(t^2)\ dt$ to a new integral in the variable $z$, and evaluate that new integral by parts.
4. Evaluate $\int s^5 e^{s^3}\ ds$ using an approach similar to that described in (c).
5. Evaluate $\int e^{2t} \cos(e^t)\ dt$. You will find it helpful to note that $e^{2t} = e^t \cdot e^t$.

## Using Integration by Parts Multiple Times

Integration by parts is well suited to integrating the product of basic functions, allowing us to trade a given integrand for a new one where one function in the product is replaced by its derivative, and the other is replaced by its antiderivative. The goal in this trade of $\int u\ dv$ for $\int v\ du$ is that the new integral be simpler to evaluate than the original one. Sometimes it is necessary to apply integration by parts more than once in order to evaluate a given integral.

### Example 5.4.2

Evaluate $\int t^2 e^t\ dt$.

*Answer available at the end of this section.*

Of course, even more than two applications of integration by parts may be necessary. In the preceding example, if the integrand had been $t^3 e^t$, we would have had to use integration by parts three times.

Next, we consider the slightly different scenario.

### Example 5.4.3

Evaluate $\int e^t \cos(t)\ dt$.

*Answer available at the end of this section.*

### Activity 5.4.4

Evaluate each of the following indefinite integrals.

1. $\int x^2 \sin(x)\ dx$
2. $\int t^3 \ln(t)\ dt$
3. $\int e^z \sin(z)\ dz$
4. $\int s^2 e^{3s}\ ds$
5. $\int t \arctan(t)\ dt$ (*Hint:* At a certain point in this problem, it is very helpful to note that $\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$.)

# Evaluating Definite Integrals Using Integration by Parts

We can use the technique of integration by parts to evaluate a definite integral.

**Example 5.4.4**

Evaluate

$$\int_0^{\pi/2} t\sin(t)\,dt.$$

*Answer available at the end of this section.*

As with any substitution technique, it is important to use notation carefully and completely, and to ensure that the end result makes sense.

## When $u$-substitution and Integration by Parts Fail to Help

Both integration techniques we have discussed apply in relatively limited circumstances. It is not hard to find examples of functions for which neither technique produces an antiderivative; indeed, there are many, many functions that appear elementary but that do not have an elementary algebraic antiderivative. For instance, neither $u$-substitution nor integration by parts proves fruitful for the indefinite integrals

$$\int e^{x^2}\,dx \quad \text{and} \quad \int x\tan(x)\,dx.$$

While there are other integration techniques, some of which we will consider briefly, none of them enables us to find an algebraic antiderivative for $e^{x^2}$ or $x\tan(x)$. We do know from the Second Fundamental Theorem of Calculus that we can construct an integral antiderivative for each function; $F(x)=\int_0^x e^{t^2}\,dt$ is an antiderivative of $f(x)=e^{x^2}$, and $G(x)=\int_0^x t\tan(t)\,dt$ is an antiderivative of $g(x)=x\tan(x)$. But finding an elementary algebraic formula that doesn't involve integrals for either $F$ or $G$ turns out not only to be impossible through $u$-substitution or integration by parts, but indeed impossible altogether. Antidifferentiation is much harder in general than differentiation.

## Summary

- Through the method of integration by parts, we can evaluate indefinite integrals that involve products of basic functions such as $\int x\sin(x)\,dx$ and $\int x\ln(x)\,dx$. Using a substitution enables us to trade one of the functions in the product for its derivative, and the other for its antiderivative, in an effort to find a different product of functions that is easier to integrate.
- If the algebraic structure of an integrand is a product of basic functions in the form $\int f(x)g'(x)\,dx$, we can use the substitution $u=f(x)$ and $dv=g'(x)\,dx$ and apply the rule

  $$\int u\,dv = uv - \int v\,du$$

  to evaluate the original integral $\int f(x)g'(x)\,dx$ by instead evaluating

  $$\int v\,du = \int f'(x)g(x)\,dx.$$

- When deciding to integrate by parts, we have to select both $u$ and $dv$. That selection is guided by the overall principle that the new integral $\int v\,du$ not be more difficult than the original integral $\int u\,dv$. In addition, it is often helpful to recognize if one of the functions present is much easier to differentiate than antidifferentiate (such as $\ln(x)$), in which case that function often is best assigned the variable $u$. In addition, $dv$ must be a function that we can antidifferentiate.

### Answers

**Example** 5.4.1

When we use integration by parts, we have a choice for $u$ and $dv$. In this problem, we can either let $u = x$ and $dv = \cos(x)\ dx$, or let $u = \cos(x)$ and $dv = x\ dx$. While there is not a universal rule for how to choose $u$ and $dv$, a good guideline is this: do so in a way that $\int v\ du$ is at least as simple as the original problem $\int u\ dv$.

This leads us to choose[1] $u = x$ and $dv = \cos(x)\ dx$, from which it follows that $du = 1\ dx$ and $v = \sin(x)$. With this substitution, the rule for integration by parts tells us that

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\cdot 1\,dx.$$

Observe that if we considered the alternate choice, and let $u = \cos(x)$ and $dv = x\ dx$, then $du = -\sin(x)\ dx$ and $v = \frac{1}{2}x^2$, from which we would write $\int x\cos(x)\ dx = \frac{1}{2}x^2\cos(x) - \int \frac{1}{2}x^2(-\sin(x))\ dx$. Thus we have replaced the problem of integrating $x\cos(x)$ with that of integrating $\frac{1}{2}x^2\sin(x)$; the latter is clearly more complicated, which shows that this alternate choice is not as helpful as the first choice.

All that remains to do is evaluate the (simpler) integral $\int \sin(x)\cdot 1\ dx$. Doing so, we find

$$\int x\cos(x)\,dx = x\sin(x) - (-\cos(x)) + C = x\sin(x) + \cos(x) + C.$$

Observe that when we get to the final stage of evaluating the last remaining antiderivative, it is at this step that we include the integration constant, $+C$.

**Example** 5.4.2

Let $u = t^2$ and $dv = e^t\ dt$. Then $du = 2t\ dt$ and $v = e^t$, and thus

$$\int t^2e^t\,dt = t^2e^t - \int 2te^t\,dt.$$

The integral on the right side is simpler to evaluate than the one on the left, but it still requires integration by parts. Now letting $u = 2t$ and $dv = e^t\ dt$, we have $du = 2\ dt$ and $v = e^t$, so that

$$\int t^2e^t\,dt = t^2e^t - \left(2te^t - \int 2e^t\,dt\right).$$

(Note the parentheses, which remind us to distribute the minus sign to the entire value of the integral $\int 2te^t\ dt$.) The final integral on the right is a basic one; evaluating that integral and distributing the minus sign, we find

$$\int t^2e^t\,dt = t^2e^t - 2te^t + 2e^t + C.$$

**Example** 5.4.3

We can choose to let $u$ be either $e^t$ or $\cos(t)$; we pick $u = \cos(t)$, and thus $dv = e^t\ dt$. With $du = -\sin(t)\ dt$ and $v = e^t$, integration by parts tells us that

$$\int e^t\cos(t)\,dt = e^t\cos(t) - \int e^t(-\sin(t))\,dt,$$

or equivalently that

$$\int e^t\cos(t)\,dt = e^t\cos(t) + \int e^t\sin(t)\,dt. \tag{5.4.14}$$

The new integral has the same algebraic structure as the original one. While the overall situation isn't necessarily better than what we started with, it hasn't gotten worse. Thus, we proceed to integrate by parts again. This time we let $u = \sin(t)$ and $dv = e^t\ dt$, so that $du = \cos(t)\ dt$ and $v = e^t$, which implies

$$\int e^t\cos(t)\,dt = e^t\cos(t) + \left(e^t\sin(t) - \int e^t\cos(t)\,dt\right). \tag{5.4.15}$$

We seem to be back where we started, as two applications of integration by parts has led us back to the original problem, $\int e^t\cos(t)\ dt$. But if we look closely at Equation (5.4.15), we see that we can use algebra to solve for the value of the desired integral. Adding $\int e^t\cos(t)\ dt$ to both sides of the equation, we have

$$2\int e^t\cos(t)\,dt = e^t\cos(t) + e^t\sin(t),$$

and therefore

$$\int e^t\cos(t)\,dt = \frac{1}{2}\left(e^t\cos(t) + e^t\sin(t)\right) + C.$$

Note that since we never actually encountered an integral we could evaluate directly, we didn't have the opportunity to add the integration constant $C$ until the final step.

**Example** 5.4.4

One option is to find an antiderivative (using indefinite integral notation) and then apply the Fundamental Theorem of Calculus to find that

$$\begin{aligned}\int_0^{\pi/2} t\sin(t)\,dt &= (-t\cos(t) + \sin(t))\Big|_0^{\pi/2}\\ &= \left(-\frac{\pi}{2}\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})\right) - (-0\cos(0) + \sin(0))\\ &= 1.\end{aligned}$$

Alternatively, we can apply integration by parts and work with definite integrals throughout. With this method, we must remember to evaluate the product $uv$ over the given limits of integration. Using the substitution $u = t$ and $dv = \sin(t)\ dt$, so that $du = dt$ and $v = -\cos(t)$, we write

$$\begin{aligned}\int_0^{\pi/2} t\sin(t)\,dt &= -t\cos(t)\Big|_0^{\pi/2} - \int_0^{\pi/2}(-\cos(t))\,dt\\ &= -t\cos(t)\Big|_0^{\pi/2} + \sin(t)\Big|_0^{\pi/2}\\ &= \left(-\frac{\pi}{2}\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})\right) - (-0\cos(0) + \sin(0))\\ &= 1.\end{aligned}$$

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# 5.5: Partial Fractions for Finding Algebraic Derivatives

## Motivating Questions

- How does the method of partial fractions enable any rational function to be antidifferentiated?
- What role have integral tables historically played in the study of calculus and how can a table be used to evaluate integrals such as $\int \sqrt{a^2+u^2}\ du$?
- What role can a computer algebra system play in the process of finding antiderivatives?

We have learned two antidifferentiation techniques: $u$-substitution and integration by parts. The former is used to reverse the chain rule, while the latter to reverse the product rule. But we have seen that each works only in specialized circumstances. For example, while $\int xe^{x^2}\ dx$ may be evaluated by $u$-substitution and $\int xe^x\ dx$ by integration by parts, neither method provides a route to evaluate $\int e^{x^2}\ dx$, and in fact an elementary algebraic antiderivative for $e^{x^2}$ does not exist. No antidifferentiation method will provide us with a simple algebraic formula for a function $F(x)$ that satisfies $F'(x) = e^{x^2}$.

In this section of the text, our main goals are to identify some classes of functions that can be antidifferentiated, and to learn some methods to do so. We should also recognize that there are many functions for which an algebraic formula for an antiderivative does not exist, and appreciate the role that computing technology can play in finding antiderivatives of other complicated functions.

## Preview Activity 5.5.1

For each of the indefinite integrals below, the main question is to decide whether the integral can be evaluated using $u$ -substitution, integration by parts, a combination of the two, or neither. For integrals for which your answer is affirmative, state the substitution(s) you would use. It is not necessary to actually evaluate any of the integrals completely, unless the integral can be evaluated immediately using a familiar basic antiderivative.

1. $\int x^2\sin(x^3)\ dx$, $\int x^2\sin(x)\ dx$, $\int \sin(x^3)\ dx$, $\int x^5\sin(x^3)\ dx$
2. $\int \frac{1}{1+x^2}\ dx$, $\int \frac{x}{1+x^2}\ dx$, $\int \frac{2x+3}{1+x^2}\ dx$, $\int \frac{e^x}{1+(e^x)^2}\ dx$,
3. $\int x\ln(x)\ dx$, $\int \frac{\ln(x)}{x}\ dx$, $\int \ln(1+x^2)\ dx$, $\int x\ln(1+x^2)\ dx$,
4. $\int x\sqrt{1-x^2}\ dx$, $\int \frac{1}{\sqrt{1-x^2}}\ dx$, $\int \frac{x}{\sqrt{1-x^2}}\ dx$, $\int \frac{1}{x\sqrt{1-x^2}}\ dx$,

## The Method of Partial Fractions

The method of partial fractions is used to integrate rational functions. It involves reversing the process of finding a common denominator.

## Example 5.5.1

Evaluate

$$\int \frac{5x}{x^2-x-2}\,dx.$$

*Answer available at the end of this section.*

It turns out that we can use the method of partial fractions to rewrite any ratinal function $R(x) = \frac{P(x)}{Q(x)}$ where the degree of the polynomial $P$ is less than [1] the degree of $Q$ as a sum of simpler rational functions of one of the following forms:

$$\frac{A}{x-c},\ \frac{A}{(x-c)^n},\ \frac{Ax+B}{x^2+k},\ \text{or}\ \frac{Ax+B}{(x^2+k)^n}$$

where $A$, $B$, and $c$ are real numbers, and $k$ is a positive real number. Because we can antidifferentiate each of these basic forms, partial fractions enables us to antidifferentiate any rational function.

If the degree of $P$ is greater than or equal to the degree of $Q$, long division may be used to write $R$ as the sum of a polynomial plus a rational function where the numerator's degree is less than the denominator's.

A computer algebra system such as *Maple*, *Mathematica*, or *WolframAlpha* can be used to find the partial fraction decomposition of any rational function. In *WolframAlpha*, entering

```
partial fraction 5x/(x^2-x-2)
```

results in the output

$$\frac{5x}{x^2-x-2} = \frac{10}{3(x-2)} + \frac{5}{3(x+1)}.$$

We will use technology to generate partial fraction decompositions of rational functions, and then evaluate the integrals using established methods.

**Activity 5.5.2**

For each of the following problems, evaluate the integral by using the partial fraction decomposition provided.

1. $\int \frac{1}{x^2-2x-3}\,dx$, given that $\frac{1}{x^2-2x-3} = \frac{1/4}{x-3} - \frac{1/4}{x+1}$
2. $\int \frac{x^2+1}{x^3-x^2}\,dx$, given that $\frac{x^2+1}{x^3-x^2} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$
3. $\int \frac{x-2}{x^4+x^2}\,dx$, given that $\frac{x-2}{x^4+x^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{-x+2}{1+x^2}$

## Using an Integral Table

Calculus has a long history, going back to Greek mathematicians in 400-300 BC. Its main foundations were first investigated and understood independently by Isaac Newton and Gottfried Wilhelm Leibniz in the late 1600s, making the modern ideas of calculus well over 300 years old. It is instructive to realize that until the late 1980s, the personal computer did not exist, so calculus (and other mathematics) had to be done by hand for roughly 300 years. In the 21st century, however, computers have revolutionized many aspects of the world we live in, including mathematics. In this section we take a short historical tour to precede discussing the role computer algebra systems can play in evaluating indefinite integrals. In particular, we consider a class of integrals involving certain radical expressions.

As seen in the short table of integrals found in Appendix A, there are many forms of integrals that involve $\sqrt{a^2 \pm w^2}$ and $\sqrt{w^2 - a^2}$. These integral rules can be developed using a technique known as *trigonometric substitution* that we choose to omit; instead, we will simply accept the results presented in the table. To see how these rules are used, consider the differences among

$$\int \frac{1}{\sqrt{1-x^2}}\,dx, \quad \int \frac{x}{\sqrt{1-x^2}}\,dx, \quad \text{and} \quad \int \sqrt{1-x^2}\,dx.$$

The first integral is a familiar basic one, and results in $\arcsin(x) + C$. The second integral can be evaluated using a standard $u$-substitution with $u = 1 - x^2$. The third, however, is not familiar and does not lend itself to $u$-substitution.

In Appendix A, we find the rule

$$(h)\ \int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right) + C.$$

Using the substitutions $a = 1$ and $u = x$ (so that $du = dx$), it follows that

$$\int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\arcsin(x) + C.$$

Whenever we are applying a rule in the table, we are doing a $u$-substitution, especially when the substitution is more complicated than setting $u = x$ as in the last example.

**Example 5.5.2**

Evaluate the integral

$$\int \sqrt{9+64x^2}\,dx.$$

*Answer available at the end of this section.*

Whenever we use a $u$-subsitution in conjunction with Appendix A, it's important that we not forget to address any constants that arise and include them in our computations, such as the $\frac{1}{8}$ that appeared in Example 5.5.12.

**Activity 5.5.3**

For each of the following integrals, evaluate the integral using $u$-substitution and/or an entry from the table found in Appendix A.

1. $\displaystyle\int \sqrt{x^2+4}\ dx$
2. $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\ dx$
3. $\displaystyle\int \frac{2}{\sqrt{16+25x^2}}\ dx$
4. $\displaystyle\int \frac{1}{x^2\sqrt{49-36x^2}}\ dx$

## Using Computer Algebra Systems

A computer algebra system (CAS) is a computer program that is capable of executing symbolic mathematics. For example, if we ask a CAS to solve the equation $ax^2 + bx + c = 0$ for the variable $x$, where $a$, $b$, and $c$ are arbitrary constants, the program will return $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Research to develop the first CAS dates to the 1960s, and these programs became publicly available in the early 1990s. Two prominent examples are the programs *Maple* and *Mathematica*, which were among the first computer algebra systems to offer a graphical user interface. Today, *Maple* and *Mathematica* are exceptionally powerful professional software packages that can execute an amazing array of sophisticated mathematical computations. They are also very expensive, as each is a proprietary program. The CAS *SAGE* is an open-source, free alternative to *Maple* and *Mathematica*.

For the purposes of this text, when we need to use a CAS, we are going to turn instead to a similar, but somewhat different computational tool, the web-based "computational knowledge engine" called *WolframAlpha*. There are two features of *WolframAlpha* that make it stand out from the CAS options mentioned above: (1) unlike *Maple* and *Mathematica*, *WolframAlpha* is free (provided we are willing to navigate some pop-up advertising); and (2) unlike any of the three, the syntax in *WolframAlpha* is flexible. Think of *WolframAlpha* as being a little bit like doing a Google search: the program will interpret what is input, and then provide a summary of options.

If we want to have *WolframAlpha* evaluate an integral for us, we can provide it syntax such as

```
integrate x^2 dx
```

to which the program responds with

$$\int x^2\,dx = \frac{x^3}{3} + \text{constant}.$$

While there is much to be enthusiastic about regarding CAS programs such as *WolframAlpha*, there are several things we should be cautious about: (1) a CAS only responds to exactly what is input; (2) a CAS can answer using powerful functions from very advanced mathematics; and (3) there are problems that even a CAS cannot do without additional human insight.

Although (1) likely goes without saying, we have to be careful with our input: if we enter syntax that defines the wrong function, the CAS will work with precisely the function we define. For example, if we are interested in evaluating the integral

$$\int \frac{1}{16-5x^2}\,dx,$$

and we mistakenly enter

```
integrate 1/16 - 5x^2 dx
```

a CAS will (correctly) reply with

$$\frac{1}{16}x - \frac{5}{3}x^3.$$

But if we are sufficiently well-versed in antidifferentiation, we will recognize that this function cannot be the one that we seek: integrating a rational function such as $\frac{1}{16-5x^2}$, we expect the logarithm function to be present in the result.

Regarding (2), even for a relatively simple integral such as $\int \frac{1}{16-5x^2}\,dx$, some CASs will invoke advanced functions rather than simple ones. For instance, if we use *Maple* to execute the command

```
int(1/(16-5*x^2), x);
```

the program responds with

$$\int \frac{1}{16-5x^2}\,dx = \frac{\sqrt{5}}{20} arctanh(\frac{\sqrt{5}}{4}x).$$

While this is correct (save for the missing arbitrary constant, which *Maple* never reports), the inverse hyperbolic tangent function is not a common nor familiar one; a simpler way to express this function can be found by using the partial fractions method, and happens to be the result reported by *WolframAlpha*:

$$\int \frac{1}{16-5x^2}\,dx = \frac{1}{8\sqrt{5}}\left(\log(4\sqrt{5}+5x) - \log(4\sqrt{5}-5x)\right) + \text{constant}.$$

Using sophisticated functions from more advanced mathematics is sometimes the way a CAS says to the user "I don't know how to do this problem." For example, if we want to evaluate

$$\int e^{-x^2}\,dx,$$

and we ask *WolframAlpha* to do so, the input

```
integrate exp(-x^2) dx
```

results in the output

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} erf(x) + \text{constant}.$$

The function "erf$(x)$" is the *error function*, which is actually defined by an integral:

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

So, in producing output involving an integral, the CAS has basically reported back to us the very question we asked.

Finally, as remarked at (3) above, there are times that a CAS will actually fail without some additional human insight. If we consider the integral

$$\int (1+x)e^x\sqrt{1+x^2e^{2x}}\,dx$$

and ask *WolframAlpha* to evaluate

```
int (1+x) * exp(x) * sqrt(1+x^2 * exp(2x)) dx,
```

the program thinks for a moment and then reports

(*no result found in terms of standard mathematical functions*)

But in fact this integral is not that difficult to evaluate. If we let $u = xe^x$, then $du = (1+x)e^x\ dx$, which means that the preceding integral has form

$$\int (1+x)e^x\sqrt{1+x^2e^{2x}}\,dx = \int \sqrt{1+u^2}\,du,$$

which is a straightforward one for any CAS to evaluate.

So, we should proceed with some caution: while any CAS is capable of evaluating a wide range of integrals (both definite and indefinite), there are times when the result can mislead us. We must think carefully about the meaning of the output, whether it is consistent with what we expect, and whether or not it makes sense to proceed.

# Summary

- We can antidifferentiate any rational function with the method of partial fractions. Any polynomial function can be factored into a product of linear and irreducible quadratic terms, so any rational function may be written as the sum of a polynomial plus rational terms of the form $\frac{A}{(x-c)^n}$ (where $n$ is a natural number) and $\frac{Bx+C}{x^2+k}$ (where $k$ is a positive real number).
- Until the development of compute algebra systems, integral tables enabled students of calculus to evaluate integrals such as $\int \sqrt{a^2+u^2}\ du$, where $a$ is a positive real number. A short table of integrals may be found in Appendix A.
- Computer algebra systems can play an important role in finding antiderivatives, though we must be cautious to use correct input, to watch for unusual or unfamiliar advanced functions that the CAS may cite in its result, and to consider the possibility that a CAS may need further assistance or insight from us in order to answer a particular question.

## Answers

**Example** 5.5.1

If we factor the denominator, we can see how $R$ might be the sum of two fractions of the form $\frac{A}{x-2} + \frac{B}{x+1}$, so we suppose that

$$\frac{5x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

and look for the constants $A$ and $B$.

Multiplying both sides of this equation by $(x-2)(x+1)$, we find that

$$5x = A(x+1) + B(x-2).$$

Since we want this equation to hold for every value of $x$, we can use insightful choices of specific $x$-values to help us find $A$ and $B$. Taking $x = -1$, we have

$$5(-1) = A(0) + B(-3),$$

so $B = \frac{5}{3}$. Choosing $x = 2$, it follows

$$5(2) = A(3) + B(0),$$

so $A = \frac{10}{3}$. Thus,

$$\int \frac{5x}{x^2-x-2}\,dx = \int \frac{10/3}{x-2} + \frac{5/3}{x+1}\,dx.$$

This integral is straightforward to evaluate, and hence we find that

$$\int \frac{5x}{x^2-x-2}\,dx = \frac{10}{3}\ln|x-2| + \frac{5}{3}\ln|x+1| + C.$$

**Example** 5.5.2

Here, we want to use Rule (c) from the table, but we now set $a = 3$ and $u = 8x$. We also choose the "+" option in the rule. With this substitution, it follows that $du = 8dx$, so $dx = \frac{1}{8}du$. Applying the substitution,

$$\int \sqrt{9+64x^2}\,dx = \int \sqrt{9+u^2}\cdot\frac{1}{8}\,du = \frac{1}{8}\int \sqrt{9+u^2}\,du.$$

By Rule (c), we now find that

$$\begin{aligned}\int \sqrt{9+64x^2}\,dx &= \frac{1}{8}\left(\frac{u}{2}\sqrt{u^2+9}+\frac{9}{2}\ln|u+\sqrt{u^2+9}|+C\right)\\ &= \frac{1}{8}\left(\frac{8x}{2}\sqrt{64x^2+9}+\frac{9}{2}\ln|8x+\sqrt{64x^2+9}|+C\right).\end{aligned}$$

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# 5.6: Numerical Integration (Simpson's Rule)*

**Motivating Questions**

- How do we accurately evaluate a definite integral such as $\int_0^1 e^{-x^2}\,dx$ when we cannot use the First Fundamental Theorem of Calculus because the integrand lacks an elementary algebraic antiderivative? Are there ways to generate accurate estimates without using extremely large values of $n$ in Riemann sums?
- What is the Trapezoid Rule, and how is it related to left, right, and middle Riemann sums?
- How are the errors in the Trapezoid Rule and Midpoint Rule related, and how can they be used to develop an even more accurate rule?

When we first explored finding the net signed area bounded by a curve, we developed the concept of a Riemann sum as a helpful estimation tool and a key step in the definition of the definite integral. Recall that the left, right, and middle Riemann sums of a function $f$ on an interval $[a, b]$ are given by

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x, \tag{5.6.1}$$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x, \tag{5.6.2}$$

$$M_n = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x, \tag{5.6.3}$$

where $x_0 = a$, $x_i = a + i\Delta x$, $x_n = b$, and $\Delta x = \frac{b-a}{n}$. For the middle sum, we defined $\overline{x}_i = (x_{i-1} + x_i)/2$.

A Riemann sum is a sum of (possibly signed) areas of rectangles. The value of $n$ determines the number of rectangles, and our choice of left endpoints, right endpoints, or midpoints determines the heights of the rectangles. We can see the similarities and differences among these three options in Figure 5.6.1, where we consider the function $f(x) = \frac{1}{20}(x-4)^3 + 7$ on the interval $[1, 8]$, and use 5 rectangles for each of the Riemann sums.

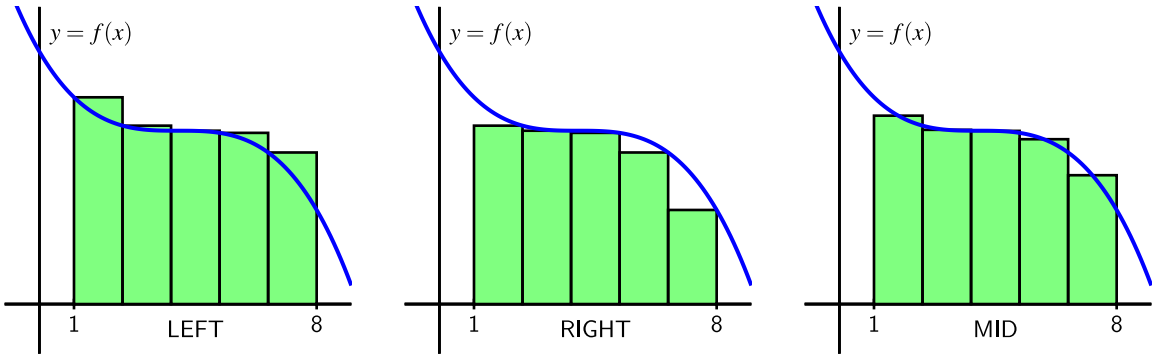



Figure 5.6.1: Left, right, and middle Riemann sums for $y = f(x)$ on $[1, 8]$ with 5 subintervals.

While it is a good exercise to compute a few Riemann sums by hand, just to ensure that we understand how they work and how varying the function, the number of subintervals, and the choice of endpoints or midpoints affects the result, using computing technology is the best way to determine $L_n$, $R_n$, and $M_n$. Any computer algebra system will offer this capability; as we saw in Preview Activity 4.3.1, a straightforward option that is freely available online is the applet[1] at http://gvsu.edu/s/a9. Note that we can adjust the formula for $f(x)$, the window of $x$- and $y$-values of interest, the number of subintervals, and the method. (See Preview Activity 4.3.1 for any needed reminders on how the applet works.)

Marc Renault, Shippensburg University

In this section we explore several different alternatives for estimating definite integrals. Our main goal is to develop formulas to estimate definite integrals accurately without using a large numbers of rectangles.

**Preview Activity 5.6.1**

As we begin to investigate ways to approximate definite integrals, it will be insightful to compare results to integrals whose exact values we know. To that end, the following sequence of questions centers on $\int_0^3 x^2 \, dx$.

1. Use the applet at http://gvsu.edu/s/a9 with the function $f(x) = x^2$ on the window of $x$ values from $0$ to $3$ to compute $L_3$, the left Riemann sum with three subintervals.
2. Likewise, use the applet to compute $R_3$ and $M_3$, the right and middle Riemann sums with three subintervals, respectively.
3. Use the Fundamental Theorem of Calculus to compute the exact value of $I = \int_0^3 x^2 \, dx$.
4. We define the *error* that results from an approximation of a definite integral to be the approximation's value minus the integral's exact value. What is the error that results from using $L_3$? From $R_3$? From $M_3$?
5. In what follows in this section, we will learn a new approach to estimating the value of a definite integral known as the Trapezoid Rule. The basic idea is to use trapezoids, rather than rectangles, to estimate the area under a curve. What is the formula for the area of a trapezoid with bases of length $b_1$ and $b_2$ and height $h$?
6. Working by hand, estimate the area under $f(x) = x^2$ on $[0, 3]$ using three subintervals and three corresponding trapezoids. What is the error in this approximation? How does it compare to the errors you calculated in (d)?

## The Trapezoid Rule

So far, we have used the simplest possible quadrilaterals (that is, rectangles) to estimate areas. It is natural, however, to wonder if other familiar shapes might serve us even better.

An alternative to $L_n$, $R_n$, and $M_n$ is called the *Trapezoid Rule*. Rather than using a rectangle to estimate the (signed) area bounded by $y = f(x)$ on a small interval, we use a trapezoid. For example, in Figure 5.6.2, we estimate the area under the curve using three subintervals and the trapezoids that result from connecting the corresponding points on the curve with straight lines.

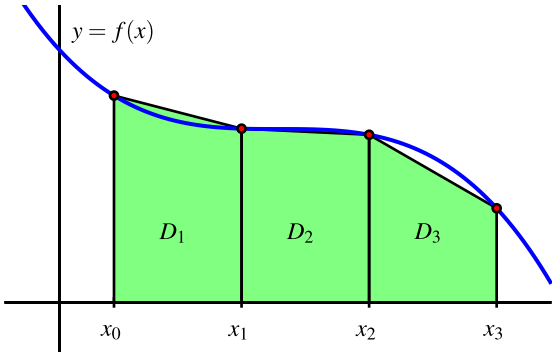



Figure 5.6.2: Estimating $\int_a^b f(x) \, dx$ using three subintervals and trapezoids, rather than rectangles, where $a = x_0$ and $b = x_3$.

The biggest difference between the Trapezoid Rule and a Riemann sum is that on each subinterval, the Trapezoid Rule uses two function values, rather than one, to estimate the (signed) area bounded by the curve. For instance, to compute $D_1$, the area of the trapezoid on $[x_0, x_1]$, we observe that the left base has length $f(x_0)$, while the right base has length $f(x_1)$. The height of the trapezoid is $x_1 - x_0 = \Delta x = \frac{b-a}{3}$. The area of a trapezoid is the average of the bases times the height, so we have

$$D_1 = \frac{1}{2}(f(x_0) + f(x_1)) \cdot \Delta x.$$

Using similar computations for $D_2$ and $D_3$, we find that $T_3$, the trapezoidal approximation to $\int_a^b f(x) \, dx$ is given by

$$\begin{aligned} T_3 &= D_1 + D_2 + D_3 \\ &= \frac{1}{2}(f(x_0) + f(x_1)) \cdot \Delta x + \frac{1}{2}(f(x_1) + f(x_2)) \cdot \Delta x + \frac{1}{2}(f(x_2) + f(x_3)) \cdot \Delta x. \end{aligned}$$

Because both left and right endpoints are being used, we recognize within the trapezoidal approximation the use of both left and right Riemann sums. Rearranging the expression for $T_3$ by removing factors of $\frac{1}{2}$ and $\Delta x$, grouping the left endpoint and right endpoint evaluations of $f$, we see that

$$T_3 = \frac{1}{2}\left[f(x_0) + f(x_1) + f(x_2)\right]\Delta x + \frac{1}{2}\left[f(x_1) + f(x_2) + f(x_3)\right]\Delta x. \tag{5.6.4}$$

We now observe that two familiar sums have arisen. The left Riemann sum $L_3$ is $L_3 = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x$, and the right Riemann sum is $R_3 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$. Substituting $L_3$ and $R_3$ for the corresponding expressions in Equation 5.6.4, it follows that $T_3 = \frac{1}{2}\left[L_3 + R_3\right]$. We have thus seen a very important result: using trapezoids to estimate the (signed) area bounded by a curve is the same as averaging the estimates generated by using left and right endpoints.

### The Trapezoid Rule

The trapezoidal approximation, $T_n$, of the definite integral $\int_a^b f(x)\ dx$ using $n$ subintervals is given by the rule

$$\begin{aligned} T_n &= \left[\frac{1}{2}(f(x_0) + f(x_1)) + \frac{1}{2}(f(x_1) + f(x_2)) + \cdots + \frac{1}{2}(f(x_{n-1}) + f(x_n))\right]\Delta x. \\ &= \sum_{i=0}^{n-1} \frac{1}{2}(f(x_i) + f(x_{i+1}))\Delta x. \end{aligned}$$

Moreover, $T_n = \frac{1}{2}\left[L_n + R_n\right]$.

### Activity 5.6.2

In this activity, we explore the relationships among the errors generated by left, right, midpoint, and trapezoid approximations to the definite integral $\int_1^2 \frac{1}{x^2}\ dx$.

1. Use the First FTC to evaluate $\int_1^2 \frac{1}{x^2}\ dx$ exactly.
2. Use appropriate computing technology to compute the following approximations for $\int_1^2 \frac{1}{x^2}\ dx$: $T_4$, $M_4$, $T_8$, and $M_8$.
3. Let the *error* that results from an approximation be the approximation's value minus the exact value of the definite integral. For instance, if we let $E_{T,4}$ represent the error that results from using the trapezoid rule with 4 subintervals to estimate the integral, we have

   $$E_{T,4} = T_4 - \int_1^2 \frac{1}{x^2}\,dx.$$

   Similarly, we compute the error of the midpoint rule approximation with 8 subintervals by the formula

   $$E_{M,8} = M_8 - \int_1^2 \frac{1}{x^2}\,dx.$$

   Based on your work in (a) and (b) above, compute $E_{T,4}$, $E_{T,8}$, $E_{M,4}$, $E_{M,8}$.
4. Which rule consistently over-estimates the exact value of the definite integral? Which rule consistently under-estimates the definite integral?
5. What behavior(s) of the function $f(x) = \frac{1}{x^2}$ lead to your observations in (d)?

## Comparing the Midpoint and Trapezoid Rules

We know from the definition of the definite integral that if we let $n$ be large enough, we can make any of the approximations $L_n$, $R_n$, and $M_n$ as close as we'd like (in theory) to the exact value of $\int_a^b f(x)\ dx$. Thus, it may be natural to wonder why we ever use

any rule other than $L_n$ or $R_n$ (with a sufficiently large $n$ value) to estimate a definite integral. One of the primary reasons is that as $n \to \infty$, $\Delta x = \frac{b-a}{n} \to 0$, and thus in a Riemann sum calculation with a large $n$ value, we end up multiplying by a number that is very close to zero. Doing so often generates roundoff error, because representing numbers close to zero accurately is a persistent challenge for computers.

Hence, we explore ways to estimate definite integrals to high levels of precision, but without using extremely large values of $n$. Paying close attention to patterns in errors, such as those observed in Activity 5.6.2, is one way to begin to see some alternate approaches.

To begin, we compare the errors in the Midpoint and Trapezoid rules. First, consider a function that is concave up on a given interval, and picture approximating the area bounded on that interval by both the Midpoint and Trapezoid rules using a single subinterval.

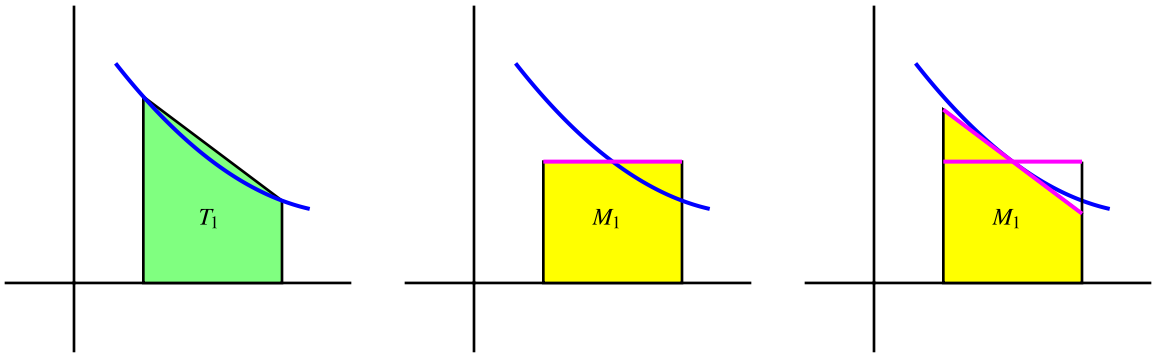



Figure 5.6.3: Estimating $\int_a^b f(x)\ dx$ using a single subinterval: at left, the trapezoid rule; in the middle, the midpoint rule; at right, a modified way to think about the midpoint rule.

As seen in Figure 5.6.3, it is evident that whenever the function is concave up on an interval, the Trapezoid Rule with one subinterval, $T_1$, will overestimate the exact value of the definite integral on that interval. From a careful analysis of the line that bounds the top of the rectangle for the Midpoint Rule (shown in magenta), we see that if we rotate this line segment until it is tangent to the curve at the midpoint of the interval (as shown at right in Figure 5.6.3), the resulting trapezoid has the same area as $M_1$, and this value is less than the exact value of the definite integral. Thus, when the function is concave up on the interval, $M_1$ underestimates the integral's true value.

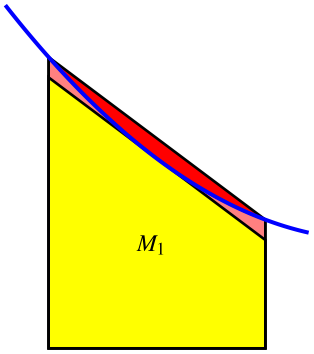



Figure 5.6.4: Comparing the error in estimating $\int_a^b f(x)\ dx$ using a single subinterval: in red, the error from the Trapezoid rule; in light red, the error from the Midpoint rule.

These observations extend easily to the situation where the function's concavity remains consistent but we use larger values of $n$ in the Midpoint and Trapezoid Rules. Hence, whenever $f$ is concave up on $[a, b]$, $T_n$ will overestimate the value of $\int_a^b f(x)\ dx$, while $M_n$ will underestimate $\int_a^b f(x)\ dx$. The reverse observations are true in the situation where $f$ is concave down.

Next, we compare the size of the errors between $M_n$ and $T_n$. Again, we focus on $M_1$ and $T_1$ on an interval where the concavity of $f$ is consistent. In Figure 5.6.4, where the error of the Trapezoid Rule is shaded in red, while the error of the Midpoint Rule is shaded lighter red, it is visually apparent that the error in the Trapezoid Rule is more significant. To see how much more significant, let's consider two examples and some particular computations.

If we let $f(x) = 1 - x^2$ and consider $\int_0^1 f(x)\ dx$, we know by the First FTC that the exact value of the integral is

$$\int_0^1 (1 - x^2)\, dx = x - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3}.$$

Using appropriate technology to compute $M_4$, $M_8$, $T_4$, and $T_8$, as well as the corresponding errors $E_{M,4}$, $E_{M,8}$, $E_{T,4}$, and $E_{T,8}$, as we did in Activity 5.6.2, we find the results summarized in Table 5.6.5. We also include the approximations and their errors for the example $\int_1^2 \frac{1}{x^2}\ dx$ from Activity 5.6.2.

*Table 5.6.5: Calculations of $T_4$, $M_4$, $T_8$, and $M_8$, along with corresponding errors, for the definite integrals $\int_0^1 (1 - x^2)\ dx$ and $\int_1^2 \frac{1}{x^2}\ dx$.*

| Rule | $\int_0^1 (1 - x^2)\ dx = 0.\overline{6}$ | error | $\int_1^2 \frac{1}{x^2}\ dx = 0.5$ | error |
|---|---|---|---|---|
| $T_4$ | 0.65625 | −0.0104166667 | 0.5089937642 | 0.0089937642 |
| $M_4$ | 0.671875 | 0.0052083333 | 0.4955479365 | −0.0044520635 |
| $T_8$ | 0.6640625 | −0.0026041667 | 0.5022708502 | 0.0022708502 |
| $M_8$ | 0.66796875 | 0.0013020833 | 0.4988674899 | −0.0011325101 |

For a given function $f$ and interval $[a, b]$, $E_{T,4} = T_4 - \int_a^b f(x)\ dx$ calculates the difference between the approximation generated by the Trapezoid Rule with $n = 4$ and the exact value of the definite integral. If we look at not only $E_{T,4}$, but also the other errors generated by using $T_n$ and $M_n$ with $n = 4$ and $n = 8$ in the two examples noted in Table 5.6.5, we see an evident pattern. Not only is the sign of the error (which measures whether the rule generates an over- or under-estimate) tied to the rule used and the function's concavity, but the magnitude of the errors generated by $T_n$ and $M_n$ seems closely connected. In particular, the errors generated by the Midpoint Rule seem to be about half the size (in absolute value) of those generated by the Trapezoid Rule.

That is, we can observe in both examples that $E_{M,4} \approx -\frac{1}{2}E_{T,4}$ and $E_{M,8} \approx -\frac{1}{2}E_{T,8}$. This property of the Midpoint and Trapezoid Rules turns out to hold in general: for a function of consistent concavity, the error in the Midpoint Rule has the opposite sign and approximately half the magnitude of the error of the Trapezoid Rule. Written symbolically,

$$E_{M,n} \approx -\frac{1}{2}E_{T,n}.$$

This important relationship suggests a way to combine the Midpoint and Trapezoid Rules to create an even more accurate approximation to a definite integral.

## Simpson's Rule

When we first developed the Trapezoid Rule, we observed that it is an average of the Left and Right Riemann sums:

$$T_n = \frac{1}{2}(L_n + R_n).$$

If a function is always increasing or always decreasing on the interval $[a, b]$, one of $L_n$ and $R_n$ will over-estimate the true value of $\int_a^b f(x)\ dx$, while the other will under-estimate the integral. Thus, the errors that result from $L_n$ and $R_n$ will have opposite signs; so averaging $L_n$ and $R_n$ eliminates a considerable amount of the error present in the respective approximations. In a similar way, it makes sense to think about averaging $M_n$ and $T_n$ in order to generate a still more accurate approximation.

We've just observed that $M_n$ is typically about twice as accurate as $T_n$. So we use the weighted average

$$S_{2n} = \frac{2M_n + T_n}{3}. \tag{5.6.5}$$

The rule for $S_{2n}$ giving by Equation 5.6.5 is usually known as *Simpson's Rule*.[2] Note that we use "$S_{2n}$" rather that "$S_n$" since the $n$ points the Midpoint Rule uses are different from the $n$ points the Trapezoid Rule uses, and thus Simpson's Rule is using $2n$ points at which to evaluate the function. We build upon the results in Table 5.6.5 to see the approximations generated by Simpson's Rule. In particular, in Table 5.6.6, we include all of the results in Table 5.6.5, but include additional results for $S_8 = \frac{2M_4+T_4}{3}$ and $S_{16} = \frac{2M_8+T_8}{3}$.

Thomas Simpson was an 18th century mathematician; his idea was to extend the Trapezoid rule, but rather than using straight lines to build trapezoids, to use quadratic functions to build regions whose area was bounded by parabolas (whose areas he could find exactly). Simpson's Rule is often developed from the more sophisticated perspective of using interpolation by quadratic functions.

*Table 5.6.6: Table 5.6.5 updated to include $S_8$, $S_{16}$, and the corresponding errors.*

| Rule | $\int_0^1 (1-x^2)\ dx = 0.\overline{6}$ | error | $\int_1^2 \frac{1}{x^2}\ dx = 0.5$ | error |
|---|---|---|---|---|
| $T_4$ | 0.65625 | $-0.0104166667$ | 0.5089937642 | 0.0089937642 |
| $M_4$ | 0.671875 | 0.0052083333 | 0.4955479365 | $-0.0044520635$ |
| $S_8$ | 0.6666666667 | 0 | 0.5000298792 | 0.0000298792 |
| $T_8$ | 0.6640625 | $-0.0026041667$ | 0.5022708502 | 0.0022708502 |
| $M_8$ | 0.66796875 | 0.0013020833 | 0.4988674899 | $-0.0011325101$ |
| $S_{16}$ | 0.6666666667 | 0 | 0.5000019434 | 0.0000019434 |

The results seen in Table 5.6.6 are striking. If we consider the $S_{16}$ approximation of $\int_1^2 \frac{1}{x^2}\ dx$, the error is only $E_{S,16} = 0.0000019434$. By contrast, $L_8 = 0.5491458502$, so the error of that estimate is $E_{L,8} = 0.0491458502$. Moreover, we observe that generating the approximations for Simpson's Rule is almost no additional work: once we have $L_n$, $R_n$, and $M_n$ for a given value of $n$, it is a simple exercise to generate $T_n$, and from there to calculate $S_{2n}$. Finally, note that the error in the Simpson's Rule approximations of $\int_0^1 (1-x^2)\ dx$ is zero![3]

Similar to how the Midpoint and Trapezoid approximations are exact for linear functions, Simpson's Rule approximations are exact for quadratic and cubic functions. See additional discussion on this issue later in the section and in the exercises.

These rules are not only useful for approximating definite integrals such as $\int_0^1 e^{-x^2}\ dx$, for which we cannot find an elementary antiderivative of $e^{-x^2}$, but also for approximating definite integrals when we are given a function through a table of data.

## Activity 5.6.3

A car traveling along a straight road is braking and its velocity is measured at several different points in time, as given in the following table. Assume that $v$ is continuous, always decreasing, and always decreasing at a decreasing rate, as is suggested by the data.

*Table 5.6.7 Data for the braking car.*

| seconds, $t$ | Velocity in ft/sec, $v(t)$ |
|---|---|
| 0 | 100 |
| 0.3 | 99 |
| 0.6 | 96 |
| 0.9 | 90 |
| 1.2 | 80 |
| 1.5 | 50 |
| 1.8 | 0 |

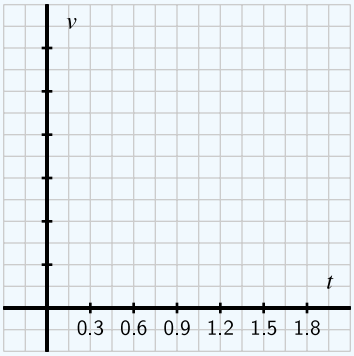



Figure 5.6.8: Axes for plotting the data in Activity 5.6.3.

1. Plot the given data on the set of axes provided in Figure 5.6.8 with time on the horizontal axis and the velocity on the vertical axis.
2. What definite integral will give you the exact distance the car traveled on $[0, 1.8]$?
3. Estimate the total distance traveled on $[0, 1.8]$ by computing $L_3$, $R_3$, and $T_3$. Which of these under-estimates the true distance traveled?
4. Estimate the total distance traveled on $[0, 1.8]$ by computing $M_3$. Is this an over- or under-estimate? Why?
5. Using your results from (c) and (d), improve your estimate further by using Simpson's Rule.
6. What is your best estimate of the average velocity of the car on $[0, 1.8]$? Why? What are the units on this quantity?

## Overall observations regarding $L_n$, $R_n$, $T_n$, $M_n$, and $S_{2n}$.

As we conclude our discussion of numerical approximation of definite integrals, it is important to summarize general trends in how the various rules over- or under-estimate the true value of a definite integral, and by how much. To revisit some past observations and see some new ones, we consider the following activity.

### Activity 5.6.4

Consider the functions $f(x) = 2 - x^2$, $g(x) = 2 - x^3$, and $h(x) = 2 - x^4$, all on the interval $[0, 1]$. For each of the questions that require a numerical answer in what follows, write your answer exactly in fraction form.

1. On the three sets of axes provided in Figure 5.6.9, sketch a graph of each function on the interval $[0, 1]$, and compute $L_1$ and $R_1$ for each. What do you observe?
2. Compute $M_1$ for each function to approximate $\int_0^1 f(x)\ dx$, $\int_0^1 g(x)\ dx$, and $\int_0^1 h(x)\ dx$, respectively.
3. Compute $T_1$ for each of the three functions, and hence compute $S_2$ for each of the three functions.
4. Evaluate each of the integrals $\int_0^1 f(x)\ dx$, $\int_0^1 g(x)\ dx$, and $\int_0^1 h(x)\ dx$ exactly using the First FTC.
5. For each of the three functions $f$, $g$, and $h$, compare the results of $L_1$, $R_1$, $M_1$, $T_1$, and $S_2$ to the true value of the corresponding definite integral. What patterns do you observe?

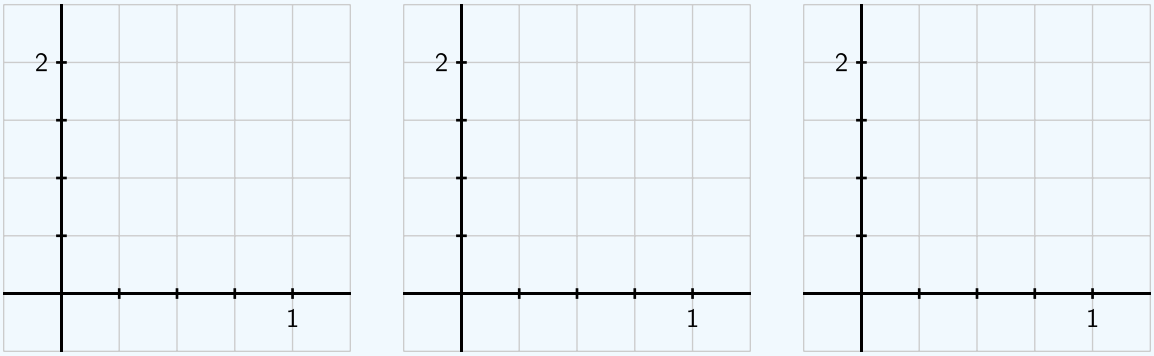



Figure 5.6.9: Axes for plotting the functions in Activity 5.6.4.

The results seen in Activity 5.6.4 generalize nicely. For instance, if $f$ is decreasing on $[a, b]$, $L_n$ will over-estimate the exact value of $\int_a^b f(x)\ dx$, and if $f$ is concave down on $[a, b]$, $M_n$ will over-estimate the exact value of the integral. An excellent exercise is to write a collection of scenarios of possible function behavior, and then categorize whether each of $L_n$, $R_n$, $T_n$, and $M_n$ is an over- or under-estimate.

Finally, we make two important notes about Simpson's Rule. When T. Simpson first developed this rule, his idea was to replace the function $f$ on a given interval with a quadratic function that shared three values with the function $f$. In so doing, he guaranteed that this new approximation rule would be exact for the definite integral of any quadratic polynomial. In one of the pleasant surprises of numerical analysis, it turns out that even though it was designed to be exact for quadratic polynomials, Simpson's Rule is exact for any cubic polynomial: that is, if we are interested in an integral such as $\int_2^5 (5x^3 - 2x^2 + 7x - 4)\ dx$, $S_{2n}$ will always be exact, regardless of the value of $n$. This is just one more piece of evidence that shows how effective Simpson's Rule is as an approximation tool for estimating definite integrals. [4]

One reason that Simpson's Rule is so effective is that $S_{2n}$ benefits from using $2n + 1$ points of data. Because it combines $M_n$, which uses $n$ midpoints, and $T_n$, which uses the $n + 1$ endpoints of the chosen subintervals, $S_{2n}$ takes advantage of the maximum amount of information we have when we know function values at the endpoints and midpoints of $n$ subintervals.

## Summary

- For a definite integral such as $\int_0^1 e^{-x^2}\ dx$ when we cannot use the First Fundamental Theorem of Calculus because the integrand lacks an elementary algebraic antiderivative, we can estimate the integral's value by using a sequence of Riemann sum approximations. Typically, we start by computing $L_n$, $R_n$, and $M_n$ for one or more chosen values of $n$.
- The Trapezoid Rule, which estimates $\int_a^b f(x)\ dx$ by using trapezoids, rather than rectangles, can also be viewed as the average of Left and Right Riemann sums. That is, $T_n = \frac{1}{2}(L_n + R_n)$.
- The Midpoint Rule is typically twice as accurate as the Trapezoid Rule, and the signs of the respective errors of these rules are opposites. Hence, by taking the weighted average $S_{2n} = \frac{2M_n + T_n}{3}$, we can build a much more accurate approximation to $\int_a^b f(x)\ dx$ by using approximations we have already computed. The rule for $S_{2n}$ is known as Simpson's Rule, which can also be developed by approximating a given continuous function with pieces of quadratic polynomials.

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# 5.7: Finding Antiderivatives and Evaluating Integrals (Exercises)

## 5.1: Construction Accurate Graphs of Antiderivatives

### Exercises 5.1.5 Exercises

#### 1. Definite integral of a piecewise linear function

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#### 2. A smooth function that starts out at 0

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#### 3. A piecewise constant function

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#### 4. Another piecewise linear function

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## 5

A moving particle has its velocity given by the quadratic function $v$ pictured in Figure 5.1.7. In addition, it is given that $A_1 = \frac{7}{6}$ and $A_2 = \frac{8}{3}$, as well as that for the corresponding position function $s$, $s(0) = 0.5$.

1. Use the given information to determine $s(1)$, $s(3)$, $s(5)$, and $s(6)$.
2. On what interval(s) is $s$ increasing? On what interval(s) is $s$ decreasing?
3. On what interval(s) is $s$ concave up? On what interval(s) is $s$ concave down?
4. Sketch an accurate, labeled graph of $s$ on the axes at right in Figure 5.1.7.
5. Note that $v(t) = -2 + \frac{1}{2}(t-3)^2$. Find a formula for $s$.

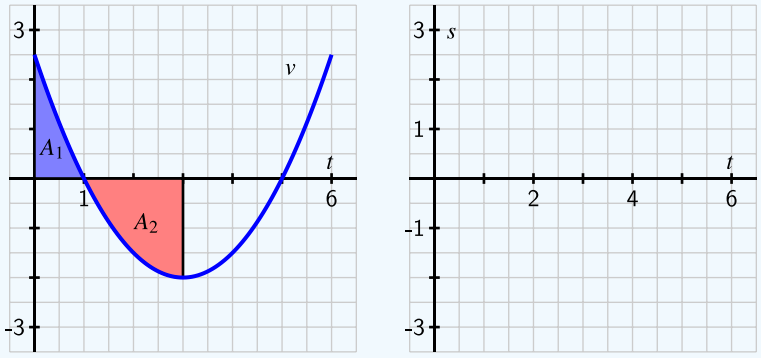



Figure 5.1.7. At left, the given graph of $v$. At right, axes for plotting $s$.

## 6

A person exercising on a treadmill experiences different levels of resistance and thus burns calories at different rates, depending on the treadmill's setting. In a particular workout, the rate at which a person is burning calories is given by the piecewise constant function $c$ pictured in Figure 5.1.8. Note that the units on $c$ are “calories per minute.”

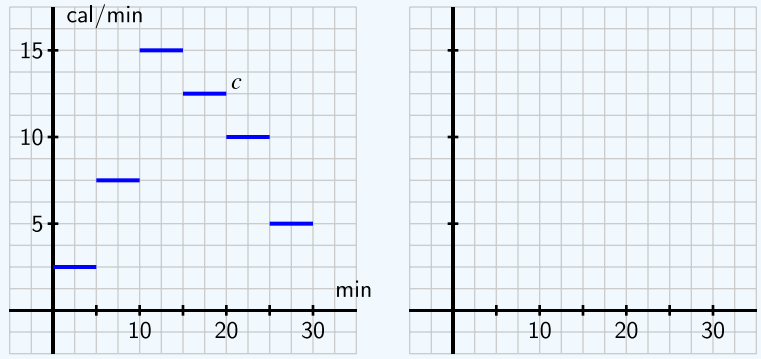



Figure 5.1.8. At left, the given graph of $c$. At right, axes for plotting $C$.

1. Let $C$ be an antiderivative of $c$. What does the function $C$ measure? What are its units?
2. Assume that $C(0) = 0$. Determine the exact value of $C(t)$ at the values $t = 5, 10, 15, 20, 25, 30$.
3. Sketch an accurate graph of $C$ on the axes provided at right in Figure 5.1.8. Be certain to label the scale on the vertical axis.
4. Determine a formula for $C$ that does not involve an integral and is valid for $5 \leq t \leq 10$.

## 7.

Consider the piecewise linear function $f$ given in Figure 5.1.9. Let the functions $A$, $B$, and $C$ be defined by the rules $A(x) = \int_{-1}^{x} f(t)\ dt$, $B(x) = \int_{0}^{x} f(t)\ dt$, and $C(x) = \int_{1}^{x} f(t)\ dt$.

1. For the values $x = -1, 0, 1, \ldots, 6$, make a table that lists corresponding values of $A(x)$, $B(x)$, and $C(x)$.
2. On the axes provided in Figure 5.1.9, sketch the graphs of $A$, $B$, and $C$.
3. How are the graphs of $A$, $B$, and $C$ related?
4. How would you best describe the relationship between the function $A$ and the function $f$?

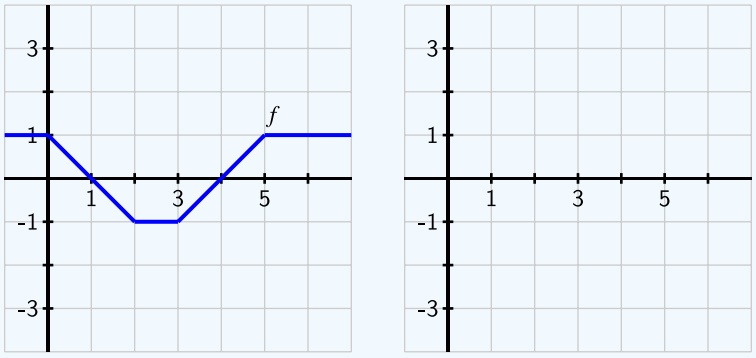



Figure 5.1.9. At left, the given graph of $f$. At right, axes for plotting $A$, $B$, and $C$.

# 5.2: The Second Fundamental Theorem of Calculus

## 1. A definite integral starting at 3

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## 2. Variable in the lower limit

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## 3. Approximating a function with derivative $e^{-x^2/5}$

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## 4

Let $g$ be the function pictured at left in Figure 5.2.6, and let $F$ be defined by $F(x) = \int_2^x g(t)\ dt$. Assume that the shaded areas have values $A_1 = 4.29$, $A_2 = 12.75$, $A_3 = 0.36$, and $A_4 = 1.79$. Assume further that the portion of $A_2$ that lies between $x = 0.5$ and $x = 2$ is $6.06$.

Sketch a carefully labeled graph of $F$ on the axes provided, and include a written analysis of how you know where $F$ is zero, increasing, decreasing, CCU, and CCD.

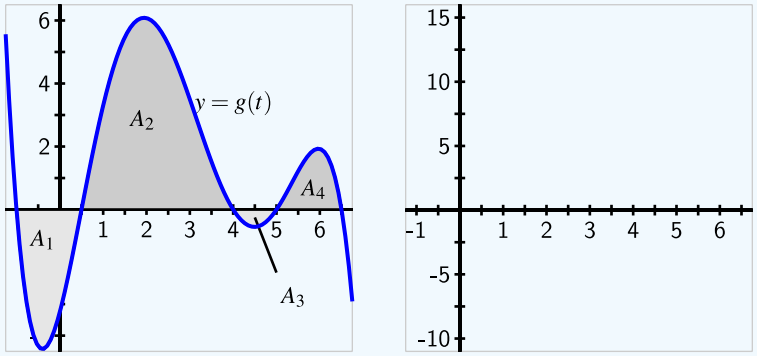



Figure 5.2.6. At left, the graph of $g$. At right, axes for plotting $F$.

## 5

The tide removes sand from the beach at a small ocean park at a rate modeled by the function

$$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right)$$

A pumping station adds sand to the beach at rate modeled by the function

$$S(t) = \frac{15t}{1+3t}$$

Both $R(t)$ and $S(t)$ are measured in cubic yards of sand per hour, $t$ is measured in hours, and the valid times are $0 \le t \le 6$. At time $t = 0$, the beach holds 2500 cubic yards of sand.

1. What definite integral measures how much sand the tide will remove during the time period $0 \le t \le 6$? Why?
2. Write an expression for $Y(x)$, the total number of cubic yards of sand on the beach at time $x$. Carefully explain your thinking and reasoning.
3. At what instantaneous rate is the total number of cubic yards of sand on the beach at time $t = 4$ changing?
4. Over the time interval $0 \le t \le 6$, at what time $t$ is the amount of sand on the beach least? What is this minimum value? Explain and justify your answers fully.

### 6

When an aircraft attempts to climb as rapidly as possible, its climb rate (in feet per minute) decreases as altitude increases, because the air is less dense at higher altitudes. Given below is a table showing performance data for a certain single engine aircraft, giving its climb rate at various altitudes, where $c(h)$ denotes the climb rate of the airplane at an altitude $h$.

*Table 5.2.7. Data for the climbing aircraft.*

| $h$ (feet) | 0 | 1000 | 2000 | 3000 | 4000 | 5000 | 6000 | 7000 | 8000 | 9000 | 10,000 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ (ft/min) | 925 | 875 | 830 | 780 | 730 | 685 | 635 | 585 | 535 | 490 | 440 |

Let a new function $m$, that also depends on $h$, (say $y = m(h)$) measure the number of minutes required for a plane at altitude $h$ to climb the next foot of altitude.

1. Determine a similar table of values for $m(h)$ and explain how it is related to the table above. Be sure to discuss the units on $m$.
2. Give a careful interpretation of a function whose derivative is $m(h)$. Describe what the input is and what the output is. Also, explain in plain English what the function tells us.
3. Determine a definite integral whose value tells us exactly the number of minutes required for the airplane to ascend to 10,000 feet of altitude. Clearly explain why the value of this integral has the required meaning.
4. Determine a formula for a function $M(h)$ whose value tells us the exact number of minutes required for the airplane to ascend to $h$ feet of altitude.
5. Estimate the values of $M(6000)$ and $M(10000)$ as accurately as you can. Include units on your results.

## 5.3 Integration by Substitution

### 1. Product involving 4th power of a polynomial

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### 2. Product involving $\sin(x^6)$

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### 3. Fraction involving $\ln^9$

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## 4. Fraction involving $e^{5x}$

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## 5. Fraction involving $e^{5\sqrt{y}}$

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## 6. Definite integral involving $e^{-cos(q)}$

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## 7

This problem centers on finding antiderivatives for the basic trigonometric functions other than $\sin(x)$ and $\cos(x)$.

1. Consider the indefinite integral $\int \tan(x)\ dx$. By rewriting the integrand as $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and identifying an appropriate function-derivative pair, make a $u$-substitution and hence evaluate $\int \tan(x)\ dx$.
2. In a similar way, evaluate $\int \cot(x)\ dx$.
3. Consider the indefinite integral

$$\int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\, dx.$$

   Evaluate this integral using the substitution $u = \sec(x) + \tan(x)$.

4. Simplify the integrand in (c) by factoring the numerator. What is a far simpler way to write the integrand?
5. Combine your work in (c) and (d) to determine $\int \sec(x)\ dx$.
6. Using (c)-(e) as a guide, evaluate $\int \csc(x)\ dx$.

## 8

Consider the indefinite integral $\int x\sqrt{x-1}\ dx$.

1. At first glance, this integrand may not seem suited to substitution due to the presence of $x$ in separate locations in the integrand. Nonetheless, using the composite function $\sqrt{x-1}$ as a guide, let $u = x - 1$. Determine expressions for both $x$ and $dx$ in terms of $u$.

2. Convert the given integral in $x$ to a new integral in $u$.
3. Evaluate the integral in (b) by noting that $\sqrt{u} = u^{1/2}$ and observing that it is now possible to rewrite the integrand in $u$ by expanding through multiplication.
4. Evaluate each of the integrals $\int x^2\sqrt{x-1}\ dx$ and $\int x\sqrt{x^2-1}\ dx$. Write a paragraph to discuss the similarities among the three indefinite integrals in this problem and the role of substitution and algebraic rearrangement in each.

### 9

Consider the indefinite integral $\int \sin^3(x)\ dx$.

1. Explain why the substitution $u = \sin(x)$ will not work to help evaluate the given integral.
2. Recall the Fundamental Trigonometric Identity, which states that $\sin^2(x) + \cos^2(x) = 1$. By observing that $\sin^3(x) = \sin(x) \cdot \sin^2(x)$, use the Fundamental Trigonometric Identity to rewrite the integrand as the product of $\sin(x)$ with another function.
3. Explain why the substitution $u = \cos(x)$ now provides a possible way to evaluate the integral in (b).
4. Use your work in (a)-(c) to evaluate the indefinite integral $\int \sin^3(x)\ dx$.
5. Use a similar approach to evaluate $\int \cos^3(x)\ dx$.

### 10

For the town of Mathland, MI, residential power consumption has shown certain trends over recent years. Based on data reflecting average usage, engineers at the power company have modeled the town's rate of energy consumption by the function

$$r(t) = 4 + \sin(0.263t + 4.7) + \cos(0.526t + 9.4).$$

Here, $t$ measures time in hours after midnight on a typical weekday, and $r$ is the rate of consumption in megawatts at time $t$. Units are critical throughout this problem.

The unit *megawatt* is itself a rate, which measures energy consumption per unit time. A *megawatt-hour* is the total amount of energy that is equivalent to a constant stream of 1 megawatt of power being sustained for 1 hour.

1. Sketch a carefully labeled graph of $r(t)$ on the interval [0,24] and explain its meaning. Why is this a reasonable model of power consumption?
2. Without calculating its value, explain the meaning of $\int_0^{24} r(t)\ dt$. Include appropriate units on your answer.
3. Determine the exact amount of energy Mathland consumes in a typical day.
4. What is Mathland's average rate of power consumption in a given 24-hour period? What are the units on this quantity?

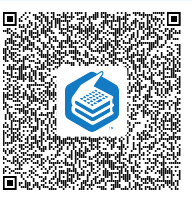
The unit *megawatt* is itself a rate, which measures energy consumption per unit time. A *megawatt-hour* is the total amount of energy that is equivalent to a constant stream of 1 megawatt of power being sustained for 1 hour.

## 5.4: Integration by Parts

### 1. Choose which method to use

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## 2. Product involving $\cos(5x)$

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## 3. Product involving $e^{8z}$

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## 4. Definite integral of $te^{-t}$

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## 5

Let $f(t) = te^{-2t}$ and $F(x) = \int_0^x f(t)\ dt$.

1. Determine $F'(x)$.
2. Use the First FTC to find a formula for $F$ that does not involve an integral.
3. Is $F$ an increasing or decreasing function for $x > 0$? Why?

## 6

Consider the indefinite integral given by $\int e^{2x}\cos(e^x)\ dx$.

1. Noting that $e^{2x} = e^x \cdot e^x$, use the substitution $z = e^x$ to determine a new, equivalent integral in the variable $z$.
2. Evaluate the integral you found in (a) using an appropriate technique.
3. How is the problem of evaluating $\int e^{2x}\cos(e^{2x})\ dx$ different from evaluating the integral in (a)? Do so.
4. Evaluate each of the following integrals as well, keeping in mind the approach(es) used earlier in this problem:
   - $\int e^{2x}\sin(e^x)\ dx$
   - $\int e^{3x}\sin(e^{3x})\ dx$
   - $\int xe^{x^2}\cos(e^{x^2})\sin(e^{x^2})\ dx$

7

For each of the following indefinite integrals, determine whether you would use $u$-substitution, integration by parts, neither*, or both to evaluate the integral. In each case, write one sentence to explain your reasoning, and include a statement of any substitutions used. (That is, if you decide in a problem to let $u = e^{3x}$, you should state that, as well as that $du = 3e^{3x}\ dx$.) Finally, use your chosen approach to evaluate each integral. (* one of the following problems does not have an elementary antiderivative and you are not expected to actually evaluate this integral; this will correspond with a choice of "neither" among those given.)

1. $\int x^2 \cos(x^3)\ dx$
2. $\int x^5 \cos(x^3)\ dx$ (Hint: $x^5 = x^2 \cdot x^3$)
3. $\int x \ln(x^2)\ dx$
4. $\int \sin(x^4)\ dx$
5. $\int x^3 \sin(x^4)\ dx$
6. $\int x^7 \sin(x^4)\ dx$

# 5.5: Other Options for Finding Algebraic Derivatives

## 1. Partial fractions: linear over difference of squares

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## 2. Partial fractions: constant over product

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## 3. Partial fractions: linear over quadratic

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## 4. Partial fractions: cubic over 4th degree

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## 5. Partial fractions: quadratic over factored cubic

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## 6

For each of the following integrals involving rational functions, (1) use a CAS to find the partial fraction decomposition of the integrand; (2) evaluate the integral of the resulting function without the assistance of technology; (3) use a CAS to evaluate the original integral to test and compare your result in (2).

1. $\displaystyle\int \frac{x^3+x+1}{x^4-1}\,dx$
2. $\displaystyle\int \frac{x^5+x^2+3}{x^3-6x^2+11x-6}\,dx$
3. $\displaystyle\int \frac{x^2-x-1}{(x-3)^3}\,dx$

## 7

For each of the following integrals involving radical functions, (1) use an appropriate $u$-substitution along with Appendix A to evaluate the integral without the assistance of technology, and (2) use a CAS to evaluate the original integral to test and compare your result in (1).

1. $\displaystyle\int \frac{1}{x\sqrt{9x^2+25}}\,dx$
2. $\displaystyle\int x\sqrt{1+x^4}\,dx$
3. $\displaystyle\int e^x\sqrt{4+e^{2x}}\,dx$
4. $\displaystyle\int \frac{\tan(x)}{\sqrt{9-\cos^2(x)}}\,dx$

## 8

Consider the indefinite integral given by

$$\int \frac{\sqrt{x+\sqrt{1+x^2}}}{x}\,dx.$$

1. Explain why $u$-substitution does not offer a way to simplify this integral by discussing at least two different options you might try for $u$.
2. Explain why integration by parts does not seem to be a reasonable way to proceed, either, by considering one option for $u$ and $dv$.
3. Is there any line in the integral table in Appendix A that is helpful for this integral?
4. Evaluate the given integral using *WolframAlpha*. What do you observe?

## 5.6: Numerical Integration

Two notes about how Exercise 5.6.6.1 is coded: (i) as explained in the problem header, you need responses to every single entry before you can get individual parts marked as correct; if you enter only an answer for (a) and submit, (a) will be marked wrong regardless. And (ii), in this problem, the notation "SIMP(2)" is actually what we have called "SIMP(4)" in our previous work. Different authors use different notation, and the author of this WeBWorK exercise chooses to write "SIMP(n)" where we have written "SIMP(2n)" in Section 5.6.

### 1. Various methods for $e^x$ numerically

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### 2. Comparison of methods for increasing concave down function

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### 3. Comparing accuracy for two similar functions

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### 4. Identifying and comparing methods

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## 5

Consider the definite integral $\int_0^1 x\tan(x)\ dx$.

1. Explain why this integral cannot be evaluated exactly by using either $u$-substitution or by integrating by parts.
2. Using appropriate subintervals, compute $L_4$, $R_4$, $M_4$, $T_4$, and $S_8$.
3. Which of the approximations in (b) is an over-estimate to the true value of $\int_0^1 x\tan(x)\ dx$? Which is an under-estimate? How do you know?

## 6

For an unknown function $f(x)$, the following information is known.

- $f$ is continuous on $[3,6]$;
- $f$ is either always increasing or always decreasing on $[3,6]$;
- $f$ has the same concavity throughout the interval $[3,6]$;
- As approximations to $\int_3^6 f(x)\ dx$, $L_4 = 7.23$, $R_4 = 6.75$, and $M_4 = 7.05$.

1. Is $f$ increasing or decreasing on $[3,6]$? What data tells you?
2. Is $f$ concave up or concave down on $[3,6]$? Why?
3. Determine the best possible estimate you can for $\int_3^6 f(x)\ dx$, based on the given information.

## 7

The rate at which water flows through Table Rock Dam on the White River in Branson, MO, is measured in cubic feet per second (CFS). As engineers open the floodgates, flow rates are recorded according to the following chart.

*Table 5.6.16. Water flow data.*

| seconds, $t$ | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| flow in CFS, $r(t)$ | 2000 | 2100 | 2400 | 3000 | 3900 | 5100 | 6500 |

1. What definite integral measures the total volume of water to flow through the dam in the 60 second time period provided by the table above?
2. Use the given data to calculate $M_n$ for the largest possible value of $n$ to approximate the integral you stated in (a). Do you think $M_n$ over- or under-estimates the exact value of the integral? Why?
3. Approximate the integral stated in (a) by calculating $S_n$ for the largest possible value of $n$, based on the given data.
4. Compute $\frac{1}{60}S_n$ and $\frac{2000+2100+2400+3000+3900+5100+6500}{7}$. What quantity do both of these values estimate? Which is a more accurate approximation?

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# 6: ACTIVE CALCULUS - USING DEFINITE INTEGRALS


6.1: Using Definite Integrals to Find Area and Length

6.2: Using Definite Integrals to Find Volume

6.3: Density, Mass, and Center of Mass

6.4: Physics Applications - Work, Force, and Pressure

6.5: Improper Integrals

6.6: Using Definite Integrals (Exercises)



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# 6.1: Using Definite Integrals to Find Area and Length

## Motivating Questions

- How can we use definite integrals to measure the area between two curves?
- How do we decide whether to integrate with respect to $x$ or with respect to $y$ when we try to find the area of a region?
- How can a definite integral be used to measure the length of a curve?

Early on in our work with the definite integral, we learned that for an object moving along an axis, the area under a non-negative velocity function $v$ between $a$ and $b$ tells us the distance the object traveled on that time interval, and that area is given precisely by the definite integral $\int_a^b v(t)\ dt$. In general, for any nonnegative function $f$ on an interval $[a, b]$, $\int_a^b f(x)\ dx$ measures the area bounded by the curve and the $x$-axis between $x = a$ and $x = b$.

Next, we will explore how definite integrals can be used to represent other physically important properties. In Preview Activity $6.1.1$, we investigate how a single definite integral may be used to represent the area between two curves.

## Preview Activity $6.1.1$

Consider the functions given by $f(x) = 5 - (x - 1)^2$ and $g(x) = 4 - x$.

1. Use algebra to find the points where the graphs of $f$ and $g$ intersect.
2. Sketch an accurate graph of $f$ and $g$ on the axes provided, labeling the curves by name and the intersection points with ordered pairs.
3. Find and evaluate exactly an integral expression that represents the area between $y = f(x)$ and the $x$-axis on the interval between the intersection points of $f$ and $g$.
4. Find and evaluate exactly an integral expression that represents the area between $y = g(x)$ and the $x$-axis on the interval between the intersection points of $f$ and $g$.
5. What is the exact area between $f$ and $g$ between their intersection points? Why?

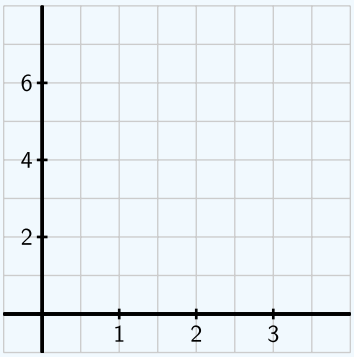



Figure $6.1.1$. Axes for plotting $f$ and $g$ in Preview Activity $6.1.1$.

## The Area Between Two Curves

In Preview Activity $6.1.1$, we saw a natural way to think about the area between two curves: it is the area beneath the upper curve minus the area below the lower curve.

## Example 6.1.1

Find the area bounded between the graphs of $f(x) = (x-1)^2 + 1$ and $g(x) = x + 2$.

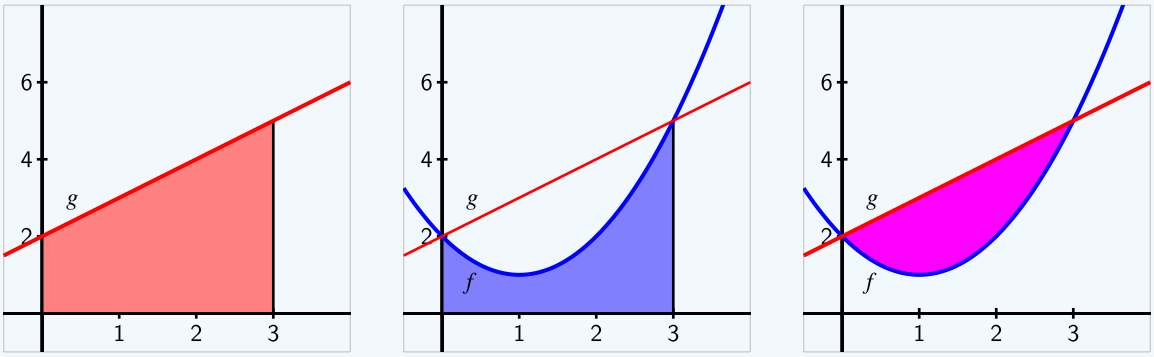



Figure 6.1.3. The areas bounded by the functions $f(x) = (x-1)^2 + 1$ and $g(x) = x + 2$ on the interval $[0, 3]$.

*Answer available at the end of this section.*

We can also think of the area this way: if we slice up the region between two curves into thin vertical rectangles (in the same spirit as we originally sliced the region between a single curve and the $x$-axis in Section 4.2), we see (as shown in Figure 6.1.4) that the height of a typical rectangle is given by the difference between the two functions, $g(x) - f(x)$, and its width is $\Delta x$. Thus the area of the rectangle is

$$A_{\text{rect}} = (g(x) - f(x))\Delta x.$$

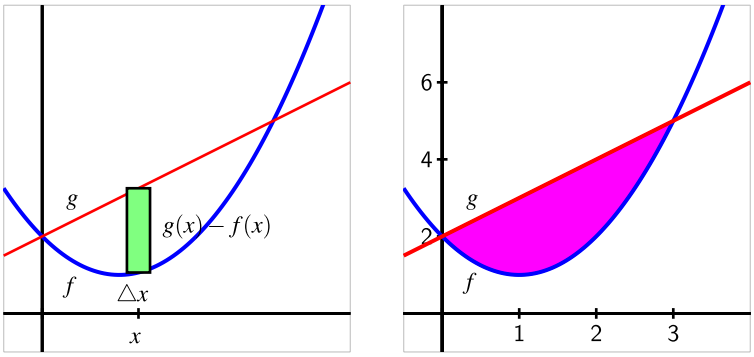



Figure 6.1.4. The area bounded by the functions $f(x) = (x-1)^2 + 1$ and $g(x) = x + 2$ on the interval $[0, 3]$.

The area between the two curves on $[0, 3]$ is thus approximated by the Riemann sum

$$A \approx \sum_{i=1}^{n} (g(x_i) - f(x_i))\Delta x,$$

and as we let $n \to \infty$, it follows that the area is given by the single definite integral

$$A = \int_0^3 (g(x) - f(x))\, dx. \tag{6.1.1}$$

In many applications of the definite integral, we will find it helpful to think of a "representative slice" and use the definite integral to add these slices. Here, the integral sums the areas of thin rectangles.

Finally, it doesn't matter whether we think of the area between two curves as the difference between the area bounded by the individual curves (as in Equation 6.1.2) or as the limit of a Riemann sum of the areas of thin rectangles between the curves (as in Equation 6.1.1). These two results are the same, since the difference of two integrals is the integral of the difference:

$$\int_0^3 g(x)\,dx - \int_0^3 f(x)\,dx = \int_0^3 (g(x) - f(x))\,dx.$$

Our work so far in this section illustrates the following general principle.

**Note**

If two curves $y = g(x)$ and $y = f(x)$ intersect at $(a, g(a))$ and $(b, g(b))$, and for all $x$ such that $a \le x \le b$, $g(x) \ge f(x)$, then the area between the curves is $A = \int_a^b (g(x) - f(x))\ dx$.

**Activity 6.1.2**

In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area.

1. The finite region bounded by $y = \sqrt{x}$ and $y = \frac{1}{4}x$.
2. The finite region bounded by $y = 12 - 2x^2$ and $y = x^2 - 8$.
3. The area bounded by the $y$-axis, $f(x) = \cos(x)$, and $g(x) = \sin(x)$, where we consider the region formed by the first positive value of $x$ for which $f$ and $g$ intersect.
4. The finite regions between the curves $y = x^3 - x$ and $y = x^2$.

## Finding Area with Horizontal Slices

At times, the shape of a region may dictate that we use horizontal rectangular slices, instead of vertical ones.

**Example 6.1.2**

Find the area of the region bounded by the parabola $x = y^2 - 1$ and the line $y = x - 1$, shown at left in Figure 6.1.6.

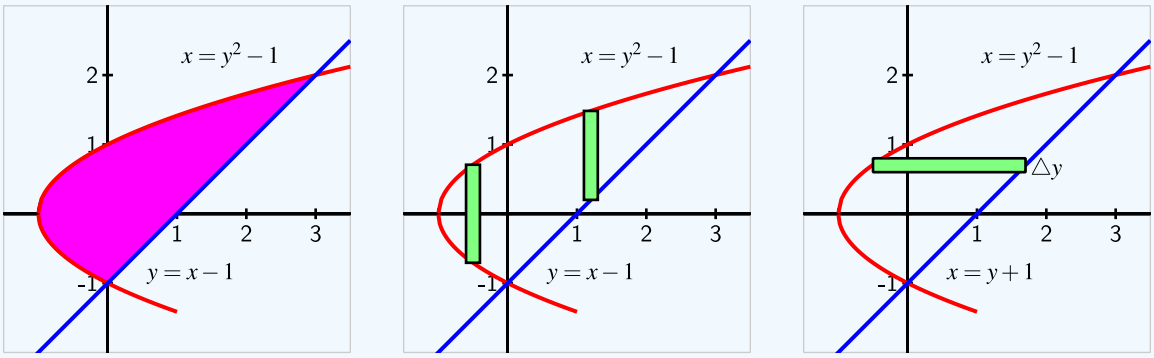



Figure 6.1.6. The area bounded by the functions $x = y^2 - 1$ and $y = x - 1$ (at left), with the region sliced vertically (center) and horizontally (at right).

*Answer available at the end of this section.*

Just as with the use of vertical rectangles of thickness $\Delta x$, we have a general principle for finding the area between two curves, which we state as follows.

**Note**

If two curves $x = g(y)$ and $x = f(y)$ intersect at $(g(c), c)$ and $(g(d), d)$, and for all $y$ such that $c \le y \le d$, $g(y) \ge f(y)$, then the area between the curves is

$$A = \int_{y=c}^{y=d} (g(y) - f(y))\, dy.$$

**Activity 6.1.3**

In each of the following problems, our goal is to determine the area of the region described. For each region, (i) determine the intersection points of the curves, (ii) sketch the region whose area is being found, (iii) draw and label a representative slice, and (iv) state the area of the representative slice. Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area. *Note well:* At the step where you draw a representative slice, you need to make a choice about whether to slice vertically or horizontally.

1. The finite region bounded by $x = y^2$ and $x = 6 - 2y^2$.
2. The finite region bounded by $x = 1 - y^2$ and $x = 2 - 2y^2$.
3. The area bounded by the $x$-axis, $y = x^2$, and $y = 2 - x$.
4. The finite regions between the curves $x = y^2 - 2y$ and $y = x$.

## Finding the Length of a Curve

We can also use the definite integral to find the length of a portion of a curve. We use the same fundamental principle: we slice the curve up into small pieces whose lengths we can easily approximate. Specifically, we subdivide the curve into small approximating line segments, as shown at left in Figure 6.1.7.

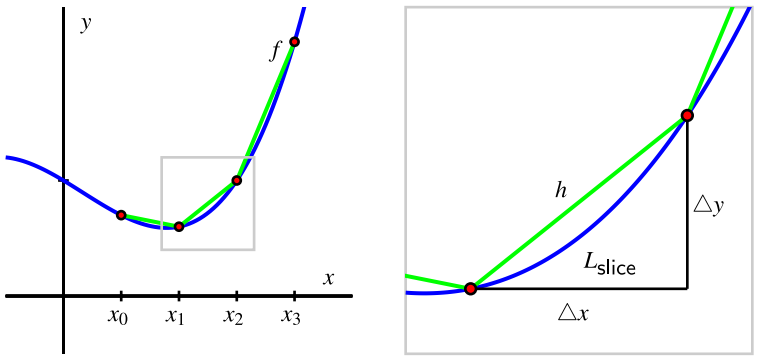



Figure 6.1.7. At left, a continuous function $y = f(x)$ whose length we seek on the interval $a = x_0$ to $b = x_3$. At right, a close up view of a portion of the curve.

We estimate the length $L_{\text{slice}}$ of each portion of the curve on a small interval of length $\Delta x$. We use the right triangle with legs parallel to the coordinate axes and hypotenuse connecting the endpoints of the slice, as seen at right in Figure 6.1.7. The length, $h$, of the hypotenuse approximates the length, $L_{\text{slice}}$, of the curve between the two selected points. Thus,

$$L_{\text{slice}} \approx h = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Next we use algebra to rearrange the expression for the length of the hypotenuse into a form that we can integrate. By removing

a factor of $(\Delta x)^2$, we find

$$\begin{aligned} L_{\text{slice}} &\approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right)} \\ &= \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}} \cdot \Delta x. \end{aligned}$$

Then, as $n \to \infty$ and $\Delta x \to 0$, we have that $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = f'(x)$. Thus, we can say that

$$L_{\text{slice}} \approx \sqrt{1 + f'(x)^2} \Delta x.$$

Taking a Riemann sum of all of these slices and letting $n \to \infty$, we arrive at the following fact.

**Note**

Given a differentiable function $f$ on an interval $[a, b]$, the total arc length, $L$, along the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

**Activity 6.1.4**

Each of the following questions somehow involves the arc length along a curve.

1. Use the definition and appropriate computational technology to determine the arc length along $y = x^2$ from $x = -1$ to $x = 1$.
2. Find the arc length of $y = \sqrt{4 - x^2}$ on the interval $-2 \le x \le 2$. Find this value in two different ways: (a) by using a definite integral, and (b) by using a familiar property of the curve.
3. Determine the arc length of $y = xe^{3x}$ on the interval $[0, 1]$.
4. Will the integrals that arise calculating arc length typically be ones that we can evaluate exactly using the First FTC, or ones that we need to approximate? Why?
5. A moving particle is traveling along the curve given by $y = f(x) = 0.1x^2 + 1$, and does so at a constant rate of 7 cm/sec, where both $x$ and $y$ are measured in cm (that is, the curve $y = f(x)$ is the path along which the object actually travels; the curve is not a "position function"). Find the position of the particle when $t = 4$ sec, assuming that when $t = 0$, the particle's location is $(0, f(0))$.

## Summary

- To find the area between two curves, we think about slicing the region into thin rectangles. If, for instance, the area of a typical rectangle on the interval $x = a$ to $x = b$ is given by $A_{\text{rect}} = (g(x) - f(x))\Delta x$, then the exact area of the region is given by the definite integral

$$A = \int_a^b (g(x) - f(x))\, dx.$$

- The shape of the region usually dictates whether we should use vertical rectangles of thickness $\Delta x$ or horizontal rectangles of thickness $\Delta y$. We want the height of the rectangle given by the difference between two curves: if those curves are best thought of as functions of $y$, we use horizontal rectangles, whereas if those curves are best viewed as functions of $x$, we use vertical rectangles.
- The arc length, $L$, along the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

## Answers

**Example** 6.1.1

In Figure 6.1.3, we see that the graphs intersect at $(0, 2)$ and $(3, 5)$. We can find these intersection points algebraically by solving the system of equations given by $y = x + 2$ and $y = (x - 1)^2 + 1$: substituting $x + 2$ for $y$ in the second equation yields $x + 2 = (x - 1)^2 + 1$, so $x + 2 = x^2 - 2x + 1 + 1$, and thus

$$x^2 - 3x = x(x - 3) = 0,$$

from which it follows that $x = 0$ or $x = 3$. Using $y = x + 2$, we find the corresponding $y$-values of the intersection points.

On the interval $[0, 3]$, the area beneath $g$ is

$$\int_0^3 (x + 2)\, dx = \frac{21}{2},$$

while the area under $f$ on the same interval is

$$\int_0^3 [(x - 1)^2 + 1]\, dx = 6.$$

Thus, the area between the curves is

$$A = \int_0^3 (x + 2)\, dx - \int_0^3 [(x - 1)^2 + 1]\, dx = \frac{21}{2} - 6 = \frac{9}{2}. \tag{6.1.2}$$

**Example** 6.1.2

By solving the second equation for $x$ and writing $x = y + 1$, we find that $y + 1 = y^2 - 1$. Hence the curves intersect where $y^2 - y - 2 = 0$. Thus, we find $y = -1$ or $y = 2$, so the intersection points of the two curves are $(0, -1)$ and $(3, 2)$.

If we attempt to use vertical rectangles to slice up the area (as in the center graph of Figure 6.1.6), we see that from $x = -1$ to $x = 0$ the curves that bound the top and bottom of the rectangle are one and the same. This suggests, as shown in the rightmost graph in the figure, that we try using horizontal rectangles.

Note that the width of a horizontal rectangle depends on $y$. Between $y = -1$ and $y = 2$, the right end of a representative rectangle is determined by the line $x = y + 1$, and the left end is determined by the parabola, $x = y^2 - 1$. The thickness of the rectangle is $\Delta y$.

Therefore, the area of the rectangle is

$$A_{\text{rect}} = [(y + 1) - (y^2 - 1)]\Delta y,$$

and the area between the two curves on the $y$-interval $[-1, 2]$ is approximated by the Riemann sum

$$A \approx \sum_{i=1}^{n} [(y_i + 1) - (y_i^2 - 1)]\Delta y.$$

Taking the limit of the Riemann sum, it follows that the area of the region is

$$A = \int_{y=-1}^{y=2} [(y + 1) - (y^2 - 1)]\, dy. \tag{6.1.3}$$

We emphasize that we are integrating with respect to $y$; this is because we chose to use horizontal rectangles whose widths depend on $y$ and whose thickness is denoted $\Delta y$. It is a straightforward exercise to evaluate the integral in Equation 6.1.3 and find that $A = \frac{9}{2}$.

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# 6.2: Using Definite Integrals to Find Volume

## Motivating Questions

- How can we use a definite integral to find the volume of a three-dimensional solid of revolution that results from revolving a two-dimensional region about a particular axis?
- In what circumstances do we integrate with respect to $y$ instead of integrating with respect to $x$?
- What adjustments do we need to make if we revolve about a line other than the $x$- or $y$-axis?

Just as we can use definite integrals to add the areas of rectangular slices to find the exact area that lies between two curves, we can also use integrals to find the volume of regions whose cross-sections have a particular shape.

In particular, we can determine the volume of solids whose cross-sections are all thin cylinders (or washers) by adding up the volumes of these individual slices. We first consider a familiar shape in Preview Activity 6.2.1: a circular cone.

## Preview Activity 6.2.1

Consider a circular cone of radius 3 and height 5, which we view horizontally as pictured in Figure 6.2.1. Our goal in this activity is to use a definite integral to determine the volume of the cone.

