BUILDING TOWARD CALCULUS



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Algebra and Trigonometry: Building Toward Calculus

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Course Content

Unit 1 — Functions (FN)

- **FN1** — Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.
- **FN2** — Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.
- **FN3** — Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.
- **FN4** — Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.
- **FN5** — Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.
- **FN6** — Find the inverse of a one-to-one function.

Unit 1 — Linear Functions (LF)

- **LF1** — Determine the average rate of change of a given function over a given interval. Find the slope of a line.
- **LF2** — Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.
- **LF3** — Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.
- **LF5** — Build linear models from verbal descriptions, and use the models to establish conclusions, including by contextualizing the meaning of slope and intercept parameters.

Unit 1 — Polynomial and Rational Functions (PR1–PR2, quadratics)

- **PR1** — Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.
- **PR2** — Use quadratic models to solve an application problem and establish conclusions.

Unit 2 — Trigonometric Functions (TR)

- **TR1** — Convert between degrees and radians. Draw angles in standard position.
- **TR2** — Identify and find coterminal angles. Find the length of a circular arc.
- **TR3** — Use a right triangle to evaluate trigonometric functions.
- **TR4** — Find exact values of trigonometric functions of special angles (30°, 45°, and 60°).
- **TR5** — Use reference angles, signs, and definitions to determine exact values of trigonometric functions.

Unit 2 — Periodic Functions (PF)

- **PF1** — Determine the basic properties of the graphs of sine and cosine, including amplitude, period, and phase shift.
- **PF2** — Graph trigonometric functions including tangent, cotangent, secant, and cosecant functions and transformations of these functions, and determine the domain and range.
- **PF3** — Determine the inverse sine, cosine, and tangent values; graph inverse trig functions and determine the limitations on the domain and range.

Unit 3 — Exponential and Logarithmic Functions (EL)

- **EL1** — Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base *e*).
- **EL2** — Graph exponential functions and determine the domain, range, and asymptotes.
- **EL3** — Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms.
- **EL4** — Graph logarithmic functions and determine the domain, range, and asymptotes.
- **EL5** — Use properties of logarithms to condense or expand logarithmic expressions.
- **EL6** — Solve exponential and logarithmic equations.

Unit 4 — Trigonometric Equations (TE)

- **TE1** — Use the sum and difference, the double-angle, and power-reducing formulas for cosine, sine, and tangent and the Pythagorean identities.
- **TE2** — Symbolically verify identities by using various trig identities.
- **TE3** — Find all solutions to a trigonometric equation.
- **TE4** — Use trigonometric functions and the Pythagorean Theorem to solve right triangles.
- **TE5** — Use the law of sines and/or the law of cosines to solve non-right triangles.
- **TE6** — Solve applications that require the use of trigonometry to find a side or angle.

Unit 5 — Polynomial and Rational Functions (PR3–PR9)

- **PR3** — Determine the zeros and their multiplicities of a polynomial in factored form. Describe and graph the behavior of a polynomial function at the intercepts and the ends.
- **PR4** — Rewrite a rational function as a polynomial plus a proper rational function (polynomial long division).
- **PR5** — Determine the zeros of a polynomial function with real coefficients.
- **PR6** — Solve rational equations.
- **PR7** — Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.
- **PR8** — Solve quadratic inequalities and express the solution graphically and with interval notation.
- **PR9** — Solve rational inequalities and express the solution graphically and using interval notation.

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1: RELATING CHANGING QUANTITIES

Functions are tools for describing and understanding relationships and change. In this unit, students learn what a function is and how to read, notate, and interpret its graph; how linear and quadratic functions model real-world situations; and how combining, composing, inverting, and transforming a function changes its behavior and meaning. The focus is on interpreting meaning — not just calculating answers — as a foundation for calculus and STEM problem-solving.

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1.1: Introduction to Functions (FN1)

1.1.1: Functions (FN1)

FN1 — Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.

Think about texting a friend: type "yes" and hit send, and every single time, "yes" is what arrives on their end — never "no," never "maybe," always the same three letters for that same input. That reliability is the entire idea behind a function: a process where one input always, dependably, produces exactly one output. It sounds almost too simple to name, but distinguishing genuine functions from relationships that don't behave this way — a menu where two items happen to cost the same, a chart where one x-value quietly maps to two different y-values — turns out to matter enormously once you start building models of the world.

You've already worked with relationships between quantities — ordered pairs, tables of x- and y-values — in your prior algebra coursework. This section asks you to look at those same familiar objects with a sharper question in mind: not just what the relationship is, but whether it's a function at all.

Once you can reliably identify a function, Section 1.2 turns to function notation — the shorthand $f(x)$ that lets you talk about functions precisely — and the visual features of a function's graph that notation makes it possible to describe.

> **Looking ahead to calculus:** Nearly every question calculus asks — how fast is this changing? what does this approach? what's the total accumulated effect? — is a question about a function's behavior. Before any of that is possible, you need to be confident you're looking at a genuine function to begin with, with each input tied to exactly one output. The vertical line test you're about to use is your first version of a check working mathematicians rely on constantly.

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1.1.1: Functions (FN1)

This section corresponds to APC slot 1.1 (Changing in tandem) & 1.2 (Functions: modeling relationships) in the Math 172 flow.

FN1 Learning outcome: Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.

Relations and Functions

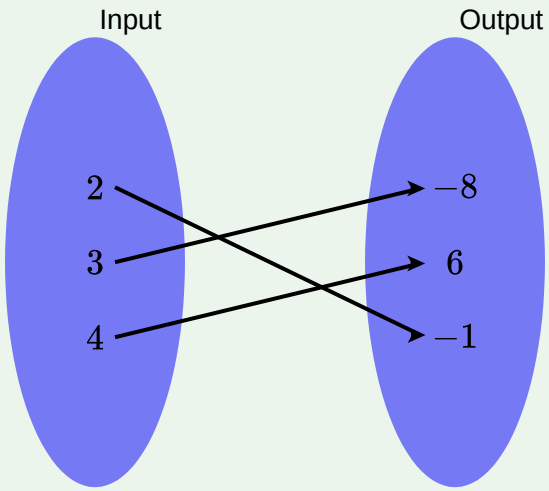
Definition: Relation

A **relation** is a relationship between sets of values. Relations in mathematics are usually represented as ordered pairs: (input, output) or (x, y). When observing relations, we often refer to the x-values as the **domain** and the y-values as the **range**.

Definition: Mapping Notation

Mapping Notation (also known as an arrow diagram) is a way to show relationships visually between sets. For example, suppose you are given the following ordered pairs: $(3, -8), (4, 6)$, and $(2, -1)$. Each of the x-values "map onto" a y-value and can be visualized in the following way:

Notice that an arrow is used to indicate which x-value is mapped onto its corresponding y-value.



Activity 1.1.1.1*

Use mapping notation to create a visual representation of the following relation.

$$(-1, 5), (2, 6), (4, -2) \tag{1.1.1.1}$$

(a)

What is the domain?

A. $\{5, 6, -2\}$
B. $\{-1, 2, 4\}$
C. $\{-2, -1, 2, 4, 5, 6\}$

Answer available at the end of this section.

(b)

What is the range?

A. $\{5, 6, -2\}$
B. $\{-1, 2, 4\}$
C. $\{-2, -1, 2, 4, 5, 6\}$

Answer available at the end of this section.

Activity 1.1.1.2

Use mapping notation to create a visual representation of the following relation.

$$(6, 4), (3, 4), (6, 5) \tag{1.1.1.2}$$

(a)

What is the domain?

A. $\{3, 6\}$
B. $\{6, 3, 6\}$
C. $\{3, 4, 5, 6\}$
D. $\{4, 5\}$

Answer available at the end of this section.

(b)

What is the range?

A. $\{3, 6\}$
B. $\{6, 3, 6\}$
C. $\{3, 4, 5, 6\}$
D. $\{4, 5\}$

Answer available at the end of this section.

Activity 1.1.1.3

Use mapping notation to create a visual representation of the following relation.

$$(1, 2), (-5, 2), (-7, 2) \tag{1.1.1.3}$$

(a)

What is the domain?

A. $\{2, 2, 2\}$
B. $\{-7, -5, 1, 2\}$
C. $\{-7, -5, 1\}$
D. $\{2\}$

Answer available at the end of this section.

(b)

What is the range?

A. $\{2, 2, 2\}$
B. $\{-7, -5, 1, 2\}$
C. $\{-7, -5, 1\}$
D. $\{2\}$

Answer available at the end of this section.

Activity 1.1.1.4

Use mapping notation to create a visual representation of the following relation.

$$(3, -2), (-4, -1), (3, 5) \tag{1.1.1.4}$$

(a)

What is the domain?

A. $\{-4, -2, -1, 3, 5\}$
B. $\{-4, 3\}$
C. $\{-2, -1, 5\}$
D. $\{-4, 3, 3\}$

Answer available at the end of this section.

(b)

What is the range?

A. $\{-4, -2, -1, 3, 5\}$
B. $\{-4, 3\}$
C. $\{-2, -1, 5\}$
D. $\{-4, 3, 3\}$

Answer available at the end of this section.

Activity 1.1.1.5

Now that you have created a visual representation for each of *Activty 1-4*, what similarities do you see? What differences?

Answer available at the end of this section.

Activity 1.1.1.6

Consider the menus shown below for two different fast food restaurants. We'll consider the Items as inputs (x-values) and the Cost as outputs (y-values).

McRonald's Item	Cost
Nuggets	$5

Burger	$6
Fries	$3
Nachos	$6

Burger Queen's Item	Cost
Nuggets	$4
Burger	$8
Fries	$2
Nuggets	$7

(a)

How much would it cost to get nuggets and fries at McRonald's?

A. $8
B. $9
C. $11
D. $12

Answer available at the end of this section.

(b)

How much would it cost to get nuggets and fries at Burger Queen?

A. $6
B. $9
C. $10
D. $12

Answer available at the end of this section.

(c)

Notice that burgers and nachos cost the same amount at McRonald's. Is that an issue? Explain your reasoning.

Answer available at the end of this section.

(d)

Notice on the Burger Queen menu that nuggets cost $4 in one spot but $7 in another spot. Is that an issue? Explain your reasoning.

Answer available at the end of this section.

Definition: Function

A **function** is a relation where every input (or x-value) is mapped onto *exactly one* output (or y-value).

Note that all functions are relations but not all relations are functions!

Note

We see that in *Activity 6*, the McRonald's menu is a function. There is no confusion in determining the cost of a given menu item. However, Burger Queen's menu is not a function. There is a discrepancy in the cost of nuggets, and the menu could cause confusion.

Activity 1.1.1.7

Relations can be expressed in multiple ways (ordered pairs, tables, and verbal descriptions).

(a)

Let's revisit some of the sets of ordered pairs we've previously explored in Activties 1-4. Which of the following sets of ordered pairs represent a function?

A. $(-1, 5), (2, 6), (4, -2)$
B. $(6, 4), (3, 4), (6, 5)$
C. $(1, 2), (-5, 2), (-7, 2)$
D. $(-1, 2), (-1, 9), (1, 9)$

Answer available at the end of this section.

(b)

Note that relations can be expressed in a table. A table of values is shown below. Is this an example of a function? Why or why not?

x	y
-5	-2
-4	-5
-2	8
8	-4
8	1

Answer available at the end of this section.

(c)

Relations can also be expressed in words. Suppose you are looking at the amount of time you spend studying versus the grade you earn in your Algebra class. Is this an example of a function? Why or why not?

Answer available at the end of this section.

Note

Notice that when trying to determine if a relation is a function, we often have to rely on looking at the domain and range values. Thus, it is important to be able to idenfity the domain and range of any relation!

Activity 1.1.1.8

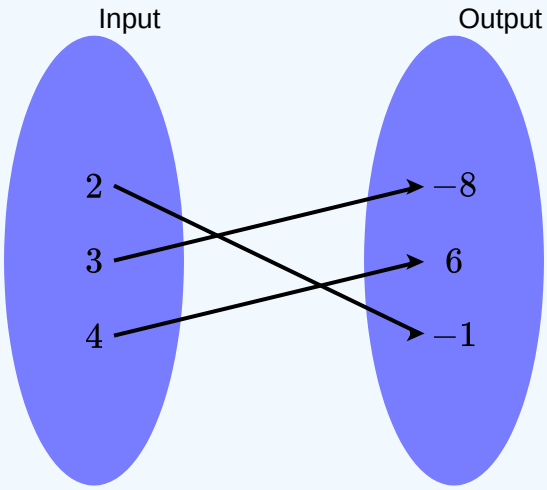
For each of the given functions, determine the domain and range.

(a)

$(-4, 3), (-1, 8), (7, 4), (1, 9)$

Answer available at the end of this section.

(b)



Answer available at the end of this section.

(c)

x	y
-2	5
0	4
3	6
8	1

Answer available at the end of this section.

(d)

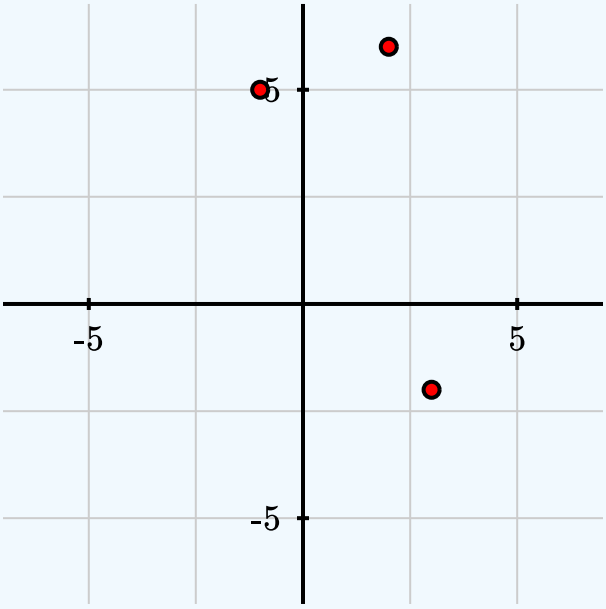
The amount of time you spend studying versus the grade you earn in your Algebra class.

Answer available at the end of this section.

Activity 1.1.1.9

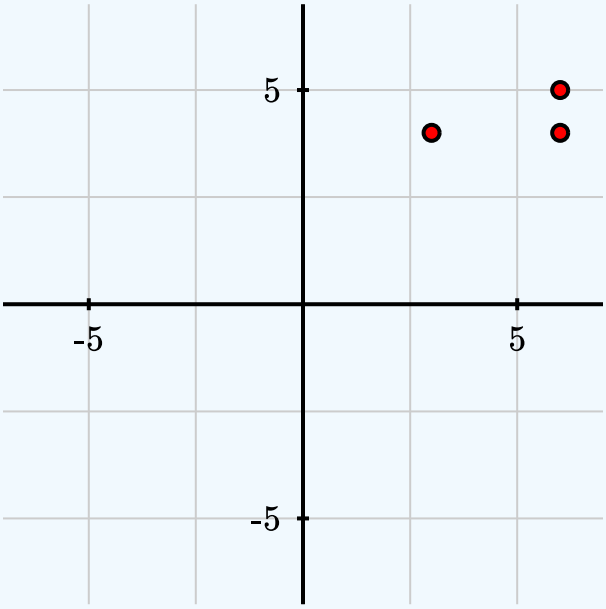
Determine whether each of the following relations is a function.

(a)



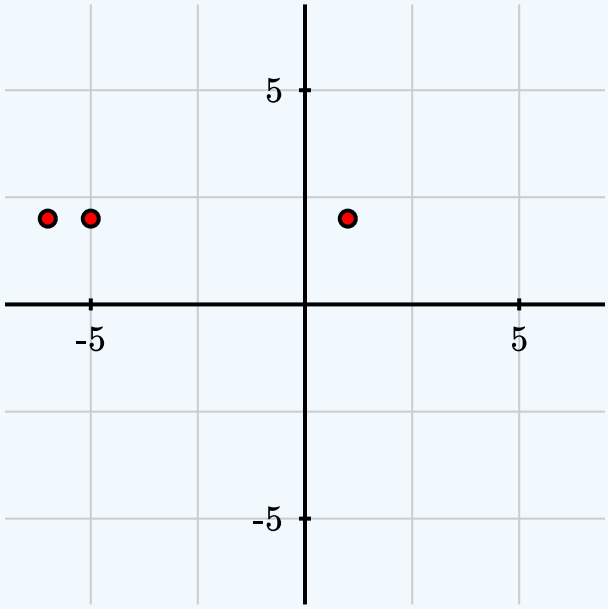
Answer available at the end of this section.

(b)



Answer available at the end of this section.

(c)



Answer available at the end of this section.

Note

You probably noticed (in *Activity 9*) that when the graph has points that "line up" or are on top of each other, they have the same x-values. When this occurs, this shows that the same x-value has two different outputs (y-values) and that the relation is not a function.

Fact: Vertical Line Test

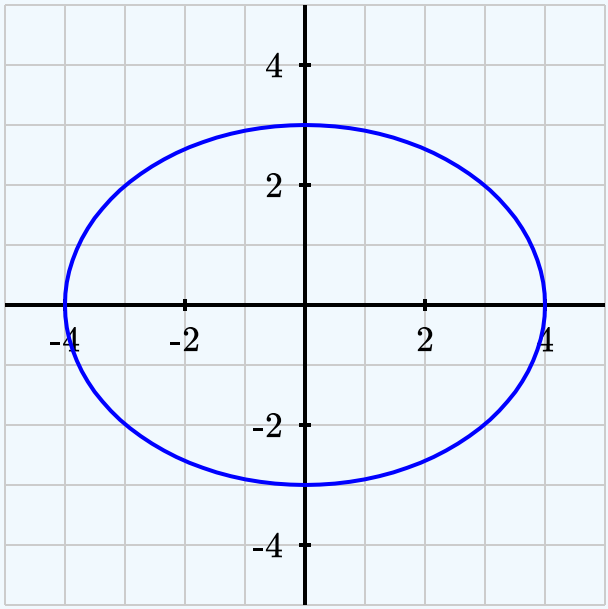
The **vertical line test** is a method used to determine whether a relation described by a graph is a function.

If there is any vertical line that intersects the graph in multiple places, the relation described by that graph is not a function.

Activity 1.1.1.10

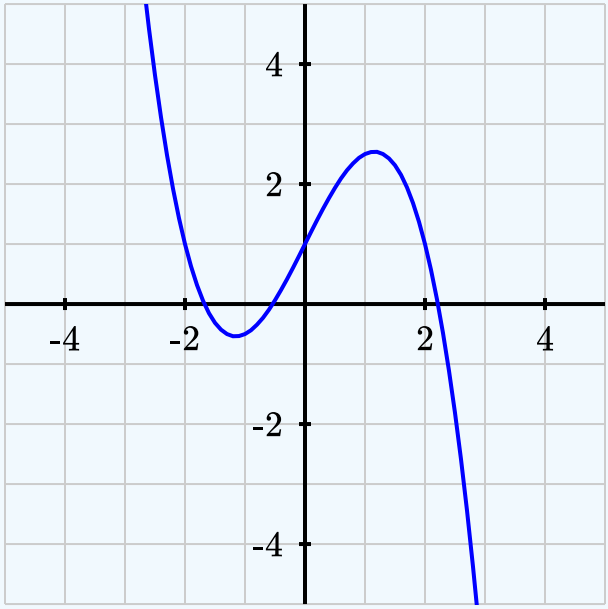
Use the vertical line test (*see Actiivty 9*) to determine whether each graph of a relation represents a function.

(a)



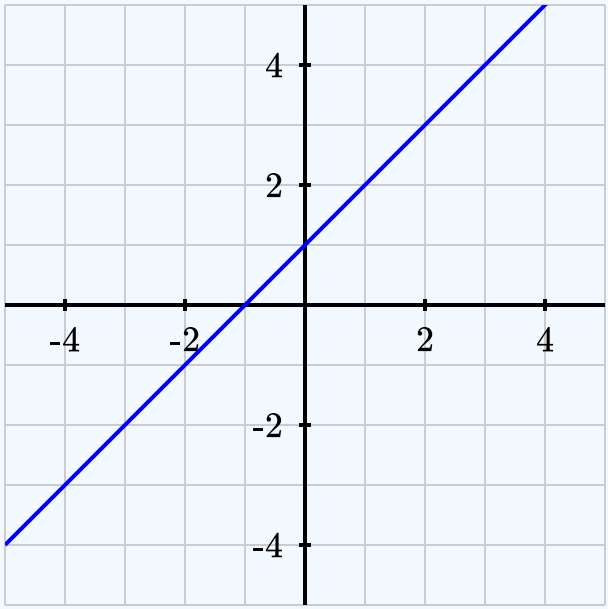
Answer available at the end of this section.

(b)



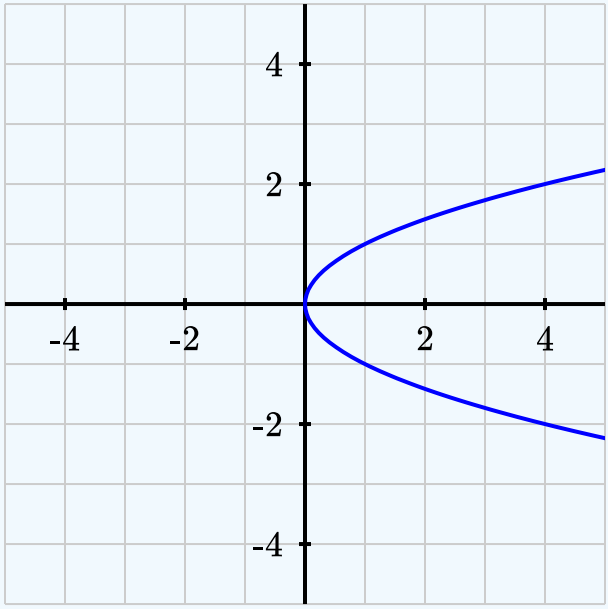
Answer available at the end of this section.

(c)



Answer available at the end of this section.

(d)



Answer available at the end of this section.

Activity 1.1.1.11

Let's explore how to determine whether an equation represents a function.

(a)

Suppose you are given the equation $x = y^2$.

- If $x = 4$, what kind of y-values would you get for $x = y^2$?
- Based on this information, do you think $x = y^2$ is a function?

Answer available at the end of this section.

(b)

Suppose you are given the equation $y = 3x^2 + 2$.

- If $x = 4$, what kind of y-values would you get for $y = 3x^2 + 2$?
- Based on this information, do you think $y = 3x^2 + 2$ is a function?

Answer available at the end of this section.

(c)

Suppose you are given the equation $x^2 + y^2 = 25$.

- If $x = 4$, what kind of y-values would you get for $x^2 + y^2 = 25$?
- Based on this information, do you think $x^2 + y^2 = 25$ is a function?

Answer available at the end of this section.

(d)

Suppose you are given the equation $y = -4x - 3$.

- If $x = 4$, what kind of y-values would you get for $y = -4x - 3$?
- Based on this information, do you think $y = -4x - 3$ is a function?

Answer available at the end of this section.

(e)

How can you look at an equation to determine whether or not it is a function?

Answer available at the end of this section.

Note

Notice that *(see Activity 11)* shows that equations with a y^2 term generally do not define functions. This is because to solve for a squared variable, you must consider both positive and negative inputs. For example, both $2^2 = 4$ and $(-2)^2 = 4$.

Activity 1.1.1.12

It's important to be able to determine the domain of any equation, especially when thinking about functions. Answer the following questions given the equation $y = \sqrt{x - 2}$.

(a)

Which of the following values of x would cause y to be undefined (if any)?

A. -2
B. 0
C. 2
D. 4

E. none of the above

Answer available at the end of this section.

(b)

Based on this information, for which of the following values of x would y be defined?

A. -2
B. 0
C. 2
D. 4
E. none of the above

Answer available at the end of this section.

(c)

There are many more x- values than just those found above that, when plugged in, give a defined value for y. How can we represent the domain of this equation in interval notation?

A. $(-\infty, 2)$
B. $(-\infty, 2]$
C. $[2, \infty)$
D. $(2, \infty)$
E. $(-\infty, \infty)$

Answer available at the end of this section.

Activity 1.1.1.13

Answer the following questions given the equation $y = -5x + 1$.

(a)

Which of the following values of x would cause y to be undefined (if any)?

A. -2
B. 0
C. 4
D. -5
E. none of the above

Answer available at the end of this section.

(b)

Based on this information, for which of the following values of x would y be defined?

A. -2
B. 0
C. 4
D. -5
E. none of the above

Answer available at the end of this section.

(c)

How can we represent the domain of this equation in interval notation?

A. $(-\infty, 0)$
B. $(0, \infty)$
C. $(-5, 1)$
D. $(-\infty, \infty)$

Answer available at the end of this section.

Activity 1.1.1.14

Answer the following questions given the equation $y = \dfrac{3}{x-5}$.

(a)

Which of the following values of x would cause y to be undefined (if any)?

A. -3
B. 0
C. -4
D. 5
E. none of the above

Answer available at the end of this section.

(b)

Based on this information, for which of the following values of x would y be defined?

A. -3
B. 0
C. -4
D. 5
E. none of the above

Answer available at the end of this section.

(c)

How can we represent the domain of this equation in interval notation?

A. $(-\infty, 5)$
B. $(5, \infty)$
C. $(-5, 5)$
D. $(-\infty, 5) \text{U} (5, \infty)$

Answer available at the end of this section.

Activity 1.1.1.15

Find the domain of each of the following functions. Write your answer in interval notation.

(a)

$f(x) = \dfrac{x+3}{(x-2)(x+5)}$

Answer available at the end of this section.

(b)

$f(x) = \sqrt{2x - 5}$

Answer available at the end of this section.

(c)

$f(x) = \dfrac{2}{\sqrt{4 - x}}$

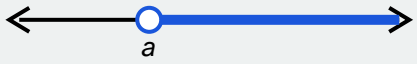
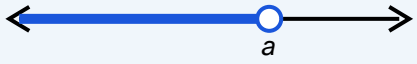
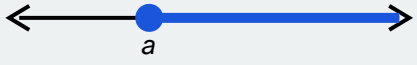
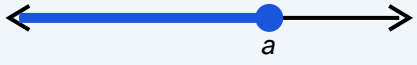
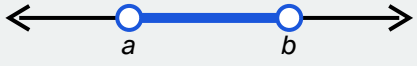
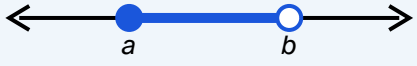
Answer available at the end of this section.

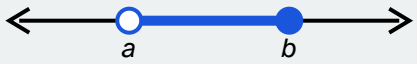
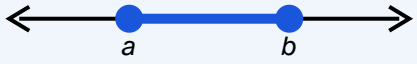
Note

When determining the domain of an equation, it is often easier to first find values of x that make the function undefined. Once you have those values, then you know that x can be any value but those.

Summary of Inerval Notation

Interval Notation Summary

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a

Inequality	Interval Notation	Graph on Number Line	Description
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Check Your Understanding

Work these on your own to confirm you can apply the definition of a function before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.1.1.1

Does the following set of ordered pairs define a function? Explain why or why not.

$$(-3, 7),\ (0, 4),\ (2, 7),\ (5, 1)$$

Answer available at the end of this section.

Check Your Understanding 1.1.1.2

Does the following table define a function? Explain why or why not.

x	y
1	4
3	9
3	2
6	5

Answer available at the end of this section.

Check Your Understanding 1.1.1.3

Does the equation $y = 2x - 5$ define y as a function of x? Explain why or why not.

Answer available at the end of this section.

Answers

Activity 1.1.1.1*

B

Activity 1.1.1.1*

A

Activity 1.1.1.2

A

Activity 1.1.1.2

D

Activity 1.1.1.3

C

Activity 1.1.1.3

D

Activity 1.1.1.4

B

Activity 1.1.1.4

C

Activity 1.1.1.5

Students may mention that some have repeated x-values, some have repeated y-values, and some don't.

Activity 1.1.1.6

A

Activity 1.1.1.6

A, B, or not able to be determined. Notice that that menu is not a function since nuggets are a repeated item.

Activity 1.1.1.6

No. It's okay for two items to have the same price.

Activity 1.1.1.6

Yes. It's not clear how much nuggets cost.

Activity 1.1.1.7

A and C

Activity 1.1.1.7

This relation is not a function. The x-value, 8, has two outputs.

Activity 1.1.1.7

Yes, this is an example of a function.

Activity 1.1.1.8

Domain: $\{-4, -1, 1, 7\}$; Range: $\{3, 4, 8, 9\}$

Activity 1.1.1.8

Domain: $\{2, 3, 4\}$; Range: $\{-8, -1, 6\}$

Activity 1.1.1.8

Domain: $\{-2, 0, 3, 8\}$; Range: $\{1, 4, 5, 6\}$

Activity 1.1.1.8

Domain: {time studying}; Range: {grade in Algebra class}

Activity 1.1.1.9

This relation is a function.

Activity 1.1.1.9

This relation IS NOT a function.

Activity 1.1.1.9

This relation is a function.

Activity 1.1.1.10

This graph does not pass the vertical line test and is therefore NOT a function.

Activity 1.1.1.10

This graph passes the vertical line test and is therefore a function.

Activity 1.1.1.10

This graph passes the vertical line test and is therefore a function.

Activity 1.1.1.10

This graph does not pass the vertical line test and is therefore NOT a function.

Activity 1.1.1.11

If $x = 4$, y could be -2 or 2. Given that there are two outputs for a given x, then $x = y^2$ is not a function.

Activity 1.1.1.11

If $x = 4$, $y = 50$. For every input (x-value), there is exactly one output (y-value) so $y = 3x^2 + 2$ is a function.

Activity 1.1.1.11

If $x = 4$, y could be -3 or 3. Given that there are two outputs for a given x, then $x^2 + y^2 = 25$ is not a function.

Activity 1.1.1.11

If $x = 4$, $y = -19$. For every input (x-value), there is exactly one output (y-value) so $y = -4x - 3$ is a function.

Activity 1.1.1.11

Students may have different answers. Ideally, you want students to see that they can check whether a given relation is a function in more than one way. They can test values (as this activity did by leading them to plug in values) or by using the vertical line test on a graph.

Activity 1.1.1.12

A and B

Activity 1.1.1.12

C and D

Activity 1.1.1.12

C

Activity 1.1.1.13

E

Activity 1.1.1.13

A, B, C, and D

Activity 1.1.1.13

D

Activity 1.1.1.14

D

Activity 1.1.1.14

A, B, and C

Activity 1.1.1.14

D

Activity 1.1.1.15

$(-\infty, -5) \cup (-5, 2) \cup (2, \infty)$

Activity 1.1.1.15

$\left[\frac{5}{2}, \infty\right)$

Activity 1.1.1.15

$(-\infty, 4)$

Check Your Understanding 1.1.1.1

Yes, this defines a function. Each input (x-value) $-3, 0, 2, 5$ is paired with exactly one output. It is fine that the output 7 is repeated for two different inputs -3 and 2 — a function only requires that no *input* is repeated with different outputs.

Check Your Understanding 1.1.1.2

No, this does not define a function. The input $x = 3$ is paired with two different outputs (9 and 2). For a relation to be a function, each input must have exactly one output.

Check Your Understanding 1.1.1.3

Yes. For every value of x you substitute, the equation produces exactly one value of y. Since no input can produce two different outputs, the equation defines y as a function of x.

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1.1.1.1: FN1 Online Exercises

FN1 — Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.	FN1 CheckIt Exercises

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1.1.2: FN1 FAQ and Resources

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1.2: Introduction to Function Notation (FN2) and Characteristics of a Function's Graph (FN3)

1.2.1: Function Notation (FN2)

1.2.2: Characteristics of a Function's Graph (FN3)

FN2 Function Notation

- Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.

FN3 Characteristics of a Function's Graph

- Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.

Picture your bank balance over the course of a month. A paycheck lands and the balance jumps up; rent and bills go out and it drops back down; by the day before the next paycheck, it might sit close to zero. Every day of the month has exactly one balance attached to it — that's what makes the balance a function of the day, and function notation gives us a compact way to talk about it: instead of writing "on day 15 my balance was $412," we write $B(15) = 412$. Once we have that notation, the graph of B lets us ask sharper questions — on which days is the balance climbing or falling, when does it peak, and does it ever cross zero?

This builds directly on Section 1.1, where you confirmed that a relationship like this — one balance for each day — passes the vertical line test and qualifies as a function in the first place. Here, we go a step further: not just recognizing a function, but reading one closely enough to describe its shape.

Section 1.3 picks up right where the "climbing or falling" question leaves off, turning a vague "the balance went up" into a precise number: the average rate of change between two days.

> **Looking ahead to calculus:** Later, calculus gives you a way to pinpoint a peak value exactly, without scanning a graph by eye — and it's the same tool that generates the formulas behind physical models in the first place, like a tank's volume written as a function of its depth. Learning to read maxima, minima, and intercepts off a graph now is what will make those calculus tools meaningful later.

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1.2.1: Function Notation (FN2)

This section corresponds to APC slot 1.2 (Functions: modeling relationships) in the Math 172 flow.

FN2 Learning outcome: Use and interpret function notation to evaluate a function for a given input value, and find a corresponding input value given an output value.

Function Notation

Function Notation

As we saw in the last section, we can represent functions in many ways, like using a set of ordered pairs, a graph, a description, or an equation. When describing a function with an equation, we will often use function notation.

If y is written as a function of x, like in the equation

$$y = x + 5 \tag{1.2.1.1}$$

, we can replace the y with $f(x)$ and get the function notation

$$f(x) = x + 5 \tag{1.2.1.2}$$

.

The x is the input variable, and $f(x)$ is the y-value or output that corresponds to x.

Generally, we use the letter f for functions. Other letters are okay as well; $g(x)$ and $h(x)$ are common. If we are using multiple functions at one time, we often denote them with different letters so we can refer to one without any confusion as to which function we mean.

Activity 1.2.1.1

Rewrite the following equations using function notation. In each case, assume y is a function of the variable x.

(a)

$y = 2x + 14$

Answer available at the end of this section.

(b)

$y + x = 3x^2 - 5$

Answer available at the end of this section.

(c)

$\frac{2}{x} - x^4 = y - 5$

Answer available at the end of this section.

Activity 1.2.1.2

Let $f(x) = 3x^2 - 4x + 1$. Find the value of $f(x)$ for the given values of x.

Answer available at the end of this section.

Note

If we are asked to find the value of $f(x)$ for a certain x-value, say $x = 5$, we use the notation $f(5)$ to indicate that.

Activity 1.2.1.3

A projectile is fired into the air from an initial height of 144 feet. The height $h(t)$ after t seconds is defined by the function

$$h(t) = -16t^2 + 128t + 144 \tag{1.2.1.3}$$

.

(a)

Find $h(4)$.

A. 400
B. 912
C. $-64\ t^2 + 512\ t + 576$
D. $-64t^2 + 128t + 144$

Answer available at the end of this section.

(b)

Describe what you've just found using the context of the situation.

A. After 400 seconds, the height of the projectile is 4 feet.
B. After 4 seconds, the height of the projectile is 400 feet.
C. After 4 feet, the height of the projectile is 400 seconds.
D. After 400 feet, the height of the projectile is 4 seconds.

Answer available at the end of this section.

Activity 1.2.1.4

Let $f(x)$, $g(x)$, and $h(x)$ be defined as shown.

$$\begin{aligned} f(x) &= 3x^2 - 4x + 1 \\ g(x) &= \sqrt{13 - x^2} \\ h(x) &= \frac{x^2 - 6x + 8}{x^2 - 4x + 3} \end{aligned} \tag{1.2.1.4}$$

Find the following, if they exist.

(a)

$f(-4)$, $f(0)$, and $f(2)$

Answer available at the end of this section.

(b)

$g(0)$, $g(2)$, and $g(8)$

Answer available at the end of this section.

(c)

$h(3)$, $h(4)$, and $h(10)$

Answer available at the end of this section.

Note

Sometimes functions are made up of multiple functions put together. We call these **piecewise functions**. Each piece is defined for only a certain interval, and these intervals do not overlap. When evaluating a piecewise function at a given x-value, we first need to find the interval that includes the x-value, and then plug in to the corresponding function piece.

Activity 1.2.1.5

Let $f(x)$ be a piecewise function as shown below.

$$f(x) = \begin{cases} x^2 + 3, & x < 5 \\ 9 - 2x, & x \geq 5 \end{cases} \tag{1.2.1.5}$$

(a)

On which interval from the piecewise function does the value $x = 1$ belong?

A. $x < 5$
B. $x \leq 5$
C. $x > 5$
D. $x \geq 5$

Answer available at the end of this section.

(b)

Find $f(1)$.

A. 3
B. 4
C. 5
D. 6
E. 7

Answer available at the end of this section.

(c)

On which interval from the piecewise function does the value $x = 5$ belong?

A. $x < 5$
B. $x \leq 5$
C. $x > 5$
D. $x \geq 5$

Answer available at the end of this section.

(d)

Find $f(5)$.

A. -10
B. -5
C. -1
D. 17
E. 28

Answer available at the end of this section.

Note

We've been practicing evaluating functions at specific numeric values. It's also possible to evaluate a function given an expression involving variables.

Activity 1.2.1.6

Let $g(x) = x^2 - 3x$.

(a)

Find $g(a)$.

A. $(ax)^2 - 3ax$
B. $a^2 - 3a$
C. $a(x^2 - 3x)$
D. $ax^2 - 3ax$
E. $a - 3$

Answer available at the end of this section.

(b)

Find $g(x + h)$.

A. $x^2 - 3x + h$
B. $(x + h)^2 - 3x$
C. $(x + h)^2 - 3(x + h)$
D. $x^2 - 3(x + h)$

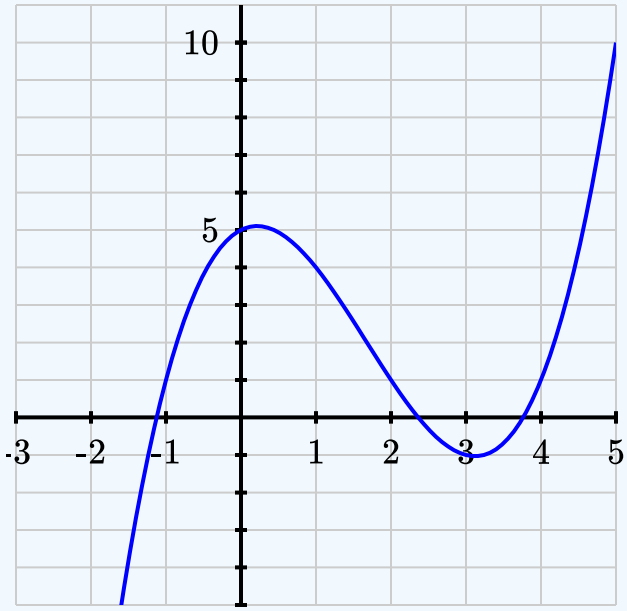
Answer available at the end of this section.

Note

We should also be able to look at a graph of a function and evaluate it for different values of x. The next activity explores that.

Activity 1.2.1.7

Let $f(x)$ be the function graphed below.



(a)

Find $f(1)$.

A. -4
B. -2
C. 0
D. 2
E. 4

Answer available at the end of this section.

(b)

Find $f(3)$.

A. -1
B. 0
C. 1
D. 2
E. 3
F. 4

Answer available at the end of this section.

(c)

For which x-value(s) does $f(x) = 1$?

A. -1
B. 0
C. 1
D. 2
E. 3
F. 4

Answer available at the end of this section.

(d)

For which x-value(s) does $f(x) = 4$. Estimate as needed!

Answer available at the end of this section.

Activity 1.2.1.8

In these activities, we are flipping the question around. This time we know what the function equals at some x-value, and we want to recover that x-value (or values!).

(a)

Let $h(x) = 5x + 7$. Find the x-value(s) such that $h(x) = -13$.

Answer available at the end of this section.

(b)

Let $f(x) = x^2 - 3x - 9$. Find the x-value(s) such that $f(x) = 9$.

Answer available at the end of this section.

Activity 1.2.1.9

Ellie has $\$13$ in her piggy bank, and she gets an additional $\$1.50$ each week for her allowance. Assuming she does not spend any money, the total amount of allowance, $A(w)$, she has after w weeks can be modeled by the function

$$A(w) = 13 + 1.50w \tag{1.2.1.6}$$

.

(a)

How much money will be in her piggy bank after 5 weeks?

Answer available at the end of this section.

(b)

After how many weeks will she have $\$40$ in her piggy bank?

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can use and interpret function notation before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.2.1.1

Let $f(x) = 3x - 7$. Evaluate $f(4)$.

Answer available at the end of this section.

Check Your Understanding 1.2.1.2

Let $g(x) = x^2 - 2x$. Evaluate $g(-3)$.

Answer available at the end of this section.

Check Your Understanding 1.2.1.3

Let $f(x) = 2x + 1$. For what input value x is $f(x) = 11$?

Answer available at the end of this section.

Answers

Activity 1.2.1.1

$f(x) = 2x + 14$

Activity 1.2.1.1

$f(x) = 3x^2 - x - 5$

Activity 1.2.1.1

$f(x) = \frac{2}{x} - x^4 + 5$

Activity 1.2.1.2

x $f(x)$ -5 96 $-\frac{1}{2}$ $\frac{15}{4}$ 01 25 10261

Activity 1.2.1.3

A

Activity 1.2.1.3

B

Activity 1.2.1.4

$f(-4) = 65$, $f(0) = 1$, and $f(2) = 5$

Activity 1.2.1.4

$g(0) = \sqrt{13}$, $g(2) = 3$, and $g(8)$ is not defined

Activity 1.2.1.4

$h(3)$ is not defined, $h(4) = 0$, and $h(10) = \frac{16}{21}$

Activity 1.2.1.5

A

Activity 1.2.1.5

B

Activity 1.2.1.5

D

Activity 1.2.1.5

C

Activity 1.2.1.6

B

Activity 1.2.1.6

C

Activity 1.2.1.7

E

Activity 1.2.1.7

A

Activity 1.2.1.7

A, D, and F

Activity 1.2.1.7

approximately $-0.5, 1$, and 4.5

Activity 1.2.1.8

$x = -4$

Activity 1.2.1.8

$x = -3, 6$

Activity 1.2.1.9

$\$20.50$

Activity 1.2.1.9

18 weeks

Check Your Understanding 1.2.1.1

$f(4) = 3(4) - 7 = 12 - 7 = 5$. The notation $f(4)$ asks for the output when the input is 4.

Check Your Understanding 1.2.1.2

$g(-3) = (-3)^2 - 2(-3) = 9 + 6 = 15$. Be careful with the negative input: $(-3)^2 = 9$, not -9.

Check Your Understanding 1.2.1.3

Set $2x + 1 = 11$, so $2x = 10$ and $x = 5$. This reverses the usual question: instead of "given the input, find the output," we are given the output 11 and asked to find the input.

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1.2.1.1: FN2 Online Exercises

FN2 — Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.	FN2 CheckIt Exercises

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1.2.2: Characteristics of a Function's Graph (FN3)

This section corresponds to APC slot 1.2 (Functions: modeling relationships) in the Math 172 flow.

FN3 Learning outcome: Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.

Characteristics of a Function's Graph

Overview

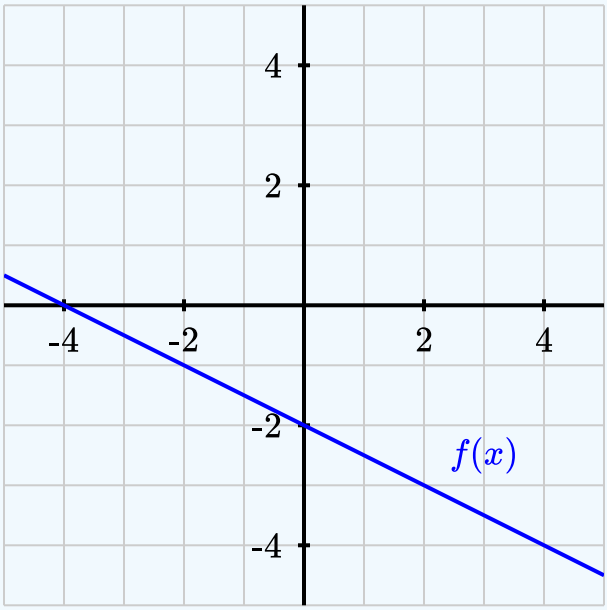
In this section, we will be looking at different kinds of graphs and will identify various characteristics. These ideas can span all kinds of functions, so you will see these come up multiple times!

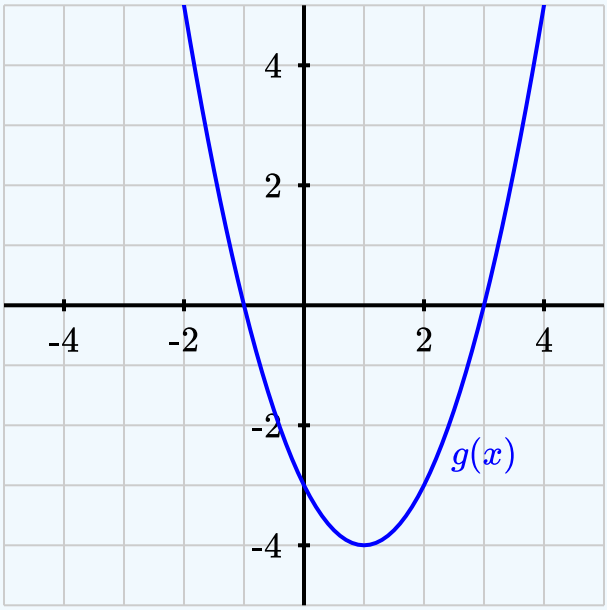
Definition: X and Y Intercepts

One of the easiest things to identify from a graph are the **intercepts**, which are points at which the graph crosses the axes. An x**-intercept** is a point at which the graph crosses the x-axis and a y**-intercept** is a point at which the graph crosses the y -axis. Because intercepts are points, they are typically written as an ordered pair: (x, y).

Activity 1.2.2.1

Use the following graphs to answer the questions.





(a)

What are the x-intercept(s) of $f(x)$?

A. $(0, -4)$

B. $(-2, 0)$
C. $(-4, 0)$
D. $(0, -2)$

Answer available at the end of this section.

(b)

What are the x-intercept(s) of $g(x)$?

A. $(0, -3)$
B. $(-1, 0)$
C. $(3, 0)$
D. $(-3, 0)$

Answer available at the end of this section.

(c)

What are the y-intercept(s) of $f(x)$?

A. $(0, -4)$
B. $(-2, 0)$
C. $(-4, 0)$
D. $(0, -2)$

Answer available at the end of this section.

(d)

What are the y-intercept(s) of $g(x)$?

A. $(0, -3)$
B. $(-1, 0)$
C. $(3, 0)$
D. $(-3, 0)$

Answer available at the end of this section.

(e)

Sketch a graph of a function with the following intercepts:

- x-intercepts: $(-2, 0)$ and $(6, 0)$
- y-intercept: $(0, 4)$

Answer available at the end of this section.

(f)

Sketch a graph of a function with the following intercepts:

- x-intercept: $(-1, 0)$
- y-intercept: $(0, 6)$ and $(0, -2)$

Answer available at the end of this section.

Note

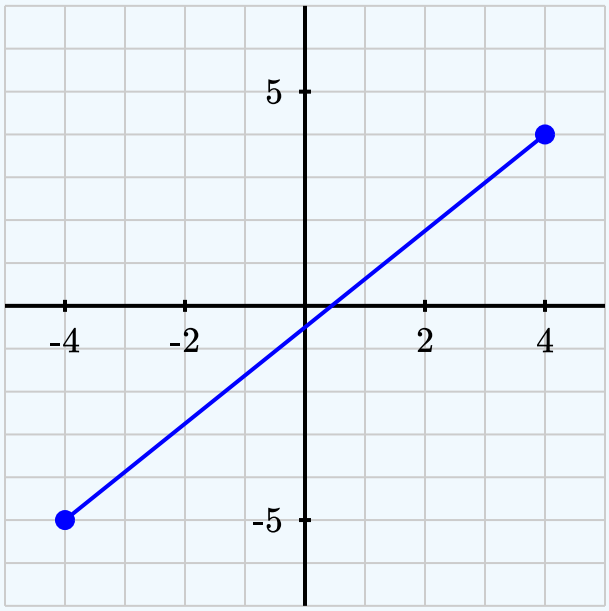
Notice in the previous activity, that a function can have multiple x-intercepts, but only one y-intercept. Having more than one y-intercept would create a graph that is not a function!

Definition: Domain and Range of Graphs

The **domain** refers to the set of possible input values and the **range** refers to the set of possible output values. If given a graph, however, it would be impossible to list out all the values for the domain and range so we use interval notation to represent the set of values.

Activity 1.2.2.2

Use the following graph to answer the questions below.



(a)

Draw on the x-axis all the values in the domain.

Answer available at the end of this section.

(b)

What interval represents the domain you drew in part (a)?

A. $[4, -4]$
B. $[-4, 4]$
C. $(-4, 4)$
D. $(4, -4)$

Answer available at the end of this section.

(c)

Draw on the y-axis all the values in the range.

Answer available at the end of this section.

(d)

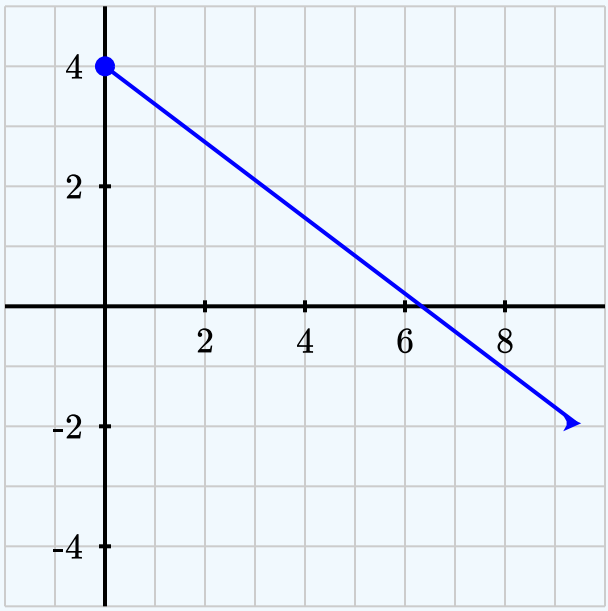
What interval represents the range you drew in part (c)?

A. $(-5, 4)$
B. $[-4, 4]$
C. $[-5, 4]$
D. $(4, -5)$

Answer available at the end of this section.

Activity 1.2.2.3

Use the following graph to answer the questions below.



(a)

What is the domain of this graph?

A. $[4, \infty)$
B. $(-\infty, 0]$
C. $(-\infty, 4]$
D. $[0, \infty)$

Answer available at the end of this section.

(b)

What is the range of this graph?

A. $[4, \infty)$
B. $(-\infty, 0]$

C. $(-\infty, 4]$
D. $[0, \infty)$

Answer available at the end of this section.

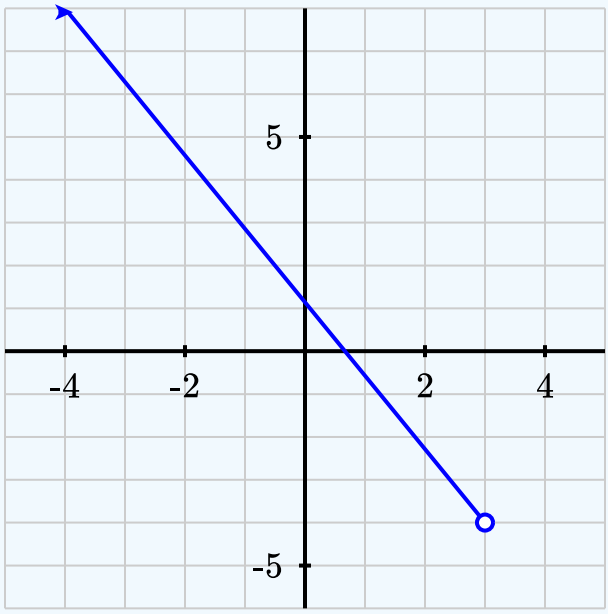
Note

When writing your intervals for domain and range, notice that you will need to write them from the smallest values to the highest values. For example, we wouldn't write $[4, -\infty)$ as an interval because $-\infty$ is smaller than 4.

For domain, read the graph from left to right. For range, read the graph from bottom to top.

Activity 1.2.2.4

Use the following graph to answer the questions below.



(a)

What is the domain of this graph?

A. $(-\infty, 3)$
B. $(\infty, -4]$
C. $(-4, \infty)$
D. $(-\infty, 3]$

Answer available at the end of this section.

(b)

What is the range of this graph?

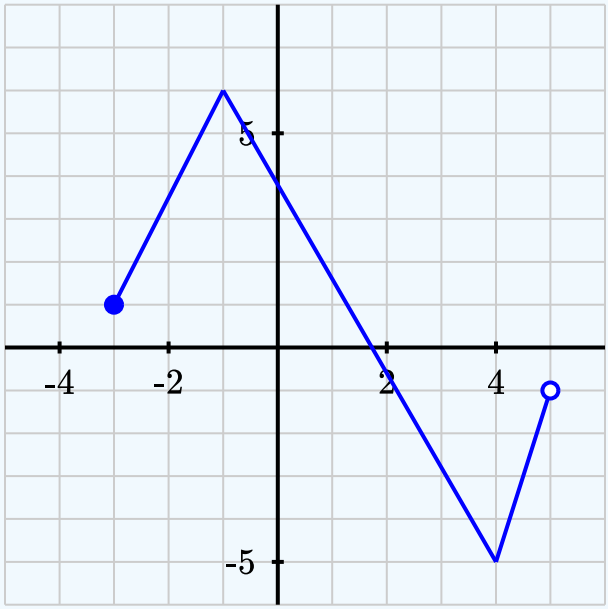
A. $(-\infty, 3)$
B. $(\infty, -4]$
C. $(-4, \infty)$

D. $(-\infty, 3]$

Answer available at the end of this section.

Activity 1.2.2.5

Use the following graph to answer the questions below.



(a)

What is the domain of this graph?

A. $(-3, 5)$
B. $(-5, 7)$
C. $[-5, 7]$
D. $[-3, 5)$

Answer available at the end of this section.

(b)

What is the range of this graph?

A. $(-3, 5)$
B. $(-5, 6)$
C. $[-5, 6]$
D. $[-3, 5)$

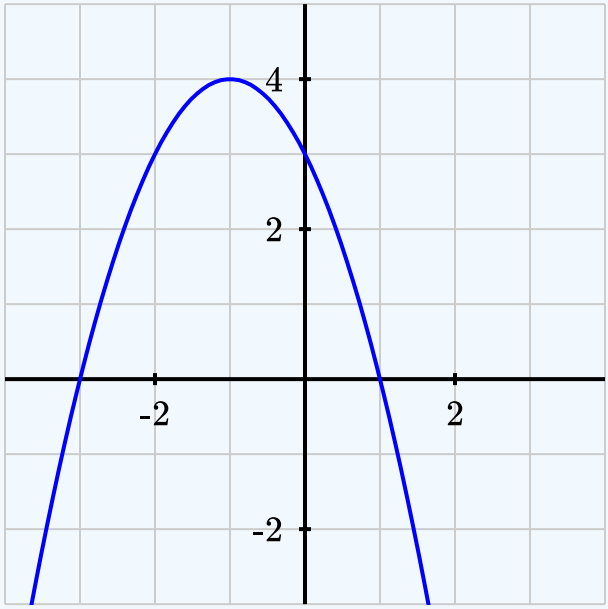
Answer available at the end of this section.

Note

Notice that finding the domain and range can be tricky! Be sure to pay attention to the x- and y-values of the entire graph - not just the endpoints!

Activity 1.2.2.6

In this activity, we will look at where the function is increasing and decreasing. Use the following graph to answer the questions below.



(a)

Where do you think the graph is increasing?

Answer available at the end of this section.

(b)

Which interval best represents where the function is increasing?

A. $(-\infty, -1]$
B. $(-\infty, -1)$
C. $(-1, \infty)$
D. $[-1, \infty)$

Answer available at the end of this section.

(c)

Where do you think the graph is decreasing?

Answer available at the end of this section.

(d)

Which interval best represents where the function is decreasing?

A. $(-\infty, -1]$
B. $(-\infty, -1)$
C. $(-1, \infty)$
D. $[-1, \infty)$

Answer available at the end of this section.

(e)

Based on what you see on the graph, do you think this graph has any maxima or minima?

Answer available at the end of this section.

Definition: Intervals of Increase and Decrease and Constant

As you noticed in *(see related activity)*, functions can increase or decrease (or even remain constant!) for a period of time. The **interval of increase** is when the y-values of the function increase as the x-values increase. The **interval of decrease** is when the y-values of the function decrease as the x-values increase. The function is constant when the y-values remain constant as x-values increase (also known as the **constant interval**).

The easiest way to identify these intervals is to read the graph from left to right and look at what is happening to the y-values.

Definition: Max and Min

The **maximum**, or **global maximum**, of a graph is the point where the y-coordinate has the largest value. The **minimum**, or **global minimum** is the point on the graph where the y-coordinate has the smallest value.

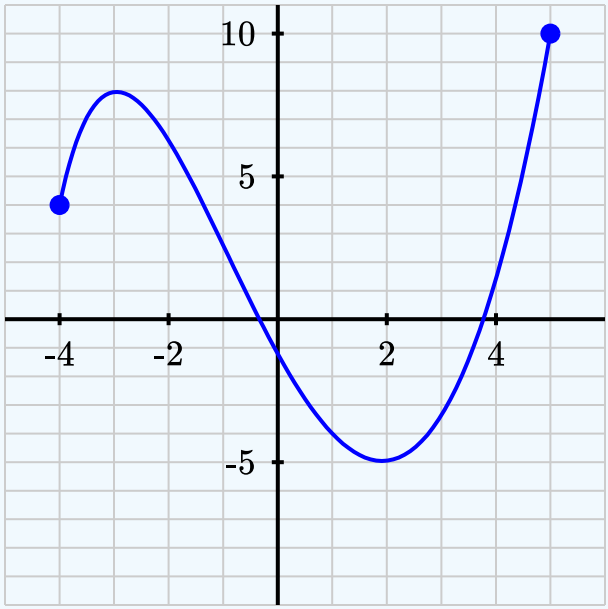
Graphs can also have **local maximums** and **local minimums**. A local maximum point is a point where the function value (i.e, y-value) is larger than all others in some neighborhood around the point. Similarly, a local minimum point is a point where the function value (i.e, y-value) is smaller than all others in some neighborhood around the point.

Note

Global extrema are sometimes referred to as absolute extrema, while local extrema are sometimes referred to as relative extrema.

Activity 1.2.2.7

Use the following graph to answer the questions below.



(a)

At what value of x is there a global maximum?

A. $x = -4$
B. $x = -3$
C. $x = 2$
D. $x = 5$

Answer available at the end of this section.

(b)

What is the global maximum value?

A. 10
B. 8
C. 4
D. -5

Answer available at the end of this section.

(c)

At approximately what value of x is there a global minimum?

A. $x \approx -4$
B. $x \approx -3$
C. $x \approx 2$
D. $x \approx 5$

Answer available at the end of this section.

(d)

What is the global minimum value?

A. 10
B. 8
C. 4
D. -5

Answer available at the end of this section.

(e)

At approximately what value of x is there a local maximum?

A. $x \approx -4$
B. $x \approx -3$
C. $x \approx 2$
D. $x \approx 5$

Answer available at the end of this section.

(f)

What is the local maximum value?

A. 10
B. 8
C. 4
D. -5

Answer available at the end of this section.

(g)

At approximately what value of x is there a local minimum?

A. $x \approx -4$
B. $x \approx -3$
C. $x \approx 2$
D. $x \approx 5$

Answer available at the end of this section.

(h)

What is the local minimum value?

A. 10
B. 8
C. 4
D. -5

Answer available at the end of this section.

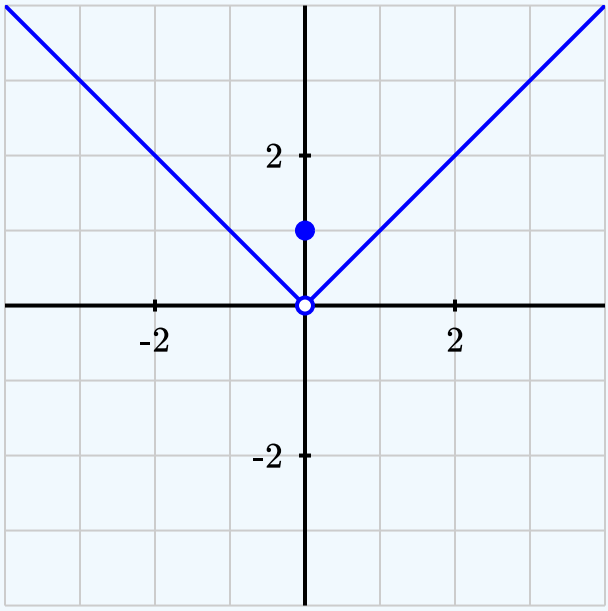
Note

Notice that in the previous activity, there are two ways we talk about max and min. We might want to know the location of where the max or min are (i.e., determining at which x-value the max or min occurs at) or we might want to know what the max or min values are (i.e., the y-value).

Also, note that in *the pervious activity*, a local minimum is also a global minimum.

Activity 1.2.2.8

Sometimes, it is not always clear what the maxima or minima are or if they exist. Consider the following graph of $f(x)$:



(a)

What is the value of $f(0)$?

A. 1
B. 0
C. $f(0)$ does not exist

Answer available at the end of this section.

(b)

What is the local minimum value of $f(x)$?

A. 1
B. 0
C. There is no local minimum

Answer available at the end of this section.

(c)

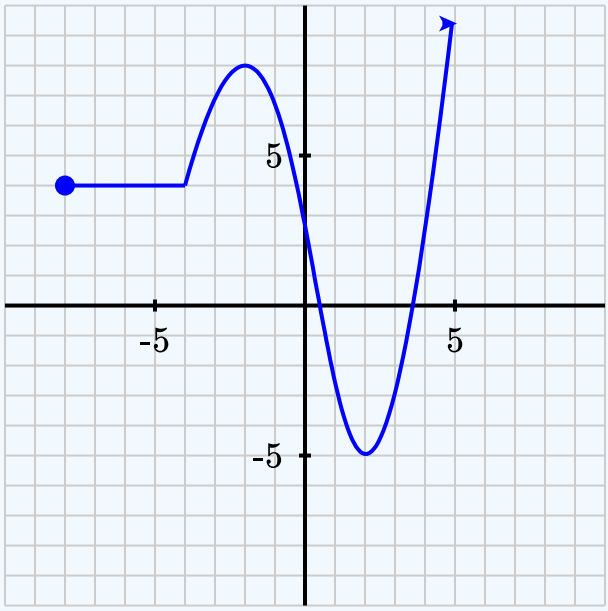
What is the global minimum value of $f(x)$?

A. 1
B. 0
C. There is no global minimum

Answer available at the end of this section.

Activity 1.2.2.9

Use the following graph to answer the questions below.



(a)

What is the domain?

Answer available at the end of this section.

(b)

What is the range?

Answer available at the end of this section.

(c)

What is the x-intercept(s)?

Answer available at the end of this section.

(d)

What is the y-intercept?

Answer available at the end of this section.

(e)

Where is the function increasing?

Answer available at the end of this section.

(f)

Where is the function decreasing?

Answer available at the end of this section.

(g)

Where is the constant interval?

Answer available at the end of this section.

(h)

At what x-values do the local maxima occur?

Answer available at the end of this section.

(i)

At what x-values do the local minima occur?

Answer available at the end of this section.

(j)

What are the global max and min?

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can read a function's graph before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding $1.2.2.1$

Suppose the graph of a function f is a solid parabola opening upward with vertex at $(2, -3)$, extending upward without bound in both directions. Give the domain and range in interval notation.

Answer available at the end of this section.

Check Your Understanding $1.2.2.2$

A function g has a graph that crosses the y-axis at $y = 4$ and crosses the x-axis at $x = -2$ and $x = 5$. List the y-intercept and the x-intercepts as ordered pairs.

Answer available at the end of this section.

Check Your Understanding $1.2.2.3$

A function h is defined on the interval $[-4, 6]$. Its graph decreases from $x = -4$ to $x = 1$, then increases from $x = 1$ to $x = 6$. The lowest point on the graph is $(1, -5)$, and the highest point is $(6, 8)$. Identify the intervals of increase and decrease, and state the maximum and minimum values.

Answer available at the end of this section.

Answers

Activity 1.2.2.1

C

Activity 1.2.2.1

B and C

Activity 1.2.2.1

D

Activity 1.2.2.1

A

Activity 1.2.2.1

Answers will vary. Students should make sure that their sketch is a function.

Activity 1.2.2.1

You could draw a variety of graphs, but they would not be functions.

Activity 1.2.2.2

Students should shade all values of x from -4 to 4. The intent here is to help students visualize that the domain consists of more than x-values that are integers.

Activity 1.2.2.2

B

Activity 1.2.2.2

Students should shade all values of y from -5 to 4. The intent here is to help students visualize that the range consists of more than y-values that are integers.

Activity 1.2.2.2

C

Activity 1.2.2.3

D

Activity 1.2.2.3

C

Activity 1.2.2.4

A

Activity 1.2.2.4

C

Activity 1.2.2.5

D

Activity 1.2.2.5

C

Activity 1.2.2.6

The function is increasing from $(-\infty, -1)$. Instructors can emphasize this by having students think about where the function is "going up." It may be helpful to also note that the function is no longer "going up" once you get to the "top". This could help students think about how to write their answer in interval notation (with parentheses and not brackets).

Activity 1.2.2.6

B

Activity 1.2.2.6

The function is decreasing from $(-1, \infty)$. Instructors can emphasize this by having students think about where the function is "going down." It may be helpful to also note that the function is no longer "going down" once you get to the "bottom." In this case, the "bottom" does not exist...This might be a good opportunity to discuss with students how to address this when writing the range in interval notation.

Activity 1.2.2.6

C

Activity 1.2.2.6

The intent here is for students to visually see there that the graph reaches a high point (i.e., the maximum). To extend this thinking, instructors could also ask students to draw a sketch (or discuss) when a graph would have a minimum.

Activity 1.2.2.7

D

Activity 1.2.2.7

A

Activity 1.2.2.7

C

Activity 1.2.2.7

D

Activity 1.2.2.7

B

Activity 1.2.2.7

B

Activity 1.2.2.7

C

Activity 1.2.2.7

D

Activity 1.2.2.8

A

Activity 1.2.2.8

C

Activity 1.2.2.8

C

Activity 1.2.2.9

$[-8, \infty)$

Activity 1.2.2.9

$[-5, \infty)$

Activity 1.2.2.9

Approximately $(0.5, 0)$ and $(3.5, 0)$

Activity 1.2.2.9

Approximately $(0, 2.5)$

Activity 1.2.2.9

$(-4, -2) \cup (2, \infty)$

Activity 1.2.2.9

$(-2, 2)$

Activity 1.2.2.9

$(-8, -4)$

Activity 1.2.2.9

$x = -2$

Activity 1.2.2.9

$x = 2$

Activity 1.2.2.9

Global minimum is -5. There is no global maximum.

Check Your Understanding 1.2.2.1

Domain: $(-\infty, \infty)$ — the parabola extends horizontally forever in both directions.

Range: $[-3, \infty)$ — the lowest output value is -3 (at the vertex), and the graph extends upward without bound. The bracket at -3 indicates the value -3 is included.

Check Your Understanding 1.2.2.2

y-intercept: $(0, 4)$ — the y-intercept always has $x = 0$.

x-intercepts: $(-2, 0)$ and $(5, 0)$ — the x-intercepts always have $y = 0$.

Check Your Understanding 1.2.2.3

Decreasing on $(-4, 1)$; increasing on $(1, 6)$. We use open intervals because $x = 1$ is the turning point (neither increasing nor decreasing at that instant).

Minimum value: -5 (occurring at $x = 1$). Maximum value: 8 (occurring at $x = 6$).

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1.2.2.1: FN3 Online Exercises

FN3 — Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.	FN3 CheckIt Exercises

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1.2.3: FN2, FN3 FAQ and Resources

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1.3: Introduction to Slope and Average Rate of Change (LF1)

1.3.1: Slope and Average Rate of Change (LF1)

LF1 Slope and Average Rate of Change

- Determine the average rate of change of a given function over a given interval. Find the slope of a line.

Go back to that bank balance from Section 1.2 for a moment. Suppose it was $180 on day 5 and $420 on day 20. Saying "the balance went up by $240" doesn't tell you whether that happened in one dramatic jump or crept up slowly day by day — but dividing that $240 by the 15 days it took gives you a single number, $16 per day, that summarizes the whole stretch. That number is the average rate of change, and geometrically, it's the slope of the straight line connecting the two points on the balance's graph.

This leans on the graph-reading habits from Section 1.2 — you already know how to locate $B(5)$ and $B(20)$ on the balance's graph; average rate of change just asks you to connect those two points and measure the steepness between them.

Section 1.4 takes that same slope idea and builds it into a full equation of a line, letting you write down not just how fast a quantity is changing, but a formula that predicts its value at any point in between.

> **Looking ahead to calculus:** Later, calculus asks what happens to this calculation as the two days get closer and closer together — instead of the average rate of change between day 5 and day 20, what's the rate of change at the exact instant of day 12? That question turns the slope of a line through two points into the slope of a curve at a single point, which is exactly what a derivative measures.

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1.3.1: Slope and Average Rate of Change (LF1)

This section corresponds to APC slot 1.3 (Average rate of change) in the Math 172 flow — the anchor section for Unit 1's essential question.

LF1 Learning outcome: Determine the average rate of change of a given function over a given interval. Find the slope of a line.

Slope and Average Rate of Change

Overview

This section will explore ideas around average rate of change and slope. To help us get started, let's take a look at a context in which these ideas can be helpful.

Activity 1.3.1.1

Robert came home one day after school to a very hot house! When he got home, the temperature on the thermostat indicated that it was 85 degrees! Robert decided that was too hot for him, so he turned on the air conditioner. The table of values below indicate the temperature of his house after turning on the air conditioner.

Time (minutes)	Temperature (degrees Fahrenheit)
0	85
1	84.3
2	83.6
3	82.9
4	82.2
5	81.5
6	80.8

(a)

How much did the temperature change from 0 to 2 minutes?

A. The temperature decreased by 0.7 degrees
B. The temperature decreased by 1.4 degrees
C. The temperature increased by 0.7 degrees
D. The temperature increased by 1.4 degrees

Answer available at the end of this section.

(b)

How much did the temperature change from 4 to 6 minutes?

A. The temperature decreased by 0.7 degrees
B. The temperature decreased by 1.4 degrees
C. The temperature increased by 0.7 degrees

D. The temperature increased by 1.4 degrees

Answer available at the end of this section.

(c)

If Robert wanted to know how much the temperature was decreasing each minute, how could he figure that out?

Answer available at the end of this section.

(d)

How would you describe the overall behavior of the temperature of Robert's house?

Answer available at the end of this section.

Note

Notice in *(see related activity)* that the temperature appears to be decreasing at a constant rate (i.e., the temperature decreased 1.4 degrees for every 2-minute interval). Upon further investigation, you might have also noticed that the temperature decreased by 0.7 degrees every minute.

Activity 1.3.1.2

Refer back to the data Robert collected of the temperature of his house after turning on the air conditioner (*(see related activity)*).

(a)

If this pattern continues, what will the temperature be after 8 minutes?

A. 80.1
B. 78.7
C. 80.8
D. 79.4

Answer available at the end of this section.

(b)

If this pattern continues, how long will it take for Robert's house to reach 78 degrees?

A. 12 minutes
B. 9 minutes
C. 10 minutes
D. 11 minutes

Answer available at the end of this section.

Note

An average rate of change helps us to see and understand how a function is generally behaving. For example, in *(see related activity)* and *(see related activity)*, we began to see how the temperature of Robert's house was decreasing every minute the air conditioner was on. In other words, when looking at average rate of change, we are comparing how one quantity is changing with respect to something else changing.

Definition: Average Rate of Change

The **average rate of change** of a function on a given interval measures how much the function's value changes per unit on that interval. For a function $f(x)$ on the interval $[a, b]$, it is calculated by the following expression:

Recall that we can use function notation to describe x- and y-values. $f(b)$, for instance represents the y-value when plugging in a value for x (or b).

$$\frac{f(b) - f(a)}{b - a}. \tag{1.3.1.1}$$

Recall that we can use function notation to describe x- and y-values. $f(b)$, for instance represents the y-value when plugging in a value for x (or b).

Note

Notice that to calculate the average rate of change over an interval $[a, b]$, we are using the two endpoints of the interval, namely $(a, f(a))$ and $(b, f(b))$.

Activity 1.3.1.3

Use the table below to answer the questions.

x	$f(x)$
-5	28
-4	19
-3	12
-2	7
-1	4

(a)

Applying *(see related activity)*, what is the average rate of change when $x = -5$ to $x = -2$?

A. $\frac{1}{7}$
B. -3
C. -7
D. 7

Answer available at the end of this section.

(b)

What is the average rate of change on the interval $[-4, -1]$?

A. -5
B. -3
C. 5
D. 3

Answer available at the end of this section.

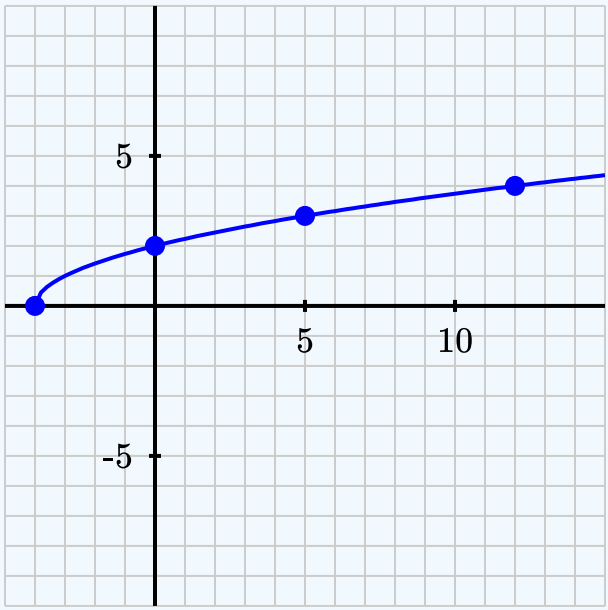
(c)

Does this function have a constant average rate of change?

Answer available at the end of this section.

Activity 1.3.1.4

Use the graph to calculate the average rate of change on the given intervals.



(a)

Applying *(see related activity)*, what is the average rate of change on the interval $[-4, 0]$?

A. $-\frac{1}{2}$
B. $\frac{1}{2}$
C. -2
D. 2

Answer available at the end of this section.

(b)

What is the average rate of change on the interval $[0, 12]$?

A. $\frac{1}{6}$
B. -6
C. $-\frac{1}{6}$
D. 6

Answer available at the end of this section.

Activity 1.3.1.5

Just like with tables and graphs, you should be able to find the average rate of change when given a function. For this activity, use the function

$$f(x) = -3x^2 - 1 \tag{1.3.1.2}$$

to answer the following questions.

(a)

Applying *(see related activity)*, what is the average rate of change on the interval $[-2, 3]$?

A. $\frac{41}{5}$
B. -3
C. $-\frac{1}{3}$
D. $\frac{5}{41}$

Answer available at the end of this section.

(b)

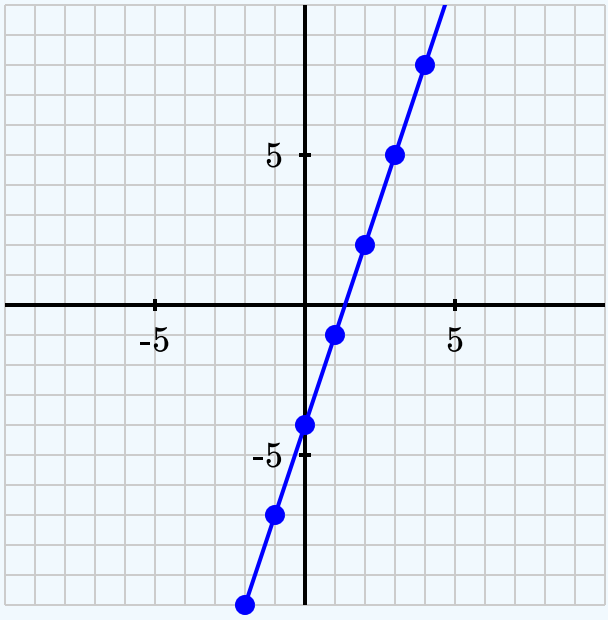
What is the average rate of change on the interval $[0, 4]$?

A. $\frac{2}{25}$
B. -50
C. $-\frac{1}{12}$
D. -12

Answer available at the end of this section.

Activity 1.3.1.6

Use the given graph of the function, $f(x) = 3x - 4$, to investigate the average rate of change of a linear function.



(a)

What is the average rate of change on the interval $[-2, 0]$?

A. $\frac{1}{3}$
B. 3
C. $-\frac{1}{7}$
D. -7

Answer available at the end of this section.

(b)

What is the average rate of change on the interval $[-1, 5]$? Notice that you cannot see the point at $x = 5$. How could you use the equation of the line to determine the y-value when $x = 5$?

A. 3
B. $\frac{1}{3}$
C. $\frac{2}{3}$
D. -3

Answer available at the end of this section.

(c)

Based on your observations in parts (a) and (b), what do you think will be the average rate of change on the interval $[5, 25]$?

Answer available at the end of this section.

Note

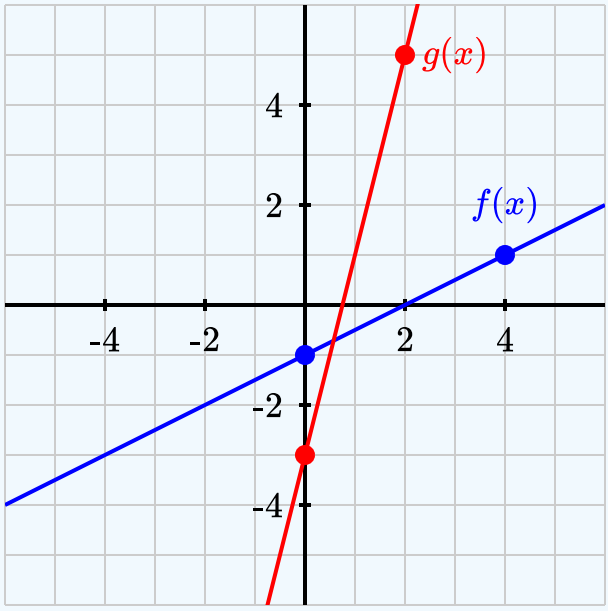
Notice in *(see related activity)*, the average rate of change was the same regardless of which interval you were given. But in *(see related activity)*, the average rate of change was not the same across different intervals.

Definition: Slope

The **slope** of a line is a constant that represents the direction and steepness of the line. For a linear function, the slope never changes - meaning it has a constant average rate of change.

Activity 1.3.1.7

The steepness of a line depends on the vertical and horizontal distances between two points on the line. Use the graph below to compare the steepness, or slope, of the two lines.



(a)

What is the vertical distance between the two points on the line $y = g(x)$ (the red line)?

A. 2
B. 4
C. 8
D. $\frac{1}{2}$

Answer available at the end of this section.

(b)

What is the horizontal distance between the two points on the line $y = g(x)$ (the red line)?

A. 2
B. 4
C. 8
D. $\frac{1}{2}$

Answer available at the end of this section.

(c)

Using information from parts (a) and (b), what value could we use to describe the steepness of the line $y = g(x)$ (the red line)?

A. 2
B. 4
C. 8
D. $\frac{1}{2}$

Answer available at the end of this section.

(d)

What is the vertical distance between the two points on the line $y = f(x)$ (the blue line)?

A. 2
B. 4
C. 8
D. $\frac{1}{2}$

Answer available at the end of this section.

(e)

What is the horizontal distance between the two points on the line $y = f(x)$ (the blue line)?

A. 2
B. 4
C. 8
D. $\frac{1}{2}$

Answer available at the end of this section.

(f)

Using information from parts (d) and (e), what value could we use to describe the steepness of the line $y = f(x)$ (the blue line)?

A. 2
B. 4
C. 8
D. $\frac{1}{2}$

Answer available at the end of this section.

(g)

Which line is the steepest?

Answer available at the end of this section.

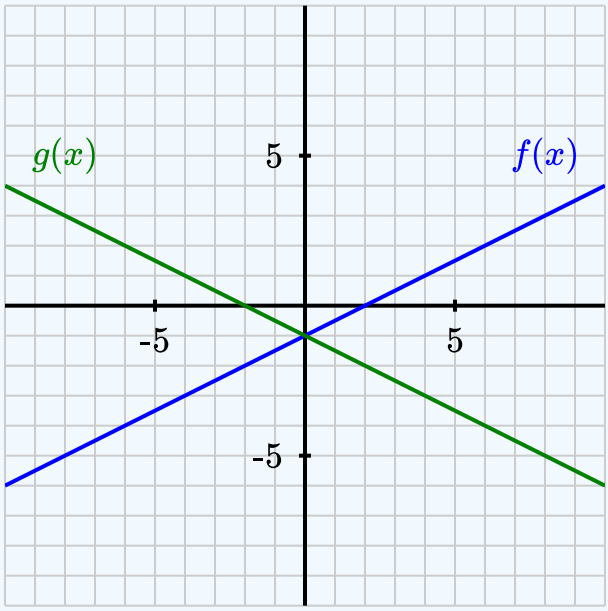
Note

The steepness, or slope, of a line can be found by the change in y (the vertical distance between two points on the line) divided by the change in x (the horizontal distance between two points on the line). Slope can be calculated as “rise over run”.

Slope is a way to describe the steepness of a line. The red line in *(see related activity)* has a larger value for it's slope than the blue line. Thus, the red line is steeper than the blue line.

Activity 1.3.1.8

Now that we know how to find the slope (or steepness) of a line, let's look at other properties of slope. Use the graph below to answer the following questions.



(a)

What is the slope of the line $y = f(x)$ (the blue line)?

A. $\frac{1}{2}$
B. 2
C. $-\frac{1}{2}$
D. -2

Answer available at the end of this section.

(b)

What is the slope of $y = g(x)$ (the green line)?

A. $\frac{1}{2}$
B. 2
C. $-\frac{1}{2}$
D. -2

Answer available at the end of this section.

(c)

How are the slopes of the lines similar?

Answer available at the end of this section.

(d)

How are the slopes of the lines different?

Answer available at the end of this section.

Note

Notice in *(see related activity)* that the slope does not just indicate how steep a line is, but also its direction. A negative slope indicates that the line is decreasing (from left to right) and a positive slope indicates that the line is increasing (from left to right).

Activity 1.3.1.9

Suppose $(-3, 7)$ and $(7, 2)$ are two points on a line.

(a)

Plot these points on a graph and find the slope by using "rise over run."

A. $\frac{1}{2}$
B. 2
C. $-\frac{1}{2}$
D. -2

Answer available at the end of this section.

(b)

Now calculate the slope by using the change in y over the change in x.

A. $\frac{1}{2}$
B. 2
C. $-\frac{1}{2}$
D. -2

Answer available at the end of this section.

(c)

What do you notice about the slopes you got in parts (a) and (b)?

Answer available at the end of this section.

Note

We can calculate slope (m) by finding the change in y and dividing by the change in x. Mathematically, this means that when given (x_1, y_1) and (x_2, y_2),

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.3.1.3}$$

.

Activity 1.3.1.10

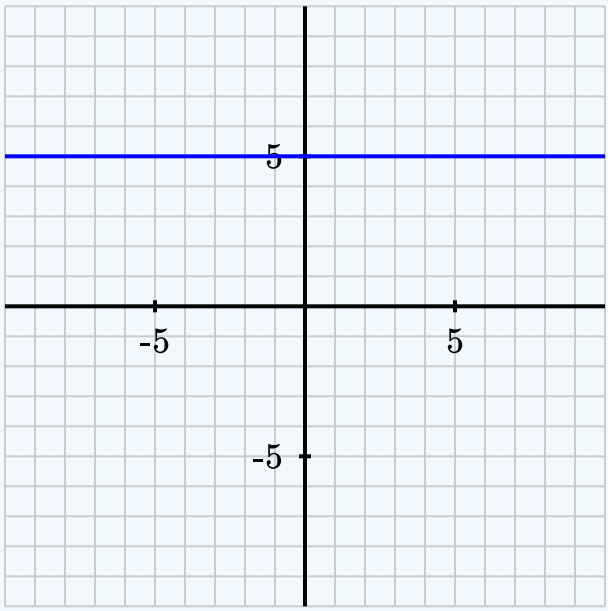
Calculate the slope of each representation of a line using the slope formula.

(a)

x	$f(x)$
-2	-7
-1	-4
0	-1
1	2
2	5

Answer available at the end of this section.

(b)



Answer available at the end of this section.

(c)

$(-4, 7)$ and $(-4, 1)$

Answer available at the end of this section.

Note

In *(see related activity)*, there were slopes that were 0 and undefined. When a line is vertical, the slope is undefined. This means that there is only a vertical distance between two points and there is no horizontal distance. When a line is horizontal, the slope is 0. This means that the line never rises vertically, giving a vertical distance of zero.

Check Your Understanding

Work these on your own to confirm you can find slope and average rate of change before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.3.1.1

Find the slope of the line through the points $(-2, 5)$ and $(4, -3)$.

Answer available at the end of this section.

Check Your Understanding 1.3.1.2

Let $f(x) = x^2 + 1$. Find the average rate of change of f on the interval $[1, 4]$.

Answer available at the end of this section.

Check Your Understanding 1.3.1.3

A car's distance from home (in miles) is given by $d(t)$, where t is time in hours. If $d(1) = 40$ and $d(3) = 130$, find the average rate of change of d on $[1, 3]$ and explain what it means in this context.

Answer available at the end of this section.

Answers

Activity 1.3.1.1

B

Activity 1.3.1.1

B

Activity 1.3.1.1

Choosing any two points that have a time difference of 1 minute should show that the temperature decreases by 0.7 degrees every minute.

Activity 1.3.1.1

Overall, the temperature of Robert's house is decreasing (the air conditioner appears to be working!). Notice that in parts (a) and (b), the temperature decreased 1.4 degrees in 2 minutes.

Activity 1.3.1.2

D

Activity 1.3.1.2

C

Activity 1.3.1.3

C

Activity 1.3.1.3

A

Activity 1.3.1.3

No. This would be a great opportunity for students to make observations of the average rate of change across different intervals.

Activity 1.3.1.4

B

Activity 1.3.1.4

A

Activity 1.3.1.5

B

Activity 1.3.1.5

D

Activity 1.3.1.6

B

Activity 1.3.1.6

A

Activity 1.3.1.6

The average rate of change is 3. Based on the previous parts, it shows that the average rate of change does not change for this function.

Activity 1.3.1.7

C

Activity 1.3.1.7

A

Activity 1.3.1.7

B. This could be a good place to discuss where 4 comes from (change of y over the change in x).

Activity 1.3.1.7

A

Activity 1.3.1.7

B

Activity 1.3.1.7

D. This could be a good place to discuss where $\frac{1}{2}$ comes from (change of y over the change in x).

Activity 1.3.1.7

The line $y = g(x)$ (the red line). The slope of $y = g(x)$ is 4 and the slope of $y = f(x)$ (the blue line) is $\frac{1}{2}$. Because 4 is larger than $\frac{1}{2}$, the line $y = g(x)$ (the red line) is steeper.

Activity 1.3.1.8

A

Activity 1.3.1.8

C

Activity 1.3.1.8

Students will probably notice that both slopes have the same steepness $(\frac{1}{2})$.

Activity 1.3.1.8

Although both slopes have the same steepness $(\frac{1}{2})$, one line has a positive slope ($y = f(x)$, the blue line) and the other line has a negative slope ($y = g(x)$, the green line).

Activity 1.3.1.9

C

Activity 1.3.1.9

C

Activity 1.3.1.9

Students should notice that they get the same answer. Both methods can be used to find the slope.

Activity 1.3.1.10

$m = 3$

Activity 1.3.1.10

$m = 0$

Activity 1.3.1.10

The slope is undefined.

Check Your Understanding 1.3.1.1

$m = \frac{-3-5}{4-(-2)} = \frac{-8}{6} = -\frac{4}{3}$. The line falls about 4 units for every 3 units it moves right.

Check Your Understanding 1.3.1.2

$\text{AROC} = \frac{f(4)-f(1)}{4-1} = \frac{17-2}{3} = \frac{15}{3} = 5$. On average, f increases 5 units of output for every 1 unit of input on this interval.

Check Your Understanding 1.3.1.3

$\text{AROC} = \frac{130-40}{3-1} = \frac{90}{2} = 45$. This means the car traveled an average of 45 miles per hour between $t = 1$ and $t = 3$ — average rate of change here is the average speed over that time interval.

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1.3.1.1: LF1 Online Exercises

LF1 — Determine the average rate of change of a given function over a given interval. Find the slope of a line.	LF1 CheckIt Exercises

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1.3.2: LF1 FAQ and Resources

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1.4: Introduction to Equations of Lines (LF2), Graphs of Linear Equations (LF3), and Linear Models and Meanings (LF5)

1.4.1: Equations of Lines (LF2)

1.4.2: Graphs of Linear Equations (LF3)

1.4.3: Linear Models and Meanings (LF5)

LF2 Equations of Lines

- Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.

LF3 Graphs of Linear Equations

- Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.

LF5 Linear Models and Meanings

- Build linear models from verbal descriptions, and use the models to establish conclusions, including by contextualizing the meaning of slope and intercept parameters.

Your phone plan includes 5 GB of data for $25 a month, then charges $3 for every extra GB once you go over. That single sentence is enough to write down your entire monthly bill as a function, graph it, and predict what any month will cost before it happens — because a constant rate of change, once you know it plus one starting point, pins a line down completely. This section works that pinning-down process from three angles: finding the equation of a line from whatever information you're given, drawing its graph, and — most usefully — building a line from a real situation and reading meaning back out of its slope and intercept.

You already found slope and average rate of change in Section 1.3; here you'll use that single number, together with one point, to write down and graph an entire line — and to recognize when a real quantity (a cost, a population, a fee) is behaving linearly in the first place.

Section 1.5 turns to situations where the rate of change isn't constant — quadratic models — where a single slope number is no longer enough to describe the whole picture.

> **Looking ahead to calculus:** Point-slope form isn't just a convenient way to write a line — it's the exact template calculus later uses to describe a tangent line: a single point plus a single rate, extended outward. Building a line from a point and a rate here is quite literally rehearsing the move you'll make constantly once "rate of change" stops being constant and starts being a rate measured at a single instant.

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1.4.1: Equations of Lines (LF2)

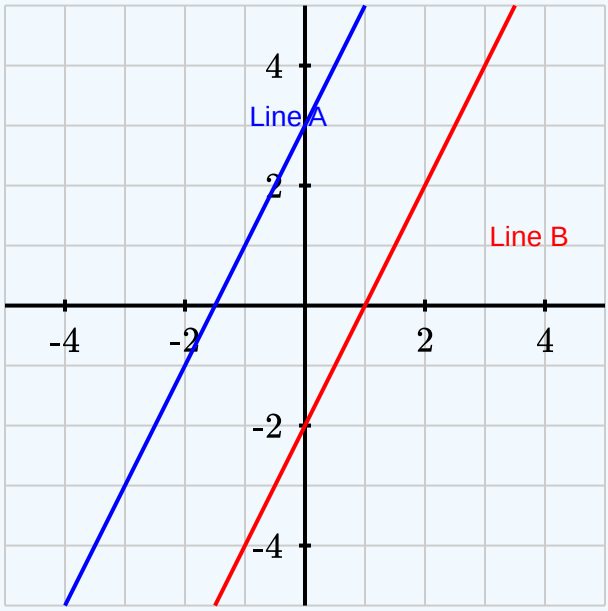
This section corresponds to APC slot 1.4 (Linear functions) in the Math 172 flow.

Learning outcome: Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.

Equations of Lines

Activity 1.4.1.1

Consider the graph of two lines.



(a)

Find the slope of line A.

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

Answer available at the end of this section.

(b)

Find the slope of line B.

A. 1

B. 2

C. $\frac{1}{2}$

D. -2

Answer available at the end of this section.

(c)

Find the y-intercept of line A.

A. -2

B. -1.5

C. 1

D. 3

Answer available at the end of this section.

(d)

Find the y-intercept of line B.

A. -2

B. -1.5

C. 1

D. 3

Answer available at the end of this section.

(e)

What is the same about the two lines?

Answer available at the end of this section.

(f)

What is different about the two lines?

Answer available at the end of this section.

Overview

Notice that in *(see related activity)* the lines have the same slope but different y-intercepts. It is not enough to just know one piece of information to determine a line, you need both a slope and a point.

Definition: Slope-Intercept Form

Linear functions can be written in **slope-intercept form**

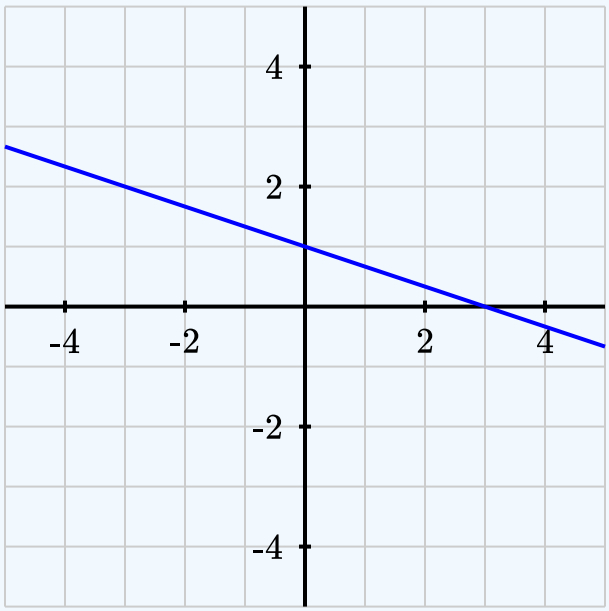
$$f(x) = mx + b \tag{1.4.1.1}$$

where b is the y-intercept (or starting value) and m is the slope (or constant rate of change).

Activity 1.4.1.2

Write the equation of each line in slope-intercept form.

(a)



A. $y = -3x + 1$

B. $y = -x + 3$

C. $y = -\frac{1}{3}x + 1$

D. $y = -\frac{1}{3}x + 3$

Answer available at the end of this section.

(b)

The slope is 4 and the y-intercept is $(0, -3)$.

A. $f(x) = 4x - 3$

B. $f(x) = 3x - 4$

C. $f(x) = -4x + 3$

D. $f(x) = 4x + 3$

Answer available at the end of this section.

(c)

Two points on the line are $(0, 1)$ and $(2, 4)$.

A. $y = 2x + 1$

B. $y = -\frac{3}{2}x + 4$

C. $y = \frac{3}{2}x + 1$

D. $y = \frac{3}{2}x + 4$

Answer available at the end of this section.

(d)

x	$f(x)$
-2	-8
0	-2
1	1
4	10

A. $f(x) = -3x - 2$

B. $f(x) = -\frac{1}{3}x - 2$

C. $f(x) = 3x + 1$

D. $f(x) = 3x - 2$

Answer available at the end of this section.

Activity 1.4.1.3

Let's try to write the equation of a line given two points that don't include the y-intercept.

(a)

Plot the points $(2, 1)$ and $(-3, 4)$.

Answer available at the end of this section.

(b)

Find the slope of the line joining the points.

A. $-\frac{5}{3}$

B. $-\frac{3}{5}$

C. $\frac{3}{5}$

D. -3

Answer available at the end of this section.

(c)

When you draw a line connecting the two points, it's often hard to draw an accurate enough graph to determine the y-intercept of the line exactly. Let's use the slope-intercept form and one of the given points to solve for the y-intercept. Try using the slope and one of the points on the line to solve the equation $y = mx + b$ for b.

A. 2

B. $\frac{11}{5}$

C. $\frac{5}{2}$

D. 3

Answer available at the end of this section.

(d)

Write the equation of the line in slope-intercept form.

Answer available at the end of this section.

Note

It would be nice if there was another form of the equation of a line that works for any points and does not require the y -intercept.

Definition: Point-Slope Form

Linear functions can be written in **point-slope form**

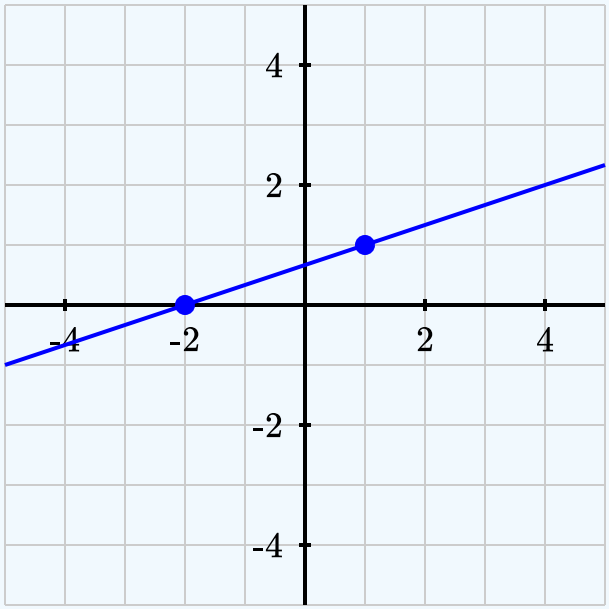
$$y - y_0 = m(x - x_0) \tag{1.4.1.2}$$

where (x_0, y_0) is any point on the line and m is the slope.

Activity 1.4.1.4

Write an equation of each line in point-slope form.

(a)



A. $y = \frac{1}{3}x + \frac{2}{3}$

B. $y - 1 = 3(x - 1)$

C. $y - 1 = \frac{1}{3}(x - 1)$

D. $y + 2 = \frac{1}{3}(x + 2)$

E. $y = \frac{1}{3}(x + 2)$

Answer available at the end of this section.

(b)

The slope is 4 and $(-1, -7)$ is a point on the line.

A. $y + 7 = 4(x + 1)$

B. $y - 7 = 4(x - 1)$

C. $y + 1 = 4(x + 7)$

D. $y - 4 = 7(x - 1)$

Answer available at the end of this section.

(c)

Two points on the line are $(1, 0)$ and $(2, -4)$.

A. $y = -4x + 1$

B. $y - 0 = -2(x - 1)$

C. $y + 4 = -4(x - 2)$

D. $y+4=-3(x-2)$

Answer available at the end of this section.

(d)

x	$f(x)$
-2	-8
1	1
4	10

A. $y+8=3(x-2)$

B. $y-1=-\frac{1}{3}(x-1)$

C. $y+8=-\frac{1}{3}(x+2)$

D. $y-10=3(x-4)$

Answer available at the end of this section.

Activity 1.4.1.5

Consider again the two points from *(see related activity)*, $(2,1)$ and $(-3,4)$.

(a)

Use point-slope form to find an equation of the line.

A. $y=-\frac{3}{5}x+\frac{11}{5}$

B. $y-1=-\frac{3}{5}(x-2)$

C. $y-4=-\frac{3}{5}(x+3)$

D. $y-2=-\frac{3}{5}(x-1)$

Answer available at the end of this section.

(b)

Solve the point-slope form of the equation for y to rewrite the equation in slope-intercept form. Identify the slope and intercept of the line.

Answer available at the end of this section.

Note

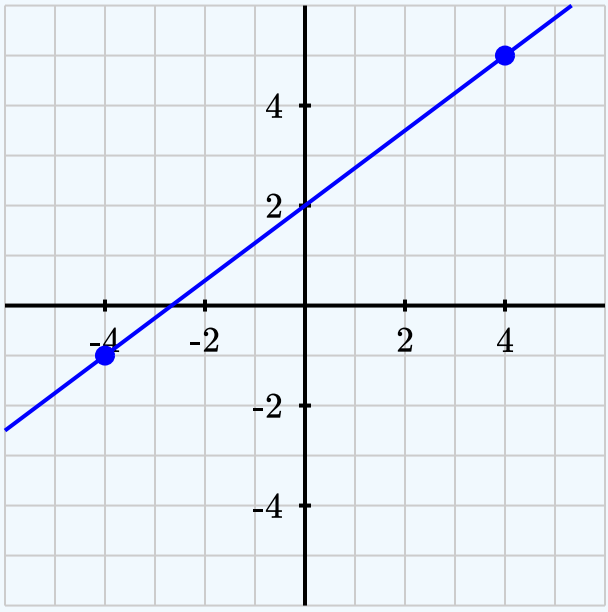
Notice that it was possible to use either point to find an equation of the line in point-slope form. But, when rewritten in slope-intercept form the equation is unique.

Activity 1.4.1.6

For each of the following lines, determine which form (point-slope or slope-intercept) would be “easier” and why. Then, write the equation of each line.

(a)

The line whose graph is given below.



Answer available at the end of this section.

(b)

The line whose slope is $-\frac{1}{2}$ and passes through the point $(1, -3)$.

Answer available at the end of this section.

(c)

The line that passes through the points $(0, 3)$ and $(2, 0)$.

Answer available at the end of this section.

Note

It is always possible to use both forms to write the equation of a line and they are both valid. Although, sometimes the given information lends itself to make one form easier.

Activity 1.4.1.7

Write the equation of each line.

(a)

The slope is 0 and $(-1, -7)$ is a point on the line.

A. $y = -7$

B. $y = 7x$

C. $y = -x$

D. $x = -1$

Answer available at the end of this section.

(b)

Two points on the line are $(3, 0)$ and $(3, 5)$.

A. $y = 3x + 3$

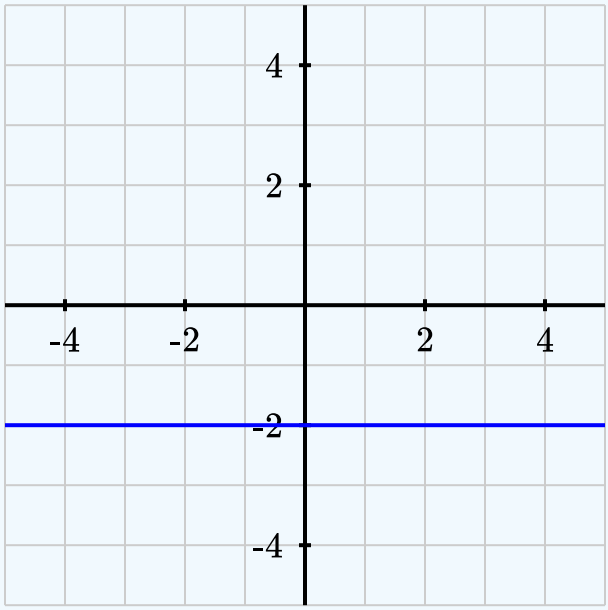
B. $y = 3x + 5$

C. $x = 3$

D. $y = 3$

Answer available at the end of this section.

(c)



A. $x = -2$

B. $y - 2 = x$

C. $y = -2x - 2$

D. $y = -2$

Answer available at the end of this section.

Definition: Horizontal and Vertical Lines

A **horizontal line** has a slope of zero and has the form $y = k$ where k is a constant. A **vertical line** has an undefined slope and has the form $x = h$ where h is a constant.

Definition: Standard Form of a Line

The equation of a line can also be written in **standard form**. Standard form looks like $Ax + By = C$.

Note

It is possible to rearrange a line written in standard form to slope-intercept form by solving for y.

Activity 1.4.1.8

Given a line in standard form

$$5x + 4y = 2. \tag{1.4.1.3}$$

Find the slope and y-intercept of the line.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can find and interpret equations of lines before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.4.1.1

Find the equation of the line through $(1, 4)$ and $(3, -2)$. Write your answer in slope-intercept form.

Answer available at the end of this section.

Check Your Understanding 1.4.1.2

Find the equation of the line with slope $\frac{2}{3}$ that passes through $(-3, 5)$. Write your answer in point-slope form, then convert to slope-intercept form.

Answer available at the end of this section.

Check Your Understanding 1.4.1.3

A line has equation $3x - 2y = 8$. Find its slope and y-intercept.

Answer available at the end of this section.

Answers

Activity 1.4.1.1

B

Activity 1.4.1.1

B

Activity 1.4.1.1

D

Activity 1.4.1.1

A

Activity 1.4.1.1

Lines A and B have the same slope.

Activity 1.4.1.1

Lines A and B have different y-intercepts.

Activity 1.4.1.2

C

Activity 1.4.1.2

A

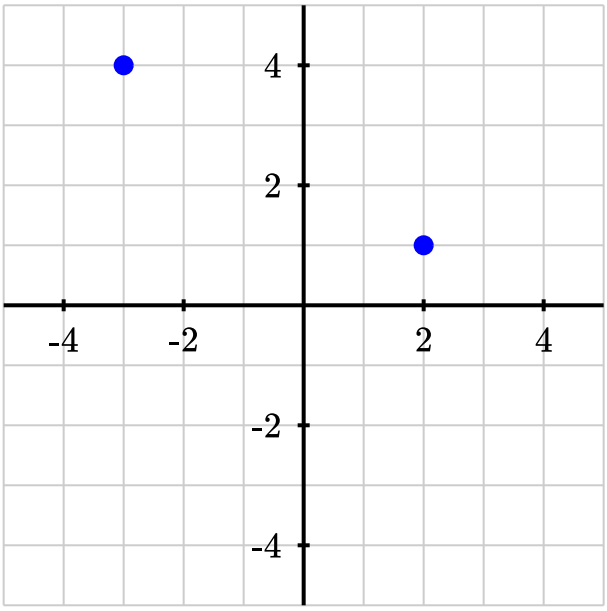
Activity 1.4.1.2

C

Activity 1.4.1.2

D

Activity 1.4.1.3



Activity 1.4.1.3

B

Activity 1.4.1.3

B

Activity 1.4.1.3

$y = -\frac{3}{5}x + \frac{11}{5}$

Activity 1.4.1.4

C or E

Activity 1.4.1.4

A

Activity 1.4.1.4

C

Activity 1.4.1.4

D

Activity 1.4.1.5

B or C

Activity 1.4.1.5

The slope-intercept form is: $y = -\frac{3}{5}x + \frac{11}{5}$. The slope is $-\frac{3}{5}$ and the y-intercept is $\frac{11}{5}$.

Activity 1.4.1.6

Slope-intercept: $y = \frac{3}{4}x + 2$

Activity 1.4.1.6

Point-slope: $y + 3 = -\frac{1}{2}(x - 1)$

Activity 1.4.1.6

Slope-intercept: $y = -\frac{3}{2}x + 3$

Activity 1.4.1.7

A

Activity 1.4.1.7

C

Activity 1.4.1.7

D

Activity 1.4.1.8

Slope: $-\frac{5}{4}$

y-intercept: $\frac{2}{5}$

Check Your Understanding 1.4.1.1

$m = \frac{-2-4}{3-1} = \frac{-6}{2} = -3$. Using point-slope with $(1, 4)$: $y - 4 = -3(x - 1)$, which simplifies to $y = -3x + 7$.

Check Your Understanding 1.4.1.2

Point-slope: $y - 5 = \frac{2}{3}(x + 3)$. Slope-intercept: $y = \frac{2}{3}x + 2 + 5 = \frac{2}{3}x + 7$.

Check Your Understanding 1.4.1.3

Solve for y: $-2y = -3x + 8$, so $y = \frac{3}{2}x - 4$. Slope: $\frac{3}{2}$. Y-intercept: $(0, -4)$.

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1.4.1.1: LF2 Online Exercises

LF2 — Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.	LF2 CheckIt Exercises

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1.4.2: Graphs of Linear Equations (LF3)

This section corresponds to APC slot 1.4 (Linear functions) in the Math 172 flow — graphing fluency practice.

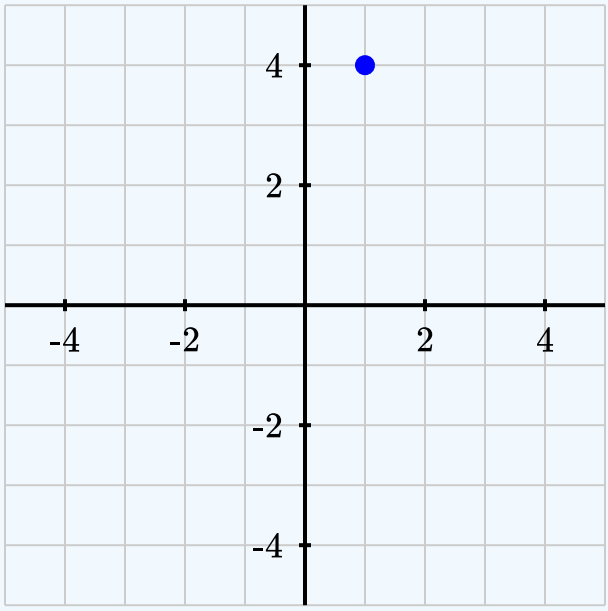
LF3 Learning outcome: Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.

Graphing Linear Equations

Activity 1.4.2.1

(a)

Draw a line that goes through the point $(1, 4)$.



Answer available at the end of this section.

(b)

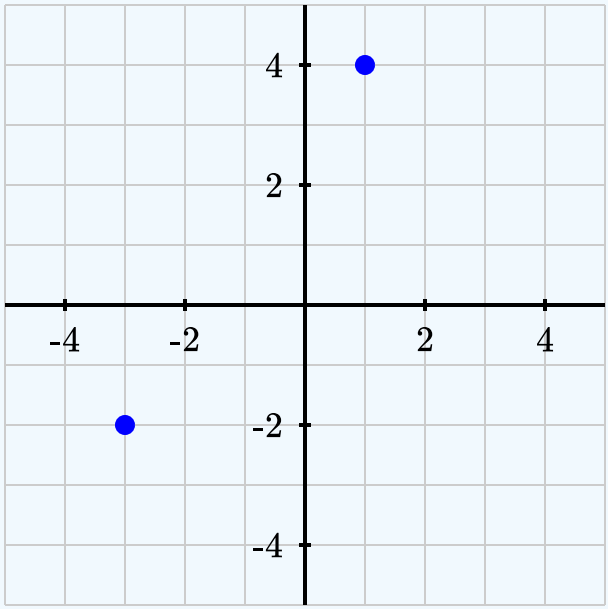
Was this the only possible line that goes through the point $(1, 4)$?

A. Yes. The line is unique.
B. No. There is exactly one more line possible.
C. No. There are a lot of lines that go through $(1, 4)$.
D. No. There are an infinite number of lines that go through $(1, 4)$.

Answer available at the end of this section.

(c)

Now draw a line that goes through the points $(1, 4)$ and $(-3, -2)$.



Answer available at the end of this section.

(d)

Was this the only possible line that goes through the points $(1, 4)$ and $(-3, -2)$?

A. Yes. The line is unique.
B. No. There is exactly one more line possible.
C. No. There are a lot of lines that go through $(1, 4)$ and $(-3, -2)$.
D. No. There are an infinite number of lines that go through $(1, 4)$ and $(-3, -2)$.

Answer available at the end of this section.

Activity 1.4.2.2

(a)

Graph the line containing the points $(-7, 1)$ and $(6, -2)$.

Answer available at the end of this section.

(b)

Graph the line containing the points $(-3, 0)$ and $(0, 8)$.

Answer available at the end of this section.

(c)

Graph the line given by the table below.

x	y
-3	-12
-2	-9

-1	-6
0	-3
1	0
2	3

Answer available at the end of this section.

(d)

Let's say you are given a table that listed six points that are on the same line. How many of those points are necessary to use to graph the line?

A. One point is enough.
B. Two points are enough.
C. Three points are enough.
D. You need to plot all six points.
E. You can use however many you want.

Answer available at the end of this section.

Overview

In *(see related activity)*, we were given at least two points in each question. However, sometimes we are not directly given two points to graph a line. Instead we are given some combination of characteristics about the line that will help us *find* two points. These characteristics could include a point, the intercepts, the slope, or an equation.

Activity 1.4.2.3

A line has a slope of $-\frac{1}{3}$ and its y-intercept is 4.

(a)

We were given the y-intercept. What point does that correspond to?

A. $(4, 0)$
B. $(0, 4)$
C. $\left(4, -\frac{1}{3}\right)$
D. $\left(-\frac{1}{3}, 4\right)$

Answer available at the end of this section.

(b)

After we plot the y-intercept, how can we use the slope to find another point?

A. Start at the y-intercept, then move up one space and to the left three spaces to find another point.
B. Start at the y-intercept, then move up one space and to the right three spaces to find another point.
C. Start at the y-intercept, then move down one space and to the left three spaces to find another point.
D. Start at the y-intercept, then move down one space and to the right three spaces to find another point.

Answer available at the end of this section.

(c)

Graph the line that has a slope of $-\frac{1}{3}$ and its y-intercept is 4.

Answer available at the end of this section.

Activity 1.4.2.4

A line is given by the equation $y = -2x + 5$.

(a)

What form is the equation given in?

A. Standard form
B. Point-slope form
C. Slope-intercept form
D. The form it is in doesn't have a name.

Answer available at the end of this section.

(b)

The form gives us one point right away: the y-intercept. Which of the following is the y-intercept?

A. $(-2, 0)$
B. $(0, -2)$
C. $(5, 0)$
D. $(0, 5)$

Answer available at the end of this section.

(c)

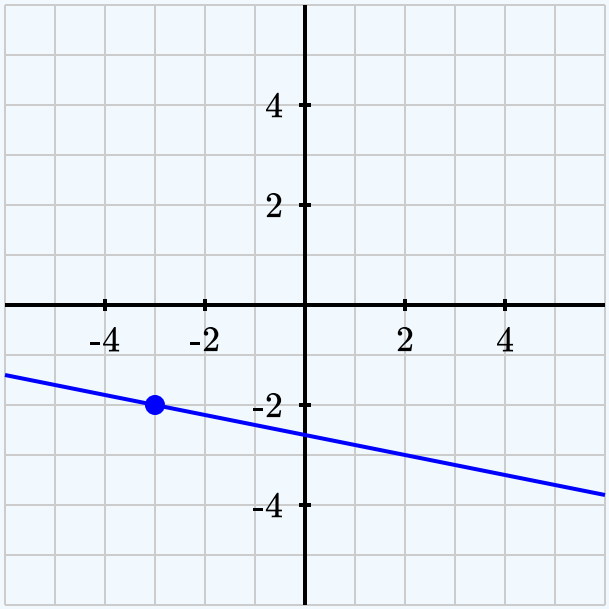
After we plot the y-intercept, we can use the slope to find another point. Find another point and graph the resulting line.

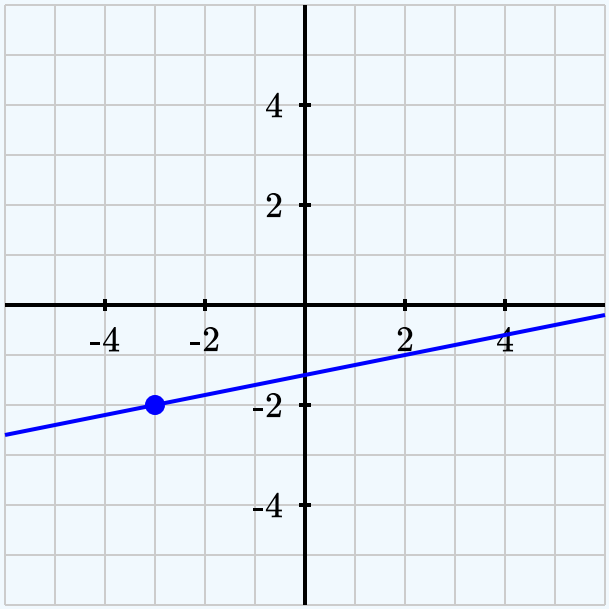
Answer available at the end of this section.

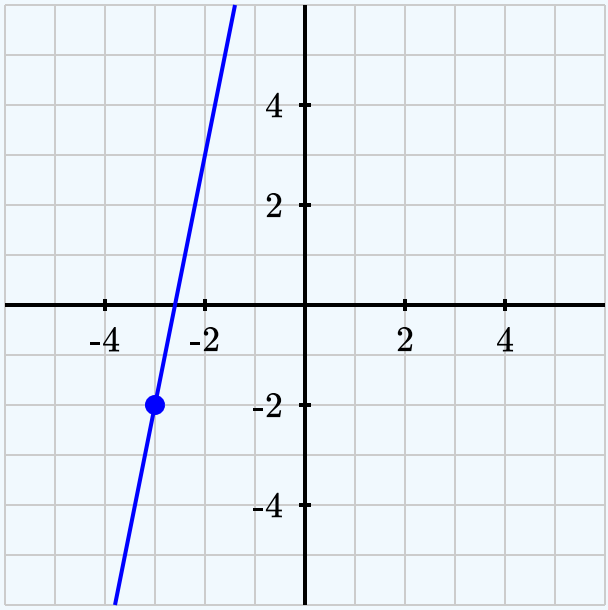
Activity 1.4.2.5

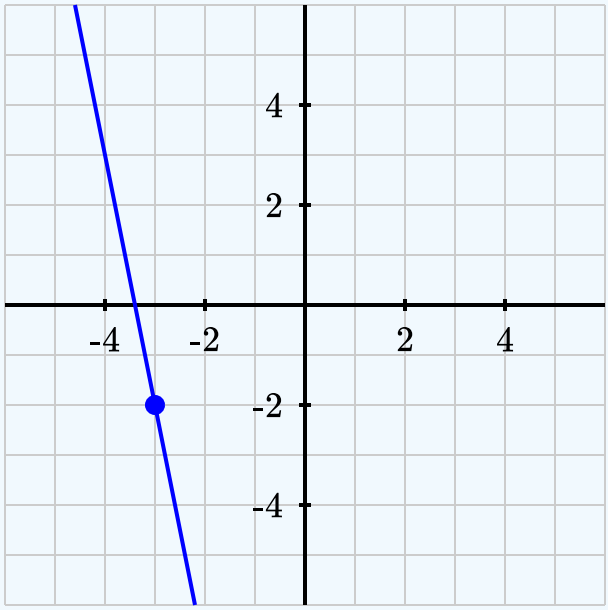
A line contains the point $(-3, -2)$ and has slope $\frac{1}{5}$. Which of the following is the graph of that line?

