CALCULUS FOR ENGINEERING TECHNOLOGY



Vismar Jay Alfaro
University of Science and Technology of Southern Philippines- Jasaan

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Calculus 1

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1: ALGEBRA REVIEW

This College Algebra text will cover a combination of classical algebra and analytic geometry, with an introduction to the transcendental exponential and $\log$ arithmic functions. There are also chapters devoted to the theory of functions, combinatorics and trigonometry. If mathematics is the language of science, then algebra is the grammar of that language. Like grammar, algebra provides a structure to mathematical notation, in addition to its uses in problem solving and its ability to change the appearance of an expression without changing the value.

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1.1: Algebraic Simplification

When algebraic techniques are presented as skills in isolation, they are much simpler to understand and practice. However the problem solving process in any context involves deciding which skills to use when. Most College Algebra students will have practiced problems in the form:

$$\begin{gathered}(x+7)(x-2)=? \\ \text{or} \\ (2x+1)^2=?\end{gathered} \tag{1.1.1}$$

The problems in this section deal with a combination of these processes which are often encountered as parts of more complex problems.

Example 1.1.1

Simplify:

$$3(x-1)(2x+5)-(x+4)^2$$

Solution

In this example, the simplification involves two expressions: $3(x-1)(2x+5)$ and $(x+4)^2$. The $(x+4)^2$ is preceded by a negative (or subtraction) sign. This textbook will often treat $-x$ and $+(-x)$ as equivalent statements, since subtraction is defined as the addition of a negative.

We will simplify each expression separately and then look to combine like terms.

$$3(x-1)(2x+5)-(x+4)^2=3\left(2x^2+3x-5\right)-\left(x^2+8x+16\right)$$

Notice that the results of both multiplications remain inside of parentheses. This is because each one has something that must be distributed.

In the case of $(2x^2+3x-5)$, there is a 3 which must be distributed, resulting in $6x^2+9x-15$. In the case of $(x^2+8x+16)$ there is a negative sign or -1 which must be distributed, resulting in $-x^2-8x-16$. It is important in these situations that the negative sign be distributed to all terms in the parentheses.

So

$$\begin{aligned}3(x-1)(2x+5)-(x+4)^2&=3\left(2x^2+3x-5\right)-\left(x^2+8x+16\right)\\&=6x^2+9x-15-x^2-8x-16\\&=5x^2+x-31\end{aligned} \tag{1.1.2}$$

Example 1.1.2

Simplify:

$$2(x+3)^2-4(3x-1)(x+2)$$

Solution

This example shows some of the same processes as the previous example. There are again two expressions that must be simplified, each of which has a coefficient that must be distributed. It is often helpful to wait until after multiplying the binomials before distributing the coefficient. However, as is often true in mathematics, there are several different approaches that may be taken in simplifying this problem.

If someone prefers to first distribute the coefficient before multiplying the binomials, then the coefficient must only be distributed to ONE of the binomials, but not both. For example, in multiplying $3 * 2 * 5 = 30$, we can first multiply $2 * 5 = 10$ and then $3 * 10 = 30$. Each factor is multiplied only once.
In the example above we can proceed as we did with the previous example:

$$\begin{aligned} 2(x+3)^2 - 4(3x-1)(x+2) &= 2\left(x^2+6x+9\right) - 4\left(3x^2+5x-2\right) \\ &= 2x^2+12x+18-12x^2-20x+8 \\ &= -10x^2-8x+26 \end{aligned} \tag{1.1.3}$$

Or, we can choose to distribute the 4 first:

$$\begin{aligned} 2(x+3)^2 - 4(3x-1)(x+2) &= 2\left(x^2+6x+9\right) - (12x-4)(x+2) \\ &= 2x^2+12x+18-\left(12x^2+20x-8\right) \\ &= 2x^2+12x+18-12x^2-20x+8 \\ &= -10x^2-8x+26 \end{aligned} \tag{1.1.4}$$

Or, we can distribute the 4 as a negative. If we do this, then the sign in front of the parentheses will be positive:

$$\begin{aligned} 2(x+3)^2 - 4(3x-1)(x+2) &= 2\left(x^2+6x+9\right) + (-12x+4)(x+2) \\ &= 2x^2+12x+18+\left(-12x^2-20x+8\right) \\ &= -10x^2-8x+26 \end{aligned} \tag{1.1.5}$$

Distributing the 2 in front of the squared binomial must also be handled carefully if you choose to do this. If you distribute the 2 before squaring the $(x+3)$, then the 2 will be squared as well. If you choose to distribute the 2, the $(x+3)^2$ must be written out as $(x+3)(x+3)$

$$\begin{aligned} 2(x+3)^2 - 4(3x-1)(x+2) &= 2(x+3)(x+3) - 4(3x-1)(x+2) \\ &= (2x+6)(x+3) - 4\left(3x^2+5x-2\right) \\ &= 2x^2+12x+18-12x^2-20x+8 \\ &= -10x^2-8x+26 \end{aligned} \tag{1.1.6}$$

Most examples in this text will distribute the coefficients as the last step before combining like terms for a final answer.

Example 1.1.3

Simplify:

$$3x[5-(2x+7)] + (3x-2)^2 - (x-5)(x+4)$$

This example has three expressions that should be simplified separately before combining like terms. In the first expression $3x[5-(2x+7)]$, we should simplify inside the brackets before distributing the $3x$

$$\begin{aligned} &3x[5-(2x+7)] + (3x-2)^2 - (x-5)(x+4) \\ =&3x[5-2x-7] + (3x-2)^2 - (x-5)(x+4) \\ =&3x[-2x-2] + (3x-2)(3x-2) - \left(x^2-x-20\right) \\ =&-6x^2-6x+\left(9x^2-12x+4\right) - x^2+x+20 \\ =&-6x^2-6x+9x^2-12x+4-x^2+x+20 \\ =&2x^2-17x+24 \end{aligned} \tag{1.1.7}$$

Exercises 1.1.1

Simplify each expression.

1. $(x-2)[2x-2(3+x)] - (x+5)^2$

2. $3x^2 - [7x - 2(2x - 1)(3 - x)]$
3. $(a + b)^2 - (a + b)(a - b) - [a(2b - 2) - (b^2 - 2a)]$
4. $5x - 3(x - 2)(x + 7) + 3(x - 2)^2$
5. $(m + 3)(m - 1) - (m - 2)^2 + 4$
6. $(a - 1)(a - 2) - (a - 2)(a - 3) + (a - 3)(a - 4)$
7. $2a^2 - 3(a + 1)(a - 2) - [7 - (a - 1)]^2$
8. $2(x - 5)(3x + 1) - (2x - 1)^2$
9. $6y + (3y + 1)(y + 2) - (y - 3)(y - 8)$
10. $6x - 4(x + 10)(x - 1) + (x + 1)^2$

Answers available at the end of this section.

Answers

Exercises 1.1.1

1. $-x^2 - 16x - 13$
2.
3. $3b^2$
4.
5. $6m - 3$
6.
7. $-2a^2 + 19a - 58$
8.
9. $2y^2 + 24y - 22$
10.

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1.2: Factoring

This section will review three of the most common types of factoring - factoring out a Greatest Common Factor, Trinomial Factoring and factoring a Difference of Squares.

Greatest Common Factor

Factoring out a greatest common factor essentially undoes the distributive multiplication that often occurs in mathematical expressions. This factor may be monomial or polynomial, but in these examples, we will explore monomial common factors.

In multiplying $3xy^2(5x - 2y) = 15x^2y^2 - 6xy^3$ the monomial term $3xy^2$ is multiplied or distributed to both terms inside the parentheses. The process of **factorization** undoes this multiplication.

Example 1.2.1

Factor

$$7x^2 + 14x$$

Solution

This expression has two terms. The coefficients share a common factor of 7 and the only variable involved in this expression is x. The highest power of the variable that is shared by both terms is x^1, so this is the power of x that can be factored out of both terms. The greatest common factor is $7x$

$$7x^2 + 14x = 7x(x + 2)$$

It isn't necessary to find the greatest common factor right away. In more complicated problems, the factoring can be accomplished in pieces, similar in fashion to reducing fractions.

Example 1.2.2

Factor

$$42x^2y^6 + 98xy^3 - 210x^3y^2$$

Solution

This expression has three terms. It's not immediately clear what the greatest common factor of the coefficients is, but they're all even numbers, so we could at least divide them all by 2. The $98xy^3$ term has an x^1, which means that this is the hightest power of x that we could factor out of all the terms. The $210x^3y^2$ has a y^2, which is the highest power of y that can be factored out of all the terms. So we can at least proceed with these factors:

$$\begin{aligned} 42x^2y^6 + 98xy^3 - 210x^3y^2 &= 2xy^2 * 21xy^4 + 2xy^2 * 49y - 2xy^2 * 105x^2 \\ &= 2xy^2\left(21xy^4 + 49y - 105x^2\right) \end{aligned} \tag{1.2.1}$$

Now, we didn't try very hard to find the greatest common factor in the beginning of this problem, so it's important that we continue to question whether or not there are any remaining common factors. The 21 and 49 clearly share a common factor of 7, so it would make sense to see if 105 is divisible by 7 as well. If we divide 105 by 7, we see that $105 = 7*15$. So, we can also factor out a common factor of 7 from the remaining terms in the parentheses.

$$\begin{aligned} 2xy^2\left(21xy^4+49y-105x^2\right) &= 2xy^2\left(7*3xy^4+7*7y-7*15x^2\right) \\ &= 7*2xy^2\left(3xy^4+7y-15x^2\right) \\ &= 14xy^2\left(3xy^4+7y-15x^2\right) \end{aligned} \tag{1.2.2}$$

Trinomial Factoring $(a = 1)$

Trinomial factoring undoes the multiplication of two binomials, and it comes in two flavors - simple and complex. The simplest form of trinomial factoring involves a trinomial expression in the form $ax^2 + bx + c$ in which the value of a is 1 This makes the task of factorization simpler than if the value of a is not 1.

Example 1.2.3

Factor $x^2 + 7x + 10$

Solution

In this example, the value of a is 1, which makes this type of trinomial factoring a little less difficult that it would otherwise be. Whether or not the value of a is 1 the fundamental issue that governs this type of factoring is the $+$ or $-$ sign of the constant term. In this problem, the constant term is positive. That means that we need to find factors of 10 that add up to 7. This is relatively straightforward:

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

A companion problem to this one is $x^2 - 7x + 10$. Notice that, in this case, the sign of the constant term is still positive, which means that we still need factors of 10 that add up to 7. This means we still need to use 2 and 5. However, in this case, instead of the +10 being produced from a multiplication of (+2)(+5) it is the result of multiplying $(-2)(-5)$. This is what makes the 7 in the second example negative:

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

Example 1.2.4

Factor $x^2 + 3x - 10$

Solution

In this case, the sign of the constant term is negative. That means that we need to find factors of 10 that have a difference of 3. This is still 5 and 2.

$$x^2 + 3x - 10 = (x - 2)(x + 5)$$

The multiplication of the (-2) and the (+5) produce the (-10) and the fact that the 2 and 5 have opposite signs creates the difference that gives us $(+3)$. A companion problem to this one is $x^2 - 3x - 10$. In this case, the sign of the constant term is still negative, which means that we still need factors of 10 that have a difference of 3. This means we still need to use 2 and 5. However in this case, instead of the (+3) as the coefficient of the middle term, we'll need a (-3) . To do this we simply reverse the signs of the 2 and 5 from the previous problem:

$$x^2 - 3x - 10 = (x + 2)(x - 5)$$

Now the (+2)(-5) gives us (-10), but the $+2x - 5x$ gives us $(-3x)$ instead of $(+3x)$.

Example 1.2.5

Factor $x^2 + 11x - 42$

Solution

In this problem, the sign of the constant term is negative. That means that we need factors of 42 that have a difference of 11. A systematic exploration of all the factor pairs of 42 can help us to find the correct pair:

1	42
2	21
3	14
6	7

Here, we can see that the factors 3 and 14 have a difference of 11. This means that we will use these factors in our answer: $(x \quad 3)(x$
14). In determining how to place the $+$ and $-$ signs in the parentheses, we can refer back to the original problem: $x^2 + 11x - 42$. If we want a difference of $(+11x)$, then we'll need to have $a(+14)$ and $a(-3)$

$$x^2 + 11x - 42 = (x - 3)(x + 14)$$

Example 1.2.6

Factor $x^2 + 28x + 96$

Solution

In this problem, the sign of the constant term is positive. That means that we need factors of 96 that add up to 28. A systematic exploration of all the factor pairs of 96 can help us to find the correct pair:

1	96
2	48
3	32
4	24
6	16
8	12

Here, we can see that the factors 4 and 24 add up to 28. This means that we will use these factors in our answer: $(x \quad 4)(x \quad 24)$. In determining how to place the $+$ and $-$ signs in the parentheses, we can refer back to the original problem:

$x^2 + 28x + 96$. If we want 4 and 24 to add up to $(+28)$, then they should both be positive:

$$x^2 + 28x + 96 = (x + 4)(x + 24)$$

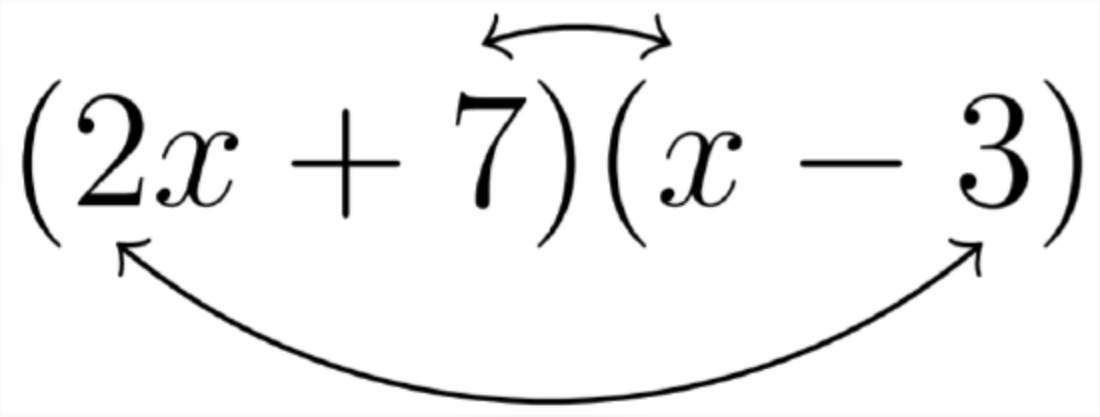
In building the charts of factor pairs in the previous two problems, nothing more difficult than dividing the constant term by the numbers $1, 2, 3, 4, 5, 6, \ldots$ and so on can help you to find the full list of factor pairs. If you don't get a whole number when dividing - for instance $96 \div 5 = 19.2$, then this number is not included in the list of factor pairs.

Trinomial Factoring $(a \neq 1)$

If the value of a is not 1, this means that, if the trinomial is factorable, at least one of its binomial factors also has a coefficient other than 1. For instance:

$$(2x + 7)(x - 3) = 2x^2 + 1x - 21$$

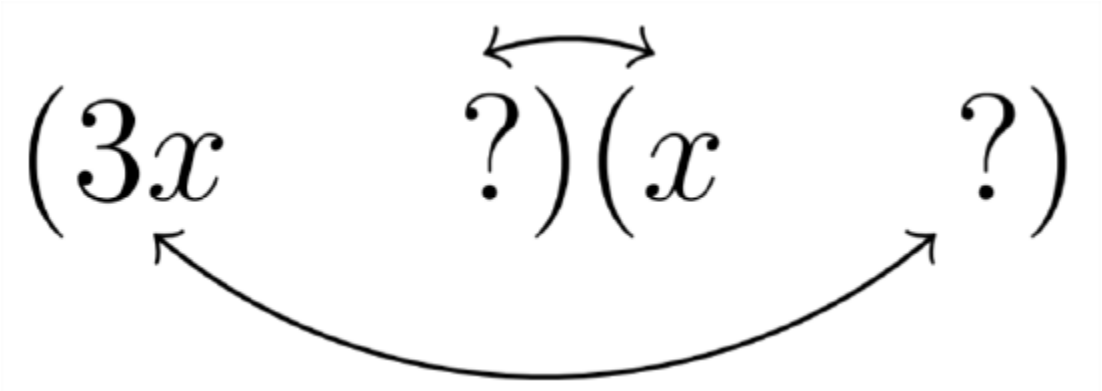
If we were to try undo this multiplication through the process of trinomial factoring, we should look to the sign of the constant term. In this example, the sign is negative. This still means that we will need to find factor pairs that produce a difference of $(+1x)$ as the middle term. However, in this scenario, it is not just the factors of 21 that are involved in producing the $(+1x)$, but the combination of the factors of 21 and the factors of the leading coefficient 2. The middle term $(+1x)$ comes from the multiplication of the $(2x)(-3)$ and the multiplication of (+7)$(+1x)$



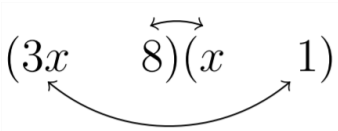
$$\begin{aligned}(2x+7)(x-3) &= 2x^2-6x+7x-21 \\ &= 2x^2+x-21\end{aligned} \tag{1.2.3}$$

In trying to factor a trinomial like $2x^2+x-21$, we need to take this into consideration. For example, if we were to factor $3x^2-10x+8$, we should first still look to the sign of the constant term, which, in this case, is positive. That means we want factor pairs that will add up to 10. But we have to take into consideration the interaction of the factors of the 3 with the factors of the 8. The 3 is a prime number, which means that we don't have a choice - it can only be split up into $3*1$, so we can start:

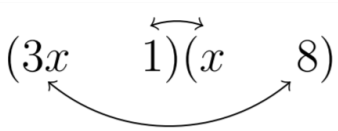
$$\text{Factor } 3x^2+10x+8 \tag{1.2.4}$$



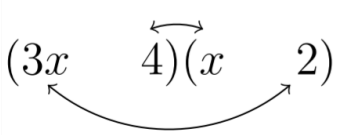
Our options for filling in the question marks will come from the factors of 8 , either $8*1$ or $4*2$. The process is by trial and error:



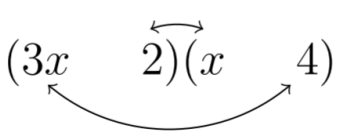
$$3x+8x=11x$$



$$24x+1x=25x$$

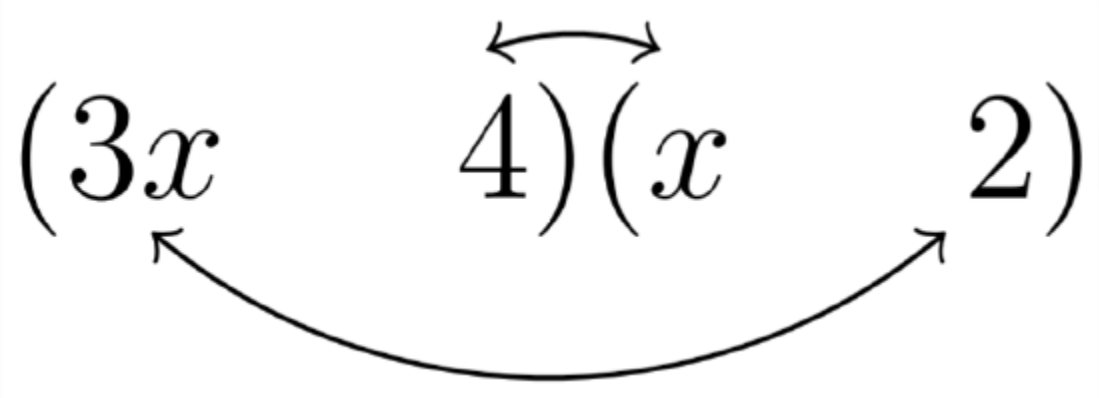


$$6x+4x=10x$$



$$12x+2x=14x$$

We can see that the choice above:



gives us the required $10x$ as the middle term. since the original problem was $3x^2 + 10x + 8$ we'll want to fill in the signs as both positive:

$$3x^2 + 10x + 8 = (3x + 4)(x + 2) \tag{1.2.5}$$

A second method for handling this type of factoring depends on how the factors of the leading coefficient and constant term interact with each other to produce the middle term. In this process, given the problem $3x^2 + 10 + 8$, we can multiply the first and last coefficient and then look at the factor pairs of the product:

$$3 * 8 = 24 \tag{1.2.6}$$

1	24
2	12
3	8
4	6

We can see that the factor pair of 24 that adds up to 10 is $6 * 4$. We proceed by splitting the $10x$ into $6x + 4x$ and then factor by grouping. If you are uncomfortable with factoring by grouping, then this is probably not a good method to try. However, if you are comfortable with factoring by grouping, the rest of the process is relatively straightforward:

$$3x^2 + 10x + 8 = 3x^2 + 6x + 4x + 8 \tag{1.2.7}$$

We then factor a common factor from the first two terms and the last two terms separately, and then factor out the common binomial factor of $(x + 2)$

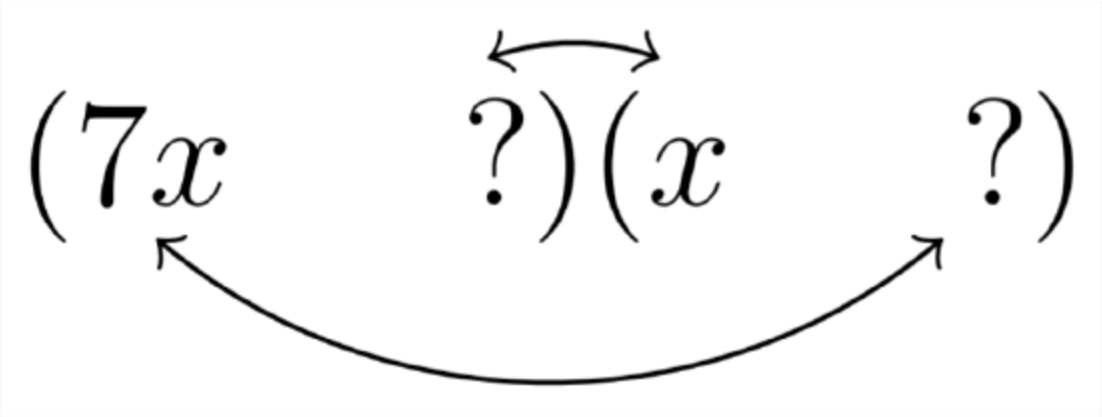
$$\begin{aligned} 3x^2 + 10x + 8 &= 3x^2 + 6x + 4x + 8 \\ &= 3x(x + 2) + 4(x + 2) \\ &= (x + 2)(3x + 4) \end{aligned} \tag{1.2.8}$$

Example 1.2.7

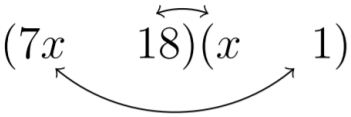
Factor $7x^2 - 5x - 18$

Solution

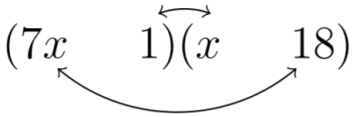
In this example, the sign of the constant term is negative, which means that we'll need factor pairs that produce a difference of 5. The leading coefficient is 7, which is prime, so, again, the only way to split up the 7 is $7 * 1$



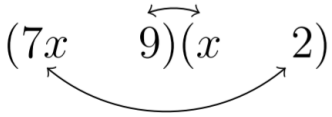
The options for filling in the question marks come from the factors of 18, for which there are three possibilities: $18 * 1, 9 * 2$, or $6 * 3$. We'll try each of these factor pairs in place of the question marks:



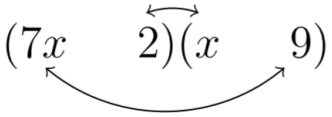
$$18x - 7x = 11x$$



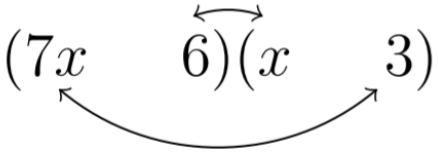
$$126x - 1x = 125x$$



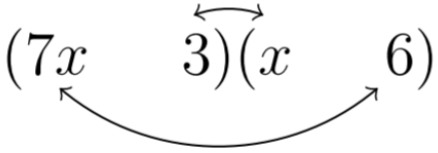
$$14x - 9x = 5x$$



$$63x - 2x = 61x$$

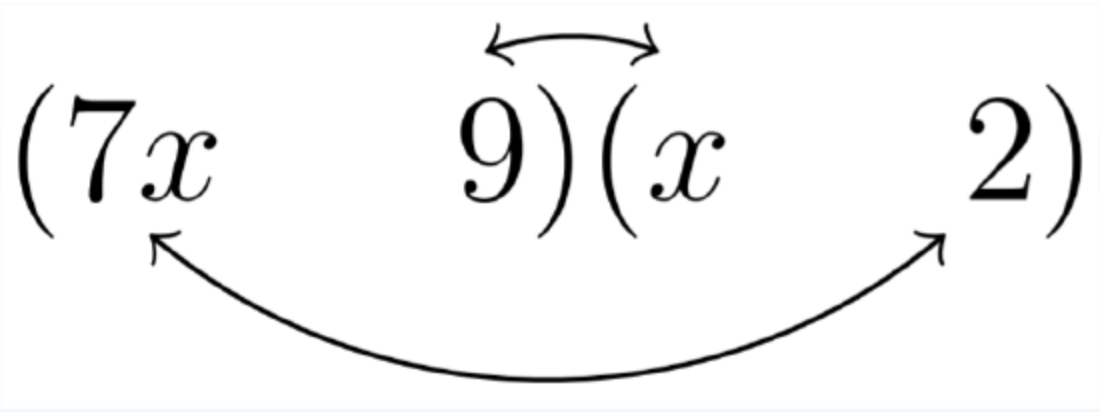


$$21x - 6x = 15x$$



$$42x - 3x = 39x$$

The choice above:



gives us the required $5x$ as the middle term. since we're looking for a $(-5x)$, we'll make the 14 negative and the 9 positive: $-14x + 9x = -5x$

$$7x^2 - 5x - 18 = (7x + 9)(x - 2)$$

If we want to try the other method for factoring $7x^2 - 5x - 18$, we would multiply $7 * 18 = 126$, and then work to find factor pairs of 126 that have a difference of 5

1	126
2	63
3	42
6	21
7	18
9	14

Here, the last factor pair, $9 * 14$, has a difference of 5. So then we proceed to factor by grouping:

$$\begin{aligned} 7x^2 - 5x - 18 &= 7x^2 + 9x - 14x - 18 \\ &= x(7x + 9) - 2(7x + 9) \\ &= (7x + 9)(x - 2) \end{aligned} \tag{1.2.9}$$

Notice that when the -2 was factored out from the last two terms $-14x - 18$, we ended up with $-2(7x + 9)$, because $(-2) * (+9) = -18$. This is also important because in order to factor out the common binomial factor of $(7x + 9)$, this binomial must be exactly the same in both terms.

Difference of Squares

Factoring a difference of squares is actually a special form of trinomial factoring. If we consider a trinomial of the form $ax^2 + bx + c$, where c is a perfect square and negative, we will find something interesting about the possible values of b that make the trinomial factorable.
Example

$$\text{Consider } x^2 + bx - 36 \tag{1.2.10}$$

For this expression to be factorable, the middle coefficient b would need to be equal to the difference of any of the factor pairs of 36. If we look at the possible factor pairs, we see the following:

1	36
2	18
3	12
4	9
6	6

This means that the possible values for b that would make this expression factorable are:

$$\begin{aligned} 36-1=35 &\rightarrow x^2+35x-36=(x+36)(x-1) \\ 18-2=16 &\rightarrow x^2+16x-36=(x+18)(x-2) \\ 12-3=9 &\rightarrow x^2+9x-36=(x+12)(x-3) \\ 9-4=5 &\rightarrow x^2+5x-36=(x+9)(x-4) \\ 6-6=0 &\rightarrow x^2+0x-36=x^2-36=(x+6)(x-6) \end{aligned} \tag{1.2.11}$$

As we see, factoring $x^2 - 36$ means that the factors of the perfect square $36 = 6 * 6$ will cancel each other out leaving $0x$ in the middle. If there is a perfect square as the leading coefficient, then this number should be square rooted as well:

$$16x^2 - 25 = (4x + 5)(4x - 5) \tag{1.2.12}$$

In the example above, the $+20x$ and $-20x$ as the middle terms cancel each other out leaving just $16x^2 - 25$

These three types of factoring can also be combined with each other as we see in the following examples.

Example 1.2.6

Factor $2x^2 - 50$

Solution

This is not a trinomial because it doesn't have three terms. It is also not a difference of squares because 2 and 50 are not perfect squares. However, there is a common factor of 2 which we can factor out:

$$2x^2 - 50 = 2\left(x^2 - 25\right) \tag{1.2.13}$$

The expression inside the parentheses is a difference of squares and should be factored:

$$2x^2 - 50 = 2\left(x^2 - 25\right) = 2(x + 5)(x - 5) \tag{1.2.14}$$

Example 1.2.7

Factor $24 - 2x - x^2$

Solution

Here the sign of the x^2 term is negative. For this problem we can factor out $a - 1$ and proceed as we did with the previous problems in which the leading coefficient was positive or we can factor it as it is:

$$24 - 2x - x^2 = -\left(x^2 + 2x - 24\right) = -(x + 6)(x - 4) \tag{1.2.15}$$

If we want to factor it as it is, we should be aware that the constant term is positive and the quadratic term is negative, which means that we will want the factors of 24 to have a difference of 2

$$24 - 2x - x^2 = (6 + x)(4 - x) \tag{1.2.16}$$

Example 1.2.8

Factor $6x^2 + 12x + 6$

Solution

First, we notice that this expression has a common factor of 6. If we factor out the 6, then we should be left with an easier problem:

$$6x^2 + 12x + 6 = 6\left(x^2 + 2x + 1\right) = 6(x + 1)(x + 1) = 6(x + 1)^2 \tag{1.2.17}$$

Exercises 1.2.1

Factor each expression completely.

1) $8a^2b^3 + 24a^2b^2$
2) $19x^2y - 38x^2y^3$
3) $13t^8 + 26t^4 - 39t^2$
4) $5y^5 + 25y^4 - 20y^3$
5) $45m^4n^5 + 36mn^6 + 81m^2n^3$
6) $125x^3y^5 + 60x^4y^4 - 85x^5y^2$

Factor each trinomial into the product of two binomials.

7) $a^2 + 3a + 2$
8) $y^2 - 8y - 48$
9) $x^2 - 6x - 27$
10) $t^2 - 13t + 42$
11) $m^2 + 3m - 54$
12) $x^2 + 11x + 24$

Factor completely. Remember to look first for a common factor. If the polynomial is prime, state this.

13) $a^2 - 9$
14) $y^2 - 121$
15) $-49 + k^2$
16) $-64 + t^2$
17) $6x^2 - 54$
18) $25y^2 - 4$
19) $200 - 2a^2$
20) $3m^2 - 12$
21) $98 - 8k^2$
22) $-80w^2 + 45$

23) $5y^2 - 80$
24) $-4a^2 + 64$
25) $8y^2 - 98$
26) $24a^2 - 54$
27) $36k - 49k^3$
28) $16y - 81y^3$

Factor each trinomial completely. Remember to look first for a common factor. If the polynomial is prime, state this.
29) $3y^2 - 15y + 16$
30) $8a^2 - 14a + 3$
31) $9x^2 - 18x + 8$
32) $6a^2 - 17a + 12$
33) $2x^2 + 7x + 6$
34) $2m^2 + 13m - 18$
35) $20y^2 + 22y + 6$
36) $36x^2 + 81x + 45$
37) $24a^2 - 42a + 9$
38) $48x^2 - 74x - 10$

Factor each expression completely.
39) $30 + 7y - y^2$
40) $45 + 4a - a^2$
41) $24 - 10x - x^2$
42) $36 - 9x - x^2$
43) $84 - 8x - x^2$
44) $72 - 6a - a^2$
45) $6y^2 + 24y + 15$
46) $10y^2 - 75y + 35$
47) $20ax^2 - 36ax - 8a$

Answer available at the end of this section.

Answers

Exercises 1.2.1

1) $8a^2b^2(b+3)$
3) $13t^2\left(t^6 + 2t^2 - 3\right)$
5) $9mn^3\left(5m^3n^2 + 4n^3 + 9m\right)$
7) $(a+2)(a+1)$
9) $(x-9)(x+3)$
11) $(m+9)(m-6)$
13) $(a+3)(a-3)$
15) $(k+7)(k-7)$
17) $6(x+3)(x-3)$
19) $2(10+a)(10-a)$
21) $2(7+2k)(7-2k)$
23) $5(y+4)(y-4)$
25) $2(2y+7)(2y-7)$
27) $k(6+7k)(6-7k)$
29) PRIME
31) $(3x-4)(3x-2)$
33) $(2x+3)(x+2)$
35) $2(5y+3)(2y+1)$
37) $3(2a-3)(4a-1)$
39) $(10-y)(3+y)$
41) $(12+x)(2-x)$
43) $(14+x)(6-x)$
45) $3\left(2y^2 + 8y + 5\right)$
47) $4a(5x+1)(x-2)$

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1.3: Quadratic Equations

Quadratic equations are equations of the second degree. The solution of quadratic equations has a long history in mathematics going back several thousand years to the geometric solutions produced by the Babylonian culture. The Indian mathematician Brahmagupta used "rhetorical algebra" (algebra written out in words) in the 7 th century to produce solutions to quadratic equations and Arab mathematicians of 9 th and 10 th centuries followed simlar methods. Leonardo of Pisa (also known as Fibonacci) included information on the Arab approach to solving quadratic equations in his book *Liber Abaci*, published in 1202

The quadratic formula is generally used to solve quadratic equations in standard form: $ax^2 + bx + c = 0$. The solutions for this are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.3.1}$$

Now, the question is - why does this formula give solutions to the standard quadratic equation? We can proceed as we normally do in solving linear equations - that is, by getting the x by itself. The only problem here is that instead of just x, there are also terms involving x^2. This is where the process of completing the square comes in handy.

We can begin with the quadratic equation in standard form:

$$ax^2 + bx + c = 0 \tag{1.3.2}$$

Just as it is easier to factor a quadratic trinomial if the leading coefficient is 1, this process of completing the square is also easier if the leading coefficient is $1.\text{So}$ next we will divide through on both sides of this equation by a.

$$\begin{aligned} \tfrac{ax^2+bx+c}{a} &= \tfrac{0}{a} \\ \tfrac{ax^2}{a} + \tfrac{bx}{a} + \tfrac{c}{a} &= \tfrac{0}{a} \\ x^2 + \tfrac{b}{a}x + \tfrac{c}{a} &= 0 \end{aligned} \tag{1.3.3}$$

Then, we will move the $\frac{c}{a}$ to the other side of the equation to clear out some room for completing the square:

$$\begin{aligned} x^2 + \tfrac{b}{a}x + \tfrac{c}{a} &= 0 \\ -\tfrac{c}{a} &= -\tfrac{c}{a} \\ x^2 + \tfrac{b}{a}x &= -\tfrac{c}{a} \end{aligned} \tag{1.3.4}$$

Now we need to complete the square. If you are already familiar with this process, you may wish to skip the following explanation.

If we look at what happens when we square a binomial like $(x+3)^2$, we will begin to notice a pattern.

$$\begin{aligned} (x+3)^2 &= (x+3)(x+3) = x^2 + 6x + 9 \\ (x+4)^2 &= (x+4)(x+4) = x^2 + 8x + 16 \\ (x+5)^2 &= (x+5)(x+5) = x^2 + 10x + 25 \\ (x+6)^2 &= (x+6)(x+6) = x^2 + 12x + 36 \end{aligned} \tag{1.3.5}$$

Our goal in the derivation of the Quadratic Formula is to rewrite the expression $x^2 + \frac{b}{a}x$ as a perfect square in the form $(x+\quad)^2$. The reason that we want to do this is that writing an expression as a binomial squared eliminates the problem of having both an x and an x^2, which was preventing us from getting the x by itself in the standard quadratic equation.

If we can figure out what should take the place of the blanks in the statement:

$$x^2 + \frac{b}{a}x + \quad = (x + \quad)^2 \tag{1.3.6}$$

then we will be well on our way to deriving the quadratic formula.

If we re-examine the sample perfect binomial squares from the previous page, we note a useful pattern. This is that the blank in

the parentheses $(x+\quad)^2$ is filled by a number that is one-half the value of the linear coefficient - or the coefficient of the x^1 term. Notice that in $x^2+6x+9=(x+3)^2, 3$ is half of 6, in $x^2+8x+16=(x+4)^2$, the 4 is half of 8, and so on. If we want to write $x^2+\frac{b}{a}x+\quad$ as a perfect square in the form $(x+\quad)^2$, the blank in the parentheses should be filled by:

$$\frac{1}{2}*\frac{b}{a}=\frac{b}{2a} \tag{1.3.7}$$

Now, it's not true that $x^2+\frac{b}{a}x+\quad=\left(x+\frac{b}{2a}\right)^2$

We're missing the constant term on the left. However, if we return to our perfect square examples, we can see that the constant term is always the square of the term inside inside the parentheses. So, we can restate our problem now as:

$$\begin{aligned} x^2+\tfrac{b}{a}x+\left(\tfrac{b}{2a}\right)^2&=\left(x+\tfrac{b}{2a}\right)^2\\ x^2+\tfrac{b}{a}x+\tfrac{b^2}{4a^2}&=\left(x+\tfrac{b}{2a}\right)^2 \end{aligned} \tag{1.3.8}$$

So, if we return to our originial problem, we were saying that:

$$\begin{aligned} x^2+\tfrac{b}{a}x+\tfrac{c}{a}&=0\\ -\tfrac{c}{a}&=-\tfrac{c}{a}\\ x^2+\tfrac{b}{a}x&=-\tfrac{c}{a} \end{aligned} \tag{1.3.9}$$

We can add $\frac{b^2}{4a^2}$ to both sides of this equation and then restate the left hand side as a perfect square of a binomial:

$$\begin{aligned} x^2+\tfrac{b}{a}x\quad&=-\tfrac{c}{a}\\ +\tfrac{b^2}{4a^2}&=+\tfrac{b^2}{4a^2}\\ x^2+\tfrac{b}{a}x+\tfrac{b^2}{4a^2}&=-\tfrac{c}{a}+\tfrac{b^2}{4a^2}\\ \left(x+\tfrac{b}{2a}\right)^2&=-\tfrac{c}{a}+\tfrac{b^2}{4a^2} \end{aligned} \tag{1.3.10}$$

The last tricky bit of this derivation is adding the two fractions on the right hand side. The common denominator for these fractions is $4a^2$, so we'll need to multiply the $-\frac{c}{a}$ by $\frac{4a}{4a}$ to get $-\frac{4ac}{4a^2}$. Then the right hand side will be $\frac{b^2-4ac}{4a^2}$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \tag{1.3.11}$$

Then, we can take the square root of both sides and get the x by itself:

$$\begin{aligned} \sqrt{\left(x+\tfrac{b}{2a}\right)^2}&=\pm\sqrt{\tfrac{b^2-4ac}{4a^2}}\\ x+\tfrac{b}{2a}&=\tfrac{\pm\sqrt{b^2-4ac}}{\sqrt{4a^2}}\\ x+\tfrac{b}{2a}&=\tfrac{\pm\sqrt{b^2-4ac}}{2a} \end{aligned} \tag{1.3.12}$$

Subtracting $\frac{b}{2a}$ from both sides is easy since we already have a common denominator:

$$\begin{aligned} x+\tfrac{b}{2a}&=\tfrac{\pm\sqrt{b^2-4ac}}{2a}\\ -\tfrac{b}{2a}&=-\tfrac{b}{2a}\\ x&=\tfrac{-b\pm\sqrt{b^2-4ac}}{2a} \end{aligned} \tag{1.3.13}$$

At the College Algebra level, it is often useful to program the quadratic formula onto a graphing calculator both for easy use and also to learn a little bit about programming. The following program is a simple example of this for the TI84 series of graphing calculators. Graphing calculators also often have built-in polynomial solver feature that can be used to solve quadratics.

Press the "prgm" key in the top middle of the calculator keypad. This will bring up a screen that shows EXEC EDIT NEW across the top. Arrow over across the top to "NEW," and then select 1: Create New.

This will bring up a screen asking you to name the program. You should see PROGRAM and then underneath it, "Name=". The

alpha lock is on automatically, so any key you press will type the letter associated with it. Name your program and press ENTER. You should see PROGRAM: Name, with whatever name you've chosen for your program. Underneath this you will se a colon: This is where you will enter the commands for the program.

First we need to enter the values for A, B and C from the quadratic equation into the calculator. To do this, press the "prgm" key again. Across the top of the screen you should see CTL I/O COLOR EXEC.

Arrow over to I/O. This is the "input/output" menu. Choose number 2:Prompt. This will return you to the program screen where you will see :Prompt under the name of the program. After :Prompt, type A, B, C. You'll need to use the "alpha" key to access the letters and the comma is right above the 7 key.

PROGRAM: Name (whatever name you've chosen should show here) :Prompt A, B, C
On the next line of the program, we will take the values of A, B and C and use them to calculate the values of the roots of the equation. Type in the following:
PROGRAM: Name :Prompt $\mathrm{A}, \mathrm{B}, \mathrm{C}$
: $(-B + \sqrt{(B^2 - 4AC)})/(2A) \to R$
: $(-B - \sqrt{(B^2 - 4AC)})/(2A) \to S$

In typing these two lines it's important that when you type $-B$, you use the negative key next to the decimal point, rather than the subtraction key. The calculator is very picky about this. When you're typing the $B^2 - 4AC$, you'll need to use the subtraction key on the far right of the keypad.

Also notice the double parentheses - one set for the numerator of the fraction and one set for the square root. If you don't type this in correctly it will produce Wrong answers. The arrow in the formula stores the values of the answer in the variables R and S, and the arrow is produced by the "sto $\to$ " key just above the ON button in the lower left of the keyboard.

Now that we've given the calculator the values for A, B and C and then had the calculator find the roots of the eqatiuon, we need to display the answers. If you press the prgm key and arrow over to the I/O menu again, you can choose 3:Disp. This will Display the answers that we've stored as R and S.

PROGRAM: Name :Prompt A, B, C : $(-B + \sqrt{B^2 - 4AC}))/(2A) \to R$
: $(-B - \sqrt{B^2 - 4AC}))/(2A) \to S$
: Disp R, S
Now we can test the program with some simple equations. To run the program, press the program key and choose the program you've created either by selecting it and pressing enter, or by pressing the number for the program in the list. This should bring you back to the calculation screen, where you can run the program by pressing enter. The calculator should then ask you for the values of A, B and
C

Solve for x : $\quad 2x^2 - x - 1 = 0$
In this example the values for A, B and C are:
$A = 2$
$B = -1$
$C = -1$
Again, it's important that you use the negative sign key next to the decimal point for the values of any negative coefficients and not the subtraction key. The calculator should return values of 1 and -0.5 as the solutions.

Solve for
x : $\quad x^2 + x + 1 = 0$
In this example the values for A, B and C are:
$A = 1$
$B = 1$
$C = 1$
The calculator should return values of $-0.5 \pm 0.8660254038i$ as solutions.