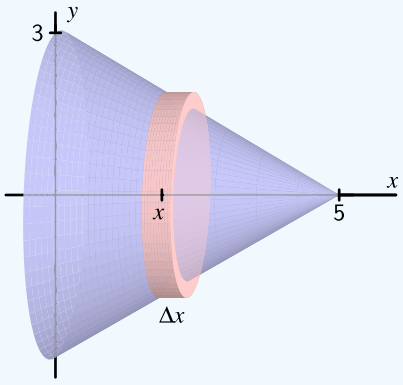



Figure 6.2.1. The circular cone described in Preview Activity 6.2.1.

1. Find a formula for the linear function $y = f(x)$ that is pictured in Figure 6.2.1.
2. For the representative slice of thickness $\Delta x$ that is located horizontally at a location $x$ (somewhere between $x = 0$ and $x = 5$), what is the radius of the representative slice? Note that the radius depends on the value of $x$.
3. What is the volume of the representative slice you found in (b)?
4. What definite integral will sum the volumes of the thin slices across the full horizontal span of the cone? What is the exact value of this definite integral?
5. Compare the result of your work in (d) to the volume of the cone that comes from using the formula $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$.

## Volume of a Solid of Revolution

A solid of revolution is a three dimensional solid that can be generated by revolving one or more curves around a fixed axis. For example, the circular cone in Figure 6.2.1 is the solid of revolution generated by revolving the portion of the line $y = 3 - \frac{3}{5}x$ from $x = 0$ to $x = 5$ about the $x$-axis. Notice that if we slice a solid of revolution perpendicular to the axis of revolution, the resulting cross-section is a circle.

We first consider solids whose slices are thin cylinders. Recall that the volume of a cylinder is given by $V = \pi r^2 h$.

### Example 6.2.1

Find the volume of the solid of revolution generated when the region $R$ bounded by $y = 4 - x^2$ and the $x$-axis is revolved about the $x$-axis.

*Answer available at the end of this section.*

For a solid such as the one in Example 6.2.1, where each slice is a cylindrical disk, we first find the volume of a typical slice (noting particularly how this volume depends on $x$), and then integrate over the range of $x$-values that bound the solid. Often, we will be content with simply finding the integral that represents the volume; if we desire a numeric value for the integral, we typically use a calculator or computer algebra system to find that value.

This method for finding the volume of a solid of revolution is often called the *disk method*.

### The Disk Method

If $y = r(x)$ is a nonnegative continuous function on $[a, b]$, then the volume of the solid of revolution generated by revolving the curve about the $x$-axis over this interval is given by

$$V = \int_a^b \pi r(x)^2 \, dx.$$

A different type of solid can emerge when two curves are involved, as we see in the following example.

### Example 6.2.2

Find the volume of the solid of revolution generated when the finite region $R$ that lies between $y = 4 - x^2$ and $y = x + 2$ is revolved about the $x$-axis.

*Answer available at the end of this section.*

This method for finding the volume of a solid of revolution generated by two curves is often called the *washer method*.

### The Washer Method

If $y = R(x)$ and $y = r(x)$ are nonnegative continuous functions on $[a, b]$ that satisfy $R(x) \geq r(x)$ for all $x$ in $[a, b]$, then the volume of the solid of revolution generated by revolving the region between them about the $x$-axis over this interval is given by

$$V = \int_a^b \pi[R(x)^2 - r(x)^2] \, dx.$$

### Activity 6.2.2

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find.

1. The region $S$ bounded by the $x$-axis, the curve $y = \sqrt{x}$, and the line $x = 4$; revolve $S$ about the $x$-axis.
2. The region $S$ bounded by the $y$-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve $S$ about the $x$-axis.

3. The finite region $S$ bounded by the curves $y = \sqrt{x}$ and $y = x^3$; revolve $S$ about the $x$-axis.
4. The finite region $S$ bounded by the curves $y = 2x^2 + 1$ and $y = x^2 + 4$; revolve $S$ about the $x$-axis.
5. The region $S$ bounded by the $y$-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve $S$ about the $y$-axis. How is this problem different from the one posed in part (b)?

## Revolving about the $y$-axis

When we revolve a given region about the $y$-axis, the representative slices now have thickness $\Delta y$, which means that we must integrate with respect to $y$.

**Example 6.2.3**

Find the volume of the solid of revolution generated when the region $R$ that lies between $y = \sqrt{x}$ and $y = x^4$ is revolved about the $y$-axis.

*Answer available at the end of this section.*

**Activity 6.2.3**

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find.

1. The region $S$ bounded by the $y$-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve $S$ about the $y$-axis.
2. The region $S$ bounded by the $x$-axis, the curve $y = \sqrt{x}$, and the line $x = 4$; revolve $S$ about the $y$-axis.
3. The finite region $S$ in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$; revolve $S$ about the $x$-axis.
4. The finite region $S$ in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$; revolve $S$ about the $y$-axis.
5. The finite region $S$ bounded by the curves $x = (y - 1)^2$ and $y = x - 1$; revolve $S$ about the $y$-axis

## Revolving about Horizontal and Vertical Lines other than the Coordinate Axes

It is possible to revolve a region around any horizontal or vertical line. Doing so adjusts the radii of the cylinders or washers involved by a constant value. A careful, well-labeled plot of the solid of revolution will usually reveal how the different axis of revolution affects the definite integral.

**Example 6.2.4**

Find the volume of the solid of revolution generated when the finite region $S$ that lies between $y = x^2$ and $y = x$ is revolved about the line $y = -1$.

*Answer available at the end of this section.*

**Activity 6.2.4**

In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find. For each prompt, use the finite region $S$ in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$.

1. Revolve $S$ about the line $y = -2$.
2. Revolve $S$ about the line $y = 4$.
3. Revolve $S$ about the line $x = -1$.
4. Revolve $S$ about the line $x = 5$.

## Summary

- We can use a definite integral to find the volume of a three-dimensional solid of revolution that results from revolving a two-dimensional region about a particular axis by taking slices perpendicular to the axis of revolution which will then be circular disks or washers.
- If we revolve about a vertical line and slice perpendicular to that line, then our slices are horizontal and of thickness $\Delta y$. This leads us to integrate with respect to $y$, as opposed to with respect to $x$ when we slice a solid vertically.
- If we revolve about a line other than the $x$- or $y$-axis, we need to carefully account for the shift that occurs in the radius of a typical slice. Normally, this shift involves taking a sum or difference of the function along with the constant connected to the equation for the horizontal or vertical line; a well-labeled diagram is usually the best way to decide the new expression for the radius.

## Answers

**Example** 6.2.1

First, we observe that $y = 4 - x^2$ intersects the $x$-axis at the points $(-2, 0)$ and $(2, 0)$. When we revolve the region $R$ about the $x$-axis, we get the three-dimensional solid pictured in Figure 6.2.1\3.

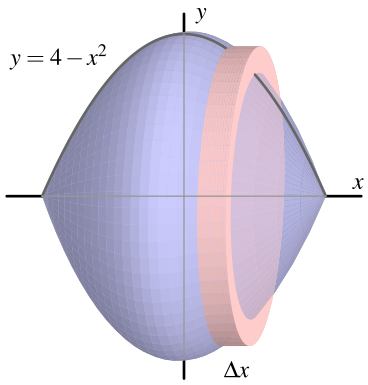



Figure 6.2.3. The solid of revolution in Example 6.2.1.

We slice the solid into vertical slices of thickness $\Delta x$ between $x = -2$ and $x = 2$. A representative slice is a cylinder of height $\Delta x$ and radius $4 - x^2$. Hence, the volume of the slice is

$$V_{\text{slice}} = \pi(4 - x^2)^2 \Delta x.$$

Using a definite integral to sum the volumes of the representative slices, it follows that

$$V = \int_{-2}^{2} \pi(4 - x^2)^2 \, dx.$$

It is straightforward to evaluate the integral and find that the volume is $V = \frac{512}{15}\pi$.

**Example** 6.2.2

First, we must determine where the curves $y = 4 - x^2$ and $y = x + 2$ intersect. Substituting the expression for $y$ from the second equation into the first equation, we find that $x + 2 = 4 - x^2$. Rearranging, it follows that

$$x^2 + x - 2 = 0,$$

and the solutions to this equation are $x = -2$ and $x = 1$. The curves therefore cross at $(-2, 0)$ and $(1, 1)$.

When we revolve the region $R$ about the $x$-axis, we get the three-dimensional solid pictured at left in Figure 6.2.5.

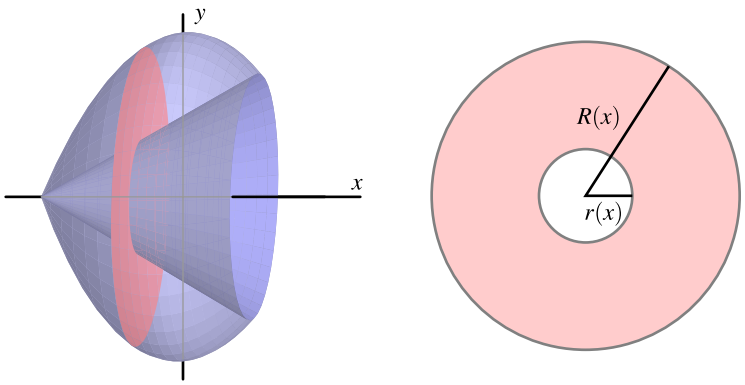



Figure 6.2.5. At left, the solid of revolution in Example 6.2.2. At right, a typical slice with inner radius $r(x)$ and outer radius $R(x)$.

Immediately we see a major difference between the solid in this example and the one in Example 6.2.1: here, the three-dimensional solid of revolution isn't "solid" because it has open space in its center along the axis of revolution. If we slice the solid perpendicular to the axis of revolution, we observe that the resulting slice is not a solid disk, but rather a *washer*, as pictured at right in Figure 6.2.5. At a given location $x$ between $x = -2$ and $x = 1$, the small radius $r(x)$ of the inner circle is determined by the curve $y = x + 2$, so $r(x) = x + 2$. Similarly, the big radius $R(x)$ comes from the function $y = 4 - x^2$, and thus $R(x) = 4 - x^2$.

To find the volume of a representative slice, we compute the volume of the outer disk and subtract the volume of the inner disk. Since

$$\pi R(x)^2 \Delta x - \pi r(x)^2 \Delta x = \pi[R(x)^2 - r(x)^2]\Delta x,$$

it follows that the volume of a typical slice is

$$V_{\text{slice}} = \pi[(4 - x^2)^2 - (x + 2)^2]\Delta x.$$

Using a definite integral to sum the volumes of the respective slices across the integral, we find that

$$V = \int_{-2}^{1} \pi[(4 - x^2)^2 - (x + 2)^2]\, dx.$$

Evaluating the integral, we find that the volume of the solid of revolution is $V = \frac{108}{5}\pi$.

**Example** 6.2.3

These two curves intersect when $x = 1$, hence at the point $(1, 1)$. When we revolve the region $R$ about the $y$-axis, we get the three-dimensional solid pictured at left in Figure 6.2.7.

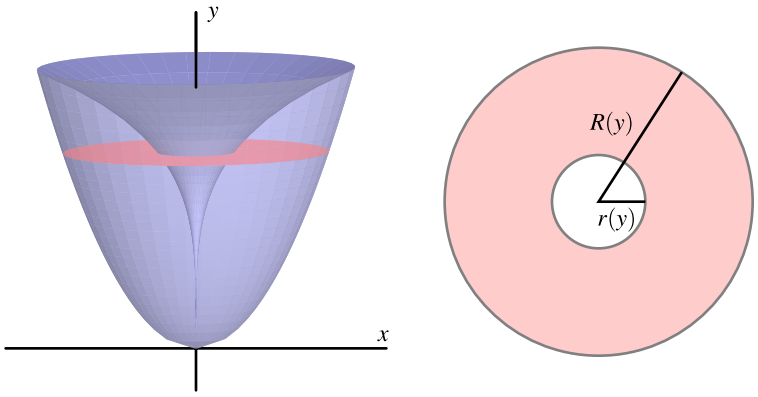



Figure 6.2.7: At left, the solid of revolution in Example 6.2.3. At right, a typical slice with inner radius $r(y)$ and outer radius $R(y)$.

Note that the slices are cylindrical washers only if taken perpendicular to the $y$-axis. We slice the solid horizontally, starting at $y = 0$ and proceeding up to $y = 1$. The thickness of a representative slice is $\Delta y$, so we must express the integrand in terms of $y$. The inner radius is determined by the curve $y = \sqrt{x}$, so we solve for $x$ and get $x = y^2 = r(y)$. In the same way, we solve the curve $y = x^4$ (which governs the outer radius) for $x$ in terms of $y$, and hence $x = \sqrt[4]{y}$. Therefore, the volume of a typical slice is

$$V_{\text{slice}} = \pi[R(y)^2 - r(y)^2] = \pi[(\sqrt[4]{y})^2 - (y^2)^2]\Delta y.$$

We use a definite integral to sum the volumes of all the slices from $y = 0$ to $y = 1$. The total volume is

$$V = \int_{y=0}^{y=1} \pi\left[(\sqrt[4]{y})^2 - (y^2)^2\right]\, dy.$$

It is straightforward to evaluate the integral and find that $V = \frac{7}{15}\pi$.

**Example** 6.2.4

Graphing the region between the two curves in the first quadrant between their points of intersection ($(0, 0)$ and $(1, 1)$) and then revolving the region about the line $y = -1$, we see the solid shown in Figure 6.2.9. Each slice of the solid perpendicular to the axis of revolution is a washer, and the radii of each washer are governed by the curves $y = x^2$ and $y = x$. But we also see that there is one added change: the axis of revolution adds a fixed length to each radius. The inner radius of a typical slice, $r(x)$, is given by $r(x) = x^2 + 1$, while the outer radius is $R(x) = x + 1$.

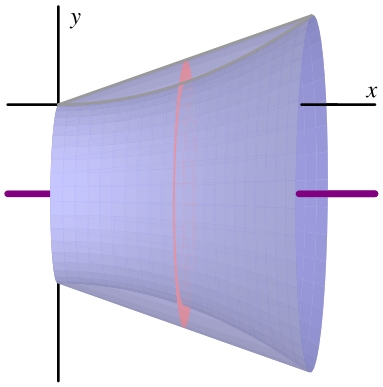



Figure 6.2.9. The solid of revolution described in Example 6.2.4.

Therefore, the volume of a typical slice is

$$V_{\text{slice}} = \pi[R(x)^2 - r(x)^2]\Delta x = \pi\left[(x+1)^2 - (x^2+1)^2\right]\Delta x.$$

Finally, we integrate to find the total volume, and

$$V = \int_0^1 \pi\left[(x+1)^2 - (x^2+1)^2\right]\,dx = \frac{7}{15}\pi.$$

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# 6.3: Density, Mass, and Center of Mass

## Motivating Questions

- How are mass, density, and volume related?
- How is the mass of an object with varying density computed?
- What is is the center of mass of an object, and how are definite integrals used to compute it?

Studying the units on the integrand and variable of integration helps us understand the meaning of a definite integral. For instance, if $v(t)$ is the velocity of an object moving along an axis, measured in feet per second, and $t$ measures time in seconds, then both the definite integral and its Riemann sum approximation,

$$\int_a^b v(t)\,dt \approx \sum_{i=1}^{n} v(t_i)\Delta t,$$

have units given by the product of the units of $v(t)$ and $t$:

$$(\text{feet/sec}) \cdot (\text{sec}) = \text{feet}.$$

Thus, $\int_a^b v(t)\ dt$ measures the total change in position of the moving object in feet.

Unit analysis will be particularly helpful to us in what follows.

## Preview Activity 6.3.1

In each of the following scenarios, we consider the distribution of a quantity along an axis.

1. Suppose that the function $c(x) = 200 + 100e^{-0.1x}$ models the density of traffic on a straight road, measured in cars per mile, where $x$ is number of miles east of a major interchange, and consider the definite integral $\int_0^2 (200 + 100e^{-0.1x})\ dx$.
   1. What are the units on the product $c(x) \cdot \Delta x$?
   2. What are the units on the definite integral and its Riemann sum approximation given by
   $$\int_0^2 c(x)\,dx \approx \sum_{i=1}^{n} c(x_i)\Delta x?$$
   3. Evaluate the definite integral $\int_0^2 c(x)\ dx = \int_0^2 (200 + 100e^{-0.1x})\ dx$ and write one sentence to explain the meaning of the value you find.
2. On a 6 foot long shelf filled with books, the function $B$ models the distribution of the weight of the books, in pounds per inch, where $x$ is the number of inches from the left end of the bookshelf. Let $B(x)$ be given by the rule $B(x) = 0.5 + \frac{1}{(x+1)^2}$.
   1. What are the units on the product $B(x) \cdot \Delta x$?
   2. What are the units on the definite integral and its Riemann sum approximation given by
   $$\int_{12}^{36} B(x)\,dx \approx \sum_{i=1}^{n} B(x_i)\Delta x?$$
   3. Evaluate the definite integral $\int_0^{72} B(x)\ dx = \int_0^{72} \left(0.5 + \frac{1}{(x+1)^2}\right)\ dx$ and write one sentence to explain the meaning of the value you find.

## Density

The *mass* of a quantity, typically measured in metric units such as grams or kilograms, is a measure of the amount of the quantity. In a corresponding way, the *density* of an object measures the distribution of mass per unit volume. For instance, if a brick has

mass 3 kg and volume 0.002 $m^3$, then the density of the brick is

$$\frac{3\text{kg}}{0.002\text{m}^3} = 1500\frac{\text{kg}}{\text{m}^3}.$$

As another example, the mass density of water is 1000 kg/$m^3$. Each of these relationships demonstrate the following general principle.

**Note**

For an object of constant density $d$, with mass $m$ and volume $V$,

$$d = \frac{m}{V}, \text{ or } m = d \cdot V.$$

But what happens when the density is not constant?

The formula $m = d \cdot V$ is reminiscent of two other equations that we have used in our work: for a body moving in a fixed direction, distance = rate $\cdot$ time, and, for a rectangle, its area is given by $A = l \cdot w$. These formulas hold when the principal quantities involved, such as the rate the body moves and the height of the rectangle, are *constant*. When these quantities are not constant, we have turned to the definite integral for assistance. By working with small slices on which the quantity of interest (such as velocity) is approximately constant, we can use a definite integral to add up the values on the pieces.

For example, if we have a nonnegative velocity function that is not constant, over a short time interval $\Delta t$ we know that the distance traveled is approximately $v(t)\Delta t$, since $v(t)$ is almost constant on a small interval. Similarly, if we are thinking about the area under a nonnegative function $f$ whose value is changing, on a short interval $\Delta x$ the area under the curve is approximately the area of the rectangle whose height is $f(x)$ and whose width is $\Delta x$: $f(x)\Delta x$. Both of these principles are represented visually in Figure 6.3.1

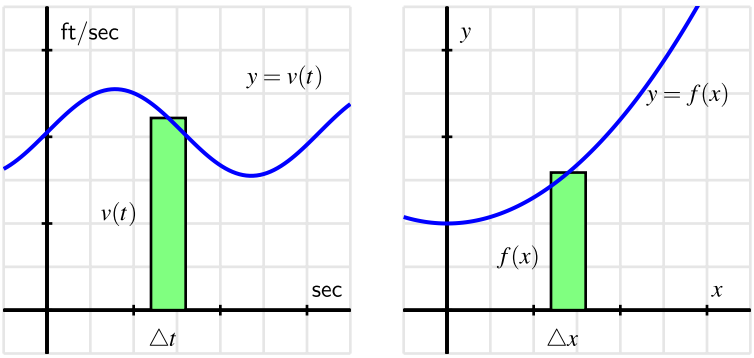



Figure 6.3.1 At left, estimating a small amount of distance traveled, $v(t)\Delta t$, and at right, a small amount of area under the curve, $f(x)\Delta x$.

In a similar way, if the density of some object is not constant, we can use a definite integral to compute the overall mass of the object. We will focus on problems where the density varies in only one dimension, say along a single axis.

Let's consider a thin bar of length $b$ whose left end is at the origin, where $x = 0$, and assume that the bar has constant cross-sectional area of 1 $cm^2$. We let $\rho(x)$ represent the mass density function of the bar, measured in grams per cubic centimeter. That is, given a location $x$, $\rho(x)$ tells us approximately how much mass will be found in a one-centimeter wide slice of the bar at $x$.

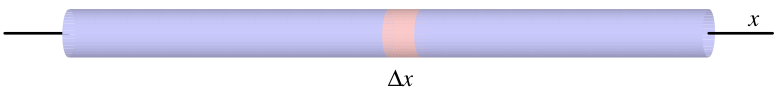



Figure 6.3.2 A thin bar of constant cross-sectional area 1 $cm^2$ with density function $\rho(x)$ g/$cm^3$.

The volume of a thin slice of the bar of width $\Delta x$, as pictured in Figure 6.3.2, is the cross-sectional area times $\Delta x$. Since the cross-sections each have constant area 1 cm$^2$, it follows that the volume of the slice is $1\Delta x$ cm$^3$. And because mass is the product of density and volume, we see that the mass of this slice is approximately

$$\text{mass}_{\text{slice}} \approx \rho(x)\ \frac{\text{g}}{\text{cm}^3} \cdot 1\Delta x\ \text{cm}^3 = \rho(x) \cdot \Delta x\ \text{g}.$$

The corresponding Riemann sum (and the integral that it approximates),

$$\sum_{i=1}^{n} \rho(x_i)\Delta x \approx \int_0^b \rho(x)\,dx,$$

therefore measure the mass of the bar between $0$ and $b$. (The Riemann sum is an approximation, while the integral will be the exact mass.)

For objects whose cross-sectional area is constant and whose mass is distributed relative to horizontal location, $x$, it makes sense to think of the density function $\rho(x)$ with units "mass per unit length," such as g/cm. Thus, when we compute $\rho(x) \cdot \Delta x$ on a small slice $\Delta x$, the resulting units are g/cm · cm = g, which thus measures the mass of the slice. The general principle follows.

**Note**

For an object of constant cross-sectional area whose mass is distributed along a single axis according to the function $\rho(x)$ (whose units are units of mass per unit of length), the total mass, $M$, of the object between $x = a$ and $x = b$ is given by

$$M = \int_a^b \rho(x)\,dx.$$

**Activity 6.3.2**

Consider the following situations in which mass is distributed in a non-constant manner.

1. Suppose that a thin rod with constant cross-sectional area of 1 cm$^2$ has its mass distributed according to the density function $\rho(x) = 2e^{-0.2x}$, where $x$ is the distance in cm from the left end of the rod, and the units on $\rho(x)$ are g/cm. If the rod is 10 cm long, determine the exact mass of the rod.
2. Consider the cone that has a base of radius 4 m and a height of 5 m. Picture the cone lying horizontally with the center of its base at the origin and think of the cone as a solid of revolution.
   1. Write and evaluate a definite integral whose value is the volume of the cone.
   2. Next, suppose that the cone has uniform density of 800 kg/m$^3$. What is the mass of the solid cone?
   3. Now suppose that the cone's density is not uniform, but rather that the cone is most dense at its base. In particular, assume that the density of the cone is uniform across cross sections parallel to its base, but that in each such cross section that is a distance $x$ units from the origin, the density of the cross section is given by the function $\rho(x) = 400 + \frac{200}{1+x^2}$, measured in kg/m$^3$. Determine and evaluate a definite integral whose value is the mass of this cone of non-uniform density. Do so by first thinking about the mass of a given slice of the cone $x$ units away from the base; remember that in such a slice, the density will be *essentially constant*.
3. Let a thin rod of constant cross-sectional area 1 cm$^2$ and length 12 cm have its mass be distributed according to the density function $\rho(x) = \frac{1}{25}(x-15)^2$, measured in g/cm. Find the exact location $z$ at which to cut the bar so that the two pieces will each have identical mass.

## Weighted Averages

The concept of an average is a natural one, and one that we have used repeatedly as part of our understanding of the meaning of the definite integral. If we have $n$ values $a_1, a_2, \ldots, a_n$, we know that their average is given by

$$\frac{a_1 + a_2 + \cdots + a_n}{n},$$

and for a quantity being measured by a function $f$ on an interval $[a, b]$, the average value of the quantity on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

As we continue to think about problems involving the distribution of mass, it is natural to consider the idea of a *weighted* average, where certain quantities involved are counted more in the average.

A common use of weighted averages is in the computation of a student's GPA, where grades are weighted according to credit hours. Let's consider the scenario in Table 6.3.1

*Table* 6.3.1 *A college student's semester grades.*

| class | grade | grade points | credits |
|---|---|---|---|
| chemistry | B+ | 3.3 | 5 |
| calculus | A- | 3.7 | 4 |
| history | B- | 2.7 | 3 |
| psychology | B- | 2.7 | 3 |

If all of the classes were of the same weight (i.e., the same number of credits), the student's GPA would simply be calculated by taking the average

$$\frac{3.3+3.7+2.7+2.7}{4} = 3.1.$$

But since the chemistry and calculus courses have higher weights (of 5 and 4 credits respectively), we actually compute the GPA according to the weighted average

$$\frac{3.3\cdot 5+3.7\cdot 4+2.7\cdot 3+2.7\cdot 3}{5+4+3+3} = 3.1\overline{6}.$$

The weighted average reflects the fact that chemistry and calculus, as courses with higher credits, have a greater impact on the students' grade point average. Note particularly that in the weighted average, each grade gets multiplied by its weight, and we divide by the sum of the weights.

In the following activity, we explore further how weighted averages can be used to find the balancing point of a physical system.

## Activity 6.3.3

For quantities of equal weight, such as two children on a teeter-totter, the balancing point is found by taking the average of their locations. When the weights of the quantities differ, we use a weighted average of their respective locations to find the balancing point.

1. Suppose that a shelf is 6 feet long, with its left end situated at $x = 0$. If one book of weight 1 lb is placed at $x_1 = 0$, and another book of weight 1 lb is placed at $x_2 = 6$, what is the location of $\overline{x}$, the point at which the shelf would (theoretically) balance on a fulcrum?
2. Now, say that we place four books on the shelf, each weighing 1 lb: at $x_1 = 0$, at $x_2 = 2$, at $x_3 = 4$, and at $x_4 = 6$. Find $\overline{x}$, the balancing point of the shelf.
3. How does $\overline{x}$ change if we change the location of the third book? Say the locations of the 1-lb books are $x_1 = 0$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 6$.
4. Next, suppose that we place four books on the shelf, but of varying weights: at $x_1 = 0$ a 2-lb book, at $x_2 = 2$ a 3-lb book, at $x_3 = 4$ a 1-lb book, and at $x_4 = 6$ a 1-lb book. Use a weighted average of the locations to find $\overline{x}$, the balancing point of the shelf. How does the balancing point in this scenario compare to that found in (b)?
5. What happens if we change the location of one of the books? Say that we keep everything the same in (d), except that $x_3 = 5$. How does $\overline{x}$ change?
6. What happens if we change the weight of one of the books? Say that we keep everything the same in (d), except that the book at $x_3 = 4$ now weighs 2 lbs. How does $\overline{x}$ change?

7. Experiment with a couple of different scenarios of your choosing where you move one of the books to the left, or you decrease the weight of one of the books.
8. Write a couple of sentences to explain how adjusting the location of one of the books or the weight of one of the books affects the location of the balancing point of the shelf. Think carefully here about how your changes should be considered relative to the location of the balancing point $\overline{x}$ of the current scenario.

## Center of Mass

In Activity $6.3.3$, we saw that the balancing point of a system of point-masses[1] (such as books on a shelf) is found by taking a weighted average of their respective locations. In the activity, we were computing the *center of mass* of a system of masses distributed along an axis, which is the balancing point of the axis on which the masses rest.

In the activity, we actually used *weight* rather than *mass*. Since weight is proportional to mass, the computations for the balancing point result in the same location regardless of whether we use weight or mass. The gravitational constant is present in both the numerator and denominator of the weighted average.

### Center of Mass (point-masses)

For a collection of $n$ masses $m_1, \ldots, m_n$ that are distributed along a single axis at the locations $x_1, \ldots, x_n$, the center of mass is given by

$$\overline{x} = \frac{x_1 m_1 + x_2 m_2 + \cdots + x_n m_n}{m_1 + m_2 + \cdots + m_n}.$$

Now consider a thin bar over which density is distributed continuously. If the density is constant, it is obvious that the balancing point of the bar is its midpoint. But if density is not constant, we must compute a weighted average. Let's say that the function $\rho(x)$ tells us the density distribution along the bar, measured in g/cm. If we slice the bar into small sections, we can think of the bar as holding a collection of adjacent point-masses. The mass $m_i$ of a slice of thickness $\Delta x$ at location $x_i$, is $m_i \approx \rho(x_i)\Delta x$.

If we slice the bar into $n$ pieces, we can approximate its center of mass by

$$\overline{x} \approx \frac{x_1 \cdot \rho(x_1)\Delta x + x_2 \cdot \rho(x_2)\Delta x + \cdots + x_n \cdot \rho(x_n)\Delta x}{\rho(x_1)\Delta x + \rho(x_2)\Delta x + \cdots + \rho(x_n)\Delta x}.$$

Rewriting the sums in sigma notation, we have

$$\overline{x} \approx \frac{\sum_{i=1}^{n} x_i \cdot \rho(x_i)\Delta x}{\sum_{i=1}^{n} \rho(x_i)\Delta x}. \tag{6.3.1}$$

The greater the number of slices, the more accurate our estimate of the balancing point will be. The sums in Equation $(6.3.1)$ can be viewed as Riemann sums, so in the limit as $n \to \infty$, we find that the center of mass is given by the quotient of two integrals.

### Center of Mass (continuous mass distribution)

For a thin rod of density $\rho(x)$ distributed along an axis from $x = a$ to $x = b$, the center of mass of the rod is given by

$$\overline{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}.$$

Note that the denominator of $\overline{x}$ is the mass of the bar, and that this quotient of integrals is simply the continuous version of the weighted average of locations, $x$, along the bar.

**Activity 6.3.4**

Consider a thin bar of length 20 cm whose density is distributed according to the function $\rho(x) = 4 + 0.1x$, where $x = 0$ represents the left end of the bar. Assume that $\rho$ is measured in g/cm and $x$ is measured in cm.

1. Find the total mass, $M$, of the bar.
2. Without doing any calculations, do you expect the center of mass of the bar to be equal to 10, less than 10, or greater than 10? Why?
3. Compute $\overline{x}$, the exact center of mass of the bar.
4. What is the average density of the bar?
5. Now consider a different density function, given by $p(x) = 4e^{0.020732x}$, also for a bar of length 20 cm whose left end is at $x = 0$. Plot both $\rho(x)$ and $p(x)$ on the same axes. Without doing any calculations, which bar do you expect to have the greater center of mass? Why?
6. Compute the exact center of mass of the bar described in (e) whose density function is $p(x) = 4e^{0.020732x}$. Check the result against the prediction you made in (e).

## Summary

- For an object of constant density $D$, with volume $V$ and mass $m$, we know that $m = D \cdot V$.
- If an object with constant cross-sectional area (such as a thin bar) has its density distributed along an axis according to the function $\rho(x)$, then we can find the mass of the object between $x = a$ and $x = b$ by

$$m = \int_a^b \rho(x)\,dx.$$

- For a system of point-masses distributed along an axis, say $m_1, \ldots, m_n$ at locations $x_1, \ldots, x_n$, the center of mass, $\overline{x}$, is given by the weighted average

$$\overline{x} = \frac{\sum_{i=1}^n x_i m_i}{\sum_{i=1}^n m_i}.$$

If instead we have mass continuously distributed along an axis, such as by a density function $\rho(x)$ for a thin bar of constant cross-sectional area, the center of mass of the portion of the bar between $x = a$ and $x = b$ is given by

$$\overline{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}.$$

In each situation, $\overline{x}$ represents the balancing point of the system of masses or of the portion of the bar.

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# 6.4: Physics Applications - Work, Force, and Pressure

## Motivating Questions

- How do we measure the work accomplished by a varying force that moves an object a certain distance?
- What is the total force exerted by water against a dam?
- How are both of the above concepts and their corresponding use of definite integrals similar to problems we have encountered in the past involving formulas such as “distance equals rate times time” and “mass equals density times volume”?

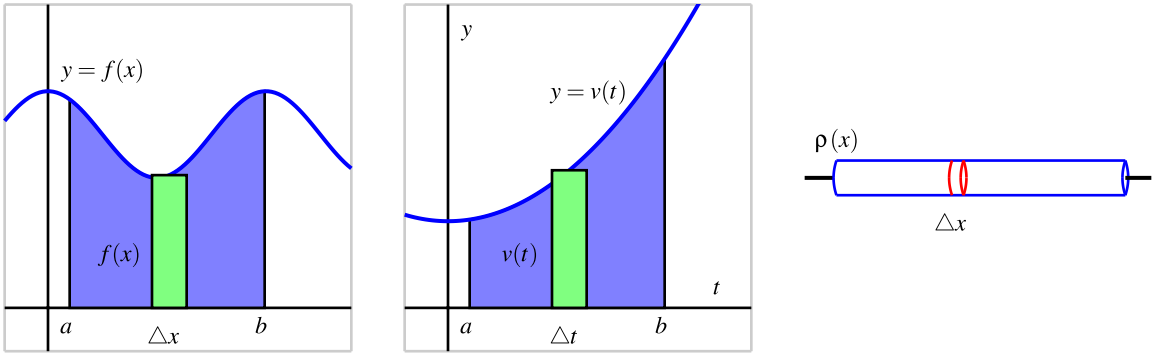



Figure 6.4.1 Three settings where we compute the accumulation of a varying quantity: the area under $y = f(x)$, the distance traveled by an object with velocity $y = v(t)$, and the mass of a bar with density function $y = \rho(x)$.

We have seen several different circumstances where the definite integral enables us to measure the accumulation of a quantity that varies, provided the quantity is approximately constant over small intervals. For instance, to find the area bounded by a nonnegative curve $y = f(x)$ and the $x$-axis on an interval $[a, b]$, we take a representative slice of width $\Delta x$ that has area $A_{\text{slice}} = f(x)\Delta x$. As we let the width of the representative slice tend to zero, we find that the exact area of the region is

$$A = \int_a^b f(x)\,dx.$$

In a similar way, if we know the velocity $v(t)$ of a moving object and we wish to know the distance the object travels on an interval $[a, b]$ where $v(t)$ is nonnegative, we can use a definite integral to generalize the fact that $d = r \cdot t$ when the rate, $r$, is constant. On a short time interval $\Delta t$, $v(t)$ is roughly constant, so for a small slice of time, $d_{\text{slice}} = v(t)\Delta t$. As the width of the time interval $\Delta t$ tends to zero, the exact distance traveled is given by the definite integral

$$d = \int_a^b v(t)\,dt.$$

Finally, if we want to determine the mass of an object of non-constant density, because $M = D \cdot V$ (mass equals density times volume, provided that density is constant), we can consider a small slice of an object on which the density is approximately constant, and a definite integral may be used to determine the exact mass of the object. For instance, if we have a thin rod whose cross sections have constant density, but whose density is distributed along the $x$ axis according to the function $y = \rho(x)$, it follows that for a small slice of the rod that is $\Delta x$ thick, $M_{\text{slice}} = \rho(x)\Delta x$. In the limit as $\Delta x \to 0$, we then find that the total mass is given by

$$M = \int_a^b \rho(x)\,dx.$$

All three of these situations are similar in that we have a basic rule ($A = l \cdot w$, $d = r \cdot t$, $M = D \cdot V$) where one of the two quantities being multiplied is no longer constant; in each, we consider a small interval for the other variable in the formula, calculate the

approximate value of the desired quantity (area, distance, or mass) over the small interval, and then use a definite integral to sum the results as the length of the small intervals is allowed to approach zero. It should be apparent that this approach will work effectively for other situations where we have a quantity that varies.

We next turn to the notion of *work*: from physics, a basic principle is that work is the product of force and distance. For example, if a person exerts a force of 20 pounds to lift a 20-pound weight 4 feet off the ground, the total work accomplished is

$$W = F \cdot d = 20 \cdot 4 = 80 \text{ foot-pounds.}$$

If force and distance are measured in English units (pounds and feet), then the units of work are *foot-pounds*. If we work in metric units, where forces are measured in Newtons and distances in meters, the units of work are *Newton-meters*.

Of course, the formula $W = F \cdot d$ only applies when the force is constant over the distance $d$. In Preview Activity $6.4.1$, we explore one way that we can use a definite integral to compute the total work accomplished when the force exerted varies.

### Preview Activity $6.4.1$

A bucket is being lifted from the bottom of a 50-foot deep well; its weight (including the water), $B$, in pounds at a height $h$ feet above the water is given by the function $B(h)$. When the bucket leaves the water, the bucket and water together weigh $B(0) = 20$ pounds, and when the bucket reaches the top of the well, $B(50) = 12$ pounds. Assume that the bucket loses water at a constant rate (as a function of height, $h$) throughout its journey from the bottom to the top of the well.

1. Find a formula for $B(h)$.
2. Compute the value of the product $B(5)\Delta h$, where $\Delta h = 2$ feet. Include units on your answer. Explain why this product represents the approximate work it took to move the bucket of water from $h = 5$ to $h = 7$.
3. Is the value in (b) an over- or under-estimate of the actual amount of work it took to move the bucket from $h = 5$ to $h = 7$? Why?
4. Compute the value of the product $B(22)\Delta h$, where $\Delta h = 0.25$ feet. Include units on your answer. What is the meaning of the value you found?
5. More generally, what does the quantity $W_{\text{slice}} = B(h)\Delta h$ measure for a given value of $h$ and a small positive value of $\Delta h$?
6. Evaluate the definite integral $\int_0^{50} B(h)\ dh$. What is the meaning of the value you find? Why?

## Work

Because work is calculated by the rule $W = F \cdot d$ whenever the force $F$ is constant, it follows that we can use a definite integral to compute the work accomplished by a varying force. For example, suppose that a bucket whose weight at height $h$ is given by $B(h) = 12 + 8e^{-0.1h}$ is being lifted in a 50-foot well.

In contrast to the problem in the preview activity, this bucket is not leaking at a constant rate; but because the weight of the bucket and water is not constant, we have to use a definite integral to determine the total work done in lifting the bucket. At a height $h$ above the water, the approximate work to move the bucket a small distance $\Delta h$ is

$$W_{\text{slice}} = B(h)\Delta h = (12 + 8e^{-0.1h})\Delta h.$$

Hence, if we let $\Delta h$ tend to 0 and take the sum of all of the slices of work accomplished on these small intervals, it follows that the total work is given by

$$W = \int_0^{50} B(h)\, dh = \int_0^{50} (12 + 8e^{-0.1h})\, dh.$$

While it is a straightforward exercise to evaluate this integral exactly using the First Fundamental Theorem of Calculus, in applied settings such as this one we will typically use computing technology. Here, it turns out that $W = \int_0^{50} (12 + 8e^{-0.1h})\ dh \approx 679.461$ foot-pounds.

Our work in Preview Activity $6.4.1$ and in the most recent discussion above employs the following important general principle.

**Note**

For an object being moved in the positive direction along an axis with location $x$ by a force $F(x)$, the total work to move the object from $a$ to $b$ is given by

$$W = \int_a^b F(x)\,dx.$$

**Activity 6.4.2**

Consider the following situations in which a varying force accomplishes work.

1. Suppose that a heavy rope hangs over the side of a cliff. The rope is 200 feet long and weighs 0.3 pounds per foot; initially the rope is fully extended. How much work is required to haul in the entire length of the rope? (Hint: set up a function $F(h)$ whose value is the weight of the rope remaining over the cliff after $h$ feet have been hauled in.)
2. A leaky bucket is being hauled up from a 100 foot deep well. When lifted from the water, the bucket and water together weigh 40 pounds. As the bucket is being hauled upward at a constant rate, the bucket leaks water at a constant rate so that it is losing weight at a rate of 0.1 pounds per foot. What function $B(h)$ tells the weight of the bucket after the bucket has been lifted $h$ feet? What is the total amount of work accomplished in lifting the bucket to the top of the well?
3. Now suppose that the bucket in (b) does not leak at a constant rate, but rather that its weight at a height $h$ feet above the water is given by $B(h) = 25 + 15e^{-0.05h}$. What is the total work required to lift the bucket 100 feet? What is the average force exerted on the bucket on the interval $h = 0$ to $h = 100$?
4. From physics, *Hooke*'s *Law* for springs states that the amount of force required to hold a spring that is compressed (or extended) to a particular length is proportionate to the distance the spring is compressed (or extended) from its natural length. That is, the force to compress (or extend) a spring $x$ units from its natural length is $F(x) = kx$ for some constant $k$ (which is called the *spring constant*.) For springs, we choose to measure the force in pounds and the distance the spring is compressed in feet. Suppose that a force of 5 pounds extends a particular spring 4 inches (1/3 foot) beyond its natural length.
    1. Use the given fact that $F(1/3) = 5$ to find the spring constant $k$.
    2. Find the work done to extend the spring from its natural length to 1 foot beyond its natural length.
    3. Find the work required to extend the spring from 1 foot beyond its natural length to 1.5 feet beyond its natural length.

## Work: Pumping Liquid from a Tank

In certain geographic locations where the water table is high, residential homes with basements have a peculiar feature: in the basement, one finds a large hole in the floor, and in the hole, there is water. For example, in Figure 6.4.2 we see a *sump crock*[1]. A sump crock provides an outlet for water that may build up beneath the basement floor to prevent flooding the basement.

Image credit to www.warreninspect.com/basement-moisture.

In the crock we see a floating pump. This pump is activated by elevation, so when the water level reaches a particular height, the pump turns on and pumps water out of the crock, hence relieving the water buildup beneath the foundation. One of the questions we'd like to answer is: how much work does a sump pump accomplish?



Figure 6.4.2. A sump crock.

## Example 6.4.1

Suppose that a sump crock has the shape of a frustum of a cone, as pictured in Figure 6.4.3. The crock has a diameter of 3 feet at its surface, a diameter of 1.5 feet at its base, and a depth of 4 feet. In addition, suppose that the sump pump is set up so that it pumps the water vertically up a pipe to a drain that is located at ground level just outside a basement window. To accomplish this, the pump must send the water to a location 9 feet above the surface of the sump crock. How much work is required to empty the sump crock if it is initially completely full?

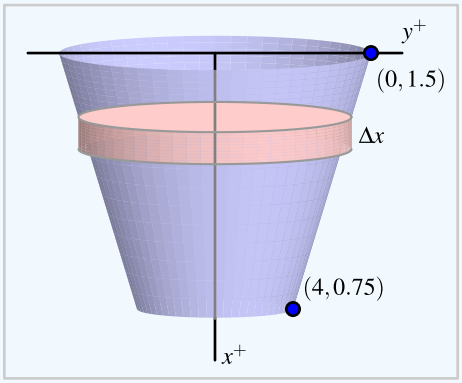



Figure 6.4.3. A sump crock with approximately cylindrical cross-sections.

*Answer available at the end of this section.*

The preceding example demonstrates the standard approach to finding the work required to empty a tank filled with liquid. The main task in each such problem is to determine the volume of a representative slice, followed by the force exerted on the slice, as well as the distance such a slice moves. In the case where the units are metric, there is one key difference: in the metric setting, rather than weight, we normally first find the mass of a slice. For instance, if distance is measured in meters, the mass density of water is 1000 kg/m$^3$. In that setting, we can find the mass of a typical slice (in kg). To determine the force required to move it, we use $F = ma$, where $m$ is the object's mass and $a$ is the gravitational constant $a = 9.81$ N/kg. That is, in metric units, the weight density of water is 9810 N/m$^3$.

## Activity 6.4.3

In each of the following problems, determine the total work required to accomplish the described task. In parts (b) and (c), a key step is to find a formula for a function that describes the curve that forms the side boundary of the tank.

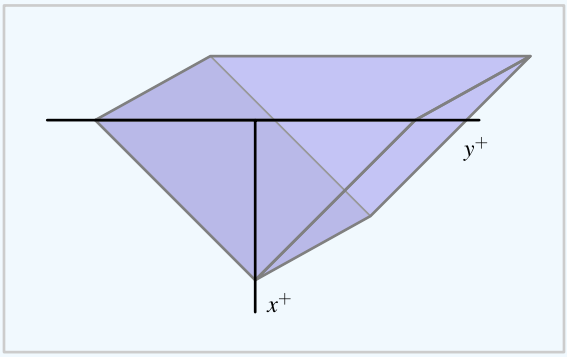



Figure 6.4.4. A trough with triangular ends, as described in Activity 6.4.3, part (c).

1. Consider a vertical cylindrical tank of radius 2 meters and depth 6 meters. Suppose the tank is filled with 4 meters of water of mass density 1000 kg/m$^3$, and the top 1 meter of water is pumped over the top of the tank.
2. Consider a hemispherical tank with a radius of 10 feet. Suppose that the tank is full to a depth of 7 feet with water of weight density 62.4 pounds/ft$^3$, and the top 5 feet of water are pumped out of the tank to a tanker truck whose height is 5 feet above the top of the tank.
3. Consider a trough with triangular ends, as pictured in Figure 6.4.4, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft$^3$, and a pump is used to empty the tank until the water remaining in the tank is 1 foot deep.

# Force due to Hydrostatic Pressure

When building a dam, engineers need to know how much force water will exert against the face of the dam. This force comes from water pressure. The pressure a force exerts on a region is measured in units of force per unit of area: for example, the air pressure in a tire is often measured in pounds per square inch (PSI). Hence, we see that the general relationship is given by

$$P = \frac{F}{A}, \text{ or } F = P \cdot A,$$

where $P$ represents pressure, $F$ represents force, and $A$ the area of the region being considered. Of course, in the equation $F = PA$, we assume that the pressure is constant over the entire region $A$.

We know from experience that the deeper one dives underwater while swimming, the greater the pressure exerted by the water. This is because at a greater depth, there is more water right on top of the swimmer: it is the force that "column" of water exerts that determines the pressure the swimmer experiences. The total water pressure is found by computing the total weight of the column of water that lies above a region of area 1 square foot at a fixed depth. At a depth of $d$ feet, a rectangular column has volume $V = 1 \cdot 1 \cdot d$ ft$^3$, so the corresponding weight of the water overhead is $62.4d$. This is the amount of force being exerted on a 1 square foot region at a depth $d$ feet underwater, so the pressure exerted by water at depth $d$ is $P = 62.4d$ (lbs/ft$^2$).

Because pressure is force per unit area, or $P = \frac{F}{A}$, we can compute the total force from a variable pressure by integrating $F = PA$.

## Example 6.4.2

Consider a trapezoid-shaped dam that is 60 feet wide at its base and 90 feet wide at its top, and assume the dam is 25 feet tall with water that rises to within 5 feet of the top of its face. Water weighs 62.4 pounds per cubic foot. How much force does the water exert against the dam?

*Answer available at the end of this section.*

### Activity 6.4.4

In each of the following problems, determine the total force exerted by water against the surface that is described.

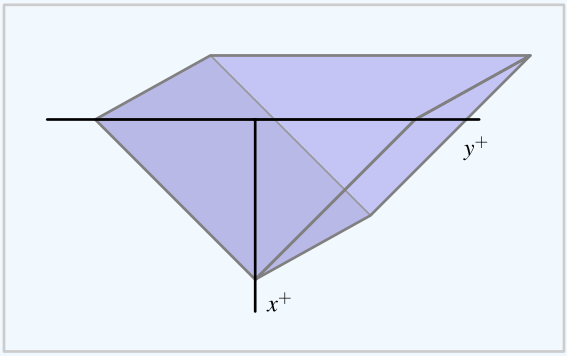



Figure 6.4.6. A trough with triangular ends, as described in Activity 6.4.3, part (c).

1. Consider a rectangular dam that is 100 feet wide and 50 feet tall, and suppose that water presses against the dam all the way to the top.
2. Consider a semicircular dam with a radius of 30 feet. Suppose that the water rises to within 10 feet of the top of the dam.
3. Consider a trough with triangular ends, as pictured in Figure 6.4.6, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ $ft^3$. How much force does the water exert against one of the triangular ends?

Although there are many different formulas involving work, force, and pressure, the fundamental ideas behind these problems are similar to others we've encountered in applications of the definite integral. We slice the quantity of interest into more manageable pieces and then use a definite integral to add them up.

## Summary

- To measure the work done by a varying force in moving an object, we divide the problem into pieces on which we can use the formula $W = F \cdot d$, and then use a definite integral to sum the work done on each piece.
- To find the total force exerted by water against a dam, we use the formula $F = P \cdot A$ to measure the force exerted on a slice that lies at a fixed depth, and then use a definite integral to sum the forces across the appropriate range of depths.
- Because work is computed as the product of force and distance (provided force is constant), and the force water exerts on a dam can be computed as the product of pressure and area (provided pressure is constant), problems involving these concepts are similar to earlier problems we did using definite integrals to find distance (via "distance equals rate times time") and mass ("mass equals density times volume").

## Answers

**Example** 6.4.1

It is helpful to think of the depth below the surface of the crock as being the independent variable, so we let the positive $x$-axis point down, and the positive $y$-axis to the right, as pictured in the figure. Because the pump sits on the surface of the water, it makes sense to think about the pump moving the water one "slice" at a time, where it takes a thin slice from the surface, pumps it out of the tank, and then proceeds to pump the next slice below.

Each slice of water is cylindrical in shape. We see that the radius of each slice varies according to the linear function $y = f(x)$ that passes through the points $(0, 1.5)$ and $(4, 0.75)$, where $x$ is the depth of the particular slice in the tank; it is a straightforward exercise to find that $f(x) = 1.5 - 0.1875x$. Now we think about the problem in several steps:

1. determining the volume of a typical slice;
2. finding the weight [2] of a typical slice (and thus the force that must be exerted on it);
3. deciding the distance that a typical slice moves;

4. and computing the work to move a representative slice.

Once we know the work it takes to move one slice, we use a definite integral over an appropriate interval to find the total work.

We assume that the weight density of water is 62.4 pounds per cubic foot.

Consider a representative cylindrical slice at a depth of $x$ feet below the top of the crock. The approximate volume of that slice is given by

$$V_{\text{slice}} = \pi f(x)^2 \Delta x = \pi(1.5 - 0.1875x)^2 \Delta x.$$

Since water weighs 62.4 lb/ft$^3$, the approximate weight of a representative slice is

$$F_{\text{slice}} = 62.4 \cdot V_{\text{slice}} = 62.4\pi(1.5 - 0.1875x)^2 \Delta x.$$

This is also the approximate force the pump must exert to move the slice.

Because the slice is located at a depth of $x$ feet below the top of the crock, the slice being moved by the pump must move $x$ feet to get to the level of the basement floor, and then, as stated in the problem description, another 9 feet to reach the drain at ground level. Hence, the total distance a representative slice travels is

$$d_{\text{slice}} = x + 9.$$

Finally, the work to move a representative slice is given by

$$W_{\text{slice}} = F_{\text{slice}} \cdot d_{\text{slice}} = 62.4\pi(1.5 - 0.1875x)^2 \Delta x \cdot (x + 9).$$

We sum the work required to move slices throughout the tank (from $x = 0$ to $x = 4$), let $\Delta x \to 0$, and hence

$$W = \int_0^4 62.4\pi(1.5 - 0.1875x)^2 (x + 9)\, dx.$$

When evaluated using appropriate technology, the integral shows that the total work is $W = 3463.2\pi$ foot-pounds.

**Example** 6.4.2

First, we sketch a picture of the dam, as shown in Figure 6.4.5. Note that, as in problems involving the work to pump out a tank, we let the positive $x$-axis point down.

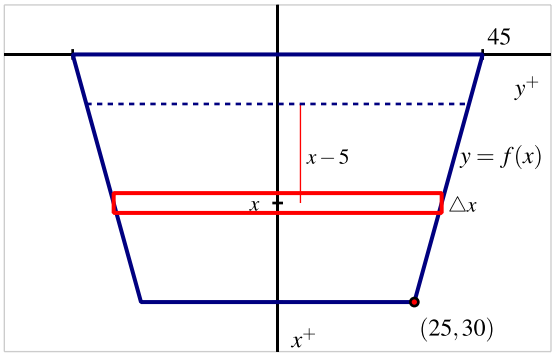



Figure 6.4.5. A trapezoidal dam that is 25 feet tall, 60 feet wide at its base, 90 feet wide at its top, with the water line 5 feet down from the top of its face.

Pressure is constant at a fixed depth, so we consider a slice of water at constant depth on the face, as shown in the figure. The area of this slice is approximately the area of the rectangle pictured. Since the width of that rectangle depends on the variable $x$, we find a formula for the line that represents one side of the dam. It is straightforward to find that $y = 45 - \frac{3}{5}x$. Hence, the approximate area of a representative slice is

$$A_{\text{slice}} = 2f(x)\Delta x = 2(45 - \frac{3}{5}x)\Delta x.$$

At any point on this slice, the depth is approximately constant, so the pressure can be considered constant. Because the water rises to within 5 feet of the top of the dam, the depth of any point on the representative slice is approximately $(x - 5)$. Now, since pressure is given by $P = 62.4d$, we have that at any point on the slice

$$P_{\text{slice}} = 62.4(x - 5).$$

Knowing both the pressure and area, we can find the force the water exerts on the slice. Using $F = PA$, it follows that

$$F_{\text{slice}} = P_{\text{slice}} \cdot A_{\text{slice}} = 62.4(x - 5) \cdot 2(45 - \frac{3}{5}x)\Delta x.$$

Finally, we use a definite integral to sum the forces over the appropriate range of $x$-values. Since the water rises to within 5 feet of the top of the dam, we start at $x = 5$ and take slices all the way to the bottom of the dam, where $x = 25$. Hence,

$$F = \int_{x=5}^{x=25} 62.4(x - 5) \cdot 2(45 - \frac{3}{5}x)\, dx.$$

Using technology to evaluate the integral, we find $F = 848640$ pounds.

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# 6.5: Improper Integrals

## Motivating Questions

- What are improper integrals and why are they important?
- What does it mean to say that an improper integral converges or diverges?
- What are some typical improper integrals that we can classify as convergent or divergent?

Another important application of the definite integral measures the likelihood of certain events. For instance, consider a company that manufactures incandescent light bulbs. Based on a large volume of test results, they have determined that the fraction of light bulbs that fail between times $t = a$ and $t = b$ of use (where $t$ is measured in months) is given by

$$\int_a^b 0.3e^{-0.3t}\,dt.$$

For example, the fraction of light bulbs that fail during their third month of use is given by

$$\begin{aligned}\int_2^3 0.3e^{-0.3t}\,dt &= -e^{-0.3t}\Big|_2^3 \\ &= -e^{-0.9} + e^{-0.6} \\ &\approx 0.1422.\end{aligned}$$

Thus about 14.22% of all lightbulbs fail between $t = 2$ and $t = 3$. Clearly we could adjust the limits of integration to measure the fraction of light bulbs that fail during any time period of interest.

## Preview Activity 6.5.1

A company with a large customer base has a call center that receives thousands of calls a day. After studying the data that represents how long callers wait for assistance, they find that the function $p(t) = 0.25e^{-0.25t}$ models the time customers wait in the following way: the fraction of customers who wait between $t = a$ and $t = b$ minutes is given by

$$\int_a^b p(t)\,dt.$$

Use this information to answer the following questions.

1. Determine the fraction of callers who wait between 5 and 10 minutes.
2. Determine the fraction of callers who wait between 10 and 20 minutes.
3. Next, let's study the fraction who wait up to a certain number of minutes:
   1. What is the fraction of callers who wait between 0 and 5 minutes?
   2. What is the fraction of callers who wait between 0 and 10 minutes?
   3. Between 0 and 15 minutes? Between 0 and 20?
4. Let $F(b)$ represent the fraction of callers who wait between $0$ and $b$ minutes. Find a formula for $F(b)$ that involves a definite integral, and then use the First FTC to find a formula for $F(b)$ that does not involve a definite integral.
5. What is the value of the limit $\lim_{b\to\infty} F(b)$? What is its meaning in the context of the problem?

## Improper Integrals Involving Unbounded Intervals

In view of the above examples, we see that we may want to integrate over an interval whose upper limit grows without bound. For example, to find the fraction of light bulbs that fail *eventually*, we wish to find

$$\lim_{b\to\infty}\int_0^b 0.3e^{-0.3t}\,dt,$$

for which we will also use the notation

$$\int_0^\infty 0.3e^{-0.3t}\,dt. \tag{6.5.1}$$

Such an integral can be interpreted as the area of an unbounded region, as pictured at right in Figure 6.5.1.

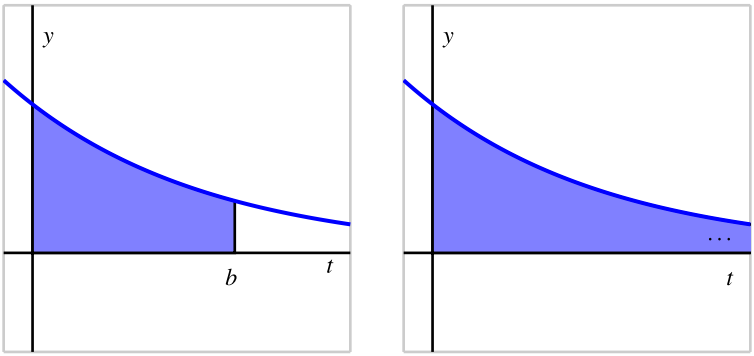



Figure 6.5.1. At left, the area bounded by $p(t) = 0.3e^{-0.3t}$ on the finite interval $[0, b]$; at right, the result of letting $b \to \infty$. By "..." in the righthand figure, we mean that the region extends to the right without bound.

We call an integral for which the interval of integration is unbounded *improper*. For instance, the integrals

$$\int_1^\infty \frac{1}{x^2}\,dx, \quad \int_{-\infty}^0 \frac{1}{1+x^2}\,dx, \quad \text{and} \int_{-\infty}^\infty e^{-x^2}\,dx$$

are all improper because they have limits of integration that involve $\infty$. To evaluate an improper integral we replace it with a limit of proper integrals. That is,

$$\int_0^\infty f(x)\,dx = \lim_{b\to\infty}\int_0^b f(x)\,dx.$$

We first attempt to evaluate $\int_0^b f(x)\ dx$ using the First FTC, and then evaluate the limit. Is it even possible for the area of an unbounded region to be finite? The following activity explores this issue and others in more detail.

## Activity 6.5.2

In this activity we explore the improper integrals $\int_1^\infty \frac{1}{x}\ dx$ and $\int_1^\infty \frac{1}{x^{3/2}}\ dx$.

1. First we investigate $\int_1^\infty \frac{1}{x}\ dx$.
   1. Use the First FTC to determine the exact values of $\int_1^{10} \frac{1}{x}\ dx$, $\int_1^{1000} \frac{1}{x}\ dx$, and $\int_1^{100000} \frac{1}{x}\ dx$. Then, use your computational device to compute a decimal approximation of each result.
   2. Use the First FTC to evaluate the definite integral $\int_1^b \frac{1}{x}\ dx$ (which results in an expression that depends on $b$).
   3. Now, use your work from (ii.) to evaluate the limit given by
   $$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx.$$
2. Next, we investigate $\int_1^\infty \frac{1}{x^{3/2}}\ dx$.
   1. Use the First FTC to determine the exact values of $\int_1^{10} \frac{1}{x^{3/2}}\ dx$, $\int_1^{1000} \frac{1}{x^{3/2}}\ dx$, and $\int_1^{100000} \frac{1}{x^{3/2}}\ dx$. Then, use your calculator to compute a decimal approximation of each result.
   2. Use the First FTC to evaluate the definite integral $\int_1^b \frac{1}{x^{3/2}}\ dx$ (which results in an expression that depends on $b$).

3. Now, use your work from (ii.) to evaluate the limit given by
$$\lim_{b\to\infty}\int_1^b \frac{1}{x^{3/2}}\,dx.$$

3. Plot the functions $y=\frac{1}{x}$ and $y=\frac{1}{x^{3/2}}$ on the same coordinate axes for the values $x=0\ldots 10$. How would you compare their behavior as $x$ increases without bound? What is similar? What is different?
4. How would you characterize the value of $\int_1^\infty \frac{1}{x}\,dx$? of $\int_1^\infty \frac{1}{x^{3/2}}\,dx$? What does this tell us about the respective areas bounded by these two curves for $x\geq 1$?

## Convergence and Divergence

Activity $6.5.2$ suggests that $\lim_{b\to\infty}\int_1^b f(x)\,dx$ is either finite or infinite (or it doesn't exist). With these possibilities in mind, we introduce the following terminology.

**Note**

If $f(x)$ is nonnegative for $x\geq a$, then we say that the improper integral $\int_a^\infty f(x)\,dx$ ***converges*** provided that
$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$
exists and is finite. Otherwise, we say that $\int_a^\infty f(x)\,dx$ ***diverges***.

We will restrict our interest to improper integrals for which the integrand is nonnegative. Also, we require that $\lim_{x\to\infty} f(x)=0$, for if $f$ does not approach $0$ as $x\to\infty$, then it is impossible for $\int_a^\infty f(x)\,dx$ to converge.

**Activity $6.5.3$**

Determine whether each of the following improper integrals converges or diverges. For each integral that converges, find its exact value.

1. $\int_1^\infty \frac{1}{x^2}\,dx$
2. $\int_0^\infty e^{-x/4}\,dx$
3. $\int_2^\infty \frac{9}{(x+5)^{2/3}}\,dx$
4. $\int_4^\infty \frac{3}{(x+2)^{5/4}}\,dx$
5. $\int_0^\infty xe^{-x/4}\,dx$
6. $\int_1^\infty \frac{1}{x^p}\,dx$, where $p$ is a positive real number

## Improper Integrals Involving Unbounded Integrands

An integral is also called improper if the integrand is unbounded on the interval of integration. For example, consider
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Because $f(x)=\frac{1}{\sqrt{x}}$ has a vertical asymptote at $x=0$, $f$ is not continuous on $[0,1]$, and the integral represents the area of the unbounded region shown at right in Figure $6.5.2$.

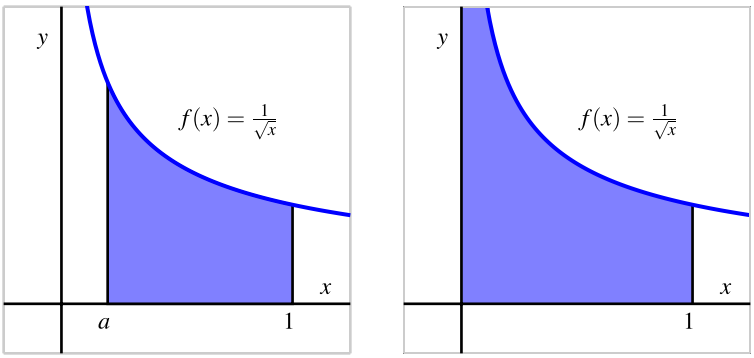



Figure 6.5.2. At left, the area bounded by $f(x) = \frac{1}{\sqrt{x}}$ on the finite interval $[a, 1]$; at right, the result of letting $a \to 0^+$, where we see that the shaded region will extend vertically without bound.

We address the problem of the integrand being unbounded by replacing the improper integral with a limit of proper integrals. For example, to evaluate $\int_0^1 \frac{1}{\sqrt{x}}\ dx$, we replace $0$ with $a$ and let $a$ approach 0 from the right. Thus,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx.$$

We evaluate the proper integral $\int_a^1 \frac{1}{\sqrt{x}}\ dx$, and then take the limit. We will say that the improper integral converges if this limit exists, and diverges otherwise. In this example, we observe that

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx \\ &= \lim_{a\to 0^+} 2\sqrt{x}\,\Big|_a^1 \\ &= \lim_{a\to 0^+} 2\sqrt{1} - 2\sqrt{a} \\ &= 2,\end{aligned}$$

so the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\ dx$ converges (to the value 2).

We have to be particularly careful with unbounded integrands, for they may arise in ways that may not initially be obvious. Consider, for instance, the integral

$$\int_1^3 \frac{1}{(x-2)^2}\,dx.$$

At first glance we might think that we can simply apply the Fundamental Theorem of Calculus by antidifferentiating $\frac{1}{(x-2)^2}$ to get $-\frac{1}{x-2}$ and then evaluating from $1$ to $3$. Were we to do so, we would be erroneously applying the FTC because $f(x) = \frac{1}{(x-2)^2}$ fails to be continuous throughout the interval, as seen in Figure 6.5.3.

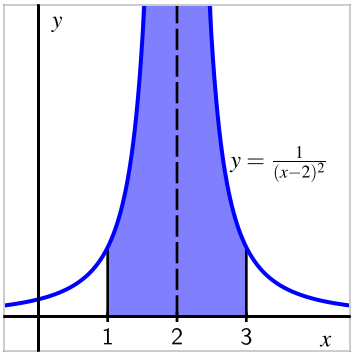



Figure 6.5.3. The function $f(x) = \frac{1}{(x-2)^2}$ on an interval including $x = 2$.

Such an incorrect application of the FTC leads to an impossible result ($-2$), which would itself suggest that something we did must be wrong. Instead, we must address the vertical asymptote at $x = 2$ by writing

$$\int_1^3 \frac{1}{(x-2)^2}\,dx = \lim_{a\to 2^-} \int_1^a \frac{1}{(x-2)^2}\,dx + \lim_{b\to 2^+} \int_b^3 \frac{1}{(x-2)^2}\,dx.$$

We then evaluate two separate limits of proper integrals. For instance, doing so for the integral with $a$ approaching 2 from the left, we find

$$\begin{aligned}
\int_1^2 \frac{1}{(x-2)^2}\,dx &= \lim_{a\to 2^-} \int_1^a \frac{1}{(x-2)^2}\,dx \\
&= \lim_{a\to 2^-} -\frac{1}{(x-2)}\Big|_1^a \\
&= \lim_{a\to 2^-} -\frac{1}{(a-2)} + \frac{1}{1-2} \\
&= \infty,
\end{aligned}$$

since $\frac{1}{a-2} \to -\infty$ as $a$ approaches 2 from the left. Thus, the improper integral $\int_1^2 \frac{1}{(x-2)^2}\,dx$ diverges; similar work shows that $\int_2^3 \frac{1}{(x-2)^2}\,dx$ also diverges. From either of these two results, we can conclude that that the original integral, $\int_1^3 \frac{1}{(x-2)^2}\,dx$ diverges, too.

## Activity 6.5.4

For each of the following definite integrals, decide whether the integral is improper or not. If the integral is proper, evaluate it using the First FTC. If the integral is improper, determine whether or not the integral converges or diverges; if the integral converges, find its exact value.

1. $\int_0^1 \frac{1}{x^{1/3}}\,dx$
2. $\int_0^2 e^{-x}\,dx$
3. $\int_1^4 \frac{1}{\sqrt{4-x}}\,dx$
4. $\int_{-2}^2 \frac{1}{x^2}\,dx$
5. $\int_0^{\pi/2} \tan(x)\,dx$
6. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

## Summary

- An integral $\int_a^b f(x)\ dx$ can be improper if at least one of $a$ or $b$ is $\pm\infty$, making the interval unbounded, or if $f$ has a vertical asymptote at $x = c$ for some value of $c$ that satisfies $a \leq c \leq b$. One reason that improper integrals are important is that certain probabilities can be represented by integrals that involve infinite limits.
- When we encounter an improper integral, we work to understand it by replacing the improper integral with a limit of proper integrals. For instance, we write

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

  and then work to determine whether the limit exists and is finite. For any improper integral, if the resulting limit of proper integrals exists and is finite, we say the improper integral converges. Otherwise, the improper integral diverges.

- An important class of improper integrals is given by

$$\int_1^\infty \frac{1}{x^p}\,dx$$

  where $p$ is a positive real number. We can show that this improper integral converges whenever $p > 1$, and diverges whenever $0 < p \leq 1$. A related class of improper integrals is $\int_0^1 \frac{1}{x^p}\ dx$, which converges for $0 < p < 1$, and diverges for $p \geq 1$.

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# 6.6: Using Definite Integrals (Exercises)

## 6.1: Using Definite Integrals to Find Area and Length

### 1. Area between two power functions

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### 2. Area between two trigonometric functions

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### 3. Area between two curves

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### 4. Arc length of a curve

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### 5

Find the exact area of each described region.

1. The finite region between the curves $x = y(y-2)$ and $x = -(y-1)(y-3)$.
2. The region between the sine and cosine functions on the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$.
3. The finite region between $x = y^2 - y - 2$ and $y = 2x - 1$.
4. The finite region between $y = mx$ and $y = x^2 - 1$, where $m$ is a positive constant.

### 6

Let $f(x) = 1 - x^2$ and $g(x) = ax^2 - a$, where $a$ is an unknown positive real number. For what value(s) of $a$ is the area between the curves $f$ and $g$ equal to 2?

### 7

Let $f(x) = 2 - x^2$. Recall that the average value of any continuous function $f$ on an interval $[a, b]$ is given by $\frac{1}{b-a} \int_a^b f(x)\ dx$.

1. Find the average value of $f(x) = 2 - x^2$ on the interval $[0, \sqrt{2}]$. Call this value $r$.
2. Sketch a graph of $y = f(x)$ and $y = r$. Find their intersection point(s).
3. Show that on the interval $[0, \sqrt{2}]$, the amount of area that lies below $y = f(x)$ and above $y = r$ is equal to the amount of area that lies below $y = r$ and above $y = f(x)$.
4. Will the result of (c) be true for any continuous function and its average value on any interval? Why?

## 6.2 Using Definite Integrals to Find Volume

### 1. Solid of revolution from one function about the $x$-axis

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### 2. Solid of revolution from one function about the $y$-axis

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### 3. Solid of revolution from two functions about the $x$-axis

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## 4. Solid of revolution from two functions about a horizontal line

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## 5. Solid of revolution from two functions about a different horizontal line

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## 6. Solid of revolution from two functions about a vertical line

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## 7

Consider the curve $f(x) = 3\cos(\frac{x^3}{4})$ and the portion of its graph that lies in the first quadrant between the $y$-axis and the first positive value of $x$ for which $f(x) = 0$. Let $R$ denote the region bounded by this portion of $f$, the $x$-axis, and the $y$-axis.

1. Set up a definite integral whose value is the exact arc length of $f$ that lies along the upper boundary of $R$. Use technology appropriately to evaluate the integral you find.
2. Set up a definite integral whose value is the exact area of $R$. Use technology appropriately to evaluate the integral you find.
3. Suppose that the region $R$ is revolved around the $x$-axis. Set up a definite integral whose value is the exact volume of the solid of revolution that is generated. Use technology appropriately to evaluate the integral you find.
4. Suppose instead that $R$ is revolved around the $y$-axis. If possible, set up an integral expression whose value is the exact volume of the solid of revolution and evaluate the integral using appropriate technology. If not possible, explain why.

## 8

Consider the curves given by $y = \sin(x)$ and $y = \cos(x)$. For each of the following problems, you should include a sketch of the region/solid being considered, as well as a labeled representative slice.

1. Sketch the region $R$ bounded by the $y$-axis and the curves $y = \sin(x)$ and $y = \cos(x)$ up to the first positive value of $x$ at which they intersect. What is the exact intersection point of the curves?
2. Set up a definite integral whose value is the exact area of $R$.
3. Set up a definite integral whose value is the exact volume of the solid of revolution generated by revolving $R$ about the $x$-axis.

4. Set up a definite integral whose value is the exact volume of the solid of revolution generated by revolving $R$ about the $y$ -axis.
5. Set up a definite integral whose value is the exact volume of the solid of revolution generated by revolving $R$ about the line $y = 2$.
6. Set up a definite integral whose value is the exact volume of the solid of revolution generated by revolving $R$ about the line $x = -1$.

**9**

Consider the finite region $R$ that is bounded by the curves $y = 1 + \frac{1}{2}(x-2)^2$, $y = \frac{1}{2}x^2$, and $x = 0$.

1. Determine a definite integral whose value is the area of the region enclosed by the two curves.
2. Find an expression involving one or more definite integrals whose value is the volume of the solid of revolution generated by revolving the region $R$ about the line $y = -1$.
3. Determine an expression involving one or more definite integrals whose value is the volume of the solid of revolution generated by revolving the region $R$ about the $y$-axis.
4. Find an expression involving one or more definite integrals whose value is the perimeter of the region $R$.

## 6.3 Density, Mass, and Center of Mass

### 1. Center of mass for a linear density function

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### 2. Center of mass for a nonlinear density function

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### 3. Interpreting the density of cars on a road

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## 4. Center of mass in a point-mass system

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## 5

Let a thin rod of length $a$ have density distribution function $\rho(x) = 10e^{-0.1x}$, where $x$ is measured in cm and $\rho$ in grams per centimeter.

1. If the mass of the rod is 30 g, what is the value of $a$?
2. For the 30g rod, will the center of mass lie at its midpoint, to the left of the midpoint, or to the right of the midpoint? Why?
3. For the 30g rod, find the center of mass, and compare your prediction in (b).
4. At what value of $x$ should the 30g rod be cut in order to form two pieces of equal mass?

## 6

Consider two thin bars of constant cross-sectional area, each of length 10 cm, with respective mass density functions $\rho(x) = \frac{1}{1+x^2}$ and $p(x) = e^{-0.1x}$.

1. Find the mass of each bar.
2. Find the center of mass of each bar.
3. Now consider a new 10 cm bar whose mass density function is $f(x) = \rho(x) + p(x)$.
    1. Explain how you can easily find the mass of this new bar with little to no additional work.
    2. Similarly, compute $\int_0^{10} xf(x)\ dx$ as simply as possible, in light of earlier computations.
    3. True or false: the center of mass of this new bar is the average of the centers of mass of the two earlier bars. Write at least one sentence to say why your conclusion makes sense.

## 7

Consider the curve given by $y = f(x) = 2xe^{-1.25x} + (30 - x)e^{-0.25(30-x)}$.

1. Plot this curve in the window $x = 0 \ldots 30$, $y = 0 \ldots 3$ (with constrained scaling so the units on the $x$ and $y$ axis are equal), and use it to generate a solid of revolution about the $x$-axis. Explain why this curve could generate a reasonable model of a baseball bat.
2. Let $x$ and $y$ be measured in inches. Find the total volume of the baseball bat generated by revolving the given curve about the $x$-axis. Include units on your answer.
3. Suppose that the baseball bat has constant weight density, and that the weight density is $0.6$ ounces per cubic inch. Find the total weight of the bat whose volume you found in (b).
4. Because the baseball bat does not have constant cross-sectional area, we see that the amount of weight concentrated at a location $x$ along the bat is determined by the volume of a slice at location $x$. Explain why we can think about the function $\rho(x) = 0.6\pi f(x)^2$ (where $f$ is the function given at the start of the problem) as being the weight density function for how the weight of the baseball bat is distributed from $x = 0$ to $x = 30$.
5. Compute the center of mass of the baseball bat.

# 6.4 Physics Applications: Work, Force, and Pressure

## Exercises 6.4.5 Exercises

### 1. Work to empty a conical tank

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### 2. Work to empty a cylindrical tank

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### 3. Work to empty a rectangular pool

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### 4. Work to empty a cylindrical tank to differing heights

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### 5. Force due to hydrostatic pressure

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### 6

Consider the curve $f(x) = 3\cos(\frac{x^3}{4})$ and the portion of its graph that lies in the first quadrant between the $y$-axis and the first positive value of $x$ for which $f(x) = 0$. Let $R$ denote the region bounded by this portion of $f$, the $x$-axis, and the $y$-axis. Assume that $x$ and $y$ are each measured in feet.

1. Picture the coordinate axes rotated $90$ degrees clockwise so that the positive $x$-axis points straight down, and the positive $y$-axis points to the right. Suppose that $R$ is rotated about the $x$ axis to form a solid of revolution, and we consider this solid as a storage tank. Suppose that the resulting tank is filled to a depth of $1.5$ feet with water weighing $62.4$ pounds per cubic foot. Find the amount of work required to lower the water in the tank until it is $0.5$ feet deep, by pumping the water to the top of the tank.
2. Again picture the coordinate axes rotated 90 degrees clockwise so that the positive $x$-axis points straight down, and the positive $y$-axis points to the right. Suppose that $R$, together with its reflection across the $x$-axis, forms one end of a storage tank that is 10 feet long. Suppose that the resulting tank is filled completely with water weighing $62.4$ pounds per cubic foot. Find a formula for a function that tells the amount of work required to lower the water by $h$ feet.
3. Suppose that the tank described in (b) is completely filled with water. Find the total force due to hydrostatic pressure exerted by the water on one end of the tank.

### 7

A cylindrical tank, buried on its side, has radius $3$ feet and length $10$ feet. It is filled completely with water whose weight density is $62.4$ lbs/ft$^3$, and the top of the tank is two feet underground.

1. Set up, but do not evaluate, an integral expression that represents the amount of work required to empty the top half of the water in the tank to a truck whose tank lies 4.5 feet above ground.
2. With the tank now only half-full, set up, but do not evaluate an integral expression that represents the total force due to hydrostatic pressure against one end of the tank.

## 6.5: Improper Integrals

### 1. An improper integral on a finite interval

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### 2. An improper integral on an infinite interval

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## 3. An improper integral involving a ratio of exponential functions

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## 4. A subtle improper integral

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## 5. An improper integral involving a ratio of trigonometric functions

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## 6

Determine, with justification, whether each of the following improper integrals converges or diverges.

1. $\int_e^\infty \frac{\ln(x)}{x}\ dx$
2. $\int_e^\infty \frac{1}{x\ln(x)}\ dx$
3. $\int_e^\infty \frac{1}{x(\ln(x))^2}\ dx$
4. $\int_e^\infty \frac{1}{x(\ln(x))^p}\ dx$, where $p$ is a positive real number
5. $\int_0^1 \frac{\ln(x)}{x}\ dx$
6. $\int_0^1 \ln(x)\ dx$

## 7

Sometimes we may encounter an improper integral for which we cannot easily evaluate the limit of the corresponding proper integrals. For instance, consider $\int_1^\infty \frac{1}{1+x^3}\ dx$. While it is hard (or perhaps impossible) to find an antiderivative for $\frac{1}{1+x^3}$, we can still determine whether or not the improper integral converges or diverges by comparison to a simpler one. Observe that for all $x > 0$, $1 + x^3 > x^3$, and therefore

$$\frac{1}{1+x^3} < \frac{1}{x^3}.$$

It therefore follows that

$$\int_1^b \frac{1}{1+x^3}\,dx < \int_1^b \frac{1}{x^3}\,dx$$

for every $b > 1$. If we let $b \to \infty$ so as to consider the two improper integrals $\int_1^\infty \frac{1}{1+x^3}\,dx$ and $\int_1^\infty \frac{1}{x^3}\,dx$, we know that the larger of the two improper integrals converges. And thus, since the smaller one lies below a convergent integral, it follows that the smaller one must converge, too. In particular, $\int_1^\infty \frac{1}{1+x^3}\,dx$ must converge, even though we never explicitly evaluated the corresponding limit of proper integrals. We use this idea and similar ones in the exercises that follow.

1. Explain why $x^2 + x + 1 > x^2$ for all $x \geq 1$, and hence show that $\int_1^\infty \frac{1}{x^2+x+1}\,dx$ converges by comparison to $\int_1^\infty \frac{1}{x^2}\,dx$.
2. Observe that for each $x > 1$, $\ln(x) < x$. Explain why

   $$\int_2^b \frac{1}{x}\,dx < \int_2^b \frac{1}{\ln(x)}\,dx$$

   for each $b > 2$. Why must it be true that $\int_2^b \frac{1}{\ln(x)}\,dx$ diverges?
3. Explain why $\sqrt{\frac{x^4+1}{x^4}} > 1$ for all $x > 1$. Then, determine whether or not the improper integral

   $$\int_1^\infty \frac{1}{x} \cdot \sqrt{\frac{x^4+1}{x^4}}\,dx$$

   converges or diverges.

---

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# 7: ACTIVE CALCULUS - DIFFERENTIAL EQUATIONS


7.1: An Introduction to Differential Equations

7.2: Qualitative Behavior of Solutions to Differential Equations*

7.3: Euler's Method*

7.4: Separable Differential Equations

7.5: Modeling with Differential Equations*

7.6: Population Growth and the Logistic Equation*

7.7: Differential Equations (Exercises)



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# 7.1: An Introduction to Differential Equations

## Motivating Questions

- What is a differential equation and what kinds of information can it tell us?
- How do differential equations arise in the world around us?
- What do we mean by a solution to a differential equation?

In previous chapters, we have seen that a function's derivative tells us the rate at which the function is changing. The Fundamental Theorem of Calculus helped us determine the total change of a function over an interval from the function's rate of change. For instance, an object's velocity tells us the rate of change of that object's position. By integrating the velocity over a time interval, we can determine how much the position changes over that time interval. If we know where the object is at the beginning of that interval, we have enough information to predict where it will be at the end of the interval.

In this chapter, we introduce the concept of *differential equations*. A differential equation is an equation that provides a description of a function's derivative, which means that it tells us the function's rate of change. Using this information, we would like to learn as much as possible about the function itself. Ideally we would like to have an algebraic description of the function. As we'll see, this may be too much to ask in some situations, but we will still be able to make accurate approximations.

## Preview Activity 7.1.1

The position of a moving object is given by the function $s(t)$, where $s$ is measured in feet and $t$ in seconds. We determine that the velocity is $v(t) = 4t + 1$ feet per second.

1. How much does the position change over the time interval $[0, 4]$?
2. Does this give you enough information to determine $s(4)$, the position at time $t = 4$? If so, what is $s(4)$? If not, what additional information would you need to know to determine $s(4)$?
3. Suppose you are told that the object's initial position $s(0) = 7$. Determine $s(2)$, the object's position 2 seconds later.
4. If you are told instead that the object's initial position is $s(0) = 3$, what is $s(2)$?
5. If we only know the velocity $v(t) = 4t + 1$, is it possible that the object's position at all times is $s(t) = 2t^2 + t - 4$? Explain how you know.
6. Are there other possibilities for $s(t)$? If so, what are they?
7. If, in addition to knowing the velocity function is $v(t) = 4t + 1$, we know the initial position $s(0)$, how many possibilities are there for $s(t)$?

## What is a differential equation?

A differential equation is an equation that describes the derivative, or derivatives, of a function that is unknown to us. For instance, the equation

$$\frac{dy}{dx} = x \sin x$$

describes the derivative of a function $y(x)$ that is unknown to us.

As many important examples of differential equations involve quantities that change in time, the independent variable in our discussion will frequently be time $t$. In the preview activity, we considered the differential equation

$$\frac{ds}{dt} = 4t + 1.$$

Knowing the velocity and the starting position of a moving object, we were able to find its position at any later time.

Because differential equations describe the derivative of a function, they give us information about how that function changes. Our goal will be to use this information to predict the value of the function in the future; in this way, differential equations provide us with something like a crystal ball.

Differential equations arise frequently in our every day world. For instance, you may hear a bank advertising:

> *Your money will grow at a 3% annual interest rate with us.*

This innocuous statement is really a differential equation. Let's translate: $A(t)$ will be amount of money you have in your account at time $t$. The rate at which your money grows is the derivative $dA/dt$, and we are told that this rate is $0.03A$. This leads to the differential equation

$$\frac{dA}{dt} = 0.03A.$$

This differential equation has a slightly different feel than the previous equation $\frac{ds}{dt} = 4t + 1$. In the earlier example, the rate of change depends only on the independent variable $t$, and we may find $s(t)$ by integrating the velocity $4t + 1$. In the banking example, however, the rate of change depends on the dependent variable $A$, so we'll need some new techniques in order to find $A(t)$.

### Activity 7.1.2

Express the following statements as differential equations. In each case, you will need to introduce notation to describe the important quantities in the statement so be sure to clearly state what your notation means.

1. The population of a town grows continuously at an annual rate of 1.25%.
2. A radioactive sample loses mass at a rate of 5.6% of its mass every day.
3. You have a bank account that continuously earns 4% interest every year. At the same time, you withdraw money continually from the account at the rate of \$1000 per year.
4. A cup of hot chocolate is sitting in a 70° room. The temperature of the hot chocolate cools continuously by 10% of the difference between the hot chocolate's temperature and the room temperature every minute.
5. A can of cold soda is sitting in a 70° room. The temperature of the soda warms continuously at the rate of 10% of the difference between the soda's temperature and the room's temperature every minute.

## Differential equations in the world around us

Differential equations give a natural way to describe phenomena we see in the real world. For instance, physical principles are frequently expressed as a description of how a quantity changes. A good example is Newton's Second Law, which says:

> *The product of an object's mass and acceleration equals the force applied to it.*

For instance, when gravity acts on an object near the earth's surface, it exerts a force equal to $mg$, the mass of the object times the gravitational constant $g$. We therefore have

$$ma = mg, \text{ or}$$
$$\frac{dv}{dt} = g,$$

where $v$ is the velocity of the object, and $g = 9.8$ meters per second squared. Notice that this physical principle does not tell us what the object's velocity is, but rather how the object's velocity changes.

### Activity 7.1.3

Shown below are two graphs depicting the velocity of falling objects. On the left is the velocity of a skydiver, while on the right is the velocity of a meteorite entering the Earth's atmosphere.

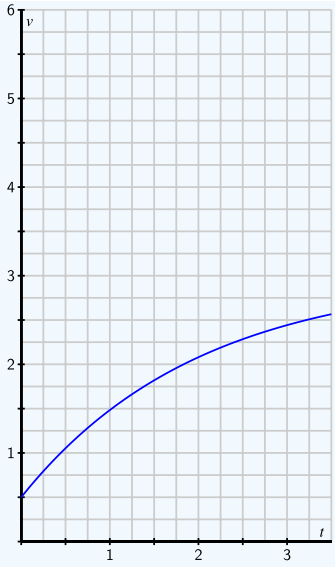



Figure 7.1.1: A skydiver's velocity.




Figure 7.1.2: A meteorite's velocity.

a. Begin with the skydiver's velocity and use the given graph to measure the rate of change $dv/dt$ when the velocity is $v = 0.5, 1.0, 1.5, 2.0,$ and $2.5$. Plot your values on the graph below. You will want to think carefully about this: you are plotting the derivative $dv/dt$ as a function of *velocity*.

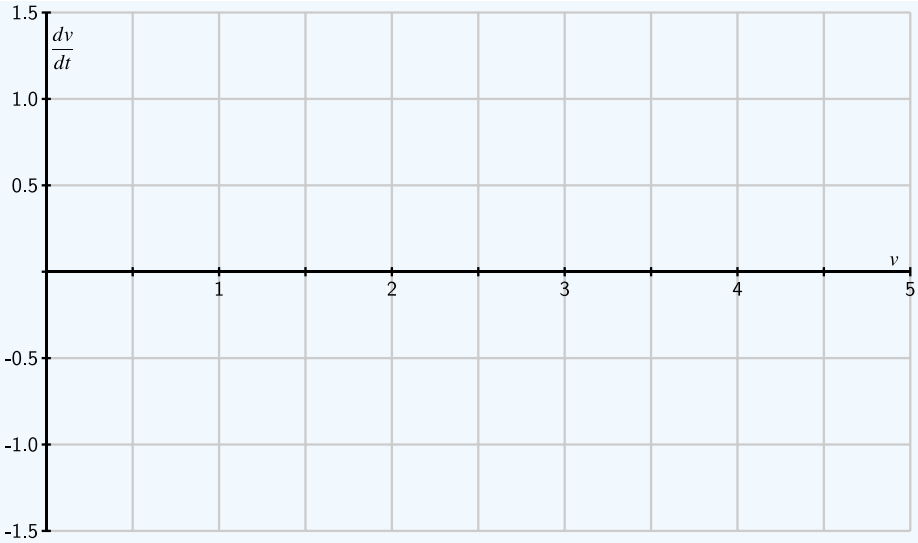



Figure 7.1.3: Plot the rates of change for parts (a) and (b) here.

b. Now do the same thing with the meteorite's velocity: use the given graph to measure the rate of change $dv/dt$ when the velocity is $v = 3.5, 4.0, 4.5,$ and $5.0$. Plot your values on the graph above.

c. You should find that all your points lie on a line. Write the equation of this line being careful to use proper notation for the quantities on the horizontal and vertical axes.

d. The relationship you just found is a differential equation. Write a complete sentence that explains its meaning.

e. By looking at the differential equation, determine the values of the velocity for which the velocity increases.

f. By looking at the differential equation, determine the values of the velocity for which the velocity decreases.

g. By looking at the differential equation, determine the values of the velocity for which the velocity remains constant.

The point of this activity is to demonstrate how differential equations model processes in the real world. In this example, two factors influence the velocities: gravity and wind resistance. The differential equation describes how these factors influence the rate of change of the velocities.

## Solving a differential equation

A differential equation describes the derivative, or derivatives, of a function that is unknown to us. By a *solution* to a differential equation, we mean simply a function that satisies this description.

For instance, the first differential equation we looked at is

$$\frac{ds}{dt} = 4t + 1,$$

which describes an unknown function $s(t)$. We may check that $s(t) = 2t^2 + t$ is a solution because it satisfies this description. Notice that $s(t) = 2t^2 + t + 4$ is also a solution.

If we have a candidate for a solution, it is straightforward to check whether it is a solution or not. Before we demonstrate, however, let's consider the same issue in a simpler context. Suppose we are given the equation $2x^2 - 2x = 2x + 6$ and asked whether $x = 3$ is a solution. To answer this question, we could rewrite the variable $x$ in the equation with the symbol $\square$:

$$2\square^2 - 2\square = 2\square + 6.$$

To determine whether $x = 3$ is a solution, we can investigate the value of each side of the equation separately when the value $3$ is placed in $\square$ and see if indeed the two resulting values are equal. Doing so, we observe that

$$2\square^2 - 2\square = 2 \cdot 3^2 - 2 \cdot 3 = 12,$$

and

$$2\square + 6 = 2 \cdot 3 + 6 = 12.$$

Therefore, $x = 3$ is indeed a solution.

We will do the same thing with differential equations. Consider the differential equation

$$\frac{dv}{dt} = 1.5 - 0.5v, \text{ or}$$
$$\frac{d\square}{dt} = 1.5 - 0.5\square.$$

Let's ask whether $v(t) = 3 - 2e^{-0.5t}$ is a solution [1]. Using this formula for $v$, observe first that

$$\frac{dv}{dt} = \frac{d\square}{dt} = \frac{d}{dt}[3 - 2e^{-0.5t}] = -2e^{-0.5t} \cdot (-0.5) = e^{-0.5t}$$

and

$$1.5 - 0.5v = 1.5 - 0.5\square = 1.5 - 0.5(3 - 2e^{-0.5t}) = 1.5 - 1.5 + e^{-0.5t} = e^{-0.5t}.$$

At this time, don't worry about why we chose this function; we will learn techniques for finding solutions to differential equations soon enough.

Since $\frac{dv}{dt}$ and $1.5 - 0.5v$ agree for all values of $t$ when $v = 3 - 2e^{-0.5t}$, we have indeed found a solution to the differential equation.

## Activity 7.1.4

Consider the differential equation

$$\frac{dv}{dt} = 1.5 - 0.5v.$$

Which of the following functions are solutions of this differential equation?

1. $v(t) = 1.5t - 0.25t^2$.
2. $v(t) = 3 + 2e^{-0.5t}$.
3. $v(t) = 3$.
4. $v(t) = 3 + Ce^{-0.5t}$ where $C$ is any constant.

This activity shows us something interesting. Notice that the differential equation has infinitely many solutions, which are parametrized by the constant $C$ in $v(t) = 3 + Ce^{-0.5t}$. In Figure 7.1.4, we see the graphs of these solutions for a few values of $C$, as labeled.