A.
B.
C.
D.


4
2
-4
-2
2
4
-2
-4


4
2
-4
-2
2
4
-2
-4





Answer available at the end of this section.

Activity 1.4.2.6

A line is given by the equation $y - 6 = -4(x + 2)$.

(a)

What form is the equation given in?

A. Standard form
B. Point-slope form
C. Slope-intercept form
D. The form it is in doesn't have a name.

Answer available at the end of this section.

(b)

The form gives us one point right away. Which of the following is a point on the line?

A. $(-2, -6)$
B. $(-2, 6)$
C. $(2, -6)$
D. $(2, 6)$

Answer available at the end of this section.

(c)

After we plot this point, we can use the slope to find another point. Find another point and graph the resulting line.

Answer available at the end of this section.

Activity 1.4.2.7

Recall from *(see related activity)* that the equation of a horizontal line has the form $y = k$ where k is a constant and a vertical line has the form $x = h$ where h is a constant.

(a)

Which type of line has a slope of zero?

A. Horizontal
B. Vertical

Answer available at the end of this section.

(b)

Which type of line has an undefined slope?

A. Horizontal
B. Vertical

Answer available at the end of this section.

(c)

Graph the vertical line that goes through the point $(4, -2)$.

Answer available at the end of this section.

(d)

What is the equation of the vertical line through the point $(4, -2)$?

A. $x = 4$
B. $y = 4$
C. $x = -2$
D. $y = -2$

Answer available at the end of this section.

(e)

Graph the horizontal line that goes through the point $(4, -2)$.

Answer available at the end of this section.

(f)

What is the equation of the horizontal line through the point $(4, -2)$?

A. $x = 4$
B. $y = 4$
C. $x = -2$
D. $y = -2$

Answer available at the end of this section.

Activity 1.4.2.8

Graph each line described below.

(a)

The line containing the points $(-3, 4)$ and $(5, -2)$.

Answer available at the end of this section.

(b)

The line whose x-intercept is -2 and whose y-intercept is 7.

Answer available at the end of this section.

(c)

The line whose slope is $\frac{2}{5}$ that goes through the point $(4, 6)$.

Answer available at the end of this section.

(d)

The line whose slope is $-\frac{1}{3}$ and whose y-intercept is -4.

Answer available at the end of this section.

(e)

The vertical line through the point $(-2, -7)$.

Answer available at the end of this section.

(f)

The horizontal line through the point $(-6, 3)$.

Answer available at the end of this section.

(g)

The line with equation $y = -\frac{5}{3}x - 6$.

Answer available at the end of this section.

(h)

The line with equation $y - 5 = \frac{7}{2}(x - 2)$.

Answer available at the end of this section.

(i)

The line with equation $3x - 6y = 8$.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can graph a line from different kinds of information before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.4.2.1

Describe how you would graph the line $y = -\frac{1}{2}x + 3$ using its slope and y-intercept. Give the coordinates of two points on the line.

Answer available at the end of this section.

Check Your Understanding 1.4.2.2

A line passes through $(-1, 4)$ with slope 3. Give the coordinates of two more points on the line without finding its equation first.

Answer available at the end of this section.

Check Your Understanding 1.4.2.3

A table shows two points on a line: when $x = 2$, $y = 9$; when $x = 5$, $y = 3$. Find the slope and describe how to graph the line from this table.

Answer available at the end of this section.

Answers

Activity 1.4.2.1

Many answers possible.

Activity 1.4.2.1

D

(Though C is also true. D is just more descriptive.)

Activity 1.4.2.1

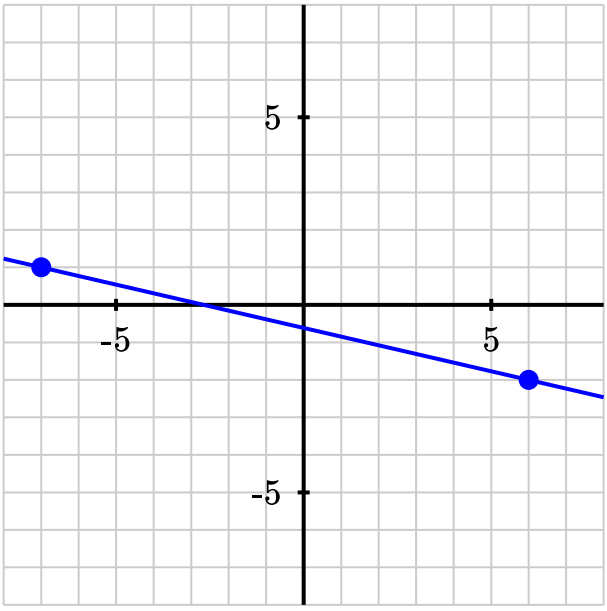
Just the one possible line.

Activity 1.4.2.1

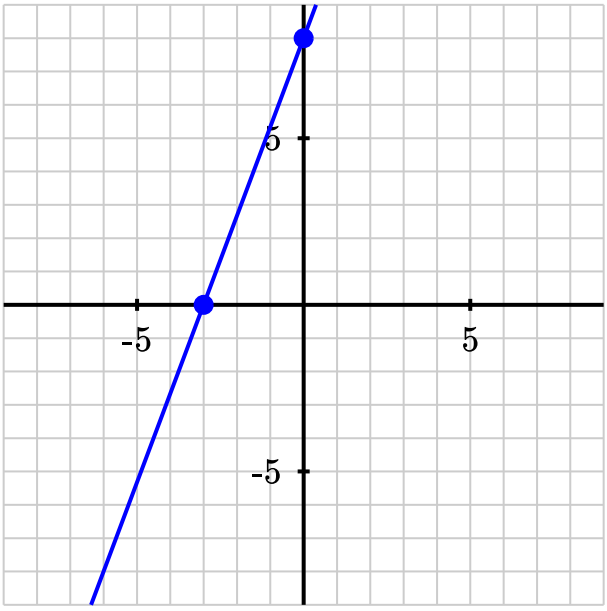
A

(Discussion could include that we would have to make the line curve to connect the points in more than one way. But, a line has to have the same slope everywhere.)

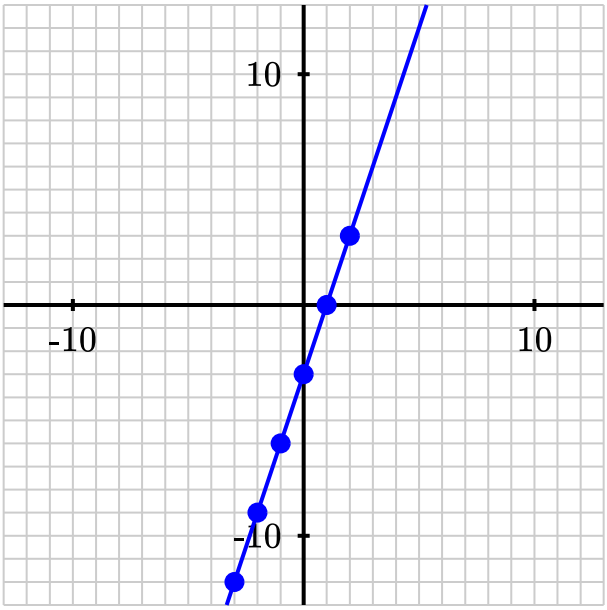
Activity 1.4.2.2



Activity 1.4.2.2



Activity 1.4.2.2



Activity 1.4.2.2

B

Discussion could include pointing out that only two are *necessary*, but we can include more if we want. Also, including more is a good way to catch an error in plotting. One will stick out!

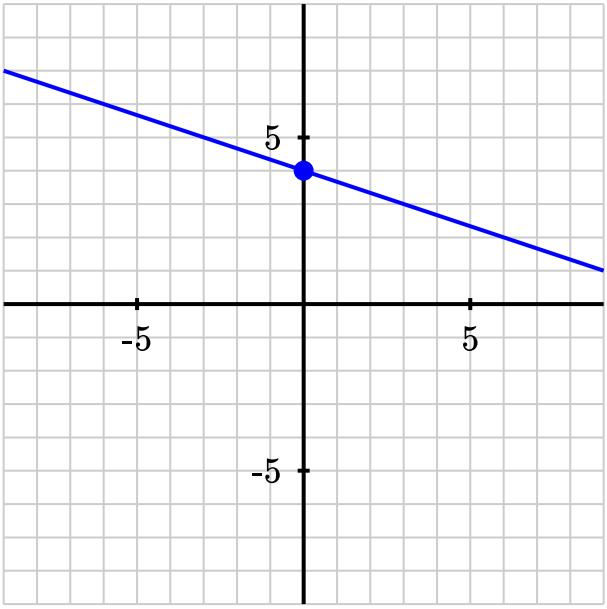
Activity 1.4.2.3

B

Activity 1.4.2.3

A and D

Activity 1.4.2.3



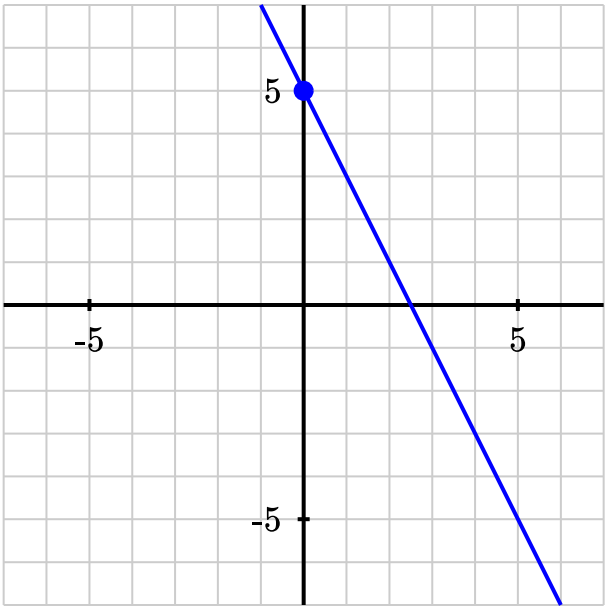
Activity 1.4.2.4

C

Activity 1.4.2.4

D

Activity 1.4.2.4



Activity 1.4.2.5

B

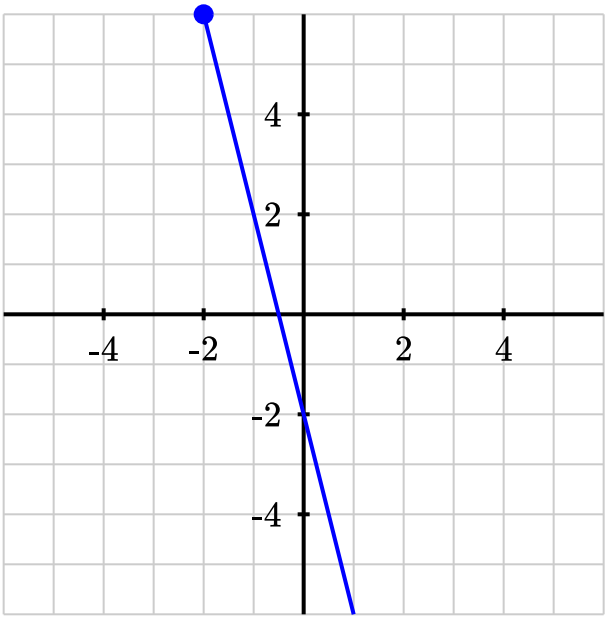
Activity 1.4.2.6

B

Activity 1.4.2.6

B

Activity 1.4.2.6



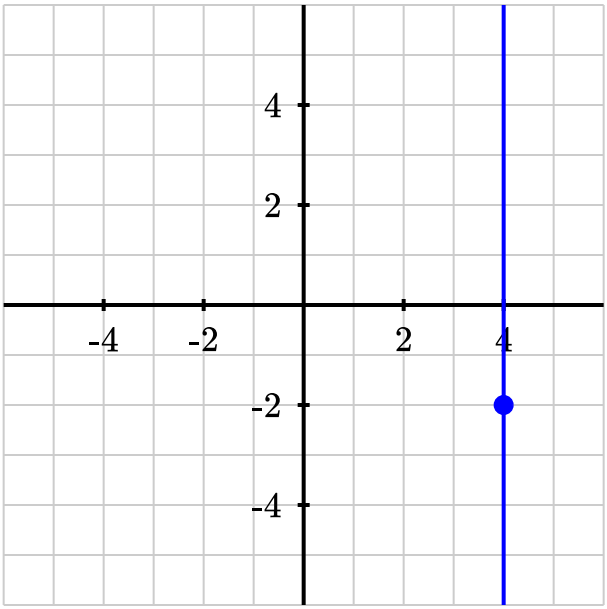
Activity 1.4.2.7

A

Activity 1.4.2.7

B

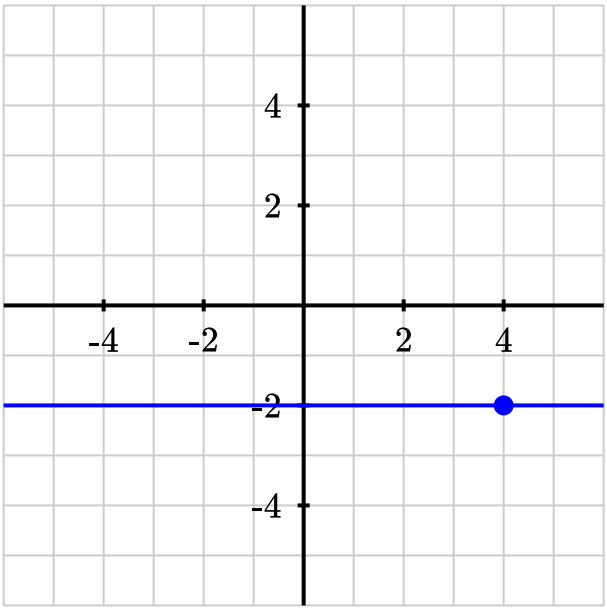
Activity 1.4.2.7



Activity 1.4.2.7

A

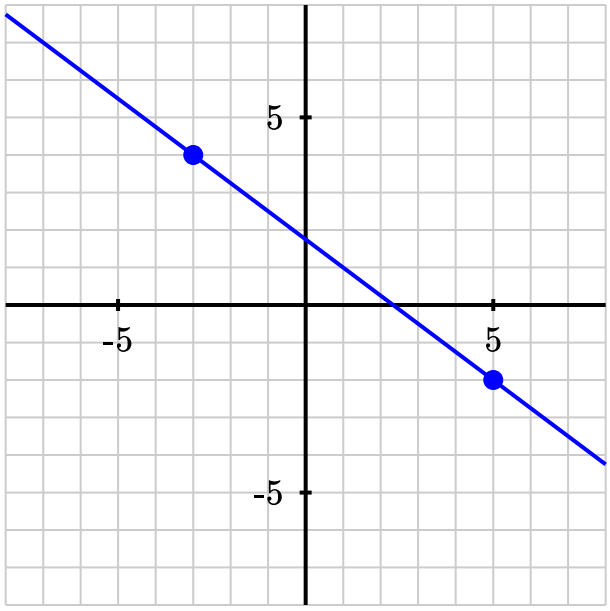
Activity 1.4.2.7



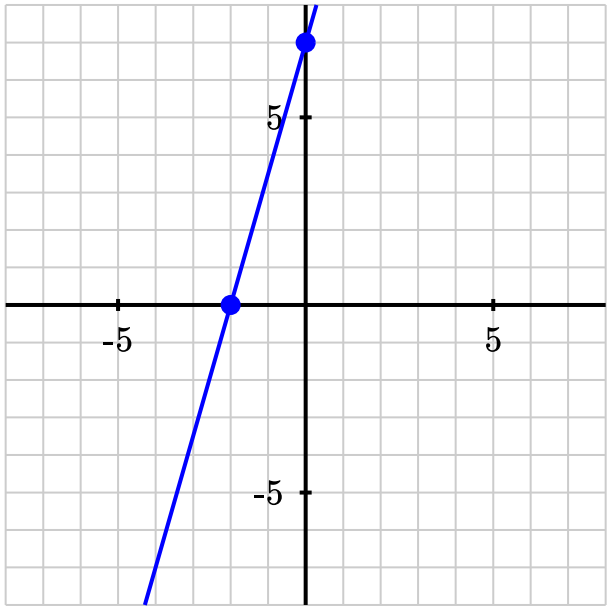
Activity 1.4.2.7

D

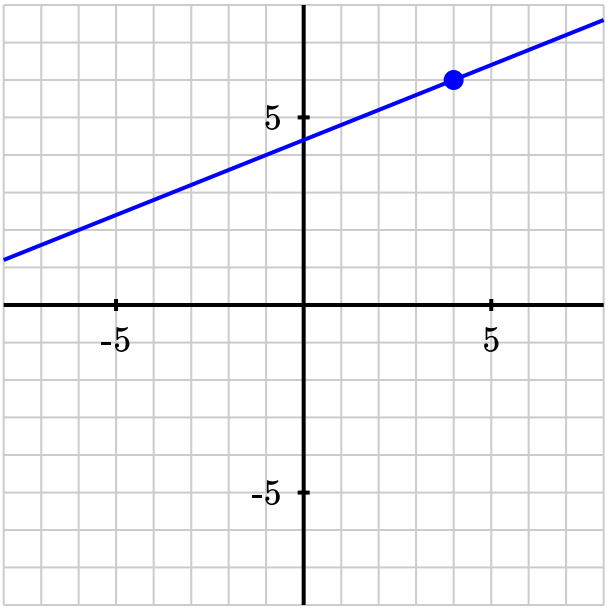
Activity 1.4.2.8



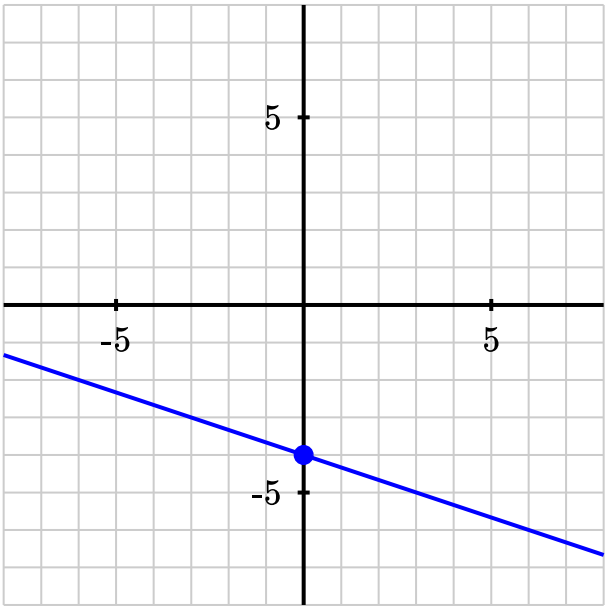
Activity 1.4.2.8



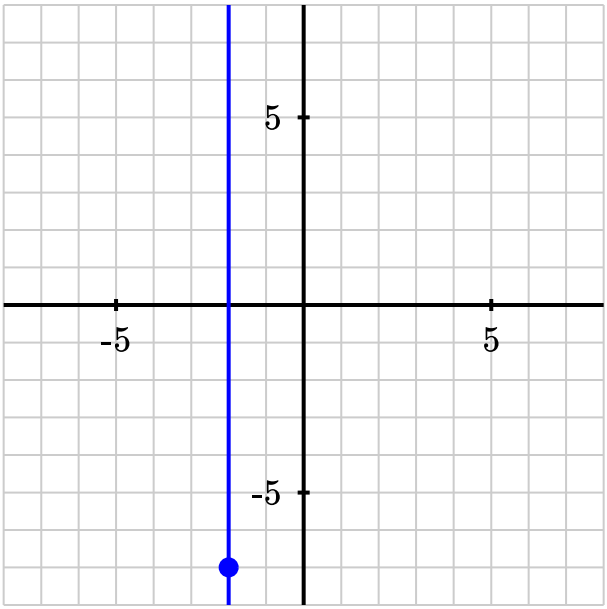
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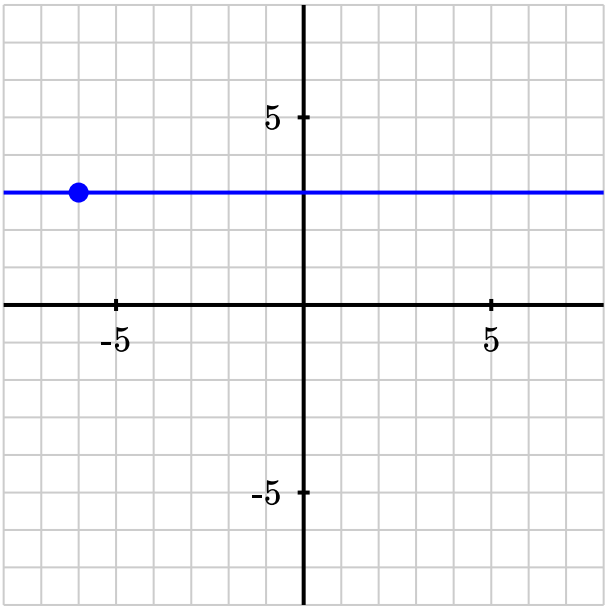
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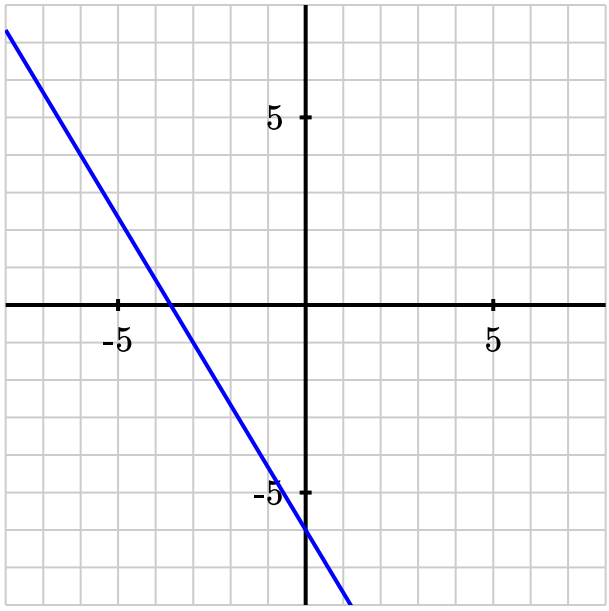
Activity 1.4.2.8



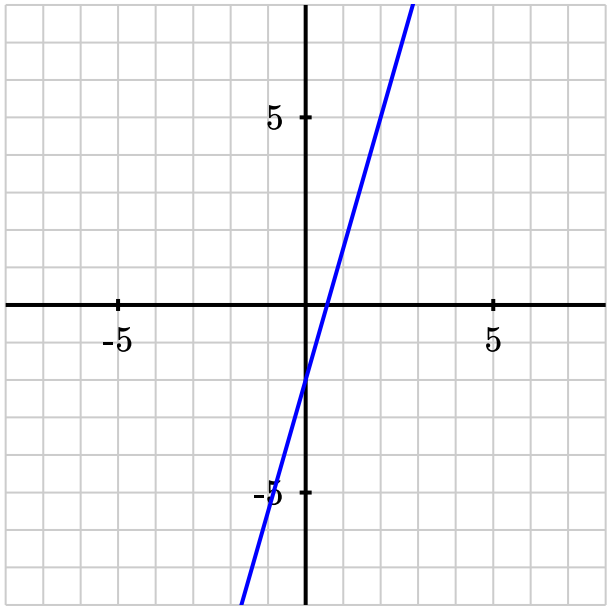
Activity 1.4.2.8



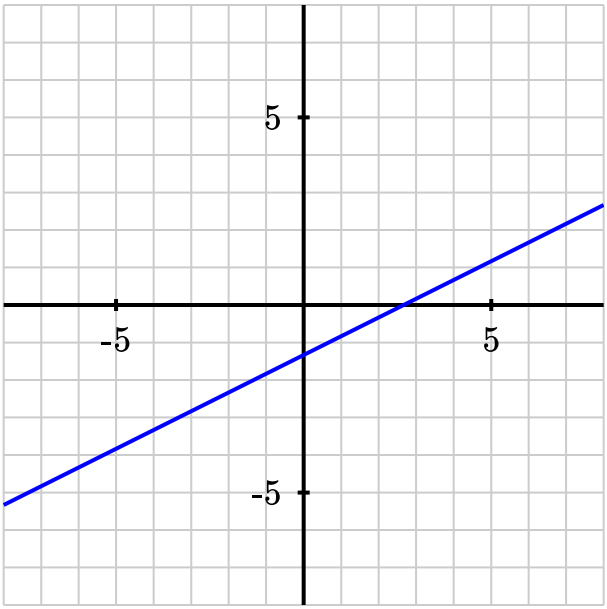
Activity 1.4.2.8



Activity 1.4.2.8



Activity 1.4.2.8



Check Your Understanding 1.4.2.1

Start at the y-intercept $(0, 3)$. The slope $-\frac{1}{2}$ means: from that point, move right 2 and down 1 to reach a second point at $(2, 2)$. Plotting $(0, 3)$ and $(2, 2)$ and drawing the line through them gives the graph.

Check Your Understanding 1.4.2.2

Slope $3 = \frac{3}{1}$ means: from $(-1, 4)$, move right 1 and up 3 to get $(0, 7)$; repeat to get $(1, 10)$. Any two of $(-1, 4)$, $(0, 7)$, $(1, 10)$ plot the same line.

Check Your Understanding 1.4.2.3

$m = \frac{3-9}{5-2} = \frac{-6}{3} = -2$. Plot the two given points $(2, 9)$ and $(5, 3)$ directly from the table, then draw the line through them — no equation is needed to graph from two known points.

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1.4.2.1: LF3 Online Exercises

LF3 — Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.	LF3 CheckIt Exercises

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1.4.3: Linear Models and Meanings (LF5)

This section corresponds to APC slot 1.4 (Linear functions) in the Math 172 flow — modeling and interpretation.

LF5 Learning outcome: Build linear models from verbal descriptions, and use the models to establish conclusions, including by contextualizing the meaning of slope and intercept parameters.

Linear Models

Overview

We begin by revisiting *(see related activity)* from *(see related activity)*.

Activity 1.4.3.1

Ellie has $\$13$ in her piggy bank, and she gets an additional $\$1.50$ each week for her allowance. Assuming she does not spend any money, the total amount of allowance, $A(w)$, she has after w weeks can be modeled by the function

$$A(w) = 13 + 1.50w \tag{1.4.3.1}$$

.

(a)

How much money will be in her piggy bank after 5 weeks?

Answer available at the end of this section.

(b)

After how many weeks will she have $\$40$ in her piggy bank?

Answer available at the end of this section.

Note

The function in the previous activity is an example of a **linear model**. A linear model is a linear function that describes, or models, a real-life application. In this section we will build and use linear models.

Activity 1.4.3.2

Jack bought a package containing 40 cookies. Each day he takes two in his lunch to work.

(a)

How many cookies are left in the package after 3 days?

A. 46
B. 42
C. 38
D. 36
E. 34

Answer available at the end of this section.

(b)

Fill out the following table to represent the number of cookies left in the package after the given number of days.

number of days	number of cookies left
0	
2	
5	
10	
20	

Answer available at the end of this section.

(c)

What is the y-intercept? Explain what it represents in the context of the problem.

Answer available at the end of this section.

(d)

What is the rate of change? Explain what it represents in the context of the problem.

Answer available at the end of this section.

(e)

Write a linear function to model the situation. Let d represent the number of days elapsed and $C(d)$ represent the number of cookies in the package. (Hint: Use the previous two questions to help!)

Answer available at the end of this section.

(f)

Find $C(6)$. Explain what that means in the context of the problem.

Answer available at the end of this section.

(g)

How many days will it take to empty the package? What does this correspond to on the graph?

Answer available at the end of this section.

Activity $1.4.3.3$

Daisy's Doughnut Shop sells delicious doughnuts. Each month, they incur a fixed cost of $\$2000$ for rent, insurance, and other expenses. Then, for each doughnut they produce, it costs them an additional $\$0.25$.

(a)

In January, Daisy's Doughnut Shop produced 1000 doughnuts. What was their total monthly cost to run the shop?

A. $2000.25
B. $2002.50
C. $2025.00
D. $2250.00
E. $4500.00

Answer available at the end of this section.

(b)

Fill out the following table to represent the cost for producing various amounts of doughnuts.

number of doughnuts	cost
0	
500	
1000	
1500	
2000	

Answer available at the end of this section.

(c)

What is the y-intercept? Explain what it represents in the context of the problem.

Answer available at the end of this section.

(d)

What is the rate of change? Explain what it represents in the context of the problem.

Answer available at the end of this section.

(e)

Write a linear function to model the situation. Let x represent the number of doughnuts produced and $C(x)$ represent the total cost. (Hint: Use the previous two questions to help!)

Answer available at the end of this section.

(f)

Find $C(1300)$. Explain what that means in the context of the problem.

Answer available at the end of this section.

(g)

Find the x-intercept. Explain what it means in the context of the problem.

Answer available at the end of this section.

Activity 1.4.3.4

A taxi costs $\$5.00$ up front, and then charges $\$0.73$ per mile traveled.

(a)

Write a linear function to model this situation.

Answer available at the end of this section.

(b)

How much will it cost for a 13 mile taxi ride?

Answer available at the end of this section.

(c)

If the taxi ride cost $\$11.06$, how many miles did it travel?

Answer available at the end of this section.

Activity 1.4.3.5

When reporting the weather, temperature is given in degrees Fahrenheit (F) or degrees Celsius (C). The two scales are related linearly, which means we can find a linear model to describe their relationship. This model lets us convert between the two scales.

(a)

Water freezes at $0C$ and $32F$. Water boils at $100C$ and $212F$. Use this information to write two ordered pairs.

Answer available at the end of this section.

(b)

Use the two points to write a linear model for this situation. Use C and F as your variables.

Answer available at the end of this section.

(c)

If the temperature outside is $25C$, what is the temperature in Fahrenheit?

Answer available at the end of this section.

(d)

If the temperature outside is $50F$, what is the temperature in Celsius?

Answer available at the end of this section.

(e)

What temperature value is the same in Fahrenheit as it is in Celsius?

Answer available at the end of this section.

Activity 1.4.3.6

Erin needs to print t-shirts for her company retreat. She has found two businesses that produce quality shirts, but they have different pricing structures. Shirts-R-Us requires a $\$60$ set up fee, then charges $\$7$ for each shirt produced. Graphix! has no set up fee and charges $\$8.50$ per shirt produced.

(a)

Write a linear function $S(x)$ that models the pricing structure for producing x shirts at Shirts-R-Us.

Answer available at the end of this section.

(b)

Write a linear function $G(x)$ that models the pricing structure for producing x shirts at Graphix!.

Answer available at the end of this section.

(c)

At which business would it be less expensive to buy 1 shirt? 10 shirts? 100 shirts? Explain your reasoning.

Answer available at the end of this section.

(d)

It depended on how many shirts were needed to determine which business was less expensive. For what range of number of shirts should Erin choose Shirts-R-Us and for what range should she choose Graphix!?

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can build and interpret linear models before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.4.3.1

A gym charges a $\$40$ sign-up fee plus $\$25$ per month. Write a linear model $C(m)$ for the total cost after m months, and interpret the slope and y-intercept in context.

Answer available at the end of this section.

Check Your Understanding 1.4.3.2

A candle is 8 inches tall when lit and burns down at a constant rate of 0.5 inches per hour. Write a linear model $H(t)$ for its height after t hours, and use it to predict the height after 6 hours.

Answer available at the end of this section.

Check Your Understanding 1.4.3.3

A plumber's total charge follows the model $P(h) = 60h + 75$, where h is hours worked. A customer is told the job will cost $\$255$. How many hours is the plumber planning to work?

Answer available at the end of this section.

Answers

Activity 1.4.3.1
$\$20.50$

Activity 1.4.3.1
18 weeks

Activity 1.4.3.2
E

Activity 1.4.3.2
number of days number of cookies left $0\ 40\ 2\ 38\ 5\ 30\ 10\ 20\ 20\ 0$

Activity 1.4.3.2
40; It represents the initial amount of cookies in the package.

Activity 1.4.3.2
-2; It represents how the number of cookies in the package changes per day.

Activity 1.4.3.2
$C(d) = 40 - 2d$

Activity 1.4.3.2
28; After 6 days, there will be 28 cookies left in the package.

Activity 1.4.3.2
20; This corresponds to the x-intercept.

Activity 1.4.3.3
D

Activity 1.4.3.3

Activity 1.4.3.3
$\$2000$; It is the fixed cost that must be paid regardless.

Activity 1.4.3.3
0.25; This how much the total cost changes for each additional doughnut produced.

Activity 1.4.3.3
$C(x) = 2000 + 0.25x$

Activity 1.4.3.3
$\$2325$; The cost to produce 1300 doughnuts is $\$2325$.

Activity 1.4.3.3
-8000; This doesn't make sense in the context in the problem. Daisy can't produce a negative number of doughnuts.

Activity 1.4.3.4
$C(x) = 5 + 0.73x$

Activity 1.4.3.4
$\$14.49$

Activity 1.4.3.4
8.3 miles

Activity 1.4.3.5
$(0, 32)$ and $(100, 212)$

Activity 1.4.3.5
$F = \frac{9}{5}C + 32$

Activity 1.4.3.5
$77F$

Activity 1.4.3.5
$10C$

Activity 1.4.3.5

$-40C = -40F$

Activity 1.4.3.6

$S(x) = 60 + 7x$

Activity 1.4.3.6

$G(x) = 8.50x$

Activity 1.4.3.6

One shirt: $S(1) = 67$ and $G(1) = 8.50$, so less expensive at Graphix!

10 shirts: $S(10) = 130$ and $G(10) = 85$, so less expensive at Graphix!

100 shirts: $S(100) = 760$ and $G(100) = 850$, so less expensive at Shirts-R-Us.

Activity 1.4.3.6

The businesses charge the same amount to produce 40 shirts. For less, choose Graphix! For more, choose Shirts-R-Us.

Check Your Understanding 1.4.3.1

$C(m) = 25m + 40$. The slope 25 means the cost increases $\$25$ for every additional month. The y-intercept 40 means there is a one-time $\$40$ charge before any monthly cost is added (at $m = 0$).

Check Your Understanding 1.4.3.2

$H(t) = 8 - 0.5t$. After 6 hours: $H(6) = 8 - 0.5(6) = 8 - 3 = 5$ inches. The slope is negative because the height is decreasing over time.

Check Your Understanding 1.4.3.3

Set $60h + 75 = 255$, so $60h = 180$ and $h = 3$. The plumber is planning to work 3 hours. This reverses the usual direction: instead of finding cost from hours, we find hours from a known cost.

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1.4.3.1: LF5 Online Exercises

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1.4.4: LF2, LF3, LF5 FAQ and Resources

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1.5: Introduction to Graphing Quadratic Functions (PR1) and Quadratic Models and Meaning (PR2)

1.5.1: Graphing Quadratic Functions (PR1)

1.5.2: Quadratic Models and Meanings (PR2)

PR1 Graphing Quadratic Functions

- Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

PR2 Quadratic Models and Meanings

- Use quadratic models to solve an application problem and establish conclusions.

Imagine a streaming service deciding what to charge for a subscription. Charge too little, and even with plenty of subscribers, the company barely brings in any money; charge too much, and subscribers cancel, so revenue falls again. Somewhere in between sits a price that brings in the most revenue — the peak of the curve, called the vertex. That curve is also symmetric around the peak: raise the price by five dollars or drop it by five dollars, and revenue falls off by about the same amount either way.

This builds on the linear models from Section 1.4, where a constant rate of change produced a straight line. Here the rate of change itself isn't constant — revenue climbs, slows, and then falls — and that changing rate is what produces a parabola instead of a line.

Section 1.6 picks up the idea of building new functions out of old ones — fittingly, since a revenue function like this one is itself built by combining a price function and a quantity function, exactly the kind of combining and composing you'll formalize there.

> **Looking ahead to calculus:** If you calculate the average rate of change of a quadratic model over several small intervals — the way you did back in Section 1.3 — something interesting happens: those rates of change themselves seem to line up along a straight line. Calculus is what turns that observation into a solid rule, and it's also what lets you find a parabola's maximum or minimum without the vertex formula at all — by asking exactly where that rate of change hits zero.

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1.5.1: Graphing Quadratic Functions (PR1)

This section corresponds to APC slot 1.5 (Quadratic functions) in the Math 172 flow.

Learning outcome: Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.

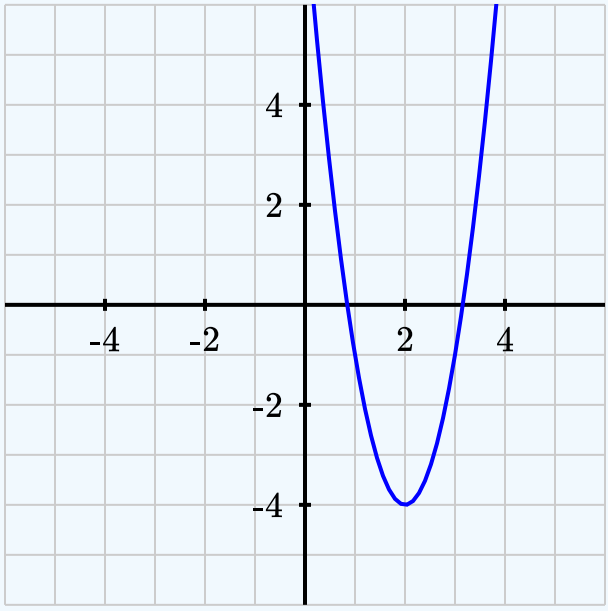
Graphing Quadratic Functions

Observation

Quadratic functions have many different applications in the real world. For example, say we want to identify a point at which the maximum profit or minimum cost occurs. Before we can interpret some of these situations, however, we will first need to understand how to read the graphs of quadratic functions to locate these least and greatest values.

Activity 1.5.1.1

Use the graph of the quadratic function $f(x) = 3(x-2)^2 - 4$ to answer the questions below.



(a)

Make a table for values of $f(x)$ corresponding to the given x-values. What is happening to the y-values as the x-values increase? Do you notice any other patterns of the y-values of the table?

Answer available at the end of this section.

(b)

At which point (x, y) does $f(x)$ have a minimum value? That is, is there a point on the graph that is lower than all other points?

A. The minimum value appears to occur near $(0, 8)$.

B. The minimum value appears to occur near $\left(-\frac{1}{5}, 10\right)$.

C. The minimum value appears to occur near $(2, -4)$.

D. There is no minimum value of this function.

Answer available at the end of this section.

(c)

At which point (x, y) does $f(x)$ have a maximum value? That is, is there a point on the graph that is higher than all other points?

A. The maximum value appears to occur near $(-2, 44)$.

B. The maximum value appears to occur near $\left(-\frac{1}{5}, 10\right)$.

C. The maximum value appears to occur near $(2, -4)$.

D. There is no maximum value of this function.

Answer available at the end of this section.

Definition: Vertex Form of a Quadratic Function

The point at which a quadratic function has a maximum or minimum value is called the **vertex**. The **vertex form** of a quadratic function is given by

$$f(x) = a(x-h)^2 + k \tag{1.5.1.1}$$

, where (h, k) is the **vertex** of the parabola.

Definition: Axis of Symmetry of a Quadratic Function

The **axis of symmetry**, also known as the line of symmetry, is the line that makes the shape of an object symmetrical. For a quadratic function, the axis of symmetry always passes through the vertex (h, k) and so is the vertical line $x = h$.

Activity 1.5.1.2

Use the given quadratic function, $f(x) = 3(x-2)^2 - 4$, to answer the following:

(a)

Applying *(see related activity)*, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -4)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 4)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -4)$; axis of symmetry: $x = -2$

D. vertex: $(2, 4)$; axis of symmetry: $x = 2$

Answer available at the end of this section.

(b)

Compare what you got in part a with the values you found in *(see related activity)*. What do you notice?

Answer available at the end of this section.

Definition: Standard Form of a Quadratic Function

The **standard form** of a quadratic function is given by

$$f(x) = ax^2 + bx + c \tag{1.5.1.2}$$

, where a, b, and c are real coefficients.

Activity 1.5.1.3

Completing the square (see *(see related activity)*) is a very useful tool or method to convert the quadratic equation in standard form into vertex form. Let's start with the standard form of $y = 2x^2 - 4x + 7$ and convert it into vertex form.

(a)

Before we begin, isolate the terms with x on one side of the equation. What equation do you now have?

Answer available at the end of this section.

(b)

Notice that the a-value is not 1. Before we complete the square, we will want to factor out the coefficient of the x^2 term. What does your equation look like when factoring out 2?

Answer available at the end of this section.

(c)

Now apply the “completing the square” steps to determine the constant term that is added to both sides of the equation.

Answer available at the end of this section.

(d)

Factor the perfect square trinomial.

Answer available at the end of this section.

(e)

Now isolate y and then determine the vertex and axis of symmetry.

Answer available at the end of this section.

Note

In *(see related activity)*, we were able to convert from standard form to vertex form. If you were to do this for the general form of $y = ax^2 + bx + c$, you will see that the x-value of the vertex will always be of the form $-\frac{b}{2a}$. For instance, suppose we start with

$$y = ax^2 + bx + c \tag{1.5.1.3}$$

. Following similar steps as we did in *(see related activity)*, we would get:

$$y - c = ax^2 + bx \tag{1.5.1.4}$$

$$y - c = a\left(x^2 + \frac{b}{a}x\right) \tag{1.5.1.5}$$

$$y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) \tag{1.5.1.6}$$

$$y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x + \frac{b}{2a}\right)^2 \tag{1.5.1.7}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \tag{1.5.1.8}$$

In the vertex form, $y = a(x-h)^2 + k$, the vertex (h, k) can be identified as:

$$h = -\frac{b}{2a} \tag{1.5.1.9}$$

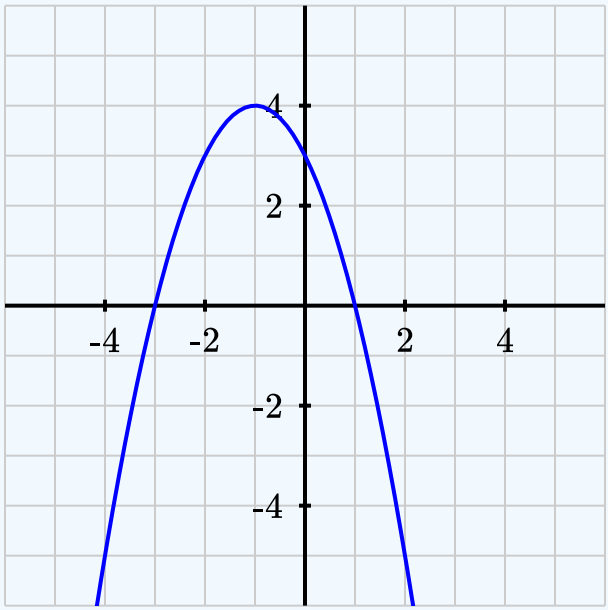
$$k = c - \frac{b^2}{4a} \tag{1.5.1.10}$$

Observation

Just as with the vertex form of a quadratic, we can use the standard form of a quadratic $(f(x) = ax^2 + bx + c)$ to find the **axis of symmetry** and the **vertex** by using the values of a, b, and c. Given the standard form of a quadratic, the axis of symmetry is the vertical line $x = -\frac{b}{2a}$ and the vertex is at the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Activity 1.5.1.4

Use the graph of the quadratic function to answer the questions below.



(a)

Which of the following quadratic functions could be the graph shown in the figure?

A. $f(x) = x^2 + 2x + 3$

B. $f(x) = -(x+1)^2 + 4$

C. $f(x) = -x^2 - 2x + 3$

D. $f(x) = (x+1)^2 + 4$

Answer available at the end of this section.

(b)

What is the maximum or minimum value?

A. -1

B. 4

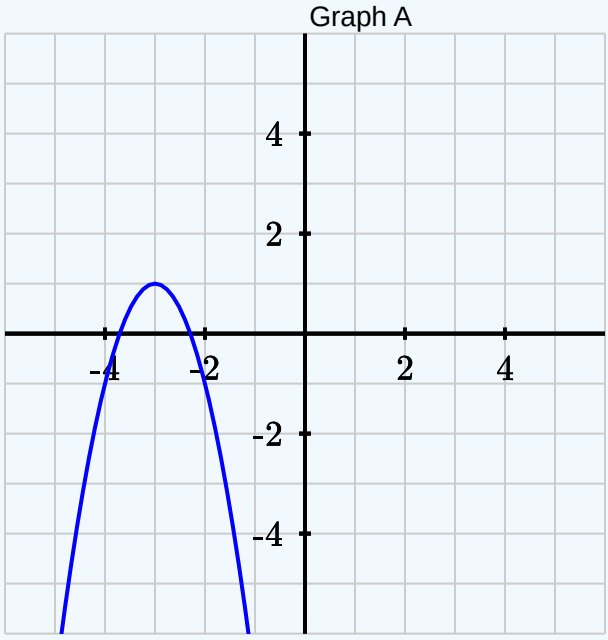
C. -3

D. 1

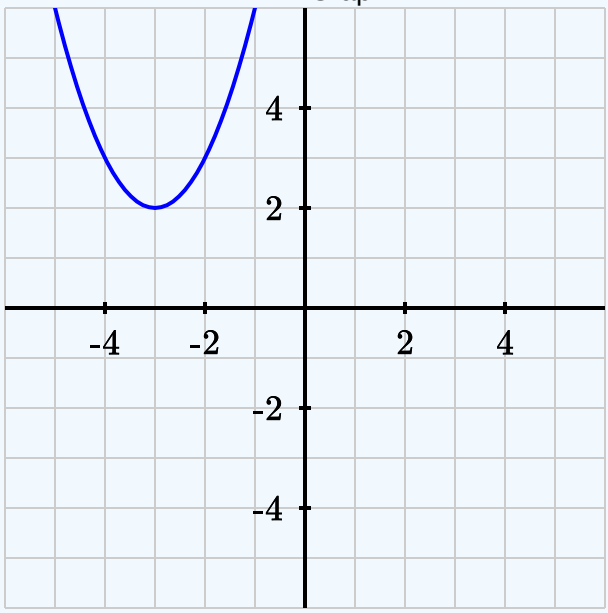
Answer available at the end of this section.

Activity 1.5.1.5

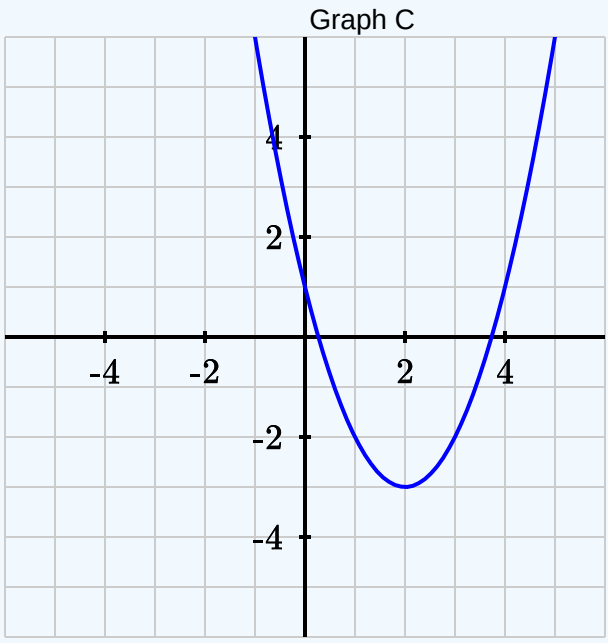
Consider the following four graphs of quadratic functions:



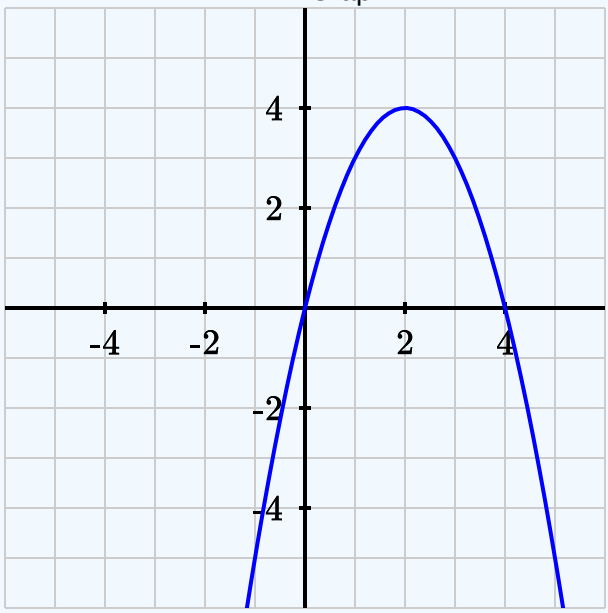
Graph B



Graph C



Graph D



(a)

Which of the graphs above have a maximum?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Answer available at the end of this section.

(b)

Which of the graphs above have a minimum?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Answer available at the end of this section.

(c)

Which of the graphs above have an axis of symmetry at $x = 2$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Answer available at the end of this section.

(d)

Which of the graphs above represents the function $f(x) = -(x-2)^2 + 4$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Answer available at the end of this section.

(e)

Which of the graphs above represents the function $f(x) = x^2 - 4x + 1$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Answer available at the end of this section.

Note

Notice that the maximum or minimum value of the quadratic function is the y-value of the vertex.

Activity 1.5.1.6

A quadratic function $f(x)$ has a maximum value of 7 and axis of symmetry at $x = -2$.

(a)

Sketch a graph of a function that meets the criteria for $f(x)$.

Answer available at the end of this section.

(b)

Was your graph the only possible answer? Try to sketch another graph that meets this criteria.

Answer available at the end of this section.

Note

Other points, such as x- and y-intercepts, may be helpful in sketching a more accurate graph of a quadratic function.

Activity 1.5.1.7

Consider the following two quadratic functions $f(x) = x^2 - 4x + 20$ and $g(x) = 2x^2 - 8x + 24$ and answer the following questions:

(a)

Applying *(see related activity)*, what is the vertex and axis of symmetry of $f(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

Answer available at the end of this section.

(b)

Applying *(see related activity)*, what is the vertex and axis of symmetry of $g(x)$?

A. vertex: $(2, -16)$; axis of symmetry: $x = 2$

B. vertex: $(-2, 16)$; axis of symmetry: $x = -2$

C. vertex: $(-2, -16)$; axis of symmetry: $x = -2$

D. vertex: $(2, 16)$; axis of symmetry: $x = 2$

Answer available at the end of this section.

(c)

What do you notice about $f(x)$ and $g(x)$?

Answer available at the end of this section.

(d)

Now graph both $f(x)$ and $g(x)$ and draw a sketch of each graph on one coordinate plane. How are they similar/different?

Answer available at the end of this section.

Note

Notice that in *(see related activity)*, two different functions could have the same vertex and axis of symmetry. When $|a| > 0$, the graph narrows. When $0 < |a| < 1$, the graph widens (refer back to *(see related activity)*).

Activity 1.5.1.8

Answer the following for each quadratic function below.

(a)

$f(x) = x^2 + 2x - 3$

Answer available at the end of this section.

(b)

$f(x) = -5(x - 3)^2 + 20$

Answer available at the end of this section.

(c)

$f(x) = 5x^2 + 30x + 40$

Answer available at the end of this section.

(d)

$f(x) = 2(x + 4)^2 - 3$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can graph quadratic functions and identify their key features before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.5.1.1

Let $f(x) = (x - 3)^2 - 4$. Identify the vertex, the axis of symmetry, and state whether the vertex is a maximum or minimum.

Answer available at the end of this section.

Check Your Understanding 1.5.1.2

Let $g(x) = -2x^2 + 8x - 3$. Find the axis of symmetry using the standard-form formula, then find the maximum or minimum value.

Answer available at the end of this section.

Check Your Understanding 1.5.1.3

A quadratic function has vertex $(-1, 6)$ and passes through the point $(1, -2)$. Write the function in vertex form.

Answer available at the end of this section.

Answers

Activity 1.5.1.1

x $f(x)$ -2 44 -1 23 0 8 1 -1 2 -4 3 -1 4 8 5 23

Activity 1.5.1.1

C

Activity 1.5.1.1

D

Activity 1.5.1.2

A

Activity 1.5.1.2

Students will probably notice that the vertex is also the minimum of the quadratic function and that the axis of symmetry goes through the minimum.

Activity 1.5.1.3

$y - 7 = 2x^2 - 4x$

Activity 1.5.1.3

$y - 7 = 2(x^2 - 2x)$

Activity 1.5.1.3

By using the "completing the square" steps, students should get:

$$y - 7 + (2) = 2(x^2 - 2x + (1)) \tag{1.5.1.11}$$

$$y - 5 = 2(x^2 - 2x + 1) \tag{1.5.1.12}$$

Activity 1.5.1.3

$$y - 5 = 2(x - 1)^2 \tag{1.5.1.13}$$

Activity 1.5.1.3

$y = 2(x - 1)^2 + 5$

Axis of Symmetry: $x = 1$

Vertex: $(1, 5)$

Activity 1.5.1.4

B or C because they both have a negative in the front, which would indicate that the graph is pointing downward.

Activity 1.5.1.4

B. Instructors might need to distinguish what a max or min value is versus where the max or min value is located.

Activity 1.5.1.5

A and D

Activity 1.5.1.5

B and C

Activity 1.5.1.5

C and D

Activity 1.5.1.5

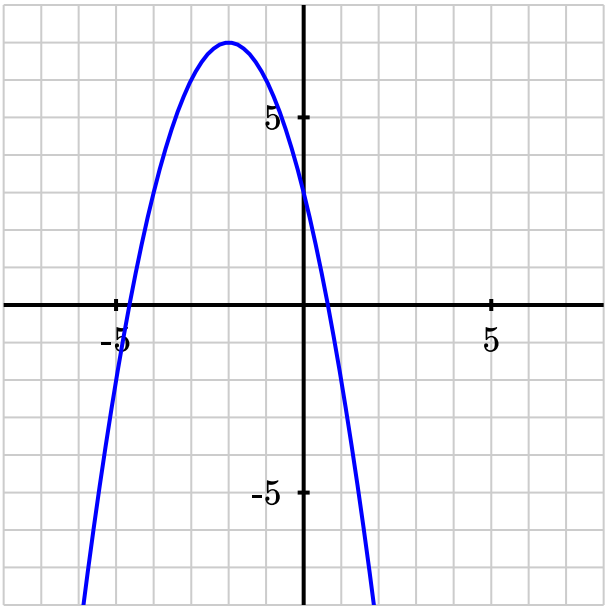
D

Activity 1.5.1.5

C

Activity 1.5.1.6

Note that this is just one possible sketch students might draw.



Activity 1.5.1.6

No. There are many possible graphs students could sketch, but each would have a maximum point at $(-2, 7)$. The graph could be transformed by horizontal stretches or compressions.

Activity 1.5.1.7

D

Activity 1.5.1.7

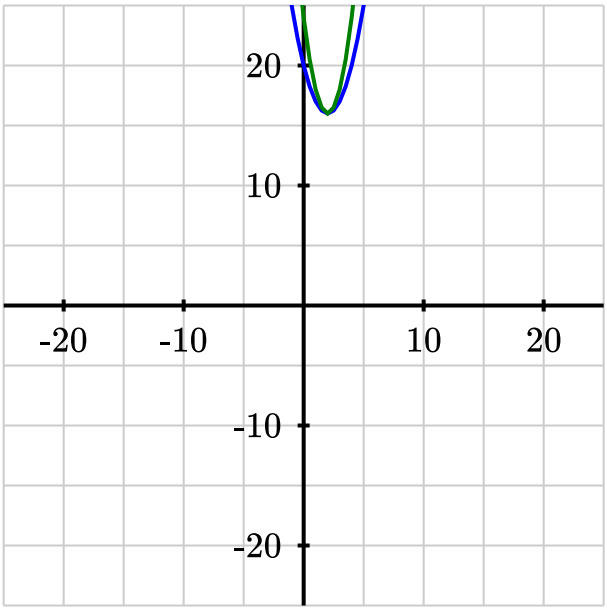
D

Activity 1.5.1.7

They have the same vertex and axis of symmetry.

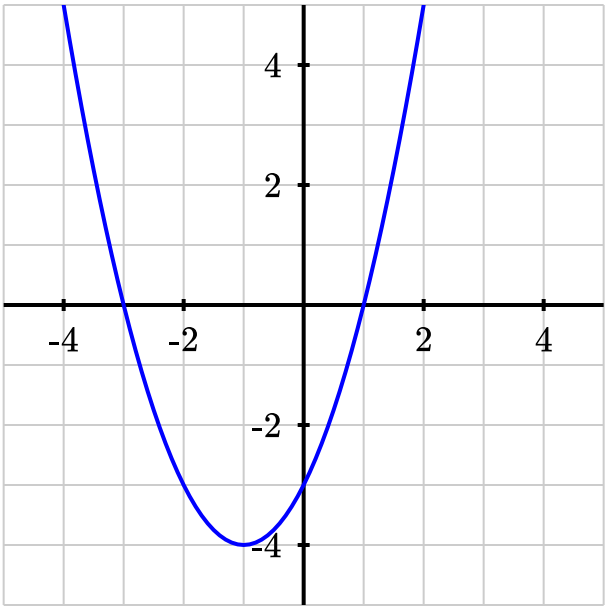
Activity 1.5.1.7

By graphing the two functions, students should be able to see that although $f(x)$ and $g(x)$ have the same vertex and axis of symmetry, they are different functions/graphs. The graph of $g(x)$ is "skinnier."



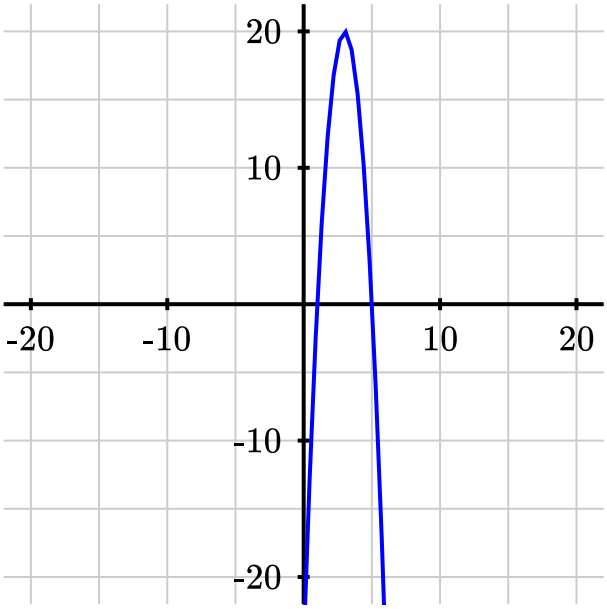
Activity 1.5.1.8

1. upwards
2. $(-1, -4)$; minimum
3. $x = -1$
4.



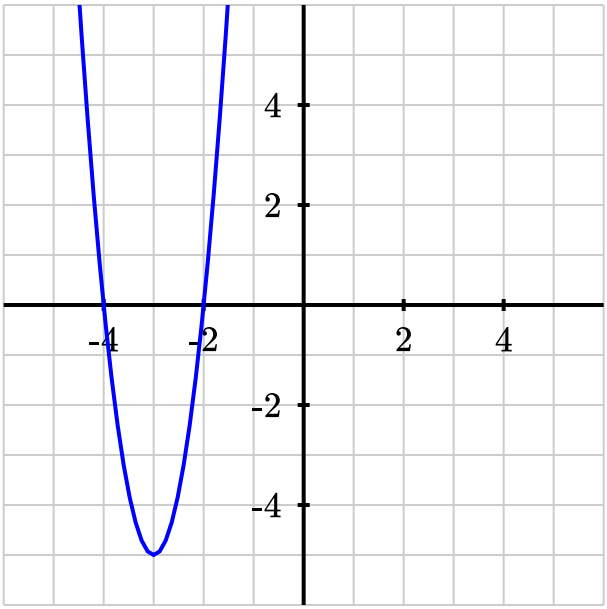
Activity 1.5.1.8

1. downwards
2. $(3, 20)$; maximum
3. $x = 3$
4.



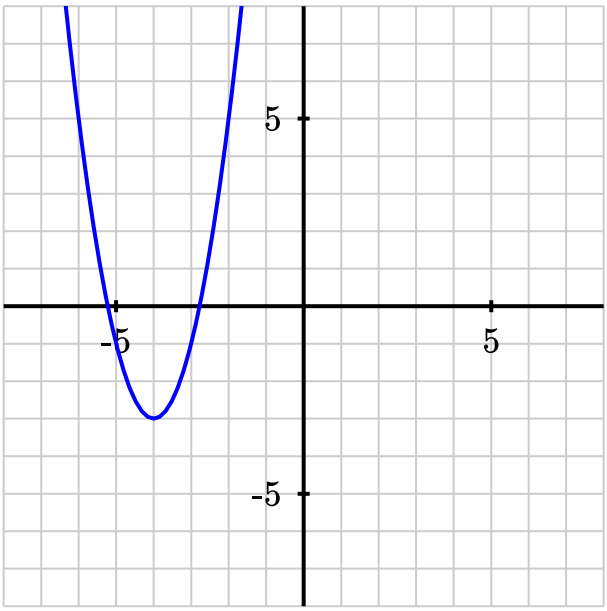
Activity 1.5.1.8

1. upwards
2. $(-3, -5)$; minimum
3. $x = -3$
4.



Activity 1.5.1.8

1. upwards
2. $(-4, -3)$; minimum
3. $x = -4$
4.



Check Your Understanding 1.5.1.1

Vertex: $(3, -4)$. Axis of symmetry: $x = 3$. Since the function is written in vertex form with a positive leading coefficient (the parabola opens upward), the vertex is a minimum.

Check Your Understanding 1.5.1.2

Axis of symmetry: $x = -\frac{b}{2a} = -\frac{8}{2(-2)} = 2$. Evaluate $g(2) = -2(4) + 16 - 3 = -8 + 16 - 3 = 5$. Since $a = -2 < 0$, the parabola opens downward, so $(2, 5)$ is a maximum.

Check Your Understanding 1.5.1.3

Vertex form: $f(x) = a(x + 1)^2 + 6$. Substitute $(1, -2)$: $-2 = a(2)^2 + 6$, so $-8 = 4a$ and $a = -2$. The function is $f(x) = -2(x + 1)^2 + 6$.

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1.5.1.1: PR1 Online Exercises

PR1 — Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.	PR1 CheckIt Exercises

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1.5.2: Quadratic Models and Meanings (PR2)

This section corresponds to APC slot 1.5 (Quadratic functions) in the Math 172 flow — modeling and interpretation.

Learning outcome: Use quadratic models to solve an application problem and establish conclusions.

Quadratic Models

Observation

Objects launched into the air follow a path that can be described by a quadratic function. We can also use quadratic functions to model area, profit, and population. Knowing the key components of a quadratic function allow us to find maximum profit, the point where an object hits the ground, or how much of an object to make for a minimum cost.

Activity 1.5.2.1

A water balloon is tossed vertically from a fifth story window. It's height $h(t)$, in feet, at a time t, in seconds, is modeled by the function

$$h(t) = -16t^2 + 40t + 50\backslash\text{amp} \tag{1.5.2.1}$$

(a)

Complete the following table. Do all the values have meaning in terms of the model?

t	$h(t)$
0	
1	
2	
3	
4	
5	

Answer available at the end of this section.

(b)

Compute the slope of the line joining $t = 0$ and $t = 1$. Then, compute the slope of the line joining $t = 1$ and $t = 2$. What do you notice about the slopes?

Answer available at the end of this section.

(c)

What is the meaning of $h(0) = 50$?

A. The initial height of the water balloon is 50 feet.
B. The water balloon reaches a maximum height of 50 feet.
C. The water balloon hits the ground after 50 seconds.

D. The water balloon travels 50 feet before hitting the ground.

Answer available at the end of this section.

(d)

Find the vertex of the quadratic function $h(t)$.

A. $(0, 50)$
B. $(1, 74)$
C. $(1.25, 75)$
D. $(3.4, 0)$

Answer available at the end of this section.

(e)

What is the meaning of the vertex?

A. The water balloon reaches a maximum height of 50 feet at the start.
B. After 1 second, the water balloon reaches a maximum height of 74 feet.
C. After 1.25 seconds, the water balloon reaches the maximum height.
D. After 3.4 seconds, the water balloon hits the ground.

Answer available at the end of this section.

Activity $1.5.2.2$

The population of a small city is given by the function $P(t) = -50t^2 + 1200t + 32000$, where t is the number of years after 2015.

(a)

When will the population of the city reach a maximum?

A. 2020
B. 2022
C. 2025
D. 2027

Answer available at the end of this section.

(b)

Determine when the population of the city is increasing and when it is decreasing.

Answer available at the end of this section.

(c)

When will the population of the city reach $36{,}000$ people?

A. 2019
B. 2025
C. 2027
D. 2035

Answer available at the end of this section.

Activity 1.5.2.3

The unit price of an item affects its supply and demand. That is, if the unit price increases, the demand for the item will usually decrease. For example, an online streaming service currently has 84 million subscribers at a monthly charge of $\$6$. Market research has suggested that if the owners raise the price to $\$8$, they would lose 4 million subscribers. Assume that subscriptions are linearly related to the price.

(a)

Which of the following represents a linear function which relates the price of the streaming service p to the number of subscribers Q (in millions)?

A. $Q(p) = -2p$
B. $Q(p) = -2p + 84$
C. $Q(p) = -2p - 4$
D. $Q(p) = -2p + 96$

Answer available at the end of this section.

(b)

Using the fact that revenue R is price times the number of items sold, $R = pQ$, which of the following represents the revenue in terms of the price?

A. $R(p) = -2p^2$
B. $R(p) = -2p^2 + 84p$
C. $R(p) = -2p^2 - 4p$
D. $R(p) = -2p^2 + 96p$

Answer available at the end of this section.

(c)

What price should the streaming service charge for a monthly subscription to maximize their revenue?

A. $\$10$
B. $\$19.50$
C. $\$24$
D. $\$28.25$

Answer available at the end of this section.

(d)

How many subscribers would the company have at this price?

A. 39.5 million
B. 48 million
C. 57 million
D. 76 million

Answer available at the end of this section.

(e)

What is the maximum revenue?

A. 760 million

B. 1112 million
C. 1152 million
D. 1116 million

Answer available at the end of this section.

Activity 1.5.2.4

The owner of a ranch decides to enclose a rectangular region with 240 feet of fencing. To help the fencing cover more land, he plans to use one side of his barn as part of the enclosed region. What is the maximum area the rancher can enclose?

(a)

Draw a picture to represent the fenced area against the barn. Use w to represent the length of fence parallel to the barn and l to represent the two sides perpendicular to the barn.

Answer available at the end of this section.

(b)

Find an equation for the area of the fence in terms of the length l. It may be useful to find an equation for the total amount of fencing in terms of the length l and width w.

A. $A = lw$
B. $A = l^2$
C. $A = l(240 - 2l)$
D. $A = l\left(120 - \frac{l}{2}\right)$

Answer available at the end of this section.

(c)

Use the area equation to find the maximum area the rancher can enclose.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can use quadratic models to solve applied problems before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.5.2.1

A ball is thrown upward, and its height in feet after t seconds is modeled by $h(t) = -16t^2 + 64t + 5$. Find the maximum height the ball reaches, and the time at which it occurs.

Answer available at the end of this section.

Check Your Understanding 1.5.2.2

A rectangular garden is enclosed by 40 feet of fencing. If w is the width, the area is modeled by $A(w) = w(20 - w)$. Find the width that maximizes the area, and state the maximum area.

Answer available at the end of this section.

Check Your Understanding 1.5.2.3

A company's weekly profit (in dollars) from selling x units is modeled by $P(x) = -5x^2 + 400x - 2000$. How many units should the company sell to maximize profit, and what is that maximum profit?

Answer available at the end of this section.

Answers

Activity 1.5.2.1

After completing the table, students might notice that there are some negative values, which does not make sense because the height of the balloon cannot be negative. t $h(t)$ 0 50 1 74 2 66 3 26 4 -46 5 -150

Activity 1.5.2.1

The slope of the line joining $t = 0$ and $t = 1$ is 24. The slope of the line joining $t = 1$ and $t = 2$ is -8. One slope is positive and one slope is negative.

Activity 1.5.2.1

A

Activity 1.5.2.1

C

Activity 1.5.2.1

C

Activity 1.5.2.2

D

Activity 1.5.2.2

The population increases until 2027 (i.e., until it reaches its maximum) and then will decrease. So, the population increases from 2015 till 2027 and then decreases from 2027.

Activity 1.5.2.2

A

Activity 1.5.2.3

D

Activity 1.5.2.3

D

Activity 1.5.2.3

C

Activity 1.5.2.3

B

Activity 1.5.2.3

C

Activity 1.5.2.4

Instructors might use to this time to make sure students are setting up their picture correctly and to find an equation that represents the amount of fencing that would be used: $2l + w = 240$.

Activity 1.5.2.4

C

Activity 1.5.2.4

Using the area equation in part (b), we can see that the vertex is $(60, 7200)$, which indicates that the maximum area the rancher can enclose is $7{,}200$ square feet.

Check Your Understanding 1.5.2.1

The time of the maximum is at the axis of symmetry: $t = -\dfrac{64}{2(-16)} = 2$ seconds. Maximum height: $h(2) = -16(4) + 128 + 5 = -64 + 128 + 5 = 69$ feet.

Check Your Understanding 1.5.2.2

Expand: $A(w) = -w^2 + 20w$. Axis of symmetry: $w = -\dfrac{20}{2(-1)} = 10$. Maximum area: $A(10) = 10(20 - 10) = 100$ square feet, occurring when the width is 10 feet.

Check Your Understanding 1.5.2.3

Axis of symmetry: $x = -\dfrac{400}{2(-5)} = 40$ units. Maximum profit: $P(40) = -5(1600) + 16000 - 2000 = -8000 + 16000 - 2000 = 6000$ dollars.

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1.5.2.1: PR2 Online Exercises

PR2 — Use quadratic models to solve an application problem and establish conclusions.	PR2 CheckIt Exercises

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1.5.3: PR1, PR2 FAQ and Resources

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1.6: Introduction to Combining and Composing Functions (FN5)

1.6.1: Combining and Composing Functions (FN5)

FN5 Combining and Composing Functions

- Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.

Back in Section 1.5, a streaming service's revenue depended on its subscription price — but that revenue function didn't come from nowhere. It was built by multiplying two smaller functions together: price times quantity. That's combining functions. Composing works differently: instead of operating on two functions side by side, you feed the output of one directly into the input of another. If production cost depends on how many units get made, and the number of units depends on how many hours the line runs, then cost as a function of hours is a composition — one function's output feeding straight into the next function's input.

This leans on the notation habits from Section 1.2 as much as the modeling from Section 1.5 — evaluating $(f+g)(3)$ or $(f \circ g)(3)$ is just careful bookkeeping once you're comfortable finding $f(3)$ and $g(3)$ separately.

Section 1.7 turns this around: instead of building a new function by composing two functions together, you'll look for the one special function that undoes another — the inverse — where composing a function with its inverse just hands you back exactly what you started with.

> **Looking ahead to calculus:** Once you can compose functions, calculus asks how a composed function changes — if cost depends on units, and units depend on hours, how fast does cost change per hour? The chain rule answers this by finding the rate of change of the outer function and the rate of change of the inner function separately, then multiplying them together — which only makes sense because you've already learned to see a composed function as two functions linked, not one tangled mess.

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1.6.1: Combining and Composing Functions (FN5)

This section corresponds to APC slots 1.6 (Composite functions) & 1.9 (Combining functions) in the Math 172 flow.

FN5 Learning outcome: Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.

Combining and Composing Functions

Activity 1.6.1.1

Let $f(x) = x^2 - 3x$ and $g(x) = x^3 - 4x^2 + 7$.

(a)

Which of the following seems likely to be the most simplified form of $f(x) + g(x)$?

A. $x^2 - 3x + x^3 - 4x^2 + 7$
B. $x^5 - 7x^3 + 7$
C. $-x^3 + 5x^2 - 3x - 7$
D. $x^3 - 3x^2 - 3x + 7$

Answer available at the end of this section.

(b)

Which of the following seems likely to be the most simplified form of $f(x) - g(x)$?

A. $x^3 - 3x^2 - 3x + 7$
B. $-x^3 + 5x^2 - 3x - 7$
C. $-x^3 - 3x^2 - 3x + 7$
D. $x^2 - 3x - x^3 + 4x^2 - 7$

Answer available at the end of this section.

Activity 1.6.1.2

Let $f(x) = \sqrt{x+1}$ and $g(x) = 5x$.

(a)

Which of the following seems likely to be the most simplified form of $f(x) \cdot g(x)$?

A. $\sqrt{5x+1}$
B. $5\sqrt{x+1}$
C. $\sqrt{5x^2+5x}$
D. $5x\sqrt{x+1}$

Answer available at the end of this section.

(b)

Which of the following seems likely to be the most simplified form of $\dfrac{f(x)}{g(x)}$?

A. $\dfrac{5x}{\sqrt{x+1}}$
B. $\dfrac{\sqrt{x+1}}{5x}$
C. $\sqrt{\dfrac{x}{5x}+\dfrac{1}{5x}}$
D. $\sqrt{\dfrac{5x}{x}+\dfrac{5x}{1}}$

Answer available at the end of this section.

Overview

In *(*Activity 1.6.1.1*)* and *(*Activity 1.6.1.2*)*, we have found the sum, difference, product, and quotient of two functions. We can use the following notation for these newly created functions:

$$\begin{aligned}(f+g)(x)\amp = f(x)+g(x)\\ (f-g)(x)\amp = f(x)-g(x)\\ (f\cdot g)(x)\amp = f(x)\cdot g(x)\\ \left(\frac{f}{g}\right)(x)\amp = \frac{f(x)}{g(x)}\end{aligned} \tag{1.6.1.1}$$

With $\left(\dfrac{f}{g}\right)(x)$, we note that the quotient is only defined when $g(x)\neq 0$.

Activity 1.6.1.3

Let $f(x)=\dfrac{1}{3x-5}$.

(a)

Find $f(4)$.

A. $\dfrac{4}{3x-5}$
B. $\dfrac{1}{4(3x-5)}$
C. $\dfrac{1}{7}$
D. 7

Answer available at the end of this section.

(b)

If you were asked to find $f(x^3-2)$, how do you think you would proceed?

A. Multiply the original function $\dfrac{1}{3x-5}$ by x^3-2.
B. Plug the expression x^3-2 in for all the x-values in $\dfrac{1}{3x-5}$.
C. Plug the original function $\dfrac{1}{3x-5}$ in for all the x-values in x^3-2.
D. Multiply $3x-5$ by x^3-2.

Answer available at the end of this section.

(c)

Find $f(x^3-2)$.

A. $\dfrac{1}{3x-5}\cdot(x^3-2)$
B. $\dfrac{1}{3(x^3-2)-5}$
C. $\left(\dfrac{1}{3x-5}\right)^3-2$
D. $(3x-5)(x^3-2)$

Answer available at the end of this section.

(d)

What if we gave the expression x^3-2 a name? Let's define $g(x)=x^3-2$. What's another way we could denote $f(x^3-2)$?

A. $f(x)\cdot g(x)$
B. $g(f(x))$
C. $f(g(x))$
D. $\dfrac{f(x)}{g(x)}$

Answer available at the end of this section.

Definition: Composition of Functions

Given the functions $f(x)$ and $g(x)$, we define the **composition of f and g** to be the new function $h(x)$ given by

$$h(x)=f(g(x)) \tag{1.6.1.2}$$

. We also sometimes use the notation

$$f\circ g \tag{1.6.1.3}$$

or

$$(f\circ g)(x) \tag{1.6.1.4}$$

to refer to $f(g(x))$.

Note

When discussing the composite function $f(g(x))$, also written as $(f\circ g)(x)$, we often call $g(x)$ the "inner function" and $f(x)$ the "outer function". It is important to note that the inner function is actually the first function that gets applied to a given input, and then the outer function is applied to the output of the inner function.

Activity 1.6.1.4

Let $f(x)=\dfrac{1}{3x-5}$ and $g(x)=x^3-2$.

(a)

Find $f(g(x))$.

A. $\dfrac{x^3-2}{3x-5}$
B. $\dfrac{1}{(3x-5)(x^3-2)}$

C. $\dfrac{1}{3(x^3-2)-5}$
D. $\left(\dfrac{1}{3x-5}\right)^3-2$

Answer available at the end of this section.

(b)

Find $g(f(x))$.

A. $\dfrac{x^3-2}{3x-5}$
B. $\dfrac{1}{(3x-5)(x^3-2)}$
C. $\dfrac{1}{3(x^3-2)-5}$
D. $\left(\dfrac{1}{3x-5}\right)^3-2$

Answer available at the end of this section.

Note

We can also evaluate the composition of two functions at a particular value just as we did with one function. For example, we may be asked to find something like $f(g(2))$ or $(g \circ f)(-3)$.

Activity 1.6.1.5

Let $f(x)=2x^3$ and $g(x)=\sqrt{6-x}$.

(a)

Find $f(g(2))$.

A. 14
B. 16
C. 18
D. 20
E. undefined

Answer available at the end of this section.

(b)

Find $(g \circ f)(-3)$.

A. 50
B. 54
C. $\sqrt{60}$
D. $\sqrt{-48}$
E. undefined

Answer available at the end of this section.

(c)

Find $(f \circ g)(10)$.

A. $2(\sqrt{-4})^3$
B. 16
C. $\sqrt{-1994}$
D. -16
E. undefined

Answer available at the end of this section.

Note

As we saw in *(see related activity)*, in order for a composite function to make sense, we need to ensure that the range of the inner function lies within the domain of the outer function so that the resulting composite function is defined at every possible input.

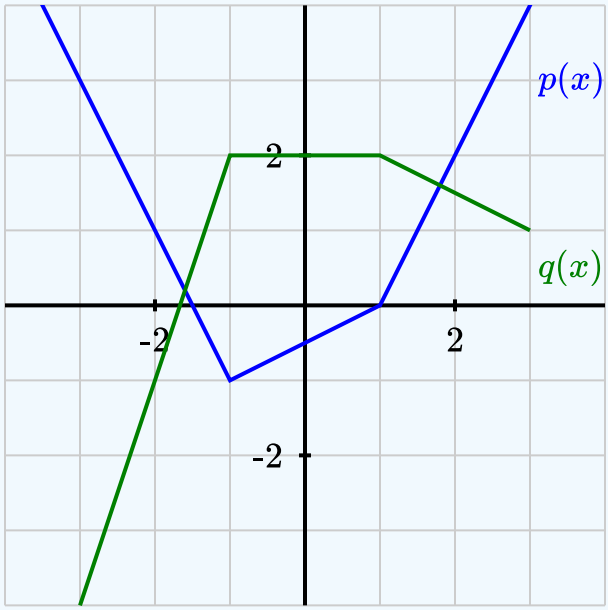
Note

In addition to the possibility that functions are given by formulas, functions can be given by tables or graphs. We can think about composite functions in these settings as well, and the following activities prompt us to consider functions given in this way.

Activity 1.6.1.6

Let functions p and q be given by the graphs below.

Find each of the following. If something is not defined, explain why.



(a)

$(p \circ q)(0)$

Answer available at the end of this section.

(b)

$q(p(0))$

Answer available at the end of this section.

(c)

$p(p(1))$

Answer available at the end of this section.

(d)

$(q \circ p)(-3)$

Answer available at the end of this section.

(e)

Find two values of x such that $q(p(x)) = 2$.

Answer available at the end of this section.

Activity 1.6.1.7

Let functions f and g be given by the tables below. Find each of the following. If something is not defined, explain why.

x	$f(x)$
0	6
1	4
2	3
3	4
4	7

x	$g(x)$
0	1
1	3
2	0
3	5
4	2

(a)

$(f \circ g)(2)$

Answer available at the end of this section.

(b)

$(g \circ f)(3)$

Answer available at the end of this section.

(c)

$g(f(4))$

Answer available at the end of this section.

(d)

For what value(s) of x is $f(g(x)) = 4$?

Answer available at the end of this section.

(e)

What are the domain and range of $(f \circ g)(x)$?

Answer available at the end of this section.

Note

Now that we've seen how to compose functions, let's take a look at how composition of functions can be applicable in our lives.

Activity 1.6.1.8

Suppose you have a $\$50$-off coupon and there is a 15%-off sale on a TV. If you are allowed to apply both the coupon and sale price of the TV, which one should you apply first? Let's investigate to determine the better deal.

(a)

Write a cost function, $C(p)$, for the price of the TV, p, if you applied the $\$50$-off coupon.

Answer available at the end of this section.

(b)

Write a cost function, $S(p)$, for the price of the TV, p, if you applied the 15%-off sale.

Answer available at the end of this section.

(c)

Suppose you apply the coupon first and then the 15%-off sale. Write a composite function to represent this situation.

Answer available at the end of this section.

(d)

Suppose you apply the 15%-off sale first and then the coupon. Write a composite function to represent this situation.

Answer available at the end of this section.

(e)

Suppose the original cost of the TV is $\$500$. Determine the cost of the TV if you applied the coupon first.

Answer available at the end of this section.

(f)

Suppose the original cost of the TV is $\$500$. Determine the cost of the TV if you applied the 15%-off sale first.

Answer available at the end of this section.

(g)

Which is the better deal and how much would you save?

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can combine and compose functions before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.6.1.1

Let $f(x) = x^2 + 1$ and $g(x) = 3x - 2$. Find $(f + g)(x)$ and $(fg)(x)$.

Answer available at the end of this section.

Check Your Understanding 1.6.1.2

Let $f(x) = \sqrt{x}$ and $g(x) = x + 5$. Find $(f \circ g)(4)$ and $(g \circ f)(4)$, and explain why they are different.

Answer available at the end of this section.

Check Your Understanding 1.6.1.3

Let $f(x) = 2x - 1$ and $g(x) = x^2$. Find $(f \circ g)(x)$ as a simplified expression in terms of x.

Answer available at the end of this section.

Answers

Activity 1.6.1.1
D

Activity 1.6.1.1
B

Activity 1.6.1.2
D

Activity 1.6.1.2
B

Activity 1.6.1.3
C

Activity 1.6.1.3
B

Activity 1.6.1.3
B

Activity 1.6.1.3

C

Activity 1.6.1.4

C

Activity 1.6.1.4

D

Activity 1.6.1.5

B

Activity 1.6.1.5

C

Activity 1.6.1.5

E

Activity 1.6.1.6

2

Activity 1.6.1.6

2

Activity 1.6.1.6

$-\frac{1}{2}$

Activity 1.6.1.6

0

Activity 1.6.1.6

x-values on the interval $[-2, 1.5]$

Activity 1.6.1.7

6

Activity 1.6.1.7

2

Activity 1.6.1.7

Not defined because $f(4) = 7$ and 7 isn't in the domain of $g(x)$.

Activity 1.6.1.7

$x = 0, 1$

Activity 1.6.1.7

Domain: $\{0, 1, 2, 3, 4\}$ and Range: $\{3, 4, 6\}$

Activity 1.6.1.8

$C(p) = p - 50$

Activity 1.6.1.8

$S(p) = p - .15p$

$S(p) = .85p$

Activity 1.6.1.8

$S(C(p)) = S(p - 50)$

$S(C(p)) = .85(p - 50)$

$S(C(p)) = .85p - 42.5$

Activity 1.6.1.8

$C(S(p)) = C(.85p)$

$S(C(p)) = (.85p) - 50$

Activity 1.6.1.8

From part (c), we know that: $S(C(p)) = .85p - 42.5$

$S(C(500)) = .85(500) - 42.5$

$S(C(500)) = 382.50$

When applying the coupon first, the reduced cost of the TV is $\$382.50$.

Activity 1.6.1.8

From part (d), we know that: $S(C(p)) = (.85p) - 50$

$S(C(500)) = (.85(500)) - 50$

$S(C(500)) = 375$

When applying the 15%-off sale first, the reduced cost of the TV is $\$375$.

Activity 1.6.1.8

When applying the coupon first, the TV would cost $\$382.50$. If applying the 15%-off sale first, however, the TV would cost $\$375$. Applying the 15%-off sale first is the better deal, saving you $\$125$ (versus $\$117.50$)!

Check Your Understanding 1.6.1.1

$(f+g)(x) = f(x) + g(x) = x^2 + 1 + 3x - 2 = x^2 + 3x - 1.$

$(fg)(x) = f(x) \cdot g(x) = (x^2+1)(3x-2) = 3x^3 - 2x^2 + 3x - 2.$

Check Your Understanding 1.6.1.2

$(f \circ g)(4) = f(g(4)) = f(9) = 3$. $(g \circ f)(4) = g(f(4)) = g(2) = 7$. They differ because composition applies the inner function first, and $f \circ g$ means "do g first, then f" — the order of operations matters, just as it does with most non-commutative operations.

Check Your Understanding 1.6.1.3

$(f \circ g)(x) = f(g(x)) = f(x^2) = 2(x^2) - 1 = 2x^2 - 1$. Substitute the entire inner function $g(x)$ everywhere x appears in f.

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1.6.1.1: FN5 Online Exercises

FN5 — Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.	FN5 CheckIt Exercises

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1.6.2: FN5 FAQ and Resources

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1.7: Introduction to Finding the Inverse Function (FN6)

1.7.1: Finding the Inverse Function (FN6)

FN6 Finding the Inverse Function

- Find the inverse of a one-to-one function.

A weather app shows the temperature in Fahrenheit, but a recipe from a European cookbook calls for an oven set to 200°C. You already have a function that converts Celsius to Fahrenheit — but what you need right now runs the other way, turning Fahrenheit into Celsius. That's an inverse function: the same relationship, reversed, with input and output swapping places. Finding it usually means swapping the variables in an equation and solving for whichever one used to stand alone — though not every function qualifies; a function only has an inverse if it's one-to-one, meaning no two different inputs ever produced the same output.

This builds directly on the composition skills from Section 1.6 — evaluating $f(g(x))$, tracking which function applies first — but turns them toward a very particular goal: finding the one function whose composition with the original hands you back exactly the input you started with.

Section 1.8 turns to transformations more broadly, and an inverse function's graph is itself one specific transformation: a reflection across the line $y = x$, since swapping input and output swaps the coordinates of every point on the graph.

> **Looking ahead to calculus:** Deciding whether a function has an inverse comes down to whether it's always increasing or always decreasing — and calculus offers a clean way to check that by looking at the sign of the rate of change rather than eyeballing a graph. That same reversing idea is also what produces two of the most important functions in calculus: the natural logarithm, which is simply the inverse of the exponential function, and the inverse trig functions you'll meet later in this book.

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1.7.1: Finding the Inverse Function (FN6)

This section corresponds to APC slot 1.7 (Inverse functions) in the Math 172 flow.

FN6 Learning outcome: Find the inverse of a one-to-one function.

Finding the Inverse Function

Overview

A function is a process that converts a collection of inputs to a corresponding collection of outputs. One question we can ask is: for a particular function, can we reverse the process and think of the original function's outputs as the inputs?

Activity 1.7.1.1

Temperature can be measured using many different units such as Fahrenheit, Celsius, and Kelvin. Fahrenheit is what is usually reported on the news each night in the United States, while Celsius is commonly used for scientific work. We will begin by converting between these two units. To convert from degrees Fahrenheit to Celsius use the following formula.

$$C = \frac{5}{9}(F - 32) \tag{1.7.1.1}$$

(a)

Room temperature is around 68 degrees Fahrenheit. Use the above equation to convert this temperature to Celsius.

A. 5.8
B. 20
C. 155.4
D. 293

Answer available at the end of this section.

(b)

Solve the equation $C = \frac{5}{9}(F - 32)$ for F in terms of C.

A. $F = \frac{5}{9}C + 32$
B. $F = \frac{5}{9}C - 32$
C. $F = \frac{9}{5}(C + 32)$
D. $F = \frac{9}{5}C + 32$

Answer available at the end of this section.

(c)

Alternatively, 20 degrees Celsius is a fairly comfortable temperature. Use your solution for F in terms of C to convert this temperature to Fahrenheit.

A. 43.1
B. -20.9

C. 93.6
D. 68

Answer available at the end of this section.

Note

Notice that when you converted 68 degrees Fahrenheit, you got a value of 20 degrees Celsius. Alternatively, when you converted 20 degrees Celsius, you got 68 degrees Fahrenheit. This indicates that the equation you were given for C and the equation you found for F are inverses.

Definition: Inverse Function

Let f be a function. If there exists a function g such that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x \tag{1.7.1.2}$$

for all x, then we say f has an **inverse function**, or that g is the **inverse of** f. When a given function f has an inverse function, we usually denote it as f^{-1}, which is read as "f inverse".

Note

An inverse is a function that "undoes" another function. For any input in the domain, the function g will reverse the process of f.

Activity 1.7.1.2

Let $f(x) = 2x + 7$ and $g(x) = \dfrac{x-7}{2}$.

(a)

Compute $f(g(x))$.

Answer available at the end of this section.

(b)

Compute $g(f(x))$.

Answer available at the end of this section.

(c)

What can you conclude about $f(x)$ and $g(x)$?

Answer available at the end of this section.

Activity 1.7.1.3

It is important to note that in *(see related activity)* we say "if there exists a function," but we don't guarantee that this is always the case. How can we determine whether a function has a corresponding inverse or not? Consider the following two functions f and g represented by the tables.

x	$f(x)$
0	6
1	4
2	3
3	4
4	6

x	$g(x)$
0	3
1	1
2	4
3	2
4	0

(a)

Use the definition of $g(x)$ in *(see related activity)* to find an x such that $g(x) = 4$.

A. $x = 0$
B. $x = 1$
C. $x = 2$
D. $x = 3$
E. $x = 4$

Answer available at the end of this section.

(b)

Is it possible to reverse the input and output rows of the function $g(x)$ and have the new table result in a function?

Answer available at the end of this section.

(c)

Use the definition of $f(x)$ in *(see related activity)* to find an x such that $f(x) = 4$.

A. $x = 0$
B. $x = 1$
C. $x = 2$
D. $x = 3$
E. $x = 4$

Answer available at the end of this section.

(d)

Is it possible to reverse the input and output rows of the function $f(x)$ and have the new table result in a function?

Answer available at the end of this section.

Note

Some functions, like $f(x)$ in *(see related activity)*, have a given output value that corresponds to two or more input values: $f(0) = 6$ and $f(4) = 6$. If we attempt to reverse the process of this function, we have a situation where the new input 6 would correspond to two potential outputs.

Definition: One-to-One Function

A **one-to-one function** is a function in which each output value corresponds to exactly one input.

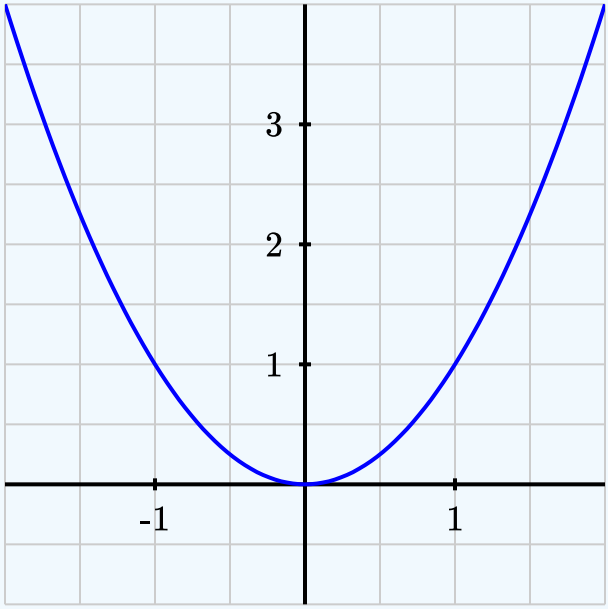
Note

A function must be one-to-one in order to have an inverse.

Activity 1.7.1.4

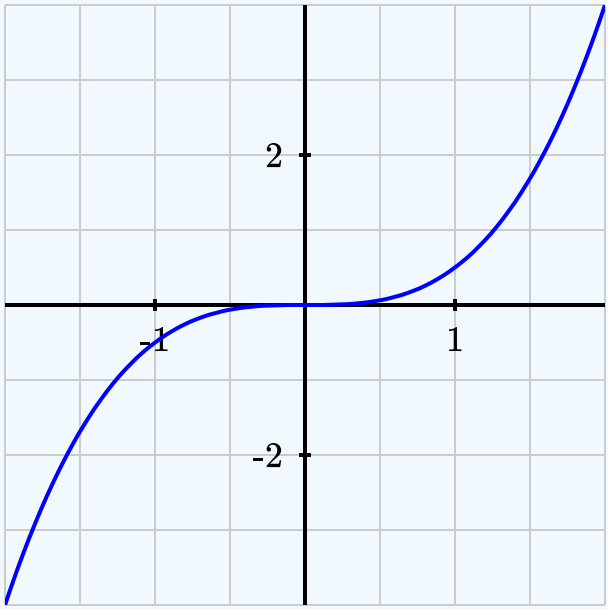
For each of the following graphs, determine if they represent a function that is one-to-one or not. If they are not one-to-one, what outputs have the same input?

(a)



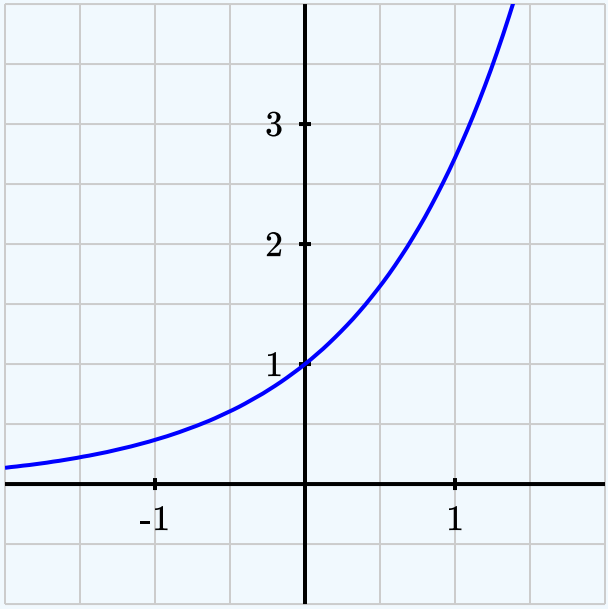
Answer available at the end of this section.

(b)



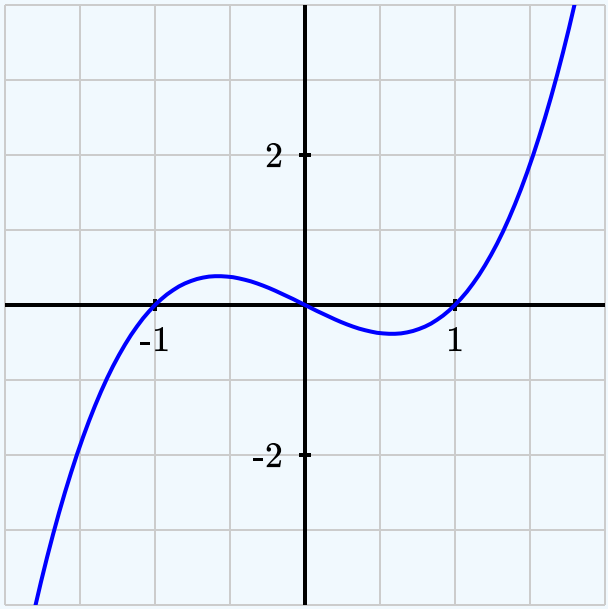
Answer available at the end of this section.

(c)



Answer available at the end of this section.

(d)



Answer available at the end of this section.

(e)

For each graph that was not one-to-one, draw a line connecting the points where two inputs had the same output. What do you notice about the lines?

Answer available at the end of this section.

Observation

When two outputs have the same input, this means that a horizontal line intersects the graph in two places. This leads us to the **horizontal line test** for one-to-one functions.

Theorem: Horizontal Line Test

A function is one-to-one precisely when its graph intersects horizontal lines at most once.

If a graph intersects a horizontal line two or more times, the function is not a one-to-one function.

Activity 1.7.1.5

Consider the function $f(x) = \dfrac{x-5}{3}$.

(a)

When you evaluate this expression for a given input value of x, what operations do you perform and in what order?

A. divide by 3, subtract 5
B. subtract 5, divide by 3
C. add 5, multiply by 3
D. multiply by 3, add 5

Answer available at the end of this section.

(b)

When you “undo” this expression to solve for a given output value of y, what operations do you perform and in what order?

A. divide by 3, subtract 5
B. subtract 5, divide by 3
C. add 5, multiply by 3
D. multiply by 3, add 5

Answer available at the end of this section.

(c)

This set of operations reverses the process for the original function, so can be considered the inverse function. Write an equation to express the inverse function f^{-1}.

A. $f^{-1}(x) = \frac{x}{3} - 5$
B. $f^{-1}(x) = \frac{x-5}{3}$
C. $f^{-1}(x) = 5(x+3)$
D. $f^{-1}(x) = 3x + 5$

Answer available at the end of this section.

(d)

Check your answer to the previous question by finding $f(f^{-1}(x))$ and $f^{-1}(f(x))$.

Answer available at the end of this section.

Observation

To find the inverse of a one-to-one function, perform the reverse operations in the opposite order.

Activity 1.7.1.6

Let's look at an alternate method for finding an inverse by solving the function for x and then interchanging the x and y.

$$h(x) = \frac{x}{x+1} \tag{1.7.1.3}$$

(a)

Interchange the variables x and y.

A. $y = \frac{x}{x+1}$
B. $x = \frac{y}{x+1}$
C. $x = \frac{y}{y+1}$
D. $x = \frac{x}{y+1}$

Answer available at the end of this section.

(b)

Eliminate the denominator.

A. $y(x+1)=x$
B. $x(x+1)=y$
C. $x(y+1)=x$
D. $x(y+1)=y$

Answer available at the end of this section.

(c)

Distribute and gather the y terms together.

A. $yx+y=x$
B. $x^2+x=y$
C. $xy-y=-x$
D. $xy=0$

Answer available at the end of this section.

(d)

Write the inverse function, by factoring and solving for y.

A. $h^{-1}(x)=\dfrac{x}{x-1}$
B. $h^{-1}(x)=\dfrac{x}{1-x}$
C. $h^{-1}(x)=\dfrac{-x}{1-x}$
D. $h^{-1}(x)=\dfrac{x+1}{x}$

Answer available at the end of this section.

Activity 1.7.1.7

Find the inverse of each function, using either method. Check your answer using function composition.

(a)

$g(x)=\dfrac{4x-1}{7}$

A. $g^{-1}(x)=\dfrac{7x+1}{4}$
B. $g^{-1}(x)=\dfrac{7x}{4}+1$
C. $g^{-1}(x)=\dfrac{4x+1}{7}$
D. $g^{-1}(x)=\dfrac{7}{4x-1}$

Answer available at the end of this section.

(b)

$f(x)=2x^3-3$

A. $f^{-1}(x)=\frac{1}{2}x^{\frac{1}{3}}+3$
B. $f^{-1}(x)=\left(\frac{1}{2}x\right)^{\frac{1}{3}}+3$

C. $f^{-1}(x) = \left(\frac{1}{2}x + 3\right)^{\frac{1}{3}}$
D. $f^{-1}(x) = \frac{1}{2}(x+3)^{\frac{1}{3}}$

Answer available at the end of this section.

Activity 1.7.1.8

Consider the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$.

(a)

Compute each of the following

1. $\left(\sqrt{9}\right)^2$
2. $\left(\sqrt{25}\right)^2$
3. $\left(\sqrt{17}\right)^2$

Answer available at the end of this section.

(b)

Compute each of the following

1. $\sqrt{3^2}$
2. $\sqrt{11^2}$
3. $\sqrt{37^2}$
4. $\sqrt{(-4)^2}$

Answer available at the end of this section.

(c)

Are the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ one-to-one?

Answer available at the end of this section.

(d)

For which values of x is it true that $f(g(x)) = x$?

Answer available at the end of this section.

(e)

For which values of x is it true that $g(f(x)) = x$?

Answer available at the end of this section.

Observation

While $f(x) = x^2$ is not a one-to-one function and thus cannot have its inverse, we can **restrict the domain** to find an invertible function. In this case, considering $f_0(x) = x^2$ defined only on the interval $[0, \infty)$, $f_0(x)$ is a one-to-one function with inverse $f_0^{-1}(x) = \sqrt{x}$.

Note

When finding inverses algebraically, it is tempting to write $\sqrt{x^2} = x$, but this only true for non-negative x-values. In general, $\sqrt{x^2} = |x|$.

Check Your Understanding

Work these on your own to confirm you can find the inverse of a one-to-one function before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.7.1.1

Let $f(x) = 4x - 7$. Find $f^{-1}(x)$.

Answer available at the end of this section.

Check Your Understanding 1.7.1.2

Let $g(x) = \sqrt[3]{x - 2}$. Find $g^{-1}(x)$, then check your answer by verifying $g(g^{-1}(x)) = x$.

Answer available at the end of this section.

Check Your Understanding 1.7.1.3

Explain why the function $h(x) = x^2$ does not have an inverse function over all real numbers, but $h(x) = x^2$ restricted to $x \geq 0$ does.

Answer available at the end of this section.

Answers

Activity 1.7.1.1
B

Activity 1.7.1.1
D

Activity 1.7.1.1
D

Activity 1.7.1.2
$f(g(x)) = x$

Activity 1.7.1.2
$g(f(x)) = x$

Activity 1.7.1.2
$f(x)$ and $g(x)$ are inverses of each other.

Activity 1.7.1.3
C

Activity 1.7.1.3
Yes. If you reverse the input and output values, you will see that every input goes to exactly one output (which defines a function).

Activity 1.7.1.3
B and D

Activity 1.7.1.3
If you reverse the input and output values, the resulting table is NOT a function. This is because the new input $x = 4$ will be mapping onto two y-values (1 and 3).

Activity 1.7.1.4

This graph is not one-to-one

Activity 1.7.1.4

This graph is one-to-one

Activity 1.7.1.4

This graph is one-to-one

Activity 1.7.1.4

This graph is not one-to-one

Activity 1.7.1.4

A line between two points whose output is the same have the same y-value, so the line is horizontal.

Activity 1.7.1.5

B

Activity 1.7.1.5

D

Activity 1.7.1.5

D

Activity 1.7.1.5

By completing this task, students should find that when they do the composition (in both directions), they will get an output of x.

Activity 1.7.1.6

C

Activity 1.7.1.6

D

Activity 1.7.1.6

C

Activity 1.7.1.6

B

Activity 1.7.1.7

A

Activity 1.7.1.7

D

Activity 1.7.1.8

1. 9
2. 25
3. 17

Activity 1.7.1.8

1. 3
2. 11
3. 37
4. 4

Activity 1.7.1.8

$f(x) = x^2$ is not one-to-one but $g(x) = \sqrt{x}$ is.

Activity 1.7.1.8

$(\sqrt{x})^2 = x$ for all x in the interval $[0, \infty)$.

Activity 1.7.1.8

$\sqrt{x^2} = x$ for all x in the interval $[0, \infty)$.

Check Your Understanding 1.7.1.1

Replace $f(x)$ with y: $y = 4x - 7$. Swap x and y: $x = 4y - 7$. Solve for y: $x + 7 = 4y$, so $y = \dfrac{x+7}{4}$. Thus $f^{-1}(x) = \dfrac{x+7}{4}$.

Check Your Understanding 1.7.1.2

$y = \sqrt[3]{x-2}$. Swap: $x = \sqrt[3]{y-2}$. Cube both sides: $x^3 = y - 2$, so $y = x^3 + 2$. Thus $g^{-1}(x) = x^3 + 2$. Check: $g(g^{-1}(x)) = \sqrt[3]{(x^3+2)-2} = \sqrt[3]{x^3} = x$. ✓

Check Your Understanding 1.7.1.3

$h(x) = x^2$ is not one-to-one over all reals: both $x = 2$ and $x = -2$ give $h(x) = 4$, so a single output would have to map back to two different inputs, which isn't a function. Restricting the domain to $x \geq 0$ makes it one-to-one (each output now comes from exactly one input), so the restricted function does have an inverse.

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1.7.1.1: FN6 Online Exercises

FN6 — Find the inverse of a one-to-one function.	FN6 CheckIt Exercises

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1.7.2: FN6 FAQ and Resources

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1.8: Introduction to Transformations of Functions (FN4)

1.8.1: Transformations of Functions (FN4)

FN4 Transformation of Functions

- Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.

Think about editing a photo on your phone: you can drag it to reposition it, flip it, or pinch to resize it — and it's still recognizably the same photo. Function transformations work the same way. Starting from a function $f(x)$, you can shift its graph up, down, left, or right; reflect it over the x-axis or y-axis; or stretch and compress it vertically or horizontally, and the result is still built from the same underlying shape. Each move shows up in a specific, predictable place in the formula: adding outside the function shifts vertically, adding inside shifts horizontally (in the opposite direction you'd guess), and multiplying stretches or compresses depending on where it happens.

Section 1.7 asked what happens when you swap a function's input and output entirely, which produces its inverse. This section asks a related but gentler question: what happens to a graph under simpler moves — shifting, reflecting over an axis, stretching — where input and output keep their original roles. It's worth noticing, though not something these activities cover directly, that an inverse's graph is also a reflection, just across the diagonal line $y = x$ rather than either axis.

Chapter 2 puts these moves to immediate use: building the sine and cosine graphs from the unit circle turns out to be an exercise in transforming a parent function, where amplitude is a vertical stretch, period is a horizontal stretch, and phase shift is exactly the horizontal shift you'll practice here.

> **Looking ahead to calculus:** Recognizing a function as a transformation of something simpler — a parent function, shifted or stretched — is exactly the skill that makes the chain rule fast instead of painful. Once you can see $f(3x + 1)$ as f applied to a shifted, compressed input, differentiating it becomes a matter of differentiating the parent function and accounting for that inner transformation, rather than starting from scratch.

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1.8.1: Transformations of Functions (FN4)

This section corresponds to APC slot 1.8 (Transformations) in the Math 172 flow.

Learning outcome: Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.

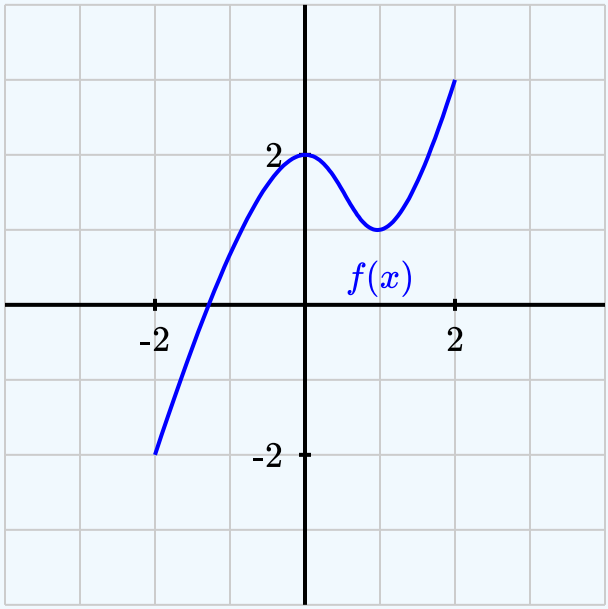
Transformation of Functions

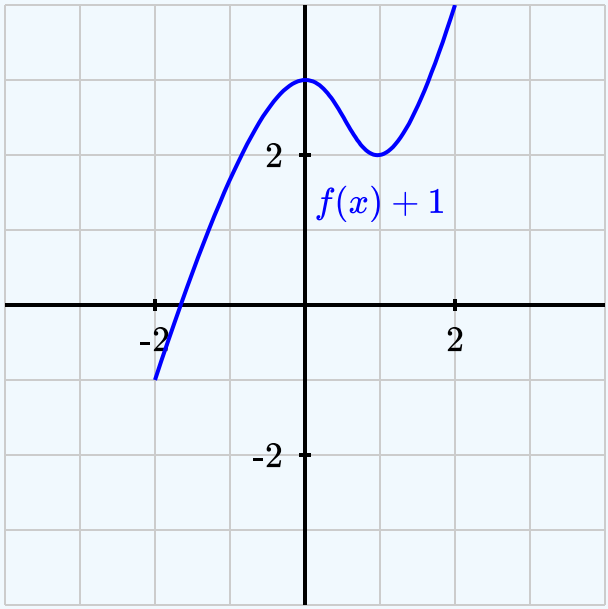
Overview

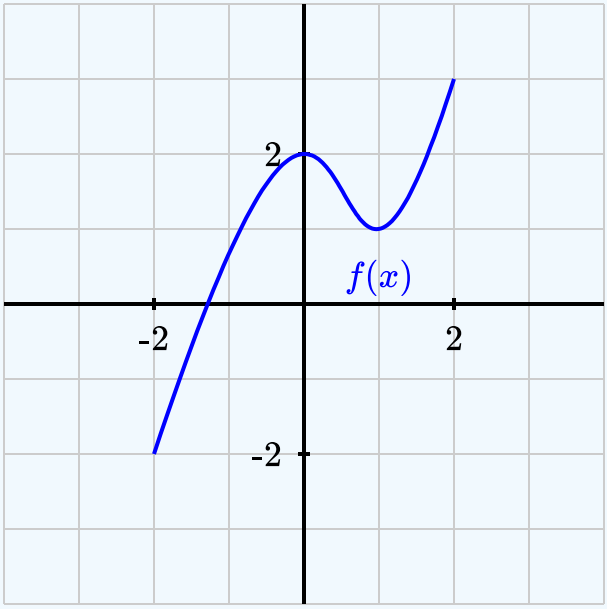
Informally, a transformation of a given function is an algebraic process by which we change the function to a related function that has the same fundamental shape, but may be shifted, reflected, and/or stretched in a systematic way.

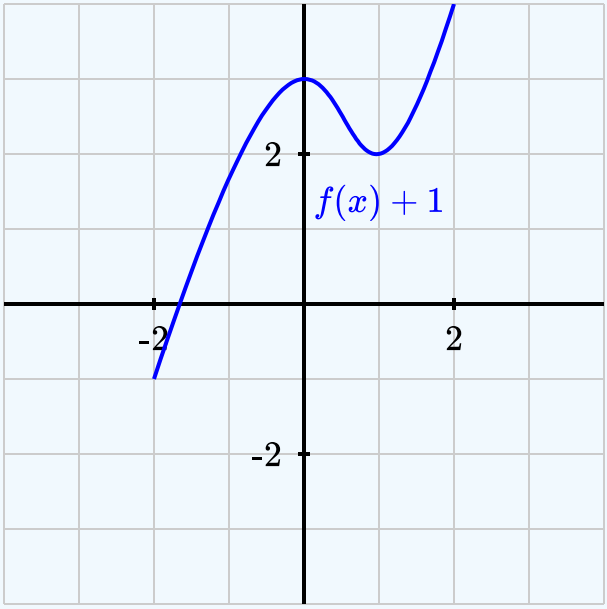
Activity 1.8.1.1

Consider the following two graphs.









(a)

How is the graph of $f(x)+1$ related to that of $f(x)$?

A. Shifted up 1 unit

B. Shifted left 1 unit

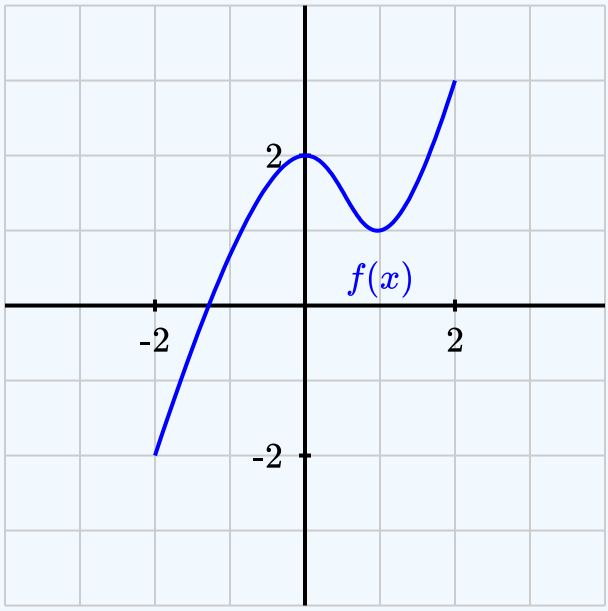
C. Shifted down 1 unit

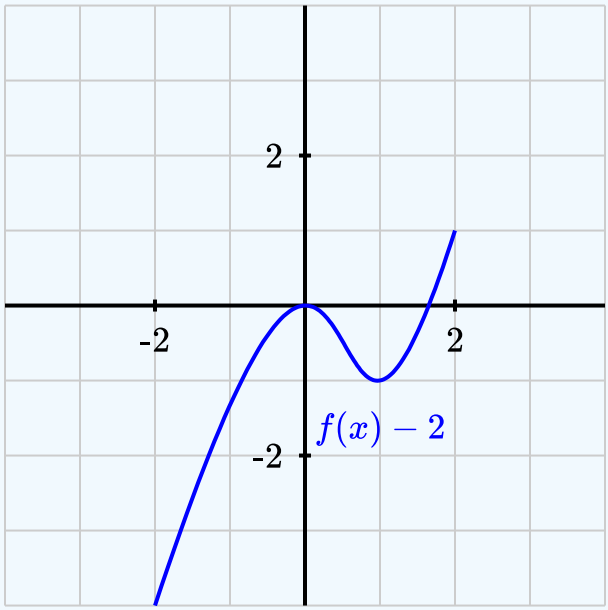
D. Shifted right 1 unit

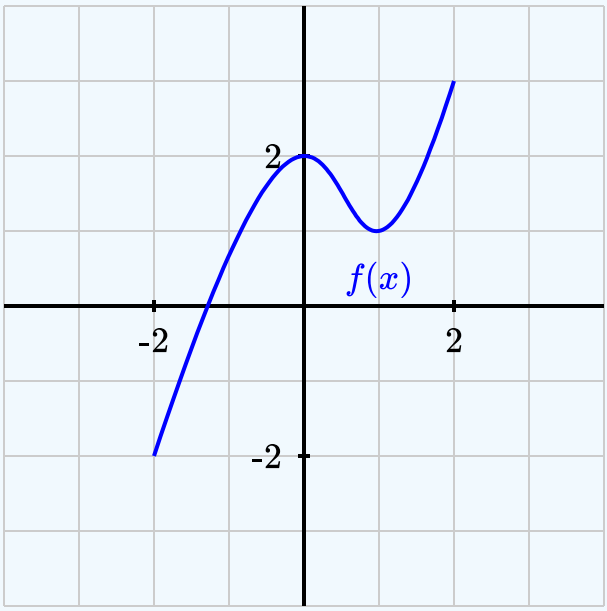
Answer available at the end of this section.

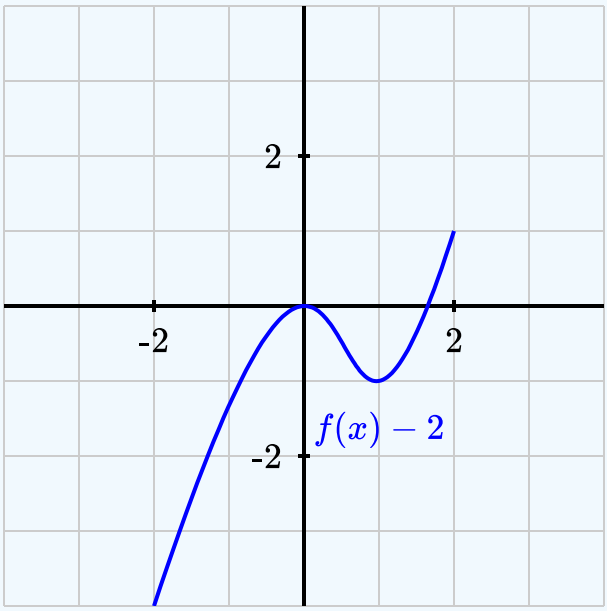
Activity 1.8.1.2

Consider the following two graphs.









(a)

How is the graph of $f(x) - 2$ related to that of $f(x)$?

A. Shifted up 2 units

B. Shifted left 2 units

C. Shifted down 2 units

D. Shifted right 2 units

Answer available at the end of this section.