If you get an error message saying "NONREAL AnswerS," you'll need to adjust the calculator setting to allow for complex valued answers. You can do this by presing the "mode" key in the top left of the keypad and arrowing down to the line that reads "REAL a+bi $\mathrm{re}^{\wedge}(\theta i)$." You can then arrow over to "a+bi" and press enter. This will allow the calculator to compute complex valued answers. Something very important to remember about the quadratic formula is that the equation must be in standard form in order to

identify the values of A, B and C to use in the formula. For example in the equation:

$$3x^2 - 7 = 2x \tag{1.3.14}$$

it is important to understand that the values of A, B and C come from the standard form of the equation and not the present form of the equation. There are several pitfalls to watch out for in this equation. First of all, the $2x$ is on the opposite side of the equation from the other terms. That means that the value of B IS $\mathrm{NOT} + 2$. Also, if we were to move the $2x$ to the other side to put the equation in standard form, it is not the order of the terms, but degree of the variable that determines whether a coefficient is identified as A, B or C

In moving the $2x$ to the other side of the equation, I have seen students put the term they're adding to that side as the last term. There is nothing wrong about this, but if you do that, you must be careful about identifying the values of A, B and C

$$\begin{aligned} 3x^2 - 7 &= 2x \\ -2x &= -2x \\ 3x^2 - 7 - 2x &= 0 \end{aligned} \tag{1.3.15}$$

There is nothing wrong about the way the equation above is written depsite the fact that it is not in "standard form." The important thing to remember is that "A" is not the coefficient of whichever term is listed first. It is the coefficient of the quradratic, or x^2 term. Likewise, "B" is not the coefficient of the second term, but rather the coefficient of the linear, or x^1 term. And "C" is not whichever number comes last, but rather the value of the constant term. So in the equation above, however it is written, the value of A is $+3, \mathrm{B}$ is -2 and C is -7

Exercises 1.3.1

Solve for x in each equation. Round any irrational values to the nearest 1000 th.

1) $x^2 + 7x = 2$
2) $5x^2 - 3x = 4$
3) $\frac{3}{4}x^2 = \frac{7}{8}x + \frac{1}{2}$
4) $\frac{2}{3}x^2 - \frac{1}{3} = \frac{5}{9}x$
5) $2x^2 + (\sqrt{5})x - 3 = 0$
6) $3x^2 + x - \sqrt{2} = 0$
7) $2.58x^2 - 3.75x - 2.83 = 0$
8) $3.73x^2 + 9.74x + 2.34 = 0$
9) $5.3x^2 + 7.08x + 1.02 = 0$
10) $3.04x^2 + 1.35x + 1.234 = 0$
11) $7x(x + 2) + 5 = 3x(x + 1)$
12) $5x(x - 1) - 7 = 4x(x - 2)$
13) $14(x - 4) - (x + 2) = (x + 2)(x - 4)$
14) $11(x - 2) + (x - 5) = (x + 2)(x - 6)$

Answer available at the end of this section.

Answers

Exercises 1.3.1

1) $x \approx 0.275, -7.275$
3) $x \approx 1.587, -0.420$
5) $x \approx 0.787, -1.905$
7) $x \approx -0.548, 2.002$
9) $x \approx -0.164, -1.172$
11) $x \approx -0.575, -2.175$
13) $x = 10, 5$

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1.4: Complex Numbers

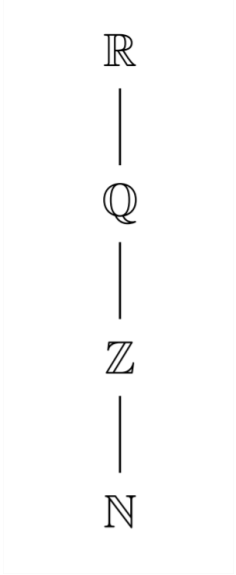
Our number system can be subdivided in many different ways. The most basic form of mathematics is counting and almost all human cultures have words to represent numbers (the Pirahã of South America are a notable exception). Thus the most basic set of numbers is the set of counting numbers represented by the double barred $\mathrm{N}: \mathbb{N}=\{1,2,3,4,5,6,7, \ldots\}$ (we will set aside the debate as to whether or not zero should be included in this set).

If we try to subtract a larger counting number from a smaller counting number we find that there are no members in the set of counting numbers to represent the answer in this situation. This extends the set of natural numbers to the set of integers: $\mathbb{Z}$ $=\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}$. The integers are represneted by the double barred Z, for the German word for numbers-"zahlen." In the earliest appearances of negative numbers in the Chinese and Indian mathematical systems, negative values were often used to represent debt. Because Greek mathematics was based on Geometry, they did not use negative numbers.

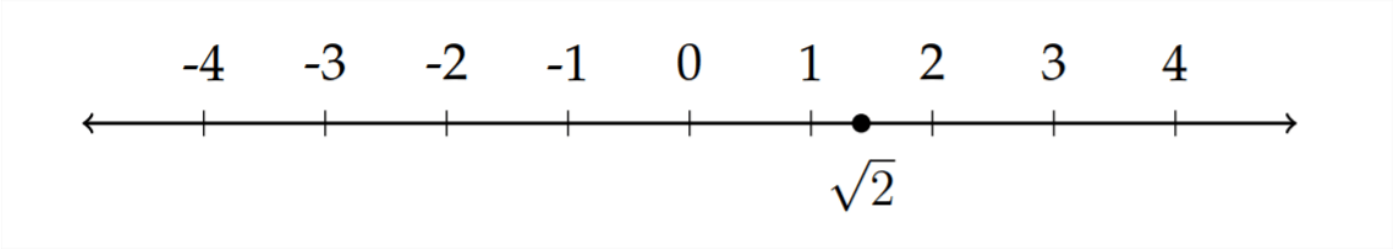
Moving to multiplication and division, if we question the value of $8 \div 2=4$ versus $8 \div 3=?$, we once again must expand our conception of numbers to allow for an answer to the second question $8 \div 3=?$. Understanding ratios of whole numbers or Rational numbers allows solutions to such problems. The set of Rational numbers is represented by the double barred Q, to represent a quotient:
$\mathbb{Q}=\left\{\frac{a}{b}: a, b \in \mathbb{Z}\right\}$

The Greek understanding of numbers mostly stopped here. They felt that all quantities could be represented as the ratio of whole numbers. The length of the diagonal of a square whose sides are of length 1 produced considerable consternation among the Pythagoreans as a result of this. Using the Pythagorean Theorem for the diagonal of a square whose sides are of length 1 shows that the diagonal would be $c^2=1^2+1^2=2$, thus $c=\sqrt{2}$. This number cannot be represented as a ratio of whole numbers. This new class of numbers adds the set
of irrational numbers to the existing set of rational numbers to create the Real numbers, represented with a double barred $\mathrm{R}: \mathbb{R}$.

This hierarchy of numbers is often represented in the following diagram:



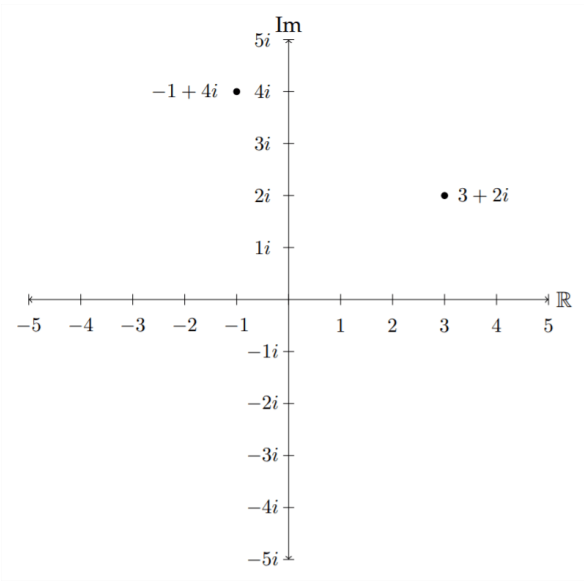
One of the best ways to conceptualize the Real number system is on the number line - every point on the number line corresponds to a unique Real number and every Real number corresponds to a unique position on the Real number line.



After the development of the printing press in the 15 th century, Fibonacci's Liber Abaci was translated into Italian from Latin and read throughout Italy. As a result, Italy became a thriving center of mathematics until the 17 th century, when the center of European mathematics moved north to France, Germany and England.

Throughout the 1500 's Italian mathematicians such as Girolamo Cardano, Raphael Bombelli and Niccolo Fontana Tartaglia worked to extend the ideas in Fibonacci's book. They produced formulas to solve cubic (x^3), and quartic (x^4) degree equations. In solving some of these equations they found that their formulas sometimes produced negative values under a square root. None of the known number systems could accomodate this possibility. In Cardano's book on algebra Ars Magna, he encounters a problem which involves the square root of a negative number. He says, "It is clear that this case is impossible. Nevertheless we will work thus..." and he proceeds to compute a valid complex solution to the problem. Mathematicians eventually defined the complex unit $\sqrt{-1} = i$ and then

devised a system in which all complex numbers are a combination of a Real part
(a) and an "imaginary" part (bi).
The complex numbers are a two dimensional number system represented by the double barred $C : \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, where i is the complex unit defined as $i = \sqrt{-1}$. Throughout the late 1700 's and early 1800 's mathematicians gradually moved towards a geometrical interpretation of the two-dimensional complex numbers. What is today known as the "Argand diagram" represents the real valued portion of a complex number along a horizontal axis and the multiple of the complex unit along the vertical axis.



Expressing square roots of negative numbers

Square roots of negative quantities are generally expressed as a multiple of i

Example 1.4.1

$$\begin{aligned}\sqrt{-4} &= 2i \\ \sqrt{-25} &= 5i \\ \sqrt{-7} &\approx 2.646i\end{aligned} \tag{1.4.1}$$

Adding, subtracting and multiplying with complex numbers

Calculating with complex numbers has many similarities to working with variables. The real part and imginary part are treated separately for addition and subtraction, but can be multiplied and divided.

Example 1.4.2

Compute the following:

$$\begin{aligned}(6-4i)+(-2+7i) &= 4+3i \\ (-9+2i)-(-4+6i) &= -9+2i+4-6i = -5-4i \\ 3(10+i) &= 30+3i \\ -7i(-5+8i) &= 35i-56i^2\end{aligned} \tag{1.4.2}$$

Now we encounter an interesting fact about complex numbers and, in particular, the complex unit i. By definition, $i=\sqrt{-1}$. Therefore, if we square i we should get -1. In the last example problem above, we can replace the i^2 with -1 to finish the problem.

$$\begin{aligned}-7i(-5+8i) &= 35i-56i^2 \\ &= 35i-56(-1) \\ &= 35i+56 \\ &= 56+35i\end{aligned} \tag{1.4.3}$$

Example 1.4.3

Compute the following:

$$\begin{aligned}(8-5i)(1-4i) &= 8-32i-5i+20i^2 \\ &= 8-37i+20(-1) \\ &= 8-37i-20 \\ &= -12-37i\end{aligned}$$

$$\begin{aligned}(9+2i)^2 &= (9+2i)(9+2i) \\ &= 81+18i+18i+4i^2 \\ &= 81+36i+4(-1) \\ &= 81+36i-4 \\ &= 77+36i\end{aligned} \tag{1.4.4}$$

Powers of i

The powers of i follow an interesting pattern based on the definition that $i^2=-1$
We can see that $i^1=i$ and that $i^2=-1$, as a result, $i^3=i^2*i^1=-1*i=-i$
In a similar fashion, $i^4=i^2*i^2=(-1)(-1)=1$
This means that $i^5=i^4*i=1*i=i$
If we put all of this information together we get the following:
$i^1=i$

$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i^1 = i$
$i^6 = i^2 = -1$
$i^7 = i^3 = -i$
$i^8 = i^4 = 1$
In other words, every power of i is equivalent to either $i, -1, -i,$ or 1. To determine which of these values a power of i is equivalent to, we need to find the remainder of the exponent when it is divided by 4

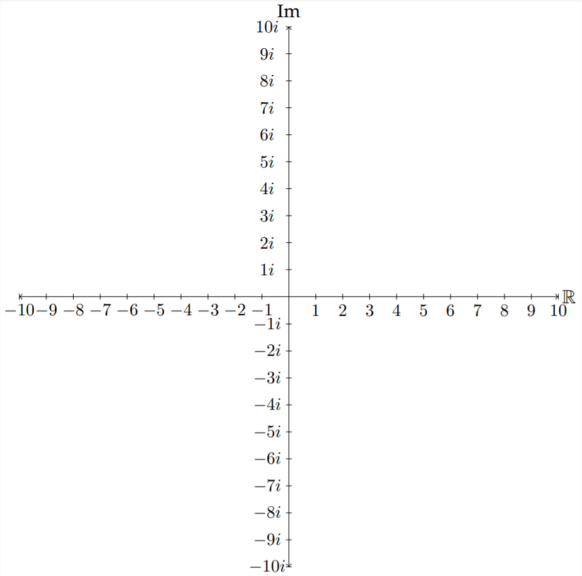
Example 1.4.4

Simplify i^{38}

Solution

since every $i^4 = 1$, then $i^{38} = i^{36} * i^2 = (i^4)^9 * i^2 = 1^9 * i^2 = i^2 = -1$
since 38 is 2 more than a multiple of 4, then $i^{38} = i^2 = -1$

Exercise 1.4.1



Graph the following complex numbers:

1) $2 + 5i$
2) $4 - 3i$
3) $-2 + 6i$
4) $-3 - 5i$
5) 4
6) $-2i$
7) $7 - i$
8) $-1 + i$
9) $-8 + 4i$
10) $8 + 3i$
11) $7i$
12) $-5 - 9i$

Express each quantity in terms of i. Round irrational values to the nearest 1000 th.

13) $\sqrt{-36}$
14) $\sqrt{-81}$
15) $\sqrt{-100}$
16) $\sqrt{-49}$
17) $\sqrt{-4}$
18) $\sqrt{-25}$
19) $\sqrt{-2}$
20) $\sqrt{-6}$
21) $\sqrt{-10}$
22) $\sqrt{-31}$
23) $\sqrt{-5}$
24) $\sqrt{-3}$

Perform the indicated operation and simplify

25) $(6+7i)+(5+3i)$
26) $(4-5i)+(3+9i)$
27) $(9+8i)-(1-2i)$
28) $(2+i)-(6-4i)$
29) $(7-4i)-(5-3i)$
30) $(8+i)-(4+3i)$
31) $(7i)(6i)$
32) $(4i)(-8i)$
33) $(-2i)(5i)$
34) $(12i)(3i)$
35) $(1+i)(3+2i)$
36) $(1+5i)(4+3i)$
37) $(6-5i)(2-3i)$
38) $(8-3i)(2+i)$
39) $(-3+4i)(-1-2i)$
40) $(-7-i)(3-5i)$
41) $(4-2i)^2$
42) $(-5+i)^2$
43) $(3+i)(3-i)$
44) $(2+6i)(2-6i)$
45) $(9-4i)(9+4i)$
46) $(5+2i)(5-2i)$

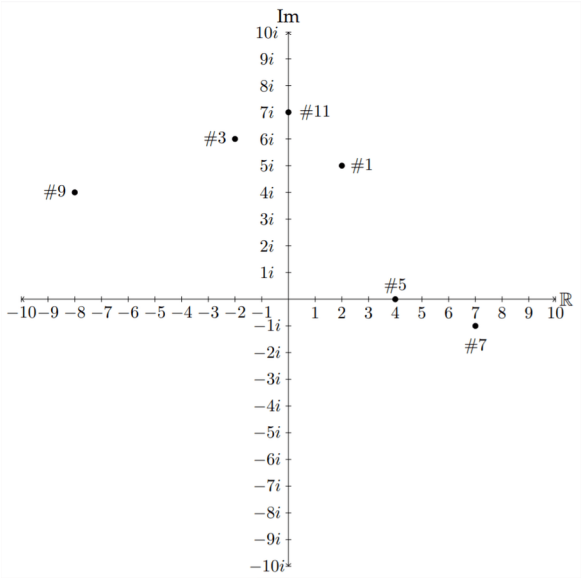
Express as either $i, -1, -i$, or 1

47) i^3
48) i^7
49) i^{21}
50) i^{13}
51) i^{29}
52) i^{56}
53) i^{72}
54) i^{35}
55) i^{66}
56) i^{103}
57) i^{16}
58) i^{53}
59) i^{11}
60) i^{42}
61) i^{70}
62) i^9

Answer available at the end of this section.

Answers

Exercise 1.4.1



13) $6i$
15) $10i$
17) $2i$
19) $1.414i$
21) $3.162i$
23) $2.236i$
25) $11 + 10i$
27) $8 + 10i$
29) $2 - i$
31) -42
33) 10
35) $1 + 5i$
37) $-3 - 28i$
39) $11 + 2i$
41) $12 - 16i$
43) 10
45) 97
47) $-i$
49) i
51) i
53) 1
55) -1
57) 1
59) $-i$
61) -1

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1.5: Quadratic Equations with Complex Roots

In Section 1.3, we considered the solution of quadratic equations that had two real-valued roots. This was due to the fact that in calculating the roots for each equation, the portion of the quadratic formula that is square rooted ($b^2 - 4ac$, often called the discriminant) was always a positive number.

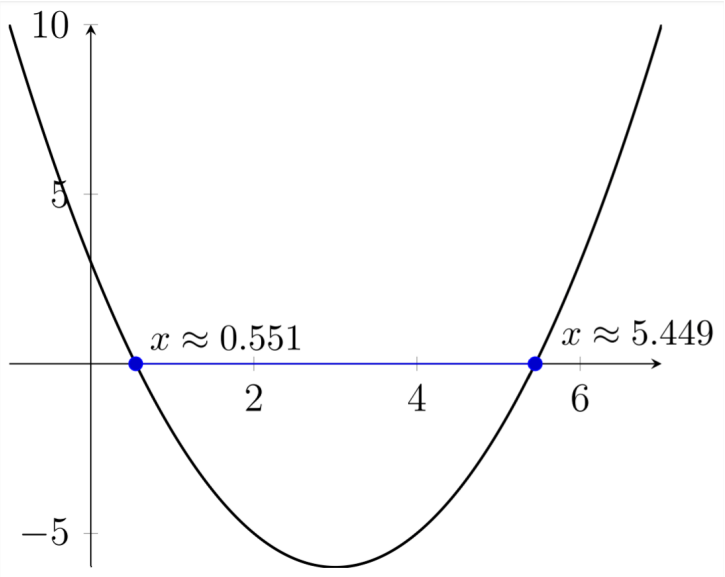
For example, in using the quadratic formula to calculate the the roots of the equation $x^2 - 6x + 3 = 0$, the discriminant is positive and we will end up with two real-valued roots:

$$\begin{aligned} x^2 - 6x + 3 &= 0 \\ a = 1, b = -6, c &= 3 \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2*1} \\ &= \frac{6 \pm \sqrt{36 - 12}}{2} \\ &= \frac{6 \pm \sqrt{24}}{2} \\ &= \frac{6 \pm 4.899}{2} \end{aligned} \tag{1.5.1}$$

$$\begin{aligned} &\approx \frac{6+4.899}{2} && \approx \frac{6-4.899}{2} \\ &\approx \frac{10.899}{2} && \approx \frac{1.101}{2} \\ &\approx 5.449 && \approx 0.551 \end{aligned} \tag{1.5.2}$$

When we added and subtracted the square root of 24 to 6 in the quadratic formula, this created two answers, and they were real-valued because the square root of 24 is real-valued.

Another way to see this is graphically. If we graph $y = x^2 - 6x + 3$ and find the x values that make $y = 0$, these will appear along the x -axis, and will be the same values that solve the equation $x^2 - 6x + 3 = 0$



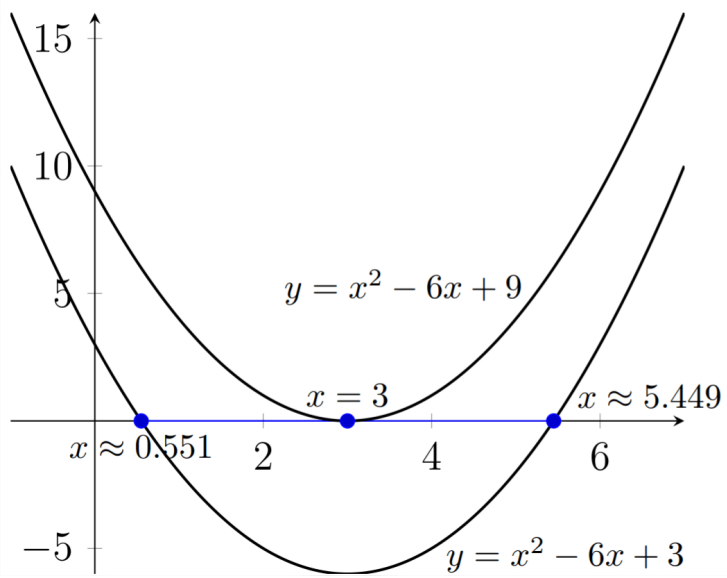
If we consider a related, but slightly different equation to start with, these relationships between the roots, the discriminant and the graphical intersections will be slightly different.

$$\begin{aligned} x^2 - 6x + 9 &= 0 \\ a = 1, b = -6, c &= 9 \end{aligned} \tag{1.5.3}$$

$$
\begin{aligned}
x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(9)}}{2 * 1} \\
&= \frac{6 \pm \sqrt{36 - 36}}{2} \\
&= \frac{6 \pm \sqrt{0}}{2} \\
&= \frac{6}{2} = 3
\end{aligned} \tag{1.5.4}
$$

Because the discriminant was 0 in this problem, we only get one real-valued answer.

Graphically, the additional 6 that was added to the original equation to change it from $x^2 - 6x + 3$ to $x^2 - 6x + 9$ shifts every y value on the graph up 6 units.

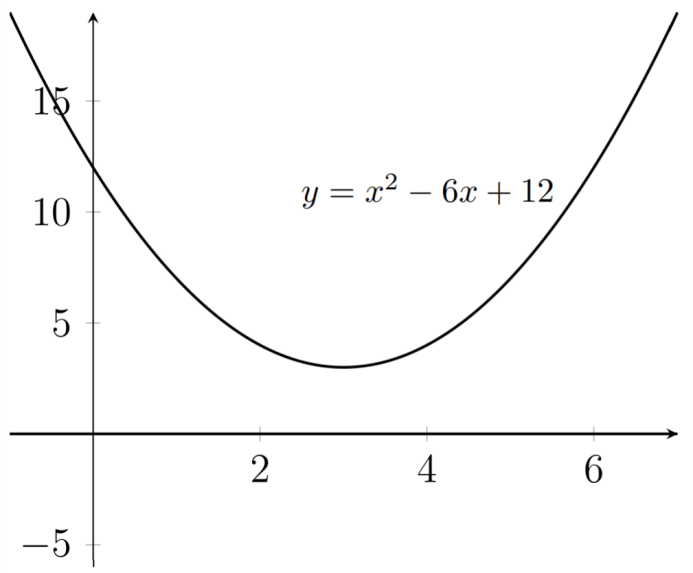


If we add an additional three units to the constant term of this quadratic equation, we encounter a third possibility.

$$
\begin{aligned}
x^2 - 6x + 12 &= 0 \\
a = 1, b = -6, c &= 12 \\
= \tfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(12)}}{2*1} & \\
= \tfrac{6 \pm \sqrt{36-48}}{2} & \\
= \tfrac{6 \pm \sqrt{-12}}{2} & \\
= \tfrac{6}{2} \pm \tfrac{6 \cdot 464i}{2} & \\
\approx 3 \pm 1.732i &
\end{aligned} \tag{1.5.5}
$$

Here the discriminant is negative, which leads to two complex-valued answers. If the equation has real-valued coefficients, the complex roots will always come in conjugate pairs. Complex conjugates share the same real-valued part and have opposite signs in their complex-valued (or imaginary) parts: $a \pm bi$

Graphically, the previous problem was one step away from not intersecting the x -axis at all and the additional three units that we added on to get $y = x^2 - 6x + 12$ moves the graph entirely away from the x -axis. Because the roots are complex-valued, we don't see any roots on the x -axis. The x -axis contains only real numbers.



since the calculator has been programmed for the quadratic formula, the focus of the problems in this section will be on putting them into standard form.

Example 1.5.1

Solve for x
$(2x+1)(x+5) - 2x(x+7) = 5(x+3)^2$

Solution

$$\begin{aligned}(2x+1)(x+5) - 2x(x+7) &= 5(x+3)^2\\ 2x^2+11x+5-2x^2-14x &= 5(x+3)(x+3)\\ -3x+5 &= 5\left(x^2+6x+9\right)\\ -3x+5 &= 5x^2+30x+45\\ 0 &= 5x^2+33x+40\\ x &= 5, b = 33, c = 40\\ x &= -5, -1.6\end{aligned} \tag{1.5.6}$$

The fact that the roots of this equation were rational numbers means that the equation could have been solved by factoring.

$$\begin{aligned}0 &= 5x^2+33x+40\\ 0 &= (5x+8)(x+5)\\ 5x &= -8 \qquad x+5=0\\ 5x+8 &= 0 \qquad x=-5\\ x &= -1.6\end{aligned} \tag{1.5.7}$$

Example 1.5.1

Solve for x
Misplaced &

Solution

$$\begin{aligned} x^2-4x+4+3\left(4x^2+3x-1\right) &= 7\left(x^2-1\right) \\ x^2-4x+4+12x^2+9x-3 &= 7x^2-7 \\ 13x^2+5x+1 &= 7x^2-7 \\ 6x^2+5x+8 &= 0 \\ a=6, b=5, c &= 8 \\ x \approx -0.41\overline{6} \pm 1.077i &\approx -\frac{5}{12} \pm 1.077i \end{aligned} \tag{1.5.8}$$

Exercise 1.5.1

Solve for x in each equation. Round any irrational values to the nearest 1000 th.

1) $3x^2-3x=4$
2) $4x^2-2x=7$
3) $5x^2=3-7x$
4) $3x^2=21-14x$
5) $6x^2+1=2x$
6) $5x-3x^2=17$
7) $(5x-1)(2x+3)=3x-20$
8) $(x+4)(3x-1)=9x-5$
9) $(x-2)^2=8x(x-1)+10$
10) $(2x-3)^2=2x-7x^2$
11) $(x+5)(x-6)=(2x-1)(x-4)$
12) $(3x-4)(x+2)=(2x-5)(x+5)$

Answer available at the end of this section.

Answers

Exercise 1.5.1

1) $x \approx 1.758, -0.758$
3) $x \approx 0.344, -1.744$
5) $x \approx 0.1\overline{6} \pm 0.373i$
7) $x \approx -0.5 \pm 1.204i$
9) $x \approx 0.286 \pm 0.881i$
11) $x \approx 4 \pm 4.243i$

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1.6: Multiplying and Dividing Rational Expressions

Reducing Rational Expressions
A rational expression is simply an algebraic fraction, and our first consideration will be to reduce these expressions to lowest terms in the same way that we reduce numerical fractions to lowest terms. When we reduce $\frac{6}{15}$ to $\frac{2}{5}$ by canceling the common factor of three, we are removing a redundant factor of 1 in the form of $\frac{3}{3}$

$$\begin{aligned} \frac{6}{15} &= \frac{3*2}{3*5} = \frac{3}{3} * \frac{2}{5} \\ &= 1 * \frac{2}{5} \\ &= \frac{2}{5} \end{aligned} \tag{1.6.1}$$

Similarly, if there is a common factor that can be factored out of an algebraic fraction, this also can be canceled.

$$\begin{aligned} \frac{21x+14}{7x+7} &= \frac{7(3x+2)}{7(x+1)} \\ &= \frac{7}{7} * \frac{3x+2}{x+1} \\ &= 1 * \frac{3x+2}{x+1} \\ &= \frac{3x+2}{x+1} \end{aligned} \tag{1.6.2}$$

It's important to remeber that only common factors can be canceled. This means that the first priority in each problem will be to identify the factors of the numerator and denominator to see if they share any common factors.

Example
Reduce to lowest terms.

$$\frac{x^2+4x-12}{x^2-4}$$

$$\begin{aligned} \frac{x^2+4x-12}{x^2-4} &= \frac{(x+6)(x-2)}{(x+2)(x-2)} \\ &= \frac{(x+6)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} \\ &= \frac{x+6}{x+2} \end{aligned} \tag{1.6.3}$$

Notice that we can't cancel the 6 and the 2 in the final answer because they aren't factors. The plus signs in the numerator and denominator prevent us from cancelling the 6 and the 2.

In the previous examples, we saw that cancelling out common factors in the numerator and denominator was actually a process of eliminating a redundant factor of 1. In the following example, we'll see a slightly different form of cancelling.
Example
Reduce to lowest terms.

$$\begin{aligned} &\frac{16-x^2}{x^2+x-20} \\ &\frac{16-x^2}{x^2+x-20} = \frac{(4+x)(4-x)}{(x+5)(x-4)} \end{aligned} \tag{1.6.4}$$

In this problem, there are no common factors, but we can do some cancelling. We can see that $(4-x)$ and $(x-4)$ are not the same expression. In the first binomial, the 4 is positive and the x is negative, whereas in the second binomial, the 4 is negative and the x is positive. So, we know that $\frac{4-x}{x-4} \neq 1$. However, if we factor $\mathrm{a}(-1)$ out of the numerator, we will see an interesting phenomenon:

$$\begin{aligned}\frac{4-x}{x-4} &= \frac{-1(-4+x)}{x-4} \\ &= \frac{-1(x-4)}{x-4} \\ &= -1 * \frac{x-4}{x-4} \\ &= -1 * 1 = -1\end{aligned} \tag{1.6.5}$$

Therefore, although $\frac{4-x}{x-4} \neq 1$, we can say that $\frac{4-x}{x-4} = -1$. This will allow us to cancel $(4-x)$ and $(x-4)$ and replace them with (-1)

$$\begin{aligned}\frac{16-x^2}{x^2+x-20} &= \frac{(4+x)(4-x)}{(x+5)(x-4)} \\ &= \frac{(4+x)\,\cancel{(4-x)}}{(x+5)\,\cancel{(x-4)}\,(-1)}\end{aligned} \tag{1.6.6}$$

In the final answer, the (-1) can be placed in the denominator or the numerator, but not both. It can also be placed in front of the fraction.

$$\begin{aligned}\frac{4+x}{-1(x+5)} &= \frac{-1(4+x)}{x+5} \\ &= -\frac{4+x}{x+5}\end{aligned} \tag{1.6.7}$$

Multiplying and Dividing Rational Expressions

In multiplying and dividing rational expressions, it is often easier to identify and cancel out common factors before multiplying rather than afterwards. Multiplying rational expressions works the same way that multiplying numerical fractions does - multiply straight across the top and straight across the bottom. As a result, any factor in either numerator of the problem will end up in the numerator of the answer. Likewise, any factor in either denominator of the problem will end up in the denominator of the answer. Thus, any factor in either numerator can be cancelled with any factor in either denominator.

Example

Multiply. Express your answer in simplest form.

$$\begin{aligned}&\frac{x^2+5x+6}{25-x^2} * \frac{x^2-2x-15}{x^2+6x+9} \\ \frac{x^2+5x+6}{25-x^2} * \frac{x^2-2x-15}{x^2+6x+9} &= \frac{(x+2)(x+3)}{(5+x)(5-x)} * \frac{(x-5)(x+3)}{(x+3)(x+3)} \\ &= \frac{(x+2)\,\cancel{(x+3)}}{(5+x)\,\cancel{(5-x)}\,(-1)} * \frac{\cancel{(x-5)}\,\cancel{(x+3)}}{\cancel{(x+3)}\,\cancel{(x+3)}} \\ &= -\frac{x+2}{x+5}\end{aligned} \tag{1.6.8}$$

Dividing rational expressions works in much the same way that dividing numerical fractions does. We multiply by the reciprocal. There are several ways to demonstrate that this is a valid definition for dividing. First, it is important to understand that the fraction bar is the same as a "divided by" symbol:

$$\frac{8}{2} = 8 \div 2 = 4 \tag{1.6.9}$$

The same is true for dividing fractions:

$$\frac{1}{3} \div \frac{2}{5} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} \tag{1.6.10}$$

We can take the complex fraction $\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}$ and multiply it by 1 without changing its value:

$$\begin{aligned}\frac{1}{3}\div\frac{2}{5}&=\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}*1\\&=\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}\end{aligned}\tag{1.6.11}$$

We can multiply by any form of 1 we want to and not change the value of the result.

$$\begin{aligned}\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}*1&=\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}\\\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}*\frac{9}{9}&=\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}\\\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}*\frac{12}{12}&=\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}\end{aligned}\tag{1.6.12}$$

With a carefully chosen form of 1, we can transform the division problem into a multiplication problem.

$$\begin{aligned}\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}*\frac{\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)}&=\frac{\frac{1}{3}*\frac{5}{2}}{\frac{2}{5}*\frac{5}{2}}\\&=\frac{\frac{1}{3}*\frac{5}{2}}{1}\\&=\frac{1}{3}*\frac{5}{2}\\&=\frac{5}{6}\end{aligned}\tag{1.6.13}$$

In this way, we can redefine division as multiplication by a reciprocal.

Example
Divide the expressions. Express your answer in lowest form.

$$\frac{2x^2-x-3}{x^2-x-12}\div\frac{x^2+5x+4}{16-x^2}$$

$$\begin{aligned}\frac{2x^2-x-3}{x^2-x-12}\div\frac{x^2+5x+4}{16-x^2}&=\frac{2x^2-x-3}{x^2-x-12}*\frac{16-x^2}{x^2+5x+4}\\&=\frac{(2x-3)(x+1)}{(x-4)(x+3)}*\frac{(4+x)(4-x)}{(x+1)(x+4)}\\&=\frac{(2x-3)\,\cancel{(x+1)}}{\cancel{(x-4)}\,(x+3)}*\frac{\cancel{(4+x)}\,\cancel{(4-x)}\,(-1)}{\cancel{(x+1)}\,\cancel{(x+4)}}\\&=-\frac{2x-3}{x+3}\end{aligned}\tag{1.6.14}$$

Exercises 1.6
Reduce each expression to lowest terms.

1) $\frac{3x+9}{x^2-9}$
2) $\frac{4x^2+8x}{12x+24}$
3) $\frac{x^2-2x}{6-3x}$
4) $\frac{15x^2+24x}{3x^2}$
5) $\frac{24x^2}{12x^2-6x}$
6) $\frac{x^2+4x+4}{x^2-4}$
7) $\frac{25-y^2}{2y^2-8y-10}$
8) $\frac{3y^2-y-2}{3y^2+5y+2}$
9) $\frac{x^2+4x-5}{x^2-2x+1}$
10) $\frac{x-x^2}{x^2+x-2}$
11) $\frac{x^2+5x-14}{2-x}$
12) $\frac{2x^2+5x-3}{1-2x}$

Multiply or divide the expressions in each problem.

Express your answers in lowest terms.

13) $\frac{3x+6}{5x^2} * \frac{x}{x^2-4}$

14) $\frac{4x^2}{x^2-16} * \frac{7x-28}{6x}$

15) $\frac{2a^2-7a+6}{4a^2-9} * \frac{4a^2+12a+9}{a^2-a-2}$

16) $\frac{4a^2-4a-3}{8a+4a^2} * \frac{16a^2}{4a^2-6a}$

17) $\frac{x^2-y^2}{(x+y)^3} * \frac{(x+y)^2}{(x-y)^2}$

18) $\frac{2x^2+x-3}{x^2-1} * \frac{2x-2}{2x^2+5x+3}$

19) $\frac{6x}{x^2-4} \div \frac{3x-9}{2x+4}$

20) $\frac{12x}{5x+20} \div \frac{4x^2}{x^2-16}$

21) $\frac{9x^2+3x-2}{6x^2-2x} \div \frac{3x+2}{6x^2}$

22) $\frac{2a^2-5a-3}{4a^2+2a} \div \frac{2a+1}{4a}$

23) $\frac{x^2+7x+6}{x^2+x-6} \div \frac{x^2+5x-6}{x^2+5x+6}$

24) $\frac{x^2+7x+10}{x^2-x-30} \div \frac{3x^2+7x+2}{9x^2-1}$

25) $\frac{2x^2-x-28}{3x^2-x-2} \div \frac{4x^2+16x+7}{3x^2+11x+6}$

26) $\frac{9x^2+3x-2}{12x^2+5x-2} \div \frac{9x^2-6x+1}{8x^2-10x-3}$

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1.7: Adding and Subtracting Rational Expressions

Just as we do with numerical fractions, we will need to have common denominators in order to add or subtract algebraic fractions. When we add $\frac{1}{2}+\frac{1}{3}$, we make a common denominator of 6 so that we can add them together.

$$\begin{aligned}\frac{1}{2}+\frac{1}{3} &= \frac{1}{2}*\frac{3}{3}+\frac{1}{3}*\frac{2}{2}\\ &= \frac{3}{6}+\frac{2}{6}\\ &= \frac{5}{6}\end{aligned} \tag{1.7.1}$$

Because the denominators, 2 and 3, are prime and don't share any common factors, the common denominator is simply $3*2=6$. We can see a similar result in adding algebraic fractions.

Example
Add the fractions. Express your answer in lowest terms.

$$\begin{aligned}&\frac{2}{x}+\frac{x}{x-3}\\ \frac{2}{x}+\frac{x}{x-3} &= \frac{2}{x}*\frac{x-3}{x-3}+\frac{x}{x-3}*\frac{x}{x}\\ &= \frac{2(x-3)}{x(x-3)}+\frac{x*x}{x(x-3)}\\ &= \frac{2x-6+x^2}{x(x-3)} = \frac{x^2+2x-6}{x(x-3)}\end{aligned} \tag{1.7.2}$$

It's important to be aware that in subtraction, the negative sign representing subtraction must be distributed to all terms in the second numerator.
Example
Subtract the given expressions. Express your answer in lowest terms.

$$\begin{aligned}&\frac{6}{x+1}-\frac{x+5}{x-2}\\ \frac{6}{x+1}-\frac{x+5}{x-2} &= \frac{6}{x+1}*\frac{x-2}{x-2}-\frac{x+5}{x-2}*\frac{x+1}{x+1}\\ &= \frac{6(x-2)-(x+5)(x+1)}{(x+1)(x-2)}\\ &= \frac{6x-12-(x^2+6x+5)}{(x+1)(x-2)}\\ &= \frac{6x-12-x^2-6x-5}{(x+1)(x-2)}\\ &= \frac{-x^2-17}{(x+1)(x-2)}\end{aligned} \tag{1.7.3}$$

In other situations, the denominators may share a common factor. In this case, we can turn one of the denominators into the other one:

Example
Add the given fractions.
$\frac{7}{x^2+8x+15}+\frac{2}{x+3}$

$$\frac{7}{x^2+8x+15}+\frac{2}{x+3} = \frac{7}{(x+3)(x+5)}+\frac{2}{x+3} \tag{1.7.4}$$

We can turn $(x+3)$ into $x^2+8x+15$ by multiplying by $(x+5)$

$$\begin{aligned}\frac{7}{(x+3)(x+5)}+\frac{2}{x+3} &= \frac{7}{(x+3)(x+5)}+\frac{2}{x+3}*\frac{x+5}{x+5}\\ &= \frac{7}{(x+3)(x+5)}+\frac{2(x+5)}{(x+3)(x+5)}\\ &= \frac{7+2x+10}{(x+3)(x+5)}\\ &= \frac{2x+17}{(x+3)(x+5)}\end{aligned} \tag{1.7.5}$$

Sometimes, the answer we end up with is not in lowest terms:

Example
Add the fractions.

$$\frac{x}{x+2}+\frac{8}{x^2+8x+12}$$

$$\begin{aligned}\frac{x}{x+2}+\frac{8}{x^2+8x+12} &= \frac{x}{x+2}+\frac{8}{(x+2)(x+6)}\\ &= \frac{x}{x+2}*\frac{x+6}{x+6}+\frac{8}{(x+2)(x+6)}\\ &= \frac{x(x+6)}{(x+2)(x+6)}+\frac{8}{(x+2)(x+6)}=\frac{x(x+6)+8}{(x+2)(x+6)}\end{aligned} \tag{1.7.6}$$

$$\frac{x(x+6)+8}{(x+2)(x+6)}=\frac{x^2+6x+8}{(x+2)(x+6)} \tag{1.7.7}$$

The numerator is factorable:

$$\begin{aligned}\frac{x^2+6x+8}{(x+2)(x+6)} &= \frac{(x+2)(x+4)}{(x+2)(x+6)}\\ &= \frac{\cancel{(x+2)}\,(x+4)}{\cancel{(x+2)}\,(x+6)}\\ &= \frac{x+4}{x+6}\end{aligned} \tag{1.7.8}$$

Add or subtract the given expressions.

1) $\frac{1}{x-1}-\frac{1}{x}$
2) $\frac{3}{y-6}-\frac{1}{y}$
3) $\frac{2}{x-3}+\frac{4}{x+3}$
4) $\frac{3}{x+4}-\frac{4}{x-2}$
5) $\frac{3}{k+2}-\frac{k-4}{k+5}$
6) $\frac{a+1}{a}-\frac{a}{a+1}$
7) $\frac{2y}{y^2-25}-\frac{y}{y-5}$
8) $\frac{x}{x^2-1}+\frac{4}{x+1}$
9) $\frac{1}{x-3}+\frac{x}{x+1}$
10) $\frac{9y}{y-4}-\frac{y+1}{y+5}$

Add or subtract the given expressions. Express your answers in lowest terms.