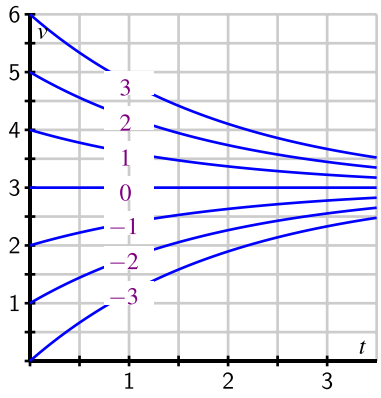



Figure 7.1.4. The family of solutions to the differential equation $\frac{dv}{dt} = 1.5 - 0.5v$.

Notice that the value of $C$ is connected to the initial value of the velocity $v(0)$, since $v(0) = 3 + C$. In other words, while the differential equation describes how the velocity changes as a function of the velocity itself, this is not enough information to determine the velocity uniquely: we also need to know the initial velocity. For this reason, differential equations will typically have infinitely many solutions, one corresponding to each initial value. We have seen this phenomenon before: given the velocity of a moving object $v(t)$, we cannot uniquely determine the object's position function unless we also know its initial position.

If we are given a differential equation and an initial value for the unknown function, we say that we have an *initial value problem*. For instance,

$$\frac{dv}{dt} = 1.5 - 0.5v, \ v(0) = 0.5$$

is an initial value problem. In this problem, we know the value of $v$ at one time and we know how $v$ is changing. Consequently, there should be exactly one function $v$ that satisfies the initial value problem.

This demonstrates the following important general property of initial value problems.

**Note**

Initial value problems that are “well behaved” have exactly one solution, which exists in some interval around the initial point.

We won't worry about what “well behaved” means—it is a technical condition that will be satisfied by all the differential equations we consider.

To close this section, we note that differential equations may be classified based on certain characteristics they may possess. You may see many different types of differential equations in a later course in differential equations. For now, we would like to introduce a few terms that are used to describe differential equations.

A *first-order* differential equation is one in which only the first derivative of the function occurs. For this reason,

$$\frac{dv}{dt} = 1.5 - 0.5v$$

is a first-order equation while

$$\frac{d^2y}{dt^2} = -10y$$

is a second-order equation.

A differential equation is *autonomous* if the independent variable does not appear in the description of the derivative. For instance,

$$\frac{dv}{dt} = 1.5 - 0.5v$$

is autonomous because the description of the derivative $dv/dt$ does not depend on time. The equation

$$\frac{dy}{dt} = 1.5t - 0.5y,$$

however, is not autonomous.

## Summary

- A differential equation is simply an equation that describes the derivative(s) of an unknown function.
- Physical principles, as well as some everyday situations, often describe how a quantity changes, which lead to differential equations.
- A solution to a differential equation is a function whose derivatives satisfy the equation's description. Differential equations typically have infinitely many solutions, parametrized by the initial values.

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# 7.2: Qualitative Behavior of Solutions to Differential Equations*

## Motivating Questions

- What is a slope field?
- How can we use a slope field to obtain qualitative information about the solutions of a differential equation?
- What are stable and unstable equilibrium solutions of an autonomous differential equation?

In earlier work, we have used the tangent line to the graph of a function $f$ at a point $a$ to approximate the values of $f$ near $a$. The usefulness of this approximation is that we need to know very little about the function; armed with only the value $f(a)$ and the derivative $f'(a)$, we may find the equation of the tangent line and the approximation

$$f(x) \approx f(a) + f'(a)(x - a).$$

Remember that a first-order differential equation gives us information about the derivative of an unknown function. Since the derivative at a point tells us the slope of the tangent line at this point, a differential equation gives us crucial information about the tangent lines to the graph of a solution. We will use this information about the tangent lines to create a *slope field* for the differential equation, which enables us to sketch solutions to initial value problems. Our aim will be to understand the solutions qualitatively. That is, we would like to understand the basic nature of solutions, such as their long-range behavior, without precisely determining the value of a solution at a particular point.

## Preview Activity 7.2.1

Let's consider the initial value problem

$$\frac{dy}{dt} = t - 2, \quad y(0) = 1.$$

a. Use the differential equation to find the slope of the tangent line to the solution $y(t)$ at $t = 0$. Then use the initial value to find the equation of the tangent line at $t = 0$. Sketch this tangent line over the interval $-0.25 \leq t \leq 0.25$ on the axes provided in Figure 7.2.1.

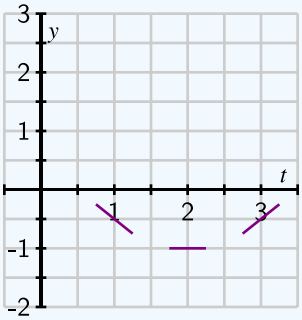



Figure 7.2.1. Grid for plotting partial tangent lines.

b. Also shown in Figure 7.2.1 are the tangent lines to the solution $y(t)$ at the points $t = 1, 2$, and $3$ (we will see how to find these later). Use the graph to measure the slope of each tangent line and verify that each agrees with the value specified by the differential equation.

c. Using these tangent lines as a guide, sketch a graph of the solution $y(t)$ over the interval $0 \leq t \leq 3$ so that the lines are tangent to the graph of $y(t)$.

d. Use the Fundamental Theorem of Calculus to find $y(t)$, the solution to this initial value problem.

e. Graph the solution you found in (d) on the axes provided, and compare it to the sketch you made using the tangent lines.

## Slope fields

Preview Activity $7.2.1$ shows that we can sketch the solution to an initial value problem if we know an appropriate collection of tangent lines. We can use the differential equation to find the slope of the tangent line at any point of interest, and hence plot such a collection.

Let's continue looking at the differential equation $\frac{dy}{dt} = t - 2$. If $t = 0$, this equation says that $dy/dt = 0 - 2 = -2$. Note that this value holds regardless of the value of $y$. We will therefore sketch tangent lines for several values of $y$ and $t = 0$ with a slope of $-2$, as shown in Figure $7.2.2$.

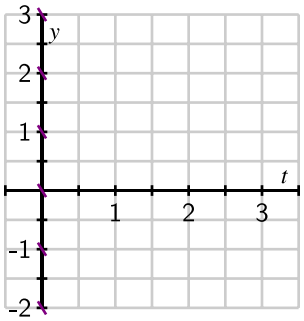



Figure $7.2.2$: Tangent lines at points with $t = 0$.

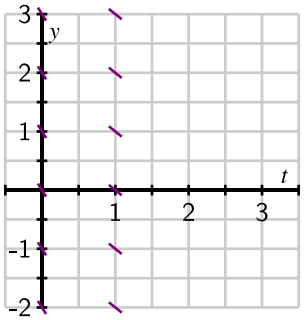



Figure $7.2.3$: Adding tangent lines at points with $t = 1$.

Let's continue in the same way: if $t = 1$, the differential equation tells us that $dy/dt = 1 - 2 = -1$, and this holds regardless of the value of $y$. We now sketch tangent lines for several values of $y$ and $t = 1$ with a slope of $-1$ in Figure $7.2.3$.

Similarly, we see that when $t = 2$, $dy/dt = 0$ and when $t = 3$, $dy/dt = 1$. We may therefore add to our growing collection of tangent line plots to achieve Figure $7.2.4$.

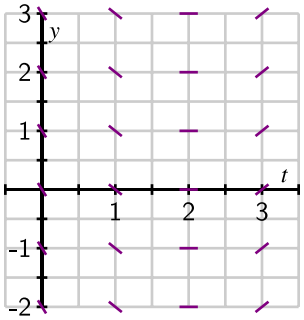



Figure $7.2.4$: Adding tangent lines at points with $t = 2$ and $t = 3$.

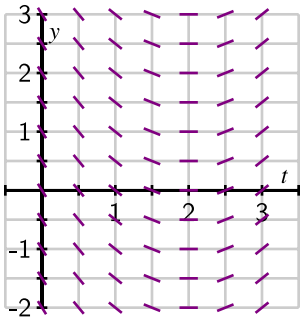



Figure 7.2.5: A completed slope field.

In Figure 7.2.4, we begin to see the solutions to the differential equation emerge. For the sake of even greater clarity, we add more tangent lines to provide the more complete picture shown at right in Figure 7.2.5.

Figure 7.2.5 is called a *slope field* for the differential equation. It allows us to sketch solutions just as we did in the preview activity. We can begin with the initial value $y(0) = 1$ and start sketching the solution by following the tangent line. Whenever the solution passes through a point at which a tangent line is drawn, that line is tangent to the solution. This principle leads us to the sequence of images in Figure 7.2.6.

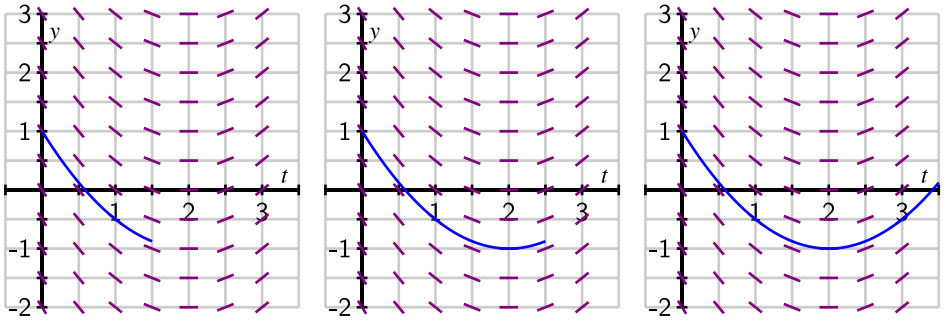



Figure 7.2.6. A sequence of images that show how to sketch the IVP solution that satisfies $y(0) = 1$.

In fact, we can draw solutions for any initial value. Figure 7.2.7 shows solutions for several different initial values for $y(0)$.

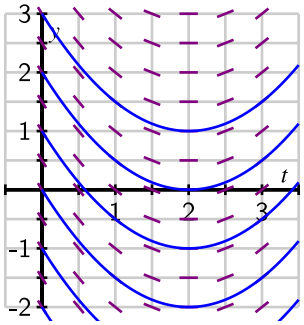



Figure 7.2.7. Different solutions to $\frac{dy}{dt} = t - 2$ that correspond to different initial values.

Just as we did for the equation $\frac{dy}{dt} = t - 2$, we can construct a slope field for any differential equation of interest. The slope field provides us with visual information about how we expect solutions to the differential equation to behave.

## Activity 7.2.2

Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y-4).$$

1. Make a plot of $\frac{dy}{dt}$ versus $y$ on the axes provided in Figure 7.2.8. Looking at the graph, for what values of $y$ does $y$ increase and for what values of $y$ does $y$ decrease?

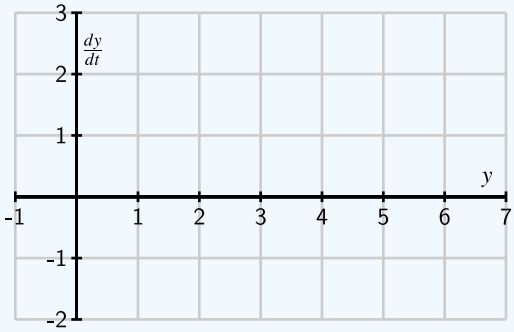



Figure 7.2.8: Axes for plotting $\frac{dy}{dt}$ versus $y$.

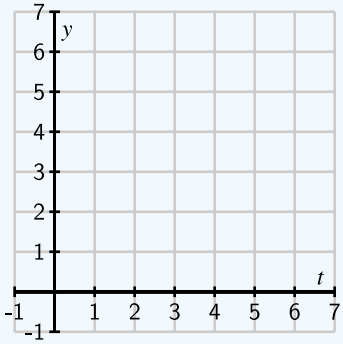



Figure 7.2.9: Axes for plotting the slope field for $\frac{dy}{dt} = -\frac{1}{2}(y-4)$.

2. Next, sketch the slope field for this differential equation on the axes provided in Figure 7.2.9.
3. Use your work in (b) to sketch (on the same axes in Figure 7.2.19.) solutions that satisfy $y(0) = 0$, $y(0) = 2$, $y(0) = 4$ and $y(0) = 6$.
4. Verify that $y(t) = 4 + 2e^{-t/2}$ is a solution to the given differential equation with the initial value $y(0) = 6$. Compare its graph to the one you sketched in (c).
5. What is special about the solution where $y(0) = 4$?

## Equilibrium solutions and stability

As our work in Activity 7.2.2 demonstrates, first-order autonomous equations may have solutions that are constant. These are simple to detect by inspecting the differential equation $dy/dt = f(y)$: constant solutions necessarily have a zero derivative, so $dy/dt = 0 = f(y)$.

For example, in Activity 7.2.2, we considered the equation $\frac{dy}{dt} = f(y) = -\frac{1}{2}(y-4)$. Constant solutions are found by setting $f(y) = -\frac{1}{2}(y-4) = 0$, which we immediately see implies that $y = 4$.

Values of $y$ for which $f(y) = 0$ in an autonomous differential equation $\frac{dy}{dt} = f(y)$ are called *equilibrium solutions* of the differential equation.

### Activity 7.2.3

Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{2}y(y-4).$$

1. Make a plot of $\frac{dy}{dt}$ versus $y$ on the axes provided in Figure 7.2.10. Looking at the graph, for what values of $y$ does $y$ increase and for what values of $y$ does $y$ decrease?

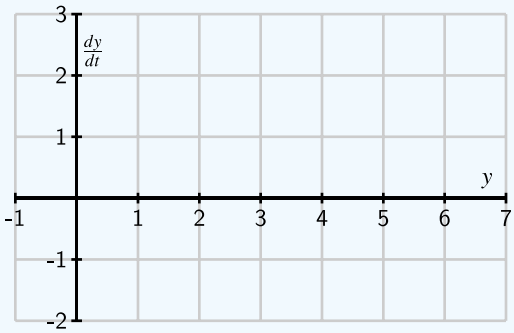



Figure 7.2.10: Axes for plotting $dy/dt$ vs $y$ for $\frac{dy}{dt} = -\frac{1}{2}y(y-4)$.

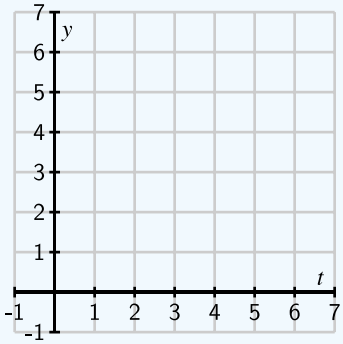



Figure 7.2.11: Axes for plotting the slope field for $\frac{dy}{dt} = -\frac{1}{2}y(y-4)$.

2. Identify any equilibrium solutions of the given differential equation.
3. Now sketch the slope field for the given differential equation on the axes provided in Figure 7.2.11.
4. Sketch the solutions to the given differential equation that correspond to initial values $y(0) = -1, 0, 1, \ldots, 5$.
5. An equilibrium solution $\overline{y}$ is called *stable* if nearby solutions converge to $\overline{y}$. This means that if the initial condition varies slightly from $\overline{y}$, then $\lim_{t\to\infty} y(t) = \overline{y}$. Conversely, an equilibrium solution $\overline{y}$ is called *unstable* if nearby solutions are pushed away from $\overline{y}$. Using your work above, classify the equilibrium solutions you found in (b) as either stable or unstable.
6. Suppose that $y(t)$ describes the population of a species of living organisms and that the initial value $y(0)$ is positive. What can you say about the eventual fate of this population?
7. Now consider a general autonomous differential equation of the form $dy/dt = f(y)$. Remember that an equilibrium solution $\overline{y}$ satisfies $f(\overline{y}) = 0$. If we graph $dy/dt = f(y)$ as a function of $y$, for which of the differential equations represented in Figure 7.2.12 and Figure 7.2.13 is $\overline{y}$ a stable equilibrium and for which is $\overline{y}$ unstable? Why?

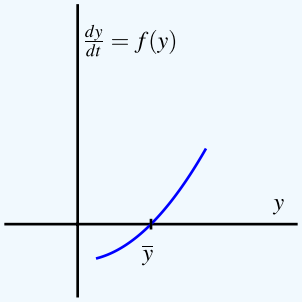



Figure 7.2.12: Plot of $\frac{dy}{dt}$ as a function of $y$.

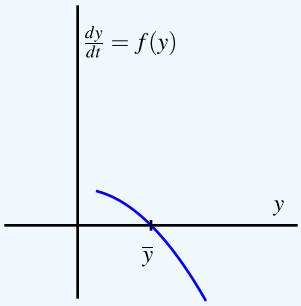



Figure 7.2.13: Plot of $\frac{dy}{dt}$ as a different function of $y$.

## Summary

- A slope field is a plot created by graphing the tangent lines of many different solutions to a differential equation.
- Once we have a slope field, we may sketch the graph of solutions by drawing a curve that is always tangent to the lines in the slope field.
- Autonomous differential equations sometimes have constant solutions that we call equilibrium solutions. These may be classified as stable or unstable, depending on the behavior of nearby solutions.

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# 7.3: Euler's Method*

## Motivating Questions

- What is Euler's method and how can we use it to approximate the solution to an initial value problem?
- How accurate is Euler's method?

In Section 7.2, we saw how a slope field can be used to sketch solutions to a differential equation. In particular, the slope field is a plot of a large collection of tangent lines to a large number of solutions of the differential equation, and we sketch a single solution by simply following these tangent lines. With a little more thought, we can use this same idea to approximate numerically the solutions of a differential equation.

## Preview Activity 7.3.1

Consider the initial value problem

$$\frac{dy}{dt} = \frac{1}{2}(y+1),\ y(0) = 0.$$

a. Use the differential equation to find the slope of the tangent line to the solution $y(t)$ at $t = 0$. Then use the given initial value to find the equation of the tangent line at $t = 0$.

b. Sketch the tangent line on the axes provided in Figure 7.3.1 on the interval $0 \le t \le 2$ and use it to approximate $y(2)$, the value of the solution at $t = 2$.

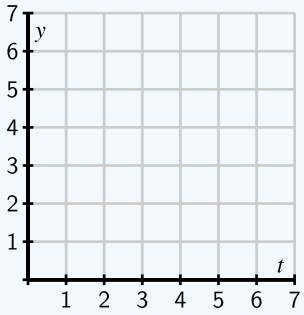



Figure 7.3.1. Grid for plotting the tangent line.

c. Assuming that your approximation for $y(2)$ is the actual value of $y(2)$, use the differential equation to find the slope of the tangent line to $y(t)$ at $t = 2$. Then, write the equation of the tangent line at $t = 2$.

d. Add a sketch of this tangent line on the interval $2 \le t \le 4$ to your plot Figure 7.3.1; use this new tangent line to approximate $y(4)$, the value of the solution at $t = 4$.

e. Repeat the same step to find an approximation for $y(6)$.

## Euler's Method

Preview Activity 7.3.1 demonstrates an algorithm known as Euler's[1] Method, which generates a numerical approximation to the solution of an initial value problem. In this algorithm, we will approximate the solution by taking horizontal steps of a fixed size that we denote by $\Delta t$.

"Euler" is pronounced "Oy-ler." Among other things, Euler is the mathematician credited with the famous number $e$; if you incorrectly pronounce his name "You-ler," you fail to appreciate his genius and legacy.

Before explaining the algorithm in detail, let's remember how we compute the slope of a line: the slope is the ratio of the vertical change to the horizontal change, as shown in Figure 7.3.2.

In other words, $m = \frac{\Delta y}{\Delta t}$. Solving for $\Delta y$, we see that the vertical change is the product of the slope and the horizontal change, or

$$\Delta y = m\Delta t.$$

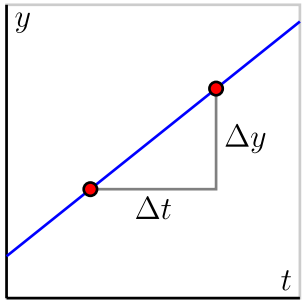


Figure 7.3.2. The role of slope in Euler's Method.

Now, suppose that we would like to solve the initial value problem

$$\frac{dy}{dt} = t - y,\ y(0) = 1.$$

There is an algorithm by which we can find an algebraic formula for the solution to this initial value problem, and we can check that this solution is $y(t) = t - 1 + 2e^{-t}$. But we are instead interested in generating an approximate solution by creating a sequence of points $(t_i, y_i)$, where $y_i \approx y(t_i)$. For this first example, we choose $\Delta t = 0.2$.

Since we know that $y(0) = 1$, we will take the initial point to be $(t_0, y_0) = (0, 1)$ and move horizontally by $\Delta t = 0.2$ to the point $(t_1, y_1)$. Thus, $t_1 = t_0 + \Delta t = 0.2$. Now, the differential equation tells us that the slope of the tangent line at this point is

$$m = \left.\frac{dy}{dt}\right|_{(0,1)} = 0 - 1 = -1,$$

so to move along the tangent line by taking a horizontal step of size $\Delta t = 0.2$, we must also move vertically by

$$\Delta y = m\Delta t = -1 \cdot 0.2 = -0.2.$$

We then have the approximation $y(0.2) \approx y_1 = y_0 + \Delta y = 1 - 0.2 = 0.8$. At this point, we have executed one step of Euler's method, as seen graphically in Figure 7.3.3.

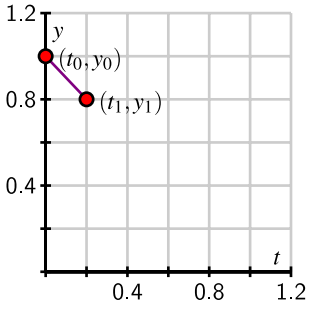


Figure 7.3.3. One step of Euler's method.

Now we repeat this process: at $(t_1, y_1) = (0.2, 0.8)$, the differential equation tells us that the slope is

$$m = \left.\frac{dy}{dt}\right|_{(0.2,0.8)} = 0.2 - 0.8 = -0.6.$$

If we move forward horizontally by $\Delta t$ to $t_2 = t_1 + \Delta = 0.4$, we must move vertically by

$$\Delta y = -0.6 \cdot 0.2 = -0.12.$$

We consequently arrive at $y_2 = y_1 + \Delta y = 0.8 - 0.12 = 0.68$, which gives $y(0.2) \approx 0.68$. Now we have completed the second step of Euler's method, as shown in Figure 7.3.4.

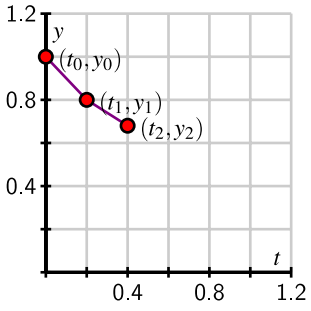


Figure 7.3.4. Two steps of Euler's method.

If we continue in this way, we may generate the points $(t_i, y_i)$ shown in Figure 7.3.5. Because we can find a formula for the actual solution $y(t)$ to this differential equation, we can graph $y(t)$ and compare it to the points generated by Euler's method, as shown in Figure 7.3.6.

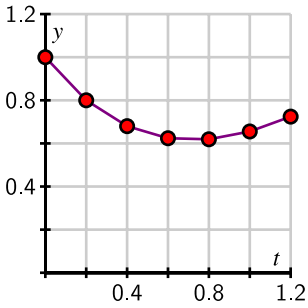


Figure 7.3.5: The points and piecewise linear approximate solution generated by Euler's method.

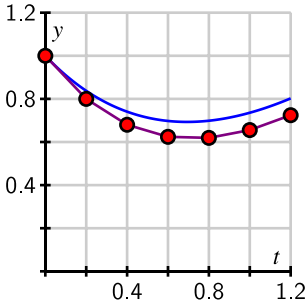


Figure 7.3.6: The approximate solution compared to the exact solution (shown in blue).

Because we need to generate a large number of points $(t_i, y_i)$, it is convenient to organize the implementation of Euler's method in a table as shown. We begin with the given initial data.

| $t_i$ | $y_i$ | $dy/dt$ | $\Delta y$ |
|---|---|---|---|
| 0.0000 | 1.0000 | | |

From here, we compute the slope of the tangent line $m = dy/dt$ using the formula for $dy/dt$ from the differential equation, and then we find $\Delta y$, the change in $y$, using the rule $\Delta y = m\Delta t$.

| $t_i$ | $y_i$ | $dy/dt$ | $\Delta y$ |
|---|---|---|---|
| 0.0000 | 1.0000 | $-1.0000$ | $-0.2000$ |

Next, we increase $t_i$ by $\Delta t$ and $y_i$ by $\Delta y$ to get

| $t_i$ | $y_i$ | $dy/dt$ | $\Delta y$ |
|---|---|---|---|
| 0.0000 | 1.0000 | $-1.0000$ | $-0.2000$ |
| 0.2000 | 0.8000 | | |

We continue the process for however many steps we decide, eventually generating a table like Table 7.3.1.

*Table* 7.3.1. *Euler's method for 6 steps with* $\Delta t = 0.2$.

| $t_i$ | $y_i$ | $dy/dt$ | $\Delta y$ |
|---|---|---|---|
| 0.0000 | 1.0000 | $-1.0000$ | $-0.2000$ |
| 0.2000 | 0.8000 | $-0.6000$ | $-0.1200$ |
| 0.4000 | 0.6800 | $-0.2800$ | $-0.0560$ |
| 0.6000 | 0.6240 | $-0.0240$ | $-0.0048$ |
| 0.8000 | 0.6192 | 0.1808 | 0.0362 |
| 1.0000 | 0.6554 | 0.3446 | 0.0689 |
| 1.2000 | 0.7243 | 0.4757 | 0.0951 |

## Activity 7.3.2

Consider the initial value problem

$$\frac{dy}{dt} = 2t - 1, \; y(0) = 0$$

1. Use Euler's method with $\Delta t = 0.2$ to approximate the solution at $t_i = 0.2, 0.4, 0.6, 0.8,$ and $1.0$. Record your work in the following table, and sketch the points $(t_i, y_i)$ on the axes provided.

   *Table* 7.3.2. *Table for recording results of Euler's method.*

   | $t_i$ | $y_i$ | $dy/dt$ | $\Delta y$ |
   |---|---|---|---|
   | 0.0000 | 0.0000 | | |
   | 0.2000 | | | |
   | 0.4000 | | | |
   | 0.6000 | | | |
   | 0.8000 | | | |
   | 1.0000 | | | |

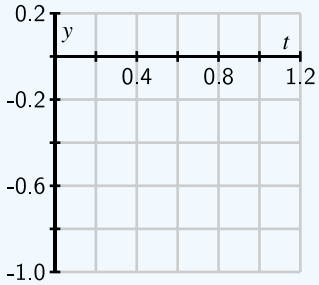



Figure 7.3.7. Grid for plotting points generated by Euler's method.

2. Find the exact solution to the original initial value problem and use this function to find the error in your approximation at each one of the points $t_i$.
3. Explain why the value $y_5$ generated by Euler's method for this initial value problem produces the same value as a left Riemann sum for the definite integral $\int_0^1 (2t-1)\,dt$.
4. How would your computations differ if the initial value was $y(0) = 1$? What does this mean about different solutions to this differential equation?

## Activity 7.3.3

Consider the differential equation $\frac{dy}{dt} = 6y - y^2$.

1. Sketch the slope field for this differential equation on the axes provided in Figure 7.3.8.

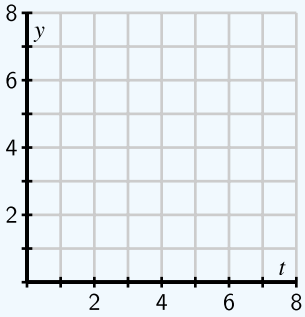



Figure 7.3.8. Grid for plotting the slope field of the given differential equation.

2. Identify any equilibrium solutions and determine whether they are stable or unstable.
3. What is the long-term behavior of the solution that satisfies the initial value $y(0) = 1$?
4. Using the initial value $y(0) = 1$, use Euler's method with $\Delta t = 0.2$ to approximate the solution at $t_i = 0.2, 0.4, 0.6, 0.8$, and $1.0$. Record your results in Table 7.3.3 and sketch the corresponding points $(t_i, y_i)$ on the axes provided in Figure 7.3.9. Note the different horizontal scale on the axes in Figure 7.3.9 compared to Figure 7.3.8.

*Table 7.3.2. Table for recording results of Euler's method with $\Delta t = 0.2$.*

| $t_i$ | $y_i$ | $dy/dt$ | $\Delta y$ |
|---|---|---|---|
| 0.0 | 1.0000 | | |
| 0.2 | | | |
| 0.4 | | | |
| 0.6 | | | |
| 0.8 | | | |

| $t_i$ | $y_i$ | $dy/dt$ | $\Delta y$ |
|---|---|---|---|
| 1.0 | | | |

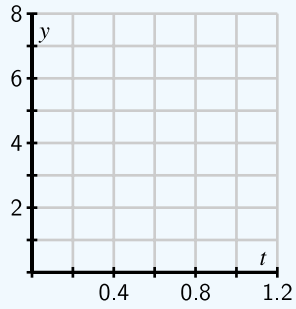



Figure 7.3.9. Axes for plotting the results of Euler's method.

5. What happens if we apply Euler's method to approximate the solution with $y(0) = 6$?

## The error in Euler's method

Since we are approximating the solutions to an initial value problem using tangent lines, we should expect that the error in the approximation will be smaller when the step size is smaller. Consider the initial value problem

$$\frac{dy}{dt} = y,\ y(0) = 1,$$

whose solution we can easily find.

The question posed by this initial value problem is "what function do we know that is the same as its own derivative and has value 1 when $t = 0$?" It is not hard to see that the solution is $y(t) = e^t$. We now apply Euler's method to approximate $y(1) = e$ using several values of $\Delta t$. These approximations will be denoted by $E_{\Delta t}$, and we'll use them to see how accurate Euler's Method is.

To begin, we apply Euler's method with a step size of $\Delta t = 0.2$. In that case, we find that $y(1) \approx E_{0.2} = 2.4883$. The error is therefore

$$y(1) - E_{0.2} = e - 2.4883 \approx 0.2300.$$

Repeatedly halving $\Delta t$ gives the following results, expressed in both tabular and graphical form.

*Table 7.3.3. Errors that correspond to different $\Delta t$ values.*

| $\Delta t$ | $E_{\Delta t}$ | Error |
|---|---|---|
| 0.200 | 2.4883 | 0.2300 |
| 0.100 | 2.5937 | 0.1245 |
| 0.050 | 2.6533 | 0.0650 |
| 0.025 | 2.6851 | 0.0332 |

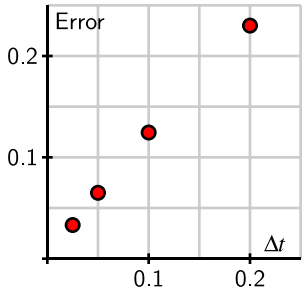



Figure 7.3.10. A plot of the error as a function of $\Delta t$.

Notice, both numerically and graphically, that the error is roughly halved when $\Delta t$ is halved. This example illustrates the following general principle.

If Euler's method is used to approximate the solution to an initial value problem at a point $\bar{t}$, then the error is proportional to $\Delta t$. That is,

$$y(\bar{t}) - E_{\Delta t} \approx K\Delta t$$

for some constant of proportionality $K$.

## Summary

- Euler's method is an algorithm for approximating the solution to an initial value problem by following the tangent lines while we take horizontal steps across the $t$-axis.
- If we wish to approximate $y(\bar{t})$ for some fixed $\bar{t}$ by taking horizontal steps of size $\Delta t$, then the error in our approximation is proportional to $\Delta t$.

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# 7.4: Separable Differential Equations

## Motivating Questions

- What is a separable differential equation?
- How can we find solutions to a separable differential equation?
- Are some of the differential equations that arise in applications separable?

In Sections 7.2 and 7.3, we have seen several ways to approximate the solution to an initial value problem. Given the frequency with which differential equations arise in the world around us, we would like to have some techniques for finding explicit algebraic solutions of certain initial value problems. In this section, we focus on a particular class of differential equations (called *separable*) and develop a method for finding algebraic formulas for their solutions.

A *separable differential equation* is a differential equation whose algebraic structure allows the variables to be separated in a particular way. For instance, consider the equation

$$\frac{dy}{dt} = ty.$$

We would like to separate the variables $t$ and $y$ so that all occurrences of $t$ appear on the right-hand side, and all occurrences of $y$ appear on the left, multiplied by $dy/dt$. For this example, we divide both sides by $y$ so that

$$\frac{1}{y}\frac{dy}{dt} = t.$$

Note that when we attempt to separate the variables in a differential equation, we require that one side is a product in which the derivative $dy/dt$ is one factor and the other factor is solely an expression involving $y$.

Not every differential equation is separable. For example, if we consider the equation

$$\frac{dy}{dt} = t - y,$$

it may seem natural to separate it by writing

$$y + \frac{dy}{dt} = t.$$

As we will see, this will not be helpful, since the left-hand side is not a product of a function of $y$ with $\frac{dy}{dt}$.

## Preview Activity 7.4.1

In this preview activity, we explore whether certain differential equations are separable or not, and then revisit some key ideas from earlier work in integral calculus.

1. Which of the following differential equations are separable? If the equation is separable, write the equation in the revised form $g(y)\frac{dy}{dt} = h(t)$.
   1. $\frac{dy}{dt} = -3y.$
   2. $\frac{dy}{dt} = ty - y.$
   3. $\frac{dy}{dt} = t + 1.$
   4. $\frac{dy}{dt} = t^2 - y^2.$
2. Explain why any autonomous differential equation is guaranteed to be separable.
3. Why do we include the term "$+C$" in the expression

$$\int x\ dx = \frac{x^2}{2} + C?$$

4. Suppose we know that a certain function $f$ satisfies the equation
$$\int f'(x)\ dx = \int x\ dx.$$
What can you conclude about $f$?

## Solving separable differential equations

Before we discuss a general approach to solving a separable differential equation, it is instructive to consider an example.

**Example 7.4.1**

Find all functions $y$ that are solutions to the differential equation
$$\frac{dy}{dt} = \frac{t}{y^2}.$$

*Answer available at the end of this section.*

The strategy of Example 7.4.1 may be applied to any differential equation of the form $\frac{dy}{dt} = g(y) \cdot h(t)$, and any differential equation of this form is said to be *separable*. We work to solve a separable differential equation by writing
$$\frac{1}{g(y)}\frac{dy}{dt} = h(t),$$
and then integrating both sides with respect to $t$. After integrating, we try to solve algebraically for $y$ in order to write $y$ as a function of $t$.

**Example 7.4.2**

Solve the differential equation
$$\frac{dy}{dt} = 3y.$$

*Answer available at the end of this section.*

There is one more technical point to make. Notice that $y = 0$ is an equilibrium solution to this differential equation. In solving the equation above, we begin by dividing both sides by $y$, which is not allowed if $y = 0$. To be perfectly careful, therefore, we should consider the equilibrium solutions separately. In this case, notice that the final form of our solution captures the equilibrium solution by allowing $C = 0$.

**Activity 7.4.2**

Suppose that the population of a town is growing continuously at an annual rate of 3% per year.

1. Let $P(t)$ be the population of the town in year $t$. Write a differential equation that describes the annual growth rate.
2. Find the solutions of this differential equation.
3. If you know that the town's population in year 0 is 10,000, find the population $P(t)$.
4. How long does it take for the population to double? This time is called the *doubling time*.
5. Working more generally, find the doubling time if the annual growth rate is $k$ times the population.

### Activity 7.4.3

Suppose that a cup of coffee is initially at a temperature of $105^\circ$ F and is placed in a $75^\circ$ F room. Newton's law of cooling says that

$$\frac{dT}{dt} = -k(T - 75),$$

where $k$ is a constant of proportionality.

1. Suppose you measure that the coffee is cooling at one degree per minute at the time the coffee is brought into the room. Use the differential equation to determine the value of the constant $k$.
2. Find all the solutions of this differential equation.
3. What happens to all the solutions as $t \to \infty$? Explain how this agrees with your intuition.
4. What is the temperature of the cup of coffee after 20 minutes?
5. How long does it take for the coffee to cool to $80^\circ$?

### Activity 7.4.4

Solve each of the following differential equations or initial value problems.

1. $\dfrac{dy}{dt} - (2 - t)y = 2 - t$
2. $\dfrac{1}{t}\dfrac{dy}{dt} = e^{t^2 - 2y}$
3. $y' = 2y + 2,\ y(0) = 2$
4. $y' = 2y^2,\ y(-1) = 2$
5. $\frac{dy}{dt} = \frac{-2ty}{t^2+1},\ y(0) = 4$

## Summary

- A separable differential equation is one that may be rewritten with all occurrences of the dependent variable multiplying the derivative and all occurrences of the independent variable on the other side of the equation.
- We may find the solutions to certain separable differential equations by separating variables, integrating with respect to $t$, and ultimately solving the resulting algebraic equation for $y$.
- This technique allows us to solve many important differential equations that arise in the world around us. For instance, questions of growth and decay and Newton's Law of Cooling give rise to separable differential equations. Later, we will learn in Section 7.6 that the important logistic differential equation is also separable.

---

## Answers

**Example** 7.4.1

We begin by separating the variables and writing

$$y^2 \frac{dy}{dt} = t.$$

Integrating both sides of the equation with respect to the independent variable $t$ shows that

$$\int y^2 \frac{dy}{dt}\, dt = \int t\, dt.$$

Next, we notice that the left-hand side allows us to change the variable of antidifferentiation[1] from $t$ to $y$. In particular, $dy = \frac{dy}{dt}\, dt$, so we now have

$$\int y^2\, dy = \int t\, dt.$$

This is why we required that the left-hand side be written as a product in which $dy/dt$ is one of the terms.

This equation says that two families of antiderivatives are equal to each other. Therefore, when we find representative antiderivatives of both sides, we know they must differ by an arbitrary constant $C$. Antidifferentiating and including the integration constant $C$ on the right, we find that

$$\frac{y^3}{3} = \frac{t^2}{2} + C.$$

It is not necessary to include an arbitrary constant on both sides of the equation; we know that $y^3/3$ and $t^2/2$ are in the same family of antiderivatives and must therefore differ by a single constant.

Finally, we solve the last equation above for $y$ as a function of $t$, which gives

$$y(t) = \sqrt[3]{\frac{3}{2}\,t^2 + 3C}.$$

Of course, the term $3C$ on the right-hand side represents 3 times an unknown constant. It is, therefore, still an unknown constant, which we will rewrite as $C$. We thus conclude that the funtion

$$y(t) = \sqrt[3]{\frac{3}{2}\,t^2 + C}$$

is a solution to the original differential equation for any value of $C$.

Notice that because this solution depends on the arbitrary constant $C$, we have found an infinite family of solutions. This makes sense because we expect to find a unique solution that corresponds to any given initial value.

For example, if we want to solve the initial value problem

$$\frac{dy}{dt} = \frac{t}{y^2},\ y(0) = 2,$$

we know that the solution has the form $y(t) = \sqrt[3]{\frac{3}{2}\,t^2 + C}$ for some constant $C$. We therefore must find the appropriate value for $C$ that gives the initial value $y(0) = 2$. Hence,

$$2 = y(0) = \sqrt[3]{\frac{3}{2}\,0^2 + C} = \sqrt[3]{C},$$

which shows that $C = 2^3 = 8$. The solution to the initial value problem is then

$$y(t) = \sqrt[3]{\frac{3}{2}\,t^2 + 8}.$$

**Example** 7.4.2

Following the same strategy as in Example 7.4.1, we have

$$\frac{1}{y}\frac{dy}{dt} = 3.$$

Integrating both sides with respect to $t$,

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int 3\,dt,$$

and thus

$$\int \frac{1}{y}\,dy = \int 3\,dt.$$

Antidifferentiating and including the integration constant, we find that

$$\ln|y| = 3t + C.$$

Finally, we need to solve for $y$. Here, one point deserves careful attention. By the definition of the natural logarithm function, it follows that

$$|y| = e^{3t+C} = e^{3t}e^{C}.$$

Since $C$ is an unknown constant, $e^C$ is as well, though we do know that it is positive (because $e^x$ is positive for any $x$). When we remove the absolute value in order to solve for $y$, however, this constant may be either positive or negative. To account for a possible $+$ or $-$, we denote this updated constant by $C$ to obtain

$$y(t) = Ce^{3t}.$$

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# 7.5: Modeling with Differential Equations*

## Motivating Questions

- How can we use differential equations to describe phenomena in the world around us?
- How can we use differential equations to better understand these phenomena?

We have seen several ways that differential equations arise in the natural world, from the growth of a population to the temperature of a cup of coffee. In this section, we look more closely at how differential equations give us a natural way to describe various phenoma. As we'll see, the key is to understand the different factors that cause a quantity to change.

## Preview Activity 7.5.1

Any time that the rate of change of a quantity is related to the amount of a quantity, a differential equation naturally arises. In the following two problems, we see two such scenarios; for each, we want to develop a differential equation whose solution is the quantity of interest.

1. Suppose you have a bank account in which money grows at an annual rate of 3%.
   1. If you have \$10,000 in the account, at what rate is your money growing?
   2. Suppose that you are also withdrawing money from the account at \$1,000 per year. What is the rate of change in the amount of money in the account? What are the units on this rate of change?
2. Suppose that a water tank holds 100 gallons and that a salty solution, which contains 20 grams of salt in every gallon, enters the tank at 2 gallons per minute.
   1. How much salt enters the tank each minute?
   2. Suppose that initially there are 300 grams of salt in the tank. How much salt is in each gallon at this point in time?
   3. Finally, suppose that evenly mixed solution is pumped out of the tank at the rate of 2 gallons per minute. How much salt leaves the tank each minute?
   4. What is the total rate of change in the amount of salt in the tank?

## Developing a differential equation

Preview Activity 7.5.1 demonstrates the kind of thinking we will be doing in this section. In each of the two examples we considered, there is a quantity, such as the amount of money in the bank account or the amount of salt in the tank, that is changing due to several factors. The governing differential equation states that the total rate of change is the difference between the rate of increase and the rate of decrease.

## Example 7.5.1

In the Great Lakes region, rivers flowing into the lakes carry a great deal of pollution in the form of small pieces of plastic averaging 1 millimeter in diameter. In order to understand how the amount of plastic in Lake Michigan is changing, construct a model for how this type pollution has built up in the lake.

*Answer available at the end of this section.*

There are many important lessons to learn from Example 7.5.1. Foremost is how we can develop a differential equation by thinking about the “total rate = rate in - rate out” model. In addition, we note how we can bring together all of our available understanding (plotting $\frac{dP}{dt}$ vs. $P$, creating a slope field, solving the differential equation) to see how the differential equation describes the behavior of a changing quantity.

We can also explore what happens when certain aspects of the problem change. For instance, let's suppose we are at a time when the plastic pollution entering Lake Michigan has stabilized at $1.5 \cdot 10^5$ cubic meters, and that new legislation is passed to prevent

this type of pollution entering the lake. So, there is no longer any inflow of plastic pollution to the lake. How does the amount of plastic pollution in Lake Michigan now change? For example, how long does it take for the amount of plastic pollution in the lake to halve?

Resetting $t = 0$ at this time, we now have the initial value problem

$$\frac{dP}{dt} = -\frac{1}{100}P,\ P(0) = 1.5 \cdot 10^5.$$

It is a straightforward and familiar exercise to find that the solution to this equation is $P(t) = 1.5 \cdot 10^5 e^{-t/100}$. The time that it takes for half of the pollution to flow out of the lake is given by $T$ where $P(T) = 0.75 \cdot 10^5$. Thus, we must solve the equation

$$0.75 \cdot 10^5 = 1.5 \cdot 10^5 e^{-T/100},$$

or

$$\frac{1}{2} = e^{-T/100}.$$

It follows that

$$T = -100 \ln\left(\frac{1}{2}\right) \approx 69.3\text{years}.$$

In the upcoming activities, we explore some other natural settings in which differential equations model changing quantities.

## Activity 7.5.2

Suppose you have a bank account that grows by 5% every year. Let $A(t)$ be the amount of money in the account in year $t$.

1. What is the rate of change of $A$ with respect to $t$?
2. Suppose that you are also withdrawing \$10,000 per year. Write a differential equation that expresses the total rate of change of $A$.
3. Sketch a slope field for this differential equation, find any equilibrium solutions, and identify them as either stable or unstable. Write a sentence or two that describes the significance of the stability of the equilibrium solution.
4. Suppose that you initially deposit \$100,000 into the account. How long does it take for you to deplete the account?
5. What is the smallest amount of money you would need to have in the account to guarantee that you never deplete the money in the account?
6. If your initial deposit is \$300,000, how much could you withdraw every year without depleting the account?

## Activity 7.5.3

A dose of morphine is absorbed from the bloodstream of a patient at a rate proportional to the amount in the bloodstream.

1. Write a differential equation for $M(t)$, the amount of morphine in the patient's bloodstream, using $k$ as the constant proportionality.
2. Assuming that the initial dose of morphine is $M_0$, solve the initial value problem to find $M(t)$. Use the fact that the half-life for the absorption of morphine is two hours to find the constant $k$.
3. Suppose that a patient is given morphine intravenously at the rate of 3 milligrams per hour. Write a differential equation that combines the intravenous administration of morphine with the body's natural absorption.
4. Find any equilibrium solutions and determine their stability.
5. Assuming that there is initially no morphine in the patient's bloodstream, solve the initial value problem to determine $M(t)$. What happens to $M(t)$ after a very long time?
6. To what rate should a doctor reduce the intravenous rate so that there is eventually 7 milligrams of morphine in the patient's bloodstream?

# Summary

- Differential equations arise in a situation when we understand how various factors cause a quantity to change.
- We may use the tools we have developed so far—slope fields, Euler's methods, and our method for solving separable equations—to understand a quantity described by a differential equation.

## Answers

**Example** 7.5.1

First, some basic facts about Lake Michigan.

- The volume of the lake is $5 \cdot 10^{12}$ cubic meters.
- Water flows into the lake at a rate of $5 \cdot 10^{10}$ cubic meters per year. It flows out of the lake at the same rate.
- Each cubic meter flowing into the lake contains roughly $3 \cdot 10^{-8}$ cubic meters of plastic pollution.

Let's denote the amount of pollution in the lake by $P(t)$, where $P$ is measured in cubic meters of plastic and $t$ in years. Our goal is to describe the rate of change of this function; so we want to develop a differential equation describing $P(t)$.

First, we will measure how $P(t)$ increases due to pollution flowing into the lake. We know that $5 \cdot 10^{10}$ cubic meters of water enters the lake every year and each cubic meter of water contains $3 \cdot 10^{-8}$ cubic meters of pollution. Therefore, pollution enters the lake at the rate of

$$\left(5 \cdot 10^{10} \frac{m^3 \text{ water}}{\text{year}}\right)\left(3 \cdot 10^{-8} \frac{m^3 \text{ plastic}}{m^3 \text{ water}}\right) = 1.5 \cdot 10^3 \text{cubic m of plastic per year.}$$

Second, we will measure how $P(t)$ decreases due to pollution flowing out of the lake. If the total amount of pollution is $P$ cubic meters and the volume of Lake Michigan is $5 \cdot 10^{12}$ cubic meters, then the concentration of plastic pollution in Lake Michigan is

$$\frac{P}{5 \cdot 10^{12}} \text{cubic m of plastic per cubic m of water.}$$

Since $5 \cdot 10^{10}$ cubic meters of water flow out each year[1], then the plastic pollution leaves the lake at the rate of

$$\left(\frac{P}{5 \cdot 10^{12}} \frac{m^3 \text{ plastic}}{m^3 \text{ water}}\right)\left(5 \cdot 10^{10} \frac{m^3 \text{ water}}{\text{year}}\right) = \frac{P}{100} \text{cubic m of plastic per year.}$$

and we assume that each cubic meter of water that flows out carries with it the plastic pollution it contains

The total rate of change of $P$ is thus the difference between the rate at which pollution enters the lake and the rate at which pollution leaves the lake; that is,

$$\begin{aligned} \frac{dP}{dt} &= 1.5 \cdot 10^3 - \frac{P}{100} \\ &= \frac{1}{100}(1.5 \cdot 10^5 - P). \end{aligned}$$

We have now found a differential equation that describes the rate at which the amount of pollution is changing. To understand the behavior of $P(t)$, we apply some of the techniques we have recently developed.

Because the differential equation is autonomous, we can sketch $dP/dt$ as a function of $P$ and then construct a slope field, as shown in Figure 7.5.2 and Figure 7.5.3 .

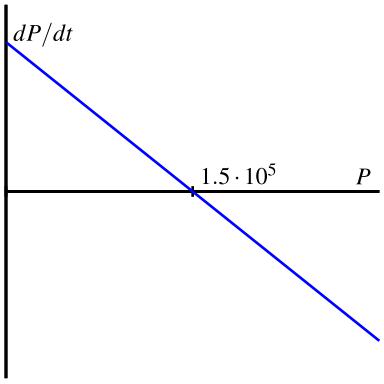



Figure 7.5.2: Plot of $\frac{dP}{dt}$ vs. $P$ for $\frac{dP}{dt} = \frac{1}{100}(1.5 \cdot 10^5 - P)$.

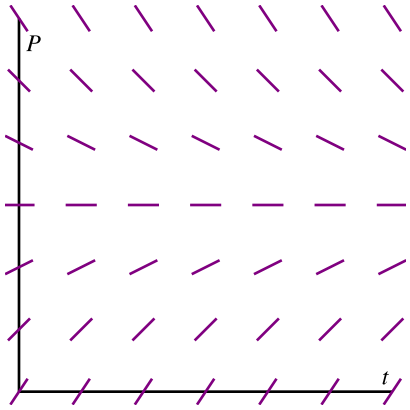



Figure 7.5.3:Plot of the slope field for $\frac{dP}{dt} = \frac{1}{100}(1.5 \cdot 10^5 - P)$.

These plots both show that $P = 1.5 \cdot 10^5$ is a stable equilibrium. Therefore, we should expect that the amount of pollution in Lake Michigan will stabilize near $1.5 \cdot 10^5$ cubic meters of pollution.

Next, assuming that there is initially no pollution in the lake, we will solve the initial value problem

$$\frac{dP}{dt} = \frac{1}{100}(1.5 \cdot 10^5 - P), \;\; P(0) = 0.$$

Separating variables, we find that

$$\frac{1}{1.5 \cdot 10^5 - P}\frac{dP}{dt} = \frac{1}{100}.$$

Integrating with respect to $t$, we have

$$\int \frac{1}{1.5 \cdot 10^5 - P}\frac{dP}{dt}\, dt = \int \frac{1}{100}\, dt,$$

and thus changing variables on the left and antidifferentiating on both sides, we find that

$$\int \frac{dP}{1.5 \cdot 10^5 - P} = \int \frac{1}{100}\, dt$$

$$-\ln|1.5 \cdot 10^5 - P| = \frac{1}{100}t + C$$

Finally, multiplying both sides by $-1$ and using the definition of the logarithm, we find that

$$1.5 \cdot 10^5 - P = Ce^{-t/100}. \tag{7.5.1}$$

This is a good time to determine the constant $C$. Since $P = 0$ when $t = 0$, we have

$$1.5 \cdot 10^5 - 0 = Ce^0 = C,$$

so $C = 1.5 \cdot 10^5$.

Using this value of $C$ in Equation (7.5.1) and solving for $P$, we arrive at the solution

$$P(t) = 1.5 \cdot 10^5(1 - e^{-t/100}).$$

Superimposing the graph of $P$ on the slope field we saw in Figure 7.5.3, we see, as shown in Figure 7.5.4

We see that, as expected, the amount of plastic pollution stabilizes around $1.5 \cdot 10^5$ cubic meters.

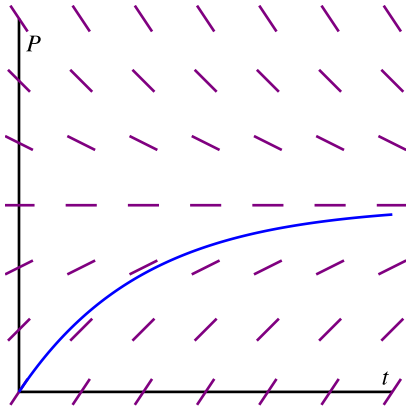



Figure 7.5.4. The solution $P(t)$ and the slope field for the differential equation $\frac{dP}{dt} = \frac{1}{100}(1.5 \cdot 10^5 - P)$.

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# 7.6: Population Growth and the Logistic Equation*

## Motivating Questions

- How can we use differential equations to realistically model the growth of a population?
- How can we assess the accuracy of our models?

The growth of the earth's population is one of the pressing issues of our time. Will the population continue to grow? Or will it perhaps level off at some point, and if so, when? In this section, we look at two ways in which we may use differential equations to help us address these questions.

Before we begin, let's consider again two important differential equations that we have seen in earlier work this chapter.

## Preview Activity 7.6.1

Recall that one model for population growth states that a population grows at a rate proportional to its size.

1. We begin with the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P.$$

Sketch a slope field below as well as a few typical solutions on the axes provided.

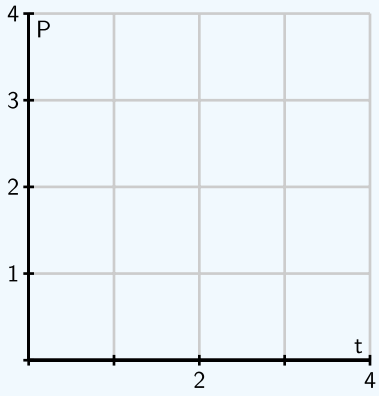



Figure 7.6.1

2. Find all equilibrium solutions of the equation $\frac{dP}{dt} = \frac{1}{2}P$ and classify them as stable or unstable.
3. If $P(0)$ is positive, describe the long-term behavior of the solution to $\frac{dP}{dt} = \frac{1}{2}P$.
4. Let's now consider a modified differential equation given by

$$\frac{dP}{dt} = \frac{1}{2}P(3 - P).$$

As before, sketch a slope field as well as a few typical solutions on the following axes provided.

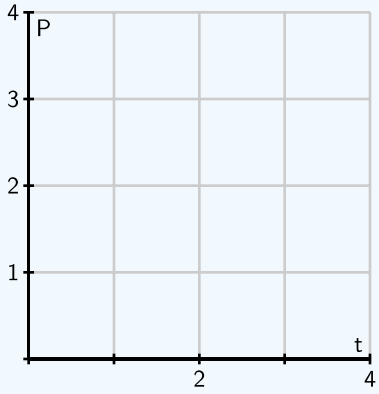



Figure 7.6.2

5. Find any equilibrium solutions and classify them as stable or unstable.
6. If $P(0)$ is positive, describe the long-term behavior of the solution.

## 7.6.1 The earth's population

We will now begin studying the earth's population. To get started, in Table 7.6.1 are some data for the earth's population in recent years that we will use in our investigations.

*Table 7.6.1. Some recent population data for planet Earth.*

| Year | 1998 | 1999 | 2000 | 2001 | 2002 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Pop (billions)** | 5.932 | 6.008 | 6.084 | 6.159 | 6.234 | 6.456 | 6.531 | 6.606 | 6.681 | 6.756 | 6.831 |

### Activity 7.6.2

Our first model will be based on the following assumption:

> *The rate of change of the population is proportional to the population.*

On the face of it, this seems pretty reasonable. When there is a relatively small number of people, there will be fewer births and deaths so the rate of change will be small. When there is a larger number of people, there will be more births and deaths so we expect a larger rate of change.

If $P(t)$ is the population $t$ years after the year 2000, we may express this assumption as

$$\frac{dP}{dt} = kP$$

where $k$ is a constant of proportionality.

1. Use the data in the table to estimate the derivative $P'(0)$ using a central difference. Assume that $t = 0$ corresponds to the year 2000.
2. What is the population $P(0)$?
3. Use your results from (a) and (b) to estimate the constant of proportionality $k$ in the differential equation.
4. Now that we know the value of $k$, we have the initial value problem

$$\frac{dP}{dt} = kP, \ P(0) = 6.084.$$

Find the solution to this initial value problem.

5. What does your solution predict for the population in the year 2010? Is this close to the actual population given in the table?
6. When does your solution predict that the population will reach 12 billion?
7. What does your solution predict for the population in the year 2500?
8. Do you think this is a reasonable model for the earth's population? Why or why not? Explain your thinking using a couple of complete sentences.

Our work in Activity 7.6.2 shows that that the exponential model is fairly accurate for years relatively close to 2000. However, if we go too far into the future, the model predicts increasingly large rates of change, which causes the population to grow arbitrarily large. This does not make much sense since it is unrealistic to expect that the earth would be able to support such a large population.

The constant $k$ in the differential equation has an important interpretation. Let's rewrite the differential equation $\frac{dP}{dt} = kP$ by solving for $k$, so that we have

$$k = \frac{dP/dt}{P}.$$

We see that $k$ is the ratio of the rate of change to the population; in other words, it is the contribution to the rate of change from a single person. We call this the *per capita growth rate*.

In the exponential model we introduced in Activity 7.6.2, the per capita growth rate is constant. This means that when the population is large, the per capita growth rate is the same as when the population is small. It is natural to think that the per capita growth rate should decrease when the population becomes large, since there will not be enough resources to support so many people. We expect it would be a more realistic model to assume that the per capita growth rate depends on the population $P$.

In the previous activity, we computed the per capita growth rate in a single year by computing $k$, the quotient of $\frac{dP}{dt}$ and $P$ (which we did for $t = 0$). If we return to the data in Table 7.6.1 and compute the per capita growth rate over a range of years, we generate the data shown in Figure 7.6.3, which shows how the per capita growth rate is a function of the population, $P$.

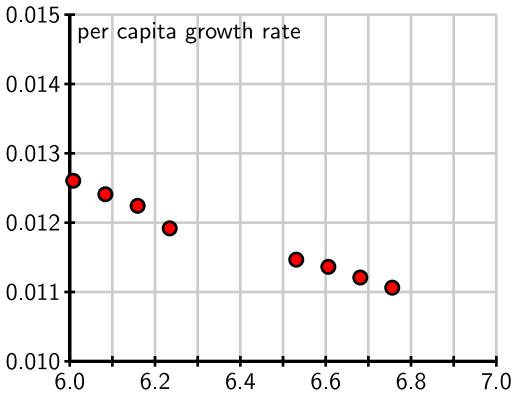



Figure 7.6.3. A plot of per capita growth rate vs. population $P$.

From the data, we see that the per capita growth rate appears to decrease as the population increases. In fact, the points seem to lie very close to a line, which is shown at two different scales in Figure 7.6.4.

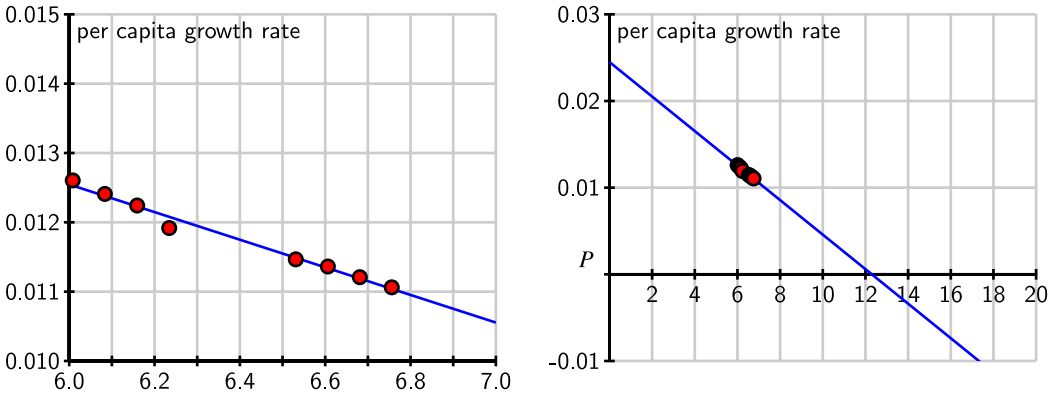



Figure 7.6.4. The line that approximates per capita growth as a function of population, $P$.

Looking at this line carefully, we can find its equation to be

$$\frac{dP/dt}{P} = 0.025 - 0.002P.$$

If we multiply both sides by $P$, we arrive at the differential equation

$$\frac{dP}{dt} = P(0.025 - 0.002P).$$

Graphing the dependence of $dP/dt$ on the population $P$, we see that this differential equation demonstrates a quadratic relationship between $\frac{dP}{dt}$ and $P$, as shown in Figure 7.6.5.

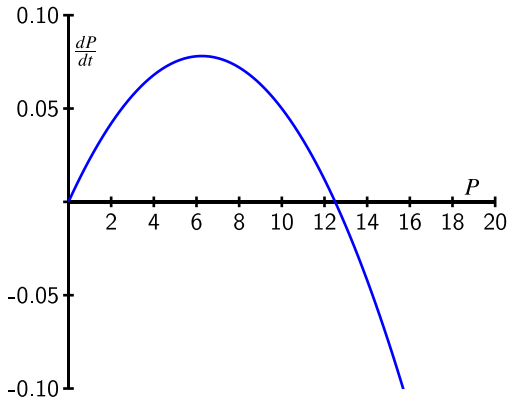



Figure 7.6.5. A plot of $\frac{dP}{dt}$ vs. $P$ for the differential equation $\frac{dP}{dt} = P(0.025 - 0.002P)$.

The equation $\frac{dP}{dt} = P(0.025 - 0.002P)$ is an example of the *logistic equation*, and is the second model for population growth that we will consider. We expect that it will be more realistic, because the per capita growth rate is a decreasing function of the population.

Indeed, the graph in Figure 7.6.5 shows that there are two equilibrium solutions, $P = 0$, which is unstable, and $P = 12.5$, which is a stable equilibrium. The graph shows that any solution with $P(0) > 0$ will eventually stabilize around 12.5. Thus, our model predicts the world's population will eventually stabilize around 12.5 billion.

A prediction for the long-term behavior of the population is a valuable conclusion to draw from our differential equation. We would, however, also like to answer some quantitative questions. For instance, how long will it take to reach a population of 10 billion? To answer this question, we need to find an explicit solution of the equation.

## Solving the logistic differential equation

Since we would like to apply the logistic model in more general situations, we state the logistic equation in its more general form,

$$\frac{dP}{dt} = kP(N-P). \tag{7.6.1}$$

The equilibrium solutions here are $P = 0$ and $1 - \frac{P}{N} = 0$, which shows that $P = N$. The equilibrium at $P = N$ is called the *carrying capacity* of the population for it represents the stable population that can be sustained by the environment.

We now solve the logistic equation (7.6.1). The equation is separable, so we separate the variables

$$\frac{1}{P(N-P)}\frac{dP}{dt} = k,$$

and integrate to find that

$$\int \frac{1}{P(N-P)}\, dP = \int k\, dt.$$

To find the antiderivative on the left, we use the partial fraction decomposition

$$\frac{1}{P(N-P)} = \frac{1}{N}\left[\frac{1}{P} + \frac{1}{N-P}\right].$$

Now we are ready to integrate, with

$$\int \frac{1}{N}\left[\frac{1}{P} + \frac{1}{N-P}\right]\, dP = \int k\, dt.$$

On the left, observe that $N$ is constant, so we can remove a factor of $\frac{1}{N}$ and antidifferentiate to find that

$$\frac{1}{N}(\ln|P| - \ln|N-P|) = kt + C.$$

Multiplying both sides of this last equation by $N$ and using a rule of logarithms, we next find that

$$\ln\left|\frac{P}{N-P}\right| = kNt + C.$$

From the definition of the logarithm, replacing $e^C$ with $C$, and letting $C$ absorb the absolute value signs, we now know that

$$\frac{P}{N-P} = Ce^{kNt}.$$

At this point, all that remains is to determine $C$ and solve algebraically for $P$.

If the initial population is $P(0) = P_0$, then it follows that $C = \frac{P_0}{N-P_0}$, so

$$\frac{P}{N-P} = \frac{P_0}{N-P_0}e^{kNt}.$$

We will solve this equation for $P$ by multiplying both sides by $(N-P)(N-P_0)$ to obtain

$$\begin{aligned} P(N-P_0) &= P_0(N-P)e^{kNt} \\ &= P_0Ne^{kNt} - P_0Pe^{kNt}. \end{aligned}$$

Solving for $P_0Ne^{kNt}$, expanding, and factoring, it follows that

$$\begin{aligned} P_0Ne^{kNt} &= P(N-P_0) + P_0Pe^{kNt} \\ &= P(N - P_0 + P_0e^{kNt}). \end{aligned}$$

Dividing to solve for $P$, we see that

$$P = \frac{P_0Ne^{kNt}}{N - P_0 + P_0e^{kNt}}.$$

Finally, we choose to multiply the numerator and denominator by $\frac{1}{P_0}e^{-kNt}$ to obtain

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt}+1}.$$

While that was a lot of algebra, notice the result: we have found an explicit solution to the logistic equation.

## Solution to the Logistic Equation

The solution to the initial value problem

$$\frac{dP}{dt} = kP(N-P),\ P(0) = P_0,$$

is

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt}+1}. \tag{7.6.1}$$

For the logistic equation describing the earth's population that we worked with earlier in this section, we have

$$k = 0.002, N = 12.5, \text{ and } P_0 = 6.084.$$

This gives the solution

$$P(t) = \frac{12.5}{1.0546e^{-0.025t}+1},$$

whose graph is shown in Figure 7.6.6.

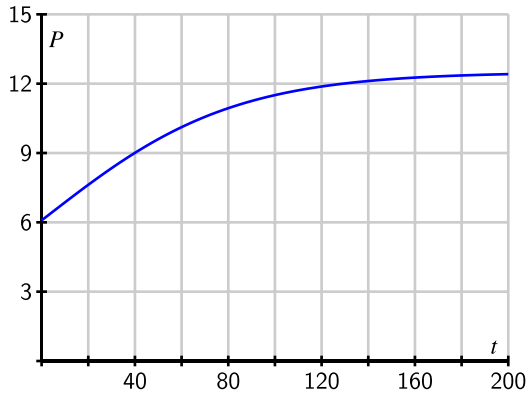



Figure 7.6.6. The solution to the logistic equation modeling the earth's population.

The graph shows the population leveling off at 12.5 billion, as we expected, and that the population will be around 10 billion in the year 2050. These results, which we have found using a relatively simple mathematical model, agree fairly well with predictions made using a much more sophisticated model developed by the United Nations.

The logistic equation is good for modeling any situation in which limited growth is possible. For instance, it could model the spread of a flu virus through a population contained on a cruise ship, the rate at which a rumor spreads within a small town, or the behavior of an animal population on an island. Through our work in this section, we have completely solved the logistic equation, regardless of the values of the constants $N$, $k$, and $P_0$. Anytime we encounter a logistic equation, we can apply the formula we found in Equation 7.6.1.

## Activity 7.6.3

Consider the logistic equation

$$\frac{dP}{dt} = kP(N - P)$$

with the graph of $\frac{dP}{dt}$ vs. $P$ shown in Figure 7.6.7.

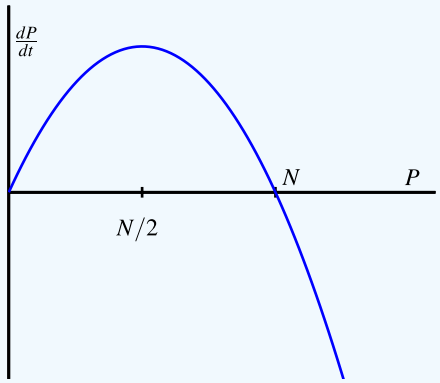



Figure 7.6.7. Plot of $\frac{dP}{dt}$ vs. $P$.

1. At what value of $P$ is the rate of change greatest?
2. Consider the model for the earth's population that we created. At what value of $P$ is the rate of change greatest? How does that compare to the population in recent years?
3. According to the model we developed, what will the population be in the year 2100?
4. According to the model we developed, when will the population reach 9 billion?
5. Now consider the general solution to the general logistic initial value problem that we found, given by

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt} + 1}.$$

Verify algebraically that $P(0) = P_0$ and that $\lim_{t\to\infty} P(t) = N$.

# Summary

- If we assume that the rate of growth of a population is proportional to the population, we are led to a model in which the population grows without bound and at a rate that grows without bound.
- By assuming that the per capita growth rate decreases as the population grows, we are led to the logistic model of population growth, which predicts that the population will eventually stabilize at the carrying capacity.

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# 7.7: Differential Equations (Exercises)

## 7.1: An Introduction to Differential Equations

### 1. Matching solutions with equations

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### 2. Finding constant to complete solution

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### 3. Choosing solution of $dy/dt = k(1 - Ay)$

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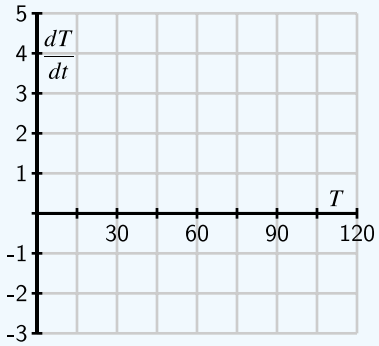
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### 4.

Suppose that $T(t)$ represents the temperature of a cup of coffee set out in a room, where $T$ is expressed in degrees Fahrenheit and $t$ in minutes. A physical principle known as Newton's Law of Cooling tells us that

$$\frac{dT}{dt} = -\frac{1}{15}T + 5.$$

1. Supposes that $T(0) = 105$. What does the differential equation give us for the value of $\frac{dT}{dt}|_{T=105}$? Explain in a complete sentence the meaning of these two facts.
2. Is $T$ increasing or decreasing at $t = 0$?
3. What is the approximate temperature at $t = 1$?
4. On the graph below, make a plot of $dT/dt$ as a function of $T$.



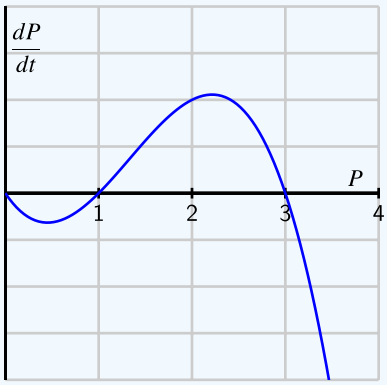

5. For which values of $T$ does $T$ increase? For which values of $T$ does $T$ decrease?
6. What do you think is the temperature of the room? Explain your thinking.
7. Verify that $T(t) = 75 + 30e^{-t/15}$ is the solution to the differential equation with initial value $T(0) = 105$. What happens to this solution after a long time?

## 5.

Suppose that the population of a particular species is described by the function $P(t)$, where $P$ is expressed in millions. Suppose further that the population's rate of change is governed by the differential equation

$$\frac{dP}{dt} = f(P)$$

where $f(P)$ is the function graphed below.




1. For which values of the population $P$ does the population increase?
2. For which values of the population $P$ does the population decrease?
3. If $P(0) = 3$, how will the population change in time?
4. If the initial population satisfies $0 < P(0) < 1$, what will happen to the population after a very long time?
5. If the initial population satisfies $1 < P(0) < 3$, what will happen to the population after a very long time?
6. If the initial population satisfies $3 < P(0)$, what will happen to the population after a very long time?
7. This model for a population's growth is sometimes called “growth with a threshold.” Explain why this is an appropriate name.

## 6

In this problem, we test further what it means for a function to be a solution to a given differential equation.

1. Consider the differential equation

$$\frac{dy}{dt} = y - t.$$

Determine whether the following functions are solutions to the given differential equation.

1. $y(t) = t + 1 + 2e^t$
2. $y(t) = t + 1$
3. $y(t) = t + 2$

2. When you weigh bananas in a scale at the grocery store, the height $h$ of the bananas is described by the differential equation

$$\frac{d^2h}{dt^2} = -kh$$

where $k$ is the *spring constant*, a constant that depends on the properties of the spring in the scale. After you put the bananas in the scale, you (cleverly) observe that the height of the bananas is given by $h(t) = 4\sin(3t)$. What is the value of the spring constant?

# 7.2: Qualitative Behavior of Solutions to DE's

## Exercises 7.2.4 Exercises

### 1. Graphing equilibrium solutions

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### 2. Sketching solution curves

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### 3. Matching equations with direction fields

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## 4. Describing equilibrium solutions

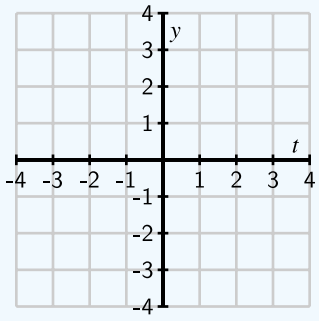
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## 5

Consider the differential equation

$$\frac{dy}{dt} = t - y.$$



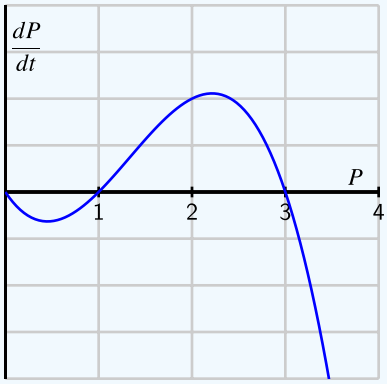

1. Sketch a slope field on the axes at right.
2. Sketch the solutions whose initial values are $y(0) = -4, -3, \ldots, 4$.
3. What do your sketches suggest is the solution whose initial value is $y(0) = -1$? Verify that this is indeed the solution to this initial value problem.
4. By considering the differential equation and the graphs you have sketched, what is the relationship between $t$ and $y$ at a point where a solution has a local minimum?

## 6

Consider the situation from problem 2 of Section 7.1: Suppose that the population of a particular species is described by the function $P(t)$, where $P$ is expressed in millions. Suppose further that the population's rate of change is governed by the differential equation

$$\frac{dP}{dt} = f(P)$$

where $f(P)$ is the function graphed below.




1. Sketch a slope field for this differential equation. You do not have enough information to determine the actual slopes, but you should have enough information to determine where slopes are positive, negative, zero, large, or small, and hence determine the qualitative behavior of solutions.
2. Sketch some solutions to this differential equation when the initial population $P(0) > 0$.
3. Identify any equilibrium solutions to the differential equation and classify them as stable or unstable.
4. If $P(0) > 1$, what is the eventual fate of the species? if $P(0) < 1$?
5. Remember that we referred to this model for population growth as "growth with a threshold." Explain why this characterization makes sense by considering solutions whose inital value is close to 1.

## 7

The population of a species of fish in a lake is $P(t)$ where $P$ is measured in thousands of fish and $t$ is measured in months. The growth of the population is described by the differential equation

$$\frac{dP}{dt} = f(P) = P(6 - P).$$

1. Sketch a graph of $f(P) = P(6 - P)$ and use it to determine the equilibrium solutions and whether they are stable or unstable. Write a complete sentence that describes the long-term behavior of the fish population.
2. Suppose now that the owners of the lake allow fishers to remove 1000 fish from the lake every month (remember that $P(t)$ is measured in *thousands* of fish). Modify the differential equation to take this into account. Sketch the new graph of $dP/dt$ versus $P$. Determine the new equilibrium solutions and decide whether they are stable or unstable.
3. Given the situation in part (b), give a description of the long-term behavior of the fish population.
4. Suppose that fishermen remove $h$ thousand fish per month. How is the differential equation modified?
5. What is the largest number of fish that can be removed per month without eliminating the fish population? If fish are removed at this maximum rate, what is the eventual population of fish?

## 8

Let $y(t)$ be the number of thousands of mice that live on a farm; assume time $t$ is measured in years.[1]

1. The population of the mice grows at a yearly rate that is twenty times the number of mice. Express this as a differential equation.
2. At some point, the farmer brings $C$ cats to the farm. The number of mice that the cats can eat in a year is

   $$M(y) = C\frac{y}{2 + y}$$

   thousand mice per year. Explain how this modifies the differential equation that you found in part a).
3. Sketch a graph of the function $M(y)$ for a single cat $C = 1$ and explain its features by looking, for instance, at the behavior of $M(y)$ when $y$ is small and when $y$ is large.

4. Suppose that $C = 1$. Find the equilibrium solutions and determine whether they are stable or unstable. Use this to explain the long-term behavior of the mice population depending on the initial population of the mice.
5. Suppose that $C = 60$. Find the equilibrium solutions and determine whether they are stable or unstable. Use this to explain the long-term behavior of the mice population depending on the initial population of the mice.
6. What is the smallest number of cats you would need to keep the mice population from growing arbitrarily large?

This problem is based on an ecological analysis presented in a research paper by C.S. Hollings: The Components of Predation as Revealed by a Study of Small Mammal Predation of the European Pine Sawfly, *Canadian Entomology 91*: 283-320.

## 7.3: Euler's Method

### 1. A few steps of Euler's method

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### 2. Using Euler's method for a solution of $y' = 4y$

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### 3. Using Euler's method with different time steps

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### 4

Newton's Law of Cooling says that the rate at which an object, such as a cup of coffee, cools is proportional to the difference in the object's temperature and room temperature. If $T(t)$ is the object's temperature and $T_r$ is room temperature, this law is expressed at

$$\frac{dT}{dt} = -k(T - T_r),$$

where $k$ is a constant of proportionality. In this problem, temperature is measured in degrees Fahrenheit and time in minutes.

1. Two calculus students, Alice and Bob, enter a 70° classroom at the same time. Each has a cup of coffee that is 100°. The differential equation for Alice has a constant of proportionality $k = 0.5$, while the constant of proportionality for Bob is $k = 0.1$. What is the initial rate of change for Alice's coffee? What is the initial rate of change for Bob's coffee?

2. What feature of Alice's and Bob's cups of coffee could explain this difference?
3. As the heating unit turns on and off in the room, the temperature in the room is

$$T_r = 70 + 10 \sin t.$$

Implement Euler's method with a step size of $\Delta t = 0.1$ to approximate the temperature of Alice's coffee over the time interval $0 \leq t \leq 50$. This will most easily be performed using a spreadsheet such as *Excel*. Graph the temperature of her coffee and room temperature over this interval.

4. In the same way, implement Euler's method to approximate the temperature of Bob's coffee over the same time interval. Graph the temperature of his coffee and room temperature over the interval.
5. Explain the similarities and differences that you see in the behavior of Alice's and Bob's cups of coffee.

## 5

We have seen that the error in approximating the solution to an initial value problem is proportional to $\Delta t$. That is, if $E_{\Delta t}$ is the Euler's method approximation to the solution to an initial value problem at $\bar{t}$, then

$$y(\bar{t}) - E_{\Delta t} \approx K \Delta t$$

for some constant of proportionality $K$.

In this problem, we will see how to use this fact to improve our estimates, using an idea called *accelerated convergence*.

1. We will create a new approximation by assuming the error is *exactly* proportional to $\Delta t$, according to the formula

$$y(\bar{t}) - E_{\Delta t} = K \Delta t.$$

Using our earlier results from the initial value problem $dy/dt = y$ and $y(0) = 1$ with $\Delta t = 0.2$ and $\Delta t = 0.1$, we have

$$y(1) - 2.4883 = 0.2K$$
$$y(1) - 2.5937 = 0.1K.$$

This is a system of two linear equations in the unknowns $y(1)$ and $K$. Solve this system to find a new approximation for $y(1)$. (You may remember that the exact value is $y(1) = e = 2.71828 \ldots$.)

2. Use the other data, $E_{0.05} = 2.6533$ and $E_{0.025} = 2.6851$ to do similar work as in (a) to obtain another approximation. Which gives the better approximation? Why do you think this is?
3. Let's now study the initial value problem

$$\frac{dy}{dt} = t - y, \; y(0) = 0.$$

Approximate $y(0.3)$ by applying Euler's method to find approximations $E_{0.1}$ and $E_{0.05}$. Now use the idea of accelerated convergence to obtain a better approximation. (For the sake of comparison, you want to note that the actual value is $y(0.3) = 0.0408$.)

## 6

In this problem, we'll modify Euler's method to obtain better approximations to solutions of initial value problems. This method is called the *Improved Euler's method*.

In Euler's method, we walk across an interval of width $\Delta t$ using the slope obtained from the differential equation at the left endpoint of the interval. Of course, the slope of the solution will most likely change over this interval. We can improve our approximation by trying to incorporate the change in the slope over the interval.

Let's again consider the initial value problem $dy/dt = y$ and $y(0) = 1$, which we will approximate using steps of width $\Delta t = 0.2$. Our first interval is therefore $0 \le t \le 0.2$. At $t = 0$, the differential equation tells us that the slope is 1, and the approximation we obtain from Euler's method is that $y(0.2) \approx y_1 = 1 + 1(0.2) = 1.2$.

This gives us some idea for how the slope has changed over the interval $0 \le t \le 0.2$. We know the slope at $t = 0$ is 1, while the slope at $t = 0.2$ is 1.2, trusting in the Euler's method approximation. We will therefore refine our estimate of the initial slope to be the average of these two slopes; that is, we will estimate the slope to be $(1 + 1.2)/2 = 1.1$. This gives the new approximation $y(1) = y_1 = 1 + 1.1(0.2) = 1.22$.

The first few steps look like what is found in Table 7.3.15.

*Table 7.3.15. The first several steps of the improved Euler's method*

| $t_i$ | $y_i$ | **Slope at** $(t_{i+1}, y_{i+1})$ | **Average slope** |
|---|---|---|---|
| 0.0 | 1.0000 | 1.2000 | 1.1000 |
| 0.2 | 1.2200 | 1.4640 | 1.3420 |
| 0.4 | 1.4884 | 1.7861 | 1.6372 |
| ⋮ | ⋮ | ⋮ | ⋮ |

1. Continue with this method to obtain an approximation for $y(1) = e$.
2. Repeat this method with $\Delta t = 0.1$ to obtain a better approximation for $y(1)$.
3. We saw that the error in Euler's method is proportional to $\Delta t$. Using your results from parts (a) and (b), what power of $\Delta t$ appears to be proportional to the error in the Improved Euler's Method?

# 7.4: Separable Differential Equations

## 1. Initial value problem for $dy/dx = x^8 y$

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## 3. Initial value problem for $dy/dt = y^2(8+t)$

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## 4. Initial value problem for $du/dt = e^{6u+10t}$

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## 5. Initial value problem for $dy/dx = 170yx^{16}$

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## 6

The mass of a radioactive sample decays at a rate that is proportional to its mass.

1. Express this fact as a differential equation for the mass $M(t)$ using $k$ for the constant of proportionality.
2. If the initial mass is $M_0$, find an expression for the mass $M(t)$.
3. The *half-life* of the sample is the amount of time required for half of the mass to decay. Knowing that the half-life of Carbon-14 is 5730 years, find the value of $k$ for a sample of Carbon-14.
4. How long does it take for a sample of Carbon-14 to be reduced to one-quarter its original mass?
5. Carbon-14 naturally occurs in our environment; any living organism takes in Carbon-14 when it eats and breathes. Upon dying, however, the organism no longer takes in Carbon-14. Suppose that you find remnants of a pre-historic firepit. By analyzing the charred wood in the pit, you determine that the amount of Carbon-14 is only 30% of the amount in living trees. Estimate the age of the firepit. [2]

This approach is the basic idea behind radiocarbon dating.

## 7

Consider the initial value problem

$$\frac{dy}{dt} = -\frac{t}{y},\ y(0) = 8$$

1. Find the solution of the initial value problem and sketch its graph.
2. For what values of $t$ is the solution defined?
3. What is the value of $y$ at the last time that the solution is defined?

4. By looking at the differential equation, explain why we should not expect to find solutions with the value of $y$ you noted in (c).

### 8

Suppose that a cylindrical water tank with a hole in the bottom is filled with water. The water, of course, will leak out and the height of the water will decrease. Let $h(t)$ denote the height of the water. A physical principle called *Torricelli's Law* implies that the height decreases at a rate proportional to the square root of the height.

1. Express this fact using $k$ as the constant of proportionality.
2. Suppose you have two tanks, one with $k = -1$ and another with $k = -10$. What physical differences would you expect to find?
3. Suppose you have a tank for which the height decreases at $20$ inches per minute when the water is filled to a depth of $100$ inches. Find the value of $k$.
4. Solve the initial value problem for the tank in part (c), and graph the solution you determine.
5. How long does it take for the water to run out of the tank?
6. Is the solution that you found valid for all time $t$? If so, explain how you know this. If not, explain why not.

### 9

The *Gompertz equation* is a model that is used to describe the growth of certain populations. Suppose that $P(t)$ is the population of some organism and that

$$\frac{dP}{dt} = -P \ln\left(\frac{P}{3}\right) = -P(\ln P - \ln 3).$$

1. Sketch a slope field for $P(t)$ over the range $0 \leq P \leq 6$.
2. Identify any equilibrium solutions and determine whether they are stable or unstable.
3. Find the population $P(t)$ assuming that $P(0) = 1$ and sketch its graph. What happens to $P(t)$ after a very long time?
4. Find the population $P(t)$ assuming that $P(0) = 6$ and sketch its graph. What happens to $P(t)$ after a very long time?
5. Verify that the long-term behavior of your solutions agrees with what you predicted by looking at the slope field.

## 7.5: Modeling with Differential Equations

### 1. Mixing problem

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### 2. Mixing problem

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## 3. Population growth problem

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## 4. Radioactive decay problem

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## 5. Investment problem

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## 6

Congratulations, you just won the lottery! In one option presented to you, you will be paid one million dollars a year for the next 25 years. You can deposit this money in an account that will earn 5% each year.

1. Set up a differential equation that describes the rate of change in the amount of money in the account. Two factors cause the amount to grow—first, you are depositing one millon dollars per year and second, you are earning 5% interest.
2. If there is no amount of money in the account when you open it, how much money will you have in the account after 25 years?
3. The second option presented to you is to take a lump sum of 10 million dollars, which you will deposit into a similar account. How much money will you have in that account after 25 years?
4. Do you prefer the first or second option? Explain your thinking.
5. At what time does the amount of money in the account under the first option overtake the amount of money in the account under the second option?

## 7

When a skydiver jumps from a plane, gravity causes her downward velocity to increase at the rate of $g \approx 9.8$ meters per second squared. At the same time, wind resistance causes her velocity to decrease at a rate proportional to the velocity.

1. Using $k$ to represent the constant of proportionality, write a differential equation that describes the rate of change of the skydiver's velocity.
2. Find any equilibrium solutions and decide whether they are stable or unstable. Your result should depend on $k$.
3. Suppose that the initial velocity is zero. Find the velocity $v(t)$.

4. A typical terminal velocity for a skydiver falling face down is 54 meters per second. What is the value of $k$ for this skydiver?
5. How long does it take to reach 50% of the terminal velocity?

### 8

During the first few years of life, the rate at which a baby gains weight is proportional to the reciprocal of its weight.

1. Express this fact as a differential equation.
2. Suppose that a baby weighs 8 pounds at birth and 9 pounds one month later. How much will he weigh at one year?
3. Do you think this is a realistic model for a long time?

### 9

Suppose that you have a water tank that holds 100 gallons of water. A briny solution, which contains 20 grams of salt per gallon, enters the tank at the rate of 3 gallons per minute.

At the same time, the solution is well mixed, and water is pumped out of the tank at the rate of 3 gallons per minute.

1. Since 3 gallons enters the tank every minute and 3 gallons leaves every minute, what can you conclude about the volume of water in the tank.
2. How many grams of salt enters the tank every minute?
3. Suppose that $S(t)$ denotes the number of grams of salt in the tank in minute $t$. How many grams are there in each gallon in minute $t$?
4. Since water leaves the tank at 3 gallons per minute, how many grams of salt leave the tank each minute?
5. Write a differential equation that expresses the total rate of change of $S$.
6. Identify any equilibrium solutions and determine whether they are stable or unstable.
7. Suppose that there is initially no salt in the tank. Find the amount of salt $S(t)$ in minute $t$.
8. What happens to $S(t)$ after a very long time? Explain how you could have predicted this only knowing how much salt there is in each gallon of the briny solution that enters the tank.

# 7.6: Population Growth and the Logistic Equation

## Exercises 7.6.4 Exercises

### 1. Analyzing a logistic equation

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## 2. Analyzing a logistic model

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## 3. Finding a logistic function for an infection model

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## 4. Analyzing a population growth model

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## 5

The logistic equation may be used to model how a rumor spreads through a group of people. Suppose that $p(t)$ is the fraction of people that have heard the rumor on day $t$. The equation

$$\frac{dp}{dt} = 0.2p(1-p)$$

describes how $p$ changes. Suppose initially that one-tenth of the people have heard the rumor; that is, $p(0) = 0.1$.

1. What happens to $p(t)$ after a very long time?
2. Determine a formula for the function $p(t)$.
3. At what time is $p$ changing most rapidly?
4. How long does it take before 80% of the people have heard the rumor?

## 6

Suppose that $b(t)$ measures the number of bacteria living in a colony in a Petri dish, where $b$ is measured in thousands and $t$ is measured in days. One day, you measure that there are 6,000 bacteria and the per capita growth rate is 3. A few days later, you measure that there are 9,000 bacteria and the per capita growth rate is 2.

1. Assume that the per capita growth rate $\frac{db/dt}{b}$ is a linear function of $b$. Use the measurements to find this function and write a logistic equation to describe $\frac{db}{dt}$.
2. What is the carrying capacity for the bacteria?
3. At what population is the number of bacteria increasing most rapidly?

4. If there are initially 1,000 bacteria, how long will it take to reach 80% of the carrying capacity?

**7**

Suppose that the population of a species of fish is controlled by the logistic equation

$$\frac{dP}{dt} = 0.1P(10 - P),$$

where $P$ is measured in thousands of fish and $t$ is measured in years.

1. What is the carrying capacity of this population?
2. Suppose that a long time has passed and that the fish population is stable at the carrying capacity. At this time, humans begin harvesting 20% of the fish every year. Modify the differential equation by adding a term to incorporate the harvesting of fish.
3. What is the new carrying capacity?
4. What will the fish population be one year after the harvesting begins?
5. How long will it take for the population to be within 10% of the carrying capacity?

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# 8: ACTIVE CALCULUS - SEQUENCES AND SERIES


8.1: Sequences

8.2: Geometric Series

8.3: Series of Real Numbers

8.4: Alternating Series

8.5: Taylor Polynomials and Taylor Series

8.6: Power Series

8.7: Sequences and Series (Exercises)



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# 8.1: Sequences

## Motivating Questions

- What is a sequence?
- What does it mean for a sequence to converge?
- What does it mean for a sequence to diverge?

We encounter sequences every day. Your monthly utility payments, the annual interest you earn on investments, the amount you spend on groceries each week; all are examples of sequences. Other sequences with which you may be familiar include the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$, where each term is the sum of the two preceding terms, and the triangular numbers $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$, the number of vertices in the triangles shown in Figure 8.1.1.

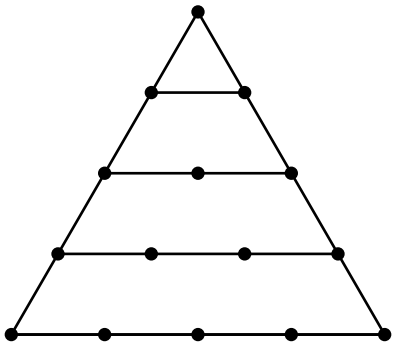


Figure 8.1.1. Triangular numbers

Sequences of integers are of such interest to mathematicians and others that they have a journal [1] devoted to them and an on-line encyclopedia [2] that catalogs a huge number of integer sequences and their connections. Sequences are also used in digital recordings and images.

The Journal of Integer Sequences at http://www.cs.uwaterloo.ca/journals/JIS/
The On-Line Encyclopedia of Integer Sequences at http://oeis.org/

Our studies in calculus have dealt with continuous functions. Sequences model *discrete* instead of continuous information. We will study ways to represent and work with discrete information in this chapter as we investigate *sequences* and *series*, and ultimately see key connections between the discrete and continuous.