Note

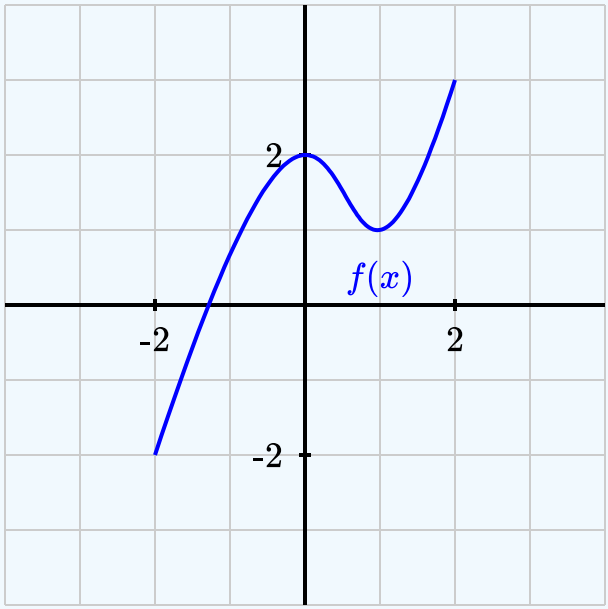
Notice that in *(see related activity)* and *(see related activity)*, the y-values of the transformed graph are changed while the x -values remain the same.

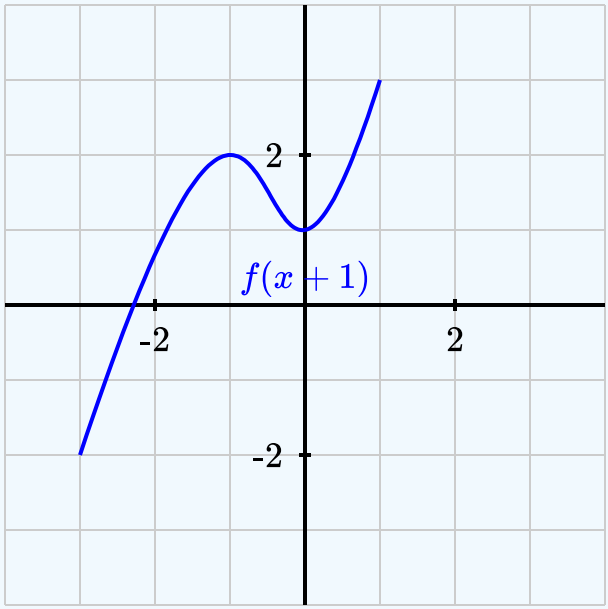
Definition: Vertical Shift

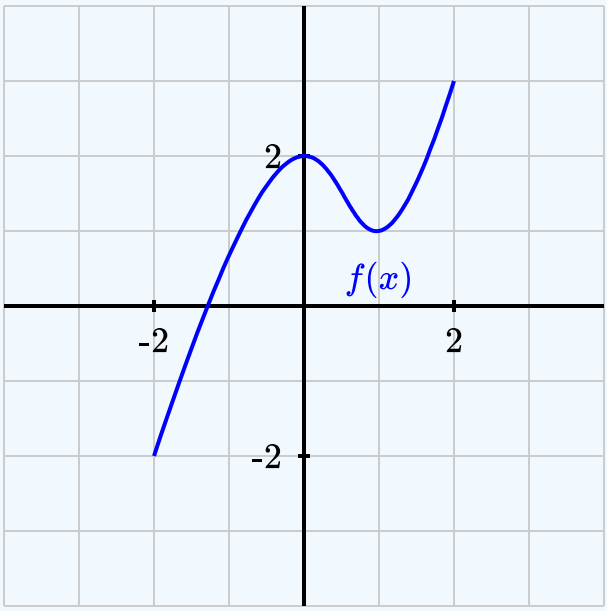
Given a function $f(x)$ and a constant c, the transformed function $g(x) = f(x) + c$ is a **vertical translation** of the graph of $f(x)$. That is, all the outputs change by c units. If c is positive, the graph will shift up. If c is negative, the graph will shift down.

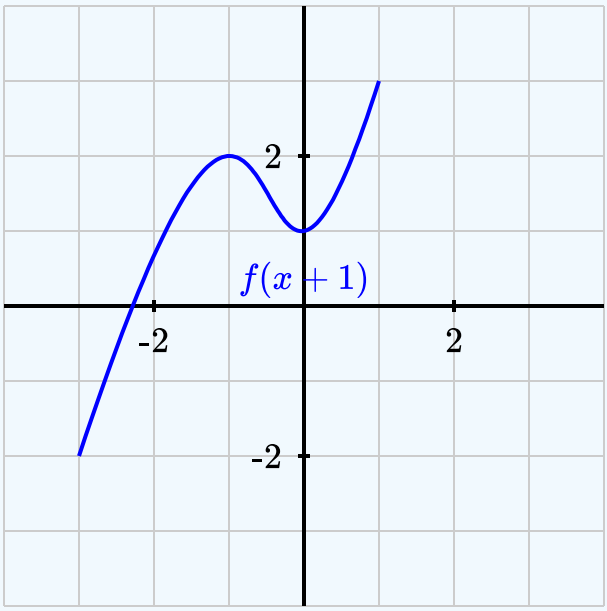
Activity 1.8.1.3

Consider the following two graphs.


2
$f(x)$
-2
2
-2


2
$f(x+1)$
-2
2
-2





(a)

How is the graph of $f(x+1)$ related to that of $f(x)$?

A. Shifted up by 1 unit

B. Shifted left 1 unit

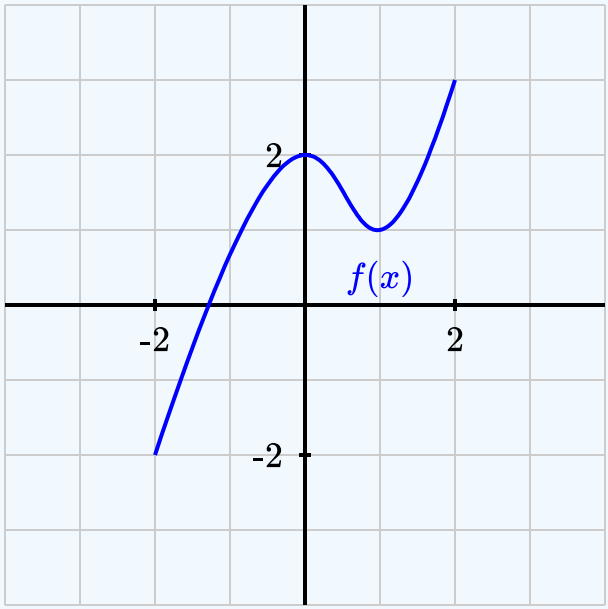
C. Shifted down 1 unit

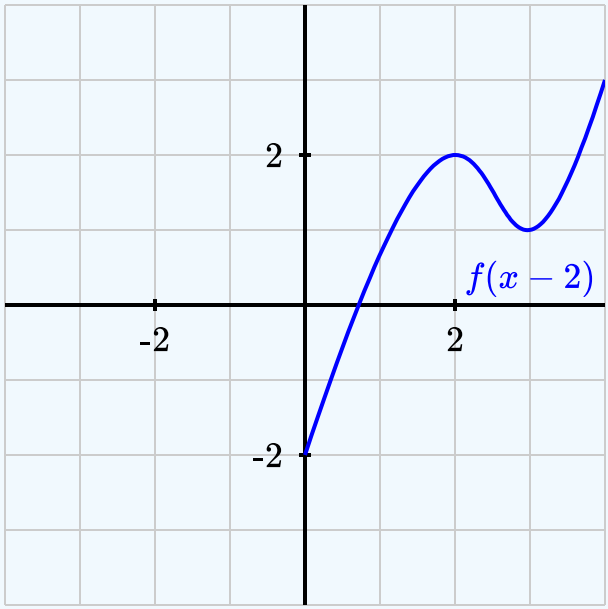
D. Shifted right 1 unit

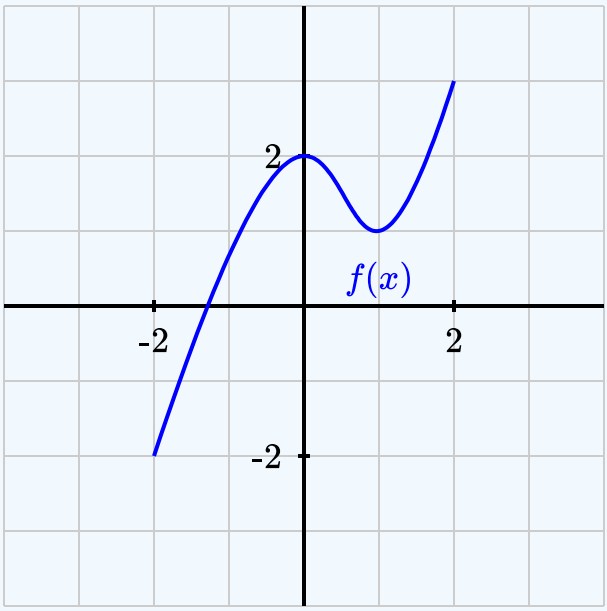
Answer available at the end of this section.

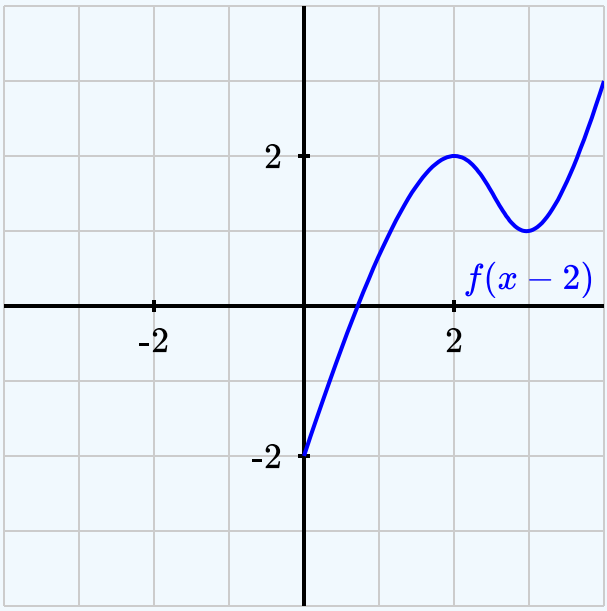
Activity 1.8.1.4

Consider the following two graphs.


2
-2
2
-2
$f(x)$


2
-2
2
-2
$f(x-2)$





(a)

How is the graph of $f(x-2)$ related to that of $f(x)$?

A. Shifted up by 2 units

B. Shifted left 2 units

C. Shifted down 2 units

D. Shifted right 2 units

Answer available at the end of this section.

Note

Notice that in *(see related activity)* and *(see related activity)*, the x-values of the transformed graph are changed while the y-values remain the same.

Definition: Horizontal Shift

Given a function $f(x)$ and a constant c, the transformed function $g(x) = f(x+c)$ is a **horizontal translation** of the graph of $f(x)$. If c is positive, the graph will shift left. If c is negative, the graph will shift right.

Activity 1.8.1.5

Describe how the graph of the function is a transformation of the graph of the original function f.

(a)

$f(x-4)+1$

A. Shifted down 4 units

B. Shifted left 4 units

C. Shifted down 1 unit

D. Shifted right 4 units

E. Shifted up 1 unit

Answer available at the end of this section.

(b)

$f(x+3)-2$

A. Shifted down 2 units

B. Shifted left 3 units

C. Shifted up 3 units

D. Shifted right 3 units

E. Shifted up 2 units

Answer available at the end of this section.

Activity 1.8.1.6

For each of the following, use the information given to find another point on the graph.

(a)

If $(2,3)$ is a point on the graph of $f(x)$, what point must be on the graph of $f(x)+2$?

A. $(4,3)$

B. $(2,5)$

C. $(4,5)$

D. $(2,1)$

Answer available at the end of this section.

(b)

If $(-1,6)$ is a point on the graph of $g(x)$, what point must be on the graph of $g(x-4)$?

A. $(-1,10)$

B. $(3,6)$

C. $(-1,2)$

D. $(-5,6)$

Answer available at the end of this section.

(c)

If $(-2,-5)$ is a point on the graph of $h(x)$, what point must be on the graph of $h(x+1)-5$?

A. $(-3,-10)$

B. $(-3,0)$

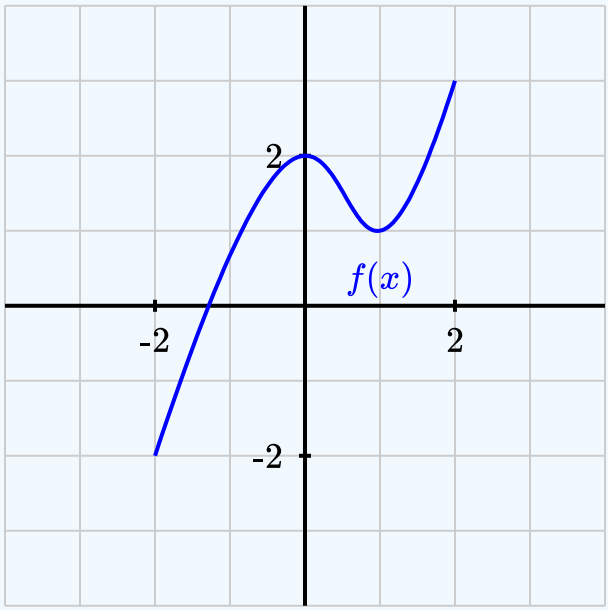
C. $(-1,-10)$

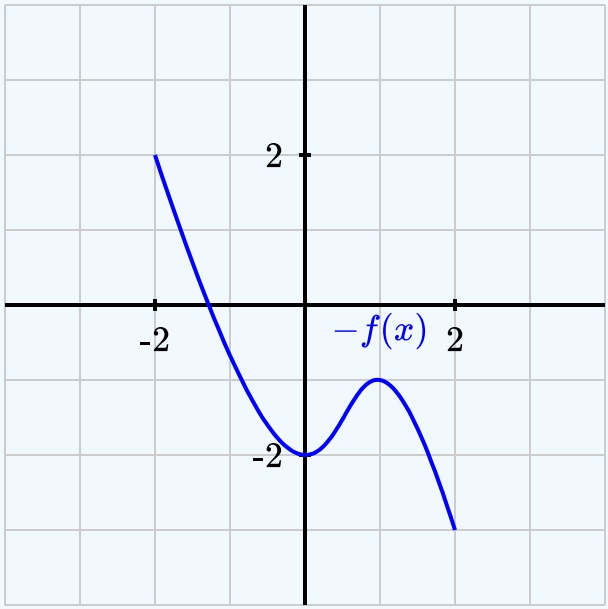
D. $(-1,0)$

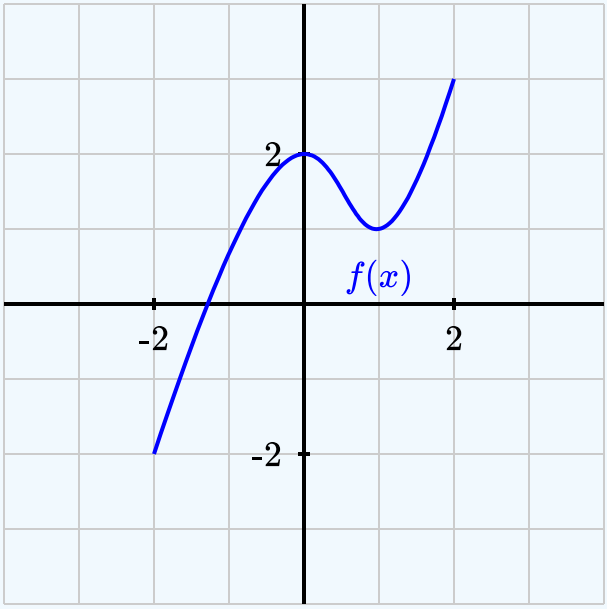
Answer available at the end of this section.

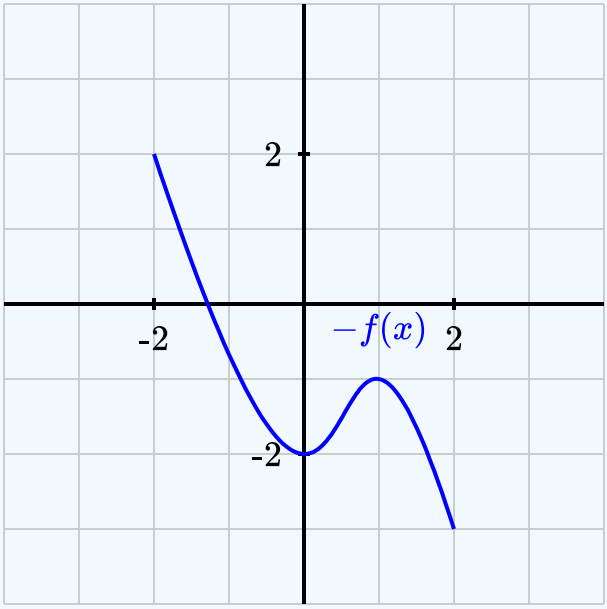
Activity 1.8.1.7

Consider the following two graphs.









(a)

How is the graph of $-f(x)$ related to that of $f(x)$?

A. Shifted down 2 units

B. Reflected over the x-axis

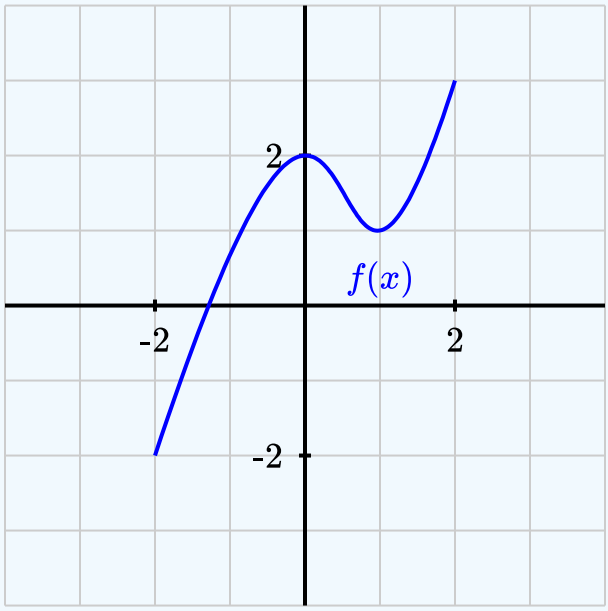
C. Reflected over the y-axis

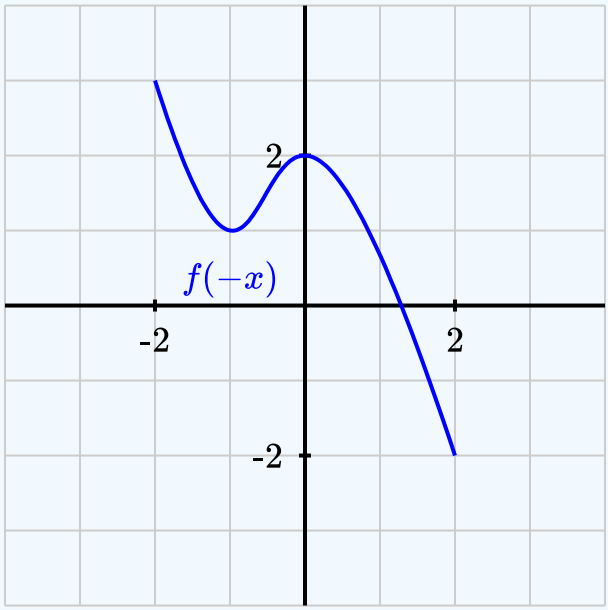
D. Shifted right 2 units

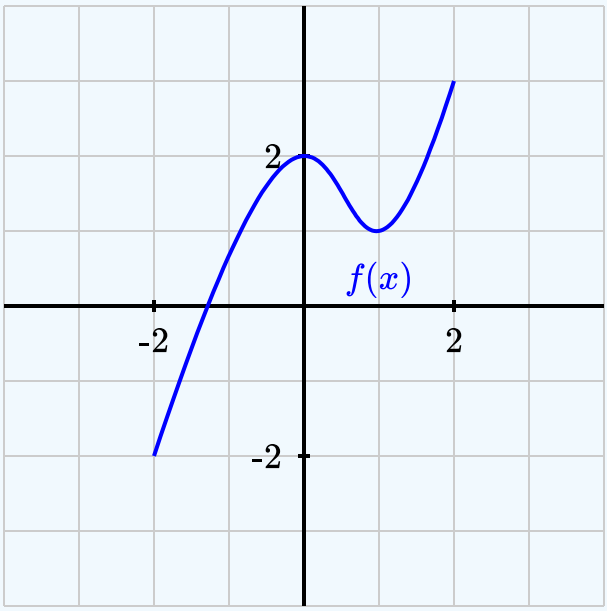
Answer available at the end of this section.

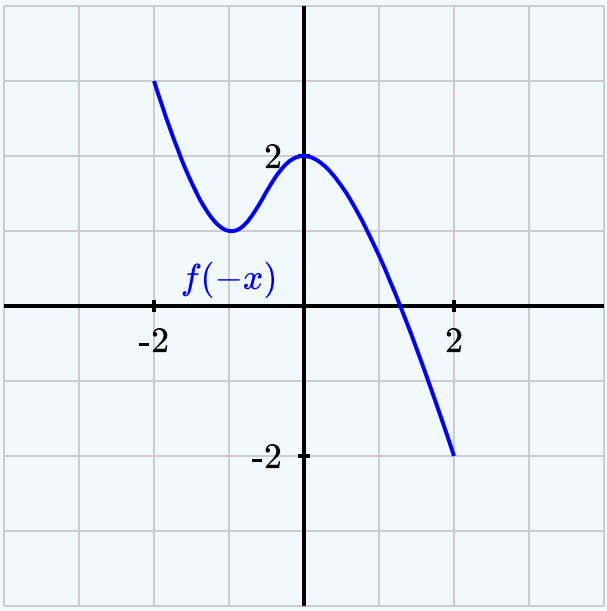
Activity 1.8.1.8

Consider the following two graphs.









(a)

How is the graph of $f(-x)$ related to that of $f(x)$?

A. Shifted down 2 units

B. Reflected over the x-axis

C. Reflected over the y-axis

D. Shifted left 2 units

Answer available at the end of this section.

Note

Notice that in *(see related activity)*, the y-values of the transformed graph are changed while the x-values remain the same. While in *(see related activity)*, the x-values of the transformed graph are changed while the y-values remain the same.

Definition: Vertical Reflection

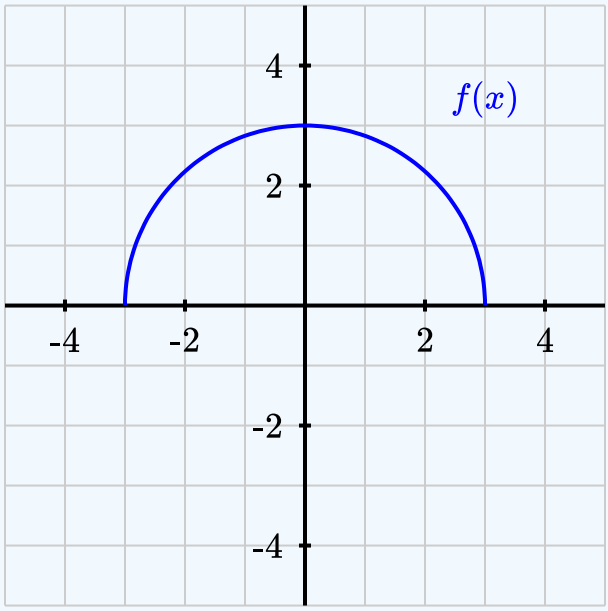
Given a function $f(x)$, the transformed function $g(x) = -f(x)$ is a **vertical reflection** of the graph of $f(x)$. That is, all the outputs are multiplied by -1. The new graph is a reflection of the old graph about the x-axis.

Definition: Horizontal Reflection

Given a function $f(x)$, the transformed function $y = g(x) = f(-x)$ is a **horizontal reflection** of the graph of $f(x)$. That is, all the inputs are multiplied by -1. The new graph is a reflection of the old graph about the y-axis.

Activity 1.8.1.9

Consider the following graph of the function $f(x)$.



(a)

How is the graph of $-f(x+2)+3$ related to that of $f(x)$?

A. Shifted up 2 units

B. Shifted up 3 units

C. Reflected over the x-axis

D. Reflected over the y-axis

E. Shifted left 3 units

F. Shifted left 2 units

Answer available at the end of this section.

(b)

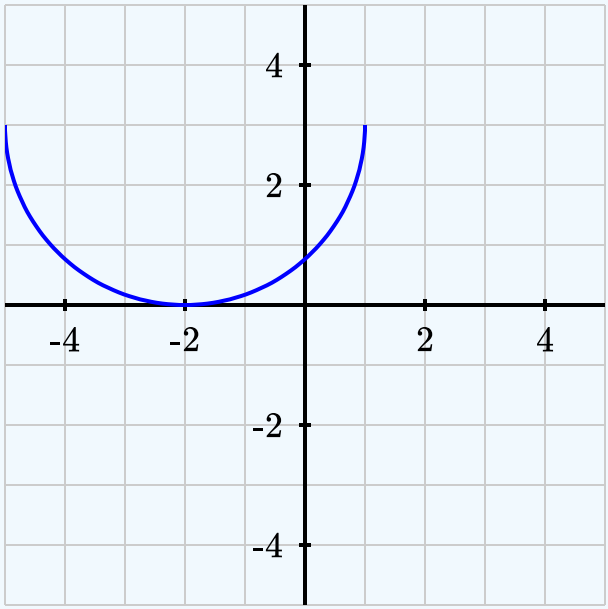
Which of the following represents the graph of the transformed function $g(x)=-f(x+2)+3$?

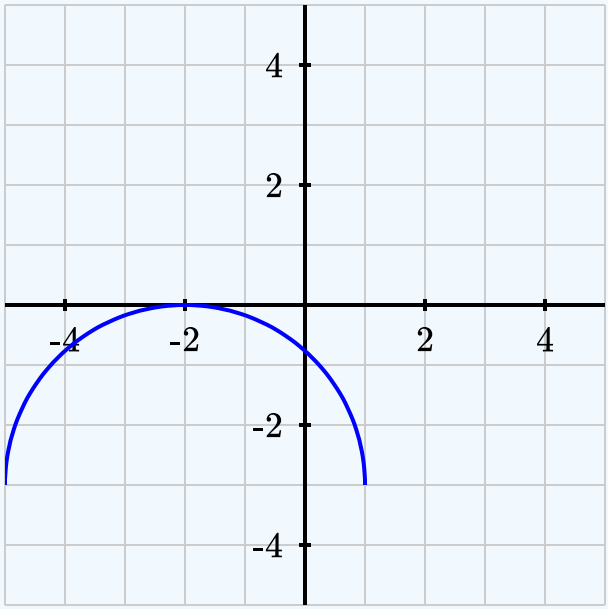
A.

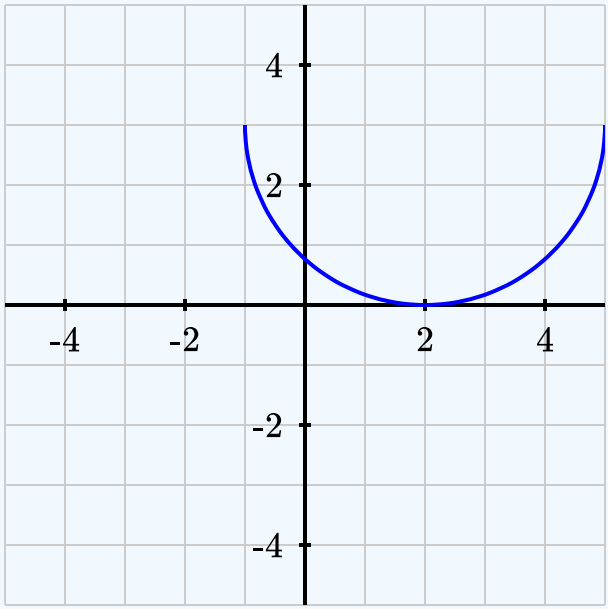
B.

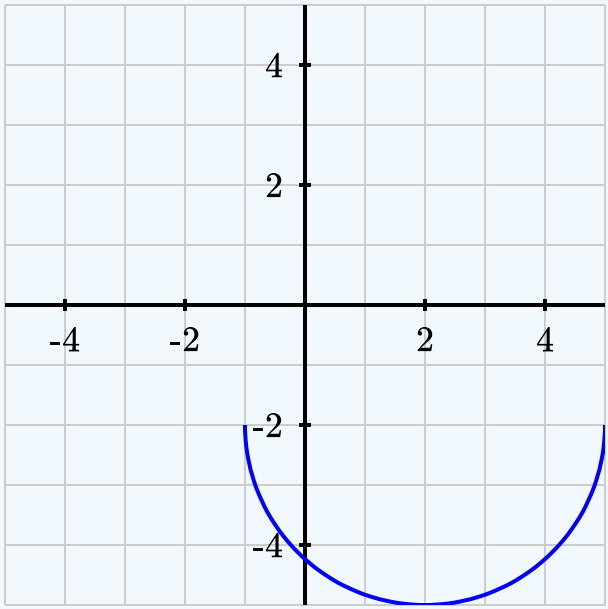
C.

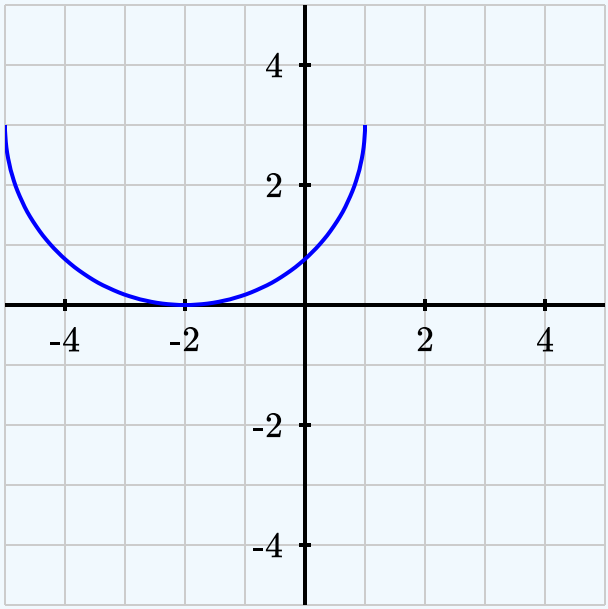
D.

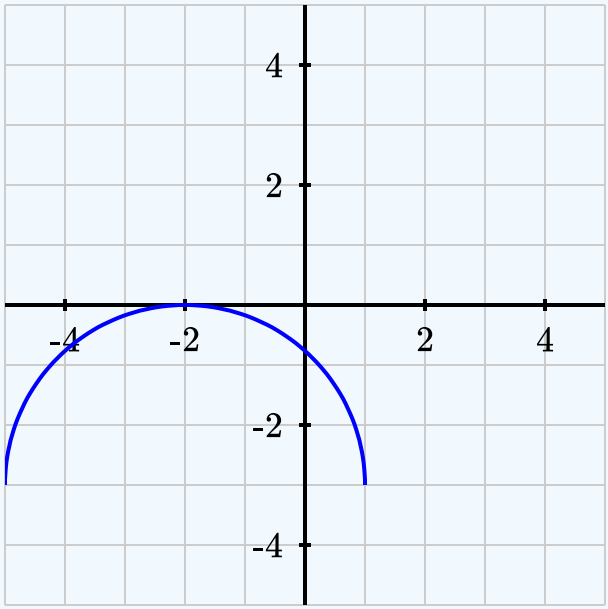

4
2
-4
-2
2
4
-2
-4

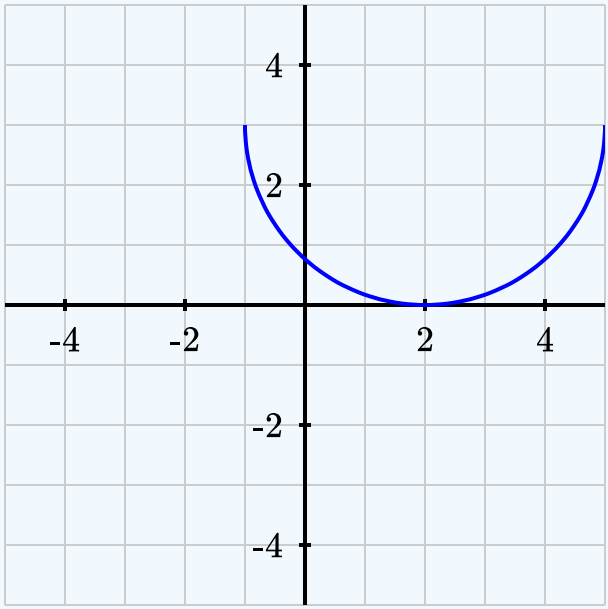

4
2
-4
-2
2
4
-2
-4

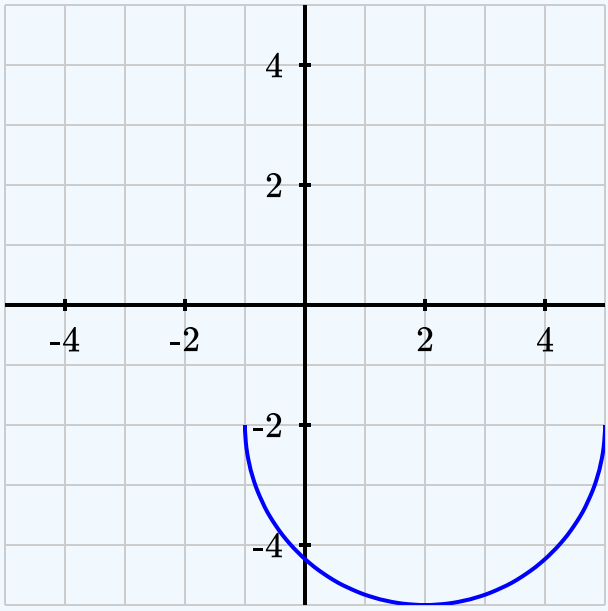

4
2
-4
-2
2
4
-2
-4


4
2
-4
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4
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-4





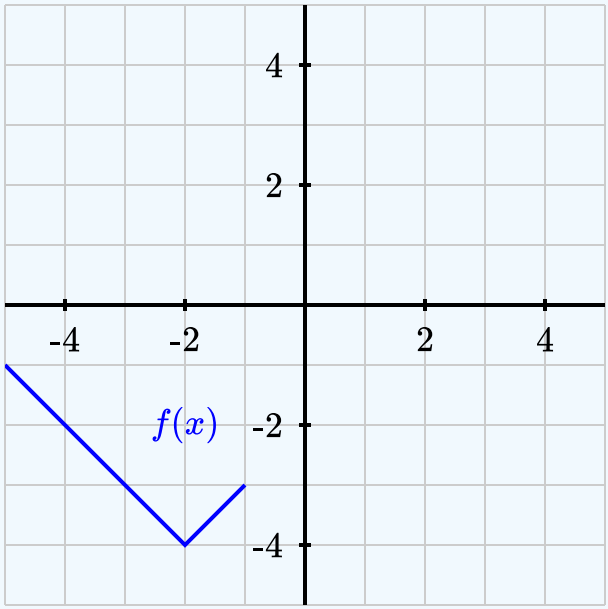
Answer available at the end of this section.

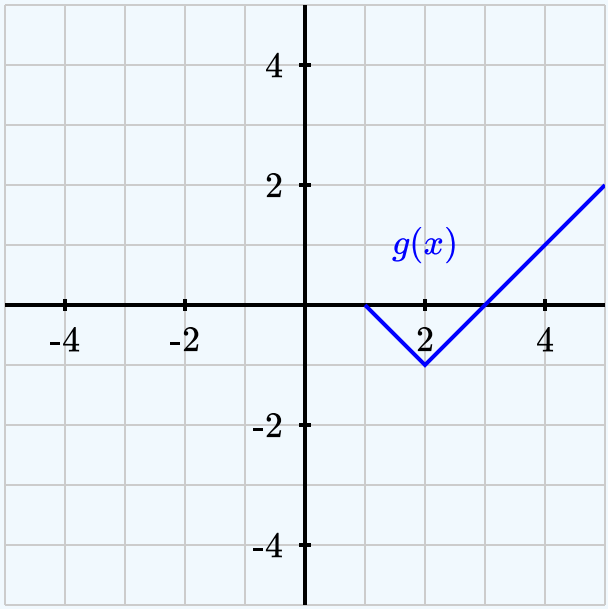
Note

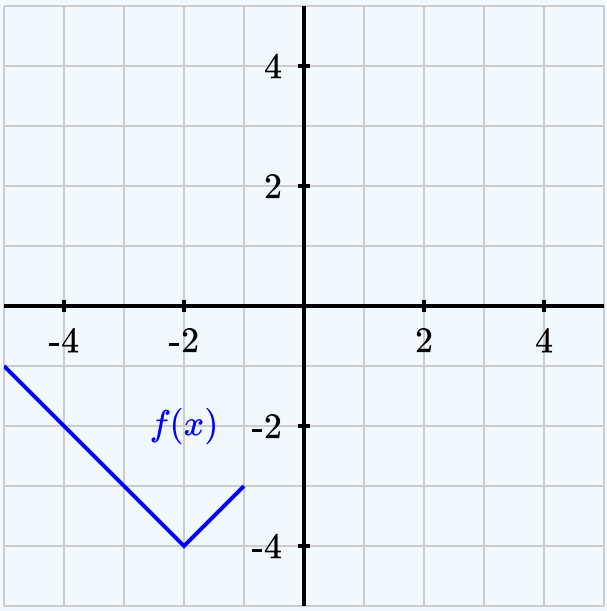
Notice that in *(see related activity)* the resulting graph is different if you perform the reflection first and then the vertical shift, versus the other order. When combining transformations, it is very important to consider the order of the transformations. Be sure to follow the order of operations.

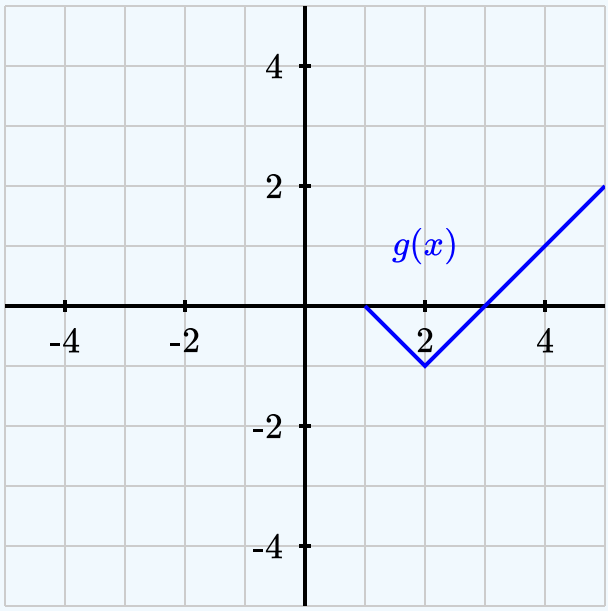
Activity 1.8.1.10

Consider the following two graphs.


4
2
-4
-2
2
4
$f(x)$
-2
-4


4
2
$g(x)$
-4
-2
2
4
-2
-4





(a)

How is the graph of $g(x)$ related to that of $f(x)$?

A. Shifted up 3 units

B. Shifted up 1 unit

C. Reflected over the x-axis

D. Reflected over the y-axis

E. Shifted left 1 unit

F. Shifted right 4 units

Answer available at the end of this section.

(b)

List the order the transformations must be applied.

Answer available at the end of this section.

(c)

Write an equation for the graphed function $g(x)$ using transformations of the graph $f(x)$.

A. $g(x) = -f(x) + 3$

B. $g(x) = f(-x) + 3$

C. $g(x) = f(-x + 3)$

D. $g(x) = -f(x + 3)$

Answer available at the end of this section.

Activity 1.8.1.11

For each of the following, use the information given to find another point on the graph.

(a)

If $(1, 6)$ is a point on the graph of $f(x)$, what point must be on the graph of $-f(x-2)$?

A. $(-3, 6)$

B. $(-1, 6)$

C. $(-1, 4)$

D. $(3, -6)$

Answer available at the end of this section.

(b)

If $(-2, -4)$ is a point on the graph of $g(x)$, what point must be on the graph of $g(-x)+3$?

A. $(-2, -7)$

B. $(2, -1)$

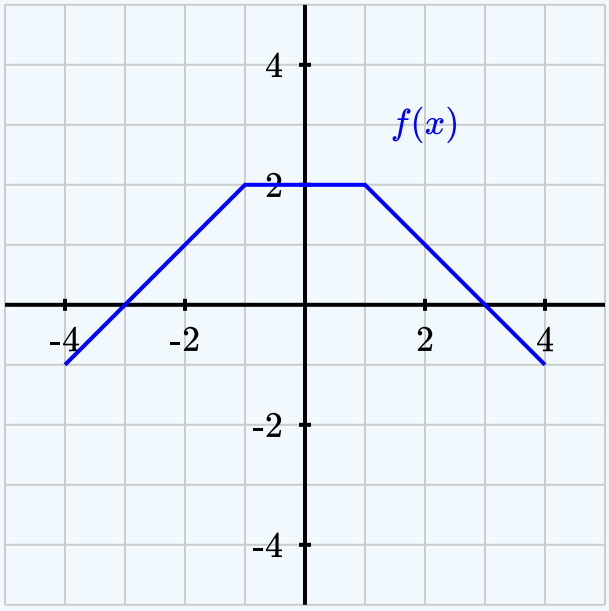
C. $(2, -7)$

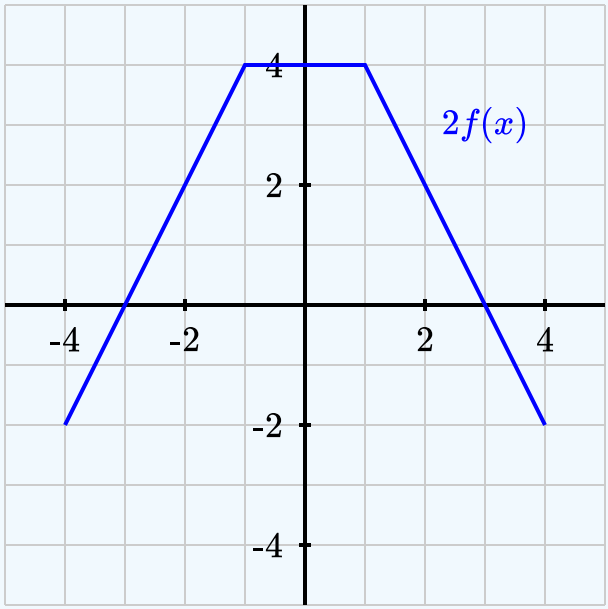
D. $(-2, 7)$

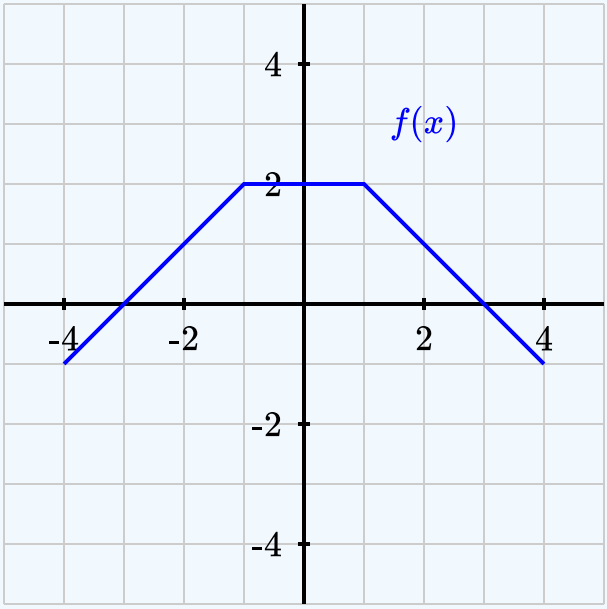
Answer available at the end of this section.

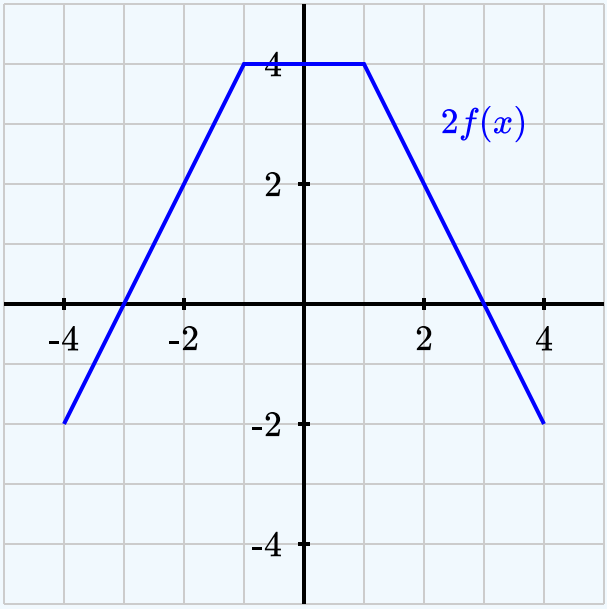
Activity 1.8.1.12

Consider the following two graphs.









(a)

Consider the y-value of the two graphs at $x = 1$. How do they compare?

A. The y-value of $2f(x)$ is twice that of $f(x)$.

B. The y-value of $2f(x)$ is half that of $f(x)$.

C. The y-value of $2f(x)$ and $f(x)$ are the same.

D. The y-value of $2f(x)$ is negative that of $f(x)$.

Answer available at the end of this section.

(b)

How is the graph of $2f(x)$ related to that of $f(x)$?

A. Vertically stretched by a factor of 2

B. Vertically compressed by a factor of 2

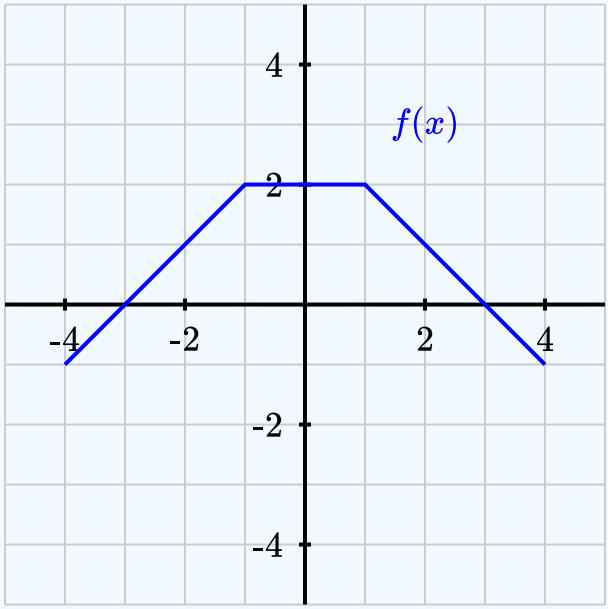
C. Horizontally stretched by a factor of 2

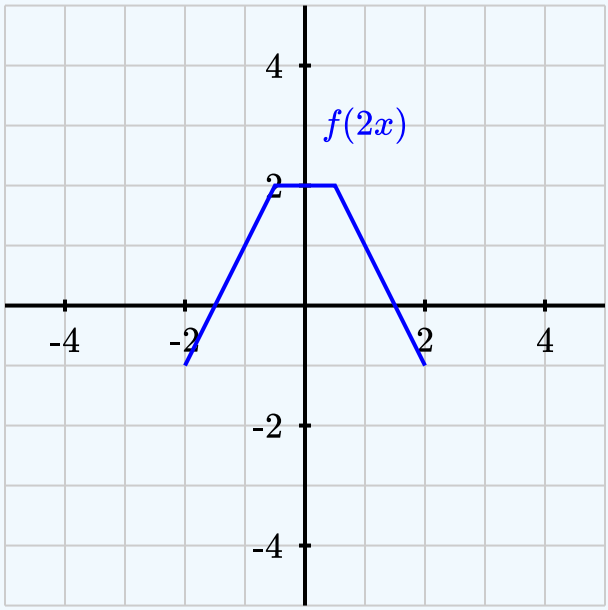
D. Horizontally compressed by a factor of 2

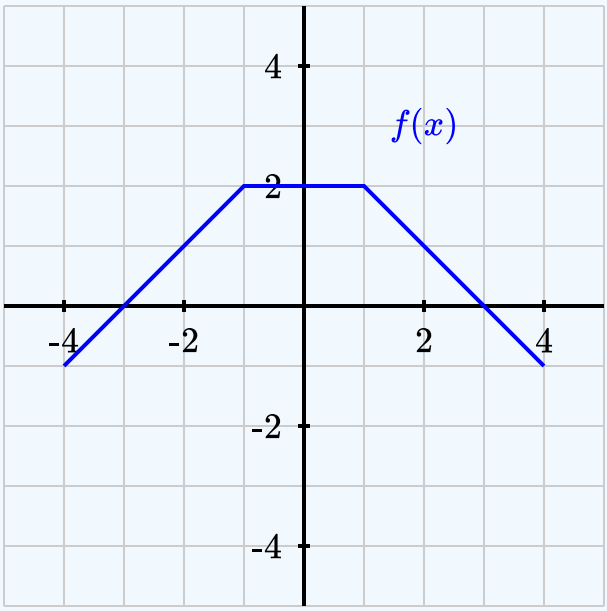
Answer available at the end of this section.

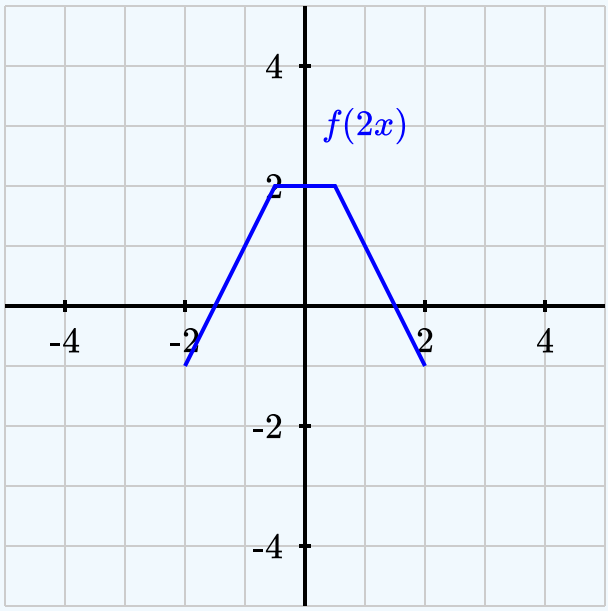
Activity 1.8.1.13

Consider the following two graphs.









(a)

Consider an x-value of the two graphs where $y = 1$. How do they compare?

A. The x-value of $f(2x)$ is twice that of $f(x)$.

B. The x-value of $f(2x)$ is half that of $f(x)$.

C. The x-value of $f(2x)$ and $f(x)$ are the same.

D. The x-value of $f(2x)$ is negative that of $f(x)$.

Answer available at the end of this section.

(b)

How is the graph of $f(2x)$ related to that of $f(x)$?

A. Vertically stretched by a factor of 2

B. Vertically compressed by a factor of 2

C. Horizontally stretched by a factor of 2

D. Horizontally compressed by a factor of 2

Answer available at the end of this section.

Note

Notice that in *(see related activity)* the y-values are doubled while the x-values remain the same. While, in *(see related activity)* the x-values are cut in half while the y-values remain the same.

Definition: Vertical Stretch

Given a function $f(x)$, the transformed function $g(x) = af(x)$ is a **vertical stretch** or **vertical compression** of the graph of $f(x)$. That is, all the outputs are multiplied by a. If $a > 1$, the new graph is a vertical stretch of the old graph away from the x-axis. If $0 < a < 1$, the new graph is a vertical compression of the old graph towards the x-axis. Points on the x-axis are unchanged.

Definition: Horizontal Stretch

Given a function $f(x)$, the transformed function $g(x) = f(ax)$ is a **horizontal stretch** or **horizontal compression** of the graph of $f(x)$. That is, all the inputs are divided by a. If $a > 1$, the new graph is a horizontal compression of the old graph toward the y-axis. If $0 < a < 1$, the new graph is a horizontal stretch of the old graph away from the y-axis. Points on the y-axis are unchanged.

Note

We often use a set of basic functions with which to begin transformations. We call these parent functions. A library of these is collected in *(see related activity)*, but here are a few of the most common ones.

Activity 1.8.1.14

Consider the function $g(x) = 3\sqrt{-x} + 2$

(a)

Identify the parent function $f(x)$.

A. $f(x) = x^2$

B. $f(x) = |x|$

C. $f(x) = \sqrt{x}$

D. $f(x) = x$

Answer available at the end of this section.

(b)

Graph the parent function $f(x)$.

Answer available at the end of this section.

(c)

How is the graph of $g(x)$ related to that of the parent function $f(x)$?

A. Reflected over the x-axis

B. Reflected over the y-axis

C. Shifted down 2 units

D. Shifted up 2 units

E. Vertically stretched by a factor of 3

F. Horizontally compressed by a factor of 3

Answer available at the end of this section.

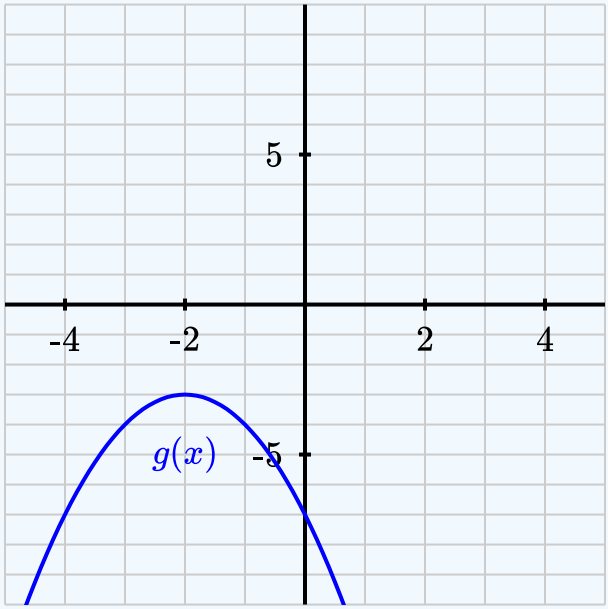
(d)

Graph the transformed function $g(x)$.

Answer available at the end of this section.

Activity 1.8.1.15

Consider the following graph of the function $g(x)$.



(a)

Identify the parent function.

A. $f(x) = x^2$

B. $f(x) = |x|$

C. $f(x) = \sqrt{x}$

D. $f(x) = x$

Answer available at the end of this section.

(b)

How is the graph of $g(x)$ related to that of the parent function $f(x)$?

A. Reflected over the x-axis

B. Reflected over the y-axis

C. Shifted down 3 units

D. Shifted up 3 units

E. Shifted left 2 units

F. Shifted right 2 units

Answer available at the end of this section.

(c)

Write an equation to represent the transformed function $g(x)$.

A. $g(x) = -(x-2)^2 - 3$

B. $g(x) = -(x+2)^2 + 3$

C. $g(x) = (-x+2)^2 - 3$

D. $g(x) = -(x+2)^2 - 3$

Answer available at the end of this section.

Activity 1.8.1.16

For each of the following, write a formula for the new function $g(x)$ when the graph of $f(x)$ is transformed as described.

(a)

$f(x) = |x|$ is shifted 3 units down

A. $g(x) = |x-3|$

B. $g(x) = |x| + 3$

C. $g(x) = |x| - 3$

D. $g(x) = |x+3|$

Answer available at the end of this section.

(b)

$f(x) = x^2$ is shifted 2 units left and reflected over the x-axis

A. $h(x) = -\sqrt{x-2}$

B. $h(x) = -(x-2)^2$

C. $h(x) = \sqrt{-x+2}$

D. $h(x) = -(x+2)^2$

Answer available at the end of this section.

(c)

$f(x) = x^3$ is reflected over the y-axis, vertically stretched by a factor of 2, and shifted 1 unit up

A. $g(x) = 2(-x)^3 + 1$

B. $g(x) = (-2x)^3 - 1$

C. $g(x) = -(2x)^3 + 1$

D. $g(x) - 2x^3 + 1$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply and describe transformations of functions before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 1.8.1.1

The graph of $f(x) = x^2$ is shifted 3 units left and 5 units down. Write the equation of the transformed function.

Answer available at the end of this section.

Check Your Understanding 1.8.1.2

Let $f(x) = \sqrt{x}$. Describe, in words, the transformation that produces $g(x) = -\sqrt{x} + 2$, then sketch how it differs from the graph of f.

Answer available at the end of this section.

Check Your Understanding 1.8.1.3

The graph of $h(x) = |x|$ is stretched vertically by a factor of 3 and reflected horizontally. Write the equation of the transformed function, and explain whether the horizontal reflection changes anything visible for this particular parent function.

Answer available at the end of this section.

Answers

Activity 1.8.1.1
A

Activity 1.8.1.2
C

Activity 1.8.1.3
B

Activity 1.8.1.4
D

Activity 1.8.1.5
D and E

Activity 1.8.1.5
A and B

Activity 1.8.1.6
B

Activity 1.8.1.6

B

Activity 1.8.1.6

A

Activity 1.8.1.7

B

Activity 1.8.1.8

C

Activity 1.8.1.9

B, C and F

Activity 1.8.1.9

A

Activity 1.8.1.10

A and D

Activity 1.8.1.10

Reflect over the y-axis and then shift up 4 units.

Activity 1.8.1.10

B

Activity 1.8.1.11

D

Activity 1.8.1.11

B

Activity 1.8.1.12

A

Activity 1.8.1.12

A

Activity 1.8.1.13

B

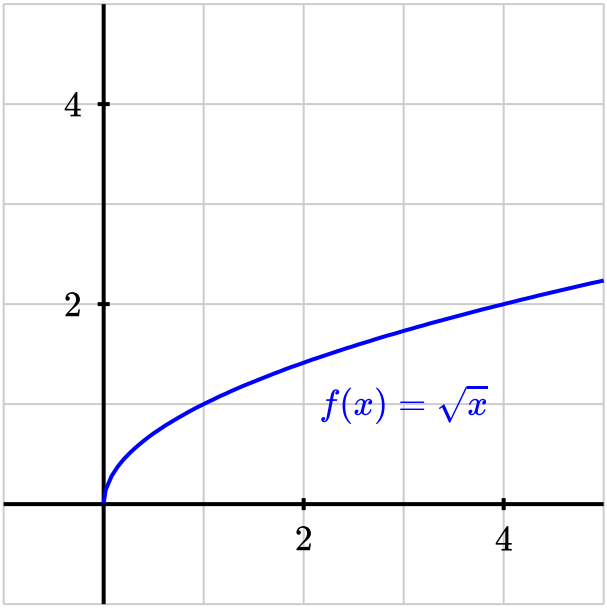
Activity 1.8.1.13

D

Activity 1.8.1.14

C

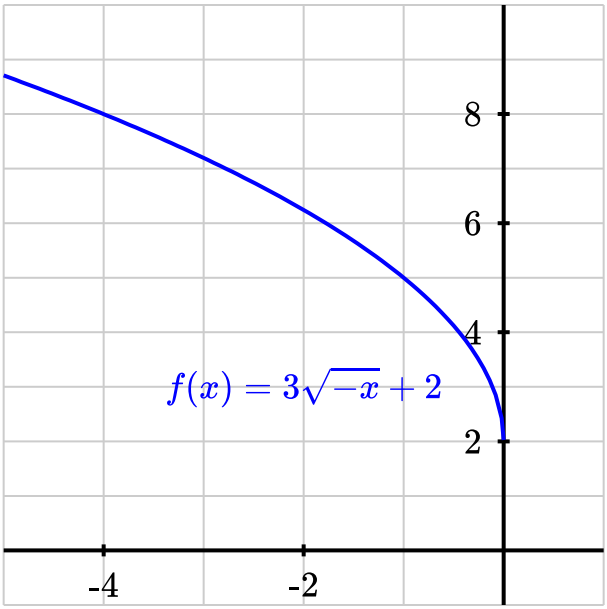
Activity 1.8.1.14



Activity 1.8.1.14

B, D, and E

Activity 1.8.1.14



Activity 1.8.1.15

A

Activity 1.8.1.15

A, B, and D

Activity 1.8.1.15

D

Activity 1.8.1.16

C

Activity 1.8.1.16

D

Activity 1.8.1.16

A

Check Your Understanding 1.8.1.1

A horizontal shift left by 3 replaces x with $x + 3$; a vertical shift down by 5 subtracts 5 from the output. The transformed function is $g(x) = (x + 3)^2 - 5$.

Check Your Understanding 1.8.1.2

The negative sign in front of the square root reflects the graph vertically (over the x-axis), and the $+2$ shifts the result up 2 units. So g looks like f flipped upside down and raised 2 units.

Check Your Understanding 1.8.1.3

A vertical stretch by 3 multiplies the output by 3; a horizontal reflection replaces x with $-x$. The transformed function is $k(x) = 3|-x| = 3|x|$. Because $|x|$ is symmetric about the y-axis, the horizontal reflection produces an identical graph to the unreflected version — a good example of how a transformation can be algebraically present but visually invisible for a symmetric parent function.

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1.8.1.1: FN4 Online Exercises

FN4 — Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.	FN4 CheckIt Exercises

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1.8.2: FN4 FAQ and Resources

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2: CIRCULAR FUNCTIONS

Many real-world phenomena repeat in predictable cycles. In this unit, students build the unit circle from degree and radian measure and arc length, then use it to generate the sine and cosine graphs. Emphasis is placed on connecting a geometric object (the circle) to an algebraic and graphical one (a periodic function) — the same representational flexibility that calculus will later demand of rates of change.

2.1: Introduction to Degree and Radian Measure + Angle Position and Arc Length (TR1 / TR2)

2.1.1: Degree and Radian Measure (TR1)

2.1.1.1: TR1 Online Exercises

2.1.2: Angle Position and Arc Length (TR2)

2.1.2.1: TR2 Online Exercises

2.1.3: TR1 FAQ and Resources

2.1.4: TR2 FAQ and Resources

2.2: Introduction to The Unit Circle (TR 5)

2.2.1: The Unit Circle (TR 5)

2.2.1.1: TR5 Online Exercises

2.2.2: TR5 FAQ and Resources

2.3: Introduction to Properties of Sine and Cosine Graphs (PF1)

2.3.1: Properties of Sine and Cosine Graphs (PF1)

2.3.1.1: PF1 Online Exercises

2.3.2: PF1 FAQ and Resources

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2.1: Introduction to Degree and Radian Measure + Angle Position and Arc Length (TR1 / TR2)

2.1.1 Degree and Radian Measure (TR1)

Learning Outcomes:

- Convert between degrees and radians. Draw angles in standard position.

2.1.2 Angle Position and Arc Length (TR2)

Learning Outcomes:

- Identify and find coterminal angles. Find the length of a circular arc.
-

Watch the minute hand of a clock. After 60 minutes it's swept all the way around and landed back exactly where it started, even though it never stopped rotating. If you tracked total rotation since midnight instead of just position, the angle would keep climbing well past 360°, even as the hand's position on the face repeats every hour. Two angles that point the same direction despite representing different total amounts of rotation are called coterminal — and how far the tip of the hand has actually traveled around the circle depends on the angle, measured in radians rather than degrees, times the length of the hand itself.

Chapter 1 built the general toolkit for functions — treating the input x as a value that could be anything, with no particular reason to repeat. An angle behaves differently: keep rotating, and you eventually land back on a position you've already visited. That wrap-around behavior is new, and coterminal angles are the first formal way to describe it.

Section 2.2 narrows the focus to one particular circle — the unit circle, with radius 1 — where the arc length formula collapses to something remarkably simple, and where the coordinates of a point at a given angle become the sine and cosine functions themselves.

> **Looking ahead to calculus:** Radians aren't just a different unit for the same idea — they're the unit that makes calculus work cleanly. When angles are measured in radians, the derivative of sine turns out to be simply cosine; measured in degrees, an extra conversion factor creeps into every derivative and the formulas get messier. That's the real reason precalculus insists on radians: it's setting up the units calculus actually needs.

This page titled 2.1: Introduction to Degree and Radian Measure + Angle Position and Arc Length (TR1 / TR2) is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Team Based Inquiry Learning.

2.1.1: Degree and Radian Measure (TR1)

Learning Outcomes:

- Convert between degrees and radians. Draw angles in standard position.

Degree and Radian Measure (TR1)

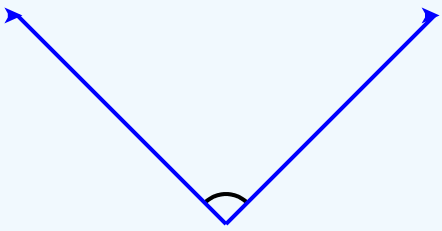
Definition: Angle and Vertex

An **angle** is formed by joining two rays at their starting points. The point where they are joined is called the **vertex** of the angle. The measure of an angle describes the amount of rotation between the two rays.

Activity 2.1.1.1

An angle that is rotated all the way around back to its starting point measures 360°, like a circle. Use this to estimate the measure of the given angles.

(a)



A. 45°

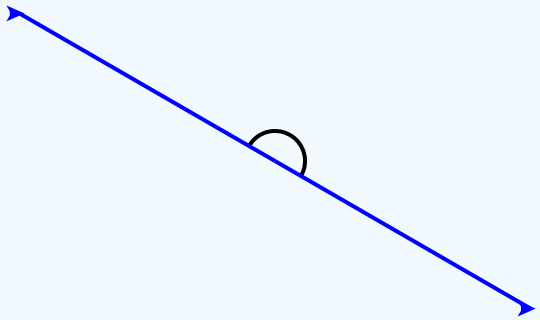
B. 90°

C. 135°

D. 180°

Answer available at the end of this section.

(b)



A. 45°

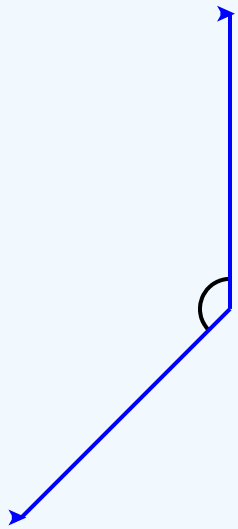
B. 90°

C. 135°

D. 180°

Answer available at the end of this section.

(c)



A. 45°

B. 90°

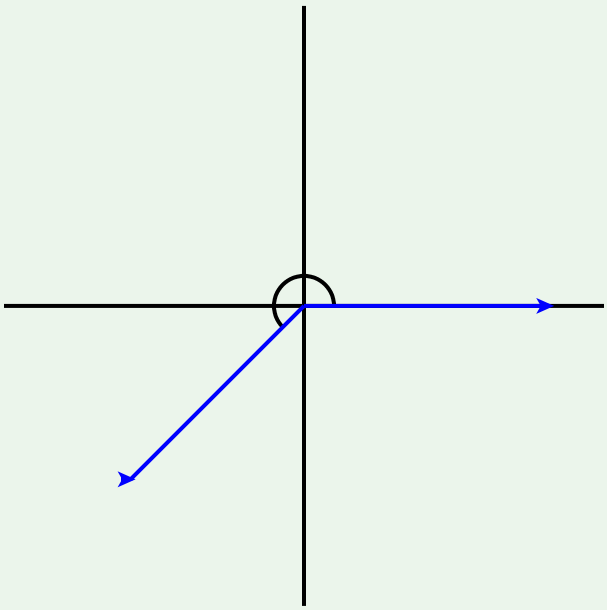
C. 135°

D. 180°

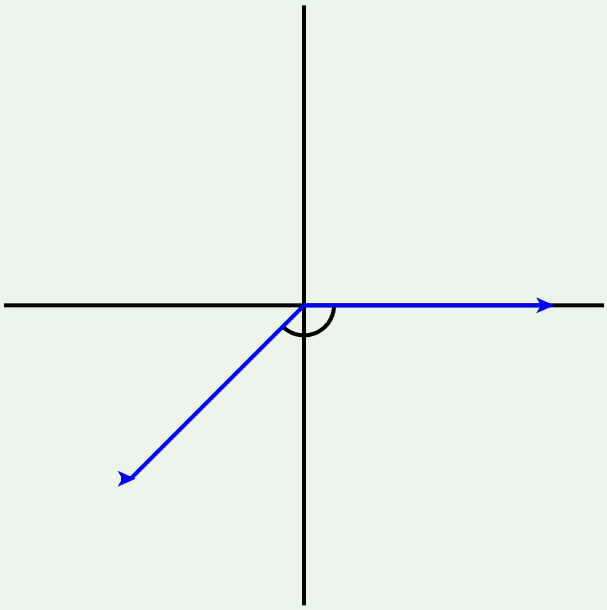
Answer available at the end of this section.

Definition: Standard position

An angle is in **standard position** if its vertex is located at the origin and its initial side extends along the positive x-axis.



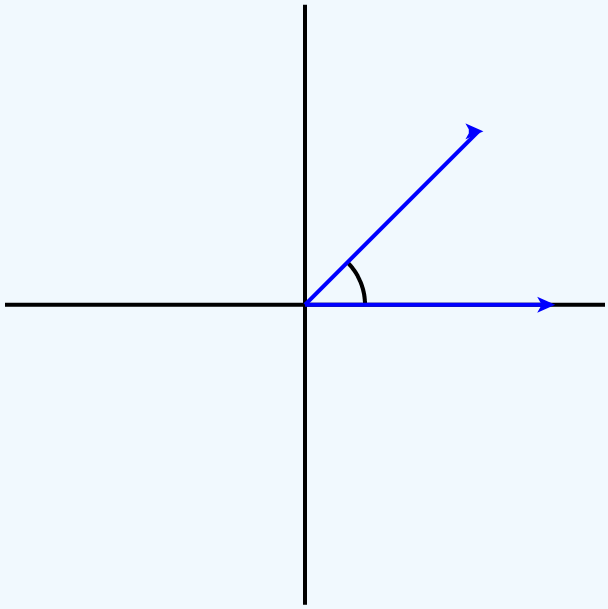
An angle measured counterclockwise from the initial side has a positive measure, while an angle measured clockwise from the initial side has a negative measure.



Activity 2.1.1.2

Estimate the measure of the angles drawn in standard position.

(a)



A. 45°

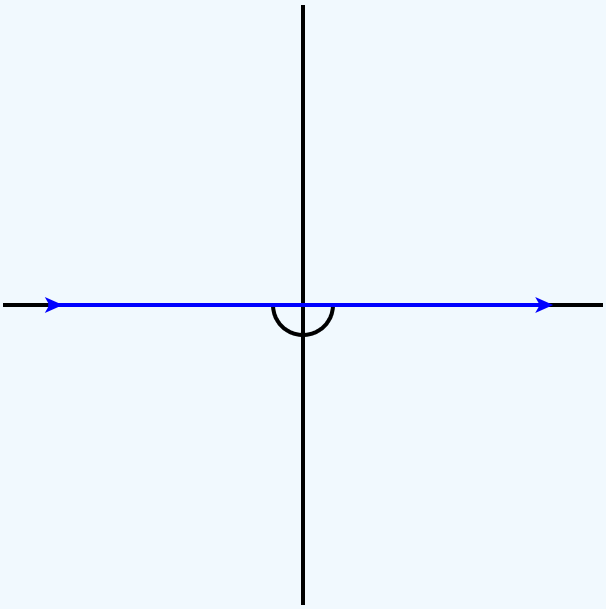
B. 90°

C. 135°

D. 180°

Answer available at the end of this section.

(b)



A. 180°

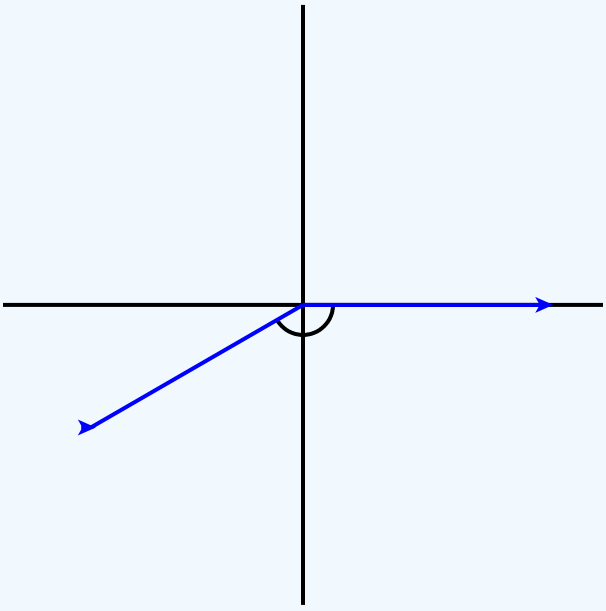
B. 90°

C. -180°

D. -90°

Answer available at the end of this section.

(c)



A. 30°

B. -150°

C. -210°

D. 210°

Answer available at the end of this section.

(d)

Draw an angle of measure -225° in standard position.

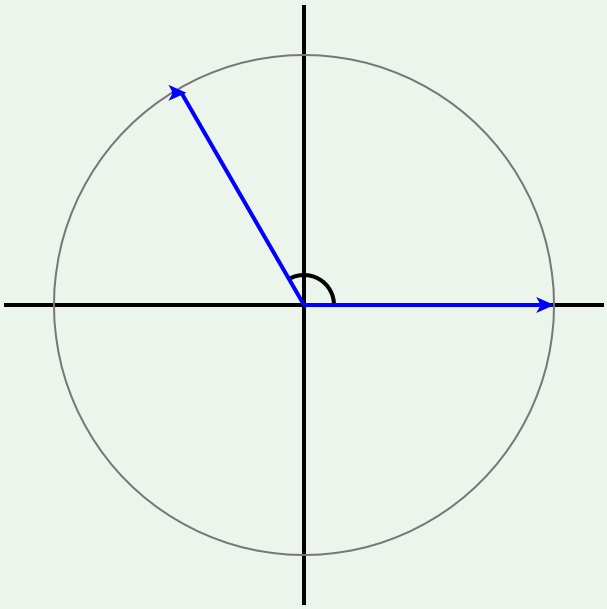
Answer available at the end of this section.

Note

Degrees are not the only way to measure an angle. We can also describe the angle's measure by the amount of the circumference of the circle that the angle's rotation created. We'll need to define a few terms to help us come up with this new measurement.

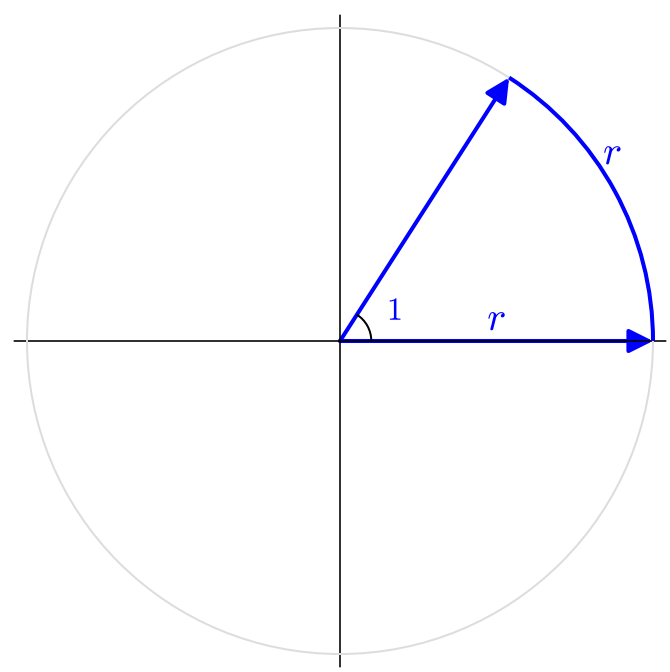
Definition: Central angle

A **central angle** is an angle whose vertex is at the center of a circle.



Definition: Radian

One **radian** is the measure of a central angle of a circle that intersects an arc the same length as the radius.



Note

Recall that the circumference of a circle is given by $C = 2\pi r$, where r is the radius of the circle. That means if we rotate through an entire circle, the circumference is $2\pi r$ which implies that the angle was 2π radians. Thus 2π radians is the same measure as 360°.

Activity 2.1.1.3

We now know that one turn around the circle measures 360° and also 2π radians. Use this information to set up a proportion to find the equivalent radian measure of the following angles that are given in degrees.

(a)

180°

A. $\frac{\pi}{4}$

B. π

C. $\frac{3\pi}{4}$

D. $\frac{\pi}{2}$

Hint:

Try setting up a proportion! $\frac{180^\circ}{360^\circ} = \frac{x}{2\pi}$

Answer available at the end of this section.

(b)

45°

A. $\frac{\pi}{4}$

B. π

C. $\frac{3\pi}{4}$

D. $\frac{\pi}{2}$

Answer available at the end of this section.

Activity 2.1.1.4

Continue using the fact that one turn around the circle measures 360° and also 2π radians. Use this information to set up a proportion to find the equivalent degree measure of the following angles that are given in radians.

(a)

$\frac{\pi}{2}$

A. 45°

B. 90°

C. 180°

D. 360°

Hint:

Try setting up a proportion! $\frac{x}{360^\circ} = \frac{\frac{\pi}{2}}{2\pi}$

Answer available at the end of this section.

(b)

$\frac{3\pi}{4}$

A. 45°

B. 90°

C. 135°

D. 180°

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 2.1.1.1

Convert the following angle measures as described.

(a) Convert 135° to radians. Leave your answer in terms of π.

Answer available at the end of this section.

(b) Convert $-\frac{2\pi}{3}$ radians to degrees.

Answer available at the end of this section.

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Answers

Activity 2.1.1.1
B

Activity 2.1.1.1
D

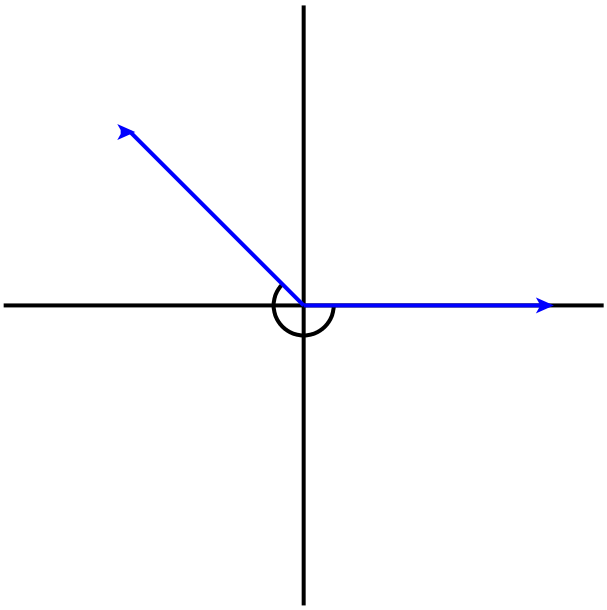
Activity 2.1.1.1
C

Activity 2.1.1.2
A

Activity 2.1.1.2
C

Activity 2.1.1.2
B

Activity 2.1.1.2



Activity 2.1.1.3
B

Activity 2.1.1.3
A

Activity 2.1.1.4

B

Activity 2.1.1.4

C

Check Your Understanding 2.1.1.1

To convert from degrees to radians, multiply by $\frac{\pi}{180^\circ}$:

$$135^\circ \cdot \frac{\pi}{180^\circ} = \frac{135\pi}{180} = \frac{3\pi}{4} \text{ radians} \tag{2.1.1.1}$$

Check Your Understanding 2.1.1.1

To convert from radians to degrees, multiply by $\frac{180^\circ}{\pi}$:

$$-\frac{2\pi}{3} \cdot \frac{180^\circ}{\pi} = -\frac{2 \cdot 180^\circ}{3} = -120^\circ \tag{2.1.1.2}$$

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2.1.1.1: TR1 Online Exercises

TR1 — Convert between degrees and radians. Draw angles in standard position.	TR1 CheckIt Exercises

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2.1.2: Angle Position and Arc Length (TR2)

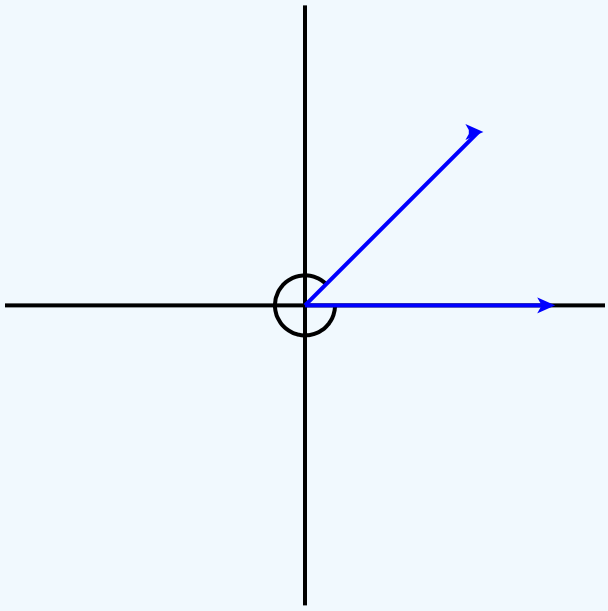
Learning Outcomes:

- Identify and find coterminal angles. Find the length of a circular arc.

Angle Position and Arc Length (TR2)

Activity 2.1.2.1

Consider the angle given below:



Which of the following angles describe the plotted angle?

A. -45°
B. -135°
C. -225°
D. -315°

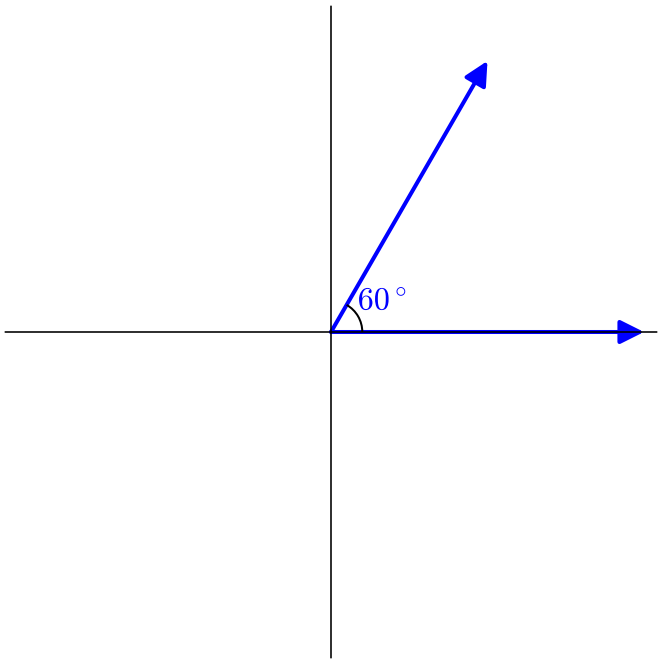
Answer available at the end of this section.

Definition: Coterminal angles

Two angles are called **coterminal angles** if they have the same terminal side when drawn in standard position.

Activity 2.1.2.2

Consider the angle given below:



(a)

Find two angles larger than 60° that are coterminal to 60°.

Answer available at the end of this section.

(b)

Find two angles smaller than 60° that are coterminal to 60°.

Answer available at the end of this section.

Note

For any angle θ, the angle $\theta + k \cdot 360^\circ$ is coterminal to θ for any integer k.

Note

Since there are many coterminal angles for any given angle, it is convenient to systematically choose one for every angle. For a given angle, we typically choose the smallest positive coterminal angle to work with instead.

Definition: Principal angle

If θ is an angle, there is a unique angle α with $0 \leq \alpha < 360^\circ$ (or $0 \leq \alpha < 2\pi$) such that α and θ are coterminal. This angle α is called the **principal angle** of θ.

Activity 2.1.2.3

Find the principal angles for each of the following angles.

(a)

540°

Answer available at the end of this section.

(b)

-600°

Answer available at the end of this section.

(c)

$\frac{11\pi}{3}$

Answer available at the end of this section.

(d)

$\frac{-7\pi}{5}$

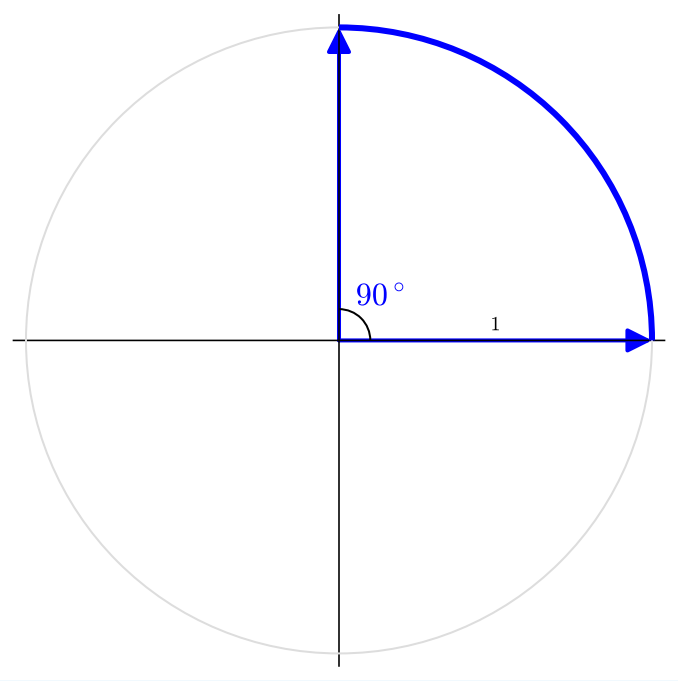
Answer available at the end of this section.

Note

Recall that the circumference of a circle of radius r is $2\pi r$. We will use this to determine the lengths of arcs on a circle.

Activity 2.1.2.4

Consider the portion of a circle of radius 1 graphed below.



(a)

What is the circumference of an entire circle of radius 1?

A. $\frac{\pi}{2}$
B. π
C. $\frac{3\pi}{2}$
D. 2π

Answer available at the end of this section.

(b)

What portion of the entire circle is the sector graphed above?

A. $\frac{1}{4}$
B. $\frac{1}{3}$
C. $\frac{1}{2}$
D. $\frac{3}{4}$

Answer available at the end of this section.

(c)

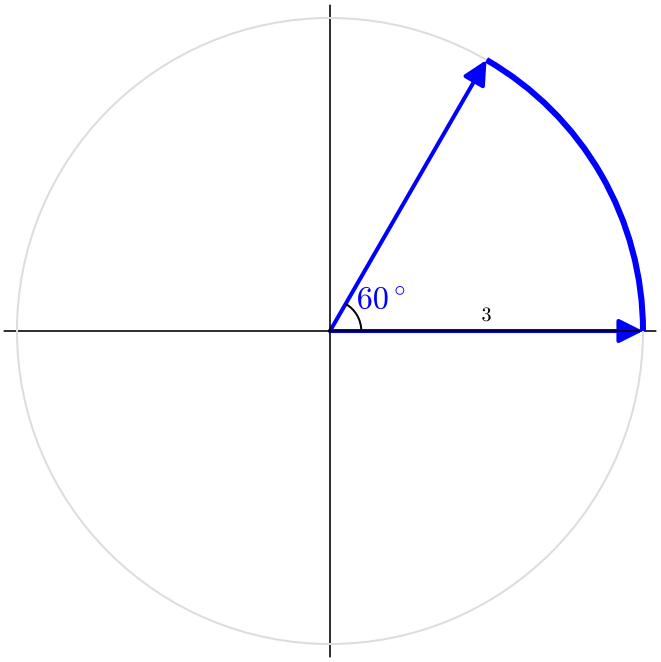
Use proportions to determine the length of the arc displayed above.

A. $\frac{\pi}{2}$
B. π
C. $\frac{3\pi}{2}$
D. 2π

Answer available at the end of this section.

Activity $2.1.2.5$

Consider the portion of a circle of radius 3 graphed below.



(a)

What is the circumference of an entire circle of radius 3?

A. π
B. 3π
C. 6π
D. 9π

Answer available at the end of this section.

(b)

What portion of the entire circle is the sector graphed above?

A. $\frac{1}{12}$
B. $\frac{1}{6}$
C. $\frac{1}{4}$
D. $\frac{1}{3}$

Answer available at the end of this section.

(c)

Use proportions to determine the length of the arc displayed above.

A. $\frac{\pi}{2}$
B. π
C. 2π
D. 3π

Answer available at the end of this section.

Note

For a sector of angle θ and radius r, we can use proportions to find the length of the arc, s. If θ is measured in degrees, we have $s = \frac{\theta}{360^\circ}(2\pi r)$, which simplifies to

$$s = \frac{\theta}{180^\circ}\pi r \tag{2.1.2.1}$$

. In radians, the formula is even nicer: $s = \frac{\theta}{2\pi}(2\pi r)$, which simplifies to

$$s = \theta r \tag{2.1.2.2}$$

.

Activity 2.1.2.6

Find the lengths of the arcs described below.

(a)

The length of the arc of a sector of measure 120° of a circle of radius 4.

Answer available at the end of this section.

(b)

The length of the arc of a sector of measure 270° of a circle of radius 2.

Answer available at the end of this section.

(c)

The length of the arc of a sector of measure $\frac{5\pi}{6}$ of a circle of radius 3.

Answer available at the end of this section.

(d)

The length of the arc of a sector of measure $\frac{11\pi}{12}$ of a circle of radius 6.

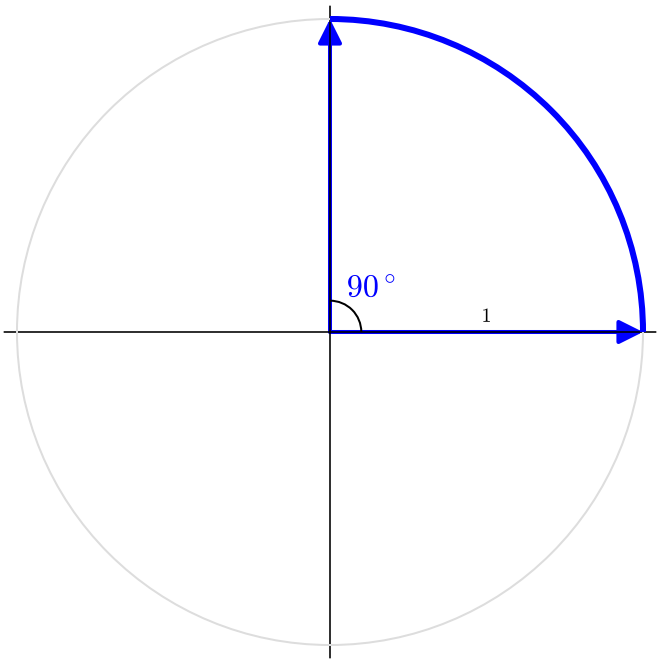
Answer available at the end of this section.

Note

Recalling that the area of a circle of radius r is πr^2, we can use this same idea of proportions to find the area of a sector of a circle.

Activity 2.1.2.7

Consider the portion of a circle of radius 1 graphed below.



(a)

What is the area of an entire circle of radius 1?

A. $\frac{\pi}{2}$
B. π
C. $\frac{3\pi}{2}$
D. 2π

Answer available at the end of this section.

(b)

What portion of the entire circle is the sector graphed above?

A. $\frac{1}{4}$
B. $\frac{1}{3}$
C. $\frac{1}{2}$
D. $\frac{3}{4}$

Answer available at the end of this section.

(c)

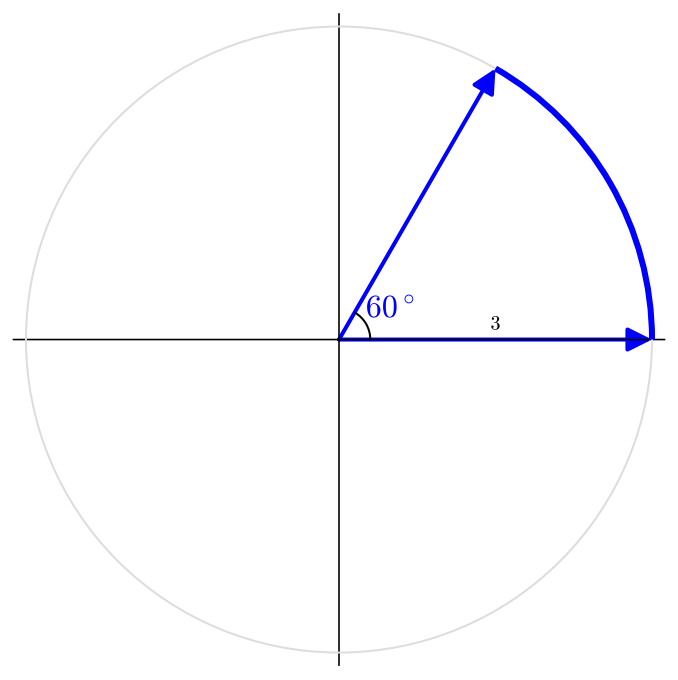
Use proportions to determine the area of the arc displayed above.

A. $\frac{\pi}{4}$
B. $\frac{\pi}{2}$
C. π
D. $\frac{3\pi}{2}$

Answer available at the end of this section.

Activity 2.1.2.8

Consider the portion of a circle of radius 3 graphed below.



(a)

What is the area of an entire circle of radius 3?

A. π
B. 3π
C. 6π
D. 9π

Answer available at the end of this section.

(b)

What portion of the entire circle is the sector graphed above?

A. $\frac{1}{12}$
B. $\frac{1}{6}$
C. $\frac{1}{4}$
D. $\frac{1}{3}$

Answer available at the end of this section.

(c)

Use proportions to determine the area of the sector displayed above.

A. $\frac{\pi}{2}$
B. π
C. $\frac{3\pi}{2}$
D. 2π

Answer available at the end of this section.

Note

For a sector of angle θ and radius r, we can use proportions to find the area of the arc. If θ is measured in degrees, we have $A = \frac{\theta}{360^\circ}(\pi r^2)$. In radians, the formula is even nicer: $A = \frac{\theta}{2\pi}(\pi r^2)$, which simplifies to

$$A = \frac{1}{2}\theta r^2 \tag{2.1.2.3}$$

.

Activity 2.1.2.9

Find the areas of each sector described below.

(a)

The sector with central angle 120° in a circle of radius 4.

Answer available at the end of this section.

(b)

The sector with central angle 270° in a circle of radius 2.

Answer available at the end of this section.

(c)

The sector with central angle $\frac{5\pi}{6}$ in a circle of radius 3.

Answer available at the end of this section.

(d)

The sector with central angle $\frac{11\pi}{12}$ in a circle of radius 6.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 2.1.2.1

Find the principal angle (smallest positive coterminal angle) for the given angles.

(a) 420°

Answer available at the end of this section.

(b) $-\frac{7\pi}{4}$

Answer available at the end of this section.

Check Your Understanding 2.1.2.2

Solve the following circular geometry problems.

(a) Find the length of the arc of a sector with a central angle of 150° in a circle of radius 6.

Answer available at the end of this section.

(b) Find the area of a sector with a central angle of $\frac{3\pi}{4}$ radians in a circle of radius 4.

Answer available at the end of this section.

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Answers

Activity 2.1.2.1
D.

Activity 2.1.2.2
420°, 780°, among others.

Activity 2.1.2.2
-300°, -660°, among others.

Activity 2.1.2.3
180°

Activity 2.1.2.3
120°

Activity 2.1.2.3
$\frac{5\pi}{3}$

Activity 2.1.2.3
$\frac{3\pi}{5}$

Activity 2.1.2.4
D.

Activity 2.1.2.4
A.

Activity 2.1.2.4
A

Activity 2.1.2.5
C.

Activity 2.1.2.5
B.

Activity 2.1.2.5
B.

Activity 2.1.2.6
$\frac{8\pi}{3}$

Activity 2.1.2.6
3π

Activity 2.1.2.6
$\frac{5\pi}{2}$

Activity 2.1.2.6

$\frac{11\pi}{2}$

Activity 2.1.2.7

B.

Activity 2.1.2.7

A.

Activity 2.1.2.7

A.

Activity 2.1.2.8

D.

Activity 2.1.2.8

B.

Activity 2.1.2.8

C.

Activity 2.1.2.9

$\frac{16\pi}{3}$

Activity 2.1.2.9

3π

Activity 2.1.2.9

$\frac{15\pi}{4}$

Activity 2.1.2.9

$\frac{33\pi}{2}$

Check Your Understanding 2.1.2.1

Subtract 360° to find the coterminal angle in the interval $[0, 360^\circ)$:

$$420^\circ - 360^\circ = 60^\circ \tag{2.1.2.4}$$

Check Your Understanding 2.1.2.1

Add 2π (or $\frac{8\pi}{4}$) to find the coterminal angle in the interval $[0, 2\pi)$:

$$-\frac{7\pi}{4} + \frac{8\pi}{4} = \frac{\pi}{4} \tag{2.1.2.5}$$

Check Your Understanding 2.1.2.2

Using the arc length formula for degrees $s = \frac{\theta}{180^\circ}\pi r$:

$$s = \frac{150^\circ}{180^\circ}\pi \cdot 6 = \frac{5}{6}\pi \cdot 6 = 5\pi \tag{2.1.2.6}$$

Check Your Understanding 2.1.2.2

Using the sector area formula for radians $A = \frac{1}{2}\theta r^2$:

$$A = \frac{1}{2} \cdot \frac{3\pi}{4} \cdot 4^2 = \frac{3\pi}{8} \cdot 16 = 6\pi \tag{2.1.2.7}$$

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2.1.2.1: TR2 Online Exercises

TR2 — Identify and find coterminal angles. Find the length of a circular arc.	TR2 CheckIt Exercises

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2.1.3: TR1 FAQ and Resources

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2.1.4: TR2 FAQ and Resources

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2.2: Introduction to The Unit Circle (TR 5)

2.1.1: The Unit Circle (TR 5)

TR5 The Unit Circle

- Use reference angles, signs and definitions to determine exact values of trigonometric functions.

Go back to the tip of that clock's minute hand, but imagine the clock face is a circle of radius exactly 1 — the unit circle. As the hand sweeps around, the tip traces a point (x, y), and those coordinates aren't arbitrary: $x = \cos\theta$ and $y = \sin\theta$, where θ is the angle swept. That's what sine and cosine really mean — not "opposite over hypotenuse" in a fixed triangle, but coordinates of a point moving around a circle. Knowing the coordinates for one angle in the first quadrant, reflecting across the axes hands you sine and cosine for every related angle in the other three quadrants for free.

Section 2.1 gave you the tools to measure how far around the circle an angle takes you — in radians, and via arc length. On the unit circle specifically, radius 1 makes that arc length formula collapse to something clean: the arc swept out by an angle θ has length θ itself, which is part of why radians are the natural unit here.

Section 2.3 takes these coordinate values and unwraps them along a horizontal axis instead of a circle, turning the repeating up-and-down motion of $y = \sin\theta$ into the sine graph you'll recognize — with amplitude, period, and phase shift describing the same circular motion from a different vantage point.

> **Looking ahead to calculus:** The fact that sine and cosine come from walking around a circle shows up again once you start taking derivatives. Differentiate sine, and you get cosine; differentiate cosine, and you get negative sine; differentiate that, and you get negative cosine; one more derivative brings you back to sine. Four derivatives, and you've gone all the way around — the same cyclical structure as the circle you're using right now to define these functions in the first place.

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2.2.1: The Unit Circle (TR 5)

Learning Outcomes:

- Use reference angles, signs and definitions to determine exact values of trigonometric functions.

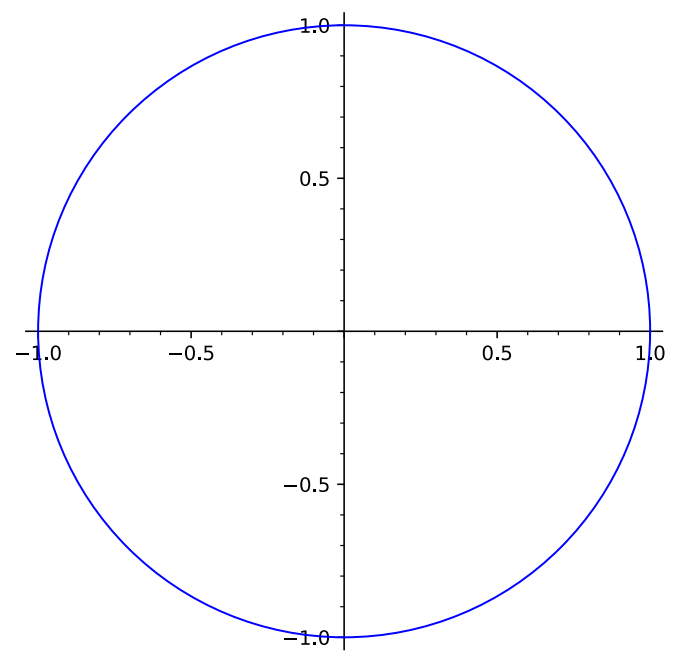
The Unit Circle (TR5)

Note

In *(see related activity)*, we learned how to find the exact values of the six trigonometric ratios for the special acute angles 30°, 45°, and 60°. In this section, we will use that knowledge and expand to finding the exact trig values of any multiple of those angles.

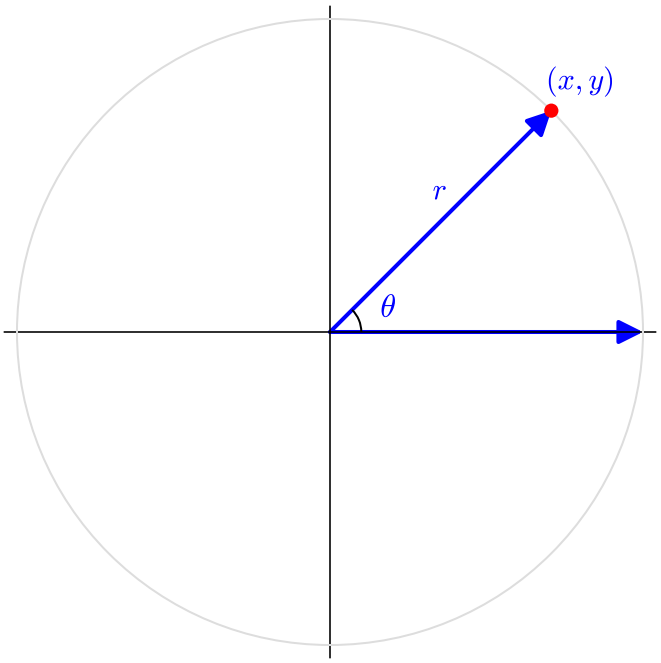
Definition: Unit circle

The **unit circle** is the circle of radius 1 centered at the origin on the coordinate plane.



Activity 2.2.1.1

Let θ be the angle shown below in standard form. Notice that the terminal side intersects with the unit circle. (Note: We will assume a circle drawn in this context is the unit circle unless told otherwise.) We will label that point of intersection as (x, y).



(a)

What is the length of line segment r, whose endpoints are the origin and the point (x, y) ?

A. 1
B. 2
C. 3
D. cannot be determined

Answer available at the end of this section.

(b)

We will now create a right triangle using the previous line segment r as the hypotenuse. Draw in a line segment of length x and another of length y to create such a triangle.

Answer available at the end of this section.

(c)

Using the triangle you've just created, find $\cos\theta$.

A. $\frac{x}{y}$
B. $\frac{1}{x}$
C. $\frac{x}{1}$
D. $\frac{1}{y}$
E. $\frac{y}{1}$

Answer available at the end of this section.

(d)

Using that same triangle, find $\sin\theta$.

A. $\frac{x}{y}$
B. $\frac{1}{\frac{x}{y}}$
C. $\frac{1}{y}$
D. $\frac{y}{1}$
E. $\frac{y}{1}$

Answer available at the end of this section.

(e)

Solve for x in one of the equations you've found above to determine an expression for the x-value of the point (x, y) .

A. $y\cos\theta$
B. $y\sin\theta$
C. $\cos\theta$
D. $\sin\theta$
E. $\frac{1}{\cos\theta}$
F. $\frac{1}{\sin\theta}$

Answer available at the end of this section.

(f)

Solve for y in one of the equations you've found above to determine an expression for the y-value of the point (x, y) .

A. $y\cos\theta$
B. $y\sin\theta$
C. $\cos\theta$
D. $\sin\theta$
E. $\frac{1}{\cos\theta}$
F. $\frac{1}{\sin\theta}$

Answer available at the end of this section.

Note

From the previous activity, we have found a connection between the sine and cosine values of an angle θ and the coordinates (x, y) of the point at which that angle intersects the unit circle. Namely,

$$x = \cos\theta \text{ and } y = \sin\theta \tag{2.2.1.1}$$

Activity 2.2.1.2

Consider each angle θ given below. Find the coordinates (x, y) for the point at which θ intersects the unit circle.

(a)

$\theta = \frac{\pi}{4}$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
D. $(0, 1)$
E. $(1, 0)$

Answer available at the end of this section.

(b)

$\theta = 30^\circ$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
D. $(0, 1)$
E. $(1, 0)$

Answer available at the end of this section.

(c)

$\theta = \frac{\pi}{3}$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
D. $(0, 1)$
E. $(1, 0)$

Answer available at the end of this section.

(d)

$\theta = 0^\circ$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
D. $(0, 1)$
E. $(1, 0)$

Answer available at the end of this section.

(e)

$\theta = \frac{\pi}{2}$

A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
B. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
C. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
D. $(0, 1)$
E. $(1, 0)$

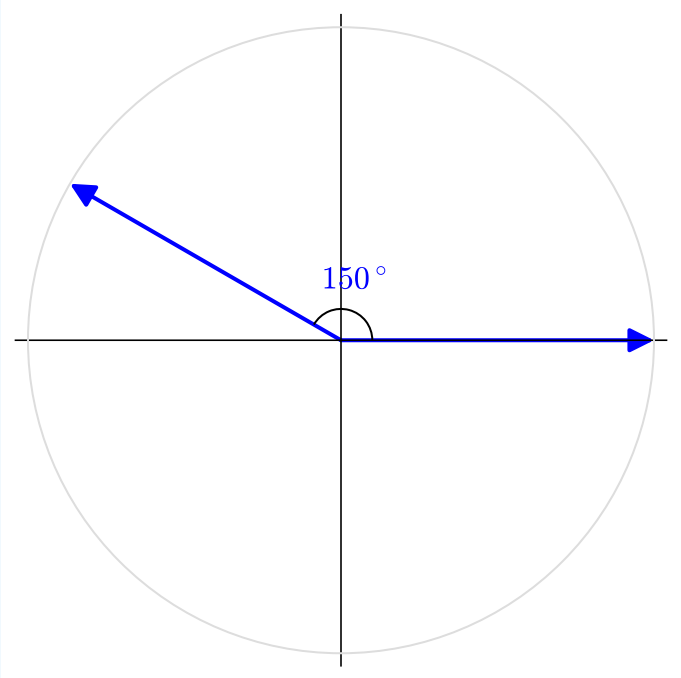
Answer available at the end of this section.

Note

In *(see related activity)*, we found (x, y)-coordinates (and thus the sine and cosine values) for angles that terminated either in Quadrant 1 or on an axis adjacent to Quadrant 1. We'll now expand to angles that terminate elsewhere, using our knowledge of the cosine and sine values of angles in the first quadrant along with how reflections over the x and y axes affect the signs of the coordinates. (See *(see related activity)* for a reminder on how these reflections work.)

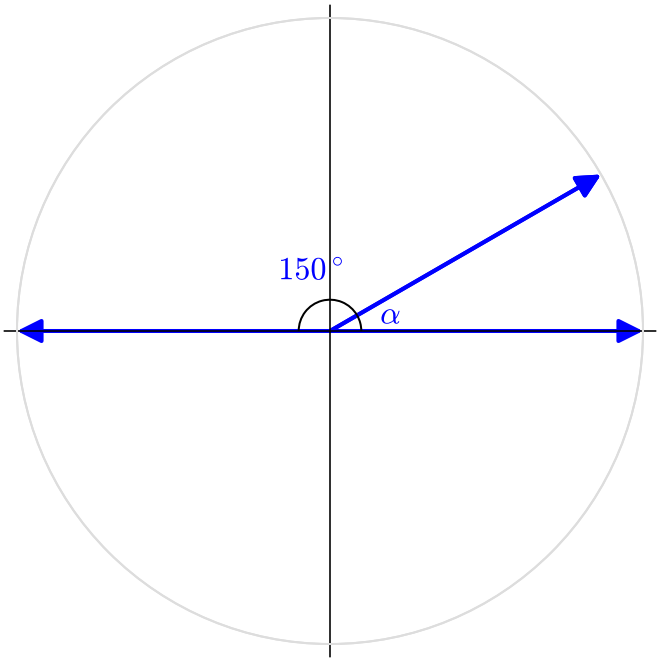
Activity 2.2.1.3

Let's consider the angle $\theta = 150°$, drawn below with the unit circle.



(a)

If we reflect this angle across the y-axis, we can obtain an angle α in the first quadrant. What is the measure of α?

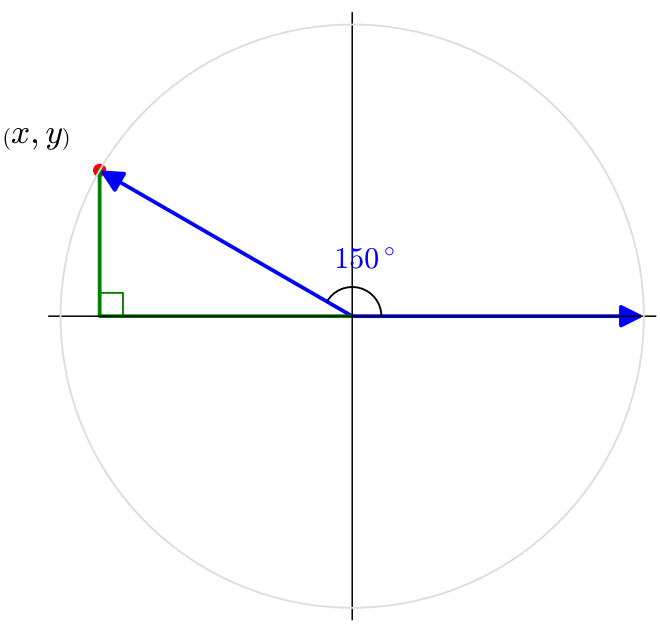


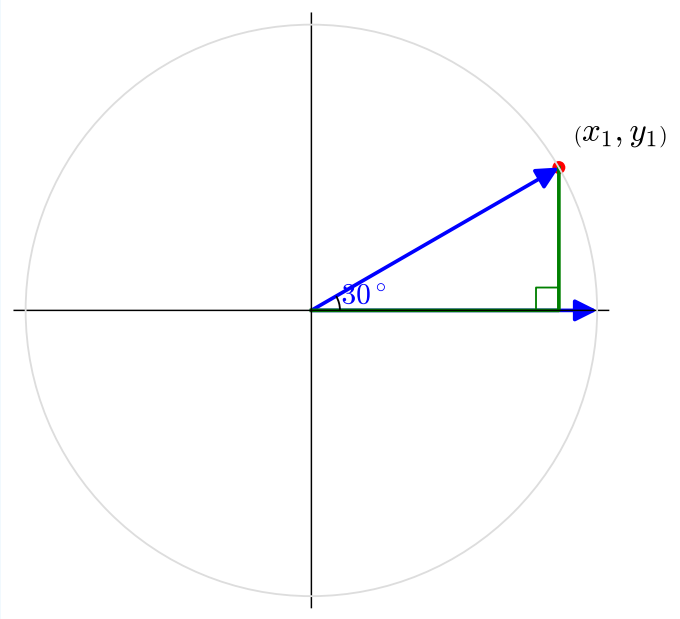
A. 0°
B. 30°
C. 45°
D. 60°
E. 75°

Answer available at the end of this section.

(b)

We can find the sine and cosines values of our original angle, $\theta = 150^\circ$, by using the angle $\alpha = 30^\circ$ to help. Find the point (x_1, y_1), where the terminal side of the 30° angle intersects the unit circle.





A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
B. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
C. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
D. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
E. $\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Answer available at the end of this section.

(c)

How does the point you've just found compare with the point (x, y), where the terminal edge of $\theta = 150^\circ$ intersects the unit circle?

A. The x-values and the y-values are switched with each other.
B. The x-values will be the same, but the y-values will have opposite signs.
C. The y-values will be the same, but the x-values will have opposite signs.
D. The x-values and the y-values will both have opposite signs.

Answer available at the end of this section.

(d)

What are the cosine and sine values of $\theta = 150^\circ$?

A. $\cos 150^\circ = \frac{\sqrt{3}}{2}$ and $\sin 150^\circ = \frac{1}{2}$
B. $\cos 150^\circ = \frac{1}{2}$ and $\sin 150^\circ = \frac{\sqrt{3}}{2}$
C. $\cos 150^\circ = \frac{1}{2}$ and $\sin 150^\circ = -\frac{\sqrt{3}}{2}$

D. $\cos 150^\circ = -\dfrac{\sqrt{3}}{2}$ and $\sin 150^\circ = \dfrac{1}{2}$

E. $\cos 150^\circ = \dfrac{\sqrt{3}}{2}$ and $\sin 150^\circ = -\dfrac{1}{2}$

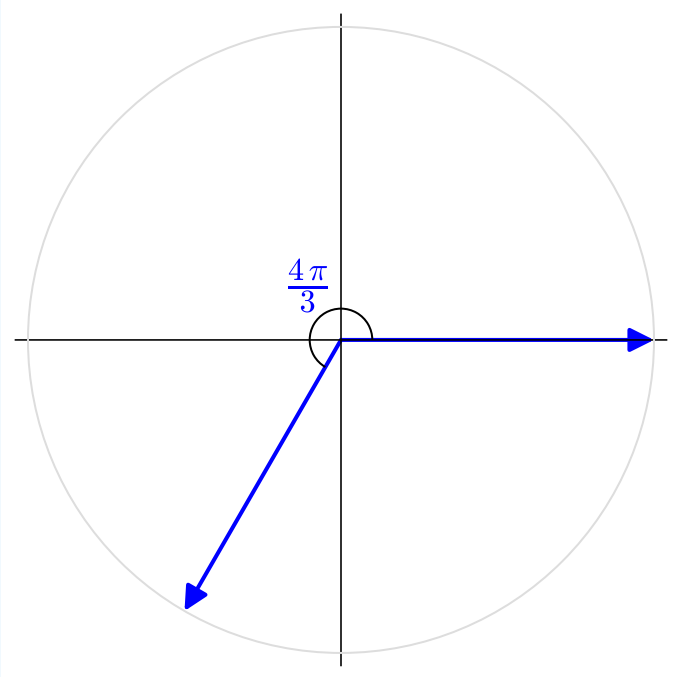
Answer available at the end of this section.

Definition: Reference angle

The **reference angle** for a given angle θ is the angle in the first quadrant obtained from reflecting θ. Equivalently, it is the smallest angle between the terminal side of θ and the x-axis.

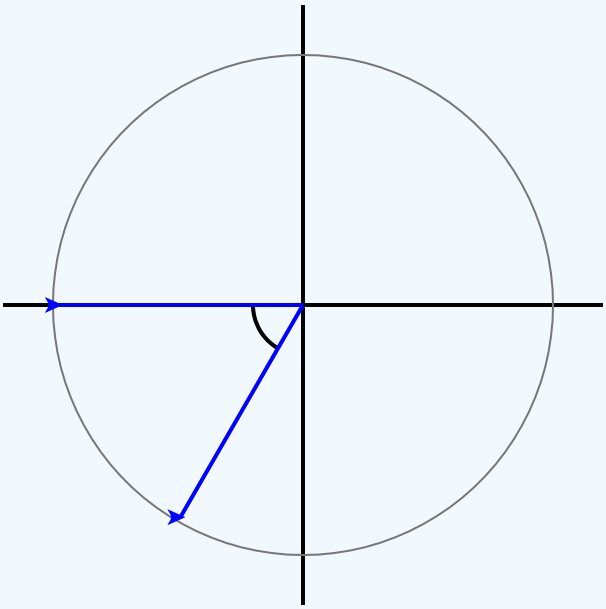
Activity 2.2.1.4

Let's consider the angle $\theta = \dfrac{4\pi}{3}$, drawn below with the unit circle.



(a)

The angle below represents the reference angle for $\theta = \dfrac{4\pi}{3}$, which is the smallest angle between the terminal side of θ and the x-axis. What is the measure of this reference angle?