11) $\frac{b}{b+1}-\frac{b+1}{2b+2}$
12) $\frac{4x+1}{8x-12}+\frac{x-3}{2x-3}$
13) $\frac{2}{a^2+4a+3}+\frac{1}{a+3}$
14) $\frac{1}{y+6}-\frac{4}{y^2+8y+12}$
15) $\frac{x+1}{2x+4}-\frac{x^2}{2x^2-8}$
16) $\frac{x+1}{x+2}-\frac{x^2+1}{x^2-x-6}$
17) $\frac{2x}{x^2-3x+2}+\frac{2x}{x-1}-\frac{x}{x-2}$
18) $\frac{3x+3}{2x^2-x-1}+\frac{1}{2x+1}$
19) $\frac{4a}{a-2}-\frac{3a}{a-3}+\frac{4a}{a^2-5a+6}$

20) $\frac{2}{y-3} - \frac{8-4y}{y^2-8y+15}$

21) $\frac{2x}{x-1} + \frac{3x}{x+1} - \frac{x+3}{x^2-1}$

22) $\frac{a}{a-1} - \frac{2}{a+2} + \frac{3(a-2)}{a^2+a-2}$

23) $\frac{x}{x-1} + \frac{x+7}{x^2-1} - \frac{x-2}{x+1}$

24) $\frac{2y+5}{y^2-16} - \frac{y-9}{y^2-y-12}$

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1.8: Complex Fractions

1.8 Complex Fractions
Complex fractions involve simplifying a rational expression which has a complicated numerator and/or denominator
Example
Simplify.

$$\frac{3+\frac{x}{x+2}}{1-\frac{x+3}{x-1}} \tag{1.8.1}$$

There are a variety of ways to approach this problem. One of the most straightforward ways to simplify the expression above is to create common denominators for the numerator and the denominator so that each one is a single fractional expression:

$$\begin{aligned}\frac{3+\frac{x}{x+2}}{1-\frac{x+3}{x-1}} &= \frac{\frac{3}{1}*\frac{x+2}{x+2}+\frac{x}{x+2}}{\frac{1}{1}*\frac{x-1}{x-1}-\frac{x+3}{x-1}} \\ &= \frac{\left(\frac{3x+6+x}{x+2}\right)}{\left(\frac{x-1-(x+3)}{x-1}\right)}\end{aligned} \tag{1.8.2}$$

$= \frac{\left(\frac{4x+6}{x+2}\right)}{\left(\frac{-4}{x-1}\right)}$ (Now this is a division problem)

$$\begin{aligned}&= \frac{4x+6}{x+2}*\frac{x-1}{-4} = \frac{2(2x+3)}{x+2}*\frac{x-1}{-4} \\ &= \frac{\cancel{2}(2x+3)}{x+2}*\frac{x-1}{\cancel{-4}(-2)} = \frac{(2x+3)(x-1)}{-2(x+2)}\end{aligned} \tag{1.8.3}$$

Simplifying complex fractions uses all of the previous concepts about rational expressions which we've covered in this chapter.
Example
Simplify.

$$\frac{x-\frac{x}{x+3}}{1+\frac{2}{x}} \tag{1.8.4}$$

$\frac{x-\frac{x}{x+3}}{1+\frac{2}{x}} = \frac{\frac{x}{1}*\frac{x+3}{x+3}-\frac{x}{x+3}}{\frac{1}{1}*\frac{x}{x}+\frac{2}{x}}$ creating common denominators

$$\begin{aligned}&= \frac{\left(\frac{x(x+3)-x}{x+3}\right)}{\left(\frac{x+2}{x}\right)} \\ &= \frac{\left(\frac{x^2+3x-x}{x+3}\right)}{\left(\frac{x+2}{x}\right)} = \frac{\left(\frac{x^2+2x}{x+3}\right)}{\left(\frac{x+2}{x}\right)} \text{ dividing fractions} \\ &= \frac{x^2+2x}{x+3}*\frac{x}{x+2} = \frac{x(x+2)}{x+3}*\frac{x}{x+2}\end{aligned} \tag{1.8.5}$$

$= \frac{x\,\cancel{(x+2)}}{x+3}*\frac{x}{\cancel{x+2}}$ factor and cancel to reduce to lowest terms

$$= \frac{x^2}{x+3} \tag{1.8.6}$$

Exercises 1.8
Simplify each complex fraction. Express your answer in lowest terms.
1) $\frac{1}{\left(x+\frac{y}{2}\right)}$
2) $\frac{\left(\frac{1}{x}+\frac{1}{y}\right)}{\left(\frac{y}{x}-\frac{x}{y}\right)}$
3) $\frac{\left(1+\frac{m}{n}\right)}{\left(1-\frac{n^2}{m^2}\right)}$
4) $\frac{\left(\frac{1}{x}-\frac{1}{y}\right)}{\left(\frac{1}{x^2}-\frac{1}{y^2}\right)}$

5) $\frac{\left(\frac{x}{y}-\frac{x-y}{x+y}\right)}{\left(\frac{y}{x}+\frac{x-y}{x+y}\right)}$

6) $\frac{\left(\frac{7}{a+1}-\frac{3}{a}\right)}{\left(\frac{3}{a}+\frac{1}{a-1}\right)}$

7) $\frac{\left(x-\frac{1}{2x+1}\right)}{\left(1-\frac{2}{2x+1}\right)}$

8) $\frac{\left(\frac{1}{2x-2}-\frac{1}{x}\right)}{\left(\frac{2}{x}-\frac{1}{x-1}\right)}$

9) $\frac{\left(x+\frac{4}{x+4}\right)}{\left(x-\frac{4x+4}{x+4}\right)}$

10) $\frac{\left(x-\frac{x+6}{x+2}\right)}{\left(x-\frac{4x+15}{x+2}\right)}$

11) $\frac{\left(\frac{1}{x+2}-\frac{1}{x-3}\right)}{\left(1+\frac{1}{x^2-x-6}\right)}$

12) $\frac{\left(1-\frac{1}{x+1}\right)}{\left(1+\frac{1}{x-1}\right)}$

13) $\frac{\left(\frac{1}{a-b}-\frac{3}{a+b}\right)}{\left(\frac{2}{b-a}+\frac{4}{b+a}\right)}$

14) $\frac{\left(\frac{3}{y^2-4}\right)}{\left(\frac{1}{y+2}-\frac{1}{y-2}\right)}$

15) $\frac{\left(n+2-\frac{5}{n-2}\right)}{\left(1-\frac{1}{(n-2)^2}\right)}$

16) $\frac{\left(4+\frac{1}{x+1}\right)}{\left(16-\frac{1}{(x+1)^2}\right)}$

17) $\frac{\left(2+\frac{x-2}{1-x^2}\right)}{\left(2-\frac{3}{x+1}\right)}$

18) $\frac{\left(\frac{1}{2x-1}-\frac{1}{2x+1}\right)}{\left(4-\frac{1}{x^2}\right)}$

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1.9: Rational Equations

1.9 Rational Equations
We can also use the skills we have covered in previous sections to solve equations involving rational expressions. There are three main methods of solution that will be explored in this section - multiplying on both sides to clear a denominator, cross-multiplying, and making common denominators. Each of these techniques is actually the same process, but approached from a slightly different perspective.
Clearing a denominator

Often, if the denominator is a single variable, it can be easy and striaghtforward to multiply on both sides by the denominator to cancel it out.

Example
Solve for x

$$x + \frac{4}{x} = 7 \tag{1.9.1}$$

If we multiply on both sides by x, it will clear the variable from the denominator:

$$\begin{aligned} x + \tfrac{4}{x} &= 7 \\ x * \left(x + \tfrac{4}{x}\right) &= (7) * x \quad \text{Multiply on both sides by } x \end{aligned} \tag{1.9.2}$$

$$x * x + x * \frac{4}{x} = 7x \quad \text{Distribute the } x \tag{1.9.3}$$

$$\begin{aligned} x^2 + 4 &= 7x \\ x^2 - 7x + 4 &= 0 \end{aligned} \quad \text{Standard form} \tag{1.9.4}$$

$$x \approx 0.628, 6.372 \tag{1.9.5}$$

If a problem is stated simply as the equality of two fractions, cross-multiplying can be a useful method of solution.

Example
Solve for x

$$\begin{aligned} \frac{x}{2x+3} &= \frac{7}{x-4} \\ \frac{x}{2x+3} &= \frac{7}{x-4} \\ x(x-4) &= 7(2x+3) \\ x^2 - 4x &= 14x + 21 \\ x^2 - 18x - 21 &= 0 \\ x &\approx 19.100, -1.100 \end{aligned} \tag{1.9.6}$$

Cross-multiplying is really just a short-cut method of clearing out the denominators by multiplying on both sides by both denominators:

$$\begin{aligned} \frac{x}{2x+3} &= \frac{7}{x-4} \\ (x-4)(2x+3) * \frac{x}{2x+3} &= \frac{7}{x-4} * (x-4)(2x+3) \\ (x-4)\cancel{(2x+3)} * \frac{x}{\cancel{2x+3}} &= \frac{7}{\cancel{x-4}} * \cancel{(x-4)}(2x+3) \\ x(x-4) &= 7(2x+3) \end{aligned} \tag{1.9.7}$$

Then the equation is ready to be solved as shown above - but by just cross-mulitiplying, we skip directly to the solution portion of the problem.

Sometimes, it is helpful to create a common denominator in order to set up a situation where cross-multiplying can be used.

Example
Solve for x

$$\begin{aligned}
\frac{1}{x+6}+\frac{4}{x-2}&=\frac{3}{x+1}\\
\frac{1}{x+6}+\frac{4}{x-2}&=\frac{3}{x+1}\\
\frac{1}{x+6}*\frac{x-2}{x-2}+\frac{4}{x-2}*\frac{x+6}{x+6}&=\frac{3}{x+1}\\
\frac{1(x-2)+4(x+6)}{(x+6)(x-2)}&=\frac{3}{x+1}\\
\frac{x-2+4x+24}{(x+6)(x-2)}&=\frac{3}{x+1}\\
\frac{5x+22}{(x+6)(x-2)}&=\frac{3}{x+1}\\
(5x+22)(x+1)=3(x+6)(x-2)&=3\left(x^2+4x-12\right)\\
5x^2+27x+22&=3x^2+12x-36\\
2x^2+15x+58&=0\\
x&\approx -3.75\pm 3.865i
\end{aligned} \tag{1.9.8}$$

Example
Solve for x

$$\begin{aligned}
\tfrac{2}{x-2}+\tfrac{x}{2x-1}&=4\\
\tfrac{2}{x-2}+\tfrac{x}{2x-1}&=4\\
\tfrac{2}{x-2}*\tfrac{2x-1}{2x-1}+\tfrac{x}{2x-1}*\tfrac{x-2}{x-2}&=4\\
\tfrac{2(2x-1)+x(x-2)}{(x-2)(2x-1)}&=4\\
\tfrac{4x-2+x^2-2x}{(x-2)(2x-1)}&=4\\
\tfrac{x^2+2x-2}{(x-2)(2x-1)}&=\tfrac{4}{1}\\
1\left(x^2+2x-2\right)=4(x-2)(2x-1)&=4\left(2x^2-5x+2\right)\\
x^2+2x-2&=8x^2-20x+8\\
\frac{2}{x}&=7x^2-22x+10\\
x&=2.592,0.551
\end{aligned} \tag{1.9.9}$$

In some situations, we can create a single common denominator for every fraction in the problem and then clear them all at once.

Example
Solve for x

$$\begin{aligned}
\frac{2}{x+3}-\frac{4}{3x-1}&=\frac{x}{3x^2+8x-3}\\
\frac{2}{x+3}-\frac{4}{3x-1}&=\frac{x}{3x^2+8x-3}\\
\frac{2}{x+3}*\frac{3x-1}{3x-1}-\frac{4}{3x-1}*\frac{x+3}{x+3}&=\frac{x}{3x^2+8x-3}\\
\frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)}&=\frac{x}{(x+3)(3x-1)}\\
2(3x-1)-4(x+3)&=x
\end{aligned} \tag{1.9.10}$$

The missing step above is the clearing of both denominators:

$$(x+3)(3x-1)*\frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)}=\frac{x}{(x+3)(3x-1)}*(x+3)(3x-1) \tag{1.9.11}$$

$$\begin{aligned}\cancel{(x+3)}\ \cancel{(3x-1)}*\frac{2(3x-1)-4(x+3)}{\cancel{(x+3)}\ \cancel{(3x-1)}}&=\frac{x}{\cancel{(x+3)}\ \cancel{(3x-1)}}*\cancel{(x+3)}\ \cancel{(3x-1)}\\ 2(3x-1)-4(x+3)&=x\end{aligned} \tag{1.9.12}$$

As is true in the process of cross-multiplying, it isn't necessary to actually cancel out the denominators in completing the problem.

$$\begin{aligned}\frac{2}{x+3}-\frac{4}{3x-1}&=\frac{x}{3x^2+8x-3}\\ \frac{2}{x+3}*\frac{3x-1}{3x-1}-\frac{4}{3x-1}*\frac{x+3}{x+3}&=\frac{x}{3x^2+8x-3}\\ \frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)}&=\frac{x}{(x+3)(3x-1)}\\ 2(3x-1)-4(x+3)&=x\\ 6x-2-4x-12&=x\\ 2x-14&=x\\ x&=14\end{aligned} \tag{1.9.13}$$

Exercises 1.9

1) $x+\frac{5}{x}=-6$
2) $x+\frac{6}{x}=-7$
3) $y-\frac{5}{y}=2$
4) $\frac{7}{a}+1=a$
5) $\frac{9}{2y+4}=\frac{3}{y}$
6) $\frac{4}{3n+7}=\frac{1}{2}$
7) $\frac{x}{x+3}=\frac{8}{x+6}$
8) $\frac{y-2}{2}=\frac{5}{y-5}$
9) $\frac{2}{n}=\frac{n}{5n+12}$
10) $\frac{x}{4-x}=\frac{2}{x}$
11) $\frac{5x}{14x+3}=\frac{1}{x}$
12) $\frac{a}{8a+3}=\frac{1}{3a}$
13) $\frac{9}{x-1}-\frac{2}{x+4}=\frac{1}{x+2}$
14) $\frac{1}{x-2}+\frac{4}{x+5}=\frac{1}{x-3}$
15) $\frac{5}{3x+2}+\frac{1}{x-1}=\frac{3}{x+2}$
16) $\frac{1}{y-2}-\frac{4}{2y+5}=\frac{6}{y-1}$
17) $\frac{5}{x+1}+\frac{1}{x+2}=3$
18) $\frac{1}{2x-1}-\frac{2}{x+7}=1$
19) $\frac{6}{y-4}-\frac{1}{y+2}=3$
20) $\frac{10}{a+1}+\frac{3}{a-2}=2$
21) $\frac{3a}{a^2-2a-15}-\frac{a}{a+3}=\frac{2a}{a-5}$
22) $\frac{u^2+2}{u^2+u-2}-\frac{3u}{u+2}=\frac{-2u-1}{u-1}$
23) $\frac{4}{2x-1}+\frac{2}{x+3}=\frac{5}{2x^2+5x-3}$
24) $\frac{5}{x+5}-\frac{2}{x^2+2x-15}=\frac{2}{x-3}$
25) $\frac{5}{y-2}-\frac{3}{2y-1}=\frac{4}{2y^2-5y+2}$
26) $\frac{x+2}{x-1}+\frac{x+4}{x}=\frac{2x+1}{x^2-x}$
27) $\frac{x}{x+2}+\frac{x+1}{x^2-7x-18}=\frac{5}{x-9}$
28) $\frac{2a}{a+7}-\frac{a}{a+3}=\frac{5}{a^2+10a+21}$
29) $\frac{x-1}{2x+1}-\frac{2x-3}{x+3}=\frac{3}{2x^2+7x+3}$
30) $\frac{y}{y+4}+\frac{6}{y+1}=\frac{y^2+4}{y^2+5y+4}$

Addendum - Word Problems

Following are a selection of word problems - some from ancient times, some from the Renaissance and Enlightenment, and some from the 19 th and 20 th century.

1) A teacher agreed to work 9 months for $\$562.50$ and board. At the end of the term, on account of two months absence caused by illness, he received only $\$409.50$ for his seven months work. If the teacher used all nine months of his board during the term, what was his board per month? (American 1892)

2) A servant is promised $\$100$ plus a cloak as wages for a year. After seven months, he leaves and receives $\$20$ plus the cloak. How much is the cloak worth? (Clavius, German 1608)

3) The sales tax on garments is $\frac{1}{20}$ of their value. A certain man buys 42 garments, paying in copper coins. Two garments and 10 copper coins are paid as tax. What is the price of a garment, O learned one? (Ancient India)

4) Two wine merchants enter Paris, one of them with 64 casks of wine, the other with 20 casks (all of the same value). since they do not have enough money to pay the customs duties, the first pays 5 casks of wine and 40 francs, and the second pays 2 casks of wine and receives 40 francs change. What is the price of each cask of wine and what is the duty on each cask? (Chuquet, French 1484)

5) One of two men had 12 fish and the other had 13 fish, and all of the fish were of the same price. From the first man, a customs agent took away 1 fish and
12 denarii for payment. From the other man he took 2 fish and gave him back 7 denarii as change. Find the customs fee and the price of each fish. (Fibonacci, Italian 1202)

6) Two traders transporting sheepskins approach their country's border. The first trader has 100 sheepskins and the border guard takes 10 sheepskins plus $\$25$ as a tariff. The second trader has 42 sheepskins and for a tariff, the border guard takes 7 sheepskins but returns $\$14$ change. What is the tariff per sheepskin and what is the value of each sheepskin?

7) Two people have a certain amount of money. The first says to the second, "If you give me 5 denarii, I will have 7 times what you have left." The second says to the first, "If you give me 7 denarii, I will have 5 times what you have left." How much money does each have? Round to the nearest 10 th. (Leonardo, Italian c. 1500)

8) Two different scenarios from Ancient Greece:
Two friends were walking. One said to the other, "If I had 10 more coins, I would have 3 times as much money as you." The other said, "If I had 10 more coins, I would have five times as much as you." How many coins does each have?

Two friends were walking. One said to the other, "If you give me 10 of your coins, I would have 3 times as much money as you." The other said, "If you give me 10 of your coins, I would have five times as much as you." How many coins does each have?

9) Andy and Betty together have $\$6$ less than Christine. If Betty gives $\$5$ to Andy, then Andy will have half as much as Christine. If, instead, Andy gives $\$5$ to Betty, then Andy will have one-third as much as Betty. How much does each person have to begin with?

10) Three friends $(\mathrm{A}, \mathrm{B}$ and $\mathrm{C})$ each have a certain amount of money. A says, "I have as much as B plus one-third as much as C." B says, "I have as much as C plus one-third as much as A." Csays, "I have 10 more than one-third of B." How much does each person have? (Ancient Greece)

11) On a test, 39 more pupils passed than failed. On the next test, 7 who passed the first test failed and one-third of those who failed the first test passed the second. As a result, 31 more passed the second test than failed it. What was the record of passing and failing on the first test?

12) At two stations, A and B, six miles apart on the railway, the prices of coal are $\$20$ per ton and $\$24$ per ton respectively. The rates of cartage of coal are $\$2.00$ per ton per mile from A and $\$3.00$ per ton per mile from B. At a certain customer's home, on the railroad from A to B, the cost of coal is the same whether delivered from A or B. Find the distance to this home from A.

13) There were three-fourths as many women as there were men on the train. At the next station six men and eight women got off the train, and twelve men and five women got on. There were then three-fifths as many women as men on the train. How many men and how many women were originally on the train?

14) If a theater could put 5 more seats in a row, it would need 20 rows less, but if each row had 3 fewer seats, it would take 20 rows more to seat the same number. How many people will it seat?

15) An audience of 450 people is seated in rows, with the same number of people in each row. It would take 5 rows less if 3 more people were seated in each row. In how many rows are the people seated?

16) If evergreens are planted 4 feet closer together it will take 44 more trees for a certain piece of road, but if they are planted 6 feet farther apart, it will take 44 fewer trees for the same length of road. How many miles is the piece of road? (use $5280 \text{ feet} = 1 \text{ mile}$)

17) A movie theater owner found that by raising the price of each ticket by
$\$1.00, 200$ fewer people attened and she broke even, but that if she lowered the price by $\$1.00$ per person, 550 people attended and she increased her receipts by
$\$1000$. What is the usual rate per person?

18) The brine pipes in an 84 -foot width artificial hockey rink are equally spaced. If the space between each pair of pipes were increased by 1 inch, then 84 fewer lengths of pipe would be needed. What is the distance between the pipes now?

19) A living room shelf is 36 inches long and contains a certain number of books of uniform width. If each book were one-half inch narrower, the shelf would hold six more books. How many books of the wider variety does it hold?

20) I am a brazen lion; my spouts are my two eyes, my mouth and flat of my right foot. My right eye fills a jar in two days, my left eye in three and my foot in four. My mouth is capable of filling it in six hours. Tell me how long all four together will take to fill it. (Ancient Greece)

21) A man wishes to have 500 rubii of grain ground. He goes to a mill that has five stones. The first stone grinds 7 rubii of grain in an hour, the second grinds 5 rubii in an hour, the third 4 rubii in an hour, the fourth grinds 3 rubii per hour and the fifth grinds 1 rubii per hour. In how long will the grain be ground and how much is done by each stone? (Clavius, German 1583)

22) If two men and three boys can plow an acre in one-sixth of a day, how long would it require three men and two boys to plow it? (Edward Brooks, American 1873)
23) A cobbler can cut leather for ten pairs of shoes in one day. He can finish 5 pairs of shoes in one day. How many pairs of shoes can he both cut and finish in one day? (Ancient Egypt)

24) Four waterspouts are filling a tank. Of the four spouts, one can fill the tank in one day, the second takes two days, the third takes three days and the fourth takes four days. How long will it take all four spouts working together to fill the tank? (Ancient Greece)

25) If, in one day, a person can make 30 arrows or fletch [put feathers on] 20 arrows, how many arrows can this person both make and fletch in one day? (Ancient China)

26) One military horse and one ordinary horse can pull a load of 40 dan. Two ordinary horses and one inferior horse can pull the same load of 40 dan as can three inferior horses and one military horse. How much can each horse pull individually? (Ancient China)

27) A barrel of water has several holes in it. The first hole empties the full barrel in 3 days. The second hole empties the full barrel in 5 days. The third hole empties the full barrel in 20 hours and the last hole empties the full barrel in 12 hours. With all the holes open, how long will it take to empty the barrel? (Levi ben Gershon, French 1321)

28) A certain lion could eat a sheep in 4 hours; a leopard could do so in 5 hours; and a bear in 6 hours. How many hours would it take for all three animals to devour a sheep if it were thrown in among them? (Fibonacci, Italian 1202)

29) Two ships are some distance apart, which journey the first can complete in 5 days and the other in 7 days - it is sought in how many days they will meet if they begin the journey at the same time. (Fibonacci, Italian 1202)

30) Sarah, alone, can paint the garage in 24 hours, her sister Jenny, alone, can paint the same garage in 12 hours. With the aid of their mother, the three together can paint the garage in 4 hours. How long would it take their mother, working alone, to paint the garage?

31) It required 75 workers 38 days to build an embankment to be used for flood control. Had 18 workers been removed to another job at the very start of operations, how much longer would it have taken to build the embankment?

32) Mark, alone, requires 6 hours to paint a fence; however, his younger brother, who alone could do it in 9 hours, helps him. If they start work at 8: 30 am, at what time should they finish the work?

33) A group decides to build a cabin together. The job can be done by 3 skilled workers in 4 days or by 5 amateurs in 6 days.

How long would it take if they all work together?

34) If it requires 18 workers 50 days to build a piece of road, how many days sooner would it be done if 7 more workers were hired at the beginning of operations?

35) A contractor estimated that a certain piece of work would be done by 9 carpenters in 8 hours or by 16 amateurs in 9 hours. The contractor wishes to get the job done as quickly as possible and uses both professional carpenters and amateurs on the same job. Four carpenters and 4 amateurs begin work at 6 am. Allowing 45 minuntes for lunch, at what time should they finish the job?

36) Mrs. Ellis alone can do a piece of work in 6 days. Her oldest daughter takes 2 days longer; her youngest daughter takes twice as long as her mother. How long will it take to complete the job if all three work together?

37) If 5 men and 2 boys work together, a piece of work can be completed in one day and if 3 men and 6 boys work together, it can be completed in one day. How long would it take a boy to do the work alone?

38) A coal company can fill a certain order from one mine in 3 weeks and from a second mine in 5 weeks. How many weeks would be required to fill the order if both mines were used?

39) If 25 skilled workers work for 8 days, they can complete the construction of a concrete dam; 12 skilled workers and 15 untrained workers together can complete the dam in 10 days. How long would it take an untrained worker alone to complete the work on the dam?

40) A and B working together can complete a piece of work in 30 days. After they have both worked 18 days, however, A leaves and B finishes the work alone in 20 more days. Find the time in which each can do the work alone.

41) A dump cart can haul enough gravel to fill a pit in 6 days. A truck can do the same work in 2 days. How long would it take two dump carts and one truck working together to fill the pit?

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2: POLYNOMIAL AND RATIONAL FUNCTIONS

This chapter will explore the solution of equations and inequalities involving both polynomial and rational functions, primarily through the examination of their graphical representations. We will also explore the use of polynomial long division and synthetic division in breaking down polynomials into their prime factors and the relationship between factors and roots.

2.1: Representing Intervals

2.2: Solution by Graphing

2.3: Solution of Polynomial Inequalities by Graphing

2.4: Solution of Rational Inequalities by Graphing

2.5: Finding Factors from Roots

2.6: Polynomial Long Division

2.7: Synthetic Division

2.8: Roots and Factorization of Polynomials

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2.1: Representing Intervals

Many of the problems in this chapter will have solutions that must be expressed as an interval. This means a range of x values that will satisfy the original problem. In this section, we will introduce the translation of graphical intervals into algebraic notation.

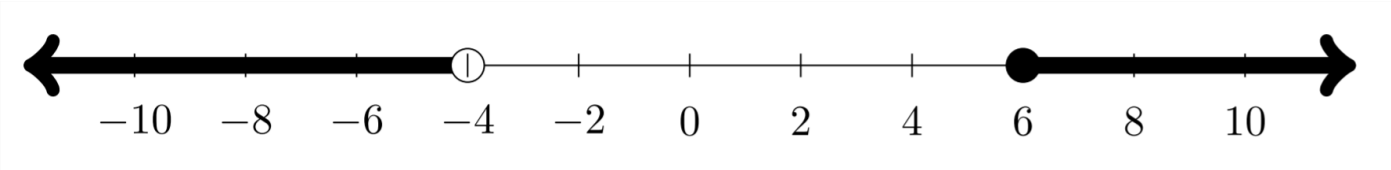
For example, in the diagram below, we would represent the interval shown on the graph as $x < -3$



In this diagram, we would represent the interval shown on the graph as $1 \leq x < 7$.

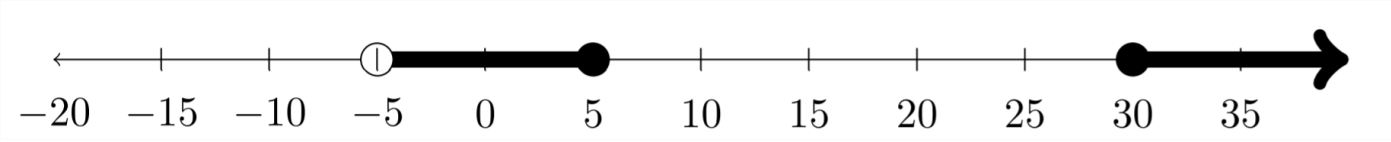


In the next diagram, we have two separate intervals to represent-and we will need two separate statements to represent them.



These intervals would be represented as $x < -4$ OR $x \geq 6$
Sometimes students try to represent the intervals above as $6 \leq x < -4$, however, this expression would represent a single interval where x is both less than -4 and, at the same time, greater than or equal to $+6$. This is simply not possible, and would result in the empty set, which is the reason that the OR portion is needed in the correct answer.

Example
Represent the intervals indicated on the graph below:

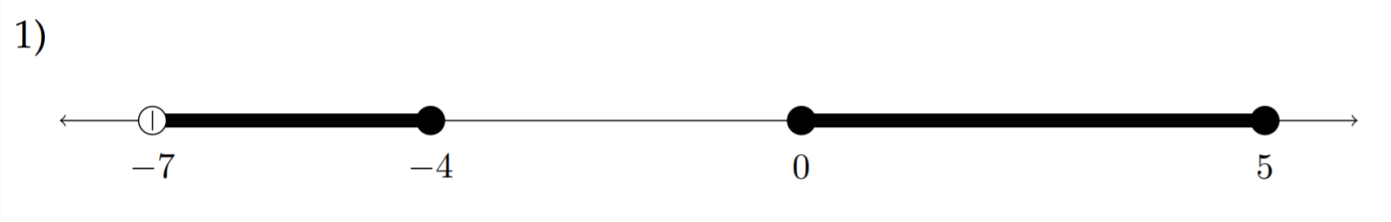


On this graph, there is one interval beginning at -5 and ending at +5 and another beginning at 30 and continuing to infinity. Thus, these intervals would be represented as \(-5

Students familiar with another form of interval notation may wish to represent this interval as $(-5, 5] \cup [30, \infty)$. Both forms of notation accomplish the same goal.

Exercises 2.1
In each problem below, represent the intervals indicated on the graph.

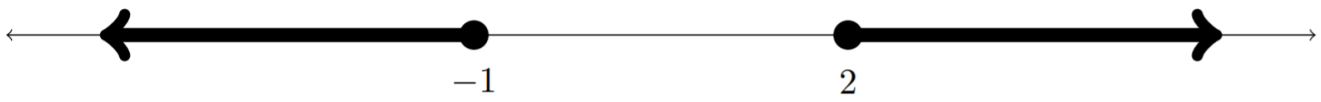
1)



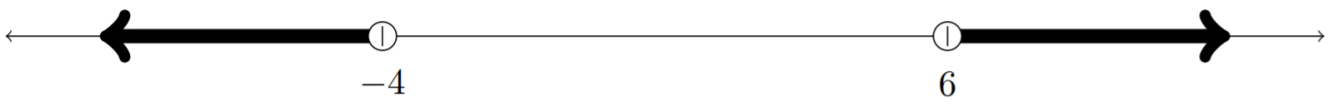
2)



3)



4)



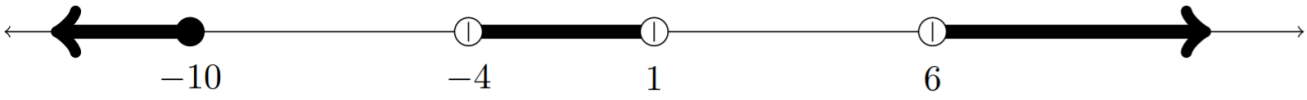
5)



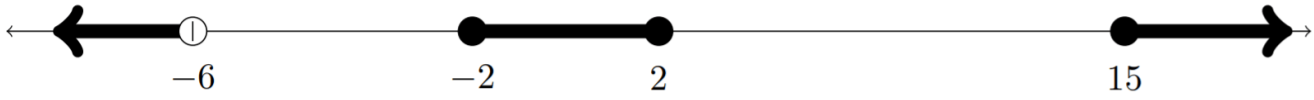
6)



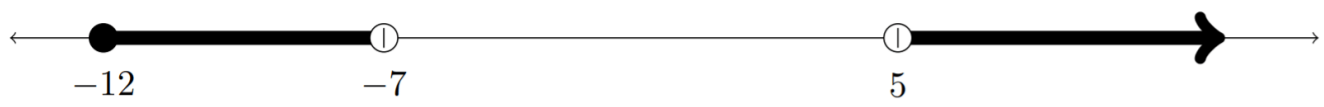
7)



8)



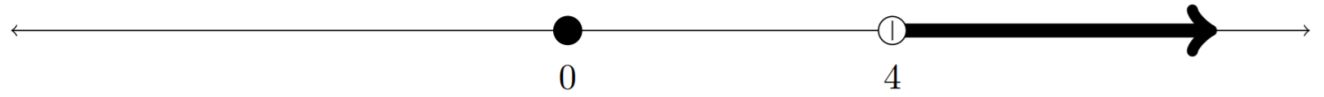
9)



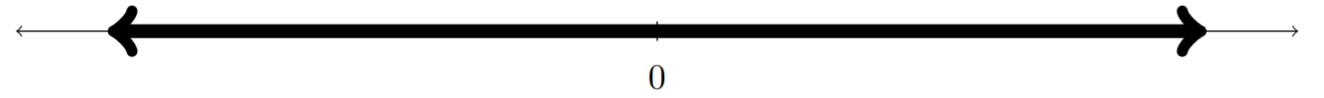
10)



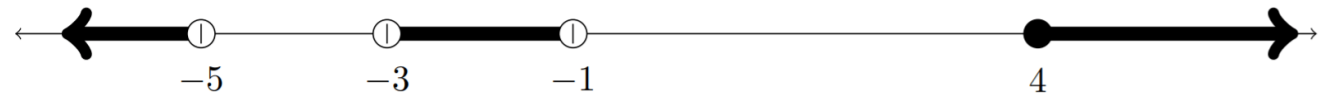
11)



12)



13)



14)



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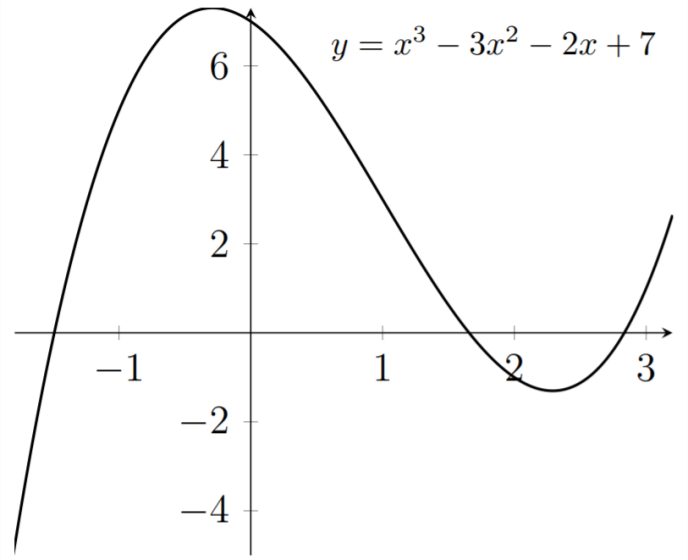
2.2: Solution by Graphing

In previous courses the solution of linear equations is covered, generally by separating the variables and constants on opposite sides of the equation to isolate the variable. In the previous chapter we examined the solution of quadratic equations, in which the variable is isolated using the technique of completing the square. The major difference between these methods of solution is that, in the solving of quadratic equations, we must contend with several different powers of the variable which makes it considerably more difficult to isolate the variable. There are formulas like the quadratic formula available to solve cubic (x^3) and quartic (x^4) equations, however these formulas are somewhat cumbersome and archaic. The primary method for solving eqautions of degree higher than 2 is solution by graphing or by algorithm.

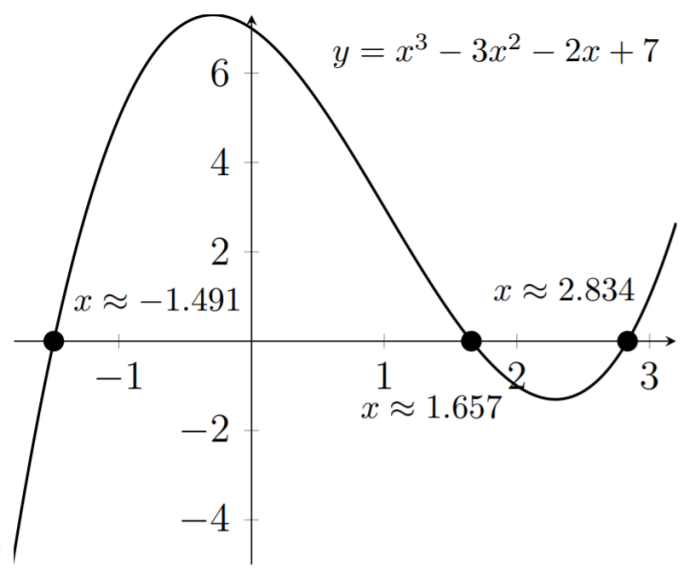
Solution by algorithm is a very interesting process as there are many different algorithms available. Which algorithm is most appropriate often depends on the types of equations being solved and the technology available to solve them. Two major types of algorithms that rely on the graphincal representation of an equation are called "Double False Position" and "The Newton-Raphson Method." Many commonly available pieces of techonolgy use one of these methods. since we have graphing calculators available to us, we will focus on solution by graphing.

Whether using a TI (Texas Instruments) or Casio graphing calculator, or a software based graphing ustility such as Graph or Desmos, the solution of these equations focuses on finding the x -intercepts of the graph since this is where the y value is 0. The specific processes for solving equations using each of these different tools will be covered in class.
Example
Solve for x
$x^3 - 3x^2 = 2x - 7$
The first step is to move all terms to one side of the equation and set them equal to zero.
$x^3 - 3x^2 - 2x + 7 = 0$

Then we graph the function and look for which x values will make the y value equal to 0



In this graph, we can see three roots, or x -intercepts, where the graph crosses the x -axis. These are the x -values that make the y -value equal to 0. We can use the available technology to find these x -values.



So, we can see that the solutions to the eqaution $x^3 - 3x^2 - 2x + 7 = 0$ are $x \approx -1.491, 1.657, 2.834$.
We can check these answers by plugging them back into the original equation.

$$\begin{aligned}(-1.491)^3 - 3 * (-1.491)^2 - 2 * (-1.491) + 7 &= -3.3146 - 3 * 2.223 + 2.982 + 7 \\ &= -3.3146 - 6.669 + 2.982 + 7 \\ &= -0.0016\end{aligned} \tag{2.2.1}$$

The result is not exactly 0 since our answers have been rounded off to the 1000 th place. If we wanted greater accuracy, then we should include greater accuracy in the value of our answers.

$$\begin{aligned}(1.657)^3 - 3 * (1.657)^2 - 2 * (1.657) + 7 &= 4.5495 - 3 * 2.7456 - 3.314 + 7 \\ &= 4.5495 - 8.2368 - 3.314 + 7 \\ &= -0.0013\end{aligned} \tag{2.2.2}$$

and

$$\begin{aligned}(2.834)^3 - 3 * (2.834)^2 - 2 * (2.834) + 7 &= 22.7614 - 3 * 8.03155 - 5.668 + 7 \\ &= 22.7614 - 24.09465 - 5.668 + 7 \\ &= -0.00125\end{aligned} \tag{2.2.3}$$

Exercises 2.2

Solve for x

1) $3x^3 - 7x^2 - x + 1 = 0$
2) $24x^4 + 5x^2 - 13x + 3 = 0$
3) $2x^3 - 2x^2 - 28x + 51 = 0$
4) $2x^3 + 5x^2 - 15x + 7 = 0$
5) $x^4 - 4x^3 + x^2 + 6x + 1 = 0$
6) $x^4 + 2x^3 + x^2 - x - 6 = 0$
7) $x^4 - 5x^3 - 3x^2 + 17x - 9 = 0$
8) $2x^3 - 5x - 3 = 0$
9) $x^4 - 4x^3 - 7x^2 - 36x - 18 = 0$
10) $6x^3 - 25x^2 + 21x + 10 = 0$
11) $2x^4 - 5x^3 + x^2 + 4x - 2 = 0$
12) $x^4 + x^3 - 5x^2 + x - 4 = 0$

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2.3: Solution of Polynomial Inequalities by Graphing

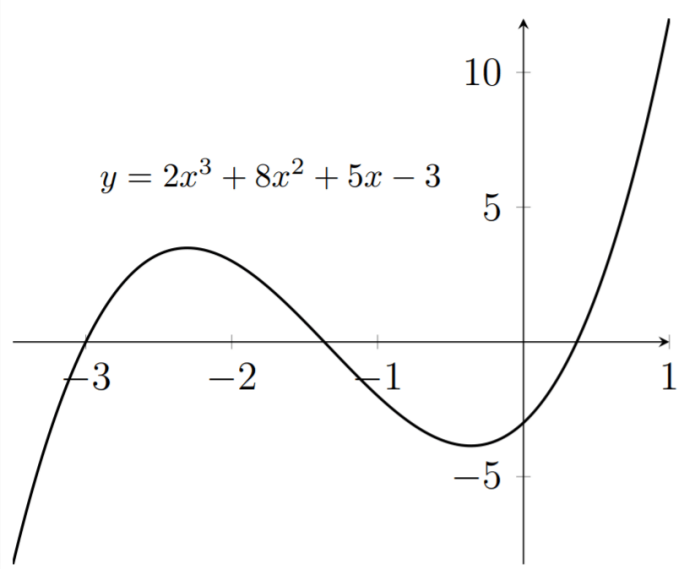
In this section, we will combine the concepts of the previous two sections to solve polynomial inequalities. In Section 2.2, we solved equations by graphing and finding the x -values which made $y = 0$. In solving an inequality, we will be concerned with finding the range of x values that make y either greater than or less than 0, depending on the given problem.

Example

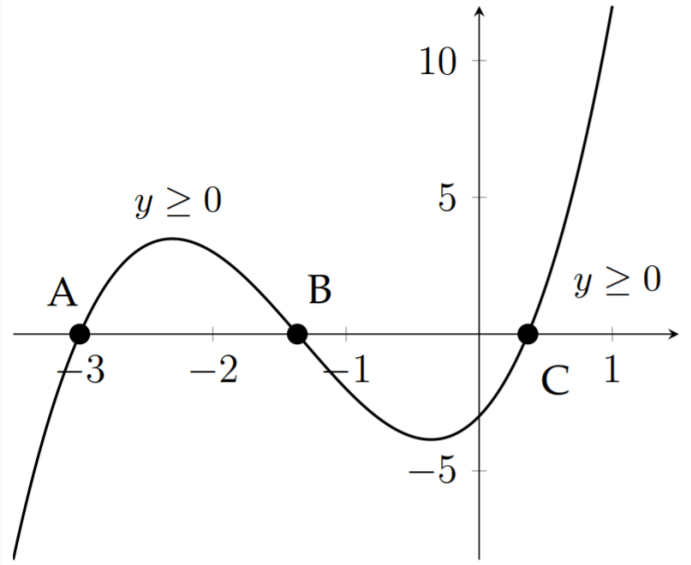
Solve the given inequality.

$2x^3 + 8x^2 + 5x - 3 \geq 0$

First, we graph the function:



Then we identify the intervals of x -values that make the y value greater than or equal to zero, as indicated in the problem.



The indicated roots of the function $(A, B \text{ and } C)$ are the x -values that make y equal to zero. These points divide the graph between the regions where y is greater than zero and the regions where y is less than zero. The solution to the given inequlaity $2x^3 + 8x^2 + 5x - 3 \geq 0$ are $\mathrm{A} \leq x \leq \mathrm{B}$ OR $x \geq \mathrm{C}$

When we find the values of A, B and $\mathrm{C}: \mathrm{A} = -3, \mathrm{B} \approx -1.366$ and $\mathrm{C} \approx 0.366$, we can complete the solution to the problem.

$2x^3 + 8x^2 + 5x - 3 \geq 0$

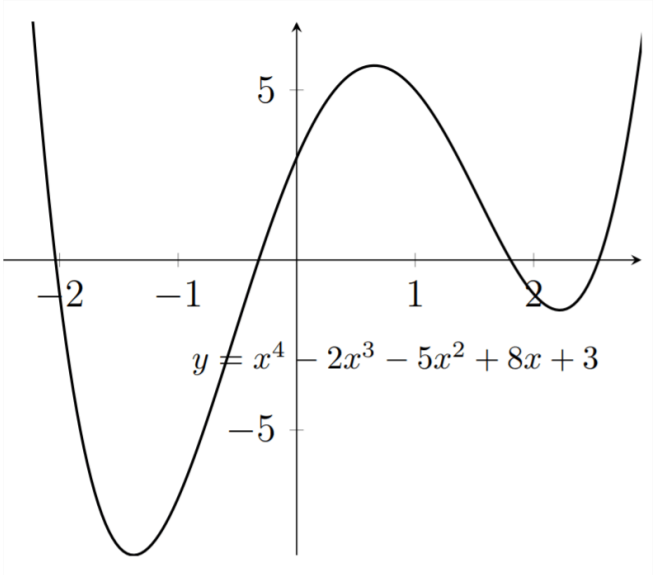
$-3 \leq x \leq -1.366$ OR $x \geq 0.366$

Example

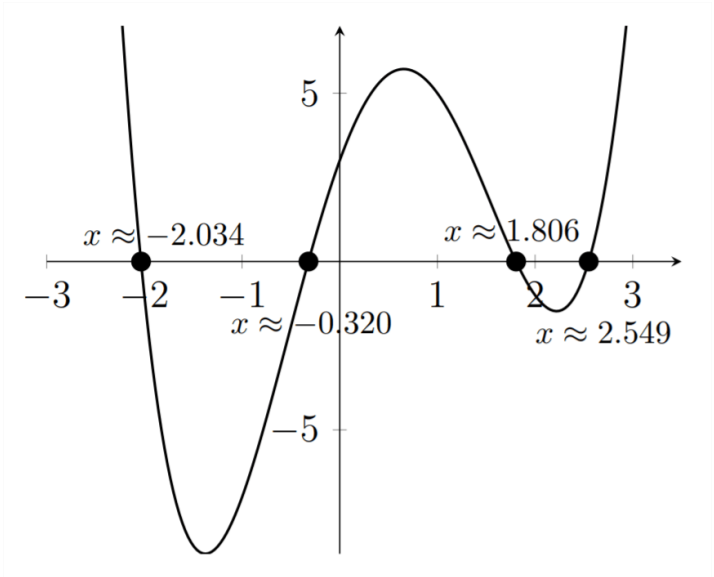
Solve the given inequality.