## Preview Activity 8.1.1

Suppose you receive $\$5000$ through an inheritance. You decide to invest this money into a fund that pays $8\%$ annually, compounded monthly. That means that each month your investment earns $\frac{0.08}{12} \cdot P$ additional dollars, where $P$ is your principal balance at the start of the month. So in the first month your investment earns

$$5000\left(\frac{0.08}{12}\right)$$

or $\$33.33$. If you reinvest this money, you will then have $\$5033.33$ in your account at the end of the first month. From this point on, assume that you reinvest all of the interest you earn.

1. How much interest will you earn in the second month? How much money will you have in your account at the end of the second month?

2. Complete Table 8.1.2 to determine the interest earned and total amount of money in this investment each month for one year.

*Table* 8.1.1. *Interest*

| Month | Interest earned | Total amount of money in the account |
|---|---|---|
| 0 | \$0.00 | \$5000.00 |
| 1 | \$33.33 | \$5033.33 |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| 11 | | |
| 12 | | |

3. As we will see later, the amount of money $P_n$ in the account after month $n$ is given by

$$P_n = 5000\left(1 + \frac{0.08}{12}\right)^n.$$

Use this formula to check your calculations in Table 8.1.2. Then find the amount of money in the account after 5 years.

4. How many years will it be before the account has doubled in value to \$10000?

## Sequences

As Preview Activity 8.1.1 illustrates, many discrete phenomena can be represented as lists of numbers (like the amount of money in an account over a period of months). We call any such list a *sequence*. A sequence is nothing more than list of terms in some order. We often list the entries of the sequence with subscripts,

$$s_1, s_2, \ldots, s_n \ldots,$$

where the subscript denotes the position of the entry in the sequence.

**Definition 8.1.1**

A sequence is a list of terms $s_1, s_2, s_3, \ldots$ in a specified order.

We can think of a sequence as a function $f$ whose domain is the set of positive integers where $f(n) = s_n$ for each positive integer $n$. This alternative view will be be useful in many situations.

We often denote the sequence

$$s_1, s_2, s_3, \ldots$$

by $\{s_n\}$. The value $s_n$ (alternatively $s(n)$) is called the $n$th *term* in the sequence. If the terms are all $0$ after some fixed value of $n$, we say the sequence is finite. Otherwise the sequence is infinite. With infinite sequences, we are often interested in their end behavior and the idea of *convergent* sequences.

## Activity 8.1.2

1. Let $s_n$ be the $n$th term in the sequence $1, 2, 3, \ldots$. Find a formula for $s_n$ and use appropriate technological tools to draw a graph of entries in this sequence by plotting points of the form $(n, s_n)$ for some values of $n$. Most graphing calculators can plot sequences; directions follow for the TI-84.
   - In the MODEmenu, highlight SEQin the FUNCline and press ENTER.
   - In the Y=menu, you will now see lines to enter sequences. Enter a value for nMin (where the sequence starts), a function for u(n) (the $n$th term in the sequence), and the value of u(nMin).
   - Set your window coordinates (this involves choosing limits for $n$ as well as the window coordinates XMin, XMax, YMin, and YMax.
   - The GRAPHkey will draw a plot of your sequence.

   Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

2. Let $s_n$ be the $n$th term in the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Find a formula for $s_n$. Draw a graph of some points in this sequence. Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.
3. Let $s_n$ be the $n$th term in the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$. Find a formula for $s_n$. Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

Next we formalize the ideas from Activity 8.1.2.

## Activity 8.1.3

1. Recall our earlier work with limits involving infinity in Section 2.8. State clearly what it means for a continuous function $f$ to have a limit $L$ as $x \to \infty$.
2. Given that an infinite sequence of real numbers is a function from the integers to the real numbers, apply the idea from part (a) to explain what you think it means for a sequence $\{s_n\}$ to have a limit as $n \to \infty$.
3. Based on your response to the part (b), decide if the sequence $\left\{\frac{1+n}{2+n}\right\}$ has a limit as $n \to \infty$. If so, what is the limit? If not, why not?

In Activities 8.1.2 and 8.1.3 we investigated a sequence $\{s_n\}$ that has a limit as $n$ goes to infinity. More formally, we make the following definition.

## Definition 8.1.2

A sequence $\{s_n\}$ ***converges*** or is a ***convergent sequence*** provided that there is a number $L$ such that we can make $s_n$ as close to $L$ as we want by taking $n$ sufficiently large. In this situation, we call $L$ the ***limit*** of the convergent sequence and write

$$\lim_{n\to\infty} s_n = L.$$

If the sequence $\{s_n\}$ does not converge, we say that the sequence $\{s_n\}$ ***diverges***.

The idea of sequence having a limit as $n \to \infty$ is the same as the idea of a continuous function having a limit as $x \to \infty$. The only difference is that sequences are discrete instead of continuous.

## Activity 8.1.4

Use graphical and/or algebraic methods to determine whether each of the following sequences converges or diverges.

1. $\left\{\dfrac{1+2n}{3n-2}\right\}$
2. $\left\{\dfrac{5+3^n}{10+2^n}\right\}$
3. $\left\{\frac{10^n}{n!}\right\}$ (where ! is the *factorial* symbol and $n! = n(n-1)(n-2)\cdots(2)(1)$ for any positive integer $n$ (as convention we define $0!$ to be 1)).

# Summary

- A sequence is a list of objects in a specified order. We will typically work with sequences of real numbers. We can think of a sequence as a function from the positive integers to the set of real numbers.
- A sequence $\{s_n\}$ of real numbers converges to a number $L$ if we can make every value of $s_k$ for $k \geq n$ as close as we want to $L$ by choosing $n$ sufficiently large.
- A sequence diverges if it does not converge.

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# 8.2: Geometric Series

## Motivating Questions

- What is a geometric series?
- What is a partial sum of a geometric series? What is a simplified form of the $n$th partial sum of a geometric series?
- Under what conditions does a geometric series converge? What is the sum of a convergent geometric series?

Many important sequences are generated by addition. In Preview Activity 8.2.1, we see an example of a sequence that is connected to a sum.

## Preview Activity 8.2.1

Warfarin is an anticoagulant that prevents blood clotting; often it is prescribed to stroke victims in order to help ensure blood flow. The level of warfarin has to reach a certain concentration in the blood in order to be effective.

Suppose warfarin is taken by a particular patient in a 5 mg dose each day. The drug is absorbed by the body and some is excreted from the system between doses. Assume that at the end of a 24 hour period, 8% of the drug remains in the body. Let $Q(n)$ be the amount (in mg) of warfarin in the body before the $(n+1)$st dose of the drug is administered.

1. Explain why $Q(1) = 5 \times 0.08$ mg.
2. Explain why $Q(2) = (5 + Q(1)) \times 0.08$ mg. Then show that
$$Q(2) = (5 \times 0.08)\,(1 + 0.08)\,\text{mg}.$$
3. Explain why $Q(3) = (5 + Q(2)) \times 0.08$ mg. Then show that
$$Q(3) = (5 \times 0.08)\left(1 + 0.08 + 0.08^2\right)\text{mg}.$$
4. Explain why $Q(4) = (5 + Q(3)) \times 0.08$ mg. Then show that
$$Q(4) = (5 \times 0.08)\left(1 + 0.08 + 0.08^2 + 0.08^3\right)\text{mg}.$$
5. There is a pattern that you should see emerging. Use this pattern to find a formula for $Q(n)$, where $n$ is an arbitrary positive integer.
6. Complete Table 8.2.1 with values of $Q(n)$ for the provided $n$-values (reporting $Q(n)$ to 10 decimal places). What appears to be happening to the sequence $Q(n)$ as $n$ increases?

*Table 8.2.1. Values of $Q(n)$ for selected values of $n$*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $Q(n)$ | 0.40 | | | | | | | | | |

## Geometric Series

In Preview Activity 8.2.1 we encountered the sum

$$(5 \times 0.08)\left(1 + 0.08 + 0.08^2 + 0.08^3 + \cdots + 0.08^{n-1}\right)$$

for the long-term level of Warfarin in the patient's system. This sum has the form

$$a + ar + ar^2 + \cdots + ar^{n-1} \tag{8.2.1}$$

where $a = 5 \times 0.08$ and $r = 0.08$. Such a sum is called a *finite geometric series* with ratio $r$.

## Activity 8.2.2

Let $a$ and $r$ be real numbers (with $r \neq 1$) and let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

In this activity we will find a shortcut formula for $S_n$ that does not involve a sum of $n$ terms.

1. Multiply $S_n$ by $r$. What does the resulting sum look like?
2. Subtract $rS_n$ from $S_n$ and explain why
$$S_n - rS_n = a - ar^n. \tag{8.2.2}$$
3. Solve equation (8.2.2) for $S_n$ to find a simple formula for $S_n$ that does not involve adding $n$ terms.

The sum of the terms of a sequence is called a *series*. We summarize the result of Activity 8.2.2 in the following way.

## Note

A finite geometric series $S_n$ is a sum of the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}, \tag{8.2.3}$$

where $a$ and $r$ are real numbers such that $r \neq 1$. The finite geometric series $S_n$ can be written more simply as

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}. \tag{8.2.4}$$

We now apply Equation (8.2.4) to the example involving Warfarin from Preview Activity 8.2.1. Recall that

$$Q(n) = (5 \times 0.08)\left(1 + 0.08 + 0.08^2 + 0.08^3 + \cdots + 0.08^{n-1}\right) \text{mg},$$

so $Q(n)$ is a geometric series with $a = 5 \times 0.08 = 0.4$ and $r = 0.08$. Thus,

$$Q(n) = 0.4\left(\frac{1-0.08^n}{1-0.08}\right) = \frac{1}{2.3}\left(1-0.08^n\right).$$

Notice that as $n$ goes to infinity, the value of $0.08^n$ goes to 0. So,

$$\lim_{n\to\infty} Q(n) = \lim_{n\to\infty} \frac{1}{2.3}\left(1-0.08^n\right) = \frac{1}{2.3} \approx 0.435.$$

Therefore, the long-term level of Warfarin in the blood under these conditions is $\frac{1}{2.3}$, which is approximately 0.435 mg.

To determine the long-term effect of Warfarin, we considered a finite geometric series of $n$ terms, and then considered what happened as $n$ was allowed to grow without bound. In this sense, we were actually interested in an infinite geometric series (the result of letting $n$ go to infinity in the finite sum).

## Definition 8.2.1

An infinite geometric series is an infinite sum of the form

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n. \tag{8.2.5}$$

The value of $r$ in the geometric series (8.2.5) is called the *common ratio* of the series because the ratio of the $(n+1)$st term, $ar^n$, to the $n$th term, $ar^{n-1}$, is always $r$:

$$\frac{ar^n}{ar^{n-1}} = r.$$

Geometric series are common in mathematics and arise naturally in many different situations. As a familiar example, suppose we want to write the number with repeating decimal expansion

$$N = 0.121\overline{212}$$

as a rational number. Observe that

$$\begin{aligned} N &= 0.12 + 0.0012 + 0.000012 + \cdots \\ &= \left(\frac{12}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right)^2 + \cdots. \end{aligned}$$

This is an infinite geometric series with $a = \frac{12}{100}$ and $r = \frac{1}{100}$.

By using the formula for the value of a finite geometric sum, we can also develop a formula for the value of an infinite geometric series. We explore this idea in the following activity.

## Activity 8.2.3

Let $r \neq 1$ and $a$ be real numbers and let

$$S = a + ar + ar^2 + \cdots ar^{n-1} + \cdots$$

be an infinite geometric series. For each positive integer $n$, let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Recall that

$$S_n = a\frac{1 - r^n}{1 - r}.$$

1. What should we allow $n$ to approach in order to have $S_n$ approach $S$?
2. What is the value of $\lim_{n\to\infty} r^n$ for $|r| > 1$? for $|r| < 1$? Explain.
3. If $|r| < 1$, use the formula for $S_n$ and your observations in (a) and (b) to explain why $S$ is finite and find a resulting formula for $S$.

We can now find the value of the geometric series

$$N = \left(\frac{12}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right)^2 + \cdots.$$

Using $a = \frac{12}{100}$ and $r = \frac{1}{100}$, we see that

$$N = \frac{12}{100}\left(\frac{1}{1 - \frac{1}{100}}\right) = \frac{12}{100}\left(\frac{100}{99}\right) = \frac{4}{33}.$$

The sum of a finite number of terms of an infinite geometric series is often called a *partial sum* of the series. Thus,

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k.$$

is called the $n$th partial sum of the series $\sum_{k=0}^{\infty} ar^k$. We summarize our recent work with geometric series as follows.

**Note**

- An infinite geometric series is an infinite sum of the form

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n, \tag{8.2.6}$$

  where $a$ and $r$ are real numbers such that $r \neq 0$.

- The $n$th partial sum $S_n$ of an infinite geometric series is

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

- If $|r| < 1$, then using the fact that $S_n = a\frac{1-r^n}{1-r}$, it follows that the sum $S$ of the infinite geometric series (8.2.6) is

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-r^n}{1-r} = \frac{a}{1-r}$$

**Activity 8.2.4**

The formulas we have derived for an infinite geometric series and its partial sum have assumed we begin indexing the sums at $n = 0$. If instead we have a sum that does not begin at $n = 0$, we can factor out common terms and use the established formulas. This process is illustrated in the examples in this activity.

1. Consider the sum

$$\sum_{k=1}^{\infty} (2)\left(\frac{1}{3}\right)^k = (2)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{3}\right)^2 + (2)\left(\frac{1}{3}\right)^3 + \cdots.$$

   Remove the common factor of $(2)\left(\frac{1}{3}\right)$ from each term and hence find the sum of the series.

2. Next let $a$ and $r$ be real numbers with $-1 < r < 1$. Consider the sum

$$\sum_{k=3}^{\infty} ar^k = ar^3 + ar^4 + ar^5 + \cdots.$$

   Remove the common factor of $ar^3$ from each term and find the sum of the series.

3. Finally, we consider the most general case. Let $a$ and $r$ be real numbers with $-1 < r < 1$, let $n$ be a positive integer, and consider the sum

$$\sum_{k=n}^{\infty} ar^k = ar^n + ar^{n+1} + ar^{n+2} + \cdots.$$

   Remove the common factor of $ar^n$ from each term to find the sum of the series.

## Summary

- An infinite geometric series is an infinite sum of the form

$$\sum_{k=0}^{\infty} ar^k$$

  where $a$ and $r$ are real numbers and $r \neq 0$.

- The $n$th partial sum of the geometric series $\sum_{k=0}^{\infty} ar^k$ is

$$S_n = \sum_{k=0}^{n-1} ar^k.$$

A formula for the $n$th partial sum of a geometric series is

$$S_n = a\frac{1-r^n}{1-r}.$$

If $|r| < 1$, the infinite geometric series $\sum_{k=0}^{\infty} ar^k$ has the finite sum $\frac{a}{1-r}$.

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# 8.3: Series of Real Numbers

## Motivating Questions

- What is an infinite series?
- What is the $n$th partial sum of an infinite series?
- How do we add up an infinite number of numbers? In other words, what does it mean for an infinite series of real numbers to converge?
- What does it mean for an infinite series of real numbers to diverge?

In Section 8.2, we encountered infinite geometric series. For example, by writing

$$N = 0.1212121212\cdots = \frac{12}{100} + \frac{12}{100}\cdot\frac{1}{100} + \frac{12}{100}\cdot\frac{1}{100^2} + \cdots$$

as a geometric series, we found a way to write $N$ as a single fraction: $N = \frac{4}{33}$. In this section, we explore other types of infinite sums. In Preview Activity 8.3.1 we see how one such sum is related to the famous number $e$.

## Preview Activity 8.3.1

Have you ever wondered how your calculator can produce a numeric approximation for complicated numbers like $e$, $\pi$ or $\ln(2)$? After all, the only operations a calculator can really perform are addition, subtraction, multiplication, and division, the operations that make up polynomials. This activity provides the first steps in understanding how this process works. Throughout the activity, let $f(x) = e^x$.

1. Find the tangent line to $f$ at $x = 0$ and use this linearization to approximate $e$. That is, find a formula $L(x)$ for the tangent line, and compute $L(1)$, since $L(1) \approx f(1) = e$.
2. The linearization of $e^x$ does not provide a good approximation to $e$ since 1 is not very close to 0. To obtain a better approximation, we alter our approach a bit. Instead of using a straight line to approximate $e$, we put an appropriate bend in our estimating function to make it better fit the graph of $e^x$ for $x$ close to 0. With the linearization, we had both $f(x)$ and $f'(x)$ share the same value as the linearization at $x = 0$. We will now use a quadratic approximation $P_2(x)$ to $f(x) = e^x$ centered at $x = 0$ which has the property that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$.
    1. Let $P_2(x) = 1 + x + \frac{x^2}{2}$. Show that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. Then, use $P_2(x)$ to approximate $e$ by observing that $P_2(1) \approx f(1)$.
    2. We can continue approximating $e$ with polynomials of larger degree whose higher derivatives agree with those of $f$ at 0. This turns out to make the polynomials fit the graph of $f$ better for more values of $x$ around 0. For example, let $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$. Show that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. Use $P_3(x)$ to approximate $e$ in a way similar to how you did so with $P_2(x)$ above.

## Infinite Series

Preview Activity 8.3.1 shows that an approximation to $e$ using a linear polynomial is 2, an approximation to $e$ using a quadratic polynomial is 2.5, and an approximation using a cubic polynomial is 2.6667. If we continue this process we can obtain approximations from quartic (degree 4) and quintic (degree 5) polynomials, giving us the following approximations to $e$:

| | | |
|---|---|---|
| **linear** | $1 + 1$ | $2$ |
| **quadratic** | $1 + 1 + \frac{1}{2}$ | $2.5$ |
| **cubic** | $1 + 1 + \frac{1}{2} + \frac{1}{6}$ | $2.\overline{6}$ |
| **quartic** | $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$ | $2.708\overline{3}$ |

| | | |
|---|---|---|
| **quintic** | $1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}$ | $2.71\overline{6}$ |

We see an interesting pattern here. The number $e$ is being approximated by the sum

$$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\cdots+\frac{1}{n!} \tag{8.3.1}$$

for increasing values of $n$.

We can use summation notation as a shorthand [1] for writing the sum in Equation (8.3.1) to get

$$e \approx 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\cdots+\frac{1}{n!}=\sum_{k=0}^{n}\frac{1}{k!}. \tag{8.3.2}$$

Note that $0!$ appears in Equation (8.3.2). By definition, $0!=1$.

We can calculate this sum using as large an $n$ as we want, and the larger $n$ is the more accurate the approximation (8.3.2) is. This suggests that we can write the number $e$ as the infinite sum

$$e=\sum_{k=0}^{\infty}\frac{1}{k!}. \tag{8.3.3}$$

This sum is an example of an *infinite series*. Note that the series (8.3.3) is the sum of the terms of the infinite sequence $\left\{\frac{1}{n!}\right\}$. In general, we use the following notation and terminology.

**Definition 8.3.1**

An ***infinite series*** of real numbers is the sum of the entries in an infinite sequence of real numbers. In other words, an infinite series is sum of the form

$$a_1+a_2+\cdots+a_n+\cdots=\sum_{k=1}^{\infty}a_k,$$

where $a_1, a_2, \ldots,$ are real numbers.

We use summation notation to identify a series. If the series adds the entries of a sequence $\{a_n\}_{n\geq 1}$, we will write the series as

$$\sum_{k\geq 1}a_k$$

or

$$\sum_{k=1}^{\infty}a_k.$$

Each of these notations is simply shorthand for the infinite sum $a_1+a_2+\cdots+a_n+\cdots$.

Is it even possible to sum an infinite list of numbers? That doing so is possible in some situations shouldn't come as a surprise. We have already examined the special case of geometric series in the previous section. Moreover, the definite integral is defineed as the limit of a sequence of finite sums, which provides insight into how we'll think of infinite sums generally. As we investigate other infinite sums, there are two key questions we seek to answer: (1) is the sum finite? and (2) if yes, what is its value?

**Activity 8.3.2**

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

While it is physically impossible to add an infinite collection of numbers, we can, of course, add any finite collection of them. In what follows, we investigate how understanding how to find the $n$th partial sum (that is, the sum of the first $n$ terms) enables us to make sense of the infinite sum.

1. Sum the first two numbers in this series. That is, find a numeric value for
$$\sum_{k=1}^{2} \frac{1}{k^2}$$
2. Next, add the first three numbers in the series.
3. Continue adding terms in this series to complete the list below. Carry each sum to at least 8 decimal places.
   - $\sum_{k=1}^{1} \frac{1}{k^2} = 1$
   - $\sum_{k=1}^{2} \frac{1}{k^2} =$
   - $\sum_{k=1}^{3} \frac{1}{k^2} =$
   - $\sum_{k=1}^{4} \frac{1}{k^2} =$
   - $\sum_{k=1}^{5} \frac{1}{k^2} =$
   - $\sum_{k=1}^{6} \frac{1}{k^2} =$
   - $\sum_{k=1}^{7} \frac{1}{k^2} =$
   - $\sum_{k=1}^{8} \frac{1}{k^2} =$
   - $\sum_{k=1}^{9} \frac{1}{k^2} =$
   - $\sum_{k=1}^{10} \frac{1}{k^2} =$
4. The sums in the table in part c form a sequence whose $n$th term is $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Based on your calculations in the table, do you think the sequence $\{S_n\}$ converges or diverges? Explain. How do you think this sequence $\{S_n\}$ is related to the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

The example in Activity 8.3.2 illustrates how we define the sum of an infinite series. We construct a new sequence of numbers (called the *sequence of partial sums*) where the $n$th term in the sequence consists of the sum of the first $n$ terms of the series. If this sequence converges, the corresponding infinite series is said to converge, and we say that we can find the sum of the series. More formally, we have the following definition.

**Definition 8.3.2**

The ***$n$th partial sum*** of the series $\sum_{k=1}^{\infty} a_k$ is the finite sum $S_n = \sum_{k=1}^{n} a_k$.

In other words, the $n$th partial sum $S_n$ of a series is the sum of the first $n$ terms in the series,

$$S_n = a_1 + a_2 + \cdots + a_n.$$

We investigate the behavior of the series by examining the sequence

$$S_1, S_2, \ldots, S_n, \ldots$$

of its partial sums. If the sequence of partial sums converges to some finite number, we say that the corresponding series *converges*. Otherwise, we say the series *diverges*. From our work in Activity 8.3.2, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

appears to converge to some number near 1.54977. We formalize the concept of convergence and divergence of an infinite series in the following definition.

**Definition 8.3.3**

The infinite series

$$\sum_{k=1}^{\infty} a_k$$

***converges*** (or is ***convergent***) if the sequence $\{S_n\}$ of partial sums converges, where

$$S_n = \sum_{k=1}^{n} a_k.$$

If $\lim_{n\to\infty} S_n = S$, then we call $S$ the sum of the series $\sum_{k=1}^{\infty} a_k$. That is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S.$$

If the sequence of partial sums does not converge, then the series

$$\sum_{k=1}^{\infty} a_k$$

***diverges*** (or is ***divergent***).

The early terms in a series do not influence whether or not the series converges or diverges. Rather, the convergence or divergence of a series

$$\sum_{k=1}^{\infty} a_k$$

is determined by what happens to the terms $a_k$ for very large values of $k$. To see why, suppose that $m$ is some constant larger than 1. Then

$$\sum_{k=1}^{\infty} a_k = (a_1 + a_2 + \cdots + a_m) + \sum_{k=m+1}^{\infty} a_k.$$

Since $a_1 + a_2 + \cdots + a_m$ is a finite number, the series $\sum_{k=1}^{\infty} a_k$ will converge if and only if the series $\sum_{k=m+1}^{\infty} a_k$ converges. Because the starting index of the series doesn't affect whether the series converges or diverges, we will often just write

$$\sum a_k$$

when we are interested in questions of convergence/divergence as opposed to the exact sum of a series.

In Section 8.2 we encountered the special family of infinite geometric series. Recall that a geometric series has the form $\sum_{k=0}^{\infty} ar^k$, where $a$ and $r$ are real numbers (and $r \neq 1$). We found that the $n$th partial sum $S_n$ of a geometric series is given by the convenient formula

$$S_n = \frac{1 - r^n}{1 - r},$$

and thus a geometric series converges if $|r| < 1$. Geometric series diverge for all other values of $r$.

It is generally a difficult question to determine if a given nongeometric series converges or diverges. There are several tests we can use that we will consider in the following sections.

## The Divergence Test

The first question we ask about any infinite series is usually "Does the series converge or diverge?" There is a straightforward way to check that certain series diverge, and we explore this test in the next activity.

### Activity 8.3.3

If the *series* $\sum a_k$ converges, then an important result necessarily follows regarding the *sequence* $\{a_n\}$. This activity explores this result.

Assume that the series $\sum_{k=1}^{\infty} a_k$ converges and has sum equal to $L$.

1. What is the $n$th partial sum $S_n$ of the series $\sum_{k=1}^{\infty} a_k$?
2. What is the $(n-1)$st partial sum $S_{n-1}$ of the series $\sum_{k=1}^{\infty} a_k$?
3. What is the difference between the $n$th partial sum and the $(n-1)$st partial sum of the series $\sum_{k=1}^{\infty} a_k$?
4. Since we are assuming that $\sum_{k=1}^{\infty} a_k = L$, what does that tell us about $\lim_{n\to\infty} S_n$? Why? What does that tell us about $\lim_{n\to\infty} S_{n-1}$? Why?
5. Combine the results of the previous two parts of this activity to determine $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$.

The result of Activity 8.3.3 is the following important conditional statement:

> If the series $\sum_{k=1}^{\infty} a_k$ converges, then the sequence $\{a_k\}$ of $k$th terms converges to 0.

It is logically equivalent to say that if the sequence $\{a_k\}$ does not converge to 0, then the series $\sum_{k=1}^{\infty} a_k$ cannot converge. This statement is called the Divergence Test.

### The Divergence Test

If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum a_k$ diverges.

### Activity 8.3.4

Determine if the Divergence Test applies to the following series. If the test does not apply, explain why. If the test does apply, what does it tell us about the series?

1. $\sum \dfrac{k}{k+1}$
2. $\sum (-1)^k$
3. $\sum \dfrac{1}{k}$

*Note well:* be very careful with the Divergence Test. This test only tells us what happens to a series if the terms of the corresponding sequence do not converge to 0. If the sequence of the terms of the series does converge to 0, the Divergence Test does not apply: indeed, as we will soon see, a series whose terms go to zero may either converge or diverge.

## The Integral Test

The Divergence Test settles the questions of divergence or convergence of series $\sum a_k$ in which $\lim_{k\to\infty} a_k \neq 0$. Determining the convergence or divergence of series $\sum a_k$ in which $\lim_{k\to\infty} a_k = 0$ turns out to be more complicated. Often, we have to investigate

the sequence of partial sums or apply some other technique.

Next, we consider the ***harmonic series*** [2]

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

The first 9 partial sums of this series are shown following.

$$\sum_{k=1}^{1} \frac{1}{k} = 1 \qquad \sum_{k=1}^{4} \frac{1}{k} = 2.083333333 \qquad \sum_{k=1}^{7} \frac{1}{k} = 2.592857143$$

$$\sum_{k=1}^{2} \frac{1}{k} = 1.5 \qquad \sum_{k=1}^{5} \frac{1}{k} = 2.283333333 \qquad \sum_{k=1}^{8} \frac{1}{k} = 2.717857143$$

$$\sum_{k=1}^{3} \frac{1}{k} = 1.833333333 \qquad \sum_{k=1}^{6} \frac{1}{k} = 2.450000000 \qquad \sum_{k=1}^{9} \frac{1}{k} = 2.828968254$$

This series is called harmonic because each term in the series after the first is the harmonic mean of the term before it and the term after it. The harmonic mean of two numbers $a$ and $b$ is $\frac{2ab}{a+b}$. See "What's Harmonic about the Harmonic Series", by David E. Kullman (in the College Mathematics Journal, Vol. 32, No. 3 (May, 2001), 201-203) for an interesting discussion of the harmonic mean.

This information alone doesn't seem to be enough to tell us if the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converges or diverges. The partial sums could eventually level off to some fixed number or continue to grow without bound. Even if we look at larger partial sums, such as $\sum_{n=1}^{1000} \frac{1}{k} \approx 7.485470861$, the result isn't particularly convincing one way or another. The Integral Test is one way to determine whether or not the harmonic series converges, and we explore this test further in the next activity.

## Activity 8.3.5

Consider the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Recall that the harmonic series will converge provided that its sequence of partial sums converges. The $n$th partial sum $S_n$ of the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &= 1(1) + (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{3}\right) + \cdots + (1)\left(\frac{1}{n}\right). \end{aligned}$$

Through this last expression for $S_n$, we can visualize this partial sum as a sum of areas of rectangles with heights $\frac{1}{m}$ and bases of length 1, as shown in Figure 8.3.1, which uses the 9th partial sum.

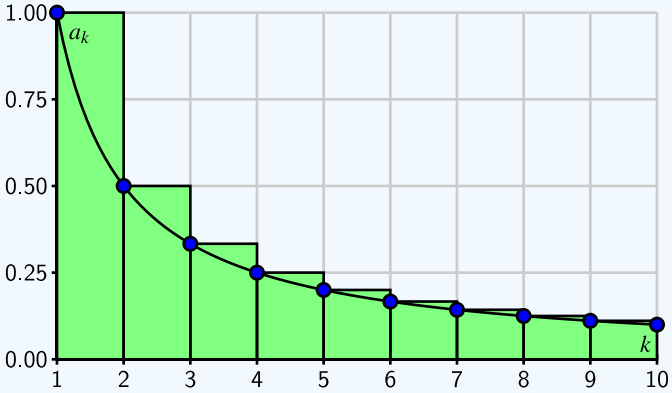


Figure 8.3.1. A picture of the 9th partial sum of the harmonic series as a sum of areas of rectangles.

The graph of the continuous function $f$ defined by $f(x) = \frac{1}{x}$ is overlaid on this plot.

1. Explain how this picture represents a particular Riemann sum.
2. What is the definite integral that corresponds to the Riemann sum you considered in (a)?
3. Which is larger, the definite integral in (b), or the corresponding partial sum $S_9$ of the series? Why?
4. If instead of considering the 9th partial sum, we consider the $n$th partial sum, and we let $n$ go to infinity, we can then compare the series $\sum_{k=1}^{\infty} \frac{1}{k}$ to the improper integral $\int_1^{\infty} \frac{1}{x}\ dx$. Which of these quantities is larger? Why?
5. Does the improper integral $\int_1^{\infty} \frac{1}{x}\ dx$ converge or diverge? What does that result, together with your work in (d), tell us about the series $\sum_{k=1}^{\infty} \frac{1}{k}$?

The ideas from Activity 8.3.5 hold more generally. Suppose that $f$ is a continuous decreasing function and that $a_k = f(k)$ for each value of $k$. Consider the corresponding series $\sum_{k=1}^{\infty} a_k$. The partial sum

$$S_n = \sum_{k=1}^{n} a_k$$

can always be viewed as a left hand Riemann sum of $f(x)$, using rectangles with bases of width 1 and heights given by the values $a_k$. A representative picture is shown at left in Figure 8.3.2. Since $f$ is a decreasing function, we have that

$$S_n > \int_1^n f(x)\ dx.$$

Taking the limit as $n$ goes to infinity shows that

$$\sum_{k=1}^{\infty} a_k > \int_1^{\infty} f(x)\ dx.$$

Therefore, if the improper integral $\int_1^{\infty} f(x)\ dx$ diverges, so does the series $\sum_{k=1}^{\infty} a_k$.

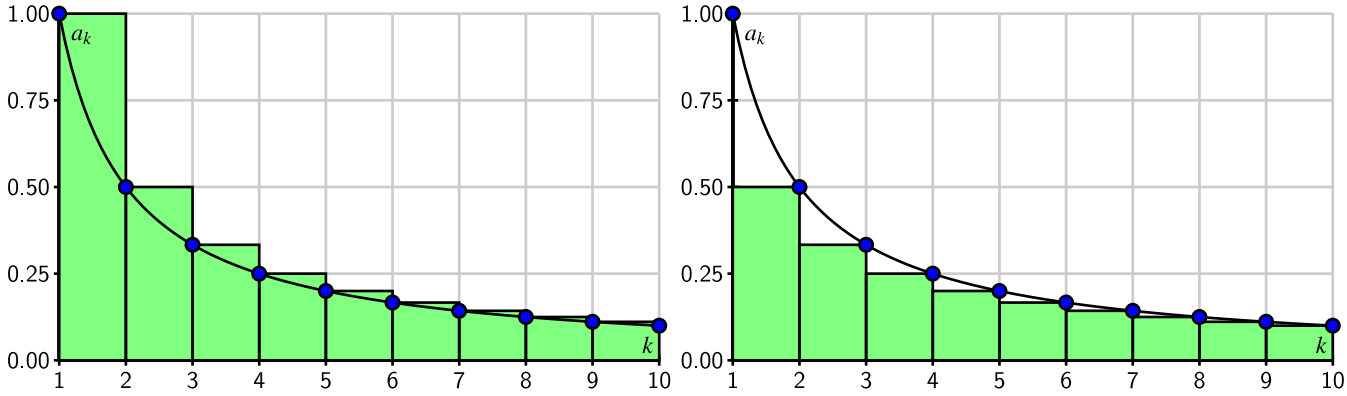



Figure 8.3.2. Comparing an improper integral to a series

What's more, if we look at the right hand Riemann sums of $f$ on $[1, n]$ as shown at right in Figure 8.3.5, we see that

$$\int_1^{\infty} f(x)\ dx > \sum_{k=2}^{\infty} a_k.$$

So if $\int_1^{\infty} f(x)\ dx$ converges, then so does $\sum_{k=2}^{\infty} a_k$, which also means that the series $\sum_{k=1}^{\infty} a_k$ also converges. Our preceding discussion has demonstrated the truth of the Integral Test.

### The Integral Test

Let $f$ be a real valued function and assume $f$ is decreasing and positive for all $x$ larger than some number $c$. Let $a_k = f(k)$ for each positive integer $k$.

1. If the improper integral $\int_c^\infty f(x)\ dx$ converges, then the series $\sum_{k=1}^\infty a_k$ converges.
2. If the improper integral $\int_c^\infty f(x)\ dx$ diverges, then the series $\sum_{k=1}^\infty a_k$ diverges.

The Integral Test compares a given infinite series to a natural, corresponding improper integral and says that the infinite series and corresponding improper integral both have the same convergence status. In the next activity, we apply the Integral Test to determine the convergence or divergence of a class of important series.

### Activity 8.3.6

The series $\sum \frac{1}{k^p}$ are special series called $p$-series. We have already seen that the $p$-series with $p = 1$ (the harmonic series) diverges. We investigate the behavior of other $p$-series in this activity.

1. Evaluate the improper integral $\int_1^\infty \frac{1}{x^2}\ dx$. Does the series $\sum_{k=1}^\infty \frac{1}{k^2}$ converge or diverge? Explain.
2. Evaluate the improper integral $\int_1^\infty \frac{1}{x^p}\ dx$ where $p > 1$. For which values of $p$ can we conclude that the series $\sum_{k=1}^\infty \frac{1}{k^p}$ converges?
3. Evaluate the improper integral $\int_1^\infty \frac{1}{x^p}\ dx$ where $p < 1$. What does this tell us about the corresponding $p$-series $\sum_{k=1}^\infty \frac{1}{k^p}$?
4. Summarize your work in this activity by completing the following statement.

   The $p$-series $\sum_{k=1}^\infty \frac{1}{k^p}$ converges if and only if .

## The Limit Comparison Test

The Integral Test allows us to determine the convergence of an entire family of series: the $p$-series. However, we have seen that it is often difficult to integrate functions, so the Integral Test is not one that we can use all of the time. In fact, even for a relatively simple series such as $\sum \frac{k^2+1}{k^4+2k+2}$, the Integral Test is not an option. In what follows we develop a test that applies to series of rational functions by comparing their behavior to the behavior of $p$-series.

### Activity 8.3.7

Consider the series $\sum \frac{k+1}{k^3+2}$. Since the convergence or divergence of a series only depends on the behavior of the series for large values of $k$, we might examine the terms of this series more closely as $k$ gets large.

1. By computing the value of $\frac{k+1}{k^3+2}$ for $k = 100$ and $k = 1000$, explain why the terms $\frac{k+1}{k^3+2}$ are essentially $\frac{k}{k^3}$ when $k$ is large.
2. Let's formalize our observations in (a) a bit more. Let $a_k = \frac{k+1}{k^3+2}$ and $b_k = \frac{k}{k^3}$. Calculate

   $$\lim_{k\to\infty} \frac{a_k}{b_k}.$$

   What does the value of the limit tell you about $a_k$ and $b_k$ for large values of $k$? Compare your response from part (a).

3. Does the series $\sum \frac{k}{k^3}$ converge or diverge? Why? What do you think that tells us about the convergence or divergence of the series $\sum \frac{k+1}{k^3+2}$? Explain.

Activity 8.3.7 illustrates how we can compare one series with positive terms to another whose convergence status we know. Suppose we have two series $\sum a_k$ and $\sum b_k$ with positive terms and we know the convergence status of the series $\sum a_k$. Recall that the convergence or divergence of a series depends only on the terms of the series for large values of $k$, so if we know that $a_k$ and $b_k$ are proportional for large $k$, then the two series $\sum a_k$ and $\sum b_k$ should behave the same way. In other words, if there is a positive finite constant $c$ such that

$$\lim_{k\to\infty} \frac{b_k}{a_k} = c,$$

then $b_k \approx ca_k$ for large values of $k$. So

$$\sum b_k \approx \sum ca_k = c\sum a_k.$$

Since multiplying by a nonzero constant does not affect the convergence or divergence of a series, it follows that the series $\sum a_k$ and $\sum b_k$ either both converge or both diverge. The formal statement of this fact is called the Limit Comparison Test.

**The Limit Comparison Test**

Let $\sum a_k$ and $\sum b_k$ be series with positive terms. If

$$\lim_{k\to\infty} \frac{b_k}{a_k} = c$$

for some positive (finite) constant $c$, then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

The Limit Comparison Test shows that if we have a series $\sum \frac{p(k)}{q(k)}$ of rational functions where $p(k)$ is a polynomial of degree $m$ and $q(k)$ a polynomial of degree $l$, then the series $\sum \frac{p(k)}{q(k)}$ will behave like the series $\sum \frac{k^m}{k^l}$. So this test allows us to determine the convergence or divergence of series whose terms are rational functions.

**Activity 8.3.8**

Use the Limit Comparison Test to determine the convergence or divergence of the series

$$\sum \frac{3k^2+1}{5k^4+2k+2}.$$

by comparing it to the series $\sum \frac{1}{k^2}$.

## The Ratio Test

The Limit Comparison Test works well if we can find a series with known behavior to compare. But such series are not always easy to find. In this section we will examine a test that allows us to examine the behavior of a series by comparing it to a geometric series, without knowing in advance which geometric series we need.

**Activity 8.3.9**

Consider the series defined by

$$\sum_{k=1}^{\infty} \frac{2^k}{3^k-k}. \tag{8.3.4}$$

This series is not a geometric series, but this activity will illustrate how we might compare this series to a geometric one. Recall that a series $\sum a_k$ is geometric if the ratio $\frac{a_{k+1}}{a_k}$ is always the same. For the series in (8.3.4), note that $a_k = \frac{2^k}{3^k-k}$.

1. To see if $\sum \frac{2^k}{3^k-k}$ is comparable to a geometric series, we analyze the ratios of successive terms in the series. Complete Table 8.3.6, listing your calculations to at least 8 decimal places.

   *Table 8.3.1. Ratios of successive terms in the series* $\sum \frac{2^k}{3^k-k}$

| $k$ | 5 | 10 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|

| $\frac{a_{k+1}}{a_k}$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|

2. Based on your calculations in Table 8.3.1, what can we say about the ratio $\frac{a_{k+1}}{a_k}$ if $k$ is large?
3. Do you agree or disagree with the statement: "the series $\sum \frac{2^k}{3^k-k}$ is approximately geometric when $k$ is large"? If not, why not? If so, do you think the series $\sum \frac{2^k}{3^k-k}$ converges or diverges? Explain.

We can generalize the argument in Activity 8.3.9 in the following way. Consider the series $\sum a_k$. If

$$\frac{a_{k+1}}{a_k} \approx r$$

for large values of $k$, then $a_{k+1} \approx ra_k$ for large $k$ and the series $\sum a_k$ is approximately the geometric series $\sum ar^k$ for large $k$. Since the geometric series with ratio $r$ converges only for $-1 < r < 1$, we see that the series $\sum a_k$ will converge if

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$$

for a value of $r$ such that $|r| < 1$. This result is known as the Ratio Test.

### The Ratio Test

Let $\sum a_k$ be an infinite series. Suppose

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = r.$$

1. If $0 \le r < 1$, then the series $\sum a_k$ converges.
2. If $1 < r$, then the series $\sum a_k$ diverges.
3. If $r = 1$, then the test is inconclusive.

*Note well:* The Ratio Test looks at the limit of the ratio of consecutive terms of a given series; in so doing, the test is asking, "is this series approximately geometric?" If so, the test uses the limit of the ratio of consecutive terms to determine if the given series converges.

We have now encountered several tests for determining convergence or divergence of series.

- The Divergence Test can be used to show that a series diverges, but never to prove that a series converges.
- We used the Integral Test to determine the convergence status of an entire class of series, the $p$-series.
- The Limit Comparison Test works well for series that involve rational functions and which can therefore by compared to $p$ -series.
- Finally, the Ratio Test allows us to compare our series to a geometric series; it is particularly useful for series that involve $n$ th powers and factorials.
- Two other tests, the Direct Comparison Test and the Root Test, are discussed in the exercises.

One of the challenges of determining whether a series converges or diverges is finding which test answers that question.

### Activity 8.3.10

Determine whether each of the following series converges or diverges. Explicitly state which test you use.

1. $\sum \frac{k}{2^k}$
2. $\sum \frac{k^3+2}{k^2+1}$
3. $\sum \frac{10^k}{k!}$

4. $\sum \dfrac{k^3 - 2k^2 + 1}{k^6 + 4}$

## Summary

- An infinite series is a sum of the elements in an infinite sequence. In other words, an infinite series is a sum of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

  where $a_k$ is a real number for each positive integer $k$.

- The $n$th partial sum $S_n$ of the series $\sum_{k=1}^{\infty} a_k$ is the sum of the first $n$ terms of the series. That is,

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

- The sequence $\{S_n\}$ of partial sums of a series $\sum_{k=1}^{\infty} a_k$ tells us about the convergence or divergence of the series. In particular
  - The series $\sum_{k=1}^{\infty} a_k$ converges if the sequence $\{S_n\}$ of partial sums converges. In this case we say that the series is the limit of the sequence of partial sums and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n.$$

  - The series $\sum_{k=1}^{\infty} a_k$ diverges if the sequence $\{S_n\}$ of partial sums diverges.

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# 8.4: Alternating Series

## Motivating Questions

- What is an alternating series?
- Under what conditions does an alternating series converge? Why?
- How well does the $n$th partial sum of a convergent alternating series approximate the actual sum of the series? Why?

So far, we've considered series with exclusively nonnegative terms. Next, we consider series that have some negative terms. For instance, the geometric series

$$2 - \frac{4}{3} + \frac{8}{9} - \cdots + 2\left(-\frac{2}{3}\right)^n + \cdots,$$

has $a = 2$ and $r = -\frac{2}{3}$, so that every other term alternates in sign. This series converges to

$$S = \frac{a}{1-r} = \frac{2}{1-\left(-\frac{2}{3}\right)} = \frac{6}{5}.$$

In Preview Activity $8.4.1$ and our following discussion, we investigate the behavior of similar series where consecutive terms have opposite signs.

## Preview Activity $8.4.1$

Preview Activity 8.3.1 showed how we can approximate the number $e$ with linear, quadratic, and other polynomial approximations. We use a similar approach in this activity to obtain linear and quadratic approximations to $\ln(2)$. Along the way, we encounter a type of series that is different than most of the ones we have seen so far. Throughout this activity, let $f(x) = \ln(1+x)$.

1. Find the tangent line to $f$ at $x = 0$ and use this linearization to approximate $\ln(2)$. That is, find $L(x)$, the tangent line approximation to $f(x)$, and use the fact that $L(1) \approx f(1)$ to estimate $\ln(2)$.
2. The linearization of $\ln(1+x)$ does not provide a very good approximation to $\ln(2)$ since $1$ is not that close to $0$. To obtain a better approximation, we alter our approach; instead of using a straight line to approximate $\ln(2)$, we use a quadratic function to account for the concavity of $\ln(1+x)$ for $x$ close to $0$. With the linearization, both the function's value and slope agree with the linearization's value and slope at $x = 0$. We will now make a quadratic approximation $P_2(x)$ to $f(x) = \ln(1+x)$ centered at $x = 0$ with the property that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$.
    1. Let $P_2(x) = x - \frac{x^2}{2}$. Show that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. Use $P_2(x)$ to approximate $\ln(2)$ by using the fact that $P_2(1) \approx f(1)$.
    2. We can continue approximating $\ln(2)$ with polynomials of larger degree whose derivatives agree with those of $f$ at $0$. This makes the polynomials fit the graph of $f$ better for more values of $x$ around $0$. For example, let $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. Show that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. Taking a similar approach to preceding questions, use $P_3(x)$ to approximate $\ln(2)$.
    3. If we used a degree $4$ or degree $5$ polynomial to approximate $\ln(1+x)$, what approximations of $\ln(2)$ do you think would result? Use the preceding questions to conjecture a pattern that holds, and state the degree $4$ and degree $5$ approximation.

## The Alternating Series Test

Preview Activity $8.4.1$ gives us several approximations to $\ln(2)$. The linear approximation is $1$, and the quadratic approximation is $1 - \frac{1}{2} = \frac{1}{2}$. If we continue this process, cubic, quartic (degree $4$), quintic (degree $5$), and higher degree polynomials give us the approximations to $\ln(2)$ in Table $8.4.1$.

*Table* 8.4.1.

| | | |
|---|---|---|
| **linear** | $1$ | $1$ |
| **quadratic** | $1-\frac{1}{2}$ | $0.5$ |
| **cubic** | $1-\frac{1}{2}+\frac{1}{3}$ | $0.8\overline{3}$ |
| **quartic** | $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$ | $0.58\overline{3}$ |
| **quintic** | $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$ | $0.78\overline{3}$ |

The pattern here shows that $\ln(2)$ can be approximated by the partial sums of the infinite series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} \tag{8.4.1}$$

where the alternating signs are indicated by the factor $(-1)^{k+1}$. We call such a series an *alternating series*.

Using computational technology, we find that the sum of the first 100 terms in this series is 0.6881721793. As a comparison, $\ln(2) \approx 0.6931471806$. This shows that even though the series (8.4.1) converges to $\ln(2)$, it must do so quite slowly, since the sum of the first 100 terms isn't particularly close to $\ln(2)$. We will investigate the issue of how quickly an alternating series converges later in this section.

**Definition 8.4.1**

An alternating series is a series of the form

$$\sum_{k=0}^{\infty}(-1)^k a_k,$$

where $a_k > 0$ for each $k$.

We have some flexibility in how we write an alternating series; for example, the series

$$\sum_{k=1}^{\infty}(-1)^{k+1} a_k,$$

whose index starts at $k=1$, is also alternating. As we will soon see, there are several very nice results that hold for alternating series, while alternating series can also demonstrate some unusual behaivior.

It is important to remember that most of the series tests we have seen in previous sections apply only to series with nonnegative terms. Alternating series require a different test.

**Activity 8.4.2**

Remember that, by definition, a series converges if and only if its corresponding sequence of partial sums converges.

1. Calculate the first few partial sums (to 10 decimal places) of the alternating series
$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}.$$
Label each partial sum with the notation $S_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$ for an appropriate choice of $n$.
2. Plot the sequence of partial sums from part (a). What do you notice about this sequence?

Activity 8.4.2 illustrates the general behavior of any convergent alternating series. We see that the partial sums of the alternating

harmonic series oscillate around a fixed number that turns out to be the sum of the series.

Recall that if $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum a_k$ diverges by the Divergence Test. From this point forward, we will thus only consider alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

in which the sequence $a_k$ consists of positive numbers that decrease to $0$. The $n$th partial sum $S_n$ is

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k.$$

Notice that

- $S_2 = a_1 - a_2$, and since $a_1 > a_2$ we have $0 < S_2 < S_1$.
- $S_3 = S_2 + a_3$ and so $S_2 < S_3$. But $a_3 < a_2$, so $S_3 < S_1$. Thus, $0 < S_2 < S_3 < S_1$.
- $S_4 = S_3 - a_4$ and so $S_4 < S_3$. But $a_4 < a_3$, so $S_2 < S_4$. Thus, $0 < S_2 < S_4 < S_3 < S_1$.
- $S_5 = S_4 + a_5$ and so $S_4 < S_5$. But $a_5 < a_4$, so $S_5 < S_3$. Thus, $0 < S_2 < S_4 < S_5 < S_3 < S_1$.

This pattern continues as illustrated in Figure $8.4.1$ (with $n$ odd) so that each partial sum lies between the previous two partial sums.

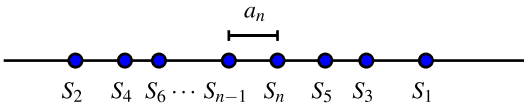



Figure $8.4.1$. Partial sums of an alternating series

Note further that the absolute value of the difference between the $(n-1)$st partial sum $S_{n-1}$ and the $n$th partial sum $S_n$ is

$$|S_n - S_{n-1}| = a_n.$$

Because the sequence $\{a_n\}$ converges to $0$, the distance between successive partial sums becomes as close to zero as we'd like, and thus the sequence of partial sums converges (even though we don't know the exact value to which it converges).

The preceding discussion has demonstrated the truth of the Alternating Series Test.

### The Alternating Series Test

Given an alternating series $\sum (-1)^k a_k$, if the sequence $\{a_k\}$ of positive terms decreases to $0$ as $k \to \infty$, then the alternating series converges.

Note that if the limit of the sequence $\{a_k\}$ is not $0$, then the alternating series diverges.

### Activity $8.4.3$

Which series converge and which diverge? Justify your answers.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+2}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2k}{k+5}$

3. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$

## Estimating Alternating Sums

If the series converges, the argument for the Alternating Series Test also provides us with a method to determine how close the $n$ th partial sum $S_n$ is to the actual sum of the series. To see how this works, let $S$ be the sum of a convergent alternating series, so

$$S = \sum_{k=1}^{\infty} (-1)^k a_k.$$

Recall that the sequence of partial sums oscillates around the sum $S$ so that

$$|S - S_n| < |S_{n+1} - S_n| = a_{n+1}.$$

Therefore, the value of the term $a_{n+1}$ provides an error estimate for how well the partial sum $S_n$ approximates the actual sum $S$. We summarize this fact in the statement of the Alternating Series Estimation Theorem.

### Alternating Series Estimation Theorem

If the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ has positive terms $a_k$ that decrease to zero as $k \to \infty$, and $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ is the $n$th partial sum of the alternating series, then

$$\left| \sum_{k=1}^{\infty} (-1)^{k+1} a_k - S_n \right| \leq a_{n+1}.$$

### Example 8.4.1

Determine how well the $100$th partial sum $S_{100}$ of

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

approximates the sum of the series.

*Answer available at the end of this section.*

### Activity 8.4.4

Determine the number of terms it takes to approximate the sum of the convergent alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$$

to within $0.0001$.

## Absolute and Conditional Convergence

A series such as

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots \tag{8.4.2}$$

whose terms are neither all nonnegative nor alternating is different from any series that we have considered so far. The behavior of such a series can be rather complicated, but there is an important connection between a series with some negative terms and series with all positive terms.

## Activity 8.4.5

1. Explain why the series
$$1-\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}-\frac{1}{81}-\frac{1}{100}+\cdots$$
must have a sum that is less than the series
$$\sum_{k=1}^{\infty}\frac{1}{k^2}.$$
2. Explain why the series
$$1-\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}-\frac{1}{81}-\frac{1}{100}+\cdots$$
must have a sum that is greater than the series
$$\sum_{k=1}^{\infty}-\frac{1}{k^2}.$$
3. Given that the terms in the series
$$1-\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}-\frac{1}{81}-\frac{1}{100}+\cdots$$
converge to 0, what do you think the previous two results tell us about the convergence status of this series?

As the example in Activity 8.4.5 suggests, if a series $\sum a_k$ has some negative terms but $\sum |a_k|$ converges, then the original series, $\sum a_k$, must also converge. That is, if $\sum |a_k|$ converges, then so must $\sum a_k$.

As we just observed, this is the case for the series (8.4.2), because the corresponding series of the absolute values of its terms is the convergent $p$-series $\sum \frac{1}{k^2}$. But there are series, such as the alternating harmonic series $\sum (-1)^{k+1}\frac{1}{k}$, that converge while the corresponding series of absolute values, $\sum \frac{1}{k}$, diverges. We distinguish between these behaviors by introducing the following language.

## Definition 8.4.6

Consider a series $\sum a_k$.

1. The series $\sum a_k$ ***converges absolutely*** (or is ***absolutely convergent***) provided that $\sum |a_k|$ converges.
2. The series $\sum a_k$ ***converges conditionally*** (or is ***conditionally convergent***) provided that $\sum |a_k|$ diverges and $\sum a_k$ converges.

In this terminology, the series (8.4.2) converges absolutely while the alternating harmonic series is conditionally convergent.

## Activity 8.4.6

1. Consider the series $\sum (-1)^k \frac{\ln(k)}{k}$.
    1. Does this series converge? Explain.
    2. Does this series converge absolutely? Explain what test you use to determine your answer.
2. Consider the series $\sum (-1)^k \frac{\ln(k)}{k^2}$.
    1. Does this series converge? Explain.

2. Does this series converge absolutely? Hint: Use the fact that $\ln(k) < \sqrt{k}$ for large values of $k$ and then compare to an appropriate $p$-series.

Conditionally convergent series turn out to be very interesting. If the sequence $\{a_n\}$ decreases to 0, but the series $\sum a_k$ diverges, the conditionally convergent series $\sum (-1)^k a_k$ is right on the borderline of being a divergent series. As a result, any conditionally convergent series converges very slowly. Furthermore, some very strange things can happen with conditionally convergent series, as illustrated in some of the exercises.

# Summary of Tests for Convergence of Series

We have discussed several tests for convergence/divergence of series in our sections and in exercises. We close this section of the text with a summary of all the tests we have encountered, followed by an activity that challenges you to decide which convergence test to apply to several different series.

**Geometric Series**

The geometric series $\sum ar^k$ with ratio $r$ converges for $-1 < r < 1$ and diverges for $|r| \geq 1$.

The sum of the convergent geometric series $\sum_{k=0}^{\infty} ar^k$ is $\frac{a}{1-r}$.

**Divergence Test**

If the sequence $a_n$ does not converge to 0, then the series $\sum a_k$ diverges.

This is the first test to apply because the conclusion is simple. However, if $\lim_{n\to\infty} a_n = 0$, no conclusion can be drawn.

**Integral Test**

Let $f$ be a positive, decreasing function on an interval $[c, \infty)$ and let $a_k = f(k)$ for each positive integer $k \geq c$.

- If $\int_c^{\infty} f(t)\ dt$ converges, then $\sum a_k$ converges.
- If $\int_c^{\infty} f(t)\ dt$ diverges, then $\sum a_k$ diverges.

Use this test when $f(x)$ is easy to integrate.

**Direct Comp. Test**

(see Ex 4 in Section 8.3)

Let $0 \leq a_k \leq b_k$ for each positive integer $k$.

- If $\sum b_k$ converges, then $\sum a_k$ converges.
- If $\sum a_k$ diverges, then $\sum b_k$ diverges.

Use this test when you have a series with known behavior that you can compare to — this test can be difficult to apply.

**Limit Comp. Test**

Let $a_n$ and $b_n$ be sequences of positive terms. If

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L$$

for some positive finite number $L$, then the two series $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

Easier to apply in general than the comparison test, but you must have a series with known behavior to compare.

Useful to apply to series of rational functions.

**Ratio Test**

Let $a_k \neq 0$ for each $k$ and suppose

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = r.$$

- If $r < 1$, then the series $\sum a_k$ converges absolutely.
- If $r > 1$, then the series $\sum a_k$ diverges.
- If $r = 1$, then test is inconclusive.

This test is useful when a series involves factorials and powers.

**Root Test**

(see Exercise 2 in Section 8.3)

Let $a_k \geq 0$ for each $k$ and suppose

$$\lim_{k\to\infty} \sqrt[k]{a_k} = r.$$

- If $r < 1$, then the series $\sum a_k$ converges.
- If $r > 1$, then the series $\sum a_k$ diverges.
- If $r = 1$, then test is inconclusive.

In general, the Ratio Test can usually be used in place of the Root Test. However, the Root Test can be quick to use when $a_k$ involves $k$th powers.

**Alt. Series Test**

If $a_n$ is a positive, decreasing sequence so that $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum (-1)^{k+1} a_k$ converges.

This test applies only to alternating series — we assume that the terms $a_n$ are all positive and that the sequence $\{a_n\}$ is decreasing.

**Alt. Series Est.**

Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ be the $n$th partial sum of the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$. Assume $a_n > 0$ for each positive integer $n$, the sequence $a_n$ decreases to 0 and $\lim_{n\to\infty} S_n = S$. Then it follows that $|S - S_n| < a_{n+1}$.

This bound can be used to determine the accuracy of the partial sum $S_n$ as an approximation of the sum of a convergent alternating series.

## Activity 8.4.7

For (a)-(j), use appropriate tests to determine the convergence or divergence of the following series. Throughout, if a series is a convergent geometric series, find its sum.

1. $\displaystyle\sum_{k=3}^{\infty} \frac{2}{\sqrt{k-2}}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{1+2k}$
3. $\displaystyle\sum_{k=0}^{\infty} \frac{2k^2+1}{k^3+k+1}$

4. $\sum_{k=0}^{\infty} \frac{100^k}{k!}$
5. $\sum_{k=1}^{\infty} \frac{2^k}{5^k}$
6. $\sum_{k=1}^{\infty} \frac{k^3-1}{k^5+1}$
7. $\sum_{k=2}^{\infty} \frac{3^{k-1}}{7^k}$
8. $\sum_{k=2}^{\infty} \frac{1}{k^k}$
9. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+1}}$
10. $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$
11. Determine a value of $n$ so that the $n$th partial sum $S_n$ of the alternating series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ approximates the sum to within $0.001$.

## Summary

- An alternating series is a series whose terms alternate in sign. It has the form
$$\sum (-1)^k a_k$$
where $a_k$ is a positive real number for each $k$.
- The sequence of partial sums of a convergent alternating series oscillates around the sum of the series if the sequence of $n$th terms converges to $0$. That is why the Alternating Series Test shows that the alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges whenever the sequence $\{a_n\}$ of $n$th terms decreases to $0$.
- The difference between the $n-1$st partial sum $S_{n-1}$ and the $n$th partial sum $S_n$ of a convergent alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$ is $|S_n - S_{n-1}| = a_n$. Since the partial sums oscillate around the sum $S$ of the series, it follows that
$$|S - S_n| < a_n.$$
So the $n$th partial sum of a convergent alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$ approximates the actual sum of the series to within $a_n$.

## Answers

**Example** $8.4.1$

If we let $S$ be the sum of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, then we know that
$$|S_{100} - S| < a_{101}.$$
Now
$$a_{101} = \frac{1}{101} \approx 0.0099,$$
so the $100$th partial sum is within $0.0099$ of the sum of the series. We have discussed the fact (and will later verify) that
$$S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2),$$
and so $S \approx 0.693147$ while
$$S_{100} = \sum_{k=1}^{100} \frac{(-1)^{k+1}}{k} \approx 0.6881721793.$$
We see that the actual difference between $S$ and $S_{100}$ is approximately $0.0049750013$, which is indeed less than $0.0099$.

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# 8.5: Taylor Polynomials and Taylor Series

## Motivating Questions

What is a Taylor polynomial? For what purposes are Taylor polynomials used?
What is a Taylor series?
How do we determine the accuracy when we use a Taylor polynomial to approximate a function?

So far, each infinite series we have discussed has been a series of real numbers, such as

$$1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}+\cdots=\sum_{k=0}^{\infty}\frac{1}{2^k} \tag{8.5.1}$$

In the remainder of this chapter, we will include series that involve a variable. For instance, if in the geometric series in Equation (8.5.1) we replace the ratio $r=\frac{1}{2}$ with the variable $x$, we have the infinite (still geometric) series

$$1+x+x^2+\cdots+x^k+\cdots=\sum_{k=0}^{\infty}x^k \tag{8.5.2}$$

Here we see something very interesting: because a geometric series converges whenever its ratio $r$ satisfies $|r|<1$, and the sum of a convergent geometric series is $\frac{a}{1-r}$, we can say that for $|x|<1$

$$1+x+x^2+\cdots+x^k+\cdots=\frac{1}{1-x}. \tag{8.5.3}$$

Equation (8.5.3) states that the non-polynomial function $\frac{1}{1-x}$ on the right is equal to the infinite polynomial expresssion on the left. Because the terms on the left get very small as $k$ gets large, we can truncate the series and say, for example, that

$$1+x+x^2+x^3\approx\frac{1}{1-x}$$

## Preview Activity 8.5.1

Preview Activity 8.3.1 showed how we can approximate the number $e$ using linear, quadratic, and other polynomial functions; we then used similar ideas in Preview Activity 8.4.1 to approximate $\ln(2)$. In this activity, we review and extend the process to find the "best" quadratic approximation to the exponential function $e^x$ around the origin. Let $f(x)=e^x$ throughout this activity.

a. Find a formula for $P_1(x)$, the linearization of $f(x)$ at $x=0$. (We label this linearization $P_1$ because it is a first degree polynomial approximation.) Recall that $P_1(x)$ is a good approximation to $f(x)$ for values of $x$ close to 0 . Plot $f$ and $P_1$ near $x=0$ to illustrate this fact.
b. Since $f(x)=e^x$ is not linear, the linear approximation eventually is not a very good one. To obtain better approximations, we want to develop a different approximation that "bends" to make it more closely fit the graph of $f$ near $x=0$. To do so, we add a quadratic term to $P_1(x)$. In other words, we let

$$P_2(x)=P_1(x)+c_2x^2$$

for some real number $c_2$. We need to determine the value of $c_2$ that makes the graph of $P_2(x)$ best fit the graph of $f(x)$ near $x=0$.

Remember that $P_1(x)$ was a good linear approximation to $f(x)$ near 0 ; this is because $P_1(0)=f(0)$ and $P_1'(0)=f'(0)$. It is therefore reasonable to seek a value of $c_2$ so that

$$P_2(0) = f(0), \quad P_2'(0) = f'(0), \quad \text{and } P_2''(0) = f''(0).$$

Remember, we are letting $P_2(x) = P_1(x) + c_2x^2$.

i. Calculate $P_2(0)$ to show that $P_2(0) = f(0)$.
ii. Calculate $P_2'(0)$ to show that $P_2'(0) = f'(0)$.
iii. Calculate $P_2''(x)$. Then find a value for $c_2$ so that $P_2''(0) = f''(0)$.
iv. Explain why the condition $P_2''(0) = f''(0)$ will put an appropriate "bend" in the graph of $P_2$ to make $P_2$ fit the graph of $f$ around $x = 0$.

# Taylor Polynomials

Preview Activity $8.5.1$ illustrates the first steps in the process of approximating functions with polynomials. Using this process we can approximate trigonometric, exponential, logarithmic, and other nonpolynomial functions as closely as we like (for certain values of $x$ ) with polynomials. This is extraordinarily useful in that it allows us to calculate values of these functions to whatever precision we like using only the operations of addition, subtraction, multiplication, and division, which can be easily programmed in a computer.
We next extend the approach in Preview Activity 8.5.1 to arbitrary functions at arbitrary points. Let $f$ be a function that has as many derivatives as we need at a point $x = a$. Recall that $P_1(x)$ is the tangent line to $f$ at $(a, f(a))$ and is given by the formula

$$P_1(x) = f(a) + f'(a)(x - a).$$

$P_1(x)$ is the linear approximation to $f$ near $a$ that has the same slope and function value as $f$ at the point $x = a$.
We next want to find a quadratic approximation

$$P_2(x) = P_1(x) + c_2(x - a)^2$$

so that $P_2(x)$ more closely models $f(x)$ near $x = a$. Consider the following calculations of the values and derivatives of $P_2(x)$ :

$P_2(x) = P_1(x) + c_2(x - a)^2 \quad P_2(a) = P_1(a) = f(a)$
$P_2'(x) = P_1'(x) + 2c_2(x - a) \quad P_2'(a) = P_1'(a) = f'(a)$
$P_2''(x) = 2c_2 \quad P_2''(a) = 2c_2.$

To make $P_2(x)$ fit $f(x)$ better than $P_1(x)$, we want $P_2(x)$ and $f(x)$ to have the same concavity at $x = a$, in addition to having the same slope and function value. That is, we want to have

$$P_2''(a) = f''(a).$$

This implies that

$$2c_2 = f''(a)$$

and thus

$$c_2 = \frac{f''(a)}{2}.$$

Therefore, the quadratic approximation $P_2(x)$ to $f$ centered at $x = a$ is

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2.$$

This approach extends naturally to polynomials of higher degree. We define polynomials

$$P_3(x) = P_2(x) + c_3(x - a)^3,$$
$$P_4(x) = P_3(x) + c_4(x - a)^4,$$
$$P_5(x) = P_4(x) + c_5(x - a)^5,$$

and in general

$$P_n(x) = P_{n-1}(x) + c_n(x-a)^n.$$

The defining property of these polynomials is that for each $n$, $P_n(x)$ and all its first $n$ derivatives must agree with those of $f$ at $x = a$. In other words we require that

$$P_n^{(k)}(a) = f^{(k)}(a)$$

for all $k$ from 0 to $n$.
To see the conditions under which this happens, suppose

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n.$$

Then

$$\begin{aligned} P_n^{(0)}(a) &= c_0 \\ P_n^{(1)}(a) &= c_1 \\ P_n^{(2)}(a) &= 2c_2 \\ P_n^{(3)}(a) &= (2)(3)c_3 \\ P_n^{(4)}(a) &= (2)(3)(4)c_4 \\ P_n^{(5)}(a) &= (2)(3)(4)(5)c_5 \end{aligned}$$

and, in general,

$$P_n^{(k)}(a) = (2)(3)(4)\cdots(k-1)(k)c_k = k!c_k.$$

So having $P_n^{(k)}(a) = f^{(k)}(a)$ means that $k!c_k = f^{(k)}(a)$ and therefore

$$c_k = \frac{f^{(k)}(a)}{k!}$$

for each value of $k$. Using this expression for $c_k$, we have found the formula for the polynomial approximation of $f$ that we seek. Such a polynomial is called a Taylor polynomial.

**Definition: 8.5.1: Taylor Polynomials**

The $n$th order Taylor polynomial of $f$ centered at $x = a$ is given by

$$\begin{aligned} P_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

This degree $n$ polynomial approximates $f(x)$ near $x = a$ and has the property that $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, \ldots, n$.

**Example 8.5.1**

Example 8.5.1. Determine the third order Taylor polynomial for $f(x) = e^x$, as well as the general $n$th order Taylor polynomial for $f$ centered at $x = 0$.

**Solution**

We know that $f'(x) = e^x$ and so $f''(x) = e^x$ and $f'''(x) = e^x$. Thus,

$$f(0) = f'(0) = f''(0) = f'''(0) = 1.$$

So the third order Taylor polynomial of $f(x) = e^x$ centered at $x = 0$ is

$$\begin{aligned} P_3(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}. \end{aligned}$$

In general, for the exponential function $f$ we have $f^{(k)}(x) = e^x$ for every positive integer $k$. Thus, the $k$ th term in the $n$th order Taylor polynomial for $f(x)$ centered at $x = 0$ is

$$\frac{f^{(k)}(0)}{k!}(x-0)^k = \frac{1}{k!}x^k.$$

Therefore, the $n$th order Taylor polynomial for $f(x) = e^x$ centered at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{1}{n!}x^n = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

## Activity 8.5.2

We have just seen that the $n$th order Taylor polynomial centered at $a = 0$ for the exponential function $e^x$ is

$$\sum_{k=0}^{n} \frac{x^k}{k!}.$$

In this activity, we determine small order Taylor polynomials for several other familiar functions, and look for general patterns.

a. Let $f(x) = \frac{1}{1-x}$.

i. Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\frac{1}{1-x}$ centered at $0$ .
ii. Based on your results from part (i), determine a general formula for $f^{(k)}(0)$

b. Let $f(x) = \cos(x)$.

i. Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\cos(x)$ centered at $0$ .
ii. Based on your results from part (i), find a general formula for $f^{(k)}(0)$. (Think about how $k$ being even or odd affects the value of the $k$ th derivative.)

c. Let $f(x) = \sin(x)$.

i. Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\sin(x)$ centered at $0$ .
ii. Based on your results from part (i), find a general formula for $f^{(k)}(0)$. (Think about how $k$ being even or odd affects the value of the $k$ th derivative.)

It is possible that an $n$th order Taylor polynomial is not a polynomial of degree $n$; that is, the order of the approximation can be different from the degree of the polynomial. For example, in Activity 8.5.3 we found that the second order Taylor polynomial $P_2(x)$ centered at $0$ for $\sin(x)$ is $P_2(x) = x$. In this case, the second order Taylor polynomial is a degree 1 polynomial.

# Taylor Series

In Activity 8.5.2 we saw that the fourth order Taylor polynomial $P_4(x)$ for $\sin(x)$ centered at $0$ is

$$P_4(x) = x - \frac{x^3}{3!}.$$

The pattern we found for the derivatives $f^{(k)}(0)$ describe the higher-order Taylor polynomials, e.g.,

$$P_5(x) = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!},$$
$$P_7(x) = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!},$$
$$P_9(x) = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \tfrac{x^9}{9!},$$

and so on. It is instructive to consider the graphical behavior of these functions; Figure 8.5.1 shows the graphs of a few of the Taylor polynomials centered at 0 for the sine function.

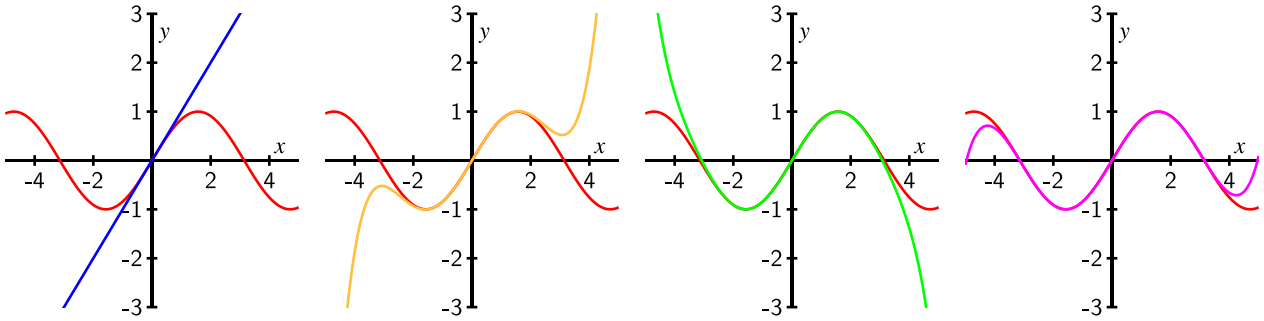



Figure 8.5.1: The order $1, 5, 7$, and 9 Taylor polynomials centered at $x = 0$ for $f(x) = \sin(x)$.

Notice that $P_1(x)$ is close to the sine function only for values of $x$ that are close to 0 , but as we increase the degree of the Taylor polynomial the Taylor polynomials provide a better fit to the graph of the sine function over larger intervals. This illustrates the general behavior of Taylor polynomials: for any sufficiently well-behaved function, the sequence $\{P_n(x)\}$ of Taylor polynomials converges to the function $f$ on larger and larger intervals (though those intervals may not necessarily increase without bound). If the Taylor polynomials ultimately converge to $f$ on its entire domain, we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

**Definition: 8.5.2**

Let $f$ be a function all of whose derivatives exist at $x = a$. The Taylor series for $f$ centered at $x = a$ is the series $T_f(x)$ defined by

$$T_f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{8.5.4}$$

In the special case where $a = 0$ in Definition 8.5.3, the Taylor series is also called the Maclaurin series for $f$. From Example 8.5.1 we know the $n$th order Taylor polynomial centered at 0 for the exponential function $e^x$; thus, the Maclaurin series for $e^x$ is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

**Activity 8.5.3**

In Activity \({8{.}5{.}\eatSpaces{2} \endSpaces}\ we determined small order Taylor polynomials for a few familiar functions, and also found general patterns in the derivatives evaluated at 0 . Use that information to write the Taylor series centered at 0 for the following functions.

a. $f(x) = \frac{1}{1-x}$
b. $f(x) = \cos(x)$ (You will need to carefully consider how to indicate that many of the coefficients are 0 . Think about a general way to represent an even integer.)
c. $f(x) = \sin(x)$ (You will need to carefully consider how to indicate that many of the coefficients are 0 . Think about a general way to represent an odd integer.)
d. $f(x) = \frac{1}{1+x}$

### Activity 8.5.4

a. Plot the graphs of several of the Taylor polynomials centered at 0 (of order at least 5) for $e^x$ and convince yourself that these Taylor polynomials converge to $e^x$ for every value of $x$.
b. Draw the graphs of several of the Taylor polynomials centered at 0 (of order at least 6) for $\cos(x)$ and convince yourself that these Taylor polynomials converge to $\cos(x)$ for every value of $x$. Write the Taylor series centered at 0 for $\cos(x)$.
c. Draw the graphs of several of the Taylor polynomials centered at 0 for $\frac{1}{1-x}$. Based on your graphs, for what values of $x$ do these Taylor polynomials appear to converge to $\frac{1}{1-x}$ ? How is this situation different from what we observe with $e^x$ and $\cos(x)$ ? In addition, write the Taylor series centered at 0 for $\frac{1}{1-x}$.

The Maclaurin series for $e^x, \sin(x), \cos(x)$, and $\frac{1}{1-x}$ will be used frequently, so we should be certain to know and recognize them well.

## The Interval of Convergence of a Taylor Series

In the previous section (in Figure 8.5.1 and Activity 8.5.4) we observed that the Taylor polynomials centered at 0 for $e^x, \cos(x)$, and $\sin(x)$ converged to these functions for all values of $x$ in their domain, but that the Taylor polynomials centered at 0 for $\frac{1}{1-x}$ converge to $\frac{1}{1-x}$ on the interval $(-1, 1)$ and diverge for all other values of $x$. So the Taylor series for a function $f(x)$ does not need to converge for all values of $x$ in the domain of $f$.
Our observations suggest two natural questions: can we determine the values of $x$ for which a given Taylor series converges? And does the Taylor series for a function $f$ actually converge to $f(x)$ ?

### Example 8.5.1

Graphical evidence suggests that the Taylor series centered at 0 for $e^x$ converges for all values of $x$. To verify this, use the Ratio Test to determine all values of $x$ for which the Taylor series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{8.5.5}$$

converges absolutely.

**Solution**

Recall that the Ratio Test applies only to series of nonnegative terms. In this example, the variable $x$ may have negative values. But we are interested in absolute convergence, so we apply the Ratio Test to the series

$$\sum_{k=0}^{\infty} \left| \frac{x^k}{k!} \right| = \sum_{k=0}^{\infty} \frac{|x|^k}{k!}.$$

Now, observe that

$$\begin{aligned}\lim_{k\to\infty}\frac{a_{k+1}}{a_k} &= \lim_{k\to\infty}\frac{\frac{|x|^{k+1}}{(k+1)!}}{\frac{|x|^k}{k}}\\ &= \lim_{k\to\infty}\frac{|x|^{k+1}k!}{|x|^k(k+1)!}\\ &= \lim_{k\to\infty}\frac{|x|}{k+1}\\ &= 0\end{aligned}$$

for any value of $x$. So the Taylor series $(8.5.4)$ converges absolutely for every value of $x$, and thus converges for every value of $x$.

One question still remains: while the Taylor series for $e^x$ converges for all $x$, what we have done does not tell us that this Taylor series actually converges to $e^x$ for each $x$. We'll return to this question when we consider the error in a Taylor approximation near the end of this section.

We can apply the main idea from Example 8.5.4 in general. To determine the values of $x$ for which a Taylor series

$$\sum_{k=0}^{\infty} c_k(x-a)^k$$

centered at $x=a$ will converge, we apply the Ratio Test with $a_k = \left|c_k(x-a)^k\right|$. The series converges if $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} < 1$.

Observe that

$$\frac{a_{k+1}}{a_k} = |x-a|\frac{|c_{k+1}|}{|c_k|}$$

so when we apply the Ratio Test, we get

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}|x-a|\frac{|c_{k+1}|}{|c_k|}.$$

Note suppose that

$$\lim_{k\to\infty}\frac{|c_{k+1}|}{|c_k|} = L$$

so that

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = |x-a|\cdot L.$$

There are three possibilities for $L: L$ can be $0$ , it can be a finite positive value, or it can be infinite. Based on this value of $L$, we can determine for which values of $x$ the original Taylor series converges.

- If $L=0$, then the Taylor series converges on $(-\infty,\infty)$.
- If $L$ is infinite, then the Taylor series converges only at $x=a$.
- If $L$ is finite and nonzero, then the Taylor series converges absolutely for all $x$ that satisfy

  $$|x-a|\cdot L < 1$$

  or for all $x$ such that

  $$|x-a| < \frac{1}{L},$$

  which is the interval

  $$\left(a-\frac{1}{L}, a+\frac{1}{L}\right).$$

Because the Ratio Test is inconclusive when the $|x-a| \cdot L = 1$, the endpoints $a \pm \frac{1}{L}$ have to be checked separately.

It is important to notice that the set of $x$ values at which a Taylor series converges is always an interval centered at $x = a$. For this reason, the set on which a Taylor series converges is called the interval of convergence. Half the length of the interval of convergence is called the radius of convergence. If the interval of convergence of a Taylor series is infinite, then we say that the radius of convergence is infinite.

**Activity 8.5.4**

a. Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \frac{1}{1-x}$ centered at $x = 0$.
b. Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \cos(x)$ centered at $x = 0$.
c. Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \sin(x)$ centered at $x = 0$.

The Ratio Test allows us to determine the set of $x$ values for which a Taylor series converges absolutely. However, just because a Taylor series for a function $f$ converges, we cannot be certain that the Taylor series actually converges to $f(x)$. To show why and where a Taylor series does in fact converge to the function $f$, we next consider the error that is present in Taylor polynomials.

## Error Approximations for Taylor Polynomials

We now know how to find Taylor polynomials for functions such as $\sin(x)$, and how to determine the interval of convergence of the corresponding Taylor series. We next develop an error bound that will tell us how well an $n$th order Taylor polynomial $P_n(x)$ approximates its generating function $f(x)$. This error bound will also allow us to determine whether a Taylor series on its interval of convergence actually equals the function $f$ from which the Taylor series is derived. Finally, we will be able to use the error bound to determine the order of the Taylor polynomial $P_n(x)$ that we will ensure that $P_n(x)$ approximates $f(x)$ to the desired degree of accuracy.

For this argument, we assume throughout that we center our approximations at 0 (but a similar argument holds for approximations centered at $a$ ). We define the exact error, $E_n(x)$, that results from approximating $f(x)$ with $P_n(x)$ by

$$E_n(x) = f(x) - P_n(x).$$

We are particularly interested in $|E_n(x)|$, the distance between $P_n$ and $f$.
Because

$$P_n^{(k)}(0) = f^{(k)}(0)$$

for $0 \leq k \leq n$, we know that

$$E_n^{(k)}(0) = 0$$

for $0 \leq k \leq n$. Furthermore, since $P_n(x)$ is a polynomial of degree less than or equal to $n$, we know that

$$P_n^{(n+1)}(x) = 0.$$

Thus, since $E_n^{(n+1)}(x) = f^{(n+1)}(x) - P_n^{(n+1)}(x)$, it follows that

$$E_n^{(n+1)}(x) = f^{(n+1)}(x)$$

for all $x$.

Suppose that we want to approximate $f(x)$ at a number $c$ close to 0 using $P_n(c)$. If we assume $\left|f^{(n+1)}(t)\right|$ is bounded by some number $M$ on $[0,c]$, so that

$$\left|f^{(n+1)}(t)\right| \leq M$$

for all $0 \leq t \leq c$, then we can say that

$$\left|E_n^{(n+1)}(t)\right| = \left|f^{(n+1)}(t)\right| \leq M$$

for all $t$ between 0 and $c$. Equivalently,

$$-M \leq E_n^{(n+1)}(t) \leq M \tag{8.5.6}$$

on $[0,c]$. Next, we integrate the three terms in Inequality 8.5.5 from $t=0$ to $t=x$, and thus find that

$$\int_0^x -M dt \leq \int_0^x E_n^{(n+1)}(t)dt \leq \int_0^x M dt$$

for every value of $x$ in $[0,c]$. Since $E_n^{(n)}(0)=0$, the First FTC tells us that

$$-Mx \leq E_n^{(n)}(x) \leq Mx$$

for every $x$ in $[0,c]$.
Integrating this last inequality, we obtain

$$\int_0^x -Mtdt \leq \int_0^x E_n^{(n)}(t)dt \leq \int_0^x Mt\_dt$$

and thus

$$-M\frac{x^2}{2} \leq E_n^{(n-1)}(x) \leq M\frac{x^2}{2}$$

for all $x$ in $[0,c]$.
Integrating $n$ times, we arrive at

$$-M\frac{x^{n+1}}{(n+1)!} \leq E_n(x) \leq M\frac{x^{n+1}}{(n+1)!}$$

for all $x$ in $[0,c]$. This enables us to conclude that

$$|E_n(x)| \leq M\frac{|x|^{n+1}}{(n+1)!}$$

for all $x$ in $[0,c]$, and we have found a bound on the approximation's error, $E_n$.
Our work above was based on the approximation centered at $a=0$; the argument may be generalized to hold for any value of $a$, which results in the following theorem.

**Definition: 8.5.3: The Lagrange Error Bound**

The Lagrange Error Bound for $P_n(x)$.

Let $f$ be a continuous function with $n+1$ continuous derivatives. Suppose that $M$ is a positive real number such that $\left|f^{(n+1)}(x)\right| \leq M$ on the interval $[a,c]$. If $P_n(x)$ is the $n$th order Taylor polynomial for $f(x)$ centered at $x=a$, then

$$|P_n(c) - f(c)| \leq M\frac{|c-a|^{n+1}}{(n+1)!}.$$

We can use this error bound to tell us important information about Taylor polynomials and Taylor series, as we see in the following examples and activities.

## Example 8.5.5

Determine how well the 10th order Taylor polynomial $P_{10}(x)$ for $\sin(x)$, centered at 0 , approximates $\sin(2)$.

**Solution**

To answer this question we use $f(x) = \sin(x), c = 2, a = 0$, and $n = 10$ in the Lagrange error bound formula. We also need to find an appropriate value for $M$. Note that the derivatives of $f(x) = \sin(x)$ are all equal to $\pm\sin(x)$ or $\pm\cos(x)$. Thus,

$$\left|f^{(n+1)}(x)\right| \leq 1$$

for any $n$ and $x$. Therefore, we can choose $M$ to be 1 . Then

$$|P_{10}(2) - f(2)| \leq (1)\frac{|2-0|^{11}}{(11)!} = \frac{2^{11}}{(11)!} \approx 0.00005130671797.$$

So $P_{10}(2)$ approximates $\sin(2)$ to within at most $0.00005130671797$. A computer algebra system tells us that

$$P_{10}(2) \approx 0.9093474427 \text{ and } \sin(2) \approx 0.9092974268$$

with an actual difference of about $0.0000500159$.

## Activity 8.5.6

Let $P_n(x)$ be the $n$th order Taylor polynomial for $\sin(x)$ centered at $x = 0$. Determine how large we need to choose $n$ so that $P_n(2)$ approximates $\sin(2)$ to 20 decimal places.

## Example 8.5.6

Show that the Taylor series for $\sin(x)$ actually converges to $\sin(x)$ for all $x$.

**Solution**

Recall from the previous example that since $f(x) = \sin(x)$, we know

$$\left|f^{(n+1)}(x)\right| \leq 1$$

for any $n$ and $x$. This allows us to choose $M = 1$ in the Lagrange error bound formula. Thus,

$$|P_n(x) - \sin(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \tag{8.5.7}$$

for every $x$.
We showed in earlier work that the Taylor series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for every value of $x$. Because the terms of any convergent series must approach zero, it follows that

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

for every value of $x$. Thus, taking the limit as $n \to \infty$ in the inequality 8.5.6, it follows that

$$\lim_{n\to\infty} |P_n(x) - \sin(x)| = 0.$$

As a result, we can now write

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for every real number $x$.

**Activity 8.5.7**

a. Show that the Taylor series centered at 0 for $\cos(x)$ converges to $\cos(x)$ for every real number $x$.
b. Next we consider the Taylor series for $e^x$.
   i. Show that the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every nonnegative value of $x$.
   ii. Show that the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every negative value of $x$.
   iii. Explain why the Taylor series centered at 0 for $e^x$ converges to $e^x$ for every real number $x$. Recall that we earlier showed that the Taylor series centered at 0 for $e^x$ converges for all $x$, and we have now completed the argument that the Taylor series for $e^x$ actually converges to $e^x$ for all $x$.
c. Let $P_n(x)$ be the $n$th order Taylor polynomial for $e^x$ centered at 0 . Find a value of $n$ so that $P_n(5)$ approximates $e^5$ correct to 8 decimal places.

## Summary

- We can use Taylor polynomials to approximate functions. This allows us to approximate values of functions using only addition, subtraction, multiplication, and division of real numbers. The $n$th order Taylor polynomial centered at $x = a$ of a function $f$ is

$$\begin{aligned} P_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

- The Taylor series centered at $x = a$ for a function $f$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{8.5.8}$$

  The $n$th order Taylor polynomial centered at $a$ for $f$ is the $n$th partial sum of its Taylor series centered at $a$. So the $n$th order Taylor polynomial for a function $f$ is an approximation to $f$ on the interval where the Taylor series converges; for the values of $x$ for which the Taylor series converges to $f$ we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{8.5.9}$$

- The Lagrange Error Bound shows us how to determine the accuracy in using a Taylor polynomial to approximate a function. More specifically, if $P_n(x)$ is the $n$th order Taylor polynomial for $f$ centered at $x = a$ and if $M$ is an upper bound for $\left|f^{(n+1)}(x)\right|$ on the interval $[a, c]$, then

$$|P_n(c) - f(c)| \leq M\frac{|c-a|^{n+1}}{(n+1)!}. \tag{8.5.10}$$

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# 8.6: Power Series

## Motivating Questions

- What is a power series?
- What are some important uses of power series?
- What is the connection between power series and Taylor series?

We have noted in our work with Taylor polynomials and Taylor series that polynomial functions are the simplest possible functions in mathematics, in part because they require only addition and multiplication to evaluate. From the point of view of calculus, polynomials are especially nice: we can easily differentiate or integrate any polynomial. In light of our work in Section 8.5, we now know that several important non-polynomials have polynomial-like expansions. For example, for any real number $x$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

There are two settings where other series like the one for $e^x$ arise: we may be given an expression such as

$$1 + 2x + 3x^2 + 4x^3 + \cdots$$

and we seek the values of $x$ for which the expression makes sense. Or we may be trying to find an unknown function $f$ that has expression

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots,$$

and we try to determine the values of the constants $a_0, a_1, \ldots$. The latter situation is explored in Preview Activity 8.6.1.

## Preview Activity 8.6.1

In Chapter 7, we learned some of the many important applications of differential equations, and learned some approaches to solve or analyze them. Here, we consider an important approach that will allow us to solve a wider variety of differential equations.

Let's consider the familiar differential equation from exponential population growth given by

$$y' = ky, \tag{8.6.1}$$

where $k$ is the constant of proportionality. While we can solve this differential equation using methods we have already learned, we take a different approach now that can be applied to a much larger set of differential equations. For the rest of this activity, let's assume that $k = 1$. We will use our knowledge of Taylor series to find a solution to the differential equation (8.6.1).

To do so, we assume that we have a solution $y = f(x)$ and that $f(x)$ has a Taylor series that can be written in the form

$$y = f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where the coefficients $a_k$ are undetermined. Our task is to find the coefficients.

1. Assume that we can differentiate a power series term by term. By taking the derivative of $f(x)$ with respect to $x$ and substituting the result into the differential equation (8.6.1), show that the equation

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} a_k x^k$$

must be satisfied in order for $f(x) = \sum_{k=0}^{\infty} a_k x^k$ to be a solution of the DE.

2. Two series are equal if and only if they have the same coefficients on like power terms. Use this fact to find a relationship between $a_1$ and $a_0$.
3. Now write $a_2$ in terms of $a_1$. Then write $a_2$ in terms of $a_0$.
4. Write $a_3$ in terms of $a_2$. Then write $a_3$ in terms of $a_0$.
5. Write $a_4$ in terms of $a_3$. Then write $a_4$ in terms of $a_0$.
6. Observe that there is a pattern in (b)-(e). Find a general formula for $a_k$ in terms of $a_0$.
7. Write the series expansion for $y$ using only the unknown coefficient $a_0$. From this, determine what familiar functions satisfy the differential equation (8.6.1). (*Hint*: Compare to a familiar Taylor series.)

## Power Series

As Preview Activity 8.6.1 shows, it can be useful to treat an unknown function as if it has a Taylor series, and then determine the coefficients from other information. In other words, we define a function as an infinite series of powers of $x$ and then determine the coefficients based on something besides a formula for the function. This method allows us to approximate solutions to many different types of differential equations, even if we cannot solve them explicitly. This is different from our work with Taylor series since we are not using an original function $f$ to generate the coefficients of the series.

### Definition 8.6.1: Power Series

A ***power series*** centered at $x = a$ is a function of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k \tag{8.6.2}$$

where $\{c_k\}$ is a sequence of real numbers and $x$ is an independent variable.

A power series defines a function $f$ whose domain is the set of $x$ values for which the power series converges. We therefore write

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k.$$

It turns out that[1], on its interval of convergence, every power series is in fact the Taylor series of the function it defines, so all of the techniques we developed in the previous section can be applied to power series as well.

See Exercise 8.6.4 in this section.

### Example 8.6.1

Consider the power series defined by

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{2^k}.$$

What are $f(1)$ and $f\left(\frac{3}{2}\right)$? Find a general formula for $f(x)$ and determine the values for which this power series converges.

*Answer available at the end of this section.*

As we did for Taylor series, we define the interval of convergence of a power series (8.6.2) to be the set of values of $x$ for which the series converges. And as we did with Taylor series, we typically use the Ratio Test to find the values of $x$ for which the power series converges absolutely, and then check the endpoints separately if the radius of convergence is finite.

**Example 8.6.2**

Let $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$. Determine the interval of convergence of this power series.

*Answer available at the end of this section.*

**Activity 8.6.2**

Determine the interval of convergence of each power series.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{3k}$
2. $\displaystyle\sum_{k=1}^{\infty} kx^k$
3. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2(x+1)^k}{4^k}$
4. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{(2k)!}$
5. $\displaystyle\sum_{k=1}^{\infty} k!x^k$

## Manipulating Power Series

We know power series expansions for important functions such as $\sin(x)$ and $e^x$. Often, we can use a known power series expansion to find a power series for a different, but related, function. The next activity demonstrates one way to do this.

**Activity 8.6.3**

Our goal in this activity is to find a power series expansion for $f(x) = \frac{1}{1+x^2}$ centered at $x = 0$.