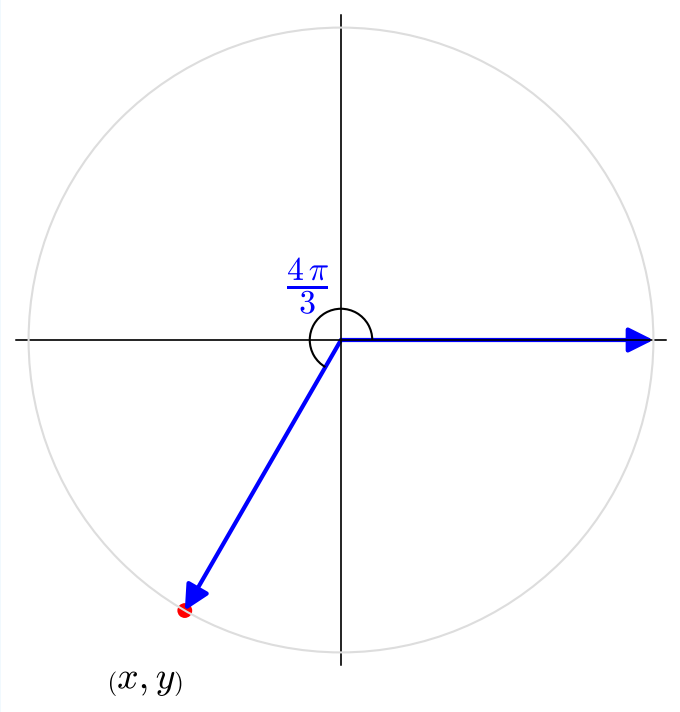


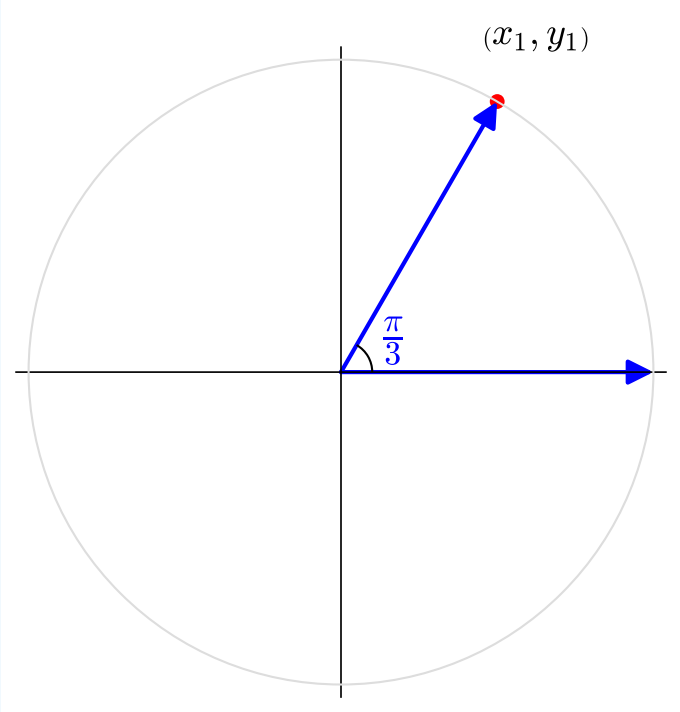
A. $\frac{\pi}{2}$
B. $\frac{\pi}{3}$
C. $\frac{\pi}{4}$
D. $\frac{\pi}{5}$
E. $\frac{\pi}{6}$

Answer available at the end of this section.

(b)

We can find the sine and cosines values of our original angle, $\theta = \frac{4\pi}{3}$, by using the reference angle to help. Find the point (x_1, y_1), where the terminal side of the angle $\frac{\pi}{3}$ intersects the unit circle.





A. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

B. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

C. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

D. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

E. $\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Answer available at the end of this section.

(c)

How does the point you've just found compare with the point (x, y), where the terminal edge of $\theta = \frac{4\pi}{3}$ intersects the unit circle?

A. The x-values and the y-values are switched with each other.
B. The x-values will be the same, but the y-values will have opposite signs.
C. The y-values will be the same, but the x-values will have opposite signs.
D. The x-values and the y-values will both have opposite signs.

Answer available at the end of this section.

(d)

What are the cosine and sine values of $\theta = \frac{4\pi}{3}$?

A. $\cos\frac{4\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = \frac{1}{2}$
B. $\cos\frac{4\pi}{3} = -\frac{1}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$
C. $\cos\frac{4\pi}{3} = \frac{1}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$
D. $\cos\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = \frac{1}{2}$
E. $\cos\frac{4\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = -\frac{1}{2}$

Answer available at the end of this section.

Activity 2.2.1.5

Find $\sin\theta$ and $\cos\theta$ for each angle given.

(a)

$\theta = \frac{\pi}{4}$

Answer available at the end of this section.

(b)

$\theta = \frac{2\pi}{3}$

Answer available at the end of this section.

(c)

$\theta = \frac{11\pi}{6}$

Answer available at the end of this section.

(d)

$\theta = 135^\circ$

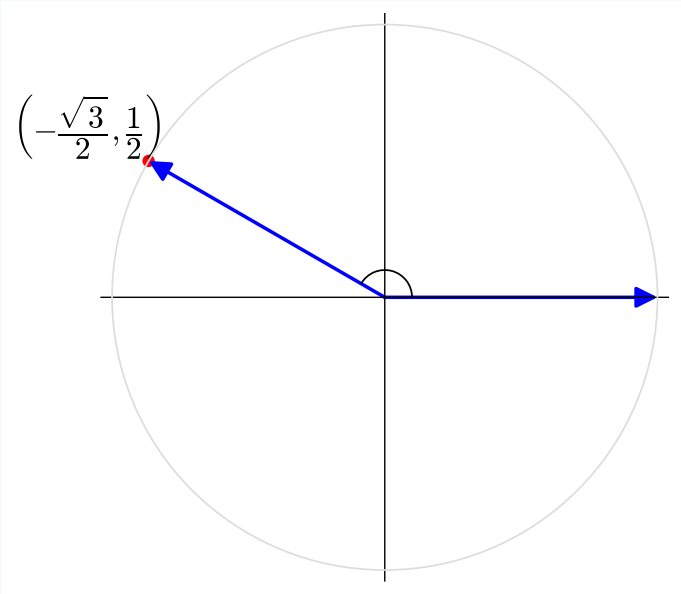
Answer available at the end of this section.

Activity 2.2.1.6

Find the following for each angle graphed below.

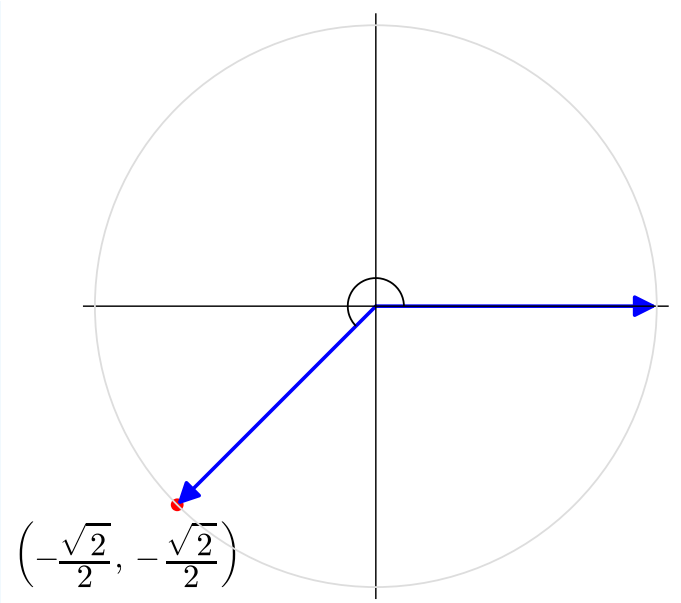
- θ in radians and degrees
- $\sin\theta$
- $\cos\theta$
- $\tan\theta$
- $\sec\theta$
- $\csc\theta$
- $\cot\theta$

(a)



Answer available at the end of this section.

(b)



Answer available at the end of this section.

Note

So far we've only dealt with angles that are part of a special right triangle (30-60-90 or 45-45-90) or are a multiple of one of these angles, but we can extend to other angles as well.

Activity 2.2.1.7

A point (x, y) lies on the unit circle in Quadrant IV. Its x-coordinate is $\frac{3}{4}$.

(a)

Draw a sketch of the angle θ whose terminal side intersects the unit circle as described above.

Answer available at the end of this section.

(b)

What sign will the y-coordinate be?

A. positive
B. negative

Answer available at the end of this section.

(c)

Find the exact value of the y-coordinate.

A. $\frac{7}{16}$
B. $-\frac{7}{16}$

C. $\frac{\sqrt{7}}{4}$

D. $-\frac{\sqrt{7}}{4}$

Hint:

Use the Pythagorean Theorem to help.

Answer available at the end of this section.

(d)

Find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$.

Answer available at the end of this section.

Activity 2.2.1.8

Let θ be the angle whose terminal side intersects the unit circle at the point described in each situation below. Find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$.

(a)

The point $\left(\frac{4}{5}, y\right)$ that lies on the unit circle in Quadrant I.

Answer available at the end of this section.

(b)

The point $\left(-\frac{35}{37}, \frac{12}{37}\right)$.

Answer available at the end of this section.

(c)

The point $\left(x, -\frac{\sqrt{11}}{6}\right)$ that lies on the unit circle in Quadrant III.

Answer available at the end of this section.

(d)

The point $\left(\frac{12}{13}, -\frac{5}{13}\right)$.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 2.2.1.1

Solve the following problems using reference angles and the unit circle.

(a) Find the reference angle for $\theta = 315^\circ$ and $\theta = -\frac{5\pi}{6}$.

Answer available at the end of this section.

(b) Evaluate the exact values of $\sin\left(\frac{2\pi}{3}\right)$ and $\cos\left(-\frac{3\pi}{4}\right)$.

Answer available at the end of this section.

(c) A point (x, y) lies on the unit circle in Quadrant III. Its y-coordinate is $-\frac{2}{5}$. Find the exact value of x and evaluate $\tan\theta$.

Answer available at the end of this section.

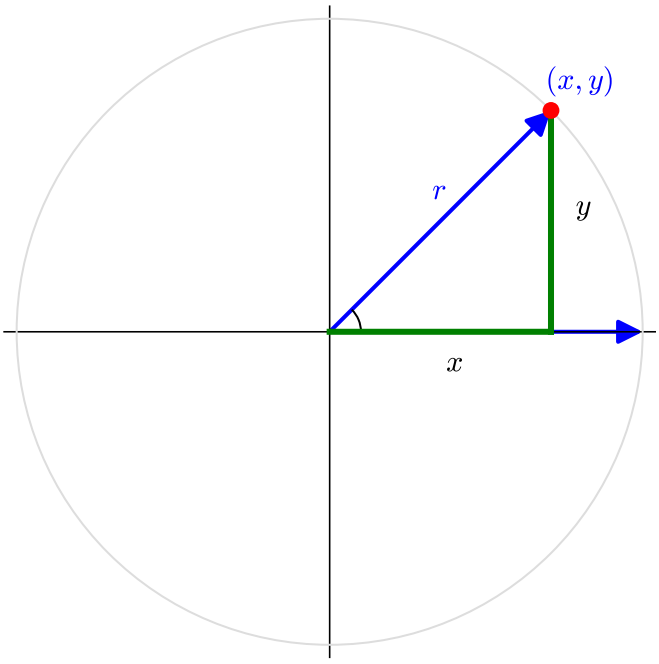
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Answers

Activity 2.2.1.1

A

Activity 2.2.1.1



Activity 2.2.1.1

C

Activity 2.2.1.1

E

Activity 2.2.1.1

C

Activity 2.2.1.1

D

Activity 2.2.1.2

B

Activity 2.2.1.2

C

Activity 2.2.1.2

A

Activity 2.2.1.2

E

Activity 2.2.1.2

D

Activity 2.2.1.3

B

Activity 2.2.1.3

B

Activity 2.2.1.3

C

Activity 2.2.1.3

D

Activity 2.2.1.4

B

Activity 2.2.1.4

A

Activity 2.2.1.4

D

Activity 2.2.1.4

B

Activity 2.2.1.5

$\sin\theta = \frac{\sqrt{2}}{2}$ and $\cos\theta = \frac{\sqrt{2}}{2}$

Activity 2.2.1.5

$\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = -\frac{1}{2}$

Activity 2.2.1.5

$\sin\theta = -\frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2}$

Activity 2.2.1.5

$\sin\theta = \frac{\sqrt{2}}{2}$ and $\cos\theta = -\frac{\sqrt{2}}{2}$

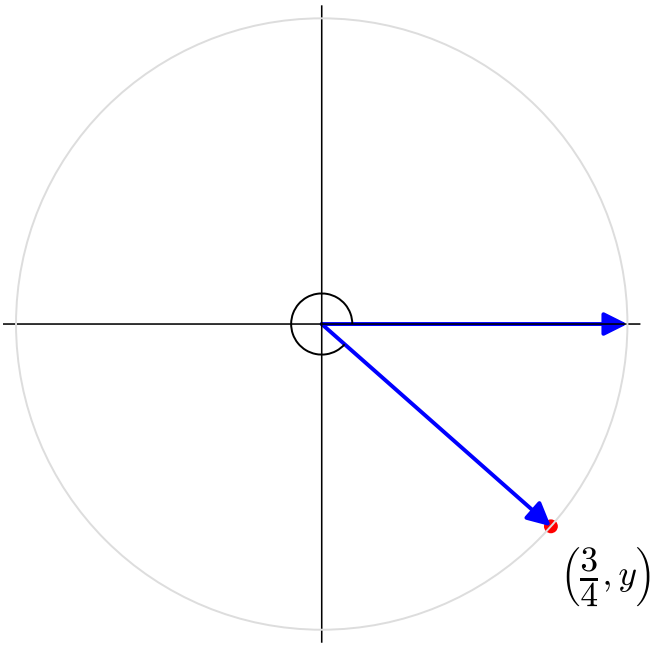
Activity 2.2.1.6

- $\theta = \frac{5\pi}{6}$ or 150°
- $\sin\theta = \frac{1}{2}$
- $\cos\theta = -\frac{\sqrt{3}}{2}$
- $\tan\theta = -\frac{\sqrt{3}}{3}$ or $-\frac{1}{\sqrt{3}}$
- $\sec\theta = -\frac{2\sqrt{3}}{3}$ or $-\frac{2}{\sqrt{3}}$
- $\csc\theta = 2$
- $\cot\theta = -\sqrt{3}$

Activity 2.2.1.6

- $\theta = \frac{5\pi}{4}$ or 225°
- $\sin\theta = -\frac{\sqrt{2}}{2}$
- $\cos\theta = -\frac{\sqrt{2}}{2}$
- $\tan\theta = 1$
- $\sec\theta = -\sqrt{2}$ or $-\frac{2}{\sqrt{2}}$
- $\csc\theta = -\sqrt{2}$ or $-\frac{2}{\sqrt{2}}$
- $\cot\theta = 1$

Activity 2.2.1.7



Activity 2.2.1.7

B

Activity 2.2.1.7

D

Activity 2.2.1.7

- $\sin\theta = -\frac{\sqrt{7}}{4}$
- $\cos\theta = \frac{3}{4}$
- $\tan\theta = -\frac{\sqrt{7}}{3}$
- $\sec\theta = \frac{4}{3}$
- $\csc\theta = -\frac{4}{\sqrt{7}}$ or $-\frac{4\sqrt{7}}{7}$
- $\cot\theta = -\frac{3}{\sqrt{7}}$ or $-\frac{3\sqrt{7}}{7}$

Activity 2.2.1.8

- $\sin\theta = \frac{3}{5}$
- $\cos\theta = \frac{4}{5}$
- $\tan\theta = \frac{3}{4}$
- $\sec\theta = \frac{5}{4}$
- $\csc\theta = \frac{5}{3}$
- $\cot\theta = \frac{4}{3}$

Activity 2.2.1.8

- $\sin\theta = \frac{12}{37}$
- $\cos\theta = -\frac{35}{37}$
- $\tan\theta = -\frac{12}{35}$
- $\sec\theta = -\frac{37}{35}$
- $\csc\theta = \frac{37}{12}$
- $\cot\theta = -\frac{35}{12}$

Activity 2.2.1.8

- $\sin\theta = -\frac{\sqrt{11}}{6}$
- $\cos\theta = -\frac{5}{6}$

- $\tan\theta = \dfrac{\sqrt{11}}{5}$
- $\sec\theta = -\dfrac{6}{5}$
- $\csc\theta = -\dfrac{6}{\sqrt{11}}$ or $-\dfrac{6\sqrt{11}}{11}$
- $\cot\theta = \dfrac{5}{\sqrt{11}}$ or $\dfrac{5\sqrt{11}}{11}$

Activity 2.2.1.8

- $\sin\theta = -\dfrac{5}{13}$
- $\cos\theta = \dfrac{12}{13}$
- $\tan\theta = -\dfrac{5}{12}$
- $\sec\theta = \dfrac{13}{12}$
- $\csc\theta = -\dfrac{13}{5}$
- $\cot\theta = -\dfrac{12}{5}$

Check Your Understanding 2.2.1.1

For $\theta = 315^\circ$ in Quadrant IV, the reference angle is:

$$360^\circ - 315^\circ = 45^\circ \tag{2.2.1.2}$$

For $\theta = -\dfrac{5\pi}{6}$ (coterminal with $\dfrac{7\pi}{6}$ in Quadrant III), the reference angle is the angle to the negative x-axis:

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6} \tag{2.2.1.3}$$

Check Your Understanding 2.2.1.1

For $\dfrac{2\pi}{3}$ in Quadrant II, the reference angle is $\dfrac{\pi}{3}$. Since sine is positive in Quadrant II:

$$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \tag{2.2.1.4}$$

For $-\dfrac{3\pi}{4}$ in Quadrant III, the reference angle is $\dfrac{\pi}{4}$. Since cosine is negative in Quadrant III:

$$\cos\left(-\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \tag{2.2.1.5}$$

Check Your Understanding 2.2.1.1

Using the equation of the unit circle $x^2 + y^2 = 1$:

$$x^2 + \left(-\frac{2}{5}\right)^2 = 1 \implies x^2 + \frac{4}{25} = 1 \implies x^2 = \frac{21}{25} \tag{2.2.1.6}$$

Since the point is in Quadrant III, the x-coordinate must be negative:

$$x = -\frac{\sqrt{21}}{5} \tag{2.2.1.7}$$

Now, evaluate $\tan\theta = \dfrac{y}{x}$:

$$\tan\theta = \frac{-2/5}{-\sqrt{21}/5} = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21} \tag{2.2.1.8}$$

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2.2.1.1: TR5 Online Exercises

TR5 — Use reference angles, signs, and definitions to determine exact values of trigonometric functions.	TR5 CheckIt Exercises

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2.2.2: TR5 FAQ and Resources

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2.3: Introduction to Properties of Sine and Cosine Graphs (PF1)

PF1 Learning Outcomes:

- Determine the basic properties of the graphs of sine and cosine, including amplitude, period, and phase shift.

2.3.1: Properties of Sine and Cosine Graphs (PF1)

Picture unrolling that clock face into a straight line — instead of watching the minute hand's height as it goes around a circle, plot that same height against time as it ticks forward. The result is the sine graph: it rises to a peak, falls to a trough, and repeats every time the hand completes a full revolution. How high the peak reaches is the amplitude; how long one full revolution takes is the period; and if the clock didn't start at the top of the hour, that starting offset is the phase shift.

This directly extends Section 2.2's definition of $\sin\theta$ as a y-coordinate on the unit circle — you're graphing exactly that value, just against θ instead of watching it move around a circle. And underneath, amplitude, period, and phase shift are the same vertical stretch, horizontal stretch, and horizontal shift from Section 1.8's transformations, now applied to sine and cosine instead of a generic parent function.

Chapter 3 turns to a completely different kind of repeated behavior: instead of a quantity that cycles between a fixed high and low forever, Section 3.1 begins with exponential functions, where each step multiplies by the same factor and the quantity grows — or shrinks — without any such ceiling.

> **Looking ahead to calculus:** Every parameter you're identifying here — amplitude, period, phase shift — reappears the moment you take a derivative. Differentiating $A\sin(Bx - C)$ doesn't just flip sine to cosine; the B from the period calculation comes along for the ride and multiplies the whole result, a preview of the chain rule you'll meet properly once functions get nested inside each other like this.

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2.3.1: Properties of Sine and Cosine Graphs (PF1)

Learning Outcomes:

- Determine the basic properties of the graphs of sine and cosine, including amplitude, period, and phase shift.

Properties of Sine and Cosine Graphs (PF1)

Note

In the last module, we learned about finding values of trigonometric functions. Now, we will learn about the graphs of these functions.

Activity 2.3.1.1

We'll begin with the graph of the sine function, $f(x) = \sin x$.

(a)

Fill in the missing values in the table below for $f(x) = \sin x$. Find the exact values, then express as a decimal, approximated to two decimal places if needed. (Notice that the values in the table are all the standard angles found on the unit circle!)

x	$\sin x$ **(exact)**	$\sin x$ **(as a decimal)**
0		
$\frac{\pi}{6}$	$\frac{1}{2}$	
$\frac{\pi}{4}$		≈ 0.71
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	
$\frac{3\pi}{4}$		
$\frac{5\pi}{6}$		0.5
π		
$\frac{7\pi}{6}$	$-\frac{1}{2}$	
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	

$\frac{4\pi}{3}$		≈ -0.87
$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	
$\frac{7\pi}{4}$		
$\frac{11\pi}{6}$		-0.5
2π		

Answer available at the end of this section.

(b)

Plot these values on a coordinate plane to approximate the graph of $f(x) = \sin x$. Then sketch in the graph of the sine curve using the points as a guide.

Answer available at the end of this section.

(c)

What is the range of the function?

Answer available at the end of this section.

Activity 2.3.1.2

Let's change our function a bit and look at $g(x) = 3\sin x$.

(a)

Fill in the table below.

x	$f(x) = \sin x$	$g(x) = 3\sin x$
0		
$\frac{\pi}{6}$		
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		

Answer available at the end of this section.

(b)

Which of the following best describes how $g(x)$ is related to $f(x) = \sin x$?

A. The x-values in $g(x)$ are three times the x-values of $f(x)$.
B. The x-values in $g(x)$ are one third of the x-values of $f(x)$.
C. The y-values in $g(x)$ are three times the y-values of $f(x)$.
D. The y-values in $g(x)$ are one third of the y-values of $f(x)$.

Answer available at the end of this section.

(c)

What is the range of $g(x)$?

Answer available at the end of this section.

Definition: Amplitude

The **amplitude** of a sine curve is the vertical distance from the center of the curve to the maximum (or minimum) value.

We can also think of the **amplitude** as the value of the vertical stretch or compression.

When written as a function $f(x) = A \sin x$, the **amplitude** is $|A|$.

Activity 2.3.1.3

(a)

We only found $f(x) = \sin x$ for some values of x in the table in *(see related activity)*, but those did not represent the entire domain. For which values of x can you find $\sin x$? (That is, what is the domain of $f(x) = \sin x$?)

Answer available at the end of this section.

(b)

Coterminal angles will have the same sine values. How do we know if two angles are coterminal?

A. The difference between them is a multiple of $\frac{\pi}{2}$.
B. The difference between them is a multiple of π.
C. The difference between them is a multiple of $\frac{3\pi}{2}$.
D. The difference between them is a multiple of 2π.

Answer available at the end of this section.

(c)

How often will the sine values repeat?

A. Every $\frac{\pi}{2}$ radians.
B. Every π radians.
C. Every $\frac{2\pi}{2}$ radians.
D. Every 2π radians.

Answer available at the end of this section.

(d)

Extend the graph you made in *(see related activity)* in both the positive and negative direction to show the repeated sine values.

Answer available at the end of this section.

Definition: Period

The **period** of a sine function is the minimum value for which the y-values begin repeating.

The **period** for $f(x) = \sin x$, the standard sine curve, is 2π.

Activity 2.3.1.4

Now let's look at $h(x) = \sin 2x$.

(a)

Think back to the types of transformations a function can have. (See *(see related activity)* if you need a reminder!) What kind of transformation is happening in $h(x)$ compared the parent function $f(x) = \sin x$?

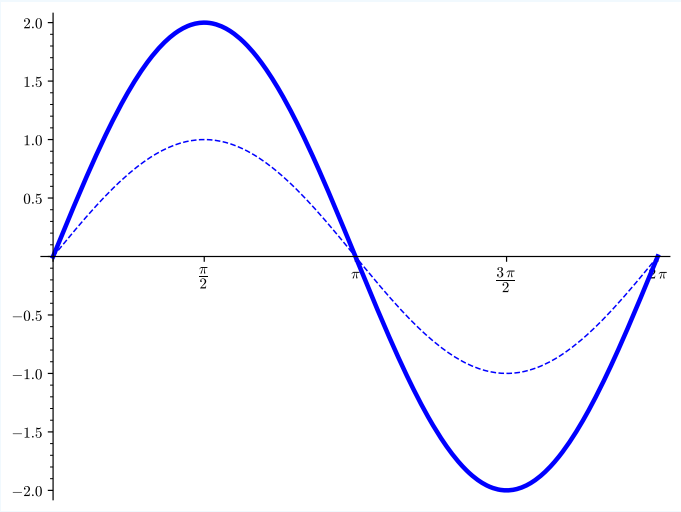
A. A vertical stretch/compression.
B. A horizontal stretch/compression.
C. A vertical shift.
D. A horizontal shift.

Answer available at the end of this section.

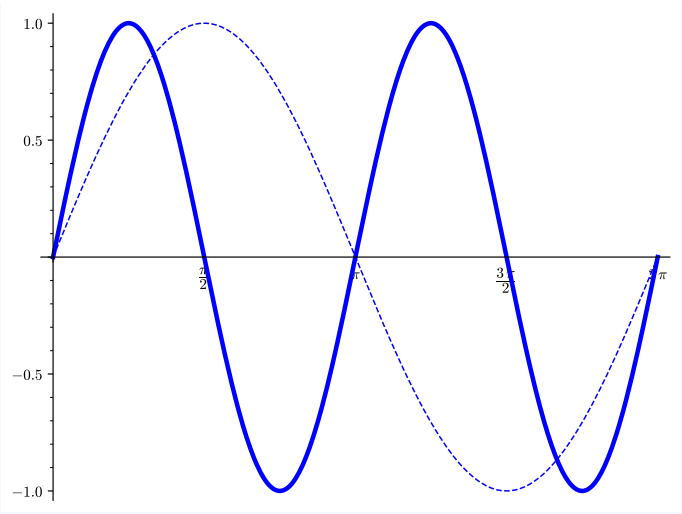
(b)

Which of the following graphs represents one cycle of $h(x) = \sin 2x$. (To help compare the functions, one cycle of $f(x) = \sin x$ is shown as a dashed line on each graph.)

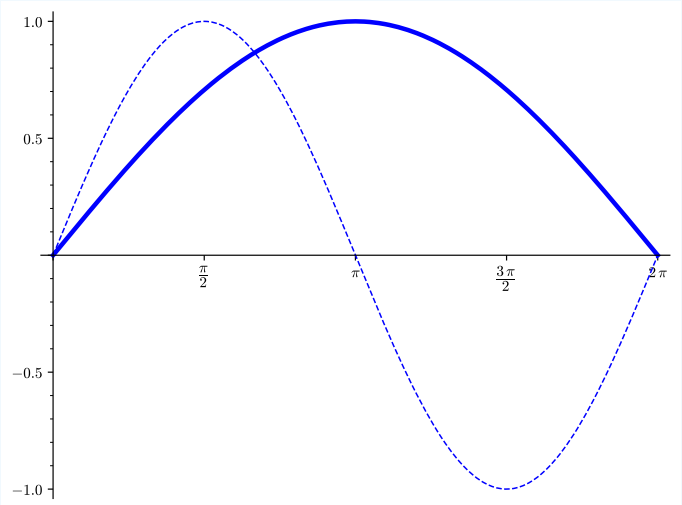
A.



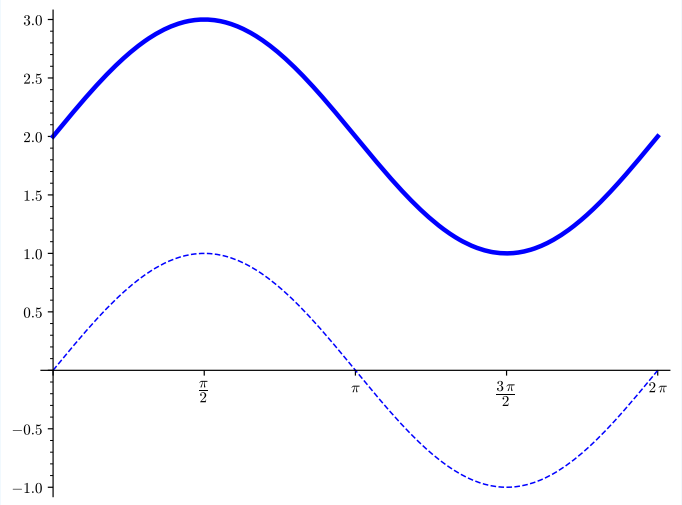
B.



C.



D.



Answer available at the end of this section.

Activity 2.3.1.5

Consider $j(x) = \sin \frac{1}{2}x$.

(a)

What type of transformation is happening in $j(x)$ compared the parent function $f(x) = \sin x$?

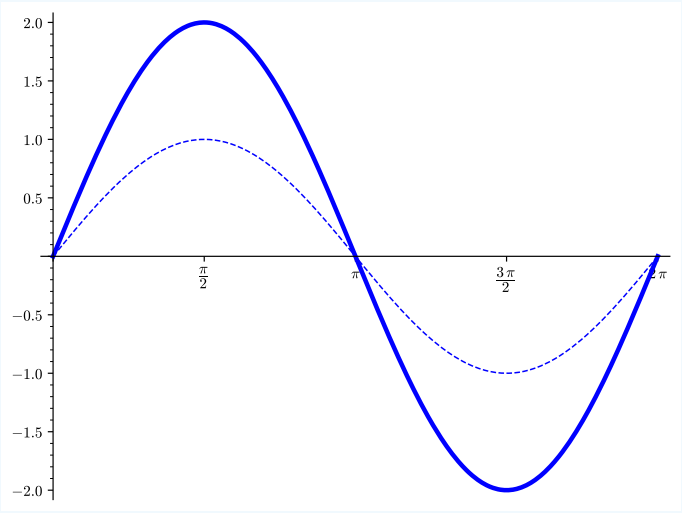
A. A vertical stretch/compression.
B. A horizontal stretch/compression.
C. A vertical shift.
D. A horizontal shift.

Answer available at the end of this section.

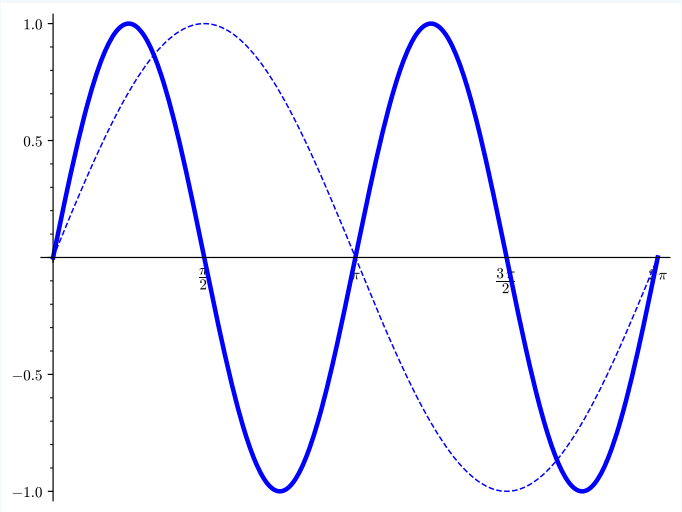
(b)

Which of the following graphs represents one cycle of $j(x) = \sin \frac{1}{2}x$. (To help compare the functions, one cycle of $f(x) = \sin x$ is shown as a dashed line on each graph.)

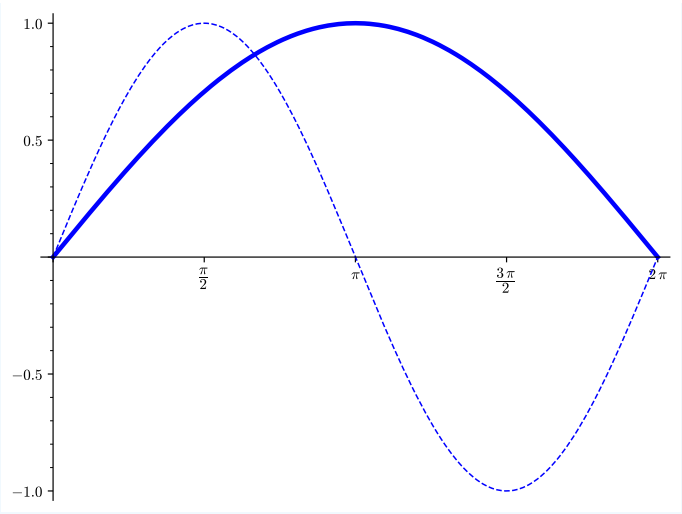
A.



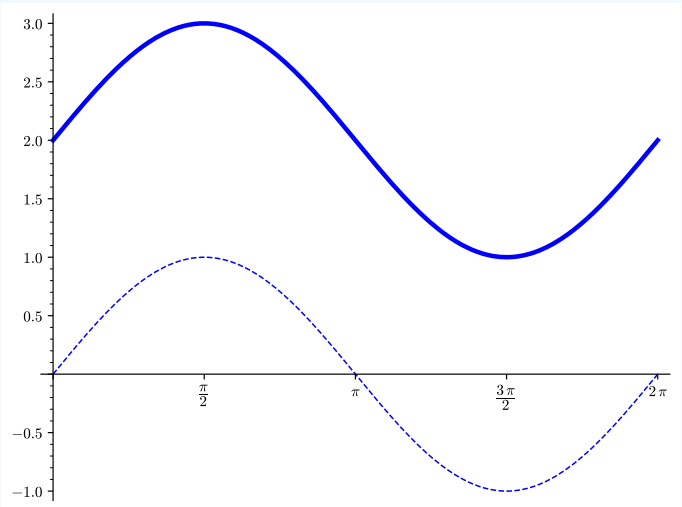
B.



C.



D.



Answer available at the end of this section.

Note

When written as a function $f(x) = \sin Bx$, the **period** is $\frac{2\pi}{|B|}$.

Activity 2.3.1.6

Consider $k(x) = \sin\left(x + \frac{\pi}{2}\right)$.

(a)

What type of transformation is happening in $k(x)$ compared the parent function $f(x) = \sin x$?

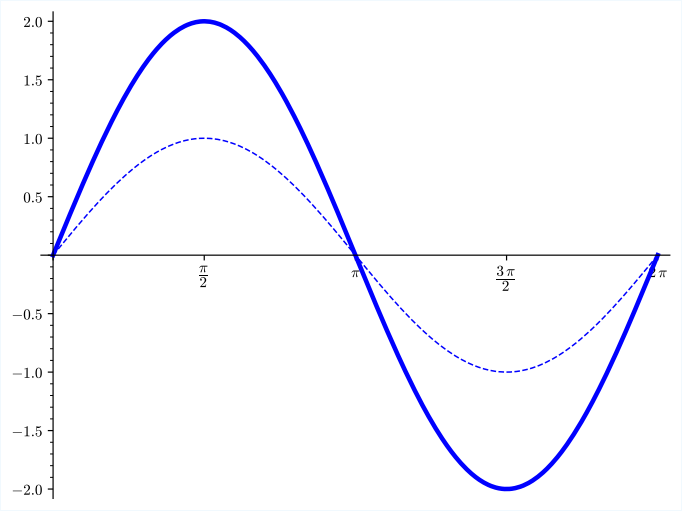
A. A vertical stretch/compression.
B. A horizontal stretch/compression.
C. A vertical shift.
D. A horizontal shift.

Answer available at the end of this section.

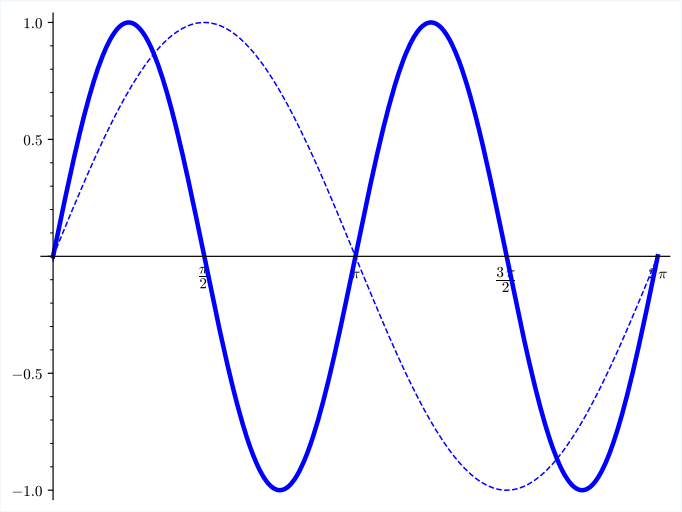
(b)

Which of the following graphs represents one cycle of $k(x) = \sin\left(x + \frac{\pi}{2}\right)$. (To help compare the functions, one cycle of $f(x) = \sin x$ is shown as a dashed line on each graph.)

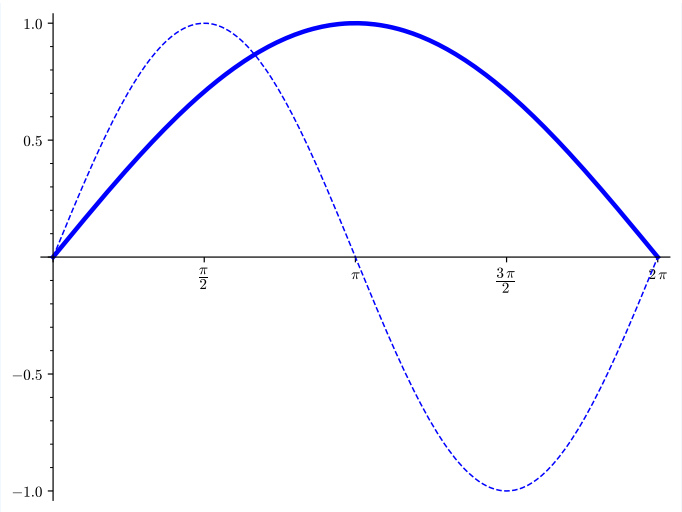
A.



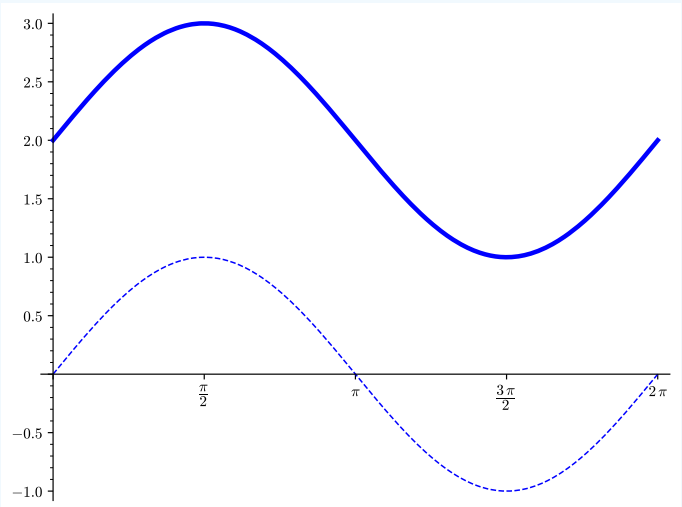
B.



C.



D.



Answer available at the end of this section.

Definition: Phase shift

The **phase shift** is the amount which a sine function is shifted horizontally from the standard sine curve.

The **phase shift** for $f(x) = \sin(x - C)$ is C, or C units to the right. The **phase shift** for $f(x) = \sin(x + C)$ is $-C$, or C units to the left.

A function can have both a horizontal shift and a change in period. In that case, it could be written as $f(x) = \sin(Bx - C)$. Here the phase shift would be $\frac{C}{B}$. You can think of solving the equation $Bx - C = 0$ for x. A positive value would represent a shift to the right and a negative value would represent a shift to the left.

Activity 2.3.1.7

Let's now turn our focus to the cosine function, $f(x) = \cos x$.

(a)

Fill in the missing values in the table below for $f(x) = \cos x$. Find the exact values, then express as a decimal, approximated to two decimal places if needed. (Notice that the values in the table are all the standard angles found on the unit circle!)

x	$\cos x$ **(exact)**	$\cos x$ **(as a decimal)**
0		
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	
$\frac{\pi}{4}$		≈ 0.71
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		0
$\frac{2\pi}{3}$	$\frac{-1}{2}$	
$\frac{3\pi}{4}$		
$\frac{5\pi}{6}$		$\approx -.87$
π		
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	
$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	
$\frac{4\pi}{3}$		-0.5
$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	$\frac{1}{2}$	
$\frac{7\pi}{4}$		
$\frac{11\pi}{6}$		≈ 0.87
2π		

Answer available at the end of this section.

(b)

Plot these values on a coordinate plane to approximate the graph of $f(x) = \cos x$. Then sketch in the graph of the cosine curve using the points as a guide.

Answer available at the end of this section.

(c)

What is the range of the function?

Answer available at the end of this section.

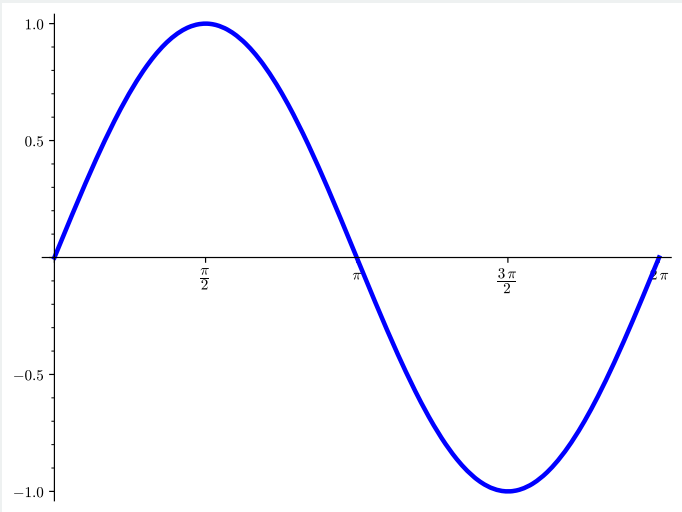
Note

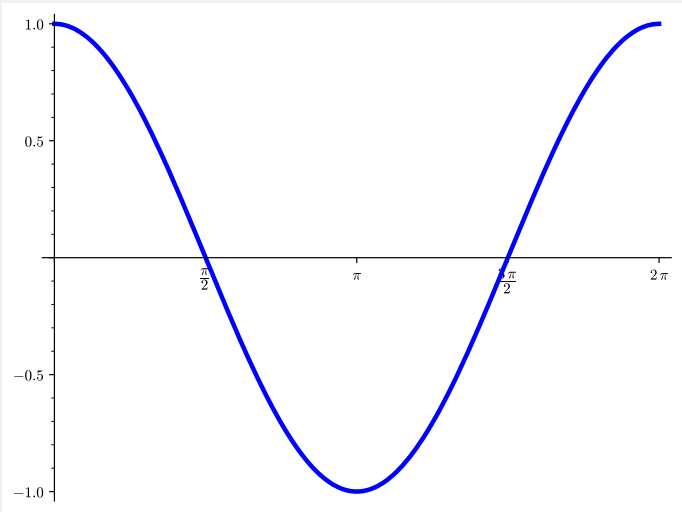
The cosine function, $f(x) = \cos x$, is equivalent to the sine function shifted to the left $\frac{\pi}{2}$ units, $g(x) = \sin\left(x + \frac{\pi}{2}\right)$.

Because of this, all of the methods we used to find amplitude, period, and phase shift for the sine function apply to the cosine function as well.

Note

Now that we can graph both the standard sine and cosine curves, we can add them to our list of parent functions in *(see related activity)*. We also show them graphed below on the interval $[0, 2\pi]$.

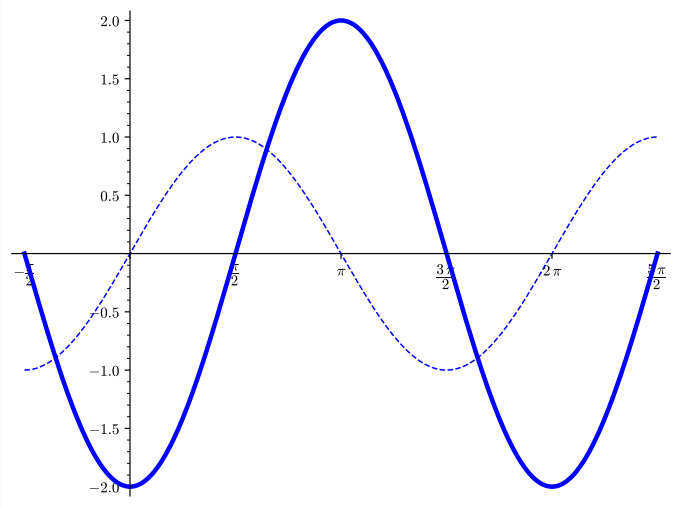




Activity 2.3.1.8

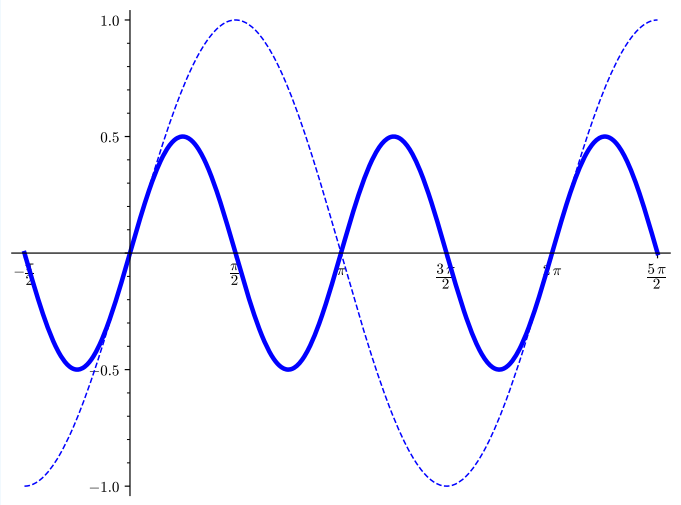
Find the amplitude, period, and phase shift of each of the following sine functions shown as a solid line. To help, $f(x) = \sin x$ is shown as a dotted line.

(a)



Answer available at the end of this section.

(b)



Answer available at the end of this section.

Activity 2.3.1.9

Find the amplitude, period, and phase shift of each of the following functions.

(a)

$f(x) = \frac{3}{2}\cos(x + \pi)$

Answer available at the end of this section.

(b)

$f(x) = 3\sin(2x - \pi)$

Answer available at the end of this section.

(c)

$f(x) = 4\sin\left(3x + \frac{\pi}{2}\right)$

Answer available at the end of this section.

(d)

$f(x) = \frac{1}{2}\cos\left(2x + \frac{\pi}{4}\right)$

Answer available at the end of this section.

Activity 2.3.1.10

Find an expression for each of the following transformations of the sine and cosine functions.

(a)

$\sin(x)$ with a phase shift left by π and an amplitude of 3.

Answer available at the end of this section.

(b)

$\cos(x)$ with a phase shift right by $\frac{\pi}{2}$ and a period of π.

Answer available at the end of this section.

(c)

$\sin(x)$ with a phase shift right by $\frac{\pi}{6}$, a period of $\frac{\pi}{3}$, and an amplitude of 5.

Answer available at the end of this section.

Activity 2.3.1.11

To graph $f(x) = 3\sin(2x - \pi) - 3$, lets apply one transformation at a time.

(a)

Which of the following is equivalent to $f(x) = 3\sin(2x - \pi) - 3$?

A. $3\sin\left(2\left(x - \pi\right)\right) - 3$
B. $3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) - 3$
C. $3\sin\left(2\left(x - \frac{\pi}{3}\right)\right) - 2$
D. $3\sin\left(2\left(x - 2\pi\right)\right) - 3$

Answer available at the end of this section.

(b)

Graph the function $f_1(x) = \sin(2x)$.

Answer available at the end of this section.

(c)

Graph the function $f_2(x) = \sin\left(2\left(x - \frac{\pi}{2}\right)\right)$.

Answer available at the end of this section.

(d)

Graph the function $f_3(x) = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right)$.

Answer available at the end of this section.

(e)

Graph the function $f_4(x) = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right) - 3$.

Answer available at the end of this section.

Activity 2.3.1.12

Graph each of the following functions.

(a)

$2\sin(3x - \pi) + 4$

Answer available at the end of this section.

(b)

$4\cos(2x + 2\pi) + 1$

Answer available at the end of this section.

(c)

$-3\sin\left(4x - \frac{\pi}{2}\right) + 1$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 2.3.1.1

Solve the following problems on the features of trigonometric graphs and transformations.

(a) Find the amplitude, midline, and period of the function $f(x) = -3\sin(2x) + 4$.

Answer available at the end of this section.

(b) Write the equation of a cosine function that has amplitude 5, period 8, is shifted right 3 units, and is shifted down 1 unit.

Answer available at the end of this section.

(c) Identify the domain, range, and midline of the function $g(x) = 2\cos(x) - 3$.

Answer available at the end of this section.

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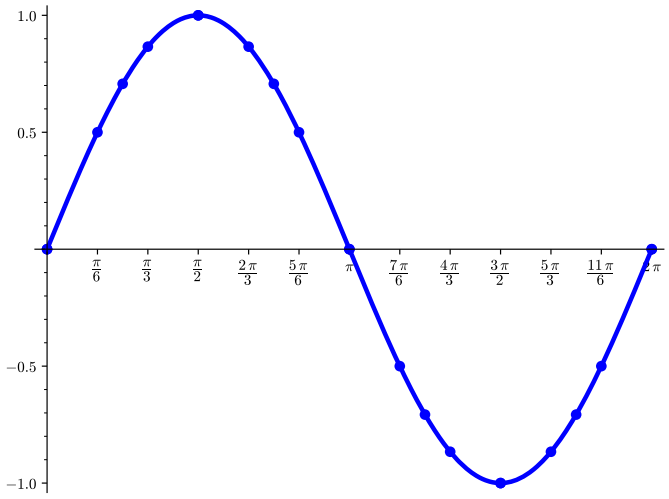
Answers

Activity 2.3.1.1

x	$\sin x$ **(exact)**	$\sin x$ **(as a decimal)**
0	0	0
$\frac{\pi}{6}$	$\frac{1}{2}$	0.5
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	≈ 0.71
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	≈ 0.87
$\frac{\pi}{2}$	1	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	≈ 0.87
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	≈ 0.71
$\frac{5\pi}{6}$	$\frac{1}{2}$	0.5
π	0	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$	-0.5
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	≈ -0.71

$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	≈ -0.87
$\frac{3\pi}{2}$	-1	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	≈ -0.87
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	≈ -0.71
$\frac{11\pi}{6}$	$-\frac{1}{2}$	-0.5
2π	0	0

Activity 2.3.1.1



Activity 2.3.1.1

$[-1, 1]$

Activity 2.3.1.2

x	$f(x) = \sin x$	$g(x) = 3\sin x$
0	0	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{3}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{3\sqrt{3}}{2}$
$\frac{\pi}{2}$	1	3

Activity 2.3.1.2

C

Activity 2.3.1.2

$[-3, 3]$

Activity 2.3.1.3

$(-\infty, \infty)$

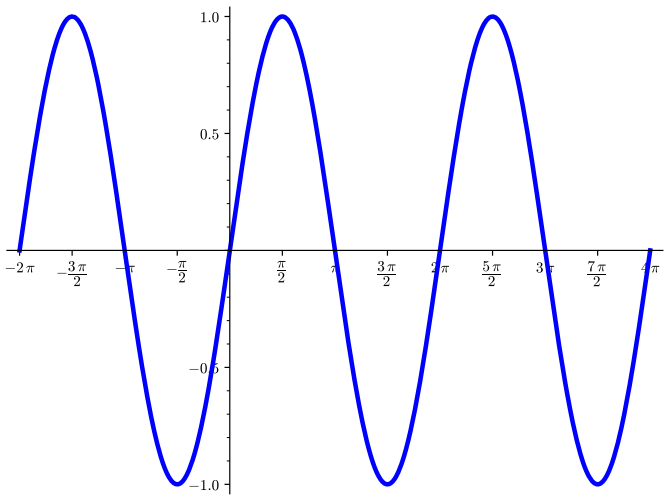
Activity 2.3.1.3

D

Activity 2.3.1.3

D

Activity 2.3.1.3



Activity 2.3.1.4

B

Activity 2.3.1.4

B

Activity 2.3.1.5

B

Activity 2.3.1.5

C

Activity 2.3.1.6

D

Activity 2.3.1.6

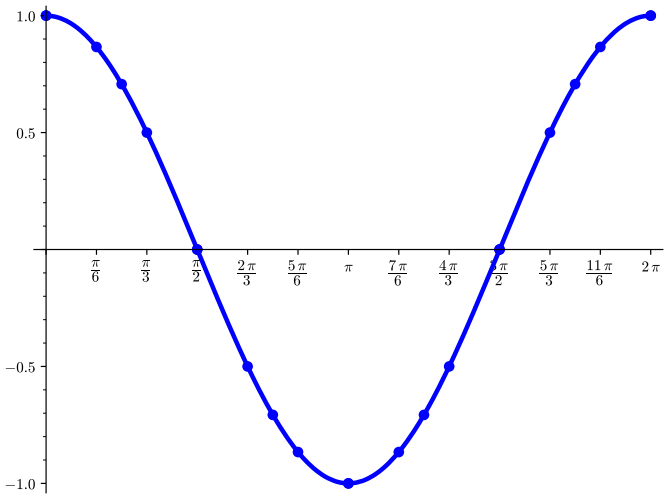
B

Activity 2.3.1.7

x	$\cos x$ **(exact)**	$\cos x$ **(as a decimal)**
0	1	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	≈ 0.87
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	≈ 0.71
$\frac{\pi}{3}$	$\frac{1}{2}$	0.5
$\frac{\pi}{2}$	0	0
$\frac{2\pi}{3}$	$\frac{-1}{2}$	-0.5

$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	≈ 0.71
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\approx -.87$
π	-1	-1
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	≈ -0.87
$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	≈ -0.71
$\frac{4\pi}{3}$	$-\frac{1}{2}$	-0.5
$\frac{3\pi}{2}$	0	0
$\frac{5\pi}{3}$	$\frac{1}{2}$	0.5
$\frac{7\pi}{4}$	$\frac{2}{2}$	≈ 0.71
$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	≈ 0.87
2π	1	1

Activity 2.3.1.7



Activity 2.3.1.7

$[-1, 1]$

Activity 2.3.1.8

Amplitude: 2

Period: 2π

Phase shift: $\frac{\pi}{2}$ or $-\frac{3\pi}{2}$

Activity 2.3.1.8

Amplitude: $\frac{1}{2}$

Period: π

Phase shift: $-\frac{\pi}{2}$ or $\frac{\pi}{2}$

Activity 2.3.1.9

Amplitude: $\frac{3}{2}$

Period: 2π

Phase shift: $-\pi$ (or π to the left)

Activity 2.3.1.9

Amplitude: 3

Period: π

Phase shift: $\frac{\pi}{2}$ (or $\frac{\pi}{2}$ to the right)

Activity 2.3.1.9

Amplitude: 4

Period: $\frac{2\pi}{3}$

Phase shift: $-\frac{\pi}{6}$ (or $\frac{\pi}{6}$ to the left)

Activity 2.3.1.9

Amplitude: $\frac{1}{2}$

Period: π

Phase shift: $-\frac{\pi}{8}$ (or $\frac{\pi}{8}$ to the left)

Activity 2.3.1.10

$3\sin(x+\pi)$

Activity 2.3.1.10

$\cos\left(2\left(x-\frac{\pi}{2}\right)\right)$

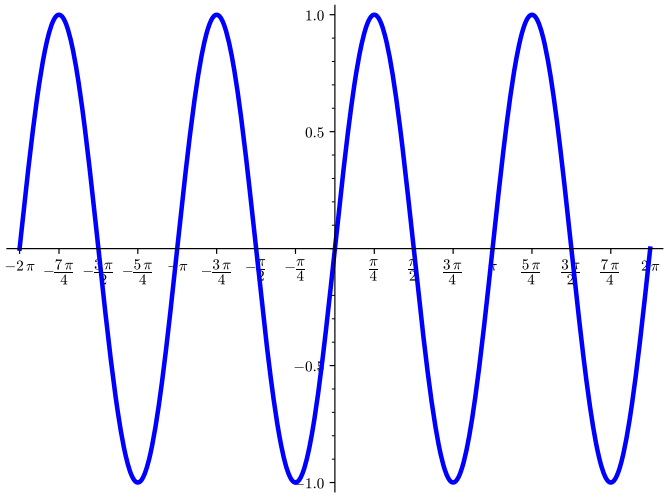
Activity 2.3.1.10

$5\sin\left(6\left(x+\frac{\pi}{6}\right)\right)$

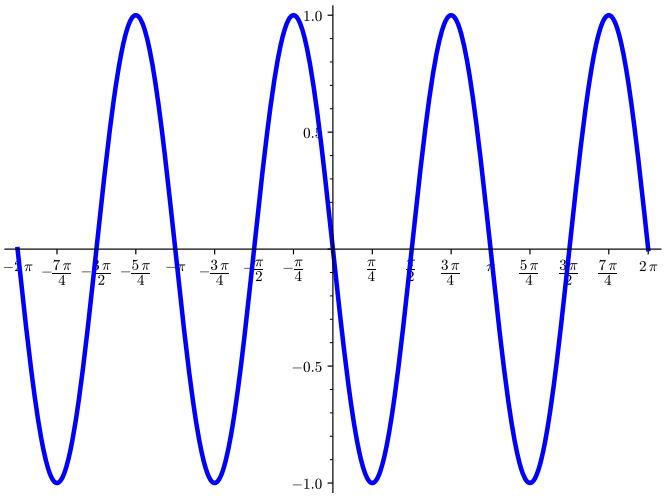
Activity 2.3.1.11

B.

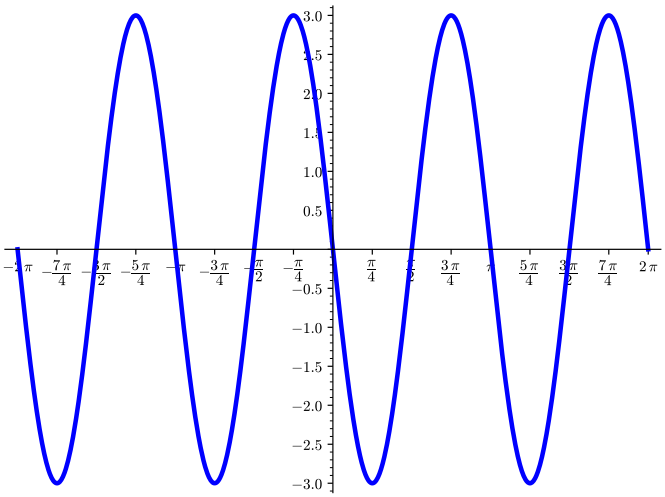
Activity 2.3.1.11



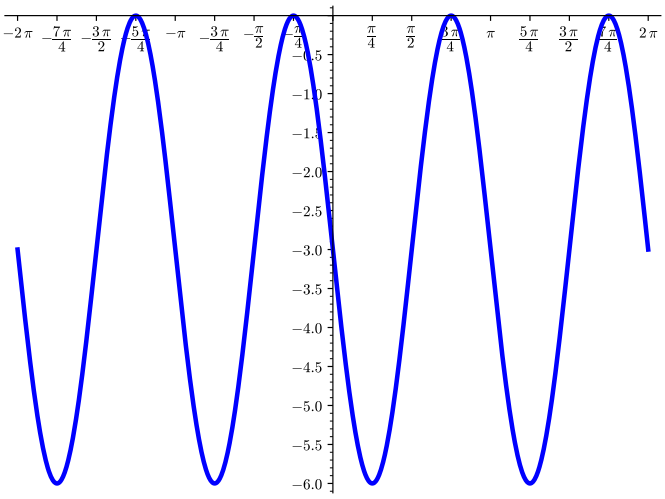
Activity 2.3.1.11



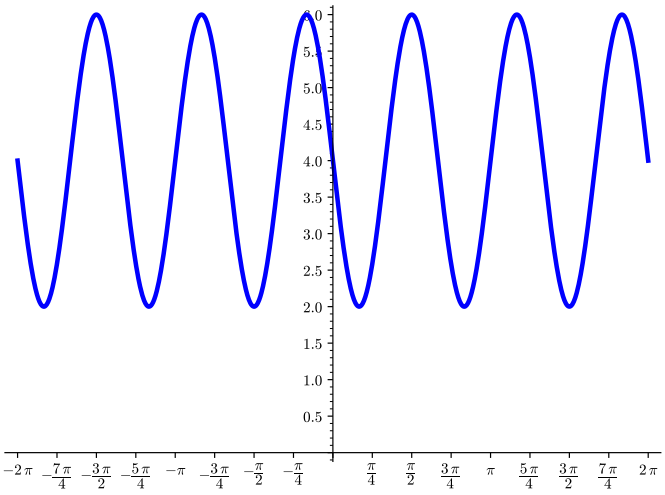
Activity 2.3.1.11



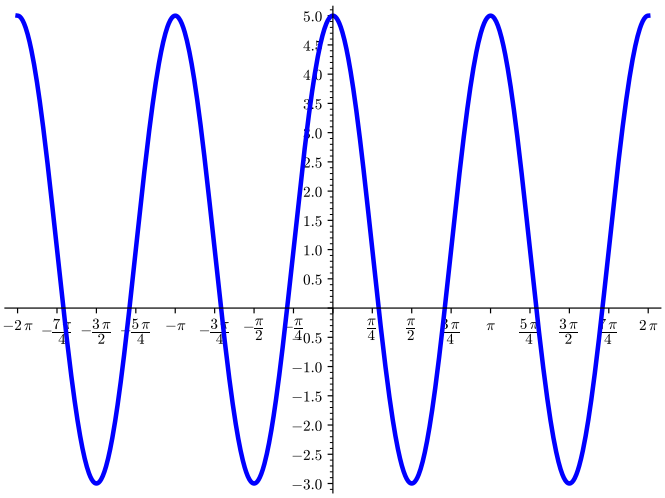
Activity 2.3.1.11



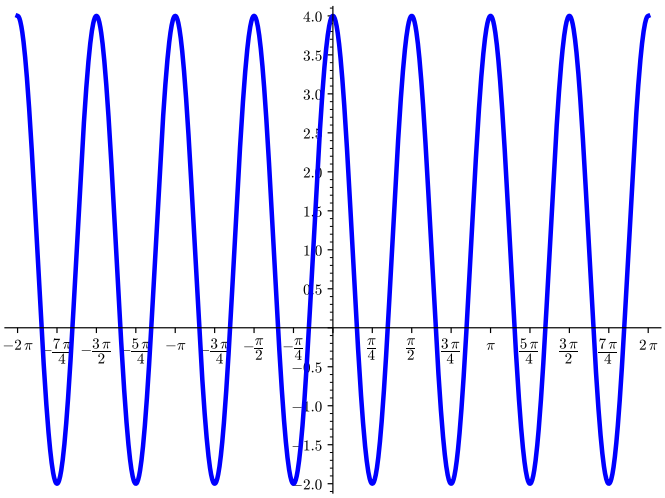
Activity 2.3.1.12



Activity 2.3.1.12



Activity 2.3.1.12



Check Your Understanding 2.3.1.1

Comparing $f(x) = -3\sin(2x) + 4$ to the general form $f(x) = A\sin(B(x-h)) + k$:

- **Midline:** The vertical shift is $k = 4$, so the midline is $y = 4$.
- **Amplitude:** The amplitude is $|A| = |-3| = 3$.
- **Period:** The horizontal frequency parameter is $B = 2$. The period is:

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi \tag{2.3.1.1}$$

Check Your Understanding 2.3.1.1

We use the general form $f(x) = A\cos(B(x-h)) + k$:

- **Amplitude:** $|A| = 5 \implies A = 5$ (assuming positive orientation).
- **Period:** The period is 8, so we solve for B:

$$8 = \frac{2\pi}{B} \implies B = \frac{2\pi}{8} = \frac{\pi}{4} \tag{2.3.1.2}$$

- **Horizontal Shift (Phase Shift):** Shifted right 3 units, so $h = 3$.
- **Vertical Shift (Midline):** Shifted down 1 unit, so $k = -1$.

Putting it all together, we get:

$$f(x) = 5\cos\left(\frac{\pi}{4}(x-3)\right) - 1 \tag{2.3.1.3}$$

Check Your Understanding 2.3.1.1

For the function $g(x) = 2\cos(x) - 3$:

- **Domain:** The cosine function is defined for all real numbers, so the domain is $(-\infty, \infty)$.
- **Midline:** The vertical shift is $k = -3$, so the midline is $y = -3$.
- **Range:** The amplitude is 2, so the function oscillates 2 units above and below the midline $y = -3$:

$$\text{Minimum} = -3 - 2 = -5 \tag{2.3.1.4}$$

$$\text{Maximum} = -3 + 2 = -1 \tag{2.3.1.5}$$

Thus, the range is $[-5, -1]$.

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2.3.1.1: PF1 Online Exercises

PF1 — Determine the basic properties of the graphs of sine and cosine, including amplitude, period, and phase shift.	PF1 CheckIt Exercises

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2.3.2: PF1 FAQ and Resources

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3: CONTINUOUS GROWTH WITH EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Many real-world phenomena are nonlinear, and exponential and logarithmic functions reveal a distinct pattern of growth, decay, and constraint that linear and quadratic models cannot capture. In this unit, students construct and interpret exponential and logarithmic functions, examine their inverse relationship, and solve equations that involve them — skills essential for modeling growth processes in science, finance, and technology.

3.1: Introduction to Exponentials (EL1)

3.1.1: Exponentials (EL1)

3.1.1.1: EL1 Online Exercises

3.1.2: EL1 FAQ and Resources

3.2: Introduction to Graphs of Exponential Functions (EL2)

3.2.1: Graphs of Exponential Functions (EL2)

3.2.1.1: EL2 Online Exercises

3.3: Introduction to Logarithms (EL3)

3.3.1: Logarithms (EL3)

3.3.1.1: EL3 Online Exercises

3.4: Introduction to Graphs of Logarithmic Functions (EL4)

3.4.1: Graphs of Logarithmic Functions (EL4)

3.4.1.1: EL4 Online Exercises

3.5: Introduction to Properties and Applications of Logarithms (EL5 and EL6)

3.5.1: Properties of Logarithms

3.5.1.1: EL5 Online Exercises

3.5.2: Solving Exponential and Logarithmic Equations

3.5.2.1: EL6 Online Exercises

3.6: Modeling with Exponential and Logarithmic Functions (EL7)

3.6.1: Modeling with Exponential and Logarithmic Functions (EL7)

3.6.1.1: EL7 Online Exercises

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3.1: Introduction to Exponentials (EL1)

3.1: Introduction to Exponentials (EL1)

EL1: Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base).

A single sheet of paper is about 0.1 millimeters thick. Fold it in half and it's 0.2 millimeters; fold it again, 0.4. That doesn't sound like much — but keep folding, doubling the thickness every time, and by the 42nd fold the stack would reach the moon. That's the signature of exponential growth: instead of adding the same amount each step, you multiply by the same factor. A function $f(x) = ab^x$ captures this — a is where you start, b is the factor multiplied at every step, and whether b is bigger or smaller than 1 determines growth or decay.

This is the natural counterpart to Section 1.3, where a linear function's defining feature was a constant difference — the same amount added for every step. An exponential function is defined by a constant ratio instead — the same factor multiplied for every step — and that single change is what separates steady, additive growth from the runaway growth of a folding sheet of paper.

Section 3.2 turns this into a picture: graphing $f(x) = ab^x$ reveals a horizontal asymptote, a domain of all real numbers, and a range that depends entirely on whether the function is growing or decaying — with the transformation vocabulary from Section 1.8 (shifts, reflections, stretches) doing most of the work.

> **Looking ahead to calculus:** An exponential growth function doesn't just increase — it increases at an increasing rate, climbing more steeply the further along it gets. That seems obvious from a graph, but proving it rigorously takes calculus. Exponential functions are unique in another way calculus cares about deeply: the natural base e is defined so that the function e^x is its own rate of change — a property no other function has.

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3.1.1: Exponentials (EL1)

Learning Outcomes:

- Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base e).

Introduction to Exponentials (EL1)

Note

Linear functions have a constant rate of change - that is a constant change in output for every change in input. Let's consider functions which do not fit this model - those which grow more rapidly and change by a varying amount for every change in input.

Activity 3.1.1.1

You have two job offers on the horizon. One has offered to pay you $\$10{,}000$ per month while the other is offering $\$0.01$ the first month, $\$0.02$ the second month, $\$0.04$ the third month and doubles every month. Which job would you rather take?

(a)

Make a table representing how much money you will be paid each month for the first two years from the first job - paying $\$10{,}000$ per month.

Answer available at the end of this section.

(b)

Make a table representing how much money you will be paid each month for the first two years from the second job - paying $\$0.01$ the first month and doubling every month after.

Answer available at the end of this section.

(c)

Which job is earning more money per month after one year?

Answer available at the end of this section.

(d)

Which job is earning more money per month after 18 months?

Answer available at the end of this section.

(e)

According to your tables, does the second job ever earn more money per month than the first job?

Answer available at the end of this section.

Note

This idea of a function that grows very rapidly by a factor, ratio, or percent each time, like the second job in *(see related activity)*, is considered exponential growth.

Definition: Exponential function

Let a be a non-zero real number and $b \neq 1$ a positive real number. An **exponential function** takes the form

$$f(x) = ab^x \tag{3.1.1.1}$$

where a is the initial value and b is the base.

Activity 3.1.1.2

Evaluate the following exponential functions.

(a)

$f(x) = 4^x$ for $f(3)$

Answer available at the end of this section.

(b)

$f(x) = \left(\frac{1}{3}\right)^x$ for $f(3)$

Answer available at the end of this section.

(c)

$f(x) = 3 \cdot (5)^x$ for $f(-2)$

Answer available at the end of this section.

(d)

$f(x) = -2^{3x-4}$ for $f(4)$

Answer available at the end of this section.

Note

Notice that in *(see related activity)* part (a) the output value is larger than the base, representing an increasing function, while in part (b) the output value is smaller than the base.

Activity 3.1.1.3

Consider two exponential functions $f(x) = 100(2)^x$ and $g(x) = 100\left(\frac{1}{2}\right)^x$.

(a)

Fill in the table of values for $f(x)$.

x	$f(x)$
0	
1	
2	
3	
4	

Answer available at the end of this section.

(b)

Fill in the table of values for $g(x)$.

x	$g(x)$
0	
1	
2	
3	
4	

Answer available at the end of this section.

(c)

How do the values in the tables compare?

Answer available at the end of this section.

Note

In *(see related activity)*, the only difference between the two exponential functions was the base. $f(x)$ has a base of 2, while $g(x)$ has a base of $\frac{1}{2}$. Let's use this fact to update *(see related activity)*.

Note

An exponential function of the form $f(x) = ab^x$ will grow (or increase) if $b > 1$ and decay (or decrease) if $0 < b < 1$.

Activity 3.1.1.4

For each year t, the population of a certain type of tree in a forest is represented by the function $F(t) = 856 \cdot (0.93)^t$.

(a)

How many of that certain type of tree are in the forest initially?

Answer available at the end of this section.

(b)

Is the number of trees of that type growing or decaying?

Answer available at the end of this section.

Activity 3.1.1.5

To begin creating equations for exponential functions using a and b, let's compare a linear function and an exponential function. The tables show outputs for two different functions r and s that correspond to equally spaced input.

x	$r(x)$	x	$s(x)$
0	12	0	12
3	10	3	9
6	8	6	6.75
9	6	9	5.0625

(a)

Which function is linear?

Answer available at the end of this section.

(b)

What is the initial value of the linear function?

Answer available at the end of this section.

(c)

What is the slope of the linear function?

Answer available at the end of this section.

(d)

What is the initial value of the exponential function?

Answer available at the end of this section.

(e)

What is the ratio of consecutive outputs in the exponential function?

A. $\frac{4}{3}$
B. $\frac{3}{4}$
C. $-\frac{4}{3}$
D. $-\frac{3}{4}$

Answer available at the end of this section.

Note

In a linear function the differences are constant, while in an exponential function the ratios are constant.

Activity 3.1.1.6

Find an equation for an exponential function passing through the points $(0, 4)$ and $(1, 6)$.

(a)

Find the initial value.

A. 0
B. 4
C. 1
D. 6

Answer available at the end of this section.

(b)

Find the common ratio.

A. $\frac{3}{2}$
B. 6
C. $\frac{2}{3}$
D. $\frac{1}{6}$

Answer available at the end of this section.

(c)

Find the equation.

A. $f(x) = 6\left(\frac{3}{2}\right)^x$
B. $f(x) = 4 \cdot (6)^x$
C. $f(x) = 4\left(\frac{2}{3}\right)^x$
D. $f(x) = 4 \cdot \left(\frac{3}{2}\right)^x$

Answer available at the end of this section.

Note

Recall the negative rule of exponents which states that for any nonzero real number a and natural number n

$$a^{-n} = \frac{1}{a^n} \tag{3.1.1.2}$$

Activity 3.1.1.7

Let's consider the two exponential functions $f(x) = 2^{-x}$ and $g(x) = \left(\frac{1}{2}\right)^x$.

(a)

Fill in the table of values for $f(x)$.

x	$f(x)$
-2	
-1	
0	
1	
2	

Answer available at the end of this section.

(b)

Fill in the table of values for $g(x)$.

x	$g(x)$
-2	
-1	
0	
1	
2	

Answer available at the end of this section.

(c)

What do you notice about the two functions?

Answer available at the end of this section.

(d)

Use *(see related activity)* and other properties of exponents to try and rewrite $f(x)$ as $g(x)$.

Answer available at the end of this section.

Note

Similar to how π arises naturally in geometry, there is an irrational number called e that arises naturally when working with exponentials. We usually use the approximation $e \approx 2.718282$. e is also found on most scientific and graphing calculators.

See *(see related activity)* for an exploration of how e arises naturally.

Activity 3.1.1.8

Use a calculator to evaluate the following exponentials involving the base e.

(a)

$f(x) = -2e^x - 2$ for $f(-2)$

A. -0.0366
B. -2.2707
C. -1.7293
D. -16.778

Answer available at the end of this section.

(b)

$f(x) = \frac{1}{3}e^{x+1}$ for $f(-1)$

A. 1
B. 0
C. 1.122
D. $\frac{1}{3}$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 3.1.1.1

Solve the following problems to test your understanding of exponential functions.

(a) Which of the following functions represents an exponential function? Explain your choice.

- $f(x) = 3x^2$
- $g(x) = 4(1.5)^x$
- $h(x) = 2x + 5$

Answer available at the end of this section.

(b) Find the equation of the exponential function $f(x) = a \cdot b^x$ that passes through the points $(0, 6)$ and $(2, 24)$.

Answer available at the end of this section.

(c) Evaluate the exponential function $f(x) = 150e^{-0.1x}$ at $x = 10$. Show both the exact value and a decimal approximation rounded to two decimal places.

Answer available at the end of this section.

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Answers

Activity 3.1.1.1

month	Job 1
1	10,000
2	10,000
3	10,000
4	10,000
5	10,000
6	10,000
7	10,000
8	10,000
9	10,000
10	10,000
11	10,000
12	10,000
13	10,000
14	10,000
15	10,000

16	10,000
17	10,000
18	10,000
19	10,000
20	10,000
21	10,000
22	10,000
23	10,000
24	10,000

Activity 3.1.1.1

month	Job 2
1	0.01
2	0.02
3	0.04
4	0.08
5	0.16
6	0.32
7	0.64
8	1.28
9	2.56
10	5.12
11	10.24
12	20.48
13	40.96
14	81.92
15	163.84
16	327.68

17	655.36
18	1,310.72
19	2,621.24
20	5,242.88
21	10,485.76
22	20,971.52
23	41,943.04
24	83,886.08

Activity 3.1.1.1

Job 1 is earning $\$10{,}000$ per month. Job 2 is earning $\$20.48$ per month.

Activity 3.1.1.1

Job 1 is earning $\$10{,}000$ per month. Job 2 is earning $\$1{,}310.72$ per month.

Activity 3.1.1.1

Yes! After 21 months, Job 1 is earning $\$10{,}000$ per month. Job 2 is earning $\$10{,}485.76$ per month.

Activity 3.1.1.2

$f(3) = 64$

Activity 3.1.1.2

$f(3) = \frac{1}{27}$

Activity 3.1.1.2

$f(-2) = \frac{3}{25}$

Activity 3.1.1.2

$f(4) = -256$

Activity 3.1.1.3

x	$f(x)$
0	100
1	200
2	400
3	800
4	1600

Activity 3.1.1.3

x	$f(x)$
0	100
1	50
2	25

3	12.5
4	6.25

Activity 3.1.1.3

The values of $f(x)$ are getting larger while those of $g(x)$ are getting smaller.

Activity 3.1.1.4

856 trees

Activity 3.1.1.4

Decaying $b = 0.93 < 1$

Activity 3.1.1.5

$r(x)$ since the outputs decrease by 2 every time.

Activity 3.1.1.5

12

Activity 3.1.1.5

$-\frac{2}{3}$

Activity 3.1.1.5

12

Activity 3.1.1.5

B

Activity 3.1.1.6

B

Activity 3.1.1.6

A

Activity 3.1.1.6

D: $f(x) = 4\left(\frac{3}{2}\right)^x$

Activity 3.1.1.7

x	$f(x)$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Activity 3.1.1.7

x	$g(x)$
-2	4
-1	2
0	1
1	$\frac{1}{2}$

2	$\frac{1}{4}$

Activity 3.1.1.7

Their values are the same.

Activity 3.1.1.7

$$f(x) = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x = g(x) \tag{3.1.1.3}$$

Activity 3.1.1.8

B

Activity 3.1.1.8

D

Check Your Understanding 3.1.1.1

The function $g(x) = 4(1.5)^x$ is an exponential function because it is in the form $y = a \cdot b^x$ where the base $b = 1.5 > 0$, $b \neq 1$, and the variable x is in the exponent. Meanwhile, $f(x)$ is a quadratic function (power function) and $h(x)$ is a linear function.

Check Your Understanding 3.1.1.1

Since the y-intercept is $(0, 6)$, we substitute $x = 0$ and $y = 6$:

$$6 = a \cdot b^0 \implies a = 6 \tag{3.1.1.4}$$

Using the point $(2, 24)$ with $a = 6$:

$$24 = 6 \cdot b^2 \implies b^2 = 4 \implies b = 2 \tag{3.1.1.5}$$

(Note that the base b must be positive, so we reject $b = -2$.)

Thus, the equation is $f(x) = 6(2)^x$.

Check Your Understanding 3.1.1.1

We substitute $x = 10$ into the function:

$$f(10) = 150e^{-0.1(10)} = 150e^{-1} = \frac{150}{e} \tag{3.1.1.6}$$

Using the approximation $e \approx 2.71828$:

$$f(10) \approx \frac{150}{2.71828} \approx 55.18 \tag{3.1.1.7}$$

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3.1.1.1: EL1 Online Exercises

EL1 — Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base e).	EL1 CheckIt Exercises

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3.1.2: EL1 FAQ and Resources

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3.2: Introduction to Graphs of Exponential Functions (EL2)

3.2: Introduction to Graphs of Exponential Functions (EL2)

EL2: Graph exponential functions and determine the domain, range, and asymptotes.

Picture graphing that paper-folding scenario from Section 3.1: thickness on the vertical axis, number of folds on the horizontal axis. For the first several folds, the graph barely lifts off the axis — the stack is still thinner than a fingernail. Then, almost without warning, it shoots upward, climbing faster and faster. That's the shape of every exponential growth graph: it hugs the x-axis on one side, seeming almost flat, and rockets upward on the other. The line it hugs but never touches is called a horizontal asymptote, and for the basic function $f(x) = b^x$, that asymptote is always $y = 0$.

This section leans hard on Section 1.8's transformation vocabulary, exactly as promised — shifting $f(x) = b^x$ up or down moves that asymptote away from $y = 0$, reflecting over the x-axis flips growth into a mirror image below it, and reflecting over the y-axis is what turns growth into decay (or the reverse) without changing the base at all.

Section 3.3 asks the reverse question: given the output of an exponential function, what input produced it? That question is exactly what a logarithm answers, and its graph — a reflection of this one across the line $y = x$ — will look strikingly familiar.

> **Looking ahead to calculus:** A horizontal asymptote is precalculus's way of describing something calculus makes precise with limits. Saying the graph of $f(x) = b^x$ "approaches $y = 0$ but never reaches it as x heads to negative infinity" is exactly what calculus means by writing $\lim_{x \to -\infty} b^x = 0$ — the same idea, just with notation built to handle far stranger asymptotic behavior than a simple exponential curve.

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3.2.1: Graphs of Exponential Functions (EL2)

Learning Outcomes:

- Graph exponential functions and determine the domain, range, and asymptotes.

Graphs of Exponential Functions (EL2)

Activity 3.2.1.1

Consider the function $f(x) = 2^x$.

(a)

Fill in the table of values for $f(x)$. Then plot the points on a graph.

x	$f(x)$
-2	
-1	
0	
1	
2	

Answer available at the end of this section.

(b)

What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $f(x) \to -\infty$.
B. As $x \to \infty$, $f(x) \to -2$.
C. As $x \to \infty$, $f(x) \to 0$.
D. As $x \to \infty$, $f(x) \to 2$.
E. As $x \to \infty$, $f(x) \to \infty$.

Answer available at the end of this section.

(c)

What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $f(x) \to -\infty$.
B. As $x \to -\infty$, $f(x) \to -2$.
C. As $x \to -\infty$, $f(x) \to 0$.
D. As $x \to -\infty$, $f(x) \to 2$.
E. As $x \to -\infty$, $f(x) \to \infty$.

Answer available at the end of this section.

(d)

Complete the graph you started in *(see related activity)*, connecting the points and including the end behavior you've just determined.

Answer available at the end of this section.

(e)

Does your graph seem to have any asymptotes?

A. No. There are no asymptotes.
B. There is a vertical asymptote but no horizontal one.
C. There is a horizontal asymptote but no vertical one.
D. The graph has both a horizontal and vertical asymptote.

Answer available at the end of this section.

(f)

What the equation for each asymptote of $f(x)$? Select all that apply.

A. There are no asymptotes.
B. $x = 0$
C. $x = 3$
D. $y = 0$
E. $y = 3$

Answer available at the end of this section.

(g)

Find the domain and range of $f(x)$. Write your answers using interval notation.

Answer available at the end of this section.

(h)

Find the interval(s) where $f(x)$ is increasing and the interval(s) where $f(x)$ is decreasing. Write your answers using interval notation.

Answer available at the end of this section.

Note

The graph of an exponential function $f(x) = b^x$ where $b > 1$ has the following characteristics:

- Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- It is an exponential growth function; that is it is increasing on $(-\infty, \infty)$.
- There is a horizontal asymptote at $y = 0$. There is no vertical asymptote.
- There is a y-intercept at $(0, 1)$. There is no x-intercept.

Activity 3.2.1.2

Consider the function $g(x) = \left(\frac{1}{2}\right)^x$.

(a)

Fill in the table of values for $g(x)$. Then plot the points on a graph.

x	$g(x)$
-2	
-1	
0	
1	
2	

Answer available at the end of this section.

(b)

What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $g(x) \to -\infty$.
B. As $x \to \infty$, $g(x) \to -2$.
C. As $x \to \infty$, $g(x) \to 0$.
D. As $x \to \infty$, $g(x) \to 2$.
E. As $x \to \infty$, $g(x) \to \infty$.

Answer available at the end of this section.

(c)

What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $g(x) \to -\infty$.
B. As $x \to -\infty$, $g(x) \to -2$.
C. As $x \to -\infty$, $g(x) \to 0$.
D. As $x \to -\infty$, $g(x) \to 2$.
E. As $x \to -\infty$, $g(x) \to \infty$.

Answer available at the end of this section.

(d)

Complete the graph you started in *(see related activity)*, connecting the points and including the end behavior you've just determined.

Answer available at the end of this section.

(e)

What are the equations of the asymptote(s) of the graph?

Answer available at the end of this section.

(f)

Find the domain and range of $f(x)$. Write your answers using interval notation.

Answer available at the end of this section.

(g)

Find the interval(s) where $f(x)$ is increasing and the interval(s) where $f(x)$ is decreasing. Write your answers using interval notation.

Answer available at the end of this section.

Activity 3.2.1.3

Consider the two exponential functions we've just graphed: $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$.

(a)

How are the graphs of $f(x)$ and $g(x)$ similar?

Answer available at the end of this section.

(b)

How are the graphs of $f(x)$ and $g(x)$ different?

Answer available at the end of this section.

Note

We can now update *(see related activity)* so that it includes all values of the base of an exponential function.

The graph of an exponential function $f(x) = b^x$ has the following characteristics:

- Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- Remember, exponential functions are only defined when $b > 0$ and $b \neq 1$ as we saw in *(see related activity)*.

  If $b > 1$, $f(x)$ is increasing on $(-\infty, \infty)$ and is an exponential growth function. If $0 < b < 1$, $f(x)$ is decreasing on $(-\infty, \infty)$ and is an exponential decay function.

- There is a horizontal asymptote at $y = 0$. There is no vertical asymptote.
- There is a y-intercept at $(0, 1)$. There is no x-intercept.

Activity 3.2.1.4

Let's look at a third exponential function, $h(x) = 2^{-x}$.

(a)

Before plotting any points or graphing, what do you think the graph might look like? What sort of characteristics might it have?

Answer available at the end of this section.

(b)

Fill in the table of values for $h(x)$. Then plot the points on a graph.

x	$h(x)$
-2	
-1	
0	
1	
2	

Answer available at the end of this section.

(c)

This function $h(x)$ looks to be the same as a function we looked at previously. Use properties of exponents to rewrite $h(x)$ in a different way.

Answer available at the end of this section.

(d)

In addition to plotting points, we can use transformations to graph. If we consider $f(x) = 2^x$ to be the parent function, what transformation is needed to graph $h(x) = 2^{-x}$?

A. A vertical stretch.
B. A horizontal stretch.
C. A reflection over the x-axis.
D. A reflection over the y-axis.

Answer available at the end of this section.

Note

For a reminder of transformations, see *Section 1.8* and its definitions.

Activity 3.2.1.5

Let $f(x) = 4^x$.

(a)

Graph $f(x)$.

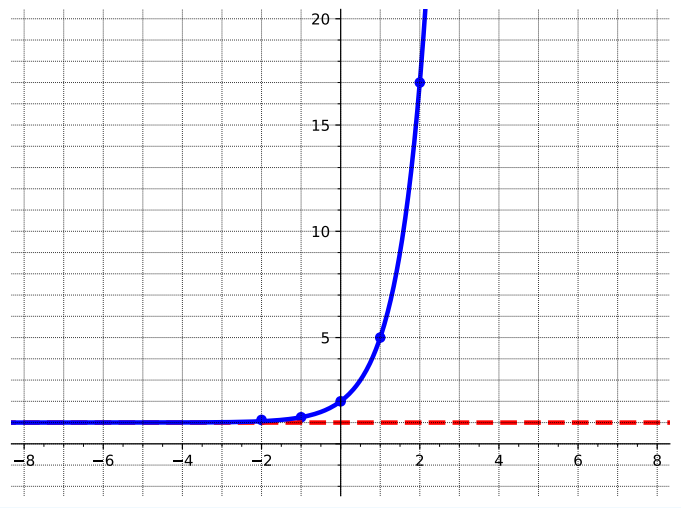
Answer available at the end of this section.

(b)

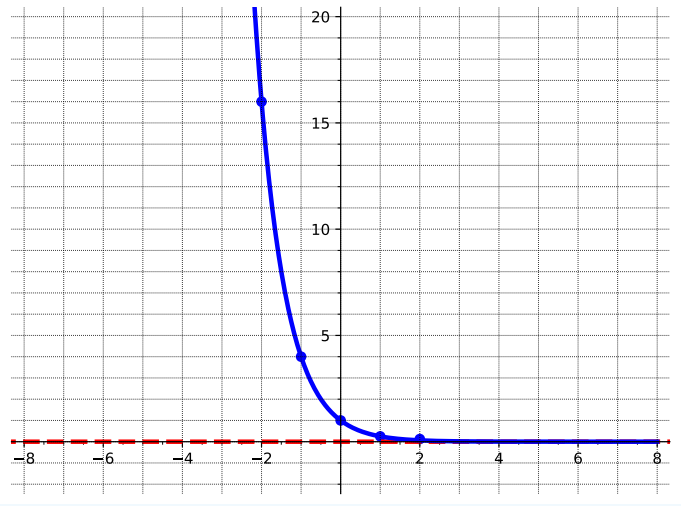
Match the following functions to their graphs.

- $g(x) = -4^x$
- $h(x) = 4^{-x}$
- $j(x) = 4^{x+1}$
- $k(x) = 4^x + 1$

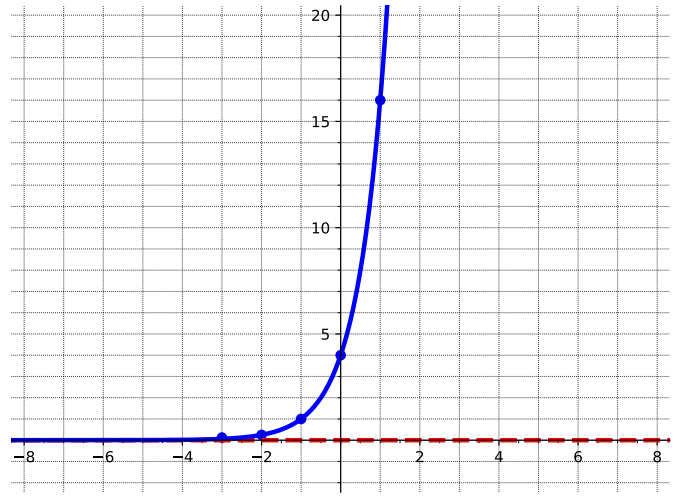
A.



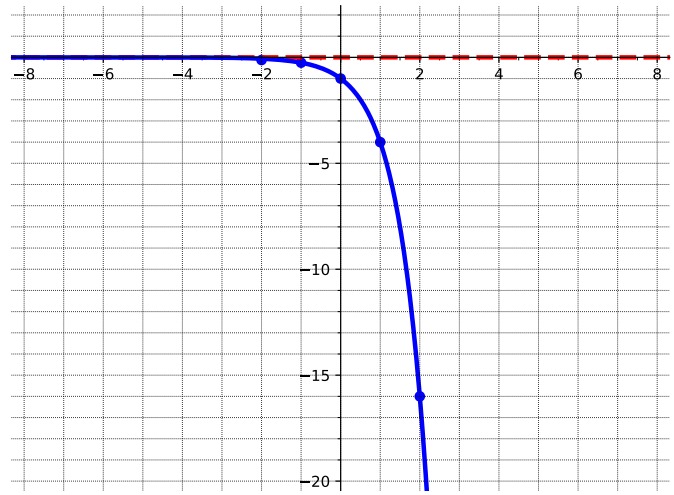
B.



C.



D.



Answer available at the end of this section.

(c)

Find the domain, range, and equation of the asymptote for the parent function $(f(x))$ and each of the four transformations $(g(x), h(x), j(x), \text{ and } k(x))$.

Answer available at the end of this section.

(d)

Which of the transformations affected the domain of the exponential function? Select all that apply.

A. A vertical shift.
B. A horizontal shift.
C. A reflection over the x-axis.
D. A reflection over the y-axis.
E. None of these.

Answer available at the end of this section.

(e)

Which of the transformations affected the range of the exponential function? Select all that apply.

A. A vertical shift.
B. A horizontal shift.
C. A reflection over the x-axis.
D. A reflection over the y-axis.
E. None of these.

Answer available at the end of this section.

(f)

Which of the transformations affected the asymptote of the exponential function? Select all that apply.

A. A vertical shift.
B. A horizontal shift.
C. A reflection over the x-axis.
D. A reflection over the y-axis.
E. None of these.

Answer available at the end of this section.

Activity 3.2.1.6

Consider the function $f(x) = e^x$.

(a)

Graph $f(x) = e^x$. First find $f(0)$ and $f(1)$. Then use what you know about the characteristics of exponential graphs to sketch the rest. Then state the domain, range, and equation of the asymptote. (Recall that $e \approx 2.72$ to help estimate where to put your points.)

Answer available at the end of this section.

(b)

Sketch the graph of $g(x) = e^{x-2}$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Answer available at the end of this section.

(c)

Sketch the graph of $h(x) = -3e^x$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Answer available at the end of this section.

(d)

Sketch the graph of $g(x) = e^{-x} - 4$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Answer available at the end of this section.

Activity 3.2.1.7

Graph each of the following exponential functions. Include any asymptotes with a dotted line. State the domain, the range, the equation of the asymptote, and whether it is growth or decay.

(a)

$f(x) = 3^x$

Answer available at the end of this section.

(b)

$f(x) = 6^{-x}$

Answer available at the end of this section.

(c)

$f(x) = \frac{1}{5}^{x-2}$

Answer available at the end of this section.

(d)

$f(x) = \frac{1}{3}^{x} + 4$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 3.2.1.1

Solve the following problems to test your understanding of graphing exponential functions.

(a) Find the horizontal asymptote, domain, and range of the function $f(x) = 2^x - 3$. Describe the behavior of its graph.

Answer available at the end of this section.

(b) Let $f(x) = 3^x$. Write the equation of the function $g(x)$ that results from shifting $f(x)$ left by 2 units, vertically stretching by a factor of 5, and shifting up by 4 units.

Answer available at the end of this section.

(c) An exponential function has a horizontal asymptote of $y = 5$ and passes through the points $(0, 8)$ and $(1, 14)$. Find its equation in the form $f(x) = a \cdot b^x + k$.

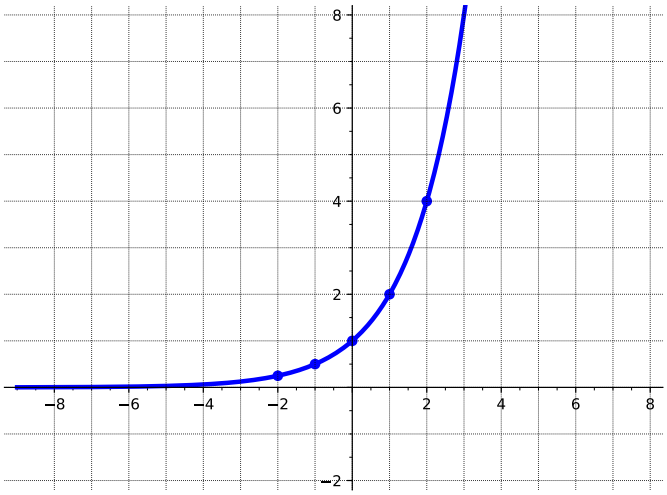
Answer available at the end of this section.

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Answers

Activity 3.2.1.1

x	$f($
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4



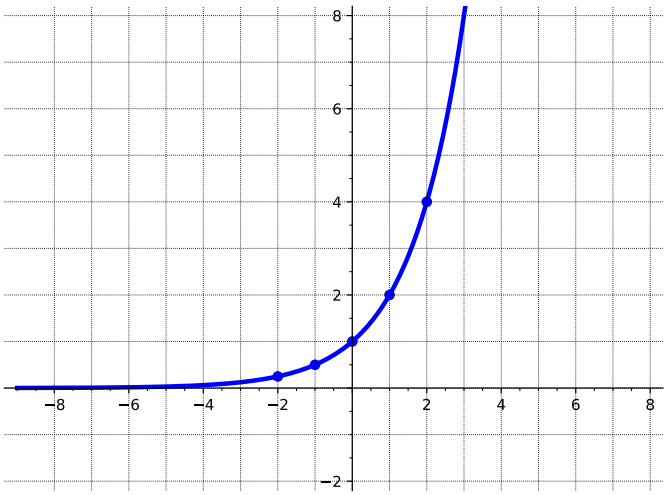
Activity 3.2.1.1

E

Activity 3.2.1.1

C

Activity 3.2.1.1



Activity 3.2.1.1

C

Activity 3.2.1.1

D

Activity 3.2.1.1

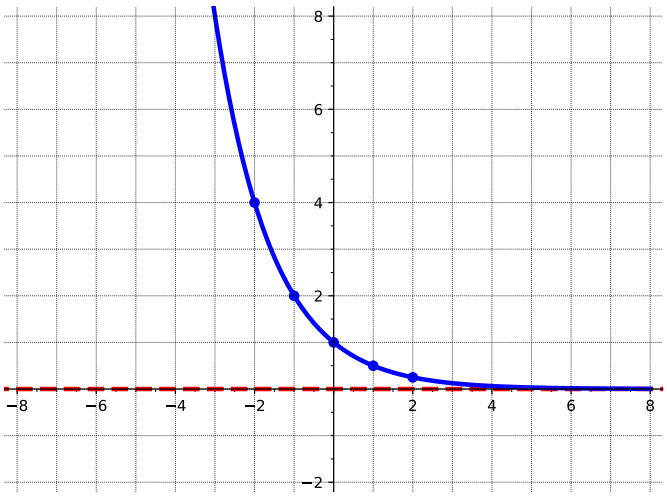
Domain: $(-\infty, \infty)$, Range: $(0, \infty)$

Activity 3.2.1.1

Increasing: $(-\infty, \infty)$, Decreasing: nowhere

Activity 3.2.1.2

x	$g(x$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$



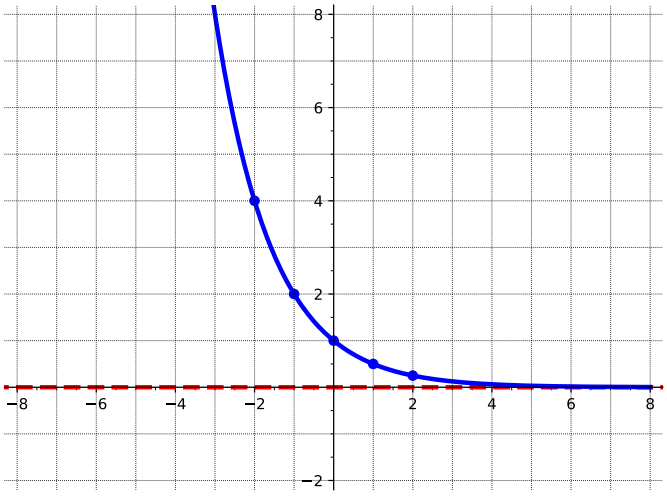
Activity 3.2.1.2

C

Activity 3.2.1.2

E

Activity 3.2.1.2



Activity 3.2.1.2

$y = 0$

Activity 3.2.1.2

Domain: $(-\infty, \infty)$, Range: $(0, \infty)$

Activity 3.2.1.2

Increasing: nowhere, Decreasing: $(-\infty, \infty)$

Activity 3.2.1.3

Answers could include basic shape, asymptote, domain, range.

Activity 3.2.1.3

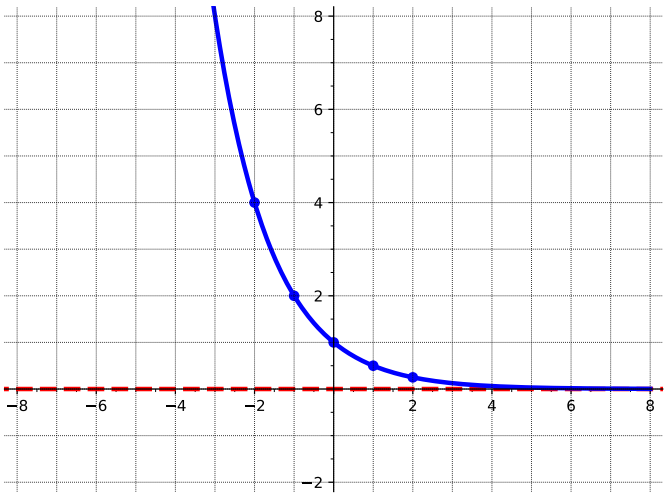
Answers could include reflection over y-axis, one is increasing, one is decreasing.

Activity 3.2.1.4

With the 2 as the base, students may assume exponential growth. But the negative in the exponent may change their mind if they remember transformations!

Activity 3.2.1.4

x	$h($
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$



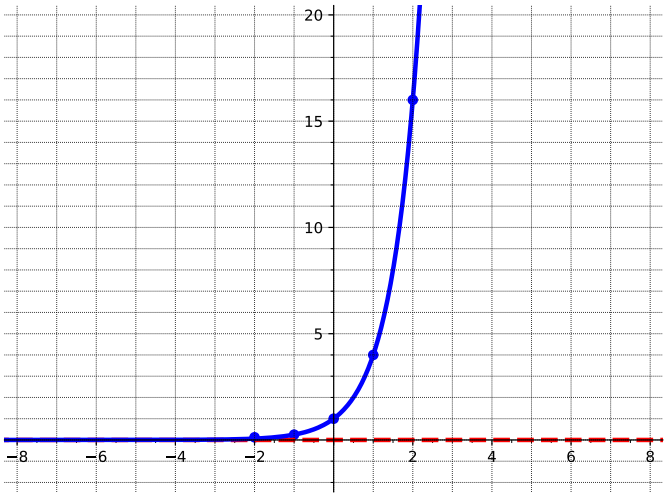
Activity 3.2.1.4

$2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$

Activity 3.2.1.4

D

Activity 3.2.1.5



Activity 3.2.1.5

- $g(x) = -4^x$ is D
- $h(x) = 4^{-x}$ is B
- $j(x) = 4^{x+1}$ is C
- $k(x) = 4^x + 1$ is A

Activity 3.2.1.5

$f(x)$:

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Asymptote: $y = 0$

$g(x)$:

- Domain: $(-\infty, \infty)$
- Range: $(-\infty, 0)$
- Asymptote: $y = 0$

$h(x)$:

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Asymptote: $y = 0$

$j(x)$:

- Domain: $(-\infty, \infty)$

- Range: $(0, \infty)$
- Asymptote: $y = 0$

$k(x)$:

- Domain: $(-\infty, \infty)$
- Range: $(1, \infty)$
- Asymptote: $y = 1$

Activity 3.2.1.5

E

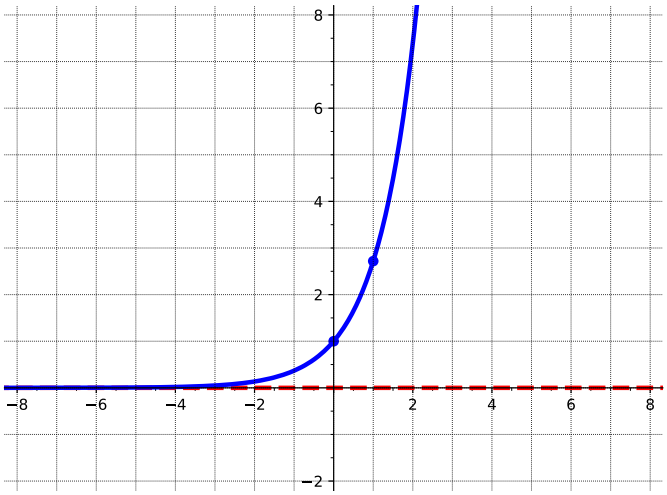
Activity 3.2.1.5

A and C

Activity 3.2.1.5

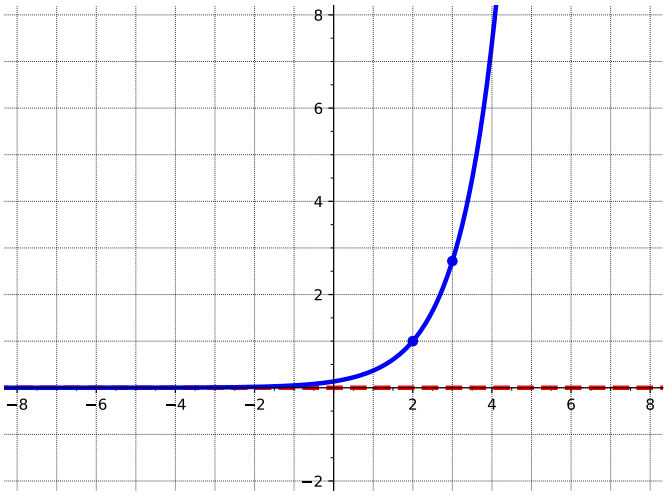
A

Activity 3.2.1.6



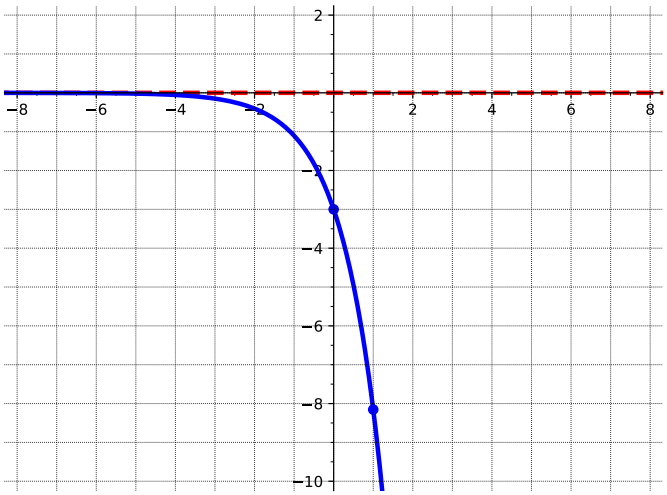
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Asymptote: $y = 0$

Activity 3.2.1.6



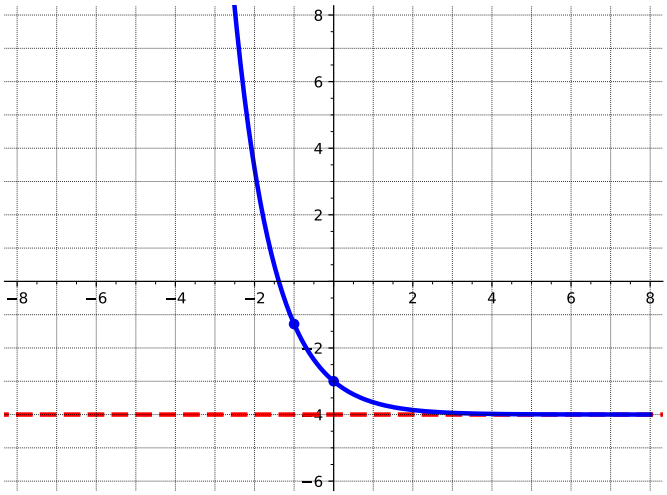
- Transformation: shift right 2
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Asymptote: $y = 0$

Activity 3.2.1.6



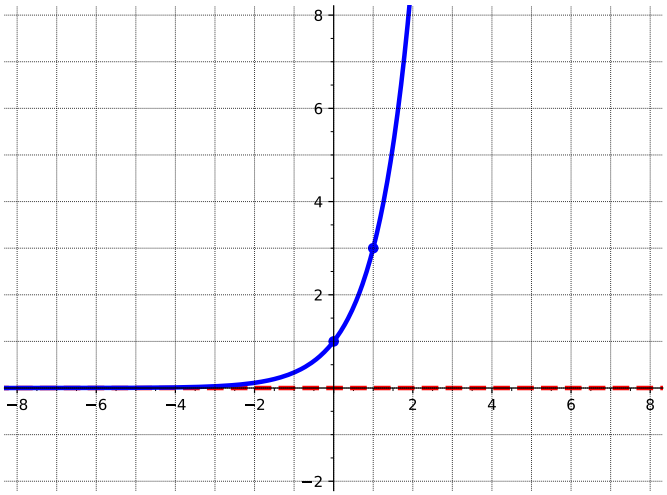
- Transformations: vertical stretch of 3, reflection over x-axis
- Domain: $(-\infty, \infty)$
- Range: $(-\infty, 0)$
- Asymptote: $y = 0$

Activity 3.2.1.6



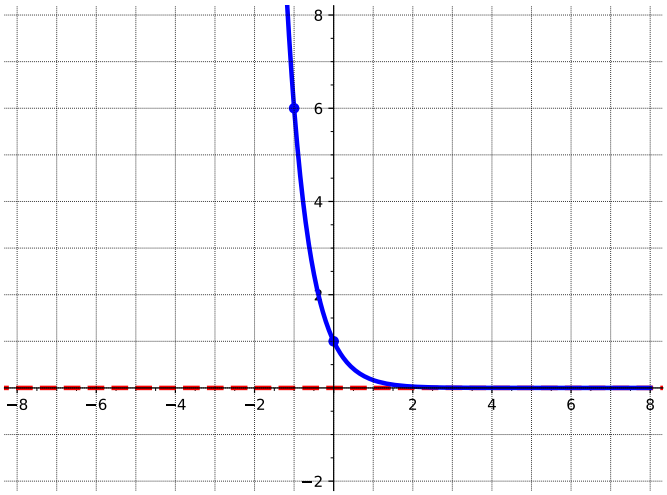
- Transformations: reflection over y-axis, shift down 4
- Domain: $(-\infty, \infty)$
- Range: $(-4, \infty)$
- Asymptote: $y = -4$

Activity 3.2.1.7



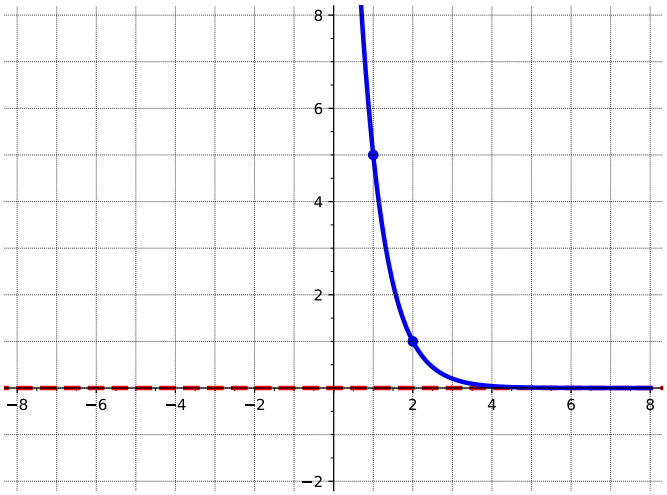
- Exponential growth
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Asymptote: $y = 0$

Activity 3.2.1.7



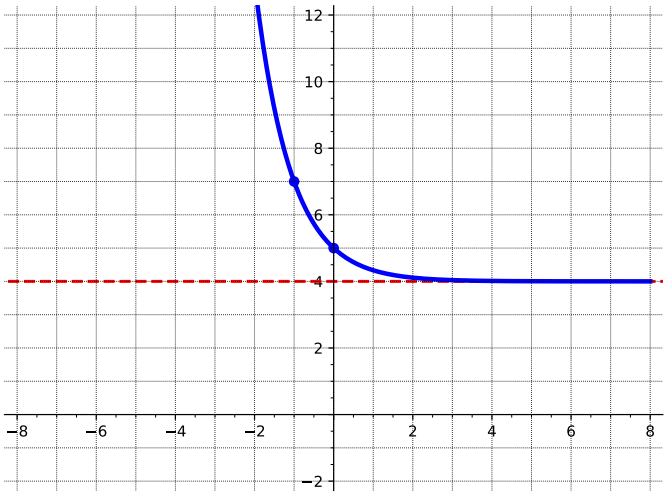
- Exponential decay
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Asymptote: $y = 0$

Activity 3.2.1.7



- Exponential decay
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Asymptote: $y = 0$

Activity 3.2.1.7



- Exponential decay
- Domain: $(-\infty, \infty)$
- Range: $(4, \infty)$
- Asymptote: $y = 4$

Check Your Understanding 3.2.1.1

- **Horizontal Asymptote:** The vertical shift is $k = -3$, so the horizontal asymptote is the line $y = -3$.
- **Domain:** The exponential function is defined for all real numbers, so the domain is $(-\infty, \infty)$.
- **Range:** Since the base $b = 2 > 1$ and there is no vertical reflection, the range is $(-3, \infty)$.
- **Graph Behavior:** The graph is an increasing exponential curve that starts near the horizontal asymptote $y = -3$ on the left and rises rapidly on the right, passing through the y-intercept $(0, -2)$ and the point $(1, -1)$.

Check Your Understanding 3.2.1.1

We apply the transformations step-by-step to the parent function $f(x) = 3^x$:

1. Shifting left by 2 units: Replace x with $x + 2 \implies 3^{x+2}$.
2. Vertically stretching by a factor of 5: Multiply the base term by $5 \implies 5 \cdot 3^{x+2}$.
3. Shifting up by 4 units: Add $4 \implies g(x) = 5 \cdot 3^{x+2} + 4$.

Thus, the final equation is $g(x) = 5 \cdot 3^{x+2} + 4$.

Check Your Understanding 3.2.1.1

We use the given properties to find the values of a, b, and k:

- Since the horizontal asymptote is $y = 5$, we have $k = 5$.
- Using the point $(0, 8)$:

$$8 = a \cdot b^0 + 5 \implies 8 = a + 5 \implies a = 3 \tag{3.2.1.1}$$

- Using the point $(1, 14)$ with $a = 3$ and $k = 5$:

$$14 = 3 \cdot b^1 + 5 \implies 9 = 3b \implies b = 3 \tag{3.2.1.2}$$

Thus, the equation of the function is $f(x) = 3 \cdot 3^x + 5$ (or $f(x) = 3^{x+1} + 5$).

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3.2.1.1: EL2 Online Exercises

EL2 — Graph exponential functions and determine the domain, range, and asymptotes.	EL2 CheckIt Exercises

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3.3: Introduction to Logarithms (EL3)

3.3.1: Logarithms (EL3)

EL3 Introduction to Logarithms → *APC 3.4 What a logarithm measures*

- Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms. .

Go back to that folding paper from Section 3.1. You know how to answer "how thick is the stack after 20 folds" — plug 20 into an exponential function and evaluate. Flip the question around, though: "how many folds until the stack reaches a mile high?" Now you know the output and need the exponent, and exponent rules alone won't get you there. That's the gap a logarithm fills: $\log_b(x)$ asks "what power do I raise b to, to get x?" — the precise inverse of evaluating b^x. And since an exponential's output is always positive, that question only makes sense when x is positive — you'll never be asked what power of 10 gives you -5.

This is literally the move from Section 1.7 — the section opens by recalling that idea directly: a function has an inverse when another function undoes its process. Applied to the exponentials from Section 3.1 (including the number e, introduced there), a logarithm is defined so $\log_b(x) = y$ exactly when $b^y = x$ — which is also why the domain of one matches the range of the other.

Section 3.4 turns this inverse relationship into a picture: since exponential and logarithmic functions are inverses, their graphs are reflections of each other across the line $y = x$ — the positive-inputs-only restriction you just worked through becomes the graph's vertical asymptote, and the horizontal asymptote from Section 3.2 disappears entirely.

> **Looking ahead to calculus:** The natural logarithm, $\ln(x) = \log_e(x)$, turns out to be one of the most important functions in calculus — not for what it does, but for a gap it fills. Every power function x^n has a clean antiderivative except x^{-1}; $\ln(x)$ is precisely the function that patches that hole, and its derivative, $\frac{1}{x}$, is remarkably simple for something built from exponentials.

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3.3.1: Logarithms (EL3)

Learning Outcomes:

- Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms.

Introduction to Logarithms (EL3)

In *(see related activity)*, we introduced the idea of an inverse function. The fundamental idea is that f has an inverse function if and only if there exists another function g such that f and g "undo" one another's respective processes. In other words, the process of the function f is reversible, and reversing f generates a related function g.

Activity 3.3.1.1

Let $P(t)$ be the function given by $P(t) = 10^t$.

(a)

Fill in the table of values for $P(t)$.

t	$y = P(t)$
-3	
-2	
-1	
0	
1	
2	
3	

Answer available at the end of this section.

(b)

Do you think P will have an inverse function? Why or why not?

Answer available at the end of this section.

(c)

Since P has an inverse function, we know there exists some other function, say L, such that $y = P(t)$ represent the same relationship between t and y as $t = L(y)$. In words, this means that L reverses the process of raising to the power of 10, telling us the *power* to which we need to raise 10 to produce a desired result. Fill in the table of values for $L(y)$.

y	$L(y)$
10^{-3}	
10^{-2}	
10^{-1}	
10^{0}	
10^{1}	
10^{2}	
10^{3}	

Answer available at the end of this section.

(d)

What are the domain and range of P?

Answer available at the end of this section.

(e)

What are the domain and range of L?

Answer available at the end of this section.

Note

The powers of 10 function $P(t)$ has an inverse L. This new function L is called the base 10 logarithm. But, we could have done a similar procedure with any base, which leads to the following definition.

Definition: Base b logarithm

The **base** b **logarithm** of a number is the exponent we must raise b to get that number. We represent this function as $y = \log_b(x)$.

We read the logarithmic expression as "The logarithm with base b of x is equal to y," or "log base b of x is y."

Note

There are some logarithms that occur so often, we sometimes write them without noting the base. They are the common logarithm and the natural logarithm.

The **common logarithm** is a logarithm with base 10 and is written without a base.

$$\log_{10}(x) = \log(x) \tag{3.3.1.1}$$

The **natural logarithm** is a logarithm with base e and has its own notation.

Recall the number e was introduced in *(see related activity)*

$$\log_e(x) = \ln(x) \tag{3.3.1.2}$$

Note

We can use *(see related activity)* to express the relationship between logarithmic form and exponential form as follows:

$$\log_b(x) = y \iff b^y = x \tag{3.3.1.3}$$

whenever $b > 0, b \neq 1$.

Activity 3.3.1.2

Write the following logarithmic equations in exponential form.

(a)

$\log_7(\sqrt{7}) = \frac{1}{2}$

A. $7^{\frac{1}{2}} = \sqrt{7}$
B. $7^{\sqrt{7}} = \frac{1}{2}$
C. $\sqrt{7}^{\frac{1}{2}} = 7$
D. $\frac{1}{2}^{7} = \sqrt{7}$

Answer available at the end of this section.