$x^4 - 2x^3 - 5x^2 + 8x + 3 \leq 0$

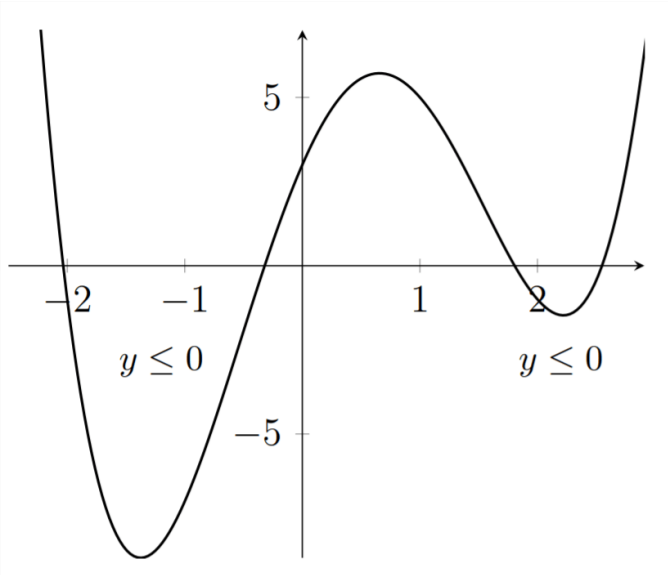
First, we graph the function:



In this problem, we're looking for the intervals of x values that make y less than or equal to zero. First, we identify the roots of the function:

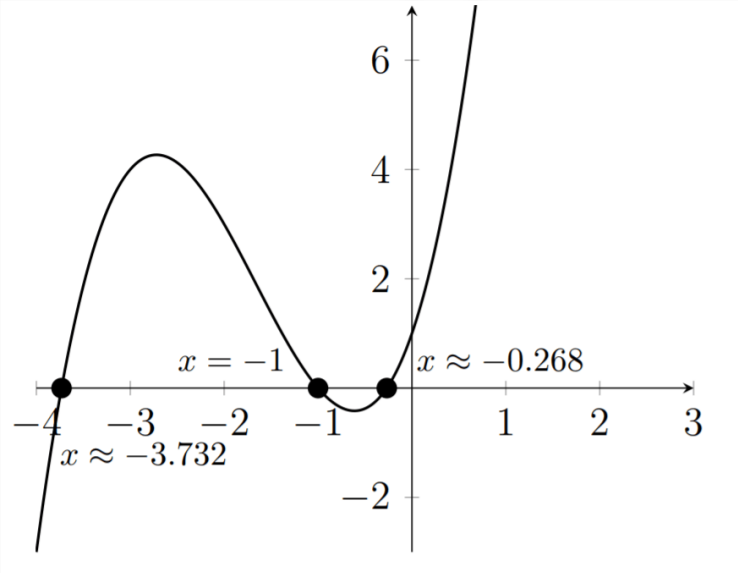


Next, we'll identify the intervals where the y values are less than zero:



So, the solution to the original inequality is:
$x^4 - 2x^3 - 5x^2 + 8x + 3 \leq 0$
$-2.034 \leq x \leq -0.320$ OR $1.806 \leq x \leq 2.549$
In the next example we'll be looking to identify both the intervals where y is greater than zero, and the intervals where y is less than zero.

Example
Determine the interval(s) for which $x^3 + 5x^2 + 5x + 1 \geq 0$
Determine the interval(s) for which $x^3 + 5x^2 + 5x + 1 < 0$
Once again, we'll start by graphing the function to find the roots:



Now that we've indentified the roots, we can determine where the y values are greater than zero and where they're less than zero.

For $y \geq 0$, we can see that this corresponds to: $-3.732 \leq x \leq -1$ OR $x \geq -0.268$
For $y < 0$, we can see that this corresponds to: \(x<-3.732 \mathrm{OR}-1

Exercises 2.3
1) Determine the interval(s) for which $x^3 - 4x^2 + 2x + 3 \geq 0$
Determine the interval(s) for which $x^3 - 4x^2 + 2x + 3 < 0$
2) Determine the interval(s) for which $4x^3 - 4x^2 - 19x + 10 \geq 0$

Determine the interval(s) for which $4x^3 - 4x^2 - 19x + 10 < 0$
3) Determine the interval(s) for which $x^3 - 2.5x^2 - 7x - 1.5 \geq 0$
Determine the interval(s) for which $x^3 - 2.5x^2 - 7x - 1.5 < 0$
4) Determine the interval(s) for which $x^3 - 3.5x^2 + 0.5x + 5 \geq 0$
Determine the interval(s) for which $x^3 - 3.5x^2 + 0.5x + 5 < 0$
5) Determine the interval(s) for which $6x^4 - 13x^3 + 2x^2 - 4x + 15 \geq 0$
Determine the interval(s) for which $6x^4 - 13x^3 + 2x^2 - 4x + 15 < 0$
6) Determine the interval(s) for which $x^4 - x^3 - x^2 + 3x - 5 \geq 0$
Determine the interval(s) for which $x^4 - x^3 - x^2 + 3x - 5 < 0$
7) Determine the interval(s) for which $3x^4 + 3x^3 - 14x^2 - x + 3 \geq 0$
Determine the interval(s) for which $3x^4 + 3x^3 - 14x^2 - x + 3 < 0$
8) Determine the interval(s) for which $4x^4 - 4x^3 - 7x^2 + 4x + 3 \geq 0$
Determine the interval(s) for which $4x^4 - 4x^3 - 7x^2 + 4x + 3 < 0$

Determine the interval(s) that satisfy each inequality.

9) $x^3 + x^2 - 5x + 3 \leq 0$
10) $x^3 - 7x + 6 > 0$
11) $x^3 - 13x + 12 > 0$
12) $x^4 - 10x^2 + 9 < 0$
13) $6x^4 - 9x^2 - 4x + 12 \geq 0$
14) $x^4 - 5x^3 + 20x - 16 > 0$
15) $x^3 - 2x^2 - 7x + 6 \leq 0$
16) $x^4 - 6x^3 + 2x^2 - 5x + 2 \leq 0$
17) $2x^4 + 3x^3 - 2x^2 - 4x + 2 > 0$
18) $x^5 + 5x^4 - 4x^3 + 3x^2 - 2 \leq 0$

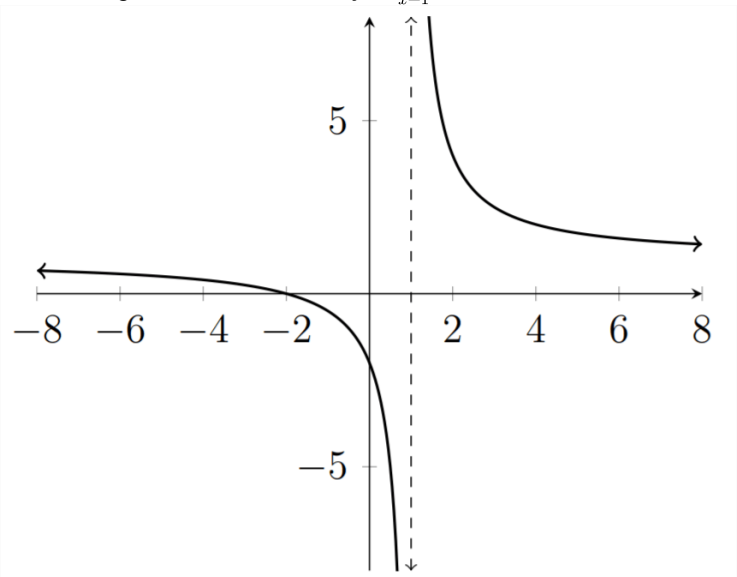
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2.4: Solution of Rational Inequalities by Graphing

In the previous section, we saw how to solve polynomial inequalities by graphing. In this section, we will use similar methods to solve rational inequalities. Rational inequalities involve ratios of polynomials or fractions. Because these types of problems involve fractions, the graphs of the functions that we work with will have what are known as asymptotes. This word comes from a Greek root having to do with two lines that come very close to each other but never meet.

The vertical asymptotes of a graph will appear at places where the original expression has a zero denominator. This means that the function is not defined at those x values and so, rather than having a y value at that point, the graph has an asymptote.

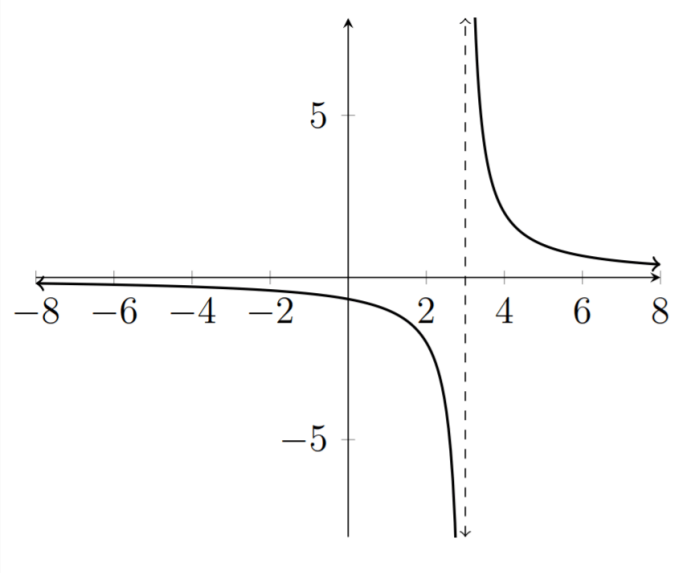
Example
Below is a graph of the function $y = \frac{x+2}{x-1}$



Rather than having a y value at the point where $x = 1$, the dotted line indicates the asymptote where the function is not defined. In the previous section, we were interested in finding the roots of the function because these are the places where $y = 0$, and can be the dividing points between where the y values are greater than zero $(y > 0)$ and the where the y values are less than zero $(y < 0)$

The importance of the asymptotes in analyzing rational functions is that, like the roots, these represent x values that can be the dividing points between where $y > 0$ and where $y < 0$

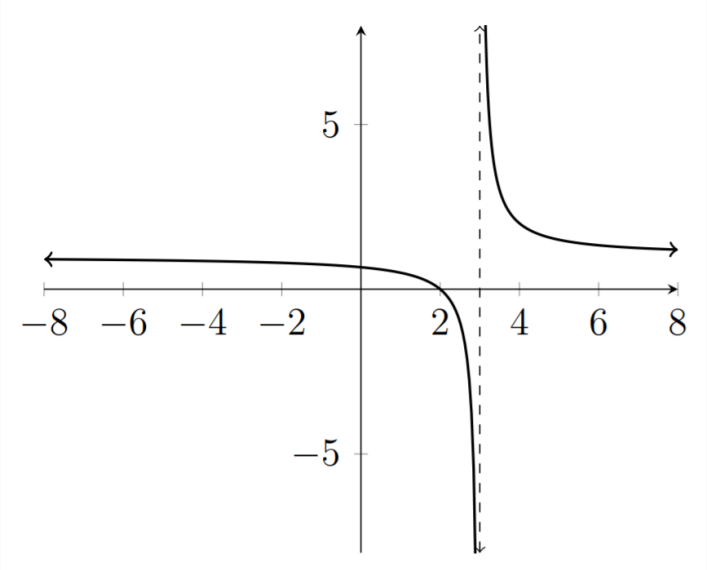
Example
Solve the given inequality.
$\frac{2}{x-3} > 0$
First we examine the graph:



Notice that the asymptote for this graph occurs at the value $x = 3$, because this is the x value that creates a zero denominator. Also notice that the y values switch from being negative to being positive across the asymptote.

There are no roots for this function because there are no x values that make $y = 0$ For a fraction to be zero, the numerator must equal zero. In this example the numerator is 2 and no value of x will make it equal zero. Therefore the only possible dividing point on the graph is $x = 3$, and the solution to the inequality is $x > 3$

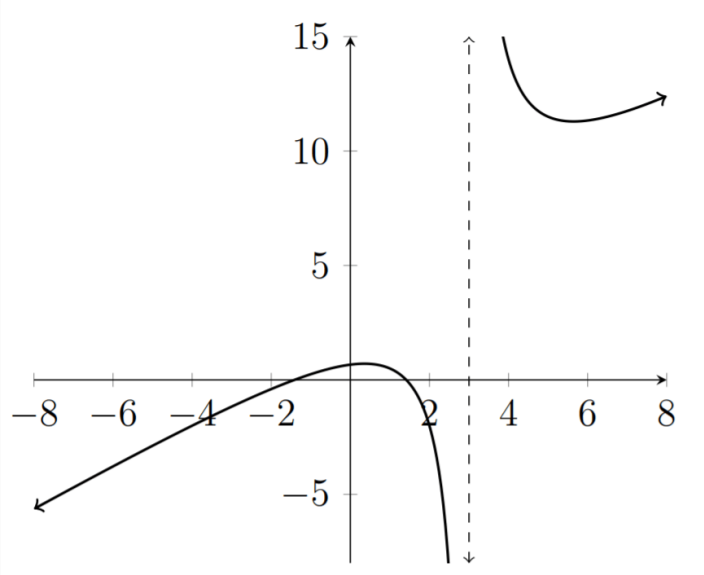
Example
Solve the given inequality.
$\frac{x-2}{x-3} > 0$



In this inequality, there is again an asymptote at $x = 3$, but there is also a root at the value $x = 2$, because when $x = 2.y = \frac{2-2}{2-3} = \frac{0}{-1} = 0$. So we have two dividing points to consider, $x = 2$ and $x = 3$. We can see from the graph that $y > 0$ for $x < 2$ or $x > 3$, so that is the solution to the given inequality.

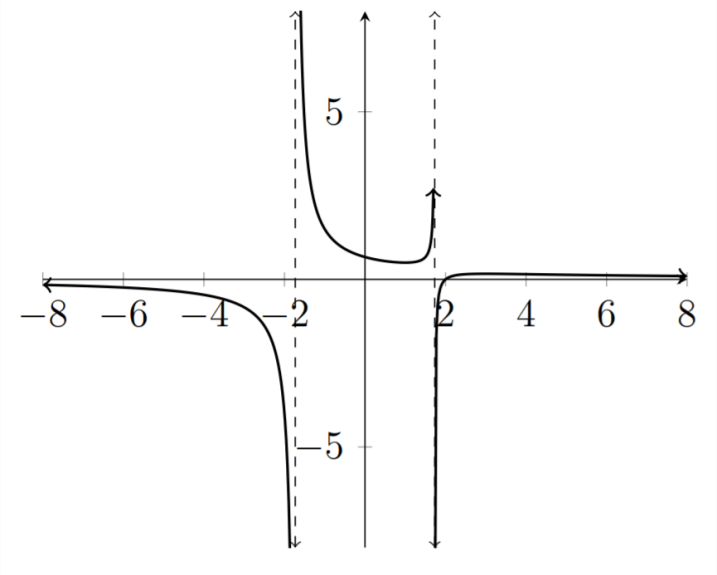
Example
Solve the given inequality.
$\frac{x^2-2}{x-3} > 0$
In this problem, we have the same asymptote as the previous two problems: $x =$
3. However, in this inequality, there are two roots, because there are two x values that make the numerator equal zero.

$x^2 - 2 = 0$ means that $x^2 = 2$ and $x = \pm\sqrt{2} \approx \pm 1.414$
We can see these roots on the graph.



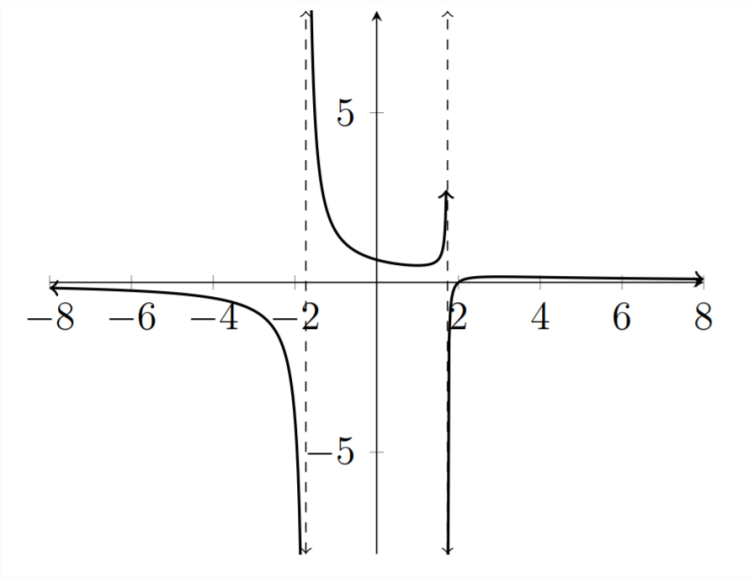
In the graph above, we can see the asymptote at $x = 3$ and the two roots at $x \approx 1.414, -1.414$
The x values that make $y > 0$ are \(-1.4143\)

Example
Solve the given inequality.
$\frac{x-2}{x^2-3} > 0$

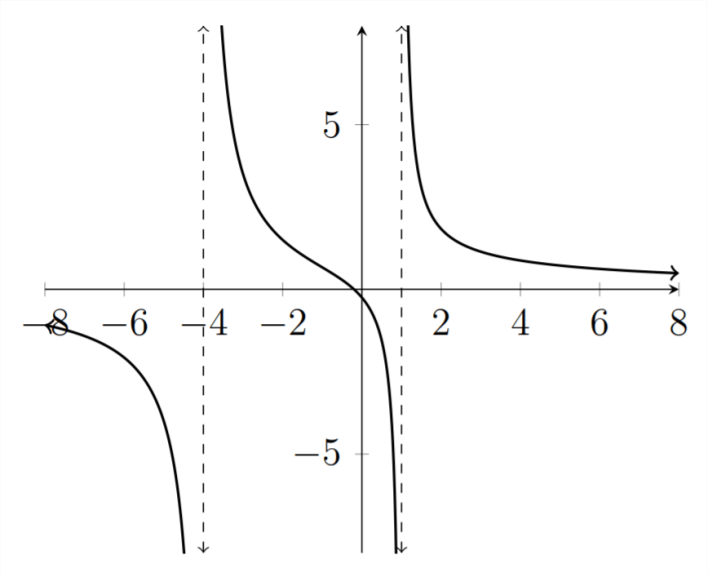


The roots for this function are the x values that make the numerator equal zero:
$x - 2 = 0$, therefore $x = 2$, and we can see this root on the graph.

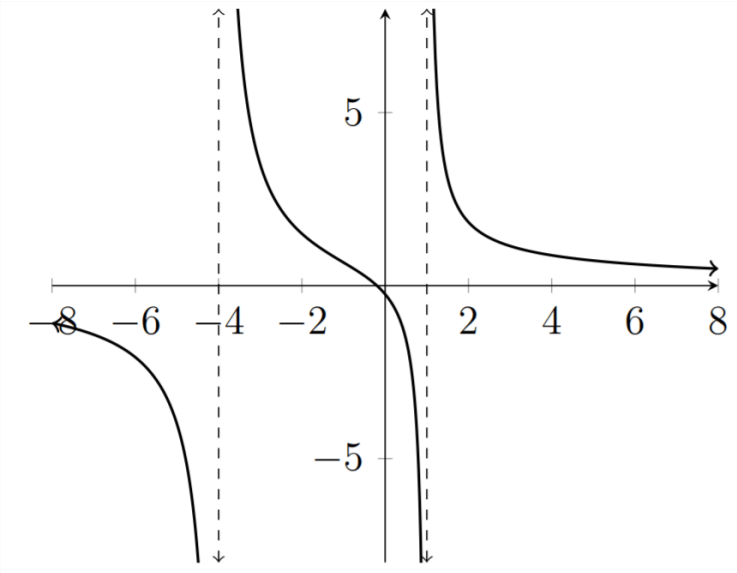
The asymptotes for the function are the x values that make the denominator equal zero:
$x^2 - 3 = 0$ means that $x^2 = 3$ and $x = \pm\sqrt{3} \approx \pm 1.732$



Therefore the solution for the given inequality is:
\(-1.7322\)
Example
Solve the given inequality
$\frac{5x+1}{x^2+3x-4} < 0$



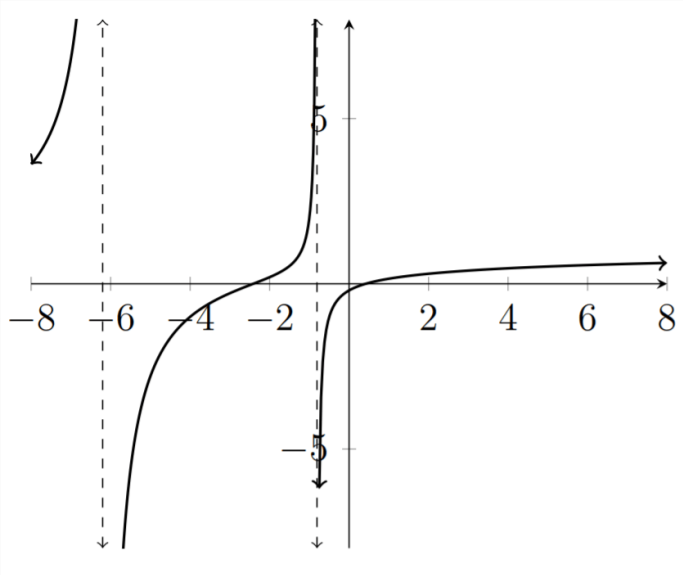
Roots
$5x + 1 = 0$
$5x = -1$
$x = -0.2 = -\frac{1}{5}$
Asymptotes
$x^2 + 3x - 4 = 0$
$(x + 4)(x - 1) = 0$
$x = -4, 1$



If we combine the algebraic analysis above with what we see in the graph, then we know that the dividing points important to the solution of this inequality are at $x = -4, -0.2, 1$. The intervals where the y values are less than zero are $x < -4$ \(\mathrm{OR}-0.2

Example

$\frac{x^2+2x-1}{x^2+7x+5} \leq 0$



Roots

$$x^2 + 2x - 1 = 0$$
$$x \approx -2.414, 0.414 \tag{2.4.1}$$

Asymptotes

$x^2 + 7x + 5 = 0$

$$x \approx -6.193, -0.807 \tag{2.4.2}$$

We can see that the dividing points important to the solution of the inequality are $x \approx -6.193, -2.414, -0.807, 0.414$. The intervals where the y values are less than or equal to zero are $-6.193 \leq x \leq -2.414$ OR $-0.807 \leq x \leq 0.414$

Exercises 2.4

Solve each inequality.

1) $\frac{x+4}{x^2-8x+12} > 0$
2) $\frac{2x+3}{x^2-2x-35} < 0$
3) $\frac{x^2-5x-14}{x^2+3x-10} < 0$
4) $\frac{2x^2-x-3}{x^2+10x+16} > 0$
5) $\frac{3x+2}{x^2+x-5} < 0$
6) $\frac{x^2+2x+5}{x^2-3x-7} > 0$
7) $\frac{x^3+9}{x^2+x-1} > 0$
8) $\frac{x^3+9}{x^2+x+1} > 0$

Solve each inequality.

9) $\frac{x^2-2x-9}{3x+11} > 0$
10) $\frac{x^2+4x+3}{2x+1} < 0$
11) $\frac{x^2+x-5}{x^2-x-6} > 0$
12) $\frac{x^3+2}{x^2-2} > 0$
13) $\frac{x^2+2x-7}{x^2+3x-6} < 0$
14) $\frac{2x-x^2}{x^2-4x+6} < 0$
15) $\frac{x^2-7}{x^2+5x-1} < 0$
16) $\frac{x-5}{3x^2-2x-3} > 0$

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2.5: Finding Factors from Roots

One method of solving equations involves finding the factors of the polynomial expression in the equation and then setting each factor equal to zero.

$$\begin{array}{c} x^2+8x+15=0 \\ (x+5)(x+3)=0 \\ x+5=0 \quad x+3=0 \\ x=-5 \quad x=-3 \end{array} \tag{2.5.1}$$

In this process, the reasoning is that if $(x+5)$ times $(x+3)$ equals zero, then one of those expressions must be equal to zero. In setting them equal to zero, we find the solutions of $x=-5,-3$. Plugging them back into the factored expression we see the following:

$$(-5+5)(-5+3)=0*-2=0 \tag{2.5.2}$$

and

$$(-3+5)(-3+3)=2*0=0 \tag{2.5.3}$$

This process works in reverse as well. In other words, if we know a root of the function, we can find factors for the expression.
Example
Find a quadratic equation that has roots of -2 and +3

$$\begin{array}{cc} x=-2 & x=3 \\ x+2=0 & x-3=0 \\ (x+2)(x-3)=0 & \\ x^2-x-6=0 & \end{array} \tag{2.5.4}$$

Roots that are fractions are a little trickier, but really no more difficult:
Example
Find a quadratic equation that has roots of -5 and $\frac{2}{3}$

$$\begin{array}{cc} x=-5 & x=\frac{2}{3} \\ x+5=0 & 3x=2 \\ x+5=0 & 3x-2=0 \\ (x+5)(3x-2)=0 & \\ 3x^2+13x-10=0 & \end{array} \tag{2.5.5}$$

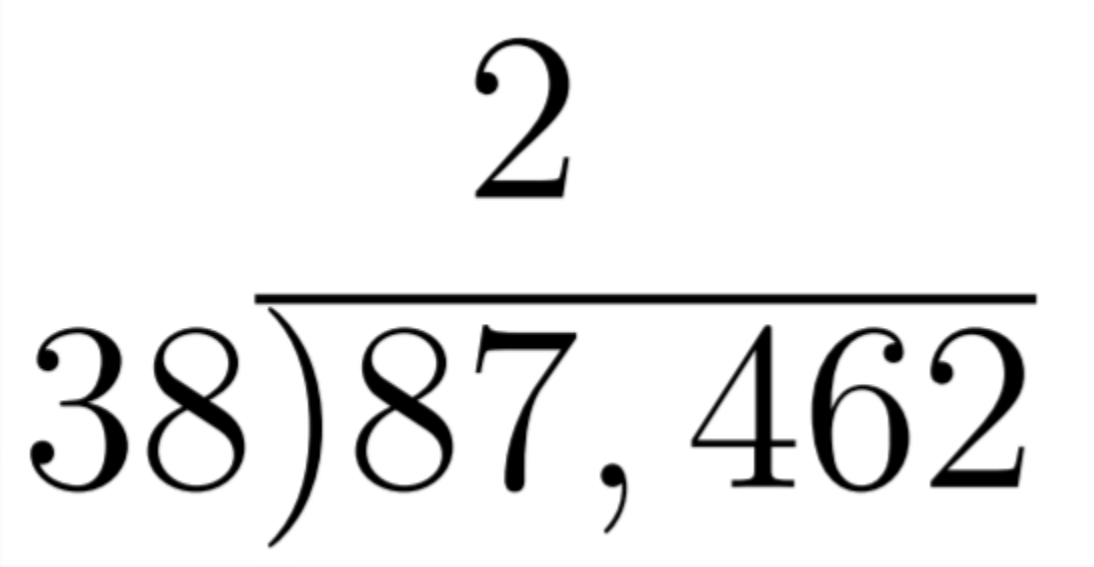
Exercises 2.5
Find a quadratic equation that has the indicated roots.
1) $4,-1$
2) $-2,7$
3) $\frac{3}{2},1$
4) $-\frac{1}{5},\frac{2}{3}$
5) $\frac{1}{3},3$
6) $-4,\frac{2}{5}$
7) $\frac{1}{2},-\frac{7}{2}$
8) $-1,\frac{3}{5}$
9) $-\frac{2}{3},-3$
10) $-\frac{2}{3},-\frac{3}{4}$
11) $-\frac{5}{2},3$
12) $-6,-2$

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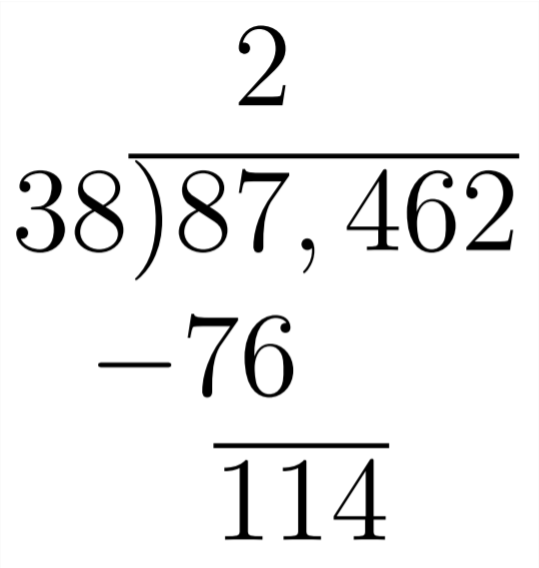
2.6: Polynomial Long Division

Polynomial long division has many similarities to numerical long division, so it is important that we understand how and why numerical long division works the way it does before discussing polynomial long division. First the HOW?

Given the numerical problem $87,462 \div 38$, the first step is to determine the highest place value in the answer.



Often the first step in numerical long division is to say "Does 38 divide into $8?$" "No." "Does 38 divide into $87?$" "Yes." This tells us that the first digit in the answer will be over the 7 in $87,462,$ and consequently will be in the thousands place. Once we know that the first digit in the answer will be in the thousands place, the next question is "How many thousands?" We can determine that $2 * 38 = 76$ but $3 * 38 = 114$ (too big), so we know that the first digit in the answer will be $2.$ Then we subtract, include the 4 and examine what is left over to continue.



Here, we see that $114 \div 38 = 3$, so we know that the next digit in the answer will be 3

$$\begin{array}{r} 2\ 3 \\ 38\overline{)87,462} \\ -76 \\ \hline 114 \\ -114 \\ \hline 0006 \end{array}$$

After including the 6, we can see that 38 does not divide evenly into 6, so we put a zero as the next digit in our answer and proceed:

$$\begin{array}{r} 2\ \ 30 \\ 38\overline{)87,462} \\ -76 \\ \hline 114 \\ -114 \\ \hline 0006 \\ 0 \\ \hline 00062 \end{array}$$

Now that we have included all the digits from our original number $87,462$, the last step is to divide 38 into 62. This goes one time with 24 left over.

So, now we have the solution to the original problem $87,462 \div 38 = 2,301\mathrm{R}24$ or
$2,301\frac{24}{38}$
The WHY? of the long division algorithm is somewhat hidden by the HOW? In the first step, we are determining which place value will hold the first digit of our answer. When we determine that 38 does divide into 87, this indicates that the first digit in our answer will be the thousands place. Dividing 38 into 87 tells us how many thousands there will be. Then we subtract:

$$\begin{array}{r} 87,462 \\ -76,000 \\ \hline 11,462 \end{array} \tag{2.6.1}$$

Now we need to determine how many times 38 will divide into $11,462$. We decided on $300 * 38 = 11,400$, then we subtract to see how much is left over:

$$\begin{array}{r} 11,462 \\ -11,400 \\ \hline 00,062 \end{array} \tag{2.6.2}$$

We can see that we won't need any tens in our answer, and that 38 divides into 62 one time with 24 left over, thus the answer is 2 thousands, 3 hundreds, no tens, 1 and a remainder of 24. To check the answer, we multiply $38 * 2301$ and add 24:

$$\begin{array}{r} 2,301 \\ \times 38 \\ \hline 18408 \\ 6903 \\ \hline 87438 \\ +24 \\ \hline 87462 \end{array} \tag{2.6.3}$$

Polynomial long division works in much the same way that numerical long division does. Given a problem $A \div B$, the goal is to find a quotient Q and remainder R so that $A = B * Q + R$
Let's look at this with the example $2x^4 + 7x^3 + 4x^2 - 2x - 1 \div x^2 + 3x + 1$ or:

$$\frac{2x^4 + 7x^3 + 4x^2 - 2x - 1}{x^2 + 3x + 1} \tag{2.6.4}$$

So, we are looking to answer the question:

$$\begin{array}{rcccc} A & = & B & * & Q & +R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & (x^2 + 3x + 1) & * & (?+?+?) & +? \end{array} \tag{2.6.5}$$

If we want to mulitply $x^2 + 3x + 1$ times something and end up with $2x^4 + 7x^3 + 4x^2 - 2x - 1$, then what we multiply by is going to have to start with $2x^2$, because

$$x^2 * 2x^2 = 2x^4 \tag{2.6.6}$$

Now we're working with this:

$$\begin{array}{rcccc} A & = & B & * & Q & +R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & (x^2 + 3x + 1) & * & (2x^2+?+?) & +? \end{array} \tag{2.6.7}$$

But the $2x^2$ doesn't just get multiplied by the x^2, it will also get multiplied by the $3x$ and the 1. So now we have:

$$\begin{array}{rcccc} A & = & B & * & Q & +R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & (x^2 + 3x + 1) & * & (2x^2+?+?) & +? \\ & & = 2x^4 + 6x^3 + 2x^2 + ????? & & & \end{array} \tag{2.6.8}$$

The issue this raises is that the next multiplication $(? * x^2 + ? * 3x + ? * 1)$ needs to add only $1x^3$ to our answer, because we need $7x^3$ and we already have $6x^3$ from the previous multiplication. That means we're going to want to multiply next by $1x$:

$$\begin{array}{rcl} A = & B * Q & +R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 = & (x^2 + 3x + 1) * (2x^2 + 1x + ?) & +? \\ & = 2x^4 + 6x^3 + 2x^2 & \\ & = 1x^3 + 3x^2 + x & \\ & = 2x^4 + 7x^3 + 5x^2 + 1x + ??? & \end{array} \tag{2.6.9}$$

In the next round of mutiplication, we're going to want to bring the $5x^2$ down to $4x^2$, so we'll need to multiply by -1

$$\begin{aligned} A &= \quad B \quad * \quad Q \quad + R \\ 2x^4+7x^3+4x^2-2x-1 &= \left(x^2+3x+1\right) * \left(2x^2+1x-1\right)+0 \\ &= 2x^4+6x^3+2x^2 \\ &= \quad 1x^3+3x^2+x \\ &= \qquad -1x^2-3x-1 \\ &= 2x^4+7x^3+5x^2-2x-1 \end{aligned} \tag{2.6.10}$$

Now we also know that the remainder is zero, because x^2+3x+1 divides evenly into $2x^4+7x^3+4x^2-2x-1$ and so:

$$2x^4+7x^3+4x^2-2x-1 = \left(x^2+3x+1\right) * \left(2x^2+1x-1\right) \tag{2.6.11}$$

This method makes the reasoning behind dividing polynomials somewhat more apparent than the long division process, but it is more cumbersome. The way that polynomial long division is usually approached is as follows:

$$x^2+3x+1\overline{)\quad 2x^4+7x^3+4x^2-2x-1}$$

Then, just as we did in the other method, we question "What should we multiply x^2 by to get $2x^4$?'' Answer: ''$2x^2$'' This is the first term in our answer:

$$\begin{array}{r} \qquad\qquad 2x^2 \\ \hline x^2+3x+1\overline{)\quad 2x^4+7x^3+4x^2-2x-1} \end{array}$$

Then we multiply $2x^2\left(x^2+3x+1\right)$ and change all the signs to see what we'll be left with:

$$\begin{array}{r} \qquad\qquad 2x^2 \\ \hline x^2+3x+1\overline{)\quad 2x^4+7x^3+4x^2-2x-1} \\ \underline{-2x^4-6x^3-2x^2} \end{array}$$

This indicates that we'll have the $2x^4$ we'll need in our answer, as well six of the seven x^3 's and two of the four x^2 's. We'll now need $1x^3$ next:

$$
\begin{array}{r}
2x^2 \\
\hline
x^2+3x+1\,\big)\;\; 2x^4+7x^3+4x^2-2x-1 \\
\underline{-\,2x^4-6x^3-2x^2} \\
x^3+2x^2-2x
\end{array}
$$

This means we'll need to multiply by $1x$:

$$
\begin{array}{r}
2x^2\;+x \\
\hline
x^2+3x+1\,\big)\;\; 2x^4+7x^3+4x^2-2x-1 \\
\underline{-\,2x^4-6x^3-2x^2} \\
x^3+2x^2-2x \\
\underline{-\,x^3-3x^2\;-x} \\
-\,x^2-3x-1
\end{array}
$$

Here, we still need to pick up a $-1x^2$, which means that our next multiplication will be with -1:

$$
\begin{array}{r}
2x^2\;+x-1 \\
\hline
x^2+3x+1\,\big)\;\; 2x^4+7x^3+4x^2-2x-1 \\
\underline{-\,2x^4-6x^3-2x^2} \\
x^3+2x^2-2x \\
\underline{-\,x^3-3x^2\;-x} \\
-\,x^2-3x-1 \\
\underline{x^2+3x+1} \\
0
\end{array}
$$

Because x^2+3x+1 divides evenly into $2x^4+7x^3+4x^2-2x-1$ we have a zero remainder. In the next example there will be a non-zero remainder:

Example
Divide:
$\frac{3x^4-8x^3+19x^2-15x+10}{x^2-x+4}$

First, we set up the problem:

$$x^2-x+4\overline{)\quad 3x^4-8x^3+19x^2-15x+10}$$

Then, we question: "What do we need to multiply x^2 by to get $3x^4$? " Answer:

$$\begin{array}{r} 3x^2 \\ \hline x^2-x+4\overline{)\quad 3x^4-8x^3+19x^2-15x+10} \end{array}$$

Then, we multiply, change signs (subtract) and combine like terms:

$$\begin{array}{r} 3x^2 \\ \hline x^2-x+4\overline{)\quad 3x^4-8x^3+19x^2-15x+10} \\ \underline{-3x^4+3x^3-12x^2} \\ -5x^3+7x^2-15x \end{array}$$

Now we'll need to multiply by $-5x$, change signs and combine like terms:

$$\begin{array}{r} 3x^2-5x \\ \hline x^2-x+4\overline{)\quad 3x^4-8x^3+19x^2-15x+10} \\ \underline{-3x^4+3x^3-12x^2} \\ -5x^3+7x^2-15x \\ \underline{5x^3-5x^2+20x} \\ 2x^2+5x+10 \end{array}$$

We need $2x^2$ so we'll need to multiply by 2 , change signs and combine like terms:

$$
\begin{array}{r}
3x^2 - 5x + 2 \\
\hline
x^2 - x + 4 \,\big)\; 3x^4 - 8x^3 + 19x^2 - 15x + 10 \\
\underline{-\,3x^4 + 3x^3 - 12x^2} \\
-\,5x^3 + 7x^2 - 15x \\
\underline{\;\;5x^3 - 5x^2 + 20x} \\
2x^2 + 5x + 10 \\
\underline{-\,2x^2 + 2x - 8} \\
7x + 2
\end{array}
$$

Because there is no positive power of x that we can multiply x^2 by to get $7x$, then this is our remainder: $7x + 2$
So:

$$
\begin{array}{cccccccc}
A & = & B & * & Q & + & R \\
3x^4 - 8x^3 + 19x^2 - 15x + 10 & = (x^2 - x + 4) & * (3x^2 - 5x + 2) & +(7x + 2)
\end{array} \tag{2.6.12}
$$

Exercises 2.6

Find the quotient in each problem.