While we could use the methods of Section 8.5 and differentiate $f(x) = \frac{1}{1+x^2}$ several times to look for patterns and find the Taylor series for $f(x)$, we seek an alternate approach because of how complicated the derivatives of $f(x)$ quickly become.

1. What is the Taylor series expansion for $g(x) = \frac{1}{1-x}$? What is the interval of convergence of this series?
2. How is $g(-x^2)$ related to $f(x)$? Explain, and hence substitute $-x^2$ for $x$ in the power series expansion for $g(x)$. Given the relationship between $g(-x^2)$ and $f(x)$, how is the resulting series related to $f(x)$?
3. For which values of $x$ will this power series expansion for $f(x)$ be valid? Why?

In a previous section we found several important Maclaurin series and their intervals of convergence. Here, we list these key functions and their corresponding expansions.

**Note**

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \text{for } -\infty < x < \infty$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \text{for } -\infty < x < \infty$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{for } -\infty < x < \infty$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{for } -1 < x < 1$$

As we saw in Activity 8.6.3, we can use these known series to find other power series expansions for related functions such as $\sin(x^2)$, $e^{5x^3}$, and $\cos(x^5)$.

**Activity 8.6.4**

Let $f$ be the function given by the power series expansion

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

1. Assume that we can differentiate a power series term by term, just like we can differentiate a (finite) polynomial. Use the fact that

   $$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

   to find a power series expansion for $f'(x)$.

2. Observe that $f(x)$ and $f'(x)$ have familiar Taylor series. What familiar functions are these? What known relationship does our work demonstrate?
3. What is the series expansion for $f''(x)$? What familiar function is $f''(x)$?

Our work in Activity 8.6.3 holds more generally. The corresponding theorem, which we will not prove, states that we can differentiate a power series for a function $f$ term by term and obtain the series expansion for $f'$, and similarly we can integrate a series expansion for a function $f$ term by term and obtain the series expansion for $\int f(x)\,dx$. For both, the radius of convergence of the resulting series is the same as the original, though it is possible that the convergence status of the various series may differ at the endpoints. The formal statement of the Power Series Differentiation and Integration Theorem follows.

**Power Series Differentiation and Integration Theorem**

Suppose $f(x)$ has a power series expansion

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

and that the series converges absolutely to $f(x)$ on the interval $-r < x < r$. Then, the power series $\sum_{k=1}^{\infty} k c_k x^{k-1}$ obtained by differentiating the power series for $f(x)$ term by term converges absolutely to $f'(x)$ on the interval $-r < x < r$. That is,

$$f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}, \text{ for } |x| < r.$$

Similarly, the power series $\sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$ obtained by integrating the power series for $f(x)$ term by term converges absolutely on the interval $-r < x < r$, and

$$\int f(x)\ dx = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1} + C, \text{ for } |x| < r.$$

This theorem validates the steps we took in Activity 8.6.4. It tells us that we can differentiate and integrate term by term on the interior of the interval of convergence, but it does not tell us what happens at the endpoints of this interval. We always need to check what happens at the endpoints separately. More importantly, we can use use the approach of differentiating or integrating a series term by term to find new series.

**Example 8.6.3**

Find a series expansion centered at $x = 0$ for $\arctan(x)$, as well as its interval of convergence.

*Answer available at the end of this section.*

**Activity 8.6.5**

Find a power series expansion for $\ln(1 + x)$ centered at $x = 0$ and determine its interval of convergence.

## Summary

- A power series is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k.$$

- We can often assume a solution to a given problem can be written as a power series, then use the information in the problem to determine the coefficients in the power series. This method allows us to approximate solutions to certain problems using partial sums of the power series; that is, we can find approximate solutions that are polynomials.
- The connection between power series and Taylor series is that they are essentially the same thing: on its interval of convergence a power series is the Taylor series of its sum.

## Answers

**Example** 8.6.1

If we evaluate $f$ at $x = 1$ we obtain the series

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

which is a geometric series with ratio $\frac{1}{2}$. So we can sum this series and find that

$$f(1) = \frac{1}{1 - \frac{1}{2}} = 2.$$

Similarly,

$$f(3/2) = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = \frac{1}{1 - \frac{3}{4}} = 4.$$

In general, $f(x)$ is a geometric series with ratio $\frac{x}{2}$, so

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \frac{1}{1 - \frac{x}{2}} = \frac{2}{2 - x}$$

provided that $-1 < \frac{x}{2} < 1$ (which ensures that the ratio is less than 1 in absolute value). Thus, the power series that defines $f(x) = \frac{2}{2-x}$ converges for $-2 < x < 2$.

**Example** 8.6.2

First we will plot some of the partial sums of this power series to get an idea of the interval of convergence. Let

$$S_n(x) = \sum_{k=1}^{n} \frac{x^k}{k^2}$$

for each $n \geq 1$. Figure 8.6.1 shows plots of $S_{10}(x)$ (in red), $S_{25}(x)$ (in blue), and $S_{50}(x)$ (in green).

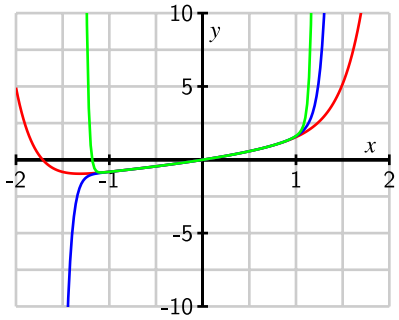



Figure 8.6.1. Graphs of some partial sums of the power series $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$.

The behavior of $S_{50}$ in particular suggests that $f(x)$ appears to be converging to a particular curve on the interval $(-1, 1)$, while growing without bound outside of that interval. Thus, the interval of convergence might be $-1 < x < 1$. To verify our conjecture, we apply the Ratio Test. Now,

$$a_k = \frac{x^k}{k^2},$$

so

$$\begin{aligned} \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{\frac{|x|^{k+1}}{(k+1)^2}}{\frac{|x|^k}{k^2}} \\ &= \lim_{k\to\infty} |x| \left(\frac{k}{k+1}\right)^2 \\ &= |x| \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2 \\ &= |x|. \end{aligned}$$

Therefore, the Ratio Test tells us that $f(x)$ converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. Because the Ratio Test is inconclusive when $|x| = 1$, we need to check $x = 1$ and $x = -1$ individually.

When $x = 1$, observe that

$$f(1) = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

This is a $p$-series with $p > 1$, which we know converges. When $x = -1$, we have

$$f(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}.$$

This is an alternating series, and since the sequence $\left\{\frac{1}{n^2}\right\}$ decreases to 0, the power series converges by the Alternating Series Test. Thus, the interval of convergence of this power series is $-1 \leq x \leq 1$.

**Example** 8.6.3

While we could differentiate $\arctan(x)$ repeatedly and look for patterns in the derivative values at $x = 0$ in an attempt to find the Maclaurin series for $\arctan(x)$ from the definition, it turns out to be far easier to use a known series in an insightful way. In Activity 8.6.3, we found that

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

for $-1 < x < 1$. Recall that

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2},$$

and therefore

$$\int \frac{1}{1+x^2}\, dx = \arctan(x) + C.$$

It follows that we can integrate the series for $\frac{1}{1+x^2}$ term by term to obtain the power series expansion for $\arctan(x)$. Doing so, we find that

$$\begin{aligned} \arctan(x) &= \int \left(\sum_{k=0}^{\infty} (-1)^k x^{2k}\right) dx \\ &= \sum_{k=0}^{\infty} \left(\int (-1)^k x^{2k}\, dx\right) \\ &= \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}\right) + C. \end{aligned}$$

The Power Series Differentiation and Integration Theorem tells us that this equality is valid for at least $-1 < x < 1$.

To find the value of the constant $C$, we can use the fact that $\arctan(0) = 0$. So

$$0 = \arctan(0) = \left(\sum_{k=0}^{\infty} (-1)^k \frac{0^{2k+1}}{2k+1}\right) + C = C,$$

and we must have $C = 0$. Therefore,

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \tag{8.6.3}$$

for at least $-1 < x < 1$.

It is a straightforward exercise to check that the power series

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

converges both when $x = -1$ and when $x = 1$; in each case, we have an alternating series with terms $\frac{1}{2k+1}$ that decrease to $0$, and thus the interval of convergence for the series expansion for $\arctan(x)$ in Equation (8.6.3) is $-1 \leq x \leq 1$.

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# 8.7: Sequences and Series (Exercises)

## 8.1: Sequences

### 1. Limits of five sequences

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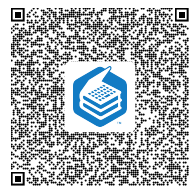
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### 2. Formula for a sequence, given first terms

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### 3. Divergent or convergent sequences

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### 4. Terms of a sequence from sampling a signal

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### 5

Finding limits of convergent sequences can be a challenge. However, there is a useful tool we can adapt from our study of limits of continuous functions at infinity to use to find limits of sequences. We illustrate in this exercise with the example of the sequence

$$\frac{\ln(n)}{n}.$$

1. Calculate the first 10 terms of this sequence. Based on these calculations, do you think the sequence converges or diverges? Why?
2. For this sequence, there is a corresponding continuous function $f$ defined by
$$f(x) = \frac{\ln(x)}{x}.$$
Draw the graph of $f(x)$ on the interval $[0, 10]$ and then plot the entries of the sequence on the graph. What conclusion do you think we can draw about the sequence $\left\{\frac{\ln(n)}{n}\right\}$ if $\lim_{x\to\infty} f(x) = L$? Explain.
3. Note that $f(x)$ has the indeterminate form $\frac{\infty}{\infty}$ as $x$ goes to infinity. What idea from differential calculus can we use to calculate $\lim_{x\to\infty} f(x)$? Use this method to find $\lim_{x\to\infty} f(x)$. What, then, is $\lim_{n\to\infty} \frac{\ln(n)}{n}$?

## 6

We return to the example begun in Preview Activity 8.1.1 to see how to derive the formula for the amount of money in an account at a given time. We do this in a general setting. Suppose you invest $P$ dollars (called the *principal*) in an account paying $r\%$ interest compounded monthly. In the first month you will receive $\frac{r}{12}$ (here $r$ is in decimal form; e.g., if we have $8\%$ interest, we write $\frac{0.08}{12}$) of the principal $P$ in interest, so you earn
$$P\left(\frac{r}{12}\right)$$
dollars in interest. Assume that you reinvest all interest. Then at the end of the first month your account will contain the original principal $P$ plus the interest, or a total of
$$P_1 = P + P\left(\frac{r}{12}\right) = P\left(1 + \frac{r}{12}\right)$$
dollars.

1. Given that your principal is now $P_1$ dollars, how much interest will you earn in the second month? If $P_2$ is the total amount of money in your account at the end of the second month, explain why
$$P_2 = P_1\left(1 + \frac{r}{12}\right) = P\left(1 + \frac{r}{12}\right)^2.$$
2. Find a formula for $P_3$, the total amount of money in the account at the end of the third month in terms of the original investment $P$.
3. There is a pattern to these calculations. Let $P_n$ the total amount of money in the account at the end of the third month in terms of the original investment $P$. Find a formula for $P_n$.

## 7

Sequences have many applications in mathematics and the sciences. In a recent paper[3] the authors write

Hui H, Farilla L, Merkel P, Perfetti R. The short half-life of glucagon-like peptide-1 in plasma does not reflect its long-lasting beneficial effects, *Eur J Endocrinol* 2002 Jun;146(6):863-9.

> The incretin hormone glucagon-like peptide-1 (GLP-1) is capable of ameliorating glucose-dependent insulin secretion in subjects with diabetes. However, its very short half-life (1.5-5 min) in plasma represents a major limitation for its use in the clinical setting.

The half-life of GLP-1 is the time it takes for half of the hormone to decay in its medium. For this exercise, assume the half-life of GLP-1 is 5 minutes. So if $A$ is the amount of GLP-1 in plasma at some time $t$, then only $\frac{A}{2}$ of the hormone will be present after $t + 5$ minutes. Suppose $A_0 = 100$ grams of the hormone are initially present in plasma.

1. Let $A_1$ be the amount of GLP-1 present after 5 minutes. Find the value of $A_1$.

2. Let $A_2$ be the amount of GLP-1 present after 10 minutes. Find the value of $A_2$.
3. Let $A_3$ be the amount of GLP-1 present after 15 minutes. Find the value of $A_3$.
4. Let $A_4$ be the amount of GLP-1 present after 20 minutes. Find the value of $A_4$.
5. Let $A_n$ be the amount of GLP-1 present after $5n$ minutes. Find a formula for $A_n$.
6. Does the sequence $\{A_n\}$ converge or diverge? If the sequence converges, find its limit and explain why this value makes sense in the context of this problem.
7. Determine the number of minutes it takes until the amount of GLP-1 in plasma is 1 gram.

## 8

Continuous data is the basis for analog information, like music stored on old cassette tapes or vinyl records. A digital signal like on a CD or MP3 file is obtained by sampling an analog signal at some regular time interval and storing that information. For example, the sampling rate of a compact disk is 44,100 samples per second. So a digital recording is only an approximation of the actual analog information. Digital information can be manipulated in many useful ways that allow for, among other things, noisy signals to be cleaned up and large collections of information to be compressed and stored in much smaller space. While we won't investigate these techniques in this chapter, this exercise is intended to give an idea of the importance of discrete (digital) techniques.

Let $f$ be the continuous function defined by $f(x) = \sin(4x)$ on the interval $[0, 10]$. A graph of $f$ is shown in Figure 8.1.5.

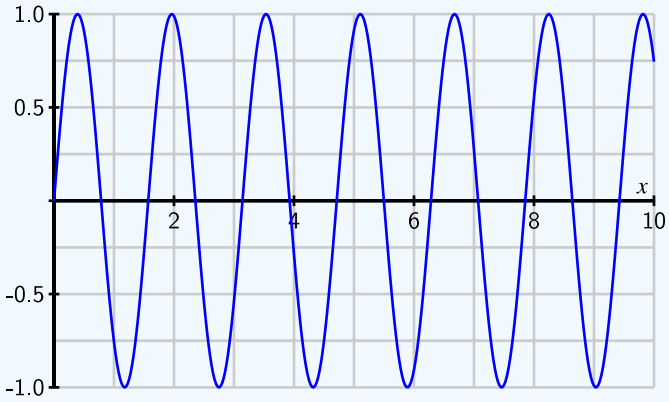



Figure 8.1.5. The graph of $f(x) = \sin(4x)$ on the interval $[0, 10]$

We approximate $f$ by *sampling*, that is by partitioning the interval $[0, 10]$ into uniform subintervals and recording the values of $f$ at the endpoints.

1. Ineffective sampling can lead to several problems in reproducing the original signal. As an example, partition the interval $[0, 10]$ into 8 equal length subintervals and create a list of points (the *sample*) using the endpoints of each subinterval. Plot your sample on graph of $f$ in Figure Figure 8.1.5. What can you say about the period of your sample as compared to the period of the original function?
2. The sampling rate is the number of samples of a signal taken per second. As the part (a) illustrates, sampling at too small a rate can cause serious problems with reproducing the original signal (this problem of inefficient sampling leading to an inaccurate approximation is called *aliasing*). There is an elegant theorem called the Nyquist-Shannon Sampling Theorem that says that human perception is limited, which allows that replacement of a continuous signal with a digital one without any perceived loss of information. This theorem also provides the *lowest* rate at which a signal can be sampled (called the Nyquist rate) without such a loss of information. The theorem states that we should sample at double the maximum desired frequency so that every cycle of the original signal will be sampled at at least two points. Recall that the frequency of a sinusoidal function is the reciprocal of the period. Identify the frequency of the function $f$ and determine the number of partitions of the interval $[0, 10]$ that give us the Nyquist rate.
3. Humans cannot typically pick up signals above 20 kHz. Explain why, then, that information on a compact disk is sampled at 44,100 Hz.

# 8.2: Geometric Series

## 1. Seventh term of a geometric sequence

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## 2. A geometric series

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## 3. A series that is not geometric

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## 4. Two sums of geometric sequences

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## 5

There is an old question that is often used to introduce the power of geometric growth. Here is one version. Suppose you are hired for a one month (30 days, working every day) job and are given two options to be paid.

*Option 1. available at the end of this section.*

1. How much will you be paid for the job in total under Option 1?
2. Complete Table 8.2.3 to determine the pay you will receive under Option 2 for the first 10 days.

*Table 8.2.3. Option 2 payments*

| Day | Pay on this day | Total amount paid to date |
|---|---|---|

| | | |
|---|---|---|
| 1 | \dollar0.01 | \dollar0.01 |
| 2 | \dollar0.02 | \dollar0.03 |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |

3. Find a formula for the amount paid on day $n$, as well as for the total amount paid by day $n$. Use this formula to determine which option (1 or 2) you should take.

## 6

Suppose you drop a golf ball onto a hard surface from a height $h$. The collision with the ground causes the ball to lose energy and so it will not bounce back to its original height. The ball will then fall again to the ground, bounce back up, and continue. Assume that at each bounce the ball rises back to a height $\frac{3}{4}$ of the height from which it dropped. Let $h_n$ be the height of the ball on the $n$th bounce, with $h_0 = h$. In this exercise we will determine the distance traveled by the ball and the time it takes to travel that distance.

1. Determine a formula for $h_1$ in terms of $h$.
2. Determine a formula for $h_2$ in terms of $h$.
3. Determine a formula for $h_3$ in terms of $h$.
4. Determine a formula for $h_n$ in terms of $h$.
5. Write an infinite series that represents the total distance traveled by the ball. Then determine the sum of this series.
6. Next, let's determine the total amount of time the ball is in the air.
    1. When the ball is dropped from a height $H$, if we assume the only force acting on it is the acceleration due to gravity, then the height of the ball at time $t$ is given by
    $$H - \frac{1}{2}gt^2.$$
    Use this formula to determine the time it takes for the ball to hit the ground after being dropped from height $H$.
    2. Use your work in the preceding item, along with that in (a)-(e) above to determine the total amount of time the ball is in the air.

## 7

Suppose you play a game with a friend that involves rolling a standard six-sided die. Before a player can participate in the game, he or she must roll a six with the die. Assume that you roll first and that you and your friend take alternate rolls. In this exercise we will determine the probability that you roll the first six.

1. Explain why the probability of rolling a six on any single roll (including your first turn) is $\frac{1}{6}$.
2. If you don't roll a six on your first turn, then in order for you to roll the first six on your second turn, both you and your friend had to fail to roll a six on your first turns, and then you had to succeed in rolling a six on your second turn. Explain why the probability of this event is

$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right).$$

3. Now suppose you fail to roll the first six on your second turn. Explain why the probability is

$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)$$

that you to roll the first six on your third turn.

4. The probability of you rolling the first six is the probability that you roll the first six on your first turn plus the probability that you roll the first six on your second turn plus the probability that your roll the first six on your third turn, and so on. Explain why this probability is

$$\frac{1}{6} + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right) + \cdots.$$

Find the sum of this series and determine the probability that you roll the first six.

## 8

The goal of a federal government stimulus package is to positively affect the economy. Economists and politicians quote numbers like "$k$ million jobs and a net stimulus to the economy of $n$ billion of dollars." Where do they get these numbers? Let's consider one aspect of a stimulus package: tax cuts. Economists understand that tax cuts or rebates can result in long-term spending that is many times the amount of the rebate. For example, assume that for a typical person, 75% of her entire income is spent (that is, put back into the economy). Further, assume the government provides a tax cut or rebate that totals $P$ dollars for each person.

1. The tax cut of $P$ dollars is income for its recipient. How much of this tax cut will be spent?
2. In this simple model, we will say that the spent portion of the tax cut/rebate from part (a) then becomes income for another person who, in turn, spends 75% of this income. After this ``second round" of spent income, how many total dollars have been added to the economy as a result of the original tax cut/rebate?
3. This second round of spending becomes income for another group who spend 75% of this income, and so on. In economics this is called the *multiplier effect*. Explain why an original tax cut/rebate of $P$ dollars will result in multiplied spending of

$$0.75P(1 + 0.75 + 0.75^2 + \cdots).$$

dollars.

4. Based on these assumptions, how much stimulus will a 200 billion dollar tax cut/rebate to consumers add to the economy, assuming consumer spending remains consistent forever.

## 9

Like stimulus packages, home mortgages and foreclosures also impact the economy. A problem for many borrowers is the adjustable rate mortgage, in which the interest rate can change (and usually increases) over the duration of the loan, causing the monthly payments to increase beyond the ability of the borrower to pay. Most financial analysts recommend fixed rate loans, ones for which the monthly payments remain constant throughout the term of the loan. In this exercise we will analyze fixed rate loans.

When most people buy a large ticket item like car or a house, they have to take out a loan to make the purchase. The loan is paid back in monthly installments until the entire amount of the loan, plus interest, is paid. With a loan, we borrow money, say $P$ dollars (called the principal), and pay off the loan at an interest rate of $r$%. To pay back the loan we make regular monthly payments, some of which goes to pay off the principal and some of which is charged as interest. In most cases, the interest is computed based on the amount of principal that remains at the beginning of the month. We assume a fixed rate loan, that is one in which we make a constant monthly payment $M$ on our loan, beginning in the original month of the loan.

Suppose you want to buy a house. You have a certain amount of money saved to make a down payment, and you will borrow the rest to pay for the house. Of course, for the privilege of loaning you the money, the bank will charge you interest on this loan, so the amount you pay back to the bank is more than the amount you borrow. In fact, the amount you ultimately pay depends on three things: the amount you borrow (called the *principal*), the interest rate, and the length of time you have to pay off the loan plus interest (called the *duration* of the loan). For this example, we assume that the interest rate is fixed at $r$ %.

To pay off the loan, each month you make a payment of the same amount (called *installments*). Suppose we borrow $P$ dollars (our principal) and pay off the loan at an interest rate of $r$% with regular monthly installment payments of $M$ dollars. So in month 1 of the loan, before we make any payments, our principal is $P$ dollars. Our goal in this exercise is to find a formula that relates these three parameters to the time duration of the loan.

We are charged interest every month at an annual rate of $r$%, so each month we pay $\frac{r}{12}$% interest on the principal that remains. Given that the original principal is $P$ dollars, we will pay $\left(\frac{0.0r}{12}\right)P$ dollars in interest on our first payment. Since we paid $M$ dollars in total for our first payment, the remainder of the payment $(M-\left(\frac{r}{12}\right)P)$ goes to pay down the principal. So the principal remaining after the first payment (let's call it $P_1$) is the original principal minus what we paid on the principal, or

$$P_1 = P - \left(M - \left(\frac{r}{12}\right)P\right) = \left(1+\frac{r}{12}\right)P - M.$$

As long as $P_1$ is positive, we still have to keep making payments to pay off the loan.

1. Recall that the amount of interest we pay each time depends on the principal that remains. How much interest, in terms of $P_1$ and $r$, do we pay in the second installment?
2. How much of our second monthly installment goes to pay off the principal? What is the principal $P_2$, or the balance of the loan, that we still have to pay off after making the second installment of the loan? Write your response in the form $P_2 = (\ )P_1 - (\ )M$, where you fill in the parentheses.
3. Show that $P_2 = \left(1+\frac{r}{12}\right)^2 P - \left[1+\left(1+\frac{r}{12}\right)\right]M$.
4. Let $P_3$ be the amount of principal that remains after the third installment. Show that
$$P_3 = \left(1+\frac{r}{12}\right)^3 P - \left[1+\left(1+\frac{r}{12}\right)+\left(1+\frac{r}{12}\right)^2\right]M.$$
5. If we continue in the manner described in the problems above, then the remaining principal of our loan after $n$ installments is
$$P_n = \left(1+\frac{r}{12}\right)^n P - \left[\sum_{k=0}^{n-1}\left(1+\frac{r}{12}\right)^k\right]M. \tag{8.2.7}$$
This is a rather complicated formula and one that is difficult to use. However, we can simplify the sum if we recognize part of it as a partial sum of a geometric series. Find a formula for the sum
$$\sum_{k=0}^{n-1}\left(1+\frac{r}{12}\right)^k. \tag{8.2.8}$$
and then a general formula for $P_n$ that does not involve a sum.
6. It is usually more convenient to write our formula for $P_n$ in terms of years rather than months. Show that $P(t)$, the principal remaining after $t$ years, can be written as
$$P(t) = \left(P - \frac{12M}{r}\right)\left(1+\frac{r}{12}\right)^{12t} + \frac{12M}{r}. \tag{8.2.9}$$
7. Now that we have analyzed the general loan situation, we apply formula (8.2.9) to an actual loan. Suppose we charge \$1,000 on a credit card for holiday expenses. If our credit card charges 20% interest and we pay only the minimum payment of \$25 each month, how long will it take us to pay off the \$1,000 charge? How much in total will we have paid on this \$1,000 charge? How much total interest will we pay on this loan?

8. Now we consider larger loans, e.g., automobile loans or mortgages, in which we borrow a specified amount of money over a specified period of time. In this situation, we need to determine the amount of the monthly payment we need to make to pay off the loan in the specified amount of time. In this situation, we need to find the monthly payment $M$ that will take our outstanding principal to $0$ in the specified amount of time. To do so, we want to know the value of $M$ that makes $P(t) = 0$ in formula (8.2.9). If we set $P(t) = 0$ and solve for $M$, it follows that

$$M = \frac{rP\left(1 + \frac{r}{12}\right)^{12t}}{12\left(\left(1 + \frac{r}{12}\right)^{12t} - 1\right)}.$$

   1. Suppose we want to borrow \$15,000 to buy a car. We take out a 5 year loan at 6.25%. What will our monthly payments be? How much in total will we have paid for this \$15,000 car? How much total interest will we pay on this loan?
   2. Suppose you charge your books for winter semester on your credit card. The total charge comes to \$525. If your credit card has an interest rate of 18% and you pay \$20 per month on the card, how long will it take before you pay off this debt? How much total interest will you pay?
   3. Say you need to borrow \$100,000 to buy a house. You have several options on the loan:
      - 30 years at 6.5%
      - 25 years at 7.5%
      - 15 years at 8.25%.
      1. What are the monthly payments for each loan?
      2. Which mortgage is ultimately the best deal (assuming you can afford the monthly payments)? In other words, for which loan do you pay the least amount of total interest?

# 8.3: Series of Real Numbers

## 1. Convergence of a sequence and its series

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## 2. Two partial sums

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## 3. Convergence of a series and its sequence

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## 4. Convergence of an integral and a related series

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## 5

In this exercise we investigate the sequence $\left\{\frac{b^n}{n!}\right\}$ for any constant $b$.

1. Use the Ratio Test to determine if the series $\sum \frac{10^k}{k!}$ converges or diverges.
2. Now apply the Ratio Test to determine if the series $\sum \frac{b^k}{k!}$ converges for any constant $b$.
3. Use your result from (b) to decide whether the sequence $\left\{\frac{b^n}{n!}\right\}$ converges or diverges. If the sequence $\left\{\frac{b^n}{n!}\right\}$ converges, to what does it converge? Explain your reasoning.

## 6

There is a test for convergence similar to the Ratio Test called the *Root Test*. Suppose we have a series $\sum a_k$ of positive terms so that $a_n \to 0$ as $n \to \infty$.

1. Assume

$$\sqrt[n]{a_n} \to r$$

as $n$ goes to infinity. Explain why this tells us that $a_n \approx r^n$ for large values of $n$.

2. Using the result of part (a), explain why $\sum a_k$ looks like a geometric series when $n$ is big. What is the ratio of the geometric series to which $\sum a_k$ is comparable?
3. Use what we know about geometric series to determine that values of $r$ so that $\sum a_k$ converges if $\sqrt[n]{a_n} \to r$ as $n \to \infty$.

## 7

The associative and distributive laws of addition allow us to add finite sums in any order we want. That is, if $\sum_{k=0}^{n} a_k$ and $\sum_{k=0}^{n} b_k$ are finite sums of real numbers, then

$$\sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (a_k + b_k).$$

However, we do need to be careful extending rules like this to infinite series.

1. Let $a_n = 1 + \frac{1}{2^n}$ and $b_n = -1$ for each nonnegative integer $n$.
    1. Explain why the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ both diverge.
    2. Explain why the series $\sum_{k=0}^{\infty} (a_k + b_k)$ converges.
    3. Explain why

$$\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k \neq \sum_{k=0}^{\infty} (a_k + b_k).$$

    This shows that it is possible to have to two divergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ but yet have the series $\sum_{k=0}^{\infty} (a_k + b_k)$ converge.

2. While part (a) shows that we cannot add series term by term in general, we can under reasonable conditions. The problem in part (a) is that we tried to add divergent series. In this exercise we will show that if $\sum a_k$ and $\sum b_k$ are convergent series, then $\sum(a_k + b_k)$ is a convergent series and
$$\sum(a_k + b_k) = \sum a_k + \sum b_k.$$
   1. Let $A_n$ and $B_n$ be the $n$th partial sums of the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, respectively. Explain why
$$A_n + B_n = \sum_{k=1}^{n}(a_k + b_k).$$
   2. Use the previous result and properties of limits to show that
$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$
   (Note that the starting point of the sum is irrelevant in this problem, so it doesn't matter where we begin the sum.)
3. Use the prior result to find the sum of the series $\sum_{k=0}^{\infty} \frac{2^k+3^k}{5^k}$.

## 8

In the Limit Comparison Test we compared the behavior of a series to one whose behavior we know. In that test we use the limit of the ratio of corresponding terms of the series to determine if the comparison is valid. In this exercise we see how we can compare two series directly, term by term, without using a limit of sequence. First we consider an example.

1. Consider the series
$$\sum \frac{1}{k^2} \text{ and } \sum \frac{1}{k^2+k}.$$
   We know that the series $\sum \frac{1}{k^2}$ is a $p$-series with $p = 2 > 1$ and so $\sum \frac{1}{k^2}$ converges. In this part of the exercise we will see how to use information about $\sum \frac{1}{k^2}$ to determine information about $\sum \frac{1}{k^2+k}$. Let $a_k = \frac{1}{k^2}$ and $b_k = \frac{1}{k^2+k}$.
   1. Let $S_n$ be the $n$th partial sum of $\sum \frac{1}{k^2}$ and $T_n$ the $n$th partial sum of $\sum \frac{1}{k^2+k}$. Which is larger, $S_1$ or $T_1$? Why?
   2. Recall that
$$S_2 = S_1 + a_2 \text{ and } T_2 = T_1 + b_2.$$
      Which is larger, $a_2$ or $b_2$? Based on that answer, which is larger, $S_2$ or $T_2$?
   3. Recall that
$$S_3 = S_2 + a_3 \text{ and } T_3 = T_2 + b_3.$$
      Which is larger, $a_3$ or $b_3$? Based on that answer, which is larger, $S_3$ or $T_3$?
   4. Which is larger, $a_n$ or $b_n$? Explain. Based on that answer, which is larger, $S_n$ or $T_n$?
   5. Based on your response to the previous part of this exercise, what relationship do you expect there to be between $\sum \frac{1}{k^2}$ and $\sum \frac{1}{k^2+k}$? Do you expect $\sum \frac{1}{k^2+k}$ to converge or diverge? Why?
2. The example in the previous part of this exercise illustrates a more general result. Explain why the Direct Comparison Test, stated here, works.

### The Direct Comparison Test

If
$$0 \le b_k \le a_k$$
for every $k$, then we must have

$$0 \leq \sum b_k \leq \sum a_k$$

1. If $\sum a_k$ converges, then $\sum b_k$ converges.
2. If $\sum b_k$ diverges, then $\sum a_k$ diverges.

**Note 8.3.7**

This comparison test applies only to series with nonnegative terms.

1. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum \frac{1}{k-1}$. Hint: Compare to the harmonic series.
2. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum \frac{k}{k^3+1}$.

## 8.4: Alternating Series

### 1. Testing convergence for an alternating series

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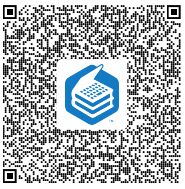
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### 2. Estimating the sum of an alternating series

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### 3. Estimating the sum of a different alternating series

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### 4. Estimating the sum of one more alternating series

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## 5

Conditionally convergent series converge very slowly. As an example, consider the famous formula [1]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}. \tag{8.4.3}$$

We will derive this formula in upcoming work.

In theory, the partial sums of this series could be used to approximate $\pi$.

1. Show that the series in (8.4.3) converges conditionally.
2. Let $S_n$ be the $n$th partial sum of the series in (8.4.3). Calculate the error in approximating $\frac{\pi}{4}$ with $S_{100}$ and explain why this is not a very good approximation.
3. Determine the number of terms it would take in the series (8.4.3) to approximate $\frac{\pi}{4}$ to 10 decimal places. (The fact that it takes such a large number of terms to obtain even a modest degree of accuracy is why we say that conditionally convergent series converge very slowly.)

## 6

We have shown that if $\sum (-1)^{k+1} a_k$ is a convergent alternating series, then the sum $S$ of the series lies between any two consecutive partial sums $S_n$. This suggests that the average $\frac{S_n + S_{n+1}}{2}$ is a better approximation to $S$ than is $S_n$.

1. Show that $\frac{S_n + S_{n+1}}{2} = S_n + \frac{1}{2}(-1)^{n+2} a_{n+1}$.
2. Use this revised approximation in (a) with $n = 20$ to approximate $\ln(2)$ given that

$$\ln(2) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}.$$

   Compare this to the approximation using just $S_{20}$. For your convenience, $S_{20} = \frac{155685007}{232792560}$.

## 7

In this exercise, we examine one of the conditions of the Alternating Series Test. Consider the alternating series

$$1 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{9} + \frac{1}{4} - \frac{1}{16} + \cdots,$$

where the terms are selected alternately from the sequences $\left\{\frac{1}{n}\right\}$ and $\left\{-\frac{1}{n^2}\right\}$.

1. Explain why the $n$th term of the given series converges to 0 as $n$ goes to infinity.
2. Rewrite the given series by grouping terms in the following manner:

$$(1 - 1) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{9}\right) + \left(\frac{1}{4} - \frac{1}{16}\right) + \cdots.$$

   Use this regrouping to determine if the series converges or diverges.

3. Explain why the condition that the sequence $\{a_n\}$ *decreases* to a limit of 0 is included in the Alternating Series Test.

## 8

Conditionally convergent series exhibit interesting and unexpected behavior. In this exercise we examine the conditionally convergent alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ and discover that addition is not commutative for conditionally convergent series. We will also encounter Riemann's Theorem concerning rearrangements of conditionally convergent series. Before we begin, we remind ourselves that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2),$$

a fact which will be verified in a later section.

1. First we make a quick analysis of the positive and negative terms of the alternating harmonic series.
   1. Show that the series $\sum_{k=1}^{\infty} \frac{1}{2k}$ diverges.
   2. Show that the series $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ diverges.
   3. Based on the results of the previous parts of this exercise, what can we say about the sums $\sum_{k=C}^{\infty} \frac{1}{2k}$ and $\sum_{k=C}^{\infty} \frac{1}{2k+1}$ for any positive integer $C$? Be specific in your explanation.
2. Recall addition of real numbers is commutative; that is

   $$a + b = b + a$$

   for any real numbers $a$ and $b$. This property is valid for any sum of finitely many terms, but does this property extend when we add infinitely many terms together?

   The answer is no, and something even more odd happens. Riemann's Theorem (after the nineteenth-century mathematician Georg Friedrich Bernhard Riemann) states that a conditionally convergent series can be rearranged to converge to *any* prescribed sum. More specifically, this means that if we choose any real number $S$, we can rearrange the terms of the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ so that the sum is $S$. To understand how Riemann's Theorem works, let's assume for the moment that the number $S$ we want our rearrangement to converge to is positive. Our job is to find a way to order the sum of terms of the alternating harmonic series to converge to $S$.

   1. Explain how we know that, regardless of the value of $S$, we can find a partial sum $P_1$

      $$P_1 = \sum_{k=1}^{n_1} \frac{1}{2k+1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n_1+1}$$

      of the positive terms of the alternating harmonic series that equals or exceeds $S$. Let

      $$S_1 = P_1.$$

   2. Explain how we know that, regardless of the value of $S_1$, we can find a partial sum $N_1$

      $$N_1 = -\sum_{k=1}^{m_1} \frac{1}{2k} = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots - \frac{1}{2m_1}$$

      so that

      $$S_2 = S_1 + N_1 \leq S.$$

   3. Explain how we know that, regardless of the value of $S_2$, we can find a partial sum $P_2$

      $$P_2 = \sum_{k=n_1+1}^{n_2} \frac{1}{2k+1} = \frac{1}{2(n_1+1)+1} + \frac{1}{2(n_1+2)+1} + \cdots + \frac{1}{2n_2+1}$$

      of the remaining positive terms of the alternating harmonic series so that

      $$S_3 = S_2 + P_2 \geq S.$$

   4. Explain how we know that, regardless of the value of $S_3$, we can find a partial sum

$$N_2 = -\sum_{k=m_1+1}^{m_2} \frac{1}{2k} = -\frac{1}{2(m_1+1)} - \frac{1}{2(m_1+2)} - \cdots - \frac{1}{2m_2}$$

of the remaining negative terms of the alternating harmonic series so that

$$S_4 = S_3 + N_2 \leq S.$$

5. Explain why we can continue this process indefinitely and find a sequence $\{S_n\}$ whose terms are partial sums of a rearrangement of the terms in the alternating harmonic series so that $\lim_{n\to\infty} S_n = S$.

# 8.5: Taylor Polynomials and Taylor Series

## 1. Determining Taylor polynomials from a function formula

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## 2. Determining Taylor polynomials from given derivative values

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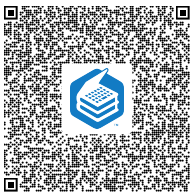
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## 3. Finding the Taylor series for a given rational function

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## 4. Finding the Taylor series for a given trigonometric function

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## 5. Finding the Taylor series for a given logarithmic function

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## 6

In this exercise we investigation the Taylor series of polynomial functions.

1. Find the 3rd order Taylor polynomial centered at $a = 0$ for $f(x) = x^3 - 2x^2 + 3x - 1$. Does your answer surprise you? Explain.
2. Without doing any additional computation, find the 4th, 12th, and 100th order Taylor polynomials (centered at $a = 0$) for $f(x) = x^3 - 2x^2 + 3x - 1$. Why should you expect this?
3. Now suppose $f(x)$ is a degree $m$ polynomial. Completely describe the $n$th order Taylor polynomial (centered at $a = 0$) for each $n$.

## 7

The examples we have considered in this section have all been for Taylor polynomials and series centered at 0, but Taylor polynomials and series can be centered at any value of $a$. We look at examples of such Taylor polynomials in this exercise.

1. Let $f(x) = \sin(x)$. Find the Taylor polynomials up through order four of $f$ centered at $x = \frac{\pi}{2}$. Then find the Taylor series for $f(x)$ centered at $x = \frac{\pi}{2}$. Why should you have expected the result?
2. Let $f(x) = \ln(x)$. Find the Taylor polynomials up through order four of $f$ centered at $x = 1$. Then find the Taylor series for $f(x)$ centered at $x = 1$.
3. Use your result from (b) to determine which Taylor polynomial will approximate $\ln(2)$ to two decimal places. Explain in detail how you know you have the desired accuracy.

## 8

We can use known Taylor series to obtain other Taylor series, and we explore that idea in this exercise, as a preview of work in the following section.

1. Calculate the first four derivatives of $\sin(x^2)$ and hence find the fourth order Taylor polynomial for $\sin(x^2)$ centered at $a = 0$.
2. Part (a) demonstrates the brute force approach to computing Taylor polynomials and series. Now we find an easier method that utilizes a known Taylor series. Recall that the Taylor series centered at 0 for $f(x) = \sin(x)$ is

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \tag{8.5.7}$$

    1. Substitute $x^2$ for $x$ in the Taylor series (8.5.7). Write out the first several terms and compare to your work in part (a). Explain why the substitution in this problem should give the Taylor series for $\sin(x^2)$ centered at 0.
    2. What should we expect the interval of convergence of the series for $\sin(x^2)$ to be? Explain in detail.

### 9

Based on the examples we have seen, we might expect that the Taylor series for a function $f$ always converges to the values $f(x)$ on its interval of convergence. We explore that idea in more detail in this exercise. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

1. Show, using the definition of the derivative, that $f'(0) = 0$.
2. It can be shown that $f^{(n)}(0) = 0$ for all $n \geq 2$. Assuming that this is true, find the Taylor series for $f$ centered at $0$.
3. What is the interval of convergence of the Taylor series centered at $0$ for $f$? Explain. For which values of $x$ the interval of convergence of the Taylor series does the Taylor series converge to $f(x)$?

## 8.6: Power Series

### 1. Finding coefficients in a power series expansion of a rational function

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### 2. Finding coefficients in a power series expansion of a function involving $\arctan(x)$

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### 3.

We can use power series to approximate definite integrals to which known techniques of integration do not apply. We will illustrate this in this exercise with the definite integral $\int_0^1 \sin(x^2)\ ds$.

1. Use the Taylor series for $\sin(x)$ to find the Taylor series for $\sin(x^2)$. What is the interval of convergence for the Taylor series for $\sin(x^2)$? Explain.
2. Integrate the Taylor series for $\sin(x^2)$ term by term to obtain a power series expansion for $\int \sin(x^2)\ dx$.
3. Use the result from part (b) to explain how to evaluate $\int_0^1 \sin(x^2)\ dx$. Determine the number of terms you will need to approximate $\int_0^1 \sin(x^2)\ dx$ to 3 decimal places.

### 4

There is an important connection between power series and Taylor series. Suppose $f$ is defined by a power series centered at $0$ so that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

1. Determine the first 4 derivatives of $f$ evaluated at 0 in terms of the coefficients $a_k$.
2. Show that $f^{(n)}(0) = n!a_n$ for each positive integer $n$.
3. Explain how the result of (b) tells us the following:

   On its interval of convergence, a power series is the Taylor series of its sum.

## 5

In this exercise we will begin with a strange power series and then find its sum. The Fibonacci sequence $\{f_n\}$ is a famous sequence whose first few terms are

$$f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, \cdots,$$

where each term in the sequence after the first two is the sum of the preceding two terms. That is, $f_0 = 0,\ f_1 = 1$ and for $n \geq 2$ we have

$$f_n = f_{n-1} + f_{n-2}.$$

Now consider the power series

$$F(x) = \sum_{k=0}^{\infty} f_k x^k.$$

We will determine the sum of this power series in this exercise.

1. Explain why each of the following is true.
   1. $xF(x) = \sum_{k=1}^{\infty} f_{k-1}x^k$
   2. $x^2F(x) = \sum_{k=2}^{\infty} f_{k-2}x^k$
2. Show that
   $$F(x) - xF(x) - x^2F(x) = x.$$
3. Now use the equation
   $$F(x) - xF(x) - x^2F(x) = x$$
   to find a simple form for $F(x)$ that doesn't involve a sum.
4. Use a computer algebra system or some other method to calculate the first 8 derivatives of $\frac{x}{1-x-x^2}$ evaluated at 0. Why shouldn't the results surprise you?

## 6

Airy's equation [2]

$$y'' - xy = 0, \tag{8.6.4}$$

can be used to model an undamped vibrating spring with spring constant $x$ (note that $y$ is an unknown function of $x$). So the solution to this differential equation will tell us the behavior of a spring-mass system as the spring ages (like an automobile shock absorber). Assume that a solution $y = f(x)$ has a Taylor series that can be written in the form

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

where the coefficients are undetermined. Our job is to find the coefficients.

The general differential equations of the form $y'' \pm k^2 xy = 0$ is called Airy's equation. These equations arise in many problems, such as the study of diffraction of light, diffraction of radio waves around an object, aerodynamics, and the buckling of a uniform column under its own weight.

1. Differentiate the series for $y$ term by term to find the series for $y'$. Then repeat to find the series for $y''$.
2. Substitute your results from part (a) into the Airy equation and show that we can write Equation (8.6.4) in the form
$$\sum_{k=2}^{\infty}(k-1)ka_k x^{k-2} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0. \tag{8.6.5}$$
3. At this point, it would be convenient if we could combine the series on the left in (8.6.5), but one written with terms of the form $x^{k-2}$ and the other with terms in the form $x^{k+1}$. Explain why
$$\sum_{k=2}^{\infty}(k-1)ka_k x^{k-2} = \sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}x^k. \tag{8.6.6}$$
4. Now show that
$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k. \tag{8.6.7}$$
5. We can now substitute (8.6.6) and (8.6.7) into (8.6.5) to obtain
$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0. \tag{8.6.8}$$
Combine the like powers of $x$ in the two series to show that our solution must satisfy
$$2a_2 + \sum_{k=1}^{\infty}\left[(k+1)(k+2)a_{k+2} - a_{k-1}\right]x^k = 0. \tag{8.6.9}$$
6. Use equation (8.6.9) to show the following:
   1. $a_{3k+2} = 0$ for every positive integer $k$,
   2. $a_{3k} = \frac{1}{(2)(3)(5)(6)\cdots(3k-1)(3k)}a_0$ for $k \geq 1$,
   3. $a_{3k+1} = \frac{1}{(3)(4)(6)(7)\cdots(3k)(3k+1)}a_1$ for $k \geq 1$.
7. Use the previous part to conclude that the general solution to the Airy equation (8.6.4) is
$$y = a_0\left(1 + \sum_{k=1}^{\infty}\frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)}\right) + a_1\left(x + \sum_{k=1}^{\infty}\frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)}\right).$$
Any values for $a_0$ and $a_1$ then determine a specific solution that we can approximate as closely as we like using this series solution.

## Answers

**5**

You can be paid $500 per day or

You can be paid 1 cent the first day, 2 cents the second day, 4 cents the third day, 8 cents the fourth day, and so on, doubling the amount you are paid each day.

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# 9: OPENSTAX V2 CHP 7- PARAMETRIC EQUATIONS AND POLAR COORDINATES

Parametric equations define a group of quantities as functions of one or more independent variables called parameters. Parametric equations are commonly used to express the coordinates of the points that make up a geometric object such as a curve or surface, in which case the equations are collectively called a parametric representation or parameterization. The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction. The reference point (analogous to the origin of a Cartesian system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate, polar angle, or azimuth


9.0: Prelude to Parametric Equations and Polar Coordinates

**9.1: Parametric Equations**

9.1.1: Exercises for Section 11.1

**9.2: Calculus of Parametric Curves**

9.2.1: Exercises for Section 11.2

**9.3: Polar Coordinates**

9.3.1: Introduction to Polar Coordinates

9.3.2: Exercises for Section 11.3

**9.4: Area and Arc Length in Polar Coordinates**

9.4.1: Exercises for Section 11.4

9.6: 6 Chapter Review Exercises



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# 9.0: Prelude to Parametric Equations and Polar Coordinates

The **chambered nautilus** is a fascinating creature. This animal feeds on hermit crabs, fish, and other crustaceans. It has a hard outer shell with many chambers connected in a spiral fashion, and it can retract into its shell to avoid predators. When part of the shell is cut away, a perfect spiral is revealed, with chambers inside that are somewhat similar to growth rings in a tree.

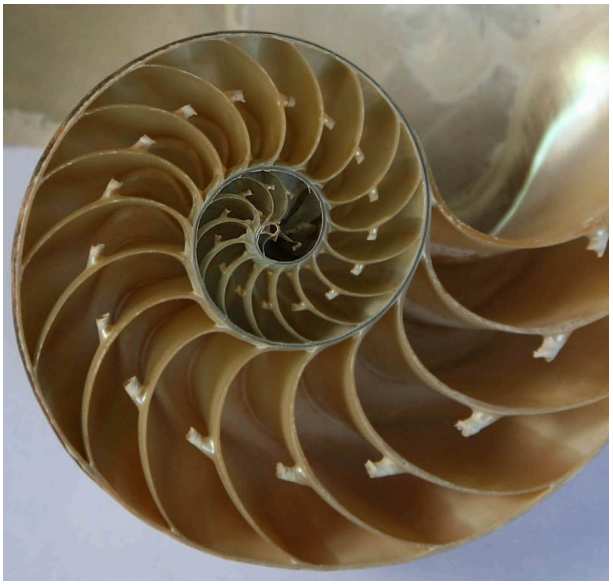


Figure 9.0.1: The chambered nautilus is a marine animal that lives in the tropical Pacific Ocean. Scientists think they have existed mostly unchanged for about 500 million years.(credit: modification of work by Jitze Couperus, Flickr)

The mathematical function that describes a spiral can be expressed using rectangular (or Cartesian) coordinates. However, if we change our coordinate system to something that works a bit better with circular patterns, the function becomes much simpler to describe. The polar coordinate system is well suited for describing curves of this type. How can we use this coordinate system to describe spirals and other radial figures?

In this chapter we also study parametric equations, which give us a convenient way to describe curves, or to study the position of a particle or object in two dimensions as a function of time. We will use parametric equations and polar coordinates for describing many topics later in this text.

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# 9.1: PARAMETRIC EQUATIONS

### 9.1.1: Exercises for Section 11.1

**Learning Objectives**

- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both $x$ and $y$ depend on, and as the parameter increases, the values of $x$ and $y$ trace out a path along a plane curve. For example, if the parameter is $t$ (a common choice), then $t$ might represent time. Then $x$ and $y$ are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

## Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 “day 1” of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

The number of the day in a year can be considered a variable that determines Earth’s position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler’s laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in Conic Sections.

An ellipse with January 1 (t = 1) at the top, April 2 (t = 92) on the left, July 1 (t = 182) on the bottom, and October 1 (t = 274) on the right. The focal points of the ellipse have F2 on the left and the Sun on the right.

Figure $9.1.1$: Earth’s orbit around the Sun in one year.

Figure $9.1.1$ depicts **Earth’s orbit** around the Sun during one year. The point labeled $F_2$ is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse (Figure $9.1.2$). Then each $x$ value on the graph is a value of position as a function of time, and each $y$ value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth’s position as a function of time.

An ellipse with January 1 (t = 1) at the top, April 2 (t = 92) on the left, July 1 (t = 182) on the bottom, and October 1 (t = 274) on the right. The focal points of the ellipse have F2 on the left and the Sun on the right. There is a line going from t = 1 to t = 182. There is also a line going from t = 92 to t = 274 that passes through F2 and the Sun. On the upper left side, there is a point marked (x(t), y(t)) with a tangent line pointing down and to the left.

Figure $9.1.2$: Coordinate axes superimposed on the orbit of Earth.

We can determine the functions for $x(t)$ and $y(t)$, thereby parameterizing the orbit of Earth around the Sun. The variable $t$ is called an independent parameter and, in this context, represents time relative to the beginning of each year.

A curve in the $(x, y)$ plane can be represented parametrically. The equations that are used to define the curve are called

*parametric equations.*

### Definition: Parametric Equations

If $x$ and $y$ are continuous functions of $t$ on an interval $I$, then the equations

$$x = x(t)$$

and

$$y = y(t)$$

are called parametric equations and $t$ is called the parameter. The set of points $(x, y)$ obtained as $t$ varies over the interval $I$ is called the graph of the parametric equations. The graph of parametric equations is called a *parametric curve* or plane curve, and is denoted by $C$.

Notice in this definition that $x$ and $y$ are used in two ways. The first is as functions of the independent variable $t$. As $t$ varies over the interval $I$, the functions $x(t)$ and $y(t)$ generate a set of ordered pairs $(x, y)$. This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered pairs, $x$ and $y$ are variables. It is important to distinguish the variables $x$ and $y$ from the functions $x(t)$ and $y(t)$.

### Example 9.1.1: Graphing a Parametrically Defined Curve

Sketch the curves described by the following parametric equations:

a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad \text{for } -3 \le t \le 2$
b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad \text{for } -2 \le t \le 3$
c. $x(t) = 4\cos t, \quad y(t) = 4\sin t, \quad \text{for } 0 \le t \le 2\pi$

**Solution a**

To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is $t$, let $t$ appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

| $t$ | $x(t)$ | $y(t)$ |
|---|---|---|
| –3 | –4 | –2 |
| –2 | –3 | 0 |
| –1 | –2 | 2 |
| 0 | –1 | 4 |
| 1 | 0 | 6 |
| 2 | 1 | 8 |

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in Figure 9.1.3. The arrows on the graph indicate the **orientation** of the graph, that is, the direction that a point moves on the graph as t varies from –3 to 2.

A straight line going from (–4, –2) through (–3, 0), (–2, 2), and (0, 6) to (1, 8) with arrow pointed up and to the right. The point (–4, –2) is marked t = –3, the point (–2, 2) is marked t = –1, and the point (1, 8) is marked t = 2. On the graph there are also written three equations: x(t) = t –1, y(t) = 2t + 4, and –3 ≤ t ≤ 2.

Figure 9.1.3: Graph of the plane curve described by the parametric equations in part a.

**Solution b**

To create a graph of this curve, again set up a table of values.

| $t$ | $x(t)$ | $y(t)$ |
|---|---|---|
| –2 | 1 | –3 |
| –1 | –2 | –1 |
| 0 | –3 | 1 |
| 1 | –2 | 3 |
| 2 | 1 | 5 |
| 3 | 6 | 7 |

The second and third columns in this table give a set of points to be plotted (Figure $9.1.4$). The first point on the graph (corresponding to $t = -2$) has coordinates $(1, -3)$, and the last point (corresponding to $t = 3$) has coordinates $(6, 7)$. As $t$ progresses from $-2$ to $3$, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

A curved line going from (1, –3) through (–3, 1) to (6, 7) with arrow pointing in that order. The point (1, –3) is marked t = –2, the point (–3, 1) is marked t = 0, and the point (6, 7) is marked t = 3. On the graph there are also written three equations: x(t) = t2 – 3, y(t) = 2t + 1, and –2 ≤ t ≤ 3.

Figure $9.1.4$: Graph of the plane curve described by the parametric equations in part b.

**Solution c**

In this case, use multiples of $\pi/6$ for $t$ and create another table of values:

| $t$ | $x(t)$ | $y(t)$ | $t$ | $x(t)$ | $y(t)$ |
|---|---|---|---|---|---|
| 0 | 4 | 0 | $\frac{7\pi}{6}$ | $-2\sqrt{3} \approx -3.5$ | -2 |
| $\frac{\pi}{6}$ | $2\sqrt{3} \approx 3.5$ | 2 | $\frac{4\pi}{3}$ | –2 | $-2\sqrt{3} \approx -3.5$ |
| $\frac{\pi}{3}$ | 2 | $2\sqrt{3} \approx 3.5$ | $\frac{3\pi}{2}$ | 0 | –4 |
| $\frac{\pi}{2}$ | 0 | 4 | $\frac{5\pi}{3}$ | 2 | $-2\sqrt{3} \approx -3.5$ |
| $\frac{2\pi}{3}$ | –2 | $2\sqrt{3} \approx 3.5$ | $\frac{11\pi}{6}$ | $2\sqrt{3} \approx 3.5$ | -2 |
| $\frac{5\pi}{6}$ | $-2\sqrt{3} \approx -3.5$ | 2 | $2\pi$ | 4 | 0 |
| $\pi$ | –4 | 0 | | | |

The graph of this plane curve appears in the following graph.

A circle with radius 4 centered at the origin is graphed with arrow going counterclockwise. The point (4, 0) is marked t = 0, the point (0, 4) is marked t = π/2, the point (–4, 0) is marked t = π, and the point (0, –4) is marked t = 3π/2. On the graph there are also written three equations: x(t) = 4 cos(t), y(t) = 4 sin(t), and 0 ≤ t ≤ 2π.

Figure $9.1.5$: Graph of the plane curve described by the parametric equations in part c.

This is the graph of a circle with radius $4$ centered at the origin, with a counterclockwise orientation. The starting point and ending points of the curve both have coordinates $(4, 0)$.

## Exercise $9.1.1$

Sketch the curve described by the parametric equations

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad \text{for } -3 \le t \le 2.$$

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables $x$ and $y$. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in Example $9.1.1b$ are

$$x(t) = t^2 - 3 \tag{9.1.1}$$
$$y(t) = 2t + 1 \tag{9.1.2}$$

over the region $-2 \leq t \leq 3$.

Solving Equation $9.1.2$ for $t$ gives

$$t = \frac{y-1}{2}.$$

This can be substituted into Equation $9.1.1$:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 \tag{9.1.3}$$
$$= \frac{y^2 - 2y + 1}{4} - 3 \tag{9.1.4}$$
$$= \frac{y^2 - 2y - 11}{4}. \tag{9.1.5}$$

Equation $9.1.5$ describes $x$ as a function of $y$. These steps give an example of eliminating the parameter. The graph of this function is a parabola opening to the right (Figure $9.1.4$). Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter $t$.

### Example $9.1.2$: Eliminating the Parameter

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}, \quad y(t) = 2t + 1, \quad \text{for } -2 \leq t \leq 6$
b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \leq t \leq 2\pi$

**Solution a**

To eliminate the parameter, we can solve either of the equations for $t$. For example, solving the first equation for $t$ gives

$$x = \sqrt{2t+4}$$
$$x^2 = 2t + 4$$
$$x^2 - 4 = 2t$$
$$t = \frac{x^2 - 4}{2}.$$

Note that when we square both sides it is important to observe that $x \geq 0$. Substituting $t = \dfrac{x^2-4}{2}$ into $y(t)$ yields

$$y(t) = 2t + 1$$
$$y = 2\left(\frac{x^2 - 4}{2}\right) + 1$$
$$y = x^2 - 4 + 1$$
$$y = x^2 - 3.$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter $t$. When $t = -2$, $x = \sqrt{2(-2) + 4} = 0$, and when $t = 6$, $x = \sqrt{2(6) + 4} = 4$. The graph of this plane curve follows.

A curved line going from (–3, 0) through (2, 1) to (4, 13) with arrow going in that order. The point (–3, 0) is marked t = –2, the point (2, 1) is marked t = 0, the point (2 times the square root of 2, 5) is marked t = 2, the point (3 times the square root of 2, 9) is marked t = 4, and the point (4, 13) is marked t = 6. On the graph there are also written three equations: x(t) = square root of the quantity (2t + 4), y(t) = 2t + 1, and –2 ≤ t ≤ 6.

Figure 9.1.6: Graph of the plane curve described by the parametric equations in part a.

**Solution b**

Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t$$

and

$$y(t) = 3\sin t$$

Solving either equation for $t$ directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by $4$ and the second equation by $3$ (and suppressing the $t$) gives us

$$\cos t = \frac{x}{4}$$

and

$$\sin t = \frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of $x$ and $y$. This gives

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semi-major axis $4$ and semi-minor axis $3$ as shown in the following graph.

An ellipse with major axis horizontal and of length 8 and with minor radius vertical and of length 6 that is centered at the origin with arrow going counterclockwise. The point (4, 0) is marked t = 0, the point (0, 3) is marked t = π/2, the point (–4, 0) is marked t = π, and the point (0, –3) is marked t = 3π/2. On the graph there are also written three equations: x(t) = 4 cos(t), y(t) = 3 sin(t), and 0 ≤ t ≤ 2π.

Figure 9.1.7: Graph of the plane curve described by the parametric equations in part b.

As $t$ progresses from $0$ to $2\pi$, a point on the curve traverses the ellipse once, in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parameterized curves to model a real-world phenomenon.

### Exercise 9.1.2

Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph.

$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad \text{for } 2 \leq t \leq 6$$

*Hint available at the end of this section.*

*Answer available at the end of this section.*

So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve.**

### Example 9.1.3: Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

**Solution**

First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing $x$ with $t$ in the equation for $y(t)$. This gives the parameterization

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of $t$.

We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on $x$; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for $x$. This gives

$$y(t) = 2(3t - 2)^2 - 2 = 2(9t^2 - 12t + 4) - 2 = 18t^2 - 24t + 8 - 2 = 18t^2 - 24t + 6.$$

Therefore, a second parameterization of the curve can be written as

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$

### Exercise 9.1.3

Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern.

Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a *cycloid* (Figure 9.1.8). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the $x$-axis at a constant height equal to the radius of the wheel. If the radius is $a$, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

for any value of $t$. Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

$$x(t) = -a \sin t$$
$$y(t) = -a \cos t.$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

$$x(t) = a(t - \sin t)$$
$$y(t) = a(1 - \cos t)$$

A series of circles with center marked and a point on the circle drawing out a curve as if the circle was rolling along a plane. The shape made seems to be half an ellipse with height the diameter of the original circle and with major axis the circumference of the circle.

Figure 9.1.8: A wheel traveling along a road without slipping; the point on the edge of the wheel traces out a cycloid.

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure 9.1.9. In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a *hypocycloid***.**

Two circles are drawn both with center at the origin and with radii 3 and 4, respectively; the circle with radius 3 has an arrow pointing in the counterclockwise direction. There is a third circle drawn with center on the circle with radius 3 and touching the circle with radius 4 at one point. That is, this third circle has radius 1. A point is drawn on this third circle, and if it were to roll along the other two circles, it would draw out a four pointed star with points at (4, 0), (0, 4), (–4, 0), and (0, –4). On the graph there are also written two equations: x(t) = 3 cos(t) + cos(3t) and y(t) = 3 sin(t) – sin(3t).

Figure 9.1.9: Graph of the hypocycloid described by the parametric equations shown.

The general parametric equations for a hypocycloid are

$$x(t) = (a - b)\cos t + b\cos(\frac{a - b}{b})t$$

$$y(t) = (a - b)\sin t - b\sin(\frac{a - b}{b})t.$$

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is $a$ and the radius of the smaller circle is $b$. Then the center of the wheel travels along a circle of radius $a - b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a - b}$.

The ratio $\frac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in Figure 9.1.10. This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. Figure 9.1.9 corresponds to $a = 4$ and $b = 1$. The result is a hypocycloid with four cusps. Figure 9.1.10 shows some other possibilities. The last two hypocycloids have irrational values for $\frac{a}{b}$. In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as **space-filling curves.**

A series of hypocycloids is given. The first is a three pointed star marked a/b = 3. The second is a four pointed star marked a/b = 4. The third is a five pointed star marked a/b = 5. None of these first three figures has lines that cross each other. The fourth figure is a five pointed star but this one has lines which cross each other and looks like the star that children first learn to draw; it is marked a/b = 5/3. A similar sort of star with seven points is next and is marked a/b = 7/3. Then a similar star with eight points is next and is marked a/b = 8/3. The next figure is a complicated series of curves that ultimately creates a small rosette in the middle; this is marked a/b = π. Lastly, there is an even more complicated series of curves that creates a large rosette with sharper florets marked a/b = the square root of 2.

Figure 9.1.10: Graph of various hypocycloids corresponding to different values of $a/b$.

## The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the folium of Descartes or the spiral of Archimedes. However, perhaps the strangest name for a curve is the **witch of Agnesi.** Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of the few recognized women mathematicians of eighteenth-century Italy. She wrote a popular book on analytic geometry, published in 1748, which included an interesting curve that had been studied by Fermat in 1630. The mathematician Guido Grandi showed in 1703 how to construct this curve, which he later called the "versoria," a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, "versiera," but in Latin, this same word means a "female goblin." When Agnesi's book was translated into English in 1801, the translator used the term "witch" for the curve, instead of rope. The name "witch of Agnesi" has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius $a$ so that the points $(0, 0)$ and $(0, 2a)$ are points on the circle (Figure 9.1.11). Let $O$ denote the origin. Choose any other point $A$ on the circle, and draw the secant line $OA$. Let $B$ denote the point at which the line $OA$ intersects the horizontal line through $(0, 2a)$. The vertical line through $B$ intersects the horizontal line through $A$ at the point $P$. As the point $A$ varies, the path that the point $P$ travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the *Cauchy distribution*. In this project you will parameterize these curves.

A circle with bottom at point O (the origin) and top at point (0, 2a) is drawn. The x axis is drawn from point O, and the y axis is drawn up from point O through (0, 2a). Parallel to the x axis is a line drawn from (0, 2a); it has point B marked to the right. A line from point B to point O passes through the circle at point A. A line is drawn parallel to the x axis from point A, and it forms a right angle with a line drawn down from point B; these lines intersect at point P. There is a curve that is symmetric about the y axis that passes through the point P. This curve has its maximum at (0, 2a) and gently decreases through the point P.

Figure 9.1.11: As the point $A$ moves around the circle, the point $P$ traces out the witch of Agnesi curve for the given circle.

1. On the figure, label the following points, lengths, and angle:
   a. $C$ is the point on the $x$-axis with the same $x$-coordinate as $A$.
   b. $x$ is the $x$-coordinate of $P$, and $y$ is the $y$-coordinate of $P$.
   c. $E$ is the point $(0, a)$.
   d. $F$ is the point on the line segment $OA$ such that the line segment $EF$ is perpendicular to the line segment $OA$.
   e. $b$ is the distance from $O$ to $F$.
   f. $c$ is the distance from $F$ to $A$.
   g. $d$ is the distance from $O$ to $B$.
   h. $\theta$ is the measure of angle $\angle COA$.

The goal of this project is to parameterize the witch using $\theta$ as a parameter. To do this, write equations for $x$ and $y$ in terms of only $\theta$.

2. Show that $d = \dfrac{2a}{\sin\theta}$.
3. Note that $x = d\cos\theta$. Show that $x = 2a\cot\theta$. When you do this, you will have parameterized the $x$-coordinate of the curve with respect to $\theta$. If you can get a similar equation for $y$, you will have parameterized the curve.
4. In terms of $\theta$, what is the angle $\angle EOA$?
5. Show that $b + c = 2a\cos\left(\frac{\pi}{2} - \theta\right)$.

6. Show that $y = 2a\cos\left(\frac{\pi}{2} - \theta\right)\sin\theta$.
7. Show that $y = 2a\sin^2\theta$. You have now parameterized the $y$-coordinate of the curve with respect to $\theta$.
8. Conclude that a parameterization of the given witch curve is

$$x = 2a\cot\theta, \quad y = 2a\sin^2\theta, \quad \text{for } -\infty < \theta < \infty.$$

9. Use your parameterization to show that the given witch curve is the graph of the function $f(x) = \dfrac{8a^3}{x^2 + 4a^2}$.

## Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, which is the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project we look at two different variations of the cycloid, called the curtate and prolate cycloids.

First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We have assumed the ant climbed onto the tire at the very edge, where the tire touches the ground. As the wheel rolls, the ant moves with the edge of the tire (Figure $9.1.12$).

As we have discussed, we have a lot of flexibility when parameterizing a curve. In this case we let our parameter t represent the angle the tire has rotated through. Looking at Figure $9.1.12$, we see that after the tire has rotated through an angle of $t$, the position of the center of the wheel, $C = (x_C, y_C)$, is given by

$$x_C = at \text{ and } y_C = a.$$

Furthermore, letting $A = (x_A, y_A)$ denote the position of the ant, we note that

$$x_C - x_A = a\sin t \text{ and } y_C - y_A = a\cos t$$

Then

$$x_A = x_C - a\sin t = at - a\sin t = a(t - \sin t)$$

$$y_A = y_C - a\cos t = a - a\cos t = a(1 - \cos t).$$

There are two figures marked (a) and (b). Figure a has a circle with point A on the circle at the origin. The circle has "spokes," with point A being at the end of one of these spokes. The circle appears to be travelling to the right on the x axis, with point A being up above the x axis in a second image of the circle drawn slightly to the right. Figure b has a circle in the first quadrant with center C. It touches the x axis at xc. A point A is drawn on the circle and a right triangle is made from this point and point C. The hypotenuse is marked a and the angle at C between A and xc is marked t. Lines are drawn to give the x and y values of A as xA and yA, respectively. Similarly, a line is drawn to give the y value of C as yC.

Figure $9.1.12$: (a) The ant clings to the edge of the bicycle tire as the tire rolls along the ground. (b) Using geometry to determine the position of the ant after the tire has rotated through an angle of $t$.

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable $t$.

After a while the ant is getting dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a **curtate cycloid** (Figure $9.1.13$). As shown in the figure, we let b denote the distance along the spoke from the center of the wheel to the ant. As before, we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant.

There are three figures marked (a), (b), and (c). Figure a has a circle with “spokes,” where point A is in the middle of one of these spokes. The circle is tangent to the x axis at the origin. The circle appears to be travelling to the right on the x axis, with point A being higher up in a second image of the circle drawn slightly to the right. Figure b shows the curve that point A would trace out, as the circle travels to the right. It is vaguely sinusoidal. Figure c has a circle in the first quadrant with center C. It touches the x axis at xc. A point A is drawn inside the circle and a right triangle is made from this point and point C. The hypotenuse is marked b, the angle at C between A and xc is marked t, and the distance from C to xc is marked a. Lines are drawn to give the x and y values of A as xA and yA, respectively. Similarly, a line is drawn to give the y value of C as yC.

Figure 9.1.13: (a) The ant climbs up one of the spokes toward the center of the wheel. (b) The ant’s path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid. (c) The new setup, now that the ant has moved closer to the center of the wheel.

1. What is the position of the center of the wheel after the tire has rotated through an angle of $t$?
2. Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.
3. On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?

Once the ant’s head clears, he realizes that the bicyclist has made a turn, and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, there is a set of train tracks nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and just catch the edge of the train wheel (without getting squished!).

The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to just hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is actually greater than the radius of the wheel (Figure 9.1.14).

The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant (Figure 9.1.14).

When the distance from the center of the wheel to the ant is greater than the radius of the wheel, his path of motion is called a prolate cycloid. A graph of a **prolate cycloid** is shown in the figure.

There are three figures marked (a), (b), and (c). Figure a has a circle and a point A that is outside the circle on the y axis (below the origin). The circle is tangent to the x axis at the origin. The circle appears to be travelling to the right on the x axis, with point A being above the x axis in a second image of the circle drawn slightly to the right. Figure b has a circle in the first quadrant with center C. It touches the x axis at xc. A point A is drawn outside the circle and a right triangle is made from this point and point C. The hypotenuse is marked b, the angle at C between A and xc is marked t, and the distance from C to xc is marked a. Lines are drawn to give the x and y values of A as xA and yA, respectively. Similarly, a line is drawn to give the y value of C as yC. Figure c shows the curve that point A would trace out, as the circle travels to the right. It is vaguely sinusoidal with an extra loop at the bottom once per revolution.

Figure 9.1.14: (a) The ant is hanging onto the flange of the train wheel. (b) The new setup, now that the ant has jumped onto the train wheel. (c) The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.
5. What do you notice about your answer to part 3 and your answer to part 4?

Notice that the ant is actually traveling backward at times (the “loops” in the graph), even though the train continues to move forward. He is probably going to be really dizzy by the time he gets home!

## Key Concepts

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

# Glossary

**cycloid**
: the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

**cusp**
: a pointed end or part where two curves meet

**orientation**
: the direction that a point moves on a graph as the parameter increases

**parameter**
: an independent variable that both $x$ and $y$ depend on in a parametric curve; usually represented by the variable $t$

**parametric curve**
: the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

**parametric equations**
: the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

**parameterization of a curve**
: rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

## Answers

**Exercise** $9.1.1$

Make a table of values for $x(t)$ and $y(t)$ using $t$ values from $-3$ to $2$.

**Exercise** $9.1.1$

A curved line going from (–7, 8) through (–1, 0) and (2, –1) to (8, 3) with arrow going in that order. The point (–7, 8) is marked t = –3, the point (2, –1) is marked t = 0, and the point (8, 3) is marked t = 2. On the graph there are also written three equations: x(t) = 3t + 2, y(t) = t2 – 1, and –3 ≤ t ≤ 2.

**Exercise** $9.1.2$

Solve one of the equations for $t$ and substitute into the other equation.

**Exercise** $9.1.2$

$x = 2 + \frac{3}{y+1}$, or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.

A curved line going from (3.5, 1) to (2.5, 5) with arrow going in that order. The point (3.5, 1) is marked t = 2 and the point (2.5, 5) is marked t = 6. On the graph there are also written three equations: x(t) = 2 + 3/t, y(t) = t – 1, and 2 ≤ t ≤ 6.

**Exercise** $9.1.3$

Follow the steps in Example $9.1.3$. Remember we have freedom in choosing the parameterization for $x(t)$.

**Exercise** $9.1.3$

One possibility is $x(t) = t, \quad y(t) = t^2 + 2t$. Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3$. There are, in fact, an infinite number of possibilities.

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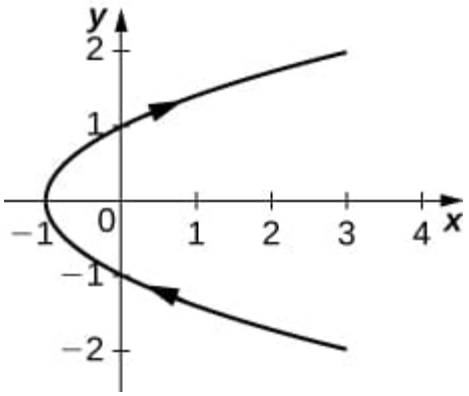
# 9.1.1: Exercises for Section 11.1

**In exercises 1 - 4, sketch the curves below by eliminating the parameter $t$. Give the orientation of the curve.**

1) $x = t^2 + 2t, \quad y = t + 1$

**Answer**

Orientation: bottom to top

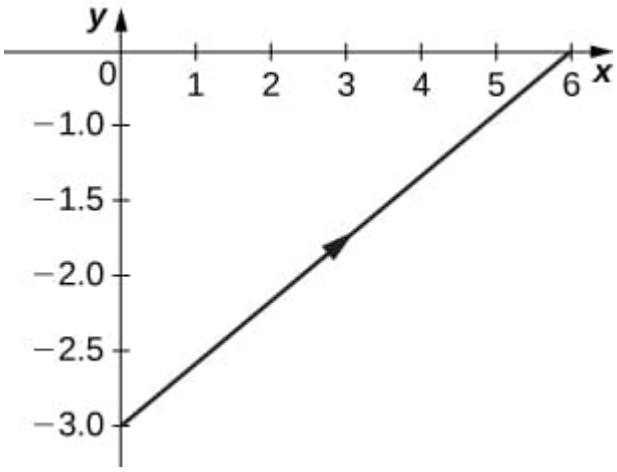



2) $x = \cos(t), \quad y = \sin(t), \quad \text{for } (0, 2\pi]$

3) $x = 2t + 4, \quad y = t - 1$

**Answer**

Orientation: left to right



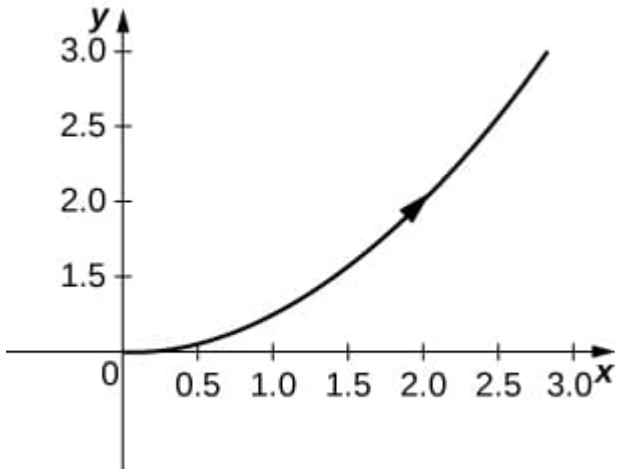

4) $x = 3 - t, \quad y = 2t - 3, \quad \text{for } 1.5 \le t \le 3$

**In exercise 5, eliminate the parameter and sketch the graph.**

5) $x = 2t^2, \quad y = t^4 + 1$

**Answer**

$$y = \frac{x^2}{4} + 1$$

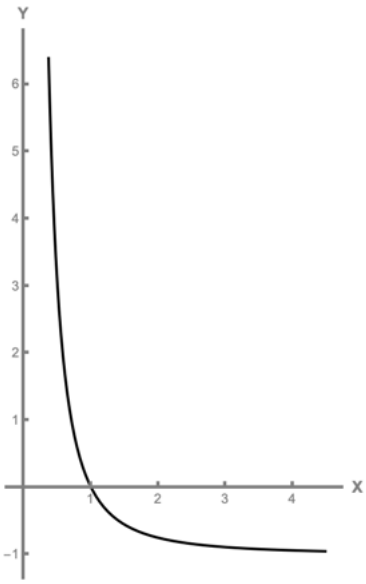



**In exercises 6 - 9, use technology (CAS or calculator) to sketch the parametric equations.**

6) [T] $x = t^2 + t, \quad y = t^2 - 1$

7) [T] $x = e^{-t}, \quad y = e^{2t} - 1$

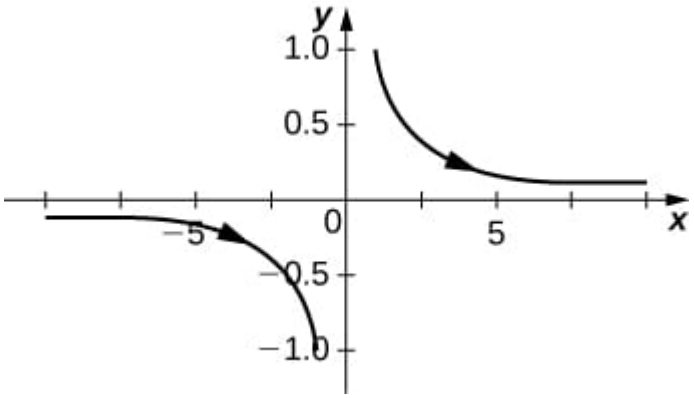
**Answer**




8) [T] $x = 3\cos t, \quad y = 4\sin t$

9) [T] $x = \sec t, \quad y = \cos t$

**Answer**

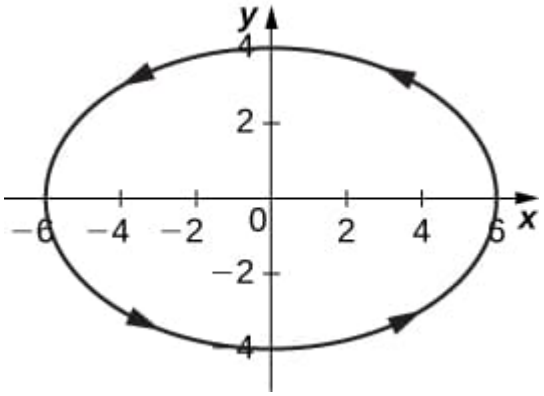



**In exercises 10 - 20, sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.**

10) $x = e^t, \quad y = e^{2t} + 1$

11) $x = 6\sin(2\theta), \quad y = -4\cos(2\theta)$

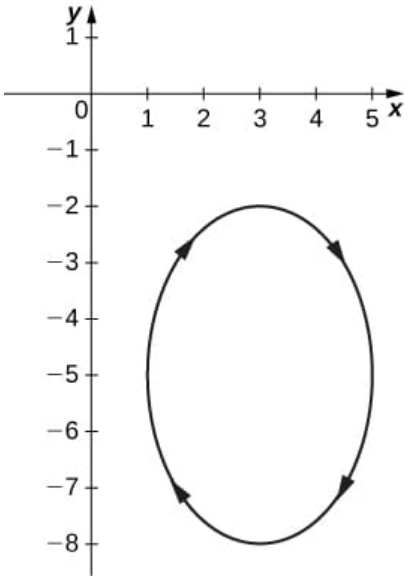
**Answer**




12) $x = \cos\theta, \quad y = 2\sin(2\theta)$

13) $x = 3 - 2\cos\theta, \quad y = -5 + 3\sin\theta$

**Answer**

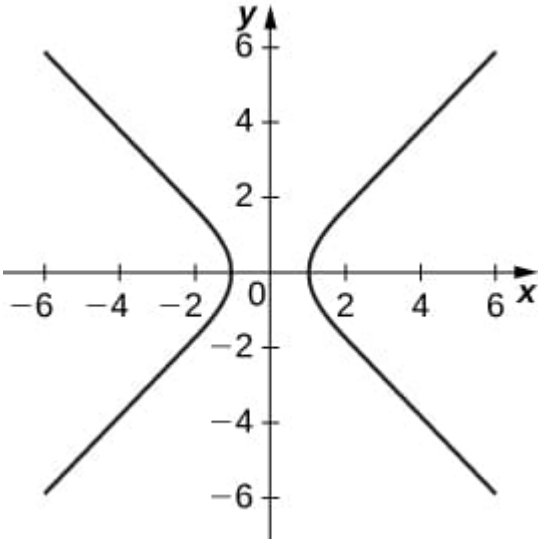



14) $x = 4 + 2\cos\theta, \quad y = -1 + \sin\theta$

15) $x = \sec t, \quad y = \tan t$

**Answer**

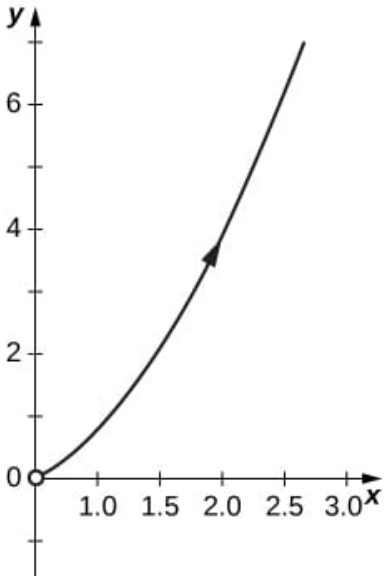
Asymptotes are $y = x$ and $y = -x$




16) $x = \ln(2t), \quad y = t^2$

17) $x = e^t, \quad y = e^{2t}$

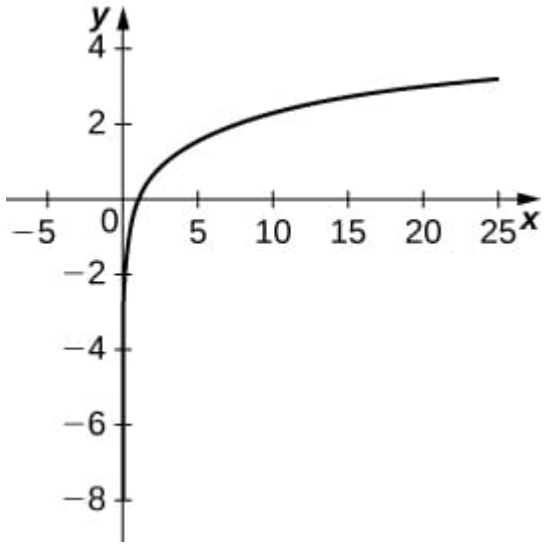
**Answer**




18) $x = e^{-2t}, \quad y = e^{3t}$

19) $x = t^3, \quad y = 3 \ln t$

**Answer**




20) $x = 4 \sec \theta, \quad y = 3 \tan \theta$

**In exercises 21 - 38, convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.**

21) $x = t^2 - 1, \quad y = \dfrac{t}{2}$

**Answer**

$x = 4y^2 - 1$; domain: $x \in [1, \infty)$.

22) $x = \dfrac{1}{\sqrt{t+1}}, \quad y = \dfrac{t}{1+t}, \quad \text{for } t > -1$

23) $x = 4 \cos \theta, \quad y = 3 \sin \theta, \quad \text{for } t \in (0, 2\pi]$

**Answer**

$\frac{x^2}{16} + \frac{y^2}{9} = 1$; domain $x \in [-4, 4]$.

24) $x = \cosh t, \quad y = \sinh t$

25) $x = 2t - 3, \quad y = 6t - 7$

**Answer**

$y = 3x + 2$; domain: all real numbers.

26) $x = t^2, \quad y = t^3$

27) $x = 1 + \cos t, \quad y = 3 - \sin t$

**Answer**

$(x-1)^2 + (y-3)^2 = 1$; domain: $x \in [0, 2]$.

28) $x = \sqrt{t}, \quad y = 2t + 4$

29) $x = \sec t, \quad y = \tan t, \quad$ for $\pi \leq t < \frac{3\pi}{2}$

**Answer**

$y = \sqrt{x^2 - 1}$; domain: $x \in (-\infty, -1]$.

30) $x = 2\cosh t, \quad y = 4\sinh t$

31) $x = \cos(2t), \quad y = \sin t$

**Answer**

$y^2 = \frac{1 - x}{2}$; domain: $x \in [-1, 1]$.

32) $x = 4t + 3, \quad y = 16t^2 - 9$

33) $x = t^2, \quad y = 2\ln t, \quad$ for $t \geq 1$

**Answer**

$y = \ln x$; domain: $x \in [1, \infty)$.

34) $x = t^3, \quad y = 3\ln t, \quad$ for $t \geq 1$

35) $x = t^n, \quad y = n\ln t, \quad$ for $t \geq 1$, where $n$ is a natural number

**Answer**

$y = \ln x$; domain: $x \in (0, \infty)$.

36) $x = \ln(5t), \quad y = \ln(t^2)$ where $1 \leq t \leq e$

37) $x = 2\sin(8t), \quad y = 2\cos(8t)$

**Answer**

$x^2 + y^2 = 4$; domain: $x \in [-2, 2]$.

38) $x = \tan t, \quad y = \sec^2 t - 1$

**In exercises 39 - 48, the pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.**

39) $x = 3t + 4, \quad y = 5t - 2$

**Answer**

line

40) $x - 4 = 5t, \quad y + 2 = t$

41) $x = 2t + 1, \quad y = t^2 - 3$

**Answer**

parabola

42) $x = 3\cos t, \quad y = 3\sin t$

43) $x = 2\cos(3t), \quad y = 2\sin(3t)$

**Answer**

circle

44) $x = \cosh t, \quad y = \sinh t$

45) $x = 3\cos t, \quad y = 4\sin t$

**Answer**

ellipse

46) $x = 2\cos(3t), \quad y = 5\sin(3t)$

47) $x = 3\cosh(4t) \quad y = 4\sinh(4t)$

**Answer**

the right branch of a horizontally opening hyperbola

48) $x = 2\cosh t, \quad y = 2\sinh t$

49) Show that $x = h + r\cos\theta, \quad y = k + r\sin\theta$ represents the equation of a circle.

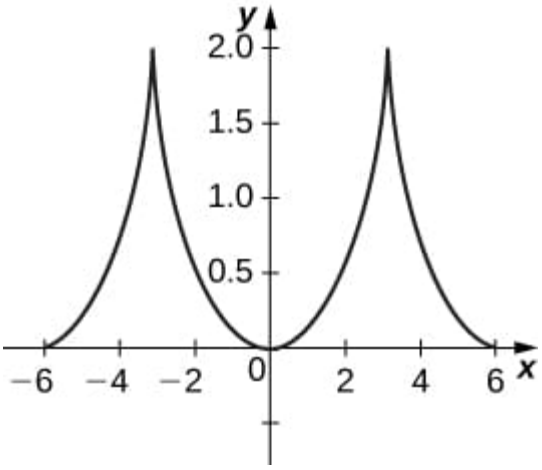
50) Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is $5$ and whose center is $(-2, 3)$.

**In exercises 51 - 53, use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.**

51) [T] $x = \theta + \sin\theta, \quad y = 1 - \cos\theta$

**Answer**

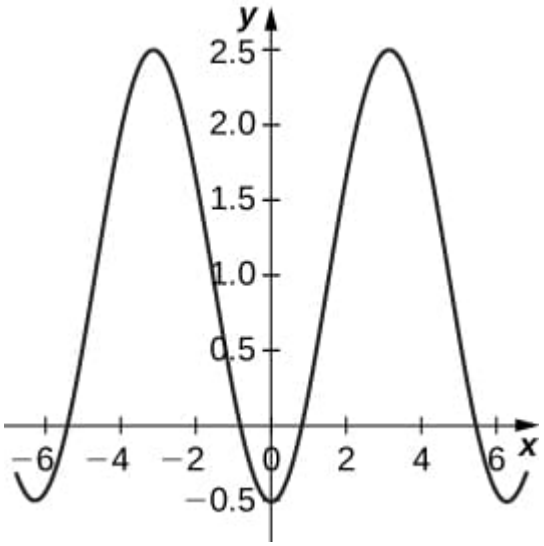
The equations represent a cycloid.




52) [T] $x = 2t - 2\sin t, \quad y = 2 - 2\cos t$

53) [T] $x = t - 0.5\sin t, \quad y = 1 - 1.5\cos t$

**Answer**




54) An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t, \quad y = -4.9t^2 + 4000, \quad \text{where } t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?

55) The trajectory of a bullet is given by $x = v_0(\cos\alpha)t, \quad y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$ where $v_0 = 500$ m/s, $g = 9.8 = 9.8\ \text{m/s}^2$, and $\alpha = 30$ degrees. When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

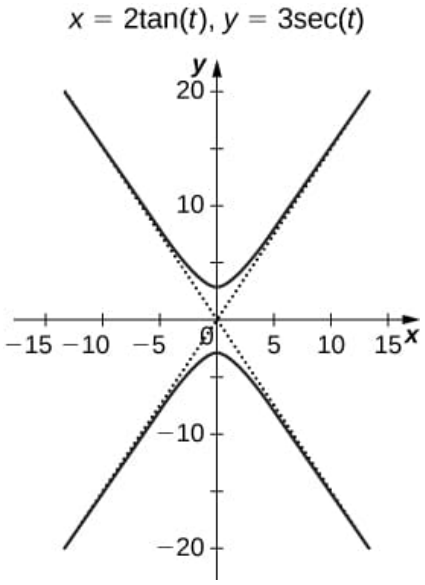
**Answer**

22,092 meters at approximately 51 seconds.

56) [T] Use technology to sketch the curve represented by $x = \sin(4t), \quad y = \sin(3t), \quad \text{for } 0 \leq t \leq 2\pi.$

57) [T] Use technology to sketch \( x=2\tan(t), \quad y=3\sec(t), \quad \text{for }–π

**Answer**



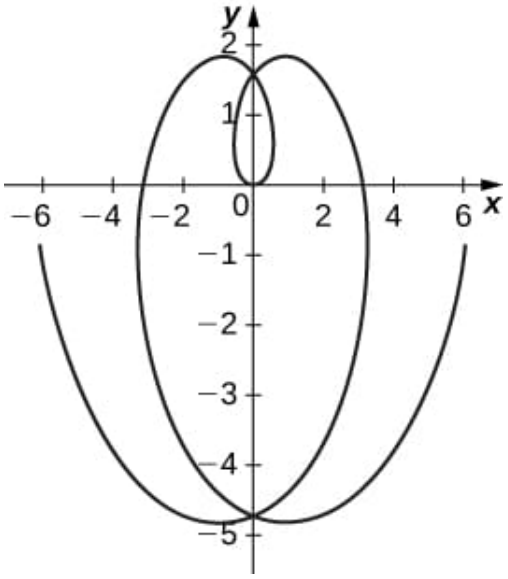

58) Sketch the curve known as an **epitrochoid**, which gives the path of a point on a circle of radius $b$ as it rolls on the outside of a circle of radius $a$. The equations are

$x = (a+b)\cos t - c \cdot \cos\left[\frac{(a+b)t}{b}\right], \quad y = (a+b)\sin t - c \cdot \sin\left[\frac{(a+b)t}{b}\right].$

Let $a = 1, \ \ b = 2, \ \ c = 1.$

59) [T] Use technology to sketch the spiral curve given by $x = t\cos(t), \quad y = t\sin(t)$ for $-2\pi \le t \le 2\pi.$

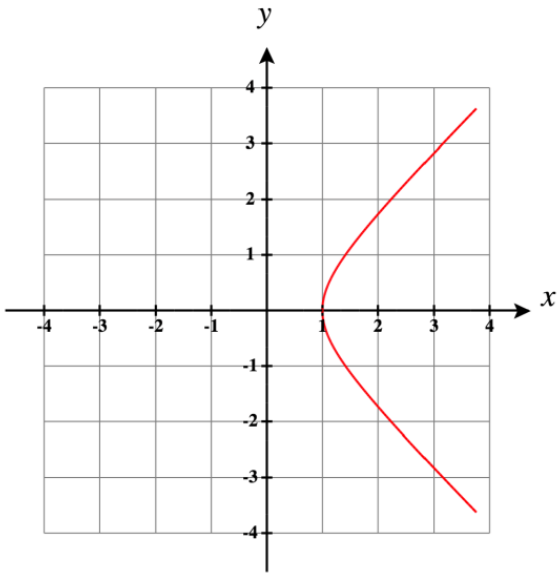
**Answer**




60) [T] Use technology to graph the curve given by the parametric equations $x = 2\cot(t), \quad y = 1 - \cos(2t), \quad \text{for } -\pi/2 \le t \le \pi/2.$ This curve is known as the witch of Agnesi.