(b)

$\log_3(m) = r$

A. $3^m = r$
B. $r^3 = m$
C. $3^r = m$
D. $m^r = 3$

Answer available at the end of this section.

(c)

$\log_2(x) = 6$

A. $2^x = 6$
B. $6^2 = x$
C. $x^x = 6$
D. $2^6 = x$

Answer available at the end of this section.

Activity 3.3.1.3

Write the following exponential equations in logarithmic form.

(a)

$5^2 = 25$

A. $\log_5(2) = 25$
B. $\log_5(25) = 2$
C. $\log_{25}(5) = 2$
D. $\log_2(25) = 5$

Answer available at the end of this section.

(b)

$3^{-1} = \frac{1}{3}$

A. $\log_3(-1) = \frac{1}{3}$
B. $\log_{\frac{1}{3}}(-1) = 3$
C. $\log_3\left(\frac{1}{3}\right) = -1$
D. $\log_{-1}\left(\frac{1}{3}\right) = 3$

Answer available at the end of this section.

(c)

$10^a = n$

A. $\log_{10}(n) = a$
B. $\log_{10}(a) = n$
C. $\log_n(10) = a$
D. $\log_a(n) = 10$

Answer available at the end of this section.

Activity 3.3.1.4

We can use the idea of converting a logarithm to an exponential to evaluate logarithms.

(a)

Consider the logarithm $\log_3(9)$. If we want to evaluate this, which question should you try and solve?

A. To what exponent must 9 be raised in order to get 3?
B. What exponent must be raised to the third in order to get 9?
C. To what exponent must 3 be raised in order to get 9?
D. What exponent must be raised to the ninth in order to get 3?

Answer available at the end of this section.

(b)

Evaluate the logarithm, $\log_3(9)$, by answering the question from part (a).

Answer available at the end of this section.

Activity 3.3.1.5

Evaluate the following logarithms.

(a)

$\log_2(8)$

A. 4
B. $\frac{1}{4}$
C. -3
D. 3

Answer available at the end of this section.

(b)

$\log_{144}(12)$

A. $\frac{1}{2}$
B. -2
C. 2
D. $-\frac{1}{2}$

Answer available at the end of this section.

(c)

$\log_{10}\left(\frac{1}{1000}\right)$

A. $\frac{1}{3}$
B. -3
C. 3
D. $-\frac{1}{3}$

Answer available at the end of this section.

(d)

$\log_e\left(e^3\right)$

A. 3
B. e^3
C. -3
D. $\frac{1}{3}$

Answer available at the end of this section.

(e)

$\log_7(1)$

A. 7
B. $\frac{1}{7}$

C. 0
D. 1

Answer available at the end of this section.

Note

Consider the results of *(see related activity)* part (d) and (e). Using the rules of exponents and the fact that exponents and logarithms are inverses, the following properties hold for any base:

- $\log_b(b^x) = x$
- $b^{\log_b(x)} = x$

In particular, note the following special cases:

- $\log_b(1) = 0$ (and therefore $\ln(1) = 0$)
- $\log_b(b) = 1$ (and therefore $\ln(e) = 1$)

Activity 3.3.1.6

Evaluate the following logarithms. Some may be done by inspection and others may require a calculator.

(a)

$\log_4\left(\frac{1}{64}\right)$

Answer available at the end of this section.

(b)

$\ln(1)$

Answer available at the end of this section.

(c)

$\ln(12)$

Answer available at the end of this section.

(d)

$\log(100)$

Answer available at the end of this section.

(e)

$\log_5(32)$

Answer available at the end of this section.

(f)

$\log_5\left(\sqrt{5}\right)$

Answer available at the end of this section.

(g)

$\log(-10)$

Answer available at the end of this section.

Note

Notice that in *(see related activity)* part (g) you were unable to evaluate the logarithm. Given that exponentials and logarithms are inverses, their domain and range are related. The range of an exponential function is $(0, \infty)$ which becomes the domain of a logarithmic function. This means that the argument of any logarithmic function must be greater than zero.

Activity 3.3.1.7

Find the domain of the function $\log_3(2x - 4)$.

(a)

Set up an inequality that you must solve to find the domain.

Answer available at the end of this section.

(b)

Solve the inequality to find the domain. Write your answer in interval notation.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 3.3.1.1

Solve the following problems to test your understanding of logarithmic functions.

(a) Convert the equations between exponential and logarithmic form:

1. Convert $\log_4(64) = 3$ to exponential form.
2. Convert $e^x = 5$ to logarithmic form.

Answer available at the end of this section.

(b) Evaluate the following logarithmic expressions exactly without a calculator:

1. $\log_5\left(\frac{1}{25}\right)$
2. $\log(100,000)$
3. $\ln(e^{-2})$

Answer available at the end of this section.

(c) Find the domain of the function $f(x) = \log_2(4x - 12)$. Write your answer in interval notation.

Answer available at the end of this section.

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Answers

Activity 3.3.1.1

t	$y = P(t)$
-3	$0.001 = 10^{-3}$
-2	$0.01 = 10^{-2}$
-1	$0.1 = 10^{-1}$
0	$1 = 10^{0}$
1	$10 = 10^{1}$
2	$100 = 10^{2}$
3	$1000 = 10^{3}$

Activity 3.3.1.1

Students may say, each input has a distinct output, the function is one-to-one.

Activity 3.3.1.1

y	$L(y)$
10^{-3}	-3
10^{-2}	-2
10^{-1}	-1
10^{0}	0
10^{1}	1
10^{2}	2
10^{3}	3

Activity 3.3.1.1

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Activity 3.3.1.1

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Activity 3.3.1.2

A

Activity 3.3.1.2

C

Activity 3.3.1.2

D

Activity 3.3.1.3

B

Activity 3.3.1.3

C

Activity 3.3.1.3

A

Activity 3.3.1.4

C

Activity 3.3.1.4

2

Activity 3.3.1.5

D

Activity 3.3.1.5

A

Activity 3.3.1.5

B

Activity 3.3.1.5

A

Activity 3.3.1.5

C

Activity 3.3.1.6

-3

Activity 3.3.1.6

0

Activity 3.3.1.6

2.485

Activity 3.3.1.6

2

Activity 3.3.1.6

2.153

Activity 3.3.1.6

$\frac{1}{2}$

Activity 3.3.1.6

Does not exist

Activity 3.3.1.7

$2x - 4 > 0$

Activity 3.3.1.7

$x > 2$ or $(2, \infty)$

Check Your Understanding 3.3.1.1

1. Using the definition $\log_b(x) = y \iff b^y = x$, we get:

$$4^3 = 64 \tag{3.3.1.4}$$

2. Using the definition with base e (natural logarithm), we get:

$$\ln(5) = x \tag{3.3.1.5}$$

(or $\log_e(5) = x$)

Check Your Understanding 3.3.1.1

1. Let $\log_5\left(\frac{1}{25}\right) = y \iff 5^y = \frac{1}{25} = 5^{-2}$, so:

$$\log_5\left(\frac{1}{25}\right) = -2 \tag{3.3.1.6}$$

2. The common logarithm has base 10. Let $\log(100,000) = y \iff 10^y = 100,000 = 10^5$, so:

$$\log(100,000) = 5 \tag{3.3.1.7}$$

3. Using the inverse property $\log_b(b^x) = x$ with base e:

$$\ln(e^{-2}) = -2 \tag{3.3.1.8}$$

Check Your Understanding 3.3.1.1

The argument of a logarithmic function must be strictly positive. Therefore, we set up the inequality:

$$4x - 12 > 0 \tag{3.3.1.9}$$

Solving for x:

$$4x > 12 \implies x > 3 \tag{3.3.1.10}$$

In interval notation, the domain is $(3, \infty)$.

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3.3.1.1: EL3 Online Exercises

EL3 — Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms.	EL3 CheckIt Exercises

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3.4: Introduction to Graphs of Logarithmic Functions (EL4)

3.4.1: Graphs of Logarithmic Functions (EL4)

EL4 Graphs of Logarithmic Functions

- Graph logarithmic functions and determine the domain, range, and asymptotes.

Back in Section 3.3, you learned that $\log_b(x)$ answers "what power do I raise b to, to get x?" Graphing that function is easiest if you lean on what you already know: rewrite $y = \log_2 x$ as its exponential form, $2^y = x$, and suddenly you can build a table by picking y-values and solving for x, instead of the other way around. The resulting graph climbs, but more and more slowly as x grows — the opposite of the exponential's climb-faster-and-faster shape from Section 3.2. And near $x = 0$, the graph plunges downward without bound, since a logarithm's input can never reach zero or go negative.

This is the inverse function from Section 3.3, made visible: $\log_2 x$ and 2^x share the same basic shape, just mirrored, with domain and range swapped and the horizontal asymptote from Section 3.2 becoming a vertical one instead. And once you're shifting, reflecting, or stretching these graphs, you're back to the transformation vocabulary from Section 1.8.

Section 3.5 shifts from the graph itself to the algebra underneath it — the product, quotient, and power rules that let you rewrite and combine logarithmic expressions, which is what unlocks the real-world applications waiting in that section's second half.

> **Looking ahead to calculus:** That ever-slowing climb you're seeing in every logarithmic graph has a name — the graph is concave down, curving beneath its own tangent lines everywhere. Calculus explains exactly why: the rate at which a logarithm grows is itself always decreasing, which is what makes the curve bend the way it does instead of climbing in a straight line or curving upward the way an exponential does.

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3.4.1: Graphs of Logarithmic Functions (EL4)

Learning Outcomes:

- Graph logarithmic functions and determine the domain, range, and asymptotes.

Graphs of Logarithmic Functions (EL4)

Activity 3.4.1.1

Consider the function $g(x) = \log_2 x$.

(a)

Since we are familiar with graphing exponential functions, we'll use that to help us graph logarithmic ones. Rewrite $g(x)$ in exponential form, replacing $g(x)$ with y.

A. $x^y = 2$
B. $2^y = x$
C. $y^2 = x$
D. $2^x = y$
E. $x^2 = y$

Answer available at the end of this section.

(b)

Fill in the table of values. Notice you are given y-values, not x-values to plug in since those are easier in the equivalent exponential form. Then plot the points on a graph.

x	y
	-2
	-1
	0
	1
	2

Answer available at the end of this section.

(c)

What seems to be happening with the graph as x goes toward infinity? Plug in large positive values of x to test your guess, then describe the end behavior.

A. As $x \to \infty$, $y \to -\infty$.
B. As $x \to \infty$, $y \to 0$.
C. As $x \to \infty$, $y \to 6$.
D. As $x \to \infty$, $y \to \infty$.

E. The graph isn't defined as $x \to \infty$.

Answer available at the end of this section.

(d)

What seems to be happening with the graph as x goes toward negative infinity? Plug in large negative values of x to test your guess, then describe the end behavior.

A. As $x \to -\infty$, $y \to -\infty$.
B. As $x \to -\infty$, $y \to 0$.
C. As $x \to -\infty$, $y \to 6$.
D. As $x \to -\infty$, $y \to \infty$.
E. The graph isn't defined as $x \to -\infty$.

Answer available at the end of this section.

(e)

What seems to be happening with the graph as we approach x-values closer and closer to zero from the positive direction?

A. As $x \to 0$ from the positive direction, $y \to -\infty$.
B. As $x \to 0$ from the positive direction, $y \to 0$.
C. As $x \to 0$ from the positive direction, $y \to \infty$.
D. As $x \to 0$ from the positive direction, the graph isn't defined.

Answer available at the end of this section.

(f)

What seems to be happening with the graph as we approach x-values closer and closer to zero from the negative direction?

A. As $x \to 0$ from the negative direction, $y \to -\infty$.
B. As $x \to 0$ from the negative direction, $y \to 0$.
C. As $x \to 0$ from the negative direction, $y \to \infty$.
D. As $x \to 0$ from the negative direction, the graph isn't defined.

Answer available at the end of this section.

(g)

Complete the graph you started in *(see related activity)*, connecting the points and including the end behavior and behavior near zero that you've just determined.

Answer available at the end of this section.

(h)

Does your graph seem to have any asymptotes?

A. No. There are no asymptotes.
B. There is a vertical asymptote but no horizontal one.
C. There is a horizontal asymptote but no vertical one.
D. The graph has both a horizontal and vertical asymptote.

Answer available at the end of this section.

(i)

What the equation for each asymptote of $f(x)$? Select all that apply.

A. There are no asymptotes.
B. $x = 0$
C. $x = 6$
D. $y = 0$
E. $y = 6$

Answer available at the end of this section.

(j)

Find the domain and range of $g(x)$. Write your answers using interval notation.

Answer available at the end of this section.

(k)

Find the interval(s) where $g(x)$ is increasing and the interval(s) where $g(x)$ is decreasing. Write your answers using interval notation.

Answer available at the end of this section.

Activity 3.4.1.2

The function we've just graphed, $g(x) = \log_2 x$, and the function $f(x) = 2^x$ (which we graphed in *(see related activity)*) are inverse functions.

(a)

How are the graphs of $f(x)$ and $g(x)$ similar?

Answer available at the end of this section.

(b)

How are the graphs of $f(x)$ and $g(x)$ different?

Answer available at the end of this section.

Note

The graph of a logarithmic function $g(x) = \log_b x$ where $b > 0$ and $b \neq 1$ has the following characteristics:

- Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- There is a vertical asymptote at $x = 0$. There is no horizontal asymptote.
- There is an x-intercept at $(1, 0)$. There is no y-intercept.

Note

Just as with other types of functions, we can use transformations to graph logarithmic functions. For a reminder of these transformations, see *(see related activity)* and the following definitions:

- *(see related activity)*
- *(see related activity)*

- *(see related activity)*
- *(see related activity)*
- *(see related activity)*
- *(see related activity)*

Activity 3.4.1.3

Let $f(x) = \log_4 x$.

(a)

Graph $f(x)$.

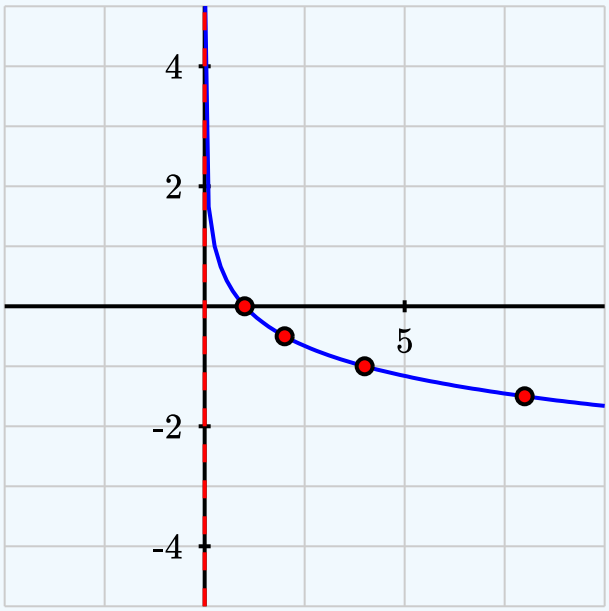
Answer available at the end of this section.

(b)

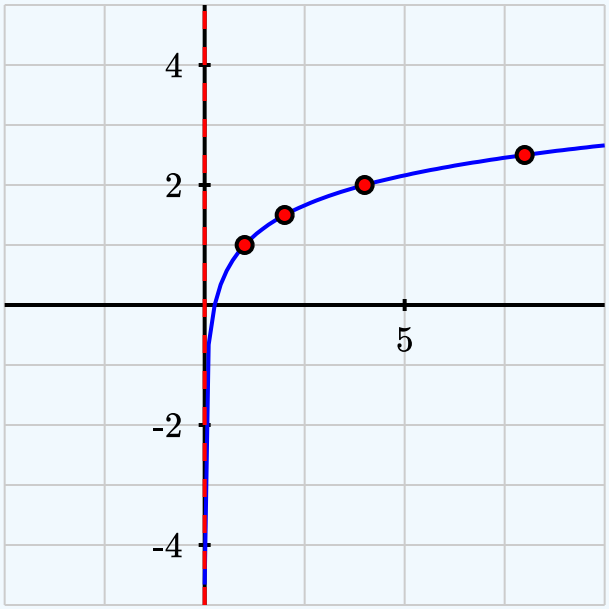
Match the following functions to their graphs.

- $g(x) = -\log_4 x$
- $h(x) = \log_4(-x)$
- $j(x) = \log_4(x+1)$
- $k(x) = \log_4(x) + 1$

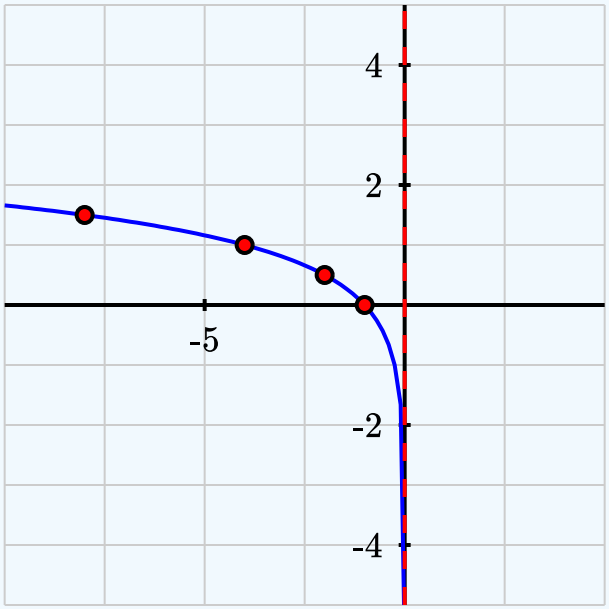
A.



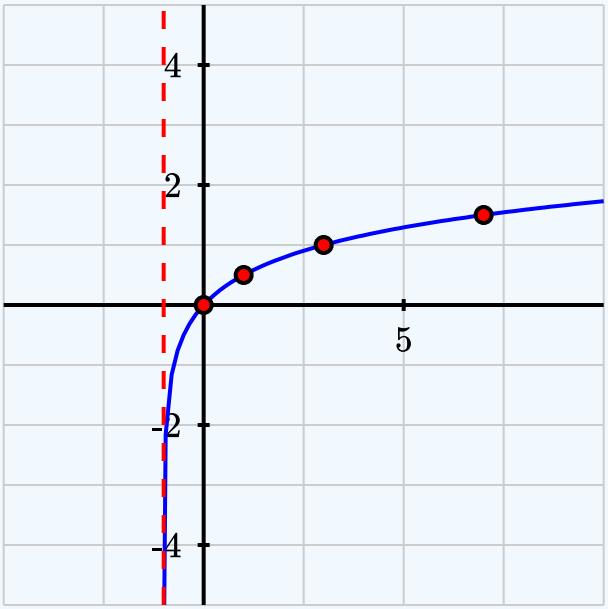
B.



C.



D.



Answer available at the end of this section.

(c)

Find the domain, range, and equation of the asymptote for the parent function $(f(x))$ and each of the four transformations $(g(x), h(x), j(x), \text{ and } k(x))$.

Answer available at the end of this section.

(d)

Which of the transformations affected the domain of the logarithmic function? Select all that apply.

A. A vertical shift.
B. A horizontal shift.
C. A reflection over the x-axis.
D. A reflection over the y-axis.
E. None of these.

Answer available at the end of this section.

(e)

Which of the transformations affected the range of the logarithmic function? Select all that apply.

A. A vertical shift.
B. A horizontal shift.
C. A reflection over the x-axis.
D. A reflection over the y-axis.
E. None of these.

Answer available at the end of this section.

(f)

Which of the transformations affected the asymptote of the logarithmic function? Select all that apply.

A. A vertical shift.
B. A horizontal shift.
C. A reflection over the x-axis.
D. A reflection over the y-axis.
E. None of these.

Answer available at the end of this section.

Activity 3.4.1.4

Consider the function $f(x) = \ln(x)$.

(a)

Graph $f(x) = \ln(x)$. First find $f(1)$ and $f(e)$. Then use what you know about the characteristics of logarithmic graphs to sketch the rest. Then state the domain, range, and equation of the asymptote. (Recall that $e \approx 2.72$ to help estimate where to put your points.)

Answer available at the end of this section.

(b)

Sketch the graph of $g(x) = \ln(x - 3)$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Answer available at the end of this section.

(c)

Sketch the graph of $h(x) = 3\ln(x)$ using transformations. State the transformation(s) used, the domain, the range, and the equation of the asymptote.

Answer available at the end of this section.

Activity 3.4.1.5

Graph each of the following logarithmic functions. Include any asymptotes with a dotted line. State the domain, the range, and the equation of the asymptote.

(a)

$f(x) = \log_3 x$

Answer available at the end of this section.

(b)

$f(x) = \log_6(-x)$

Answer available at the end of this section.

(c)

$f(x) = \log_{\frac{1}{5}} x$

Answer available at the end of this section.

(d)

$f(x) = \log_{\frac{1}{3}} x + 2$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 3.4.1.1

Solve the following problems to test your understanding of graphing logarithmic functions and their properties.

(a) Consider the function $f(x) = \log_3(x + 4) - 2$. Identify its domain, range, and the equation of its vertical asymptote.

Answer available at the end of this section.

(b) Let $g(x) = -\ln(x)$. Describe the transformation applied to the parent function $y = \ln(x)$, and state the domain, range, and vertical asymptote of $g(x)$.

Answer available at the end of this section.

(c) A logarithmic function of the form $h(x) = \log_b(x - h)$ has a vertical asymptote at $x = 2$ and passes through the point $(5, 1)$. Find the values of h and b, and write the equation of the function.

Answer available at the end of this section.

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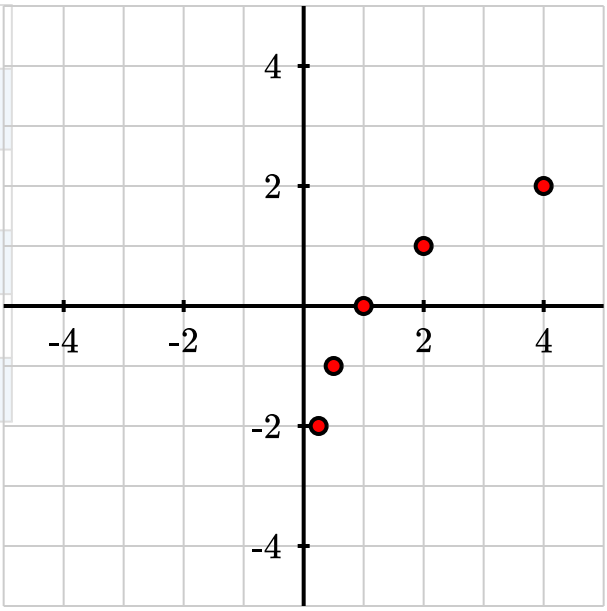
Answers

Activity 3.4.1.1

B

Activity 3.4.1.1

x	y
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2



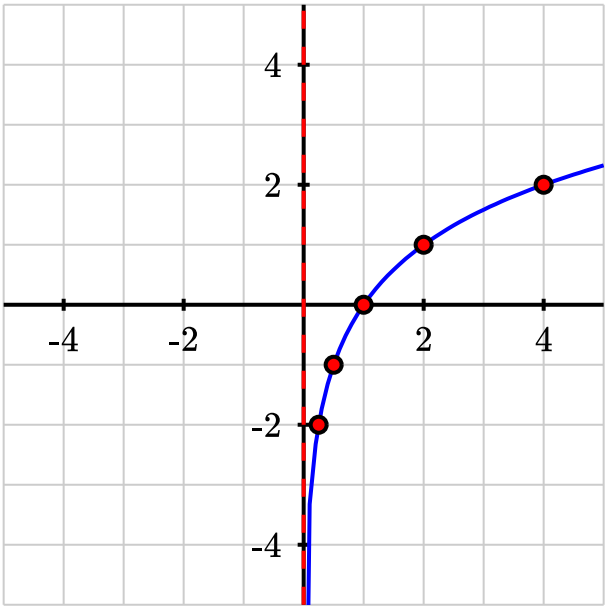
Activity 3.4.1.1
D

Activity 3.4.1.1
E

Activity 3.4.1.1
A

Activity 3.4.1.1
D

Activity 3.4.1.1



Activity 3.4.1.1
B

Activity 3.4.1.1
B

Activity 3.4.1.1
Domain: $(0, \infty)$, Range: $(-\infty, \infty)$

Activity 3.4.1.1

Increasing: $(0, \infty)$, Decreasing: nowhere

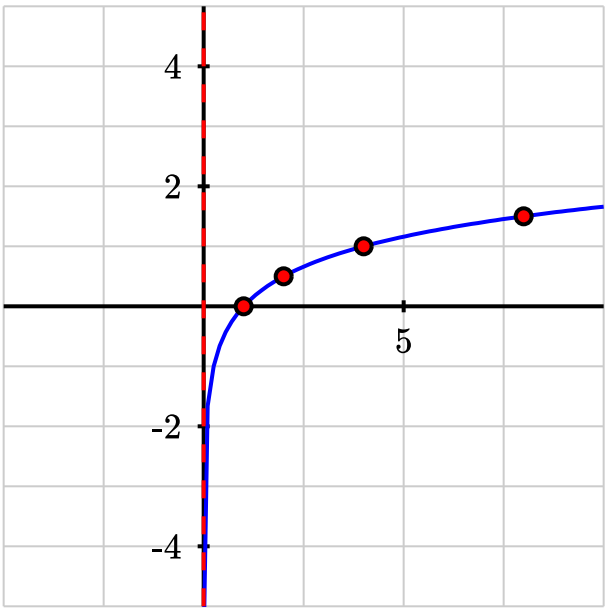
Activity 3.4.1.2

Answers could include basic shape (though mirrored), both are increasing.

Activity 3.4.1.2

Answers could include flipped x and y values, flipped asymptote, one has x-intercept and one has y-intercept, domain and range are swapped.

Activity 3.4.1.3



Activity 3.4.1.3

- $g(x) = -\log_4 x$ is A.
- $h(x) = \log_4(-x)$ is C.
- $j(x) = \log_4(x + 1)$ is D.
- $k(x) = \log_4(x) + 1$ is B.

Activity 3.4.1.3

$f(x)$:

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

$g(x)$:

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

$h(x)$:

- Domain: $(-\infty, 0)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

$j(x)$:

- Domain: $(-1, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = -1$

$k(x)$:

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

Activity 3.4.1.3

B and C

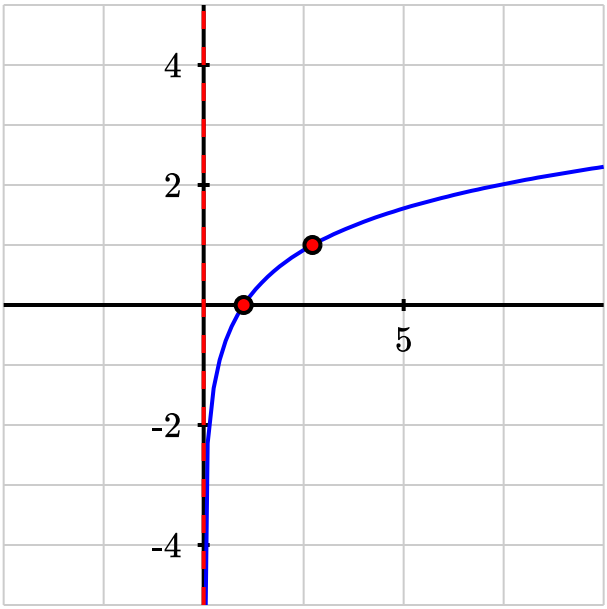
Activity 3.4.1.3

E

Activity 3.4.1.3

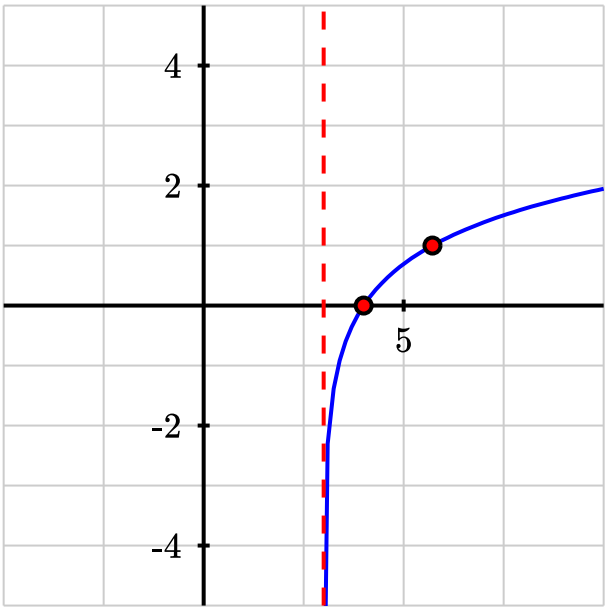
B

Activity 3.4.1.4



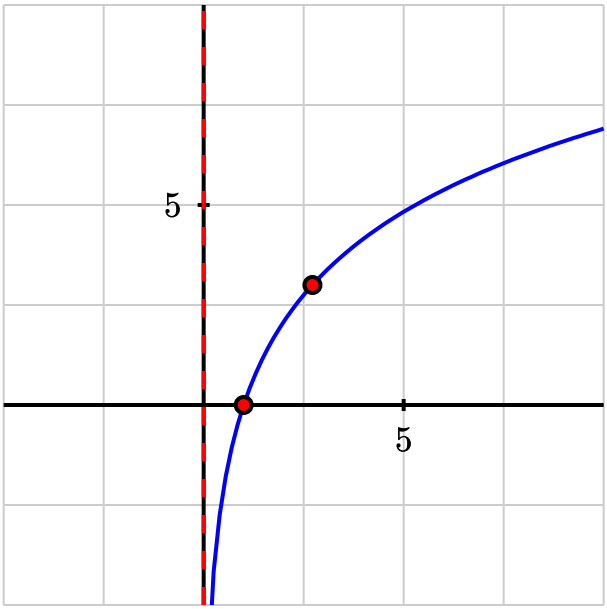
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

Activity 3.4.1.4



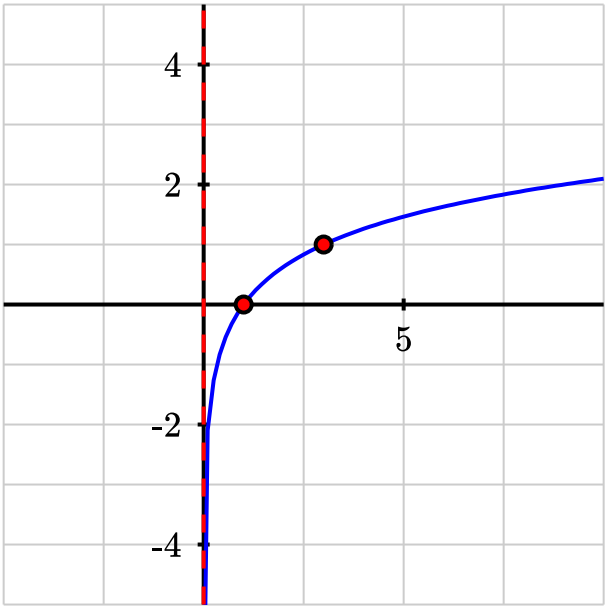
- Transformation: shift right 3
- Domain: $(3, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 3$

Activity 3.4.1.4



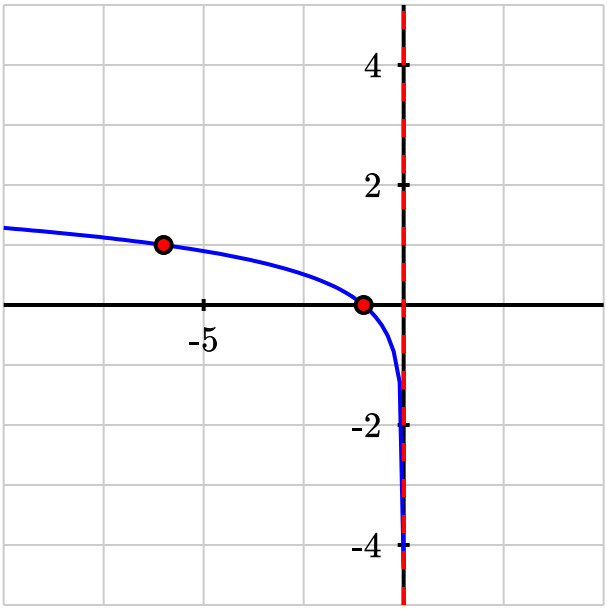
- Transformation: vertical stretch with factor of three.
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

Activity 3.4.1.5



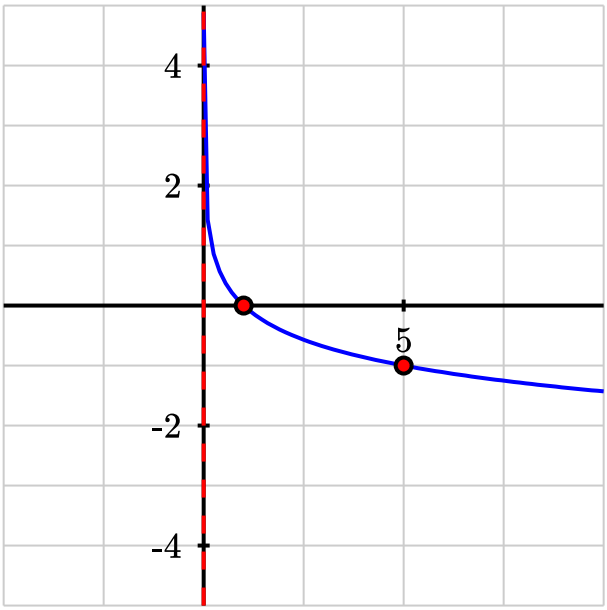
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

Activity 3.4.1.5



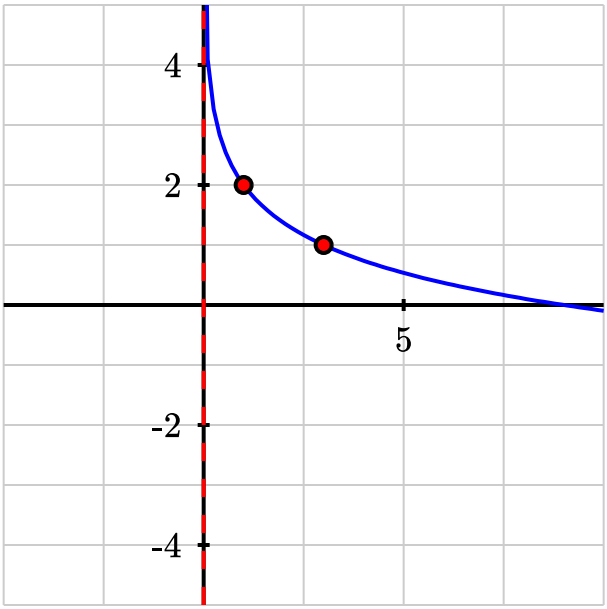
- Domain: $(-\infty, 0)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

Activity 3.4.1.5



- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

Activity 3.4.1.5



- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Asymptote: $x = 0$

Check Your Understanding 3.4.1.1

- **Domain:** The argument of the logarithm must be positive: $x + 4 > 0 \implies x > -4$. Thus, the domain is $(-4, \infty)$.
- **Range:** All logarithmic functions of this form have a range of all real numbers, so the range is $(-\infty, \infty)$.
- **Vertical Asymptote:** The vertical shift does not affect the asymptote, but the horizontal shift of 4 units left moves the vertical asymptote to the line $x = -4$.

Check Your Understanding 3.4.1.1

- **Transformation:** The negative sign in front of the logarithm reflects the graph of $y = \ln(x)$ vertically across the x-axis.
- **Domain:** The reflection does not affect the domain since the argument is still x, so the domain is $(0, \infty)$.
- **Range:** The range remains all real numbers, $(-\infty, \infty)$.
- **Vertical Asymptote:** The asymptote is unaffected by the vertical reflection, so it remains the line $x = 0$ (the y-axis).

Check Your Understanding 3.4.1.1

- Since the vertical asymptote of the parent function $y = \log_b(x)$ is $x = 0$ and the asymptote of $h(x)$ is $x = 2$, the graph has been shifted horizontally to the right by 2 units. Therefore, $h = 2$, and the function is of the form:

$$h(x) = \log_b(x - 2) \tag{3.4.1.1}$$

- Using the point $(5, 1)$, we substitute $x = 5$ and $h(5) = 1$:

$$1 = \log_b(5 - 2) \implies 1 = \log_b(3) \tag{3.4.1.2}$$

- Converting to exponential form:

$$b^1 = 3 \implies b = 3 \tag{3.4.1.3}$$

- Thus, the equation of the function is:

$$h(x) = \log_3(x - 2) \tag{3.4.1.4}$$

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3.4.1.1: EL4 Online Exercises

EL4 — Graph logarithmic functions and determine the domain, range, and asymptotes.	EL4 CheckIt Exercises

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3.5: Introduction to Properties and Applications of Logarithms (EL5 and EL6)

3.5.1: Properties of Logarithms

3.5.2: Solving Exponential and Logarithmic Equations

EL5 Properties of Logarithms

- Use properties of logarithms to condense or expand logarithmic expressions.

EL6 Solving Exponential and Logarithmic Equations

- Solve exponential and logarithmic equations.

Suppose you're trying to solve $5^x = 3$ for x — not evaluate a logarithm, but solve an equation with the unknown stuck in the exponent. Logarithmic notation alone doesn't get you there; you need rules for combining and rearranging logarithms once several of them get tangled together. This section supplies three: a product property that turns $\log(mn)$ into $\log(m) + \log(n)$, a quotient property that does the same for division, and a power property that pulls an exponent down in front of the log entirely — the exact move that solves $5^x = 3$, by taking the log of both sides and bringing that x down out of the exponent.

This builds on the log-exponential relationship from Section 3.3: because a logarithm is the inverse of a one-to-one exponential function, $\log_b M$ and $\log_b N$ can only be equal when M and N themselves are equal — the one-to-one property that opens this section, and the same invertibility idea from Section 1.7 that made a logarithm possible in the first place.

Section 3.6 puts these properties to work: modeling problems routinely land you at an equation like $P_0 e^{kt} = P$ with the unknown buried in the exponent, and solving for it is only possible because you now know how to take a log of both sides and bring that exponent down.

> **Looking ahead to calculus:** These properties reappear directly in a technique called logarithmic differentiation. When a function is an unwieldy product, quotient, or power of several pieces, taking the log of both sides first — turning multiplication into addition via the product property — often makes an otherwise painful derivative manageable, before undoing the log at the very end.

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3.5.1: Properties of Logarithms

Learning Outcomes:

- Use properties of logarithms to condense or expand logarithmic expressions.

Properties of Logarithms (EL5)

In this section, we will explore the properties of logarithms and learn how to manipulate them that will be helpful when we are ready to solve logarithmic equations.

Note

Recall that we can convert between exponential and logarithmic forms.

$$\log_b x = y \tag{3.5.1.1}$$

is equivalent to

$$b^y = x \tag{3.5.1.2}$$

.

Activity 3.5.1.1

Suppose you are given two equations $\log_b M = x$ and $\log_b N = x$.

(a)

Rewrite each logarithmic equation into an exponential equation.

Answer available at the end of this section.

(b)

Look at the exponential equations you found in part (a). What conclusion can you make about M and N?

Answer available at the end of this section.

(c)

Given that both $\log_b M$ and $\log_b N$ are both equal to x, what can you say about $\log_b M$ and $\log_b N$?

Answer available at the end of this section.

(d)

If given $\log_b M = \log_b N$, what can you say about M and N? (Refer back to the previous parts of this activity.)

Answer available at the end of this section.

For any values $M > 0$ and $N > 0$, and for any $b > 0$ with $b \neq 1$,

$$\log_b M = \log_b N \tag{3.5.1.3}$$

if and only if

$$M = N. \tag{3.5.1.4}$$

This is called the **one-to-one property of logarithms**.

Note

Notice this fact will eventually help us solve logarithmic equations. If we have an equation where each side is expressed as a single logarithm with matching bases (such as $\log_b M = \log_b N$), then it follows that the arguments (M and N) are also equal to each other.

Note

Recall that exponential functions and logarithmic functions are inverses. We know that $\log_b(b^k) = k$ and according to the law of exponents, we know that:

$$x^a \cdot x^b = x^{a+b} \tag{3.5.1.5}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{3.5.1.6}$$

$$(x^a)^b = x^{a \cdot b} \tag{3.5.1.7}$$

Consider all these as you move through the activities in this section.

Activity 3.5.1.2

Let's begin with the law of exponents to see if we can understand the product property of logs. According to the law of exponents, we know that $10^x \cdot 10^y = 10^{x+y}$. Start with this equation as you move through this activity.

(a)

Let $a = 10^x$ and $b = 10^y$. How could you rewrite the left side of the equation $10^x \cdot 10^y$?

A. $a + b$
B. $a - b$
C. 10^{x+y}
D. $a \cdot b$

Answer available at the end of this section.

(b)

Recall that $\log_b M = \log_b N$ if and only if $M = N$. Use this property to apply the logarithm to both sides of the rewritten equation from part (a). What is that equation?

A. $\log_{10}(a + b) = \log_{10}\left(10^{x+y}\right)$
B. $\log_{10}(a \cdot b) = \log_{10}\left(10^{x+y}\right)$
C. $\log_{10}(a \cdot b) = \log_{10}\left(10^{a+b}\right)$
D. $\log_{10}(a + b) = \log_{10}\left(10^{a+b}\right)$

Answer available at the end of this section.

(c)

Knowing that $\log_b(b^k) = k$, how could you simplify the right side of the equation? Recall that this property was covered in Section 3 (Introduction to logs).

A. $\log_{10}(a + b)$

B. $\log_{10}(x+y)$
C. $a+b$
D. $x+y$

Answer available at the end of this section.

(d)

Recall in part (a), we defined $10^x = a$ and $10^y = b$. What would these look like in logarithmic form?

A. $\log_{10} a = x$
B. $\log_x a = 10$
C. $\log_{10} b = y$
D. $\log_y b = 10$

Answer available at the end of this section.

(e)

Using your solutions in part (d), how can we rewrite the right side of the equation?

A. 10^{a+b}
B. $\log_{10} a - \log_{10} b$
C. $\log_{10} a + \log_{10} b$
D. $10^x + 10^y$

Answer available at the end of this section.

(f)

Combining parts (a) and (d), which equation represents $10^x \cdot 10^y = 10^{x+y}$ in terms of logarithms?

A. $\log_{10}(a+b) = 10^{a+b}$
B. $\log_{10}(a \cdot b) = \log_{10} a - \log_{10} b$
C. $\log_{10}(a \cdot b) = \log_{10} a + \log_{10} b$
D. $\log_{10}(a \cdot b) = 10^x + 10^y$

Answer available at the end of this section.

Activity 3.5.1.3

According to the law of exponents, we know that $\dfrac{10^x}{10^y} = 10^{x-y}$. Start with this equation as you move through this activity.

(a)

If we let $a = 10^x$ and $b = 10^y$ and apply a logarithm to both sides of the equation, what would be the result?

A. $\log_{10}(a-b) = \log_{10}\left(10^{x-y}\right)$
B. $\log_{10}\left(\dfrac{a}{b}\right) = \log_{10}\left(10^{x-y}\right)$
C. $\log_{10}\left(\dfrac{a}{b}\right) = \log_{10}\left(10^{a+b}\right)$
D. $\log_{10}(a-b) = \log_{10}\left(10^{a-b}\right)$

Answer available at the end of this section.

(b)

Knowing that $\log_b(b^k) = k$, how could you simplify the right side of the equation? Recall that this property was covered in *(see related activity)*.

A. $\log_{10}\left(\frac{a}{b}\right) = \log_{10}(a-b)$
B. $\log_{10}\left(\frac{a}{b}\right) = \log_{10}(x-y)$
C. $\log_{10}\left(\frac{a}{b}\right) = x-y$
D. $\log_{10}\left(\frac{a}{b}\right) = a-b$

Answer available at the end of this section.

(c)

Recall in part (a), we defined $10^x = a$ and $10^y = b$. What would these look like in logarithmic form? (Choose two.)

A. $x = \log_{10} a$
B. $x = \log_a 10$
C. $y = \log_{10} b$
D. $y = \log_b 10$

Answer available at the end of this section.

(d)

Combining these results, which equation represents $\frac{10^x}{10^y} = 10^{x-y}$ in terms of logarithms?

A. $\log_{10}\left(\frac{a}{b}\right) = \log_{10} a + \log_{10} b$
B. $\log_{10}\left(\frac{a}{b}\right) = 10^{x-y}$
C. $\log_{10}\left(\frac{a}{b}\right) = \log_{10} a - \log_{10} b$
D. $\log_{10}\left(\frac{a}{b}\right) = 10^{x+y}$

Answer available at the end of this section.

Note

In these last two activities, you explored an example of two common properties of logarithms! Let's define them.

Definition: Product property of logarithms and Quotient property of logarithms

The **product property of logarithms** states

$$\log_a(m \cdot n) = \log_a m + \log_a n \tag{3.5.1.8}$$

. The **quotient property of logarithms** states

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n \tag{3.5.1.9}$$

. Notice that for each of these properties, the base has to be the same.

Activity 3.5.1.4

There is still one more property to consider. This activity will investigate the power property. Suppose you are given

$$\log(x^3) = \log(x \cdot x \cdot x) \tag{3.5.1.10}$$

.

(a)

By applying *(see related activity)*, how could you rewrite the right-hand side of the equation?

A. $\log(3x)$
B. $\log(x) - \log(x) - \log(x)$
C. $\log(x^3)$
D. $\log(x) + \log(x) + \log(x)$

Answer available at the end of this section.

(b)

By combining "like" terms, you can simplify the right-hand side of the equation further. What equation do you have after simplifying the right-hand side?

A. $\log(x^3) = \log(3x)$
B. $\log(x^3) = -3\log(x)$
C. $\log(x^3) = \log(x^3)$
D. $\log(x^3) = 3\log(x)$

Answer available at the end of this section.

Definition: Power property of logarithms

The **power property of logarithms** states

$$\log_a(m^n) = n \cdot \log_a(m) \tag{3.5.1.11}$$

.

Activity 3.5.1.5

Apply *(see related activity)* and *(see related activity)* to expand the following. (Note: When you are asked to expand logarithmic expressions, your goal is to express a single logarithmic expression into many individual parts or components.)

(a)

$$\log_3\left(\frac{6}{19}\right) \tag{3.5.1.12}$$

A. $\log_3 6 + \log_3 19$
B. $\log_3(6 - 19)$
C. $\log_3 6 - \log_3 19$
D. $\log_3(6 + 19)$

Answer available at the end of this section.

(b)

$$\log\left((a \cdot b)^2\right) \tag{3.5.1.13}$$

A. $2(\log a + \log b)$
B. $2\log a - \log b$
C. $\log a + 2\log b$
D. $2\log a + 2\log b$

Answer available at the end of this section.

(c)

$$\ln\left(\frac{x^3}{y}\right) \tag{3.5.1.14}$$

A. $3\ln x + \ln y$
B. $3\ln x - \ln y$
C. $3(\ln x - \ln y)$
D. $\ln x^3 - \ln y$

Answer available at the end of this section.

(d)

$$\log(x \cdot y \cdot z^3) \tag{3.5.1.15}$$

A. $\log x + \log y + 3\log z$
B. $\log x + \log y + \log z^3$
C. $\log x - \log y - 3\log z$
D. $3(\log x + y + z)$

Answer available at the end of this section.

Activity 3.5.1.6

Apply what you learned in the above activities to condense into a single logarithm.

(a)

$$6\log_6 a + 3\log_6 b \tag{3.5.1.16}$$

A. $6(\log_6 a) + 3(\log_6 b)$
B. $\log_6 a^6 + \log_6 b^3$
C. $(\log_6 a)^6 + (\log_6 b)^3$
D. $\log_6 (a^6 \cdot b^3)$

Answer available at the end of this section.

(b)

$$\ln x - 4\ln y \tag{3.5.1.17}$$

A. $\ln x - \ln y^4$
B. $\ln\left(\frac{x}{y^4}\right)$
C. $\ln\left(\frac{x}{y}\right)^4$
D. $\ln(x \cdot y^4)$

Answer available at the end of this section.

(c)

$$2(\log(2x) - \log y) \tag{3.5.1.18}$$

A. $\log\left(\frac{4x^2}{y^2}\right)$

B. $\log\left(\frac{2x^2}{y^2}\right)$
C. $2 \cdot \log\left(\frac{2x}{y}\right)$
D. $\log\left(\frac{4x^2}{y}\right)$

Answer available at the end of this section.

(d)

$$\log 3 + 2\log 5 \tag{3.5.1.19}$$

A. $\log(3 \cdot 5^2)$
B. $2 \cdot \log 75$
C. $\log 75$
D. $\log 225$

Answer available at the end of this section.

Note

You might have noticed that a scientific calculator has only "log" and "ln" buttons (because those are the most common bases we use), but not all logs have base 10 or e as their bases.

Activity 3.5.1.7

Suppose you wanted to find the value of $\log_5 3$ in your calculator but you do not know how to input a base other than 10 or e (i.e., you only have the "log" and "ln" buttons on your calculator). Let's explore another helpful tool that can help us find the value of $\log_5 3$.

(a)

Let's start with the general statement, $\log_b a = x$. How can we rewrite this logarithmic equation into an exponential equation?

Answer available at the end of this section.

(b)

Now take the log of both sides of your equation and apply the power property of logarithms to bring the exponent down. What equation do you have now?

Answer available at the end of this section.

(c)

Solve for x. What does x equal?

Answer available at the end of this section.

(d)

Recall that when we started, we defined $x = \log_b a$. Substitute $\log_b a$ into your equation you got in part c for x. What is the resulting equation?

Answer available at the end of this section.

(e)

Apply what you got in part d to find the value of $\log_5 3$. What is the approximate value of $\log_5 3$?

Answer available at the end of this section.

Note

Notice in $3.5.1.7$, we were able to calculate $\log_5 3$ using logs of base 10. You should now be able to find the value of a logarithm of any base!

Definition: Change of base formula

The **change of base formula** is used to write a logarithm of a number with a given base as the ratio of two logarithms each with the same base that is different from the base of the original logarithm.

$$\log_b a = \frac{\log a}{\log b} \tag{3.5.1.20}$$

Activity $3.5.1.8$

Apply the above definition and a calculator to approximate the value of each logarithm.

(a)

$\log_2 30$

A. 0.204
B. 4.907
C. $\dfrac{\log 30}{\log 2}$
D. $\dfrac{\log 2}{\log 30}$

Answer available at the end of this section.

(b)

$\ln 183$

A. $\dfrac{\ln 183}{\ln e}$
B. 67.32
C. $\dfrac{\ln e}{\ln 183}$
D. 5.209

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 3.5.1.1

Solve the following problems to test your understanding of logarithmic properties.

(a) Expand the expression $\log_b\left(\frac{x^3y^2}{z^5}\right)$ fully using properties of logarithms. Assume all variables are positive.

Answer available at the end of this section.

(b) Condense the expression $2\ln(x)+\frac{1}{2}\ln(y)-3\ln(z)$ into a single logarithm. Assume all variables are positive.

Answer available at the end of this section.

(c) Evaluate the expression $\log_4(25)$ using the change-of-base formula with natural logarithms. Round your answer to four decimal places.

Answer available at the end of this section.

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Answers

Activity 3.5.1.1

$\log_b M = x \iff b^x = M$

$\log_b N = x \iff b^x = N$

Activity 3.5.1.1

Because $b^x = M$ and $b^x = N$, then it follows that $M = N$.

Activity 3.5.1.1

$\log_b M = \log_b N$

Activity 3.5.1.1

Based on what we have seen in parts (b) and (c), we can conclude that $M = N$.

Activity 3.5.1.2

D

Activity 3.5.1.2

B

Activity 3.5.1.2

D

Activity 3.5.1.2

A and C

Activity 3.5.1.2

C

Activity 3.5.1.2

C

Activity 3.5.1.3

B

Activity 3.5.1.3

C

Activity 3.5.1.3

A and C

Activity 3.5.1.3

C

Activity 3.5.1.4

D

Activity 3.5.1.4

D

Activity 3.5.1.5

C

Activity 3.5.1.5

A and D

Activity 3.5.1.5

B. Note that D is also correct, but it is not expanded fully.

Activity 3.5.1.5

A

Activity 3.5.1.6

D. B is also correct, but it is not condensed fully.

Activity 3.5.1.6

B

Activity 3.5.1.6

A

Activity 3.5.1.6

C. Note that A is also correct, but not condensed completely.

Activity 3.5.1.7

$b^x = a$

Activity 3.5.1.7

$x \cdot \log b = \log a$

Activity 3.5.1.7

$x = \dfrac{\log a}{\log b}$

Activity 3.5.1.7

$\log_b a = \dfrac{\log a}{\log b}$

Activity 3.5.1.7

$\log_5 3$ is approximately 0.683

Activity 3.5.1.8

B and C

Activity 3.5.1.8

A and D

Check Your Understanding 3.5.1.1

We apply the properties of logarithms step-by-step:

1. Apply the quotient rule, $\log_b\left(\dfrac{M}{N}\right) = \log_b(M) - \log_b(N)$:

$$\log_b\left(\frac{x^3y^2}{z^5}\right) = \log_b(x^3y^2) - \log_b(z^5) \tag{3.5.1.21}$$

2. Apply the product rule, $\log_b(M \cdot N) = \log_b(M) + \log_b(N)$:

$$\log_b(x^3y^2) - \log_b(z^5) = \log_b(x^3) + \log_b(y^2) - \log_b(z^5) \tag{3.5.1.22}$$

3. Apply the power rule, $\log_b(M^p) = p\log_b(M)$:

$$3\log_b(x) + 2\log_b(y) - 5\log_b(z) \tag{3.5.1.23}$$

Check Your Understanding 3.5.1.1

We apply the properties of logarithms in reverse order:

1. Apply the power rule, $p\log_b(M) = \log_b(M^p)$:

$$2\ln(x) + \frac{1}{2}\ln(y) - 3\ln(z) = \ln(x^2) + \ln\left(y^{1/2}\right) - \ln(z^3) = \ln(x^2) + \ln(\sqrt{y}) - \ln(z^3) \tag{3.5.1.24}$$

2. Apply the product rule to the positive terms:

$$\ln(x^2) + \ln(\sqrt{y}) - \ln(z^3) = \ln\left(x^2\sqrt{y}\right) - \ln(z^3) \tag{3.5.1.25}$$

3. Apply the quotient rule for subtraction:

$$\ln\left(\frac{x^2\sqrt{y}}{z^3}\right) \tag{3.5.1.26}$$

Check Your Understanding 3.5.1.1

Using the change-of-base formula $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$:

\[\log_{4}(25) = \dfrac{\ln(25)}{\ln(4)}\]

Using a calculator to find the natural logarithm values:

$$\ln(25) \approx 3.218876 \tag{3.5.1.27}$$

$$\ln(4) \approx 1.386294 \tag{3.5.1.28}$$

Dividing these values gives:

$$\log_4(25) \approx \frac{3.218876}{1.386294} \approx 2.3219 \tag{3.5.1.29}$$

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3.5.1.1: EL5 Online Exercises

EL5 — Use properties of logarithms to condense or expand logarithmic expressions.	EL5 CheckIt Exercises

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3.5.2: Solving Exponential and Logarithmic Equations

Learning Outcomes:

- Solve exponential and logarithmic equations.

Solving Exponential and Logarithmic Equations (EL6)

Note

Recall that we can convert between exponential and logarithmic forms.

$$\log_b x = y \tag{3.5.2.1}$$

is equivalent to

$$b^y = x \tag{3.5.2.2}$$

Activity 3.5.2.1

This activity will investigate ways we can solve logarithmic equations using properties and definitions from previous sections.

(a)

Given that $\log_3 9 = x$, how can we rewrite this into an exponential equation?

A. $9^x = 3$
B. $\log\left(\frac{9}{3}\right)$
C. $3^x = 9$
D. $x^3 = 9$

Answer available at the end of this section.

(b)

Now that $\log_3 9 = x$ is rewritten as an exponential equation, what is the value of x?

A. -2
B. 2
C. $-\frac{1}{2}$
D. $\frac{1}{2}$

Answer available at the end of this section.

Note

Notice in the above activity, you were able to solve a logarithmic equation by converting it into an exponential equation. This is one method in solving logarithmic equations.

Activity 3.5.2.2

For each of the following, solve the logarithmic equations by first converting them to exponential equations.

(a)

$\log_{10}(1{,}000{,}000) = x$

A. -6
B. $100{,}000$
C. 6
D. 0.00001

Answer available at the end of this section.

(b)

$\log_3(x+3) = 0$

A. 0
B. -2
C. 2
D. -1

Answer available at the end of this section.

(c)

$\log_5(2x+4) = 2$

A. 14
B. $\frac{29}{2}$
C. 16
D. $\frac{21}{2}$

Answer available at the end of this section.

Activity 3.5.2.3

Not all logarithmic equations can be solved by converting to exponential equations. In this activity, we will explore another way to solve logarithmic equations.

(a)

Suppose you are given the equation,

$$\log(4x-5) = \log(2x-1) \tag{3.5.2.3}$$

, and you brought all the logs to one side to get

$$\log(4x-5) - \log(2x-1) = 0 \tag{3.5.2.4}$$

. Using the quotient property of logs, how could you condense the left side of the equation?

A. $\log\left(\frac{4x-5}{2x-1}\right)$
B. $\log\left(\frac{2x-1}{4x-5}\right)$

C. $\dfrac{\log(4x-5)}{\log(2x-1)}$
D. $\dfrac{\log(2x-1)}{\log(4x-5)}$

Answer available at the end of this section.

(b)

Now that you have a single logarithm, convert this logarithmic form into exponential form. What does this new equation look like?

A. $\dfrac{4x-5}{2x-1}=10^0$
B. $\dfrac{2x-1}{4x-5}=1$
C. $\dfrac{2x-1}{4x-5}=10^0$
D. $\dfrac{4x-5}{2x-1}=1$

Answer available at the end of this section.

(c)

Notice that the "log" has disappeared and you now have an equation with just the variable x. Which of the following is equivalent to the equation you got in part (b)?

A. $4x-5=2x-1$
B. $4x-5=0$
C. $2x-1=0$

Answer available at the end of this section.

(d)

Compare the answer you got in part (c) to the original equation given $\log(4x-5)=\log(2x-1)$. What do you notice?

Answer available at the end of this section.

(e)

Solve the equation you got in part (d) to find the value of x.

A. -3
B. $\dfrac{5}{4}$
C. $\dfrac{1}{2}$
D. 2

Answer available at the end of this section.

Note

Notice in the above activity, that you did not have to convert the logarithmic equation into an exponential equation. A faster method, when you have a log on both sides of the equals sign, is to "drop" the logs and set the arguments equal to one another. Be careful though - you can only have one log on each side before you can "drop" them!

Activity 3.5.2.4

Apply the one-to-one property of logarithms and other properties of logarithms (i.e., product, quotient, and power) to solve the following logarithmic equations.

(a)

$\log(-2a+9) = \log(7-4a)$

Answer available at the end of this section.

(b)

$\log_9(x+6) - \log_9 x = \log_9 2$

Answer available at the end of this section.

(c)

$\log_8 2 + \log_8(4x^2) = 1$

Answer available at the end of this section.

(d)

$\ln(4x+1) - \ln 3 = 5$

Answer available at the end of this section.

Activity 3.5.2.5

In some cases, you will get equations with logs of different bases. Apply properties of logarithms to solve the following logarithmic equations.

(a)

$\log_3(x-6) = \log_9 x$

Hint:

Use the change of base formula to rewrite $\log_9 x$ *so that it has a base of* 3.

Answer available at the end of this section.

(b)

$\log_2 x = \log_8(4x)$

Answer available at the end of this section.

Activity 3.5.2.6

Now that we've looked at how to solve logarithmic equations, let's see how we can apply similar methods to solving exponential equations.

(a)

Suppose you are given the equation $2^x = 48$. There is no whole number value we can raise 2 to to get 48. What two whole numbers must x be between?

Answer available at the end of this section.

(b)

We'll use logarithms to isolate the variable in the exponent. How can we convert $2^x = 48$ into a logarithmic equation?

A. $x = \log_{48} 2$
B. $x = \log_2 48$
C. $48 = \log_2 x$
D. $2 = \log_{48} x$

Answer available at the end of this section.

(c)

Notice that the answer you got in part (b) is an exact answer for x. There will be times, though, that it will be helpful to also have an approximation for x. Which of the following is a good approximation for x?

A. $x \approx 5.585$
B. $x \approx 0.179$
C. $x \approx 24$
D. $x \approx \frac{1}{24}$

Answer available at the end of this section.

Note

Notice in the above activity we started with an exponential equation and then solved by converting the equation into a logarithmic equation. Logarithms can help us get the variable out of the exponent.

Activity 3.5.2.7

Although rewriting an exponential equation into a logarithmic equation is helpful at times, it is not the only method in solving exponential equations. In this activity, we will explore what happens when we take the log of both sides of an exponential equation and use the properties of logarithms to solve in another way.

(a)

Suppose you are given the equation:

$$3^x = 7 \tag{3.5.2.5}$$

. Take the log of both sides. What equation do you now have?

Answer available at the end of this section.

(b)

Apply the power property of logarithms (*(see related activity)*) to bring down the exponent. What equation do you now have?

Answer available at the end of this section.

(c)

Solve for x. What does x equal?

Answer available at the end of this section.

(d)

Using the change-of-base formula (*(see related activity)*), rewrite your answer from part (c) so that x is written as a single logarithm. What is the exact value of x?

Answer available at the end of this section.

(e)

If you were to solve $3^x = 7$ by converting it into a logarithmic equation, what would it look like?

Answer available at the end of this section.

(f)

What do you notice about your answer from parts (d) and (e)?

Answer available at the end of this section.

Activity 3.5.2.8

Use the method of taking the log of both sides (as you saw in *(see related activity)*) to solve $5^{2x+3} = 8$.

(a)

Take the log of both sides and use the power property of logarithms to bring down the exponent. What equation do you have now?

A. $2x + 3\log 5 = \log 8$
B. $2x + (3\log 5) = \log 8$
C. $(2x + 3) \cdot \log 5 = \log 8$
D. $2x \log 5 + 3 = \log 8$

Answer available at the end of this section.

(b)

Solve for x.

A. $x = \dfrac{\log_5 8 - 3}{2}$
B. $x = \dfrac{\log 8 - 3\log 5}{2}$
C. $x = \dfrac{\log 8}{3\log 5} - 2$
D. $x = \dfrac{\log 8 - 3}{2\log 5}$

Answer available at the end of this section.

(c)

What is the approximate value of x?

A. $x \approx -0.85$

B. $x \approx -1.5$
C. $x \approx -0.60$
D. $x \approx -1.57$

Answer available at the end of this section.

Activity 3.5.2.9

In this activity, we will explore other types of exponential equations, which will require other methods of solving.

(a)

Suppose you are given the equation

$$5^{3x} = 5^{7x-2} \tag{3.5.2.6}$$

and you decide to take the log of both sides as your first step to get

$$\log 5^{3x} = \log 5^{7x-2} \tag{3.5.2.7}$$

What would you use next to solve this equation?

A. Quotient property of logarithms
B. Power property of logarithms
C. Product property of logarithms
D. Change of base formula
E. One-to-one property of logarithms

Answer available at the end of this section.

(b)

Applying the property you chose in part a, what would the resulting equation be?

A. $3x \log 5 = 7x - 2 \log 5$
B. $3x \log 5 = 7x \log 5 - 2 \log 5$
C. $\log 5^{3x} = \log 5^{7x-2}$
D. $(3x) \log 5 = (7x - 2) \log 5$

Answer available at the end of this section.

(c)

Now that you have a logarithmic equation, divide both sides by $\log 5$ to begin to isolate the variable x. After dividing by $\log 5$, what equation do you now have?

A. $3x = \dfrac{7x}{\log 5} - 2$
B. $3x = 7x - 2$
C. $5^{3x} = 5^{7x-2}$
D. $\dfrac{3x \log 5}{\log 5} = \dfrac{(7x - 2) \log 5}{\log 5}$

Answer available at the end of this section.

(d)

Compare the equation you got in part (c) to the original equation given. What do you notice?

Answer available at the end of this section.

(e)

Solve for x.

A. 2
B. $\frac{1}{2}$
C. -2
D. $\frac{1}{2}$

Answer available at the end of this section.

Note

Notice in the above activity, it is much faster to set the exponents equal to one another. Make sure to check that the bases are equal *before* you set the exponents equal! And if the bases are not equal, you might have to use properties of exponents to help you get the bases to be the same.

Definition: One-to-one property of exponentials

When you are given an exponential equation with the same bases on both sides, you can simply set the exponents equal to one another and solve. This is known as the **one-to-one property of exponentials**.

Activity 3.5.2.10

When an exponential equation has the same base on each side, the exponents can be set equal to one another. If the bases aren't the same, we can rewrite them using properties of exponents and use the one-to-one property of exponentials.

(a)

Suppose you are given

$$5^x = 625 \tag{3.5.2.8}$$

, how could you rewrite this equation so that both sides have a base of 5?

Answer available at the end of this section.

(b)

Suppose you are given

$$4^x = 32 \tag{3.5.2.9}$$

, how could you rewrite this equation so that both sides have a base of 2?

Answer available at the end of this section.

(c)

Suppose you are given

$$3^{1-x} = \frac{1}{27} \tag{3.5.2.10}$$

, how could you rewrite this equation so that both sides have a base of 3? (Hint: you many need to revisit properties of exponents)

Answer available at the end of this section.

(d)

Suppose you are given

$$6^{\frac{x-3}{4}} = \sqrt{6} \tag{3.5.2.11}$$

, how could you rewrite this equation so that both sides have a base of 6? (Hint: you many need to revisit properties of exponents)

Answer available at the end of this section.

Activity 3.5.2.11

For each of the following, use properties of exponentials and logarithms to solve.

(a)

$6^{-2x} = 6^{2-3x}$

A. 2
B. $-\frac{2}{5}$
C. -2
D. $-\frac{5}{2}$

Answer available at the end of this section.

(b)

$\log_2 256 = x$

A. -8
B. 254
C. 8
D. 128

Answer available at the end of this section.

(c)

$5\ln(9x) = 20$

A. e^4
B. $\frac{e^4}{9}$
C. $e^{(\frac{4}{9})}$
D. $\frac{4}{\ln 9}$

Answer available at the end of this section.

(d)

$10^x = 4.23$

A. $\log 4.23$
B. 42.3

C. 1.44
D. 0.63

Answer available at the end of this section.

(e)

$\log_6(5x-5) = \log_6(3x+7)$

A. 2
B. 0
C. 6
D. 1

Answer available at the end of this section.

Activity 3.5.2.12

For each of the following, use properties of exponentials and logarithms to solve.

(a)

$\log_8 2 + \log_8 4x^2 = 1$

Answer available at the end of this section.

(b)

$5^{x+7} = 3$

Answer available at the end of this section.

(c)

$8^{\frac{x-6}{6}} = \sqrt{8}$

Answer available at the end of this section.

(d)

$\log_6(x+1) - \log_6 x = \log_6 29$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 3.5.2.1

Solve the following problems to test your understanding of solving exponential and logarithmic equations.

(a) Solve the equation $3^{2x-1} = 5^{x+2}$ for x. Give the exact answer and a decimal approximation rounded to three decimal places.

Answer available at the end of this section.

(b) Solve the equation $\log_2(x) + \log_2(x-6) = 4$ for x. Check for extraneous solutions.

Answer available at the end of this section.

(c) Solve the equation $e^{2x} - 5e^x + 6 = 0$ for x. Give the exact answer(s).

Answer available at the end of this section.

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Answers

Activity 3.5.2.1
C

Activity 3.5.2.1
B

Activity 3.5.2.2
C

Activity 3.5.2.2
B

Activity 3.5.2.2
D

Activity 3.5.2.3
A

Activity 3.5.2.3
A and D

Activity 3.5.2.3
A

Activity 3.5.2.3
Students should notice that the equation they have thus far is $4x - 5 = 2x - 1$, which is very similar to the original equation (only without the logs). Students may make a conjecture about “dropping” the logs to solve.

Activity 3.5.2.3
D

Activity 3.5.2.4
-1

Activity 3.5.2.4
6

Activity 3.5.2.4
1 and -1

Activity 3.5.2.4
$\frac{3e^5 - 1}{4}$ or 111.06

Activity 3.5.2.5
$x = 9$

Activity 3.5.2.5
$x = 0$ and $x = 2$

Activity 3.5.2.6
Between 5 and 6.

Activity 3.5.2.6
B

Activity 3.5.2.6
A

Activity 3.5.2.7

$\log 3^x = \log 7$

Activity 3.5.2.7

$x \cdot \log 3 = \log 7$

Activity 3.5.2.7

$x = \dfrac{\log 7}{\log 3}$

Activity 3.5.2.7

$x = \log_3 7$

Activity 3.5.2.7

$\log_3 7 = x$

Activity 3.5.2.7

Students should see that the two answers they got for parts (d) and (e) are the same. This might be a good time to discuss why taking the log of both sides is a valid method in solving an exponential equation.

Activity 3.5.2.8

C

Activity 3.5.2.8

A

Activity 3.5.2.8

A

Activity 3.5.2.9

B

Activity 3.5.2.9

B and D

Activity 3.5.2.9

B

Activity 3.5.2.9

Students will probably notice that their resulting equation are just the exponents of the original equation set equal to one another.

Activity 3.5.2.9

D

Activity 3.5.2.10

$5^x = 5^4$

Activity 3.5.2.10

$2^{2x} = 2^5$

Activity 3.5.2.10

$3^{1-x} = 3^{-3}$

Activity 3.5.2.10

$6^{\frac{x-3}{4}} = 6^{\frac{1}{2}}$

Activity 3.5.2.11

A

Activity 3.5.2.11

C

Activity 3.5.2.11

B

Activity 3.5.2.11

A and D

Activity 3.5.2.11

C

Activity 3.5.2.12

$x = 1, -1$

Activity 3.5.2.12

$x = \dfrac{log3}{log5} - 7$

Activity 3.5.2.12

$x = 9$

Activity 3.5.2.12

$x = \dfrac{1}{28}$

Check Your Understanding 3.5.2.1

We take the natural logarithm of both sides to solve for the exponent variable:

$$\ln(3^{2x-1}) = \ln(5^{x+2}) \tag{3.5.2.12}$$

Apply the power rule to bring the exponents to the front:

$$(2x-1)\ln(3) = (x+2)\ln(5) \tag{3.5.2.13}$$

Distribute the logarithm constants:

$$2x\ln(3) - \ln(3) = x\ln(5) + 2\ln(5) \tag{3.5.2.14}$$

Move all terms containing x to one side and constants to the other:

$$2x\ln(3) - x\ln(5) = 2\ln(5) + \ln(3) \tag{3.5.2.15}$$

Factor out x on the left side:

$$x(2\ln(3) - \ln(5)) = 2\ln(5) + \ln(3) \tag{3.5.2.16}$$

Divide by the coefficient of x to find the exact solution:

$$x = \frac{2\ln(5) + \ln(3)}{2\ln(3) - \ln(5)} \tag{3.5.2.17}$$

To find the decimal approximation, evaluate the logarithms using a calculator:

$$x = \frac{2(1.6094) + 1.0986}{2(1.0986) - 1.6094} \approx \frac{3.2189 + 1.0986}{2.1972 - 1.6094} \approx \frac{4.3175}{0.5878} \approx 7.345 \tag{3.5.2.18}$$

Check Your Understanding 3.5.2.1

First, apply the product rule of logarithms to condense the left side:

$$\log_2(x(x-6)) = 4 \tag{3.5.2.19}$$

Convert the equation to exponential form:

$$x(x-6) = 2^4 \implies x^2 - 6x = 16 \tag{3.5.2.20}$$

Set the quadratic equation to zero and solve by factoring:

$$x^2 - 6x - 16 = 0 \implies (x-8)(x+2) = 0 \tag{3.5.2.21}$$

This yields two potential solutions: $x = 8$ or $x = -2$.

We must check these in the original equation to ensure the arguments of all logarithms are positive:

- If $x = 8$: both arguments $x = 8$ and $x - 6 = 2$ are positive. (Valid)
- If $x = -2$: the argument $x = -2$ is negative, and $\log_2(-2)$ is undefined. (Extraneous)

Thus, the only valid solution is $x = 8$.

Check Your Understanding 3.5.2.1

Notice that this equation is quadratic in form. We can make a substitution letting $u = e^x$. Since $e^{2x} = (e^x)^2 = u^2$, the equation becomes:

$$u^2 - 5u + 6 = 0 \tag{3.5.2.22}$$

Solve the quadratic equation by factoring:

$$(u-2)(u-3) = 0 \implies u = 2 \text{ or } u = 3 \tag{3.5.2.23}$$

Substitute back $u = e^x$ to solve for x:

- $e^x = 2 \implies x = \ln(2)$
- $e^x = 3 \implies x = \ln(3)$

Since the exponential function is positive for all real exponents, both solutions are valid. The exact solutions are $x = \ln(2)$ and $x = \ln(3)$.

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3.5.2.1: EL6 Online Exercises

EL6 — Solve exponential and logarithmic equations.	EL6 CheckIt Exercises

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3.6: MODELING WITH EXPONENTIAL AND LOGARITHMIC FUNCTIONS (EL7)

3.6.1: Modeling with Exponential and Logarithmic Functions (EL7)

3.6.1.1: EL7 Online Exercises

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3.6.1: Modeling with Exponential and Logarithmic Functions (EL7)

Learning Outcomes:

- Solve application problems using exponential and logarithmic equations.

Modeling with Exponential and Logarithmic Functions

Note

Now that we have explored multiple methods for solving exponential and logarithmic equations, let's put those in to practice using some real-world application problems.

Activity 3.6.1.1

A coffee is sitting on Mr. Abacus's desk cooling. It cools according to the function $T = 70(0.80)^x + 20$, where x is the time elapsed in minutes and T is the temperature in degrees Celsius.

Newton's Law of Heating and Cooling states that the temperature of a body changes at a rate proportional to the difference in temperature between the body and its surroundings. In this case, we will be observing what happens to the temperature of Mr. Abacus's coffee as it's temperature adjusts to its surroundings.

(a)

What is the initial temperature of Mr. Abacus's coffee?

A. $20°C$
B. $70°C$
C. $0.80°C$
D. $90°C$

Answer available at the end of this section.

(b)

What is the temperature of Mr. Abacus's coffee after 10 minutes?

A. $7.5°C$
B. $20°C$
C. $27.5°C$
D. $76°C$

Answer available at the end of this section.

(c)

According to the function given, if Mr. Abacus leaves his coffee on his desk all day, what will the coffee eventually cool to?

A. $90°C$
B. $70°C$
C. $20°C$

D. *Hint:*

Think about what happens to the function as $x \to \infty$.

Answer available at the end of this section.

Activity 3.6.1.2

A video posted on YouTube initially had 80 views as soon as it was posted. The total number of views to date has been increasing exponentially according to the exponential growth function $y = 80e^{0.2t}$, where t represents time measured in days since the video was posted.

(a)

How many views will the video have after 3 days? Round to the nearest whole number.

A. 293 views
B. 146 views
C. 98 views
D. 82 views

Answer available at the end of this section.

(b)

How many days does it take until $2{,}500$ people have viewed this video?

A. 18 days
B. 156 days
C. 39 days
D. 17 days

Answer available at the end of this section.

Activity 3.6.1.3

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Recall that the general form of an exponential equation is $f(x) = a \cdot b^x$, where a is the initial value, b is the growth/decay factor, and x is time. We want to write a function $N(t)$ to represent the deer population after t years.

(a)

What is the initial value for the deer population?

Answer available at the end of this section.

(b)

We are not given the growth factor, so we must solve for it. Write an exponential equation using the initial population, the 2012 population, and the time elapsed.

Answer available at the end of this section.

(c)

Take your equation in part b and solve for b using logs.

Answer available at the end of this section.

(d)

Now that you have found the growth factor (from part b), what is the equation, in terms of $N(t)$ and t that represents the deer population?

Answer available at the end of this section.

(e)

If the growth continues according to this exponential function, when will the population reach 250?

Answer available at the end of this section.

Activity 3.6.1.4

The concentration of salt in ocean water, called salinity, varies as you go deeper in the ocean. Suppose $f(x) = 28.9 + 1.3\log(x + 1)$ models salinity of ocean water to depths of 1000 meters at a certain latitude, where x is the depth in meters and $f(x)$ is in grams of salt per kilogram of seawater. (Note that salinity is expressed in the unit g/kg, which is often written as ppt (part per thousand) or ‰ (permil).)

(a)

At 50 meters, what is the salinity of the seawater?

A. 31.1 ppt
B. 32.1 ppt
C. 51.6 ppt
D. 30.6 ppt

Answer available at the end of this section.

(b)

Approximate the depth (to the nearest tenth of a meter) where the salinity equals 33 ppt.

A. -1.0 meters
B. 1426.1 meters
C. -0.9 meters
D. 1424.1 meters

Answer available at the end of this section.

Activity 3.6.1.5

The first key on a piano keyboard (called A_0) corresponds to a pitch with a frequency of 27.5 cycles per second. With every successive key, going up the black and white keys, the pitch multiplies by a constant. The formula for the frequency, f of the pitch sounded when the nth note up the keyboard is played is given by

$$n = 1 + 12\log_2\frac{f}{27.5} \tag{3.6.1.1}$$

The equal tempered scale is the common musical scale used for the tuning of pianos and other instruments of relatively fixed scale. This is a 12-tone equal temperament, but there are many other temperaments.

(a)

A note has a frequency of 220 cycles per second. How many notes up the piano keyboard is this?

A. 37 notes
B. 39 notes
C. 12 notes
D. 96 notes

Answer available at the end of this section.

(b)

What frequency does the 12th note have? Round to the nearest tenth.

A. 0.1 cycles per second
B. 227.0 cycles per second
C. 51.9 cycles per second
D. 49.4 cycles per second

Answer available at the end of this section.

Note

Another application of exponential equations is compound interest. Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use compound interest. The term compounding refers to interest earned not only on the original value, but on the accumulated value of the account.

Compound interest can be calculated by using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \tag{3.6.1.2}$$

, where $A(t)$ is the account value, t is measured in years, P is the starting amount of the account (also known as the principal), r is the annual percentage rate (APR) written as a decimal, and n is the number of compounding periods in one year.

Activity $3.6.1.6$

Before we can apply the compound interest formula, we need to understand what “compounding” means. Recall that compounding refers to interest earned not only on the original value, but on the accumulated value of the account. This amount is calculated a certain number of times in a given year.

(a)

Suppose an account is compounded quarterly, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

Answer available at the end of this section.

(b)

Suppose an account is compounded daily, what value of n would we use in the formula?

A. 1

B. 2
C. 4
D. 12
E. 365

Answer available at the end of this section.

(c)

Suppose an account is compounded semi-annually, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

Answer available at the end of this section.

(d)

Suppose an account is compounded monthly, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

Answer available at the end of this section.

(e)

Suppose an account is compounded annually, what value of n would we use in the formula?

A. 1
B. 2
C. 4
D. 12
E. 365

Answer available at the end of this section.

Activity $3.6.1.7$

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily currently has $\$10{,}000$ and opens a 529 account that will earn 6% compounded semi-annually.

(a)

Which equation could we use to determine how much money Lily will have for her granddaughter after t years?

A. $A(t) = 10{,}000\left(1+\frac{6}{2}\right)^{2t}$
B. $A(t) = 10{,}000\left(1+\frac{0.06}{2}\right)^{2t}$

C. $A(t) = 10{,}000\left(1 + \dfrac{6}{\frac{1}{2}}\right)^{\frac{1}{2}t}$

D. $A(t) = 10{,}000\left(1 + \dfrac{0.06}{2}\right)^{18}$

Answer available at the end of this section.

(b)

To the nearest dollar, how much will Lily have in the account in 10 years?

A. $\$106{,}090$
B. $\$103{,}000$
C. $\$13{,}439$
D. $\$18{,}061$

Answer available at the end of this section.

(c)

How many years will it take Lily to have $\$40{,}000$ in the account for her granddaughter? Round to the nearest tenth.

A. 23.4 years
B. 46.9 years
C. 52.1 years
D. 25.6 years

Answer available at the end of this section.

Activity 3.6.1.8

For each of the following, determine the appropriate equation to use to solve the problem.

(a)

Kathy plans to purchase a car that depreciates (loses value) at a rate of 14% per year. The initial cost of the car is $\$21{,}000$. Which equation represents the value, v, of the car after 3 years?

A. $v = 21{,}000(0.14)^3$
B. $v = 21{,}000(0.86)^3$
C. $v = 21{,}000(1.14)^3$
D. $v = 21{,}000(0.86)(3)$

Answer available at the end of this section.

(b)

Mr. Smith invested $\$2{,}500$ in a savings account that earns 3% interest compounded annually. He made no additional deposits or withdrawals. Which expression can be used to determine the number of dollars in this account at the end of 4 years?

A. $2{,}500(1 + 0.03)^4$
B. $2{,}500(1 + 0.3)^4$
C. $2{,}500(1 + 0.04)^3$
D. $2{,}500(1 + 0.4)^3$

Answer available at the end of this section.

Activity 3.6.1.9

Suppose you want to invest $\$100$ in a banking account that has a 100% interest rate. Let's investigate what would happen to the amount of money you have at the end of one year in the account with varying compounding periods. Use the formula, $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$, to help you solve the following problems.

(a)

Suppose the account is compounded annually (i.e., $n = 1$). How much money would you have in the account at the end of the year?

Answer available at the end of this section.

(b)

By what factor did the money in your account grow in part a?

Answer available at the end of this section.

(c)

Suppose the account is compounded semi-annually. How much money would you have in the account at the end of the year?

Answer available at the end of this section.

(d)

By what factor did the money in your account grow in part b?

Answer available at the end of this section.

(e)

Let's investigate as the compounding periods increase. Fill in the following table.

n	$A(t)$	**Growth Factor**
1		
2		
5		
10		
100		
1,000		
10,000		
100,000		

Answer available at the end of this section.

(f)

What do you notice as the value of n increases?

Answer available at the end of this section.

Note

In *(see related activity)*, we can see that as the value of n increases, the factor by which the money increased by tended toward the value of 2.71. This value is a significant mathematical constant and is denoted by the symbol e. It is approximately equal to 2.718.

Note

For many real-world phenomena, e is used as the base for exponential functions. Exponential models that use e as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

For all real numbers t, and all positive numbers a and r, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt} \tag{3.6.1.3}$$

, where a is the initial value, r is the continuous growth rate per of unit time, and t is the elapsed time. If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt} \tag{3.6.1.4}$$

, where P is the principal or the initial invested, r is the growth or interest rate per of unit time, and t is the period or term of the investment.

Activity 3.6.1.10

Use the continuous formulas to answer the following questions.

(a)

A person invested $\$1{,}000$ in an account earning 10% per year compounded continuously. How much was in the account at the end of one year?

Answer available at the end of this section.

(b)

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 3.6.1.1

Solve the following application problems to test your understanding of exponential and logarithmic models.

(a) A certain bacteria culture doubles in size every 6 hours. Initially, there are 500 bacteria.

- Find an exponential model $P(t) = a \cdot b^t$ for the bacteria population after t hours.
- How many bacteria are present after 15 hours?
- After how many hours will the population reach 10,000 bacteria? Round to two decimal places.

Answer available at the end of this section.

(b) Carbon-14 has a half-life of 5730 years. A fossilized bone is found to contain 15% of the carbon-14 originally present. Determine the age of the fossil. Round your answer to the nearest year.

Answer available at the end of this section.

(c) The pH of a solution is defined by $\mathrm{pH} = -\log_{10}[H^+]$, where $[H^+]$ is the hydrogen ion concentration in moles per liter.

- If a solution has a pH of 3.4, find its hydrogen ion concentration $[H^+]$.
- If the hydrogen ion concentration of a solution is 2.5×10^{-6} moles/liter, find its pH. Round to one decimal place.

Answer available at the end of this section.

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Answers

Activity 3.6.1.1

D

Activity 3.6.1.1

C

Activity 3.6.1.1

C

Activity 3.6.1.2

B

Activity 3.6.1.2

A

Activity 3.6.1.3

80 deer

Activity 3.6.1.3

$180 = 80 \cdot b^6$

Activity 3.6.1.3

b is approximately 1.1447

Activity 3.6.1.3

$N(t) = 80(1.1447)^t$

Activity 3.6.1.3

It will take 8.43 years, which means the deer population will reach 250 in the year 2015.

Activity 3.6.1.4

A

Activity 3.6.1.4

D

Activity 3.6.1.5

A. This note is called A_3.

Activity 3.6.1.5

C

Activity 3.6.1.6

C

Activity 3.6.1.6

E

Activity 3.6.1.6

B

Activity 3.6.1.6

D

Activity 3.6.1.6

A

Activity 3.6.1.7

B

Activity 3.6.1.7

D

Activity 3.6.1.7

A

Activity 3.6.1.8

B

Activity 3.6.1.8

A

Activity 3.6.1.9

$\$200$

Activity 3.6.1.9

Because you initially started with $\$100$ and the account grew to $\$200$ by the end of the year, the account grew by 100% (doubled), or by a factor of 2.

Activity 3.6.1.9

$\$225$

Activity 3.6.1.9

At the end of the year, you will have $\$225$, which is 225% in growth, or a factor of 2.25.

Activity 3.6.1.9

n	$A(t)$	**Growth Factor**
1	$\$200$	2
2	$\$225$	2.25
5	$\$248.83$	≈ 2.4883
10	$\$259.37$	≈ 2.5937
100	$\$270.48$	≈ 2.7048
$1{,}000$	$\$271.69$	≈ 2.7169
$10{,}000$	$\$271.82$	≈ 2.7182
$100{,}000$	$\$271.83$	≈ 2.7183

Activity 3.6.1.9

The growth factor seems to tend towards ≈ 2.71 as the value of n increases. In other words, as the interval gets smaller, the total returns get slightly higher. If interest is calculated n times per year, at a rate of $\frac{100\%}{n}$, the total accreted wealth at the end of the first year would be slightly greater than 2.7 times the initial investment if n is sufficiently large.

Activity 3.6.1.10

$\$1{,}105.17$

Activity 3.6.1.10

59.51 mg

Check Your Understanding 3.6.1.1

Let's solve each part:

- **Model:** Since the initial population is 500, we have $a = 500$. The population doubles (factor of 2) every 6 hours, which gives the base as $2^{1/6}$. Thus, the population model is:

$$P(t) = 500(2)^{t/6} \tag{3.6.1.5}$$

- **Population at $t = 15$:** Substitute 15 for t:

$$P(15) = 500(2)^{15/6} = 500(2)^{2.5} \approx 500(5.65685) \approx 2828 \text{ bacteria} \tag{3.6.1.6}$$

- **Time to reach 10,000:** Set $P(t) = 10,000$ and solve for t:

$$10,000 = 500(2)^{t/6} \implies 20 = 2^{t/6} \tag{3.6.1.7}$$

Take the natural logarithm of both sides:

$$\ln(20) = \ln(2^{t/6}) \implies \ln(20) = \frac{t}{6}\ln(2) \tag{3.6.1.8}$$

Isolate t:

$$t = 6 \cdot \frac{\ln(20)}{\ln(2)} \approx 6 \cdot \frac{2.99573}{0.69315} \approx 25.93 \text{ hours} \tag{3.6.1.9}$$

Check Your Understanding 3.6.1.1

The radioactive decay model for Carbon-14 is given by:

$$A(t) = A_0(0.5)^{t/5730} \tag{3.6.1.10}$$

where A_0 is the original amount and t is the age in years. Since 15% remains, we set $A(t) = 0.15A_0$:

$$0.15A_0 = A_0(0.5)^{t/5730} \implies 0.15 = (0.5)^{t/5730} \tag{3.6.1.11}$$

Take the natural logarithm of both sides:

$$\ln(0.15) = \ln\left((0.5)^{t/5730}\right) \implies \ln(0.15) = \frac{t}{5730}\ln(0.5) \tag{3.6.1.12}$$

Solve for t:

$$t = 5730 \cdot \frac{\ln(0.15)}{\ln(0.5)} \approx 5730 \cdot \frac{-1.89712}{-0.69315} \approx 15,683 \text{ years} \tag{3.6.1.13}$$

The fossil is approximately 15,683 years old.

Check Your Understanding 3.6.1.1

Let's solve each part:

- **Finding $[H^+]$ from pH = 3.4:**

$$3.4 = -\log_{10}[H^+] \implies -3.4 = \log_{10}[H^+] \tag{3.6.1.14}$$

Convert to exponential form:

$$[H^+] = 10^{-3.4} \approx 3.98 \times 10^{-4} \text{ moles/liter} \tag{3.6.1.15}$$

- **Finding pH from $[H^+] = 2.5 \times 10^{-6}$:**

$$\text{pH} = -\log_{10}\left(2.5 \times 10^{-6}\right) \tag{3.6.1.16}$$

Apply logarithm properties:

$$\text{pH} = -\left(\log_{10}(2.5) + \log_{10}(10^{-6})\right) = -\left(\log_{10}(2.5) - 6\right) \approx -(0.39794 - 6) \approx 5.6 \tag{3.6.1.17}$$

The pH of the solution is approximately 5.6.

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3.6.1.1: EL7 Online Exercises

EL7 — Solve exponential and logarithmic equations.	EL7 CheckIt Exercises

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4: TRIGONOMETRY AND TRIANGLES IN MOTION

Building on the unit circle, students extend trigonometric reasoning to right triangles, oblique triangles, and identities. This unit emphasizes reasoning across multiple representations — ratio, graph, equation — and applying trigonometry to real-world and geometric contexts, reinforcing ideas that reappear throughout calculus and engineering.

4.1: Introduction to Additional Trigonometric Functions (PF2)

4.1.1: Additional Trigonometric Functions (PF2)

4.1.1.1: PF2 Online Exercises

4.2: Introduction to Inverse Trig Functions (PF3)

4.2.1: Inverse Trig Functions (PF3)

4.2.1.1: PF3 Online Exercises

4.3: Introduction to Solving Triangles (TE4, TE5, TE6)

4.3.1: Solving Right Triangles (TE4)

4.3.1.1: TE4 Online Exercises

4.3.2: Trigonometric Laws (TE5)

4.3.2.1: TE Online Exercises

4.3.3: Applications of Trig (TE6)

4.3.3.1: TE6 Online Exercises

4.4: Introduction to Other Trig and Identities (TE1, TE2, TE3)

4.4.1: Trigonometric Identities (TE1)

4.4.1.1: TE1 Online Exercises

4.4.2: Verifying Identities (TE2)

4.4.2.1: TE2 Online Exercises

4.4.3: Trigonometric Equations (TE3)

4.4.3.1: TE3 Online Exercises

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4.1: INTRODUCTION TO ADDITIONAL TRIGONOMETRIC FUNCTIONS (PF2)

4.1.1: Additional Trigonometric Functions (PF2)

4.1.1.1: PF2 Online Exercises

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4.1.1: Additional Trigonometric Functions (PF2)

This section corresponds to APC slot 4.5 (Other trig functions & identities) in the Math 172 flow.

PF2 Learning outcome: Graph trigonometric functions including tangent, cotangent, secant, and cosecant, and transformations of these functions, and determine the domain and range.

Graphing Tangent, Cotangent, Secant, and Cosecant

Overview

In the previous sections, we looked at graphs of the sine and cosine functions. We will now look at graphs of the other four trigonometric functions.

Activity 4.1.1.1

Consider the tangent function, $f(x) = \tan(x)$.

(a)

Fill in the missing values in the table below for $f(x) = \tan(x)$. Find the exact values, then express as a decimal, approximated to two decimal places if needed. (Notice that the values in the table are all the standard angles found on the unit circle!)

x	$\tan(x)$ **(exact)**	$\tan(x)$ **(as a decimal)**
0		
$\frac{\pi}{6}$	$\frac{1}{\sqrt{3}}$	
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		≈ 1.73
$\frac{\pi}{2}$		
$\frac{2\pi}{3}$		
$\frac{3\pi}{4}$		
$\frac{5\pi}{6}$		≈ -0.58
π		
$\frac{7\pi}{6}$		
$\frac{5\pi}{4}$		
$\frac{4\pi}{3}$	$\sqrt{3}$	
$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$		

$\frac{7\pi}{4}$		
$\frac{11\pi}{6}$		
2π		

Answer available at the end of this section.

(b)

What do you think is happening to the graph at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$?

A. The graph has a hole.
B. The graph has a horizontal asymptote.
C. The graph has a vertical asymptote.

Hint:

Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Answer available at the end of this section.

(c)

Plot these values on a coordinate plane to approximate the graph of $f(x) = \tan(x)$. Then sketch in the graph of the tangent curve using the points as a guide.

Answer available at the end of this section.

(d)

What is the domain of $\tan(x)$?

A. $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$
B. $\ldots \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$
C. $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$
D. $(-\infty, \infty)$

Answer available at the end of this section.

(e)

What is the range of $\tan(x)$?

A. $[-1, 1]$
B. $(-\infty, -1] \cup [1, \infty)$
C. $(-\infty, 0) \cup (0, \infty)$
D. $(-\infty, \infty)$

Answer available at the end of this section.

(f)

What is the period of $\tan(x)$?

A. $\frac{\pi}{2}$
B. π
C. 2π
D. 4π

Answer available at the end of this section.

Activity 4.1.1.2

Consider the secant function, $f(x) = \sec(x)$.

(a)

Fill in the missing values in the table below for $f(x) = \sec(x)$. Find the exact values, then express as a decimal, approximated to two decimal places if needed. (Notice that the values in the table are all the standard angles found on the unit circle!)

x	$\sec(x)$ **(exact)**	$\sec(x)$ **(as a decimal)**
0		
$\frac{\pi}{6}$		
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		
$\frac{2\pi}{3}$		
$\frac{3\pi}{4}$		≈ -1.41
$\frac{5\pi}{6}$		
π		
$\frac{7\pi}{6}$	$-\frac{2}{\sqrt{3}}$	
$\frac{5\pi}{4}$		
$\frac{4\pi}{3}$		
$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	2	
$\frac{7\pi}{4}$		
$\frac{11\pi}{6}$		
2π		

Answer available at the end of this section.

(b)

What do you think is happening to the graph at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$?

A. The graph has a hole.
B. The graph has a horizontal asymptote.
C. The graph has a vertical asymptote.

Answer available at the end of this section.

(c)

Plot these values on a coordinate plane to approximate the graph of $f(x) = \sec(x)$. Then sketch in the graph of the secant curve using the points as a guide.

Answer available at the end of this section.

(d)

What is the domain of $\sec(x)$?

A. $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$
B. $\ldots \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$
C. $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$
D. $(-\infty, \infty)$

Answer available at the end of this section.

(e)

What is the range of $\sec(x)$?

A. $[-1, 1]$
B. $(-\infty, -1] \cup [1, \infty)$
C. $(-\infty, 0) \cup (0, \infty)$
D. $(-\infty, \infty)$

Answer available at the end of this section.

(f)

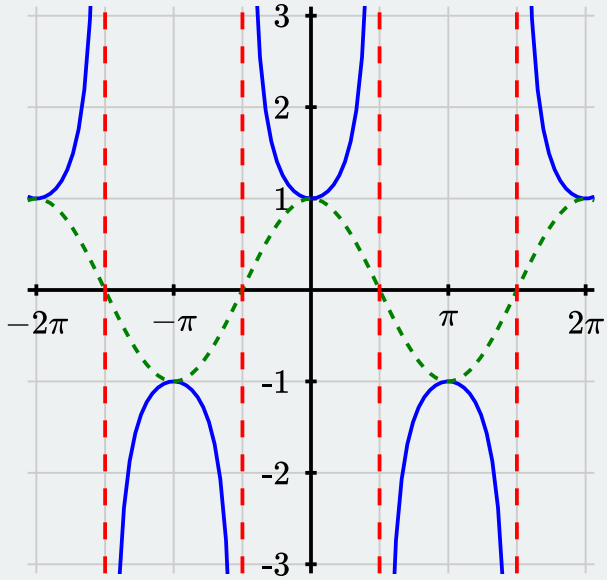
What is the period of $\sec(x)$?

A. $\frac{\pi}{2}$
B. π
C. 2π
D. 4π

Answer available at the end of this section.

Observation

Since $\sec(x) = \frac{1}{\cos(x)}$, we can see that their graphs are related: $\sec(x)$ (solid blue curve) has a vertical asymptote everywhere $\cos(x)$ (dotted green curve) has a zero, and for every point (a, b) on the graph of $\cos(x)$, the point $(a, \frac{1}{b})$ is on the graph of $\sec(x)$.



Activity 4.1.1.3

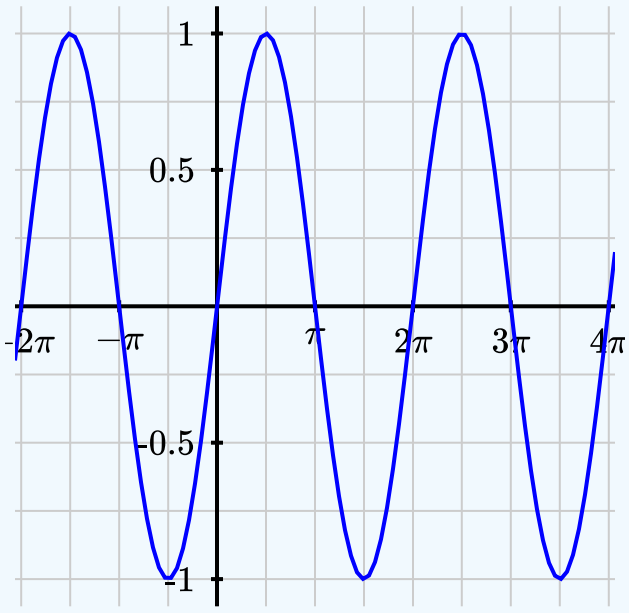
Consider the cosecant function, $f(x) = \csc(x)$. While we could make a table as in *(see related activity)*, let's instead take advantage of the fact that the graphs of $\csc(x)$ and its reciprocal $\sin(x)$ will be related in the same way as the graphs of $\sec(x)$ and its reciprocal $\cos(x)$.

(a)

Where does $\sin(x)$ have zeros?

Hint:

Recall the graph of $\sin(x)$



Answer available at the end of this section.

(b)

Where does $\csc(x)$ have vertical asymptotes?

Answer available at the end of this section.

(c)

Where does $\sin(x)$ have local maximum and minimum values?

Answer available at the end of this section.

(d)

Where does $\csc(x)$ have local maximum and minimum values?

Answer available at the end of this section.

(e)

On what intervals is $\sin(x)$ increasing and decreasing?

Answer available at the end of this section.

(f)

On what intervals is $\csc(x)$ increasing and decreasing?

Answer available at the end of this section.

(g)

Use your answers to the previous tasks to plot the graph of $y = \csc(x)$.

Hint:

It may be helpful to first sketch the graph of $\sin(x)$.

Answer available at the end of this section.

(h)

What is the domain of $\csc(x)$?

A. $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$
B. $\ldots \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$
C. $(0, \pi) \cup (\pi, 2\pi) \cup \ldots$
D. $(-\infty, \infty)$

Answer available at the end of this section.

(i)

What is the range of $\csc(x)$?

A. $[-1, 1]$
B. $(-\infty, -1] \cup [1, \infty)$

C. $(-\infty, 0) \cup (0, \infty)$
D. $(-\infty, \infty)$

Answer available at the end of this section.

(j)

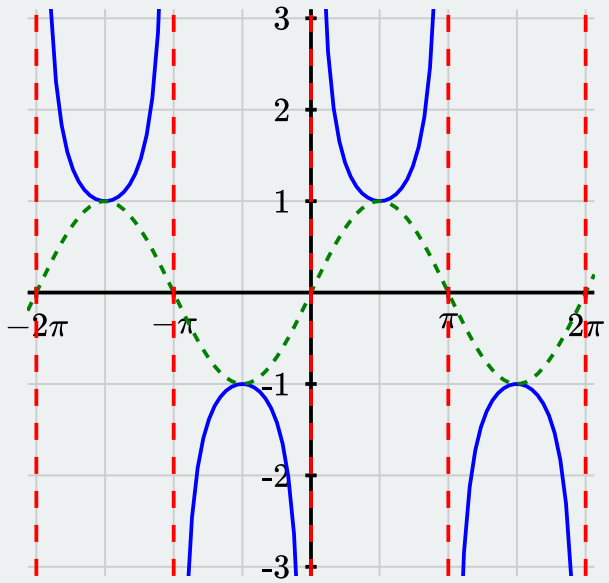
What is the period of $\csc(x)$?

A. $\frac{\pi}{2}$
B. π
C. 2π
D. 4π

Answer available at the end of this section.

Observation

Since $\csc(x) = \frac{1}{\sin(x)}$, we can see that their graphs are related: $\csc(x)$ (solid blue) has a vertical asymptote everywhere $\sin(x)$ (dotted green) has a zero, and for every point (a, b) on the graph of $\sin(x)$, the point $(a, \frac{1}{b})$ is on the graph of $\csc(x)$.



Activity 4.1.1.4

Consider the cotangent function, $f(x) = \cot(x)$.

(a)

Where does $\tan(x)$ have zeros?

Hint:

Recall the graph of $\tan(x)$

Answer available at the end of this section.

(b)

Where does $\cot(x)$ have vertical asymptotes?

Answer available at the end of this section.

(c)

Where does $\tan(x) = 1$ and where does $\tan(x) = -1$?

Answer available at the end of this section.

(d)

Where does $\cot(x) = 1$ and where does $\cot(x) = -1$?

Answer available at the end of this section.

(e)

On what intervals is $\tan(x)$ increasing and decreasing?

Answer available at the end of this section.

(f)

On what intervals is $\cot(x)$ increasing and decreasing?

Answer available at the end of this section.

(g)

Use your answers to the previous tasks to plot the graph of $y = \cot(x)$.

Hint:

It may be helpful to first sketch the graph of $\tan(x)$.

Answer available at the end of this section.

(h)

What is the domain of $\cot(x)$?

A. $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$
B. $\ldots \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$
C. $(0, \pi) \cup (\pi, 2\pi) \cup \ldots$
D. $(-\infty, \infty)$

Answer available at the end of this section.

(i)

What is the range of $\cot(x)$?

A. $[-1, 1]$
B. $(-\infty, -1] \cup [1, \infty)$

C. $(-\infty, 0) \cup (0, \infty)$
D. $(-\infty, \infty)$

Answer available at the end of this section.

(j)

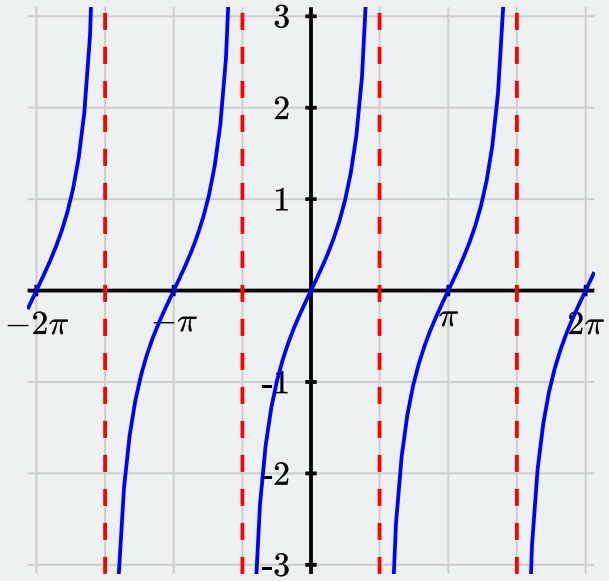
What is the period of $\cot(x)$?

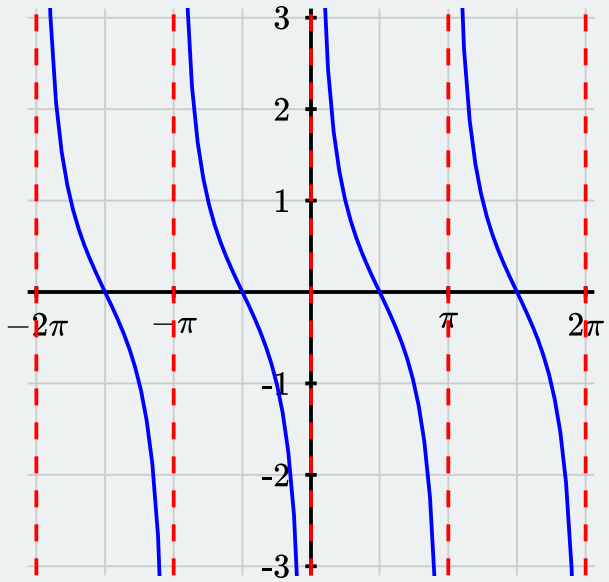
A. $\frac{\pi}{2}$
B. π
C. 2π
D. 4π

Answer available at the end of this section.

Observation

Since $\cot(x) = \frac{1}{\tan(x)}$, we can see that their graphs are related: $\cot(x)$ has a vertical asymptote everywhere $\tan(x)$ has a zero (and vice versa), and for every point (a, b) on the graph of $\tan(x)$, the point $(a, \frac{1}{b})$ is on the graph of $\cot(x)$.





Note

The graphs of all six *(see related activity)* are listed in *(see related activity)*.

Observation

Everything we learned about transformations of functions in *(see related activity)* applies equally well to trigonometric functions.

Activity 4.1.1.5

Consider the function $g(x) = \sec\left(x + \frac{\pi}{2}\right)$.

(a)

How is this graph related to the graph of $\sec(x)$?

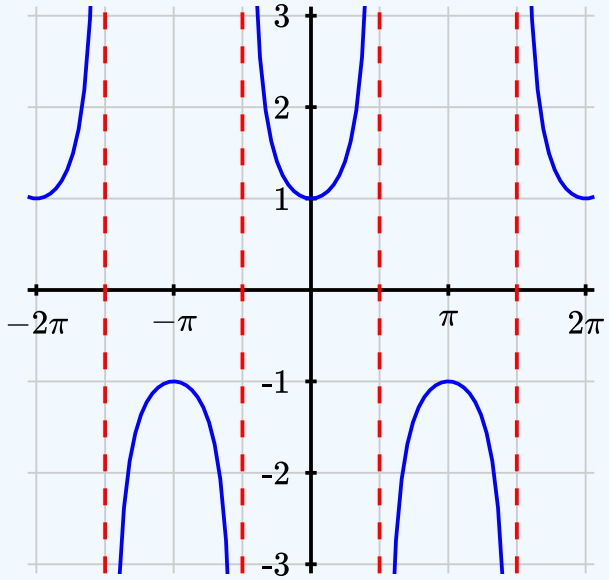
A. It is shifted left $\frac{\pi}{2}$.
B. It is shifted right $\frac{\pi}{2}$.
C. It is shifted up $\frac{\pi}{2}$.
D. It is shifted down $\frac{\pi}{2}$.

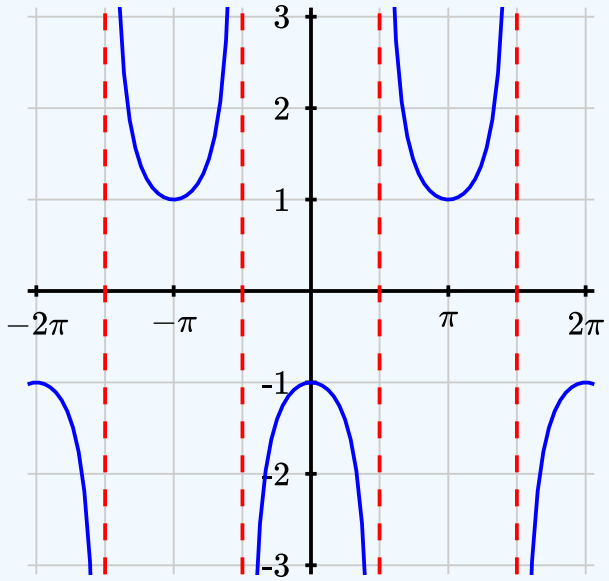
Answer available at the end of this section.

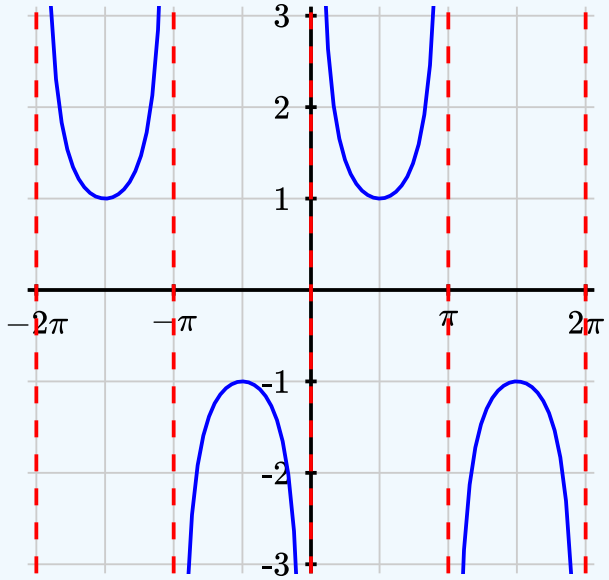
(b)

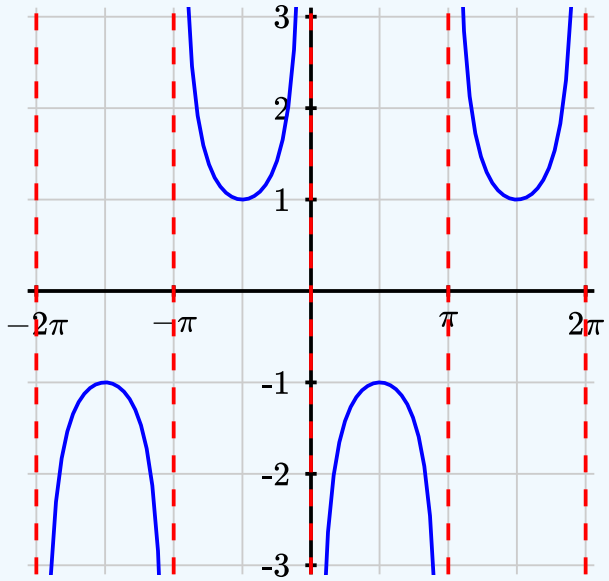
Which of the following is the graph of $g(x) = \sec\left(x + \frac{\pi}{2}\right)$?

A.
B.
C.
D.


3
2
1
-1
-2
-3
-2π
$-\pi$
π
2π


3
2
1
-1
-2
-3
-2π
$-\pi$
π
2π





Answer available at the end of this section.

(c)

What is the domain of $g(x) = \sec\left(x + \frac{\pi}{2}\right)$?

Answer available at the end of this section.

(d)

What is the range of $g(x) = \sec\left(x + \frac{\pi}{2}\right)$?

Answer available at the end of this section.

(e)

What is the period of $g(x) = \sec\left(x + \frac{\pi}{2}\right)$?

Answer available at the end of this section.

Activity 4.1.1.6

Consider the function $h(x) = \tan(2x)$

(a)

What sort of transformation of $\tan(x)$ is $\tan(2x)$?

A. Horizontal stretch
B. Horizontal compression
C. Vertical stretch
D. Vertical compression

Answer available at the end of this section.

(b)

Which of the following are different for $\tan(2x)$ than for $\tan(x)$? Select all that apply.

A. Location of vertical asymptotes
B. Period
C. Domain
D. Range

Answer available at the end of this section.

(c)

Where are the vertical asymptotes of $h(x) = \tan(2x)$?

A. $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$
B. $\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$
C. $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
D. $\ldots, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$

Hint:

Recall that $\tan(x)$ has vertical asymptotes at $\frac{\pi}{2} + \pi k$ for each integer k.

Answer available at the end of this section.

(d)

Where are the zeros of $h(x) = \tan(2x)$?

A. $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$
B. $\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$
C. $\ldots, , -\frac{\pi}{2}, 0, \frac{\pi}{2}, \ldots$

D. $\ldots, -\frac{\pi}{4}, 0, \frac{\pi}{4}, , \ldots$

Hint:

Recall that $\tan(x)$ has zeroes at πk for each integer k.

Answer available at the end of this section.

(e)

Graph $h(x) = \tan(2x)$.

Answer available at the end of this section.

(f)

What is the domain of $h(x) = \tan(2x)$?

Answer available at the end of this section.

(g)

What is the range of $h(x) = \tan(2x)$?

Answer available at the end of this section.

(h)

What is the period of $h(x) = \tan(2x)$?

Answer available at the end of this section.

Activity 4.1.1.7

Consider the function $k(x) = 3\csc\left(\frac{x}{2}\right)$.

(a)

Where are the vertical asymptotes for $k(x) = 3\csc\left(\frac{x}{2}\right)$ located?

Answer available at the end of this section.

(b)

Where are the local minima and local maxima for $k(x) = 3\csc\left(\frac{x}{2}\right)$ located?

Answer available at the end of this section.

(c)

What are the local minimum and local maximum values for $k(x) = 3\csc\left(\frac{x}{2}\right)$?

Answer available at the end of this section.

(d)

Graph $k(x) = 3\csc\left(\frac{x}{2}\right)$.

Answer available at the end of this section.

(e)

What is the domain of $k(x) = 3\csc\left(\frac{x}{2}\right)$?

Answer available at the end of this section.

(f)

What is the range of $k(x) = 3\csc\left(\frac{x}{2}\right)$?

Answer available at the end of this section.

(g)

What is the period of $k(x) = 3\csc\left(\frac{x}{2}\right)$?

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can describe the graphs, domain, and range of tangent, cotangent, secant, and cosecant before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 4.1.1.1

Explain why the graph of $y = \tan(x)$ has vertical asymptotes at $x = \frac{\pi}{2} + n\pi$ (for any integer n), and state the domain of tangent using this fact.

Answer available at the end of this section.

Check Your Understanding 4.1.1.2

State the range of $y = \sec(x)$, and explain how it relates to the range of $y = \cos(x)$.

Answer available at the end of this section.

Check Your Understanding 4.1.1.3

The graph of $y = 2\csc(x)$ is a vertical stretch of $y = \csc(x)$ by a factor of 2. Describe what happens to the vertical asymptotes of cosecant under this transformation, and explain why.

Answer available at the end of this section.

Answers

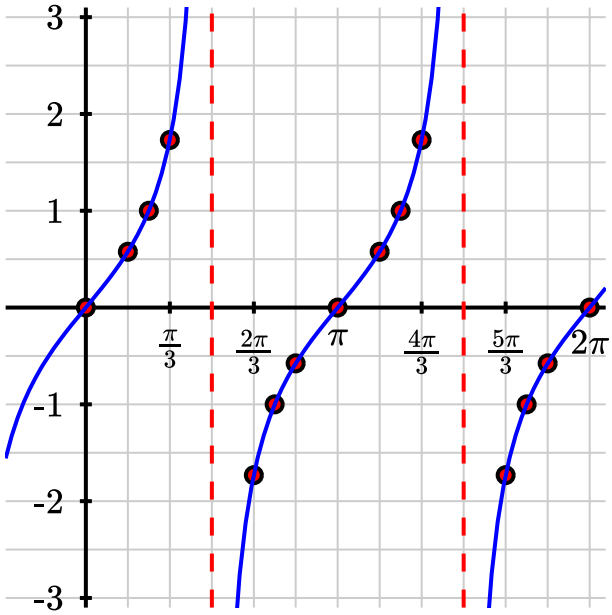
Activity 4.1.1.1

x $\tan(x)$ (exact) $\tan(x)$ (as a decimal) 0 0 0 $\frac{\pi}{6}$ $\frac{1}{\sqrt{3}}$ ≈ 0.58 $\frac{\pi}{4}$ 1 1 $\frac{\pi}{3}$ $\sqrt{3}$ ≈ 1.73 $\frac{\pi}{2}$ Undefined Undefined $\frac{2\pi}{3}$ $-\sqrt{3}$ ≈ -1.73 $\frac{3\pi}{4}$ -1 -1 $\frac{5\pi}{6}$ $-\frac{1}{\sqrt{3}}$ ≈ -0.58 π 0 0 $\frac{7\pi}{6}$ $\frac{1}{\sqrt{3}}$ ≈ 0.58 $\frac{5\pi}{4}$ 1 1 $\frac{4\pi}{3}$ $\sqrt{3}$ ≈ 1.73 $\frac{3\pi}{2}$ Undefined Undefined $\frac{5\pi}{3}$ $-\sqrt{3}$ ≈ -1.73 $\frac{7\pi}{4}$ -1 -1 $\frac{11\pi}{6}$ $-\frac{\sqrt{1}}{3}$ ≈ -0.58 2π 0 0

Activity 4.1.1.1

C.

Activity 4.1.1.1



Activity 4.1.1.1

A.

Activity 4.1.1.1

D.

Activity 4.1.1.1

B.