1) $\frac{y^3-4y^2+6y-4}{y-2}$
2) $\frac{x^3-5x^2+x+15}{x-3}$
3) $\frac{x^3-4x^2-3x-10}{x^2+x+2}$
4) $\frac{2x^3-3x^2+7x-3}{x^2-x+3}$
5) $\frac{x^4+2x^3-x^2+x+6}{x+2}$
6) $\frac{x^4+x^3+5x^2+3x+6}{x^2+x-1}$
7) $\frac{2z^3+5z+8}{z+1}$
8) $\frac{x^5+3x+2}{x^3+2x+1}$
9) $\frac{x^4+2x^3+4x^2+3x+2}{x^2+x+2}$
10) $\frac{2x^4+3x^3+3x^2-5x-3}{2x^2-x-1}$
11) $\frac{2y^5-3y^4-y^2+y+4}{y^2+1}$
12) $\frac{3x^5-4x^3+3x^2+12x-10}{x^2+2x-1}$
13) $\frac{5x^4-3x^2+2}{x^2-3x+5}$
14) $\frac{3y^3-4y^2-3}{y^2+5y+2}$

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2.7: Synthetic Division

The process for polynomial long division (like the process for numerical long division) has been separated somewhat from its logical underpinnings for a more efficient method to arrive at an answer. For particular types of polynomial long division, we can even take this abstraction one step further. Synthetic Division is a handy shortcut for polynomial long division problems in which we are dividing by a linear polynomial. This means that the highest power of x we are dividing by needs to be x^1. This limits the usefulness of Synthetic Division, but it will serve us well for certain purposes. Let's examine where the coefficients in our answer come from when we divide by a linear polynomial:

$$\begin{array}{r} \quad\quad 2x^3 \\ \hline x-5\,\big)\;2x^4-6x^3-23x^2+16x-5 \\ \quad -2x^4+10x^3 \\ \hline \quad\quad\quad 4x^3-23x^2 \end{array}$$

Notice that the first coefficient in the answer is the same as the first coefficient in the polynomial we're dividing into. This is because we're dividing by a polynomial in the form $1x^1-a$. This also makes the power of the first term in the answer one less than the power of the polynomial we are dividing into. Let's see where the subsequent coefficients in the answer come from:

$$\begin{array}{r} \quad\quad 2x^3+4x^2 \\ \hline x-5\,\big)\;2x^4-6x^3-23x^2+16x-5 \\ \quad -2x^4+10x^3 \\ \hline \quad\quad\quad 4x^3-23x^2 \\ \quad\quad\quad -4x^3+20x^2 \\ \hline \quad\quad\quad\quad\quad -3x^2+16x \end{array}$$

The next coefficient in the answer (4) comes from the combination of the -6 and the $+10$. The +10 came from multiplying the 2 in the answer by the 5 in the divisor $x-5$. The next coefficient in the answer will be -3, which comes from multiplying the 4 (in the answer) by the 5 (in the divisor) and combining it with the -23 in the polynomial we're dividing into:

$$
\begin{array}{r}
2x^3 + 4x^2 - 3x \\
x-5\overline{)\;2x^4 - 6x^3 - 23x^2 + 16x - 5} \\
\underline{-2x^4 + 10x^3} \\
4x^3 - 23x^2 \\
\underline{-4x^3 + 20x^2} \\
-3x^2 + 16x \\
\underline{3x^2 - 15x} \\
\end{array}
$$

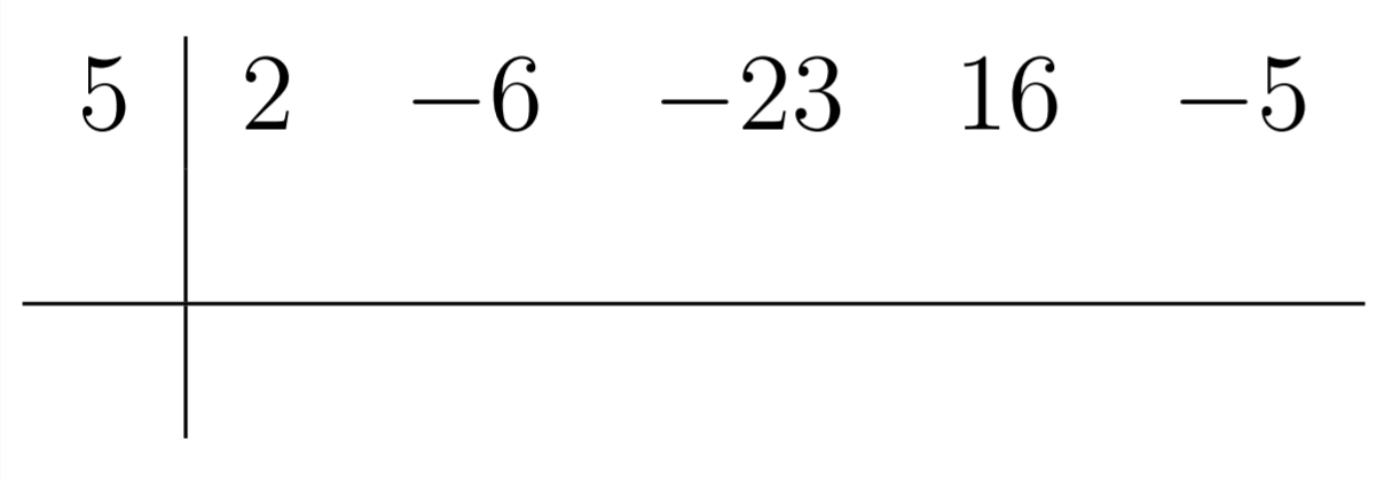
The last part of our answer will come from multiplying the -3 in the answer times the 5 in the divisor (making -15) and combining this with the +16 in the polynomial we're dividing into:

$$
\begin{array}{r}
2x^3 + 4x^2 - 3x + 1 \\
x-5\overline{)\;2x^4 - 6x^3 - 23x^2 + 16x - 5} \\
\underline{-2x^4 + 10x^3} \\
4x^3 - 23x^2 \\
\underline{-4x^3 + 20x^2} \\
-3x^2 + 16x \\
\underline{3x^2 - 15x} \\
1x - 5 \\
\underline{-x + 5} \\
0
\end{array}
$$

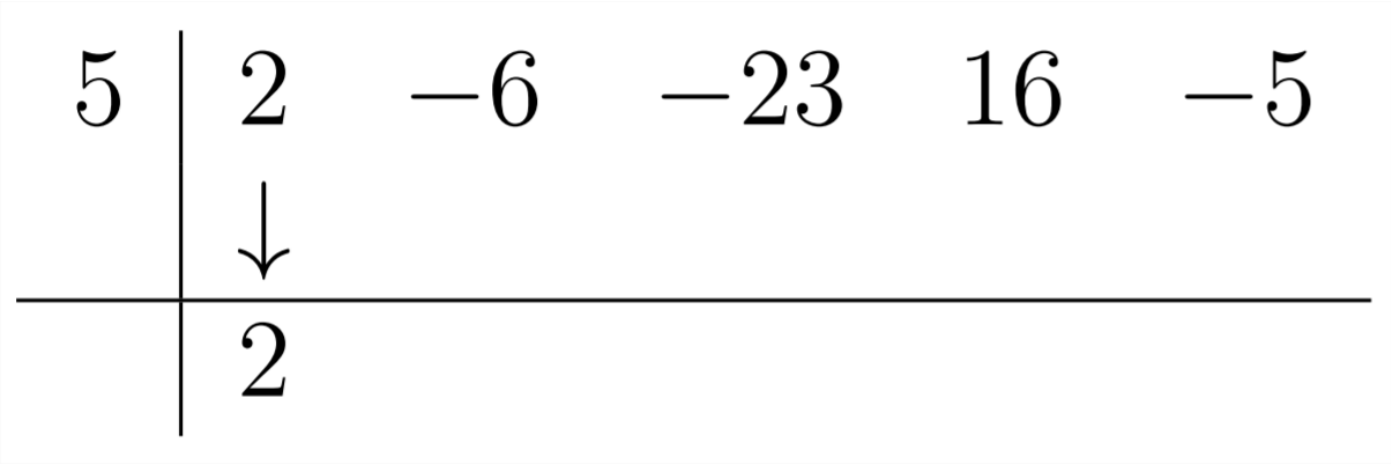
The process of Synthetic Division uses these relationships as a shortcut to finding the answer. The set-up for a Synthetic Division problem is shown below:

$$
\begin{array}{r|rrrrr}
5 & 2 & -6 & -23 & 16 & -5 \\
\hline
 & & & & & \\
\hline
 & & & & &
\end{array}
$$

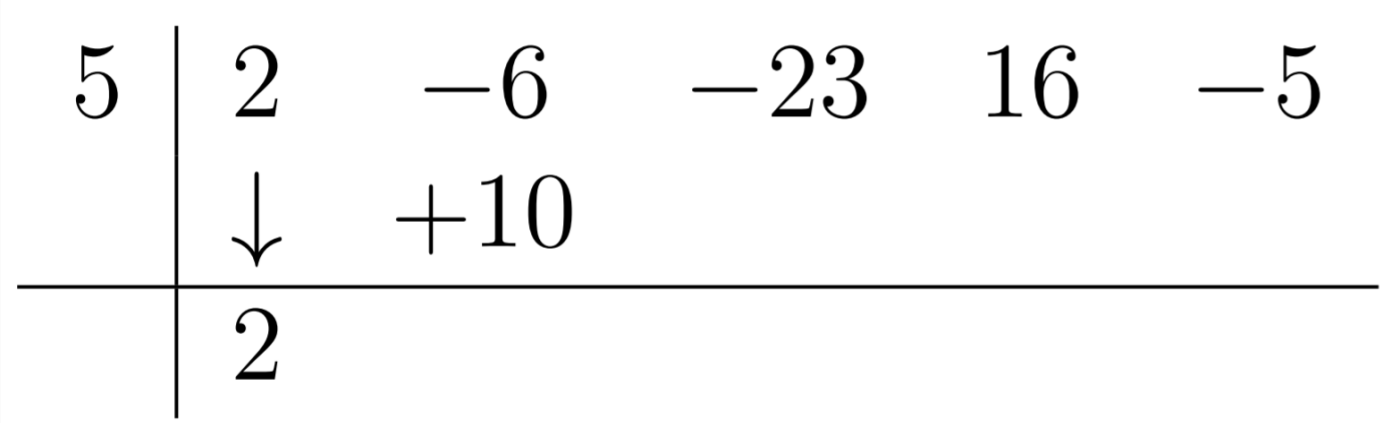
This set-up allows us to complete the division problem $\frac{2x^4-6x^3-23x^2+16x-5}{x-5}$.



The first coefficient in the answer is the same as the first coefficient in the polynomial we're dividing into:



To get the next coefficient, we multiply the 2 by the 5 to get +10 and fill this in under the -6:



Then, $-6+10=+4$, which is the next coefficient in the answer:

$$\begin{array}{c|cccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & & & \\ \hline & 2 & 4 & & & \end{array}$$

Then, we continue this process, multiplying the 4 by the 5 to get 20 and combining this with the $-23 : -23 + 20 = -3$:

$$\begin{array}{c|cccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & & \\ \hline & 2 & 4 & -3 & & \end{array}$$

Next, multiply the -3 by the 5 and combine the resulting -15 with the 16:

$$\begin{array}{c|cccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & -15 & \\ \hline & 2 & 4 & -3 & 1 & \end{array}$$

In the last step, multiply the 1 times the 5 and combine the result with the -5 in the problem to get zero:

5	2	-6	-23	16	-5
	$\downarrow$	$+10$	$+20$	-15	5
	2	4	-3	1	0

This last coefficient represents the remainder - in this case 0. The other numerals in the answer represent the coefficients for the powers of x in the answer. On the far right is the remainder, then the constant (x^0) term, then the linear (x^1) term and so on:

5	2	-6	-23	16	-5
	$\downarrow$	$+10$	$+20$	-15	5
	$2x^3$	$4x^2$	$-3x^1$	$1x^0$	0

$\frac{2x^4-6x^3-23x^2+16x-5}{x-5} = 2x^3 + 4x^2 - 3x + 1$

Let's look at another example:

Example

Use Synthetic Division to divide:

$\frac{3x^3+5x^2-9x+9}{x+3}$

since Synthetic Division is set up to divide by $x - a$, if we're dividing by $x + 3$ we'll need to use a -3 in the Synthetic Division:

-3	3	5	-9	9
	$\downarrow$			
	3			

Then, $3 * -3 = -9$

-3	3	5	-9	9
	$\downarrow$	-9		
	3	-4		

Next, $-4 * -3 = +12$

-3	3	5	-9	9
	$\downarrow$	-9	12	
	3	-4	3	

And this example also has a zero remainder:

-3	3	5	-9	9
	$\downarrow$	-9	12	-9
	3	-4	3	0

The answer here is $3x^2 - 4x + 3$

$$\frac{3x^3+5x^2-9x+9}{x+3}=3x^2-4x+3 \tag{2.7.1}$$

and

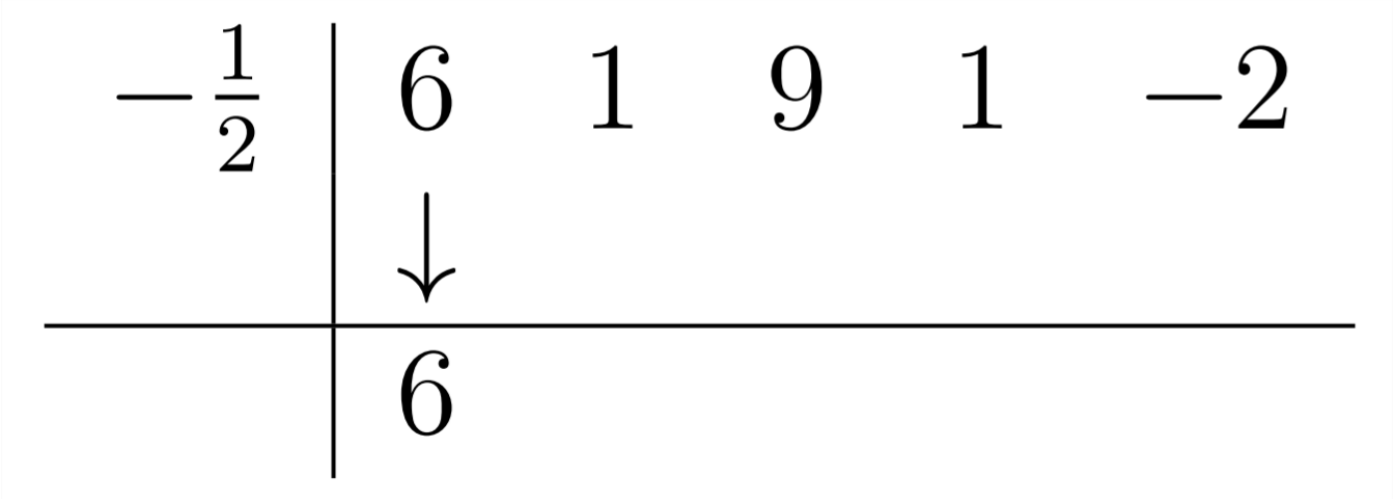
$$3x^3+5x^2-9x+9=(x+3)\left(3x^2-4x+3\right) \tag{2.7.2}$$

Let's look at an example that is a little bit different.

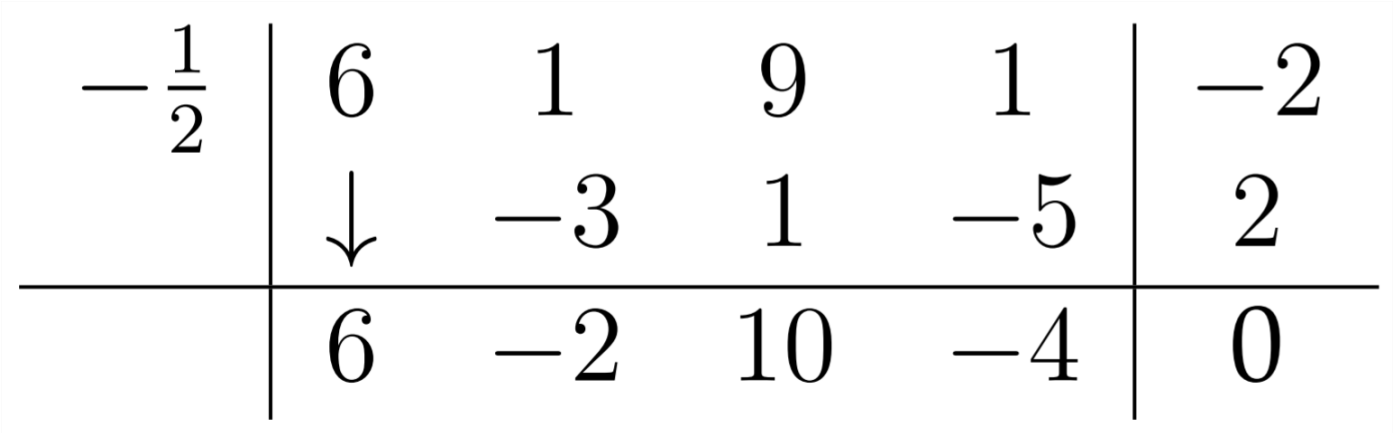
Example

Use Synthetic Division to divide:
$\frac{6x^4+x^3+9x^2+x-2}{2x+1}$

Synthetic Division is set up to handle problems in which we are dividing by $1x-a$. Clearly, this is not the case in this example, however we can work around this. Another way of looking at setting up the synthetic division is that we use the number that is the solution to $x-a=0$. When we divided by $x-5$, we used
+5. When we were dividing by $x+3$, we used -3. So, if we're going to divide by $2x+1$, we'll use $-\frac{1}{2}$ in the Synthetic Division:



Then, we proceed as usual:



Notice that, again, we have a zero remainder. Also, notice that each coefficient in our answer has a common factor of 2, which was the coefficient of the x in $2x+1$, which we originally were going to divide by. What we've done here is not division by $2x+1$, but division by $x+\frac{1}{2}$
So, in the end, our work tells us that:

$$\frac{6x^4+x^3+9x^2+x-2}{x+\frac{1}{2}}=6x^3-2x^2+10x-4 \tag{2.7.3}$$

and

$$6x^4+x^3+9x^2+x-2=\left(x+\frac{1}{2}\right)\left(6x^3-2x^2+10x-4\right) \tag{2.7.4}$$

Notice that if we factor out the common factor of 2 from our answer, we can multiply it back into the $x+\frac{1}{2}$ and get an answer for our original problem:

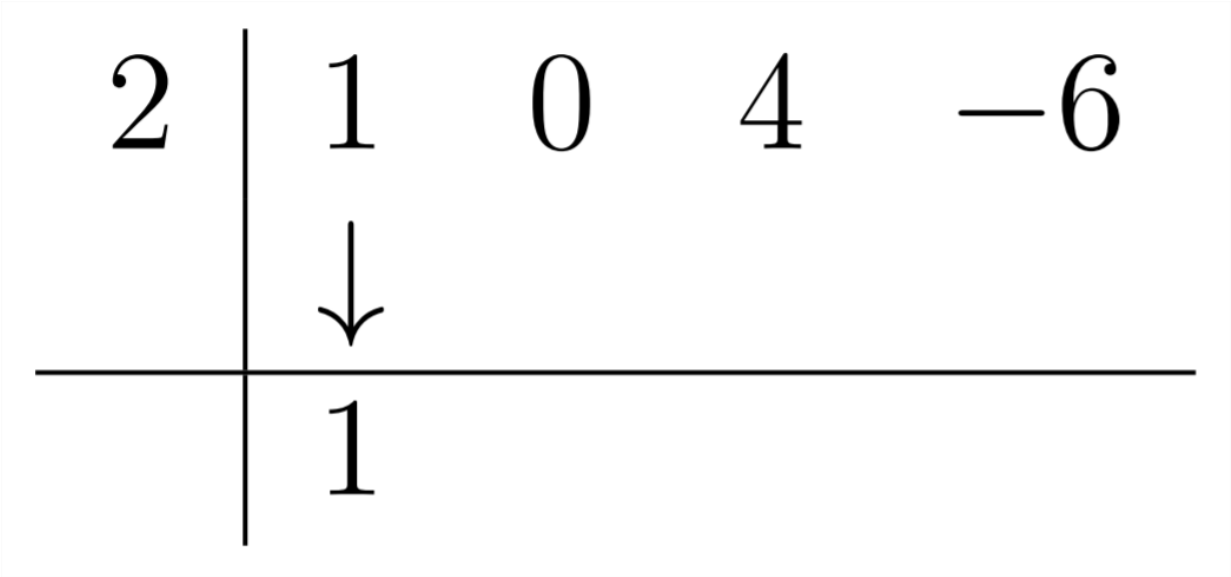
$$\begin{aligned} 6x^4+x^3+9x^2+x-2 &= \left(x+\frac{1}{2}\right)2\left(3x^3-x^2+5x-2\right) \\ &= (2x+1)\left(3x^3-x^2+5x-2\right) \end{aligned} \tag{2.7.5}$$

This means that:

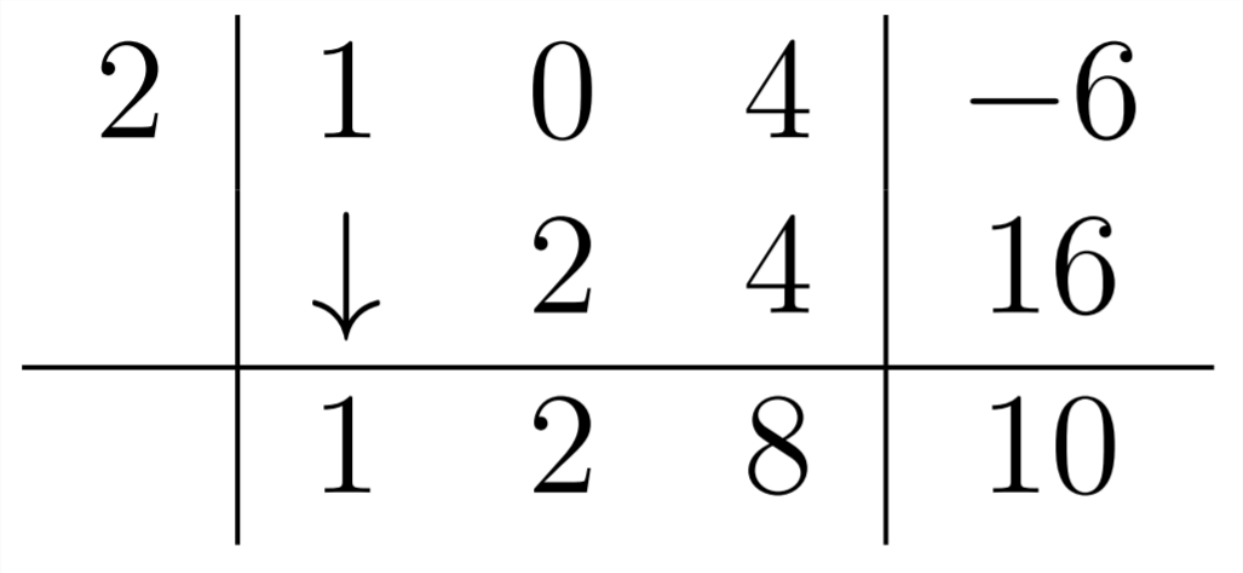
$$\frac{6x^4+x^3+9x^2+x-2}{2x+1}=3x^3-x^2+5x-2 \tag{2.7.6}$$

Another thing to understand about Synthetic Division is that if there is a missing power of x, then you should include a zero as the coefficient of that power

Example
Use Synthetic Division to divide:
$\frac{x^3+4x-6}{x-2}$
since there is no x^2 term in the polynomial we're divding into, we'll enter a zero as the coefficient for that term:



And then proceed as usual:



So the answer for this problem is $x^2 + 2x + 8$

Exercises 2.7

Use synthetic division to find the quotient in each problem.

1) $\frac{x^3-8x^2+5x+50}{x-5}$
2) $\frac{x^3+5x^2-x-14}{x+2}$
3) $\frac{x^3+12x^2+34x-7}{x+7}$
4) $\frac{x^3-10x^2+23x-6}{x-3}$
5) $\frac{x^4-15x^2+10x+24}{x+4}$
6) $\frac{x^4-3x^3+4x^2-36}{x-3}$
7) $\frac{x^4-2x^3-x+10}{x-2}$
8) $\frac{x^4-16x^2-5x-24}{x+4}$
9) $\frac{2x^4-x^3+2x-1}{2x-1}$
10) $\frac{3x^4+x^3-3x+1}{3x+1}$
11) $\frac{3x^4-8x^3+9x^2-2x-2}{3x+1}$
12) $\frac{6x^4-7x^3+5x^2-17x+10}{3x-2}$
13) $\frac{2x^3+7x^2+6x-5}{2x-1}$
14) $\frac{3x^4-x^3-21x^2-11x+6}{3x-1}$

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2.8: Roots and Factorization of Polynomials

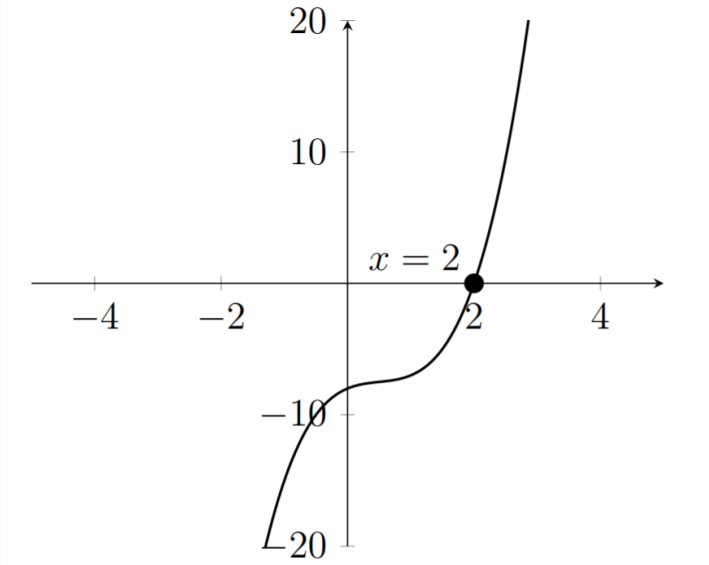
In this section we will use some of the skills we have seen in previous sections in order to find all the roots of a polynomial function (both real and complex) and also factor the polynomial as the product of prime factors with integer coefficients.

Example

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

$2x^3 - 3x^2 + 2x - 8 = 0$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see that there is a root at $x = 2$. This means that the polynomial will have a factor of $(x - 2)$. We can use Synthetic Division to find any other factors. Because $x = 2$ is a root, we should get a zero remainder:

2	2	−3	2	−8
	↓	4	2	8
	2	1	4	0

So, now we know that $2x^3 - 3x^2 + 2x - 8 = (x - 2)(2x^2 + x + 4)$. To finish the problem, we can set each factor equal to zero and find the roots:

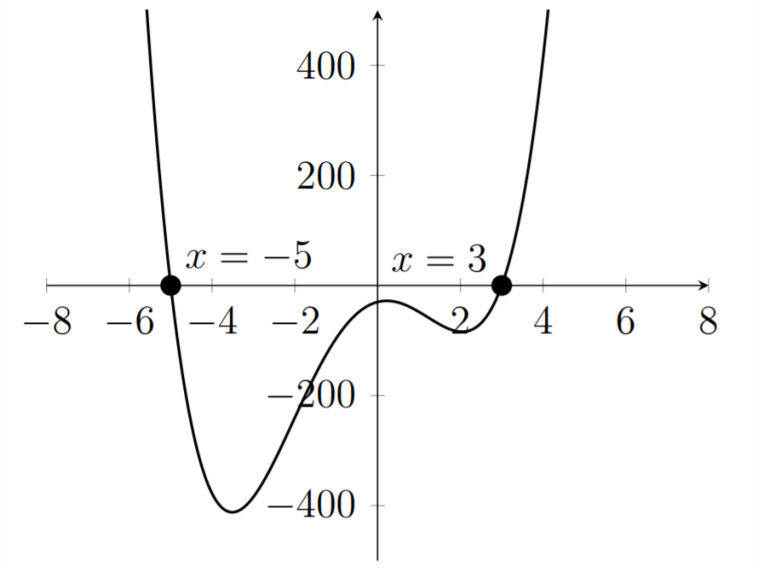
$$\begin{aligned} 2x^3 - 3^2 + 2x - 8 &= 0 \\ (x-2)\left(2x^2 + x + 4\right) &= 0 \\ x - 2 = 0 \quad & 2x^2 + x + 4 = 0 \\ x = 2 \quad & x \approx -0.25 \pm 1.392i \end{aligned} \tag{2.8.1}$$

Let's look at an example that has more than one real root:

Example
Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

$$3x^4 + 5x^3 - 45x^2 + 19x - 30 = 0 \tag{2.8.2}$$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see roots at $x = -5, 3$, which means that $(x+5)$ and $(x-3)$ are both factors of this polynomial. We'll need to divide by both of these factors to break down the polynomial. First, we divide by $(x-3)$:

3	3	5	-45	19	-30
	$\downarrow$	9	42	-9	30
	3	14	-3	10	0

And then by $(x+5)$:

3	3	5	−45	19	−30
	↓	9	42	−9	30
−5	3	14	−3	10	0
	↓	−15	5	−10	
	3	−1	2	0	

Now we know that $3x^4 + 5x^3 - 45x^2 + 19x - 30 = (x+5)(x-3)\left(3x^2 - x + 2\right)$ and so, to finish the problem:

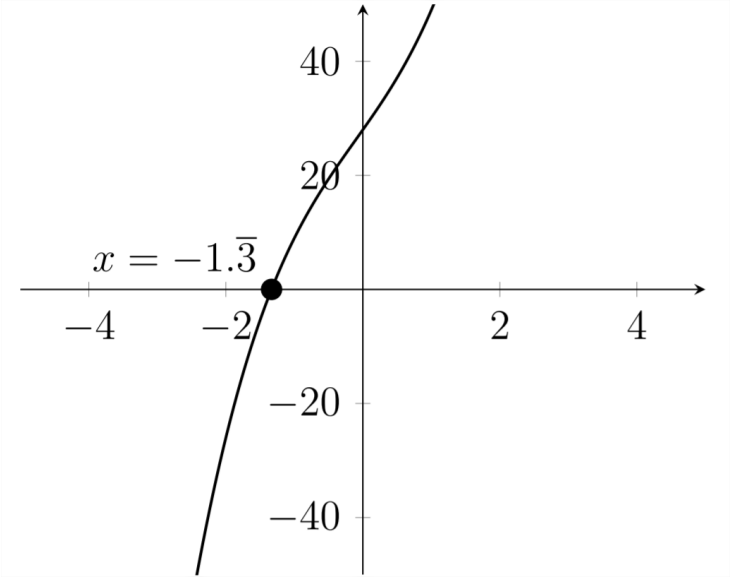
$$\begin{gathered} 3x^4 + 5x^3 - 45x^2 + 19x - 30 = 0 \\ (x+5)(x-3)\left(3x^2 - x + 2\right) = 0 \\ x+5=0 \quad x-3=0 \quad 3x^2 - x + 2 = 0 \\ x = -5 \quad x = 3 \quad x \approx \tfrac{1}{6} \pm 0.799i \end{gathered} \tag{2.8.3}$$

Next, let's look at an example where there is a root that is not a whole number:

Example

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.
$3x^3 + x^2 + 17x + 28 = 0$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see in the graph that this polynomial has a root at $x = -\frac{4}{3}$. That means that the polynomial must have a factor of $3x + 4$. We can use Synthetic Division to find the other factor for this polynomial. Because we know that $x = -\frac{4}{3}$ is a root, we should get

a zero remainder:

$$\begin{array}{c|ccc|c} -\frac{4}{3} & 3 & 1 & 17 & 28 \\ & \downarrow & -4 & 4 & -28 \\ \hline & 3 & -3 & 21 & 0 \end{array}$$

Notice that, because the root we used was a fraction, there is a common factor of 3 in the answer to our Synthetic Division. We should factor this out to obtain the
answer:
$\left(x+\frac{4}{3}\right)\left(3x^2-3x+21\right)=(3x+4)\left(x^2-x+7\right)$
So, this means that:

$$\begin{aligned} 3x^3+x^2+17x+28&=0 \\ (3x+4)\left(x^2-x+7\right)&=0 \\ 3x+4=0 \quad x^2-x+7&=0 \\ x=-\tfrac{4}{3} \quad x&\approx 0.5\pm 2.598i \end{aligned} \tag{2.8.4}$$

Exercises 2.8
Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.
Set #1
1) $x^4-3x^3+5x^2-x-10=0$
2) $3x^3-5x^2+2x-8=0$
3) $2x^4-5x^3+x^2+4x-4=0$
4) $x^4+x^3-3x^2-17x-30=0$
5) $x^4-9x^3+21x^2+21x-130=0$
6) $x^4-7x^3+14x^2-38x-60=0$
7) $x^5-9x^4+31x^3-49x^2+36x-10=0$
8) $x^4+4x^3+2x^2+12x+45=0$
9) $x^4-6x^3+12x^2-10x+3=0$
10) $x^4-6x^3+13x^2-24x+36=0$
11) $x^5-3x^4+12x^3-28x^2+27x-9=0$
12) $x^5+2x^4-3x^3-3x^2+2x+1=0$
Set #2
13) $15x^3-7x^2+13x+3=0$
14) $x^4-5x^3+3x^2-11x-20=0$
15) $6x^3+13x^2+12x+4=0$
16) $6x^3-5x^2+5x-2=0$
17) $4x^4+20x^3+29x^2+10x-15=0$
18) $3x^4-4x^3+10x^2+12x-5=0$
19) $2x^4-3x^3-6x^2-8x-3=0$
20) $12x^4-53x^3-31x^2-19x-5=0$
21) $12x^4+4x^3+x^2-3x-2=0$
22) $3x^4+13x^3-26x-40=0$

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3: EXPONENTS AND LOGARITHMS

In this chapter, we will examine concepts that are related to exponential, logarithmic and logistic relationships. In the first section, we will look at how to approach these problems from a graphical perspective. In the subsequent sections, we will examine the methods necessary to work with these problems algebraically.

3.1: Exponential and Logistic Applications

3.2: Logarithmic Notation

3.3: Solving Exponential Equations

3.4: Solving Logarithmic Equations

3.5: Applications of the Negative Exponential Function

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3.1: Exponential and Logistic Applications

There are a variety of different types of mathematical relationships. The simplest mathematical relationship is the additive relationship. This is a situation in which the value of one quantity is always a certain amount more (or less) than another quantity. A good example of an additive relationship is an age relationship. In an age relationship, the age of the older person is always the same amount more than the age of the younger person. If the older person is five years older, then the age of the older person (y) will always be equal to the age of the younger person (x) plus five: $y = x + 5$

Another type of additive relationship is seen where two quantities add up to a constant value. Let's say there is a board whose length is 20 inches. If we cut it into two pieces, with one piece being 6 inches, then the other piece will be 14 inches. If one piece is 9 inches, then the other will be 11 inches. If one piece is x inches, then the other piece (y) will be $20 - x : y = 20 - x$ or $x + y = 20$

The next type of mathematical relationshp is a multiplicative relationship. This represents a situation in which one quantity is always a multiple of the other quantity. This is commonly seen in proportional relationships. If a recipe for a cake calls for 2 cups of flour, then, if we want to make 3 cakes, we'll need 6 cups of flour. The amount of flour (y) is always two times the number of cakes we want to make $(x) : y = 2x$

If a recipe for a batch of cookies (with 20 cookies per batch) calls for 1.5 cups of sugar, then three batches would require 4.5 cups of sugar. The amount of sugar required (y) is always the number of batches (x) times $1.5 : y = 1.5x$. If we wanted to represent this relationship based on the number of cookies instead of the number of batches, we would need to adjust the formula. Given that there are 20 cookies per batch, we could adjust our formula so that we first calculate the number of batches from the number of cookies and then multiply by 1.5. If the number of cookies is x and the amount of sugar is y, then $y = 1.5 * \frac{x}{20}$ or $y = \frac{3}{40}x$
The next type of mathematical relationship is the polynomial relationship. In this type of relationship, one quantity is related to a power of another quantity. A good example of this type of relationship involves gravity. As Galileo discovered in the 16 th century, the distance that an object falls after it is dropped is not proportional to the time that it has been falling. Rather, it is proportional to the square of the time. The table below shows this type of relationship.

t	d
1	16
2	64
3	144
4	256

After one second, it looks like the distance will always be sixteen times the time the object has been falling. However, after two seconds, we can see that this relationship no longer is true. That's because this relationship is a polynomial relationship in which the distance an object has fallen (d) is proportional to the square of the time it has been falling $(t) : d = 16t^2$

Exponential Relationships
The next type of relationship is the focus of this chapter - the exponential relationship. In this situation, the rate of change of a quantity is proportional to the size of that quantity. This relationship can be explored in more depth in an integral calculus course, but we will discuss the basics here.

In a linear or proportional relationship, the slope, or rate of change, is constant. For example, in the equation $y = 3x + 1$, the slope is always three, no matter what the values of x and y are. In an exponential relationship, the rate of change (also called " y prime"

or y') is proportional to the value of y. In this case, we say that

$$y' = k * y \tag{3.1.1}$$

This is what is known as a differential equation. This is an equation in which the variable and its rate of change are related. Through the processes of differential and integral calculus, we can solve the equation above $y' = k * y$ as:

$$y = Ae^{kt} \tag{3.1.2}$$

In the equation above, A is the value of y at time $t = 0, k$ is a constant that determines how fast the quantity y increases or decreases and t plays the role of the independent variable (as x often does) and represents the time that has passed. If k is positive, then the quantity y is growing because its rate of change is positive. If k is negative, then the quantity y is decreasing because the rate of change is negative.

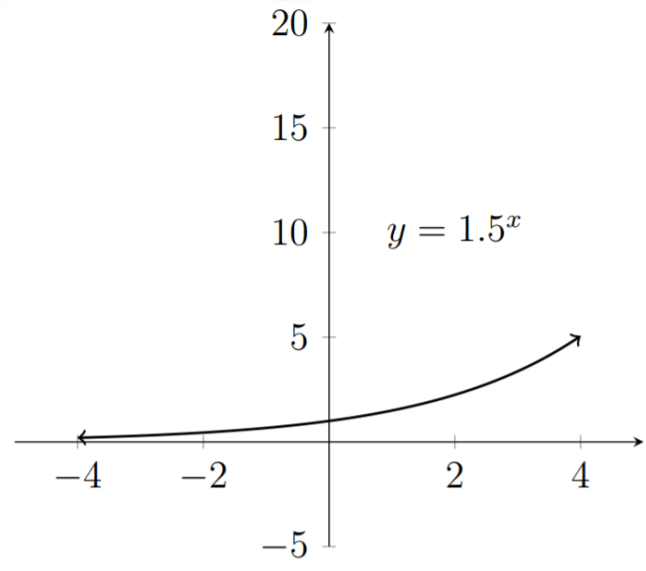
The quantity represented by e in the above equation is a mathematical constant (like π) that is often used to represent exponential relationships. The best way to understand the value of e and what it represents is directly related to fundamental questions from differential and integral calculus.

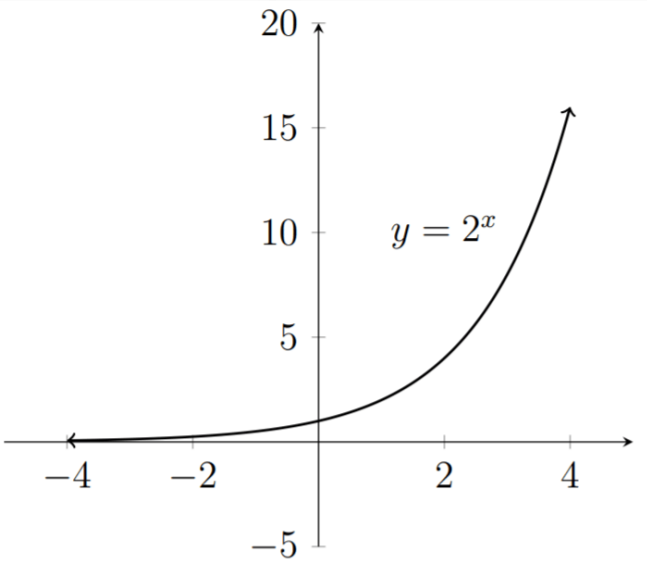
Differential Calculus is concerned primarily with the question of slopes. We discussed earlier that a linear relationship has a constant slope. Polynomial and exponential relationships have slopes that depend on the value of x and / or y. This is what makes them curves rather than lines. If we consider the slopes of some different exponential relationships, we can see one aspect of where the value for e comes from.

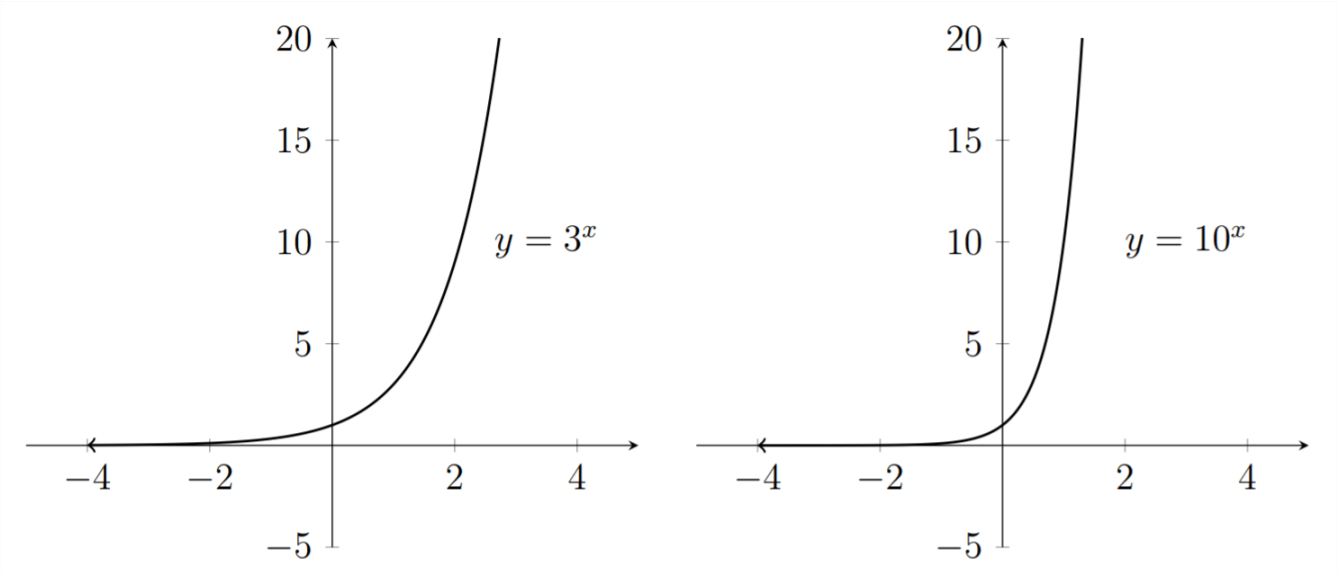
Consider the graphs of the following relationships:

$$\begin{aligned} y &= 1.5^x \\ y &= 2^x \\ y &= 3^x \\ y &= 10^x \end{aligned} \tag{3.1.3}$$

Let's look at the graphs for these functions:

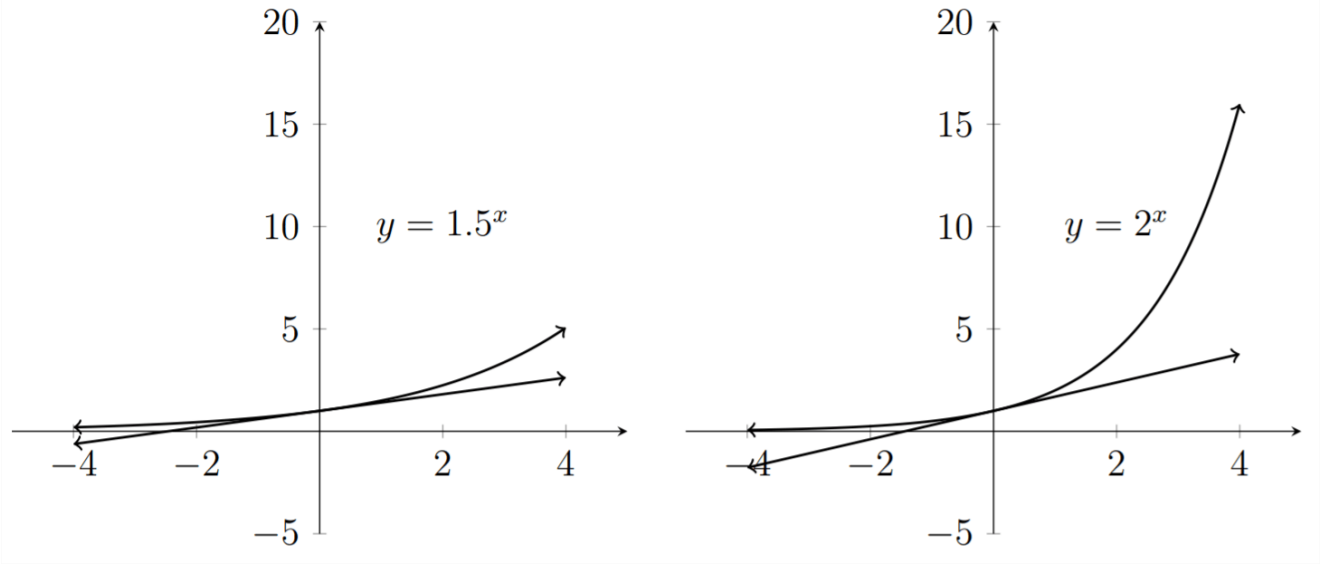


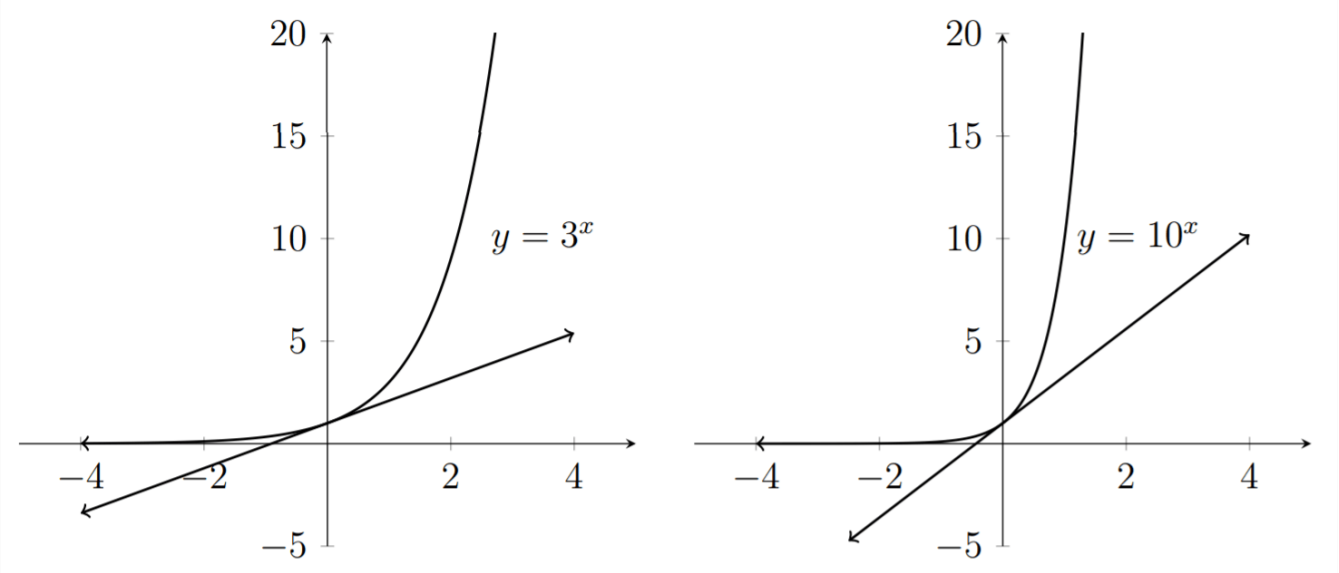




We can see that these graphs demonstrate slightly different behavior and different x and y values. One thing that they all have in common is that they all pass through the point (0,1) on the graph. This is because $1.5^0 = 1, 2^0 = 1, 3^0 = 1$ and $10^0 = 1$. Therefore the point where x is 0 and y is 1 is on all four of the graphs.