61) [T] Sketch the curve given by parametric equations $x = \cosh(t), \quad y = \sinh(t)$, for $-2 \le t \le 2.$

**Answer**




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# 9.2: CALCULUS OF PARAMETRIC CURVES

## 9.2.1: Exercises for Section 11.2

**Learning Objectives**

- Determine derivatives and equations of tangents for parametric curves.
- Find the area under a parametric curve.
- Use the equation for arc length of a parametric curve.
- Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t))$ then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

## Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

$$x(t) = 2t + 3 \tag{9.2.1}$$
$$y(t) = 3t - 4 \tag{9.2.2}$$

within $-2 \leq t \leq 3$.

The graph of this curve appears in Figure $9.2.1$. It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.

A straight line from (–1, –10) to (9, 5). The point (–1, –10) is marked t = –2, the point (3, –4) is marked t = 0, and the point (9, 5) is marked t = 3. There are three equations marked: x(t) = 2t + 3, y(t) = 3t – 4, and –2 ≤ t ≤ 3

Figure $9.2.1$: Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving Equation $9.2.1$ for $t$:

$$x(t) = 2t + 3$$

$$x - 3 = 2t$$

$$t = \frac{x-3}{2}.$$

Substituting this into $y(t)$ (Equation $9.2.2$), we obtain

$$y(t) = 3t - 4$$

$$y = 3\left(\frac{x-3}{2}\right) - 4$$

$$y = \frac{3x}{2} - \frac{9}{2} - 4$$

$$y = \frac{3x}{2} - \frac{17}{2}.$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$. Next we calculate $x\,'(t)$ and $y\,'(t)$. This gives $x\,'(t) = 2$ and $y\,'(t) = 3$. Notice that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}.$$

This is no coincidence, as outlined in the following theorem.

## Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x\,'(t)$ and $y\,'(t)$ exist, and assume that $x\,'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}. \tag{9.2.3}$$

## Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter $t$ can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t) = F'\big(x(t)\big)x'(t),$$

so

$$F'\big(x(t)\big) = \frac{y'(t)}{x'(t)}.$$

But $F\,'\big(x(t)\big) = \frac{dy}{dx}$, which proves the theorem.

□

Equation 9.2.3 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f\,'(x_0) = 0$ or $f\,'(x_0)$ does not exist. Equation 9.2.3 gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

## Example 9.2.1: Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically defined plane curves, and locate any critical points on their respective graphs.

a. $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \leq t \leq 4$
b. $x(t) = 2t + 1, \quad y(t) = t^3 - 3t + 4, \quad \text{for } -2 \leq t \leq 2$
c. $x(t) = 5\cos t, \quad y(t) = 5\sin t, \quad \text{for } 0 \leq t \leq 2\pi$

**Solution**

a. To apply Equation 9.2.3, first calculate $x\,'(t)$ and $y\,'(t)$:

$$x\,'(t) = 2t$$

$$y\,'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$ , which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex as shown.

A curved line going from (6, –7) through (–3, –1) to (13, 7) with arrow pointing in that order. The point (6, –7) is marked t = –3, the point (–3, –1) is marked t = 0, and the point (13, 7) is marked t = 4. On the graph there are also written three equations: x(t) = t2 – 3, y(t) = 2t – 1, and –3 ≤ t ≤ 4.

Figure 9.2.2: Graph of the parabola described by parametric equations in part a.

b. To apply Equation 9.2.3, first calculate $x\,'(t)$ and $y\,'(t)$:

$$x\,'(t) = 2$$

$$y\,'(t) = 3t^2 - 3.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}.$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.

A vaguely sinusoidal curve going from (–3, 2) through (–1, 6) and (3, 2) to (5, 6). The point (–3, 2) is marked t = –2, the point (–1, 6) is marked t = –1, the point (3, 2) is marked t = 1, and the point (5, 6) is marked t = 2. On the graph there are also written three equations: x(t) = 2t + 1, y(t) = t3 – 3t + 4, and –2 ≤ t ≤ 2.

Figure 9.2.3: Graph of the curve described by parametric equations in part b.

c. To apply Equation 9.2.3, first calculate $x\,'(t)$ and $y\,'(t)$:

$$x\,'(t) = -5\sin t$$

$$y\,'(t) = 5\cos t.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{5\cos t}{-5\sin t}$$

$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and $2\pi$ as critical points for *t*. Substituting each of these into $x(t)$ and $y(t)$, we obtain

| $t$ | $x(t)$ | $y(t)$ |
|---|---|---|
| 0 | 5 | 0 |
| $\frac{\pi}{2}$ | 0 | 5 |
| $\pi$ | −5 | 0 |
| $\frac{3\pi}{2}$ | 0 | −5 |
| $2\pi$ | 5 | 0 |

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations (Figure 9.2.4). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.

A circle with radius 5 centered at the origin is graphed with arrow going counterclockwise. The point (5, 0) is marked t = 0, the point (0, 5) is marked t = π/2, the point (−5, 0) is marked t = π, and the point (0, −5) is marked t = 3π/2. On the graph there are also written three equations: x(t) = 5 cos(t), y(t) = 5 sin(t), and 0 ≤ t ≤ 2π.

Figure 9.2.4: Graph of the curve described by parametric equations in part c.

## Exercise 9.2.1

Calculate the derivative $dy/dx$ for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Example 9.2.2: Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4$$

when $t = 2$.

**Solution**

First find the slope of the tangent line using Equation 9.2.3, which means calculating $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$

$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph (Figure $9.2.5$). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$y - y_0 = m(x - x_0)$$

$$y - 3 = \frac{1}{2}(x - 1)$$

$$y - 3 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{5}{2}.$$

A curved line going from (6, –7) through (–3, –1) to (13, 7) with arrow pointing in that order. The point (6, –7) is marked t = –3, the point (–3, –1) is marked t = 0, and the point (13, 7) is marked t = 4. On the graph there are also written three equations: x(t) = t2 – 3, y(t) = 2t – 1, and –3 ≤ t ≤ 4. At the point (1, 3), which is marked t = 2, there is a tangent line with equation y = x/2 + 5/2.

Figure $9.2.5$: Tangent line to the parabola described by the given parametric equations when $t = 2$.

**Exercise $9.2.2$**

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 6 \text{ when } t = 5.$$

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]. \tag{9.2.4}$$

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

we can replace the $y$ on both sides of Equation $9.2.4$ with $\frac{dy}{dx}$. This gives us

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}. \tag{9.2.5}$$

If we know $dy/dx$ as a function of $t$, then this formula is straightforward to apply

### Example 9.2.3: Finding a Second Derivative

Calculate the second derivative $d^2y/dx^2$ for the plane curve defined by the parametric equations $x(t) = t^2 - 3, \quad y(t) = 2t - 1,$ for $-3 \le t \le 4.$

**Solution**

From Example 9.2.1 we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using Equation 9.2.5, we obtain

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

### Exercise 9.2.3

Calculate the second derivative $d^2y/dx^2$ for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by these parametric equations

$$x(t) = t - \sin t$$
$$y(t) = 1 - \cos t.$$

Suppose we want to find the area of the shaded region in the following graph.

A series of half circles drawn above the x axis with x intercepts being multiples of 2π. The half circle between 0 and 2π is highlighted. On the graph there are also written two equations: x(t) = t – sin(t) and y(t) = 1 – cos(t).

Figure 9.2.6: Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions

$$x = x(t)$$
$$y = y(t)$$

where $a \le t \le b$.

We assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \le t \le b$. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and consider the following graph.

A curved line is drawn in the first quadrant. Below it are a series of rectangles marked that begin at the x axis and reach up to the curved line; the rectangle's height is determined by the location of the curved line at the leftmost point of the rectangle. These lines are noted as x(t0), x(t1), ..., x(tn).

Figure 9.2.7: Approximating the area under a parametrically defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y(x(\bar{t}_i))$ for some value $\bar{t}_i$ in the $i^{\text{th}}$ subinterval, and the width can be calculated as $x(t_i) - x(t_{i-1})$. Thus the area of the $i^{\text{th}}$ rectangle is given by

$$A_i = y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Then a Riemann sum for the area is

$$A_n = \sum_{i=1}^{n} y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

$$\begin{aligned} A_n &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left( \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \right) (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left( \frac{x(t_i) - x(t_{i-1})}{\Delta t} \right) \Delta t. \end{aligned}$$

Taking the limit as $n$ approaches infinity gives

$$A = \lim_{n \to \infty} A_n = \int_a^b y(t)x'(t)\, dt.$$

This leads to the following theorem.

## Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\, dt. \tag{9.2.6}$$

## Example 9.2.4: Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad \text{for } 0 \le t \le 2\pi.$$

**Solution**

Using Equation 9.2.6, we have

$$\begin{aligned} A &= \int_a^b y(t)x'(t)\,dt \\ &= \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt \\ &= \int_0^{2\pi} (1-2\cos t+\cos^2 t)\,dt \\ &= \int_0^{2\pi} \left(1-2\cos t+\frac{1+\cos(2t)}{2}\right)\,dt \\ &= \int_0^{2\pi} \left(\frac{3}{2}-2\cos t+\frac{\cos(2t)}{2}\right)\,dt \\ &= \frac{3t}{2}-2\sin t+\frac{\sin(2t)}{4}\,|_0^{2\pi} \\ &= 3\pi \end{aligned}$$

**Exercise 9.2.4**

Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos t + \cos(3t), \quad y(t) = 3\sin t - \sin(3t), \quad \text{for } 0 \le t \le \pi.$$

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point $A$ to point $B$ along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.

A curved line in the first quadrant with points marked for x = 1, 2, 3, 4, and 5. These points have values roughly 2.1, 2.7, 3, 2.7, and 2.1, respectively. The points for x = 1 and 5 are marked A and B, respectively.

Figure 9.2.7: Approximation of a curve by line segments.

Given a plane curve defined by the functions $x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$, we start by partitioning the interval $[a, b]$ into $n$ equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b$. The width of each subinterval is given by $\Delta t = (b-a)/n$. We can calculate the length of each line segment:

$$d_1 = \sqrt{(x(t_1)-x(t_0))^2 + (y(t_1)-y(t_0))^2}$$

$$d_2 = \sqrt{(x(t_2)-x(t_1))^2 + (y(t_2)-y(t_1))^2}$$

etc.

Then add these up. We let $s$ denote the exact arc length and $s_n$ denote the approximation by $n$ line segments:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k)-x(t_{k-1}))^2 + (y(t_k)-y(t_{k-1}))^2}. \tag{9.2.7}$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of $t$, then the Mean Value Theorem applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t_k}$ and $\tilde{t_k}$ such that

$$x(t_k) - x(t_{k-1}) = x'(\hat{t_k})(t_k - t_{k-1}) = x'(\hat{t_k})\,\Delta t$$

$$y(t_k) - y(t_{k-1}) = y'(\tilde{t_k})(t_k - t_{k-1}) = y'(\tilde{t_k})\,\Delta t.$$

Therefore Equation 9.2.7 becomes

$$\begin{aligned} s \approx \sum_{k=1}^{n} s_k &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k})\Delta t)^2 + (y'(\tilde{t_k})\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2(\Delta t)^2 + (y'(\tilde{t_k}))^2(\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \end{aligned}$$

When taking the limit, the values of $\hat{t_k}$ and $\tilde{t_k}$ are both contained within the same ever-shrinking interval of width $\Delta t$, so they must converge to the same value.

We can summarize this method in the following theorem.

## Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of $t$. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{9.2.8}$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives

$$y'(t) = F'\big(x(t)\big)x'(t).$$

Substituting this into Equation 9.2.8 gives

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + (F'(x))^2\right)}\, dt \\ &= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\ dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is the formula for arc length obtained in the Introduction to the Applications of Integration.

## Example 9.2.5: Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le \pi.$$

**Solution**

The values $t = 0$ to $t = \pi$ trace out the blue curve in Figure 9.2.8. To determine its length, use Equation 9.2.8:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} \\ &= 3\pi \text{ units.} \end{aligned}$$

Note that the formula for the arc length of a semicircle is $\pi r$ and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.

A semicircle is drawn with radius 3. There is an arrow pointing counterclockwise. On the graph there are also written three equations: x(t) = 3 cos(t), y(t) = 3 sin(t), and 0 ≤ t ≤ π.

Figure 9.2.8: The arc length of the semicircle is equal to its radius times $\pi$.

## Exercise 9.2.5

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad \text{for } 1 \le t \le 3.$$

*Hint available at the end of this section.*

*Answer available at the end of this section.*

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive $x$-axis, the parametric equations for this curve can be written as

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where $t$ represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x(t)$ and $y(t)$ using *v* as an independent

variable, so as to eliminate any confusion with the parameter $t$:

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{(\frac{dx}{dv})^2 + (\frac{dy}{dv})^2}\, dv \\ &= \int_0^t \sqrt{140^2 + (-32v+2)^2}\, dv \end{aligned}$$

The variable $v$ acts as a dummy variable that disappears after integration, leaving the arc length as a function of time $t$. To integrate this expression we can use a formula from Appendix A,

$$\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln \mid u + \sqrt{a^2+u^2} \mid + C.$$

We set $a = 140$ and $u = -32v + 2$. This gives $du = -32\ dv$, so $dv = -\frac{1}{32}\ du$. Therefore

$$\begin{aligned} \int \sqrt{140^2 + (-32v+2)^2}\, dv &= -\frac{1}{32}\int \sqrt{a^2+u^2}\, du \\ &= -\frac{1}{32}\left[\frac{(-32v+2)}{2}\sqrt{140^2+(-32v+2)^2} + \frac{140^2}{2}\ln \mid (-32v+2) + \sqrt{140^2+(-32v+2)^2} \mid + C\right] \end{aligned}$$

and

$$\begin{aligned} s(t) &= -\frac{1}{32}\left[\frac{(-32t+2)}{2}\sqrt{140^2+(-32t+2)^2} + \frac{140^2}{2}\ln\left|(-32t+2) + \sqrt{140^2+(-32t+2)^2}\right|\right] + \frac{1}{32}\left[\sqrt{140^2+2^2} + \frac{140^2}{2}\ln\left|2+\sqrt{140^2+2^2}\right|\right] \\ &= \left(\frac{t}{2} - \frac{1}{32}\right)\sqrt{1024t^2 - 128t + 19604} - \frac{1225}{4}\ln\left|(-32t+2) + \sqrt{1024t^2-128t+19604}\right| + \frac{\sqrt{19604}}{32} + \frac{1225}{4}\ln(2+\sqrt{19604}) \end{aligned}.$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to $t$. While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(u)\, du = f(x).$$

Therefore

$$\begin{aligned} s'(t) &= \frac{d}{dt}\Big[s(t)\Big] \\ &= \frac{d}{dt}\left[\int_0^t \sqrt{140^2+(-32v+2)^2}\, dv\right] \\ &= \sqrt{140^2+(-32t+2)^2} \\ &= \sqrt{1024t^2-128t+19604} \\ &= 2\sqrt{256t^2-32t+4901}. \end{aligned}$$

One third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

$$
\begin{aligned}
s\left(\frac{1}{3}\right) &= \left(\frac{1/3}{2}-\frac{1}{32}\right)\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604} \\
&\quad -\frac{1225}{4}\ln\left|\left(-32\left(\frac{1}{3}\right)+2\right)+\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604}\right| \\
&\quad +\frac{\sqrt{19604}}{32}+\frac{1225}{4}\ln(2+\sqrt{19604}) \\
&\approx 46.69 \text{ feet.}
\end{aligned}
$$

This value is just over three quarters of the way to home plate. The speed of the ball is

$$s\,'\left(\tfrac{1}{3}\right)=2\sqrt{256\left(\tfrac{1}{3}\right)^2-32\left(\tfrac{1}{3}\right)+4901}\approx 140.27 \text{ ft/s.}$$

This speed translates to approximately $95$ mph—a major-league fastball.

## Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In Curve Length and Surface Area, we derived a formula for finding the surface area of a volume generated by a function $y=f(x)$ from $x=a$ to $x=b$, revolved around the $x$-axis:

$$S=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x=x(t), \quad y=y(t), \quad \text{for } a\le t\le b$ around the $x$-axis as shown in Figure $9.2.9$.

A curve is drawn in the first quadrant with endpoints marked t = a and t = b. On this curve, there is a point marked (x(t), y(t)). There is a circle with an arrow drawn around the x axis that seems to indicate a rotation about the x axis, and there is a shape that accompanies that curve that seems to be what you would obtain if you rotated the curve about the x axis.

Figure $9.2.9$: A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

$$S=2\pi\int_a^b y(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt \tag{9.2.9}$$

provided that $y(t)$ is not negative on $[a,b]$.

### Example $9.2.6$: Finding Surface Area

Find the surface area of a sphere of radius $r$ centered at the origin.

**Solution**

We start with the curve defined by the equations

$$x(t)=r\cos t, \quad y(t)=r\sin t, \quad \text{for } 0\le t\le\pi.$$

This generates an upper semicircle of radius $r$ centered at the origin as shown in the following graph.

A semicircle is drawn with radius r. On the graph there are also written three equations: x(t) = r cos(t), y(t) = r sin(t), and 0 ≤ t ≤ π.

Figure $9.2.10$: A semicircle generated by parametric equations.

When this curve is revolved around the $x$-axis, it generates a sphere of radius $r$. To calculate the surface area of the sphere, we use Equation $9.2.9$:

$$
\begin{aligned}
S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= 2\pi \int_0^\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\
&= 2\pi \int_0^\pi r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\
&= 2\pi \int_0^\pi r\sin t\sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\
&= 2\pi \int_0^\pi r^2\sin t\, dt \\
&= 2\pi r^2\left(-\cos t\Big|_0^\pi\right) \\
&= 2\pi r^2(-\cos\pi + \cos 0) \\
&= 4\pi r^2 \text{ units}^2.
\end{aligned}
$$

This is, in fact, the formula for the surface area of a sphere.

**Exercise 9.2.6**

Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad \text{for } 0 \le t \le 1$$

is revolved around the $x$-axis.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- The area between a parametric curve and the $x$-axis can be determined by using the formula $A = \displaystyle\int_{t_1}^{t_2} y(t)x'(t)\, dt$.
- The arc length of a parametric curve can be calculated by using the formula

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

- The surface area of a volume of revolution revolved around the $x$-axis is given by

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

- If the curve is revolved around the $y$-axis, then the formula is

$$S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

## Key Equations

- **Derivative of parametric equations**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

- **Second-order derivative of parametric equations**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}$$

- **Area under a parametric curve**

$$A = \int_a^b y(t)x'(t)\,dt$$

- **Arc length of a parametric curve**

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- **Surface area generated by a parametric curve**

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

## Answers

**Exercise** 9.2.1

Calculate $x\,'(t)$ and $y\,'(t)$ and use Equation 9.2.3.

**Exercise** 9.2.1

$x\,'(t) = 2t - 4$ and $y\,'(t) = 6t^2 - 6$, so $\frac{dy}{dx} = \frac{6t^2-6}{2t-4} = \frac{3t^2-3}{t-2}$.

This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1$.

A curve going from (12, –4) through the origin and (–4, 0) to (–3, 36) with arrows in that order. The point (12, –4) is marked t = –2 and the point (–3, 36) is marked t = 3. On the graph there are also written three equations: x(t) = t2 – 4t, y(t) = 2t3 – 6t, and –2 ≤ t ≤ 3.

**Exercise** 9.2.2

Calculate $x\,'(t)$ and $y\,'(t)$ and use Equation 9.2.3.

**Exercise** 9.2.2

The equation of the tangent line is $y = 24x + 100$.

**Exercise** 9.2.3

Start with the solution from the previous exercise, and use Equation 9.2.5.

**Exercise** 9.2.3

$\frac{d^2y}{dx^2} = \frac{3t^2-12t+3}{2(t-2)^3}$. Critical points $(5, 4),\ (-3, -4)$,and $(-4, 6)$.

**Exercise** 9.2.4

Use Equation 9.2.6, along with the identities $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ and $\sin^2 t = \frac{1-\cos(2t)}{2}$.

**Exercise** 9.2.4

$A = 3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x(t)$ is a decreasing function over the interval $[0, \pi]$; that is, the curve is traced from right to left.)

**Exercise** 9.2.5

Use Equation 9.2.8.

**Exercise** 9.2.5

$s = 2(10^{3/2} - 2^{3/2}) \approx 57.589$ units

**Exercise** 9.2.6

Use Equation 9.2.9. When evaluating the integral, use a $u$-substitution.

**Exercise** 9.2.6

$A = \frac{\pi(494\sqrt{13}+128)}{1215}$ units$^2$

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# 9.2.1: Exercises for Section 11.2

**In exercises 1 - 4, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.**

1) $x = 3 + t, \quad y = 1 - t$

2) $x = 8 + 2t, \quad y = 1$

**Answer**

$m = 0$

3) $x = 4 - 3t, \quad y = -2 + 6t$

4) $x = -5t + 7, \quad y = 3t - 1$

**Answer**

$m = -\frac{3}{5}$

**In exercises 5 - 9, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.**

5) $x = 3\sin t, \quad y = 3\cos t, \quad \text{for } t = \frac{\pi}{4}$

6) $x = \cos t, \quad y = 8\sin t, \quad \text{for } t = \frac{\pi}{2}$

**Answer**

Slope $= 0; y = 8.$

7) $x = 2t, \quad y = t^3, \quad \text{for } t = -1$

8) $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}, \quad \text{for } t = 1$

**Answer**

Slope is undefined; $x = 2.$

9) $x = \sqrt{t}, \quad y = 2t, \quad \text{for } t = 4$

**In exercises 10 - 13, find all points on the curve that have the given slope.**

10) $x = 4\cos t, \quad y = 4\sin t$, slope = $0.5$

**Solution**

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4\cos t}{-4\sin t} = -\cot t.$

Setting this derivative equal to $0.5$, we obtain the equation, $\tan t = -2.$

$\tan t = -2 \implies \dfrac{y}{x} = -2 \implies y = -2x.$

Note also that this pair of parametric equations represents the circle $x^2 + y^2 = 16.$

By substitution, we find that this curve has a slope of $0.5$ at the points:

$\left(\frac{4\sqrt{5}}{5}, \frac{-8\sqrt{5}}{5}\right)$ and $\left(\frac{-4\sqrt{5}}{5}, \frac{8\sqrt{5}}{5}\right)$.

11) $x = 2\cos t, \quad y = 8\sin t$, slope= $-1$

12) $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}$, slope= $1$

**Answer**

No points possible; undefined expression.

13) $x = 2 + \sqrt{t}, \quad y = 2 - 4t$, slope= $0$

**In exercises 14 - 16, write the equation of the tangent line in Cartesian coordinates for the given parameter $t$.**

14) $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad$ for $t = 1$

**Answer**

$y = -\left(\frac{4}{e}\right)x + 5$

15) $x = t \ln t, \quad y = \sin^2 t, \quad$ for $t = \frac{\pi}{4}$

16) $x = e^t, \quad y = (t-1)^2$, at $(1, 1)$

**Answer**

$y = -2x + 3$

17) For $x = \sin(2t), \quad y = 2 \sin t$ where $0 \le t < 2\pi$. Find all values of $t$ at which a horizontal tangent line exists.

18) For $x = \sin(2t), \quad y = 2 \sin t$ where $0 \le t < 2\pi$. Find all values of $t$ at which a vertical tangent line exists.

**Answer**

A vertical tangent line exists at $t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

19) Find all points on the curve $x = 4\cos(t), \quad y = 4\sin(t)$ that have the slope of $\frac{1}{2}$.

20) Find $\frac{dy}{dx}$ for $x = \sin(t), \quad y = \cos(t)$.

**Answer**

$$\frac{dy}{dx} = -\tan(t)$$

21) Find the equation of the tangent line to $x = \sin(t), \quad y = \cos(t)$ at $t = \frac{\pi}{4}$.

22) For the curve $x = 4t, \quad y = 3t - 2$, find the slope and concavity of the curve at $t = 3$.

**Answer**

$\frac{dy}{dx} = \frac{3}{4}$ and $\frac{d^2y}{dx^2} = 0$, so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

23) For the parametric curve whose equation is $x = 4\cos\theta, \quad y = 4\sin\theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

24) Find the slope and concavity for the curve whose equation is $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta$ at $\theta = \frac{\pi}{6}$.

**Answer**

$\frac{dy}{dx} = 4, \quad \frac{d^2y}{dx^2} = -4\sqrt{3}$; the curve is concave down at $\theta = \frac{\pi}{6}$.

25) Find all points on the curve $x = t + 4, \quad y = t^3 - 3t$ at which there are vertical and horizontal tangents.

26) Find all points on the curve $x = \sec\theta, \quad y = \tan\theta$ at which horizontal and vertical tangents exist.

**Answer**

No horizontal tangents. Vertical tangents at $(1, 0)$ and $(-1, 0)$.

**In exercises 27 - 29, find $d^2y/dx^2$.**

27) $x = t^4 - 1, \quad y = t - t^2$

28) $x = \sin(\pi t), \quad y = \cos(\pi t)$

**Answer**

$d^2y/dx^2 = -\sec^3(\pi t)$

29) $x = e^{-t}, \quad y = te^{2t}$

**In exercises 30 - 31, find points on the curve at which tangent line is horizontal or vertical.**

30) $x = t(t^2 - 3), \quad y = 3(t^2 - 3)$

**Answer**

Horizontal $(0, -9)$;
Vertical $(\pm 2, -6)$.

31) $x = \dfrac{3t}{1 + t^3}, \quad y = \dfrac{3t^2}{1 + t^3}$

**In exercises 32 - 34, find $dy/dx$ at the value of the parameter.**

32) $x = \cos t, y = \sin t, \quad \text{for } t = \frac{3\pi}{4}$

**Answer**

$dy/dx = 1$

33) $x = \sqrt{t}, \quad y = 2t + 4, t = 9$

34) $x = 4\cos(2\pi s), \quad y = 3\sin(2\pi s), \quad \text{for } s = -\frac{1}{4}$

**Answer**

$dy/dx = 0$

**In exercises 35 - 36, find $d^2y/dx^2$ at the given point without eliminating the parameter.**

35) $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad \text{for } t = 2$

36) $x = \sqrt{t}, \quad y = 2t + 4, \quad \text{for } t = 1$

**Answer**

$d^2y/dx^2 = 4$

37) Find intervals for $t$ on which the curve $x = 3t^2, \quad y = t^3 - t$ is concave up as well as concave down.

38) Determine the concavity of the curve $x = 2t + \ln t, \quad y = 2t - \ln t$.

**Answer**

Concave up on $t > 0$.

39) Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$.

40) Find the area bounded by the curve $x = \cos t, \quad y = e^t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

**Answer**

$1 \text{ unit}^2$

41) Find the area enclosed by the ellipse $x = a\cos\theta, \quad y = b\sin\theta, \quad \text{for } 0 \le \theta < 2\pi$.

42) Find the area of the region bounded by $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

**Answer**

$\frac{3\pi}{2} \text{ units}^2$

**In exercises 43 - 46, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.**

43) $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad \text{for } 0 \le \theta \le \pi$

44) [T] $x = 2a\cos t - a\cos(2t), \quad y = 2a\sin t - a\sin(2t), \quad \text{for } 0 \le t < 2\pi$

**Answer**

$6\pi a^2 \text{ units}^2$

45) [T] $x = a\sin(2t), \quad y = b\sin(t), \quad \text{for } 0 \le t < 2\pi$ (the "hourglass")

46) [T] $x = 2a\cos t - a\sin(2t), \quad y = b\sin t, \quad \text{for } 0 \le t < 2\pi$ (the "teardrop")

**Answer**

$2\pi ab \text{ units}^2$

**In exercises 47 - 52, find the arc length of the curve on the indicated interval of the parameter.**

47) $x = 4t + 3, \quad y = 3t - 2, \quad \text{for } 0 \le t \le 2$

48) $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad \text{for } 0 \le t \le 1$

**Answer**

$s = \frac{1}{3}(2\sqrt{2} - 1)$ units

49) $x = \cos(2t), \quad y = \sin(2t), \quad \text{for } 0 \le t \le \frac{\pi}{2}$

50) $x = 1 + t^2, \quad y = (1 + t)^3, \quad \text{for } 0 \le t \le 1$

**Answer**

$s = 7.075$ units

51) $x = e^t\cos t, \quad y = e^t\sin t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

52) $x = a\cos^3\theta, \quad y = a\sin^3\theta$ on the interval $[0, 2\pi)$ (the hypocycloid)

**Answer**

$s = 6a$ units

53) Find the length of one arch of the cycloid $x = 4(t - \sin t), \quad y = 4(1 - \cos t).$

54) Find the distance traveled by a particle with position $(x, y)$ as $t$ varies in the given time interval: $x = \sin^2 t, \quad y = \cos^2 t, \quad \text{for } 0 \le t \le 3\pi.$

**Answer**

$6\sqrt{2}$ units

55) Find the length of one arch of the cycloid $x = \theta - \sin\theta, \quad y = 1 - \cos\theta.$

56) Show that the total length of the ellipse $x = 4\sin\theta, \quad y = 3\cos\theta$ is $L = 16\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$, where $e = \frac{c}{a}$ and $c = \sqrt{a^2 - b^2}$.

57) Find the length of the curve $x = e^t - t, \quad y = 4e^{t/2}, \quad \text{for } -8 \le t \le 3.$

**In exercises 58 - 59, find the area of the surface obtained by rotating the given curve about the $x$-axis.**

58) $x = t^3, \quad y = t^2, \quad \text{for } 0 \le t \le 1$

**Answer**

$$\frac{2\pi(247\sqrt{13}+64)}{1215}\ \text{units}^2$$

59) $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$

60) [T] Use a CAS to find the area of the surface generated by rotating $x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad \text{for } 1 \le t \le 2$ about the $x$-axis. (Answer to three decimal places.)

**Answer**

$$59.101\ \text{units}^2$$

61) Find the surface area obtained by rotating $x = 3t^2, \quad y = 2t^3, \quad \text{for } 0 \le t \le 5$ about the $y$-axis.

62) Find the area of the surface generated by revolving $x = t^2, \quad y = 2t, \quad \text{for } 0 \le t \le 4$ about the $x$-axis.

**Answer**

$$\frac{8\pi}{3}(17\sqrt{17}-1)\ \text{units}^2$$

63) Find the surface area generated by revolving $x = t^2, \quad y = 2t^2, \quad \text{for } 0 \le t \le 1$ about the $y$-axis.

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# 9.3: POLAR COORDINATES

9.3.1: Introduction to Polar Coordinates
9.3.2: Exercises for Section 11.3

Learning Objectives

- Locate points in a plane by using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves from given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a *one-to-one mapping* from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. In this section we see that in some circumstances, polar coordinates can be more useful than rectangular coordinates.

## Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider Figure $9.3.1$. The point $P$ has Cartesian coordinates $(x, y)$. The line segment connecting the origin to the point $P$ measures the distance from the origin to $P$ and has length $r$. The angle between the positive x-axis and the line segment has measure $\theta$. This observation suggests a natural correspondence between the coordinate pair $(x, y)$ and the values $r$ and $\theta$. This correspondence is the basis of the **polar coordinate system**. Note that every point in the Cartesian plane has two values (hence the term ordered pair) associated with it. In the polar coordinate system, each point also has two values associated with it: $r$ and $\theta$.

A point P(x, y) is given in the first quadrant with lines drawn to indicate its x and y values. There is a line from the origin to P(x, y) marked r and this line make an angle θ with the x axis.

Figure $9.3.1$: An arbitrary point in the Cartesian plane.

Using right-triangle trigonometry, the following equations are true for the point $P$:

$$\cos\theta = \frac{x}{r} \text{ so } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ so } y = r\sin\theta.$$

Furthermore,

$$r^2 = x^2 + y^2$$

and

$$\tan\theta = \frac{y}{x}.$$

Each point $(x, y)$ in the Cartesian coordinate system can therefore be represented as an ordered pair $(r, \theta)$ in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

Note that the equation $\tan\theta = y/x$ has an infinite number of solutions for any ordered pair $(x, y)$. However, if we restrict the solutions to values between $0$ and $2\pi$ then we can assign a unique solution to the quadrant in which the original point $(x, y)$ is

located. Then the corresponding value of $r$ is positive, so $r^2 = x^2 + y^2$.

## Converting Points between Coordinate Systems

Given a point $P$ in the plane with Cartesian coordinates $(x, y)$ and polar coordinates $(r, \theta)$, the following conversion formulas hold true:

$$x = r\cos\theta \tag{9.3.1}$$

$$y = r\sin\theta \tag{9.3.2}$$

and

$$r^2 = x^2 + y^2 \tag{9.3.3}$$

$$\tan\theta = \frac{y}{x} \tag{9.3.4}$$

.

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates. Notice that Equation 9.3.3 is the *Pythagorean theorem*. (Figure 9.3.1).

## Example 9.3.1: Converting between Rectangular and Polar Coordinates

Convert each of the following points into polar coordinates.

a. $(1, 1)$
b. $(-3, 4)$
c. $(0, 3)$
d. $(5\sqrt{3}, -5)$

Convert each of the following points into rectangular coordinates.

e. $(3, \pi/3)$
f. $(2, 3\pi/2)$
g. $(6, -5\pi/6)$

**Solution**

a. Use $x = 1$ and $y = 1$ in Equation 9.3.3:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned}$$

and via Equation 9.3.4

$$\begin{aligned} \tan\theta = \frac{y}{x} = \frac{1}{1} = 1 \\ \theta = \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in Equation 9.3.3:

$$\begin{aligned} r^2 = x^2 + y^2 = (-3)^2 + (4)^2 \\ r = 5 \end{aligned}$$

and via Equation 9.3.4

$$\tan\theta = \frac{y}{x} = -\frac{4}{3}$$

$$\theta = \arctan(-\frac{4}{3}) + \pi \approx 2.21.$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

c. Use $x = 0$ and $y = 3$ in Equation 9.3.3:

$$r^2 = x^2 + y^2 = (3)^2 + (0)^2 = 9 + 0\ r = 3$$

and via Equation 9.3.4

$$\tan\theta = \frac{y}{x} = \frac{3}{0}.$$

Direct application of the second equation leads to division by zero. Graphing the point $(0, 3)$ on the rectangular coordinate system reveals that the point is located on the positive y-axis. The angle between the positive x-axis and the positive y-axis is $\frac{\pi}{2}$. Therefore this point can be represented as $(3, \frac{\pi}{2})$ in polar coordinates.

d. Use $x = 5\sqrt{3}$ and $y = -5$ in Equation 9.3.3:

$$r^2 = x^2 + y^2 = (5\sqrt{3})^2 + (-5)^2 = 75 + 25$$

$$r = 10$$

and via Equation 9.3.4

$$\tan\theta = \frac{y}{x} = \frac{-5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\theta = -\frac{\pi}{6}.$$

Therefore this point can be represented as $(10, -\frac{\pi}{6})$ in polar coordinates.

e. Use $r = 3$ and $\theta = \frac{\pi}{3}$ in Equation 9.3.1:

$$x = r\cos\theta = 3\cos(\frac{\pi}{3}) = 3(\frac{1}{2}) = \frac{3}{2}$$

and

$$y = r\sin\theta = 3\sin(\frac{\pi}{3}) = 3(\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{2}.$$

Therefore this point can be represented as $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = \frac{3\pi}{2}$ in Equation 9.3.1:

$$x = r\cos\theta = 2\cos(\frac{3\pi}{2}) = 2(0) = 0$$

and

$$y = r\sin\theta = 2\sin(\frac{3\pi}{2}) = 2(-1) = -2.$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -\frac{5\pi}{6}$ in Equation 9.3.1:

$$x = r\cos\theta = 6\cos(-\frac{5\pi}{6}) = 6(-\frac{\sqrt{3}}{2}) = -3\sqrt{3}$$

and

$$y = r\sin\theta = 6\sin(-\frac{5\pi}{6}) = 6(-\frac{1}{2}) = -3.$$

Therefore this point can be represented as $(-3\sqrt{3}, -3)$ in rectangular coordinates.

## Exercise 9.3.1

Convert $(-8, -8)$ into polar coordinates and $(4, \frac{2\pi}{3})$ into rectangular coordinates.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

The polar representation of a point is not unique. For example, the polar coordinates $(2, \frac{\pi}{3})$ and $(2, \frac{7\pi}{3})$ both represent the point $(1, \sqrt{3})$ in the rectangular system. Also, the value of r can be negative. Therefore, the point with polar coordinates $(-2, \frac{4\pi}{3})$ also represents the point $(1, \sqrt{3})$ in the rectangular system, as we can see by using Equation 9.3.1:

$$x = r\cos\theta = -2\cos(\frac{4\pi}{3}) = -2(-\frac{1}{2}) = 1$$

and

$$y = r\sin\theta = -2\sin(\frac{4\pi}{3}) = -2(-\frac{\sqrt{3}}{2}) = \sqrt{3}.$$

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Note that the polar representation of a point in the plane also has a visual interpretation. In particular, $r$ is the directed distance that the point lies from the origin, and $\theta$ measures the angle that the line segment from the origin to the point makes with the positive $x$-axis. Positive angles are measured in a counterclockwise direction and negative angles are measured in a clockwise direction. The polar coordinate system appears in Figure 9.3.2.

A series of concentric circles is drawn with spokes indicating different values between 0 and 2π in increments of π/12. The first quadrant starts with 0 where the x axis would be, then the next spoke is marked π/12, then π/6, π/4, π/3, 5π/12, π/2, and so on into the second, third, and fourth quadrants. The polar axis is noted near the former x axis line.

Figure 9.3.2: The polar coordinate system.

The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis.** The center point is the **pole**, or origin, of the coordinate system, and corresponds to $r = 0$. The innermost circle shown in Figure 9.3.2 contains all points a distance of 1 unit from the pole, and is represented by the equation $r = 1$. Then $r = 2$ is the set of points 2 units from the pole, and so on. The line segments emanating from the pole correspond to fixed angles. To plot a point in the polar coordinate system, start with the angle. If the angle is positive, then measure the angle from the polar axis in a counterclockwise direction. If it is negative, then measure it clockwise. If the value of $r$ is positive, move that distance along

the terminal ray of the angle. If it is negative, move along the ray that is opposite the terminal ray of the given angle.

### Example 9.3.2: Plotting Points in the Polar Plane

Plot each of the following points on the polar plane.

a. $(2, \frac{\pi}{4})$
b. $(-3, \frac{2\pi}{3})$
c. $(4, \frac{5\pi}{4})$

**Solution**

The three points are plotted in Figure 9.3.3.

Three points are marked on a polar coordinate plane, specifically (2, π/4) in the first quadrant, (4, 5π/4) in the third quadrant, and (–3, 2π/3) in the fourth quadrant.

Figure 9.3.3: Three points plotted in the polar coordinate system.

### Exercise 9.3.2

Plot $(4, \frac{5\pi}{3})$ and $(-3, -\frac{7\pi}{2})$ on the polar plane.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Polar Curves

Now that we know how to plot points in the polar coordinate system, we can discuss how to plot curves. In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. In a similar fashion, we can graph a curve that is generated by a function $r = f(\theta)$.

The general idea behind graphing a function in polar coordinates is the same as graphing a function in rectangular coordinates. Start with a list of values for the independent variable ($\theta$ in this case) and calculate the corresponding values of the dependent variable $r$. This process generates a list of ordered pairs, which can be plotted in the polar coordinate system. Finally, connect the points, and take advantage of any patterns that may appear. The function may be periodic, for example, which indicates that only a limited number of values for the independent variable are needed.

### Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

1. Create a table with two columns. The first column is for $\theta$, and the second column is for $r$.
2. Create a list of values for $\theta$.
3. Calculate the corresponding $r$ values for each $\theta$.
4. Plot each ordered pair $(r, \theta)$ on the coordinate axes.
5. Connect the points and look for a pattern.

### Example 9.3.3: Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4\sin\theta$. Identify the curve and rewrite the equation in rectangular coordinates.

**Solution**

Because the function is a multiple of a sine function, it is periodic with period $2\pi$, so use values for $\theta$ between $0$ and $2\pi$. The result of steps 1–3 appear in the following table. Figure $9.3.4$ shows the graph based on this table.

*Table $9.3.1$: Table of values for this polar function.*

| $\theta$ | $r = 4\sin\theta$ | $\theta$ | $r = 4\sin\theta$ |
|---|---|---|---|
| 0 | 0 | $\pi$ | 0 |
| $\frac{\pi}{6}$ | 2 | $\frac{7\pi}{6}$ | $-2$ |
| $\frac{\pi}{4}$ | $2\sqrt{2} \approx 2.8$ | $\frac{5\pi}{4}$ | $-2\sqrt{2} \approx -2.8$ |
| $\frac{\pi}{3}$ | $2\sqrt{3} \approx 3.4$ | $\frac{4\pi}{3}$ | $-2\sqrt{3} \approx -3.4$ |
| $\frac{\pi}{2}$ | 4 | $\frac{3\pi}{2}$ | $-4$ |
| $\frac{2\pi}{3}$ | $2\sqrt{3} \approx 3.4$ | $\frac{5\pi}{3}$ | $-2\sqrt{3} \approx -3.4$ |
| $\frac{3\pi}{4}$ | $2\sqrt{2} \approx 2.8$ | $\frac{7\pi}{4}$ | $-2\sqrt{2} \approx -2.8$ |
| $\frac{5\pi}{6}$ | 2 | $\frac{11\pi}{6}$ | –2 |
| | | $2\pi$ | 0 |

On the polar coordinate plane, a circle is drawn with radius 2. It touches the origin, (2 times the square root of 2, π/4), (4, π/2), and (2 times the square root of 2, 3π/4).

Figure $9.3.4$: The graph of the function $r = 4\sin\theta$ is a circle.

This is the graph of a circle. The equation $r = 4\sin\theta$ can be converted into rectangular coordinates by first multiplying both sides by $r$. This gives the equation $r^2 = 4r\sin\theta$. Next use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

$$x^2 + y^2 - 4y = 0$$
$$x^2 + (y^2 - 4y) = 0$$
$$x^2 + (y^2 - 4y + 4) = 0 + 4$$
$$x^2 + (y-2)^2 = 4$$

This is the equation of a circle with radius 2 and center $(0, 2)$ in the rectangular coordinate system.

## Exercise $9.3.3$

Create a graph of the curve defined by the function $r = 4 + 4\cos\theta$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

The graph in Example $9.3.3$ was that of a circle. The equation of the circle can be transformed into rectangular coordinates using the coordinate transformation formulas in Equation $9.3.1$. Example $9.3.4$ gives some more examples of functions for transforming

from polar to rectangular coordinates.

## Example 9.3.4: Transforming Polar Equations to Rectangular Coordinates

Rewrite each of the following equations in rectangular coordinates and identify the graph.

a. $\theta = \dfrac{\pi}{3}$
b. $r = 3$
c. $r = 6\cos\theta - 8\sin\theta$

**Solution**:

a. Take the tangent of both sides. This gives $\tan\theta = \tan(\pi/3) = \sqrt{3}$. Since $\tan\theta = y/x$ we can replace the left-hand side of this equation by $y/x$. This gives $y/x = \sqrt{3}$, which can be rewritten as $y = x\sqrt{3}$. This is the equation of a straight line passing through the origin with slope $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan K$.

b. First, square both sides of the equation. This gives $r^2 = 9$. Next replace $r^2$ with $x^2 + y^2$. This gives the equation $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3. In general, any polar equation of the form $r = k$ where *k* is a positive constant represents a circle of radius *k* centered at the origin. (Note: when squaring both sides of an equation it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case we do not introduce new points. For example, $(-3, \dfrac{\pi}{3})$ is the same point as $(3, \dfrac{4\pi}{3})$.)

c. Multiply both sides of the equation by $r$. This leads to $r^2 = 6r\cos\theta - 8r\sin\theta$. Next use the formulas

$$r^2 = x^2 + y^2,\ \ x = r\cos\theta,\ \ y = r\sin\theta.$$

This gives

$$r^2 = 6(r\cos\theta) - 8(r\sin\theta)$$

$$x^2 + y^2 = 6x - 8y.$$

To put this equation into standard form, first move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

$$x^2 + y^2 = 6x - 8y$$

$$x^2 - 6x + y^2 + 8y = 0$$

$$(x^2 - 6x) + (y^2 + 8y) = 0$$

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 9 + 16$$

$$(x - 3)^2 + (y + 4)^2 = 25.$$

This is the equation of a circle with center at $(3, -4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

## Exercise 9.3.4

Rewrite the equation $r = \sec\theta\tan\theta$ in rectangular coordinates and identify its graph.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

We have now seen several examples of drawing graphs of curves defined by **polar equations.** A summary of some common curves is given in the tables below. In each equation, *a* and *b* are arbitrary constants.

This table has three columns and 3 rows. The first row is a header row and is given from left to right as name, equation, and example. The second row is Line passing through the pole with slope tan K; θ = K; and a picture of a straight line on the polar coordinate plane with θ = π/3. The third row is Circle; r = a cosθ + b sinθ; and a picture of a circle on the polar coordinate plane with equation r = 2 cos(t) – 3 sin(t): the circle touches the origin but has center in the third quadrant.

Figure 9.3.5

This table has three columns and 3 rows. The first row is Spiral; r = a + bθ; and a picture of a spiral starting at the origin with equation r = θ/3. The second row is Cardioid; r = a(1 + cosθ), r = a(1 – cosθ), r = a(1 + sinθ), r = a(1 – sinθ); and a picture of a cardioid with equation r = 3(1 + cosθ): the cardioid looks like a heart turned on its side with a rounded bottom instead of a pointed one. The third row is Limaçon; r = a cosθ + b, r = a sinθ + b; and a picture of a limaçon with equation r = 2 + 4 sinθ: the figure looks like a deformed circle with a loop inside of it. The seventh row is Rose; r = a cos(bθ), r = a sin(bθ); and a picture of a rose with equation r = 3 sin(2θ): the rose looks like a flower with four petals, one petal in each quadrant, each with length 3 and reaching to the origin between each petal.

Figure 9.3.6

A **cardioid** is a special case of a **limaçon** (pronounced “lee-mah-son”), in which $a = b$ or $a = -b$. The **rose i**s a very interesting curve. Notice that the graph of $r = 3\sin 2\theta$ has four petals. However, the graph of $r = 3\sin 3\theta$ has three petals as shown.

A rose with three petals, one in the first quadrant, another in the second quadrant, and the third in both the third and fourth quadrants, each with length 3. Each petal starts and ends at the origin.

Figure 9.3.7: Graph of $r = 3\sin 3\theta$.

If the coefficient of $\theta$ is even, the graph has twice as many petals as the coefficient. If the coefficient of $\theta$ is odd, then the number of petals equals the coefficient. You are encouraged to explore why this happens. Even more interesting graphs emerge when the coefficient of $\theta$ is not an integer. For example, if it is rational, then the curve is closed; that is, it eventually ends where it started (Figure 9.3.8$a$). However, if the coefficient is irrational, then the curve never closes (Figure 9.3.8$b$). Although it may appear that the curve is closed, a closer examination reveals that the petals just above the positive x axis are slightly thicker. This is because the petal does not quite match up with the starting point.

This figure has two figures. The first is a rose with so many overlapping petals that there are a few patterns that develop, starting with a sharp 10 pointed star in the center and moving out to an increasingly rounded set of petals. The second figure is a rose with even more overlapping petals, so many so that it is impossible to tell what is happening in the center, but on the outer edges are a number of sharply rounded petals.

Figure 9.3.8: Polar rose graphs of functions with (a) rational coefficient and (b) irrational coefficient. Note that the rose in part (b) would actually fill the entire circle if plotted in full.

Since the curve defined by the graph of $r = 3\sin(\pi\theta)$ never closes, the curve depicted in Figure 9.3.8$b$ is only a partial depiction. In fact, this is an example of a **space-filling curve.** A space-filling curve is one that in fact occupies a two-dimensional subset of the real plane. In this case the curve occupies the circle of radius 3 centered at the origin.

## Example 9.3.5: Describing a Spiral

Recall the **chambered nautilus** introduced in the chapter prelude. This creature displays a spiral when half the outer shell is cut away. It is possible to describe a spiral using rectangular coordinates. Figure 9.3.9 shows a spiral in rectangular coordinates. How can we describe this curve mathematically?

A spiral starting at the origin and continually increasing its radius to a point P(x, y).

Figure 9.3.9: How can we describe a spiral graph mathematically?

**Solution**

As the point *P* travels around the spiral in a counterclockwise direction, its distance d from the origin increases. Assume that the distance d is a constant multiple k of the angle $\theta$ that the line segment OP makes with the positive x-axis. Therefore $d(P, O) = k\theta$, where $O$ is the origin. Now use the distance formula and some trigonometry:

$$d(P,O) = k\theta$$

$$\sqrt{(x-0)^2+(y-0)^2} = k\arctan(\frac{y}{x})$$

$$\sqrt{x^2+y^2} = k\arctan(\frac{y}{x})$$

$$\arctan(\frac{y}{x}) = \frac{\sqrt{x^2+y^2}}{k}$$

$$y = x\tan(\frac{\sqrt{x^2+y^2}}{k}).$$

Although this equation describes the spiral, it is not possible to solve it directly for either *x* or *y*. However, if we use polar coordinates, the equation becomes much simpler. In particular, $d(P,O) = r$, and $\theta$ is the second coordinate. Therefore the equation for the spiral becomes $r = k\theta$. Note that when $\theta = 0$ we also have $r = 0$, so the spiral emanates from the origin. We can remove this restriction by adding a constant to the equation. Then the equation for the spiral becomes $r = a + k\theta$ for arbitrary constants $a$ and $k$. This is referred to as an **Archimedean spiral**, after the Greek mathematician Archimedes.

Another type of spiral is the logarithmic spiral, described by the function $r = a \cdot b^{\theta}$. A graph of the function $r = 1.2(1.25^{\theta})$ is given in Figure $9.3.10$. This spiral describes the shell shape of the chambered nautilus.

This figure has two figures. The first is a shell with many chambers that increase in size from the center out. The second is a spiral with equation r = 1.2(1.25θ).

Figure $9.3.10$: A logarithmic spiral is similar to the shape of the chambered nautilus shell. (credit: modification of work by Jitze Couperus, Flickr)

Suppose a curve is described in the polar coordinate system via the function $r = f(\theta)$. Since we have conversion formulas from polar to rectangular coordinates given by

$$x = r\cos\theta$$

$$y = r\sin\theta$$

,

it is possible to rewrite these formulas using the function

$$x = f(\theta)\cos\theta$$

$$y = f(\theta)\sin\theta.$$

This step gives a parameterization of the curve in rectangular coordinates using $\theta$ as the parameter. For example, the spiral formula $r = a + b\theta$ from Figure becomes

$$x = (a+b\theta)\cos\theta$$

$$y = (a+b\theta)\sin\theta.$$

Letting $\theta$ range from $-\infty$ to $\infty$ generates the entire spiral.

## Symmetry in Polar Coordinates

When studying **symmetry** of functions in rectangular coordinates (i.e., in the form $y = f(x)$), we talk about symmetry with respect to the *y*-axis and symmetry with respect to the origin. In particular, if $f(-x) = f(x)$ for all $x$ in the domain of $f$, then $f$ is an even function and its graph is symmetric with respect to the *y*-axis. If $f(-x) = -f(x)$ for all *x* in the domain of $f$, then f is an odd function and its graph is symmetric with respect to the origin. By determining which types of symmetry a graph exhibits, we can learn more about the shape and appearance of the graph. Symmetry can also reveal other properties of the function that

generates the graph. Symmetry in polar curves works in a similar fashion.

## Symmetry in Polar Curves and Equations

Consider a curve generated by the function $r = f(\theta)$ in polar coordinates.

i. The curve is symmetric about the polar axis if for every point $(r, \theta)$ on the graph, the point $(r, -\theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged by replacing $\theta$ with $-\theta$.
ii. The curve is symmetric about the pole if for every point $(r, \theta)$ on the graph, the point $(r, \pi + \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when replacing $r$ with $-r$, or $\theta$ with $\pi + \theta$.
iii. The curve is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$ if for every point $(r, \theta)$ on the graph, the point $(r, \pi - \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when $\theta$ is replaced by $\pi - \theta$.

The following table shows examples of each type of symmetry.

This table has three rows and two columns. The first row reads “Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis” and it has a picture of a cardioid with equation r = 2 – 2 cosθ: this cardioid has points marked (r, θ) and (r, –θ), which are symmetric about the x axis, and the entire cardioid is symmetric about the x axis. The second row reads “Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well” and it has a picture of a skewed infinity symbol with equation r2 = 9 cos(2θ – π/2): this figure has points marked (r, θ) and (–r, θ), which are symmetric about the pole, and the entire figure is symmetric about the pole. The third row reads “Symmetry with respect to the vertical line θ = π/2: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis” and there is a picture of a cardioid with equation r = 2 – 2 sinθ: this figure has points marked (r, θ) and (r, π – θ), which are symmetric about the vertical line θ = π/2, and the entire cardioid is symmetric about the vertical line θ = π/2.

## Example 9.3.6: Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

**Solution**

Suppose the point $(r, \theta)$ is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing $\theta$ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing $r$ with $-r$ and $\theta$ with $\pi - \theta$ yields

$$\begin{aligned} -r &= 3\sin(2(\pi - \theta)) \\ -r &= 3\sin(2\pi - 2\theta) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by $-1$ gives $r = 3\sin 2\theta$, which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace $r$ with $-r$, which yields $-r = 3\sin(2\theta)$. Multiplying both sides by $-1$ gives $r = -3\sin(2\theta)$, which does not agree with the original equation. Therefore the equation does not pass the test for this symmetry. However, returning to the original equation and replacing $\theta$ with $\theta + \pi$ gives

$$\begin{aligned} r &= 3\sin(2(\theta + \pi)) \\ &= 3\sin(2\theta + 2\pi) \\ &= 3(\sin 2\theta \cos 2\pi + \cos 2\theta \sin 2\pi) \\ &= 3\sin 2\theta. \end{aligned}$$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$, first replace both $r$ with $-r$ and $\theta$ with $-\theta$.

$$-r = 3\sin(2(-\theta))$$
$$-r = 3\sin(-2\theta)$$
$$-r = -3\sin 2\theta.$$

Multiplying both sides of this equation by $-1$ gives $r = 3\sin 2\theta$, which is the original equation. Therefore the graph is symmetric about the vertical line $\theta = \frac{\pi}{2}$.

This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of $\theta$ between $0$ and $\pi/2$ and then reflect the resulting graph.

*Table 9.3.2: Table of values for this polar function.*

| $\theta$ | $r = 3\sin(2\theta)$ |
|---|---|
| 0 | 0 |
| $\frac{\pi}{6}$ | $\frac{3\sqrt{3}}{2} \approx 2.6$ |
| $\frac{\pi}{4}$ | 3 |
| $\frac{\pi}{3}$ | $\frac{3\sqrt{3}}{2} \approx 2.6$ |
| $\frac{\pi}{2}$ | 0 |

This gives one petal of the rose, as shown in the following graph.

A single petal is graphed with equation r = 3 sin(2θ) for 0 ≤ θ ≤ π/2. It starts at the origin and reaches a maximum distance from the origin of 3.

Figure 9.3.11: The graph of the equation between $\theta = 0$ and $\theta = \pi/2$.

Reflecting this image into the other three quadrants gives the entire graph as shown.

A four-petaled rose is graphed with equation r = 3 sin(2θ). Each petal starts at the origin and reaches a maximum distance from the origin of 3.

Figure 9.3.12: The entire graph of the equation is called a four-petaled rose.

### Exercise 9.3.5Symmetry

Determine the symmetry of the graph determined by the equation $r = 2\cos(3\theta)$ and create a graph.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}.$$

- To sketch a polar curve from a given polar function, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

# Glossary

**angular coordinate**
: $\theta$ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial $x$-axis, measured counterclockwise

**cardioid**
: a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

**limaçon**
: the graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid

**polar axis**
: the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

**polar coordinate system**
: a system for locating points in the plane. The coordinates are $r$, the radial coordinate, and $\theta$, the angular coordinate

**polar equation**
: an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

**pole**
: the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

**radial coordinate**
: $r$ the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

**rose**
: graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$for a positive constant $a$

**space-filling curve**
: a curve that completely occupies a two-dimensional subset of the real plane

## Answers

**Exercise** 9.3.1

Use Equation 9.3.3 and Equation 9.3.1. Make sure to check the quadrant when calculating $\theta$.

**Exercise** 9.3.1

$(8\sqrt{2}, \frac{5\pi}{4})$ and $(-2, 2\sqrt{3})$

**Exercise** 9.3.2

Start with $\theta$, then use $r$.

**Exercise** 9.3.2

Two points are marked on a polar coordinate plane, specifically (–3, –7π/2) on the y axis and (4, 5π/3) in the fourth quadrant.

**Exercise** 9.3.3

Follow the problem-solving strategy for creating a graph in polar coordinates.

**Exercise** 9.3.3

The name of this shape is a cardioid, which we will study further later in this section.

The graph of r = 4 + 4 cosθ is given. It vaguely looks look a heart tipped on its side with a rounded bottom instead of a pointed one. Specifically, the graph starts at the origin, moves into the second quadrant and increases to a rounded circle-like figure. The graph is symmetric about the x axis, so it continues its rounded circle-like figure, goes into the third quadrant, and comes to a point at the origin.

**Exercise** 9.3.4

Convert to sine and cosine, then multiply both sides by cosine.

**Exercise** $9.3.4$

$y = x^2$, which is the equation of a parabola opening upward.

**Exercise** $9.3.5$**Symmetry**

Use Note.

**Exercise** $9.3.5$**Symmetry**

Symmetric with respect to the polar axis.

A three-petaled rose is graphed with equation r = 2 cos(3θ). Each petal starts at the origin and reaches a maximum distance from the origin of 2.

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# 9.3.1: Introduction to Polar Coordinates

**Learning Objectives**

In this section, you will:

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see Figure 1). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

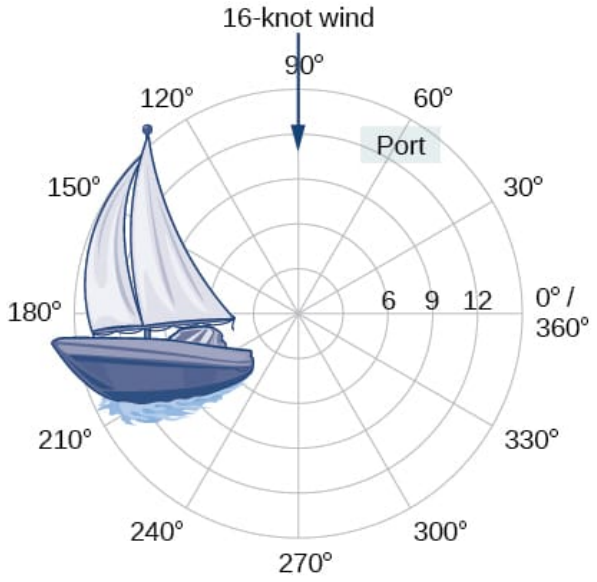
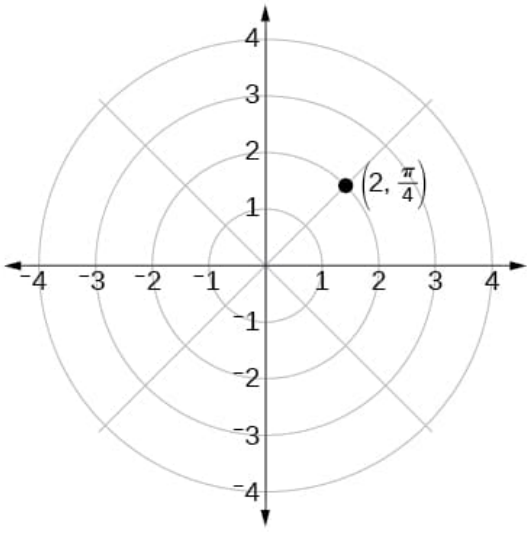



Figure 1

## Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of rectangular coordinates $(x, y)$ in the Cartesian coordinate plane. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled $(r, \theta)$ and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive *x*-axis now viewed as the polar axis and the origin as the pole. The first coordinate $r$ is the radius or length of the directed line segment from the pole. The angle $\theta$, measured in radians, indicates the direction of $r$. We move counterclockwise from the polar axis by an angle of $\theta$, and measure a directed line segment the length of $r$ in the direction of $\theta$. Even though we measure $\theta$ first and then $r$, the polar point is written with the *r*-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$, we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in Figure 2.




Polar Grid

Figure 2

**Example 1**

**Plotting a Point on the Polar Grid**

Plot the point $(3, \frac{\pi}{2})$ on the polar grid.

*Answer available at the end of this section.*

**Try It #1**

Plot the point $(2, \ \frac{\pi}{3})$ in the polar grid.

**Example 2**

**Plotting a Point in the Polar Coordinate System with a Negative Component**

Plot the point $(-2, \ \frac{\pi}{6})$ on the polar grid.

*Answer available at the end of this section.*

**Try It #2**

Plot the points $(3, -\frac{\pi}{6})$ and $(2, \frac{9\pi}{4})$ on the same polar grid.

## Converting from Polar Coordinates to Rectangular Coordinates

When given a set of polar coordinates, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables $x, \ y, \ r$, and $\theta$.

$$\cos\ \theta = \frac{x}{r} \rightarrow x = r\cos\ \theta$$
$$\sin\ \theta = \frac{y}{r} \rightarrow y = r\sin\ \theta$$

Dropping a perpendicular from the point in the plane to the *x*-axis forms a right triangle, as illustrated in Figure 5. An easy way to remember the equations above is to think of $\cos\ \theta$ as the adjacent side over the hypotenuse and $\sin\ \theta$ as the opposite side over the hypotenuse.

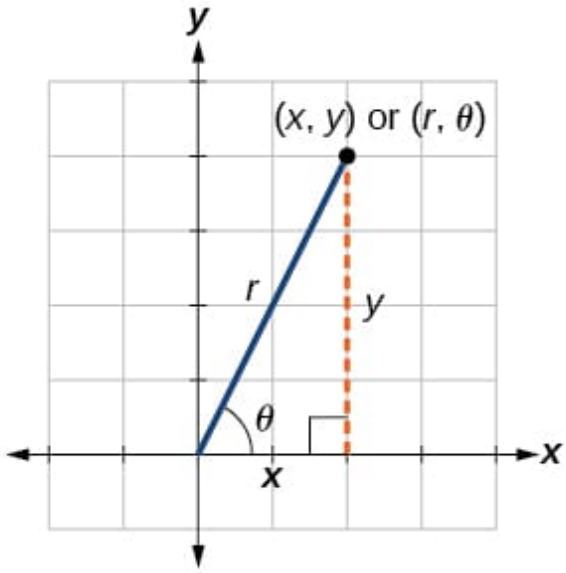



Figure 5

**Converting from Polar Coordinates to Rectangular Coordinates**

To convert polar coordinates $(r,\ \theta)$ to rectangular coordinates $(x,\ y)$, let

$$\cos\ \theta = \frac{x}{r} \rightarrow x = r\cos\ \theta$$

$$\sin\ \theta = \frac{y}{r} \rightarrow y = r\sin\ \theta$$

**How To**

**Given polar coordinates, convert to rectangular coordinates.**

1. Given the polar coordinate $(r, \theta)$, write $x = r\cos\ \theta$ and $y = r\sin\ \theta$.
2. Evaluate $\cos\ \theta$ and $\sin\ \theta$.
3. Multiply $\cos\ \theta$ by $r$ to find the *x*-coordinate of the rectangular form.
4. Multiply $\sin\ \theta$ by $r$ to find the *y*-coordinate of the rectangular form.

**Example 3**

**Writing Polar Coordinates as Rectangular Coordinates**

Write the polar coordinates $(3, \frac{\pi}{2})$ as rectangular coordinates.

*Answer available at the end of this section.*

Example 4

**Writing Polar Coordinates as Rectangular Coordinates**

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

*Answer available at the end of this section.*

Try It #3

Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

## Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

Converting from Rectangular Coordinates to Polar Coordinates

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in Figure 8.

$$\cos\ \theta = \frac{x}{r} x = r\cos\ \theta$$
$$\sin\ \theta = \frac{y}{r}\ \text{ or }\ y = r\sin\ \theta$$
$$r^2 = x^2 + y^2$$
$$\tan\ \theta = \frac{y}{x}$$

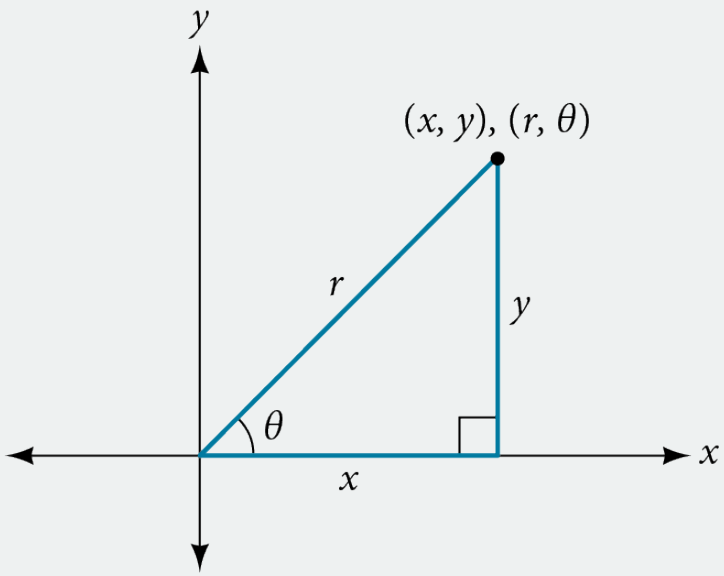



Figure 8

Example 5

**Writing Rectangular Coordinates as Polar Coordinates**

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

*Answer available at the end of this section.*

## Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2},\ \frac{5\pi}{4}\right)$ and $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2},\ \frac{\pi}{4}\right)$. The point $\left(-3\sqrt{2},\ \frac{5\pi}{4}\right)$ indicates a move further counterclockwise by $\pi$, which is directly opposite $\frac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\frac{5\pi}{4}$ is located in the third quadrant and, as $r$ is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2},\ \frac{\pi}{4}\right)$. The point $\left(3\sqrt{2},\ -\frac{7\pi}{4}\right)$ is a move further clockwise by $-\frac{7\pi}{4}$, from $\frac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

# Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How To

**Given an equation in polar form, graph it using a graphing calculator.**

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: $r_1,\ r_2,\ \ldots,\ r_6$.
3. Enter the polar equation, set equal to $r$.
4. Press **GRAPH**.

Example 6

**Writing a Cartesian Equation in Polar Form**

Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

*Answer available at the end of this section.*

To graph a circle in rectangular form, we must first solve for $y$.

$$\begin{aligned} x^2 + y^2 &= 9 \\ y^2 &= 9 - x^2 \\ y &= \pm\sqrt{9 - x^2} \end{aligned}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1 = \sqrt{9 - x^2}$ and $Y_2 = -\sqrt{9 - x^2}$. Press **GRAPH.**

Example 7

**Rewriting a Cartesian Equation as a Polar Equation**

Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

*Answer available at the end of this section.*

The Cartesian or rectangular equation is plotted on the rectangular grid, and the polar equation is plotted on the polar grid. Clearly, the graphs are identical.

Example 8

**Rewriting a Cartesian Equation in Polar Form**

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

*Answer available at the end of this section.*

Try It #4

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

## Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 9

**Graphing a Polar Equation by Converting to a Rectangular Equation**

Covert the polar equation $r = 2 \sec\ \theta$ to a rectangular equation, and draw its corresponding graph.

*Answer available at the end of this section.*

A similar discussion would demonstrate that the graph of the function $r = 2 \csc\ \theta$ will be the horizontal line $y = 2$. In fact, $r = c \csc\ \theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example 10

**Rewriting a Polar Equation in Cartesian Form**

Rewrite the polar equation $r = \frac{3}{1-2\cos\ \theta}$ as a Cartesian equation.

*Answer available at the end of this section.*

The "hour-glass" shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in Analytic Geometry.

## Analysis

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However,

the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola's standard form. To do this, we can start with the initial equation.

$$\begin{aligned}
x^2+y^2 &= (3+2x)^2 \\
x^2+y^2-(3+2x)^2 &= 0 \\
x^2+y^2-(9+12x+4x^2) &= 0 \\
x^2+y^2-9-12x-4x^2 &= 0 \\
-3x^2-12x+y^2 &= 9 && \text{Multiply through by } -1 \\
3x^2+12x-y^2 &= -9 \\
3\left(x^2+4x+\right)-y^2 &= -9 \\
3\left(x^2+4x+4\right)-y^2 &= -9+12 && \textit{Organize} \text{ terms to complete the square for } x \\
3(x+2)^2-y^2 &= 3 \\
(x+2)^2-\frac{y^2}{3} &= 1
\end{aligned}$$

**Try It #5**

Rewrite the polar equation $r = 2\sin\ \theta$ in Cartesian form.

**Example 11**

**Rewriting a Polar Equation in Cartesian Form**

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

*Answer available at the end of this section.*

**Media**

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates
- Comparing Polar and Rectangular Coordinates

# 8.3 Section Exercises

## Verbal

1.

How are polar coordinates different from rectangular coordinates?

2.

How are the polar axes different from the *x*- and *y*-axes of the Cartesian plane?

3.

Explain how polar coordinates are graphed.

4.

How are the points $(3, \frac{\pi}{2})$ and $(-3, \frac{\pi}{2})$ related?

5.

Explain why the points $(-3, \frac{\pi}{2})$ and $(3, -\frac{\pi}{2})$ are the same.

## Algebraic

For the following exercises, convert the given polar coordinates to Cartesian coordinates. Remember to consider the quadrant in which the given point is located when determining $\theta$ for the point.

6.

$(7, \frac{7\pi}{6})$

7.

$(5, \pi)$

8.

$(6, -\frac{\pi}{4})$

9.

$(-3, \frac{\pi}{6})$

10.

$(4, \frac{7\pi}{4})$

For the following exercises, convert the given Cartesian coordinates to polar coordinates with $r > 0,\ 0 \leq \theta < 2\pi$. Remember to consider the quadrant in which the given point is located.

11.

$(4, 2)$

12.

$(-4, 6)$

13.

$(3, -5)$

14.

$(-10, -13)$

15.

$(8, 8)$

For the following exercises, convert the given Cartesian equation to a polar equation.

16.

$x = 3$

17.

$y = 4$

18.

$y = 4x^2$

19.

$y = 2x^4$

20.

$x^2 + y^2 = 4y$

21.

$x^2 + y^2 = 3x$

22.

$x^2 - y^2 = x$

23.

$x^2 - y^2 = 3y$

24.

$x^2 + y^2 = 9$

25.

$x^2 = 9y$

26.

$y^2 = 9x$

27.

$9xy = 1$

For the following exercises, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

28.

$r = 3 \sin\ \theta$

29.

$r = 4 \cos\ \theta$

30.

$r = \frac{4}{\sin\ \theta + 7 \cos\ \theta}$

31.

$r = \frac{6}{\cos\ \theta + 3 \sin\ \theta}$

32.

$r = 2 \sec\ \theta$

33.

$r = 3 \csc\ \theta$

34.

$r = \sqrt{r \cos\ \theta + 2}$

35.

$r^2 = 4 \sec\ \theta\ \csc\ \theta$

36.

$r = 4$

37.

$r^2 = 4$

38.

$r = \frac{1}{4 \cos\ \theta - 3 \sin\ \theta}$

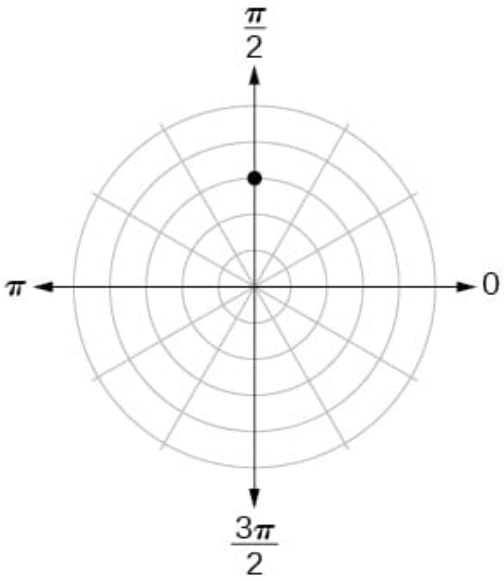
39.

$r = \frac{3}{\cos\ \theta - 5 \sin\ \theta}$

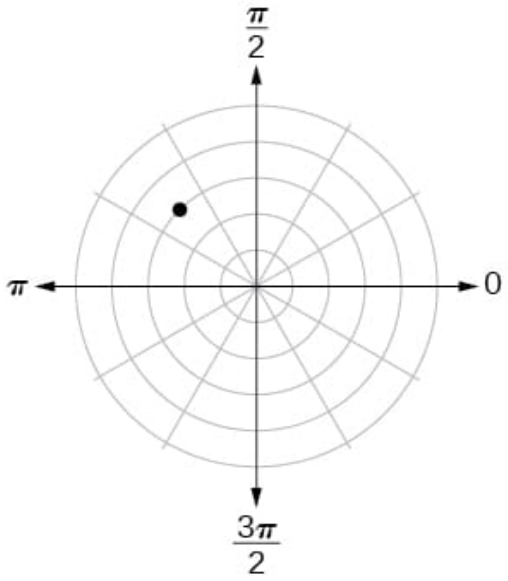
## Graphical

For the following exercises, find the polar coordinates of the point.

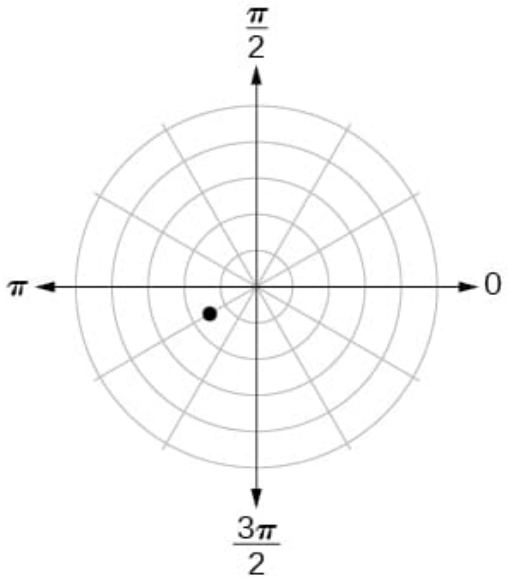
40.



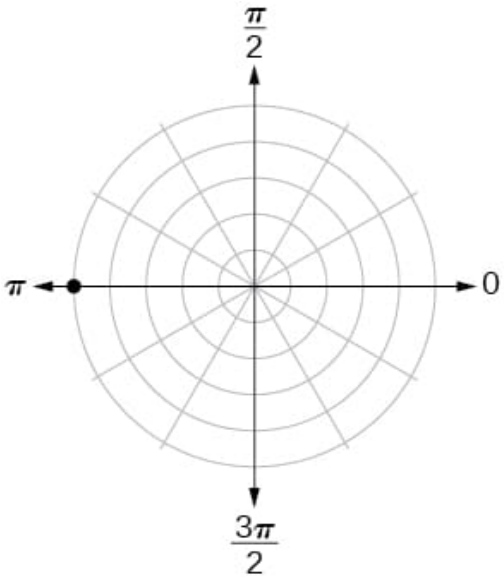

41.



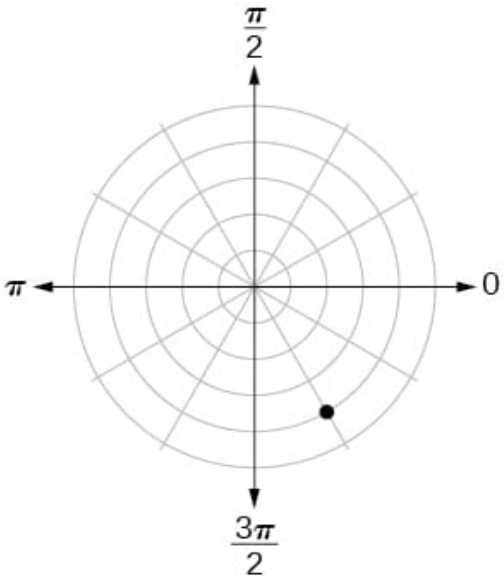

42.




43.




44.




For the following exercises, plot the points.

45.

$\left(-2, \frac{\pi}{3}\right)$

46.

$\left(-1, -\frac{\pi}{2}\right)$

47.

$\left(3.5, \frac{7\pi}{4}\right)$

48.

$\left(-4, \frac{\pi}{3}\right)$

49.

$(5, \frac{\pi}{2})$

50.

$(4, \frac{-5\pi}{4})$

51.

$(3, \frac{5\pi}{6})$

52.

$(-1.5, \frac{7\pi}{6})$

53.

$(-2, \frac{\pi}{4})$

54.

$(1, \frac{3\pi}{2})$

For the following exercises, convert the equation from rectangular to polar form and graph on the polar axis.

55.

$5x - y = 6$

56.

$2x + 7y = -3$

57.

$x^2 + (y-1)^2 = 1$

58.

$(x+2)^2 + (y+3)^2 = 13$

59.

$x = 2$

60.

$x^2 + y^2 = 5y$

61.

$x^2 + y^2 = 3x$

For the following exercises, convert the equation from polar to rectangular form and graph on the rectangular plane.

62.

$r = 6$

63.

$r = -4$

64.

$\theta = -\frac{2\pi}{3}$

65.

$\theta = \frac{\pi}{4}$

66.

$r = \sec\ \theta$

67.

$r = -10 \sin\ \theta$

68.

$r = 3 \cos\ \theta$

## Technology

69.

Use a graphing calculator to find the rectangular coordinates of $\left(2, -\frac{\pi}{5}\right)$. Round to the nearest thousandth.

70.

Use a graphing calculator to find the rectangular coordinates of $\left(-3, \frac{3\pi}{7}\right)$. Round to the nearest thousandth.

71.

Use a graphing calculator to find the polar coordinates of $(-7, 8)$ in degrees. Round to the nearest thousandth.

72.

Use a graphing calculator to find the polar coordinates of $(3, -4)$ in degrees. Round to the nearest hundredth.

73.

Use a graphing calculator to find the polar coordinates of $(-2, 0)$ in radians. Round to the nearest hundredth.

## Extensions

74.

Describe the graph of $r = a \sec\ \theta; a > 0$.

75.

Describe the graph of $r = a \sec\ \theta; a < 0$.

76.

Describe the graph of $r = a \csc\ \theta; a > 0$.

77.

Describe the graph of $r = a \csc\ \theta; a < 0$.

78.

What polar equations will give an oblique line?

For the following exercise, graph the polar inequality.

79.

$r < 4$

80.

$0 \leq \theta \leq \frac{\pi}{4}$

81.

$\theta = \frac{\pi}{4}, \ r \geq 2$

82.

$\theta = \frac{\pi}{4}, \ r \geq -3$

83.

$0 \leq \theta \leq \frac{\pi}{3}, \ r < 2$

84.

$\frac{-\pi}{6} < \theta \leq \frac{\pi}{3}, -3 < r < 2$

## Answers

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in Figure 3.

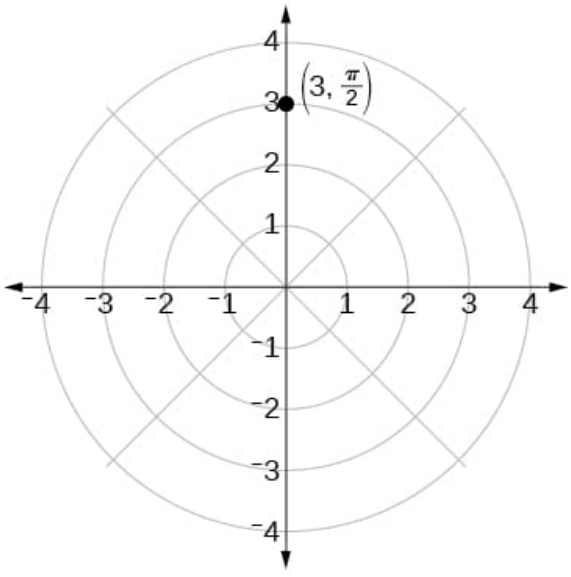



Figure 3

We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative $r$ in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See Figure 4(a). Compare this to the graph of the polar coordinate $\left(2, \frac{\pi}{6}\right)$ shown in Figure 4(b).

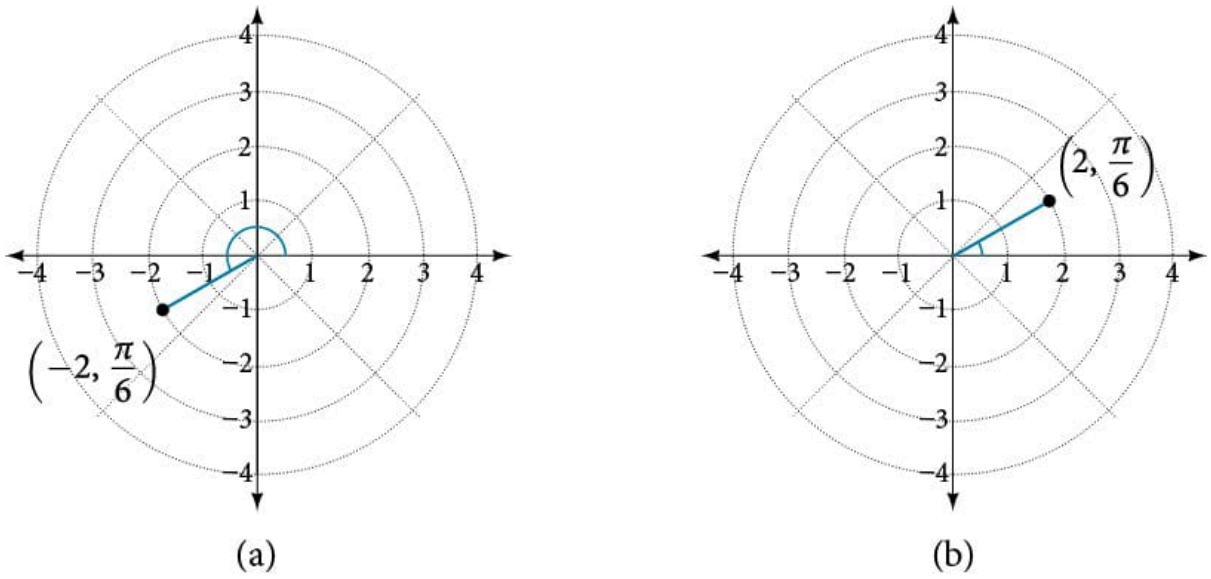



Figure 4

Use the equivalent relationships.

$$x = r\cos\ \theta$$
$$x = 3\cos\ \tfrac{\pi}{2} = 0$$
$$y = r\sin\ \theta$$
$$y = 3\sin\ \tfrac{\pi}{2} = 3$$

The rectangular coordinates are $(0, 3)$. See Figure 6.

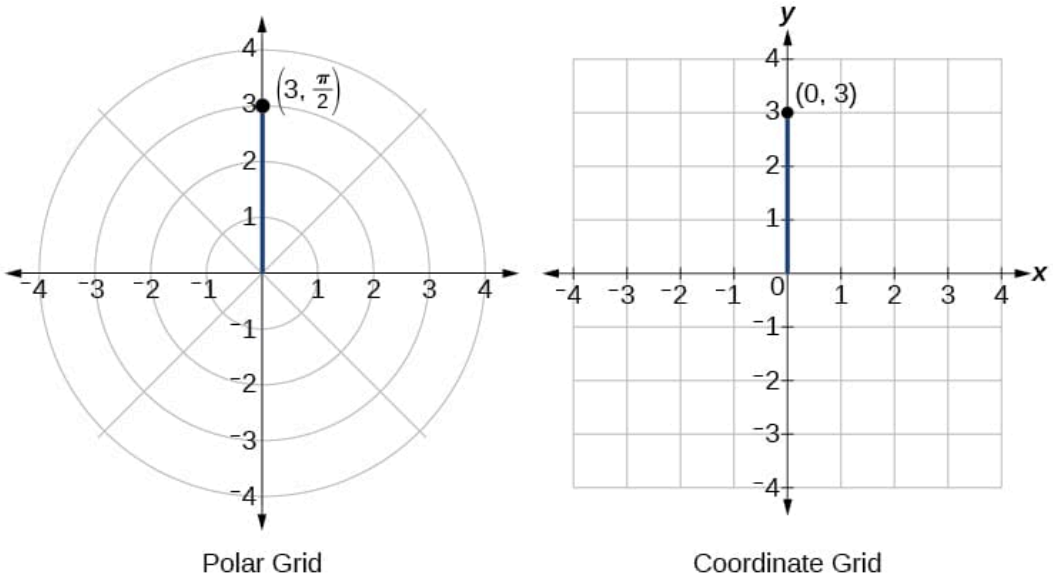



Figure 6

See Figure 7. Writing the polar coordinates as rectangular, we have

$$x = r\cos\ \theta$$
$$x = -2\cos\left(0\right) = -2$$

$$y = r\sin\ \theta$$
$$y = -2\sin\left(0\right) = 0$$

The rectangular coordinates are also $(-2, 0)$.

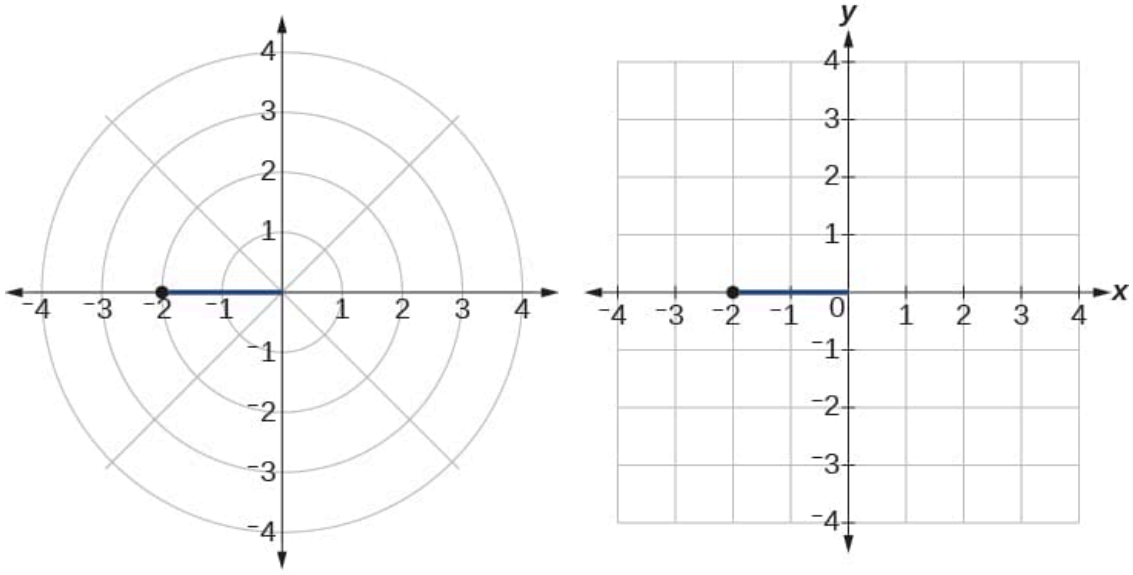



Figure 7

We see that the original point $(3,3)$ is in the first quadrant. To find $\theta$, use the formula $\tan\ \theta = \frac{y}{x}$. This gives

$$\tan\ \theta = \tfrac{3}{3}$$
$$\tan\ \theta = 1$$
$$\theta = \tan^{-1}(1)$$
$$\theta = \tfrac{\pi}{4}$$

To find $r$, we substitute the values for $x$ and $y$ into the formula $r = \sqrt{x^2 + y^2}$. We know that $r$ must be positive, as $\frac{\pi}{4}$ is in the first quadrant. Thus

$$r = \sqrt{3^2 + 3^2}$$
$$r = \sqrt{9 + 9}$$
$$r = \sqrt{18} = 3\sqrt{2}$$

So, $r = 3\sqrt{2}$ and $\theta = \frac{\pi}{4}$, giving us the polar point $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. See Figure 9.

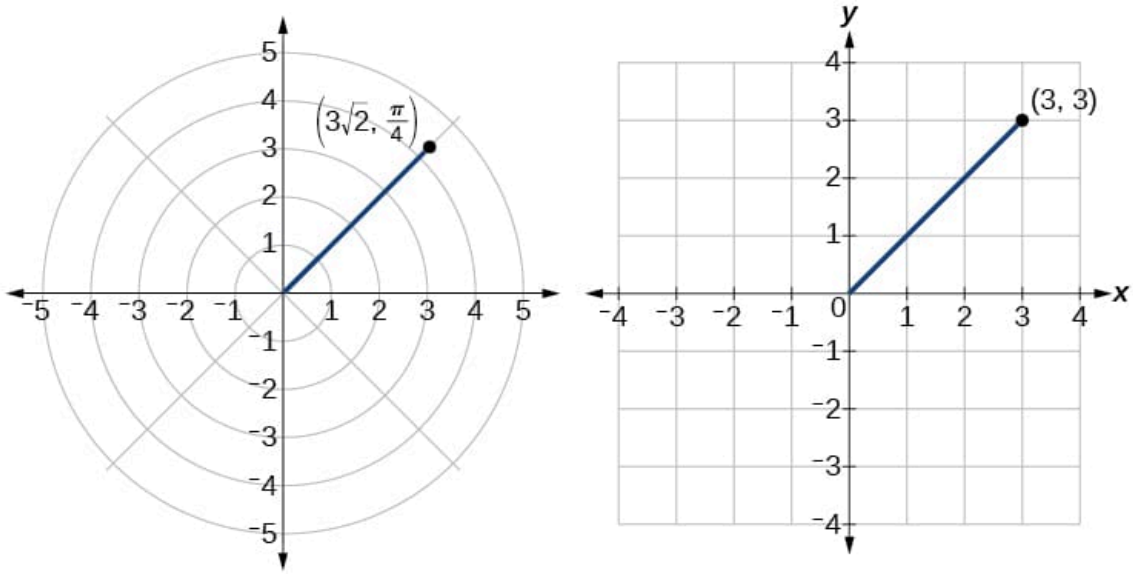



Figure 9

The goal is to eliminate $x$ and $y$ from the equation and introduce $r$ and $\theta$. Ideally, we would write the equation $r$ as a function of $\theta$. To obtain the polar form, we will use the relationships between $(x, y)$ and $(r, \theta)$. Since $x = r\cos\ \theta$ and $y = r\sin\ \theta$, we can substitute and solve for $r$.

$$(r\cos\ \theta)^2 + (r\sin\ \theta)^2 = 9$$
$$r^2\cos^2\theta + r^2\sin^2\theta = 9$$
$$r^2(\cos^2\theta + \sin^2\theta) = 9$$
$$r^2(1) = 9 \qquad \text{Substitute } \cos^2\theta + \sin^2\theta = 1.$$
$$r = \pm 3 \qquad \text{Use the square root property.}$$

Thus, $x^2 + y^2 = 9, r = 3$, and $r = -3$ should generate the same graph. See Figure 10.