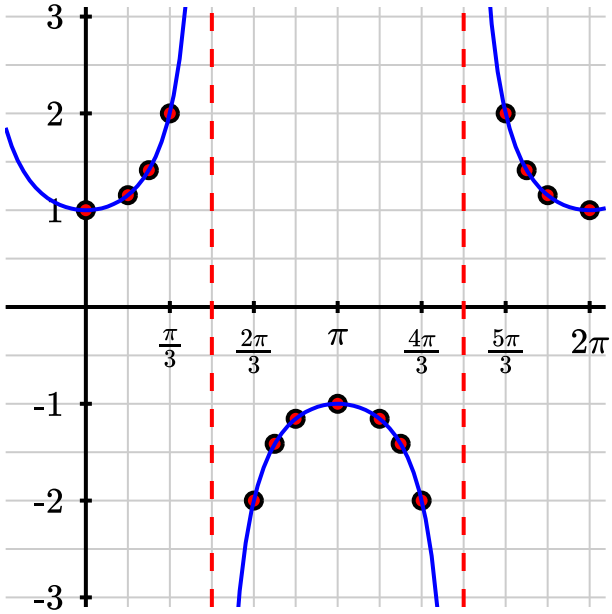
Activity 4.1.1.2

x $\sec(x)$ (exact) $\sec(x)$ (as a decimal) 0 1 1 $\frac{\pi}{6}$ $\frac{2}{\sqrt{3}}$ ≈ 1.15 $\frac{\pi}{4}$ $\sqrt{2}$ ≈ 1.41 $\frac{\pi}{3}$ 2 2 $\frac{\pi}{2}$ Undefined Undefined $\frac{2\pi}{3}$ -2 -2 $\frac{3\pi}{4}$ $-\sqrt{2}$ ≈ -1.41 $\frac{5\pi}{6}$ $-\frac{2}{\sqrt{3}}$ ≈ -1.15 π -1 -1 $\frac{7\pi}{6}$ $-\frac{2}{\sqrt{3}}$ ≈ -1.15 $\frac{5\pi}{4}$ $-\sqrt{2}$ ≈ -1.41 $\frac{4\pi}{3}$ -2 -2 $\frac{3\pi}{2}$ Undefined Undefined $\frac{5\pi}{3}$ 2 2 $\frac{7\pi}{4}$ $\sqrt{2}$ ≈ 1.41 $\frac{11\pi}{6}$ $\frac{2}{\sqrt{3}}$ ≈ 1.15 2π 1 1

Activity 4.1.1.2

C.

Activity 4.1.1.2



Activity 4.1.1.2

A.

Activity 4.1.1.2

B.

Activity 4.1.1.2

C.

Activity 4.1.1.3

$\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

Activity 4.1.1.3

$\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

Activity 4.1.1.3

Local maxima are at $\ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$.

Local minima are at $\ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$.

Activity 4.1.1.3

Local minima are at $\ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$.

Local maxima are at $\ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$.

Activity 4.1.1.3

Increasing on $\ldots \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$.

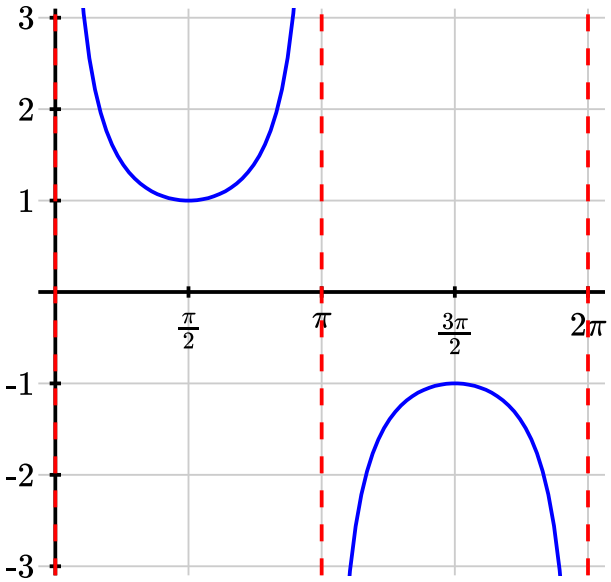
Decreasing on $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$.

Activity 4.1.1.3

Increasing on $\ldots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \ldots$.

Decreasing on $\ldots \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$.

Activity 4.1.1.3



Activity 4.1.1.3

B.

Activity 4.1.1.3

B.

Activity 4.1.1.3

C.

Activity 4.1.1.4

$\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

Activity 4.1.1.4

$\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

Activity 4.1.1.4

$\tan\left(\frac{\pi}{4} + \pi k\right) = 1$ for each integer k.

$\tan\left(\frac{3\pi}{4} + \pi k\right) = -1$ for each integer k.

Activity 4.1.1.4

$\cot\left(\frac{\pi}{4} + \pi k\right) = 1$ for each integer k.

$\cot\left(\frac{3\pi}{4} + \pi k\right) = -1$ for each integer k.

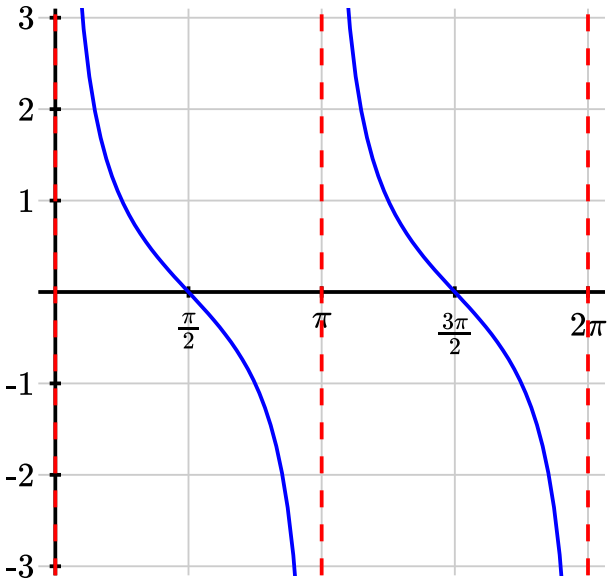
Activity 4.1.1.4

$\tan(x)$ is increasing everywhere it is defined, and decreasing nowhere.

Activity 4.1.1.4

$\cot(x)$ is decreasing everywhere it is defined, and increasing nowhere.

Activity 4.1.1.4



Activity 4.1.1.4

B.

Activity 4.1.1.4

D.

Activity 4.1.1.4

B.

Activity 4.1.1.5

A.

Activity 4.1.1.5

D.

Activity 4.1.1.5

$\ldots (-2\pi, -\pi) \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 2\pi) \cup \ldots$

Activity 4.1.1.5

$(-\infty, -1] \cup [1, \infty)$

Activity 4.1.1.5

2π

Activity 4.1.1.6

B.

Activity 4.1.1.6

The vertical asymptotes, period, and domain all change, while the range is still $(-\infty, \infty)$.

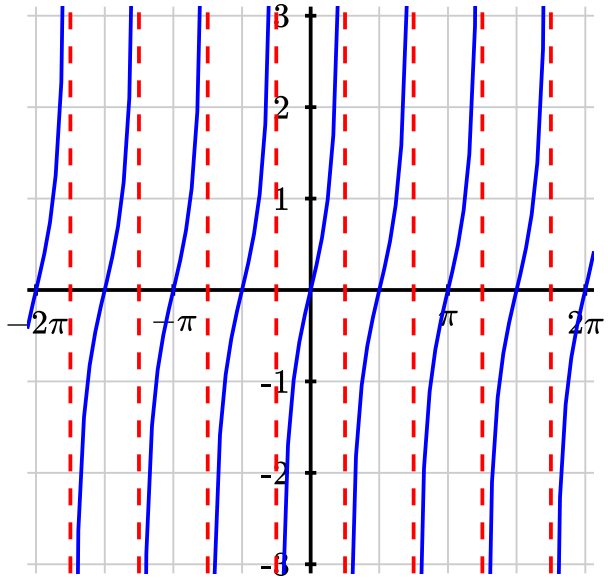
Activity 4.1.1.6

D.

Activity 4.1.1.6

C.

Activity 4.1.1.6



Activity 4.1.1.6

$\ldots \left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \ldots$

Activity 4.1.1.6

$(-\infty, \infty)$

Activity 4.1.1.6

$\frac{\pi}{2}$

Activity 4.1.1.7

$\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

Activity 4.1.1.7

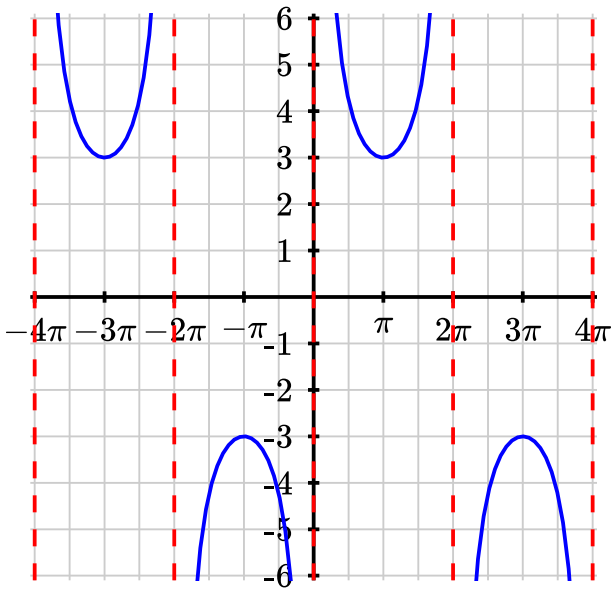
Local minima are at $\ldots, -3\pi, \pi, 5\pi, \ldots$.

Local maxima are at $\ldots, -5\pi, -\pi, 3\pi, \ldots$.

Activity 4.1.1.7

The local minimum values are 3 and the local maximum values are -3

Activity 4.1.1.7



Activity 4.1.1.7

$\ldots \cup (-4\pi, -2\pi) \cup (-2\pi, 0) \cup (0, 2\pi) \cup (2\pi, 4\pi) \cup \ldots$

Activity 4.1.1.7

$(-\infty, -3] \cup [3, \infty])$

Activity 4.1.1.7

4π

Check Your Understanding 4.1.1.1

Since $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, the function is undefined wherever $\cos(x) = 0$, which happens at $x = \dfrac{\pi}{2} + n\pi$. The domain is all real numbers except $x = \dfrac{\pi}{2} + n\pi$ for any integer n.

Check Your Understanding 4.1.1.2

The range of $\sec(x)$ is $(-\infty, -1] \cup [1, \infty)$. Since $\sec(x) = \dfrac{1}{\cos(x)}$ and $\cos(x)$ ranges over $[-1, 1]$ excluding 0, taking reciprocals of values between -1 and 1 (excluding 0) always produces a result with absolute value ≥ 1.

Check Your Understanding 4.1.1.3

The vertical asymptotes stay in exactly the same place (at every integer multiple of π, where $\sin(x) = 0$). A vertical stretch multiplies outputs, but wherever the original function is undefined (division by zero), the transformed function is still undefined — stretching a function does not change where its asymptotes occur.

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4.1.1.1: PF2 Online Exercises

PF2 — Graph trigonometric functions including tangent, cotangent, secant, and cosecant functions and transformations of these functions, and determine the domain and range.	PF2 CheckIt Exercises

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4.2: INTRODUCTION TO INVERSE TRIG FUNCTIONS (PF3)

4.2.1: Inverse Trig Functions (PF3)

4.2.1.1: PF3 Online Exercises

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4.2.1: Inverse Trig Functions (PF3)

This section corresponds to APC slot 4.3 (Inverse trig) in the Math 172 flow.

PF3 Learning outcome: Determine the inverse sine, cosine, and tangent values; graph inverse trigonometric functions and determine the limitations on the domain and range.

Inverse Trigonometric Functions

Activity 4.2.1.1

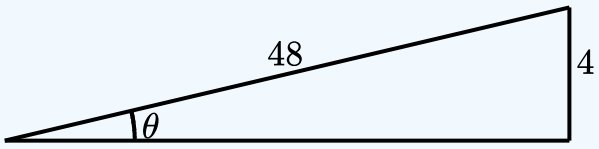
Which of the following angles satisfy $\cos(\theta) = \frac{1}{2}$?

A. $\frac{\pi}{6}$
B. $\frac{\pi}{3}$
C. $\frac{2\pi}{3}$
D. $\frac{5\pi}{6}$
E. $\frac{7\pi}{6}$
F. $\frac{4\pi}{3}$
G. $\frac{5\pi}{3}$
H. $\frac{11\pi}{6}$

Answer available at the end of this section.

Activity 4.2.1.2

A carpenter is cutting a hand rail for a ramp on his mitre saw. The ramp goes up 4 feet, and the length of the hand rail is 48 feet long. Which of the following equations determines the angle of the ramp, which the carpenter will use to set his saw?



A. $\sin(\theta) = \frac{1}{12}$
B. $\cos(\theta) = \frac{1}{12}$
C. $\tan(\theta) = \frac{1}{12}$
D. $\cot(\theta) = \frac{1}{12}$

Answer available at the end of this section.

Observation

Often in applications, in addition to finding the sine or cosine of an angle, we need to find an angle with a given sine or cosine or tangent. In other words, we need to have inverse functions of our six trigonometric functions.

Note

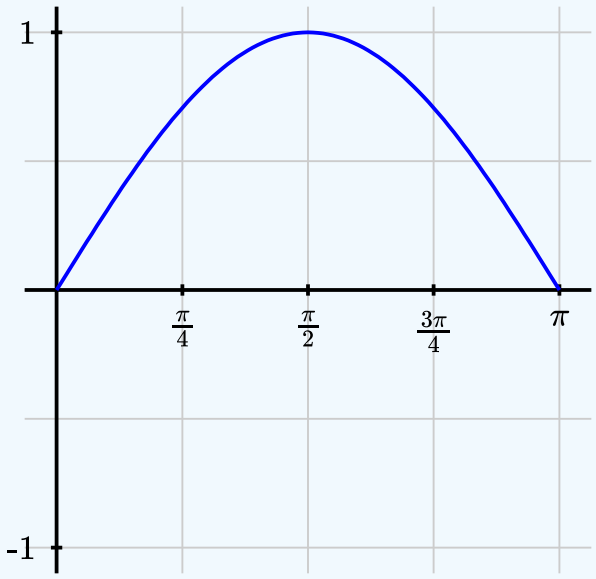
In *(see related activity)*, we saw that there are many angles with a given sine or cosine. We must systematically choose one of these to define an inverse function.

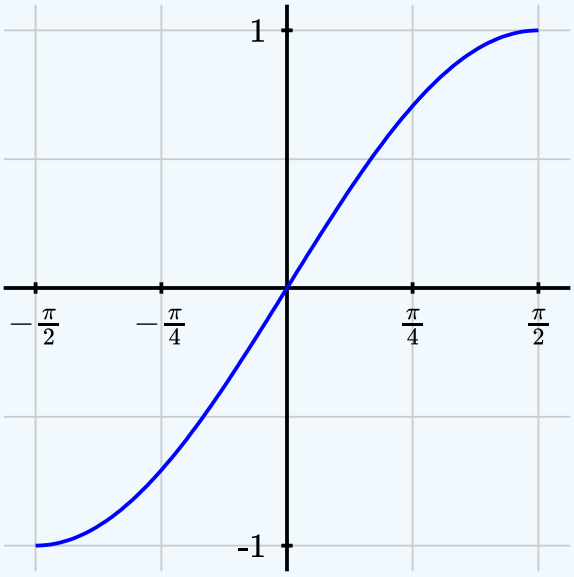
Activity 4.2.1.3

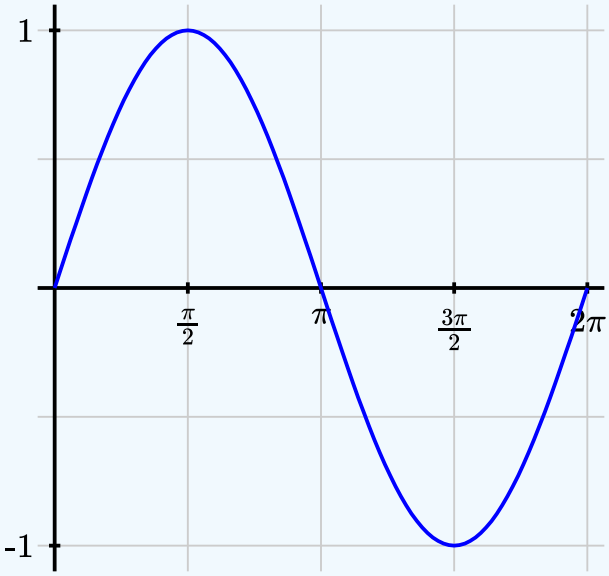
By restricting the domain, we can find a part of the sine function which is one-to-one, and thus allows us to define an inverse function.

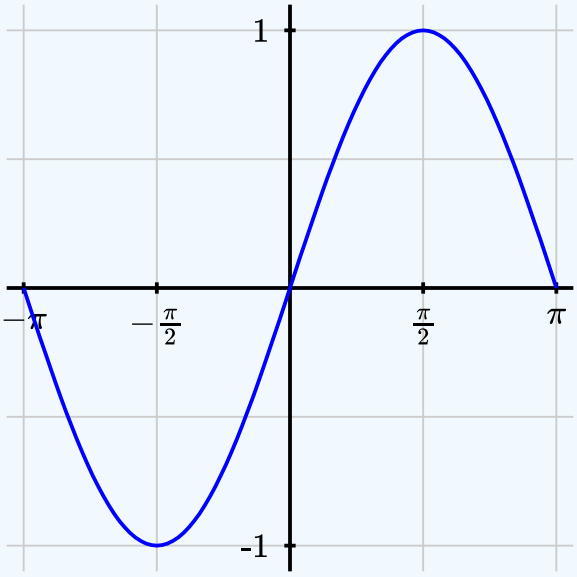
Which of the following domain restrictions is one-to-one?

A. $0 \leq x \leq \pi$
B. $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
C. $0 \leq x \leq 2\pi$
D. $-\pi \leq x \leq \pi$




1
$-\frac{\pi}{2}$
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
-1


1
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-1



Answer available at the end of this section.

Definition: Arcsin

The **arcsine** function, denoted $\arcsin(x)$, is the inverse of the restriction of $\sin(x)$ to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

In other words, $\arcsin(x)$ is the unique angle θ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ such that $\sin(\theta) = x$.

Note

Since the inverse of a function $f(x)$ is usually denoted $f^{-1}(x)$, some authors and calculators like to use the (more compact) notation $\sin^{-1}(x)$ instead of $\arcsin(x)$. This should not be confused with $(\sin(x))^{-1}$, i.e. $\csc(x)$.

Activity 4.2.1.4

Compute each of the following, without the use of technology.

(a)

$\arcsin\left(\frac{1}{2}\right)$

Answer available at the end of this section.

(b)

$\arcsin(-1)$

Answer available at the end of this section.

(c)

$\arcsin\left(\frac{\sqrt{2}}{2}\right)$

Answer available at the end of this section.

(d)

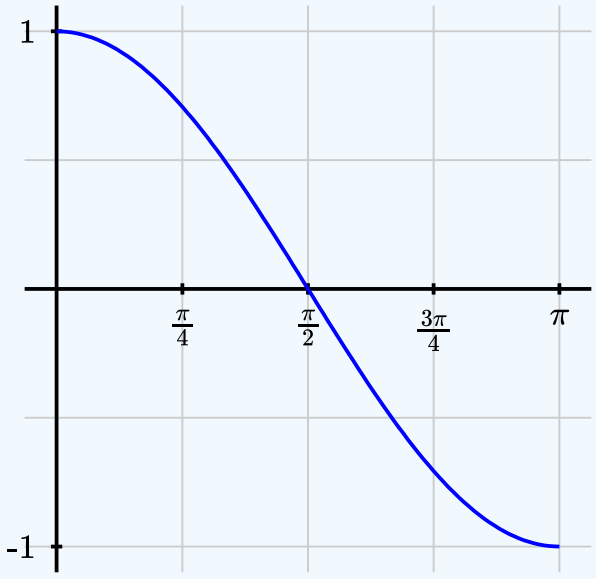
$\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

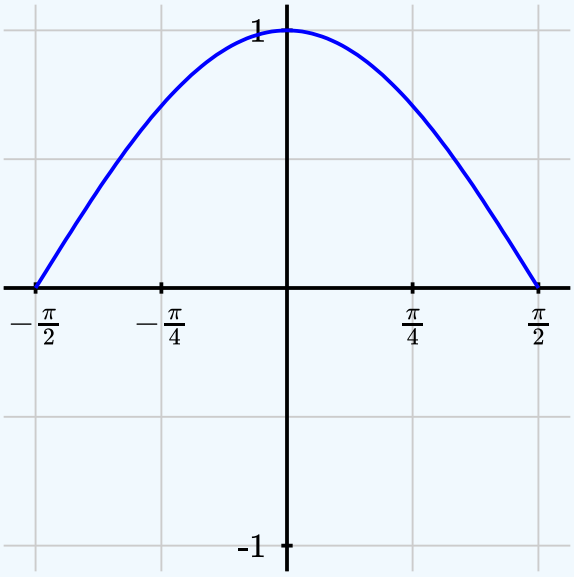
Answer available at the end of this section.

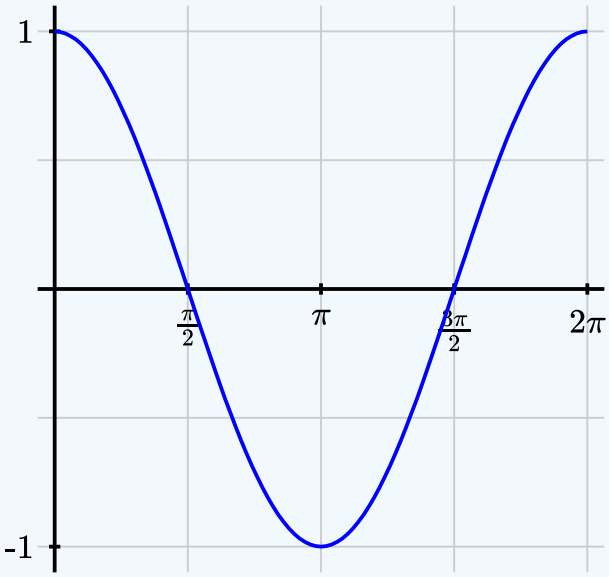
Activity 4.2.1.5

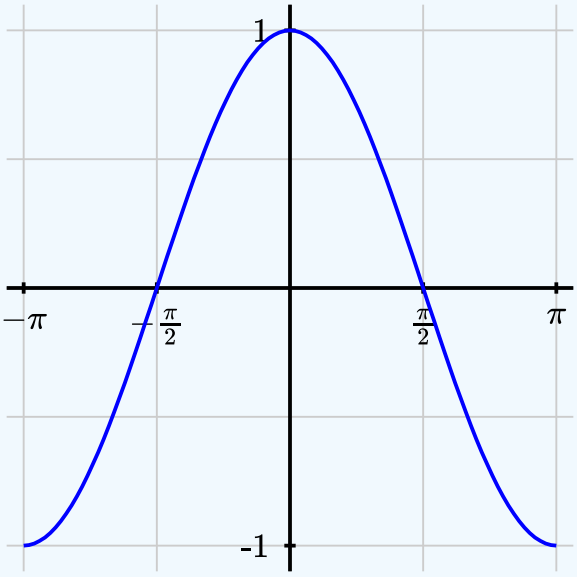
Which of the following domain restrictions of $\cos(x)$ is one-to-one?

A. $0 \leq x \leq \pi$
B. $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
C. $0 \leq x \leq 2\pi$
D. $-\pi \leq x \leq \pi$




1
$-\frac{\pi}{2}$
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
-1


1
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-1



Answer available at the end of this section.

Definition: Arccos

The **arccosine** function, denoted $\arccos(x)$, is the inverse of the restriction of $\cos(x)$ to the domain $[0, \pi]$.

In other words, $\arccos(x)$ is the unique angle θ with $0 \leq \theta \leq \pi$ such that $\cos(\theta) = x$.

Activity 4.2.1.6

Compute each of the following, without the use of technology.

(a)

$\arccos\left(\frac{1}{2}\right)$

Answer available at the end of this section.

(b)

$\arccos(-1)$

Answer available at the end of this section.

(c)

$\arccos\left(\frac{\sqrt{2}}{2}\right)$

Answer available at the end of this section.

(d)

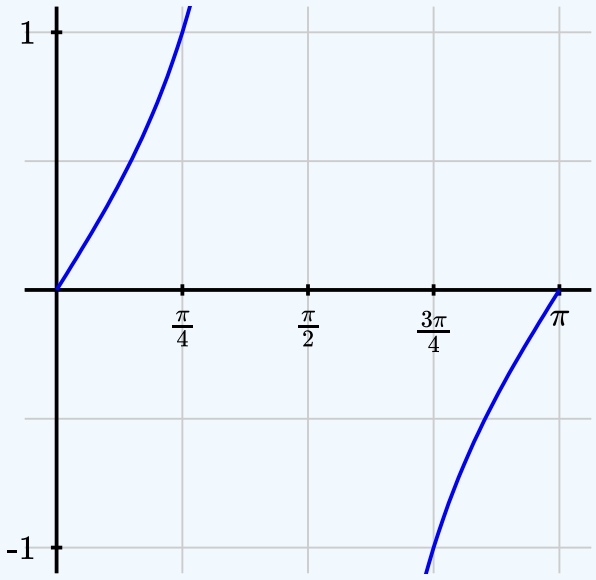
$\arccos\left(-\frac{\sqrt{3}}{2}\right)$

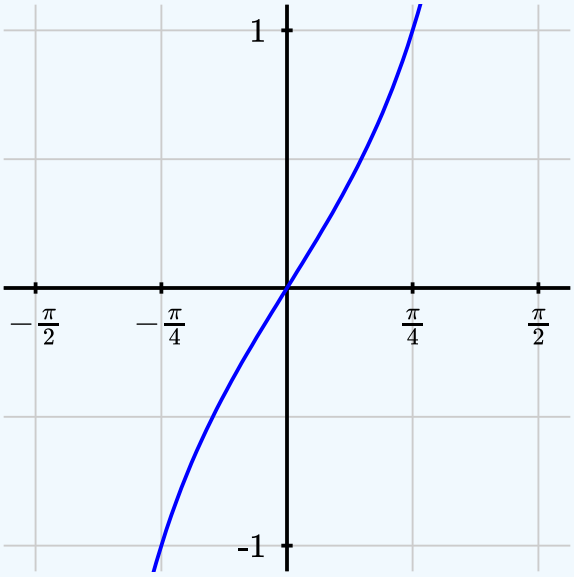
Answer available at the end of this section.

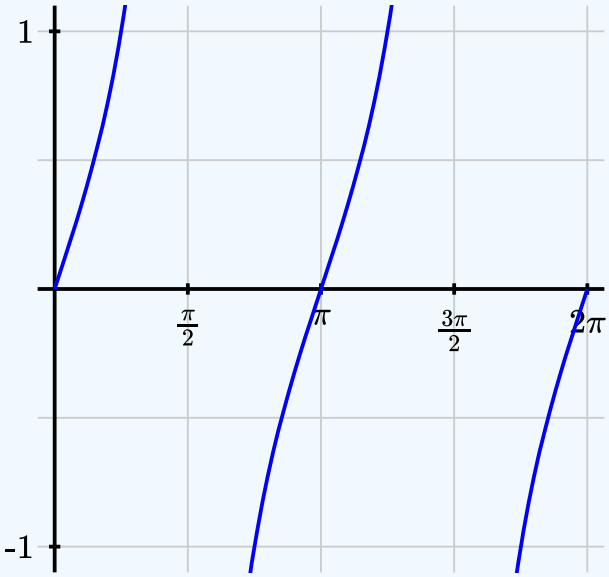
Activity 4.2.1.7

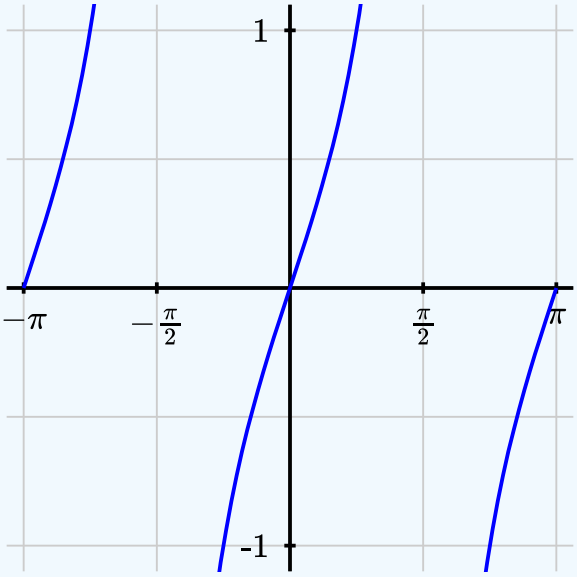
Which of the following domain restrictions of $\tan(x)$ is one-to-one?

A. $0 \leq x \leq \pi$
B. $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
C. $0 \leq x \leq 2\pi$
D. $-\pi \leq x \leq \pi$




1
$-\frac{\pi}{2}$
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
-1


1
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-1



Answer available at the end of this section.

Definition: Arctan

The **arctangent** function, denoted $\arctan(x)$, is the inverse of the restriction of $\tan(x)$ to the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

In other words, $\arctan(x)$ is the unique angle θ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ such that $\tan(\theta) = x$.

Observation

Note that while $\arcsin(x)$ and $\arccos(x)$ were defined by restricting the domain to a closed interval, since $\tan(x)$ is not defined at $-\frac{\pi}{2}$ or $\frac{\pi}{2}$, we define $\arctan(x)$ by restricting the domain to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Activity 4.2.1.8

Compute each of the following, without the use of technology.

(a)

$\arctan(1)$

Answer available at the end of this section.

(b)

$\arctan\left(-\sqrt{3}\right)$

Answer available at the end of this section.

(c)

$\arctan(0)$

Answer available at the end of this section.

(d)

$\arctan\left(\frac{\sqrt{3}}{3}\right)$

Answer available at the end of this section.

Activity 4.2.1.9

Sometimes, as in *(see related activity)*, we need to find an inverse trigonometric function that does not produce one of our special angles.

Compute each of the following using technology (e.g. a calculator).

Note

Note that most calculators use the shorter notation, e.g. $\sin^{-1}$, for this operation.

(a)

$\arcsin\left(\frac{1}{12}\right)$

Answer available at the end of this section.

(b)

$\arccos\left(-\frac{3}{5}\right)$

Answer available at the end of this section.

(c)

$\arctan(2)$

Answer available at the end of this section.

Observation

Next, we look at the graphs of $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$.

Activity 4.2.1.10

Consider the function $f(x) = \arcsin(x)$.

(a)

Complete the table of values.

x	$\arcsin(x)$
-1	
$-\frac{\sqrt{3}}{2}$	
$-\frac{\sqrt{2}}{2}$	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
$\frac{\sqrt{2}}{2}$	
$\frac{\sqrt{3}}{2}$	
1	

Hint:

Recall that $\theta = \arcsin(x)$ means $\sin(\theta) = x$.

Answer available at the end of this section.

(b)

Plot these values on a coordinate plane to approximate the graph of $f(x) = \arcsin(x)$. Then sketch the graph of the arcsine curve using the points as a guide.

Answer available at the end of this section.

(c)

What is the domain of $f(x) = \arcsin(x)$?

Answer available at the end of this section.

(d)

What is the range of $f(x) = \arcsin(x)$?

Answer available at the end of this section.

Activity 4.2.1.11

Consider the function $f(x) = \arccos(x)$.

(a)

Complete the table of values.

x	$\arccos(x)$
-1	

$-\frac{\sqrt{3}}{2}$	
$-\frac{\sqrt{2}}{2}$	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
$\frac{\sqrt{2}}{2}$	
$\frac{\sqrt{3}}{2}$	
1	

Hint:

Recall that $\theta = \arccos(x)$ means $\cos(\theta) = x$.

Answer available at the end of this section.

(b)

Plot these values on a coordinate plane to approximate the graph of $f(x) = \arccos(x)$. Then sketch the graph of the arccosine curve using the points as a guide.

Answer available at the end of this section.

(c)

What is the domain of $f(x) = \arccos(x)$?

Answer available at the end of this section.

(d)

What is the range of $f(x) = \arccos(x)$?

Answer available at the end of this section.

Activity 4.2.1.12

Consider the function $f(x) = \arctan(x)$.

(a)

Complete the table of values.

x	$\arctan(x)$
$-\sqrt{3}$	
-1	
$-\frac{\sqrt{3}}{3}$	

0	
$\frac{\sqrt{3}}{3}$	
1	
$\sqrt{3}$	

Hint:

Recall that $\theta = \arctan(x)$ means $\tan(\theta) = x$.

Answer available at the end of this section.

(b)

Plot these values on a coordinate plane to approximate the graph of $f(x) = \arctan(x)$. Then sketch the graph of the arctangent curve using the points as a guide.

Answer available at the end of this section.

(c)

What is the domain of $f(x) = \arctan(x)$?

Answer available at the end of this section.

(d)

What is the range of $f(x) = \arctan(x)$?

Answer available at the end of this section.

Activity 4.2.1.13

Sometimes when solving applied problems, we need to exactly (not approximately) evaluate expressions like $\sin\left(\arccos\left(\frac{5}{13}\right)\right)$.

(a)

Which of the following sentences describe the expression $\sin\left(\arccos\left(\frac{5}{13}\right)\right)$?

A. The angle whose cosine is the same as the sine of $\frac{5}{13}$.
B. The angle whose sine is the same as the cosine of $\frac{5}{13}$.
C. The cosine of the angle whose sine is $\frac{5}{13}$.
D. The sine of the angle whose cosine is $\frac{5}{13}$.

Answer available at the end of this section.

(b)

Let $\theta = \arccos(\frac{5}{13})$. Draw a right triangle with an angle of θ, and find the lengths of its three sides.

Answer available at the end of this section.

(c)

Find $\sin(\theta)$. Since we defined $\theta = \arccos(\frac{5}{13})$, this gives us $\sin(\arccos(\frac{5}{13}))$.

A. $\frac{5}{13}$
B. $\frac{12}{13}$
C. $\frac{5}{12}$
D. $\frac{13}{12}$

Answer available at the end of this section.

Activity 4.2.1.14

Compute each of the following.

(a)

$\tan(\arcsin(\frac{8}{17}))$

Answer available at the end of this section.

(b)

$\sec(\arctan(\frac{24}{7}))$

Answer available at the end of this section.

(c)

$\tan(\arcsin(\frac{3}{4}))$

Answer available at the end of this section.

(d)

$\cos(\arcsin(x))$

Hint:

Draw an appropriate right triangle with two sides as x and 1.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can evaluate and graph inverse trig functions before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 4.2.1.1

Find the exact value of $\arcsin\left(\frac{1}{2}\right)$, and explain why the answer is a single angle rather than infinitely many angles (even though infinitely many angles have a sine of $\frac{1}{2}$).

Answer available at the end of this section.

Check Your Understanding 4.2.1.2

State the domain and range of $y = \arccos(x)$.

Answer available at the end of this section.

Check Your Understanding 4.2.1.3

Find the exact value of $\tan\left(\arcsin\left(\frac{3}{5}\right)\right)$ without a calculator.

Answer available at the end of this section.

Answers

Activity 4.2.1.1
B,G

Activity 4.2.1.2
A.

Activity 4.2.1.3
B.

Activity 4.2.1.4
$\frac{\pi}{6}$

Activity 4.2.1.4
$-\frac{\pi}{2}$

Activity 4.2.1.4
$\frac{\pi}{4}$

Activity 4.2.1.4
$-\frac{\pi}{3}$

Activity 4.2.1.5
A.

Activity 4.2.1.6
$\frac{\pi}{3}$

Activity 4.2.1.6
π

Activity 4.2.1.6
$\frac{\pi}{4}$

Activity 4.2.1.6
$\frac{5\pi}{6}$

Activity 4.2.1.7
A. and B.

Activity 4.2.1.8
$\frac{\pi}{4}$

Activity 4.2.1.8
$-\frac{\pi}{3}$

Activity 4.2.1.8
0

Activity 4.2.1.8
$\frac{\pi}{6}$

Activity 4.2.1.9
4.78° or 0.083 radians

Activity 4.2.1.9

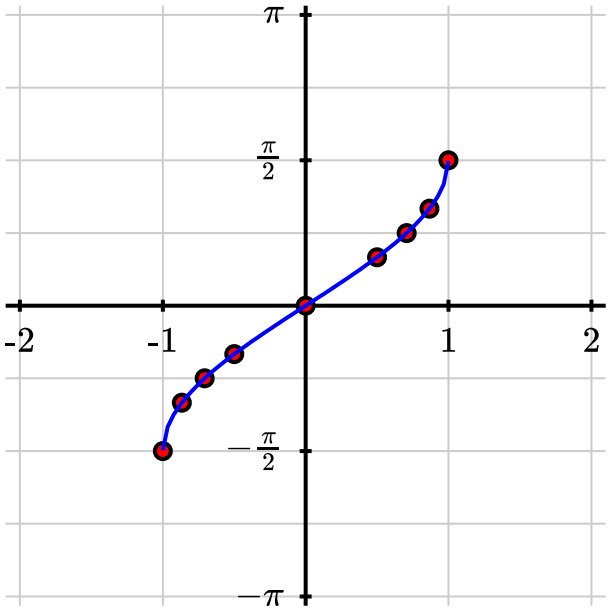
126.9° or 2.21 radians

Activity 4.2.1.9

63.43° or 1.11 radians

Activity 4.2.1.10

Activity 4.2.1.10



Activity 4.2.1.10

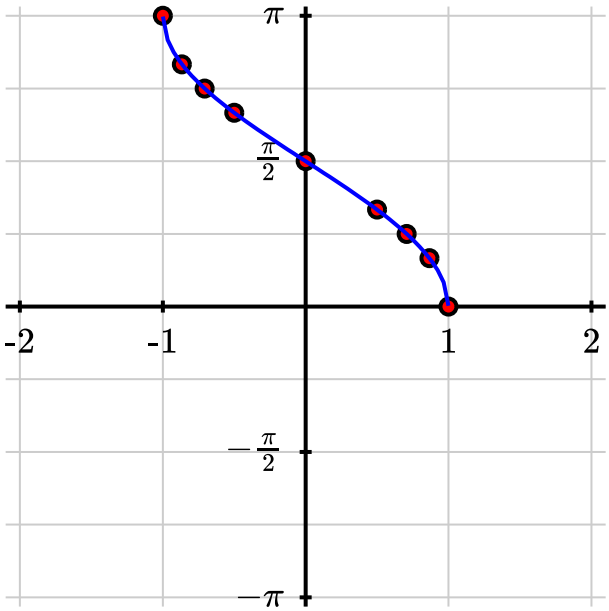
$[-1, 1]$

Activity 4.2.1.10

$[-\frac{\pi}{2}, \frac{\pi}{2}]$

Activity 4.2.1.11

Activity 4.2.1.11



Activity 4.2.1.11

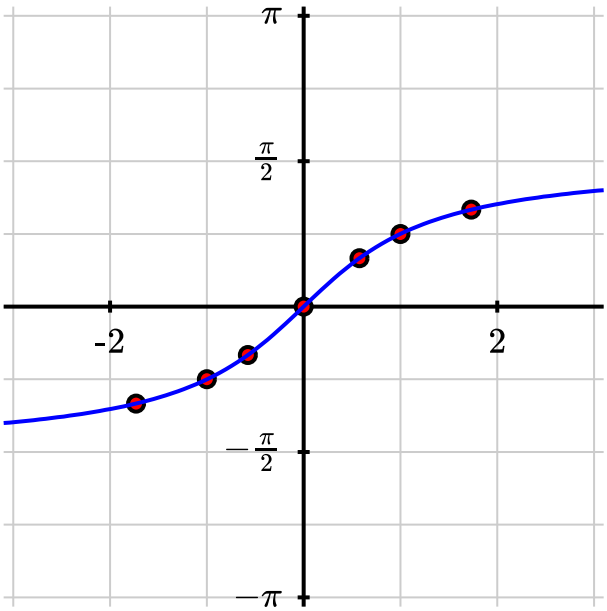
$[-1, 1]$

Activity 4.2.1.11

$[0, \pi]$

Activity 4.2.1.12

Activity 4.2.1.12



Activity 4.2.1.12

$(-\infty, \infty)$

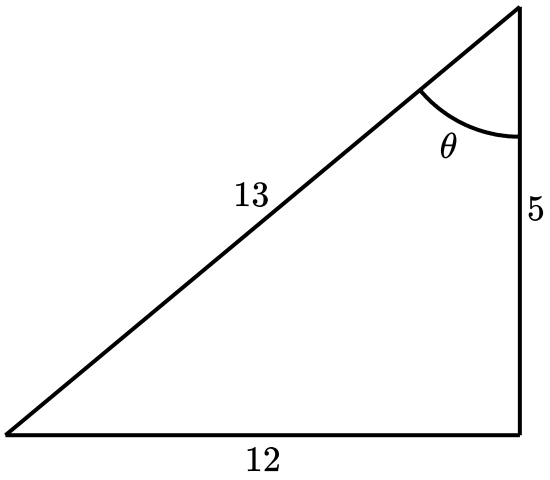
Activity 4.2.1.12

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Activity 4.2.1.13

D.

Activity 4.2.1.13



Activity 4.2.1.13

B.

Activity 4.2.1.14

$\frac{8}{15}$

Activity 4.2.1.14

$\frac{25}{7}$

Activity 4.2.1.14

$\frac{3}{\sqrt{7}}$

Activity 4.2.1.14

$\sqrt{1-x^2}$

Check Your Understanding 4.2.1.1

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Although infinitely many angles (like $\frac{\pi}{6}$, $\frac{5\pi}{6}$, etc.) have a sine of $\frac{1}{2}$, the arcsine function's range is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that it is a genuine function — each input produces exactly one output.

Check Your Understanding 4.2.1.2

Domain: $[-1, 1]$ (the same as the range of cosine, since arccos reverses cosine's inputs and outputs). Range: $[0, \pi]$ — the restricted interval chosen so that cosine is one-to-one and arccos is a genuine function.

Check Your Understanding 4.2.1.3

Let $\theta = \arcsin\left(\frac{3}{5}\right)$, so $\sin(\theta) = \frac{3}{5}$, meaning in a right triangle the opposite side is 3 and hypotenuse is 5. The adjacent side is $\sqrt{5^2 - 3^2} = 4$. So $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$.

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4.2.1.1: PF3 Online Exercises

PF3 — Determine the inverse sine, cosine, and tangent values; graph inverse trig functions and determine the limitations on the domain and range.	PF3 CheckIt Exercises

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4.3: INTRODUCTION TO SOLVING TRIANGLES (TE4, TE5, TE6)

4.3.1: Solving Right Triangles (TE4)

4.3.1.1: TE4 Online Exercises

4.3.2: Trigonometric Laws (TE5)

4.3.2.1: TE Online Exercises

4.3.3: Applications of Trig (TE6)

4.3.3.1: TE6 Online Exercises

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4.3.1: Solving Right Triangles (TE4)

This section corresponds to APC slot 4.4 (Finding angles) in the Math 172 flow.

TE4 Learning outcome: Use trigonometric functions and the Pythagorean theorem to solve right triangles.

Solving Right Triangles

Activity 4.3.1.1

Let's revisit the triangle ABC from *(see related activity)*, where $a = 35$, $b = 12$, and $c = 37$, with c being the hypotenuse of the triangle.

(a)

Find $\cos A$.

A. $\frac{12}{35}$
B. $\frac{12}{37}$
C. $\frac{37}{35}$
D. $\frac{37}{12}$

Answer available at the end of this section.

(b)

Suppose we want to know the measure of angle A. We can find the measure of angle A in three different ways by using either sine, cosine, or tangent (since all side lengths are given). For each trigonometric function, write the trigonometric ratio that can be used to find the measure of angle A.

Answer available at the end of this section.

(c)

Now that we have set up a trigonometric ratio to help find the measure of angle A, how can we use these ratios to determine how big A is?

Answer available at the end of this section.

Overview

Sometimes you will need to use inverse trigonometric functions (e.g. $\arcsin(x)$) to find the measure of an angle. Recall from *(see related activity)* that the inverse trig function keys on your calculator are usually denoted $\sin^{-1}$.

Activity 4.3.1.2

Refer back to *(see related activity)*, where you were given all the sides of a right triangle, but no angle measures. (In triangle ABC, $a = 35$, $b = 12$, and $c = 37$, with c being the hypotenuse of the triangle).

(a)

What is the trigonometric ratio for $\cos A$?

A. $\frac{35}{12}$
B. $\frac{35}{37}$
C. $\frac{12}{35}$
D. $\frac{12}{37}$

Answer available at the end of this section.

(b)

Use the inverse trig function, $\cos^{-1}$ to find the measure of angle A. (Make sure your calculator is in degree mode!)

Answer available at the end of this section.

(c)

What is the trigonometric ratio for $\sin A$?

A. $\frac{35}{12}$
B. $\frac{35}{37}$
C. $\frac{12}{35}$
D. $\frac{12}{37}$

Answer available at the end of this section.

(d)

Use the inverse trig function, $\sin^{-1}$ to find the measure of angle A. (Make sure your calculator is in degree mode!)

Answer available at the end of this section.

(e)

What is the trigonometric ratio for $\tan A$?

A. $\frac{35}{12}$
B. $\frac{35}{37}$
C. $\frac{12}{35}$
D. $\frac{12}{37}$

Answer available at the end of this section.

(f)

Use the inverse trig function, $\tan^{-1}$ to find the measure of angle A. (Make sure your calculator is in degree mode!)

Answer available at the end of this section.

(g)

Refer back to parts (b), (d), and (f). What do you notice about your answers from those parts?

Answer available at the end of this section.

(h)

Now that we know the measure of angle A, find the measure of angle B.

Answer available at the end of this section.

Note

Determining all of the side lengths and angle measures of a right triangle is known as **solving** a right triangle. In *(see related activity)* and *(see related activity)*, we were given all the sides of the triangle and used trigonometric ratios to determine the measure of the angles.

Activity 4.3.1.3

Solve the following triangles using your knowledge of right triangles, the Pythagorean Theorem and trigonometric functions. Be sure to draw a picture to help you determine the relationship between the angles and sides.

(a)

In triangle ABC, $B = 53$ and $c = 5$ meters (with c being the hypotenuse).

Answer available at the end of this section.

(b)

In triangle ABC, $A = 28$ and $b = 29.3$ miles (with c being the hypotenuse).

Answer available at the end of this section.

(c)

In triangle ABC, $a = 8$ feet, $b = 17$ feet, and $c = 15$ feet (with b being the hypotenuse).

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can solve right triangles using trigonometric functions and the Pythagorean theorem before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 4.3.1.1

A right triangle has a hypotenuse of length 10 and one angle of $35°$. Find the length of the side opposite that angle, rounded to two decimal places.

Answer available at the end of this section.

Check Your Understanding 4.3.1.2

A right triangle has legs of length 7 and 24. Find the length of the hypotenuse and the measure of the angle opposite the side of length 7, rounded to one decimal place.

Answer available at the end of this section.

Check Your Understanding 4.3.1.3

A ladder leans against a wall, making a 70° angle with the ground. If the base of the ladder is 4 feet from the wall, how long is the ladder, rounded to two decimal places? Explain which trig ratio you chose and why.

Answer available at the end of this section.

Answers

Activity 4.3.1.1

B

Activity 4.3.1.1

Students should be able to write all three trigonometric functions: $\cos A = \frac{12}{37}$, $\sin A = \frac{35}{37}$, and $\tan A = \frac{35}{12}$.

Activity 4.3.1.1

Give students the opportunity to discuss with one another on how they would try to determine the measure of angle A. Instructors might want to give a hint about how to "undo" the trigonometric function.

Activity 4.3.1.2

D

Activity 4.3.1.2

Students should get approximately 71.08.

Activity 4.3.1.2

B

Activity 4.3.1.2

Students should get approximately 71.08.

Activity 4.3.1.2

A

Activity 4.3.1.2

Students should get approximately 71.08.

Activity 4.3.1.2

Students should notice that they got the same angle measure for A regardless of which trigonometric function they used.

Activity 4.3.1.2

Angle B is approximately 18.92.

Activity 4.3.1.3

$A = 37$, $C = 90$, $a = 3$ meters, and $b = 4$ meters.

Activity 4.3.1.3

$B = 62$, $C = 90$, $a = 15.6$ miles, and $c = 33.2$ miles.

Activity 4.3.1.3

$A = 28.07$, $B = 90$, and $C = 61.93$.

Check Your Understanding 4.3.1.1

$\sin(35^\circ) = \frac{\text{opposite}}{10}$, so opposite $= 10\sin(35^\circ) \approx 10(0.5736) \approx 5.74$.

Check Your Understanding 4.3.1.2

Hypotenuse: $\sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$. Angle: $\theta = \arcsin\left(\frac{7}{25}\right) \approx 16.3^\circ$.

Check Your Understanding 4.3.1.3

The 4-foot distance is adjacent to the 70° angle, and the ladder itself is the hypotenuse, so cosine relates them: $\cos(70^\circ) = \frac{4}{L}$, giving $L = \frac{4}{\cos(70^\circ)} \approx \frac{4}{0.3420} \approx 11.70$ feet. Cosine was the right choice because it relates the adjacent side and hypotenuse, which are the two quantities given and asked for here.

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4.3.1.1: TE4 Online Exercises

TE4 — Use trigonometric functions and the Pythagorean Theorem to solve right triangles.	TE4 CheckIt Exercises

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4.3.2: Trigonometric Laws (TE5)

*This section is **supplemental** — not currently part of the core Math 172 sequence. Converted and kept available for possible future use.*

TE5 Learning outcome: Use the Law of Sines and/or the Law of Cosines to solve non-right triangles.

Law of Sines and Law of Cosines

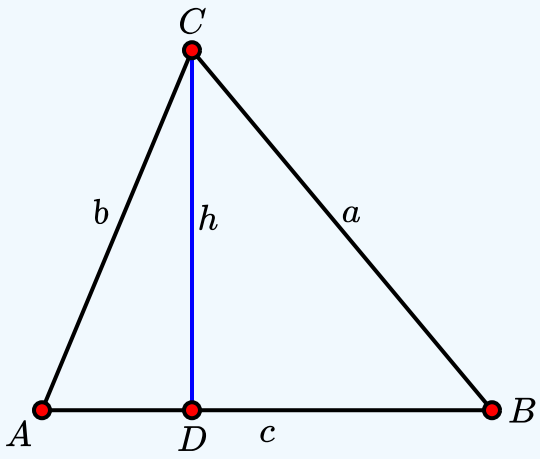
Overview

In *(see related activity)* and *(see related activity)*, we learned how to solve right triangles (and special right triangles) using trigonometric ratios. In this section, we will learn how to solve oblique (non-right triangles).

Activity 4.3.2.1

Suppose you are given the following triangle, where h is the altitude of triangle ABC. By drawing the altitude, we've now created two right triangles. If we label the point where the altitude intersects C as point D, then we can call the triangles ACD and BCD.

Triangle ABC with sides a, b, and c, with an altitude h drawn from C to D.



(a)

Find $\sin A$.

A. $\sin A = \frac{a}{h}$
B. $\sin A = \frac{h}{b}$
C. $\sin A = \frac{h}{a}$
D. $\sin A = \frac{a}{b}$

Answer available at the end of this section.

(b)

Find $\sin B$.

A. $\sin B = \frac{a}{h}$
B. $\sin B = \frac{h}{b}$
C. $\sin B = \frac{h}{a}$
D. $\sin B = \frac{a}{b}$

Answer available at the end of this section.

(c)

Take your answers from parts (a) and (b) and solve each equation for h (the altitude). What does h equal in each case?

Answer available at the end of this section.

(d)

Notice that we now have two ways to express h. Set the two expressions of h equal to one another and then rearrange this equation so that a and $\sin A$ are on the same side.

Answer available at the end of this section.

Theorem: Law of Sines

The **Law of Sines** defines the ratio of sides of a triangle and their respective sine angles are equivalent to each other.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{4.3.2.1}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{4.3.2.2}$$

When using the Law of Sines, you will only need to set one fraction to one other fraction. Note that these are proportions and so the reciprocals also hold true.

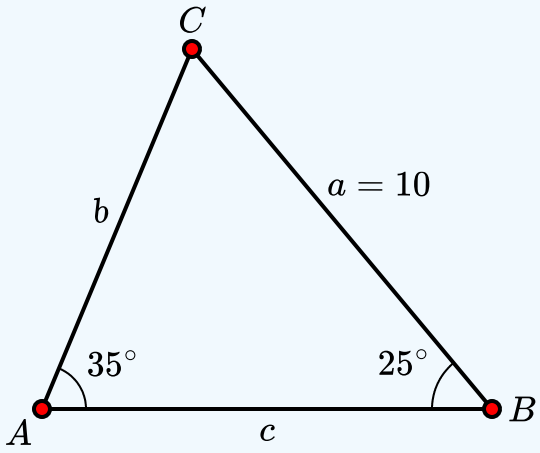
Note

1. two angles and one of their corresponding sides.
2. two sides and one of their corresponding angles.

The Law of Sines is helpful when we have

Activity 4.3.2.2

Suppose you are given triangle ABC where $A = 35$, $B = 25$, and $a = 10$ as shown in the figure below.



(a)

What parts of the triangle are missing?

Answer available at the end of this section.

(b)

What is the value of C?

A. 60°
B. 155°
C. 145°
D. 120°

Hint:

Answer available at the end of this section.

(c)

Use the Law of Sines to set up a proportion to find c. What does that proportion look like?

Answer available at the end of this section.

(d)

Solve for c from the proportion you got in part (c). What is the value of c to the nearest tenth?

A. 8.7
B. 6.6
C. 15.1
D. 0.1

Answer available at the end of this section.

(e)

We now have side b left to find. Which method would be the best method to find b?

A. Use the Law of Sines again with angle B, angle C, and side c.
B. Use the Pythagorean Theorem
C. Use the Law of Sines again with angle A, side a, and angle B.

Answer available at the end of this section.

(f)

Use the method you identified in the previous part to find b. What is the value of b to the nearest tenth?

A. 5.7
B. 4.2
C. 13.6
D. 7.4

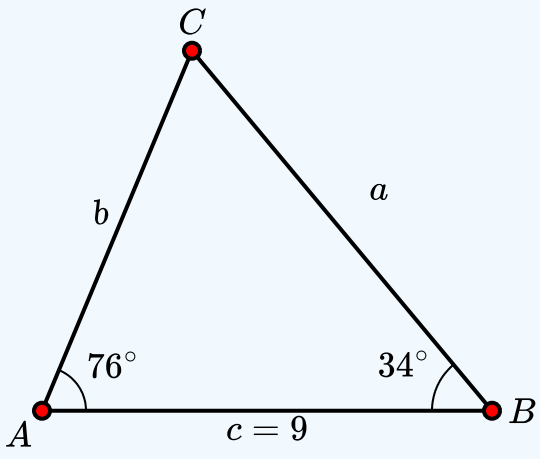
Answer available at the end of this section.

Note

The triangle in *(see related activity)* is known as an AAS (Angle-Angle-Side) triangle, which means we know two angles and one side (which is NOT between the angles) in that order. This is just one type of triangle you might see when solving oblique triangles. The next activity will show another type of triangle you might encounter.

Activity 4.3.2.3

Suppose you are given triangle ABC where $A = 76$, $B = 34$, and $c = 9$ as shown in the figure below.



(a)

What parts of the triangle are missing?

Answer available at the end of this section.

(b)

What is the value of C?

A. 70°
B. 110°
C. 104°
D. 146°

Answer available at the end of this section.

(c)

Use the Law of Sines to set up a proportion to find a. What is the value of a to the nearest tenth?

A. 8.7
B. 9.3
C. 8.5
D. 9.1

Answer available at the end of this section.

(d)

Use the Law of Sines to set up a proportion to find b. What is the value of b to the nearest tenth?

A. 5.0
B. 8.5
C. 15.1
D. 5.4

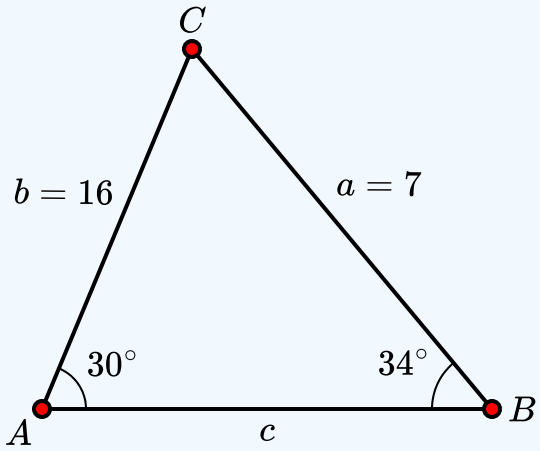
Answer available at the end of this section.

Note

The triangle in *(see related activity)* is known as an ASA (Angle-Side-Angle) triangle, which means we know two angles and a side BETWEEN the angles.

Activity $4.3.2.4$

Suppose you are given triangle ABC, where $A = 30$, $a = 7$, and $b = 16$ as shown in the figure below.



(a)

What parts of the triangle are missing?

Answer available at the end of this section.

(b)

Use the Law of Sines to set up a proportion to find $\sin B$. What is the value of $\sin B$?

A. 8.0
B. 1.14
C. 0.875
D. 0.031

Answer available at the end of this section.

(c)

Use part (b) to determine the value of B.

Answer available at the end of this section.

Note

In *(see related activity)*, we can see that no triangle can be created because $\sin B$ was equal to a value greater than 1. This is because the largest value that the sine of an angle can have is 1 (refer back to *(see related activity)*).

Activity $4.3.2.5$

Suppose you are given triangle ABC, where $A = 30°$, $a = 10$, and $b = 16$.

(a)

Draw triangle ABC and label what is known. What parts of the triangle are missing?

Answer available at the end of this section.

(b)

Use the Law of Sines to determine the value of B (to the nearest degree).

A. 143°
B. 53°
C. 80°
D. 37°

Answer available at the end of this section.

(c)

Continue solving triangle ABC. What are the missing values (to the nearest degree/whole number)?

Answer available at the end of this section.

(d)

Let's go back to part (b) when we were asked to solve for B. We needed to solve $\sin B = \frac{4}{5}$. Using the inverse sine function on our calculator, we got that $B \approx 53$. However, there is another angle between 0° and 180° whose sine is $\frac{4}{5}$. Which of the following values could also be B?

A. 143°
B. 37°
C. 80°
D. 127°

Answer available at the end of this section.

(e)

Why do you think there are two values of B for this triangle?

Answer available at the end of this section.

(f)

Continue solving this other triangle ABC. What are the missing values (to the nearest degree/whole number)?

Answer available at the end of this section.

Note

So far we have seen that when given two sides and an angle (also known as a SSA (Side-Side-Angle) triangle), we can have no solution (i.e., no triangle can be created) or two solutions (i.e., there are two possible triangles). There is still one more case we need to explore.

Activity 4.3.2.6

Suppose you are given triangle ABC, where $A = 30^\circ$, $a = 20$, and $b = 16$.

(a)

Draw triangle ABC and label what is known. What parts of the triangle are missing?

Answer available at the end of this section.

(b)

Use the Law of Sines to determine the value of B (to the nearest degree).

A. 24°
B. 39°
C. 141°
D. 156°

Answer available at the end of this section.

(c)

Choose the smaller value of B and solve triangle ABC. What are the missing values (to the nearest degree/whole number)?

Answer available at the end of this section.

(d)

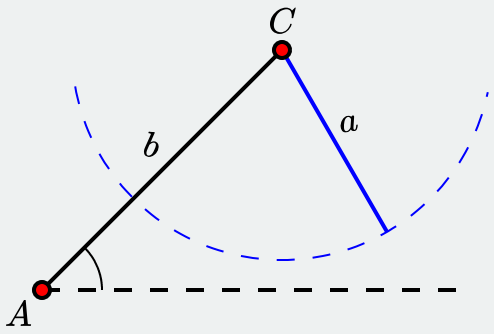
Choose the larger value of B and solve triangle ABC. What are the missing values (to the nearest degree/whole number)?

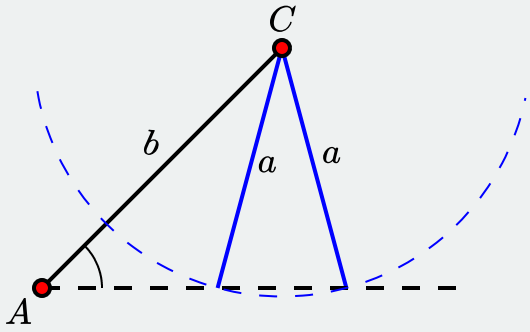
Answer available at the end of this section.

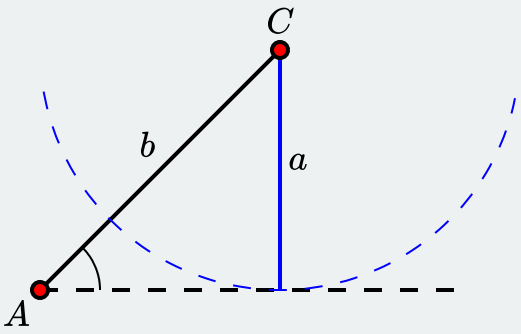
Observation

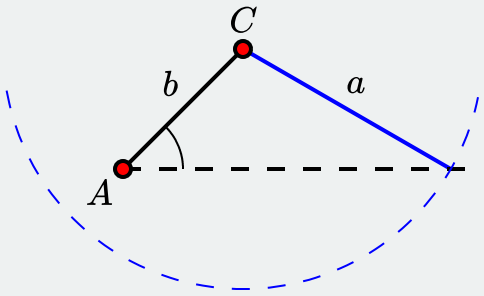
1. NO triangle exists (no solution)
2. TWO different triangle exist (2 solutions)
3. Exactly ONE triangle exists (1 solution)

In the previous three activities, we saw that when we use the Law of Sines to find an angle, an ambiguity can arise due to the sine function being positive in Quadrant I and Quadrant II. In other words, if two sides and the non-included angle are given (SSA), three situations may occur. The **Ambiguous Case of the Law of Sines** states that when using the Law of Sines to find a missing side length, the possibility of two solutions for the measure of the same side may occur.


C
b
a
A


C
b
a
a
A





Activity 4.3.2.7

State the number of possible triangles that can be formed with the given measurements. Then, solve each triangle. Round your answers to the nearest tenth.

(a)

$C = 145°$, $b = 7$, $c = 33$

Answer available at the end of this section.

(b)

$B = 84°$, $a = 18$, $b = 9$

Answer available at the end of this section.

(c)

$B = 45^\circ$, $a = 28$, $b = 27$

Answer available at the end of this section.

Activity 4.3.2.8

Suppose you are given triangle ABC, where $A = 70^\circ$, $b = 14$, and $c = 7$.

(a)

Draw triangle ABC and label what is known. What parts of the triangle are missing?

Answer available at the end of this section.

(b)

Use the Law of Sines to solve triangle ABC.

Answer available at the end of this section.

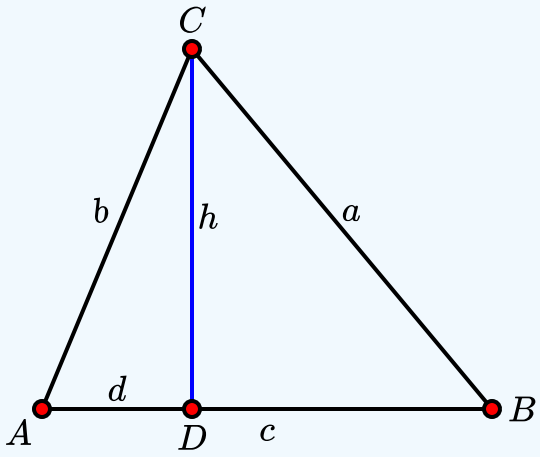
Note

Notice in *(see related activity)*, we do not currently have enough information to be able to solve for triangle ABC since the Law of Sines cannot be used. In the next activity, we will explore another method that can be used to solve oblique triangles.

Activity 4.3.2.9

Suppose you are given the following triangle, where h is the altitude of triangle ABC.

Triangle ABC with sides a, b, and c, with an altitude h drawn from C to D. The segment AD is labeled d.



(a)

Using right triangle trigonometry, find $\sin A$.

A. $\sin A = \frac{b}{h}$
B. $\sin A = \frac{h}{b}$
C. $\sin A = \frac{m}{b}$
D. $\sin A = \frac{b}{m}$

Answer available at the end of this section.

(b)

Using right triangle trigonometry, find $\cos A$.

A. $\cos A = \frac{b}{h}$
B. $\cos A = \frac{h}{b}$
C. $\cos A = \frac{m}{b}$
D. $\cos A = \frac{b}{m}$

Answer available at the end of this section.

(c)

Take your equation in part (a) and solve for h.

Answer available at the end of this section.

(d)

Take your equation in part (b) and solve for m.

Answer available at the end of this section.

(e)

Recall that we can use the Pythagorean Theorem to represent the relationship between the sides of a right triangle. For example, $h^2 + (c - m)^2 = a^2$ can be used to represent the relationship of the sides of triangle BCD. Take your equations from parts (c) and (d) to substitute m and h into that equation.

Answer available at the end of this section.

(f)

Simplify your equation from part (e).

Hint:

When simplifying the term $(c - b\cos A)^2$, don't forget that $(c - b\cos A)^2 = (c - b\cos A) * (c - b\cos A)$. You may also have to use a trig identity to simplify!

Answer available at the end of this section.

Theorem: Law of Cosines

The **Law of Cosines** relates the lengths of the sides of a triangle to the cosine of one of its angles. In other words, the square of any one side of a triangle is equal to the difference between the sum of squares of the other two sides and double the product of other sides and cosine angle included between them.

Let a, b, and c be the lengths of the three sides of a triangle and A, B, and C be the three angles of the triangle. Then, the Law of Cosines states that:

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{4.3.2.3}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{4.3.2.4}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{4.3.2.5}$$

Activity 4.3.2.10

Let's revisit *(see related activity)* to see how we can apply the Law of Cosines to solve for this triangle. Recall that we were given triangle ABC, where $A = 70°$, $b = 14$, and $c = 7$.

(a)

Draw triangle ABC and label what is known. What parts of the triangle are missing?

Answer available at the end of this section.

(b)

Use the Law of Cosines to find a (to the nearest tenth).

A. 178.0
B. 17.7
C. 13.3
D. 7.8

Answer available at the end of this section.

(c)

Now that we have another side of triangle ABC, what would you use to find angle B?

A. Law of Sines
B. Law of Cosines
C. Pythagorean Theorem

Answer available at the end of this section.

(d)

Find the measure of B and C (to the nearest degree).

Answer available at the end of this section.

Activity 4.3.2.11

Suppose you are given triangle ABC, where $a = 30$, $b = 20$, and $c = 17$.

(a)

Draw triangle ABC and label what is known. What parts of the triangle are missing?

Answer available at the end of this section.

(b)

Which law should you start with to solve triangle ABC?

Answer available at the end of this section.

(c)

What is the value of A (to the nearest degree)?

A. 109°
B. 39°
C. 108°
D. 33°

Answer available at the end of this section.

(d)

Now that you have found angle A, which law would be the best to use to find the measure of angle B?

Answer available at the end of this section.

(e)

What is the value of B (to the nearest degree)?

A. 109°
B. 39°
C. 108°
D. 33°

Answer available at the end of this section.

(f)

What is the value of C (to the nearest degree)?

A. 109°
B. 39°
C. 108°
D. 33°

Answer available at the end of this section.

Note

Notice in *(see related activity)*, you were given a triangle with three known sides. This type of triangle is known as a side-side-side (SSS) triangle.

Activity 4.3.2.12

State whether the Law of Sines or the Law of Cosines is the best choice to use to determine the indicated angle/side.

(a)

Given $B = 112^\circ$, $a = 12$, and $b = 25$, find A.

Hint:

It might help to draw a picture!

Answer available at the end of this section.

(b)

Given $B = 87^\circ$, $a = 15$, and $c = 16$, find b.

Answer available at the end of this section.

(c)

Given $a = 37$, $b = 55$, and $c = 30$, find C.

Answer available at the end of this section.

(d)

Given $A = 108^\circ$, $B = 40^\circ$, and $b = 20$, find c.

Answer available at the end of this section.

Activity 4.3.2.13

Now that we can determine which laws to use, let's go back to *(see related activity)* to solve each triangle (to the nearest degree/whole number).

(a)

$B = 112^\circ$, $a = 12$, and $b = 25$.

Answer available at the end of this section.

(b)

$B = 87^\circ$, $a = 15$, and $c = 16$.

Answer available at the end of this section.

(c)

$a = 37$, $b = 55$, and $c = 30$.

Answer available at the end of this section.

(d)

$A = 108^\circ$, $B = 40^\circ$, and $b = 20$.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can use the Law of Sines and Law of Cosines to solve non-right triangles before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 4.3.2.1

A triangle has angle $A = 40^\circ$, angle $B = 65^\circ$, and side $a = 12$ (opposite angle A). Find side b (opposite angle B), rounded to two decimal places.

Answer available at the end of this section.

Check Your Understanding 4.3.2.2

A triangle has sides $a = 8$, $b = 10$, and included angle $C = 50^\circ$ (the angle between sides a and b). Find side c, rounded to two decimal places, and explain why the Law of Sines could not be used directly here.

Answer available at the end of this section.

Check Your Understanding 4.3.2.3

A triangle has sides $a = 6$, $b = 9$, $c = 11$. Use the Law of Cosines to find angle C (opposite side c), rounded to one decimal place.

Answer available at the end of this section.

Answers

Activity 4.3.2.1
B

Activity 4.3.2.1
C

Activity 4.3.2.1
$h = b \cdot \sin A$

$h = a \cdot \sin B$

Activity 4.3.2.1

$$\frac{\sin A}{a} = \frac{\sin B}{b} \tag{4.3.2.6}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{4.3.2.7}$$

Activity 4.3.2.2

The following parts of the triangle are missing: C, b, and c.

Activity 4.3.2.2

D

Activity 4.3.2.2

$$\frac{\sin 35^\circ}{10} = \frac{\sin 120^\circ}{c} \tag{4.3.2.8}$$

or

$$\frac{10}{\sin 35^\circ} = \frac{c}{\sin 120^\circ} \tag{4.3.2.9}$$

Activity 4.3.2.2

C

Activity 4.3.2.2

A and C

Activity 4.3.2.2

D

Activity 4.3.2.3

The following parts of the triangle are missing: C, a, and b.

Activity 4.3.2.3

A

Activity 4.3.2.3

B

Activity 4.3.2.3

D

Activity 4.3.2.4

The following parts of the triangle are missing: B, C, and c.

Activity 4.3.2.4

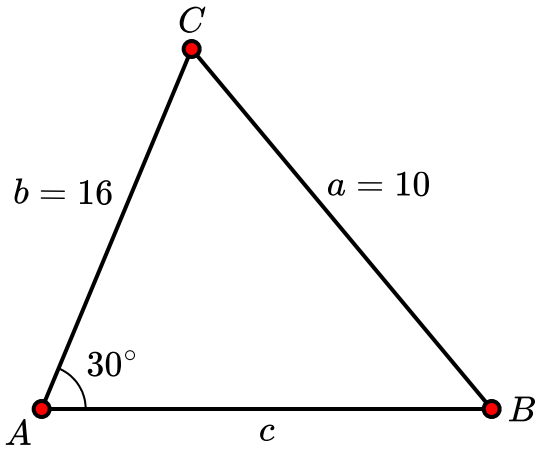
B

Activity 4.3.2.4

Students will see that when finding $\sin^{-1}(\frac{8}{7})$, that the value does not exist.

Activity 4.3.2.5

Students should be able to draw a triangle like this: The following parts of the triangle are missing: B, C, and c.



Activity 4.3.2.5

B

Activity 4.3.2.5

$C \approx 97^\circ$

$c \approx 20$

Activity 4.3.2.5

D

Activity 4.3.2.5

Because $\sin\theta$ exists in both Quadrant I and Quadrant II, there exists two possible values of θ. $\sin 30^\circ$ and $\sin 150^\circ$, for example, are both equal to $\frac{1}{2}$.

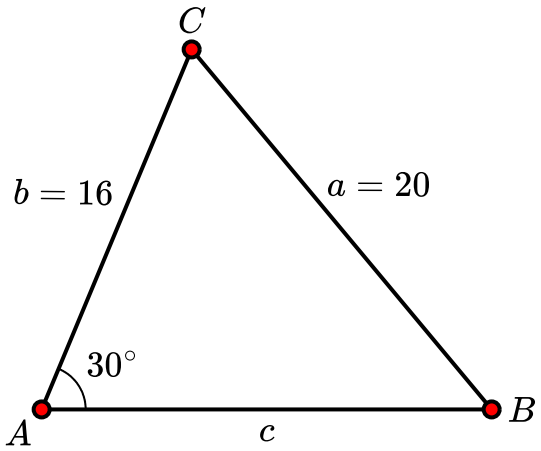
Activity 4.3.2.5

$C \approx 23^\circ$

$c \approx 8$

Activity 4.3.2.6

Students should be able to draw a triangle like this: The following parts of the triangle are missing: B, C, and c.



Activity 4.3.2.6

A and D

Activity 4.3.2.6

$C \approx 126^\circ$

$c \approx 32$

Activity 4.3.2.6

Another triangle does not exist because if B is equal to 156°, then that would mean the third angle, C, would have to be -6°. In other words, the sum of those two angles is already greater than 180°, so no triangle can exist.

Activity 4.3.2.7

One triangle: $A = 28^\circ$, $B = 7^\circ$, $a = 27$

Activity 4.3.2.7

Not a triangle

Activity 4.3.2.7

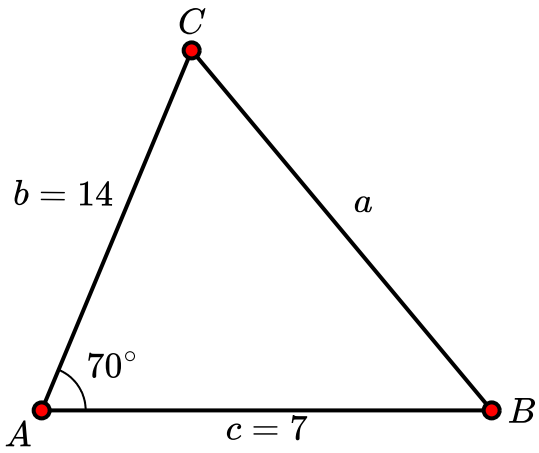
Two triangles exist:

$C = 87.8^\circ$, $A = 47.2^\circ$, $c = 38.2$

$C = 2.2^\circ$, $A = 132.8^\circ$, $c = 1.5$

Activity 4.3.2.8

Students should draw the triangle like this: The following parts of the triangle are missing: B, C, and a.



Activity 4.3.2.8

Students should recognize that the Law of Sines cannot be used to solve this triangle.

Activity 4.3.2.9

B

Activity 4.3.2.9

C

Activity 4.3.2.9

$h = b \sin A$

Activity 4.3.2.9

$m = b \cos A$

Activity 4.3.2.9

$h^2 + (c - m)^2 = a^2$

$(b \sin A)^2 + (c - (b \cos A))^2 = a^2$

Activity 4.3.2.9

$(b \sin A)^2 + (c - (b \cos A))^2 = a^2$

$b^2 \sin^2 A + c^2 - 2bc \cos A + b^2 \cos^2 A = a^2$

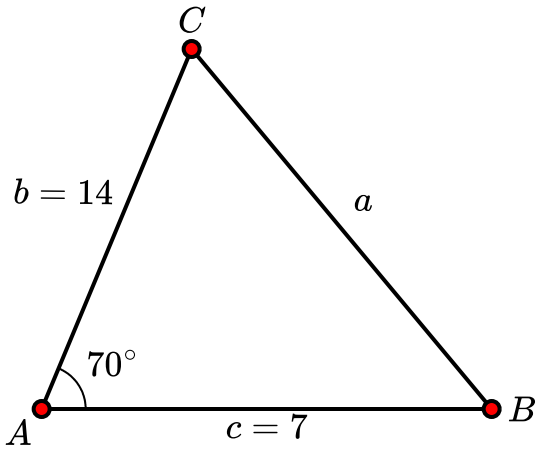
$b^2 \sin^2 A + b^2 \cos^2 A + c^2 - 2bc \cos A = a^2$

$b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A = a^2$

$b^2 + c^2 - 2bc \cos A = a^2$

Activity 4.3.2.10

Students should draw the triangle like this: The following parts of the triangle are missing: B, C, and a.



Activity 4.3.2.10

C

Activity 4.3.2.10

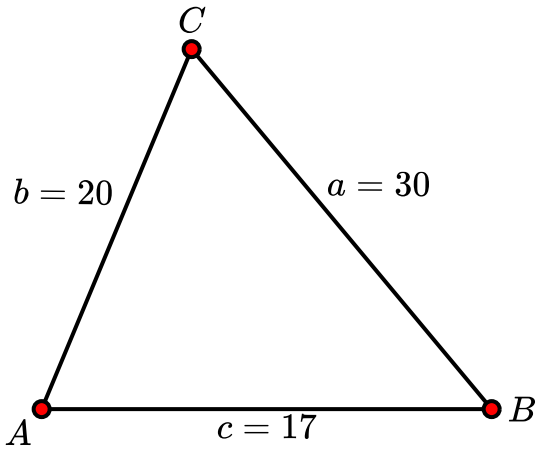
A or B. Students should discuss how either method can used, but that there is a possibility of the ambiguous case for the Law of Sines.

Activity 4.3.2.10

$B \approx 81$ and $C \approx 29^\circ$

Activity 4.3.2.11

Students should draw the triangle like this: The following parts of the triangle are missing: A, B, and C.



Activity 4.3.2.11

Law of Cosines

Activity 4.3.2.11

C

Activity 4.3.2.11

The Law of Sines might be the easiest method to use, but students could also use the Law of Cosines.

Activity 4.3.2.11

B

Activity 4.3.2.11

D

Activity 4.3.2.12

Law of Sines

Activity 4.3.2.12

Law of Cosines

Activity 4.3.2.12

Law of Cosines

Activity 4.3.2.12

Law of Sines

Activity 4.3.2.13

$A \approx 26^\circ$, $C \approx 42^\circ$, and $c \approx 18$

Activity 4.3.2.13

$b \approx 21$, $A \approx 45^\circ$, and $C \approx 48^\circ$

Activity 4.3.2.13

$C \approx 31^\circ$, $A \approx 39^\circ$, and $B \approx 110^\circ$

Activity 4.3.2.13

$C \approx 32^\circ$, $a \approx 30$, and $c \approx 16$

Check Your Understanding 4.3.2.1

By the Law of Sines, $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$, so $b = \frac{a\sin(B)}{\sin(A)} = \frac{12\sin(65^\circ)}{\sin(40^\circ)} \approx \frac{12(0.9063)}{0.6428} \approx 16.92$.

Check Your Understanding 4.3.2.2

By the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(C) = 64 + 100 - 2(8)(10)\cos(50^\circ) \approx 164 - 160(0.6428) \approx 61.15$, so $c \approx 7.82$. The Law of Sines needs a known angle paired with its opposite side; here we only know two sides and the angle between them (not opposite either given side), so the Law of Cosines is the tool that fits this information.

Check Your Understanding 4.3.2.3

Solving the Law of Cosines for the angle: $\cos(C) = \frac{a^2 + b^2 - c^2}{2ab} = \frac{36 + 81 - 121}{2(6)(9)} = \frac{-4}{108} \approx -0.0370$. So $C = \arccos(-0.0370) \approx 92.1^\circ$.

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4.3.2.1: TE Online Exercises

TE5 — Use the law of sines and/or the law of cosines to solve non-right triangles.	TE5 CheckIt Exercises

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4.3.3: Applications of Trig (TE6)

*This section is **supplemental** — not currently part of the core Math 172 sequence. Converted and kept available for possible future use.*

TE6 Learning outcome: Solve applications that require the use of trigonometry to find a side or angle.

Trigonometric Applications: Angles of Elevation and Depression

Overview

Trigonometry is the branch of mathematics that focuses on the relationships between the angles and sides of triangles. It has a wide range of applications across various fields, including science, engineering, architecture, and more. In this section, we will look at some common ways trigonometry is used.

Activity 4.3.3.1

A pilot signals to an air traffic controller that she wants to land. She wants to know what angle of descent to take when she is currently at $40{,}000$ feet. Her plane is a horizontal distance of $750{,}000$ feet from the runway, as the air traffic controller can see on the radar.

(a)

Draw a diagram to represent this situation.

Answer available at the end of this section.

(b)

The pilot wants to know what angle of descent to take in order to reach the runway. The angle of descent is the angle between the flight path and the ground. In other words, it is an angle that is formed by the horizontal line and the pilot's line of sight to the runway. Where is this angle located in your diagram?

Answer available at the end of this section.

(c)

Draw a line that represents the line of sight of the pilot. What do you notice about this line and the base of your triangle?

Answer available at the end of this section.

(d)

If we know that the pilot's line of sight is parallel to the base of the triangle you created in part (a), then the hypotenuse of the triangle could also be considered a transversal that cuts the two parallel lines. What angle of the triangle is congruent to the "angle of descent" the pilot wants to take in order to descend the plane $750{,}000$ feet from the runway?

Answer available at the end of this section.

(e)

Now that we know which angle is congruent to the "angle of descent" the pilot needs, which of the 6 trig functions could we use to find the angle at which the pilot should descend?

Answer available at the end of this section.

(f)

Find the measure of the angle the pilot needs to make her initial descent.

Answer available at the end of this section.

Note

Notice that *(see related activity)*, the angle that we needed to find was an angle that was not inside the right triangle. In these cases, it would be helpful to use prior knowledge of parallel lines and angle relationships to determine which other angle is congruent to that given angle.

Definition: Angle of Depression and Elevation

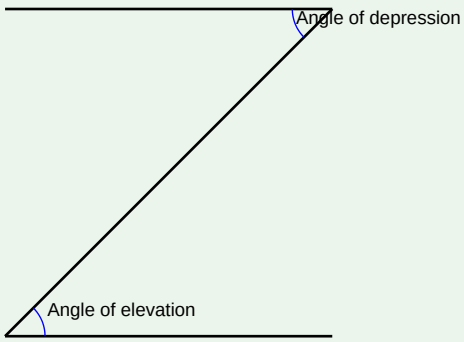
Many applications of trigonometry will include the angle of elevation and the angle of depression which are formed by two parallel lines cut by a transversal.

- An **angle of elevation** is the angle formed by a horizontal line and a line of sight to a point above the line.
- An **angle of depression** is the angle formed by a horizontal line and a line of sight to a point below the line.

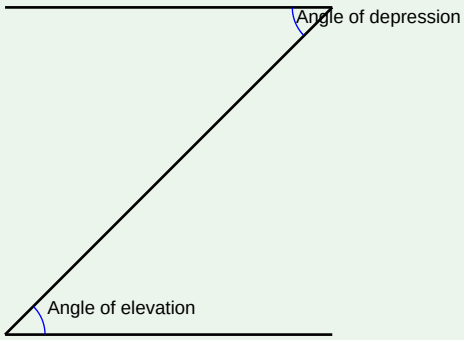
An **angle of elevation** is the angle formed by a horizontal line and a line of sight to a point above the line.

An **angle of depression** is the angle formed by a horizontal line and a line of sight to a point below the line.

The angle of elevation and the angle of depression are congruent.



Notice that because both the angle of elevation and the angle of depression are formed by horizontal lines that are parallel, the angle of elevation is congruent to the angle of depression (by the alternate interior angles theorem).



Activity 4.3.3.2

An observer who is 1.5 meters tall is standing 67 meters away from the Seattle Space Needle.

(a)

Draw a diagram to represent this situation.

Answer available at the end of this section.

(b)

If the angle of elevation from where the observers stands to the top of the Space Needle is 70, which trig function could you use to find the height of the Space Needle?

Answer available at the end of this section.

(c)

How tall is the Space Needle (to the nearest meter)?

Hint:

Don't forget to account for the height of the observer!

Answer available at the end of this section.

Activity 4.3.3.3

Use *(see related activity)* and your knowledge of right triangles to solve each of the following. It might be helpful to draw a diagram to represent the situation before solving.

(a)

Sarah's kite is flying above a field at the end of 65 meters of string. If the angle of elevation to the kite measures 70, how high is the kite above Sarah's head?

Answer available at the end of this section.

(b)

Standing on a cliff 380 meters above the sea, Sean sees an approaching ship and measures its angle of depression, obtaining 9 degrees. How far from shore is the ship (to the nearest meter)?

Answer available at the end of this section.

(c)

A 14-foot ladder is used to scale a 13-foot wall. At what angle of elevation (to the nearest degree) must the ladder be situated in order to reach the top of the wall?

Answer available at the end of this section.

(d)

A submarine starts on the surface, and dives at an angle of 13 to the surface. It goes diagonally a distance of 890 meters before reaching the bottom. How far is it along the ocean surface from the point where the submarine started to the point directly above where it reached the bottom?

Answer available at the end of this section.

Note

All applications we have done so far have involved right triangles. Let's now look at some examples of non-right triangle applications.

Activity 4.3.3.4

Airplane A is flying directly towards the airport which is 20 miles away. From Airplane A's point of view, the angle between the airport and Airplane B is 45 and from Airplane B's point of view, the angle between the airport and Airplane A is 50.

(a)

Draw a diagram to represent this situation.

(b)

Based on the information given and your diagram, how far is Airplane B from the airport (to the nearest tenth of a mile)?

Answer available at the end of this section.

Activity 4.3.3.5

Carlos, Jean, and Travis are camping in their tents. The distance between Carlos and Jean is 153 feet, the distance between Carlos and Travis is 201 feet, and the distance between Jean and Travis is 175 feet.

(a)

Draw a diagram to represent this situation.

Hint:

It might be helpful to label your triangle in terms of C, J, and T for the angles and c, j, and t for the sides of the triangle.

Answer available at the end of this section.

(b)

What type of triangle represents this situation? How can you determine whether this triangle is a right triangle?

Answer available at the end of this section.

(c)

Refer back to the previous section. Which trigonometric law (the Law of Sines or the Law of Cosines) would be the best one to use if we wanted to find the angle at which Carlos is from his friends?

Answer available at the end of this section.

(d)

Find the angle at which Carlos is located from his friends to the nearest degree.

Answer available at the end of this section.

(e)

Now that we know the angle at which Carlos is located from Jean and Travis, determine the angle (to the nearest degree) at which Travis is located from his friends by using the Law of Sines.

Answer available at the end of this section.

(f)

Find the remaining angle left of the triangle.

Answer available at the end of this section.

Note

Trigonometric functions can model relationships between different quantities that follow a periodic nature: height over time, distance over time, temperature over time and so on. Scientists observe this back-and-forth movement and collect data so they can model them using an equation or a graph. They then use this information to make predictions for the future.

Activity $4.3.3.6$

The depth of the water in meters at a certain pier varies with the tide and is modeled by the equation $d(t) = 2 + \frac{1}{2}\sin\frac{\pi}{6}t$ where t is the number of hours after 10 a.m.

(a)

How deep will the water be at high tide?

Answer available at the end of this section.

(b)

How deep will it be at low tide?

Answer available at the end of this section.

(c)

How many hours are there between two successive high tides?

Answer available at the end of this section.

(d)

How many of these tide cycles are there in a 24 hour day?

Answer available at the end of this section.

Activity $4.3.3.7$

A circular Ferris wheel is 120 meters in diameter and contains several carriages. Jesus and Allison enter a carriage at the bottom of the wheel and get off 24 minutes later after having gone around 8 times. When a carriage is at the bottom of the wheel, it is 1 meter off the ground.

(a)

What is the maximum and minimum height of Jesus and Allison's carriage?

Answer available at the end of this section.

(b)

Let $h(t)$ be a function that represents the height of the carriage t minutes after it has started moving. What is the period of $h(t)$?

Answer available at the end of this section.

(c)

Which trigonometric function would be the best to use to model this situation?

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can solve applied trigonometry problems involving angles of elevation and depression before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding $4.3.3.1$

From a point on the ground 50 feet from the base of a building, the angle of elevation to the top of the building is 38°. Find the height of the building, rounded to two decimal places.

Answer available at the end of this section.

Check Your Understanding 4.3.3.2

A drone hovers at a height of 120 feet. The angle of depression from the drone to a landmark on the ground is 25°. Find the horizontal distance from the point directly below the drone to the landmark, rounded to two decimal places.

Answer available at the end of this section.

Check Your Understanding 4.3.3.3

Two people stand on opposite sides of a flagpole. One measures the angle of elevation to the top as 42° from a distance of 30 feet; the other stands 45 feet away on the opposite side. Find the second person's angle of elevation, rounded to one decimal place, using the flagpole's height found from the first measurement.

Answer available at the end of this section.

Answers

Activity 4.3.3.1

Students should draw a right triangle where the height is 40,000 and the base is 750,000.

Activity 4.3.3.1

Students may notice that the angle they are looking for is not inside the triangle and the line of sight of the pilot is looking down at the ground.

Activity 4.3.3.1

The pilot's line of sight is parallel to the base of the triangle.

Activity 4.3.3.1

The “angle of descent” is congruent to the same angle that is created with the base of the triangle (not the right angle).

Activity 4.3.3.1

If students label the triangle correctly, they should see that they can now use $\tan^{-1}$ to find the measure of the angle because they have 40,000 as the opposite side and 750,000 as the adjacent side.

Activity 4.3.3.1

Approximately 3.05.

Activity 4.3.3.2

Students should draw a right triangle so that the base of the triangle is 67 meters, as it represents the distance the observer is from the Space Needle.

Activity 4.3.3.2

The tangent function would be the best to use because the opposite of the given angle (angle of elevation) is the height of the Space Needle and the distance from the Space Needle to the observer would be the adjacent side.

Activity 4.3.3.2

Approximately 185.5 meters.

Activity 4.3.3.3

61 meters

Activity 4.3.3.3

2,399 meters

Activity 4.3.3.3

68

Activity 4.3.3.3

Approximately 867 meters.

Activity 4.3.3.4

18.5 miles

Activity 4.3.3.5

In their diagram, students should draw a triangle with all three sides labeled with the side lengths 153, 201, and 175.

Activity 4.3.3.5

Students might try to use the Pythagorean Theorem to determine if this is a right triangle. After some time, students should see that this is a non-right triangle with all three sides given.

Activity 4.3.3.5

Because we are given all sides of this non-right triangle, the best trigonometric law to use would be the Law of Cosines.

Activity 4.3.3.5

57

Activity 4.3.3.5

47

Activity 4.3.3.5

76

Activity 4.3.3.6

Given that the amplitude of $\frac{1}{2}\sin\frac{\pi}{6}t$ is $\frac{1}{2}$, the function will always be between $\frac{1}{2}$ and $-\frac{1}{2}$. So, the maximum value $d(t)$ is $2+\frac{1}{2}$, or 2.5 meters deep at high tide.

Activity 4.3.3.6

Given that the amplitude of $\frac{1}{2}\sin\frac{\pi}{6}t$ is $\frac{1}{2}$, the function will always be between $\frac{1}{2}$ and $-\frac{1}{2}$. So, the minimum value $d(t)$ is $2-\frac{1}{2}$, or 1.5 meters deep at low tide.

Activity 4.3.3.6

This is really asking for the period. The period of a sine function is $\frac{2\pi}{b}$. In the equation given, the b value is $\frac{\pi}{6}$. Therefore, the period is $\frac{2\pi}{\frac{\pi}{6}}$, which is equal to 12. So, the time between successive high tides is 12 hours.

Activity 4.3.3.6

From part (c), we know the period is 12 hours. So, that means one cycle is 12 hours, so there are 2 cycles in a 24-hour day.

Activity 4.3.3.7

Because the diameter of the Ferris wheel is 120 meters and that the carriage is at its lowest height off the ground at the bottom of the wheel (1 meter), the minimum height is 1 and the maximum height is $1+120=121$ meters.

Activity 4.3.3.7

The period is the amount of time it takes to complete one revolution (or one cycle around the Ferris wheel). We know that it takes 24 minutes to go around 8 times, so it must take 3 minutes to go around once.

Activity 4.3.3.7

The cosine function would be the best to use because when people get on a Ferris wheel, they are starting at the minimum of a curve and then does a complete revolution when they ride the Ferris wheel. Because it starts at a minimum, it is really an upside down cosine graph (because the parent function starts at a maximum).

Check Your Understanding 4.3.3.1

$\tan(38^\circ)=\frac{\text{height}}{50}$, so height $=50\tan(38^\circ)\approx 50(0.7813)\approx 39.07$ feet.

Check Your Understanding 4.3.3.2

The angle of depression from the drone equals the angle of elevation from the landmark (alternate interior angles), so $\tan(25^\circ)=\frac{120}{\text{distance}}$, giving distance $=\frac{120}{\tan(25^\circ)}\approx\frac{120}{0.4663}\approx 257.34$ feet.

Check Your Understanding 4.3.3.3

First find the height: $\tan(42^\circ)=\frac{h}{30}$, so $h=30\tan(42^\circ)\approx 27.01$ feet. Then for the second person: $\tan(\theta)=\frac{27.01}{45}$, so $\theta=\arctan\left(\frac{27.01}{45}\right)\approx 31.0^\circ$.

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4.3.3.1: TE6 Online Exercises

TE6 — Solve applications that require the use of trigonometry to find a side or angle.	TE6 CheckIt Exercises

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4.4: INTRODUCTION TO OTHER TRIG AND IDENTITIES (TE1, TE2, TE3)

4.4.1: Trigonometric Identities (TE1)

4.4.1.1: TE1 Online Exercises

4.4.2: Verifying Identities (TE2)

4.4.2.1: TE2 Online Exercises

4.4.3: Trigonometric Equations (TE3)

4.4.3.1: TE3 Online Exercises

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4.4.1: Trigonometric Identities (TE1)

This section corresponds to APC slot 4.5 (Other trig functions & identities) in the Math 172 flow.

TE1 Learning outcome: Use the sum and difference, the double-angle, and power-reducing formulas for cosine, sine, and tangent, and the Pythagorean identities.

Trigonometric Identities

Overview

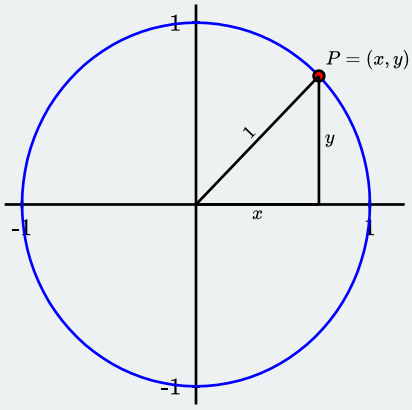
Pythagorean Identities, as we have seen in a previous section, are derived from the Pythagorean Theorem $(a^2 + b^2 = c^2)$.

For example, consider a point P on the unit circle, with coordinates (x, y). If we draw a right triangle (as shown in the figure below), the Pythagorean Theorem says that $x^2 + y^2 = 1$.

But, remember, that the x-coordinate of the point corresponds to $\cos\theta$ and the y-coordinate corresponds to $\sin\theta$. Thus, we get:

$$\sin^2\theta + \cos^2\theta = 1 \tag{4.4.1.1}$$

. Pythagorean Identities are used in solving many trigonometric problems where one trigonometric ratio is given and we are expected to find the other trigonometric ratios. The next two activities will lead us to find the other two Pythagorean Identities.



Activity 4.4.1.1

Let's begin with the Pythagorean Identity, $\cos^2\theta + \sin^2\theta = 1$.

(a)

If we divide each term by $\cos^2\theta$, what would the resulting equation be?

Answer available at the end of this section.

(b)

Recall the following trigonometric functions:

- $\sin\theta$
- $\csc\theta = \frac{1}{\sin\theta}$
- $\cos\theta$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\tan\theta$
- $\cot\theta = \frac{\sin\theta}{\cos\theta}$

Using these trigonometric relationships, rewrite each fractional term in the equation you got in part (a) as a single trigonometric function.

Answer available at the end of this section.

Note

In *(see related activity)*, we derived the second Pythagorean Identity:

$$1 + \tan^2\theta = \sec^2\theta \tag{4.4.1.2}$$

Activity 4.4.1.2

We can also derive the third Pythagorean identity from $\sin^2\theta + \cos^2\theta = 1$.

(a)

What do we get if we divide each term by $\sin^2\theta$?

Answer available at the end of this section.

(b)

Rewrite each fractional term in the equation you got in part (a) as a single trigonometric function.

Answer available at the end of this section.

Note

In *(see related activity)*, we derived the third Pythagorean Identity:

$$1 + \cot^2\theta = \csc^2\theta \tag{4.4.1.3}$$

Theorem: Pythagorean Identities

The three **Pythagorean Identities** are:

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Note that each of these identities can be written in different forms, which are all equivalent to one another. For example, $\sin^2\theta + \cos^2\theta = 1$ can be rewritten as $1 - \sin^2\theta = \cos^2\theta$ or $1 - \cos^2\theta = \sin^2\theta$. You will want to become familiar with all these forms.

Note

Other identities and formulas (in addition to the Pythagorean Identities) can be used to solve various mathematical problems. The next few activities will lead us through an exploration of other types of identities and formulas.

Activity 4.4.1.3

Refer back to the unit circle to determine the exact value of $\sin\left(\pi + \frac{\pi}{2}\right)$.

(a)

What is the value of $\sin\pi$?

A. -1
B. 0
C. 1

Answer available at the end of this section.

(b)

What is the value of $\sin\left(\frac{\pi}{2}\right)$?

A. -1
B. 0
C. 1

Answer available at the end of this section.

(c)

Based on your answers from parts (a) and (b), what do you think the value of $\sin\left(\pi + \frac{\pi}{2}\right)$ is?

Answer available at the end of this section.

(d)

Let's test your conjecture from part (c). What is the value of $\pi + \frac{\pi}{2}$?

A. $\frac{2\pi}{2}$
B. π
C. $\frac{3\pi}{2}$
D. $\frac{2\pi}{3}$

Answer available at the end of this section.

(e)

What is the value of $\sin\left(\frac{3\pi}{2}\right)$?

A. -1
B. 0
C. 1

Answer available at the end of this section.

(f)

Is it safe to assume that $\sin\left(\pi+\frac{\pi}{2}\right)=\sin\pi+\sin\left(\frac{\pi}{2}\right)$?

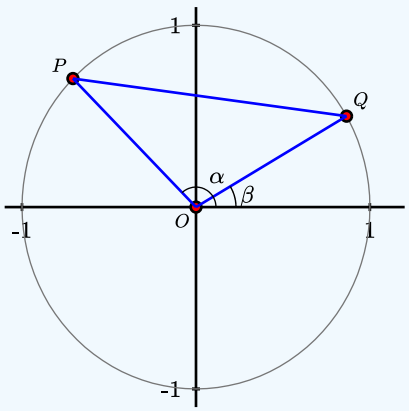
Answer available at the end of this section.

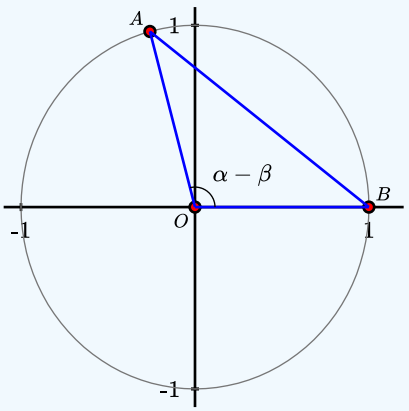
Note

Notice that in *(see related activity)*, we saw that $\sin\left(\pi+\frac{\pi}{2}\right)\neq\sin\pi+\sin\frac{\pi}{2}$. This is also true for both cosine and tangent as well - that is, if we want to find the cosine or tangent of the sum of two angles, we cannot assume that it is equal to the sum of the two trigonometric functions of each angle. For example, $\cos\left(\pi+\frac{\pi}{2}\right)\neq\cos\pi+\cos\frac{\pi}{2}$ and $\tan\left(\pi+\frac{\pi}{2}\right)\neq\tan\pi+\tan\frac{\pi}{2}$. The same is true for finding the difference of two angles.

Activity 4.4.1.4

Recall that the coordinates of points on the unit circle are given by $(\cos\theta,\sin\theta)$. In the first unit circle shown below, point P makes an angle α with the positive x-axis and has coordinates $(\cos\alpha,\sin\alpha)$ and point Q makes an angle β with the positive x-axis and has coordinates $(\cos\beta,\sin\beta)$. In the second figure, the triangle is rotated so that point B has coordinates $(1,0)$.





(a)

Note that angle AOB is equal to $(\alpha - \beta)$. What are the coordinates of point A?

Answer available at the end of this section.

(b)

Triangles POQ and AOB are rotations of one another. What can we say about the lengths of PQ and AB?

Answer available at the end of this section.

(c)

Let's use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to find the length of PQ. What do you get when you plug in the coordinates of point P and point Q?

A. $d = \sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2}$
B. $d = \sqrt{(\cos\alpha - \sin\beta)^2 + (\sin\alpha - \cos\beta)^2}$
C. $d = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$
D. $d = \sqrt{(\sin\alpha - \cos\beta)^2 + (\cos\alpha - \sin\beta)^2}$

Hint:

Remember that point P has coordinates $(\cos\alpha, \sin\alpha)$ and point Q has coordinates $(\cos\beta, \sin\beta)$.

Answer available at the end of this section.

(d)

Begin simplifying your answer from part (c) by applying the algebraic identity $(a - b)^2 = a^2 - 2ab + b^2$. What do you get when squaring the two binomials under the radical?

A. $d = \sqrt{\sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta + \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta}$
B. $d = \sqrt{\cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta}$
C. $d = \sqrt{\cos^2\alpha - \cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - \sin\alpha\sin\beta + \sin^2\beta}$
D. $d = \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta}$

Answer available at the end of this section.

(e)

Simplify your answer from part (d) even further by applying the Pythagorean Identity, $\cos^2\theta + \sin^2\theta = 1$. What does your answer from part (d) simplify to?

A. $d = \sqrt{1 - \sin\alpha\sin\beta - \cos\alpha\cos\beta}$
B. $d = \sqrt{2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta}$
C. $d = \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$
D. $d = \sqrt{2 - \sin\alpha\sin\beta - \cos\alpha\cos\beta}$

Hint:

You may need to rearrange some of your terms before applying the Pythagorean Identity.

Answer available at the end of this section.

(f)

Using the same steps as in parts (c-e), use the distance formula to find the distance between points A and B.

A. $d = \sqrt{1 - 2\cos(\alpha - \beta)}$
B. $d = \sqrt{2 - 2\cos(\alpha - \beta)}$
C. $d = \sqrt{1 - \cos(\alpha - \beta)}$
D. $d = \sqrt{-2\cos(\alpha - \beta) + 2}$

Hint:

Point A has coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$ and point B has coordinates $(1, 0)$.

Answer available at the end of this section.

(g)

From part (b), we know that the length of PQ is equal to the length of AB. Therefore,

$$\sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} = \sqrt{2 - 2\cos(\alpha - \beta)} \tag{4.4.1.4}$$

. Square both sides of this equation and solve for $\cos(\alpha - \beta)$.

Answer available at the end of this section.

Note

In *(see related activity)*, we derived the difference formula for cosine. The development of the sum/difference formulas are similar for sine and cosine and same for $\tan(\alpha - \beta)$ compared to $\tan(\alpha + \beta)$.

Theorem: Sum and Difference Formulas

The **Sum and Difference Formulas** in trigonometry are used to find the value of the trigonometric functions at specific angles where it is easier to express the angle as the sum or difference of unique angles such as $0°$, $30°$, $45°$, $60°$, $90°$, and $180°$.

The six main trigonometric sum and difference formulas are:

- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Activity 4.4.1.5

Use the unit circle and *(see related activity)* to find the exact value of $\cos 75°$.

(a)

Split $75°$ into the sum of two angles which can be found on the unit circle (use unique angles such as $0°$, $30°$, $45°$, $60°$, $90°$, and $180°$, etc.).

Answer available at the end of this section.

(b)

Rewrite $\cos 75°$ as $\cos(\alpha + \beta)$ where α and β are the two angles you found in part (a).

Answer available at the end of this section.

(c)

Apply *(see related activity)* to rewrite $\cos 75^\circ$ with four trigonometric functions.

Answer available at the end of this section.

(d)

Use the unit circle to find the value of each trigonometric function in the formula you wrote in part (c).

Answer available at the end of this section.

(e)

Simplify your answer from part (d).

Answer available at the end of this section.

Activity 4.4.1.6

Determine whether each statement is true or false by referring to *(see related activity)*.

(a)

$\sin(45^\circ - 30^\circ) = \sin 45^\circ - \sin 30^\circ$

Answer available at the end of this section.

(b)

$\cos 15^\circ = \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ$

Answer available at the end of this section.

(c)

$$\tan 75^\circ = \frac{\tan 120^\circ - \tan 45^\circ}{1 + \tan 120^\circ \tan 45^\circ}$$

Answer available at the end of this section.

Activity 4.4.1.7

Apply *(see related activity)* to find the exact value for each of the following trigonometric functions.

(a)

$\sin 165^\circ$

Answer available at the end of this section.

(b)

$\tan 195^\circ$

Answer available at the end of this section.

(c)

$\cos\left(\frac{7\pi}{12}\right)$

Answer available at the end of this section.

(d)

$\tan\left(\frac{11\pi}{12}\right)$

Answer available at the end of this section.

Activity 4.4.1.8

Apply *(see related activity)* to find the exact value for each of the following trigonometric functions.

(a)

$\cos 87^\circ \cos 33^\circ - \sin 87^\circ \sin 33^\circ$

Answer available at the end of this section.

(b)

$\sin 20^\circ \cos 80^\circ - \cos 20^\circ \sin 80^\circ$

Answer available at the end of this section.

(c)

$\sin\frac{\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}\cos\frac{\pi}{12}$

Answer available at the end of this section.

(d)

$$\frac{\tan 40^\circ - \tan 10^\circ}{1 + \tan 40^\circ \tan 10^\circ}$$

Answer available at the end of this section.

Activity 4.4.1.9

Recall from *(see related activity)* the sum formulas for sine, cosine, and tangent.

(a)

Suppose $\alpha = \beta$. Rewrite the left-hand and right-hand sides of the sum formula for sine $(\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta)$ in terms of α.

Answer available at the end of this section.

(b)

Suppose $\alpha = \beta$. Rewrite the left-hand and right-hand sides of the sum formula for cosine $(\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta)$ in terms of α.

Answer available at the end of this section.

(c)

Suppose $\alpha = \beta$. Rewrite the left-hand and right-hand sides of the sum formula for tangent $\left(\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\right)$ in terms of α.

Answer available at the end of this section.

Theorem: Double Angle Formulas

Double angle formulas are used to express the trigonometric ratios of double angles (2θ) in terms of trigonometric ratios of single angle (θ).

The **Double Angle Formulas** of sine, cosine, and tangent are:

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$
- $\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Observation

You may have noticed that $\cos(2\theta)$ has three different forms. This is because we can use Pythagorean Identities to obtain the other forms.

For example, suppose we start with $\cos(2\theta) = \cos^2\theta - \sin^2\theta$. We can substitute the first $\cos^2\theta$ with $1 - \sin^2\theta$ (think about how you can rewrite the first Pythagorean Identity, $\cos^2\theta + \sin^2\theta = 1$). We will then have $1 - \sin^2\theta - \sin^2\theta$, which simplifies to $1 - 2\sin^2\theta$. Thus, $\cos(2\theta) = 1 - 2\sin^2\theta$. Using the same method, you can get $2\cos^2\theta - 1$.

Activity 4.4.1.10

Suppose $\sin\alpha = \dfrac{2}{3}$ and α lies in Quadrant I.

(a)

Use the Pythagorean Identities and *(see related activity)* to find $\sin 2\alpha$.

Answer available at the end of this section.

(b)

Use the Pythagorean Identities and *(see related activity)* to find $\cos 2\alpha$.

Answer available at the end of this section.

(c)

Use the Pythagorean Identities and *(see related activity)* to find $\tan 2\alpha$.

Answer available at the end of this section.

Activity 4.4.1.11

The double-angle formulas can be used to derive the reduction formulas, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine.

(a)

Let's start with the double angle formula for cosine to find our first power-reduction formula: $\cos 2\theta = 1 - 2\sin^2\theta$. Use your algebra skills to solve for $\sin^2\theta$.

Answer available at the end of this section.

(b)

Given the second double angle formula,$\cos 2\theta = 2\cos^2\theta - 1$, solve for $\cos^2\theta$.

Answer available at the end of this section.

(c)

To generate the power reduction formula for tangent, let's begin with its definition: $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$. Use the formulas you created in parts (a) and (b) to rewrite $\tan^2\theta$.

Answer available at the end of this section.

Theorem: Power Reduction Formulas

The trigonometric power reduction formulas allow us to rewrite expressions involving trigonometric terms with trigonometric terms of smaller powers.

The **Power Reduction Formulas** are:

- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
- $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Activity 4.4.1.12

Rewrite $\sin^4\beta$ as an expression without powers greater than one.

(a)

How can we rewrite $\sin^4\beta$ as a function squared?

Answer available at the end of this section.

(b)

Rewrite $\sin^2\beta$ using *(see related activity)*.

Answer available at the end of this section.

(c)

Square the right-hand side of the equation you found in part (b).

Answer available at the end of this section.

(d)

Notice that you still have a cosine function being squared in your equation in part (c). Substitute the $\cos^2\beta$ using the power reduction formula in *(see related activity)* and then simplify

Answer available at the end of this section.

Activity 4.4.1.13

The next set of identities is the set of half-angle formulas, which can be derived from the reduction formulas. Let's derive the half-angle formula for $\sin\frac{\theta}{2}$.

(a)

Recall the power reduction formula: $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. Replace θ with $\frac{\theta}{2}$.

Answer available at the end of this section.

(b)

Simplify the right-hand side of your equation in part (a).

Answer available at the end of this section.

(c)

Now take the square root of both sides to solve for $\sin\theta$.

Answer available at the end of this section.

Note

The derivation for $\cos\left(\frac{\theta}{2}\right)$ and $\tan\left(\frac{\theta}{2}\right)$ is similar to that in *(see related activity)* when starting with the power-reduction formulas for each trigonometric function.

Theorem: Half Angle Formulas

The **Half Angle Formulas** are:

- $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ or $\sin\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{2}}$ depending on which quadrant θ is in.

- $\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$ or $\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}$ depending on which quadrant θ is in.
- $\tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}$ or $\tan\frac{\theta}{2}=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}$ depending on which quadrant θ is in.

$\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$ or $\sin\frac{\theta}{2}=-\sqrt{\frac{1-\cos\theta}{2}}$ depending on which quadrant θ is in.

$\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$ or $\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}$ depending on which quadrant θ is in.

$\tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}$ or $\tan\frac{\theta}{2}=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}$ depending on which quadrant θ is in.

Activity 4.4.1.14

Find $\sin 15^\circ$ using a half-angle formula in *(see related activity)*.

(a)

Which quadrant is the angle 15° in?

Answer available at the end of this section.

(b)

What angle is 15° half of?

Answer available at the end of this section.

(c)

Which half angle formula should we use to determine $\sin 15^\circ$?

Answer available at the end of this section.

(d)

Substitute θ with 30° in the half-angle formula for sine.

Answer available at the end of this section.

(e)

Use the unit circle to find the exact value of $\cos 30^\circ$ and substitute that into the equation you got in part (d).

Answer available at the end of this section.

(f)

Simplify your equation in part (e) to find the exact value of $\sin 15^\circ$.

Answer available at the end of this section.

Activity 4.4.1.15

Use the appropriate half-angle formula in *(see related activity)* to find the exact value of each of the following.

(a)

$\cos 75^\circ$

Answer available at the end of this section.

(b)

$\sin 165^\circ$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply sum/difference, double-angle, and Pythagorean identities before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 4.4.1.1

Use a sum formula to find the exact value of $\cos(75^\circ)$ by writing $75^\circ = 45^\circ + 30^\circ$.

Answer available at the end of this section.

Check Your Understanding 4.4.1.2

If $\sin(\theta) = \dfrac{3}{5}$ and θ is in the first quadrant, use a double-angle identity to find $\sin(2\theta)$.

Answer available at the end of this section.

Check Your Understanding 4.4.1.3

Verify the identity $\dfrac{1-\cos(2\theta)}{\sin(2\theta)} = \tan(\theta)$ by rewriting the left side using double-angle identities.

Answer available at the end of this section.

Answers

Activity 4.4.1.1

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{4.4.1.5}$$

Activity 4.4.1.1

$1 + \tan^2\theta = \sec^2\theta$

Activity 4.4.1.2

$\dfrac{\sin^2\theta}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$

Activity 4.4.1.2

$1 + \cot^2\theta = \csc^2\theta$

Activity 4.4.1.3

B

Activity 4.4.1.3

C

Activity 4.4.1.3

Students will probably make the conjecture that the value of $\sin\left(\pi+\frac{\pi}{2}\right)$ is just $\sin\pi+\sin\left(\frac{\pi}{2}\right)$, which equals 1.

Activity 4.4.1.3

C

Activity 4.4.1.3

A

Activity 4.4.1.3

From the previous parts, students should see that

$$\sin\pi+\sin\frac{\pi}{2}=1 \tag{4.4.1.6}$$

and

$$\sin\left(\pi+\frac{\pi}{2}\right)=-1 \tag{4.4.1.7}$$

. So from this example, we can see that it is NOT safe to assume that $\sin\left(\pi+\frac{\pi}{2}\right)=\sin\pi+\sin\left(\frac{\pi}{2}\right)$.

Activity 4.4.1.4

Point A has coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$.

Activity 4.4.1.4

Because triangles POQ and AOB are rotations of one another, we know that the lengths of PQ and AB are the same. That is, the distance from P to Q is equal to the distance from A to B.

Activity 4.4.1.4

A or C, depending on which point students started with first

Activity 4.4.1.4

A or D, depending on which point students started with first

Activity 4.4.1.4

B or C

Activity 4.4.1.4

B or D

Activity 4.4.1.4

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Activity 4.4.1.5

30° and 45°

Activity 4.4.1.5

$\cos 75^\circ=\cos(30^\circ+40^\circ)$

Activity 4.4.1.5

$\cos 75^\circ=\cos 30^\circ\cos 45^\circ-\sin 30^\circ\sin 45^\circ$

Activity 4.4.1.5

$\cos 75^\circ=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

Activity 4.4.1.5

$\cos 75^\circ=\frac{\sqrt{6}-\sqrt{2}}{4}$

Activity 4.4.1.6

False

Activity 4.4.1.6

True

Activity 4.4.1.6

True

Activity 4.4.1.7

$\frac{\sqrt{6}-\sqrt{2}}{4}$

Activity 4.4.1.7

$2-\sqrt{3}$

Activity 4.4.1.7

$\frac{\sqrt{2}-\sqrt{6}}{4}$

Activity 4.4.1.7

$\sqrt{3}-2$

Activity 4.4.1.8

$-\frac{1}{2}$

Activity 4.4.1.8

$-\frac{\sqrt{3}}{2}$

Activity 4.4.1.8

1

Activity 4.4.1.8

$\frac{\sqrt{3}}{3}$

Activity 4.4.1.9

$\sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha$

$\sin(2\alpha)=2\sin\alpha\cos\alpha$

Activity 4.4.1.9

$\cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha$

$\cos(2\alpha)=\cos^2\alpha-sin^2\alpha$

Activity 4.4.1.9

$(\tan\alpha+\alpha)=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$

$\tan(2\alpha)=\frac{2\tan\alpha}{1-\tan^2\alpha}$

Activity 4.4.1.10

Use $\sin^2\theta+\cos^2\theta=1$ to find the value of $\cos\alpha$, which is equal to $\frac{\sqrt{5}}{3}$. Then, use the double angle formula to find that $\sin 2\alpha=\frac{4\sqrt{5}}{9}$.

Activity 4.4.1.10

From part (a), we know that $\cos\alpha=\frac{\sqrt{5}}{3}$ and $\sin\alpha=\frac{2}{3}$. So, by using the double angle formula $\cos^2\alpha-\sin^2\alpha$, we know that $\cos 2\alpha=\frac{5}{9}-\frac{4}{9}$, which equals $\frac{1}{9}$.

Activity 4.4.1.10

From part (a), we know that $\cos\alpha=\frac{\sqrt{5}}{3}$ and $\sin\alpha=\frac{2}{3}$. So, $\tan\alpha=\frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}}$, which is equal to $\frac{2}{\sqrt{5}}$. By using the double angle formula $\frac{2\tan\alpha}{1-\tan^2\alpha}$, we know that $\tan 2\alpha$ $=\frac{\frac{4}{\sqrt{5}}}{1-\frac{4}{5}}$, which equals $4\sqrt{5}$.

Activity 4.4.1.11

$\sin^2\theta=\frac{1-\cos 2\theta}{2}$

Activity 4.4.1.11

$\cos^2\theta=\frac{1+\cos 2\theta}{2}$

Activity 4.4.1.11

$\tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}$

Activity 4.4.1.12

$\sin^4\beta=\left(\sin^2\beta\right)^2$

Activity 4.4.1.12

$\sin^4\beta=\left(\frac{1-\cos 2\beta}{2}\right)^2$

Activity 4.4.1.12

$\sin^4\beta=\frac{1-2\cos 2\beta+\cos^2 2\beta}{4}$

Activity 4.4.1.12

$\frac{2-4\cos\beta+1+\cos 4\beta}{8}$ or $\frac{3-4\cos 2\beta+\cos 4\beta}{8}$

Activity 4.4.1.13

$\sin^2\left(\frac{\theta}{2}\right)=\frac{1-\cos 2\left(\frac{\theta}{2}\right)}{2}$

Activity 4.4.1.13

$\sin^2\left(\frac{\theta}{2}\right)=\frac{1-\cos\theta}{2}$

Activity 4.4.1.13

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Activity 4.4.1.14

Quadrant I

Activity 4.4.1.14

30°

Activity 4.4.1.14

Because the angle 15° lies in Quadrant I, we should use the formula: $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$.