Although all four of the graphs pass through the point $(0, 1)$, they each do this in a different way. Let's look at the slope of a line tangent to each curve at the point $(0, 1)$. This is the straight line that touches the curve at the point $(0, 1)$, but nowhere else:

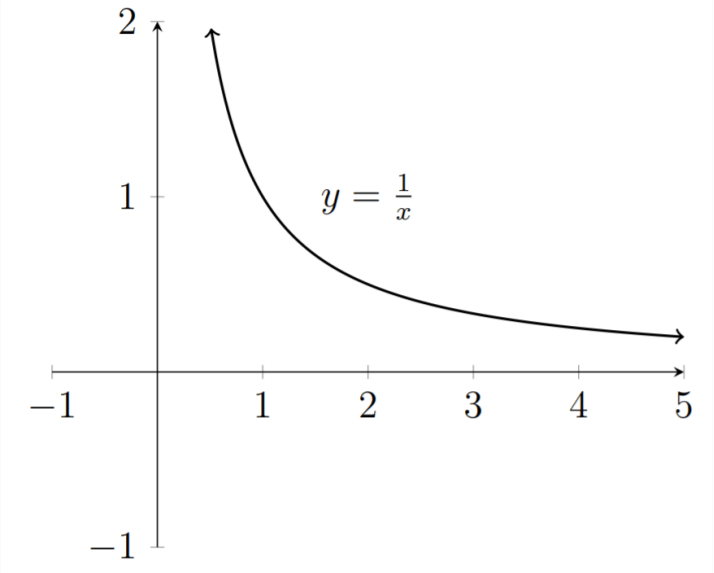




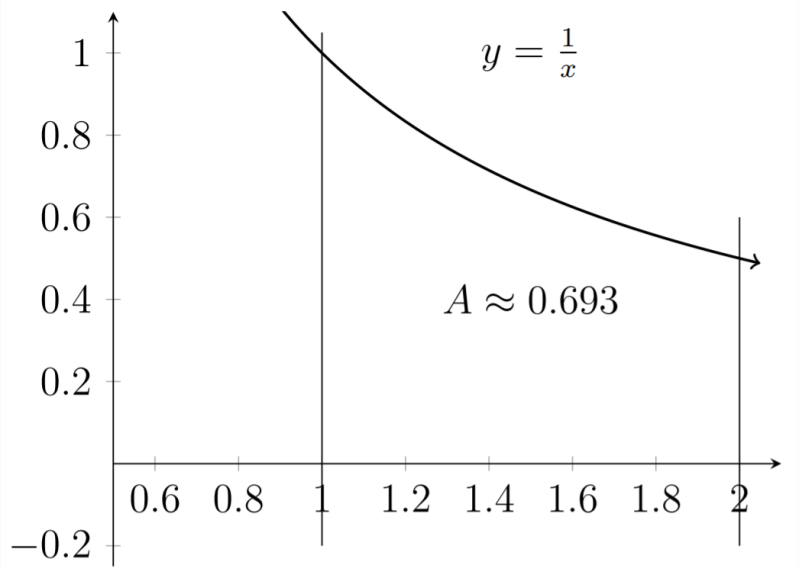
We can see that the slopes of these tangent lines are all different. In the case of $y = 2^x$, the slope of the tangent line at (0,1) is about 0.7, while for the graph of $y = 3^x$, the slope of the tangent line at (0,1) is about 1.1

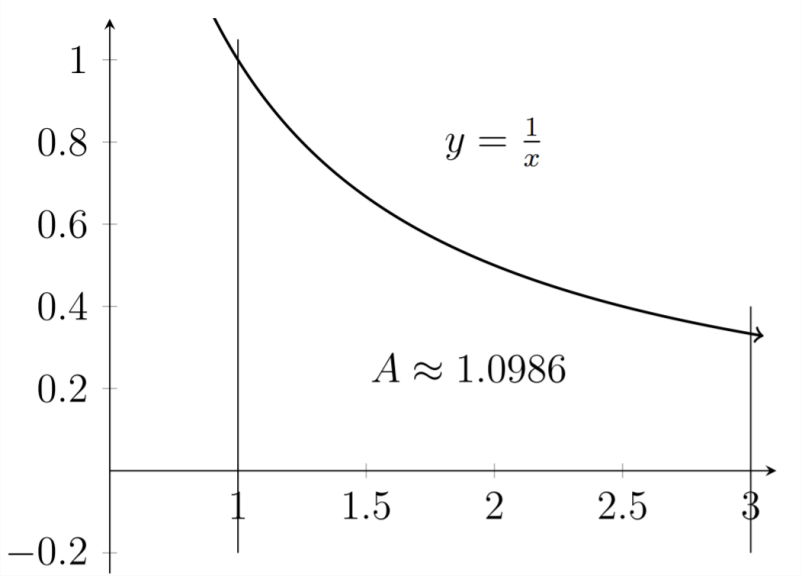
As mathematicians examined these graphs during the 17 th and 18 th centuries, they began to question what the value of the base " b'' should be in the equation $y = b^x$ so that the slope of the tangent line at the point (0,1) would be equal to exactly 1. The answer was $e \approx 2.71828$

Another way to derive the value of e uses Integral Calculus. Integral Calculus is often concerned with finding the area under a curve. This process can then be generalized and used to make many other types of calculations that are similar to finding area. Consider the graph of the curve $y = \frac{1}{x}$



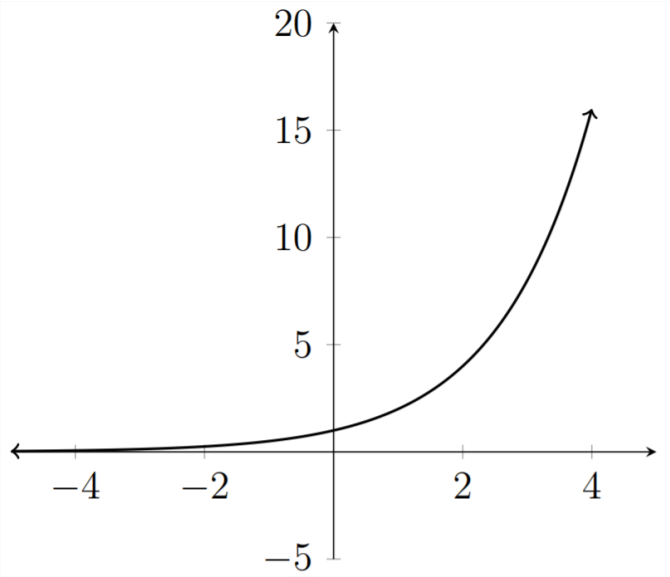
We can delineate borders on the x values and determine the area of the resulting region using the techniques of calculus:



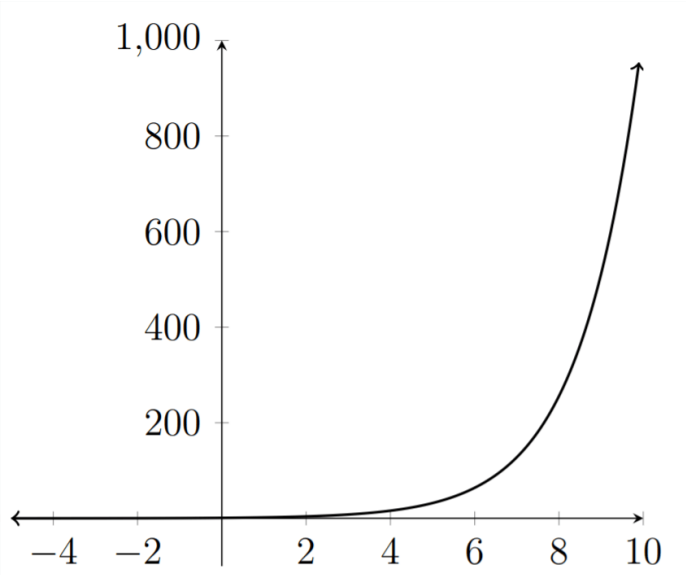


These values for the area under the curve are actually the same values as those for the slope of the tangent line in the previous graphs. If you ask the question, "Where should you draw the second vertical line so that the area under the curve is equal to exactly 1?'' then just like the slope question, the answer is $e \approx 2.71828$
This is how the value of e was determined and why it is used to represent these exponential relationships.

Logistic Relationships
Let's consider the graph of $y = 2^x$:



If we extend the x -axis out further past $x = 4$, we would see that the y values for this relationship will grow very quickly, as they continue doubling.



Some phenomena in the natural world exhibit behavior similar to the growth of this function. However, in the natural world, few, if any, things can grow unconstrained. Most growth of any kind is limited by the resources that fuel the growth. Populations often grow exponentially for a period of time, however, populations are dependent on natural resources to continue growing. As a result, the simple exponential function is only useful for modeling real-world behavior if the x -values are limited.

It was this problem with the simple exponential function that led French mathematician Pierre Verhulst to slightly adjust the differential equation that gives rise to the exponential function to make it more realistic.
The original differential equation said:

$$y' = k * y \tag{3.1.4}$$

This says that the rate of growth of y is always directly proportional to the value of
y. In other words the larger a population gets, the faster it will grow - forever. Verhulst changed this to say:

$$y' = k * y\left(1 - \frac{y}{N}\right) \tag{3.1.5}$$

This is the defining relationship for the Logistic function. Notice that when values of y are small, this is essentially the same as

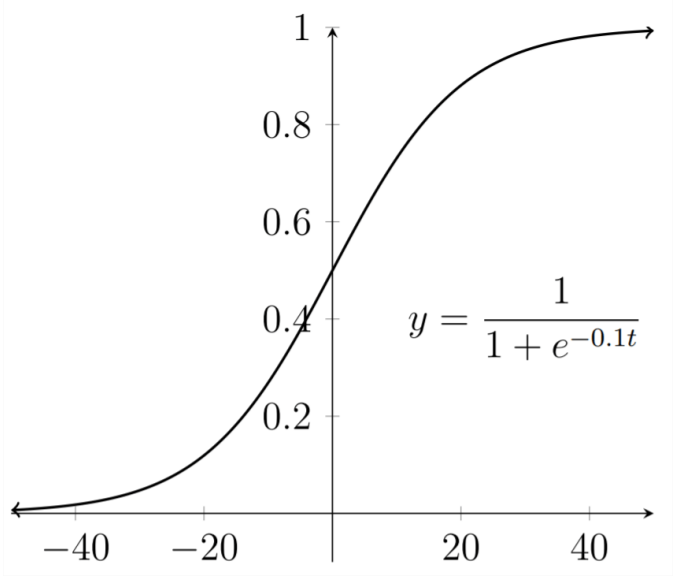
the simple exponential. If y is small, then the $(1 - \frac{y}{N})$ term will be very close to 1 and will produce behavior very much like the simple exponential.

The N in the formula is a theoretical "maximum population." As the value of y approaches this maximum value, $\frac{y}{N}$ will approach 1 and $(1 - \frac{y}{N})$ will get smaller and smaller. As it gets smaller, the factor of $(1 - \frac{y}{N})$ will slow down the growth of the function to model the pressure that is put on the resources that are driving the growth.

The solution of the Logistic equation is quite complicated and results in a standard form of:

$$y = \frac{N}{1 + be^{-kt}} \tag{3.1.6}$$

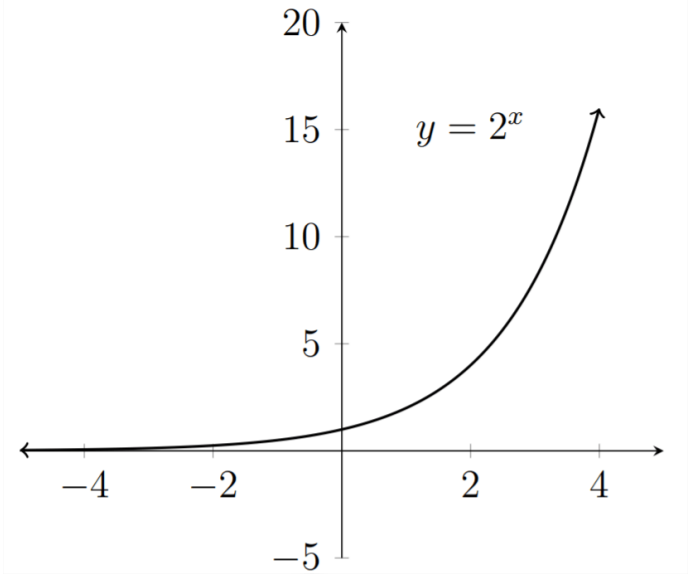
The graph for a sample logistic relationship is shown below:



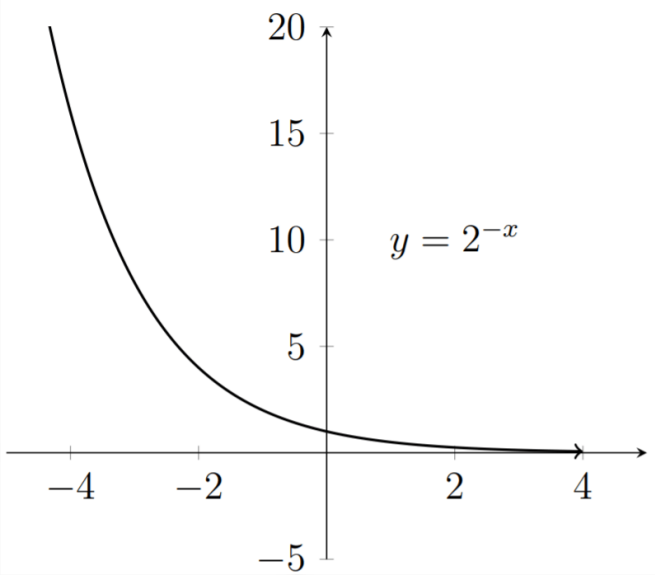
The "lazy-s" shape is characteristic of the logistic function. In the early stages, the relationship shows growth very similar to the simple exponential function but, as the function grows larger, the growth decreases and the function values stabilize. The maximum y value of N is always the horizontal asymptote for the logistic function.

Negative Exponential Relationships

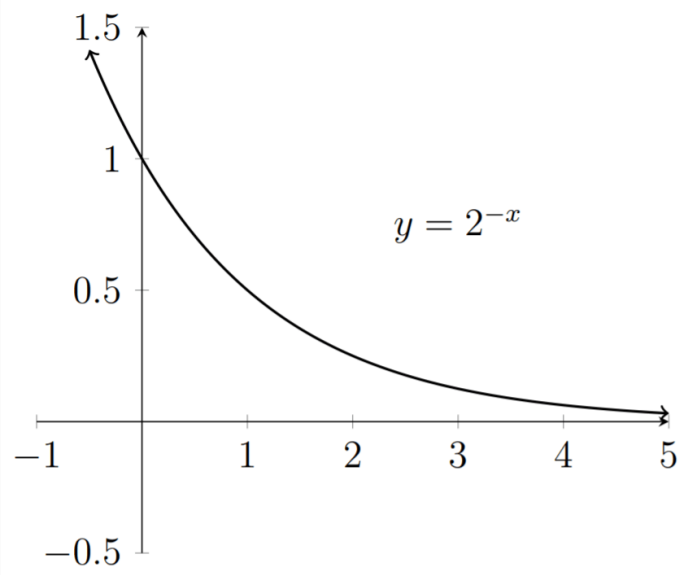
The Logistic function is very useful for modeling phenomena from the natural world. Although the simple exponential function is somewhat limited in modeling natural phenomena, the negative exponential is quite useful. Looking back to the graph of $y = 2^x$:



If we turn the graph around by changing x to $-x$ in the formula, then we will be working with the decaying tail of the graph:



Let's zoom in on the portion of the graph in the Quadrant I:



The behavior shown in the graph is quite useful for modeling radioactive decay, processes of heating and cooling, and assimilation of medication in the bloodstream. Let's look at an example.

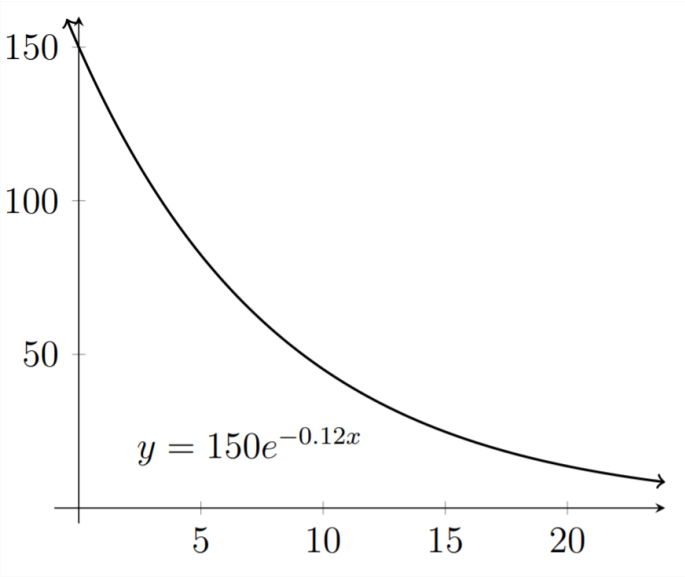
Example

If a patient is injected with 150 micrograms of medication, the amount of medication still in the bloodstream after t hours is given by the function:

$$A(t) = 150e^{-0.12t} \tag{3.1.7}$$

a) Find the amount of medication in the bloodstream after 3 hours.
Round your answer to the nearest 100 th.
b) How long will it take for the amount of medication to reach 60 micrograms?
Round your answer to the nearest 10 th of an hour.

In this section we will focus on solving these problems using the graphing calculator. In later sections, we will cover processes that can be used to solve these problems algebraically. It is helpful to understand both methods of solution.
First, let's graph the function given in the problem:

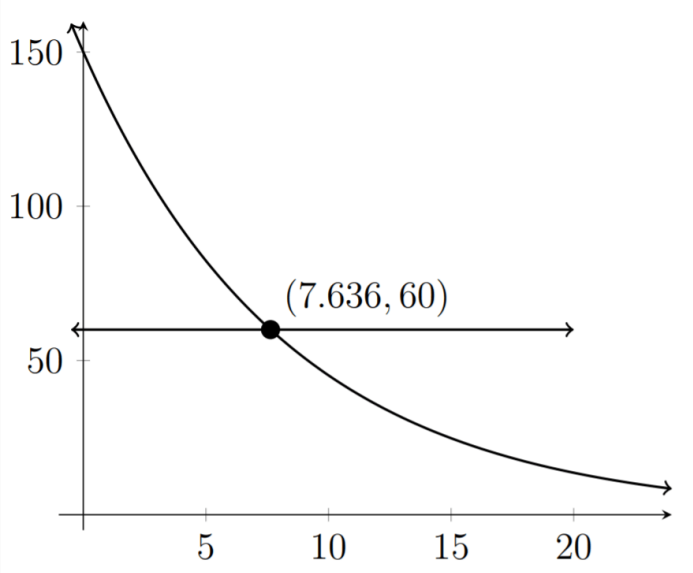


We can directly calculate the answer for part
(a) by plugging the value of 3 for t or we can use the table in the graphing calculator to find this value:

x	y
0	150
1	133.04
2	117.99
3	104.65

We can see that after 3 hours there are approximately 104.65 micrograms of medication in the patient's bloodstream.

To answer part (b) graphically, we'll graph the original function along with the horizontal line $y = 60$. When we find the intersection of these two graphs, we'll know how long it takes for there to be 60 micrograms of medication in the patient's bloodstream:



Here, we can see that it would take about 7.6 hours (or about 7 hours 36 minutes), for there to be 60 micrograms of medication in the patient's bloodstream.

Example
The deer population on a nature preserve can be modeled using the equation:

$$y = \frac{8000}{1 + 9e^{-0.2t}} \tag{3.1.8}$$

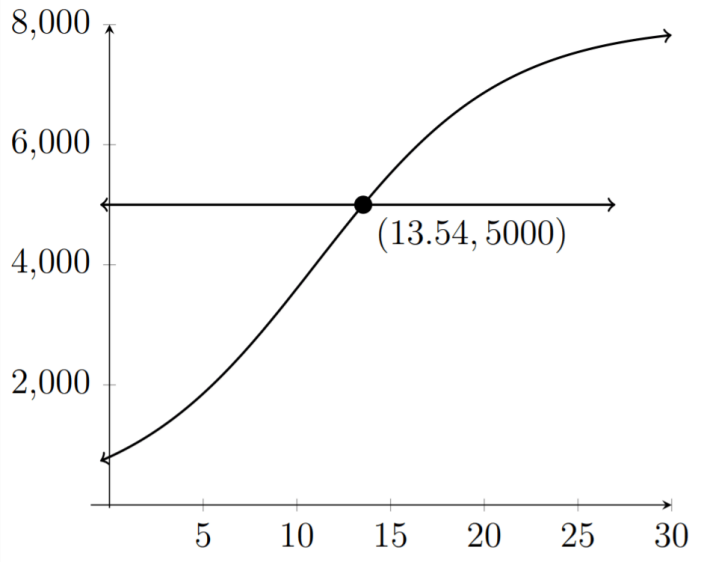
y indicates the number of deer living in the nature preserve and t represents the number of years that have passed since the initial population of deer were established there.
a) How many deer were in the initial population?
b) What is the deer population after 10 years?
Round your answer to the nearest whole number.
c) How long does it take for the population to reach $5,000$?
Round your answer to the nearest 10 th of a year.
Parts
(a) and (b) can both be answered from a table of values for the function. Before we look at the table, let's consider the question in Part (a). The problem is asking what the initial population of deer was. The means that we're looking to find out what y is when $t = 0$
Let's see what happens when we plug zero in for t in the formula:

$$\begin{aligned} y &= \frac{8000}{1 + 9e^{-0.2*0}} = \frac{8000}{1 + 9e^{0}} \\ &= \frac{8000}{1 + 9 * 1} = \frac{8000}{10} \\ &= 800 \end{aligned} \tag{3.1.9}$$

Let's look at the table of values:

t	y
0	800
5	1855.8
10	3606.8

To the nearest whole number, the deer population after 10 years is 3,607
In Part (c), we'll need to graph a horizontal line at $y = 5,000$ and find the intersection.



To the nearest 10 th, it takes about 13.5 years for the deer population to reach 5,000

Exercises 3.1
1) Medication in the bloodstream
The function $A(t) = 200e^{-0.014t}$ gives the amount of medication, in milligrams, in a patient's bloodstream t minutes after the medication has been injected.
a) Find the amount of medication in the bloodstream after 45 minutes.
Round your answer to the nearest milligram.
b) How long will it take for the amount of medication to reach 50 milligrams?
Round your answer to the nearest minute.

2) Medication in the bloodstream
The function $D(t) = 50e^{-0.2t}$ gives the amount of medication, in milligrams, in a patient's bloodstream t hours after the medication has been injected.
a) Find the amount of medication in the bloodstream after 3 hours.
Round your answer to the nearest milligram.
b) How long will it take for the amount of medication to reach 10 milligrams?

Round your answer to the nearest minute.

3) Fish Population
The number of bass in a lake can be modeled using the given equation:

$$P(t) = \frac{3600}{1 + 7e^{-0.05t}} \tag{3.1.10}$$

where t is the number of months that have passed since the lake was stocked with bass. In each question below, round your answer to the nearest whole number.
a) How many bass were in the lake immediately after it was stocked?
b) How many bass were in the lake 1 year after it was stocked?

4) Bird Population
The population of a certain species of bird is limited by the type of habitat required for nesting. The population can be modeled using the following equation:

$$P(t) = \frac{5600}{0.5 + 27.5e^{-0.044t}} \tag{3.1.11}$$

where t is the number of years. In each question below, round your answer to the nearest whole number.
a) Find the initial bird population.
b) What is the population 100 years later?

5) A Temperature Model
A cup of coffee is heated to $180^{\circ}\mathrm{F}$ and placed in a room that maintains a temperature of $65^{\circ}\mathrm{F}$. The temperature of the coffee after t minutes have passed is given by:

$$F(t) = 65 + 115e^{-0.042t} \tag{3.1.12}$$

a) Find the temperature of the coffee 10 minutes after it is placed in the room.
Round your answer to the nearest degree.
b) When will the temperature of the coffee be $100^{\circ}\mathrm{F}$?
Round your answer to the nearest tenth of a minute.

6) A Temperature Model
Soup at a temperature of $170^{\circ}\mathrm{F}$ is poured into a bowl in a room that maintains a constant temperature. The temperature of the soup decreases according to the model given by:

$$F(t) = 75 + 95e^{-0.12t} \tag{3.1.13}$$

where t is the number of minutes that have passed since the soup was poured.
a) What is the temperature of the soup after 2 minutes?
Round your answer to the nearest 10 th of a degree.
b) A certain customer prefers that the soup be cooled to $110^{\circ}\mathrm{F}$.
How long will this take?
Round your answer to the nearest 10 th of a minute.

7) Radioactive Decay
A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the equation:

$$m(t) = 13e^{-0.015t} \tag{3.1.14}$$

where the amount is measured in kilograms.
a) Find the mass at time $t = 0$
b) How much of the mass remains after 45 days?
c) How long does it take for there to be 5 kg. left?

8) Radioactive Decay
Radioactive iodine is used by doctors as a tracer in diagnosing certain thyroid gland disorders. This type of iodine decays in such

a way that the mass remaining after t days is given by:

$$m(t) = 6e^{-0.087t} \tag{3.1.15}$$

where the amount is measured in grams.
a) Find the mass at time $t = 0$
b) How much remains after 20 days?
c) How long does it take for there to be 2 grams left?
Round your answer to the nearest tenth of a day.

9) Fish population
The function:

$$P(t) = \frac{12}{1 + e^{-t}} \tag{3.1.16}$$

gives the size of a fish population in thousands at time t, measured in years.
a) Find the initial population of fish at time $t = 0$
Find the population of fish after 2 years, time $t = 2$
b) How long will it take for the population to be $10,000$?
What appears to be the maximum population for this particular model?

10) Fish population
The function:

$$P(t) = \frac{70}{1 + 2e^{-t}} \tag{3.1.17}$$

gives the size of a fish population in thousands at time t, measured in years.
a) Find the initial population of fish at time $t = 0$
Find the population of fish after 1.5 years, $t = 1.5$
b) How long will it take for the population to be $50,000$?
What appears to be the maximum population for this particular model?

11) Fish population
The function:

$$P(t) = \frac{20}{1 - 0.4e^{-0.35t}} \tag{3.1.18}$$

gives the size of a fish population in thousands at time t, measured in years.
a) Find the initial population of fish at time $t = 0$
Find the population of fish after 5 years, time $t = 5$
b) How long will it take for the population to be $25,000$?
What happens to this population over time?

12) Fish population
The function:

$$P(t) = \frac{10}{1 - 0.73e^{-0.08t}} \tag{3.1.19}$$

gives the size of a fish population in thousands at time t, measured in years.
a) Find the initial population of fish at time $t = 0$
Find the population of fish after 5 years, time $t = 5$
b) How long will it take for the population to be $25,000$?
What happens to this population over time?

13) Continuous Mixing
A 100 gallon tank of pure water has salt water with 0.5 lb per gallon added to it at a rate of 2 gallons per minute. The brine

solution is mixed thoroughly and drained at rate of 2 gallons per minute.

The equation for how many pounds of salt are in the tank at time t is given by:

$$y = 50 - 50e^{-0.02t} \tag{3.1.20}$$

where y is measured in pounds and t is measured in minutes.
a) How many pounds of salt are in the tank after 20 minutes?
b) How long does it take for there to be 30 lbs of salt in the tank?
Round your answer to the nearest tenth of a minute.

14) Continuous Mixing
A 20 gallon tank of pure water has salt water with 0.4 lb per gallon of salt added to it at a rate of 3 gallons per minute. The brine solution is mixed thoroughly and drained at rate of 3 gallons per minute.

The equation for how many pounds of salt are in the tank at time t is given by:

$$y = 8 - 8e^{-0.15t} \tag{3.1.21}$$

where y is measured in pounds and t is measured in minutes.
a) How many pounds of salt are in the tank after 15 minutes?
b) How long does it take for there to be 6 lbs of salt in the tank?
Round your answer to the nearest tenth of a minute.

15) Continuous Mixing
A tank contains 200 gallons of a 2% solution of HCl. A 5% solution of HCl is added at 5 gallons per minute. The well mixed solution is being drained at 5 gallons per minute.
The amount of HCl in the tank at any given time t in minutes is:

$$y = 10 - 6e^{-0.025t} \tag{3.1.22}$$

a) How many gallons of HCl are in the tank at $t = 25$ minutes?
b) When does the concentration of HCl in the solution reach 4%?

16) Continuous Mixing
A tank contains 100 gallons of salt water which contains a total of 25 lbs of salt. Salt water containing 0.4 lbs per gallon is added to the tank at a rate of 5 gallons per minute and the well-mixed solution is drained at the same rate.
The amount of salt in pounds in the tank at any given time t in minutes is:

$$y = 40 - 15e^{-0.05t} \tag{3.1.23}$$

a) How many pounds of salt are in the tank at $t = 20$ minutes?
b) How long does it take for there to be 30 lbs of salt in the tank?

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3.2: Logarithmic Notation

A Logarithm is an exponent. In the early $1600'$ s, the Scottish mathematician John Napier devised a method of expressing numbers in terms of their powers of ten in order simplify calculation. since the advent of digital calculators, the methods of calculation using logarithms have become obsolete, however the concept of logarithms continues to be used in many area of mathematics.

The fundamental idea of logarithmic notation is that it is simply a restatement of an exponential relationship. The definition of a logarithm says:

$$\log_b N = x \rightarrow b^x = N \tag{3.2.1}$$

The notation above would be read as "log to the base b of N equals x means that b to the x power equals N. " In this section we will focus mainly on becoming familiar with this notation. In later sections, we will learn to use this process to solve equations.

Example
Express the given statement using exponential notation:

$$\log_2 32 = 5 \tag{3.2.2}$$

If $\log_2 32 = 5$, then $2^5 = 32$

Example
Express the given statement using exponential notation:
$\log_7 4 \approx 0.7124$
If $\log_7 4 \approx 0.7124$, then $7^{0.7124} \approx 4$

If the logarithm notation appears without a base, it is usually assumed that the base should be 10

Example
Express the given statement using exponential notation:

$$\begin{gathered} \log 100 = 2 \\ \text{If } \log 100 = 2, \text{ then } 10^2 = 100 \end{gathered} \tag{3.2.3}$$

The notation $\ln N = x$ is typically used to indicate a logrithm to the base e. This means that:

$$\ln N = x \rightarrow e^x = N \tag{3.2.4}$$

Example
Express the given statement using exponential notation:
$\ln 15 \approx 2.708$
If $\ln 15 \approx 2.708$, then $e^{2.708} \approx 15$
In some cases, we would want to change an exponential statement into a logarithmic statement.

Example
Express the given statement using logarithmic notation:
$12^4 = 20,736$
If $12^4 = 20,736$ then $\log_{12} 20,736 = 4$

Example
Express the given statement using logarithmic notation:
$10^{2.5} \approx 316.23$
If $10^{2.5} \approx 316.23$ then $\log 316.23 \approx 2.5$

Example
Express the given statement using logarithmic notation:
$e^6 \approx 403.4$

If $e^6 \approx 403.4$, then $\ln 403.4 \approx 6$

Exercises 3.2

Rewrite each of the following using exponential notation.

1) $t = \log_5 9$
2) $h = \log_7 10$
3) $\log_5 25 = 2$
4) $\log_6 6 = 1$
5) $\log 0.1 = -1$
6) $\log 0.01 = -2$
7) $\log 7 \approx 0.845$
8) $\log 3 \approx 0.4771$
9) $\log_2 35 \approx 5.13$
10) $\log_{12} 50 \approx 1.5743$
11) $\ln 0.25 \approx -1.3863$
12) $\ln 0.989 \approx -0.0111$

Rewrite each of the following using logarithmic notation.

13) $10^2 = 100$
14) $10^4 = 10,000$
15) $4^{-5} = \frac{1}{1024}$
16) $5^{-3} = \frac{1}{125}$
17) $16^{\frac{3}{4}} = 8$
18) $8^{\frac{1}{3}} = 2$
19) $10^{1.3} \approx 20$
20) $10^{0.301} = 2$
21) $e^3 \approx 20.0855$
22) $e^2 \approx 7.3891$
23) $e^{-4} \approx 0.0183$
24) $e^{-2} \approx 0.1353$

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3.3: Solving Exponential Equations

Because of the fact that logarithms are exponents, the rules for working with logarithms are similar to those that govern exponential expressions. One very helpful rule of equality for working with logarithms is related to the exponential rule for raising a power to a power. We recall one of the rules of exponents as:

$$(b^x)^y = b^{x*y} \tag{3.3.1}$$

in other words

$$\left(5^2\right)^4 = \left(5^2\right)\left(5^2\right)\left(5^2\right)\left(5^2\right) = 5^{2*4} = 5^8 \tag{3.3.2}$$

In logarithmic notation, this rule works out as:

$$\log_b M^p = p * \log_b M \tag{3.3.3}$$

The reason for this comes from the rule for exponents. Let's say that $\log_b M = x$
Then:

$$\log_b M = x \tag{3.3.4}$$

this means that

$$b^x = M \tag{3.3.5}$$

and

$$\begin{gathered}(b^x)^p = (M)^p \\ \text{so} \\ b^{p*x} = M^p\end{gathered} \tag{3.3.6}$$

Now, we come back to the question of $\log_b M^p =?$. This expression $(\log_b M^p)$ is asking the question "What power do we raise b to in order to get an answer of
M^p? The result on the previous page shows that:

$$b^{px} = M^p \tag{3.3.7}$$

This means that we must raise b to the px power to get an answer of M^p. Remember that $x = \log_b M$. This means that:

$$\begin{gathered}b^{px} = M^p \\ \text{so} \\ \log_b M^p = px = p * \log_b M\end{gathered} \tag{3.3.8}$$

This statement of equality is useful if we are trying to solve equations in which the variable is an exponent.
Example
Solve for x

$$4^x = 53 \tag{3.3.9}$$

We start by taking a logarithm on both of the equation. Just as we can add to both sides of an equation, or multiply on both sides of an equation, or raise both sides of an equation to a power, we can also take the logarithm of both sides. So long as two quantities are equal, then their logarithms will also be equal.

$$\begin{aligned}4^x &= 53 \\ \log 4^x &= \log 53 \\ x \log 4 &= \log 53 \\ x &= \frac{\log 53}{\log 4} \approx 2.864\end{aligned} \tag{3.3.10}$$

since the log base 10 and log base e are both programmed into most calculators, these are the most commonly used bases for logarithms.
Example
Solve for x

$$5^{2x+3} = 17 \tag{3.3.11}$$

We start this problem in the same fashion, but this time we will use a logarithm to the base e :

$$\begin{aligned} 5^{2x+3} &= 17 \\ \ln 5^{2x+3} &= \ln 17 \\ (2x+3)\ln 5 &= \ln 17 \end{aligned} \tag{3.3.12}$$

There are several possibilities for finishing the problem from this point. We will focus on two of them that are the most useful for solving more complex problems. First, we will distribute the $\ln 5$ into the parentheses and then get the x by itself.

$$\begin{aligned} (2x+3)\ln 5 &= \ln 17 \\ x*2\ln 5 + 3\ln 5 &= \ln 17 \\ -3\ln 5 &= -3\ln 5 \\ x*2\ln 5 &= \ln 17 - 3\ln 5 \\ x &= \frac{\ln 17 - 3\ln 5}{2\ln 5} \approx -0.620 \end{aligned} \tag{3.3.13}$$

And we can check the answer by plugging it back in:

$$5^{2*(-0.0620)+3} \approx 5^{1.760} \approx 16.9897 \approx 17 \tag{3.3.14}$$

We can also approximate the logarithms in the problem and solve for an approximate answer:

$$\begin{aligned} (2x+3)\ln 5 &= \ln 17 \\ x*2\ln 5 + 3\ln 5 &= \ln 17 \\ 3.2189x + 4.8283 &\approx 2.8332 \\ -4.8283 &\approx -4.8283 \\ 3.2189x &\approx -1.9951 \\ x &\approx -0.620 \end{aligned} \tag{3.3.15}$$

If you use the method of approximating, it's important to make a good approximation. At least $4-5$ decimal places are necessary for an accurate answer.
Let's look at an example that has variables on both sides of the equation:

Example
Solve for x

$$4^{3x} = 9^{2x-1} \tag{3.3.16}$$

We'll use log base 10 in this problem.

$$\begin{aligned} 4^{3x} &= 9^{2x-1} \\ \log 4^{3x} &= \log 9^{2x-1} \\ 3x*\log 4 &= (2x-1)\log 9 \\ x*3\log 4 &= x*2\log 9 - \log 9 \end{aligned} \tag{3.3.17}$$

If we collect like terms, we'll end up with:

$$\begin{aligned} x*3\log 4 &= x*2\log 9 - \log 9 \\ \log 9 &= x*2\log 9 - x*3\log 4 \end{aligned} \tag{3.3.18}$$

At this point, if we want to get the x by itself, we need to factor out the x on the right-hand side:

$$\begin{aligned}\log 9 &= x * 2\log 9 - x * 3\log 4 \\ \log 9 &= x(2\log 9 - 3\log 4)\end{aligned} \tag{3.3.19}$$

Then divide on both sides by the coefficient in parentheses:

$$\begin{aligned}\frac{\log 9}{2\log 9 - 3\log 4} &= \frac{x\ (2\log 9 - 3\log 4)}{2\log 9 - 3\log 4} \\ \frac{\log 9}{2\log 9 - 3\log 4} &= x \\ 9.327 &\approx x\end{aligned} \tag{3.3.20}$$

Again, we can check our answer by plugging it back into the equation:

$$4^{3*9.327} \approx 4^{27.981} \approx 7.0184 * 10^{16} \tag{3.3.21}$$

$$9^{2*9.327-1} \approx 9^{17.654} \approx 7.0177 * 10^{16} \tag{3.3.22}$$

We could also have solved this equation by approximating the logarithms in the beginning.

$$\begin{aligned}4^{3x} &= 9^{2x-1} \\ \log 4^{3x} &= \log 9^{2x-1} \\ 3x * \log 4 &= (2x-1)\log 9 \\ 3x(0.60206) &\approx (2x-1)0.95424 \\ 1.80618x &\approx 1.9085x - 0.95424 \\ 0.95424 &\approx 0.10232x \\ 9.326 &\approx x\end{aligned} \tag{3.3.23}$$

This answer is less accurate than the other approximation $(9.326036 \text{ vs. } 9.327424)$ The accuracy of an answer depends upon the original approximations for the logarithms.

Solve for the indicated variable.