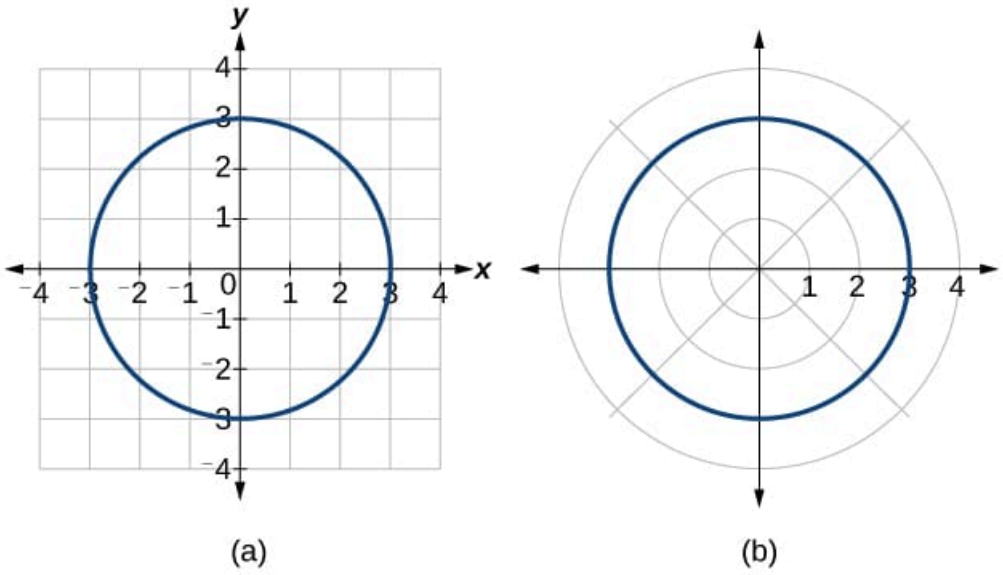



Figure 10 (a) Cartesian form $x^2+y^2=9$ (b) Polar form $r=3$

This equation appears similar to the previous example, but it requires different steps to convert the equation.

We can still follow the same procedures we have already learned and make the following substitutions:

$$\begin{aligned} r^2 &= 6y && \text{Use} x^2+y^2=r^2. \\ r^2 &= 6r\sin\ \theta && \text{Substitute } y=r\sin\ \theta. \\ r^2-6r\sin\ \theta &= 0 && \text{Set equal to 0.} \\ r(r-6\sin\ \theta) &= 0 && \text{Factor and solve.} \\ r &= 0 && \text{We reject} r=0, \text{ as it only represents one point,} (0,0). \\ r &= 6\sin\ \theta \end{aligned}$$

Therefore, the equations $x^2+y^2=6y$ and $r=6\sin\ \theta$ should give us the same graph. See Figure 11.

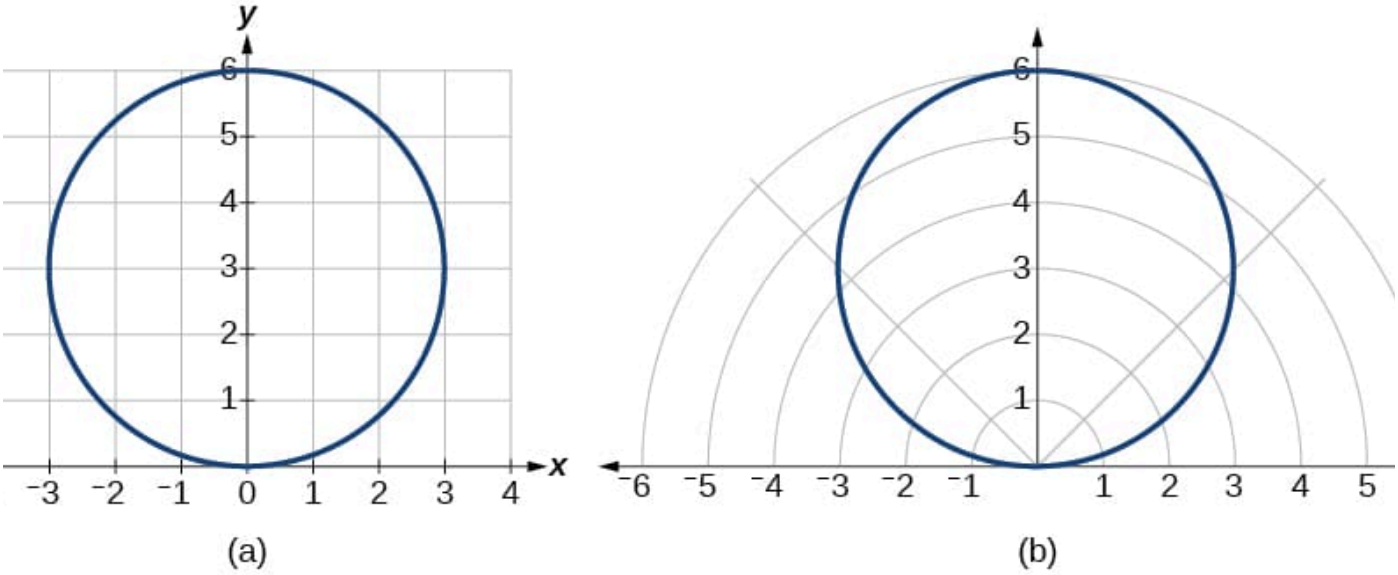



Figure 11 (a) Cartesian form $x^2+y^2=6y$ (b) polar form $r=6\sin\ \theta$

We will use the relationships $x=r\cos\ \theta$ and $y=r\sin\ \theta$.

$$\begin{aligned} y &= 3x+2 \\ r\sin\ \theta &= 3r\cos\ \theta+2 \\ r\sin\ \theta-3r\cos\ \theta &= 2 \\ r(\sin\ \theta-3\cos\ \theta) &= 2 && \text{Isolate} r. \\ r &= \tfrac{2}{\sin\ \theta-3\cos\ \theta} && \text{Solve for} r. \end{aligned}$$

The conversion is

$$\begin{aligned} r &= 2\sec\ \theta \\ r &= \tfrac{2}{\cos\ \theta} \\ r\ \cos\ \theta &= 2 \\ x &= 2 \end{aligned}$$

Notice that the equation $r = 2\sec\ \theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see Figure 12). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\ \theta$ is the standard form for a vertical line in polar form.

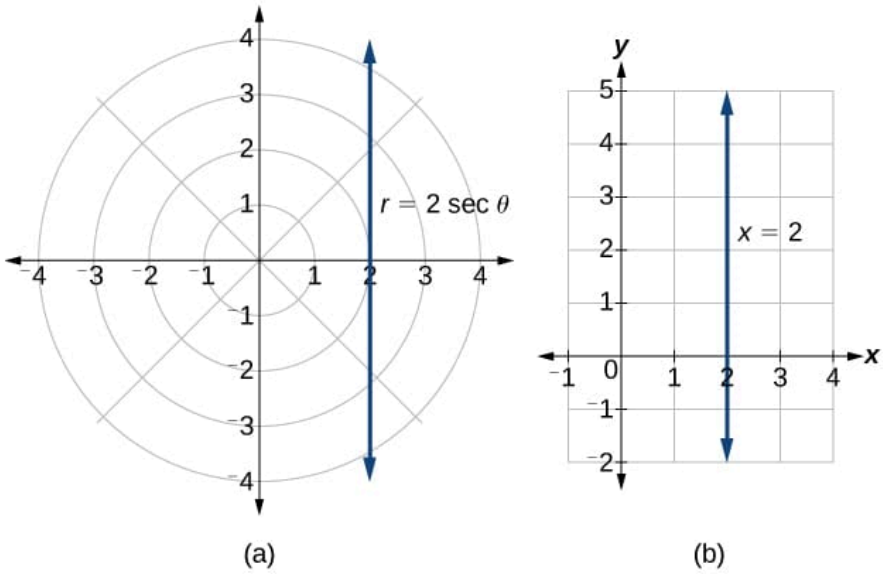



Figure 12 (a) Polar grid (b) Rectangular coordinate system

The goal is to eliminate $\theta$ and $r$, and introduce $x$ and $y$. We clear the fraction, and then use substitution. In order to replace $r$ with $x$ and $y$, we must use the expression $x^2 + y^2 = r^2$.

$$\begin{aligned} r &= \tfrac{3}{1-2\ \cos\ \theta} & \\ r\left(1-2\left(\tfrac{x}{r}\right)\right) &= 3 & \\ r-2x &= 3 & \text{Use } \cos\ \theta = \tfrac{x}{r} \text{ to eliminate } \theta \\ r-2x &= 3 & \\ r &= 3+2x & \text{Isolate } r \\ r^2 &= (3+2x)^2 & \text{Square both sides} \\ x^2+y^2 &= (3+2x)^2 & \text{Use } x^2+y^2=r^2 \end{aligned}$$

The Cartesian equation is $x^2 + y^2 = (3 + 2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate $y$.

$$\begin{aligned} x^2+y^2 &= (3+2x)^2 \\ y^2 &= (3+2x)^2 - x^2 \\ y &= \pm\sqrt{(3+2x)^2 - x^2} \end{aligned}$$

When our entire equation has been changed from $r$ and $\theta$ to $x$ and $y$, we can stop, unless asked to solve for $y$ or simplify. See Figure 13.

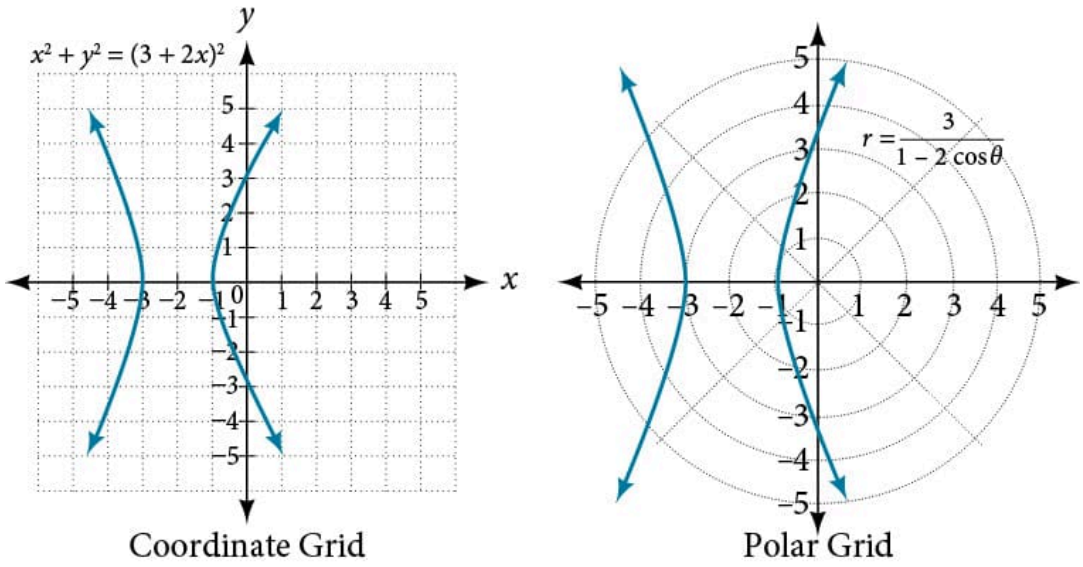



Figure 13

| | |
|---|---|
| $r = \sin(2\theta)$ | Use the double angle identity for sine. |
| $r = 2\sin\ \theta\cos\ \theta$ | Use $\cos\ \theta = \frac{x}{r}$ and $\sin\ \theta = \frac{y}{r}$. |
| $r = 2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right)$ | Simplify. |
| $r = \frac{2xy}{r^2}$ | Multiply both sides by $r^2$. |
| $r^3 = 2xy$ | |
| $\left(\sqrt{x^2 + y^2}\right)^3 = 2xy$ | As $x^2 + y^2 = r^2, r = \sqrt{x^2 + y^2}$. |

This equation can also be written as

$$(x^2 + y^2)^{\frac{3}{2}} = 2xy \text{ or } x^2 + y^2 = (2xy)^{\frac{2}{3}}$$

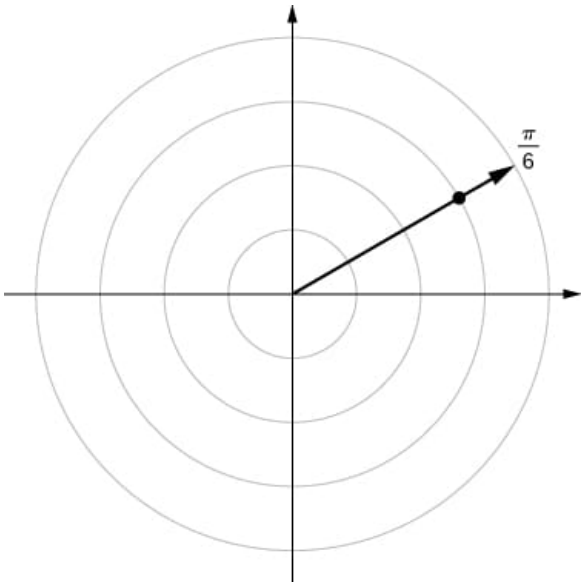
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# 9.3.2: Exercises for Section 11.3

**In exercises 1 - 7, plot the point whose polar coordinates are given by first constructing the angle $\theta$ and then marking off the distance $r$ along the ray.**

1) $\left(3, \frac{\pi}{6}\right)$

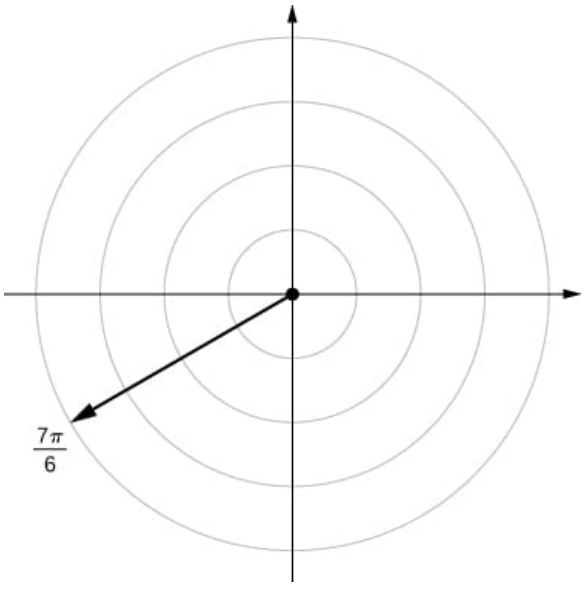
**Answer**




2) $\left(-2, \frac{5\pi}{3}\right)$

3) $\left(0, \frac{7\pi}{6}\right)$

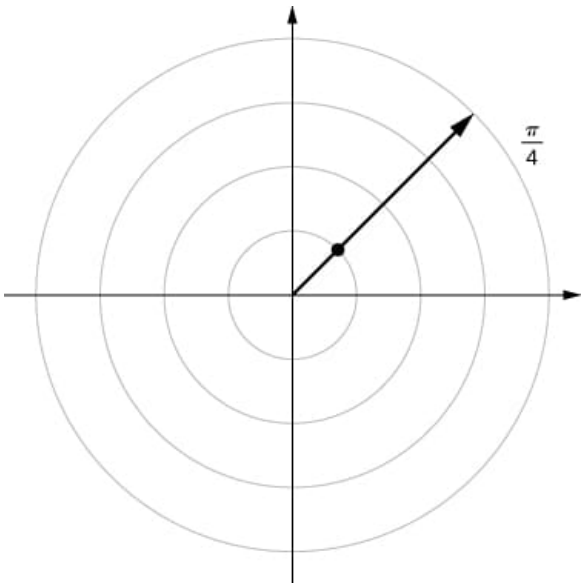
**Answer**




4) $\left(-4, \frac{3\pi}{4}\right)$

5) $\left(1, \frac{\pi}{4}\right)$

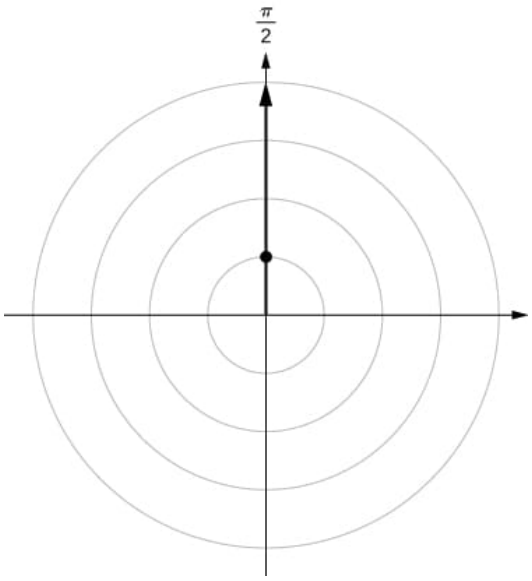
**Answer**



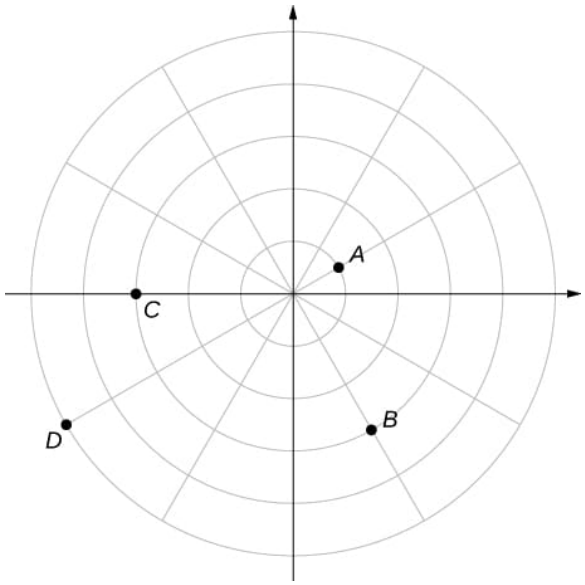

6) $(2, \frac{5\pi}{6})$

7) $(1, \frac{\pi}{2})$

**Answer**




**In exercises 8 - 11, consider the polar graph below. Give two sets of polar coordinates for each point.**




8) Coordinates of point A.

9) Coordinates of point B.

**Answer**

$B\left(3, \frac{-\pi}{3}\right)\, B\left(-3, \frac{2\pi}{3}\right)$

10) Coordinates of point C.

11) Coordinates of point D.

**Answer**

$D\left(5, \frac{7\pi}{6}\right)\, D\left(-5, \frac{\pi}{6}\right)$

**In exercises 12 - 17, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point in $(0, 2\pi]$. Round to three decimal places.**

12) $(2, 2)$

13) $(3, -4)$

**Answer**

$(5, -0.927),\ \ (-5, -0.927 + \pi)$

14) $(8, 15)$

15) $(-6, 8)$

**Answer**

$(10, -0.927),\ \ (-10, -0.927 + \pi)$

16) $(4, 3)$

17) $(3, -\sqrt{3})$

**Answer**

$(2\sqrt{3}, -0.524),\ \ (-2\sqrt{3}, -0.524 + \pi)$

**In exercises 18 - 24, find rectangular coordinates for the given point in polar coordinates.**

18) $\left(2, \frac{5\pi}{4}\right)$

19) $\left(-2, \frac{\pi}{6}\right)$

**Answer**

$(-\sqrt{3}, -1)$

20) $\left(5, \frac{\pi}{3}\right)$

21) $\left(1, \frac{7\pi}{6}\right)$

**Answer**

$\left(-\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

22) $\left(-3, \frac{3\pi}{4}\right)$

23) $\left(0, \frac{\pi}{2}\right)$

**Answer**

$(0, 0)$

24) $(-4.5, 6.5)$

**In exercises 25 - 29, determine whether the graphs of the polar equation are symmetric with respect to the $x$-axis, the $y$-axis, or the origin.**

25) $r = 3\sin(2\theta)$

**Answer**

Symmetry with respect to the *x*-axis, *y*-axis, and origin.

26) $r^2 = 9\cos\theta$

27) $r = \cos\left(\frac{\theta}{5}\right)$

**Answer**

Symmetric with respect to *x*-axis only.

28) $r = 2\sec\theta$

29) $r = 1 + \cos\theta$

**Answer**

Symmetry with respect to *x*-axis only.

**In exercises 30 - 33, describe the graph of each polar equation. Confirm each description by converting into a rectangular equation.**

30) $r = 3$

31) $\theta = \frac{\pi}{4}$

**Answer**

Line $y = x$

32) $r = \sec\theta$

33) $r = \csc\theta$

**Answer**

$y = 1$

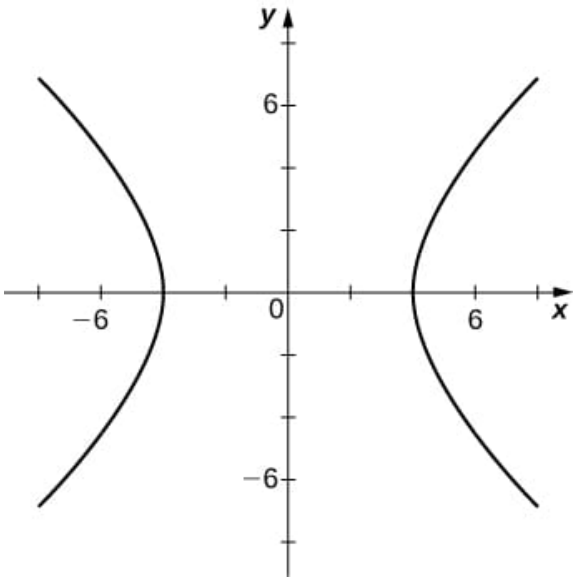
**In exercises 34 - 36, convert the rectangular equation to polar form and sketch its graph.**

34) $x^2 + y^2 = 16$

35) $x^2 - y^2 = 16$

**Answer**

Hyperbola; polar form $r^2 \cos(2\theta) = 16$ or $r^2 = 16 \sec \theta$.



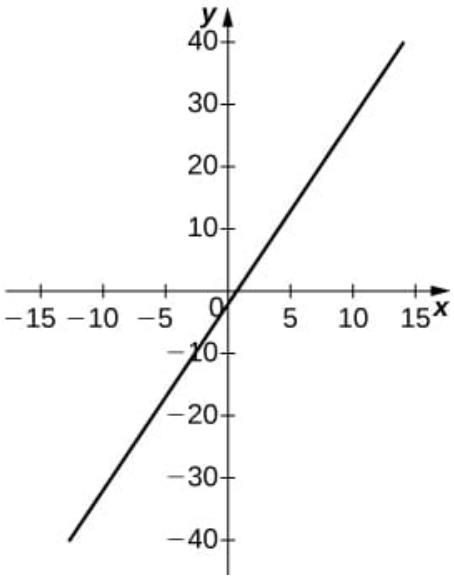

36) $x = 8$

**In exercises 37 - 38, convert the rectangular equation to polar form and sketch its graph.**

37) $3x - y = 2$

**Answer**

$r = \frac{2}{3\cos\theta - \sin\theta}$



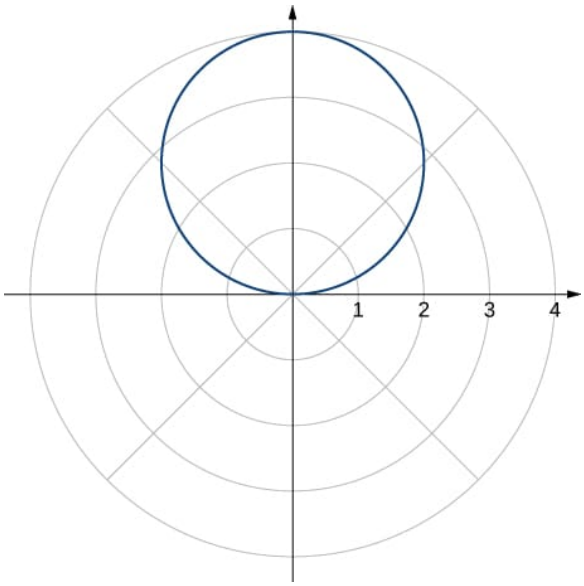

38) $y^2 = 4x$

**In exercises 39 - 43, convert the polar equation to rectangular form and sketch its graph.**

39) $r = 4 \sin \theta$

40) $x^2 + y^2 = 4y$

**Answer**

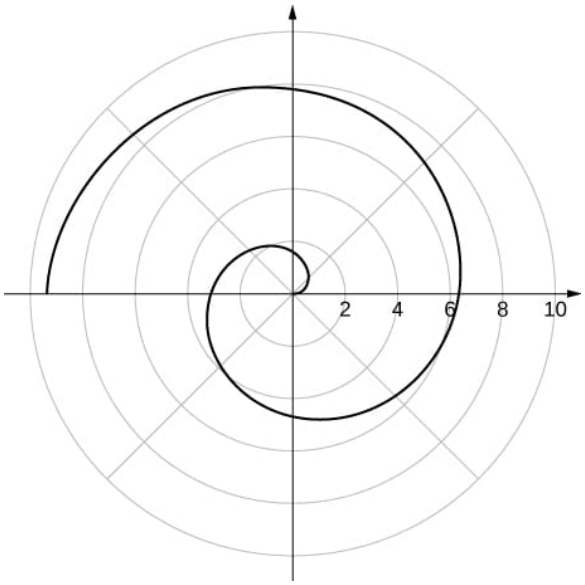



41) $r = 6 \cos \theta$

42) $r = \theta$

**Answer**

$x \tan \sqrt{x^2 + y^2} = y$



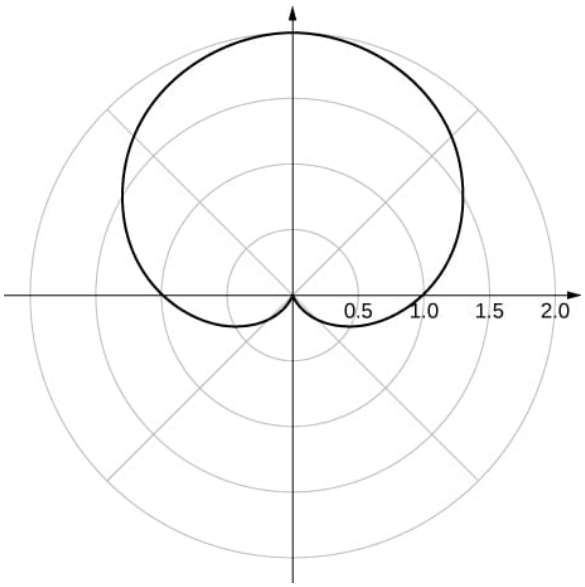

43) $r = \cot\theta \csc\theta$

**In exercises 44 - 54, sketch a graph of the polar equation and identify any symmetry.**

44) $r = 1 + \sin\theta$

**Answer**

$y$-axis symmetry

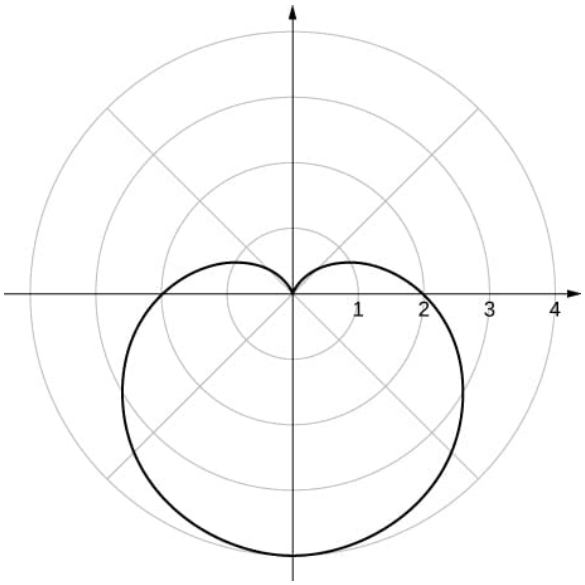



45) $r = 3 - 2\cos\theta$

46) $r = 2 - 2\sin\theta$

**Answer**

$y$-axis symmetry

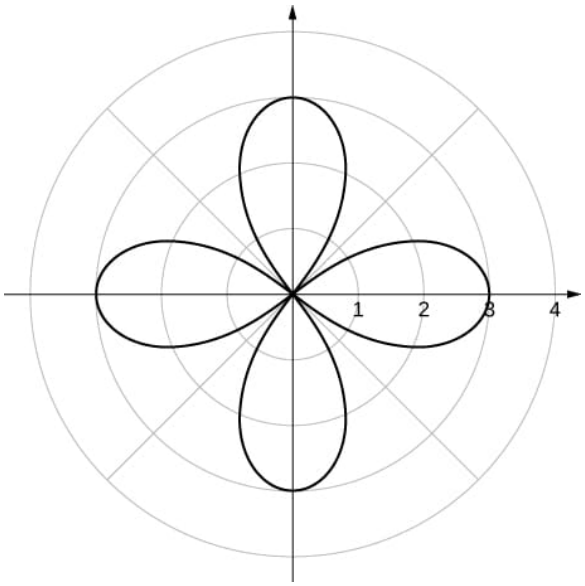



47) $r = 5 - 4\sin\theta$

48) $r = 3\cos(2\theta)$

**Answer**

$x$-and $y$-axis symmetry and symmetry about the pole

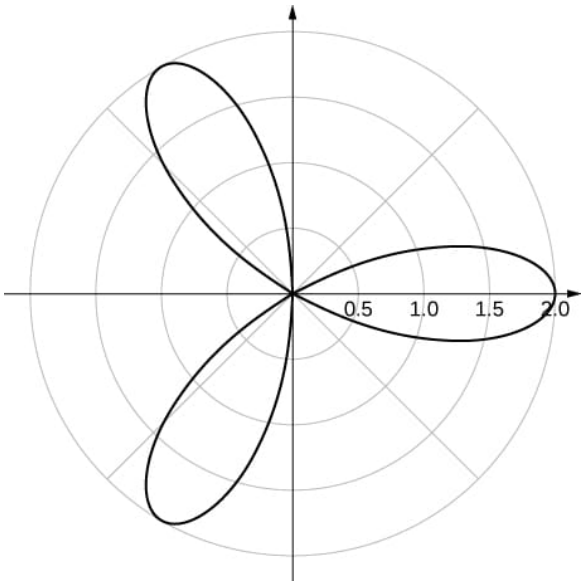



49) $r = 3\sin(2\theta)$

50) $r = 2\cos(3\theta)$

**Answer**

$x$-axis symmetry

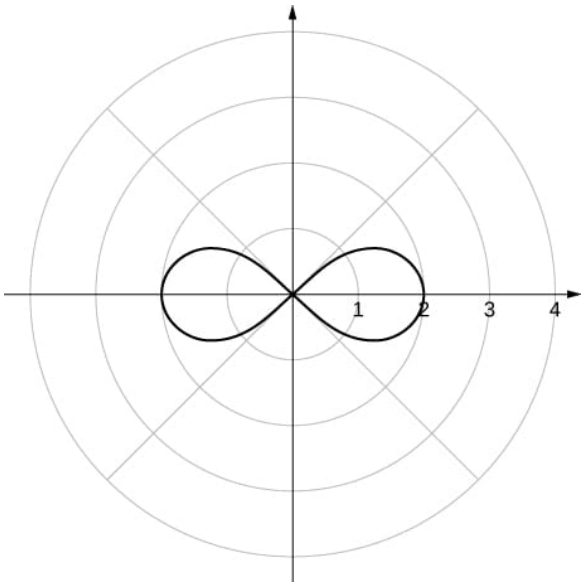



51) $r = 3\cos\left(\frac{\theta}{2}\right)$

52) $r^2 = 4\cos\left(\frac{2}{\theta}\right)$

**Answer**

$x$-and $y$-axis symmetry and symmetry about the pole

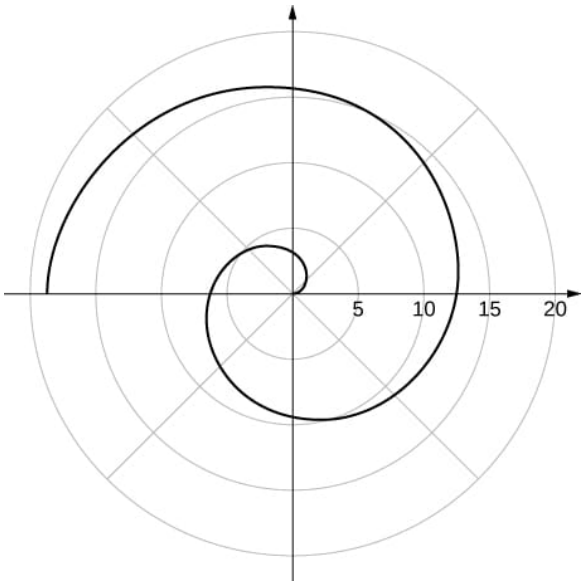



53) $r^2 = 4\sin\theta$

54) $r = 2\theta$

**Answer**

no symmetry

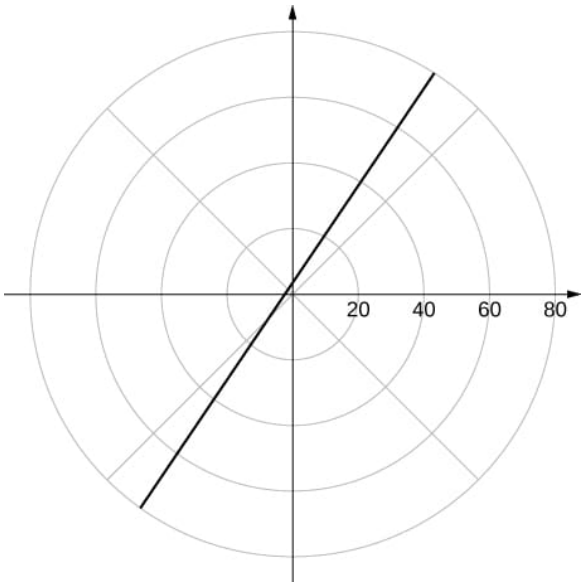



55) [T] The graph of $r = 2\cos(2\theta)\sec(\theta).$ is called a strophoid. Use a graphing utility to sketch the graph, and, from the graph, determine the asymptote.

56) [T] Use a graphing utility and sketch the graph of $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$.

**Answer**

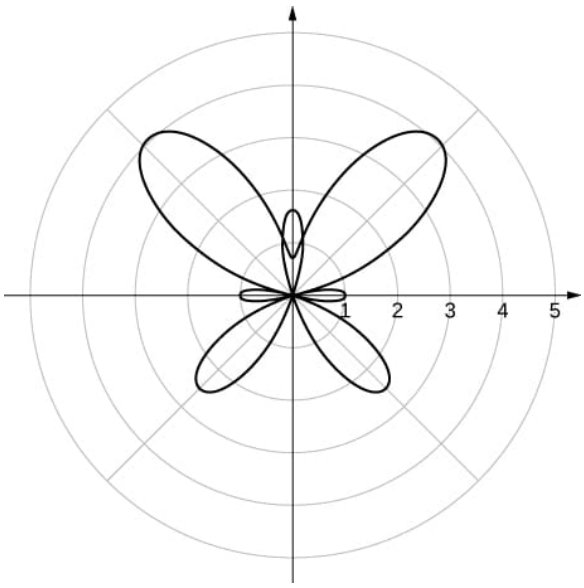
a line




57) [T] Use a graphing utility to graph $r = \frac{1}{1-\cos\theta}$.

58) [T] Use technology to graph $r = e^{\sin(\theta)} - 2\cos(4\theta)$.

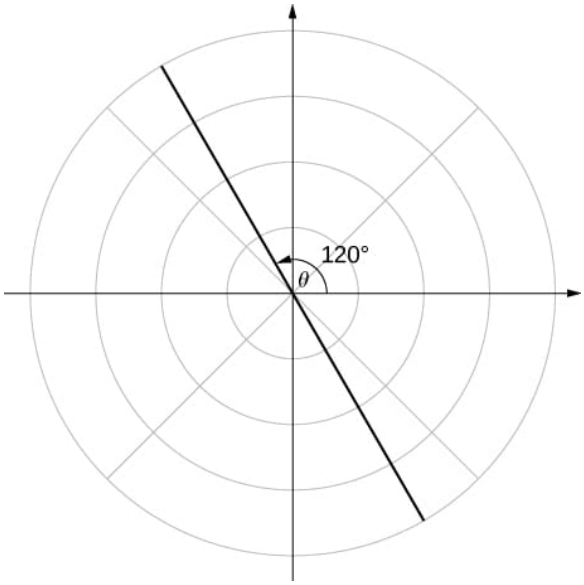
**Answer**




59) [T] Use technology to plot $r = \sin(\frac{3\theta}{7})$ (use the interval $0 \leq \theta \leq 14\pi$).

60) Without using technology, sketch the polar curve $\theta = \frac{2\pi}{3}$.

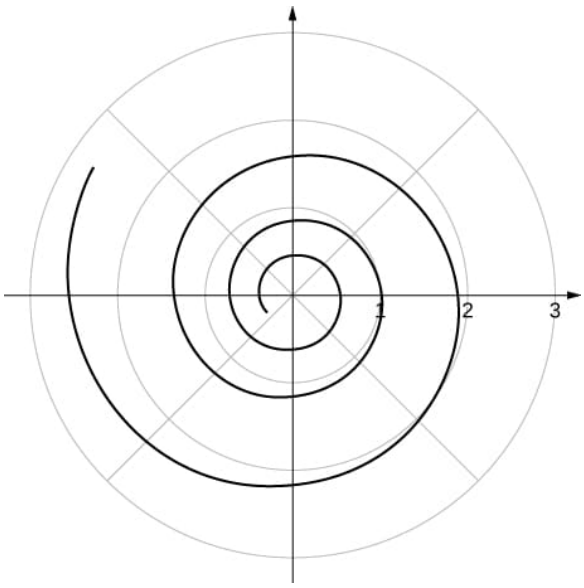
**Answer**




61) [T] Use a graphing utility to plot $r = \theta \sin \theta$ for $-\pi \leq \theta \leq \pi$.

62) [T] Use technology to plot $r = e^{-0.1\theta}$ for $-10 \leq \theta \leq 10$.

**Answer**




63) [T] There is a curve known as the “Black Hole.” Use technology to plot $r = e^{-0.01\theta}$ for $-100 \leq \theta \leq 100$.

64) [T] Use the results of the preceding two problems to explore the graphs of $r = e^{-0.001\theta}$ and $r = e^{-0.0001\theta}$ for $|\theta| > 100$.

**Answer**

Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

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# 9.4: AREA AND ARC LENGTH IN POLAR COORDINATES

### 9.4.1: Exercises for Section 11.4

**Learning Objectives**

- Apply the formula for area of a region in polar coordinates.
- Determine the arc length of a polar curve.

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval, the area between the curve and the x-axis is given by

$$A = \int_a^b f(x)dx.$$

This fact, along with the formula for evaluating this integral, is summarized in the Fundamental Theorem of Calculus. Similarly, the arc length of this curve is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

In this section, we study analogous formulas for area and arc length in the polar coordinate system.

## Areas of Regions Bounded by Polar Curves

We have studied the formulas for area under a curve defined in rectangular coordinates and parametrically defined curves. Now we turn our attention to deriving a formula for the area of a region bounded by a polar curve. Recall that the proof of the Fundamental Theorem of Calculus used the concept of a Riemann sum to approximate the area under a curve by using rectangles. For polar curves we use the Riemann sum again, but the rectangles are replaced by sectors of a circle.

Consider a curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into *n* equal-width subintervals. The width of each subinterval is given by the formula $\Delta\theta = (\beta - \alpha)/n$, and the *i*th partition point $\theta_i$ is given by the formula $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan\theta_i$ passing through the pole as shown in the following graph.

On the polar coordinate plane, a curve is drawn in the first quadrant, and there are rays from the origin that intersect this curve at a regular interval. Every time one of these rays intersects the curve, a perpendicular line is made from the ray to the next ray. The first instance of a ray-curve intersection is labeled θ = α; the last instance is labeled θ = β. The intervening ones are marked θ1, θ2, ..., θn−1.

Figure 9.4.1: A partition of a typical curve in polar coordinates.

The line segments are connected by arcs of constant radius. This defines sectors whose areas can be calculated by using a geometric formula. The area of each sector is then used to approximate the area between successive line segments. We then sum the areas of the sectors to approximate the total area. This approach gives a Riemann sum approximation for the total area. The formula for the area of a sector of a circle is illustrated in the following figure.

A circle is drawn with radius r and a sector of angle θ. It is noted that A = (1/2) θ r2.

Figure 9.4.2: *The area of a sector of a circle is given by* $A = \frac{1}{2}\theta r^2$.

Recall that the area of a circle is $A = \pi r^2$. When measuring angles in radians, 360 degrees is equal to $2\pi$ radians. Therefore a fraction of a circle can be measured by the central angle $\theta$. The fraction of the circle is given by $\frac{\theta}{2\pi}$, so the area of the sector is

this fraction multiplied by the total area:

$$A = (\frac{\theta}{2\pi})\pi r^2 = \frac{1}{2}\theta r^2.$$

Since the radius of a typical sector in Figure $9.4.1$ is given by $r_i = f(\theta_i)$, the area of the $i$th sector is given by

$$A_i = \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

Therefore a Riemann sum that approximates the area is given by

$$A_n = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} A_n = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

This gives the following theorem.

## Area of a Region Bounded by a Polar Curve

Suppose $f$ is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \tag{9.4.1}$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta. \tag{9.4.2}$$

## Example $9.4.1$: Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

**Solution**

The graph of $r = 3\sin(2\theta)$ follows.

A four-petaled rose with furthest extent 3 from the origin at π/4, 3π/4, 5π/4, and 7π/4.

Figure $9.4.3$: The graph of $r = 3\sin(2\theta)$.

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for $\theta$. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use Equation $9.4.2$ with $f(\theta) = 3\sin(2\theta), \alpha = 0$, and $\beta = \pi/2$:

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} [3\sin(2\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta) d\theta. \end{aligned}$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$ :

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} \frac{(1-\cos(4\theta))}{2}d\theta \\
&= \frac{9}{4}(\int_0^{\pi/2} 1-\cos(4\theta)d\theta) \\
&= \frac{9}{4}(\theta - \frac{\sin(4\theta)}{4}\ |_0^{\pi/2} \\
&= \frac{9}{4}(\frac{\pi}{2} - \frac{\sin 2\pi}{4}) - \frac{9}{4}(0 - \frac{\sin 4(0)}{4}) \\
&= \frac{9\pi}{8}
\end{aligned}
$$

## Exercise 9.4.1

Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

Example 9.4.1 involved finding the area inside one curve. We can also use Equation 9.4.2 to find the area between two polar curves. However, we often need to find the points of intersection of the curves and determine which function defines the outer curve or the inner curve between these two points.

## Example 9.4.2: Finding the Area between Two Polar Curves

Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.

**Solution**

First draw a graph containing both curves as shown.

A cardioid with equation r = 2 + 2 sinθ is shown, so it has its upper heart part at the origin and the rest of the cardioid is pointed up. There is a circle with radius 6 centered at (3, π/2). The area above the cardioid but below the circle is shaded orange.

Figure 9.4.4: *The region between the curves* $r = 2 + 2\sin\theta$ *and* $r = 6\sin\theta$.

To determine the limits of integration, first find the points of intersection by setting the two functions equal to each other and solving for $\theta$:

$$
\begin{aligned}
6\sin\theta &= 2 + 2\sin\theta \\
4\sin\theta &= 2 \\
\sin\theta &= \frac{1}{2}
\end{aligned}
$$

This gives the solutions $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, which are the limits of integration. The circle $r = 3\sin\theta$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin\theta$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

$$A = \text{circle} - \text{cardioid}$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}[6\sin\theta]^2 d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6}[2+2\sin\theta]^2 d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 36\sin^2\theta\ d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4+8\sin\theta+4\sin^2\theta\ d\theta$$

$$= 18\int_{\pi/6}^{5\pi/6}\frac{1-\cos(2\theta)}{2}d\theta - 2\int_{\pi/6}^{5\pi/6} 1+2\sin\theta+\frac{1-\cos(2\theta)}{2}d\theta$$

$$= 9[\theta - \frac{\sin(2\theta)}{2}]_{\pi/6}^{5\pi/6} - 2[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin(2\theta)}{4}]_{\pi/6}^{5\pi/6}$$

$$= 9(\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) - 9(\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}) - (3(\frac{5\pi}{6}) - 4\cos\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) + (3(\frac{\pi}{6}) - 4\cos\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2})$$

$$= 4\pi.$$

**Exercise 9.4.2**

Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

In Example 9.4.2 we found the area inside the circle and outside the cardioid by first finding their intersection points. Notice that solving the equation directly for $\theta$ yielded two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, in the graph there are three intersection points. The third intersection point is the origin. The reason why this point did not show up as a solution is because the origin is on both graphs but for different values of $\theta$. For example, for the cardioid we get

$$\begin{aligned} 2+2\sin\theta &= 0 \\ \sin\theta &= -1, \end{aligned}.$$

so the values for $\theta$ that solve this equation are $\theta = \frac{3\pi}{2} + 2n\pi$, where $n$ is any integer. For the circle we get

$$6\sin\theta = 0.$$

The solutions to this equation are of the form $\theta = n\pi$ for any integer value of $n$. These two solution sets have no points in common. Regardless of this fact, the curves intersect at the origin. This case must always be taken into consideration.

## Arc Length in Polar Curves

Here we derive a formula for the arc length of a curve defined in polar coordinates. In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \le \theta \le \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos\theta = f(\theta)\cos\theta$$

and

$$y = r\sin\theta = f(\theta)\sin\theta,$$

and we replace the parameter $t$ by $\theta$. Then

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

We replace $dt$ by $d\theta$, and the lower and upper limits of integration are $\alpha$ and $\beta$, respectively. Then the arc length formula becomes

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2(\cos^2\theta + \sin^2\theta) + (f(\theta))^2(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \end{aligned}$$

This gives us the following theorem.

## Arc Length of a Curve Defined by a Polar Function

Let $f$ be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The arc length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \tag{9.4.3}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{9.4.4}$$

## Example 9.4.3: Finding the Arc Length of a cardioid

Find the arc length of the cardioid $r = 2 + 2\cos\theta$.

**Solution**

When $\theta = 0, r = 2 + 2\cos 0 = 4$. Furthermore, as $\theta$ goes from $0$ to $2\pi$, the cardioid is traced out exactly once. Therefore these are the limits of integration. Using $f(\theta) = 2 + 2\cos\theta, \alpha = 0$, and $\beta = 2\pi$, Equation 9.4.3 becomes

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{[2+2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{4 + 8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\ &= 2\int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta. \end{aligned}$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1$, add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2\alpha$. Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2)$, so the integral becomes

$$\begin{aligned} L &= 2\int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= 2\int_{0}^{2\pi} \sqrt{4\cos^2(\frac{\theta}{2})}\, d\theta \\ &= 4\int_{0}^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta. \end{aligned}$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from $0$ to $\pi$ and double the answer. This strategy works because cosine is positive between $0$ and $\frac{\pi}{2}$. Thus,

$$\begin{aligned} L &= 4\int_{0}^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta \\ &= 8\int_{0}^{\pi} \cos(\frac{\theta}{2})\, d\theta \\ &= 8(2\sin(\frac{\theta}{2}))\,|_{0}^{\pi} \\ &= 16 \end{aligned}$$

### Exercise 9.4.3

Find the total arc length of the graph of $r = 3\sin\theta$.

*Hint available at the end of this section.*

*Answer available at the end of this section.*

## Key Concepts

- The area of a region in polar coordinates defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral $A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta$.
- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral $L = \int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}d\theta = \int_{\alpha}^{\beta}\sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta$.

# Key Equations

- **Area of a region bounded by a polar curve**

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

- **Arc length of a polar curve**

$$L = \int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}d\theta = \int_{\alpha}^{\beta}\sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta$$

## Answers

**Exercise** 9.4.1

Use Equation 9.4.2. Be sure to determine the correct limits of integration before evaluating.

**Exercise** 9.4.1

$A = 3\pi/2$

**Exercise** 9.4.2

Use Equation 9.4.2 and take advantage of symmetry.

**Exercise** 9.4.2

$A = \frac{4\pi}{3} + 2\sqrt{3}$

**Exercise** 9.4.3

Use Equation 9.4.3. To determine the correct limits, make a table of values.

**Exercise** 9.4.3

$s = 3\pi$

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# 9.4.1: Exercises for Section 11.4

**In exercises 1 -13, determine a definite integral that represents the area.**

1) Region enclosed by $r = 4$

2) Region enclosed by $r = 3\sin\theta$

**Answer**

$$\frac{9}{2}\int_0^{\pi} \sin^2\theta \; d\theta$$

3) Region in the first quadrant within the cardioid $r = 1 + \sin\theta$

4) Region enclosed by one petal of $r = 8\sin(2\theta)$

**Answer**

$$32\int_0^{\pi/2} \sin^2(2\theta) \; d\theta$$

5) Region enclosed by one petal of $r = cos(3\theta)$

6) Region below the polar axis and enclosed by $r = 1 - \sin\theta$

**Answer**

$$\frac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2 \; d\theta$$

7) Region in the first quadrant enclosed by $r = 2 - \cos\theta$

8) Region enclosed by the inner loop of $r = 2 - 3\sin\theta$

**Answer**

$$\int_{\sin^{-1}(2/3)}^{\pi/2} (2-3\sin\theta)^2 \; d\theta$$

9) Region enclosed by the inner loop of $r = 3 - 4\cos\theta$

10) Region enclosed by $r = 1 - 2\cos\theta$ and outside the inner loop

**Answer**

$$\int_0^{\pi} (1-2\cos\theta)^2 \; d\theta - \int_0^{\pi/3} (1-2\cos\theta)^2 \; d\theta$$

11) Region common to $r = 3\sin\theta$ and $r = 2 - \sin\theta$

12) Region common to $r = 2$ and $r = 4\cos\theta$

**Answer**

$$4\int_0^{\pi/3} \; d\theta + 16\int_{\pi/3}^{\pi/2} (\cos^2\theta) \; d\theta$$

13) Region common to $r = 3\cos\theta$ and $r = 3\sin\theta$

**In exercises 14 -26, find the area of the described region.**

14) Enclosed by $r = 6\sin\theta$

**Answer**

$9\pi \text{ units}^2$

15) Above the polar axis enclosed by $r = 2 + \sin\theta$

16) Below the polar axis and enclosed by $r = 2 - \cos\theta$

**Answer**

$\frac{9\pi}{4} \text{ units}^2$

17) Enclosed by one petal of $r = 4\cos(3\theta)$

18) Enclosed by one petal of $r = 3\cos(2\theta)$

**Answer**

$\frac{9\pi}{8} \text{ units}^2$

19) Enclosed by $r = 1 + \sin\theta$

20) Enclosed by the inner loop of $r = 3 + 6\cos\theta$

**Answer**

$\frac{18\pi - 27\sqrt{3}}{2} \text{ units}^2$

21) Enclosed by $r = 2 + 4\cos\theta$ and outside the inner loop

22) Common interior of $r = 4\sin(2\theta)$ and $r = 2$

**Answer**

$\frac{4}{3}(4\pi - 3\sqrt{3}) \text{ units}^2$

23) Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$

24) Common interior of $r = 6\sin\theta$ and $r = 3$

**Answer**

$\frac{3}{2}(4\pi - 3\sqrt{3}) \text{ units}^2$

25) Inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$

26) Common interior of $r = 2 + 2\cos\theta$ and $r = 2\sin\theta$

**Answer**

$(2\pi - 4) \text{ units}^2$

**In exercises 27 - 30, find a definite integral that represents the arc length.**

27) $r = 4\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

28) $r = 1 + \sin\theta$ on the interval $0 \le \theta \le 2\pi$

**Answer**

$$\int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\, d\theta$$

29) $r = 2\sec\theta$ on the interval $0 \le \theta \le \frac{\pi}{3}$

30) $r = e^{\theta}$ on the interval $0 \le \theta \le 1$

**Answer**

$$\sqrt{2}\int_0^1 e^\theta \, d\theta$$

**In exercises 31 - 35, find the length of the curve over the given interval.**

31) $r = 6$ on the interval $0 \le \theta \le \frac{\pi}{2}$

32) $r = e^{3\theta}$ on the interval $0 \le \theta \le 2$

**Answer**

$\frac{\sqrt{10}}{3}(e^6 - 1)$ units

33) $r = 6\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

34) $r = 8 + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

**Answer**

$32$ units

35) $r = 1 - \sin\theta$ on the interval $0 \le \theta \le 2\pi$

**In exercises 36 - 40, use the integration capabilities of a calculator to approximate the length of the curve.**

36) [T] $r = 3\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

**Answer**

$6.238$ units

37) [T] $r = \dfrac{2}{\theta}$ on the interval $\pi \le \theta \le 2\pi$

38) [T] $r = \sin^2\left(\frac{\theta}{2}\right)$ on the interval $0 \le \theta \le \pi$

**Answer**

$2$ units

39) [T] $r = 2\theta^2$ on the interval $0 \le \theta \le \pi$

40) [T] $r = \sin(3\cos\theta)$ on the interval $0 \le \theta \le \pi$

**Answer**

$4.39$ units

**In exercises 41 - 43, use the familiar formula from geometry to find the area of the region described and then confirm by using the definite integral.**

41) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

42) $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

**Answer**

$$A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2} \text{ units}^2 \text{ and } \frac{1}{2}\int_0^\pi (1 + 2\sin\theta\cos\theta) \, d\theta = \frac{\pi}{2} \text{ units}^2$$

43) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

**In exercises 44 - 46, use the familiar formula from geometry to find the length of the curve and then confirm using the definite integral.**

44) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

**Answer**

$C = 2\pi\left(\frac{3}{2}\right) = 3\pi$ units and $\int_0^{\pi} 3 \; d\theta = 3\pi$ units

45) $r = \sin\theta + \cos\theta$ on the interval $0 \leq \theta \leq \pi$

46) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \leq \theta \leq \pi$

**Answer**

$C = 2\pi(5) = 10\pi$ units and $\int_0^{\pi} 10 \; d\theta = 10\pi$ units

47) Verify that if $y = r\sin\theta = f(\theta)\sin\theta$ then $\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$.

**In exercises 48 - 56, find the slope of a tangent line to a polar curve $r = f(\theta)$. Let $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so the polar equation $r = f(\theta)$ is now written in parametric form.**

48) Use the definition of the derivative $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ and the product rule to derive the derivative of a polar equation.

**Answer**

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

49) $r = 1 - \sin\theta; \;\; \left(\frac{1}{2}, \frac{\pi}{6}\right)$

50) $r = 4\cos\theta; \;\; \left(2, \frac{\pi}{3}\right)$

**Answer**

The slope is $\frac{1}{\sqrt{3}}$.

51) $r = 8\sin\theta; \;\; \left(4, \frac{5\pi}{6}\right)$

52) $r = 4 + \sin\theta; \;\; \left(3, \frac{3\pi}{2}\right)$

**Answer**

The slope is 0.

53) $r = 6 + 3\cos\theta; \;\; (3, \pi)$

54) $r = 4\cos(2\theta)$; tips of the leaves

**Answer**

At $(4, 0)$, the slope is undefined. At $\left(-4, \frac{\pi}{2}\right)$, the slope is 0.

55) $r = 2\sin(3\theta)$; tips of the leaves

56) $r = 2\theta; \;\; \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

**Answer**

The slope is undefined at $\theta = \frac{\pi}{4}$.

57) Find the points on the interval $-\pi \leq \theta \leq \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.

58) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

**Answer**

Slope = –1.

**In exercises 59 - 62, find the slope of the tangent line to the given polar curve at the point given by the value of $\theta$.**

59) $r = 3\cos\theta, \quad \theta = \frac{\pi}{3}$

60) $r = \theta, \quad \theta = \frac{\pi}{2}$

**Answer**

Slope is $\frac{-2}{\pi}$.

61) $r = \ln\theta, \quad \theta = e$

62) [T] Use technology: $r = 2 + 4\cos\theta$ at $\theta = \frac{\pi}{6}$

**Answer**

Calculator answer: –0.836.

**In exercises 63 - 66, find the points at which the following polar curves have a horizontal or vertical tangent line.**

63) $r = 4\cos\theta$

64) $r^2 = 4\cos(2\theta)$

**Answer**

Horizontal tangent at $\left(\pm\sqrt{2}, \frac{\pi}{6}\right), \quad \left(\pm\sqrt{2}, -\frac{\pi}{6}\right)$.

65) $r = 2\sin(2\theta)$

66) The cardioid $r = 1 + \sin\theta$

**Answer**

Horizontal tangents at $\frac{\pi}{2}, \quad \frac{7\pi}{6}, \quad \frac{11\pi}{6}$.
Vertical tangents at $\frac{\pi}{6}, \quad \frac{5\pi}{6}$ and also at the pole $(0, 0)$.

67) Show that the curve $r = \sin\theta\tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote.

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# 9.6: 6 Chapter Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) The rectangular coordinates of the point $\left(4, \frac{5\pi}{6}\right)$ are $\left(2\sqrt{3}, -2\right)$.

2) The equations $x = \cosh(3t), \;\; y = 2\sinh(3t)$ represent a hyperbola.

**Answer**

True

3) The arc length of the spiral given by $r = \dfrac{\theta}{2}$ for $0 \le \theta \le 3\pi$ is $\frac{9}{4}\pi^3$ units.

4) Given $x = f(t)$ and $y = g(t)$, if $\dfrac{dx}{dy} = \dfrac{dy}{dx}$, then $f(t) = g(t) + C$, where $C$ is a constant.

**Answer**

False. Imagine $y = t + 1, \;\; x = -t + 1.$

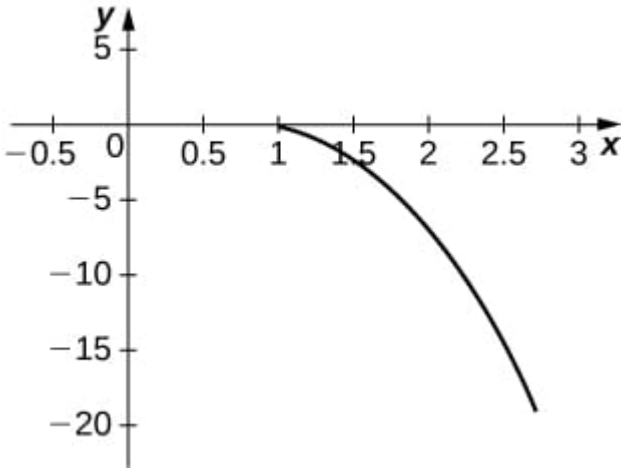
**In exercises 5 -8, sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.**

5) $x = 1 + t, \;\; y = t^2 - 1, \quad -1 \le t \le 1$

6) $x = e^t, \;\; y = 1 - e^{3t}, \quad 0 \le t \le 1$

**Answer**

$y = 1 - x^3$

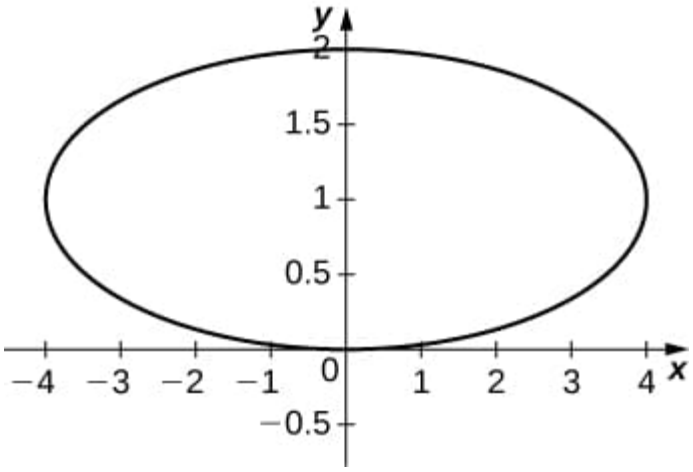



7) $x = \sin\theta, \;\; y = 1 - \csc\theta, \quad 0 \le \theta \le 2\pi$

8) $x = 4\cos\phi, \;\; y = 1 - \sin\phi, \quad 0 \le \phi \le 2\pi$

**Answer**

$\dfrac{x^2}{16} + (y-1)^2 = 1$



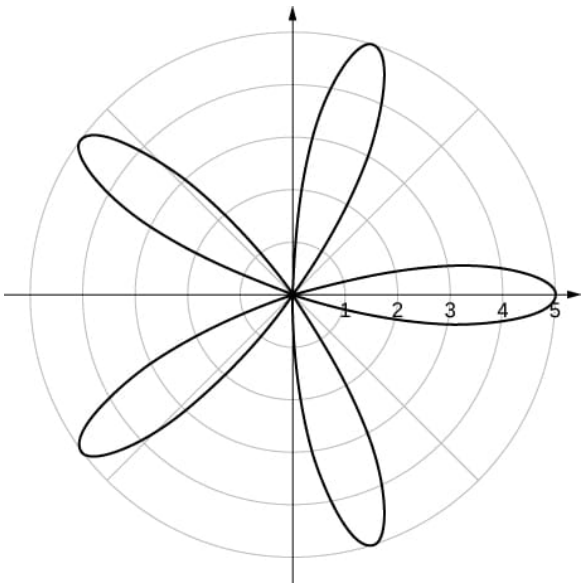

**In exercises 9 - 10, sketch the polar curve and determine what type of symmetry exists, if any.**

9) $r = 4\sin\left(\frac{\theta}{3}\right)$

10) $r = 5\cos(5\theta)$

**Answer**

Symmetric about polar axis




**In exercises 11 - 12, find the polar equation for the curve given as a Cartesian equation.**

11) $x + y = 5$

12) $y^2 = 4 + x^2$

**Answer**

$$r^2 = \frac{4}{\sin^2\theta - \cos^2\theta}$$

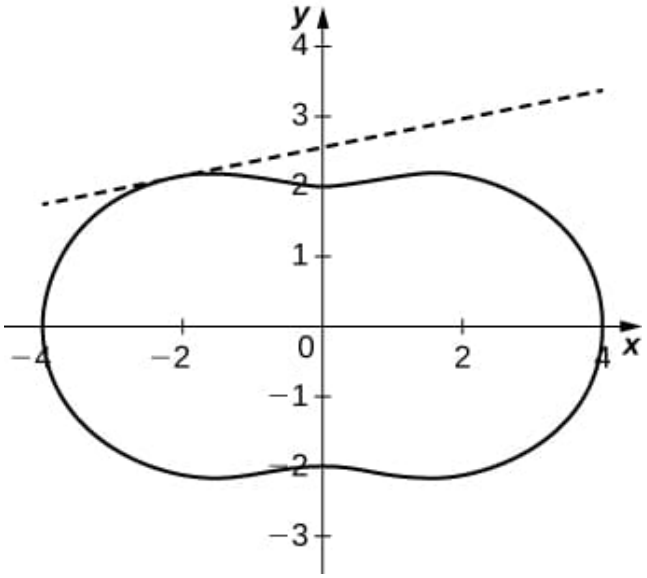
**In exercises 13 - 14, find the equation of the tangent line to the given curve. Graph both the function and its tangent line.**

13) $x = \ln(t),\ \ y = t^2 - 1,\ \ t = 1$

14) $r = 3 + \cos(2\theta),\ \ \theta = \frac{3\pi}{4}$

**Answer**

$$y = \frac{3\sqrt{2}}{2} + \frac{1}{5}\left(x + \frac{3\sqrt{2}}{2}\right)$$




15) Find $\frac{dy}{dx}$, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ of $y = (2 + e^{-t}),\ \ x = 1 - \sin t$

**In exercises 16 -17, find the area of the region.**

16) $x = t^2,\ \ y = \ln(t),\quad 0 \le t \le e$

**Answer**

$$\frac{e^2}{2} \text{ units}^2$$

17) $r = 1 - \sin\theta$ in the first quadrant

**In exercises 18 - 19, find the arc length of the curve over the given interval.**

18) $x = 3t + 4,\ \ y = 9t - 2,\quad 0 \le t \le 3$

**Answer**

$9\sqrt{10}$ units

19) $r = 6\cos\theta,\quad 0 \le \theta \le 2\pi$. Check your answer by geometry.

**In exercises 20 - 22, find the Cartesian equation describing the given shapes.**

20) A parabola with focus $(2, -5)$ and directrix $x = 6$

**Answer**

$$(y + 5)^2 = -8x + 32$$

21) An ellipse with a major axis length of 10 and foci at $(-7, 2)$ and $(1, 2)$

22) A hyperbola with vertices at $(3, -2)$ and $(-5, -2)$ and foci at $(-2, -6)$ and $(-2, 4)$

**Answer**

$$\frac{(y+1)^2}{16} - \frac{(x+2)^2}{9} = 1$$

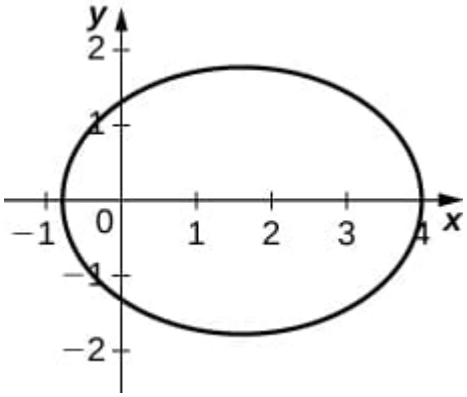
**In exercises 23 - 25, determine the eccentricity and identify the conic. Sketch the conic.**

23) $r = \dfrac{6}{1+3\cos\theta}$

24) $r = \dfrac{4}{3-2\cos\theta}$

**Answer**

$e = \frac{2}{3}$, ellipse




25) $r = \dfrac{7}{5-5\cos\theta}$

26) Determine the Cartesian equation describing the orbit of Pluto, the most eccentric orbit around the Sun. The length of the major axis is 39.26 AU and minor axis is 38.07 AU. What is the eccentricity?

**Answer**

$$\frac{y^2}{19.03^2} + \frac{x^2}{19.63^2} = 1, \quad e = 0.2447$$

27) The C/1980 E1 comet was observed in 1980. Given an eccentricity of $1.057$ and a perihelion (point of closest approach to the Sun) of $3.364$ AU, find the Cartesian equations describing the comet's trajectory. Are we guaranteed to see this comet again? (Hint: Consider the Sun at point $(0,0)$.)

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# APPENDICES

1: A- Short Table of Integrals
2: B- Answers to Activities
3: C- Answers to Selected Exercises
4: Table of Derivatives
5: Table of Integrals

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# 1: A- Short Table of Integrals

Appendix A A Short Table of Integrals

a.

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C \tag{1.1}$$

b.

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C \tag{1.2}$$

c.

$$\int \sqrt{u^2 \pm a^2}du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|u + \sqrt{u^2 \pm a^2}\right| + C \tag{1.3}$$

d.

$$\int \frac{u^2 du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2}\sqrt{u^2 \pm a^2} \mp \frac{a^2}{2}\ln\left|u + \sqrt{u^2 \pm a^2}\right| + C \tag{1.4}$$

e.

$$\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C \tag{1.5}$$

f.

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{u}{a}\right) + C \tag{1.6}$$

g.

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C \tag{1.7}$$

h.

$$\int \sqrt{a^2-u^2}du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right) + C \tag{1.8}$$

i.

$$\int \frac{u^2}{\sqrt{a^2-u^2}}du = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right) + C \tag{1.9}$$

j.

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C \tag{1.10}$$

k.

$$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2 u} + C \tag{1.11}$$

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# 2: B- Answers to Activities

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# 3: C- Answers to Selected Exercises

## Solutions C Answers to Selected Exercises

This appendix contains answers to all non-WeBWorK exercises in the text. For WeBWorK exercises, please use the HTML version of the text for access to answers and solutions.

### Chapter 1 Understanding the Derivative

#### Section 1.1 How do we measure velocity?

## Exercises

Answer

Answer

Answer

#### Section 1.2 The notion of limit

## Exercises

Answer

Answer

Answer

Answer

#### Section 1.3 The derivative of a function at a point

## Exercises

Answer

Answer

Answer

Answer

#### Section 1.4 The derivative function

## Exercises

Answer

Answer

Answer

Answer

### Section 1.5 Interpreting, estimating, and using the derivative

## Exercises

Answer

Answer

Answer

Answer

### Section 1.6 The second derivative

## Exercises

Answer

Answer

Answer

Answer

### Section 1.7 Limits, Continuity, and Differentiability

## Exercises

Answer

Answer

Answer

Answer

### Section 1.8 The Tangent Line Approximation

## Exercises

Answer

Answer

Answer

Answer

# Chapter 2 Computing Derivatives

## Section 2.1 Elementary derivative rules

### Exercises

Answer

Answer

Answer

Answer

## Section 2.2 The sine and cosine functions

### Exercises

Answer

Answer

Answer

## Section 2.3 The product and quotient rules

### Exercises

Answer

Answer

Answer

Answer

Answer

## Section 2.4 Derivatives of other trigonometric functions

### Exercises

Answer

Answer

Answer

## Section 2.5 The chain rule

### Exercises

Answer

Answer

Answer

Answer

### Section 2.6 Derivatives of Inverse Functions

## Exercises

Answer

Answer

Answer

Answer

### Section 2.7 Derivatives of Functions Given Implicitly

## Exercises

Answer

Answer

Answer

### Section 2.8 Using Derivatives to Evaluate Limits

## Exercises

Answer

Answer

Answer

Answer

## Chapter 3 Using Derivatives

### Section 3.1 Using derivatives to identify extreme values

## Exercises

Answer

Answer

Answer

Answer

### Section 3.2 Using derivatives to describe families of functions

## Exercises

Answer

Answer

Answer

### Section 3.3 Global Optimization

## Exercises

Answer

Answer

Answer

Answer

### Section 3.4 Applied Optimization

## Exercises

Answer

Answer

Answer

Answer

### Section 3.5 Related Rates

## Exercises

Answer

Answer

Answer

Answer

## Chapter 4 The Definite Integral

### Section 4.1 Determining distance traveled from velocity

## Exercises

Answer

Answer

Answer

Answer

### Section 4.2 Riemann Sums

## Exercises

Answer

Answer

Answer

Answer

### Section 4.3 The Definite Integral

## Exercises

Answer

Answer

Answer

Answer

### Section 4.4 The Fundamental Theorem of Calculus

## Exercises

Answer

Answer

Answer

Answer

Answer

## Chapter 5 Evaluating Integrals

### Section 5.1 Constructing Accurate Graphs of Antiderivatives

## Exercises

Answer

Answer

Answer

### Section 5.2 The Second Fundamental Theorem of Calculus

## Exercises

Answer

Answer

Answer

### Section 5.3 Integration by Substitution

## Exercises

Answer

Answer

Answer

Answer

### Section 5.4 Integration by Parts

## Exercises

Answer

Answer

Answer

### Section 5.5 Other Options for Finding Algebraic Antiderivatives

## Exercises

Answer

Answer

Answer

### Section 5.6 Numerical Integration

## Exercises

Answer

Answer

Answer

# Chapter 6 Using Definite Integrals

## Section 6.1 Using Definite Integrals to Find Area and Length

## Exercises

Answer

Answer

Answer

## Section 6.2 Using Definite Integrals to Find Volume

## Exercises

Answer

Answer

Answer

## Section 6.3 Density, Mass, and Center of Mass

## Exercises

Answer

Answer

Answer

## Section 6.4 Physics Applications: Work, Force, and Pressure

## Exercises

Answer

Answer

## Section 6.5 Improper Integrals

## Exercises

Answer

Answer

# Chapter 7 Differential Equations

## Section 7.1 An Introduction to Differential Equations

## Exercises

Answer

Answer

Answer

## Section 7.2 Qualitative behavior of solutions to DEs

## Exercises

Answer

Answer

Answer

Answer

## Section 7.3 Euler's method

## Exercises

Answer

Answer

Answer

## Section 7.4 Separable differential equations

## Exercises

Answer

Answer

Answer

Answer

## Section 7.5 Modeling with differential equations

## Exercises

Answer

Answer

Answer

Answer

### Section 7.6 Population Growth and the Logistic Equation

## Exercises

Answer

Answer

Answer

## Chapter 8 Sequences and Series

### Section 8.1 Sequences

## Exercises

Answer

Answer

Answer

Answer

### Section 8.2 Geometric Series

## Exercises

Answer

Answer

Answer

Answer

Answer

### Section 8.3 Series of Real Numbers

## Exercises

Answer

Answer

Answer

Answer

### Section 8.4 Alternating Series

## Exercises

Answer

Answer

Answer

### Section 8.5 Taylor Polynomials and Taylor Series

## Exercises

Answer

Answer

Answer

### Section 8.6 Power Series

## Exercises

Answer

Answer

Answer

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# 4: Table of Derivatives

## General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f\,'(x) + g\,'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f\,'(x)g(x) + f(x)g\,'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad$ for real numbers $n$
5. $\frac{d}{dx}(cf(x)) = cf\,'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f\,'(x) - g\,'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f\,'(g(x)) \cdot g\,'(x)$

## Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

## Inverse Trigonometric Functions

15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

18. $\dfrac{d}{dx}\left(\cos^{-1}x\right)=\dfrac{-1}{\sqrt{1-x^2}}$

19. $\dfrac{d}{dx}\left(\cot^{-1}x\right)=\dfrac{-1}{1+x^2}$

20. $\dfrac{d}{dx}\left(\csc^{-1}x\right)=\dfrac{-1}{|x|\sqrt{x^2-1}}$

## Exponential and Logarithmic Functions

21. $\dfrac{d}{dx}\left(e^x\right)=e^x$

22. $\dfrac{d}{dx}\left(\ln|x|\right)=\dfrac{1}{x}$

23. $\dfrac{d}{dx}\left(b^x\right)=b^x\ln b$

24. $\dfrac{d}{dx}\left(\log_b x\right)=\dfrac{1}{x\ln b}$

## Hyperbolic Functions

25. $\dfrac{d}{dx}\left(\sinh x\right)=\cosh x$

26. $\dfrac{d}{dx}\left(\tanh x\right)=\operatorname{sech}^2 x$

27. $\dfrac{d}{dx}\left(\operatorname{sech}x\right)=-\operatorname{sech}\ x\tanh x$

28. $\dfrac{d}{dx}\left(\cosh x\right)=\sinh x$

29. $\dfrac{d}{dx}\left(\coth x\right)=-\operatorname{csch}^2 x$

30. $\dfrac{d}{dx}\left(\operatorname{csch}x\right)=-\operatorname{csch}x\coth x$

## Inverse Hyperbolic Functions

31. $\dfrac{d}{dx}\left(\sinh^{-1}x\right)=\dfrac{1}{\sqrt{x^2+1}}$

32. $\dfrac{d}{dx}\left(\tanh^{-1}x\right)=\dfrac{1}{1-x^2}\quad(|x|<1)$

33. \(\quad \dfrac{d}{dx}\left(\text{sech}^{-1}\,x\right)=\dfrac{-1}{x\sqrt{1-x^2}}\quad (0

34. $\dfrac{d}{dx}\left(\cosh^{-1}x\right)=\dfrac{1}{\sqrt{x^2-1}}\quad(x>1)$

35. $\dfrac{d}{dx}\left(\coth^{-1}x\right)=\dfrac{1}{1-x^2}\quad(|x|>1)$

36. $\dfrac{d}{dx}\left(\operatorname{csch}^{-1}x\right)=\dfrac{-1}{|x|\sqrt{1+x^2}}\quad(x\neq 0)$

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# 5: Table of Integrals

## Basic Integrals

1. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln |u| + C$
3. $\int e^u \, du = e^u + C$
4. $\int a^u \, du = \frac{a^u}{\ln a} + C$
5. $\int \sin u \, du = -\cos u + C$
6. $\int \cos u \, du = \sin u + C$
7. $\int \sec^2 u \, du = \tan u + C$
8. $\int \csc^2 u \, du = -\cot u + C$
9. $\int \sec u \tan u \, du = \sec u + C$
10. $\int \csc u \cot u \, du = -\csc u + C$
11. $\int \tan u \, du = \ln |\sec u| + C$
12. $\int \cot u \, du = \ln |\sin u| + C$
13. $\int \sec u \, du = \ln |\sec u + \tan u| + C$
14. $\int \csc u \, du = \ln |\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

## Trigonometric Integrals

18. $\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

19. $\int \cos^2 u \ du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

20. $\int \tan^2 u \ du = \tan u - u + C$

21. $\int \cot^2 u \ du = -\cot u - u + C$

22. $\int \sin^3 u \ du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

23. $\int \cos^3 u \ du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

24. $\int \tan^3 u \ du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u \ du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u \ du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u \ du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u \ du = \frac{-1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u \ du$

29. $\int \cos^n u \ du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u \ du$

30. $\int \tan^n u \ du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u \ du$

31. $\int \cot^n u \ du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u \ du$

32. $\int \sec^n u \ du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u \ du$

33. $\int \csc^n u \ du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u \ du$

34. $\int \sin au \sin bu \ du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu \ du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu \ du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u \ du = \sin u - u \cos u + C$

38. $\int u \cos u \ du = \cos u + u \sin u + C$

39. $\int u^n \sin u \ du = -u^n \cos u + n\int u^{n-1} \cos u \ du$

40. $\int u^n \cos u \ du = u^n \sin u - n\int u^{n-1} \sin u \ du$

41. Erroneous nesting of equation structures

## Exponential and Logarithmic Integrals

42. $\int ue^{au}\ du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au}\ du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\ du$

44. $\int e^{au}\sin bu\ du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\ du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\ du = u\ln u - u + C$

47. $\int u^n \ln u\ du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u}\ du = \ln|\ln u| + C$

## Hyperbolic Integrals

49. $\int \sinh u\ du = \cosh u + C$

50. $\int \cosh u\ du = \sinh u + C$

51. $\int \tanh u\ du = \ln\cosh u + C$

52. $\int \coth u\ du = \ln|\sinh u| + C$

53. $\int \operatorname{sech}\ u\ du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch}\ u\ du = \ln|\tanh\frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u\ du = \tanh\ u + C$

56. $\int \operatorname{csch}^2 u\ du = -\coth\ u + C$

57. $\int \operatorname{sech}\ u\tanh u\ du = -\operatorname{sech}\ u + C$

58. $\int \operatorname{csch}\ u\coth u\ du = -\operatorname{csch}\ u + C$

## Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\ du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u \; du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u \; du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u\sin^{-1} u \; du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\cos^{-1} u \; du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1} u \; du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \cos^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \tan^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

## Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2} \; du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2} \; du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u} \; du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2} \; du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}} \; du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

## Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2-a^2} \; du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $$\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

79. $$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$$

80. $$\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

81. $$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

82. $$\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

83. $$\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$$

84. $$\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$$

## Integrals Involving $a^2 - u^2$, $a > 0$

85. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

86. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

87. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

88. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$$

89. $$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\sin^{-1}\frac{u}{a}\right) + C$$

90. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

91. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$$

92. $$\int \left(a^2-u^2\right)^{3/2}\,du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$

93. $$\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$$

## Integrals Involving $2au - u^2$, $a > 0$

94. $$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

95. $$\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$$

96. $$\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

97. $$\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$$

## Integrals Involving $a + bu$, $a \neq 0$

98. $$\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$$

99. $$\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

100. $$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

101. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

102. $$\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

103. $$\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

104. $$\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right) + C$$

105. $$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$$

106. $$\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$$

107. $$\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$$

108. $$\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & \text{if } a > 0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, & \text{if } a < 0 \end{cases}$$

109. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$$

110. $$\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$$

111. $$\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

112. $$\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$$

113. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

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# GLOSSARY

## A

**absolute convergence** | if the series $\sum_{n=1}^{\infty} |a_n|$ converges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely

**absolute error** | if $B$ is an estimate of some quantity having an actual value of $A$, then the absolute error is given by $|A - B|$

**absolute extremum** | if $f$ has an absolute maximum or absolute minimum at $c$, we say $f$ has an absolute extremum at $c$

**absolute maximum** | if $f(c) \geq f(x)$ for all $x$ in the domain of $f$, we say $f$ has an absolute maximum at $c$

**absolute minimum** | if $f(c) \leq f(x)$ for all $x$ in the domain of $f$, we say $f$ has an absolute minimum at $c$

**absolute value function** | $f(x) = \{-x, \quad \text{if } x < 0 \backslash x, \quad \text{if } x \geq 0$

**acceleration** | is the rate of change of the velocity, that is, the derivative of velocity

**acceleration vector** | the second derivative of the position vector

**algebraic function** | a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable $x$

**alternating series** | a series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n \geq 0$, is called an alternating series

**alternating series test** | for an alternating series of either form, if $b_{n+1} \leq b_n$ for all integers $n \geq 1$ and $b_n \to 0$, then an alternating series converges

**amount of change** | the amount of a function $f(x)$ over an interval $[x, x + h] is f(x + h) - f(x)$

**angular coordinate** | $\theta$ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial (x) axis, measured counterclockwise

**antiderivative** | a function $F$ such that $F'(x) = f(x)$ for all $x$ in the domain of $f$ is an antiderivative of $f$

**arc length** | the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

**arc-length function** | a function $s(t)$ that describes the arc length of curve $C$ as a function of $t$

**arc-length parameterization** | a reparameterization of a vector-valued function in which the parameter is equal to the arc length

**arithmetic sequence** | a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

**asymptotically semi-stable solution** | $y = k$ if it is neither asymptotically stable nor asymptotically unstable

**asymptotically stable solution** | $y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y), y(x_0) = c$ approaches $k$ as $x$ approaches infinity

**asymptotically unstable solution** | $y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y), y(x_0) = c$ never approaches $k$ as $x$approaches infinity

**autonomous differential equation** | an equation in which the right-hand side is a function of $y$ alone

**average rate of change** | is a function $f(x)$ over an interval $[x, x + h]$ is $\frac{f(x+h)-f(a)}{b-a}$

**average value of a function** | (or $f_{ave}$) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

**average velocity** | the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval [$t, a$] (if \(ta\)), with a position given by $s(t)$, that is $v_{ave} = \dfrac{s(t) - s(a)}{t - a}$

## B

**base** | the number $b$ in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

**binomial series** | the Maclaurin series for $f(x) = (1 + x)^r$; it is given by $(1 + x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \dfrac{r(r-1)}{2!} x^2 + \cdots + \dfrac{r(r-1)\cdots(r-n+1)}{n!} x^n + \cdots$ for $|x| < 1$

**binormal vector** | a unit vector orthogonal to the unit tangent vector and the unit normal vector

**boundary conditions** | the conditions that give the state of a system at different times, such as the position of a spring-mass system at two different times

**boundary point** | a point $P_0$ of $R$ is a boundary point if every $\delta$ disk centered around $P_0$ contains points both inside and outside $R$

**boundary-value problem** | a differential equation with associated boundary conditions

**bounded above** | a sequence $a_n$ is bounded above if there exists a constant $M$ such that $a_n \leq M$ for all positive integers $n$

**bounded below** | a sequence $a_n$ is bounded below if there exists a constant $M$ such that $M \leq a_n$ for all positive integers $n$

**bounded sequence** | a sequence $a_n$ is bounded if there exists a constant $M$ such that $|a_n| \leq M$ for all positive integers $n$

## C

**cardioid** | a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

**carrying capacity** | the maximum population of an organism that the environment can sustain indefinitely

**catenary** | a curve in the shape of the function $y = a \cdot \cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

**center of mass** | the point at which the total mass of the system could be concentrated without changing the moment

**centroid** | the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

**chain rule** | the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

**change of variables** | the substitution of a variable, such as $u$, for an expression in the integrand

**characteristic equation** | the equation $a\lambda^2 + b\lambda + c = 0$ for the differential equation $ay''+by'+cy = 0$

**circulation** | the tendency of a fluid to move in the direction of curve $C$. If $C$ is a closed curve, then the circulation of $\vecs{F}$ along $C$ is line integral $\int_C \vecs{F} \cdot \vecs{T}\, ds$, which we also denote $\oint_C \vecs{F} \cdot \vecs{T}\, ds$.

**closed curve** | a curve for which there exists a parameterization $\vecs{r}(t), a \leq t \leq b$, such that $\vecs{r}(a) = \vecs{r}(b)$, and the curve is traversed exactly once

**closed curve** | a curve that begins and ends at the same point

**closed set** | a set $S$ that contains all its boundary points

**comparison test** | If $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \geq b_n \geq 0$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

**complementary equation** | for the nonhomogeneous linear differential equation

$$a + 2(x)y''+a_1(x)y'+a_0(x)y = r(x),$$

the associated homogeneous equation, called the *complementary equation*, is

$$a_2(x)y'' + a_1(x)y'+a_0(x)y = 0$$

**component** | a scalar that describes either the vertical or horizontal direction of a vector

**component functions** | the component functions of the vector-valued function $\vecs{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ are $f(t)$ and $g(t)$, and the component functions of the vector-valued function $\vecs{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ are $f(t)$, $g(t)$ and $h(t)$

**composite function** | given two functions $f$ and $g$, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

**computer algebra system (CAS)** | technology used to perform many mathematical tasks, including integration

**concave down** | if $f$ is differentiable over an interval $I$ and $f'$ is decreasing over $I$, then $f$ is concave down over $I$

**concave up** | if $f$ is differentiable over an interval $I$ and $f'$ is increasing over $I$, then $f$ is concave up over $I$

**concavity** | the upward or downward curve of the graph of a function

**concavity test** | suppose $f$ is twice differentiable over an interval $I$; if $f'' > 0$ over $I$, then $f$ is concave up over $I$; if $f'' <$ over $I$, then $f$ is concave down over $I$

**conditional convergence** | if the series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally

**conic section** | a conic section is any curve formed by the intersection of a plane with a cone of two nappes

**connected region** | a region in which any two points can be connected by a path with a trace contained entirely inside the region

**connected set** | an open set $S$ that cannot be represented as the union of two or more disjoint, nonempty open subsets

**conservative field** | a vector field for which there exists a scalar function $f$ such that $\vecs\nabla f = \vecs F$

**constant multiple law for limits** | the limit law

$$\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x) = cL$$

**constant multiple rule** | the derivative of a constant $c$ multiplied by a function $f$ is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big) = cf\,'(x)$

**constant rule** | the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where $c$ is a constant

**constraint** | an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

**continuity at a point** | A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x\to a} f(x)$ exists, and (3) $\lim x \to a f(x) = f(a)$

**continuity from the left** | A function is continuous from the left at b if $\lim_{x\to b^-} f(x) = f(b)$

**continuity from the right** | A function is continuous from the right at a if $\lim_{x\to a^+} f(x) = f(a)$

**continuity over an interval** | a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in $(a, b)$, and it is continuous from the right at $a$ and from the left at $b$

**contour map** | a plot of the various level curves of a given function $f(x, y)$

**convergence of a series** | a series converges if the sequence of partial sums for that series converges

**convergent sequence** | a convergent sequence is a sequence $a_n$ for which there exists a real number $L$ such that $a_n$ is arbitrarily close to $L$ as long as $n$ is sufficiently large

**coordinate plane** | a plane containing two of the three coordinate axes in the three-dimensional coordinate system, named by the axes it contains: the $xy$-plane, $xz$-plane, or the $yz$-plane

**critical point** | if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of $f$

**critical point of a function of two variables** | the point $(x_0, y_0)$ is called a critical point of $f(x, y)$ if one of the two following conditions holds: 1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ 2. At least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist

**cross product** | $\vecs u \times \vecs v = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$, where $\vecs u = \langle u_1, u_2, u_3\rangle$ and $\vecs v = \langle v_1, v_2, v_3\rangle$ determinant a real number associated with a square matrix parallelepiped a three-dimensional prism with six faces that are parallelograms torque the effect of a force that causes an object to rotate triple scalar product the dot product of a vector with the cross product of two other vectors: $\vecs u \cdot (\vecs v \times \vecs w)$ vector product the cross product of two vectors.