Activity 4.4.1.14

$$\sin\left(\frac{30^\circ}{2}\right) = \sqrt{\frac{1-\cos 30^\circ}{2}}$$

Activity 4.4.1.14

$$\sin\left(\frac{30^\circ}{2}\right) = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

Activity 4.4.1.14

$$\sin\left(\frac{30^\circ}{2}\right) = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$$

$$\sin\left(\frac{30^\circ}{2}\right) = \sqrt{\frac{2-\sqrt{3}}{4}}$$

$$\sin\left(\frac{30^\circ}{2}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$$

Activity 4.4.1.15

$$\cos 75^\circ = \cos\left(\frac{150^\circ}{2}\right) = -\frac{\sqrt{2-\sqrt{3}}}{2}$$

Activity 4.4.1.15

$$\sin 165^\circ = \sin\left(\frac{330^\circ}{2}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$$

Check Your Understanding 4.4.1.1

$\cos(45^\circ + 30^\circ) = \cos(45^\circ)\cos(30^\circ) - \sin(45^\circ)\sin(30^\circ) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$.

Check Your Understanding 4.4.1.2

Since θ is in the first quadrant, $\cos(\theta) = \frac{4}{5}$ (using the Pythagorean identity). Then $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$.

Check Your Understanding 4.4.1.3

Using $1-\cos(2\theta) = 2\sin^2(\theta)$ (a power-reducing form of the double-angle identity) and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$: $\frac{2\sin^2(\theta)}{2\sin(\theta)\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$. ✓

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4.4.1.1: TE1 Online Exercises

TE1 — Use the sum and difference, the double-angle, and power-reducing formulas for cosine, sine, and tangent and the Pythagorean identities.	TE1 CheckIt Exercises

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4.4.2: Verifying Identities (TE2)

This section corresponds to APC slot 4.5 (Other trig functions & identities) in the Math 172 flow.

TE2 Learning outcome: Symbolically verify identities by using various trigonometric identities.

Verifying Trigonometric Identities

Overview

Occasionally a question may ask you to "prove the identity" or "establish the identity." In these situations, you must show the algebraic manipulations that demonstrate that the left and right side of the equation are equal. You can think of a "prove the identity" problem as a simplification problem where you know the answer: you know what the end goal of the simplification should be, and just need to show the steps to get there.

To prove an identity, start with the expression on one side of the identity and manipulate it using algebra and trigonometric identities until you have simplified it to the expression on the other side of the equation. Do not treat the identity like an equation! The proof is establishing the two expressions are equal, so work with one side at a time rather than applying an operation simultaneously to both sides of the equation.

Activity 4.4.2.1

Follow the steps to verify the identity

$$\tan\theta \cdot \cos\theta = \sin\theta \tag{4.4.2.1}$$

.

(a)

Rewrite the left-hand side so that each trigonometric function is written in terms of sine and cosine.

Answer available at the end of this section.

(b)

Now simplify the left-hand side.

Answer available at the end of this section.

(c)

Compare the simplified version of the left hand side to the original right hand side of the identity. What do you notice?

Answer available at the end of this section.

Note

As we saw in *(see related activity)*, one method that often helps in verifying identities is to rewrite everything in terms of sine and cosine to see if one side of the equation simplifies.

Activity 4.4.2.2

Verify the identity:

$$\cos\theta + \sin\theta\tan\theta = \sec\theta \tag{4.4.2.2}$$

.

(a)

Rewrite the left-hand side in terms of sine and cosine.

Answer available at the end of this section.

(b)

Now find a common denominator to add the two fractions. Add the fractions so that you have one fraction on the left-hand side.

Answer available at the end of this section.

(c)

Simplify the numerator of your fraction by using one of the Pythagorean Identities.

Answer available at the end of this section.

Activity 4.4.2.3

Verify the following identity:

$$\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \csc^2\theta \tag{4.4.2.3}$$

.

(a)

In some cases, the more complex side involves a fraction that can be split up. How can we rewrite the left side of the equation so that we end up with two fractions?

Answer available at the end of this section.

(b)

Rewrite $\tan\theta$ and $\cot\theta$ in terms of sine and cosine.

Answer available at the end of this section.

(c)

Simplify each complex fraction to verify that the left-hand side is equal to $\sec^2\theta - \csc^2\theta$.

Answer available at the end of this section.

Note

Starting with the more complex side can sometimes make the simplification easier.

Activity 4.4.2.4

Refer back to *(see related activity)* to help you verify the identity:

$$\frac{\csc\theta - \sin\theta}{\sin\theta\csc\theta} = \csc\theta - \sin\theta \tag{4.4.2.4}$$

. Justify each step by listing what you used, such as algebra, Pythagorean identity, reciprocal identities, etc.

Answer available at the end of this section.

Activity 4.4.2.5

Verify the following identity:

$$\frac{\cos^2\theta}{1+\sin\theta} = 1 - \sin\theta \tag{4.4.2.5}$$

(a)

Since the left side of the identity is more complicated, we should probably start there. We notice that the right side only involves sine. We will start by converting the cosine into something involving sine. Which identity could help us rewrite $\cos^2\theta$ into sine?

Answer available at the end of this section.

(b)

Rewrite the numerator of the left-hand side into a function of sine (use the Pythagorean Identity you found in part (a)).

Answer available at the end of this section.

(c)

Take a look at the numerator you now have. How can we factor the numerator?

Answer available at the end of this section.

(d)

Simplify by canceling out terms.

Answer available at the end of this section.

Note

As we saw in *(see related activity)*, knowing how to factor can be very useful when simplifying trigonometric identities.

Note

Using the property of conjugates is sometimes helpful in simplifying trigonometric identities. For an expression like $a+b$, the conjugate would be $a-b$. When you multiply conjugates, you often get a more useful expression. Sometimes multiplying by the conjugate will simplify an expression and help in verifying the given identity. Let's try this method in the next activity.

Activity 4.4.2.6

Verify the following identity:

$$\frac{\cos\theta}{1-\sin\theta} = \frac{1+\sin\theta}{\cos\theta} \tag{4.4.2.6}$$

.

(a)

Let's start with the left-hand side. Multiply the left-hand side by the conjugate of $1-\sin\theta$.

Hint:

When multiplying by the conjugate, do not distribute the $\cos\theta$ in the numerator.

Answer available at the end of this section.

(b)

Rewrite the denominator of your fraction in part (a) so that the denominator is written in terms of cosine.

Hint:

Use a Pythagorean Identity.

Answer available at the end of this section.

(c)

Simplify your expression from part (b) to get the right-hand side.

Answer available at the end of this section.

Note

As we've seen from the activities from this section, there are some basic tools that can be helpful when verifying trigonometric identities. Here are some suggestions as you continue to work through these types of problems.

Work on one side of the identity. It is usually better to start with the more complex side, as it is easier to simplify than to build.

Look for opportunities to

- Multiply expressions out and combine like terms.
- Factor expressions in a fraction and to cancel common factors.
- Split apart fractions.
- Rewrite multiple fractions using a common denominator, and combine the fractions into a single fraction.
- Simplify two term denominators by substituting with a Pythagorean identity.
- Simplify two term denominators by multiplying numerator and denominator by the conjugate of the binomial denominator.
- Multiply expressions out and combine like terms.
- Factor expressions in a fraction and to cancel common factors.
- Split apart fractions.
- Rewrite multiple fractions using a common denominator, and combine the fractions into a single fraction.
- Simplify two term denominators by substituting with a Pythagorean identity.

- Simplify two term denominators by multiplying numerator and denominator by the conjugate of the binomial denominator.
- Observe which functions are in the final expression, and look for opportunities to use identities that would lead to those functions.

If all else fails, try rewriting all terms to sines and cosines.

Activity 4.4.2.7

Use the tools you have learned in this section to verify each of the identities given below.

(a)

$$2\tan\theta\sec\theta = \frac{2\sin\theta}{1-\sin^2\theta}$$

Answer available at the end of this section.

(b)

$$\frac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$$

Answer available at the end of this section.

(c)

$$\tan\theta + \frac{\cos\theta}{1+\sin\theta} = \sec\theta$$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can symbolically verify trig identities before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 4.4.2.1

Verify the identity $\sec(x) - \sin(x)\tan(x) = \cos(x)$ by simplifying the left side.

Answer available at the end of this section.

Check Your Understanding 4.4.2.2

Verify the identity $\dfrac{\sin(x)}{1-\cos(x)} = \dfrac{1+\cos(x)}{\sin(x)}$ by multiplying the left side by a well-chosen form of 1.

Answer available at the end of this section.

Check Your Understanding 4.4.2.3

Explain the general strategy for verifying a trig identity: why do we typically only manipulate one side of the equation rather than performing the same operation to both sides, the way we solve an equation?

Answer available at the end of this section.

Answers

Activity 4.4.2.1

$\tan\theta\cos\theta$ can be rewritten as $\frac{\sin\theta}{\cos\theta}\cdot\cos\theta$

Activity 4.4.2.1

$\frac{\sin\theta}{\cos\theta}\cdot\cos\theta$ simplifies to $\sin\theta$

Activity 4.4.2.1

Students should notice that when we simplified the left hand side, we were able to get the right hand side. Instructors should emphasize that we were able to do this by focusing on one side (rather than trying to apply operations to both sides).

Activity 4.4.2.2

$$\cos\theta+\sin\theta\tan\theta \tag{4.4.2.7}$$

can be rewritten as

$$\cos\theta+\sin\theta\left(\frac{\sin\theta}{\cos\theta}\right) \tag{4.4.2.8}$$

.

Activity 4.4.2.2

The common denominator is $\cos\theta$. If we rewrite the left-hand side as two fractions with a common denominator, we would get

$$\frac{\cos^2\theta}{\cos\theta}+\frac{\sin^2\theta}{\cos\theta} \tag{4.4.2.9}$$

, which then combines to

$$\frac{\cos^2\theta+\sin^2\theta}{\cos\theta} \tag{4.4.2.10}$$

.

Activity 4.4.2.2

The numerator, $\cos^2\theta+\sin^2\theta$, is equal to 1, which then gives us $\frac{1}{\cos\theta}$. This term is the same as $\sec\theta$.

Activity 4.4.2.3

$$\frac{\tan\theta-\cot\theta}{\sin\theta\cos\theta} \tag{4.4.2.11}$$

can be rewritten as

$$\frac{\tan\theta}{\sin\theta\cos\theta}-\frac{\cot\theta}{\sin\theta\cos\theta} \tag{4.4.2.12}$$

Activity 4.4.2.3

$\frac{\frac{\sin\theta}{\cos\theta}}{\sin\theta\cos\theta}-\frac{\frac{\cos\theta}{\sin\theta}}{\sin\theta\cos\theta}$

Activity 4.4.2.3

$\frac{\frac{\sin\theta}{\cos\theta}}{\sin\theta\cos\theta}$ simplifies to $\frac{1}{\cos^2\theta}$, which is equivalent to $\sec^2\theta$. $-\frac{\frac{\cos\theta}{\sin\theta}}{\sin\theta\cos\theta}$ simplifies to $\frac{1}{\sin^2\theta}$, which is equivalent to $\csc^2\theta$.

Activity 4.4.2.4

We can simplify the left-hand side in the following way:

$$\frac{\csc\theta-\sin\theta}{\sin\theta\csc\theta} \tag{4.4.2.13}$$

$$\frac{\csc\theta}{\sin\theta\csc\theta}-\frac{\sin\theta}{\sin\theta\csc\theta} \tag{4.4.2.14}$$

$$\frac{1}{\sin\theta}-\frac{1}{\csc\theta} \tag{4.4.2.15}$$

$$\csc\theta-\sin\theta \tag{4.4.2.16}$$

Activity 4.4.2.5

$\sin^2\theta+\cos^2\theta=1$

Activity 4.4.2.5

$\frac{\cos^2\theta}{1+\sin\theta}$ can be rewritten as $\frac{1-\sin^2\theta}{1+\sin\theta}$

Activity 4.4.2.5

The numerator can be factored into $(1-\sin\theta)(1+\sin\theta)$. Students might see that this is a difference of squares.

Activity 4.4.2.5

From part (c), we can see that the left-hand side can be simplified to $\frac{(1-\sin\theta)(1+\sin\theta)}{1+\sin\theta}$ and that the $1+\sin\theta$ will cancel from the numerator and denominator. The result is $1-\sin\theta$, which is equal to the right-hand side.

Activity 4.4.2.6

If we multiply by $1+\sin\theta$, we will get $\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$.

Activity 4.4.2.6

Using the Pythagorean Identity, $\sin^2\theta+\cos^2\theta=1$,

$$\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta} \tag{4.4.2.17}$$

can simplify to

$$\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta} \tag{4.4.2.18}$$

.

Activity 4.4.2.6

By canceling out the $\cos\theta$ from the numerator and denominator,

$$\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta} \tag{4.4.2.19}$$

simplifies to

$$\frac{1+\sin\theta}{\cos\theta} \tag{4.4.2.20}$$

.

Activity 4.4.2.7

Starting on the right-hand side: $\frac{2\sin\theta}{1-\sin^2\theta}=\frac{2\sin\theta}{\cos^2\theta}=2\left(\frac{\sin\theta}{\cos\theta}\cdot\left(\frac{1}{\cos\theta}\right)\right)=2\tan\theta\sec\theta$

Activity 4.4.2.7

Starting on the left-hand side: $\frac{\sec^2\theta-1}{\sec^2\theta}=\frac{\tan^2\theta}{\sec^2\theta}=\frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}=\sin^2\theta$

Activity 4.4.2.7

Starting on the left-hand side: $\tan\theta+\frac{\cos\theta}{1+\sin\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{1+\sin\theta}=\frac{\sin\theta+\sin^2\theta+\cos^2\theta}{(\cos\theta)(1+\sin\theta)}=\frac{1+\sin\theta}{(\cos\theta)(1+\sin\theta)}=\sec\theta$

Check Your Understanding 4.4.2.1

Rewrite in terms of sine and cosine: $\frac{1}{\cos(x)}-\sin(x)\cdot\frac{\sin(x)}{\cos(x)}=\frac{1-\sin^2(x)}{\cos(x)}$. Since $1-\sin^2(x)=\cos^2(x)$ (Pythagorean identity), this becomes $\frac{\cos^2(x)}{\cos(x)}=\cos(x)$. ✓

Check Your Understanding 4.4.2.2

Multiply the left side by $\frac{1+\cos(x)}{1+\cos(x)}$: $\frac{\sin(x)(1+\cos(x))}{(1-\cos(x))(1+\cos(x))}=\frac{\sin(x)(1+\cos(x))}{1-\cos^2(x)}=\frac{\sin(x)(1+\cos(x))}{\sin^2(x)}=\frac{1+\cos(x)}{\sin(x)}$. ✓

Check Your Understanding 4.4.2.3

An identity is a statement claimed to be true for all valid input values — the goal is to show the two sides are already equal, not to solve for an unknown. Operating on both sides (like adding the same quantity to each) could accidentally make two unequal expressions look equal, or could obscure whether the original statement was true. Working one side at a time, using only valid algebraic and identity substitutions, keeps every step logically reversible and honestly demonstrates that the original equation was already an identity.

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4.4.2.1: TE2 Online Exercises

TE2 — Symbolically verify identities by using various trig identities.	TE2 CheckIt Exercises

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4.4.3: Trigonometric Equations (TE3)

This section corresponds to APC slot 4.5 (Other trig functions & identities) in the Math 172 flow.

TE3 Learning outcome: Find all solutions to a trigonometric equation.

Solving Trigonometric Equations

Overview

Now that we've learned about properties of and identities involving trigonometric functions, we will now turn to solving equations that include trigonometric functions.

Activity 4.4.3.1

Consider the equation $2\sin\theta - 1 = 0$.

(a)

What first step would you take to isolate $\sin\theta$?

A. Divide both sides by 2.
B. Add 1 to both sides.
C. Subtract $2\sin\theta$ from both sides.

Hint:

If given the equation $2x - 1 = 0$, what would your first step be?

Answer available at the end of this section.

(b)

Continue using algebra to isolate $\sin\theta$. What equation would you have left?

A. $\sin\theta = \frac{1}{2}$
B. $\sin\theta = -\frac{1}{2}$
C. $\sin\theta = 1$
D. $\sin\theta = -1$

Answer available at the end of this section.

(c)

For what values of θ, on the interval $[0, 2\pi)$, is the equation true?

A. $\frac{\pi}{6}$
B. $\frac{\pi}{3}$
C. $\frac{2\pi}{3}$
D. $\frac{5\pi}{6}$
E. $\frac{7\pi}{6}$
F. $\frac{4\pi}{3}$

Hint:

Use the unit circle to help you find all the values of θ.

Answer available at the end of this section.

Note

One way to solve a trigonometric equation is to isolate the trigonometric term on one side. Once it has been isolated, you can then determine values of θ that satisfies the equation.

Definition: General Solution

In many cases, you will be asked to find the solution to a trigonometric equation within a given interval (such as $[0, 2\pi)$). If, however, you are asked to find the **general solution** of a trigonometric equation (like in *(see related activity)*), you need to include all solutions. This requires the notation below, which includes all angles coterminal with the solutions in the interval $[0, 2\pi)$. If α is a solution to a trigonometric equation, then the general solution would be

$$\theta = \alpha + 2\pi n, n \in \mathbb{Z} \tag{4.4.3.1}$$

where $\mathbb{Z}$ represents the set of integers.

Activity 4.4.3.2

From *(see related activity)*, we were able to determine two values ($\frac{\pi}{6}$ and $\frac{5\pi}{6}$) that satisfy the equation $2\sin\theta - 1 = 0$ within the interval $[0, 2\pi)$. In this activity, we will explore other values of θ that also satisfy the equation.

(a)

If we know that $\frac{\pi}{6}$ is a solution to $2\sin\theta - 1 = 0$, determine whether $\frac{\pi}{6} + 2\pi$ is a solution.

Hint:

Recall that one complete revolution is 2π, so the positive x-axis can correspond to either an angle of 0 or 2π radians (or 4π, or 6π, or -2π, or -4π, etc. depending on the direction of rotation).

Answer available at the end of this section.

(b)

Determine if $\frac{\pi}{6} - 2\pi$ is a solution to the equation $2\sin\theta - 1 = 0$.

Answer available at the end of this section.

(c)

Determine if $\frac{\pi}{6} + \frac{\pi}{2}$ is a solution to the equation $2\sin\theta - 1 = 0$.

Answer available at the end of this section.

(d)

What other solutions can you find to the equation $2\sin\theta - 1 = 0$?

Answer available at the end of this section.

(e)

Refer back to parts (a) and (b) and use *(see related activity)* to find the general solution to $2\sin\theta - 1 = 0$.

Hint:

As we see in parts (a) and (b), there are more angles that solve the equation. In fact, any angle coterminal with the either of the angles we found will also be solutions. That means, we can add or subtract any multiple of 2π to the solutions to find another solution.

Answer available at the end of this section.

Activity 4.4.3.3

Consider the equation $(2\sin\theta + \sqrt{3})(\cos\theta - 1) = 0$.

(a)

Refer back to the Zero Product Property *(see related activity)*. What equations do we need to solve?

Answer available at the end of this section.

(b)

Solve $2\sin\theta + \sqrt{3} = 0$ on the interval $[0, 2\pi)$. What values of θ satisfies that equation?

A. $\frac{\pi}{6}$
B. $\frac{5\pi}{3}$
C. $\frac{5\pi}{6}$
D. $\frac{7\pi}{6}$
E. $\frac{4\pi}{3}$
F. $\frac{11\pi}{6}$

Answer available at the end of this section.

(c)

Solve $\cos\theta - 1 = 0$ on the interval $[0, 2\pi)$. What values of θ satisfies that equation?

A. 0
B. $\frac{\pi}{2}$
C. π
D. $\frac{3\pi}{2}$
E. 2π

Answer available at the end of this section.

(d)

Refer back to parts (b) and (c). What are all the solutions that satisfy the equation $(2\sin\theta + \sqrt{3})(\cos\theta - 1) = 0$ on the interval $[0, 2\pi)$?

Answer available at the end of this section.

(e)

Find the general solution of $(2\sin\theta + \sqrt{3})(\cos\theta - 1) = 0$.

Answer available at the end of this section.

Activity 4.4.3.4

Trigonometric equations can also include other trig functions. Consider the equation $4 = 5 + \tan\theta$.

(a)

Use algebra to isolate $\tan\theta$.

Answer available at the end of this section.

(b)

For what values of θ, on the interval $[0, 2\pi)$, is the equation true?

A. $\frac{\pi}{4}$
B. $\frac{3\pi}{4}$
C. π
D. $\frac{5\pi}{4}$
E. $\frac{3\pi}{2}$
F. $\frac{7\pi}{4}$

Hint:

Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Answer available at the end of this section.

(c)

What is the general solution of $4 = 5 + \tan\theta$?

Answer available at the end of this section.

Activity 4.4.3.5

Consider the equation $(\sin\theta - 1)(\csc\theta + 2) = 0$.

(a)

Refer back to the Zero Product Property. What two equations do we solve?

Answer available at the end of this section.

(b)

Solve $\sin\theta - 1 = 0$ on the interval $[0, 2\pi)$. What value(s) of θ satisfy that equation?

A. 0
B. $\frac{\pi}{2}$
C. π
D. $\frac{3\pi}{2}$

Answer available at the end of this section.

(c)

To solve $\csc\theta + 2 = 0$, isolate the $\csc\theta$ term. What does $\csc\theta$ equal?

Answer available at the end of this section.

(d)

Recall that $\csc\theta = \frac{1}{\sin\theta}$. If $\csc\theta = -2$, what does $\sin\theta$ equal?

A. $\sin\theta = 2$
B. $\sin\theta = -2$
C. $\sin\theta = \frac{1}{2}$
D. $\sin\theta = -\frac{1}{2}$

Answer available at the end of this section.

(e)

Using what you know from part (d), what value(s) of θ satisfy that equation $\csc\theta = -2$ on the interval $[0, 2\pi)$?

A. $\frac{\pi}{6}$
B. $\frac{2\pi}{3}$
C. $\frac{5\pi}{6}$
D. $\frac{7\pi}{6}$
E. $\frac{4\pi}{3}$
F. $\frac{11\pi}{6}$

Answer available at the end of this section.

(f)

Refer back to parts (b) and (e). What are all the solutions that satisfy the equation $(\sin\theta - 1)(\csc\theta + 2) = 0$ on the interval $[0, 2\pi)$?

Answer available at the end of this section.

(g)

Find the general solution of $(\sin\theta - 1)(\csc\theta + 2) = 0$.

Answer available at the end of this section.

Activity 4.4.3.6

Solve each equation on the interval $[0, 2\pi)$.

(a)

$3 + \sin\theta = 2$

Answer available at the end of this section.

(b)

$\sin\theta(\tan\theta - 1) = 0$

Hint:

Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Answer available at the end of this section.

(c)

$3\tan\theta = 3\sqrt{3}$

Answer available at the end of this section.

(d)

$\cos\theta + \sqrt{3} = -\cos\theta$

Answer available at the end of this section.

(e)

$(\sec\theta - 1)(2\cos\theta + 1) = 0$

Hint:

Recall that $\sec\theta = \frac{1}{\cos\theta}$.

Answer available at the end of this section.

Note

In some cases, it is not easy to isolate the trigonometric term like we have been doing so far. The next few activities will help us learn how to apply methods we have used before when solving algebraic equations to solve trigonometric equations.

Activity 4.4.3.7

Consider the equation $4\sin^2\theta - 2 = 0$.

(a)

Use algebra to isolate the $\sin^2\theta$ term.

Answer available at the end of this section.

(b)

To undo a squared term, you will need to take the square root (recall *(see related activity)*). Take the square root of $\sin^2\theta$ to get $\sin\theta$. What does $\sin\theta$ equal?

A. $\pm\sqrt{\frac{1}{2}}$
B. $\pm\frac{1}{\sqrt{2}}$
C. $\pm\frac{1}{2}$
D. $\pm\frac{\sqrt{2}}{2}$

Answer available at the end of this section.

(c)

Although there are multiple ways to represent $\sin\theta$, which solution(s) from part (b) do you see on the unit circle?

A. $\pm\sqrt{\frac{1}{2}}$
B. $\pm\frac{1}{\sqrt{2}}$
C. $\pm\frac{1}{2}$
D. $\pm\frac{\sqrt{2}}{2}$

Answer available at the end of this section.

(d)

Notice there are two values of $\sin\theta$ to consider, which corresponds to four values of θ that satisfy the equation $\sin\theta = \pm\frac{\sqrt{2}}{2}$ on the interval $[0, 2\pi)$. What are these values of θ?

A. $\frac{\pi}{4}$
B. $\frac{\pi}{3}$
C. $\frac{3\pi}{4}$
D. $\frac{5\pi}{6}$
E. $\frac{5\pi}{4}$
F. $\frac{7\pi}{4}$

Answer available at the end of this section.

Activity 4.4.3.8

Consider the equation $3\sec^2\theta - 4 = 0$.

(a)

Using algebra, isolate the $\sec^2\theta$ term.

Answer available at the end of this section.

(b)

Use the square root property to solve for $\sec\theta$. What does $\sec\theta$ equal?

A. $\pm\frac{4}{3}$
B. $\pm\frac{2}{3}$
C. $\pm\frac{2}{\sqrt{3}}$
D. $\pm\frac{\sqrt{3}}{2}$

Answer available at the end of this section.

(c)

Recall that $\sec\theta = \frac{1}{\cos\theta}$. Knowing this relationship, what does $\cos\theta$ equal?

A. $\pm\frac{3}{2}$
B. $\pm\frac{2}{3}$
C. $\pm\frac{2}{\sqrt{3}}$
D. $\pm\frac{\sqrt{3}}{2}$

Answer available at the end of this section.

(d)

What values of θ satisfy this equation on the interval $[0, 2\pi)$?

A. $\frac{\pi}{6}$
B. $\frac{\pi}{3}$
C. $\frac{5\pi}{6}$
D. $\frac{7\pi}{6}$
E. $\frac{4\pi}{3}$
F. $\frac{11\pi}{6}$

Answer available at the end of this section.

Note

Notice in *(see related activity)* and *(see related activity)*, we had to use the square root property to isolate the trigonometric term. The next few activities will highlight another tool to solve trigonometric equations.

Activity 4.4.3.9

In this activity, we will solve trigonometric equations by first factoring. Consider the equation

$$2\sin\theta\cos\theta - \sqrt{2}\cos\theta = 0 \tag{4.4.3.2}$$

.

(a)

One of the first methods of factoring you should always start with is GCF (Greatest Common Factor). Look at each term in the equation. What is the GCF?

Answer available at the end of this section.

(b)

Factor out the GCF. What equation do you now have?

A. $2\sin\theta - \sqrt{2} = 0$
B. $\cos\theta(2\sin\theta - \sqrt{2}\cos\theta) = 0$
C. $\sin\theta(2\cos\theta - \sqrt{2}\cos\theta) = 0$
D. $\cos\theta(2\sin\theta - \sqrt{2}) = 0$

Answer available at the end of this section.

(c)

Now use the Zero Product Property to create two equations. What two equations do you now have?

A. $2\sin\theta - \sqrt{2} = 0$
B. $\sin\theta = 0$
C. $\cos\theta = 0$
D. $2\sin\theta - \sqrt{2}\cos\theta = 0$

Answer available at the end of this section.

(d)

Solve each of the equations you got in part (c). What value(s) of θ satisfy both equations on the interval $[0, 2\pi)$?

A. 0

B. $\frac{\pi}{2}$
C. π
D. $\frac{3\pi}{2}$
E. $\frac{\pi}{4}$
F. $\frac{3\pi}{4}$

Answer available at the end of this section.

Activity 4.4.3.10

Consider the equation $2\cos^2\theta - 5\cos\theta - 3 = 0$.

(a)

Notice that this trigonometric equation looks similar to a quadratic equation. Let's we replace every $\cos\theta$ with x to get $2x^2 - 5x - 3 = 0$. What are the factors of the left hand side of this equation?

A. $(2x + 1)$
B. $(x + 3)$
C. $(2x - 1)$
D. $(x - 3)$

Answer available at the end of this section.

(b)

Now use the Zero Product Property to create two equations in terms of $\cos\theta$. What two equations do you now have?

A. $(2\cos\theta + 1)$
B. $(\cos\theta + 3)$
C. $(2\cos\theta - 1)$
D. $(\cos\theta - 3)$

Answer available at the end of this section.

(c)

Solve each of the equations you got in part (b). What values of θ satisfies both equations on the interval $[0, 2\pi)$?

A. 0
B. $\frac{\pi}{3}$
C. $\frac{2\pi}{3}$
D. $\frac{4\pi}{3}$
E. $\frac{5\pi}{3}$
F. 2π

Answer available at the end of this section.

Note

Notice in *(see related activity)*, one of the equations did not provide a value of θ that satisfies the equation. This can happen sometimes!

Activity 4.4.3.11

Solve each of the following trigonometric equations on the interval $[0, 2\pi)$.

(a)

$2\cos^2\theta + \cos\theta = 0$

Answer available at the end of this section.

(b)

$\sin^2\theta - 1 = 0$

Answer available at the end of this section.

(c)

$2\cos^2\theta - \cos\theta = 3$

Hint:

Get everything over to one side before you factor!

Answer available at the end of this section.

(d)

$2\sin^2\theta - 4 = 7\sin\theta$

Answer available at the end of this section.

(e)

$2\sec^2\theta - 2\sec\theta - 4 = 0$

Answer available at the end of this section.

Note

In this last activity, let's look at trigonometric equations that involve multiple angles.

Activity 4.4.3.12

Solve the equation $\sin(2\theta) = \frac{1}{2}$. Find the general solution, then find all solutions on the interval $[0, 2\pi)$.

(a)

Notice that this equation involves $\sin(2\theta)$, a multiple angle, rather that $\sin(2\theta)$ like we have been dealing with. Let's replace 2θ with α so it looks more like the equations we've been solving so far. What values of α would satisfy the new equation $\sin\alpha = \frac{1}{2}$ on the interval $[0, 2\pi)$?

A. $\frac{\pi}{6}$
B. $\frac{\pi}{3}$
C. $\frac{5\pi}{6}$
D. $\frac{4\pi}{3}$

E. $\frac{5\pi}{3}$
F. $\frac{11\pi}{6}$

Answer available at the end of this section.

(b)

What would the general solution of the equation $\sin\alpha = \frac{1}{2}$ be?

Answer available at the end of this section.

(c)

For $\sin\alpha = \frac{1}{2}$, we know the general solution is

$$\alpha = \frac{\pi}{6} + 2\pi(n) \tag{4.4.3.3}$$

and

$$\alpha = \frac{5\pi}{6} + 2\pi(n) \tag{4.4.3.4}$$

. And because $\alpha = 2\theta$, we can replace α with 2θ in those solutions. Therefore,

$$2\theta = \frac{\pi}{6} + 2\pi(n) \tag{4.4.3.5}$$

and

$$2\theta = \frac{5\pi}{6} + 2\pi(n) \tag{4.4.3.6}$$

. Solve each general equation to find values of θ that satisfies the equation $\sin(2\theta) = \frac{1}{2}$.

Hint:

Solve for θ.

Answer available at the end of this section.

(d)

We now need to find all values of θ that satisfy our original equation on the interval $[0, 2\pi)$. We can find these angles by substituting values of n in the general solution. We will start with $n = 0$. What values of θ result when $n = 0$? Are these angles on the interval $[0, 2\pi)$?

Answer available at the end of this section.

(e)

Now let $n = 1$. What values of θ would satisfy the equation $\sin(2\theta) = \frac{1}{2}$ on the interval $[0, 2\pi)$?

Answer available at the end of this section.

(f)

Suppose $n = 2$. What values of θ would satisfy the equation $\sin(2\theta) = \frac{1}{2}$ on the interval $[0, 2\pi)$?

Answer available at the end of this section.

(g)

Let's now try negative values of n. Suppose $n = -1$. What values of θ would satisfy the equation $\sin(2\theta) = \frac{1}{2}$ on the interval $[0, 2\pi)$?

Answer available at the end of this section.

(h)

Look back at parts (d)-(g). Notice that that once we reach values outside of $[0, 2\pi)$ in either direction, we can stop substituting values of n in that direction. What are all the solutions to $\sin(2\theta) = \frac{1}{2}$ on the interval $[0, 2\pi)$?

Answer available at the end of this section.

Observation

Recall that $y = \sin(2\theta)$ is a horizontal compression by a factor of 2 of the function $y = \sin\theta$. On an interval of 2π, we can graph two periods of $y = \sin(2\theta)$, as opposed to one cycle of $y = \sin\theta$. This compression of the graph suggests there may be twice as many x-intercepts or solutions to $\sin(2\theta) = \frac{1}{2}$ compared to $\sin\theta = \frac{1}{2}$.

Check Your Understanding

Work these on your own to confirm you can find all solutions to a trigonometric equation before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 4.4.3.1

Find all solutions to $\sin(x) = \dfrac{1}{2}$ on the interval $[0, 2\pi)$.

Answer available at the end of this section.

Check Your Understanding 4.4.3.2

Find the general solution to $\cos(x) = \dfrac{\sqrt{2}}{2}$ (all solutions, not restricted to one period).

Answer available at the end of this section.

Check Your Understanding 4.4.3.3

Find all solutions to $2\sin^2(x) - \sin(x) - 1 = 0$ on the interval $[0, 2\pi)$. (Hint: treat this as a quadratic in $\sin(x)$.)

Answer available at the end of this section.

Answers

Activity 4.4.3.1

B

Activity 4.4.3.1

A

Activity 4.4.3.1

A and D

Activity 4.4.3.2

$\frac{\pi}{6} + 2\pi$ is a solution to the equation $2\sin\theta - 1 = 0$.

Students can think of this in a couple different ways. They can use the idea that 2π is one revolution, which means that $\frac{\pi}{6} + 2\pi$ is just one revolution from $\frac{\pi}{6}$. Hence, the sine value will yield the same y-value.

Students may also find the value of $\frac{\pi}{6}+2\pi$, which is $\frac{13\pi}{6}$. Then, they can plug this value in for θ to determine if $\frac{\pi}{6}+2\pi$ is a solution.

Activity 4.4.3.2

$\frac{\pi}{6}-2\pi$ is a solution to the equation $2\sin\theta-1=0$. Students may use similar reasoning as part (a), but in this case, subtracting 2π means that one revolution is in the counter clockwise direction.

Activity 4.4.3.2

$\frac{\pi}{6}+\frac{\pi}{2}$ is NOT a solution to the equation $2\sin\theta-1=0$.

Activity 4.4.3.2

Students should see that any coterminal angle will yield the same value and thus, there will be an infinite number of solutions to this equation.

Activity 4.4.3.2

The notation we use for this is $\theta=\dfrac{\pi}{6}+2\pi n$ and $\theta=\dfrac{5\pi}{6}+2\pi n$ where n is an integer.

Activity 4.4.3.3

$2\sin\theta+\sqrt{3}=0$ and $\cos\theta-1=0$

Activity 4.4.3.3

B and E

Activity 4.4.3.3

A

Activity 4.4.3.3

$\theta=0,\frac{4\pi}{3},\frac{5\pi}{3}$

Activity 4.4.3.3

$\theta=0+2\pi(n)$

$\theta=\frac{4\pi}{3}+2\pi(n)n\in\mathbb{Z}$

$\theta=\frac{5\pi}{3}+2\pi(n)n\in\mathbb{Z}$

Activity 4.4.3.4

$\tan\theta=-1$

Activity 4.4.3.4

B and F

Activity 4.4.3.4

$\theta=\frac{3\pi}{4}+2\pi(n)$

$\theta=\frac{7\pi}{4}+2\pi(n)$

Activity 4.4.3.5

$\sin\theta-1=0$ and $\csc\theta+2=0$

Activity 4.4.3.5

B

Activity 4.4.3.5

$\csc\theta=-2$

Activity 4.4.3.5

D

Activity 4.4.3.5

D and F

Activity 4.4.3.5

$\theta=\frac{\pi}{2},\frac{7\pi}{6},\frac{11\pi}{6}$

Activity 4.4.3.5

$\theta=\frac{\pi}{2}+2\pi(n)$

$\theta=\frac{7\pi}{6}+2\pi(n)$

$\theta=\frac{11\pi}{6}+2\pi(n)$

Activity 4.4.3.6

$\theta=\frac{3\pi}{2}$

Activity 4.4.3.6

$\theta=0,\frac{\pi}{4},\frac{5\pi}{4}$

Activity 4.4.3.6

$\theta=\frac{\pi}{3},\frac{4\pi}{3}$

Activity 4.4.3.6

$\theta=\frac{5\pi}{6},\frac{7\pi}{6}$

Activity 4.4.3.6

$\theta=0,\frac{2\pi}{3},\frac{4\pi}{3}$

Activity 4.4.3.7

$\sin^2\theta=\frac{1}{2}$

Activity 4.4.3.7
A, B, and D

Activity 4.4.3.7
D

Activity 4.4.3.7
A, C, E, and F

Activity 4.4.3.8
$\sec^2\theta = \frac{4}{3}$

Activity 4.4.3.8
C

Activity 4.4.3.8
D

Activity 4.4.3.8
A, C, D, and F

Activity 4.4.3.9
$\cos\theta$

Activity 4.4.3.9
D

Activity 4.4.3.9
A and C

Activity 4.4.3.9
B, D, E, and F

Activity 4.4.3.10
A and D

Activity 4.4.3.10
A and D

Activity 4.4.3.10
C and D

Activity 4.4.3.11
$\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

Activity 4.4.3.11
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Activity 4.4.3.11
$\theta = \pi$

Activity 4.4.3.11
$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Activity 4.4.3.11
$\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Activity 4.4.3.12
A and C

Activity 4.4.3.12
$\theta = \frac{\pi}{6} + 2\pi(n)$
$\theta = \frac{5\pi}{6} + 2\pi(n)$

Activity 4.4.3.12
$\theta = \frac{\pi}{12} + \pi(n)$
$\theta = \frac{5\pi}{12} + \pi(n)$

Activity 4.4.3.12
$\theta = \frac{\pi}{12} + \pi(0) = \frac{\pi}{12}$
$\theta = \frac{5\pi}{12} + \pi(0) = \frac{5\pi}{12}$

Activity 4.4.3.12
$\theta = \frac{\pi}{12} + \pi(1) = \frac{13\pi}{12}$
$\theta = \frac{5\pi}{12} + \pi(1) = \frac{17\pi}{12}$

Activity 4.4.3.12
$\theta = \frac{\pi}{12} + \pi(2) = \frac{25\pi}{12}$, but this value is too big!
$\theta = \frac{5\pi}{12} + \pi(2) = \frac{29\pi}{12}$, but this value is too big!

Activity 4.4.3.12
$\theta = \frac{\pi}{12} + \pi(-1) = \frac{-11\pi}{12}$, but this value is too small!

$\theta = \frac{5\pi}{12} + \pi(-1) = \frac{-7\pi}{12}$, but this value is too small!

Activity 4.4.3.12

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

Check Your Understanding 4.4.3.1

$x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Both angles have reference angle $\frac{\pi}{6}$, and since sine is positive, both lie in the first and second quadrants (where sine is positive).

Check Your Understanding 4.4.3.2

$x = \frac{\pi}{4} + 2n\pi$ or $x = -\frac{\pi}{4} + 2n\pi$, for any integer n. Since cosine has period 2π, every solution repeats by adding whole-number multiples of 2π to each base solution.

Check Your Understanding 4.4.3.3

Let $u = \sin(x)$: $2u^2 - u - 1 = 0$ factors as $(2u + 1)(u - 1) = 0$, so $u = -\frac{1}{2}$ or $u = 1$. For $\sin(x) = -\frac{1}{2}$: $x = \frac{7\pi}{6}, \frac{11\pi}{6}$. For $\sin(x) = 1$: $x = \frac{\pi}{2}$. All solutions on $[0, 2\pi)$: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

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4.4.3.1: TE3 Online Exercises

TE3 — Find all solutions to a trigonometric equation.	TE3 CheckIt Exercises

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5: POLYNOMIAL AND RATIONAL FUNCTIONS

Advanced mathematical ideas are connected by structure, patterns, and representations that support calculus reasoning. In this unit, students examine polynomial and rational functions — their graphs, zeros, division, and end behavior — as a way to consolidate precalculus knowledge and build fluency with mathematical language. Analyzing the end behavior of a rational function is, in fact, already reasoning about a limit; this unit is where the course most directly previews the language and thinking of Calculus 1.

5.1: Introduction to Graphs of Polynomial Functions (PR3)

5.1.1: Graphs of Polynomial Functions (PR3)

5.1.1.1: PR3 Online Exercises

5.2: Analyzing Polynomial Functions and Their Zeros (PR4 and PR5)

5.2.1: Polynomial Long Division (PR4)

5.2.1.1: PR4 Online Exercises

5.2.2: Zeros of a Polynomial Functions (PR5)

5.2.2.1: PR5 Online Exercises

5.3: Analyzing Rational Functions (PR6 and PR7)

5.3.1: Rational Equations (PR6)

5.3.1.1: PR6 Online Exercises

5.3.2: Graphs of Rational Functions (PR7)

5.3.2.1: PR7 Online Exercises

5.4: Introduction to Quadratic Inequalities (PR8)

5.4.1: Quadratic Inequalities (PR8)

5.4.1.1: PR8 Online Exercises

5.5: Introduction to Rational Inequalities (PR9)

5.5.1: Rational Inequalities (PR9)

5.5.1.1: PR9 Online Exercises

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5.1: INTRODUCTION TO GRAPHS OF POLYNOMIAL FUNCTIONS (PR3)

5.1.1: Graphs of Polynomial Functions (PR3)

5.1.1.1: PR3 Online Exercises

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5.1.1: Graphs of Polynomial Functions (PR3)

Learning Outcomes:

- Determine the zeros and their multiplicities of a polynomial in factored form. Describe and graph the behavior of a polynomial function at the intercepts and the ends.

Graphs of Polynomial Functions (PR3)

Note

Just like with quadratic functions, we should be able to determine key characteristics that will help guide us in creating a sketch of any polynomial function. We can start by finding both x and y-intercepts and then explore other characteristics polynomial functions can have. Recall that the **zeros** of a function are the **x-intercepts** - i.e., the values of x that cross or touch the x-axis. Just like with quadratic functions, we can find the zeros of a function by setting the function equal to 0 and solving for x.

Activity 5.1.1.1

Given the function, $f(x) = (x-2)(x+1)(x-3)^2$, determine the following characteristics.

(a)

How many zeros does $f(x)$ have?

A. 1
B. 2
C. 3
D. 4

Answer available at the end of this section.

(b)

What are the zeros of $f(x)$?

A. $1, 2, 3$
B. $-1, 2, 3$
C. $1, -2, -3$
D. $-1, 2, -3$

Answer available at the end of this section.

(c)

What is the y-intercept of $f(x)$?

A. -1
B. -6
C. 6
D. -18

Answer available at the end of this section.

(d) Now that we have found both the x and y-intercepts of $f(x)$, do we have enough information to draw a possible sketch of $f(x)$? What other characteristics would be useful to help us draw an accurate sketch of $f(x)$?

Answer available at the end of this section.

Definition: End behavior

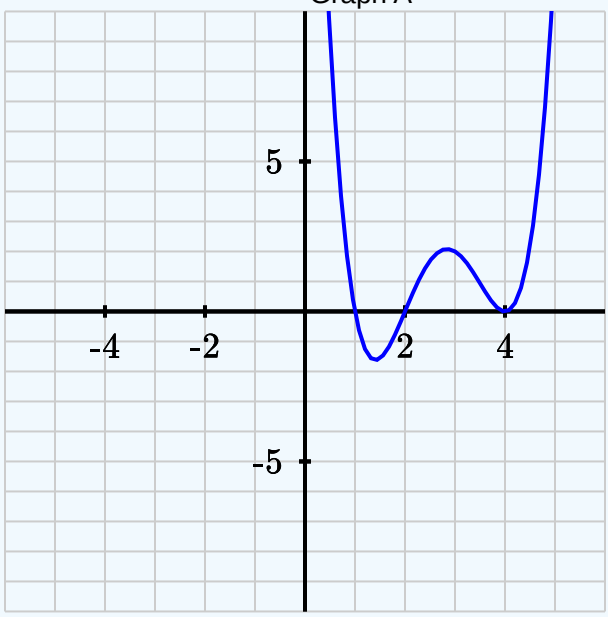
The **end behavior** of a polynomial function describes the behavior of the graph at the "ends" of the function. In other words, as we move to the right of the graph (as the x values increase), what happens to the y values? Similarly, as we move to the left of the graph (as the x values decrease), what happens to the y values?

Although we are looking at the "ends" to determine the end behavior, note that polynomials do not actually have ends. In other words, polynomial functions have y-values that exist for every x-value.

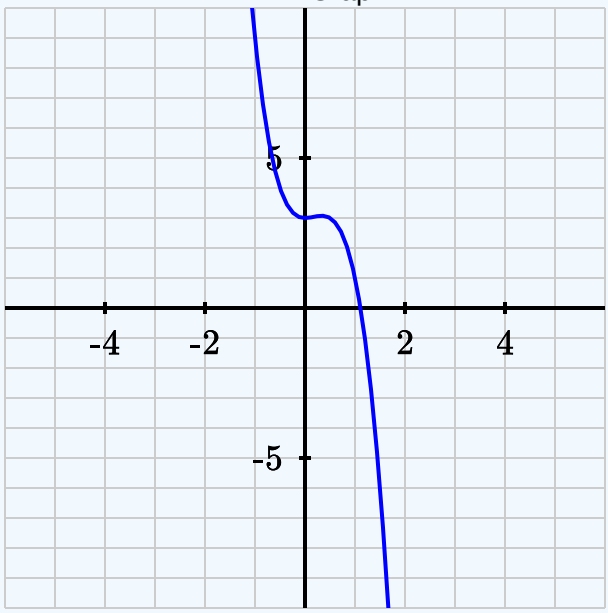
Activity 5.1.1.2

Use the graphs of the following polynomial functions to answer the questions below.

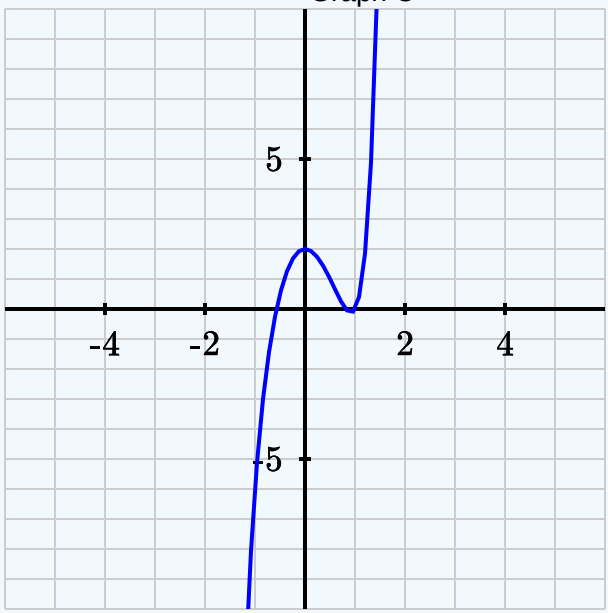
Graph A



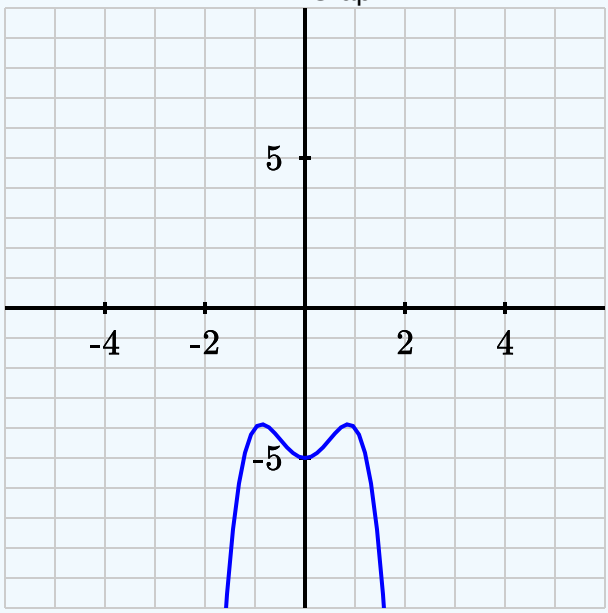
Graph B



Graph C



Graph D



(a)

How would you describe the behavior of Graph A as you approach the ends?

A. Graph A rises on the left and on the right.
B. Graph A rises on the left, but falls on the right.
C. Graph A rises on the right, but falls on the left.
D. Graph A falls on the left and on the right.

Answer available at the end of this section.

(b)

How would you describe the behavior of Graph B as you approach the ends?

A. Graph B rises on the left and on the right.
B. Graph B rises on the left, but falls on the right.
C. Graph B rises on the right, but falls on the left.
D. Graph B falls on the left and on the right.

Answer available at the end of this section.

(c)

How would you describe the behavior of Graph C as you approach the ends?

A. Graph C rises on the left and on the right.
B. Graph C rises on the left, but falls on the right.
C. Graph C rises on the right, but falls on the left.
D. Graph C falls on the left and on the right.

Answer available at the end of this section.

(d)

How would you describe the behavior of Graph D as you approach the ends?

A. Graph D rises on the left and on the right.
B. Graph D rises on the left, but falls on the right.
C. Graph D rises on the right, but falls on the left.
D. Graph D falls on the left and on the right.

Answer available at the end of this section.

Definition: Degree and Leading coefficient

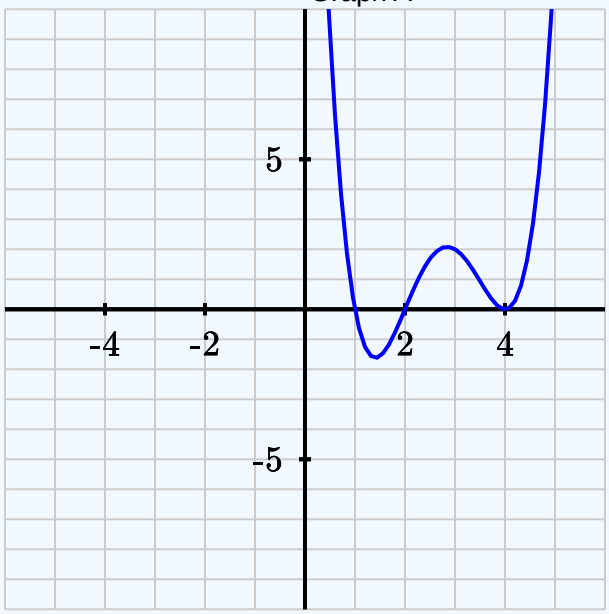
Typically, when given an equation of a polynomial function, we look at the **degree** and **leading coefficient** to help us determine the behavior of the ends. The **degree** is the highest exponential power in the polynomial. The **leading coefficient** is the number written in front of the variable with the highest exponential power.

Activity 5.1.1.3

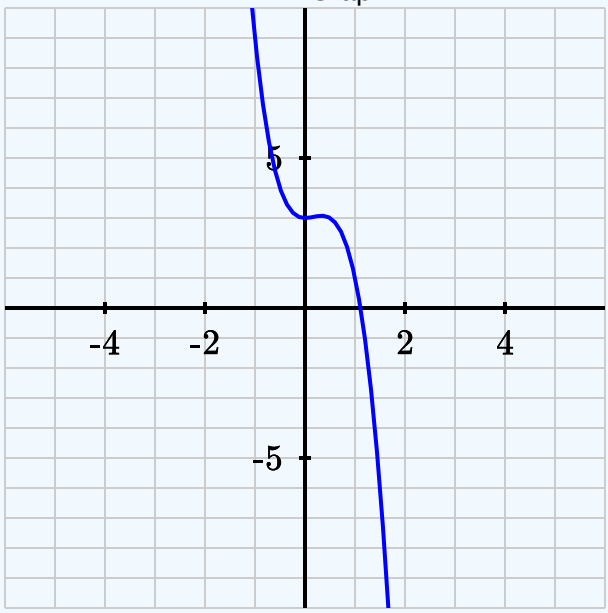
Let's refer back to the graphs in Activity 5.1.1.2 and look at the equations of those polynomial functions. Let's apply the Definition of Degree and Leading coefficient to see if we can determine how the degree and leading coefficients of those graphs affect their end behavior.

- Graph A: $f(x) = -11x^3 + 32 + 42x^2 + x^4 - 64x$
- Graph B: $g(x) = 2x^2 + 3 - 4x^3$
- Graph C: $h(x) = x^7 + 2x^3 - 5x^2 + 2$
- Graph D: $j(x) = 3x^2 - 2x^4 - 5$

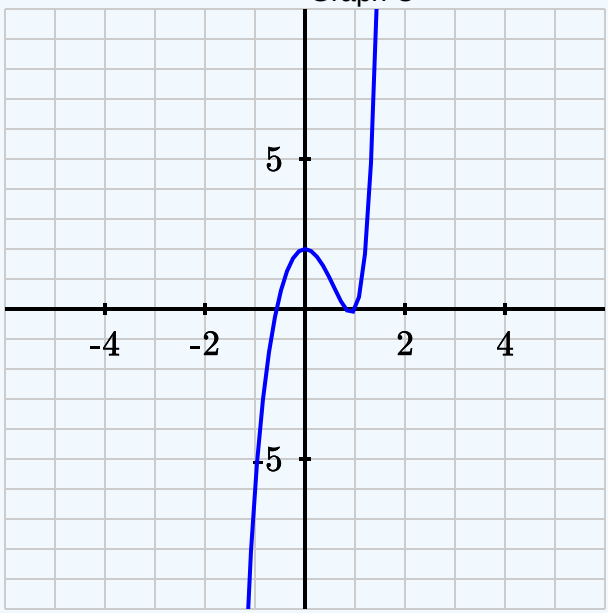
Graph A



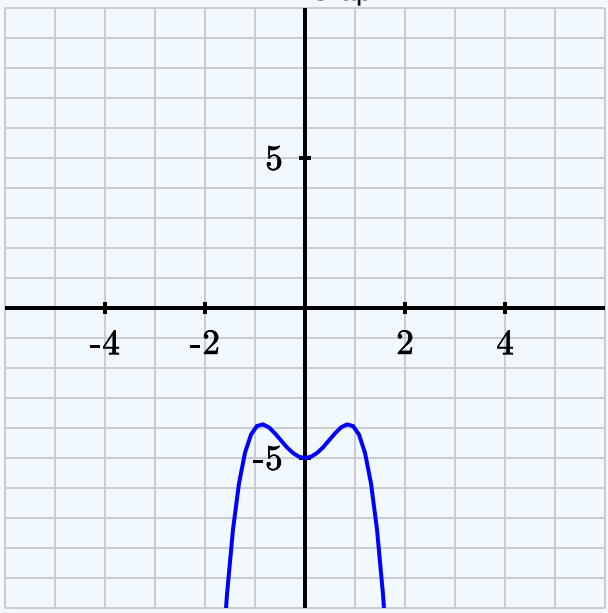
Graph B



Graph C



Graph D



(a)

What is the degree and leading coefficient of Graph A?

A. Degree: -64; Leading Coefficient: 4
B. Degree: 4; Leading Coefficient: 0
C. Degree: 1; Leading Coefficient: -64
D. Degree: 4; Leading Coefficient: 1

Answer available at the end of this section.

(b)

What is the degree and leading coefficient of Graph B?

A. Degree: 3; Leading Coefficient: -4
B. Degree: -4; Leading Coefficient: 3
C. Degree: 2; Leading Coefficient: 3
D. Degree: 3; Leading Coefficient: 4

Answer available at the end of this section.

(c)

What is the degree and leading coefficient of Graph C?

A. Degree: -5; Leading Coefficient: 2
B. Degree: 0; Leading Coefficient: 7
C. Degree: -5; Leading Coefficient: 3
D. Degree: 7; Leading Coefficient: 1

Answer available at the end of this section.

(d)

What is the degree and leading coefficient of Graph D?

A. Degree: -2; Leading Coefficient: 4
B. Degree: 3; Leading Coefficient: 2
C. Degree: 4; Leading Coefficient: -2
D. Degree: -5; Leading Coefficient: 4

Answer available at the end of this section.

(e)

Notice that Graph A and Graph D have their ends going in the same direction. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

Answer available at the end of this section.

(f)

Notice that Graph B and Graph C have their ends going in opposite directions. What conjectures can you make about the relationship between their degrees and leading coefficients with the behavior of their graphs?

Answer available at the end of this section.

Note

From Activity $5.1.1.3$, we saw that the degree and leading coefficient of a polynomial function can give us more clues about the behavior of the function. In summary, we know:

- If the degree is even, the ends of the polynomial function will be going in the same direction. If the leading coefficient is positive, both ends will be pointing up. If the leading coefficient is negative, both ends will be pointing down.
- If the degree is odd, the ends of the polynomial function will be going in opposite directions. If the leading coefficient is positive, the left end will fall and the right end will rise. If the leading coefficient is negative, the left end will rise and the right end will fall.

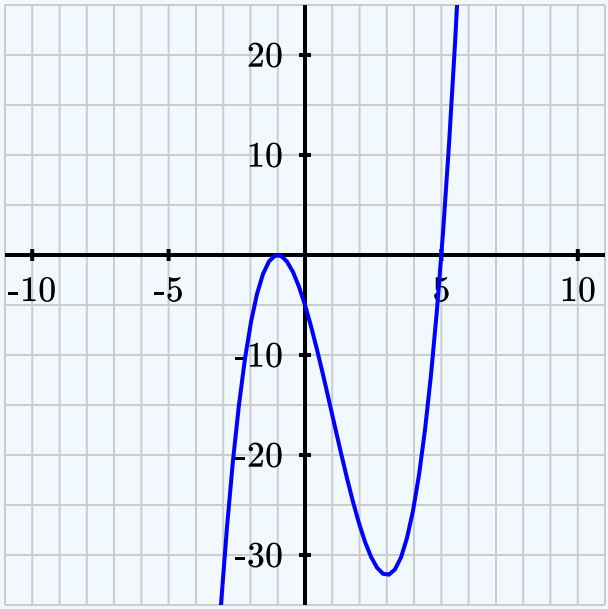
Definition: Arrow notation

When describing end behavior, mathematicians typically use **arrow notation**. Just as the name suggests, arrows are used to indicate the behavior of certain values on a graph.

For end behavior, students are often asked to determine the behavior of y-values as x-values either increase or decrease. The statement "As $x \to \infty$, $f(x) \to -\infty$" can be translated to "As x approaches infinity (or as x increases), $f(x)$ (or the y-values) go to negative infinity (i.e., it decreases)."

Activity $5.1.1.4$

Use the graph of $f(x)$ to answer the questions below.



(a)

How would you describe the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.
B. $f(x)$ rises on the left, but falls on the right.
C. $f(x)$ rises on the right, but falls on the left.
D. $f(x)$ falls on the left and on the right.

Answer available at the end of this section.

(b)

How would you describe the end behavior of $f(x)$ using arrow notation?

A. As $x \to -\infty$, $f(x) \to -\infty$

As $x \to \infty$, $f(x) \to -\infty$

B. As $x \to -\infty$, $f(x) \to -\infty$

As $x \to \infty$, $f(x) \to \infty$

C. As $x \to -\infty$, $f(x) \to \infty$

As $x \to \infty$, $f(x) \to -\infty$

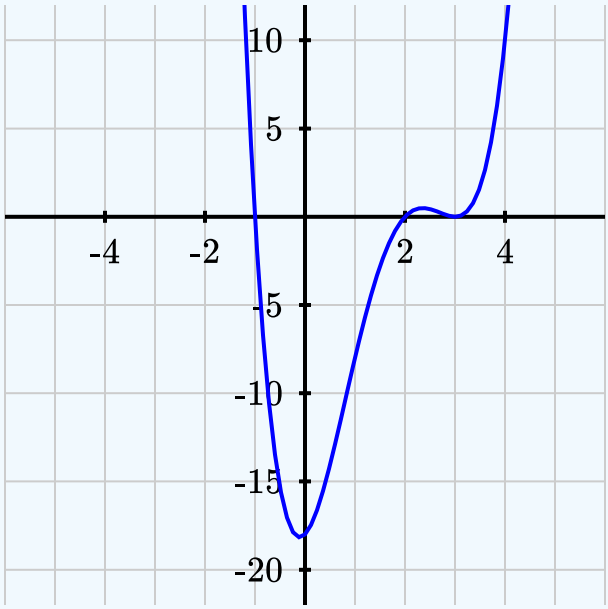
D. As $x \to -\infty$, $f(x) \to \infty$

As $x \to \infty$, $f(x) \to \infty$

Answer available at the end of this section.

Activity 5.1.1.5

Let's now look at the graph of $f(x) = (x-2)(x+1)(x-3)^2$ to answer the questions below.



(a) What are the zeros of $f(x)$?

A. $1, 2, 3$
B. $-1, 2, 3$
C. $1, -2, -3$
D. $-1, 2, -3$

Answer available at the end of this section.

(b) Describe the behavior at each zero. What do you notice?

Answer available at the end of this section.

Definition: Multiplicity

The **multiplicity** of a polynomial function is the number of times a given factor appears in the factored form of the equation of a polynomial.

The zero, $x = 3$ in Activity $5.1.1.5$ has multiplicity 2 because the factor $(x - 3)$ occurs twice (the exponent is a 2).

Activity $5.1.1.6$

Use the function, $f(x) = (x - 3)^2(x - 1)$ to answer the following questions:

(a) What are the zeros of $f(x)$?

A. $-1, -3$
B. $-1, 3$
C. $1, -3$
D. $1, 3$

Answer available at the end of this section.

(b) Using the Definition of Multiplicity, determine the multiplicities of the zeros you found in part (a).

Answer available at the end of this section.

(c)

Now graph $f(x)$ and look at the zeros on the graph. What do you notice about the behavior of the graph at each zero?

Answer available at the end of this section.

Note

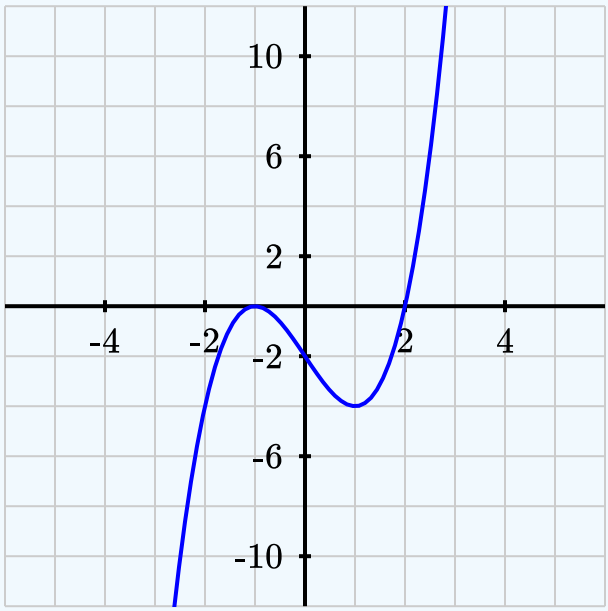
Refer back to Activity $5.1.1.5$ and Activity $5.1.1.6$. Notice that when the graph crosses the x-axis at the zero, the mulitplicity of that zero is odd. When the graph touches the x-axis at the zero, the multiplicity is even. In other words, factors of $f(x)$ with odd exponents will cross the x-axis and factors of $f(x)$ with even exponents will touch the x-axis.

Definition: Local maxima and Local minima

When graphing polynomial functions, you may notice that these functions have some "hills" and "valleys." These characteristics of the graph are known as the **local maxima** and **local minima** of the graph - similar to what we've already seen with quadratic functions. Unlike quadratic functions, however, a polynomial graph can have many local maxima/minima (quadratic functions only have one).

Activity $5.1.1.7$

Now that we know all the different characteristics of polynomials, we should also be able to identify them from a graph. Use the graph below to find the given characteristics.



(a)

What are the x-intercept(s) of the polynomial function? Select all that apply.

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$

D. $(0, -2)$

Answer available at the end of this section.

(b)

What are the y-intercept(s) of the polynomial function?

A. $(1, 0)$
B. $(-1, 0)$
C. $(2, 0)$
D. $(0, -2)$

Answer available at the end of this section.

(c)

How many zeros does this polynomial function have?

A. 0
B. 1
C. 2
D. 3

Answer available at the end of this section.

(d)

At what point is the local minimum located?

A. $(2, -4)$
B. $(-1, 0)$
C. $(-2, 0)$
D. $(1, -4)$
E. $(2, 0)$

Answer available at the end of this section.

(e)

At what point is the local maximum located?

A. $(2, -4)$
B. $(-1, 0)$
C. $(-2, 0)$
D. $(1, -4)$
E. $(2, 0)$

Answer available at the end of this section.

(f)

How do you describe the behavior of the polynomial function as $x \to \infty$?

A. the y-values go to negative infinity
B. $f(x) \to \infty$
C. the y-values go to positive infinity
D. $f(x) \to -\infty$

Answer available at the end of this section.

(g)

How do you describe the behavior of the polynomial function as $x \to -\infty$?

A. the y-values go to negative infinity
B. $f(x) \to \infty$
C. the y-values go to positive infinity
D. $f(x) \to -\infty$

Answer available at the end of this section.

Activity 5.1.1.8

Sketch the function, $f(x) = (x-2)(x+1)(x-3)^2$, by first finding the given characteristics.

(a)

Find the zeros of $f(x)$.

Answer available at the end of this section.

(b)

Find the multiplicities and describe the behavior at each zero.

Answer available at the end of this section.

(c)

Find the y-intercept of $f(x)$.

Answer available at the end of this section.

(d)

Describe the end behavior of $f(x)$.

Answer available at the end of this section.

(e)

Estimate where any local maximums and minimums may occur.

Answer available at the end of this section.

Activity 5.1.1.9

Sketch the graph of a function $f(x)$ that meets all of the following criteria. Be sure to scale your axes and label any important features of your graph.

- The x-intercepts of $f(x)$ are $0, 2$, and 5.
- 0 and 2 have even multiplicity and 5 has odd multiplicity.
- The end behavior of $f(x)$ is given as: As $x \to \infty$, $f(x) \to \infty$ and As $x \to -\infty$, $f(x) \to -\infty$

Answer available at the end of this section.

Activity 5.1.1.10

Use the given function, $f(x) = (x+1)^2(x-5)$, to answer the following questions.

(a)

What are the zeros of $f(x)$?

A. $-1, -5$
B. $-1, 5$
C. $1, -5$
D. $1, 5$

Answer available at the end of this section.

(b)

What are the multiplicities at each zero?

A. At $x = -1$, the multiplicity is even.

   At $x = 5$, the multiplicity is even.

B. At $x = -1$, the multiplicity is even.

   At $x = 5$, the multiplicity is odd.

C. At $x = -1$, the multiplicity is odd.

   At $x = 5$, the multiplicity is even.

D. At $x = -1$, the multiplicity is odd.

   At $x = 5$, the multiplicity is odd.

Answer available at the end of this section.

(c)

What is the end behavior of $f(x)$?

A. $f(x)$ rises on the left and on the right.
B. $f(x)$ rises on the left, but falls on the right.
C. $f(x)$ rises on the right, but falls on the left.
D. $f(x)$ falls on the left and on the right.

Answer available at the end of this section.

(d)

Using what you know about the zeros, multiplicities, and end behavior, where on the graph can we estimate the local maxima and minima to be?

Answer available at the end of this section.

(e)

Now look at the graph of $f(x)$. At which zero does a local maximum or local minimum occur? Explain how you know.

Answer available at the end of this section.

Note

We can estimate where these local maxima and minima occur by looking at other characteristics, such as multiplicities and end behavior.

From Activity 5.1.1.10, we saw that when the function touches the x-axis at a zero, then that zero could be either a local maximum or a local minimum of the graph. When the function crosses the x-axis, however, the local maximum or local minimum occurs between the zeros.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 5.1.1.1

Solve the following problems to test your understanding of polynomial functions, their zeros, multiplicities, and end behavior.

(a) Let $P(x) = -2(x-3)^2(x+1)^3(x-5)$. Find the zeros of $P(x)$, state the multiplicity of each zero, and describe the behavior of the graph at each x-intercept.

Answer available at the end of this section.

(b) Determine the end behavior of the polynomial function $f(x) = -3x^5 + 4x^3 - 2x^2 + 7$.

Answer available at the end of this section.

(c) A polynomial $y = P(x)$ of degree 3 has x-intercepts at $(-2, 0)$, $(1, 0)$, and $(4, 0)$, and a y-intercept at $(0, 16)$. Find the equation of the polynomial.

Answer available at the end of this section.

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Answers

Activity 5.1.1.1
C

Activity 5.1.1.1
B

Activity 5.1.1.1
D

Activity 5.1.1.1
Answers may include, x-intercepts, end behavior, max and min

Activity 5.1.1.2
A

Activity 5.1.1.2
B

Activity 5.1.1.2
C

Activity 5.1.1.2

D

Activity 5.1.1.3

D

Activity 5.1.1.3

A

Activity 5.1.1.3

D

Activity 5.1.1.3

C

Activity 5.1.1.3

Even degree go the same way

Activity 5.1.1.3

Odd degree go the opposite direction

Activity 5.1.1.4

C

Activity 5.1.1.4

B

Activity 5.1.1.5

B

Activity 5.1.1.5

At -1 and 2 the graph crosses the x-axis. At 3 the graph touches the axis and turns around.

Activity 5.1.1.6

D

Activity 5.1.1.6

3 has multiplicity 2. 1 has multiplicity 1.

Activity 5.1.1.6

3 touches the axis and turns around. 1 crosses the axis.

Activity 5.1.1.7

A and C

Activity 5.1.1.7

D

Activity 5.1.1.7

C

Activity 5.1.1.7

A

Activity 5.1.1.7

C

Activity 5.1.1.7

B and C

Activity 5.1.1.7

A and D

Activity 5.1.1.8

2, 1, and 3

Activity 5.1.1.8

2 and -1 multiplicity 1 and crosses axis. 3 multiplicity 2 and touches the axis and turns around

Activity 5.1.1.8

$(0, -18)$

Activity 5.1.1.8

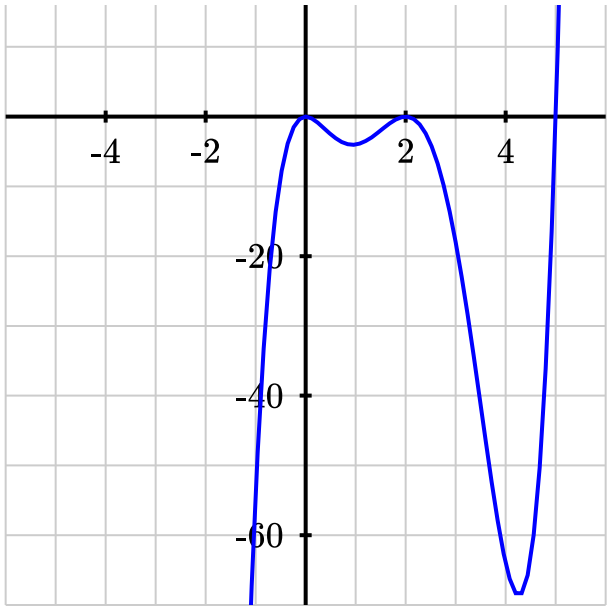
As $x \to \infty$, $y \to \infty$. As $x \to -\infty$, $y \to -\infty$.

Activity 5.1.1.8

Minimum occurs between $x = -1$ and $x = 2$. Then a maximum at $x = 3$

Activity 5.1.1.9

One such graph is



Activity 5.1.1.10

B

Activity 5.1.1.10

B

Activity 5.1.1.10

C

Activity 5.1.1.10

Between $x = -1$ and $x = 5$ since the graph must touch the x-axis at both places.

Activity 5.1.1.10

At $x = -1$ since the graph touches the axis and turns around.

Check Your Understanding 5.1.1.1

The zeros are found by setting each factor to zero:

- $x = 3$ with multiplicity 2. Since the multiplicity is even, the graph is tangent to (touches and turns around at) the x-axis at $(3, 0)$.
- $x = -1$ with multiplicity 3. Since the multiplicity is odd and greater than 1, the graph crosses the x-axis flattening out at $(-1, 0)$.
- $x = 5$ with multiplicity 1. Since the multiplicity is 1, the graph crosses the x-axis linearly at $(5, 0)$.

Check Your Understanding 5.1.1.1

The end behavior of a polynomial is determined by its leading term: $-3x^5$.

- The degree is 5 (odd).
- The leading coefficient is -3 (negative).

Therefore, as $x \to \infty$, $f(x) \to -\infty$, and as $x \to -\infty$, $f(x) \to \infty$.

Check Your Understanding 5.1.1.1

Since the degree is 3 and we are given three distinct x-intercepts, the polynomial has the factored form:

$$P(x) = a(x + 2)(x - 1)(x - 4) \tag{5.1.1.1}$$

To find the leading coefficient a, use the y-intercept $(0, 16)$:

$$16 = a(0 + 2)(0 - 1)(0 - 4) \implies 16 = a(2)(-1)(-4) \implies 16 = 8a \implies a = 2 \tag{5.1.1.2}$$

Thus, the equation of the polynomial is:

$$P(x) = 2(x + 2)(x - 1)(x - 4) \tag{5.1.1.3}$$

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5.1.1.1: PR3 Online Exercises

PR3 — Determine the zeros and their multiplicities of a polynomial in factored form. Describe and graph the behavior of a polynomial function at the intercepts and the ends.	PR3 CheckIt Exercises

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5.2: ANALYZING POLYNOMIAL FUNCTIONS AND THEIR ZEROS (PR4 AND PR5)

5.2.1: Polynomial Long Division (PR4)

5.2.1.1: PR4 Online Exercises

5.2.2: Zeros of a Polynomial Functions (PR5)

5.2.2.1: PR5 Online Exercises

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5.2.1: Polynomial Long Division (PR4)

Learning Outcomes:

- Rewrite a rational function as a polynomial plus a proper rational function.

Dividing Polynomials (PR4)

Note

We have seen previously that we can reduce rational functions by factoring, for example

$$\frac{x^2+5x+4}{x^3+3x^2+2x} = \frac{(x+1)(x+4)}{x(x+2)(x+1)} = \frac{x+4}{x(x+2)}. \tag{5.2.1.1}$$

In this section, we will explore the question: what can we do to simplify rational functions if we are not able to reduce by easily factoring?

Definition: Proper and Improper and Proper rational function

Recall that a fraction is called **proper** if its numerator is smaller than its denominator, and **improper** if the numerator is larger than the denominator (so $\frac{3}{5}$ is a proper fraction, but $\frac{32}{7}$ is an improper fraction). Similarly, we define a **proper rational function** to be a rational function where the degree of the numerator is less than the degree of the denominator.

Activity 5.2.1.1

Label each of the following rational functions as either **proper** or **improper**.

A. $\dfrac{x^3+x}{x^2+4}$

B. $\dfrac{3}{x^2+3x+4}$

C. $\dfrac{7+x^3}{x^2+x+1}$

D. $\dfrac{x^4+x+1}{x^4+4x^2}$

Answer available at the end of this section.

Note

When dealing with an improper fraction such as $\frac{32}{7}$, it is sometimes useful to rewrite this as an integer plus a proper fraction, e.g. $\frac{32}{7} = 4 + \frac{4}{7}$. Similarly, it will sometimes be useful to rewrite an improper rational function as the sum of a polynomial and a proper rational function, such as $\frac{x^3+x}{x^2+4} = x - \frac{3x}{x^2+4}$.

Activity 5.2.1.2

Consider the improper fraction $\frac{357}{11}$.

(a)

Use long division to write $\frac{357}{11}$ as an integer plus a proper fraction.

Answer available at the end of this section.

(b)

Now we will carefully redo this process in a way that we can generalize to rational functions. Note that we can rewrite 357 as $357 = 3 \cdot 10^2 + 5 \cdot 10 + 7$, and 11 as $11 = 1 \cdot 10 + 1$. By comparing the leading terms in these expansions, we see that to knock off the leading term of 357, we need to multiply 11 by $3 \cdot 10^1$.

Using the fact that $357 = 11 \cdot 30 + 27$, rewrite $\frac{357}{11}$ as $\frac{357}{11} = 30 + \frac{?}{11}$.

Answer available at the end of this section.

(c)

Note now that if we can rewrite $\frac{27}{11}$ as an integer plus a proper fraction, we will be done, since $\frac{357}{11} = 30 + \frac{27}{11}$.

Rewrite $\frac{27}{11} = ? + \frac{?}{11}$ as an integer plus a proper fraction.

Answer available at the end of this section.

(d)

Combine your work in the previous two parts to rewrite $\frac{357}{11}$ as an integer plus a proper fraction. How does this compare to what you obtained in part (a)?

Answer available at the end of this section.

Activity 5.2.1.3

Now let's consider the rational function $\frac{3x^2 + 5x + 7}{x + 1}$. We want to rewrite this as a polynomial plus a proper rational function.

(a)

Looking at the leading terms, what do we need to multiply $x + 1$ by so that it would have the same leading term as $3x^2 + 5x + 7$?

A. 3
B. x
C. $3x$
D. $3x + 5$

Answer available at the end of this section.

(b)

Rewrite $3x^2 + 5x + 7 = 3x(x + 1) + ?$, and use this to rewrite $\frac{3x^2 + 5x + 7}{x + 1} = 3x + \frac{?}{x + 1}$.

Answer available at the end of this section.

(c)

Now focusing on $\dfrac{2x+7}{x+1}$, what do we need to multiply $x+1$ by so that it would have the same leading term as $2x+7$?

A. 2
B. x
C. $2x$
D. $2x+7$

Answer available at the end of this section.

(d)

Rewrite $\dfrac{2x+7}{x+1} = 2 + \dfrac{?}{x+1}$.

Answer available at the end of this section.

(e)

Combine this with the previous parts to rewrite $\dfrac{3x^2+5x+7}{x+1} = 3x + ? + \dfrac{?}{x+1}$.

Answer available at the end of this section.

Activity 5.2.1.4

Next we will use the notation of long division to rewrite the rational function $\dfrac{3x^2+5x+7}{x+1}$ as a polynomial plus a proper rational function.

(a)

First, let's use long division notation to write the quotient.

$$\begin{array}{rl} & ? \\ \cline{2-2} \multicolumn{1}{r}{(x+1)} & \big) 3x^2+5x+7 \end{array} \tag{5.2.1.2}$$

What do we need to multiply $x+1$ by so that it would have the same leading term as $3x^2+5x+7$?

Answer available at the end of this section.

(b)

Now to rewrite $3x^2+5x+7$ as $3x(x+1)+?$, place the product $3x(x+1)$ below and subtract.

$$\begin{array}{rl} & 3x \\ \cline{2-2} \multicolumn{1}{r}{(x+1)} & \big) 3x^2+5x+7 \\ & 3x^2+3x \\ \cline{2-2} & ? \ +? \end{array} \tag{5.2.1.3}$$

Answer available at the end of this section.

(c)

Now focusing on $2x+7$, what do we need to multiply $x+1$ by so that it would have the same leading term as $2x+7$?

$$\begin{array}{rl} & 3x+? \\ \cline{2-2} \multicolumn{1}{r}{(x+1)} & \multicolumn{1}{l}{\big)\,3x^2+5x+7} \\ & 3x^2+3x \\ \cline{2-2} & 2x+7 \end{array} \tag{5.2.1.4}$$

Answer available at the end of this section.

(d)

Now, subtract $2(x+1)$ to finish the long division.

$$\begin{array}{rl} & 3x+2 \\ \cline{2-2} \multicolumn{1}{r}{(x+1)} & \multicolumn{1}{l}{\big)\,3x^2+5x+7} \\ & 3x^2+3x \\ \cline{2-2} & 2x+7 \\ & ?\ \ +? \\ \cline{2-2} & ? \end{array} \tag{5.2.1.5}$$

Answer available at the end of this section.

(e)

This long division calculation has shown that

$$3x^2+5x+7=(x+1)(3x+2)+5 \tag{5.2.1.6}$$

. Use this to rewrite $\dfrac{3x^2+5x+7}{x+1}$ as a polynomial plus a proper rational function.

Answer available at the end of this section.

Note

Note that in Activity 5.2.1.3 and Activity 5.2.1.4 we performed the same computations, but just organized our work a little differently.

Activity 5.2.1.5

Rewrite $\dfrac{x^2+1}{x-1}$ as a polynomial plus a proper rational function.

Hint:

Note that $x^2+1=x^2+0x+1$.

Answer available at the end of this section.

Activity 5.2.1.6

Rewrite $\dfrac{x^5+x^3+2x^2-6x+7}{x^2+x-1}$ as a polynomial plus a proper rational function.

Answer available at the end of this section.

Activity 5.2.1.7

Rewrite $\dfrac{3x^4 - 5x^2 + 2}{x - 1}$ as a polynomial plus a proper rational function.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 5.2.1.1

Solve the following problems to test your understanding of polynomial division, the Remainder Theorem, and the Factor Theorem.

(a) Use polynomial long division to find the quotient and remainder of the improper rational expression:

$$\frac{3x^3 - 5x^2 + 10x - 7}{3x - 2} \tag{5.2.1.7}$$

Answer available at the end of this section.

(b) Use synthetic division and the Remainder Theorem to find $P(-3)$ for the polynomial function $P(x) = 2x^4 + 5x^3 - 4x^2 + 8$.

Answer available at the end of this section.

(c) Show that $x - 2$ is a factor of $P(x) = x^3 - 6x^2 + 11x - 6$, and use this to write the polynomial in fully factored form.

Answer available at the end of this section.

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Answers

Activity 5.2.1.1

A, C, and D are improper, while B is proper.

Activity 5.2.1.2

$$\begin{array}{rr} & 32 \\ & \cline{2-4} \\ \multicolumn{1}{r}{11} & \multicolumn{1}{r}{)357} \\ & 33 \\ & \cline{2-3} \\ & 27 \\ & 22 \\ & \cline{3-4} \\ & 5 \end{array} \tag{5.2.1.8}$$

So $\dfrac{357}{11} = 32 + \dfrac{5}{11}$.

Activity 5.2.1.2

$\dfrac{357}{11} = 30 + \dfrac{27}{11}$.

Activity 5.2.1.2

$\dfrac{27}{11} = 2 + \dfrac{5}{11}$.

Activity 5.2.1.2

$$\frac{357}{11} = 30 + \frac{27}{11} = 30 + 2 + \frac{5}{11} = 32 + \frac{5}{11}.$$

Activity 5.2.1.3

C

Activity 5.2.1.3

$$\frac{3x^2+5x+7}{x+1} = 3x + \frac{2x+7}{x+1}$$

Activity 5.2.1.3

A

Activity 5.2.1.3

$$\frac{2x+7}{x+1} = 2 + \frac{5}{x+1}$$

Activity 5.2.1.3

$$\frac{3x^2+5x+7}{x+1} = 3x + 2 + \frac{5}{x+1}$$

Activity 5.2.1.4

$$\begin{array}{rl} & 3x \\ \cline{2-2} \multicolumn{1}{r}{(x+1)} & \multicolumn{1}{|l}{3x^2+5x+7} \end{array} \tag{5.2.1.9}$$

Activity 5.2.1.4

$$\begin{array}{rl} & 3x \\ \cline{2-2} \multicolumn{1}{r}{(x+1)} & \multicolumn{1}{|l}{3x^2+5x+7} \\ & 3x^2+3x \\ \cline{2-2} & \multicolumn{1}{|l}{2x+7} \end{array} \tag{5.2.1.10}$$

Activity 5.2.1.4

$$\begin{array}{rl} & 3x+2 \\ \cline{2-2} \multicolumn{1}{r}{(x+1)} & \multicolumn{1}{|l}{3x^2+5x+7} \\ & 3x^2+3x \\ \cline{2-2} & \multicolumn{1}{|l}{2x+7} \end{array} \tag{5.2.1.11}$$

Activity 5.2.1.4

$$\begin{array}{rl} & 3x+2 \\ \cline{2-2} \multicolumn{1}{r}{(x+1)} & \multicolumn{1}{|l}{3x^2+5x+7} \\ & 3x^2+3x \\ \cline{2-2} & \multicolumn{1}{|l}{2x+7} \\ & 2x+2 \\ \cline{2-2} & \multicolumn{1}{|l}{5} \end{array} \tag{5.2.1.12}$$

Activity 5.2.1.4

$$\frac{3x^2+5x+7}{x+1} = 3x + 2 + \frac{5}{x+1}$$

Activity 5.2.1.5

$$x + 1 + \frac{3}{x-1}.$$

Activity 5.2.1.6

$$x^3 - x^2 + 3x - 2 + \frac{-x+5}{x^2+x-1}.$$

Activity 5.2.1.7

$$3x^3 + 3x^2 - 2x - 2.$$

Check Your Understanding 5.2.1.1

Set up polynomial long division:

1. Divide the leading term of the numerator by the leading term of the divisor: $\frac{3x^3}{3x} = x^2$. Multiply $x^2(3x-2) = 3x^3 - 2x^2$. Subtract this from the numerator to get the new remainder: $-3x^2 + 10x$.
2. Divide the leading term: $\frac{-3x^2}{3x} = -x$. Multiply $-x(3x-2) = -3x^2 + 2x$. Subtract to get: $8x - 7$.
3. Divide the leading term: $\frac{8x}{3x} = \frac{8}{3}$. Multiply $\frac{8}{3}(3x-2) = 8x - \frac{16}{3}$. Subtract: $-7 - \left(-\frac{16}{3}\right) = -\frac{21}{3} + \frac{16}{3} = -\frac{5}{3}$.

Thus, the quotient is $x^2 - x + \frac{8}{3}$ and the remainder is $-\frac{5}{3}$. We can write this as:

$$x^2 - x + \frac{8}{3} - \frac{5}{3(3x-2)} \tag{5.2.1.13}$$

Check Your Understanding 5.2.1.1

First, write the coefficients of $P(x)$, placing a 0 for the missing x^1 term: $2, 5, -4, 0, 8$.

Set up synthetic division with $c = -3$:

$$\begin{array}{rrrrrr} -3 & 2 & 5 & -4 & 0 & 8 \\ & & -6 & 3 & 3 & -9 \\ & 2 & -1 & -1 & 3 & -1 \end{array}$$

The last value in the bottom row is the remainder, which is -1. By the Remainder Theorem, $P(-3) = -1$.

Check Your Understanding 5.2.1.1

By the Factor Theorem, $x - 2$ is a factor of $P(x)$ if and only if $P(2) = 0$:

$$P(2) = 2^3 - 6(2^2) + 11(2) - 6 = 8 - 24 + 22 - 6 = 0 \tag{5.2.1.14}$$

Since $P(2) = 0$, $x - 2$ is a factor. To find the remaining factors, use synthetic division by 2 to divide $P(x)$ by $x - 2$:

$$\begin{array}{rrrrr} 2 & 1 & -6 & 11 & -6 \\ & & 2 & -8 & 6 \\ & 1 & -4 & 3 & 0 \end{array}$$

The quotient is $x^2 - 4x + 3$. We factor this quadratic expression: $(x - 1)(x - 3)$.

Therefore, the fully factored form of the polynomial is:

$$P(x) = (x - 2)(x - 1)(x - 3) \tag{5.2.1.15}$$

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5.2.1.1: PR4 Online Exercises

PR4 — Rewrite a rational function as a polynomial plus a proper rational function (polynomial long division).	PR4 CheckIt Exercises

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5.2.2: Zeros of a Polynomial Functions (PR5)

Learning Outcomes:

- Determine the zeros of a polynomial function with real coefficients.

Theorems about Zeros (PR5)

Recall that to find the x-intercepts of a function $f(x)$, we need to find the values of x that make $f(x) = 0$. We saw in Section 2.5 that the zero product property (the Definition of Zero Product Property) was helpful when $f(x)$ is a polynomial that we can factor. Let's consider how we approach polynomials which are not in factored form or easily factorable.

Activity 5.2.2.1

Consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a)

Use polynomial division from Section 5.2.1 to divide $f(x)$ by $x - 2$. What is the remainder?

Answer available at the end of this section.

(b)

Find $f(2)$. What do you notice?

Answer available at the end of this section.

Theorem: Remainder Theorem

If a polynomial $p(x)$ is divided by $x - a$ then the remainder is equal to $p(a)$.

Activity 5.2.2.2

Again consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a)

Divide $f(x)$ by $x - 3$. What is the remainder?

Answer available at the end of this section.

(b)

Find $f(3)$. What do you notice?

Answer available at the end of this section.

Theorem: Factor Theorem

A number a is a zero of a polynomial function $p(x)$ (that is, $p(a) = 0$) if and only if $x - a$ is a factor of $p(x)$.

Note

If we know one zero, then we can divide by $x - a$ where a is a zero. After this, the quotient will have smaller degree and we can work on factoring the rest. We can "chip away" at the polynomial one zero at a time.

Activity 5.2.2.3

One more time consider the function $f(x) = x^3 - 7x^2 + 7x + 15$.

(a)

We already know from Activity 5.2.2.2 that $x - 3$ is a factor of the polynomial $f(x)$. Use division to express $f(x)$ as $(x - 3) \cdot q(x)$, where $q(x)$ is a quadratic function.

A. $q(x) = x^2 - 2x - 3$
B. $q(x) = x^2 - 10x - 37$
C. $q(x) = x^2 - 4x - 5$
D. $q(x) = x^2 + 4x - 5$

Answer available at the end of this section.

(b)

Notice that $q(x)$ is something we can factor. Factor this quadratic and find the remaining zeros.

A. -5
B. 5
C. 4
D. -1
E. 3
F. 1

Answer available at the end of this section.

Note

We were able to find all the zeros of the polynomial in Activity 5.2.2.3 because we were given one of the zeros. If we don't have a zero to help us get started (or need more than one zero for a function of higher degree), we have a couple of options.

Activity 5.2.2.4

Consider the function $f(x) = 18x^4 + 67x^3 - 81x^2 - 202x + 168$.

(a)

Graph the function. According to the graph, what value(s) seem to be zeros?

Answer available at the end of this section.

(b)

Use the Remainder Theorem to confirm that your guesses are actually zeros.

Answer available at the end of this section.

(c)

Now use these zeros along with polynomial division to rewrite the function as $f(x) = (x - a)(x - b)q(x)$ where a and b are zeros and $q(x)$ is the remaining quadratic function.

Answer available at the end of this section.

(d)

Solve the quadratic $q(x)$ to find the remaining zeros.

Answer available at the end of this section.

(e)

List all zeros of $f(x)$.

Answer available at the end of this section.

(f)

Rewrite $f(x)$ as a product of linear factors.

Answer available at the end of this section.

Note

Using the graph to find an initial zero can be helpful, but they may not always be easy to identify.

Activity $5.2.2.5$

Consider the quadratic function $f(x) = (2x - 5)(3x - 8) = 6x^2 - 31x + 40$.

(a)

What are the roots of this quadratic?

Answer available at the end of this section.

(b)

What do you notice about these roots in relation to the factors of $a = 6$ and $c = 40$ in $f(x) = 6x^2 - 31x + 40$?

Answer available at the end of this section.

Note

In Activity $5.2.2.5$ we found that the roots were both factors of the constant term divided by factors of the leading coefficient. This can be extended to polynomials of larger degree.

Theorem: Rational Root Theorem

If a polynomial $p(x) = a_n x^n + x_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then the rational zeros have the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Activity 5.2.2.6

Consider the polynomial $f(x) = 5x^3 - 2x^2 + 20x - 8$.

(a)

List the factors of the constant term.

Answer available at the end of this section.

(b)

List the factors of the leading coefficient.

Answer available at the end of this section.

(c)

Use parts (a) and (b) to list all the possible rational roots.

Answer available at the end of this section.

(d)

Use the Remainder Theorem to determine at least one root of $f(x)$.

Answer available at the end of this section.

Activity 5.2.2.7

Consider the polynomial $f(x) = 6x^4 + 5x^3 - 6x - 5$

(a)

Use the graph and the Rational Root Theorem (Theorem 5.2.2.3) to find the rational zeros of $f(x)$.

Answer available at the end of this section.

(b)

Use the roots, along with the Factor Theorem, to simplify the polynomial into linear and quadratic factors.

Answer available at the end of this section.

(c)

Find the zeros of the quadratic factor.

Answer available at the end of this section.

(d)

List the roots of the polynomial.

Answer available at the end of this section.

Note

Notice that the zeros of the quadratic factor were imaginary and are related. This also occured in the section on the Quadratic Formula.

Theorem: Conjugate Zeros Theorem

Let $p(x)$ be a polynomial function with real coefficients. If $a+bi$ is a complex zero of the function, then the conjugate $a-bi$ is also a zero of the function. These two zeroes are called **conjugate zeros**, or a **conjugate pair of zeros**.

Activity 5.2.2.8

Consider the function $f(x)=x^5+3x^4+4x^3+8x^2-16$.

(a)

Use a graphing utility to graph $f(x)$.

Answer available at the end of this section.

(b)

Find *all* the zeros of $f(x)$ and their corresponding multiplicities.

Answer available at the end of this section.

Activity 5.2.2.9

Consider the following information about a polynomial $f(x)$:

- $x=2$ is a zero with multiplicity 1
- $x=-1$ is a zero with multiplicity 2
- $x=i$ is a zero with multiplicity 1

(a)

What is the smallest possible degree of such a polynomial $f(x)$ with real coefficients?

A. 2
B. 3
C. 4
D. 5
E. 6

Answer available at the end of this section.

(b)

Write an expression for such a polynomial $f(x)$ with real coefficients of smallest possible degree.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 5.2.2.1

Solve the following problems to test your understanding of polynomial zeros, Descartes' Rule of Signs, and the Complex Conjugate Zeros Theorem.

(a) Consider the polynomial function $f(x) = 2x^3 - 3x^2 - 8x - 3$.

1. List all possible rational zeros given by the Rational Zero Theorem.
2. Find all actual real zeros of the polynomial.

Answer available at the end of this section.

(b) Use Descartes' Rule of Signs to determine the possible number of positive and negative real zeros of $P(x) = x^4 - 2x^3 - x^2 + 5x - 6$.

Answer available at the end of this section.

(c) Find a polynomial of degree 4 with real coefficients that has zeros at $x = 2$ (multiplicity 2) and $x = 1 - 2i$.

Answer available at the end of this section.

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Answers

Activity 5.2.2.1

9

Activity 5.2.2.1

$f(2) = 9$ Remainder and value are the same

Activity 5.2.2.2

0

Activity 5.2.2.2

$f(3) = 0$ When value is 0 there is no remainder

Activity 5.2.2.3

D

Activity 5.2.2.3

B and F

Activity 5.2.2.4

-4 and -2

Activity 5.2.2.4

$f(-4) = 0$ and $f(-2) = 0$

Activity 5.2.2.4

$(x + 4)(x + 2)(18x^2 - 41x + 21)$

Activity 5.2.2.4

$\frac{3}{2}$ and $\frac{7}{9}$

Activity 5.2.2.4

$-4, -2, \frac{7}{9}, \frac{3}{2}$

Activity 5.2.2.4

$(x+4)(x+2)(2x-3)(9x-7)$

Activity 5.2.2.5

$\frac{5}{2}$ and $\frac{8}{3}$

Activity 5.2.2.5

The numerator is a factor of c and the denominator is a factor of a.

Activity 5.2.2.6

$\pm 8, \pm 4, \pm 2 \pm 1$

Activity 5.2.2.6

$\pm 5, \pm 1$

Activity 5.2.2.6

$\pm 8, \pm 4, \pm 2 \pm 1, \pm\frac{8}{5}, \pm\frac{4}{5}, \pm\frac{2}{5}, \pm\frac{1}{5}$

Activity 5.2.2.6

$\frac{2}{5}$

Activity 5.2.2.7

1 and $-\frac{5}{6}$

Activity 5.2.2.7

$(6x+5)(x-1)(x^2+x+1)$

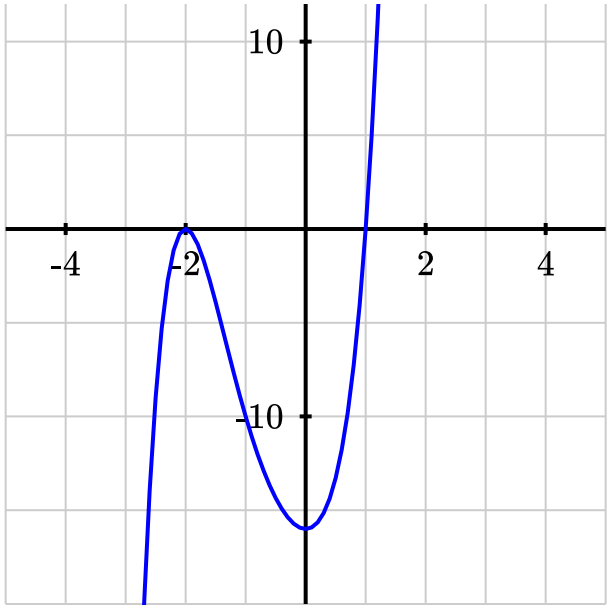
Activity 5.2.2.7

$\frac{-1 \pm \sqrt{3}}{2}$

Activity 5.2.2.7

$\frac{-1+\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}, 1,$ and $-\frac{5}{6}$

Activity 5.2.2.8



Activity 5.2.2.8

$f(x)$ has zeros at -2, 1, $-2i$, and $2i$, all of multiplicity 1 except -2 has multiplicity 2

Activity 5.2.2.9

D

Activity 5.2.2.9

$f(x) = (x-2)(x-1)^2(x^2+1) = x^5 - 4x^4 + 6x^3 - 6x^2 + 5x - 2$ is one such polynomial.

Check Your Understanding 5.2.2.1

We solve each part:

1. According to the Rational Zero Theorem, any rational zero of $f(x)$ must be of the form $\pm\frac{p}{q}$, where p is a factor of the constant term -3 (factors are $\pm 1, \pm 3$) and

q is a factor of the leading coefficient 2 (factors are $\pm 1, \pm 2$). The possible rational zeros are:

$$\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2} \tag{5.2.2.1}$$

2. We test these candidate zeros. Let's evaluate $f(-1)$:

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 8(-1) - 3 = -2 - 3 + 8 - 3 = 0 \tag{5.2.2.2}$$

Since $f(-1) = 0$, $x = -1$ is a zero. We use synthetic division to divide $f(x)$ by $x + 1$:

$$\begin{array}{rrrrr} -1 & 2 & -3 & -8 & -3 \\ & & -2 & 5 & 3 \\ & 2 & -5 & -3 & 0 \end{array} \tag{5.2.2.3}$$

The quotient is $2x^2 - 5x - 3$. We find the remaining zeros by factoring this quadratic:

$$2x^2 - 5x - 3 = (2x + 1)(x - 3) = 0 \implies x = -\frac{1}{2} \text{ or } x = 3 \tag{5.2.2.4}$$

Thus, the actual real zeros of the polynomial are:

$$x = -1, \quad x = -\frac{1}{2}, \quad x = 3 \tag{5.2.2.5}$$

Check Your Understanding 5.2.2.1

- **Positive Real Zeros:** Count the number of sign changes in the coefficients of $P(x) = x^4 - 2x^3 - x^2 + 5x - 6$:
  - From x^4 to $-2x^3$: sign changes from positive to negative (1st change).
  - From $-2x^3$ to $-x^2$: no change.
  - From $-x^2$ to $5x$: sign changes from negative to positive (2nd change).
  - From $5x$ to -6: sign changes from positive to negative (3rd change).

  Since there are 3 sign changes, there are either **3 or 1** positive real zeros.
- **Negative Real Zeros:** Count the number of sign changes in the coefficients of $P(-x)$:

$$P(-x) = (-x)^4 - 2(-x)^3 - (-x)^2 + 5(-x) - 6 = x^4 + 2x^3 - x^2 - 5x - 6 \tag{5.2.2.6}$$

  - From x^4 to $2x^3$: no change.
  - From $2x^3$ to $-x^2$: sign changes from positive to negative (1st change).
  - From $-x^2$ to $-5x$: no change.
  - From $-5x$ to -6: no change.

  Since there is exactly 1 sign change, there is **exactly 1** negative real zero.

Check Your Understanding 5.2.2.1

Since the polynomial has real coefficients, the complex zeros must occur in conjugate pairs. Thus, if $x = 1 - 2i$ is a zero, then $x = 1 + 2i$ must also be a zero.

The factors corresponding to these zeros are:

1. For the complex conjugate zeros:

$$(x - (1 - 2i))(x - (1 + 2i)) = ((x - 1) + 2i)((x - 1) - 2i) \tag{5.2.2.7}$$

$$= (x - 1)^2 - (2i)^2 = x^2 - 2x + 1 - (-4) = x^2 - 2x + 5 \tag{5.2.2.8}$$

2. For the zero $x = 2$ of multiplicity 2:

$$(x - 2)^2 = x^2 - 4x + 4 \tag{5.2.2.9}$$

To find the polynomial, multiply these two factored parts together:

$$P(x) = (x^2 - 4x + 4)(x^2 - 2x + 5) \tag{5.2.2.10}$$

$$= x^2(x^2 - 2x + 5) - 4x(x^2 - 2x + 5) + 4(x^2 - 2x + 5) \tag{5.2.2.11}$$

$$= (x^4 - 2x^3 + 5x^2) - (4x^3 - 8x^2 + 20x) + (4x^2 - 8x + 20) \tag{5.2.2.12}$$

$$= x^4 - 6x^3 + 17x^2 - 28x + 20 \tag{5.2.2.13}$$

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5.2.2.1: PR5 Online Exercises

PR5 — Determine the zeros of a polynomial function with real coefficients.	PR5 CheckIt Exercises

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5.3: ANALYZING RATIONAL FUNCTIONS (PR6 AND PR7)

5.3.1: Rational Equations (PR6)

5.3.1.1: PR6 Online Exercises

5.3.2: Graphs of Rational Functions (PR7)

5.3.2.1: PR7 Online Exercises

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5.3.1: Rational Equations (PR6)

Learning Outcomes:

- Solve rational equations.

Rational Equations (PR6)

Definition: Rational expression and Rational equation

An algebraic expression is called a **rational expression** if it can be written as the ratio of two polynomials, p and q.

An equation is called a **rational equation** if it consists of only rational expressions and constants.

Note

Technically, linear and quadratic equations are also rational equations. They are a special case where the denominator of the rational expressions is 1. We will focus in this section on cases where the denominator is not a constant; that is, rational equations where there are variables in the denominator.

With variables in the denominator, there will often be values that cause the denominator to be zero. This is a problem because division by zero is undefined. Thus, we need to be sure to exclude any values that would make those denominators equal to zero.

Note

For this section, it might be helpful to refer back to *(see related activity)*, where you solved a linear equation with fractions. When revisiting that activity, think about how you had to carefully choose a number to get the denominators to cancel out. Keep this in mind as you work through the next activity.

Activity 5.3.1.1

Which value(s) should be excluded as possible solutions to the following rational equations? Select all that apply.

(a)

$$\frac{2}{x+5} = \frac{x-3}{x-8} - 7 \tag{5.3.1.1}$$

A. -7
B. -5
C. 2
D. 3
E. 8

Answer available at the end of this section.

(b)

$$\frac{x^2-6x+8}{x^2-4x+3} = 0 \tag{5.3.1.2}$$

A. 0
B. 1
C. 2
D. 3
E. 4

Answer available at the end of this section.

Activity 5.3.1.2

Consider the rational equation

$$5 = -\frac{6}{x-2} \tag{5.3.1.3}$$

(a)

What value should be excluded as a possible solution?

A. 5
B. 6
C. -6
D. 2
E. -2

Answer available at the end of this section.

(b)

To solve, we begin by clearing out the fraction involved. What can we multiply each term by that will clear the fraction?

A. $x-5$
B. $x-6$
C. $x+6$
D. $x-2$
E. $x+2$

Answer available at the end of this section.

(c)

Multiply each term by the expression you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $5(x-5) = -6$
B. $5(x-6) = -6$
C. $5(x+6) = -6$
D. $5(x-2) = -6$
E. $5(x+2) = -6$

Answer available at the end of this section.

(d)

Solve the linear equation. Check your answer using the original rational equation.

Answer available at the end of this section.

Activity 5.3.1.3

Consider the rational equation

$$\frac{4}{x+1} = -\frac{2}{x+6} \tag{5.3.1.4}$$

(a)

What values should be excluded as possible solutions?

A. 2 and 4
B. 1 and 6
C. -1 and -6
D. 1 and 4
E. 2 and 6

Answer available at the end of this section.

(b)

To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x-2$ and $x-4$
B. $x-1$ and $x-6$
C. $x+1$ and $x+6$
D. $x-1$ and $x-4$
E. $x-2$ and $x-6$

Answer available at the end of this section.

(c)

Multiply each term by the expressions you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $4(x+1) = -2(x+6)$
B. $4(x+6) = -2(x+1)$
C. $4(x+1)(x+6) = -2(x+1)(x+6)$
D. $4(x+1) = -2(-x-6)$
E. $4(x+6) = -2(-x-1)$

Answer available at the end of this section.

(d)

Solve the linear equation. Check your answer using the original rational equation.

Answer available at the end of this section.

Note

In Activity 5.3.1.3, you may have noticed that the resulting linear equation looked like the result of cross-multiplying. This is no coincidence! Cross-multiplying is a method of clearing out fractions that works specifically when the equation is in proportional form: $\frac{a}{b} = \frac{c}{d}$.

Activity 5.3.1.4

Consider the rational equation

$$\frac{x}{x+2} = -\frac{2}{x+2} - \frac{2}{5} \tag{5.3.1.5}$$

(a)

What value(s) should be excluded as possible solutions?

Answer available at the end of this section.

(b)

To solve, we'll once again begin by clearing out the fraction involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x+2$, $x+2$, and 5
B. $x+2$ and 5
C. $x+2$
D. 5

Answer available at the end of this section.

(c)

Multiply each term by the expressions you chose and simplify. You should end up with a linear equation.

Answer available at the end of this section.

(d)

Solve the linear equation. Check your answer using the original rational equation.

Answer available at the end of this section.

Note

Activity 5.3.1.4 demonstrates why it is so important to determine excluded values and check our answers when solving rational equations. Just because a number is a solution to the *linear* equation we found, it doesn't mean it is automatically a solution to the *rational* equation we started with.

Activity 5.3.1.5

Consider the rational equation

$$\frac{2x}{x-1} - \frac{3}{x-3} = \frac{x^2-11x+18}{x^2-4x+3} \tag{5.3.1.6}$$

(a)

What values should be excluded as possible solutions? Select all that apply.

A. 0
B. 1
C. 2

D. 3
E. 9

Answer available at the end of this section.

(b)

To solve, we'll begin by clearing out any fractions involved. Which of the following should we multiply each term by to clear out all of the fractions?

A. $x-1$
B. $x-1$ and $x-3$
C. $x-1$, $x-3$, and x^2-4x+3
D. $x-1$ and x^2-4x+3
E. $x-3$ and x^2-4x+3

Answer available at the end of this section.

(c)

Multiply each term by the expressions you chose and simplify. Notice that the result is a quadratic equation. Which of the following quadratic equations does the rational equation simplify to?

A. $x^2+2x-15=0$
B. $x^2-11x+18=0$
C. $x^2-9x-9=0$
D. $x^2-13x+21=0$

Answer available at the end of this section.

(d)

Solve the quadratic equation. Check your answer using the original rational equation. What are the solutions to the rational equation?

A. $x=3$ and $x=-5$
B. $x=-3$ and $x=5$
C. $x=3$
D. $x=-5$
E. $x=-3$
F. $x=5$

Answer available at the end of this section.

Activity 5.3.1.6

Consider the rational equation

$$\frac{2x}{x-2}-\frac{x^2+21x-15}{x^2+3x-10}=\frac{-6}{x+5} \tag{5.3.1.7}$$

(a)

What values should be excluded as possible solutions?

Answer available at the end of this section.

(b)

What expression(s) should we multiply by to clear out all of the fractions?

Answer available at the end of this section.

(c)

Multiply each term by the expressions you chose and simplify. Your result should be a quadratic equation.

Answer available at the end of this section.

(d)

Solve the quadratic equation. Check your answer using the original rational equation. What are the solutions to the rational equation?

Answer available at the end of this section.

Activity 5.3.1.7

Solve the following rational equations.

(a)

$$\frac{4}{x} + 9 = 16$$

Answer available at the end of this section.

(b)

$$-5 = \frac{2}{x-4}$$

Answer available at the end of this section.

(c)

$$\frac{-3}{x-10} = \frac{x}{x-6}$$

Answer available at the end of this section.

(d)

$$\frac{x+2}{x-3} + \frac{x}{2x-1} = 6$$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 5.3.1.1

Solve the following rational equations and formulas.

(a) Solve the rational equation:

$$\frac{3}{x-2}+\frac{5}{x+2}=\frac{12}{x^2-4} \tag{5.3.1.8}$$

Identify any extraneous solutions.

Answer available at the end of this section.

(b) Solve the equation:

$$\frac{x}{x-3}-\frac{2}{x+1}=\frac{12}{x^2-2x-3} \tag{5.3.1.9}$$

Answer available at the end of this section.

(c) Solve the formula for R_1:

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2} \tag{5.3.1.10}$$

Answer available at the end of this section.

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Answers

Activity 5.3.1.1
B and E

Activity 5.3.1.1
B and D

Activity 5.3.1.2
D

Activity 5.3.1.2
D

Activity 5.3.1.2
D

Activity 5.3.1.2
$x=\frac{4}{5}$

Activity 5.3.1.3
C

Activity 5.3.1.3
C

Activity 5.3.1.3
B

Activity 5.3.1.3
$x=-\frac{13}{3}$

Activity 5.3.1.4
$x=-2$

Activity 5.3.1.4
B

Activity 5.3.1.4
$5x=-10-2(x+2)$

Activity 5.3.1.4
No solution

Activity 5.3.1.5
B and D

Activity 5.3.1.5

B

Activity 5.3.1.5

A

Activity 5.3.1.5

D

Activity 5.3.1.6

$x = 2$ and $x = -5$

Activity 5.3.1.6

$x - 2$ and $x + 5$

Activity 5.3.1.6

$x^2 - 5x + 3 = 0$

Activity 5.3.1.6

$x = \dfrac{5 \pm \sqrt{13}}{2}$

Activity 5.3.1.7

$x = \dfrac{4}{7}$

Activity 5.3.1.7

$x = \dfrac{18}{5}$

Activity 5.3.1.7

$x = 9$ and $x = -2$

Activity 5.3.1.7

$x = \dfrac{7 \pm \sqrt{29}}{3}$

Check Your Understanding 5.3.1.1

First, identify values of x that make the denominators zero: $x \neq 2$ and $x \neq -2$.

Multiply both sides of the equation by the least common multiple (LCM), which is $(x-2)(x+2) = x^2 - 4$:

$$3(x+2) + 5(x-2) = 12 \tag{5.3.1.11}$$

Distribute and combine like terms:

$$3x + 6 + 5x - 10 = 12 \implies 8x - 4 = 12 \tag{5.3.1.12}$$

$$8x = 16 \implies x = 2 \tag{5.3.1.13}$$

Since the restriction states that $x \neq 2$, the candidate solution $x = 2$ is **extraneous**. Therefore, the equation has **no solution**.

Check Your Understanding 5.3.1.1

First, factor the denominator on the right side: $x^2 - 2x - 3 = (x-3)(x+1)$. The restrictions are $x \neq 3$ and $x \neq -1$.

Multiply both sides by the LCM, which is $(x-3)(x+1)$:

$$x(x+1) - 2(x-3) = 12 \tag{5.3.1.14}$$

Expand and set the quadratic equation to zero:

$$x^2 + x - 2x + 6 = 12 \implies x^2 - x + 6 = 12 \tag{5.3.1.15}$$

$$x^2 - x - 6 = 0 \tag{5.3.1.16}$$

Factor the quadratic equation:

$$(x-3)(x+2) = 0 \implies x = 3 \text{ or } x = -2 \tag{5.3.1.17}$$

Comparing these with our restrictions: $x = 3$ is restricted, so it is **extraneous**. Therefore, the only valid solution is:

$$x = -2 \tag{5.3.1.18}$$

Check Your Understanding 5.3.1.1

Isolate the term containing R_1:

$$\frac{1}{R_1} = \frac{1}{R} - \frac{1}{R_2} \tag{5.3.1.19}$$

Find a common denominator for the right side:

$$\frac{1}{R_1} = \frac{R_2}{RR_2} - \frac{R}{RR_2} = \frac{R_2 - R}{RR_2} \tag{5.3.1.20}$$

Take the reciprocal of both sides to solve for R_1:

$$R_1 = \frac{RR_2}{R_2 - R} \tag{5.3.1.21}$$

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5.3.1.1: PR6 Online Exercises

PR6 — Solve rational equations.	PR6 CheckIt Exercises

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5.3.2: Graphs of Rational Functions (PR7)

Learning Outcomes:

- Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.

Graphs of Rational Functions (PR7)

Definition: Rational

A function r is **rational** provided that it is possible to write r as the ratio of two polynomials, p and q. That is, r is rational provided that for some polynomial functions p and q, we have

$$r(x) = \frac{p(x)}{q(x)}. \tag{5.3.2.1}$$

Note

Rational functions occur in many applications, so our goal in this lesson is to learn about their properties and be able to graph them. In particular we want to investigate the domain, end behavior, and zeros of rational functions.

Activity 5.3.2.1

Consider the rational function

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}. \tag{5.3.2.2}$$

(a)

Find $r(1)$, $r(2)$, $r(3)$, and $r(4)$.

Answer available at the end of this section.

(b)

Label each of these four values as giving us information about the DOMAIN of $r(x)$, information about the ZEROES of $r(x)$, or NEITHER.

Answer available at the end of this section.

Note

Let p and q be polynomial functions so that $r(x) = \frac{p(x)}{q(x)}$ is a rational function. The domain of r is the set of all real numbers except those for which $q(x) = 0$.

Recall that the **domain** of a function was defined in *(see related activity)*.

Activity 5.3.2.2

Let's investigate the domain of $r(x)$ more closely. We will be using the same function from the previous activity:

$$r(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}. \tag{5.3.2.3}$$

(a)

Rewrite $r(x)$ by factoring the numerator and denominator, but do not try to simplify any further. What do you notice about the relationship between the values that are not in the domain and how the function is now written?

Answer available at the end of this section.

(b)

The function was not defined for $x = 3$. Make a table for values of $r(x)$ near $x = 3$.

x	$r(x)$
2	
2.9	
2.99	
2.999	
3	undefined
3.001	
3.01	
3.1	

Answer available at the end of this section.

(c)

Which of the following describe the behavior of the graph near $x = 3$?

A. As $x \to 3$, $r(x)$ approaches a finite number
B. As $x \to 3$ from the left, $r(x) \to \infty$
C. As $x \to 3$ from the left, $r(x) \to -\infty$
D. As $x \to 3$ from the right, $r(x) \to \infty$
E. As $x \to 3$ from the right, $r(x) \to -\infty$

Answer available at the end of this section.

(d)

The function was also not defined for $x = 1$. Make a table for values of $r(x)$ near $x = 1$.

x	$r(x)$

0	
0.9	
0.99	
0.999	
1	undefined
1.001	
1.01	
1.1	

Answer available at the end of this section.

(e)

Which of the following describe the behavior of the graph near $x = 1$?

A. As $x \to 1$, $r(x)$ approaches a finite number
B. As $x \to 1$ from the left, $r(x) \to \infty$
C. As $x \to 1$ from the left, $r(x) \to -\infty$
D. As $x \to 1$ from the right, $r(x) \to \infty$
E. As $x \to 1$ from the right, $r(x) \to -\infty$

Answer available at the end of this section.

(f)

The function is behaving differently near $x = 1$ than it is near $x = 3$. Can you see anything in the factored form of $r(x)$ that may help you account for the difference?

Answer available at the end of this section.

Note

Features of a rational function

Let $r(x) = \dfrac{p(x)}{q(x)}$ be a rational function.

- If $p(a) = 0$ and $q(a) \neq 0$, then $r(a) = 0$, so r has a **zero** at $x = a$.
- If $q(a) = 0$ and $p(a) \neq 0$, then $r(a)$ is undefined and r has a **vertical asymptote** at $x = a$.
- If $p(a) = 0$ and $q(a) = 0$ and we can show that there is a finite number L such that $r(x) \to L$, then $r(a)$ is not defined and r has a **hole** at the point (a, L).

Note

Another property of rational functions we want to explore is the end behavior. This means we want to explore what happens to a given rational function $r(x)$ when x goes toward positive infinity or negative infinity.

Activity 5.3.2.3

Consider the rational function $r(x) = \frac{1}{x^3}$.

(a)

Plug in some very large positive numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches positive infinity?

A. As $x \to \infty$, $r(x) \to \infty$.
B. As $x \to \infty$, $r(x) \to -\infty$.
C. As $x \to \infty$, $r(x) \to 0$.
D. As $x \to \infty$, $r(x) \to 1$.