1) $2^x = 5$
2) $2^x = 9$
3) $3^x = 7$
4) $3^x = 20$
5) $2^{x+1} = 6$
6) $7^{x+1} = 41$
7) $5^{x+1} = 36$
8) $8^{x-2} = 6$
9) $4^{2x+3} = 50$
10) $4^{x+2} = 5^x$
11) $5^{2x+1} = 9$
12) $6^{x+4} = 10^x$
13) $7^{y+1} = 3^y$
14) $2^{x+1} = 3^{x-2}$
15) $6^{y+2} = 5^y$
16) $7^{x-3} = 3^{x+1}$
17) $6^{2x+1} = 5^{x+2}$
18) $9^{1-x} = 12^{x+1}$
19) $5^{2x-1} = 3^{x-3}$
20) $3^{x-2} = 4^{2x+1}$
21) $8^{3x-2} = 9^{x+2}$
22) $2^{2x-3} = 5^{-x-1}$
23) $10^{3x+2} = 5^{x+3}$
24) $5^{3x} = 3^{x+4}$
25) $3^{x+4} = 2^{1-3x}$
26) $4^{2x+3} = 5^{x-2}$
27) $3^{2-3x} = 4^{2x+1}$

28) $2^{2x-3} = 5^{x-2}$

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3.4: Solving Logarithmic Equations

In the previous section, we took exponential equations and used the properties of logarithms to restate them as logarithmic equations. In this section, we will take logarithmic equations and use properties of logarithms to restate them as exponential equations. In the previous section, we used the property of logarithms that said $\log_b M^p = p \log_b M$. In this section, we will make use of two additional properties of logarithms:

$$\log_b(M * N) = \log_b M + \log_b N \tag{3.4.1}$$

and

$$\log_b \frac{M}{N} = \log_b M - \log_b N \tag{3.4.2}$$

Just as our previous property of logarithms was simply a restatement of the rules of expoenents, these two properties of logarithms depend on the rules of exponents as well. since we're interested in $\log_b M$ and $\log_b N$, let's restate these in terms of exponents:
If $\log_b M = x$ then $b^x = M$ and if $\log_b N = y$ then $b^y = N$
The properties of logarithms we're interested in justifying have to do with $M * N$ and $\frac{M}{N}$, so let's look at those expressions in terms of exponents:

$$\begin{gathered} M * N = b^x * b^y = b^{x+y} \\ \text{and} \\ \frac{M}{N} = \frac{b^x}{b^y} = b^{x-y} \end{gathered} \tag{3.4.3}$$

If we're interested in $\log_b(M * N)$, then we're asking the question "What power do we raise b to in order to get $M * N$? " We can see above that raising b to the $x + y$ power gives us $M * N$. since $x = \log_b M$ and $y = \log_b N$ then $x + y = \log_b M + \log_b N$, so:

$$\log_b(M * N) = x + y = \log_b M + \log_b N \tag{3.4.4}$$

Likewise, if we're interested in $\log_b \frac{M}{N}$, we're asking the question "What power do we raise b to in order to get $\frac{M}{N}$? " since raising b to the $x - y$ power gives us $\frac{M}{N}$ and $x - y = \log_b M - \log_b N$, then:

$$\log_b \frac{M}{N} = x - y = \log_b M - \log_b N \tag{3.4.5}$$

Let's look at an example to see how we'll use this to solve equations:

Example
Solve for x

$$\log_2 x + \log_2(x - 4) = 2 \tag{3.4.6}$$

The first thing we can do here is to combine the two logarithmic statements into one. since $\log_b(M * N) = \log_b M + \log_b N$, then $\log_2 x + \log_2(x - 4) = \log_2[x(x - 4)]$

$$\begin{aligned} \log_2 x + \log_2(x - 4) &= 2 \\ \log_2[x(x - 4)] &= 2 \end{aligned} \tag{3.4.7}$$

Then we'll restate the resulting logarithmic relationship as an exponential relationship:

$$\begin{aligned} 2^2 &= x(x - 4) \\ 4 &= x^2 - 4x \\ 0 &= x^2 - 4x - 4 \\ 4.828, -0.828 &\approx x \end{aligned} \tag{3.4.8}$$

Most textbooks reject answers that result in taking the logarithm of a negative number, such as would be the case for $x \approx -0.828$. However, the logarithms of negative numbers result in complex valued answers, rather than an undefined quantity. For that reason, in this text, we will include all answers.

If a problem involves a difference of logarithms, we can use the other property of logarithms introduced in this section.
Example
Solve for x

$$\log(5x-1)-\log(x-2)=2 \tag{3.4.9}$$

Again, our first step is to restate the difference of logarithms using the property $\log_b \frac{M}{N}=\log_b M-\log_b N$:

$$\begin{aligned}\log(5x-1)-\log(x-2)&=2\\ \log\left[\frac{5x-1}{x-2}\right]&=2\end{aligned} \tag{3.4.10}$$

We're working with a logarithm in base 10 in this problem, so in our next step we'll say:

$$\begin{aligned}\log\left[\tfrac{5x-1}{x-2}\right]&=2\\ \tfrac{5x-1}{x-2}&=10^2\end{aligned} \tag{3.4.11}$$

Then multiply on both sides by $x-2$

$$\begin{aligned}10^2&=\tfrac{5x-1}{x-2}\\ (x-2)*100&=\tfrac{5x-1}{\cancel{(x-2)}}*\cancel{(x-2)}\end{aligned} \tag{3.4.12}$$

And, solve for x

$$\begin{aligned}100x-200&=5x-1\\ 95x&=199\\ x&=\tfrac{199}{95}\end{aligned} \tag{3.4.13}$$

In some equations, all of the terms are stated using logartihms. These equations often come out in a form that says $\log_b x=\log_b y$. If this is the case, we can then conclude that $x=y$

It seems reasonable that if the exponent we raise b to in order to get x is the same exponent that we raise b to in order to get y, then x and y are the same thing.
Assume:

$$\log_b x=\log_b y \tag{3.4.14}$$

then

$$b^a=x \text{ and } b^a=y \tag{3.4.15}$$

if both x and y are equal to b^a, then $x=y$

Example
Solve for x
$\log_5(4-x)=\log_5(x+8)+\log_5(2x+13)$

First, let's use the properties of logarithms to restate the equation so that there is only one logarithm on each side.

$$\begin{aligned}\log_5(4-x)&=\log_5(x+8)+\log_5(2x+13)\\ \log_5(4-x)&=\log_5[(x+8)(2x+13)]\end{aligned} \tag{3.4.16}$$

Then, we'll use the property of logarithms we just discussed:

$$\text{If } \log_b x=\log_b y \tag{3.4.17}$$

then

$$\begin{aligned} x &= y \\ \log_5(4-x) &= \log_5[(x+8)(2x+13)] \\ 4-x &= (x+8)(2x+13) \\ 0 &= 2x^2+29x+104 \\ 0 &= 2(x+5)(x+10) \\ -5,-10 &= x \end{aligned} \tag{3.4.18}$$

Exercises 3.4

Solve for the indicated variable in each equation.

1) $\log_3 5 + \log_3 x = 2$
2) $\log_4 x + \log_4 5 = 1$
3) $\log_2 x = 2 + \log_2 3$
4) $\log_5 x = 2 + \log_5 3$
5) $\log_3 x + \log_3(x-8) = 2$
6) $\log_6 x + \log_6(x-5) = 1$
7) $\log(3x+2) = \log(x-4) + 1$
8) $\log(x-1) - \log x = -0.5$
9) $\log_2 a + \log_2(a+2) = 3$
10) $\log_3 x + \log_3(x-2) = 1$
11) $\log_2 y - \log_2(y-2) = 3$
12) $\log_2 x - \log_2(x+3) = 2$
13) $\log_3 x + \log_3(x+4) = 2$
14) $\log_4 u + \log_4(u+1) = 1$
15) $\log 5 + \log x = \log 6$
16) $\ln x + \ln 4 = \ln 2$
17) $\log_7 x - \log_7 12 = \log_7 2$
18) $\log 2 - \log x = \log 8$
19) $\log_3 x - \log_3(x-2) = \log_3 4$
20) $\log_6 2 - \log_6(x-2) = \log_6 9$
21) $\log_4 x - \log_4(x-4) = \log_4(x-6)$
22) $\log_9(2x+7) - \log_9(x-1) = \log_9(x-7)$
23) $2\log_2 x = \log_2(2x-1)$
24) $2\log_4 y = \log_4(y+2)$
25) $2\log(x-3) - 3\log 2 = 1$
26) $2\log_5 7 - \log_5(x+1) = \log_5(2x-5)$

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3.5: Applications of the Negative Exponential Function

At the beginning of Chapter 3, we worked with application problems and solved them using the graphing calculator. In this section, we will revisit some of these application problems and use the solution methods discussed in the previous sections to solve these problems algebraically.

Radioactive Decay

The decay of a radioactive element into its non-radioactive form occurs following a time line dictated by the "half-life" of the element. The half-life is the amount of time that it takes for half of the existing radioactive material to decay to its non-radioactive form.

Consider the equation:

$$A(t) = A_0 e^{-kt} \tag{3.5.1}$$

where $A(t)$ is the amount of material left at time t, A_0 is the amount present at $t = 0$, and k is a constant that can be determined based on the half-life of the material.

If we know that after one half-life, there will 50% of the radioactive material remaining, then we can say that:

$$0.5A_0 = A_0 e^{-kt_h} \tag{3.5.2}$$

where t_h is the half-life. To solve this equation for k, we would first divide on both sides by A_0 :

$$\begin{aligned} \frac{0.5A_0}{A_0} &= \frac{A_0 e^{-kt_h}}{A_0} \\ 0.5 &= e^{-kt_h} \end{aligned} \tag{3.5.3}$$

Then take the natural logarithm of both sides and bring the exponent down in front of the expression as a coefficient:

$$\begin{aligned} 0.5 &= e^{-kt_h} \\ \ln(0.5) &= \ln\left(e^{-kt_h}\right) \\ \ln(0.5) &= -kt_h * \ln(e) \\ \ln(0.5) &= -kt_h * 1 \\ -\frac{\ln(0.5)}{t_h} &= k \end{aligned} \tag{3.5.4}$$

The value of k can then be used in the equation $A(t) = A_0 e^{-kt}$ to determine the amount of material left after any time t

Example

The isotope Gold-198 $\left({}^{198}\mathrm{Au}\right)$ is a type of gold sometimes used in medical applications and has a half-life of 2.7 days. How much of a 65 gram sample of ${}^{198}\mathrm{Au}$ will be left after 6 days? How long would it take for there to be 10 grams left?

If we know the half-life, we can calculate the value of the constant k

$$\begin{aligned} k &= -\frac{\ln(0.5)}{t_h} \\ k &= -\frac{\ln(0.5)}{2.7} \\ k &\approx 0.2567 \end{aligned} \tag{3.5.5}$$

Now that we know the value of k, we can directly calculate the amount of ${}^{198}\mathrm{Au}$ left after 6 days:

$$\begin{aligned} A(t) &= A_0 e^{-kt} \\ &= 65e^{-0.2567*6} \\ &\approx 13.932 \end{aligned} \tag{3.5.6}$$

So, approximately 13.932 grams of ^{198} Au would be left after 6 days.

In order to calculate how long it takes for 10 grams of ^{198} Au to be left, we'll need to solve for t in the equation $A(t) = A_0 e^{-kt}$ with $A(t) = 10$:

$$\begin{aligned} A(t) &= A_0 e^{-kt} \\ 10 &= 65e^{-0.2567t} \end{aligned} \tag{3.5.7}$$

First, we'll divide on both sides by 65

$$\begin{aligned} \frac{10}{65} &= \frac{65e^{-0.2567t}}{65} \\ \frac{10}{65} &= e^{-0.2567t} \end{aligned} \tag{3.5.8}$$

Then, take the natural logarithm on both sides:

$$\ln\left(\frac{10}{65}\right) = \ln\left(e^{-0.2567t}\right) \tag{3.5.9}$$

We'll calculate an approximate value for $\ln\left(\frac{10}{65}\right)$ and restate the right hand side of the equation using the properties of logarithms:

$$\begin{aligned} -1.872 &\approx -0.2567t * \ln e \\ -1.872 &\approx -0.2567t * 1 \\ -1.872 &\approx -0.2567t \\ 7.3 &\approx t \end{aligned} \tag{3.5.10}$$

So, it would take about 7.3 days for the 65 grams of ^{198} Au to decay to 10 grams.

Newton's Law of Cooling
Newton's Law of Cooling states that the temperature of an object can be determined using the equation:

$$T = T_a + Ce^{-kt} \tag{3.5.11}$$

where T_a is the ambient temperature of the surrounding environment. The values of the constants C and k can often be calculated from given information.

Example
A bottle of soda at room temperature $(72^\circ\mathrm{F})$ is placed in a refrigerator where the temperature is $44^\circ\mathrm{F}$

After half an hour, the soda has cooled to $61^\circ\mathrm{F}$. What is the temperature of the soda after another half hour?

First, since the soda is placed in the refrigerator where the ambient temperature is $44^\circ\mathrm{F}$, then $T_a = 44$. We also know that at $t = 0, T = 72$, which is the temperature of the soda can when it is first put in the refrigerator. This will allow us to calculate the constant C.

$$\begin{aligned} T &= T_a + Ce^{-kt} \\ 72 &= 44 + Ce^{-k*0} \\ 72 &= 44 + Ce^0 = 44 + C * 1 \\ 72 &= 44 + C \\ 28 &= C \end{aligned} \tag{3.5.12}$$

Now we know that $T_a = 44$ and $C = 28$. We can use the other piece of information from the problem to calculate the value of k. The problem states that after 30 minutes the soda has cooled off to $61^\circ\mathrm{F}$. That means that when $t = 30$ (or $t = 0.5$ depending on which units you choose) the temperature T will be $61^\circ\mathrm{F}$. We can then set up the equation to reflect this and calculate the vlaue of k :

$$\begin{aligned} T &= 44 + 28e^{-kt} \\ 61 &= 44 + 28e^{-k*30} \end{aligned} \tag{3.5.13}$$

First, we'll subtract 44 on both sides and divide by 28 :

$$\begin{aligned} 61 &= 44 + 28e^{-k*30} \\ 17 &= 28e^{-30k} \\ \frac{17}{28} &= e^{-30k} \end{aligned} \tag{3.5.14}$$

Now, we'll take the natural logarithm on both sides to bring the $-30k$ down from the exponent:

$$\begin{aligned}\ln\left(\tfrac{17}{28}\right) &= \ln\left(e^{-30k}\right)\\ \ln\left(\tfrac{17}{28}\right) &= -30k * \ln e\\ -0.499 &\approx -30k\\ 0.01663 &\approx k\end{aligned} \tag{3.5.15}$$

Now, we have the full formula for calculating temperature in this scenario:

$$T = 44 + 28e^{-0.01663t} \tag{3.5.16}$$

To find out what happens after 60 minutes, we can simply plug in 60 for t

$$\begin{aligned}T &= 44 + 28e^{-0.01663t}\\ T &= 44 + 28e^{-0.01663\cdot 60}\\ T &= 44 + 28e^{-0.9978}\\ T &\approx 44 + 28 * 0.3687\\ T &\approx 44 + 10.3236 \approx 54.3^{\circ}\mathrm{F}\end{aligned} \tag{3.5.17}$$

Exercises 3.5

1) If the half-life of radioactive cesium- 137 is 30 years, find the value of k in the equation $A(t) = A_0e^{-kt}$
a) Given a 10 gram sample of cesium-137, how much will remain after 80 years?
b) How long will it take for only 2 grams of cesium- 137 to remain?
2) The half-life for radioactive thorium- 234 is about 25 days. Use this to find the value of k in the equation $A(t) = A_0e^{-kt}$
a) How much of a 40 gram sample will remain after 60 days?
b) After how long will only 10 grams of thorium- 234 remain?
3) Given a sample of strontium- 90 , it is known that after 18 years there are $32\mathrm{mg}$ remaining and after 65 years there are $10\mathrm{mg}$ remaining. Use this information to find out how much strontium- 90 was in the sample to begin with, and also determine the half-life of strontium- 90 .
4) A 12 mg sample of radioactive polonium decays to $7.26\mathrm{mg}$ in 100 days.
a) What is the half-life of polonium?
b) How much of the 12 mg sample remains after 180 days?

5) A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling so that its temperature at time t is given by:

$$y = 65 + 145e^{-0.05t} \tag{3.5.18}$$

where t is measured in minutes and y is measured in degrees Fahrenheit.
a) What is the initial temperature of the soup?
b) What is the temperature after 10 minutes?
c) After how long will the temperature be 100°?
6) Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is $98.6^{\circ}\mathrm{F}$. Immediately following death, the body begins to cool. This process uses Newton's Law of Cooling:

$$y = T_a + Ce^{-kt} \tag{3.5.19}$$

If the ambient temperature is 60°, and the body has cooled to $72^{\circ}\mathrm{F}$ after 6 hours, use this information to determine the value of k in the equation.

7) The police discover the body of a murder victim. Critical to solving the crime is determining when the murder was committed. The coroner arrives at the murder scene at 12 Noon. She immediately takes the temperature of the body and finds it to be $94.6^{\circ}\mathrm{F}$. She then takes the temperature 1.5 hours later and finds it to be $93.4^{\circ}\mathrm{F}$. If the temperature of the room is $70^{\circ}\mathrm{F}$, when was the murder committed?

8) A roasted turkey is taken from the oven when its temperature has reached
$185^{\circ}\mathrm{F}$ and is placed on a table in a room where the temperature is $75^{\circ}\mathrm{F}$
a) If the temperature of the turkey is $150^{\circ}\mathrm{F}$ after 30 minutes, what is its temperature after 45 minutes?
b) When will the turkey cool off to $100^{\circ}\mathrm{F}$?

9) A kettle full of water is brought to a boil in a room with an ambient temperature of $20^{\circ}\mathrm{C}$. After 15 minutes, the temperature of

the water has decreased from $100^\circ\mathrm{C}$ to $75^\circ\mathrm{C}$. Find an equation using Newton's Law of Cooling to represent to temperature at time t. Find the temperature of the water after 25 minutes.

10) A cup of coffee with a temperature of $105^\circ\mathrm{F}$ is placed in a freezer with a temperature of $0^\circ\mathrm{F}$. After 5 minutes, the temperature of the coffee is $70^\circ\mathrm{F}$. Find an equation using Newton's Law of Cooling to represent to temperature at time t What will the temperature be in 10 minutes?

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4: FUNCTIONS

A function is generally defined as a relation in which each x value corresponds to one and only one y value. This assigning of only one y value to each x is known as "univalence." The univalence requirement makes some manipulations in calculus easier to work with and can be important in certain applications, but many important relations (such as the hyperbola, circle and ellipse) are not univalent and thus not functions.

A function typically describes a relationship between two sets. In our consideration of functions these two sets will typically be real or complex numbers. Functions are usually defined in one of several ways. They can be defined by a graph, (2) an algebraic relationship,
(3) a rule or (4) a table of values. More than one of these methods can be used to describe the same function.

4.1: Function Notation

4.2: Domain and Range of a Function

4.3: Maximum and Minimum Values

4.4: Transformations

4.5: Toolbox Functions

4.6: Piecewise-defined Functions

4.7: Composite functions

4.8: Inverse Functions

4.9: Optimization

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4.1: Function Notation

The notation for a function is generally the $f(x)$ notation. In learning about graphing in algebra we typically use the x and y notation, that is: $y = 6x - 1$ In function notation, the dependent variable y is replaced by the notation $f(x)$:

$$f(x) = 6x - 1 \tag{4.1.1}$$

Function values for particular values of the independent variable x can be found by substituting the appropriate x value into the formula.

$$\begin{aligned} f(9) &= 6(9) - 1 = 53 \\ f(9) &= 53 \end{aligned} \tag{4.1.2}$$

Find each of the following values for the given functions:
$f(0) \quad f(-1) \quad f(3) \quad f\left(\frac{1}{2}\right) \quad f(x+2) \quad f(x+h)$
1) $f(x) = 2x^2 - 3x + 1$
2) $f(x) = 5x^2 + x - 7$
3) $f(x) = \frac{x}{x^2-1}$
4) $f(x) = \frac{x+3}{x^2+1}$

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4.2: Domain and Range of a Function

The analysis of the behavior of functions addresses questions of when the function is increasing or decreasing, whether and where it has maximum or minimum values, where it crosses the x or y axis, and which values of x and y are to be included in the function.

The set of values available for the x, or independent variable is called the Domain of the function. The set of corresponding y values is called the Range of the function.

The linear function mentioned above $f(x) = 6x - 1$ has a domain of all real numbers and a range of all real numbers, $x \in \mathbb{R}$ and $y \in \mathbb{R}$. On the other hand, the function $f(x) = x^2$ has a domain of all real numbers, $x \in \mathbb{R}$, but its range is limited to the positive real numbers, $y \geq 0$

Considerations of the domain of a function typically refer to restrictions on which x values will generate real number values for y. The most common restrictions occur with the use of square root functions or rational functions.

The domain of the function $f(x) = \sqrt{x - 7}$ would be the set of $x \geq 7$, so that no negative values are permitted under the square root. This secures the necessary real values for y. The range for this function is $y \geq 0$

The domain of the function $f(x) = \frac{x}{2x-3}$ would be $x \in \mathbb{R}$ (all real numbers), but $x \neq \frac{3}{2}$, thus avoiding a zero denominator which is an undefined value. The range for this function would be $y \in \mathbb{R}$ (all real numbers), but $y \neq \frac{1}{2}$, due to the horizontal asymptote at $y = \frac{1}{2}$

The questions of domain and range become more interesting when considered in relation to functions defined by graphs, or in applications. In an application involving perimeter in which the perimeter of a rectangle is given as 50 feet, we know that

$$2\ell + 2w = 50 \tag{4.2.1}$$

Rewriting this as a function of w, we can say that

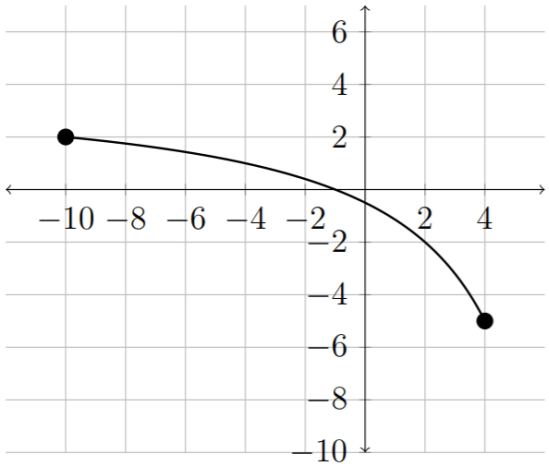
$$\ell = f(w) = 25 - w \tag{4.2.2}$$

In this function relating the length and width based on a given perimeter, we can say the domain of the function is \(0 Find the domain and range for each of the following functions:

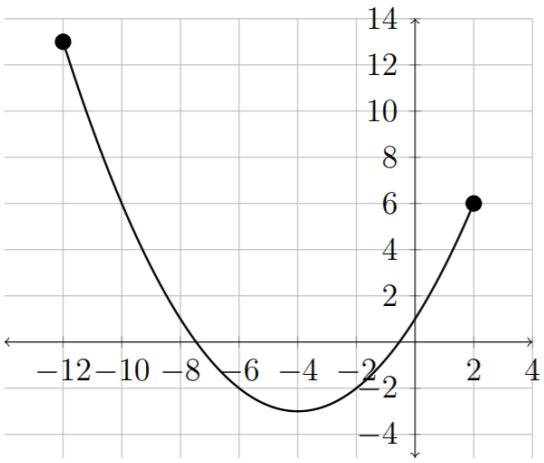
1) $f(x) = \sqrt{2x + 1}$
2) $f(x) = \sqrt{3x - 5}$
3) $\frac{x}{x+9}$
4) $f(x) = \frac{x+2}{2x-3}$

For the following graphs, assume that the endpoints for the domain and range are whole numbers.

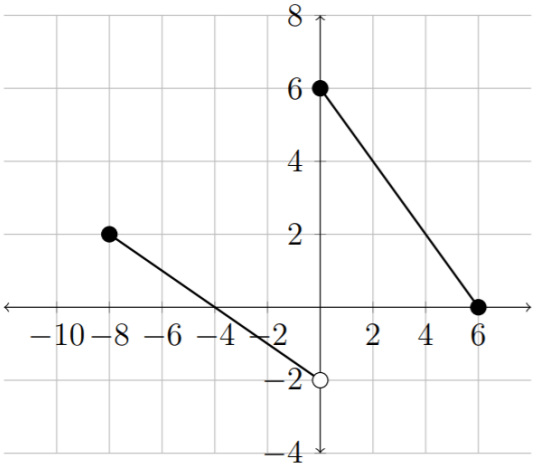
5)


6
4
2
−10 −8 −6 −4 −2
2 4
−2
−4
−6
−8
−10

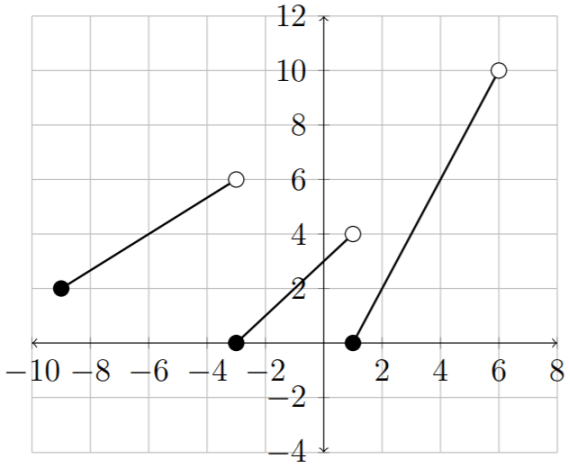
6)


14
12
10
8
6
4
2
−12 −10 −8 −6 −4 −2
2 4
−2
−4

7)


8
6
4
2
−10 −8 −6 −4 −2
2 4 6
−2
−4

8)

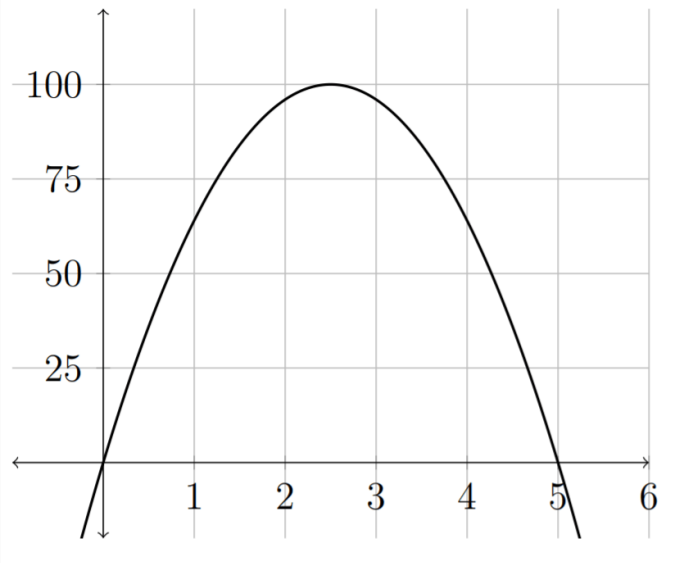


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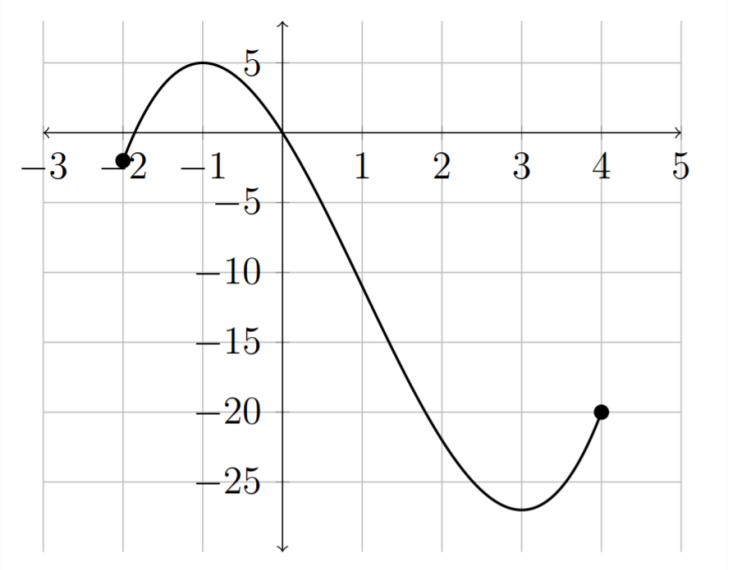
4.3: Maximum and Minimum Values

Finding the maximum and minimum values of a function can be very useful in applications. This is usually referred to as "optimization." Later on, we will examine the use of optimization in application problems. For now, we will use graphs to find the maximum and minimum values of the function.

In the example below, the maximum function value in the region shown is 100 . This occurs where $x = 2.5$

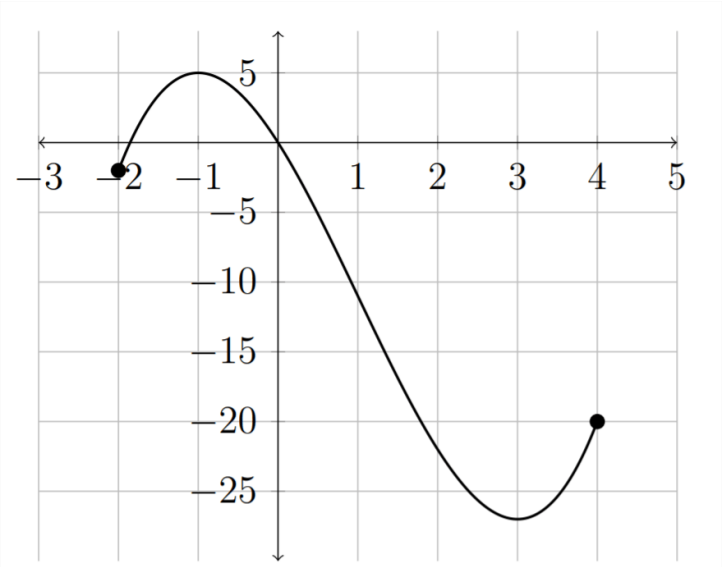


In the graph below, the function shows a maximum value of 5 at $x = -1$ and a minimum value of -27 at $x = 3$



We can use the graphing calculator to find maximum and minimum values. In calculus the maximum and minimum values can be found algebraically.

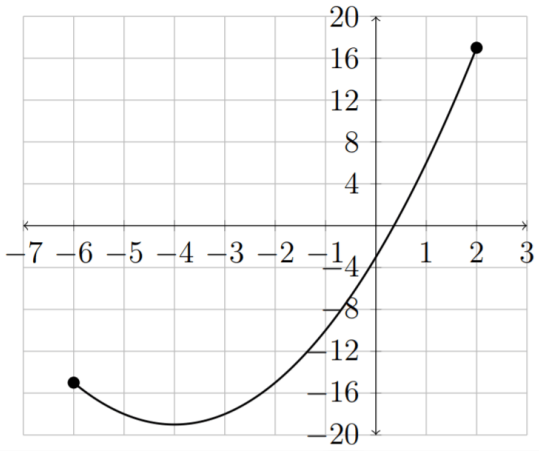
The maximum and minimum values are also helpful in determining where the function is increasing or decreasing. In the previous example:



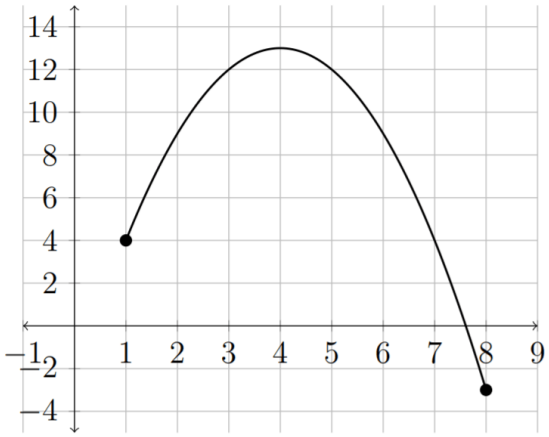
The function is increasing between $x = -2$ and $x = -1$, then again from $x = 3$ to $x = 4$. So we would say that the intervals in which the function is increasing are $-2 \leq x < -1$ and \(3 The interval in which the function is decreasing is \(-1

Determine the domain and range for each function given. Then determine the maximum and minimum values for the function and the intervals in which the function is increasing or decreasing.

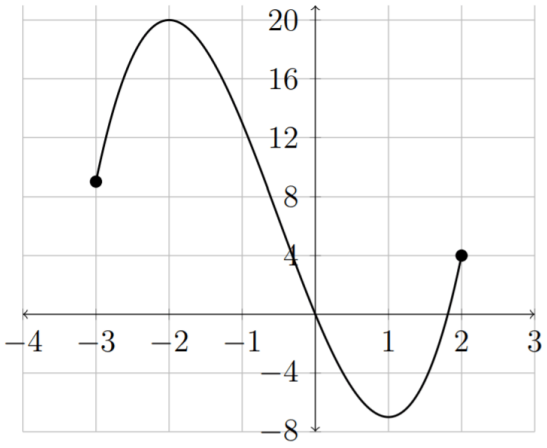
1)



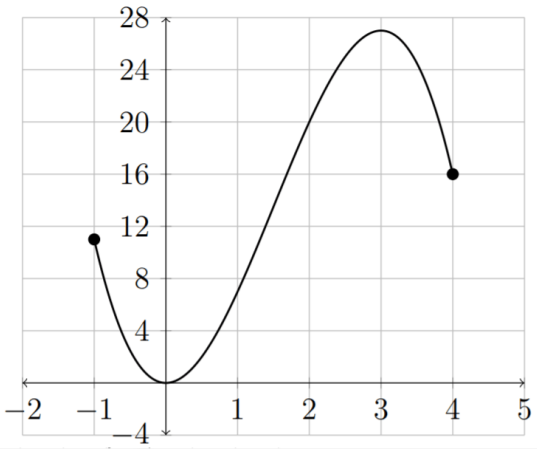
2)


14
12
10
8
6
4
2
−1
−2
−4
1
2
3
4
5
6
7
8
9

3)


20
16
12
8
4
−4
−8
−4
−3
−2
−1
1
2
3

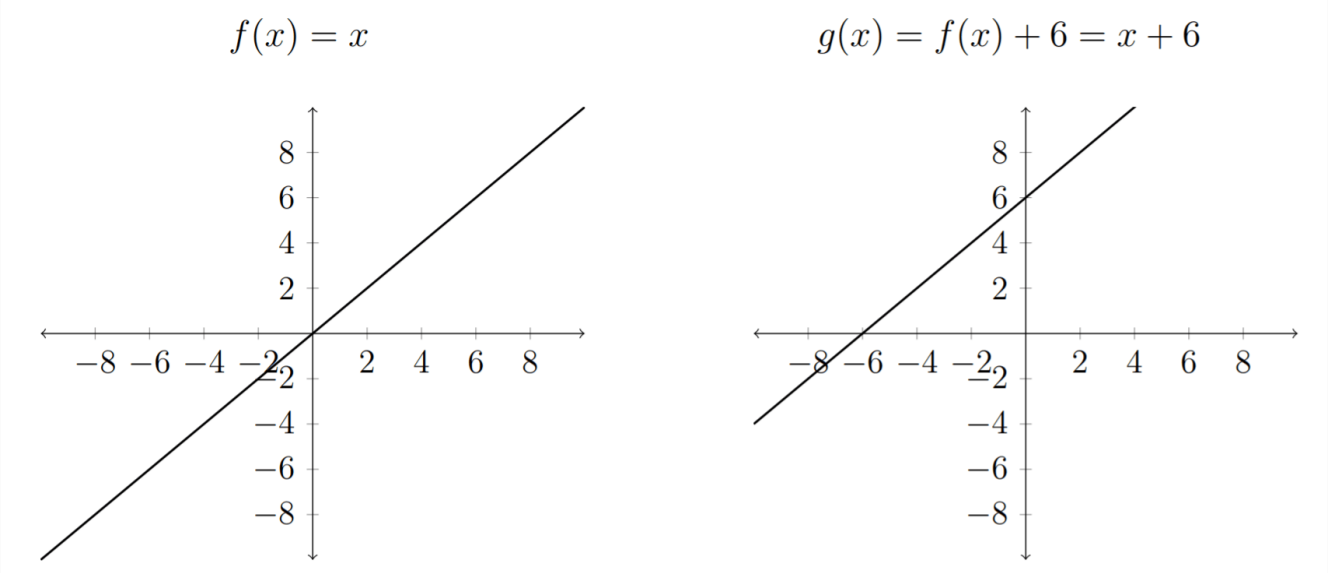
4)


28
24
20
16
12
8
4
−4
−2
−1
1
2
3
4
5

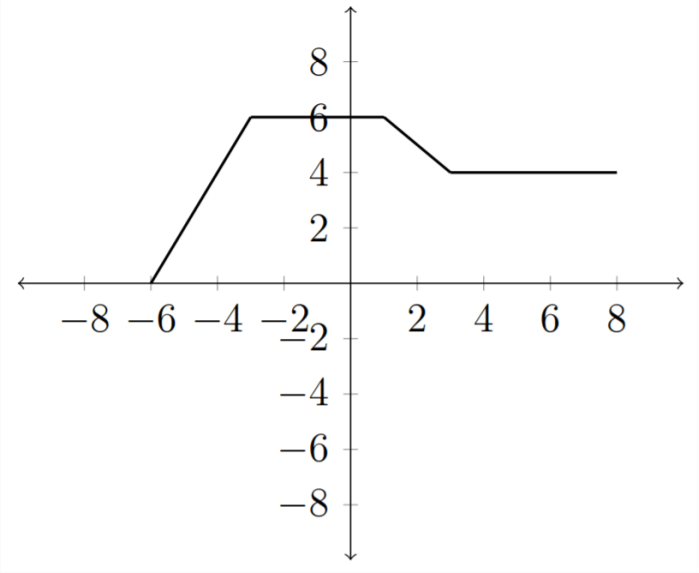
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4.4: Transformations

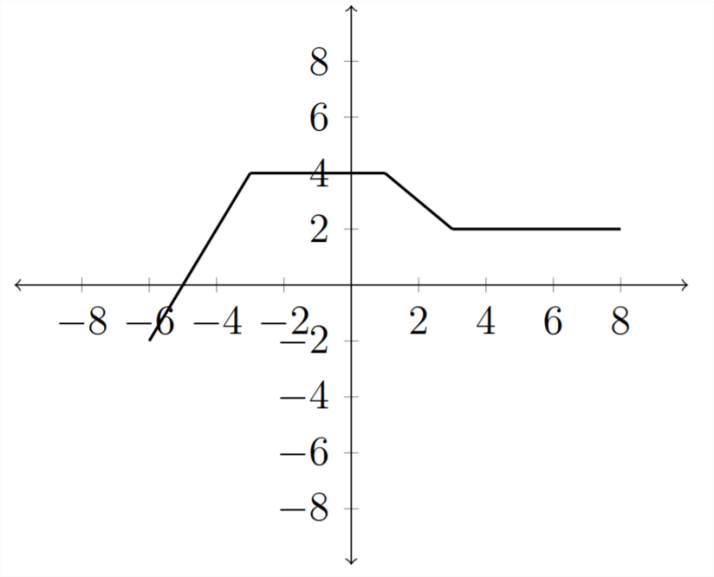
There are three major types of transformation that we will consider:
1) Horizontal and vertical shifts
2) Reflections over the x or y -axis
3) Horizontal and vertical stretches
If we take a given function, let's say $f(x) = x$, then this has the graph we see below - a straight line with a slope of 1 and a y -intercept of 0. If we add to the function $f(x) + 6 = x + 6$, then this will add 6 to all of the y -values which shifts the graph 6 places up.



Vertical Shifts
So, if we have a general function $f(x)$ that is described by a graph, we can determine a graph for $f(x) + k$, where k is some number that will either shift $f(x)$ up (if $k > 0$) or down (if $k < 0$)
For example, consider the following graph for $f(x)$

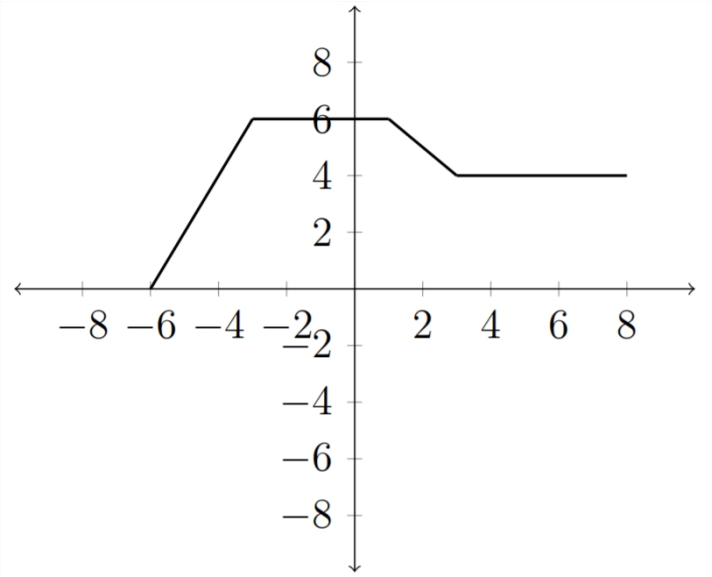


Then, if we want a graph for $f(x) - 2$, just shift all the y -values down 2 places:



This is a standard vertical shift transformation of a function.

Horizontal Shifts
A function may also be shifted horizontally by adding or subtracting a number inside the parentheses.
If we start with our original function $f(x)$



Then the transformation $f(x-2)$ will shift the graph horizontally, except that it will move in the opposite direction of the sign. The $f(x-2)$ shift will move the graph 2 places to the right whereas a shift of $f(x+2)$ will move the graph 2 places to the left.

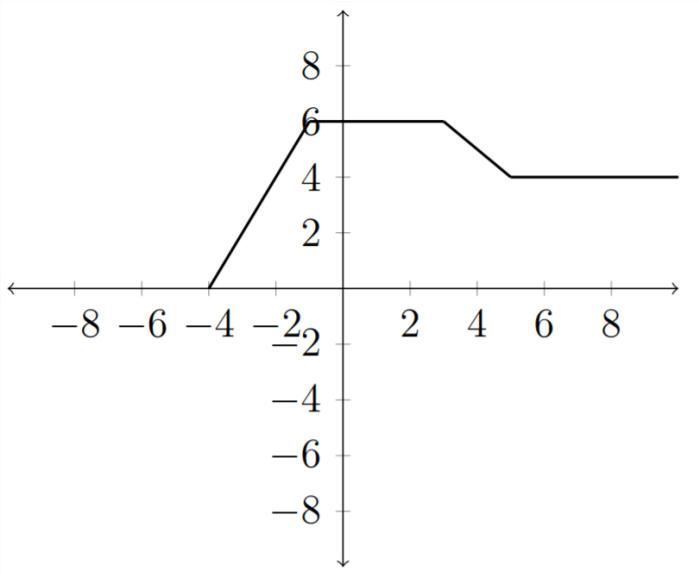
The reason why this happens will be more clear if we look at a table of values for the function:

x	$f(x)$
-6	0
-3	6
1	6
3	4
8	4

Now, if, instead of $f(x)$, we want $f(x-2)$, then we add another column to the table:

x	$x-2$	$f(x-2)$
-4	-6	0
-1	-3	6
3	1	6
1	3	4
6	8	4

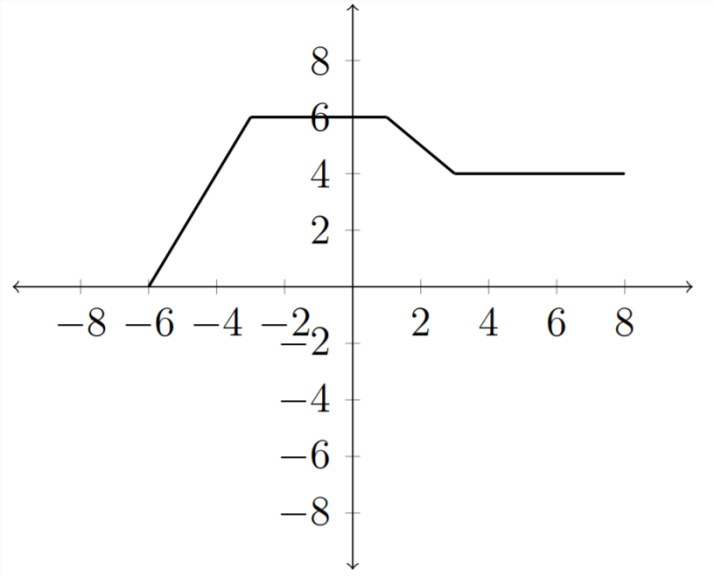
Notice that to have the same y -value as the original graph, we must go 2 places to the right so that after we subtract 2 from the x -value, we arrive back at the original x -value. So the graph of $f(x-2)$ will look like this:



Notice how the values in the graph match up with the values in the table.