**cross-section** | the intersection of a plane and a solid object

**cubic function** | a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

**curl** | the curl of vector field $\vecs F = \langle P, Q, R\rangle$, denoted $\vecs\nabla \times \vecs F$ is the "determinant" of the matrix

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

and is given by the expression $(R_y - Q_z)\ \hat{\mathbf{i}} + (P_z - R_x)\ \hat{\mathbf{j}} + (Q_x - P_y)\ \hat{\mathbf{k}}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

**curvature** | the derivative of the unit tangent vector with respect to the arc-length parameter

**cusp** | a pointed end or part where two curves meet

**cycloid** | the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

**cylinder** | a set of lines parallel to a given line passing through a given curve

**cylindrical coordinate system** | a way to describe a location in space with an ordered triple $(r, \theta, z)$, where $(r, \theta)$ represents the polar coordinates of the point's projection in the $xy$-plane, and $z$ represents the point's projection onto the $z$-axis

## D

**decreasing on the interval** $I$ | a function decreasing on the interval $I$ if, for all $x_1,\ x_2 \in I,\ f(x_1) \geq f(x_2)$ if \(x_1

**definite integral** | a primary operation of calculus; the area between the curve and the $x$-axis over a given interval is a definite integral

**definite integral of a vector-valued function** | the vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function

**degree** | for a polynomial function, the value of the largest exponent of any term

**density function** | a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

**dependent variable** | the output variable for a function

**derivative** | the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

**derivative function** | gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

**derivative of a vector-valued function** | the derivative of a vector-valued function $\vecs r(t)$ is $\vecs r\,'(t) = \lim_{\Delta t \to 0} \frac{\vecs r(t+\Delta t) - \vecs r(t)}{\Delta t}$, provided the limit exists

**difference law for limits** | the limit law

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

**difference quotient** | of a function $f(x)$ at $a$ is given by $\dfrac{f(a+h) - f(a)}{h}$ or $\dfrac{f(x) - f(a)}{x - a}$

**difference rule** | the derivative of the difference of a function $f$ and a function $g$ is the same as the difference of the derivative of $f$ and the derivative of $g$: $\dfrac{d}{dx}\big(f(x) - g(x)\big) = f\,'(x) - g\,'(x)$

**differentiable** | a function $f(x, y)$ is differentiable at $(x_0, y_0)$ if $f(x, y)$ can be expressed in the form $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y)$, where the error term $E(x, y)$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \dfrac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$

**differentiable at** $a$ | a function for which $f'(a)$ exists is differentiable at $a$

**differentiable function** | a function for which $f'(x)$ exists is a differentiable function

**differentiable on** $S$ | a function for which $f'(x)$ exists for each $x$ in the open set $S$ is differentiable on $S$

**differential** | the differential $dx$ is an independent variable that can be assigned any nonzero real number; the differential $dy$ is defined to be $dy = f'(x)\ dx$

**differential calculus** | the field of calculus concerned with the study of derivatives and their applications

**differential equation** | an equation involving a function $y = y(x)$ and one or more of its derivatives

**differential form** | given a differentiable function $y = f'(x)$, the equation $dy = f'(x)\ dx$ is the differential form of the derivative of $y$ with respect to $x$

**differentiation** | the process of taking a derivative

**direction angles** | the angles formed by a nonzero vector and the coordinate axes

**direction cosines** | the cosines of the angles formed by a nonzero vector and the coordinate axes

**direction field (slope field)** | a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

**direction vector** | a vector parallel to a line that is used to describe the direction, or orientation, of the line in space

**directional derivative** | the derivative of a function in the direction of a given unit vector

**directrix** | a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

**discontinuity at a point** | A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

**discriminant** | the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving $xy$, is called a discriminant

**discriminant** | the discriminant of the function $f(x, y)$ is given by the formula $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$

**disk method** | a special case of the slicing method used with solids of revolution when the slices are disks

**divergence** | the divergence of a vector field $\vecs F = \langle P, Q, R\rangle$, denoted $\vecs\nabla \times \vecs F$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

**divergence of a series** | a series diverges if the sequence of partial sums for that series diverges

**divergence test** | if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges

**divergent sequence** | a sequence that is not convergent is divergent

**domain** | the set of inputs for a function

**dot product or scalar product** | $\vecs u \cdot \vecs v = u_1 v_1 + u_2 v_2 + u_3 v_3$ where $\vecs u = \langle u_1, u_2, u_3\rangle$ and $\vecs v = \langle v_1, v_2, v_3\rangle$

**double integral** | of the function $f(x,y)$ over the region $R$ in the $xy$-plane is defined as the limit of a double Riemann sum, $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

**double Riemann sum** | of the function $f(x,y)$ over a rectangular region $R$ is $$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$ where $R$ is divided into smaller subrectangles $R_{ij}$ and $(x_{ij}^*, y_{ij}^*)$ is an arbitrary point in $R_{ij}$

**doubling time** | if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

## E

**eccentricity** | the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

**ellipsoid** | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; all traces of this surface are ellipses

**elliptic cone** | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$; traces of this surface include ellipses and intersecting lines

**elliptic paraboloid** | a three-dimensional surface described by an equation of the form $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$; traces of this surface include ellipses and parabolas

**end behavior** | the behavior of a function as $x\to\infty$ and $x\to-\infty$

**epsilon-delta definition of the limit** | $\lim_{x\to a} f(x)=L$ if for every $\varepsilon>0$, there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$

**equilibrium solution** | any solution to the differential equation of the form $y=c$, where $c$ is a constant

**equivalent vectors** | vectors that have the same magnitude and the same direction

**Euler's Method** | a numerical technique used to approximate solutions to an initial-value problem

**even function** | a function is even if $f(-x)=f(x)$ for all $x$ in the domain of $f$

**explicit formula** | a sequence may be defined by an explicit formula such that $a_n=f(n)$

**exponent** | the value $x$ in the expression $b^x$

**exponential decay** | systems that exhibit exponential decay follow a model of the form $y=y_0e^{-kt}$

**exponential growth** | systems that exhibit exponential growth follow a model of the form $y=y_0e^{kt}$

**extreme value theorem** | if $f$ is a continuous function over a finite, closed interval, then $f$ has an absolute maximum and an absolute minimum

## F

**Fermat's theorem** | if $f$ has a local extremum at $c$, then $c$ is a critical point of $f$

**first derivative test** | let $f$ be a continuous function over an interval $I$ containing a critical point $c$ such that $f$ is differentiable over $I$ except possibly at $c$; if $f'$ changes sign from positive to negative as $x$ increases through $c$, then $f$ has a local maximum at $c$; if $f'$ changes sign from negative to positive as $x$ increases through $c$, then $f$ has a local minimum at $c$; if $f'$ does not change sign as $x$ increases through $c$, then $f$ does not have a local extremum at $c$

**flux** | the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\vecs F$ across plane curve $C$ is line integral $\int_C \vecs F \cdot \frac{\vecs n(t)}{\|\vecs n(t)\|}\,ds$

**flux integral** | another name for a surface integral of a vector field; the preferred term in physics and engineering

**focal parameter** | the focal parameter is the distance from a focus of a conic section to the nearest directrix

**focus** | a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

**formal definition of an infinite limit** | $\lim_{x\to a} f(x)=\infty$ if for every $M>0$, there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $f(x)>M$ $\lim_{x\to a} f(x)=-\infty$ if for every $M>0$, there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $f(x)<-M$

**Frenet frame of reference** | (TNB frame) a frame of reference in three-dimensional space formed by the unit tangent vector, the unit normal vector, and the binormal vector

**frustum** | a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

**Fubini's theorem** | if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$, then the double integral of $f$ over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

**function** | a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

**function of two variables** | a function $z = f(x,y)$ that maps each ordered pair $(x,y)$ in a subset $D$ of $R^2$ to a unique real number $z$

**Fundamental Theorem for Line Integrals** | the value of line integral $\int_C \vecs{\nabla} f \cdot d\vecs{r}$ depends only on the value of $f$ at the endpoints of $C$ : $\int_C \vecs{\nabla} f \cdot d\vecs{r} = f(\vecs{r}(b)) - f(\vecs{r}(a))$

**fundamental theorem of calculus** | (also, evaluation theorem) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

**fundamental theorem of calculus** | uses a definite integral to define an antiderivative of a function

**fundamental theorem of calculus** | the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

## G

**general form** | an equation of a conic section written as a general second-degree equation

**general form of the equation of a plane** | an equation in the form $ax + by + cz + d = 0$, where $\vecs{n} = \langle a,b,c\rangle$ is a normal vector of the plane, $P = (x_0, y_0, z_0)$ is a point on the plane, and $d = -ax_0 - by_0 - cz_0$

**general solution (or family of solutions)** | the entire set of solutions to a given differential equation

**generalized chain rule** | the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

**geometric sequence** | a sequence $a_n$ in which the ratio $a_{n+1}/a_n$ is the same for all positive integers $n$ is called a geometric sequence

**geometric series** | a geometric series is a series that can be written in the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

**gradient** | the gradient of the function $f(x,y)$ is defined to be $\vecs{\nabla} f(x,y) = (\partial f/\partial x)\ \hat{\mathbf{i}} + (\partial f/\partial y)\ \hat{\mathbf{j}}$, which can be generalized to a function of any number of independent variables

**gradient field** | a vector field $\vecs{F}$ for which there exists a scalar function $f$ such that $\vecs{\nabla} f = \vecs{F}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

**graph of a function** | the set of points $(x,y)$ such that $x$ is in the domain of $f$ and $y = f(x)$

**graph of a function of two variables** | a set of ordered triples $(x,y,z)$ that satisfies the equation $z = f(x,y)$ plotted in three-dimensional Cartesian space

**Green's theorem** | relates the integral over a connected region to an integral over the boundary of the region

**grid curves** | curves on a surface that are parallel to grid lines in a coordinate plane

**growth rate** | the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

## H

**half-life** | if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

**harmonic series** | the harmonic series takes the form $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

**heat flow** | a vector field proportional to the negative temperature gradient in an object

**helix** | a three-dimensional curve in the shape of a spiral

**higher-order derivative** | a derivative of a derivative, from the second derivative to the $n^{\text{th}}$ derivative, is called a higher-order derivative

**higher-order partial derivatives** | second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

**homogeneous linear equation** | a second-order differential equation that can be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$, but $r(x) = 0$ for every value of $x$

**Hooke's law** | this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx$, where $k$ is a constant

**horizontal asymptote** | if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of $f$

**horizontal line test** | a function $f$ is one-to-one if and only if every horizontal line intersects the graph of $f$, at most, once

**hydrostatic pressure** | the pressure exerted by water on a submerged object

**hyperbolic functions** | the functions denoted $\sinh,\ \cosh,\ \tanh,\ \text{csch},\ \text{sech}$, and $\coth$, which involve certain combinations of $e^x$ and $e^{-x}$

**hyperboloid of one sheet** | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; traces of this surface include ellipses and hyperbolas

**hyperboloid of two sheets** | a three-dimensional surface described by an equation of the form $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; traces of this surface include ellipses and hyperbolas

## I

**implicit differentiation** | is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable $y$ as a function) and solving for $\frac{dy}{dx}$

**improper double integral** | a double integral over an unbounded region or of an unbounded function

**improper integral** | an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

**increasing on the interval** $I$ | a function increasing on the interval $I$ if for all $x_1,\ x_2\in I,\ \ f(x_1)\le f(x_2)$ if \(x_1

**indefinite integral** | the most general antiderivative of $f(x)$ is the indefinite integral of $f$; we use the notation $\int f(x)\ dx$ to denote the indefinite integral of $f$

**indefinite integral of a vector-valued function** | a vector-valued function with a derivative that is equal to a given vector-valued function

**independence of path** | a vector field $\vecs F$ has path independence if $\int_{C_1}\vecs F\cdot d\vecs r=\int_{C_2}\vecs F\cdot d\vecs r$ for any curves $C_1$ and $C_2$ in the domain of $\vecs F$ with the same initial points and terminal points

**independent variable** | the input variable for a function

**indeterminate forms** | When evaluating a limit, the forms $\frac{0}{0}$,$\infty/\infty, 0\cdot\infty, \infty-\infty, 0^0, \infty^0$, and $1^\infty$ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

**index variable** | the subscript used to define the terms in a sequence is called the index

**infinite discontinuity** | An infinite discontinuity occurs at a point $a$ if $\lim\limits_{x\to a^-}f(x)=\pm\infty$ or $\lim\limits_{x\to a^+}f(x)=\pm\infty$

**infinite limit** | A function has an infinite limit at a point $a$ if it either increases or decreases without bound as it approaches $a$

**infinite limit at infinity** | a function that becomes arbitrarily large as $x$ becomes large

**infinite series** | an infinite series is an expression of the form $a_1+a_2+a_3+\cdots=\sum\limits_{n=1}^{\infty}a_n$

**inflection point** | if $f$ is continuous at $c$ and $f$ changes concavity at $c$, the point $(c,f(c))$ is an inflection point of $f$

**initial point** | the starting point of a vector

**initial population** | the population at time $t=0$

**initial value problem** | a problem that requires finding a function $y$ that satisfies the differential equation $\frac{dy}{dx}=f(x)$ together with the initial condition $y(x_0)=y_0$

**initial value(s)** | a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

**initial velocity** | the velocity at time $t=0$

**initial-value problem** | a differential equation together with an initial value or values

**instantaneous rate of change** | the rate of change of a function at any point along the function $a$, also called $f\,'(a)$, or the derivative of the function at $a$

**instantaneous velocity** | The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t,a]$ and $[a,t]$ approach as the values of $t$ move closer to $a$, provided such a value exists

**integrable function** | a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as $n$ goes to infinity exists

**integral calculus** | the study of integrals and their applications

**integral test** | for a series $\sum_{n=1}^{\infty} a_n$ with positive terms $a_n$, if there exists a continuous, decreasing function $f$ such that $f(n) = a_n$ for all positive integers $n$, then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_1^{\infty} f(x)\,dx$$

either both converge or both diverge

**integrand** | the function to the right of the integration symbol; the integrand includes the function being integrated

**integrating factor** | any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

**integration by parts** | a technique of integration that allows the exchange of one integral for another using the formula $\int u\ dv = uv - \int v\ du$

**integration by substitution** | a technique for integration that allows integration of functions that are the result of a chain-rule derivative

**integration table** | a table that lists integration formulas

**interior point** | a point $P_0$ of $\mathbb{R}$ is a boundary point if there is a $\delta$ disk centered around $P_0$ contained completely in $\mathbb{R}$

**Intermediate Value Theorem** | Let $f$ be continuous over a closed bounded interval $[a, b]$ if $z$ is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

**intermediate variable** | given a composition of functions (e.g., $f(x(t), y(t))$), the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t), y(t))$, the variables $x$ and $y$ are examples of intermediate variables

**interval of convergence** | the set of real numbers $x$ for which a power series converges

**intuitive definition of the limit** | If all values of the function $f(x)$ approach the real number $L$ as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

**inverse function** | for a function $f$, the inverse function $f^{-1}$ satisfies $f^{-1}(y) = x$ if $f(x) = y$

**inverse hyperbolic functions** | the inverses of the hyperbolic functions where $\cosh$ and $\operatorname{sech}$ are restricted to the domain $[0, \infty)$;each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

**inverse trigonometric functions** | the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

**iterated integral** | for a function $f(x, y)$ over the region $R$ is a. $\int_a^b \int_c^d f(x,y)\ dx\ dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$, b. $\int_c^d \int_a^b f(x,y)\ dx\ dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$, where $a, b, c$, and $d$ are any real numbers and $R = [a, b] \times [c, d]$

**iterative process** | process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number $x_0$ and defining $x_n = F(x_{n-1})$ for $n \geq 1$

## J

**Jacobian** | the Jacobian $J(u, v)$ in two variables is a $2 \times 2$ determinant:

$$J(u,v) = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\ \frac{\partial x}{\partial v}\frac{\partial y}{\partial v}\right|;$$

the Jacobian $J(u, v, w)$ in three variables is a $3 \times 3$ determinant:

$$J(u,v,w) = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\frac{\partial z}{\partial u}\ \frac{\partial x}{\partial v}\frac{\partial y}{\partial v}\frac{\partial z}{\partial v}\ \frac{\partial x}{\partial w}\frac{\partial y}{\partial w}\frac{\partial z}{\partial w}\right|$$

**jump discontinuity** | A jump discontinuity occurs at a point $a$ if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

## K

**Kepler's laws of planetary motion** | three laws governing the motion of planets, asteroids, and comets in orbit around the Sun

## L

**L'Hôpital's rule** | If $f$ and $g$ are differentiable functions over an interval $a$, except possibly at $a$, and $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$

are infinite, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is $\infty$ or $-\infty$.

**Lagrange multiplier** | the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable $\lambda$

**lamina** | a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

**left-endpoint approximation** | an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

**level curve of a function of two variables** | the set of points satisfying the equation $f(x,y)=c$ for some real number $c$ in the range of $f$

**level surface of a function of three variables** | the set of points satisfying the equation $f(x,y,z)=c$ for some real number $c$ in the range of $f$

**limaçon** | the graph of the equation $r=a+b\sin\theta$ or $r=a+b\cos\theta$. If $a=b$ then the graph is a cardioid

**limit** | the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as $x$ approaches $a$ is the value that $f(x)$ approaches as $x$ approaches $a$

**limit at infinity** | a function that approaches a limit value $L$ as $x$ becomes large

**limit comparison test** | Suppose $a_n,b_n\geq 0$ for all $n\geq 1$. If $\lim_{n\to\infty}a_n/b_n\to L\neq 0$, then $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ both converge or both diverge; if $\lim_{n\to\infty}a_n/b_n\to 0$ and $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges. If $\lim_{n\to\infty}a_n/b_n\to\infty$, and $\sum_{n=1}^{\infty}b_n$ diverges, then $\sum_{n=1}^{\infty}a_n$ diverges.

**limit laws** | the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x\neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}g(x)=M$; let c be a constant

**limit of a sequence** | the real number LL to which a sequence converges is called the limit of the sequence

**limit of a vector-valued function** | a vector-valued function $\vecs r(t)$ has a limit $\vecs L$ as $t$ approaches $a$ if $\lim t\to a\,|\vecs r(t)-\vecs L|=0$

**limits of integration** | these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

**line integral** | the integral of a function along a curve in a plane or in space

**linear** | description of a first-order differential equation that can be written in the form $a(x)y'+b(x)y=c(x)$

**linear approximation** | the linear function $L(x)=f(a)+f'(a)(x-a)$ is the linear approximation of $f$ at $x=a$

**linear approximation** | given a function $f(x,y)$ and a tangent plane to the function at a point $(x_0,y_0)$, we can approximate $f(x,y)$ for points near $(x_0,y_0)$ using the tangent plane formula

**linear function** | a function that can be written in the form $f(x)=mx+b$

**linearly dependent** | a set of functions $f_1(x),f_2(x),\ldots,f_n(x)$ for whichthere are constants $c_1,c_2,\ldots c_n$, not all zero, such that $c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)=0$ for all \(x\) in the interval of interest

**linearly independent** | a set of functions $f_1(x),f_2(x),\ldots,f_n(x)$ for which there are no constants $c_1,c_2,\ldots c_n$, such that $c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)=0$ for all \(x\) in the interval of interest

**local extremum** | if $f$ has a local maximum or local minimum at $c$, we say $f$ has a local extremum at $c$

**local maximum** | if there exists an interval $I$ such that $f(c)\geq f(x)$ for all $x\in I$, we say $f$ has a local maximum at $c$

**local minimum** | if there exists an interval $I$ such that $f(c)\leq f(x)$ for all $x\in I$, we say $f$ has a local minimum at $c$

**logarithmic differentiation** | is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

**logarithmic function** | a function of the form $f(x)=\log_b(x)$ for some base $b>0,\ b\neq 1$ such that $y=\log_b(x)$ if and only if $b^y=x$

**logistic differential equation** | a differential equation that incorporates the carrying capacity $K$ and growth rate rr into a population model

**lower sum** | a sum obtained by using the minimum value of $f(x)$ on each subinterval

## M

**Maclaurin polynomial** | a Taylor polynomial centered at $0$; the $n^{\text{th}}$-degree Taylor polynomial for $f$ at $0$ is the $n^{\text{th}}$-degree Maclaurin polynomial for $f$

**Maclaurin series** | a Taylor series for a function $f$ at $x=0$ is known as a Maclaurin series for $f$

**magnitude** | the length of a vector

**major axis** | the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

**marginal cost** | is the derivative of the cost function, or the approximate cost of producing one more item

**marginal profit** | is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

**marginal revenue** | is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

**mass flux** | the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

**mathematical model** | A method of simulating real-life situations with mathematical equations

**mean value theorem** | if $f$ is continuous over $[a, b]$ and differentiable over $(a, b)$, then there exists $c \in (a, b)$ such that $f\,'(c) = \frac{f(b)-f(a)}{b-a}$

**mean value theorem for integrals** | guarantees that a point $c$ exists such that $f(c)$ is equal to the average value of the function

**method of cylindrical shells** | a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

**method of Lagrange multipliers** | a method of solving an optimization problem subject to one or more constraints

**method of undetermined coefficients** | a method that involves making a guess about the form of the particular solution, then solving for the coefficients in the guess

**method of variation of parameters** | a method that involves looking for particular solutions in the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$, where $y_1$ and $y_2$ are linearly independent solutions to the complementary equations, and then solving a system of equations to find $u(x)$ and $v(x)$

**midpoint rule** | a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where $m_i$ is the midpoint of the $i^{\text{th}}$ subinterval to approximate $\int_a^b f(x)\ dx$

**minor axis** | the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

**mixed partial derivatives** | second-order or higher partial derivatives, in which at least two of the differentiations are with respect to different variables

**moment** | if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the $x$- and $y$-axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\ dx$ and $M_y = \rho \int_a^b xf(x)\ dx$, respectively

**monotone sequence** | an increasing or decreasing sequence

**multivariable calculus** | the study of the calculus of functions of two or more variables

## N

**nappe** | a nappe is one half of a double cone

**natural exponential function** | the function $f(x) = e^x$

**natural logarithm** | the function $\ln x = \log_e x$

**net change theorem** | if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

**net signed area** | the area between a function and the $x$-axis such that the area below the $x$-axis is subtracted from the area above the $x$-axis; the result is the same as the definite integral of the function

**Newton's method** | method for approximating roots of $f(x) = 0$; using an initial guess $x_0$; each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

**nonelementary integral** | an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

**nonhomogeneous linear equation** | a second-order differential equation that can be written in the form $a_2(x)y\,'' + a_1(x)y\,' + a_0(x)y = r(x)$, but $r(x) \neq 0$ for some value of $x$

**normal component of acceleration** | the coefficient of the unit normal vector $\vecs{N}$ when the acceleration vector is written as a linear combination of $\vecs{T}$ and $\vecs{N}$

**normal plane** | a plane that is perpendicular to a curve at any point on the curve

**normal vector** | a vector perpendicular to a plane

**normalization** | using scalar multiplication to find a unit vector with a given direction

**number e** | as $m$ gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be $e$; the value of $e$ is approximately $2.718282$

**numerical integration** | the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

## O

**objective function** | the function that is to be maximized or minimized in an optimization problem

**oblique asymptote** | the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or$x \to -\infty$

**octants** | the eight regions of space created by the coordinate planes

**odd function** | a function is odd if $f(-x) = -f(x)$ for all $x$ in the domain of $f$

**one-sided limit** | A one-sided limit of a function is a limit taken from either the left or the right

**one-to-one function** | a function $f$ is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

**one-to-one transformation** | a transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ is said to be one-to-one if no two points map to the same image point

**open set** | a set $S$ that contains none of its boundary points

**optimization problem** | calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

**optimization problems** | problems that are solved by finding the maximum or minimum value of a function

**order of a differential equation** | the highest order of any derivative of the unknown function that appears in the equation

**orientation** | the direction that a point moves on a graph as the parameter increases

**orientation of a curve** | the orientation of a curve $C$ is a specified direction of $C$

**orientation of a surface** | if a surface has an "inner" side and an "outer" side, then an orientation is a choice of the inner or the outer side; the surface could also have "upward" and "downward" orientations

**orthogonal vectors** | vectors that form a right angle when placed in standard position

**osculating circle** | a circle that is tangent to a curve $C$ at a point $P$ and that shares the same curvature

**osculating plane** | the plane determined by the unit tangent and the unit normal vector

## P

***p*-series** | a series of the form $\sum_{n=1}^{\infty} 1/n^p$

**parallelogram method** | a method for finding the sum of two vectors; position the vectors so they share the same initial point; the vectors then form two adjacent sides of a parallelogram; the sum of the vectors is the diagonal of that parallelogram

**parameter** | an independent variable that both $x$ and $y$ depend on in a parametric curve; usually represented by the variable $t$

**parameter domain (parameter space)** | the region of the $uv$-plane over which the parameters $u$ and $v$ vary for parameterization $\vecs r(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$

**parameterization of a curve** | rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

**parameterized surface (parametric surface)** | a surface given by a description of the form $\vecs r(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$, where the parameters $u$ and $v$ vary over a parameter domain in the $uv$-plane

**parametric curve** | the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

**parametric equations** | the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

**parametric equations of a line** | the set of equations $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$ describing the line with direction vector $v = \langle a, b, c\rangle$ passing through point $(x_0, y_0, z_0)$

**partial derivative** | a derivative of a function of more than one independent variable in which all the variables but one are held constant

**partial differential equation** | an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives

**partial fraction decomposition** | a technique used to break down a rational function into the sum of simple rational functions

**partial sum** | the $kth$ partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$

**particular solution** | member of a family of solutions to a differential equation that satisfies a particular initial condition

**particular solution** | a solution $y_p(x)$ of a differential equation that contains no arbitrary constants

**partition** | a set of points that divides an interval into subintervals

**percentage error** | the relative error expressed as a percentage

**periodic function** | a function is periodic if it has a repeating pattern as the values of $x$ move from left to right

**phase line** | a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

**piecewise smooth curve** | an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

**piecewise-defined function** | a function that is defined differently on different parts of its domain

**planar transformation** | a function $T$ that transforms a region $G$ in one plane into a region $R$ in another plane by a change of variables

**plane curve** | the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$

**point-slope equation** | equation of a linear function indicating its slope and a point on the graph of the function

**polar axis** | the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

**polar coordinate system** | a system for locating points in the plane. The coordinates are $r$, the radial coordinate, and $\theta$, the angular coordinate

**polar equation** | an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

**polar rectangle** | the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) \mid a \leq r \leq b, \ \alpha \leq \theta \leq \beta\}$

**pole** | the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

**polynomial function** | a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$

**population growth rate** | is the derivative of the population with respect to time

**potential function** | a scalar function $f$ such that $\vecs{\nabla} f = \vecs{F}$

**power function** | a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

**power law for limits** | the limit law

$$\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n = L^n$$

for every positive integer n

**power reduction formula** | a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

**power rule** | the derivative of a power function is a function in which the power on $x$ becomes the coefficient of the term and the power on $x$ in the derivative decreases by 1: If $n$ is an integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

**power series** | a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x - a)^n$ is a power series centered at $x = a$

**principal unit normal vector** | a vector orthogonal to the unit tangent vector, given by the formula $\frac{\vecs{T}'(t)}{\|\vecs{T}'(t)\|}$

**principal unit tangent vector** | a unit vector tangent to a curve *C*

**product law for limits** | the limit law

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

**product rule** | the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}\big(f(x)g(x)\big) = f\,'(x)g(x) + g\,'(x)f(x)$

**projectile motion** | motion of an object with an initial velocity but no force acting on it other than gravity

**propagated error** | the error that results in a calculated quantity $f(x)$ resulting from a measurement error $dx$

## Q

**quadratic function** | a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

**quadric surfaces** | surfaces in three dimensions having the property that the traces of the surface are conic sections (ellipses, hyperbolas, and parabolas)

**quotient law for limits** | the limit law $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for M≠0

**quotient rule** | the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}$

## R

**radial coordinate** | $r$ the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

**radial field** | a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

**radians** | for a circular arc of length $s$ on a circle of radius 1, the radian measure of the associated angle $\theta$ is $s$

**radius of convergence** | if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for \(|x−a|R\), then $R$ is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers $x$, the radius of convergence is $R = \infty$

**radius of curvature** | the reciprocal of the curvature

**radius of gyration** | the distance from an object's center of mass to its axis of rotation

**range** | the set of outputs for a function

**ratio test** | for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

**rational function** | a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

**recurrence relation** | a recurrence relation is a relationship in which a term $a_n$ in a sequence is defined in terms of earlier terms in the sequence

**region** | an open, connected, nonempty subset of $\mathbb{R}^2$

**regular parameterization** | parameterization $\vecs r(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$ such that $r_u \times r_v$ is not zero for point $(u, v)$ in the parameter domain

**regular partition** | a partition in which the subintervals all have the same width

**related rates** | are rates of change associated with two or more related quantities that are changing over time

**relative error** | given an absolute error $\Delta q$ for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

**relative error** | error as a percentage of the actual value, given by

$$\text{relative error} = \left|\frac{A - B}{A}\right| \cdot 100\%$$

**remainder estimate** | for a series $\sum_{n=}^{\infty} 1a_n$ with positive terms $a_n$ and a continuous, decreasing function $f$ such that $f(n) = a_n$ for all positive integers $n$, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate: \[∫^∞_{N+1}f(x)\,dx

**removable discontinuity** | A removable discontinuity occurs at a point $a$ if $f(x)$ is discontinuous at $a$, but $\lim_{x\to a} f(x)$ exists

**reparameterization** | an alternative parameterization of a given vector-valued function

**restricted domain** | a subset of the domain of a function $f$

**riemann sum** | an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

**right-endpoint approximation** | the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

**right-hand rule** | a common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the $z$-axis in such a way that the fingers curl from the positive $x$-axis to the positive $y$-axis, the thumb points in the direction of the positive $z$-axis

***RLC* series circuit** | a complete electrical path consisting of a resistor, an inductor, and a capacitor; a second-order, constant-coefficient differential equation can be used to model the charge on the capacitor in an *RLC* series circuit

**rolle's theorem** | if $f$ is continuous over $[a, b]$ and differentiable over $(a, b)$, and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f\,'(c) = 0$

**root function** | a function of the form $f(x) = x^{1/n}$ for any integer $n \ge 2$

**root law for limits** | the limit law $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \ge 0$ if n is even

**root test** | for a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

**rose** | graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$for a positive constant $a$

**rotational field** | a vector field in which the vector at point $(x, y)$ is tangent to a circle with radius $r = \sqrt{x^2 + y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

**rulings** | parallel lines that make up a cylindrical surface

## S

**saddle point** | given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but $f$ does not have a local extremum at $(x_0, y_0)$

**scalar** | a real number

**scalar equation of a plane** | the equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ used to describe a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c\rangle$ or its alternate form $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$

**scalar line integral** | the scalar line integral of a function $f$ along a curve $C$ with respect to arc length is the integral $\int_C f\ ds$, it is the integral of a scalar function $f$ along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

**scalar multiplication** | a vector operation that defines the product of a scalar and a vector

**scalar projection** | the magnitude of the vector projection of a vector

**secant** | A secant line to a function $f(x)$ at $a$ is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{sec} = \dfrac{f(x) - f(a)}{x - a}$

**second derivative test** | suppose $f'(c) = 0$ and $f''$ is continuous over an interval containing $c$; if $f''(c) > 0$, then $f$ has a local minimum at $c$; if $f''(c) < 0$, then $f$ has a local maximum at $c$; if $f''(c) = 0$, then the test is inconclusive

**separable differential equation** | any equation that can be written in the form $y' = f(x)g(y)$

**separation of variables** | a method used to solve a separable differential equation

**sequence** | an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

**sigma notation** | (also, summation notation) the Greek letter sigma ($\Sigma$) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

**simple curve** | a curve that does not cross itself

**simple harmonic motion** | motion described by the equation $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, as exhibited by an undamped spring-mass system in which the mass continues to oscillate indefinitely

**simply connected region** | a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

**Simpson's rule** | a rule that approximates $\int_a^b f(x)\ dx$ using the area under a piecewise quadratic function. The approximation $S_n$ to $\int_a^b f(x)\ dx$ is given by

$$S_n = \frac{\Delta x}{3}\big(f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\big).$$

**skew lines** | two lines that are not parallel but do not intersect

**slicing method** | a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

**slope** | the change in $y$ for each unit change in $x$

**slope-intercept form** | equation of a linear function indicating its slope and $y$-intercept

**smooth** | curves where the vector-valued function $\vecs r(t)$ is differentiable with a non-zero derivative

**solid of revolution** | a solid generated by revolving a region in a plane around a line in that plane

**solution curve** | a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

**solution to a differential equation** | a function $y = f(x)$ that satisfies a given differential equation

**space curve** | the set of ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$

**space-filling curve** | a curve that completely occupies a two-dimensional subset of the real plane

**speed** | is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time $t$ whose velocity is given by $v(t)$

**sphere** | the set of all points equidistant from a given point known as the *center*

**spherical coordinate system** | a way to describe a location in space with an ordered triple $(\rho, \theta, \varphi)$, where $\rho$ is the distance between $P$ and the origin $(\rho \neq 0)$, $\theta$ is the same angle used to describe the location in cylindrical coordinates, and $\varphi$ is the angle formed by the positive $z$-axis and line segment $\bar{OP}$, where $O$ is the origin and $0 \leq \varphi \leq \pi$

**squeeze theorem** | states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where L is a real number, then $\lim_{x \to a} g(x) = L$

**standard equation of a sphere** | $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ describes a sphere with center $(a, b, c)$ and radius $r$

**standard form** | the form of a first-order linear differential equation obtained by writing the differential equation in the form $y' + p(x)y = q(x)$

**standard form** | an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

**standard unit vectors** | unit vectors along the coordinate axes: $\hat{\mathbf{i}} = \langle 1, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1 \rangle$

**standard-position vector** | a vector with initial point $(0, 0)$

**steady-state solution** | a solution to a nonhomogeneous differential equation related to the forcing function; in the long term, the solution approaches the steady-state solution

**step size** | the increment hh that is added to the xx value at each step in Euler's Method

**Stokes' theorem** | relates the flux integral over a surface $S$ to a line integral around the boundary $C$ of the surface $S$

**stream function** | if $\vecs F = \langle P, Q \rangle$ is a source-free vector field, then stream function $g$ is a function such that $P = g_y$ and $Q = -g_x$

**sum law for limits** | The limit law $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

**sum rule** | the derivative of the sum of a function $f$ and a function $g$ is the same as the sum of the derivative of $f$ and the derivative of $g$: $\frac{d}{dx}\big(f(x) + g(x)\big) = f\,'(x) + g\,'(x)$

**surface** | the graph of a function of two variables, $z = f(x, y)$

**surface area** | the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

**surface area** | the area of surface $S$ given by the surface integral

$$\iint_S dS$$

**surface independent** | flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

**surface integral** | an integral of a function over a surface

**surface integral of a scalar-valued function** | a surface integral in which the integrand is a scalar function

**surface integral of a vector field** | a surface integral in which the integrand is a vector field

**symmetric equations of a line** | the equations $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ describing the line with direction vector $v = \langle a, b, c \rangle$ passing through point $(x_0, y_0, z_0)$

**symmetry about the $y$-axis** | the graph of a function $f$ is symmetric about the $y$-axis if $(-x, y)$ is on the graph of $f$ whenever $(x, y)$ is on the graph

**symmetry about the origin** | the graph of a function $f$ is symmetric about the origin if $(-x, -y)$ is on the graph of $f$ whenever $(x, y)$ is on the graph

**symmetry principle** | the symmetry principle states that if a region $R$ is symmetric about a line $I$, then the centroid of $R$ lies on $I$

## T

**table of values** | a table containing a list of inputs and their corresponding outputs

**tangent** | A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with $x$-values that approach $a$; the slope of the tangent line to a graph at $a$ measures the rate of change of the function at $a$

**tangent line approximation (linearization)** | since the linear approximation of $f$ at $x = a$ is defined using the equation of the tangent line, the linear approximation of $f$ at $x = a$ is also known as the tangent line approximation to $f$ at $x = a$

**tangent plane** | given a function $f(x, y)$ that is differentiable at a point $(x_0, y_0)$, the equation of the tangent plane to the surface $z = f(x, y)$ is given by $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

**tangent vector** | to $\vecs r(t)$ at $t = t_0$ any vector $\vecs v$ such that, when the tail of the vector is placed at point $\vecs r(t_0)$ on the graph, vector $\vecs v$ is tangent to curve $C$

**tangential component of acceleration** | the coefficient of the unit tangent vector $\vecs T$ when the acceleration vector is written as a linear combination of $\vecs T$ and $\vecs N$

**Taylor polynomials** | the $n^{\text{th}}$-degree Taylor polynomial for $f$ at $x = a$ is $p_n(x) = f(a) + f\,'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$

**Taylor series** | a power series at $a$ that converges to a function $f$ on some open interval containing $a$.

**Taylor's theorem with remainder** | for a function $f$ and the $n^{\text{th}}$-degree Taylor polynomial for $f$ at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some$c$ between $x$ and $a$; if there exists an interval $I$ containing $a$ and a real number $M$ such that $| f^{(n+1)}(x) | \leq M$ for all $x$ in $I$, then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$

**telescoping series** | a telescoping series is one in which most of the terms cancel in each of the partial sums

**term** | the number $a_n$ in the sequence $a_n$ is called the $nth$ term of the sequence

**term-by-term differentiation of a power series** | a technique for evaluating the derivative of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ by evaluating the derivative of each term separately to create the new power series $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

**term-by-term integration of a power series** | a technique for integrating a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ by integrating each term separately to create the new power series $C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$

**terminal point** | the endpoint of a vector

**theorem of Pappus for volume** | this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

**three-dimensional rectangular coordinate system** | a coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple $(x, y, z)$ that plots its location relative to the defining axes

**threshold population** | the minimum population that is necessary for a species to survive

**total area** | total area between a function and the $x$-axis is calculated by adding the area above the $x$-axis and the area below the $x$-axis; the result is the same as the definite integral of the absolute value of the function

**total differential** | the total differential of the function $f(x, y)$ at $(x_0, y_0)$ is given by the formula $dz = f_x(x_0, y_0)dx + fy(x_0, y_0)dy$

**trace** | the intersection of a three-dimensional surface with a coordinate plane

**transcendental function** | a function that cannot be expressed by a combination of basic arithmetic operations

**transformation** | a function that transforms a region GG in one plane into a region RR in another plane by a change of variables

**transformation of a function** | a shift, scaling, or reflection of a function

**trapezoidal rule** | a rule that approximates $\int_a^b f(x)\ dx$ using the area of trapezoids. The approximation $T_n$ to $\int_a^b f(x)\ dx$ is given by

$$T_n = \frac{\Delta x}{2}\Big(f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\Big).$$

**tree diagram** | illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

**triangle inequality** | If $a$ and $b$ are any real numbers, then $|a + b| \leq |a| + |b|$

**triangle inequality** | the length of any side of a triangle is less than the sum of the lengths of the other two sides

**triangle method** | a method for finding the sum of two vectors; position the vectors so the terminal point of one vector is the initial point of the other; these vectors then form two sides of a triangle; the sum of the vectors is the vector that forms the third side; the initial point of the sum is the initial point of the first vector; the terminal point of the sum is the terminal point of the second vector

**trigonometric functions** | functions of an angle defined as ratios of the lengths of the sides of a right triangle

**trigonometric identity** | an equation involving trigonometric functions that is true for all angles $\theta$ for which the functions in the equation are defined

**trigonometric integral** | an integral involving powers and products of trigonometric functions

**trigonometric substitution** | an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral

**triple integral** | the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box $B$ is the limit of a Riemann sum for a function of three variables, if this limit exists

**triple integral in cylindrical coordinates** | the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, s^*_{ijk}) r^*_{ijk} \Delta r \Delta\theta \Delta z$$

**triple integral in spherical coordinates** | the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(\rho_{ijk}^{*},\theta_{ijk}^{*},\varphi_{ijk}^{*})(\rho_{ijk}^{*})^{2}\sin\,\varphi\Delta\rho\Delta\theta\Delta\varphi$$

**Type I** | a region $D$ in the $xy$- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

**Type II** | a region $D$ in the $xy$-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(h)$

## U

**unbounded sequence** | a sequence that is not bounded is called unbounded

**unit vector** | a vector with magnitude $1$

**unit vector field** | a vector field in which the magnitude of every vector is 1

**upper sum** | a sum obtained by using the maximum value of $f(x)$ on each subinterval

## V

**variable of integration** | indicates which variable you are integrating with respect to; if it is $x$, then the function in the integrand is followed by $dx$

**vector** | a mathematical object that has both magnitude and direction

**vector addition** | a vector operation that defines the sum of two vectors

**vector difference** | the vector difference $\vecs v - \vecs w$ is defined as $\vecs v + (-\vecs w) = \vecs v + (-1)\vecs w$

**vector equation of a line** | the equation $\vecs r = \vecs r_0 + t\vecs v$ used to describe a line with direction vector $\vecs v = \langle a, b, c\rangle$ passing through point $P = (x_0, y_0, z_0)$, where $\vecs r_0 = \langle x_0, y_0, z_0\rangle$, is the position vector of point $P$

**vector equation of a plane** | the equation $\vecs n \cdot \vecd PQ = 0$, where $P$ is a given point in the plane, $Q$ is any point in the plane, and $\vecs n$ is a normal vector of the plane

**vector field** | measured in $\mathbb{R}^2$, an assignment of a vector $\vecs F(x, y)$ to each point $(x, y)$ of a subset $D$ of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\vecs F(x, y, z)$ to each point $(x, y, z)$ of a subset $D$ of $\mathbb{R}^3$

**vector line integral** | the vector line integral of vector field $\vecs F$ along curve $C$ is the integral of the dot product of $\vecs F$ with unit tangent vector $\vecs T$ of $C$ with respect to arc length, $\int_C \vecs F \cdot \vecs T\ ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

**vector parameterization** | any representation of a plane or space curve using a vector-valued function

**vector projection** | the component of a vector that follows a given direction

**vector sum** | the sum of two vectors, $\vecs v$ and $\vecs w$, can be constructed graphically by placing the initial point of $\vecs w$ at the terminal point of $\vecs v$; then the vector sum $\vecs v + \vecs w$ is the vector with an initial point that coincides with the initial point of $\vecs v$, and with a terminal point that coincides with the terminal point of $\vecs w$

**vector-valued function** | a function of the form $\vecs r(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vecs r(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,where the component functions $f$, $g$, and $h$ are real-valued functions of the parameter $t$.

**velocity vector** | the derivative of the position vector

**vertex** | a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

**vertical asymptote** | A function has a vertical asymptote at $x = a$ if the limit as $x$ approaches $a$ from the right or left is infinite

**vertical line test** | given the graph of a function, every vertical line intersects the graph, at most, once

**vertical trace** | the set of ordered triples $(c, y, z)$ that solves the equation $f(c, y) = z$ for a given constant $x = c$ or the set of ordered triples $(x, d, z)$ that solves the equation $f(x, d) = z$ for a given constant $y = d$

## W

**washer method** | a special case of the slicing method used with solids of revolution when the slices are washers

**work** | the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

**work done by a force** | work is generally thought of as the amount of energy it takes to move an object; if we represent an applied force by a vector $\vecs F$ and the displacement of an object by a vector $\vecs s$, then the work done by the force is the dot product of $\vecs F$ and $\vecs s$.

## Z

**zero vector** | the vector with both initial point and terminal point $(0, 0)$

**zeros of a function** | when a real number $x$ is a zero of a function $f$, $f(x) = 0$

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# GLOSSARY

## A

**absolute convergence** | if the series $\sum_{n=1}^{\infty} |a_n|$ converges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely

**absolute error** | if $B$ is an estimate of some quantity having an actual value of $A$, then the absolute error is given by $|A - B|$

**absolute extremum** | if $f$ has an absolute maximum or absolute minimum at $c$, we say $f$ has an absolute extremum at $c$

**absolute maximum** | if $f(c) \geq f(x)$ for all $x$ in the domain of $f$, we say $f$ has an absolute maximum at $c$

**absolute minimum** | if $f(c) \leq f(x)$ for all $x$ in the domain of $f$, we say $f$ has an absolute minimum at $c$

**absolute value function** | $f(x) = \{-x, \quad \text{if } x < 0 \backslash x, \quad \text{if } x \geq 0$

**acceleration** | is the rate of change of the velocity, that is, the derivative of velocity

**acceleration vector** | the second derivative of the position vector

**algebraic function** | a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable $x$

**alternating series** | a series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n \geq 0$, is called an alternating series

**alternating series test** | for an alternating series of either form, if $b_{n+1} \leq b_n$ for all integers $n \geq 1$ and $b_n \to 0$, then an alternating series converges

**amount of change** | the amount of a function $f(x)$ over an interval $[x, x + h] is f(x + h) - f(x)$

**angular coordinate** | $\theta$ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial (x) axis, measured counterclockwise

**antiderivative** | a function $F$ such that $F'(x) = f(x)$ for all $x$ in the domain of $f$ is an antiderivative of $f$

**arc length** | the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

**arc-length function** | a function $s(t)$ that describes the arc length of curve $C$ as a function of $t$

**arc-length parameterization** | a reparameterization of a vector-valued function in which the parameter is equal to the arc length

**arithmetic sequence** | a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

**asymptotically semi-stable solution** | $y = k$ if it is neither asymptotically stable nor asymptotically unstable

**asymptotically stable solution** | $y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y), y(x_0) = c$ approaches $k$ as $x$ approaches infinity

**asymptotically unstable solution** | $y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y), y(x_0) = c$ never approaches $k$ as $x$approaches infinity

**autonomous differential equation** | an equation in which the right-hand side is a function of $y$ alone

**average rate of change** | is a function $f(x)$ over an interval $[x, x + h]$ is $\frac{f(x+h)-f(a)}{b-a}$

**average value of a function** | (or $f_{ave}$) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

**average velocity** | the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval [$t, a$] (if \(ta\)), with a position given by $s(t)$, that is $v_{ave} = \dfrac{s(t) - s(a)}{t - a}$

## B

**base** | the number $b$ in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

**binomial series** | the Maclaurin series for $f(x) = (1 + x)^r$; it is given by $(1 + x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \dfrac{r(r-1)}{2!} x^2 + \cdots + \dfrac{r(r-1)\cdots(r-n+1)}{n!} x^n + \cdots$ for $|x| < 1$

**binormal vector** | a unit vector orthogonal to the unit tangent vector and the unit normal vector

**boundary conditions** | the conditions that give the state of a system at different times, such as the position of a spring-mass system at two different times

**boundary point** | a point $P_0$ of $R$ is a boundary point if every $\delta$ disk centered around $P_0$ contains points both inside and outside $R$

**boundary-value problem** | a differential equation with associated boundary conditions

**bounded above** | a sequence $a_n$ is bounded above if there exists a constant $M$ such that $a_n \leq M$ for all positive integers $n$

**bounded below** | a sequence $a_n$ is bounded below if there exists a constant $M$ such that $M \leq a_n$ for all positive integers $n$

**bounded sequence** | a sequence $a_n$ is bounded if there exists a constant $M$ such that $|a_n| \leq M$ for all positive integers $n$

## C

**cardioid** | a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

**carrying capacity** | the maximum population of an organism that the environment can sustain indefinitely

**catenary** | a curve in the shape of the function $y = a \cdot \cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

**center of mass** | the point at which the total mass of the system could be concentrated without changing the moment

**centroid** | the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

**chain rule** | the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

**change of variables** | the substitution of a variable, such as $u$, for an expression in the integrand

**characteristic equation** | the equation $a\lambda^2 + b\lambda + c = 0$ for the differential equation $ay\,''+by\,'+cy = 0$

**circulation** | the tendency of a fluid to move in the direction of curve $C$. If $C$ is a closed curve, then the circulation of $\vecs{F}$ along $C$ is line integral $\int_C \vecs{F} \cdot \vecs{T}\, ds$, which we also denote $\oint_C \vecs{F} \cdot \vecs{T}\, ds$.

**closed curve** | a curve for which there exists a parameterization $\vecs{r}(t), a \leq t \leq b$, such that $\vecs{r}(a) = \vecs{r}(b)$, and the curve is traversed exactly once

**closed curve** | a curve that begins and ends at the same point

**closed set** | a set $S$ that contains all its boundary points

**comparison test** | If $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \geq b_n \geq 0$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

**complementary equation** | for the nonhomogeneous linear differential equation

$$a + 2(x)y''+a_1(x)y'+a_0(x)y = r(x),$$

the associated homogeneous equation, called the *complementary equation*, is

$$a_2(x)y'' + a_1(x)y'+a_0(x)y = 0$$

**component** | a scalar that describes either the vertical or horizontal direction of a vector

**component functions** | the component functions of the vector-valued function $\vecs{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ are $f(t)$ and $g(t)$, and the component functions of the vector-valued function $\vecs{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ are $f(t)$, $g(t)$ and $h(t)$

**composite function** | given two functions $f$ and $g$, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

**computer algebra system (CAS)** | technology used to perform many mathematical tasks, including integration

**concave down** | if $f$ is differentiable over an interval $I$ and $f'$ is decreasing over $I$, then $f$ is concave down over $I$

**concave up** | if $f$ is differentiable over an interval $I$ and $f'$ is increasing over $I$, then $f$ is concave up over $I$

**concavity** | the upward or downward curve of the graph of a function

**concavity test** | suppose $f$ is twice differentiable over an interval $I$; if $f'' > 0$ over $I$, then $f$ is concave up over $I$; if $f'' <$ over $I$, then $f$ is concave down over $I$

**conditional convergence** | if the series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally

**conic section** | a conic section is any curve formed by the intersection of a plane with a cone of two nappes

**connected region** | a region in which any two points can be connected by a path with a trace contained entirely inside the region

**connected set** | an open set $S$ that cannot be represented as the union of two or more disjoint, nonempty open subsets

**conservative field** | a vector field for which there exists a scalar function $f$ such that $\vecs\nabla f = \vecs F$

**constant multiple law for limits** | the limit law

$$\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x) = cL$$

**constant multiple rule** | the derivative of a constant $c$ multiplied by a function $f$ is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big) = cf\,'(x)$

**constant rule** | the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where $c$ is a constant

**constraint** | an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

**continuity at a point** | A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x\to a} f(x)$ exists, and (3) $\lim x \to a f(x) = f(a)$

**continuity from the left** | A function is continuous from the left at b if $\lim_{x\to b^-} f(x) = f(b)$

**continuity from the right** | A function is continuous from the right at a if $\lim_{x\to a^+} f(x) = f(a)$

**continuity over an interval** | a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in $(a, b)$, and it is continuous from the right at $a$ and from the left at $b$

**contour map** | a plot of the various level curves of a given function $f(x, y)$

**convergence of a series** | a series converges if the sequence of partial sums for that series converges

**convergent sequence** | a convergent sequence is a sequence $a_n$ for which there exists a real number $L$ such that $a_n$ is arbitrarily close to $L$ as long as $n$ is sufficiently large

**coordinate plane** | a plane containing two of the three coordinate axes in the three-dimensional coordinate system, named by the axes it contains: the $xy$-plane, $xz$-plane, or the $yz$-plane

**critical point** | if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of $f$

**critical point of a function of two variables** | the point $(x_0, y_0)$ is called a critical point of $f(x, y)$ if one of the two following conditions holds: 1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ 2. At least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist

**cross product** | $\vecs u \times \vecs v = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$, where $\vecs u = \langle u_1, u_2, u_3\rangle$ and $\vecs v = \langle v_1, v_2, v_3\rangle$ determinant a real number associated with a square matrix parallelepiped a three-dimensional prism with six faces that are parallelograms torque the effect of a force that causes an object to rotate triple scalar product the dot product of a vector with the cross product of two other vectors: $\vecs u \cdot (\vecs v \times \vecs w)$ vector product the cross product of two vectors.

**cross-section** | the intersection of a plane and a solid object

**cubic function** | a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

**curl** | the curl of vector field $\vecs F = \langle P, Q, R\rangle$, denoted $\vecs\nabla \times \vecs F$ is the "determinant" of the matrix

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

and is given by the expression $(R_y - Q_z)\ \hat{\mathbf{i}} + (P_z - R_x)\ \hat{\mathbf{j}} + (Q_x - P_y)\ \hat{\mathbf{k}}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

**curvature** | the derivative of the unit tangent vector with respect to the arc-length parameter

**cusp** | a pointed end or part where two curves meet

**cycloid** | the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

**cylinder** | a set of lines parallel to a given line passing through a given curve

**cylindrical coordinate system** | a way to describe a location in space with an ordered triple $(r, \theta, z)$, where $(r, \theta)$ represents the polar coordinates of the point's projection in the $xy$-plane, and $z$ represents the point's projection onto the $z$-axis

## D

**decreasing on the interval** $I$ | a function decreasing on the interval $I$ if, for all $x_1,\ \ x_2 \in I,\ \ f(x_1) \geq f(x_2)$ if \(x_1

**definite integral** | a primary operation of calculus; the area between the curve and the $x$-axis over a given interval is a definite integral

**definite integral of a vector-valued function** | the vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function

**degree** | for a polynomial function, the value of the largest exponent of any term

**density function** | a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

**dependent variable** | the output variable for a function

**derivative** | the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

**derivative function** | gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

**derivative of a vector-valued function** | the derivative of a vector-valued function $\vecs r(t)$ is $\vecs r\,'(t) = \lim_{\Delta t \to 0} \frac{\vecs r(t+\Delta t) - \vecs r(t)}{\Delta t}$, provided the limit exists

**difference law for limits** | the limit law

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

**difference quotient** | of a function $f(x)$ at $a$ is given by $\dfrac{f(a+h) - f(a)}{h}$ or $\dfrac{f(x) - f(a)}{x - a}$

**difference rule** | the derivative of the difference of a function $f$ and a function $g$ is the same as the difference of the derivative of $f$ and the derivative of $g$: $\dfrac{d}{dx}\big(f(x) - g(x)\big) = f\,'(x) - g\,'(x)$

**differentiable** | a function $f(x, y)$ is differentiable at $(x_0, y_0)$ if $f(x, y)$ can be expressed in the form $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y)$, where the error term $E(x, y)$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \dfrac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$

**differentiable at** $a$ | a function for which $f'(a)$ exists is differentiable at $a$

**differentiable function** | a function for which $f'(x)$ exists is a differentiable function

**differentiable on** $S$ | a function for which $f'(x)$ exists for each $x$ in the open set $S$ is differentiable on $S$

**differential** | the differential $dx$ is an independent variable that can be assigned any nonzero real number; the differential $dy$ is defined to be $dy = f'(x)\ dx$

**differential calculus** | the field of calculus concerned with the study of derivatives and their applications

**differential equation** | an equation involving a function $y = y(x)$ and one or more of its derivatives

**differential form** | given a differentiable function $y = f'(x)$, the equation $dy = f'(x)\ dx$ is the differential form of the derivative of $y$ with respect to $x$

**differentiation** | the process of taking a derivative

**direction angles** | the angles formed by a nonzero vector and the coordinate axes

**direction cosines** | the cosines of the angles formed by a nonzero vector and the coordinate axes

**direction field (slope field)** | a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

**direction vector** | a vector parallel to a line that is used to describe the direction, or orientation, of the line in space

**directional derivative** | the derivative of a function in the direction of a given unit vector

**directrix** | a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

**discontinuity at a point** | A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

**discriminant** | the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving $xy$, is called a discriminant

**discriminant** | the discriminant of the function $f(x, y)$ is given by the formula $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$

**disk method** | a special case of the slicing method used with solids of revolution when the slices are disks

**divergence** | the divergence of a vector field $\vecs F = \langle P, Q, R\rangle$, denoted $\vecs\nabla \times \vecs F$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

**divergence of a series** | a series diverges if the sequence of partial sums for that series diverges

**divergence test** | if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges

**divergent sequence** | a sequence that is not convergent is divergent

**domain** | the set of inputs for a function

**dot product or scalar product** | $\vecs u \cdot \vecs v = u_1 v_1 + u_2 v_2 + u_3 v_3$ where $\vecs u = \langle u_1, u_2, u_3\rangle$ and $\vecs v = \langle v_1, v_2, v_3\rangle$

**double integral** | of the function $f(x,y)$ over the region $R$ in the $xy$-plane is defined as the limit of a double Riemann sum, $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

**double Riemann sum** | of the function $f(x,y)$ over a rectangular region $R$ is $$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$ where $R$ is divided into smaller subrectangles $R_{ij}$ and $(x_{ij}^*, y_{ij}^*)$ is an arbitrary point in $R_{ij}$

**doubling time** | if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

## E

**eccentricity** | the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

**ellipsoid** | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; all traces of this surface are ellipses

**elliptic cone** | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$; traces of this surface include ellipses and intersecting lines

**elliptic paraboloid** | a three-dimensional surface described by an equation of the form $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$; traces of this surface include ellipses and parabolas

**end behavior** | the behavior of a function as $x\to\infty$ and $x\to-\infty$

**epsilon-delta definition of the limit** | $\lim_{x\to a} f(x)=L$ if for every $\varepsilon>0$, there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$

**equilibrium solution** | any solution to the differential equation of the form $y=c$, where $c$ is a constant

**equivalent vectors** | vectors that have the same magnitude and the same direction

**Euler's Method** | a numerical technique used to approximate solutions to an initial-value problem

**even function** | a function is even if $f(-x)=f(x)$ for all $x$ in the domain of $f$

**explicit formula** | a sequence may be defined by an explicit formula such that $a_n=f(n)$

**exponent** | the value $x$ in the expression $b^x$

**exponential decay** | systems that exhibit exponential decay follow a model of the form $y=y_0e^{-kt}$

**exponential growth** | systems that exhibit exponential growth follow a model of the form $y=y_0e^{kt}$

**extreme value theorem** | if $f$ is a continuous function over a finite, closed interval, then $f$ has an absolute maximum and an absolute minimum

## F

**Fermat's theorem** | if $f$ has a local extremum at $c$, then $c$ is a critical point of $f$

**first derivative test** | let $f$ be a continuous function over an interval $I$ containing a critical point $c$ such that $f$ is differentiable over $I$ except possibly at $c$; if $f'$ changes sign from positive to negative as $x$ increases through $c$, then $f$ has a local maximum at $c$; if $f'$ changes sign from negative to positive as $x$ increases through $c$, then $f$ has a local minimum at $c$; if $f'$ does not change sign as $x$ increases through $c$, then $f$ does not have a local extremum at $c$

**flux** | the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\vecs F$ across plane curve $C$ is line integral $\int_C \vecs F \cdot \frac{\vecs n(t)}{\|\vecs n(t)\|}\,ds$

**flux integral** | another name for a surface integral of a vector field; the preferred term in physics and engineering

**focal parameter** | the focal parameter is the distance from a focus of a conic section to the nearest directrix

**focus** | a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

**formal definition of an infinite limit** | $\lim_{x\to a} f(x)=\infty$ if for every $M>0$, there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $f(x)>M$ $\lim_{x\to a} f(x)=-\infty$ if for every $M>0$, there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $f(x)<-M$

**Frenet frame of reference** | (TNB frame) a frame of reference in three-dimensional space formed by the unit tangent vector, the unit normal vector, and the binormal vector

**frustum** | a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

**Fubini's theorem** | if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$, then the double integral of $f$ over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

**function** | a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

**function of two variables** | a function $z = f(x,y)$ that maps each ordered pair $(x,y)$ in a subset $D$ of $R^2$ to a unique real number $z$

**Fundamental Theorem for Line Integrals** | the value of line integral $\int_C \vecs{\nabla} f \cdot d\vecs{r}$ depends only on the value of $f$ at the endpoints of $C$ : $\int_C \vecs{\nabla} f \cdot d\vecs{r} = f(\vecs{r}(b)) - f(\vecs{r}(a))$

**fundamental theorem of calculus** | (also, evaluation theorem) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

**fundamental theorem of calculus** | uses a definite integral to define an antiderivative of a function

**fundamental theorem of calculus** | the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

## G

**general form** | an equation of a conic section written as a general second-degree equation

**general form of the equation of a plane** | an equation in the form $ax + by + cz + d = 0$, where $\vecs{n} = \langle a,b,c\rangle$ is a normal vector of the plane, $P = (x_0, y_0, z_0)$ is a point on the plane, and $d = -ax_0 - by_0 - cz_0$

**general solution (or family of solutions)** | the entire set of solutions to a given differential equation

**generalized chain rule** | the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

**geometric sequence** | a sequence $a_n$ in which the ratio $a_{n+1}/a_n$ is the same for all positive integers $n$ is called a geometric sequence

**geometric series** | a geometric series is a series that can be written in the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

**gradient** | the gradient of the function $f(x,y)$ is defined to be $\vecs{\nabla} f(x,y) = (\partial f/\partial x)\ \hat{\mathbf{i}} + (\partial f/\partial y)\ \hat{\mathbf{j}}$, which can be generalized to a function of any number of independent variables

**gradient field** | a vector field $\vecs{F}$ for which there exists a scalar function $f$ such that $\vecs{\nabla} f = \vecs{F}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

**graph of a function** | the set of points $(x,y)$ such that $x$ is in the domain of $f$ and $y = f(x)$

**graph of a function of two variables** | a set of ordered triples $(x,y,z)$ that satisfies the equation $z = f(x,y)$ plotted in three-dimensional Cartesian space

**Green's theorem** | relates the integral over a connected region to an integral over the boundary of the region

**grid curves** | curves on a surface that are parallel to grid lines in a coordinate plane

**growth rate** | the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

## H

**half-life** | if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

**harmonic series** | the harmonic series takes the form $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

**heat flow** | a vector field proportional to the negative temperature gradient in an object

**helix** | a three-dimensional curve in the shape of a spiral

**higher-order derivative** | a derivative of a derivative, from the second derivative to the $n^{\text{th}}$ derivative, is called a higher-order derivative

**higher-order partial derivatives** | second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

**homogeneous linear equation** | a second-order differential equation that can be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$, but $r(x) = 0$ for every value of $x$

**Hooke's law** | this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx$, where $k$ is a constant

**horizontal asymptote** | if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of $f$

**horizontal line test** | a function $f$ is one-to-one if and only if every horizontal line intersects the graph of $f$, at most, once

**hydrostatic pressure** | the pressure exerted by water on a submerged object

**hyperbolic functions** | the functions denoted $\sinh,\ \cosh,\ \tanh,\ \text{csch},\ \text{sech}$, and $\coth$, which involve certain combinations of $e^x$ and $e^{-x}$

**hyperboloid of one sheet** | a three-dimensional surface described by an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; traces of this surface include ellipses and hyperbolas

**hyperboloid of two sheets** | a three-dimensional surface described by an equation of the form $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; traces of this surface include ellipses and hyperbolas

## I

**implicit differentiation** | is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable $y$ as a function) and solving for $\frac{dy}{dx}$

**improper double integral** | a double integral over an unbounded region or of an unbounded function

**improper integral** | an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

**increasing on the interval** $I$ | a function increasing on the interval $I$ if for all $x_1,\ x_2\in I,\ \ f(x_1)\le f(x_2)$ if \(x_1

**indefinite integral** | the most general antiderivative of $f(x)$ is the indefinite integral of $f$; we use the notation $\int f(x)\ dx$ to denote the indefinite integral of $f$

**indefinite integral of a vector-valued function** | a vector-valued function with a derivative that is equal to a given vector-valued function

**independence of path** | a vector field $\vecs F$ has path independence if $\int_{C_1}\vecs F\cdot d\vecs r=\int_{C_2}\vecs F\cdot d\vecs r$ for any curves $C_1$ and $C_2$ in the domain of $\vecs F$ with the same initial points and terminal points

**independent variable** | the input variable for a function

**indeterminate forms** | When evaluating a limit, the forms $\frac{0}{0}$,$\infty/\infty, 0\cdot\infty, \infty-\infty, 0^0, \infty^0$, and $1^\infty$ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

**index variable** | the subscript used to define the terms in a sequence is called the index

**infinite discontinuity** | An infinite discontinuity occurs at a point $a$ if $\lim\limits_{x\to a^-}f(x)=\pm\infty$ or $\lim\limits_{x\to a^+}f(x)=\pm\infty$

**infinite limit** | A function has an infinite limit at a point $a$ if it either increases or decreases without bound as it approaches $a$

**infinite limit at infinity** | a function that becomes arbitrarily large as $x$ becomes large

**infinite series** | an infinite series is an expression of the form $a_1+a_2+a_3+\cdots=\sum\limits_{n=1}^{\infty}a_n$

**inflection point** | if $f$ is continuous at $c$ and $f$ changes concavity at $c$, the point $(c,f(c))$ is an inflection point of $f$

**initial point** | the starting point of a vector

**initial population** | the population at time $t=0$

**initial value problem** | a problem that requires finding a function $y$ that satisfies the differential equation $\frac{dy}{dx}=f(x)$ together with the initial condition $y(x_0)=y_0$

**initial value(s)** | a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

**initial velocity** | the velocity at time $t=0$

**initial-value problem** | a differential equation together with an initial value or values

**instantaneous rate of change** | the rate of change of a function at any point along the function $a$, also called $f\,'(a)$, or the derivative of the function at $a$

**instantaneous velocity** | The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t,a]$ and $[a,t]$ approach as the values of $t$ move closer to $a$, provided such a value exists

**integrable function** | a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as $n$ goes to infinity exists

**integral calculus** | the study of integrals and their applications

**integral test** | for a series $\sum_{n=1}^{\infty} a_n$ with positive terms $a_n$, if there exists a continuous, decreasing function $f$ such that $f(n) = a_n$ for all positive integers $n$, then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_1^{\infty} f(x)\,dx$$

either both converge or both diverge

**integrand** | the function to the right of the integration symbol; the integrand includes the function being integrated

**integrating factor** | any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

**integration by parts** | a technique of integration that allows the exchange of one integral for another using the formula $\int u\ dv = uv - \int v\ du$

**integration by substitution** | a technique for integration that allows integration of functions that are the result of a chain-rule derivative

**integration table** | a table that lists integration formulas

**interior point** | a point $P_0$ of $\mathbb{R}$ is a boundary point if there is a $\delta$ disk centered around $P_0$ contained completely in $\mathbb{R}$

**Intermediate Value Theorem** | Let $f$ be continuous over a closed bounded interval $[a, b]$ if $z$ is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

**intermediate variable** | given a composition of functions (e.g., $f(x(t), y(t))$), the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t), y(t))$, the variables $x$ and $y$ are examples of intermediate variables

**interval of convergence** | the set of real numbers $x$ for which a power series converges

**intuitive definition of the limit** | If all values of the function $f(x)$ approach the real number $L$ as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

**inverse function** | for a function $f$, the inverse function $f^{-1}$ satisfies $f^{-1}(y) = x$ if $f(x) = y$

**inverse hyperbolic functions** | the inverses of the hyperbolic functions where $\cosh$ and $\operatorname{sech}$ are restricted to the domain $[0, \infty)$;each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

**inverse trigonometric functions** | the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

**iterated integral** | for a function $f(x, y)$ over the region $R$ is a. $\int_a^b \int_c^d f(x,y)\ dx\ dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$, b. $\int_c^d \int_a^b f(x,y)\ dx\ dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$, where $a, b, c$, and $d$ are any real numbers and $R = [a, b] \times [c, d]$

**iterative process** | process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number $x_0$ and defining $x_n = F(x_{n-1})$ for $n \geq 1$

## J

**Jacobian** | the Jacobian $J(u, v)$ in two variables is a $2 \times 2$ determinant:

$$J(u,v) = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\ \frac{\partial x}{\partial v}\frac{\partial y}{\partial v}\right|;$$

the Jacobian $J(u, v, w)$ in three variables is a $3 \times 3$ determinant:

$$J(u,v,w) = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\frac{\partial z}{\partial u}\ \frac{\partial x}{\partial v}\frac{\partial y}{\partial v}\frac{\partial z}{\partial v}\ \frac{\partial x}{\partial w}\frac{\partial y}{\partial w}\frac{\partial z}{\partial w}\right|$$

**jump discontinuity** | A jump discontinuity occurs at a point $a$ if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

## K

**Kepler's laws of planetary motion** | three laws governing the motion of planets, asteroids, and comets in orbit around the Sun

## L

**L'Hôpital's rule** | If $f$ and $g$ are differentiable functions over an interval $a$, except possibly at $a$, and $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$

are infinite, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is $\infty$ or $-\infty$.

**Lagrange multiplier** | the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable $\lambda$

**lamina** | a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

**left-endpoint approximation** | an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

**level curve of a function of two variables** | the set of points satisfying the equation $f(x,y)=c$ for some real number $c$ in the range of $f$

**level surface of a function of three variables** | the set of points satisfying the equation $f(x,y,z)=c$ for some real number $c$ in the range of $f$

**limaçon** | the graph of the equation $r=a+b\sin\theta$ or $r=a+b\cos\theta$. If $a=b$ then the graph is a cardioid

**limit** | the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as $x$ approaches $a$ is the value that $f(x)$ approaches as $x$ approaches $a$

**limit at infinity** | a function that approaches a limit value $L$ as $x$ becomes large

**limit comparison test** | Suppose $a_n,b_n\geq 0$ for all $n\geq 1$. If $\lim_{n\to\infty}a_n/b_n\to L\neq 0$, then $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ both converge or both diverge; if $\lim_{n\to\infty}a_n/b_n\to 0$ and $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges. If $\lim_{n\to\infty}a_n/b_n\to\infty$, and $\sum_{n=1}^{\infty}b_n$ diverges, then $\sum_{n=1}^{\infty}a_n$ diverges.

**limit laws** | the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x\neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}g(x)=M$; let c be a constant

**limit of a sequence** | the real number LL to which a sequence converges is called the limit of the sequence

**limit of a vector-valued function** | a vector-valued function $\vecs r(t)$ has a limit $\vecs L$ as $t$ approaches $a$ if $\lim t\to a\,|\vecs r(t)-\vecs L|=0$

**limits of integration** | these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

**line integral** | the integral of a function along a curve in a plane or in space

**linear** | description of a first-order differential equation that can be written in the form $a(x)y'+b(x)y=c(x)$

**linear approximation** | the linear function $L(x)=f(a)+f'(a)(x-a)$ is the linear approximation of $f$ at $x=a$

**linear approximation** | given a function $f(x,y)$ and a tangent plane to the function at a point $(x_0,y_0)$, we can approximate $f(x,y)$ for points near $(x_0,y_0)$ using the tangent plane formula

**linear function** | a function that can be written in the form $f(x)=mx+b$

**linearly dependent** | a set of functions $f_1(x),f_2(x),\ldots,f_n(x)$ for whichthere are constants $c_1,c_2,\ldots c_n$, not all zero, such that $c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)=0$ for all \(x\) in the interval of interest

**linearly independent** | a set of functions $f_1(x),f_2(x),\ldots,f_n(x)$ for which there are no constants $c_1,c_2,\ldots c_n$, such that $c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)=0$ for all \(x\) in the interval of interest

**local extremum** | if $f$ has a local maximum or local minimum at $c$, we say $f$ has a local extremum at $c$

**local maximum** | if there exists an interval $I$ such that $f(c)\geq f(x)$ for all $x\in I$, we say $f$ has a local maximum at $c$

**local minimum** | if there exists an interval $I$ such that $f(c)\leq f(x)$ for all $x\in I$, we say $f$ has a local minimum at $c$

**logarithmic differentiation** | is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

**logarithmic function** | a function of the form $f(x)=\log_b(x)$ for some base $b>0,\ b\neq 1$ such that $y=\log_b(x)$ if and only if $b^y=x$

**logistic differential equation** | a differential equation that incorporates the carrying capacity $K$ and growth rate rr into a population model

**lower sum** | a sum obtained by using the minimum value of $f(x)$ on each subinterval

## M

**Maclaurin polynomial** | a Taylor polynomial centered at $0$; the $n^{\text{th}}$-degree Taylor polynomial for $f$ at $0$ is the $n^{\text{th}}$-degree Maclaurin polynomial for $f$

**Maclaurin series** | a Taylor series for a function $f$ at $x=0$ is known as a Maclaurin series for $f$

**magnitude** | the length of a vector

**major axis** | the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

**marginal cost** | is the derivative of the cost function, or the approximate cost of producing one more item

**marginal profit** | is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

**marginal revenue** | is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

**mass flux** | the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

**mathematical model** | A method of simulating real-life situations with mathematical equations

**mean value theorem** | if $f$ is continuous over $[a, b]$ and differentiable over $(a, b)$, then there exists $c \in (a, b)$ such that $f\,'(c) = \frac{f(b)-f(a)}{b-a}$

**mean value theorem for integrals** | guarantees that a point $c$ exists such that $f(c)$ is equal to the average value of the function

**method of cylindrical shells** | a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

**method of Lagrange multipliers** | a method of solving an optimization problem subject to one or more constraints

**method of undetermined coefficients** | a method that involves making a guess about the form of the particular solution, then solving for the coefficients in the guess

**method of variation of parameters** | a method that involves looking for particular solutions in the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$, where $y_1$ and $y_2$ are linearly independent solutions to the complementary equations, and then solving a system of equations to find $u(x)$ and $v(x)$

**midpoint rule** | a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where $m_i$ is the midpoint of the $i^{\text{th}}$ subinterval to approximate $\int_a^b f(x)\ dx$

**minor axis** | the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

**mixed partial derivatives** | second-order or higher partial derivatives, in which at least two of the differentiations are with respect to different variables

**moment** | if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the $x$- and $y$-axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\ dx$ and $M_y = \rho \int_a^b xf(x)\ dx$, respectively

**monotone sequence** | an increasing or decreasing sequence

**multivariable calculus** | the study of the calculus of functions of two or more variables

## N

**nappe** | a nappe is one half of a double cone

**natural exponential function** | the function $f(x) = e^x$

**natural logarithm** | the function $\ln x = \log_e x$

**net change theorem** | if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

**net signed area** | the area between a function and the $x$-axis such that the area below the $x$-axis is subtracted from the area above the $x$-axis; the result is the same as the definite integral of the function

**Newton's method** | method for approximating roots of $f(x) = 0$; using an initial guess $x_0$; each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

**nonelementary integral** | an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

**nonhomogeneous linear equation** | a second-order differential equation that can be written in the form $a_2(x)y\,'' + a_1(x)y\,' + a_0(x)y = r(x)$, but $r(x) \neq 0$ for some value of $x$

**normal component of acceleration** | the coefficient of the unit normal vector $\vecs{N}$ when the acceleration vector is written as a linear combination of $\vecs{T}$ and $\vecs{N}$

**normal plane** | a plane that is perpendicular to a curve at any point on the curve

**normal vector** | a vector perpendicular to a plane

**normalization** | using scalar multiplication to find a unit vector with a given direction

**number e** | as $m$ gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be $e$; the value of $e$ is approximately $2.718282$

**numerical integration** | the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

## O

**objective function** | the function that is to be maximized or minimized in an optimization problem

**oblique asymptote** | the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or$x \to -\infty$

**octants** | the eight regions of space created by the coordinate planes

**odd function** | a function is odd if $f(-x) = -f(x)$ for all $x$ in the domain of $f$

**one-sided limit** | A one-sided limit of a function is a limit taken from either the left or the right

**one-to-one function** | a function $f$ is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

**one-to-one transformation** | a transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ is said to be one-to-one if no two points map to the same image point

**open set** | a set $S$ that contains none of its boundary points

**optimization problem** | calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

**optimization problems** | problems that are solved by finding the maximum or minimum value of a function

**order of a differential equation** | the highest order of any derivative of the unknown function that appears in the equation

**orientation** | the direction that a point moves on a graph as the parameter increases

**orientation of a curve** | the orientation of a curve $C$ is a specified direction of $C$

**orientation of a surface** | if a surface has an "inner" side and an "outer" side, then an orientation is a choice of the inner or the outer side; the surface could also have "upward" and "downward" orientations

**orthogonal vectors** | vectors that form a right angle when placed in standard position

**osculating circle** | a circle that is tangent to a curve $C$ at a point $P$ and that shares the same curvature

**osculating plane** | the plane determined by the unit tangent and the unit normal vector

## P

***p*-series** | a series of the form $\sum_{n=1}^{\infty} 1/n^p$

**parallelogram method** | a method for finding the sum of two vectors; position the vectors so they share the same initial point; the vectors then form two adjacent sides of a parallelogram; the sum of the vectors is the diagonal of that parallelogram

**parameter** | an independent variable that both $x$ and $y$ depend on in a parametric curve; usually represented by the variable $t$

**parameter domain (parameter space)** | the region of the $uv$-plane over which the parameters $u$ and $v$ vary for parameterization $\vecs r(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$

**parameterization of a curve** | rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

**parameterized surface (parametric surface)** | a surface given by a description of the form $\vecs r(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$, where the parameters $u$ and $v$ vary over a parameter domain in the $uv$-plane

**parametric curve** | the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

**parametric equations** | the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

**parametric equations of a line** | the set of equations $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$ describing the line with direction vector $v = \langle a, b, c\rangle$ passing through point $(x_0, y_0, z_0)$

**partial derivative** | a derivative of a function of more than one independent variable in which all the variables but one are held constant

**partial differential equation** | an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives

**partial fraction decomposition** | a technique used to break down a rational function into the sum of simple rational functions

**partial sum** | the $kth$ partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$

**particular solution** | member of a family of solutions to a differential equation that satisfies a particular initial condition

**particular solution** | a solution $y_p(x)$ of a differential equation that contains no arbitrary constants

**partition** | a set of points that divides an interval into subintervals

**percentage error** | the relative error expressed as a percentage

**periodic function** | a function is periodic if it has a repeating pattern as the values of $x$ move from left to right

**phase line** | a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

**piecewise smooth curve** | an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

**piecewise-defined function** | a function that is defined differently on different parts of its domain

**planar transformation** | a function $T$ that transforms a region $G$ in one plane into a region $R$ in another plane by a change of variables

**plane curve** | the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$

**point-slope equation** | equation of a linear function indicating its slope and a point on the graph of the function

**polar axis** | the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

**polar coordinate system** | a system for locating points in the plane. The coordinates are $r$, the radial coordinate, and $\theta$, the angular coordinate

**polar equation** | an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

**polar rectangle** | the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$

**pole** | the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

**polynomial function** | a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$

**population growth rate** | is the derivative of the population with respect to time

**potential function** | a scalar function $f$ such that $\vecs{\nabla} f = \vecs{F}$

**power function** | a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

**power law for limits** | the limit law

$$\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n = L^n$$

for every positive integer n

**power reduction formula** | a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

**power rule** | the derivative of a power function is a function in which the power on $x$ becomes the coefficient of the term and the power on $x$ in the derivative decreases by 1: If $n$ is an integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

**power series** | a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x - a)^n$ is a power series centered at $x = a$

**principal unit normal vector** | a vector orthogonal to the unit tangent vector, given by the formula $\frac{\vecs{T}'(t)}{\|\vecs{T}'(t)\|}$

**principal unit tangent vector** | a unit vector tangent to a curve *C*

**product law for limits** | the limit law

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

**product rule** | the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}\big(f(x)g(x)\big) = f\,'(x)g(x) + g\,'(x)f(x)$

**projectile motion** | motion of an object with an initial velocity but no force acting on it other than gravity

**propagated error** | the error that results in a calculated quantity $f(x)$ resulting from a measurement error $dx$

## Q

**quadratic function** | a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

**quadric surfaces** | surfaces in three dimensions having the property that the traces of the surface are conic sections (ellipses, hyperbolas, and parabolas)

**quotient law for limits** | the limit law $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for M≠0

**quotient rule** | the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}$

## R

**radial coordinate** | $r$ the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

**radial field** | a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

**radians** | for a circular arc of length $s$ on a circle of radius 1, the radian measure of the associated angle $\theta$ is $s$

**radius of convergence** | if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for \(|x−a|R\), then $R$ is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers $x$, the radius of convergence is $R = \infty$

**radius of curvature** | the reciprocal of the curvature

**radius of gyration** | the distance from an object's center of mass to its axis of rotation

**range** | the set of outputs for a function

**ratio test** | for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

**rational function** | a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

**recurrence relation** | a recurrence relation is a relationship in which a term $a_n$ in a sequence is defined in terms of earlier terms in the sequence

**region** | an open, connected, nonempty subset of $\mathbb{R}^2$

**regular parameterization** | parameterization $\vecs r(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$ such that $r_u \times r_v$ is not zero for point $(u, v)$ in the parameter domain

**regular partition** | a partition in which the subintervals all have the same width

**related rates** | are rates of change associated with two or more related quantities that are changing over time

**relative error** | given an absolute error $\Delta q$ for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

**relative error** | error as a percentage of the actual value, given by

$$\text{relative error} = \left|\frac{A - B}{A}\right| \cdot 100\%$$

**remainder estimate** | for a series $\sum_{n=}^{\infty} 1a_n$ with positive terms $a_n$ and a continuous, decreasing function $f$ such that $f(n) = a_n$ for all positive integers $n$, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate: \[∫^∞_{N+1}f(x)\,dx

**removable discontinuity** | A removable discontinuity occurs at a point $a$ if $f(x)$ is discontinuous at $a$, but $\lim_{x\to a} f(x)$ exists

**reparameterization** | an alternative parameterization of a given vector-valued function

**restricted domain** | a subset of the domain of a function $f$

**riemann sum** | an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

**right-endpoint approximation** | the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

**right-hand rule** | a common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the $z$-axis in such a way that the fingers curl from the positive $x$-axis to the positive $y$-axis, the thumb points in the direction of the positive $z$-axis

***RLC* series circuit** | a complete electrical path consisting of a resistor, an inductor, and a capacitor; a second-order, constant-coefficient differential equation can be used to model the charge on the capacitor in an *RLC* series circuit

**rolle's theorem** | if $f$ is continuous over $[a, b]$ and differentiable over $(a, b)$, and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f\,'(c) = 0$

**root function** | a function of the form $f(x) = x^{1/n}$ for any integer $n \ge 2$

**root law for limits** | the limit law $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \ge 0$ if n is even

**root test** | for a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

**rose** | graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$for a positive constant $a$

**rotational field** | a vector field in which the vector at point $(x, y)$ is tangent to a circle with radius $r = \sqrt{x^2 + y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

**rulings** | parallel lines that make up a cylindrical surface

## S

**saddle point** | given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but $f$ does not have a local extremum at $(x_0, y_0)$

**scalar** | a real number

**scalar equation of a plane** | the equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ used to describe a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c\rangle$ or its alternate form $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$

**scalar line integral** | the scalar line integral of a function $f$ along a curve $C$ with respect to arc length is the integral $\int_C f\ ds$, it is the integral of a scalar function $f$ along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

**scalar multiplication** | a vector operation that defines the product of a scalar and a vector

**scalar projection** | the magnitude of the vector projection of a vector

**secant** | A secant line to a function $f(x)$ at $a$ is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{sec} = \dfrac{f(x) - f(a)}{x - a}$

**second derivative test** | suppose $f'(c) = 0$ and $f''$ is continuous over an interval containing $c$; if $f''(c) > 0$, then $f$ has a local minimum at $c$; if $f''(c) < 0$, then $f$ has a local maximum at $c$; if $f''(c) = 0$, then the test is inconclusive

**separable differential equation** | any equation that can be written in the form $y' = f(x)g(y)$

**separation of variables** | a method used to solve a separable differential equation

**sequence** | an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

**sigma notation** | (also, summation notation) the Greek letter sigma ($\Sigma$) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

**simple curve** | a curve that does not cross itself

**simple harmonic motion** | motion described by the equation $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, as exhibited by an undamped spring-mass system in which the mass continues to oscillate indefinitely

**simply connected region** | a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

**Simpson's rule** | a rule that approximates $\int_a^b f(x)\ dx$ using the area under a piecewise quadratic function. The approximation $S_n$ to $\int_a^b f(x)\ dx$ is given by

$$S_n = \frac{\Delta x}{3}\big(f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\big).$$

**skew lines** | two lines that are not parallel but do not intersect

**slicing method** | a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

**slope** | the change in $y$ for each unit change in $x$

**slope-intercept form** | equation of a linear function indicating its slope and $y$-intercept

**smooth** | curves where the vector-valued function $\vecs r(t)$ is differentiable with a non-zero derivative

**solid of revolution** | a solid generated by revolving a region in a plane around a line in that plane

**solution curve** | a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

**solution to a differential equation** | a function $y = f(x)$ that satisfies a given differential equation

**space curve** | the set of ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$

**space-filling curve** | a curve that completely occupies a two-dimensional subset of the real plane

**speed** | is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time $t$ whose velocity is given by $v(t)$

**sphere** | the set of all points equidistant from a given point known as the *center*

**spherical coordinate system** | a way to describe a location in space with an ordered triple $(\rho, \theta, \varphi)$, where $\rho$ is the distance between $P$ and the origin $(\rho \neq 0)$, $\theta$ is the same angle used to describe the location in cylindrical coordinates, and $\varphi$ is the angle formed by the positive $z$-axis and line segment $\bar{OP}$, where $O$ is the origin and $0 \leq \varphi \leq \pi$

**squeeze theorem** | states that if $f(x) \le g(x) \le h(x)$ for all $x \ne a$ over an open interval containing a and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ where L is a real number, then $\lim_{x\to a} g(x) = L$

**standard equation of a sphere** | $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ describes a sphere with center $(a, b, c)$ and radius $r$

**standard form** | the form of a first-order linear differential equation obtained by writing the differential equation in the form $y' + p(x)y = q(x)$

**standard form** | an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

**standard unit vectors** | unit vectors along the coordinate axes: $\hat{\mathbf{i}} = \langle 1, 0\rangle,\ \hat{\mathbf{j}} = \langle 0, 1\rangle$

**standard-position vector** | a vector with initial point $(0, 0)$

**steady-state solution** | a solution to a nonhomogeneous differential equation related to the forcing function; in the long term, the solution approaches the steady-state solution

**step size** | the increment hh that is added to the xx value at each step in Euler's Method

**Stokes' theorem** | relates the flux integral over a surface $S$ to a line integral around the boundary $C$ of the surface $S$

**stream function** | if $\vecs F = \langle P, Q\rangle$ is a source-free vector field, then stream function $g$ is a function such that $P = g_y$ and $Q = -g_x$

**sum law for limits** | The limit law $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$

**sum rule** | the derivative of the sum of a function $f$ and a function $g$ is the same as the sum of the derivative of $f$ and the derivative of $g$: $\frac{d}{dx}\big(f(x) + g(x)\big) = f\,'(x) + g\,'(x)$

**surface** | the graph of a function of two variables, $z = f(x, y)$

**surface area** | the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

**surface area** | the area of surface $S$ given by the surface integral

$$\iint_S dS$$

**surface independent** | flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

**surface integral** | an integral of a function over a surface

**surface integral of a scalar-valued function** | a surface integral in which the integrand is a scalar function

**surface integral of a vector field** | a surface integral in which the integrand is a vector field

**symmetric equations of a line** | the equations $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ describing the line with direction vector $v = \langle a, b, c\rangle$ passing through point $(x_0, y_0, z_0)$

**symmetry about the $y$-axis** | the graph of a function $f$ is symmetric about the $y$-axis if $(-x, y)$ is on the graph of $f$ whenever $(x, y)$ is on the graph

**symmetry about the origin** | the graph of a function $f$ is symmetric about the origin if $(-x, -y)$ is on the graph of $f$ whenever $(x, y)$ is on the graph

**symmetry principle** | the symmetry principle states that if a region $R$ is symmetric about a line $I$, then the centroid of $R$ lies on $I$

## T

**table of values** | a table containing a list of inputs and their corresponding outputs

**tangent** | A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with $x$-values that approach $a$; the slope of the tangent line to a graph at $a$ measures the rate of change of the function at $a$

**tangent line approximation (linearization)** | since the linear approximation of $f$ at $x = a$ is defined using the equation of the tangent line, the linear approximation of $f$ at $x = a$ is also known as the tangent line approximation to $f$ at $x = a$

**tangent plane** | given a function $f(x, y)$ that is differentiable at a point $(x_0, y_0)$, the equation of the tangent plane to the surface $z = f(x, y)$ is given by $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

**tangent vector** | to $\vecs r(t)$ at $t = t_0$ any vector $\vecs v$ such that, when the tail of the vector is placed at point $\vecs r(t_0)$ on the graph, vector $\vecs v$ is tangent to curve $C$

**tangential component of acceleration** | the coefficient of the unit tangent vector $\vecs T$ when the acceleration vector is written as a linear combination of $\vecs T$ and $\vecs N$

**Taylor polynomials** | the $n^{\text{th}}$-degree Taylor polynomial for $f$ at $x = a$ is $p_n(x) = f(a) + f\,'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$

**Taylor series** | a power series at $a$ that converges to a function $f$ on some open interval containing $a$.

**Taylor's theorem with remainder** | for a function $f$ and the $n^{\text{th}}$-degree Taylor polynomial for $f$ at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some$c$ between $x$ and $a$; if there exists an interval $I$ containing $a$ and a real number $M$ such that $|\, f^{(n+1)}(x)\,| \leq M$ for all $x$ in $I$, then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$

**telescoping series** | a telescoping series is one in which most of the terms cancel in each of the partial sums

**term** | the number $a_n$ in the sequence $a_n$ is called the $nth$ term of the sequence

**term-by-term differentiation of a power series** | a technique for evaluating the derivative of a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ by evaluating the derivative of each term separately to create the new power series $\displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

**term-by-term integration of a power series** | a technique for integrating a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ by integrating each term separately to create the new power series $C + \displaystyle\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

**terminal point** | the endpoint of a vector

**theorem of Pappus for volume** | this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

**three-dimensional rectangular coordinate system** | a coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple $(x, y, z)$ that plots its location relative to the defining axes

**threshold population** | the minimum population that is necessary for a species to survive

**total area** | total area between a function and the $x$-axis is calculated by adding the area above the $x$-axis and the area below the $x$-axis; the result is the same as the definite integral of the absolute value of the function

**total differential** | the total differential of the function $f(x, y)$ at $(x_0, y_0)$ is given by the formula $dz = f_x(x_0, y_0)dx + fy(x_0, y_0)dy$

**trace** | the intersection of a three-dimensional surface with a coordinate plane

**transcendental function** | a function that cannot be expressed by a combination of basic arithmetic operations

**transformation** | a function that transforms a region GG in one plane into a region RR in another plane by a change of variables

**transformation of a function** | a shift, scaling, or reflection of a function

**trapezoidal rule** | a rule that approximates $\displaystyle\int_a^b f(x)\ dx$ using the area of trapezoids. The approximation $T_n$ to $\displaystyle\int_a^b f(x)\ dx$ is given by

$$T_n = \frac{\Delta x}{2}\Big(f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\Big).$$

**tree diagram** | illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

**triangle inequality** | If $a$ and $b$ are any real numbers, then $|a+b| \leq |a| + |b|$

**triangle inequality** | the length of any side of a triangle is less than the sum of the lengths of the other two sides

**triangle method** | a method for finding the sum of two vectors; position the vectors so the terminal point of one vector is the initial point of the other; these vectors then form two sides of a triangle; the sum of the vectors is the vector that forms the third side; the initial point of the sum is the initial point of the first vector; the terminal point of the sum is the terminal point of the second vector

**trigonometric functions** | functions of an angle defined as ratios of the lengths of the sides of a right triangle

**trigonometric identity** | an equation involving trigonometric functions that is true for all angles $\theta$ for which the functions in the equation are defined

**trigonometric integral** | an integral involving powers and products of trigonometric functions

**trigonometric substitution** | an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral

**triple integral** | the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box $B$ is the limit of a Riemann sum for a function of three variables, if this limit exists

**triple integral in cylindrical coordinates** | the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, s^*_{ijk}) r^*_{ijk} \Delta r \Delta\theta \Delta z$$

**triple integral in spherical coordinates** | the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(\rho_{ijk}^{*},\theta_{ijk}^{*},\varphi_{ijk}^{*})(\rho_{ijk}^{*})^{2}\sin\,\varphi\Delta\rho\Delta\theta\Delta\varphi$$

**Type I** | a region $D$ in the $xy$- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

**Type II** | a region $D$ in the $xy$-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(h)$

## U

**unbounded sequence** | a sequence that is not bounded is called unbounded

**unit vector** | a vector with magnitude $1$

**unit vector field** | a vector field in which the magnitude of every vector is 1

**upper sum** | a sum obtained by using the maximum value of $f(x)$ on each subinterval

## V

**variable of integration** | indicates which variable you are integrating with respect to; if it is $x$, then the function in the integrand is followed by $dx$

**vector** | a mathematical object that has both magnitude and direction

**vector addition** | a vector operation that defines the sum of two vectors

**vector difference** | the vector difference $\vecs v - \vecs w$ is defined as $\vecs v + (-\vecs w) = \vecs v + (-1)\vecs w$

**vector equation of a line** | the equation $\vecs r = \vecs r_0 + t\vecs v$ used to describe a line with direction vector $\vecs v = \langle a, b, c\rangle$ passing through point $P = (x_0, y_0, z_0)$, where $\vecs r_0 = \langle x_0, y_0, z_0\rangle$, is the position vector of point $P$

**vector equation of a plane** | the equation $\vecs n \cdot \vecd PQ = 0$, where $P$ is a given point in the plane, $Q$ is any point in the plane, and $\vecs n$ is a normal vector of the plane

**vector field** | measured in $\mathbb{R}^2$, an assignment of a vector $\vecs F(x, y)$ to each point $(x, y)$ of a subset $D$ of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\vecs F(x, y, z)$ to each point $(x, y, z)$ of a subset $D$ of $\mathbb{R}^3$

**vector line integral** | the vector line integral of vector field $\vecs F$ along curve $C$ is the integral of the dot product of $\vecs F$ with unit tangent vector $\vecs T$ of $C$ with respect to arc length, $\int_C \vecs F \cdot \vecs T\ ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

**vector parameterization** | any representation of a plane or space curve using a vector-valued function

**vector projection** | the component of a vector that follows a given direction

**vector sum** | the sum of two vectors, $\vecs v$ and $\vecs w$, can be constructed graphically by placing the initial point of $\vecs w$ at the terminal point of $\vecs v$; then the vector sum $\vecs v + \vecs w$ is the vector with an initial point that coincides with the initial point of $\vecs v$, and with a terminal point that coincides with the terminal point of $\vecs w$

**vector-valued function** | a function of the form $\vecs r(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vecs r(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,where the component functions $f$, $g$, and $h$ are real-valued functions of the parameter $t$.

**velocity vector** | the derivative of the position vector

**vertex** | a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

**vertical asymptote** | A function has a vertical asymptote at $x = a$ if the limit as $x$ approaches $a$ from the right or left is infinite

**vertical line test** | given the graph of a function, every vertical line intersects the graph, at most, once

**vertical trace** | the set of ordered triples $(c, y, z)$ that solves the equation $f(c, y) = z$ for a given constant $x = c$ or the set of ordered triples $(x, d, z)$ that solves the equation $f(x, d) = z$ for a given constant $y = d$

## W

**washer method** | a special case of the slicing method used with solids of revolution when the slices are washers

**work** | the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

**work done by a force** | work is generally thought of as the amount of energy it takes to move an object; if we represent an applied force by a vector $\vecs F$ and the displacement of an object by a vector $\vecs s$, then the work done by the force is the dot product of $\vecs F$ and $\vecs s$.

## Z

**zero vector** | the vector with both initial point and terminal point $(0, 0)$

**zeros of a function** | when a real number $x$ is a zero of a function $f$, $f(x) = 0$

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