Answer available at the end of this section.

(b)

Now let's look at $r(x)$ as x tends toward negative infinity. Plug in some very large negative numbers for x to see what $r(x)$ is tending toward. Which of the following best describes the behavior of the graph as x approaches negative infinity?

A. As $x \to -\infty$, $r(x) \to \infty$.
B. As $x \to -\infty$, $r(x) \to -\infty$.
C. As $x \to -\infty$, $r(x) \to 0$.
D. As $x \to -\infty$, $r(x) \to 1$.

Answer available at the end of this section.

Note

We can generalize what we have just found to any function of the form $\frac{1}{x^n}$, where $n > 0$. Since x^n increases without bound as $x \to \infty$, we find that $\frac{1}{x^n}$ will tend to 0. In fact, the numerator can be any constant and the function will still tend to 0!

Similarly, as $x \to -\infty$, we find that $\frac{1}{x^n}$ will tend to 0 too.

Activity 5.3.2.4

Consider the rational function $r(x) = \frac{3x^2 - 5x + 1}{7x^2 + 2x - 11}$.

Observe that the largest power of x that's present in $r(x)$ is x^2. In addition, because of the dominant terms of $3x^2$ in the numerator and $7x^2$ in the denominator, both the numerator and denominator of r increase without bound as x increases without bound.

(a)

In order to understand the end behavior of r, we will start by writing the function in a different algebraic form.

Multiply the numerator and denominator of r by $\frac{1}{x^2}$. Then distribute and simplify as much as possible in both the numerator and denominator to write r in a different algebraic form. Which of the following is that new form?

A. $\frac{3x^4 - 5x^3 + x^2}{7x^4 + 2x^3 - 11x^2}$

B. $\dfrac{3-\dfrac{5}{x}+\dfrac{1}{x^2}}{7+\dfrac{2}{x}-\dfrac{11}{x^2}}$

C. $\dfrac{\dfrac{3x^2}{x^2}-\dfrac{5x}{x^2}+\dfrac{1}{x^2}}{\dfrac{7x^2}{x^2}+\dfrac{2x}{x^2}-\dfrac{11}{x^2}}$

D. $\dfrac{3x^2-5x+1}{7x^4+2x^3-11x^2}$

Answer available at the end of this section.

(b)

Now determine the end behavior of each piece of the numerator and each piece of the denominator.

Hint:

Use Note $5.3.2.4$ *to help!*

Answer available at the end of this section.

(c)

Simplify your work from the previous step. Which of the following best describes the end behavior of $r(x)$?

A. As $x \to \pm\infty$, $r(x)$ goes to 0.
B. As $x \to \pm\infty$, $r(x)$ goes to $\dfrac{3}{7}$.
C. As $x \to \pm\infty$, $r(x)$ goes to ∞.
D. As $x \to \pm\infty$, $r(x)$ goes to $-\infty$.

Answer available at the end of this section.

Note

If the end behavior of a function tends toward a specific value a, then we say that the function has a **horizontal asymptote** at $y = a$.

Activity $5.3.2.5$

Find the horizontal asymptote (if one exists) of the following rational functions. Follow the same method we used in Activity $5.3.2.4$.

(a)

$$f(x) = \frac{4x^3 - 3x^2 + 6}{9x^3 + 7x - 5}$$

A. $y = 0$
B. $y = \dfrac{4}{9}$
C. $y = -\dfrac{3}{7}$
D. $y = -\dfrac{6}{5}$
E. There is no horizontal asymptote.

Answer available at the end of this section.

(b)

$$g(x) = \frac{4x^3 - 3x^2 + 6}{9x^5 + 7x - 5}$$

A. $y = 0$
B. $y = \frac{4}{9}$
C. $y = -\frac{3}{7}$
D. $y = -\frac{6}{5}$
E. There is no horizontal asymptote.

Answer available at the end of this section.

(c)

$$h(x) = \frac{4x^5 - 3x^2 + 6}{9x^3 + 7x - 5}$$

A. $y = 0$
B. $y = \frac{4}{9}$
C. $y = -\frac{3}{7}$
D. $y = -\frac{6}{5}$
E. There is no horizontal asymptote.

Answer available at the end of this section.

Activity 5.3.2.6

Some patterns have emerged from the previous problem. Fill in the rest of the sentences below to describe how to find horizontal asymptotes of rational functions.

(a)

If the degree of the numerator is the same as the degree of the denominator, then...

Answer available at the end of this section.

(b)

If the degree of the numerator is less than the degree of the denominator, then...

Answer available at the end of this section.

(c)

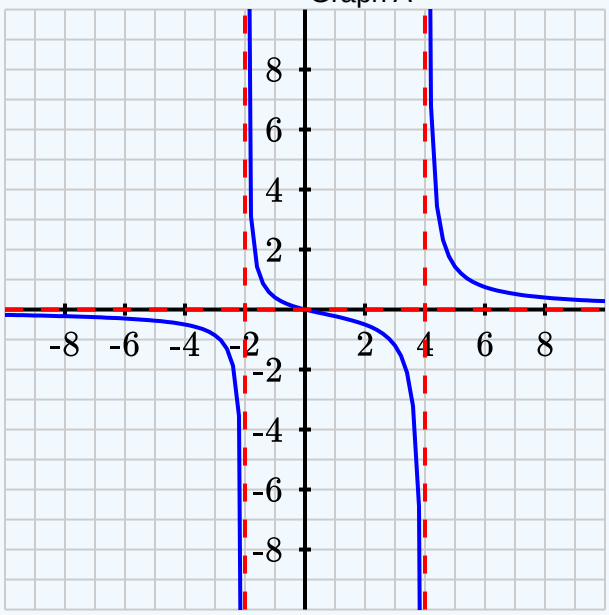
If the degree of the numerator is greater than the degree of the denominator, then...

Answer available at the end of this section.

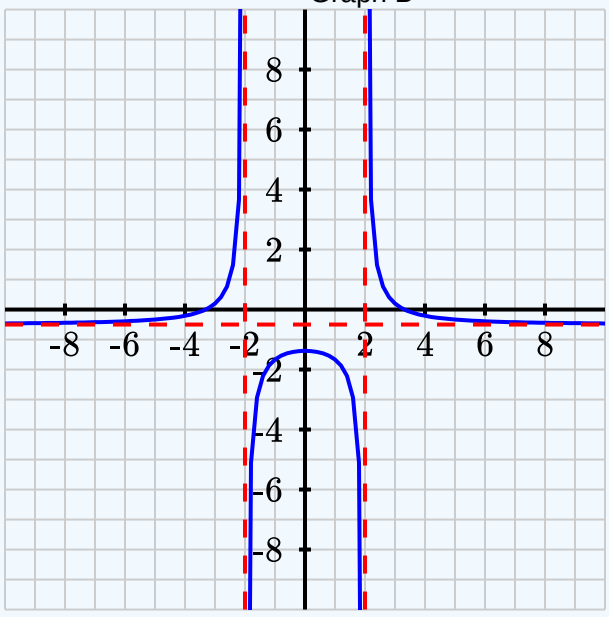
Activity 5.3.2.7

Consider the following six graphs of rational functions:

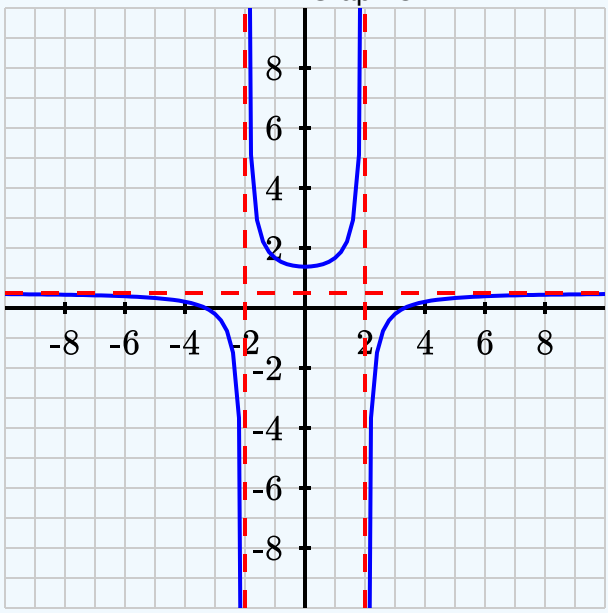
Graph A



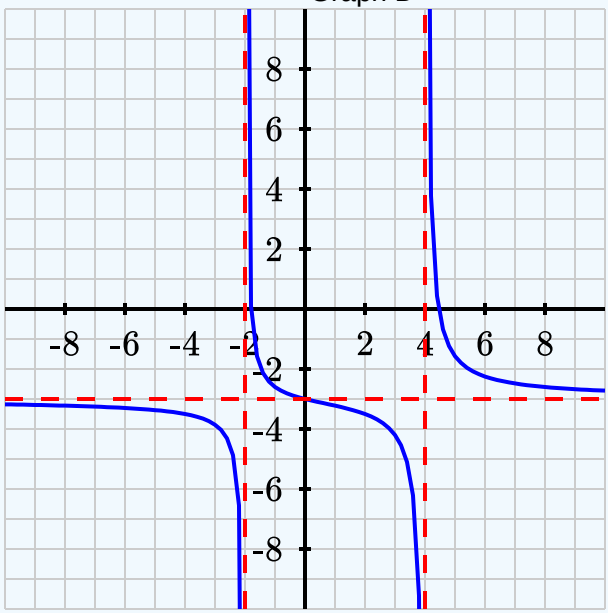
Graph B



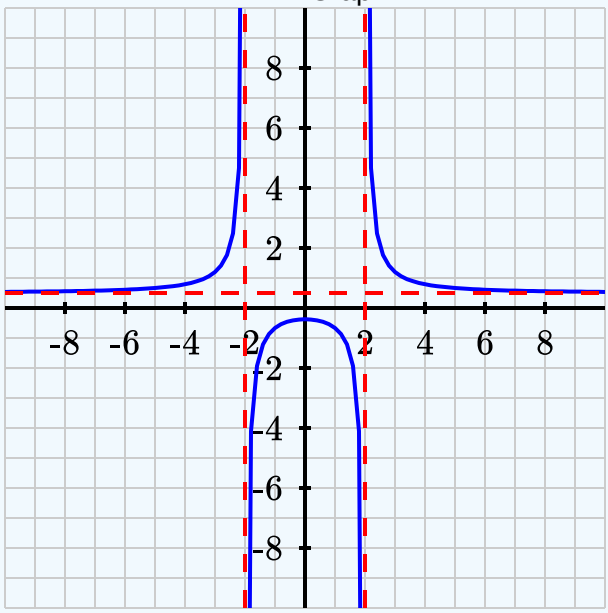
Graph C



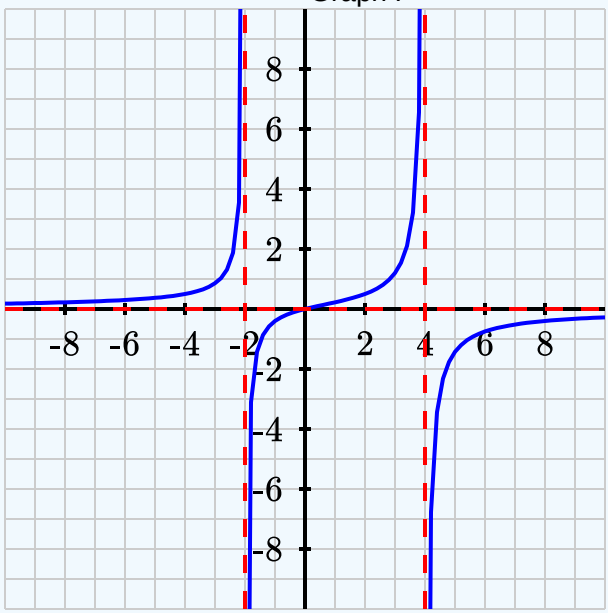
Graph D



Graph E



Graph F



(a)

Which of the graphs above represents the function $f(x) = \dfrac{2x}{x^2 - 2x - 8}$?

Answer available at the end of this section.

(b)

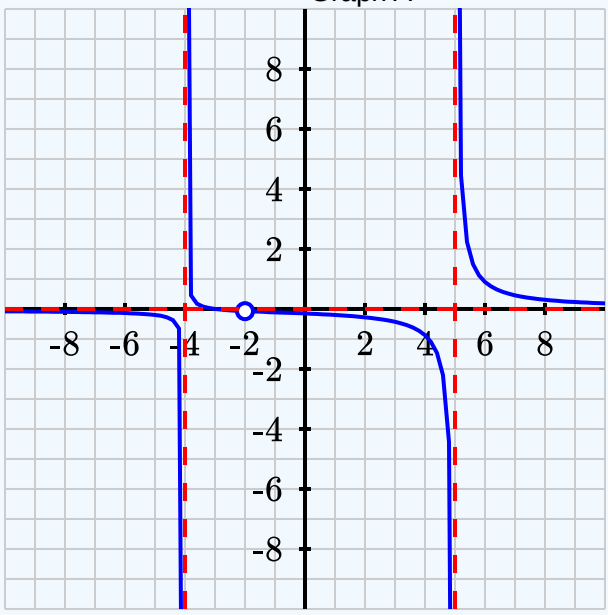
Which of the graphs above represents the function $g(x) = \dfrac{x^2+3}{2x^2-8}$?

Answer available at the end of this section.

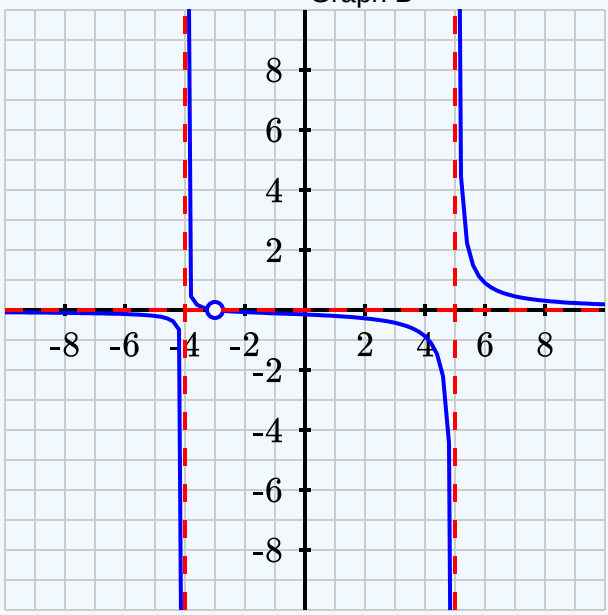
Activity 5.3.2.8

Consider the following six graphs of rational functions:

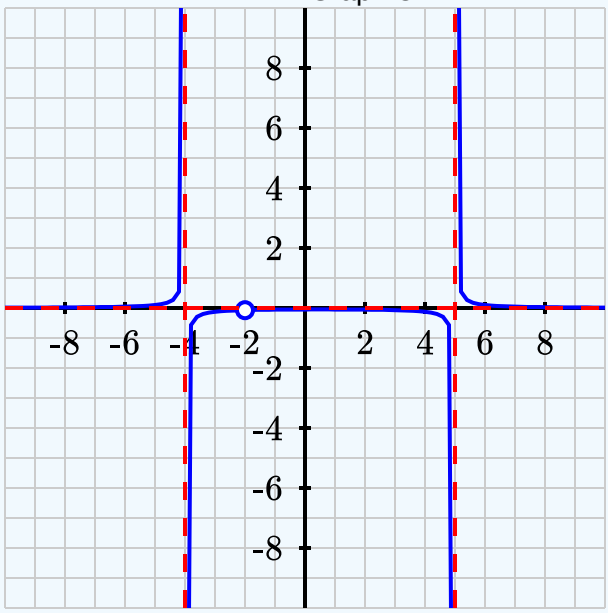
Graph A



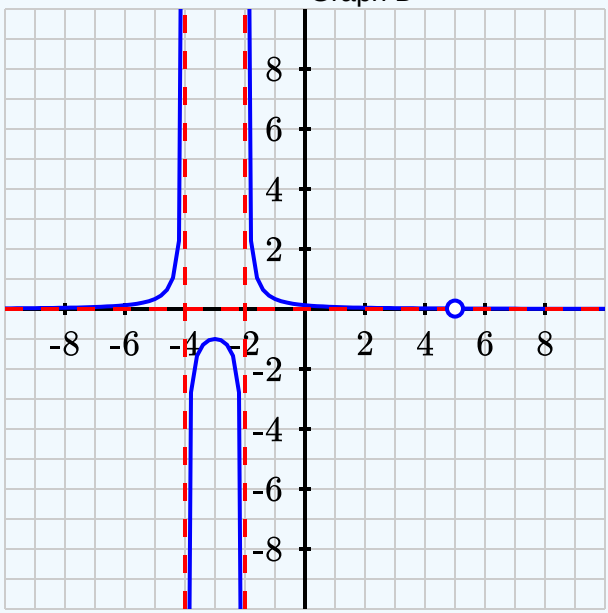
Graph B



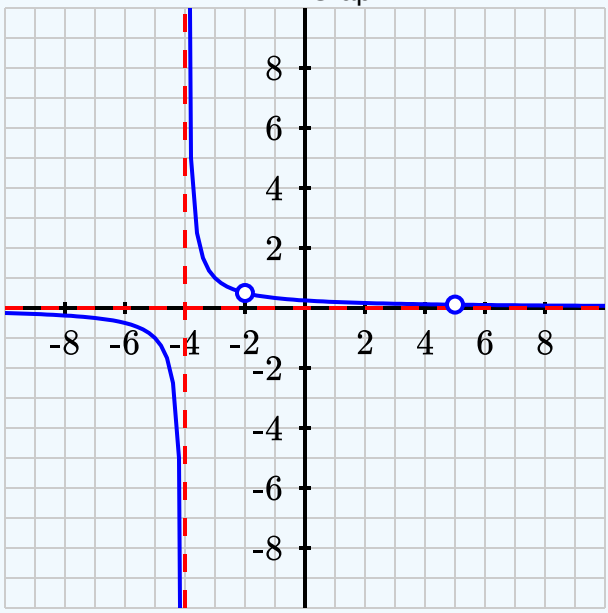
Graph C



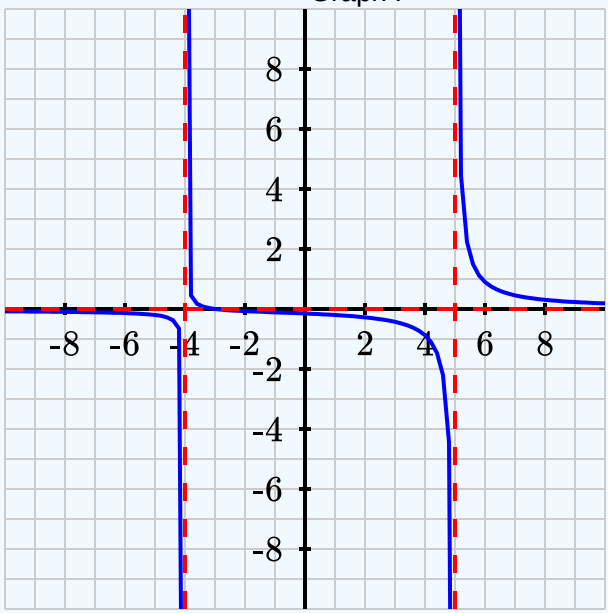
Graph D



Graph E



Graph F



(a)

Which of the graphs above represents the function $f(x) = \dfrac{x^2 + 5x + 6}{(x^2 - 3x - 10)(x + 4)}$?

Answer available at the end of this section.

(b)

Which of the graphs above represents the function $g(x) = \dfrac{x^2 - 3x - 10}{(x+2)(x^2 - x - 20)}$?

Answer available at the end of this section.

Activity 5.3.2.9

Let $f(x) = \dfrac{-(x-1)(x-4)}{2\,(x+3)^2(x-1)}$.

(a)

Find the roots of $f(x)$.

Answer available at the end of this section.

(b)

Find the y-intercept of the graph of $f(x)$.

Answer available at the end of this section.

(c)

Find any horizontal asymptotes on the graph of $f(x)$.

Answer available at the end of this section.

(d)

Find any vertical asymptotes on the graph of $f(x)$.

Answer available at the end of this section.

(e)

Find any holes on the graph of $f(x)$.

Answer available at the end of this section.

(f)

Sketch the graph of $f(x)$.

Answer available at the end of this section.

Activity 5.3.2.10

For each of the following rational functions, identify the location of any potential hole in the graph. Then, create a table of function values for input values near where the hole should be located. Use your work to decide whether or not the graph indeed has a hole, with written justification.

(a)

$r(x) = \dfrac{x^2 - 16}{x + 4}$

Answer available at the end of this section.

(b)

$$s(x) = \frac{(x-2)^2(x+3)}{x^2-5x+6}$$

Answer available at the end of this section.

(c)

$$u(x) = \frac{(x-2)^3(x+3)}{x^2-5x+6}$$

Answer available at the end of this section.

(d)

$$w(x) = \frac{(x-2)(x+3)}{(x^2-5x+6)^2}$$

Answer available at the end of this section.

Activity 5.3.2.11

Suppose you are given a function $r(x) = \frac{p(x)}{q(x)}$, and you know that $p(3) = 0$ and $q(3) = 0$. What can you conclude about the function $r(x)$ at $x = 3$?

A. $r(x)$ has a hole at $x = 3$.
B. $r(x)$ has an asymptote at $x = 3$.
C. $r(x)$ has either a hole or an asymptote at $x = 3$.
D. $r(x)$ has neither a hole nor an asymptote at $x = 3$.

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 5.3.2.1

Solve the following problems to test your understanding of rational functions, their asymptotes, intercepts, holes, and graphical representations.

(a) Consider the rational function $f(x) = \frac{2x^2-8}{x^2-x-6}$. Identify its domain, any x-intercepts and holes, and the equations of its vertical and horizontal asymptotes.

Answer available at the end of this section.

(b) For the rational function $g(x) = \frac{3x-5}{x-2}$, state the vertical asymptote and describe the behavior of the function as x approaches the vertical asymptote from both the left and the right using arrow notation.

Answer available at the end of this section.

(c) Find a rational function $y = R(x)$ of the form $R(x) = a \cdot \dfrac{x - z}{x - p}$ that has a vertical asymptote at $x = 4$, an x-intercept at $(1, 0)$, and a horizontal asymptote at $y = 3$.

Answer available at the end of this section.

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Answers

Activity 5.3.2.1

$r(1)$ and $r(3)$ do not exist. $r(2) = 0$ $r(4) = 2$

Activity 5.3.2.1

$r(1)$ and $r(3)$ give information about the domain. $r(2)$ gives information about the zeroes. $r(4)$ does not give information about either.

Activity 5.3.2.2

$r(x) = \dfrac{(x-2)(x-1)}{(x-3)(x-1)}$ The values not in the domain are zeros of the denominator.

Activity 5.3.2.2

x	$r(x)$
2	0
2.9	-9
2.99	-99
2.999	-999
3	undefined
3.001	1001
3.01	101
3.1	11

Activity 5.3.2.2

C and D

Activity 5.3.2.2

x	$r(x)$
0	0.667
0.9	0.524
0.99	0.502
0.999	0.500

1	undefined
1.001	0.499
1.01	0.497
1.1	0.474

Activity $5.3.2.2$

A

Activity $5.3.2.2$

The $x-1$ factor occurs in the numerator and denominator.

Activity $5.3.2.3$

C

Activity $5.3.2.3$

C

Activity $5.3.2.4$

B

Activity $5.3.2.4$

Numerator goes to 3 and denominator goes to 7.

Activity $5.3.2.4$

B

Activity $5.3.2.5$

B

Activity $5.3.2.5$

A

Activity $5.3.2.5$

E

Activity $5.3.2.6$

the horizontal asymptote is a non-zero number.

Activity $5.3.2.6$

there is a horizontal asymptote at 0.

Activity $5.3.2.6$

there is no horizontal asymptote.

Activity $5.3.2.7$

Graph A

Activity $5.3.2.7$

Graph E

Activity $5.3.2.8$

Graph A

Activity $5.3.2.8$

Graph E

Activity $5.3.2.9$

4

Activity $5.3.2.9$

$\left(0, \frac{2}{9}\right)$

Activity $5.3.2.9$

$y=0$

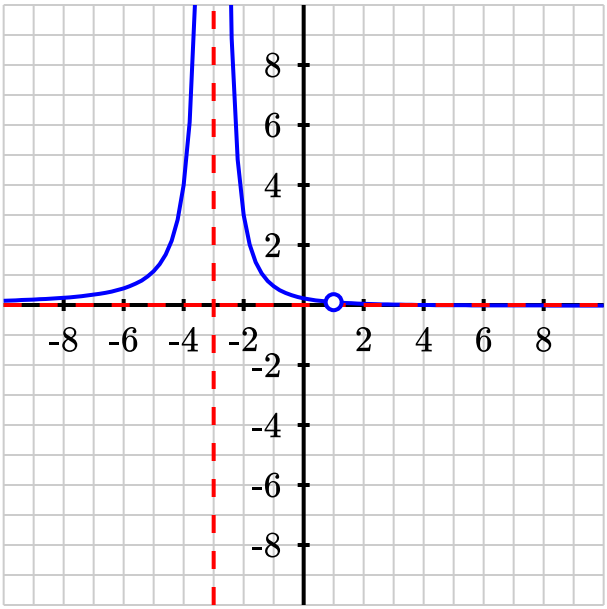
Activity $5.3.2.9$

$x=-3$.

Activity $5.3.2.9$

There is a hole at $\left(1, \frac{3}{32}\right)$.

Activity $5.3.2.9$



Activity 5.3.2.10

Hole when $x = -4$.

Activity 5.3.2.10

Hole when $x = 2$. The graph near $x = 2$ behaves like $x = 2$ is a root with odd multiplicity.

Activity 5.3.2.10

Hole when $x = 2$. The graph near $x = 2$ behaves like $x = 2$ is a root with even multiplicity.

Activity 5.3.2.10

Vertical asymptote at $x = 2$.

Activity 5.3.2.11

C

Check Your Understanding 5.3.2.1

First, factor the numerator and denominator completely:

$$f(x) = \frac{2(x-2)(x+2)}{(x-3)(x+2)} \tag{5.3.2.4}$$

- **Domain:** The denominator is zero when $x = 3$ or $x = -2$. Thus, the domain is all real numbers except 3 and -2: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.
- **Holes:** The factor $(x + 2)$ is common to both the numerator and the denominator. Thus, there is a hole at $x = -2$. To find the y-coordinate of the hole, evaluate the simplified function $g(x) = \frac{2(x-2)}{x-3}$ at $x = -2$:

$$g(-2) = \frac{2(-2-2)}{-2-3} = \frac{2(-4)}{-5} = \frac{8}{5} \tag{5.3.2.5}$$

  So, there is a hole at the point $\left(-2, \frac{8}{5}\right)$.
- **Intercepts:** Setting the numerator of the simplified function to zero gives $2(x-2) = 0 \implies x = 2$. The x-intercept is at $(2, 0)$. (Evaluating $f(0) = \frac{-8}{-6} = \frac{4}{3}$ gives the y-intercept at $\left(0, \frac{4}{3}\right)$).
- **Vertical Asymptote:** Setting the denominator of the simplified function to zero gives $x - 3 = 0 \implies x = 3$. The vertical asymptote is the line $x = 3$.
- **Horizontal Asymptote:** Since the numerator and denominator have the same degree (degree 2), the horizontal asymptote is the ratio of their leading coefficients: $y = \frac{2}{1} = 2$.

Check Your Understanding 5.3.2.1

The vertical asymptote is at $x = 2$. To describe the behavior near the asymptote:

- As $x \to 2^-$ (approaching 2 from values slightly smaller than 2, e.g., 1.99): the numerator $3x - 5 \approx 1 > 0$ and the denominator $x - 2 < 0$. Thus, the function values grow large in the negative direction:

$$\text{As } x \to 2^-, \quad g(x) \to -\infty \tag{5.3.2.6}$$

- As $x \to 2^+$ (approaching 2 from values slightly larger than 2, e.g., 2.01): the numerator $3x - 5 \approx 1 > 0$ and the denominator $x - 2 > 0$. Thus, the function values grow large in the positive direction:

$$\text{As } x \to 2^+, \quad g(x) \to \infty \tag{5.3.2.7}$$

Check Your Understanding 5.3.2.1

- A vertical asymptote at $x = 4$ implies the denominator factor is $(x - 4)$, so $p = 4$.
- An x-intercept at $(1, 0)$ implies the numerator factor is $(x - 1)$, so $z = 1$.
- This gives the function form: $R(x) = a \cdot \frac{x-1}{x-4}$.

- A horizontal asymptote at $y = 3$ means that as $x \to \pm\infty$, $R(x) \to 3$. For this form, the limit as $x \to \infty$ is a. Therefore, $a = 3$.
- Thus, the rational function is:

$$R(x) = 3 \cdot \frac{x-1}{x-4} = \frac{3x-3}{x-4} \tag{5.3.2.8}$$

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5.3.2.1: PR7 Online Exercises

PR7 — Find the domain and range, vertical and horizontal asymptotes, and intercepts of a rational function and use this information to sketch the graph.	PR7 CheckIt Exercises

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5.4: INTRODUCTION TO QUADRATIC INEQUALITIES (PR8)

5.4.1: Quadratic Inequalities (PR8)

5.4.1.1: PR8 Online Exercises

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5.4.1: Quadratic Inequalities (PR8)

Learning Outcomes:

- Solve quadratic inequalities and express the solution graphically and with interval notation.

Polynomial Inequalities (PR8)

Note

In Section 2.5 we learned how to solve quadratic equations. In this section, we use these skills to solve quadratic *inequalities*.

Definition: Quadratic inequality

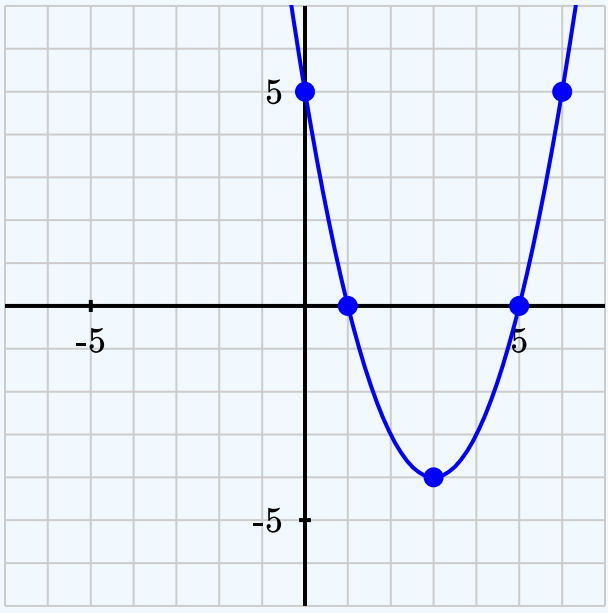
A **quadratic inequality** is an inequality that can be written in one of the following forms:

- $$ax^2 + bx + c > 0 \quad (5.4.1.1)$$
- $$ax^2 + bx + c < 0 \quad (5.4.1.2)$$
- $$ax^2 + bx + c \geq 0 \quad (5.4.1.3)$$
- $$ax^2 + bx + c \leq 0 \quad (5.4.1.4)$$

where a, b, and c are real numbers and $a \neq 0$.

Activity 5.4.1.1

Consider the graph shown below.



(a)

What is the value of $f(0)$?

A. 1
B. -4
C. 5
D. 0

Answer available at the end of this section.

(b)

What is the value of $f(3)$?

A. 1
B. -4
C. 5
D. 0

Answer available at the end of this section.

(c)

What is the value of $f(5)$?

A. 1
B. -4
C. 5
D. 0

Answer available at the end of this section.

(d)

What is the value of $f(6)$?

A. 1
B. -4
C. 5
D. 0

Answer available at the end of this section.

(e)

Notice from parts (a) - (d), that $f(x)$ can either be positive, negative, or zero depending on the value of x. What are the x -intercept(s) of $f(x)$?

A. 1
B. -4
C. 5
D. 0

Answer available at the end of this section.

(f)

Based on what you see on the graph (and your solutions to parts (a) - (d)), for what values of x would $f(x)$ be positive?

A. $x < 1$
B. $x > 1$
C. $x > 5$

D. $x < 5$
E. $1 < x < 5$

Answer available at the end of this section.

(g)

Now use interval notation to express where $x^2 - 6x + 5 > 0$.

A. $(\infty, 1] \cup [5, \infty)$
B. $[1, 5]$
C. $(\infty, 1) \cup (5, \infty)$
D. $(1, 5)$

Answer available at the end of this section.

Note

From Activity $5.4.1.1$, we saw that a function could have y-values that are positive, negative, or zero, which can then help us find values of x to solve inequalities. Let's now look at how we can solve inequalities using algebra.

Activity $5.4.1.2$

Use the Definition of Quadratic inequality and your knowledge of quadratic equations (see Section 2.5) to help answer the following questions.

(a)

Which of the following inequalities are quadratic inequalities?

A. $(x - 1)(x + 3) < 7$
B. $-4x + 3 \geq 10$
C. $2x^2 - 7x + 3 > 0$
D. $5x - 1 \leq 4x$

Answer available at the end of this section.

(b)

Given the quadratic equation, $x^2 - x - 6 = 0$, determine whether -1 is a solution.

Answer available at the end of this section.

(c)

Given the quadratic inequality, $x^2 - x - 6 \leq 0$, determine whether -1 is a solution.

Answer available at the end of this section.

(d)

Are there any other values of x that would satisfy the inequality, $x^2 - x - 6 \leq 0$?

Answer available at the end of this section.

Note

Notice from Activity 5.4.1.2, a solution to a quadratic inequality is a real number that will produce a true statement when substituted for the variable. Quadratic inequalities will often have an infinite number of solutions, which we will express in interval notation. However, it is also possible for the inequality to have no solution.

Activity 5.4.1.3

Let's look at how we can algebraically determine the solutions to a quadratic inequality using a number line. Consider the quadratic inequality

$$x^2 - 4x - 32 \geq 0 \tag{5.4.1.5}$$

.

(a)

What is the factored form of the quadratic (left-hand side of the inequality)?

A. $(x - 16)(x + 2)$
B. $(x - 4)(x + 8)$
C. $(x + 4)(x - 8)$
D. $(x - 4)(x - 8)$

Answer available at the end of this section.

(b)

Rewrite the quadratic inequality with the answer you got from part (a). What values of x would give you 0?

A. $x = 16, -2$
B. $x = 4, -8$
C. $x = 4, 8$
D. $x = -4, 8$

Hint:

Refer back to Section 2.5 and think about this as a quadratic equation (equal to 0).

Answer available at the end of this section.

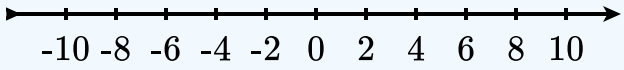
(c)

These solutions (that you got in part (b)) correspond to all of the x-intercepts of the graph, and are the only spots where the y-values on either side might change from positive to negative or negative to positive. So, with our two x-intercepts, we have divided our graph into three intervals. What are these three intervals?

Answer available at the end of this section.

(d)

Notice that the x-intercepts are solutions to the quadratic equation $x^2 - 4x - 32 = 0$. How should we mark these values of x on a number line?



Answer available at the end of this section.

(e)

Choose a value of x within each interval and substitute that value into the equation $x^2 - 4x - 32$ (or the factored form you found in part (a)). If you get a positive value, place a "+" sign above that region on the number line. Similarly, if you get a negative value, place a "-" sign above that region on the number line.

Answer available at the end of this section.

(f)

Using the number line and what you determined in part (e), shade in areas on the number line that satisfies the quadratic inequality $x^2 - 4x - 32 \geq 0$?

Answer available at the end of this section.

(g)

Using your graph, express the solution of the inequality $x^2 - 4x - 32 \geq 0$ in interval notation.

A. $(-4, 8)$
B. $[-4, 8]$
C. $(-\infty, -4) \cup (8, \infty)$
D. $(-\infty, -4] \cup [8, \infty)$

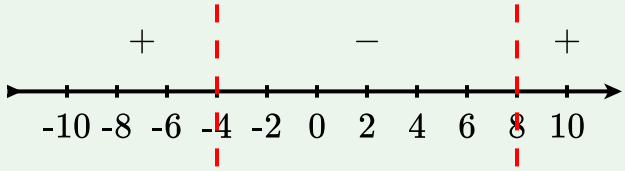
Answer available at the end of this section.

Note

Notice in Activity $5.4.1.3$, we had to begin with the solutions to the quadratic equation to determine regions of the number line to then test values of x that satisfy the inequality that was given. Creating a visual can help in determining the solutions to inequalities, especially when there are many solutions.

Definition: Sign chart

A **sign chart** is a number line representing the x-axis that shows where a function is positive or negative by using a '+' or a '-' sign to indicate which regions are positive or negative. For example, a sign chart for $f(x) = x^2 - 4x - 32$ (also seen in Activity $5.4.1.3$) is shown below.



A sign chart for the function $f(x) = x^2 - 4x - 32$.

Definition: Critical points and Boundary points

In Activity 5.4.1.3, we saw that when we place the solutions to the quadratic equation, $x^2 - 4x - 32 = 0$, on the number line, that it divided the number line into three regions.

The values in the domain of a function that separate regions that produce positive or negative results are called **critical points** or **boundary points**. These values bound the regions where the function is positive or negative.

Activity 5.4.1.4

For this activity, consider the quadratic inequality

$$2x^2 - 28 < 10x \tag{5.4.1.6}$$

.

(a)

Factor the quadratic equation $2x^2 - 28 = 10x$ using methods discussed in Section 2.5.

A. $(x - 2)(x + 7)$
B. $(2x + 4)(x - 7)$
C. $(x + 2)(x - 7)$
D. $2(x^2 - 14)$

Answer available at the end of this section.

(b)

Use the factors you found in part (a) to find the critical points.

A. $x = -2, 7$
B. $x = \sqrt{14}$
C. $x = -7, 2$
D. $x = 2, -12$

Answer available at the end of this section.

(c)

Plot the critical points you found in part (b) on a number line. Determine whether those should be included in your solution based on the inequality that was given.

Answer available at the end of this section.

(d)

Test values of x in each region that was created by the critical points and create a sign chart to show which regions are positive and negative.

Answer available at the end of this section.

(e)

Using the number line and what you determined in part (d), shade in areas on the number line that satisfies the quadratic inequality $2x^2 - 28 < 10x$?

Answer available at the end of this section.

(f)

Write your solution using interval notation.

A. $(-\infty, -2) \cup (7, \infty)$
B. $(-\infty, -7) \cup (2, \infty)$
C. $(-7, 2)$
D. $(-2, 7)$

Answer available at the end of this section.

Note

When solving quadratic inequalities, be sure to get all your terms to one side of the inequality first! Then, apply the methods we learned in Section 2.5 to determine the critical points (boundary points). From there, you can then create your sign chart to help determine the solution to the inequality.

Activity 5.4.1.5

For each of the following, determine the critical points and use a number line (and sign chart) to then find the solutions. Write your answers in interval notation.

(a)

$x^2 - 1 < 0$

Answer available at the end of this section.

(b)

$x^2 - 5x - 6 \geq 0$

Answer available at the end of this section.

(c)

$2x^2 - 7x + 3 > 0$

Answer available at the end of this section.

(d)

$4x^2 - 6x - 9 < x^2$

Answer available at the end of this section.

(e)

$-2x^2 - 10x - 10 \geq 6x + 20$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 5.4.1.1

Solve the following polynomial inequalities and write your solutions in interval notation.

(a) Solve the quadratic inequality:

$$x^2 - 2x - 8 \leq 0 \tag{5.4.1.7}$$

Answer available at the end of this section.

(b) Solve the polynomial inequality:

$$(x-1)^2(x+3)(x-5) > 0 \tag{5.4.1.8}$$

Answer available at the end of this section.

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Answers

Activity 5.4.1.1
C

Activity 5.4.1.1
B

Activity 5.4.1.1
D

Activity 5.4.1.1
C

Activity 5.4.1.1
A and C

Activity 5.4.1.1
A and C

Activity 5.4.1.1
C

Activity 5.4.1.2
A and C

Activity 5.4.1.2
By plugging in -1 into the quadratic equation, $x^2 - x - 6 = 0$, students should see that it is NOT a solution.
$(-1)^2 - (-1) - 6 = -4$

Activity 5.4.1.2
By plugging in -1 into the quadratic inequality, $x^2 - x - 6 \leq 0$, students should see that it IS a solution as it produces a true statement.
$(-1)^2 - (-1) - 6 = -4$ and $-4 \leq 0$.

Activity 5.4.1.2
Any value of x from -2 to 3 (including both -2 and 3) will satisfy the inequality. In interval notation, the solution is $[-2, 3]$ (although students may not know this at this point in the section).

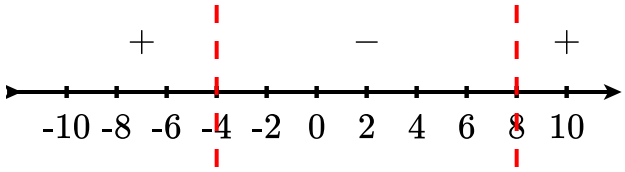
Activity 5.4.1.3
C

Activity 5.4.1.3
D

Activity 5.4.1.3
$(-\infty, -4)$, $(-4, 8)$, and $(8, \infty)$

Activity 5.4.1.3
From *(see related activity)*, students should recall that because it is a greater than or equal to statement, that the x-intercepts should be included on the number (with a bracket or solid dot).

Activity 5.4.1.3



Activity 5.4.1.3

Graph: A numberline from -10 to 10. The portion to the left of -4 is shaded, with a] at -4. The portion to the right of 8 is shaded, with a [at 8.

Activity 5.4.1.3

D

Activity 5.4.1.4

B and C (although B is not completely factored)

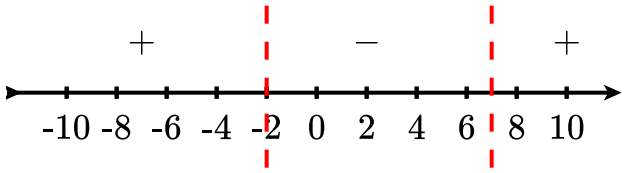
Activity 5.4.1.4

A

Activity 5.4.1.4

Because the inequality is “less than”, the critical points (or boundary points) should not be included in the solution.

Activity 5.4.1.4



Activity 5.4.1.4

Graph: A number line from -10 to 10. The portion between -2 and 7 is shaded, with a (at -2 and a) at 7.

Activity 5.4.1.4

D

Activity 5.4.1.5

$(-1, 1)$

Activity 5.4.1.5

$(-\infty, -1] \cup [6, \infty)$

Activity 5.4.1.5

$\left(-\infty, \frac{1}{2}\right] \cup [3, \infty)$

Activity 5.4.1.5

$(-1, 3)$

Activity 5.4.1.5

$[-5, -3]$

Check Your Understanding 5.4.1.1

First, find the critical values by setting the quadratic expression to zero and solving by factoring:

$$x^2 - 2x - 8 = 0 \implies (x-4)(x+2) = 0 \implies x = 4 \text{ or } x = -2 \tag{5.4.1.9}$$

These critical values divide the real number line into three intervals: $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$.

We choose a test point from each interval to determine the sign of the expression in that interval:

- For the interval $(-\infty, -2)$, test $x = -3$:

$$(-3)^2 - 2(-3) - 8 = 9 + 6 - 8 = 7 > 0 \quad \text{(Positive)} \tag{5.4.1.10}$$

- For the interval $(-2, 4)$, test $x = 0$:

$$(0)^2 - 2(0) - 8 = -8 < 0 \quad \text{(Negative)} \tag{5.4.1.11}$$

- For the interval $(4, \infty)$, test $x = 5$:

$$(5)^2 - 2(5) - 8 = 25 - 10 - 8 = 7 > 0 \quad \text{(Positive)} \tag{5.4.1.12}$$

Since the inequality is $x^2 - 2x - 8 \leq 0$, we want the interval where the expression is negative or zero. The endpoints $x = -2$ and $x = 4$ are included. The solution set in interval notation is:

$$[-2, 4] \tag{5.4.1.13}$$

Check Your Understanding $5.4.1.1$

Identify the critical values by setting the polynomial to zero: $x = 1$, $x = -3$, and $x = 5$.

These critical values divide the real number line into four intervals: $(-\infty, -3)$, $(-3, 1)$, $(1, 5)$, and $(5, \infty)$.

Note that the factor $(x - 1)^2$ has an even exponent, meaning the expression will not change signs across $x = 1$.

We choose a test point from each interval to determine the sign of the expression:

- For $(-\infty, -3)$, test $x = -4$:

$$(-4 - 1)^2(-4 + 3)(-4 - 5) = (25)(-1)(-9) = 225 > 0 \quad \text{(Positive)} \tag{5.4.1.14}$$

- For $(-3, 1)$, test $x = 0$:

$$(0 - 1)^2(0 + 3)(0 - 5) = (1)(3)(-5) = -15 < 0 \quad \text{(Negative)} \tag{5.4.1.15}$$

- For $(1, 5)$, test $x = 2$:

$$(2 - 1)^2(2 + 3)(2 - 5) = (1)(5)(-3) = -15 < 0 \quad \text{(Negative)} \tag{5.4.1.16}$$

- For $(5, \infty)$, test $x = 6$:

$$(6 - 1)^2(6 + 3)(6 - 5) = (25)(9)(1) = 225 > 0 \quad \text{(Positive)} \tag{5.4.1.17}$$

Since the inequality is strictly greater than zero (> 0), we want the positive intervals and we exclude the critical values (which make the expression 0). The solution set in interval notation is:

$$(-\infty, -3) \cup (5, \infty) \tag{5.4.1.18}$$

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5.4.1.1: PR8 Online Exercises

PR8 — Solve quadratic inequalities and express the solution graphically and with interval notation.	PR8 CheckIt Exercises

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5.5: INTRODUCTION TO RATIONAL INEQUALITIES (PR9)

5.5.1: Rational Inequalities (PR9)

5.5.1.1: PR9 Online Exercises

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5.5.1: Rational Inequalities (PR9)

Learning Outcomes:

- Solve rational inequalities and express the solution graphically and using interval notation.

Rational Inequalities (PR9)

Note

In *5.3.1* we learned how to solve rational equations. In this section, we use these skills to solve rational *inequalities*.

Definition: Rational inequality

A **rational inequality** is an inequality that contains a rational expression. Some examples of a rational inequality are shown below:

$$\frac{3}{2x} > 1 \tag{5.5.1.1}$$

$$\frac{2x-3}{x-6} \leq x \tag{5.5.1.2}$$

Activity 5.5.1.1

Use the Definition of Rational inequality and your knowledge of quadratic equations (see *(see related activity)*) to help answer the following questions.

(a)

Which of the following inequalities are rational inequalities?

A. $\frac{4x+7}{5} < -1$
B. $\frac{4x-1}{3x+2} \geq 0$
C. $2x^2 - 7x > -3$
D. $5x - 1 \leq 4x$

Answer available at the end of this section.

(b)

Given the rational equation, $\frac{x-1}{x+3} = 0$, determine whether 1 is a solution.

Answer available at the end of this section.

(c)

Given the rational equation, $\frac{x-1}{x+3} = 0$, determine whether -4 is a solution.

Answer available at the end of this section.

(d)

Given the rational inequality, $\frac{x-1}{x+3} \geq 0$, determine whether -4 is a solution.

Answer available at the end of this section.

(e)

Given the rational equation, $\frac{x-1}{x+3} = 0$, determine whether -3 is a solution.

Answer available at the end of this section.

(f)

Are there any other values of x that would satisfy the inequality, $\frac{x-1}{x+3} \geq 0$?

Answer available at the end of this section.

Note

In Activity $5.5.1.1$, we saw that there can be solutions to the rational inequality $\frac{x-1}{x+3} \geq 0$ that does NOT satisfy the rational equation $\frac{x-1}{x+3} = 0$. Just like with linear and quadratic inequalities, we can have many solutions that can satisfy rational inequalities. We do, however, need to be careful about our critical points, as we will see in the next activity.

Activity $5.5.1.2$

For this activity, let's consider the inequality

$$\frac{x-1}{x+3} \geq 0 \tag{5.5.1.3}$$

.

(a)

Focus on the rational equation $\frac{x-1}{x+3}$ (the left-hand side of the inequality). What value(s) should be excluded as possible solutions to that rational expression?

A. 1
B. -1
C. -3
D. 3

Answer available at the end of this section.

(b)

The value you got in part (a) is a critical value for this rational expression because it is a value of x where the rational expression is undefined. Critical points for rational expressions also include values that make the rational expression equal to 0. What value of x would make this rational expression equal to 0?

A. 1
B. -1
C. -3
D. 3

Answer available at the end of this section.

(c)

Use the critical points you found in parts (a) and (b) and plot them on a number line. Determine whether those should be included in your solution based on the inequality that was given.

Answer available at the end of this section.

(d)

Notice that your number line has now been divided into three regions. Test values of x in each region that was created by the critical points and create a sign chart to show which regions are positive and negative.

Hint:

Make sure you substitute values into BOTH xs in the rational expression.

Answer available at the end of this section.

(e)

Using the number line and what you determined in part (d), shade in areas on the number line that satisfies the quadratic inequality $\frac{x-1}{x+3} \geq 0$?

Answer available at the end of this section.

(f)

Write your solution using interval notation.

A. $(-\infty, -3) \cup (1, \infty)$
B. $(-\infty, -3) \cup [1, \infty)$
C. $(-3, 1)$
D. $(-3, 1)$

Answer available at the end of this section.

Note

When looking for critical points of rational inequalities, remember to look for points where the rational expression will be zero or undefined.

Activity 5.5.1.3

For each of the following, determine the critical points and use a number line (and sign chart) to then find the solutions. Write your answers in interval notation.

(a)

$\frac{x+5}{x-4} \leq 0$

Answer available at the end of this section.

(b)

$\frac{(x+3)(x+5)}{x+2} > 0$

Hint:

Notice for this rational inequality there are 3 critical points! Be sure to test all four regions on your number line.

Answer available at the end of this section.

Activity 5.5.1.4

When solving an inequality, the goal is to first get x (or whatever the variable is) on its own on one side of the inequality sign and 0 on the other side. To do this, we have to be careful of the actions we take as some actions can change the direction of the inequality. Let's revisit some of the actions we have taken previously and see how we can apply these same actions to solve the rational inequality

$$\frac{3x-10}{x-4} > 2 \tag{5.5.1.4}$$

.

(a)

When solving rational equations in *(see related activity)*, we often started by "clearing the fractions" by multiplying the denominator to both sides of the equation. Suppose we have the equation $\frac{3x-10}{x-4} = 2$. What can we multiply each term by that will clear the fraction?

A. $x+4$
B. $3x-10$
C. $x-4$
D. $x-2$

Answer available at the end of this section.

(b)

Multiply each term by the expression you chose and simplify. Which of the following linear equations does the rational equation simplify to?

A. $(x+4)(3x-10) = 2(x+4)$
B. $(3x-10) = 2(x-4)$
C. $2(3x-10) = (x-4)$
D. $(3x-10) = 2(x+4)$

Answer available at the end of this section.

(c)

What values of x would make the denominator of the rational expression positive?

Answer available at the end of this section.

(d)

What values of x would make the denominator of the rational expression negative?

Answer available at the end of this section.

(e)

Now suppose we applied the same actions as we did in parts (a) and (b) to the rational inequality $\frac{3x-10}{x-4} > 2$. If the denominator of the rational expression $\frac{3x-10}{x-4}$ was positive, how would you write the inequality as a linear inequality?

A. $(x+4)(3x-10) > 2(x+4)$

B. $(3x-10) > 2(x-4)$
C. $2(3x-10) > (x-4)$
D. $(3x-10) > (x+4)$

Answer available at the end of this section.

(f)

If the denominator of the rational expression $\frac{3x-10}{x-4}$ was negative, how would you write the inequality as a linear inequality?

A. $(x+4)(3x-10) < 2(x+4)$
B. $(3x-10) > 2(x-4)$
C. $2(3x-10) < (x-4)$
D. $(3x-10) < 2(x-4)$

Hint:

What happens to the inequality symbol when multiplying or dividing by a negative number?

Answer available at the end of this section.

(g)

Look at your answers in parts (e) and (f). Which inequality would help us solve $\frac{3x-10}{x-4} > 2$.

Answer available at the end of this section.

Note

In Activity $5.5.1.4$, we saw that it is not clear whether the denominator of a rational expression would yield a positive or negative value. Because we do not actually know whether the denominator is positive or negative, we cannot multiply the denominator to "clear the fractions" as we did before when solving rational equations. In the next activity, we will look at how we can solve rational inequalities.

Activity $5.5.1.5$

Let's reconsider the rational inequality

$$\frac{3x-10}{x-4} > 2 \tag{5.5.1.5}$$

.

(a)

From Activity $5.5.1.4$, we saw that we cannot multiply by the denominator to "clear the fraction." Our goal, however, is still to get all the xs to one side and 0 on the other side. What action can we take to get 0 on one side?

A. Add 2 to each side
B. Subtract 2 from each side
C. Multiply by 2 on each side
D. Multiply by 4 on each side

Answer available at the end of this section.

(b)

What rational inequality do you now have?

A. $\frac{3x-10}{x-4}+2<0$
B. $\frac{3x-10}{x-4}-2<0$
C. $\frac{3x-10}{x-4}+2>0$
D. $\frac{3x-10}{x-4}-2>0$

Answer available at the end of this section.

(c)

What is the common denominator of $\frac{3x-10}{x-4}$ and 2?

A. $x+4$
B. $x-4$
C. $3x-10$
D. $3x+10$

Answer available at the end of this section.

(d)

Multiply 2 by the common denominator and simplify the rational expression. What rational inequality do you now have?

A. $\frac{x-18}{x-4}>0$
B. $\frac{x-2}{x-4}>0$
C. $\frac{x-2}{x-4}>0$
D. $\frac{2x-6}{x-4}>0$

Answer available at the end of this section.

(e)

Use the inequality you got in part (d) to determine the critical points.

Hint:

Remember that for a rational inequality, the critical points are the values of x that make the rational expression equal to 0 or undefined.

Answer available at the end of this section.

(f)

Use the critical points you found in part (e) and plot them on a number line. Determine whether those should be included in your solution based on the inequality that was given.

Answer available at the end of this section.

(g)

Use the critical points to create regions on the number line for you to test values of x and create a sign chart to show which regions are positive and negative.

Answer available at the end of this section.

(h)

Using the number line and what you determined in part (d), shade in areas on the number line that satisfies the rational inequality $\frac{3x-10}{x-4} > 2$?

Answer available at the end of this section.

(i)

Write your solution using interval notation.

A. $(-\infty, 2) \cup (4, \infty)$
B. $(-\infty, 2] \cup [4, \infty)$
C. $(2, 4)$
D. $[2, 4]$

Answer available at the end of this section.

Activity 5.5.1.6

Consider the rational inequality

$$\frac{4x+3}{x+2} > x \tag{5.5.1.6}$$

.

(a)

Subtract x on both sides of the inequality to get 0 on one side. Simplify $\frac{4x+3}{x+2} - x$ into a single rational expression.

A. $\frac{4x+3}{x+2}$
B. $\frac{3x+3}{x+2}$
C. $\frac{x^2+6x+3}{x+2}$
D. $\frac{-x^2+2x+3}{x+2}$

Answer available at the end of this section.

(b)

What are the critical points of this rational inequality?

Hint:

Factor the numerator.

Answer available at the end of this section.

(c)

Use the critical points you found in part (b) and plot them on a number line. Determine whether those should be included in your solution based on the inequality that was given.

Answer available at the end of this section.

(d)

Use the critical points to create regions on the number line for you to test values of x and create a sign chart to show which regions are positive and negative.

Answer available at the end of this section.

(e)

Using the number line and what you determined in part (d), shade in areas on the number line that satisfies the rational inequality $\frac{4x+3}{x+2} > x$?

Answer available at the end of this section.

(f)

How can we express the answers to part (e) for the rational inequality using interval notation?

A. $(1, 3)$
B. $(-2, -1) \cup (3, \infty)$
C. $(-2, -1)$
D. $(-\infty, -2) \cup (-1, 3)$

Answer available at the end of this section.

Activity 5.5.1.7

For each of the following, solve the rational inequality by bringing all x terms to one side (and 0 on the other) and simplify the rational expression. Then, use critical points and the sign chart/number line to determine the solution.

(a)

$\frac{x+68}{x+8} \geq 5$

Answer available at the end of this section.

(b)

$\frac{x^2-4x+4}{x-2} > 1$

Answer available at the end of this section.

(c)

$\frac{x-8}{x} \leq 3 - x$

Answer available at the end of this section.

(d)

$$\frac{x+8}{x-2} \leq \frac{x+10}{x+5}$$

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can apply the concepts before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 5.5.1.1

Solve the following rational inequalities and write your solutions in interval notation.

(a) Solve the rational inequality:

$$\frac{x-3}{x+4} \geq 0 \tag{5.5.1.7}$$

Answer available at the end of this section.

(b) Solve the rational inequality:

$$\frac{2x+5}{x-1} < 3 \tag{5.5.1.8}$$

Answer available at the end of this section.

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Answers

Activity 5.5.1.1

B

Activity 5.5.1.1

By plugging in 1 into the rational equation, $\frac{x-1}{x+3} = 0$, students should see that it IS a solution.

$\frac{1-1}{1+3} = \frac{0}{4} = 0$

Activity 5.5.1.1

By plugging in -4 into the rational equation, $\frac{x-1}{x+3} = 0$, students should see that it is NOT a solution.

$\frac{-4-1}{-4+3} = \frac{-5}{-1} = 5$, which is NOT equal to 0.

Activity 5.5.1.1

By plugging in -4 into the rational inequality, $\frac{x-1}{x+3} \geq 0$, students should see that it IS a solution.

$\frac{-4-1}{-4+3} = \frac{-5}{-1} = 5$, which IS a solution to the inequality because it is greater than 0.

Activity 5.5.1.1

By plugging in -3 into the rational inequality, $\frac{x-1}{x+3} \geq 0$, students should see that it is NOT a solution.

$\frac{-3-1}{-3+3} = \frac{-4}{0}$, which is undefined.

Activity 5.5.1.1

Any value of x less than -3 or greater than or equal to 1 will satisfy the inequality. In interval notation, the solution is $(-\infty, -3) \cup [1, \infty)$ (although students may not know this at this point in the section).

Activity 5.5.1.2

C

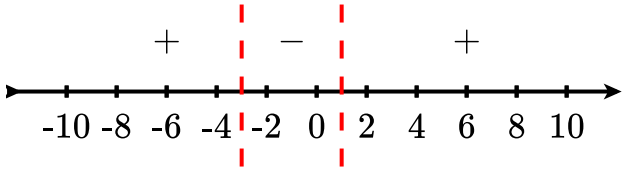
Activity 5.5.1.2

A

Activity 5.5.1.2

The value 1 IS a solution to the inequality because although it makes the rational expression equal to 0, the inequality states “greater than or equal to 0.” The other critical value, -3, however should be in the solution as it makes the rational expression undefined. So, on a number line, -3 should have a parentheses and 1 should have a bracket.

Activity 5.5.1.2



Activity 5.5.1.2

Graph: A number line from -10 to 10. The portion to the left of -3 is shaded, with a) at -3. The portion to the right of 1 is shaded, with a [at 1.

Activity 5.5.1.2

B

Activity 5.5.1.3

$[-5, 4)$

Activity 5.5.1.3

$(-5, -3)$

Activity 5.5.1.4

C

Activity 5.5.1.4

B

Activity 5.5.1.4

Any value of x greater than or equal to 4.

Activity 5.5.1.4

Any value of x less than 4.

Activity 5.5.1.4

B

Activity 5.5.1.4

D

Activity 5.5.1.4

Students should see that it is not clear which linear inequality to use because there are values of x that can make the denominator of the rational expression positive or negative.

Activity 5.5.1.5

B

Activity 5.5.1.5

D

Activity 5.5.1.5

B

Activity 5.5.1.5

C

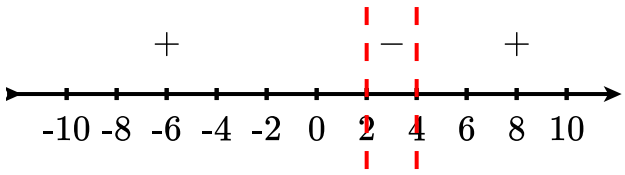
Activity 5.5.1.5

2 and 4

Activity 5.5.1.5

Both values of x should be excluded from the solution. On the number line, they would have parentheses (or open circles).

Activity 5.5.1.5



Activity 5.5.1.5

Graph: A number line from -10 to 10. The portion to the left of 2 is shaded, with a) at 2. The portion to the right of 4 is shaded, with a (at 4.

Activity 5.5.1.5

D

Activity 5.5.1.6

D

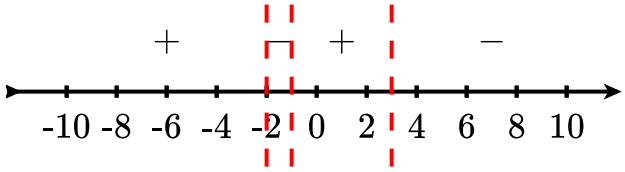
Activity 5.5.1.6

$x = -2, -1, 3$

Activity 5.5.1.6

All critical points are not included in the solution.

Activity 5.5.1.6



Activity 5.5.1.6

Graph: A number line from -10 to 10. The portion to the left of -2 is shaded, with a) at 2. The portion between -1 and 3 is shaded with a (at -1 and a) at 3.

Activity 5.5.1.6

A

Activity 5.5.1.7

$(-8, 7]$

Activity 5.5.1.7

$(3, \infty)$

Activity 5.5.1.7

$(-\infty, -2] \cup (0, 4]$

Activity 5.5.1.7

$(-\infty, -12] \cup (-5, 2)$

Check Your Understanding 5.5.1.1

First, identify the critical values by setting the numerator and denominator to zero:

- Numerator: $x - 3 = 0 \implies x = 3$
- Denominator: $x + 4 = 0 \implies x = -4$ (note: $x = -4$ is excluded from the domain)

These critical values divide the real number line into three intervals: $(-\infty, -4)$, $(-4, 3)$, and $(3, \infty)$.

We choose a test point from each interval to check the sign of the rational expression:

- For the interval $(-\infty, -4)$, test $x = -5$:

$$\frac{-5-3}{-5+4} = \frac{-8}{-1} = 8 > 0 \quad \text{(Positive)} \tag{5.5.1.9}$$

- For the interval $(-4, 3)$, test $x = 0$:

$$\frac{0-3}{0+4} = \frac{-3}{4} = -0.75 < 0 \quad \text{(Negative)} \tag{5.5.1.10}$$

- For the interval $(3, \infty)$, test $x = 4$:

$$\frac{4-3}{4+4} = \frac{1}{8} = 0.125 > 0 \quad \text{(Positive)} \tag{5.5.1.11}$$

Since the inequality is ≥ 0, we want the positive intervals. The critical value from the numerator ($x = 3$) is included (bracket), but the critical value from the denominator ($x = -4$) is excluded (parenthesis) because division by zero is undefined. The solution set is:

$$(-\infty, -4) \cup [3, \infty) \tag{5.5.1.12}$$

Check Your Understanding 5.5.1.1

First, subtract 3 from both sides to set one side to zero:

$$\frac{2x+5}{x-1} - 3 < 0 \tag{5.5.1.13}$$

Find a common denominator to combine into a single rational expression:

$$\frac{2x+5-3(x-1)}{x-1} < 0 \implies \frac{2x+5-3x+3}{x-1} < 0 \implies \frac{-x+8}{x-1} < 0 \tag{5.5.1.14}$$

Identify the critical values by setting the numerator and denominator to zero:

- Numerator: $-x + 8 = 0 \implies x = 8$
- Denominator: $x - 1 = 0 \implies x = 1$ (note: $x = 1$ is excluded from the domain)

These critical values divide the real number line into three intervals: $(-\infty, 1)$, $(1, 8)$, and $(8, \infty)$.

We choose a test point from each interval to check the sign of the rational expression $\frac{-x+8}{x-1}$:

- For $(-\infty, 1)$, test $x = 0$:

$$\frac{-0+8}{0-1} = \frac{8}{-1} = -8 < 0 \quad \text{(Negative)} \tag{5.5.1.15}$$

- For $(1, 8)$, test $x = 2$:

$$\frac{-2+8}{2-1} = \frac{6}{1} = 6 > 0 \quad \text{(Positive)} \tag{5.5.1.16}$$

- For $(8, \infty)$, test $x = 9$:

$$\frac{-9+8}{9-1} = \frac{-1}{8} = -0.125 < 0 \quad \text{(Negative)} \tag{5.5.1.17}$$

Since the inequality is strictly less than zero (< 0), we want the negative intervals and exclude both critical values. The solution set in interval notation is:

$$(-\infty, 1) \cup (8, \infty) \tag{5.5.1.18}$$

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5.5.1.1: PR9 Online Exercises

PR9 — Solve rational inequalities and express the solution graphically and using interval notation.	PR9 CheckIt Exercises

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6: FOUNDATIONAL FLUENCY

6.1: Foundational Skill Linear Equations EQ1

6.2: Foundational Skill Linear Applications EQ2

6.3: Foundational Skill Distance and Midpoint EQ3

6.4: Foundational Skill Absolute Value Equations EQ4

6.5: Foundational Skill Quadratic Equations EQ5

6.6: Foundational Skill Linear Systems LF6

6.7: Foundational Trig Ratios Skill TR3

6.8: Foundational Skill Special Rigth TrianglesTR4

6.9: Foundational Skill Similar Triangles (ST)

6.10: Foundational Skill Exponents (EX)

6.11: Foundational Skill Algebraic Fractions (AF)

6.12: Foundational Skill EQ3 Advanced Math (AM)

6.13: Introduction to Right Triangles

6.13.1: Trigonometric Ratios (TR3)

6.13.1.1: TR3 Online Exercises

6.13.2: Special Right Triangles (TR4)

6.13.2.1: TR4 Online Exercises

The Foundational Gateway builds fluency in prerequisite computational skills you'll lean thoughout this course and into Calculus.

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6.1: Foundational Skill Linear Equations EQ1

EQ1 — Solve linear equations in one variable. Solve linear inequalities in one variable and express the solution graphically and using interval notation.	EQ1 CheckIt Exercises

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6.2: Foundational Skill Linear Applications EQ2

EQ2 — Solve application problems involving linear equations.	EQ2 CheckIt Exercises

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6.3: Foundational Skill Distance and Midpoint EQ3

EQ3 — Given two points, determine the distance between them and the midpoint of the line segment connecting them.	EQ3 CheckIt Exercises

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6.4: Foundational Skill Absolute Value Equations EQ4

EQ4 — Solve linear equations involving an absolute value. Solve linear inequalities involving absolute values and express the answers graphically and using interval notation..	EQ4 CheckIt Exercises

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6.5: Foundational Skill Quadratic Equations EQ5

EQ5 — Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.	EQ5 CheckIt Exercises

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6.6: Foundational Skill Linear Systems LF6

LF6 — Solve quadratic equations using factoring, the square root property, completing the square, and the quadratic formula and express these answers in exact form.	LF6 CheckIt Exercises

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6.7: Foundational Trig Ratios Skill TR3

TR3 — Use a right triangle to evaluate trigonometric functions.	TR3 CheckIt Exercises

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6.8: Foundational Skill Special Rigth TrianglesTR4

TR4 — Find exact values of trigonometric functions of special angles (30, 45, and 60).	TR4 CheckIt Exercises

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6.9: Foundational Skill Similar Triangles (ST)

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6.13: INTRODUCTION TO RIGHT TRIANGLES

6.13.1: Trigonometric Ratios (TR3)

6.13.1.1: TR3 Online Exercises

6.13.2: Special Right Triangles (TR4)

6.13.2.1: TR4 Online Exercises

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6.13.1: Trigonometric Ratios (TR3)

This section corresponds to APC slots 4.1 (Right triangles) & 4.2 (Tangent) in the Math 172 flow — the opening section of Unit 4 (Triangles in Motion).

TR3 Learning outcome: Use a right triangle to evaluate trigonometric functions.

Right Triangle Trigonometry

Overview

In this section, we will learn how to use right triangles to evaluate trigonometric functions. Before doing that, however, let's review some key concepts of right triangles that can be helpful when solving.

Definition: Pythagorean Theorem

Given a right triangle with legs of length a and b and hypotenuse of length c, the following equation holds:

$$a^2 + b^2 = c^2 \tag{6.13.1.1}$$

This is called the **Pythagorean Theorem**.

Note

The Pythagorean Theorem is helpful because if we know the lengths of any two sides of a right triangle, we can always find the length of the third side.

Activity 6.13.1.1

Suppose two legs of a right triangle measure 3 inches and 4 inches.

(a)

Draw a picture of this right triangle and label the sides. Use x to refer to the missing side.

Answer available at the end of this section.

(b)

What is the value of x (i.e., the length of the third side)?

A. 5 inches
B. $\sqrt{5}$ inches
C. 25 inches
D. 16 inches
E. $\sqrt{7}$ inches

Answer available at the end of this section.

Activity 6.13.1.2

Suppose the hypotenuse of a right triangle is 13 cm long and one of the legs is 5 cm long.

(a)

Draw a picture of this right triangle and label the sides. Use x to refer to the missing side.

Answer available at the end of this section.

(b)

What is the value of x (i.e., the length of the third side)?

A. 144
B. 12
C. 194
D. 14

Answer available at the end of this section.

Definition: Pythagorean Triple

Pythagorean triples are integers a, b, and c that satisfy the Pythagorean Theorem. *(see related activity)* and *(see related activity)* highlight some of the most common types of Pythagorean triples: $3-4-5$ and $5-12-13$. All triangles similar to the $3-4-5$ triangle will also have side lengths that are multiples of $3-4-5$ (like $6-8-10$). Similarly, this is true for all triangles similar to the $5-12-13$ triangle.

Activity 6.13.1.3

Suppose you are given a right triangle where the hypotenuse is 11 cm long and one of the interior angles is 60.

(a)

Draw a picture of this right triangle and label the parts of the triangle given.

Answer available at the end of this section.

(b)

What is the measure of the third angle?

A. 90
B. 60
C. 30
D. 180

Answer available at the end of this section.

(c)

Suppose you are asked to find one of the sides of the right triangle. What additional information would you need to find the length of another side of the triangle?

Answer available at the end of this section.

Note

When working with right triangles, it is often helpful to refer to specific angles and sides. One way this is done is by using letters, such as A and a to show that these are an angle-side pair because every angle has a side opposite the angle in a triangle. Note that the capital letter indicates the angle, and the lower case letter indicates the side.

Another way to label the sides of a triangle is to use the relationships between a given angle within a triangle and the sides.

The **hypotenuse** of a right triangle is always the side opposite the right angle. This side also happens to be the longest side of the triangle.

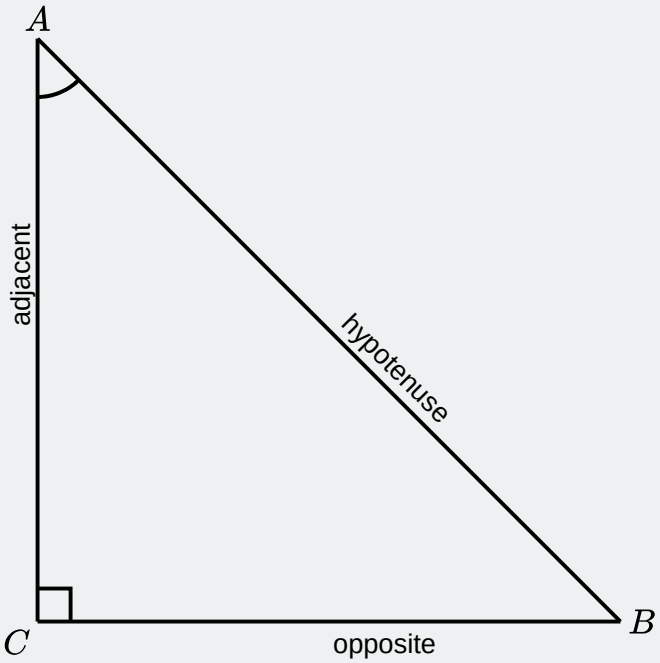
The **opposite side** is the non-hypotenuse side across from a given angle.

The **adjacent side** is the non-hypotenuse side that is next to a given angle.

When given an angle, all sides of a triangle can be labeled from that angle's perspective. For example, from angle A's perspective, the sides of a right triangle are labeled as:

From the perspective of angle

, all sides of a right triangle can be labeled.



Definition: Right Triangle Trigonometry

We will define **trigonometric functions** for a given angle θ as ratios between the side lengths of a right triangle. The first three trigonometric functions we will discuss are the sine function, the cosine function, and the tangent function.

$$\begin{aligned} \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \end{aligned} \tag{6.13.1.2}$$

Notice that these are defined according to the sides of a triangle from the perspective of an angle θ. This is why it is important to be able to label the triangle correctly!

Activity 6.13.1.4

Let ABC be a right triangle with lengths $a = 35$, $b = 12$, and $c = 37$.

(a)

Draw a right triangle and label it with angles A, B, and C, with c being the hypotenuse of the triangle.

Answer available at the end of this section.

(b)

Find $\sin A$.

A. $\frac{12}{37}$
B. $\frac{35}{37}$
C. $\frac{35}{12}$

Answer available at the end of this section.

(c)

Find $\cos A$.

A. $\frac{12}{37}$
B. $\frac{35}{37}$
C. $\frac{35}{12}$

Answer available at the end of this section.

(d)

Find $\tan A$.

A. $\frac{12}{37}$
B. $\frac{35}{37}$
C. $\frac{35}{12}$

Answer available at the end of this section.

Definition: Reciprocal Trig Functions

The three trigonometric ratios we have worked with so far (sine, cosine, and tangent) are referred to as the **basic trigonometric functions**. There are three additional functions, **cosecant**, **secant**, and **cotangent** that are found by taking the reciprocal of the basic trigonometric functions. These three ratios are referred to as the **reciprocal trigonometric functions** and can be defined as follows:

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos\theta} \tag{6.13.1.3}$$
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan\theta}$$

Activity 6.13.1.5

Suppose you are given triangle ABC, where $a = 24$, $b = 32$, and $c = 40$, with c being the hypotenuse of the triangle.

(a)

Find $\sin A$.

A. $\frac{4}{5}$
B. $\frac{3}{5}$
C. $\frac{5}{3}$
D. $\frac{5}{4}$

Answer available at the end of this section.

(b)

Find $\csc A$.

A. $\frac{4}{5}$
B. $\frac{3}{5}$
C. $\frac{5}{3}$
D. $\frac{5}{4}$

Answer available at the end of this section.

(c)

Find $\cos B$.

A. $\frac{4}{5}$
B. $\frac{3}{5}$
C. $\frac{5}{3}$
D. $\frac{5}{4}$

Answer available at the end of this section.

(d)

Find $\sec B$.

A. $\frac{4}{5}$
B. $\frac{3}{5}$
C. $\frac{5}{3}$
D. $\frac{5}{4}$

Answer available at the end of this section.

(e)

Find $\cot B$.

A. $\frac{4}{5}$
B. $\frac{4}{3}$
C. $\frac{5}{4}$
D. $\frac{3}{4}$

Answer available at the end of this section.

Activity 6.13.1.6

Suppose $\cos\theta = \frac{12}{13}$.

(a)

Draw a triangle and label one of the angles θ. Then, label each side as "opposite," "adjacent," and "hypotenuse."

Answer available at the end of this section.

(b)

Use the given information to determine the length of each side of the triangle.

Answer available at the end of this section.

(c)

Find $\csc\theta$.

Answer available at the end of this section.

(d)

Find $\cot\theta$

Answer available at the end of this section.

Activity 6.13.1.7

For each triangle given, determine which trigonometric function would be the most helpful in determining the length of the side of a triangle. Be sure to draw a picture of the triangle to help you determine the relationship between the given angle and sides. In each case assume angle C is the right angle.

(a)

In triangle ABC, $B = 37$ and $a = 11$. Which trigonometric function will best help determine the length of side c?

A. sine
B. cosine
C. tangent

Answer available at the end of this section.

(b)

In triangle ABC, $A = 32$ and $b = 13$. Which trigonometric function will best help determine the length of side a?

A. sine
B. cosine
C. tangent

Answer available at the end of this section.

(c)

In triangle ABC, angle $A = 24$ and the hypotenuse has length 14. Which trigonometric function will best help determine the length of side a?

A. sine
B. cosine
C. tangent

Answer available at the end of this section.

Activity 6.13.1.8

In triangle ABC, $B = 37$, $a = 11$, and c is the hypotenuse.

(a)

Draw a right triangle and label the angles/sides with the information given.

Answer available at the end of this section.

(b)

Which trigonometric equation will best help determine the length of side b?

A. $\sin 37^\circ = \frac{b}{11}$
B. $\tan 37^\circ = \frac{b}{11}$
C. $\cos 37^\circ = \frac{b}{11}$

Answer available at the end of this section.

(c)

Use the equation from part (b) and solve for b. Round to the nearest tenth.

Answer available at the end of this section.

Activity 6.13.1.9

Sarah is standing 500 meters from the base of the Eiffel Tower. She looks at the top of the tower at an angle of 31.

(a)

Draw a diagram to represent the situation. Use x to refer to the missing side.

Answer available at the end of this section.

(b)

Which trig function could we use to find the height of the tower?

A. sine
B. cosine
C. tangent

Answer available at the end of this section.

(c)

How could we correctly set up the trigonometric function to find the height of the Eiffel Tower?

A. $\sin 31^\circ = \frac{x}{500}$
B. $\cos 31^\circ = \frac{500}{x}$
C. $\cos 31^\circ = \frac{500}{x}$
D. $\tan 31^\circ = \frac{x}{500}$

Answer available at the end of this section.

(d)

Take your equation in part (c) and solve for x.

Answer available at the end of this section.

(e)

Find the height of the tower to the nearest hundredth.

A. 428.58 meters
B. 300.43 meters
C. 220.85 meters
D. 257.52 meters

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can evaluate trigonometric functions using a right triangle before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 6.13.1.1

A right triangle has legs of length 5 and 12. Find the length of the hypotenuse.

Answer available at the end of this section.

Check Your Understanding 6.13.1.2

In a right triangle, the side opposite angle θ has length 8 and the hypotenuse has length 17. Find $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$.

Answer available at the end of this section.

Check Your Understanding 6.13.1.3

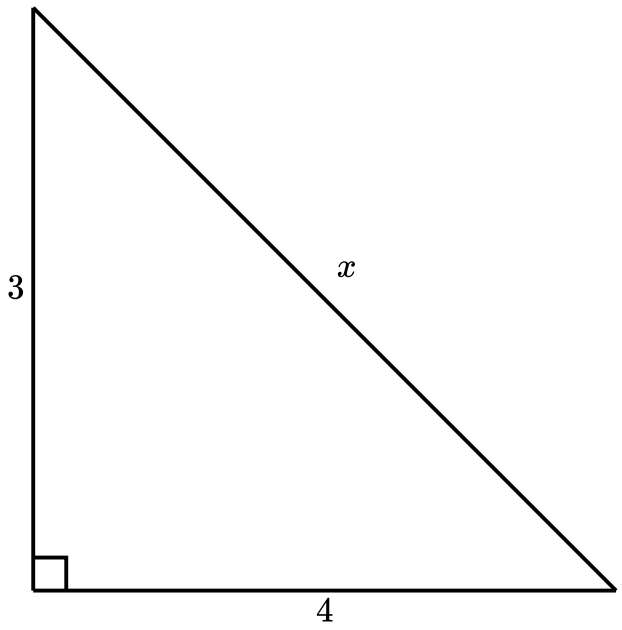
If $\cos(\theta) = \frac{3}{5}$ for an acute angle θ in a right triangle, find $\sec(\theta)$ and $\csc(\theta)$.

Answer available at the end of this section.

Answers

Activity 6.13.1.1

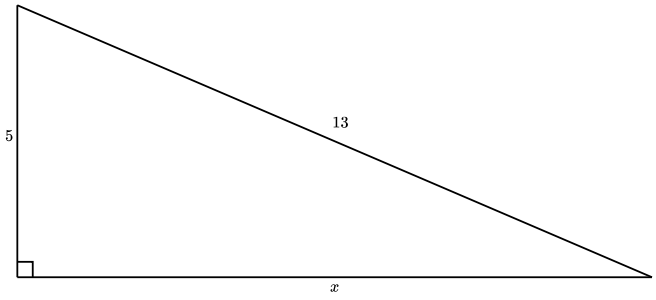
Students should draw a triangle with x representing the hypotenuse and the two legs 3 and 4 inches respectively.



Activity 6.13.1.1

A

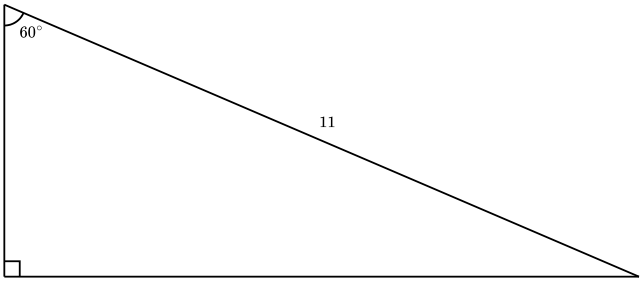
Activity 6.13.1.2



Activity 6.13.1.2

B

Activity 6.13.1.3



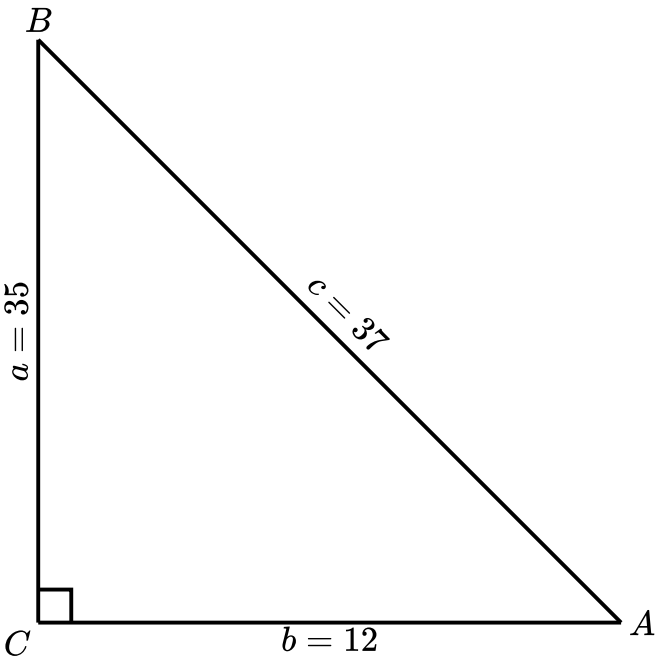
Activity 6.13.1.3

C

Activity 6.13.1.3

Students will probably notice that the Pythagorean Theorem is not helpful in this case because they only know the length of one side. This is a great opportunity to discuss how the Pythagorean Theorem is useful in finding side lengths when at least two sides are known.

Activity 6.13.1.4



Activity 6.13.1.4

B

Activity 6.13.1.4

A

Activity 6.13.1.4

C

Activity 6.13.1.5

B

Activity 6.13.1.5

C

Activity 6.13.1.5

A

Activity 6.13.1.5

D

Activity 6.13.1.5

D

Activity 6.13.1.6

Students should draw a triangle where the side across from θ is labeled “opposite,” the side next to the angle θ is labeled “adjacent,” and the “hypotenuse” as the third side across from the 90-degree angle.

Activity 6.13.1.6

Given that $\cos\theta = \frac{12}{13}$, we know that the side adjacent to θ is 12 and the hypotenuse is 13. Using the Pythagorean Theorem, the third side is 5.

Activity 6.13.1.6

$\csc\theta = \frac{13}{5}$

Activity 6.13.1.6

$\cot\theta = \frac{12}{5}$

Activity 6.13.1.7

B

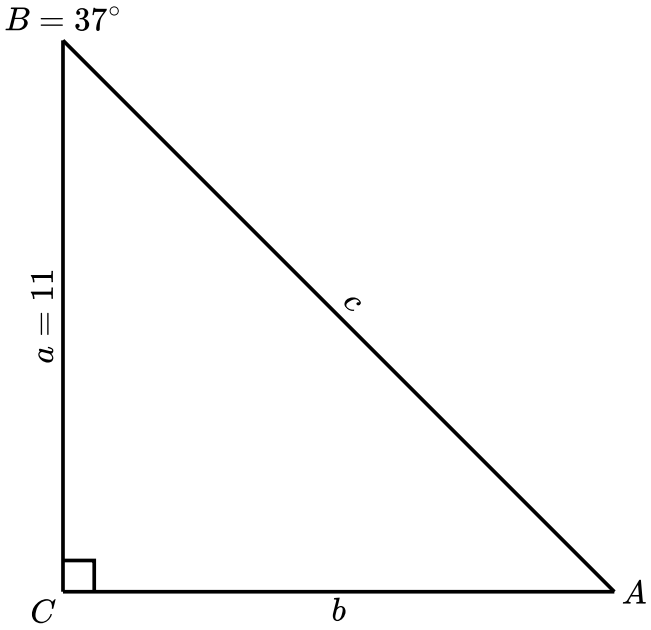
Activity 6.13.1.7

C

Activity 6.13.1.7

A

Activity 6.13.1.8



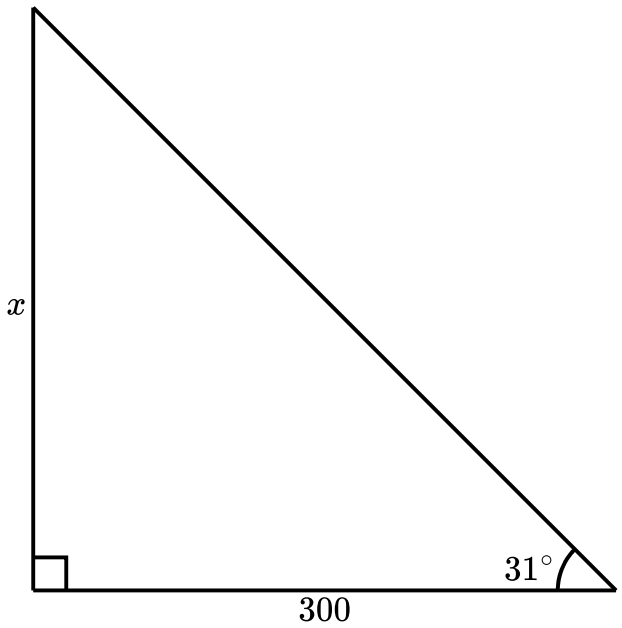
Activity 6.13.1.8

B

Activity 6.13.1.8

$b = 8.3$

Activity 6.13.1.9



Activity 6.13.1.9

C

Activity 6.13.1.9

D

Activity 6.13.1.9

$x = 500 \cdot \tan 31^\circ$

Activity 6.13.1.9

B

Check Your Understanding 6.13.1.1

By the Pythagorean theorem, $c^2 = 5^2 + 12^2 = 25 + 144 = 169$, so $c = \sqrt{169} = 13$. (This is the well-known 5-12-13 Pythagorean triple.)

Check Your Understanding 6.13.1.2

First find the adjacent side: $a^2 = 17^2 - 8^2 = 289 - 64 = 225$, so $a = 15$. Then $\sin(\theta) = \frac{8}{17}$, $\cos(\theta) = \frac{15}{17}$, $\tan(\theta) = \frac{8}{15}$.

Check Your Understanding 6.13.1.3

Since $\cos(\theta) = \frac{3}{5}$, the adjacent side is 3 and hypotenuse is 5; the opposite side is $\sqrt{5^2 - 3^2} = 4$. $\sec(\theta)$ is the reciprocal of cosine: $\sec(\theta) = \frac{5}{3}$. $\csc(\theta)$ is the reciprocal of sine ($\sin(\theta) = \frac{4}{5}$): $\csc(\theta) = \frac{5}{4}$.

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6.13.1.1: TR3 Online Exercises

TR3 — Use a right triangle to evaluate trigonometric functions.	TR3 CheckIt Exercises

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6.13.2: Special Right Triangles (TR4)

This section corresponds to APC slot 4.1 (Right triangles) in the Math 172 flow.

TR4 Learning outcome: Find exact values of trigonometric functions of special angles (30°, 45°, and 60°).

Special Angles and Exact Trig Values

Overview

Recall from the previous section that we can find values of trigonometric functions of any angle. In this section, we can take what we know about right triangles and find exact values of trigonometric functions of special angles.

Definition: Special Right Triangles

There are two special right triangle relationships that will continually appear when speaking about special angles.

- $45-45-90$ triangle
- $30-60-90$ triangle

$45-45-90$ triangle

$30-60-90$ triangle

Activity $6.13.2.1$

Let's explore the relationship of the $45-45-90$ special right triangle.

(a)

Draw a right triangle and label the angles to have 45, 45, and 90.

Answer available at the end of this section.

(b)

In a $45-45-90$ triangle, two angles are the same size. If those two angles are the same size, what do we know about the sides opposite those angles?

Answer available at the end of this section.

(c)

Suppose one of the legs of the right triangle is of length 1, how long is the other leg?

Answer available at the end of this section.

(d)

Now that we know two sides of the right triangle, use that information and the Pythagorean Theorem to find the length of the third side.

Answer available at the end of this section.

Definition: 45-45-90 Triangle Rule

From *(see related activity)*, we saw that a $45-45-90$ triangle is an isosceles right triangle, which means that two of the sides of the triangle are equal. The ratio of its legs and hypotenuse is expressed as follows:

$$\text{Leg} : \text{Leg} : \text{Hypotenuse} = 1 : 1 : \sqrt{2} \tag{6.13.2.1}$$

. In terms of x, this ratio can be expressed as

$$x : x : x\sqrt{2} \tag{6.13.2.2}$$

. Therefore, the **45-45-90 triangle rule** states that the three sides of the triangle are in the ratio $x : x : x\sqrt{2}$.

Activity 6.13.2.2

Suppose you are given a right triangle ABC, where $C = 45$, $a = 6$ cm, and b is the hypotenuse.

(a)

Draw a picture of this right triangle and label the sides.

Answer available at the end of this section.

(b)

If we apply *(see related activity)*, what would the length of c be?

Answer available at the end of this section.

(c)

If we apply *(see related activity)*, what would the length of the hypotenuse be?

Answer available at the end of this section.

Activity 6.13.2.3

For each of the following, use *(see related activity)* to find the missing side.

(a)

In triangle ABC, $A = 90°$, $c = 3$, and $B = 45°$. What is the length of a?

A. 6
B. 5
C. 3
D. $3\sqrt{2}$

Answer available at the end of this section.

(b)

In triangle ABC, $B = 90°$, $b = 6\sqrt{2}$, and $C = 45°$. What is the length of c?

A. 6

B. 5
C. 4
D. 12

Answer available at the end of this section.

(c)

In triangle ABC, $B = 90°$, $b = 4$, and $A = 45°$. What is the length of c?

A. $\sqrt{6}$
B. $2\sqrt{2}$
C. 4
D. $4\sqrt{2}$

Answer available at the end of this section.

Activity 6.13.2.4

Suppose you are given an equilateral triangle, which has three equal sides and three equal angles (60).

(a)

Draw an equilateral triangle and then draw the height from the base of the triangle to the top angle. What kind of triangles did you just create when drawing the height?

Answer available at the end of this section.

(b)

What do you notice about the relationships of the sides of the equilateral triangle to that of the $30 - 60 - 90$ triangles?

Answer available at the end of this section.

(c)

Label the angles to have 30, 60, and 90 and the side opposite of the 30 angle as having a length of 1.

Answer available at the end of this section.

(d)

Given that the side opposite the 30 angle has a length of 1, how long is the length of one side of the equilateral triangle?

Answer available at the end of this section.

(e)

Now that you know the length of two sides of the $30 - 60 - 90$ triangles, find the length of the third side using the Pythagorean Theorem.

Answer available at the end of this section.

Definition: 30-60-90 Triangle Rule

From *(see related activity)*, we saw that if a triangle has angle measures 30, 60, and 90, then the sides are in the ratio:

$$1 : \sqrt{3} : 2 \tag{6.13.2.3}$$

. In terms of x, this ratio can be expressed as

$$x : x\sqrt{3} : 2x \tag{6.13.2.4}$$

. Therefore, the **30-60-90 triangle rule** states that the three sides of the triangle are in the ratio $x : x\sqrt{3} : 2x$. Note that the shorter leg is always x, the longer leg is always $x\sqrt{3}$, and the hypotenuse is always $2x$.

Activity 6.13.2.5

Suppose you are given a right triangle ABC, where $C = 30$, $c = 7$ cm, and b is the hypotenuse.

(a)

Draw a picture of this right triangle and label the sides.

Answer available at the end of this section.

(b)

If we apply *(see related activity)*, what would the length of a be?

Answer available at the end of this section.

(c)

If we apply *(see related activity)*, what would the length of the hypotenuse be?

Answer available at the end of this section.

Activity 6.13.2.6

For each of the following, use *(see related activity)* to find the missing side.

(a)

In triangle ABC, $B = 90$,, $c = 6$, and $C = 30$. What is the length of a?

A. $6\sqrt{3}$
B. 4
C. 3
D. $6\sqrt{2}$

Answer available at the end of this section.

(b)

In triangle ABC, $C = 90$,, $a = 4\sqrt{3}$, and $B = 30$. What is the length of b?

A. 3
B. 6
C. $4\sqrt{2}$
D. 4

Answer available at the end of this section.

(c)

In triangle ABC, $B = 90$,, $a = 8\sqrt{3}$, and $A = 60$. What is the length of b?

A. 4
B. 16
C. 12
D. $8\sqrt{2}$

Answer available at the end of this section.

Note

Recall that we can find values of trigonometric functions of angles of right triangles. We can use the same idea to find values of trigonometric functions of angles of special right triangles.

Activity 6.13.2.7

Suppose you are given triangle ABC, where $C = 45$, $a = 5$, and b is the hypotenuse.

(a)

Find the measures of sides b and c.

Answer available at the end of this section.

(b)

Now that you have found all the sides and angles of triangle ABC, find the ratio that represents $\tan A$.

A. $\frac{5}{5\sqrt{2}}$
B. $\frac{5\sqrt{2}}{5}$
C. $\sqrt{2}$
D. 1

Answer available at the end of this section.

(c)

Find the ratio that represents $\sin A$.

A. $\frac{5}{5\sqrt{2}}$
B. $\frac{5\sqrt{2}}{5}$
C. $\frac{1}{\sqrt{2}}$
D. $\frac{\sqrt{2}}{2}$

Answer available at the end of this section.

(d)

What is the approximate value of $\sin A$?

Answer available at the end of this section.

Note

Some textbooks insist on **rationalizing the denominator** and will always convert to expressions like $\frac{\sqrt{2}}{2}$. However, it is useful to be able to use both, and to recognize that $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$ are the same value.

In *(see related activity)*, notice that when finding a value of a trigonometric ratio, sometimes the values are decimal approximations. For example, the ratio for $\sin A$ was $\frac{1}{\sqrt{2}}$ (or $\frac{\sqrt{2}}{2}$), which gives an approximate value of 0.707.

Definition: Exact Values of Trig Functions

The **exact values of trigonometric functions** are values of trigonometric functions of certain angles that can be expressed exactly using expressions containing real numbers and roots of real numbers. When finding trigonometric ratios, we often give an exact value, rather than an approximation.

Activity 6.13.2.8

Find the exact value of the trigonometric function you are asked to find for each of the following.

(a)

Given the triangle, ABC, with $B = 90$, $C = 45$, and $a = 7$, find $\cos A$.

A. $\frac{7}{7\sqrt{2}}$
B. $\frac{7\sqrt{2}}{7}$
C. $\frac{1}{\sqrt{2}}$
D. $\frac{\sqrt{2}}{2}$

Answer available at the end of this section.

(b)

Using your calculator, find the value of $\cos 45$ to the nearest thousandth.

Answer available at the end of this section.

(c)

Which of the following is equivalent to the value of $\cos 45$ that you found in part (b)?

A. $\frac{7}{7\sqrt{2}}$
B. $\frac{7\sqrt{2}}{7}$
C. $\frac{1}{\sqrt{2}}$
D. $\frac{\sqrt{2}}{2}$

Answer available at the end of this section.

(d)

How can using the $45 - 45 - 90$ triangle help us find the value of $\sin 45$?

Answer available at the end of this section.

Note

Notice that if you know the relationships of the sides of special right triangles, it can help you find the exact value of special angles (i.e., 30, 45, and 60). In addition, notice that the trig values of the angles were the same regardless of the size of the triangle.

Activity 6.13.2.9

For each of the following, find the exact value of the trigonometric function. Use the $45-45-90$ and $30-60-90$ trigonometric rules to help you:

$$1:1:\sqrt{2} \tag{6.13.2.5}$$

$$1:\sqrt{3}:2 \tag{6.13.2.6}$$

(a)

What is $\tan 45$?

A. $\frac{1}{\sqrt{2}}$
B. $\sqrt{2}$
C. 1
D. $\frac{\sqrt{2}}{2}$

Answer available at the end of this section.

(b)

What is $\cos 60$?

A. $\frac{1}{2}$
B. $\sqrt{3}$
C. 2
D. $\frac{\sqrt{3}}{2}$

Answer available at the end of this section.

(c)

What is $\sin 30$?

A. $\frac{\sqrt{3}}{2}$
B. $\sqrt{3}$
C. 2
D. $\frac{1}{2}$

Answer available at the end of this section.

(d)

What is $\cos 45$?

A. $\frac{1}{\sqrt{2}}$
B. $\sqrt{2}$
C. 1
D. $\frac{\sqrt{2}}{2}$

Answer available at the end of this section.

(e)

Now that you have determined $\tan 45$ (part a), what is $\cot 45$?

Answer available at the end of this section.

(f)

What is $\csc 30$?

Answer available at the end of this section.

(g)

What is $\sec 30$?

Answer available at the end of this section.

Check Your Understanding

Work these on your own to confirm you can find exact trig values for the special angles before moving on. Answers are provided so you can self-check. For more practice with randomized problems, see the separate practice page.

Check Your Understanding 6.13.2.1

Find the exact value of $\sin(45^\circ)$ and $\cos(45^\circ)$ using a 45-45-90 triangle, and explain why these two values are equal.

Answer available at the end of this section.

Check Your Understanding 6.13.2.2

Using a 30-60-90 triangle, find the exact value of $\tan(60^\circ)$.

Answer available at the end of this section.

Check Your Understanding 6.13.2.3

Without a calculator, find the exact value of $\sin(30^\circ) + \cos(60^\circ)$.

Answer available at the end of this section.

Answers

Activity 6.13.2.1

Students should be able to draw a right triangle with one 90 angle (indicated by a box) and two 45 angles.

Activity 6.13.2.1

Students should recognize that the two legs of the triangle must be the same size.

Activity 6.13.2.1

Students may notice that because the two legs are equal, then the two legs should both be equal to 1.

Activity 6.13.2.1

$1^2 + 1^2 = c^2$

$2 = c^2$

$\sqrt{2} = c$

Activity $6.13.2.2$

Students should draw a right triangle where B is the right angle and one of the legs is labeled 6 cm.

Activity $6.13.2.2$

According to the $45 - 45 - 90$ triangle rule, the other leg (the side that is also labeled c) of this right triangle is also 6 cm long.

Activity $6.13.2.2$

According to the $45 - 45 - 90$ triangle rule, if the legs of the triangle are represented by x, then the length of the hypotenuse would be $x\sqrt{2}$. So, the length of the hypotenuse would be $6 * \sqrt{2}$ or $6\sqrt{2}$.

Activity $6.13.2.3$

D

Activity $6.13.2.3$

A

Activity $6.13.2.3$

B

Activity $6.13.2.4$

Students may recognize that the equilateral triangle is made up of two $30 - 60 - 90$ triangles as the height bisects the top angle.

Activity $6.13.2.4$

Students may notice a number of things, like the side of the equilateral triangle forms the hypotenuse of each of the $30 - 60 - 90$ triangles, the base of the equilateral triangle is divided in half and is the shorter leg of the $30 - 60 - 90$ triangles, and the height is the longer leg of the $30 - 60 - 90$ triangles.

Activity $6.13.2.4$

Students should label all the angles and the side opposite the 30 angle as having a side length of 1 based on the information that was given.

Activity $6.13.2.4$

Because the side opposite of the 30 is half the length of the side of the equilateral triangle, the length of one side of the equilateral triangle is 2.

Activity $6.13.2.4$

Using the Pythagorean Theorem, students should get $\sqrt{3}$ as the length of the third side (the side opposite the 60).

Activity $6.13.2.5$

Students should draw a right triangle where B is the right angle, A is 60, and C is 30.

Activity $6.13.2.5$

According to the $30 - 60 - 90$ triangle rule, the other leg (which is also the longer leg) of this right triangle would be $x\sqrt{3}$. In this case, x is equal to 7, and so the longer leg would be $7 * \sqrt{3}$, or $7\sqrt{3}$ cm.

Activity $6.13.2.5$

According to the $30 - 60 - 90$ triangle rule, if the shortest leg of the triangle is represented by x, then the length of the hypotenuse would be $2x$. So, the length of the hypotenuse would be $2 * 7$ or 14 cm.

Activity $6.13.2.6$

A

Activity $6.13.2.6$

D

Activity $6.13.2.6$

B

Activity $6.13.2.7$

$b = 5\sqrt{2}$ and $c = 5$

Activity $6.13.2.7$

D

Activity $6.13.2.7$

A, C, and D

Activity $6.13.2.7$

0.707

Activity $6.13.2.8$

A, C, and D

Activity $6.13.2.8$

0.707

Activity $6.13.2.8$

A, C, and D

Activity $6.13.2.8$

Students should refer to the $45 - 45 - 90$ triangle they drew in part (a) to find that $\sin 45$ is equal to $\frac{\sqrt{2}}{2}$ (or approximately 0.707).

Activity $6.13.2.9$

C

Activity $6.13.2.9$

A

Activity 6.13.2.9

D

Activity 6.13.2.9

A and D

Activity 6.13.2.9

1

Activity 6.13.2.9

2

Activity 6.13.2.9

$\frac{2\sqrt{3}}{3}$

Check Your Understanding 6.13.2.1

$\sin(45^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}$. They are equal because a 45-45-90 triangle is isosceles — the two legs (opposite and adjacent to a 45° angle) have equal length, so the ratios opposite/hypotenuse and adjacent/hypotenuse are identical.

Check Your Understanding 6.13.2.2

In a 30-60-90 triangle with hypotenuse 2, the side opposite 30° is 1 and the side opposite 60° is $\sqrt{3}$. So $\tan(60^\circ) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$.

Check Your Understanding 6.13.2.3

$\sin(30^\circ) = \frac{1}{2}$ and $\cos(60^\circ) = \frac{1}{2}$ (these are actually the same ratio in a 30-60-90 triangle, since the side opposite 30° is also adjacent to 60°). So the sum is $\frac{1}{2} + \frac{1}{2} = 1$.

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6.13.2.1: TR4 Online Exercises

TR4 — Find exact values of trigonometric functions of special angles (30°, 45°, and 60°).	TR4 CheckIt Exercises

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7: HOT (HIGHER ORDER THINKING) EXERCISES

7.1: HOT Chapter 1

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7.1: HOT Chapter 1

Section 1.3 Exercises: The Average Rate of Change of a Function

Exercise 1

Let P_1 and P_2 be the populations (in hundreds) of Town 1 and Town 2, respectively. The table below shows data for these two populations for five different years.

Year	1980	1983	1987	1993	1999
P_1	49	53	57	61	65
P_2	79	72	65	58	51

Find the average rate of change of each population over each of the time intervals below.
(a) From 1980 to 1987, the average rate of change of the population of Town 1 was _____ hundred people per year, and the average rate of change of the population of Town 2 was _____ hundred people per year.
(b) From 1987 to 1999, the average rate of change of the population of Town 1 was _____ hundred people per year, and the average rate of change of the population of Town 2 was _____ hundred people per year.
(c) From 1980 to 1999, the average rate of change of the population of Town 1 was _____ hundred people per year, and the average rate of change of the population of Town 2 was _____ hundred people per year.

Exercise 2

(a) What is the average rate of change of $g(x) = -6 - 5x$ between the points $(-4, 14)$ and $(5, -31)$?
answer = _____
(b) The function g is

- increasing
- decreasing

on the interval $-4 \leq x \leq 5$.

Exercise 3

Find the average rate of change of $f(x) = 3x^2 + 7$ between each of the pairs of points below.
(a) Between $(3, 34)$ and $(5, 82)$
answer = _____
(b) Between (c, k) and (q, t)
answer = _____
(c) Between $(x,\ f(x))$ and $(x + h,\ f(x + h))$
answer = _____

Exercise 4

In 2005, you have 45 CDs in your collection. In 2008, you have 130 CDs. In 2012, you have 50 CDs. What is the average rate of change in the size of your CD collection between:
(a) 2005 and 2008? _____
(b) 2008 and 2012? _____
(c) 2005 and 2012? _____

Exercise 5

Based on the graphs of $f(x)$ and $g(x)$ below, answer the following questions. You should not approximate any of your answers.
Graphs of two functions f and g used to compare their average rates of change.
a) What is the average rate of change of $f(x)$ over the interval $2.2 \leq x \leq 6.1$? _____
b) What is the average rate of change of $g(x)$ over the interval $2.2 \leq x \leq 6.1$? _____

Exercise 6

The graph below shows the distance traveled, D (in miles) as a function of time, t (in hours).
Graph of distance traveled D as a function of time t.
(Click on the graph to get a larger version.)
a) For each of the intervals, find the values of ΔD and Δt between the indicated start and end times. Enter your answers in their respective columns in the table below.

Time Interval	ΔD	Δt
t = 1.5 to t = 4.5	_____	_____
t = 2 to t = 4.5	_____	_____
t = 1 to t = 3	_____	_____

b) Based on your results from (a) it follows that the average rate of change of D is constant, it does not depend over which interval of time you choose. What is the constant rate of change of D?
$\frac{\Delta D}{\Delta t} =$ _____
c) Which of the statements below CORRECTLY explains the significance of your answer to part (b)? Select ALL that apply (more than one may apply).

- It is the average velocity of the car over the first two hours.
- It is the total distance the car travels in five hours.
- It is how far the car will travel in a half-hour.
- It represents the car's velocity.
- It is the acceleration of the car over the five hour time interval.
- It is the slope of the line.
- None of the above

Exercise 7

Let $f(x) = 36 - x^2$.
a) Compute each of the following expressions and interpret each as an average rate of change:
(i) $\frac{f(4)-f(0)}{4-0} =$ _____
(ii) $\frac{f(6)-f(4)}{6-4} =$ _____
(iii) $\frac{f(6)-f(0)}{6-0} =$ _____
b) Based on the graph sketched below, match each of your answers in (i) - (iii) with one of the lines labeled A - F. Type the corresponding letter of the line segment next to the appropriate formula. Clearly not all letters will be used.
Graph of f(x) = 36 - x squared with labeled candidate secant lines.
(click on image to enlarge)

_____	$\frac{f(4)-f(0)}{4-0}$
_____	$\frac{f(6)-f(4)}{6-4}$
_____	$\frac{f(6)-f(0)}{6-0}$

Exercise 8

The table below gives the average temperature, T, at a depth d, in a borehole in Belleterre, Quebec.

d, depth (m)	T, temp (C°)
25	5.50
50	5.20
75	5.10
100	5.10
125	5.30
150	5.50
175	5.75
200	6.00
225	6.25
250	6.50
275	6.75
300	7.00

Evaluate $\Delta T/\Delta d$ on the following intervals
a) $125 \leq d \leq 275$ $\Delta T/\Delta d$ = _____
b) $25 \leq d \leq 125$ $\Delta T/\Delta d$ = _____
c) $75 \leq d \leq 200$ $\Delta T/\Delta d$ = _____
d) Which of the statements below correctly explains the significance of your answer to part (c)? Select all that apply (more than one may apply).

- On average, the temperature is changing at a rate of 0.0072 degrees Celsius per minute over the interval $75 \leq d \leq 200$.
- 0.0072 is the slope of the graph of at $d = 75$.
- The temperature changes by a total of 0.0072 degrees Celsius when moving from a depth 75 meters to 200 meters.
- Over the interval from 75 meters to 200 meters, the temperature changes on average at a rate of 0.0072 degrees Celsius per meter.
- The temperature is changing at a rate of 0.0072 degrees Celsius per minute when the depth is 75 meters.
- None of the above

Exercise 9

A cold can of soda is removed from a refrigerator. Its temperature F in degrees Fahrenheit is measured at 5-minute intervals, as recorded in the following table.

Table 1.3.5. Data for the soda's temperature as a function of time.

t (minutes)	0	5	10	15	20	25	30	35
F (Fahrenheit temp)	37.00	44.74	50.77	55.47	59.12	61.97	64.19	65.92

1. Determine $AV_{[0,5]}$, $AV_{[5,10]}$, and $AV_{[10,15]}$, including appropriate units. Choose one of these quantities and write a careful sentence to explain its meaning. Your sentence might look something like "On the interval ... , the temperature of the soda is ... on average by ... for each 1-unit increase in ...".

2. On which interval is there more total change in the soda's temperature: $[10, 20]$ or $[25, 35]$?
3. What can you observe about when the soda's temperature appears to be changing most rapidly?
4. Estimate the soda's temperature when $t = 37$ minutes. Write at least one sentence to explain your thinking.

Exercise 10

The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.3.6. The car's position function has units measured in thousands of feet. For instance, the point $(2, 4)$ on the graph indicates that after 2 minutes, the car has traveled 4000 feet.

Graph of the position y = s(t) of a car, measured in thousands of feet, over time in minutes.

The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

Figure 1.3.6. The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

1. In everyday language, describe the behavior of the car over the provided time interval. In particular, carefully discuss what is happening on each of the time intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, 5]$, plus provide commentary overall on what the car is doing on the interval $[0, 12]$.
2. Compute the average rate of change of s on the intervals $[3, 4]$, $[4, 6]$, and $[5, 8]$. Label your results using the notation "$AV_{[a,b]}$" appropriately, and include units on each quantity.
3. On the graph of s, sketch the three lines whose slope corresponds to the values of $AV_{[3,4]}$, $AV_{[4,6]}$, and $AV_{[5,8]}$ that you computed in (b).
4. Is there a time interval on which the car's average velocity is 5000 feet per minute? Why or why not?
5. Is there ever a time interval when the car is going in reverse? Why or why not?

Exercise 11

Consider an inverted conical tank (point down) whose top has a radius of 3 feet and that is 2 feet deep. The tank is initially empty and then is filled at a constant rate of 0.75 cubic feet per minute. Let $V = f(t)$ denote the volume of water (in cubic feet) at time t in minutes, and let $h = g(t)$ denote the depth of the water (in feet) at time t. It turns out that the formula for the function g is $g(t) = \left(\frac{t}{\pi}\right)^{1/3}$.

1. In everyday language, describe how you expect the height function $h = g(t)$ to behave as time increases.
2. For the height function $h = g(t) = \left(\frac{t}{\pi}\right)^{1/3}$, compute $AV_{[0,2]}$, $AV_{[2,4]}$, and $AV_{[4,6]}$. Include units on your results.
3. Again working with the height function, can you determine an interval $[a, b]$ on which $AV_{[a,b]} = 2$ feet per minute? If yes, state the interval; if not, explain why there is no such interval.
4. Now consider the volume function, $V = f(t)$. Even though we don't have a formula for f, is it possible to determine the average rate of change of the volume function on the intervals $[0, 2]$, $[2, 4]$, and $[4, 6]$? Why or why not?

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8: COURSE RESOURCES

8.1: Online Exercises (Full Course)

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8.1: Online Exercises (Full Course)

Math 172 — CheckIt Practice & Reassessment by Learning Outcome

Each link opens a randomized CheckIt exercise for that outcome from the TBIL Precalculus bank. Regenerate for a new version each time — useful for practice or reassessment attempts.

Unit 1 — Functions (FN)

Outcome	CheckIt Exercises
FN1 — Determine if a relation, equation, or graph defines a function using the definition as well as the vertical line test.	FN1 CheckIt Exercises
FN2 — Use and interpret function notation to evaluate a function for a given input value and find a corresponding input value given an output value.	FN2 CheckIt Exercises
FN3 — Use the graph of a function to find the domain and range in interval notation, the x- and y-intercepts, the maxima and minima, and where it is increasing and decreasing using interval notation.	FN3 CheckIt Exercises
FN4 — Apply transformations including horizontal and vertical shifts, stretches, and reflections to a function. Express the result of these transformations graphically and algebraically.	FN4 CheckIt Exercises
FN5 — Find the sum, difference, product, quotient, and composition of two or more functions and evaluate them.	FN5 CheckIt Exercises
FN6 — Find the inverse of a one-to-one function.	FN6 CheckIt Exercises

Unit 1 — Linear Functions (LF)

Outcome	CheckIt Exercises
LF1 — Determine the average rate of change of a given function over a given interval. Find the slope of a line.	LF1 CheckIt Exercises

Outcome	CheckIt Exercises
LF2 — Determine an equation for a line when given two points on the line and when given the slope and one point on the line. Express these equations in slope-intercept or point-slope form and determine the slope and y-intercept of a line given an equation.	LF2 CheckIt Exercises
LF3 — Graph a line given its equation or some combination of characteristics, such as points on the graph, a table of values, the slope, or the intercepts.	LF3 CheckIt Exercises
LF5 — Build linear models from verbal descriptions, and use the models to establish conclusions, including by contextualizing the meaning of slope and intercept parameters.	LF5 CheckIt Exercises

Unit 1 — Polynomial and Rational Functions (PR1–PR2)

Outcome	CheckIt Exercises
PR1 — Graph quadratic functions and identify their axis of symmetry, and maximum or minimum point.	PR1 CheckIt Exercises
PR2 — Use quadratic models to solve an application problem and establish conclusions.	PR2 CheckIt Exercises

Unit 2 — Trigonometric Functions (TR)

Outcome	CheckIt Exercises
TR1 — Convert between degrees and radians. Draw angles in standard position.	TR1 CheckIt Exercises
TR2 — Identify and find coterminal angles. Find the length of a circular arc.	TR2 CheckIt Exercises
TR3 — Use a right triangle to evaluate trigonometric functions.	TR3 CheckIt Exercises
TR4 — Find exact values of trigonometric functions of special angles (30°, 45°, and 60°).	TR4 CheckIt Exercises
TR5 — Use reference angles, signs, and definitions to determine exact values of trigonometric functions.	TR5 CheckIt Exercises

Unit 2 — Periodic Functions (PF)

Outcome	CheckIt Exercises
PF1 — Determine the basic properties of the graphs of sine and cosine, including amplitude, period, and phase shift.	PF1 CheckIt Exercises
PF2 — Graph trigonometric functions including tangent, cotangent, secant, and cosecant functions and transformations of these functions, and determine the domain and range.	PF2 CheckIt Exercises
PF3 — Determine the inverse sine, cosine, and tangent values; graph inverse trig functions and determine the limitations on the domain and range.	PF3 CheckIt Exercises

Unit 3 — Exponential and Logarithmic Functions (EL)

Outcome	CheckIt Exercises
EL1 — Determine if a given function is exponential. Find an equation of an exponential function. Evaluate exponential functions (including base *e*).	EL1 CheckIt Exercises
EL2 — Graph exponential functions and determine the domain, range, and asymptotes.	EL2 CheckIt Exercises
EL3 — Convert between exponential and logarithmic form. Evaluate a logarithmic function, including common and natural logarithms.	EL3 CheckIt Exercises
EL4 — Graph logarithmic functions and determine the domain, range, and asymptotes.	EL4 CheckIt Exercises
EL5 — Use properties of logarithms to condense or expand logarithmic expressions.	EL5 CheckIt Exercises
EL6 — Solve exponential and logarithmic equations.	EL6 CheckIt Exercises

Unit 4 — Trigonometric Equations (TE)

Outcome	CheckIt Exercises
TE1 — Use the sum and difference, the double-angle, and power-reducing formulas for cosine, sine, and tangent and the Pythagorean identities.	TE1 CheckIt Exercises

Outcome	CheckIt Exercises
TE2 — Symbolically verify identities by using various trig identities.	TE2 CheckIt Exercises
TE3 — Find all solutions to a trigonometric equation.	TE3 CheckIt Exercises
TE4 — Use trigonometric functions and the Pythagorean Theorem to solve right triangles.	TE4 CheckIt Exercises
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Unit 5 — Polynomial and Rational Functions (PR3–PR9)

Outcome	CheckIt Exercises
PR3 — Determine the zeros and their multiplicities of a polynomial in factored form. Describe and graph the behavior of a polynomial function at the intercepts and the ends.	PR3 CheckIt Exercises
PR4 — Rewrite a rational function as a polynomial plus a proper rational function (polynomial long division).	PR4 CheckIt Exercises
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PR6 — Solve rational equations.	PR6 CheckIt Exercises
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PR8 — Solve quadratic inequalities and express the solution graphically and with interval notation.	PR8 CheckIt Exercises
PR9 — Solve rational inequalities and express the solution graphically and using interval notation.	PR9 CheckIt Exercises

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