Reflections

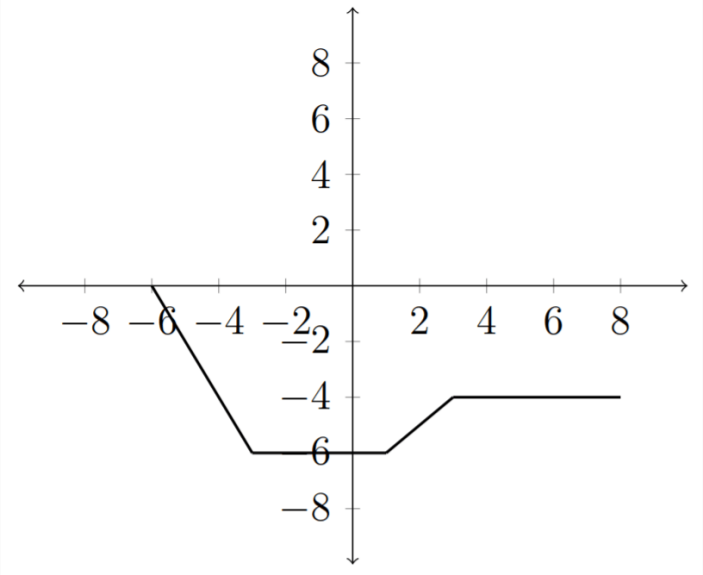
Negating the x or y values of a function will have the effect of reflecting the function over the y or x axis. If we consider the function $f(x)$ - the same one we used in the previous examples:



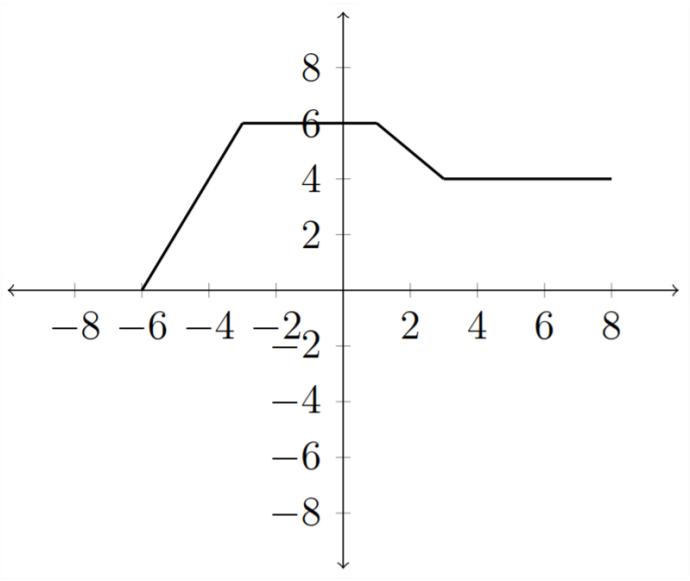
Then the graph of $y = -f(x)$ will be reflected over the x -axis. The negative sign in front of the function negates all of the y values, reflecting them over the x -axis. Lets look at this in a table of values:

x	$f(x)$	$-f(x)$
-6	0	0
-3	6	-6
1	6	-6
3	4	-4
8	4	-4

So the graph would look like this:



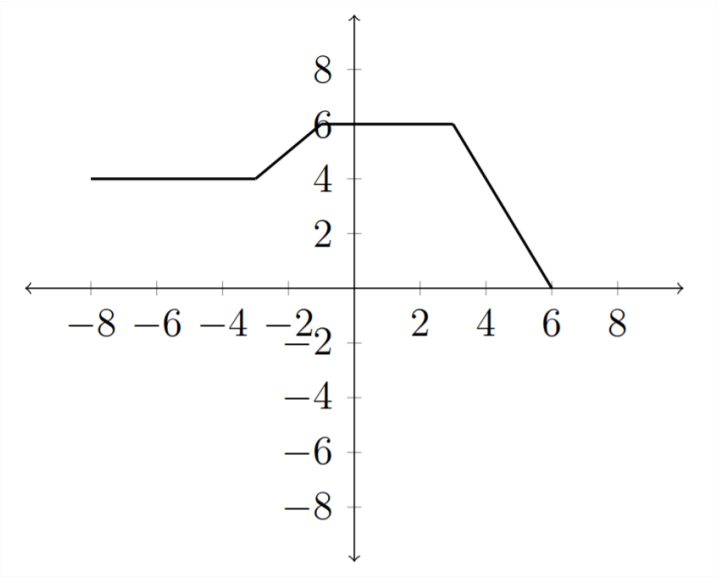
On the other hand, if we negate the x -variable $(f(-x))$, then the function values previously associated with the positive values of x would be associated with the negative values of x and vice versa. This would reflect the function over the y -axis.
If we again consider our original function and a table of values:



In the table:

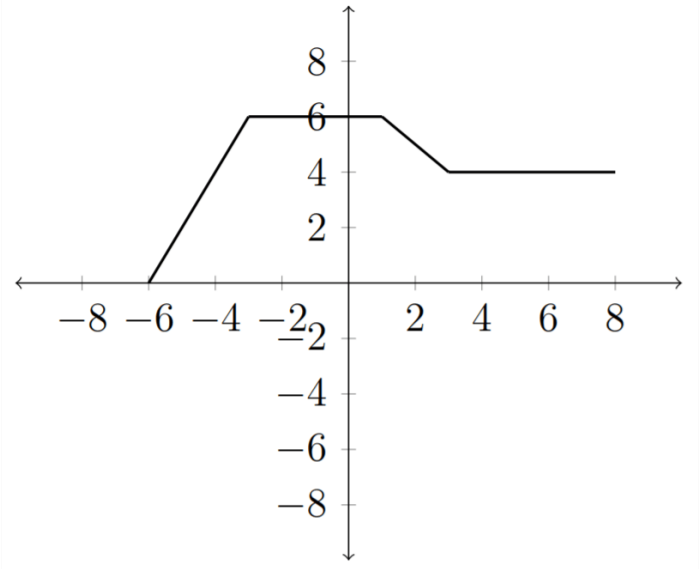
x	$-x$	$f(-x)$
-8	8	4
-3	3	4
-1	1	6
3	-3	6
6	-6	0

And the graph would be reflected over the y -axis:



Stretching and compressing graphs

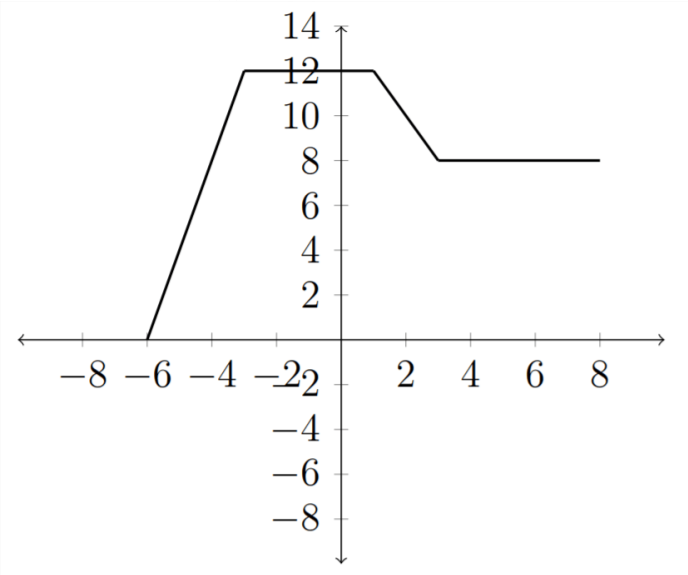
The last type of transformation we will examine is that of stretching or compressing a graph by multiplying inside or outside the parentheses. Starting with our familiar example function $y = f(x)$



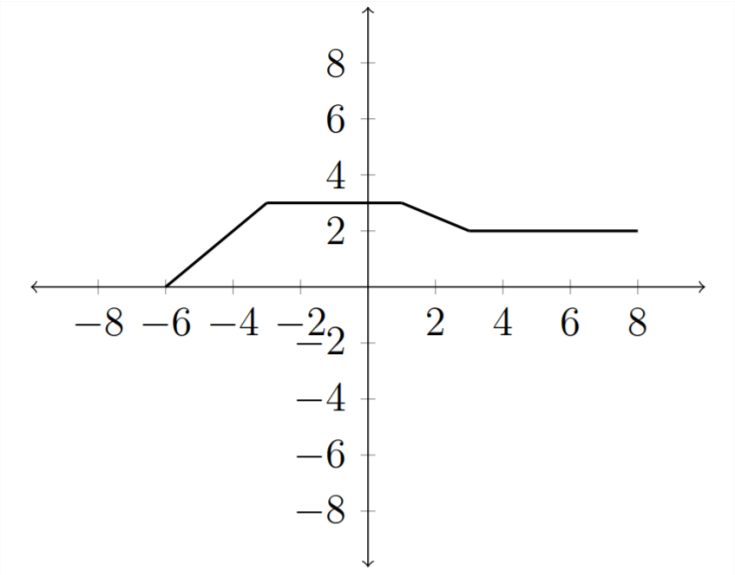
If we multiply the function by a constant outside of the parentheses: $y = 2f(x)$ then this will have the effect of multiplying all of the y values by 2. In the table:

x	$f(x)$	$2f(x)$
-6	0	0
-3	6	12
1	6	12
3	4	8
8	4	8

The graph of $y = 2f(x)$ would look like this:



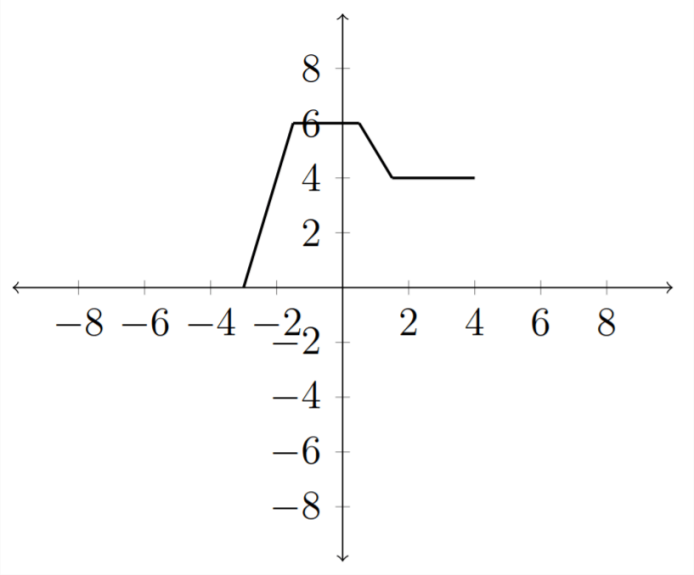
In a similar way, multiplying by a number less than 1 would compress the graph. The graph for $y = \frac{1}{2}f(x)$ is below:



Multiplying inside the parentheses affects the x variables.
If we consider the function $y = f(2x)$, then this will have the effect of compressing the graph along the x -axis:

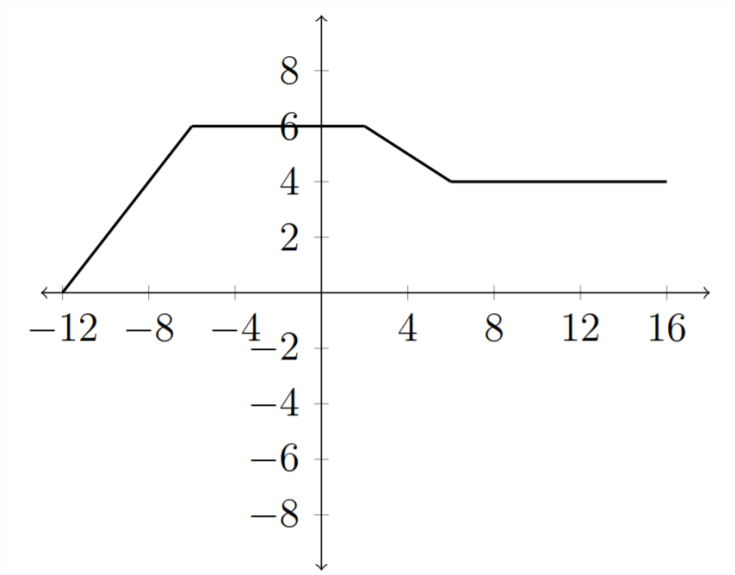
x	$2x$	$f(2x)$
-3	-6	0
-1.5	-3	6
0.5	1	6
1.5	3	4
4	8	4

Notice how each x value had to be cut in half so that when we multiplied it by 2 we ended up with the original x value. The graph would look like this:



Multiplying inside the parentheses by a number less than one would stretch the graph out.

x	$\frac{1}{2}x$	$f\left(\frac{1}{2}x\right)$
-12	-6	0
-6	-3	6
2	1	6
6	3	4
16	8	4

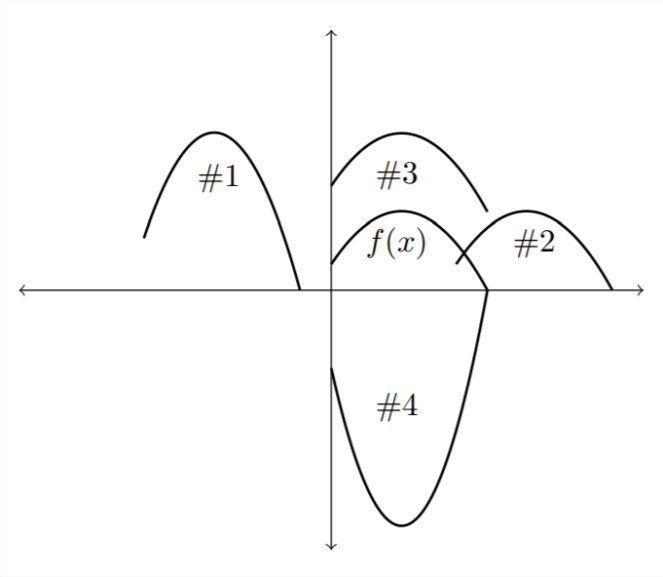


In these examples, we have considered only one transformation at a time. In the exercises you will need to consider the effect of several transformations at once.

Exercises 4.4

1) Match each of the functions on the graph to the appropriate transformation described below.

a) $f(x-4)$
b) $f(x)+3$
c) $-3f(x)$
d) $2f(x+6)$



2) Match each of the descriptions with the appropriate function transformation.

$a)$	$y=f(x-1)$	1)Shift left one unit
$b)$	$y=f(x)-1$	2)Reflect over x -axis, then shift left one unit
$c)$	$y=f(x)+1$	3)Shift right one unit
$d)$	$y=f(x+1)$	4)Reflect over x -axis, then shift up one unit
$e)$	$y=f(-x)+1$	5)Reflect over x -axis, then shift down one unit
$f)$	$y=f(-x)-1$	6)Shift down one unit
$g)$	$y=-f(x)+1$	7)Reflect over x -axis, reflect over y -axis, then shift up one unit
$h)$	$y=-f(x+1)$	8Shift left one unit, reflect over y -axis, then shift up one unit
$i)$	$y=-f(x)-1)$	9)Shift up one unit
$j)$	$y=f(-x+1)$	10)Reflect over y -axis, then shift up one unit
$k)$	$y=-f(-x)$	11)Reflect over y -axis, then shift down one unit

(4.4.1)

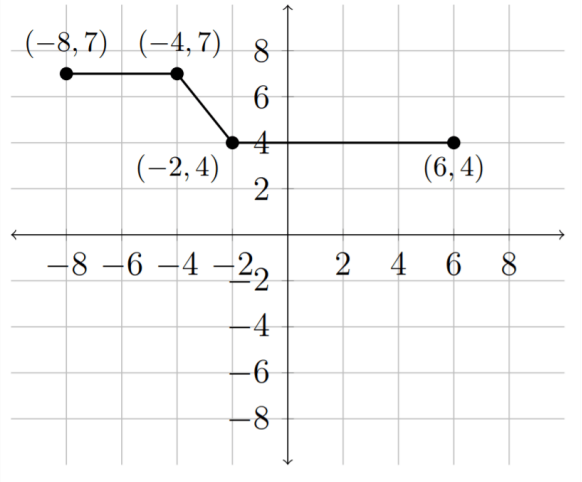
3) Match each of the descriptions with the appropriate function transformation.

$a)$	$y=f(x+2)+3$	1)Shift left 2 units, then shift down 3 units
$b)$	$y=f(x+3)+2$	2)Shift left 3 units, then shift up 2 units
$c)$	$y=f(x-2)+3$	3)Shift right 3 units, then shift up 2 units
$d)$	$y=f(x-2)-3$	4)Shift left 3 units, then shift down 2 units
$e)$	$y=f(x+2)-3$	5)Shift right 3 units, then shift down 2 units
$f)$	$y=f(x-3)+2$	6)Reflect over the y -axis, then shift up 2 units
$g)$	$y=f(x-3)-2$	7)Reflect over x -axis, then shift right 2 units
$h)$	$y=f(x+3)-2$	8)Reflect over x -axis, then shift left 2 units
$i)$	$y=-f(x+2)$	9)Shift left 2 units, then reflect over the y -axis
$j)$	$y=-f(x-2)$	10)Shift right 2 units, then shift up 3 units
$k)$	$y=f(2-x)$	11)Shift left 2 units, then shift up 3 units
$\ell)$	$y=f(-x)+2$	12)Shift right 2 units, then shift down 3 units

(4.4.2)

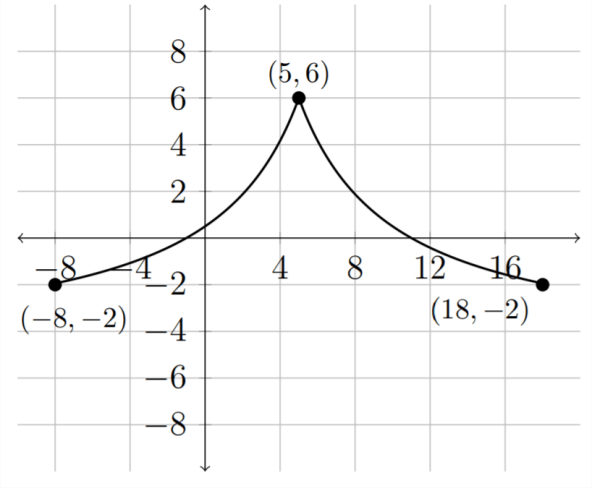
Apply the transformations indicated for each function.

4) $f(x)$



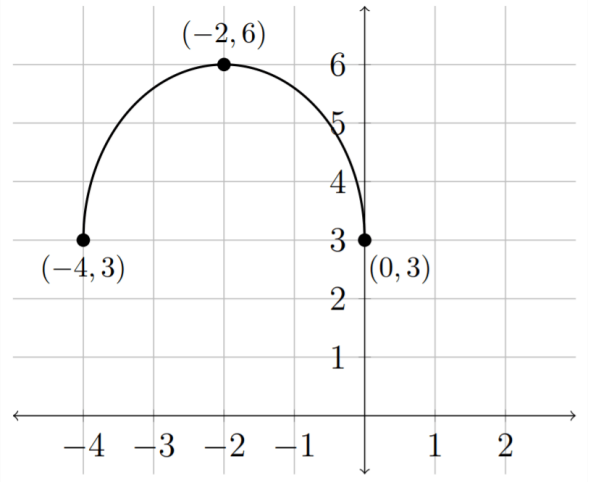
a) $f(x-3)$
b) $-f(x)+2$
c) $\frac{1}{2}f(x-1)$
d) $f(-x)+1$

5) $g(x)$



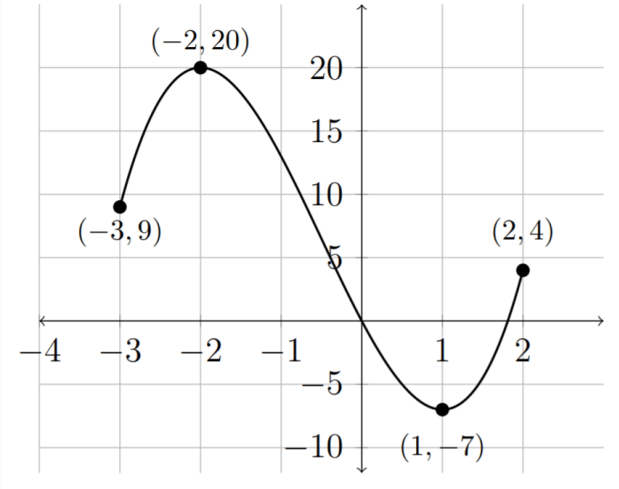
a) $g(x) - 3$
b) $-g(x) - 1$
c) $2g(x + 2)$
d) $g(2x) - 1$

6) $f(x)$



a) $f(x) + 2$
b) $-f(x - 1)$
c) $f(x - 1) - 3$
d) $\frac{1}{3}f(x) + 4$

7) $g(x)$

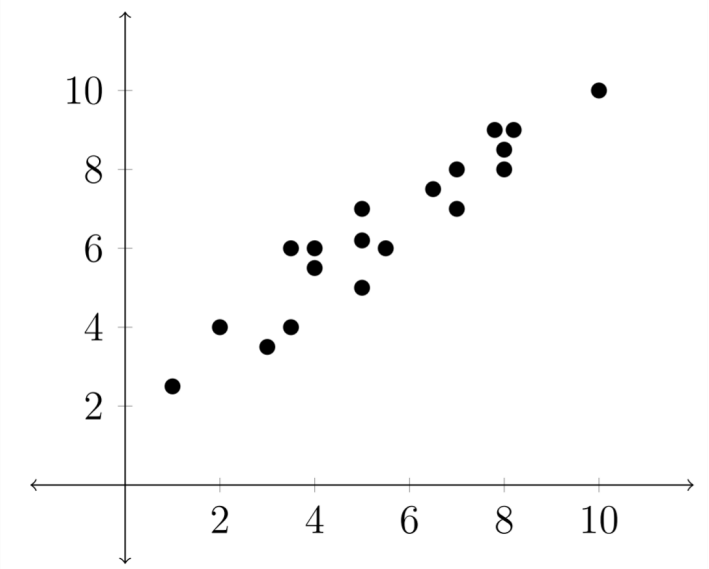


a) $g(x-2)$
b) $-g(x)+1$
c) $2g(x-1)$
d) $\frac{1}{2}g(x)-3$

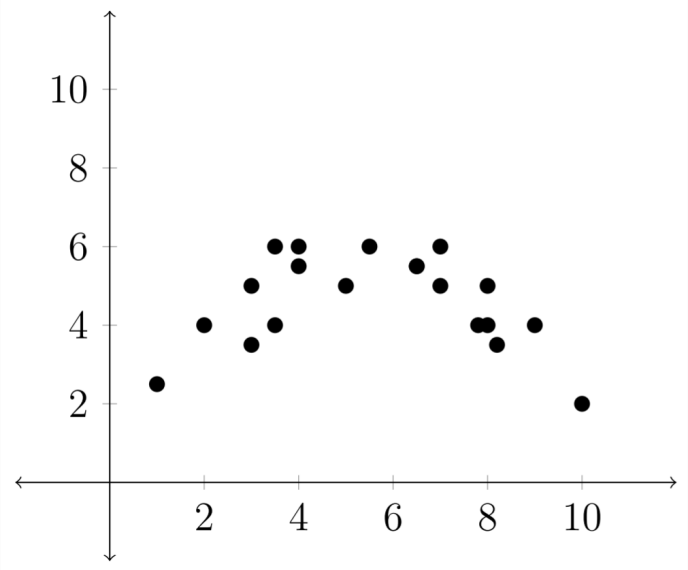
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4.5: Toolbox Functions

Learning the general shapes of some common function families can be helpful in analyzing various problems. This can also be helpful in applying the ideas of statistical regression. Statistical regression typically gathers a collection of data points and tries to fit a mathematical function to the data points. Choosing the type of function that will best fit the data is an important step in determining a suitable regression function.
Below is an example of a Linear Regression:

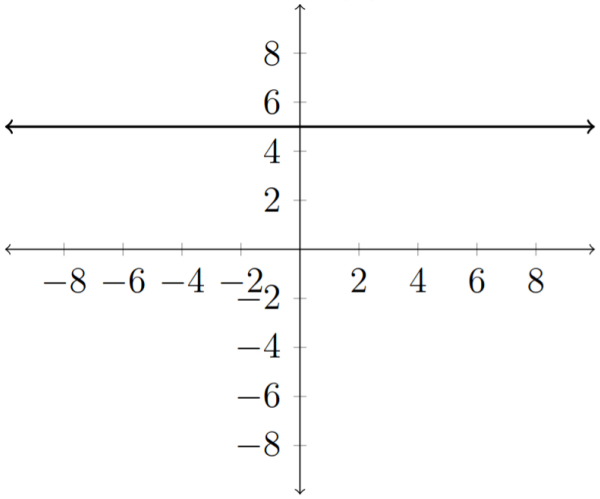


The graph below is an example of a Quadratic Regression:

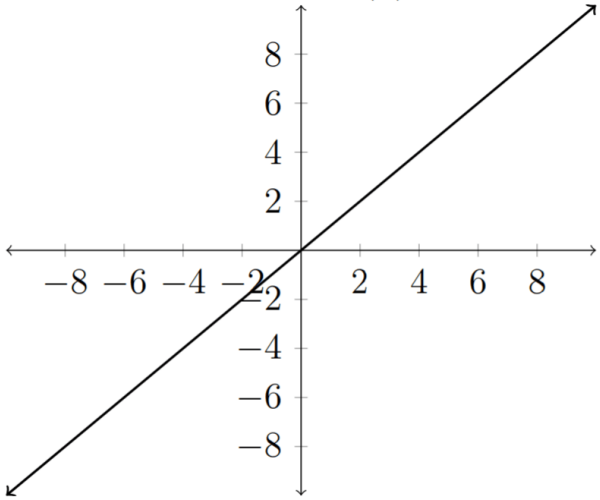


Being familiar with the typical shape of the various function families can help in analyzing experimental data.
The standard function families are:

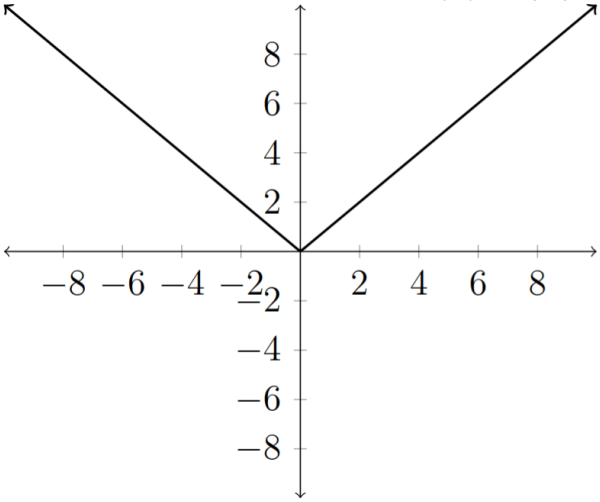
Constant function: $f(x) = c$



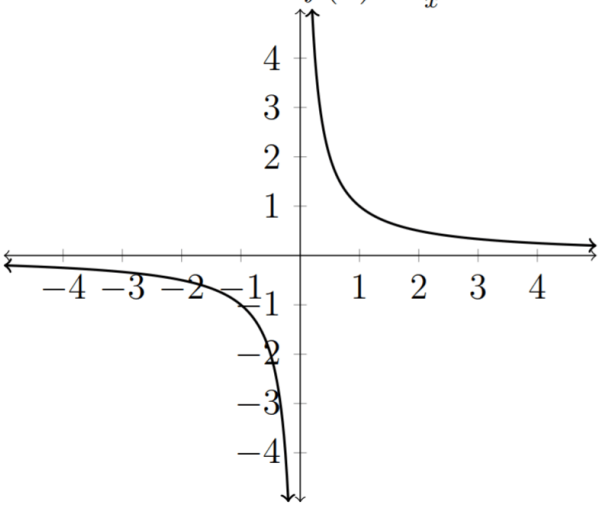
Linear function: $y = f(x) = x$



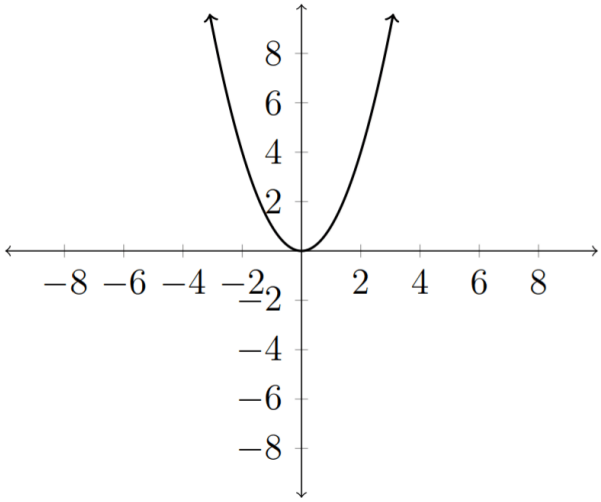
Absolute value function: $f(x) = |x|$



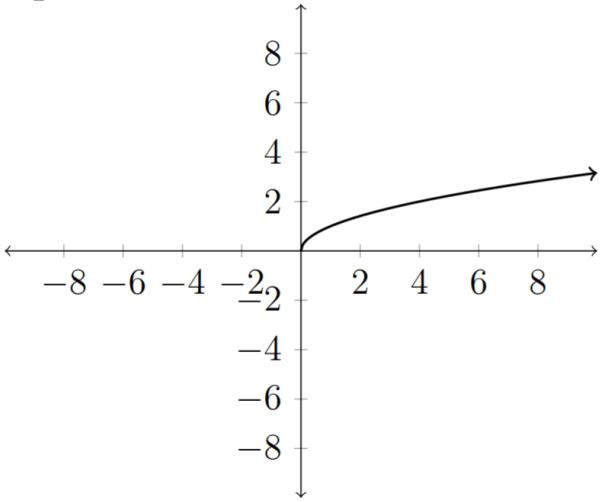
Rational function: $f(x) = \frac{1}{x}$

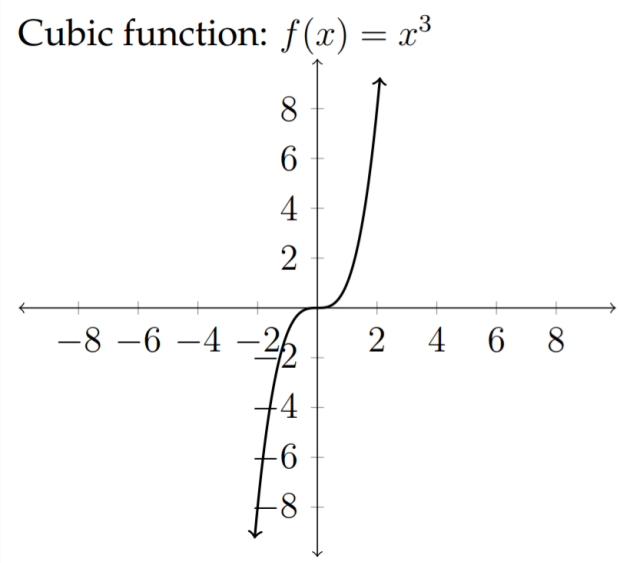


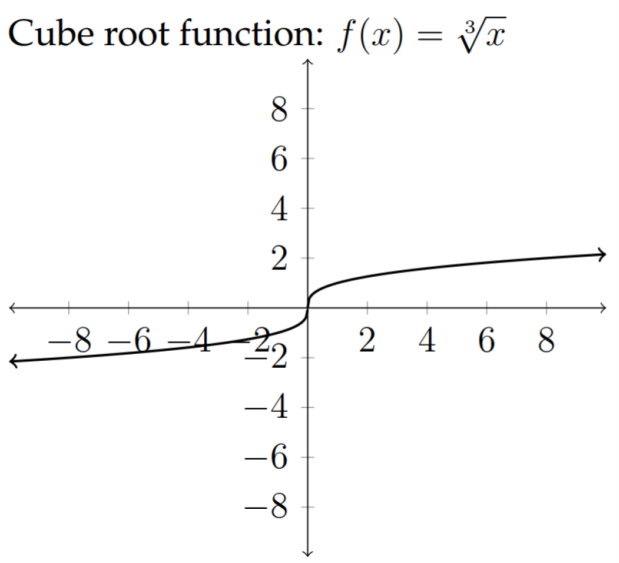
Quadratic function: $f(x) = x^2$



Square root function: $f(x) = \sqrt{x}$







Some other function families that we won't discuss are:

Exponential function: $f(x) = a^x$
Logarithmic function: $f(x) = \log_b x$
Trigonometric function: $f(x) = \sin x$

Exercises 4.5

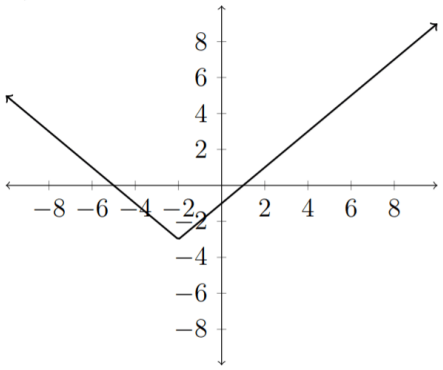
Sketch the graph for each of the following transformations.

1) $f(x) = x^2 + 3$
2) $f(x) = x^2 - 4$
3) $f(x) = (x-5)^2 + 3$
4) $f(x) = (x+1)^2 - 4$
5) $f(x) = |x| - 2$
6) $f(x) = |x| + 5$
7) $f(x) = |x+3| - 2$
8) $f(x) = |x-1| + 5$

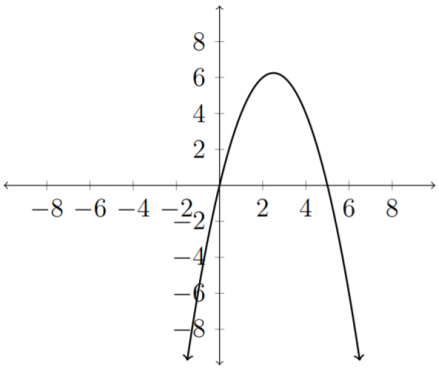
Match each of the following equations to the appropriate graph.

9) $f(x) = \sqrt{x+4} - 1$
10) $h(x) = x^2 + x - 6$
11) $g(x) = |x+2| - 3$
12) $f(x) = 4x - x^2$
13) $h(x) = -|x-2| + 1$
14) $g(x) = -\frac{3}{5}x + 2$
15) $h(x) = \sqrt[3]{x} + 4$
16) $f(x) = x^3 - 3x^2 + 3x + 2$
17) $g(x) = -\sqrt[3]{x+2}$
18) $h(x) = -\sqrt{x+1} - 3$
19) $f(x) = \frac{5}{4}x - 2$
20) $f(x) = 4x - x^3$

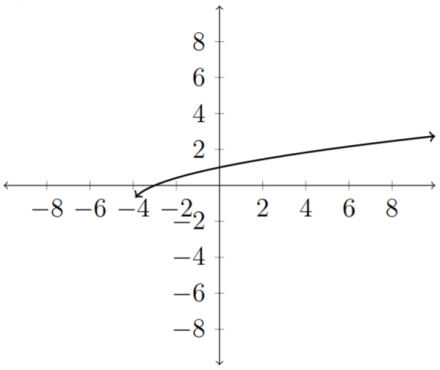
a)



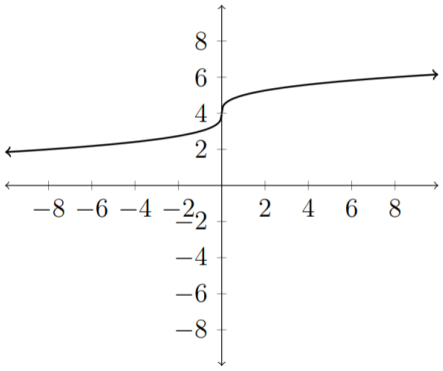
b)



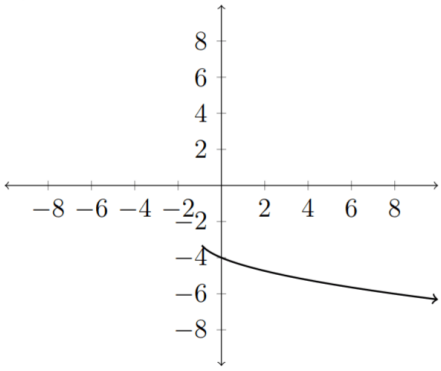
c)



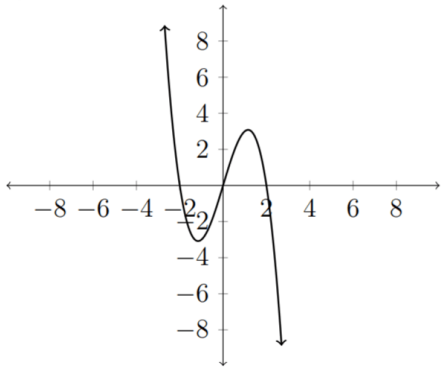
d)



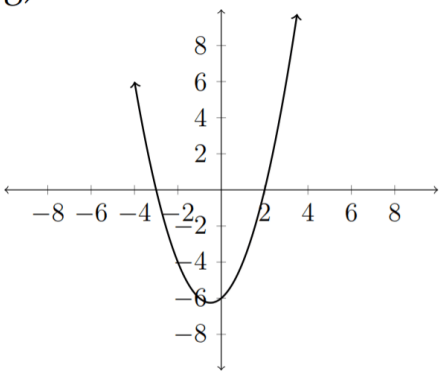
e)



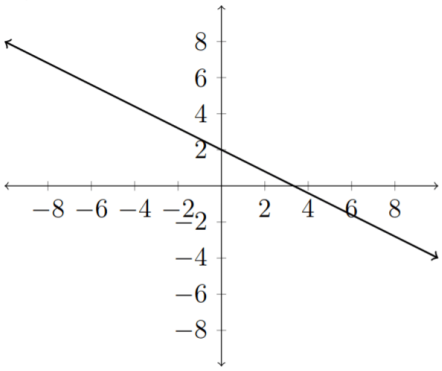
f)



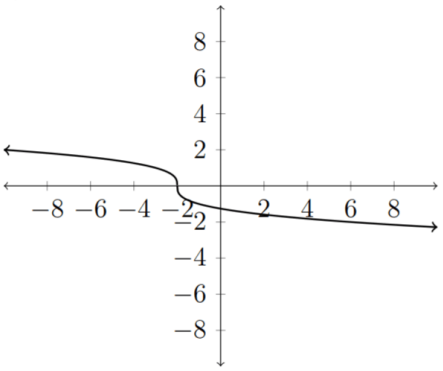
g)



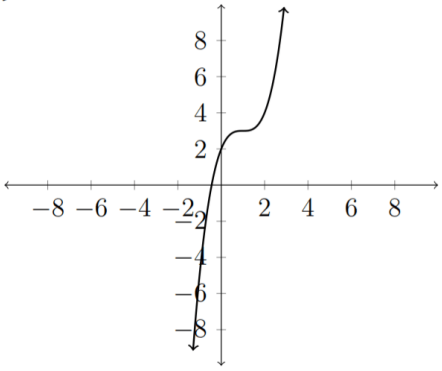
h)



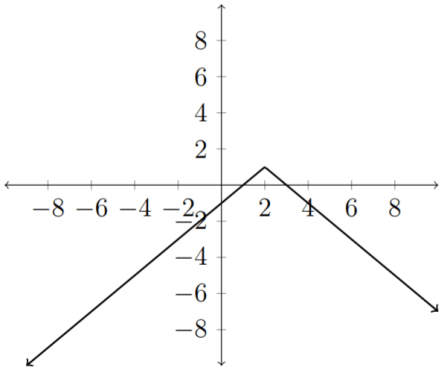
i)



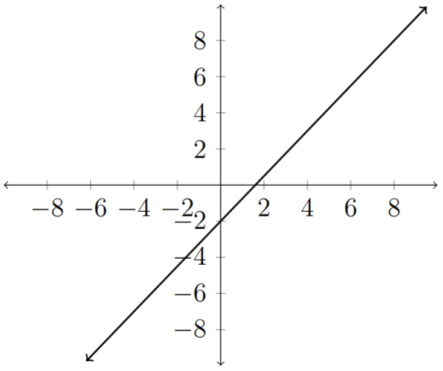
j)



k)



ℓ)



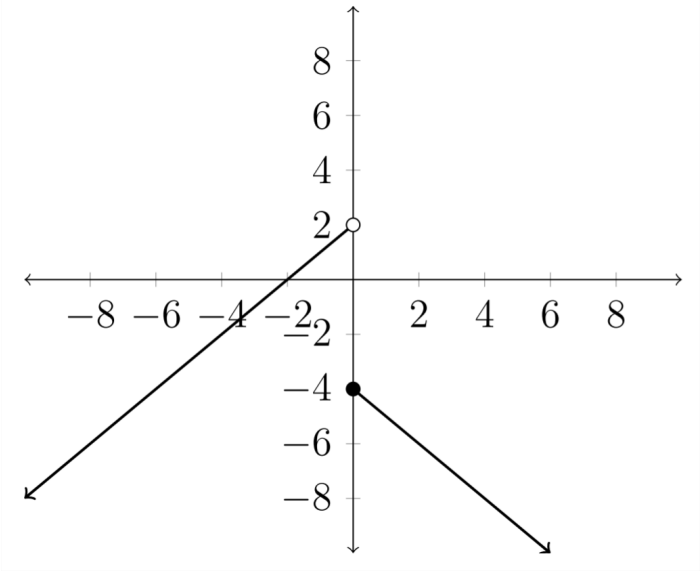
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4.6: Piecewise-defined Functions

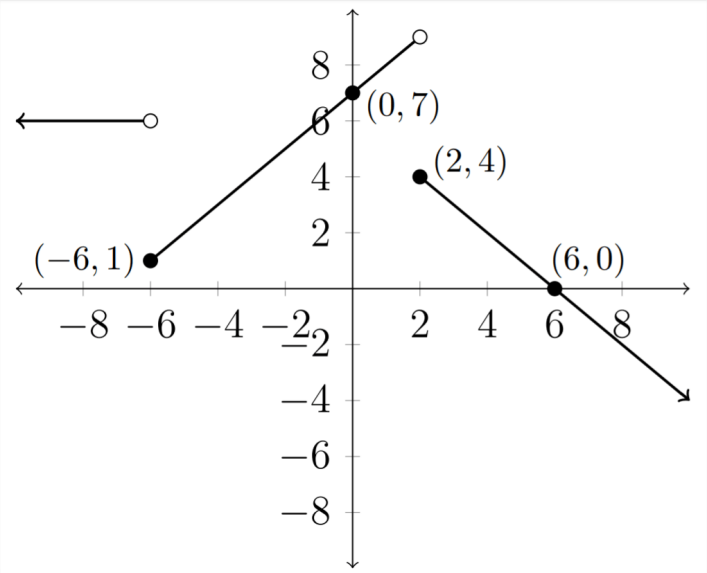
In certain situations a numerical relationship may follow one pattern of behavior for a while and then exhibit a different kind of behavior. In a situation such as this, it is helpful to use what is known as a piecewise defined function - a function that is defined in pieces.

$$f(x) = \begin{cases} x + 2 & x < 0 \\ -x - 4 & x \geq 0 \end{cases}$$

In the above example of a piecewise defined function, we see that the y values for the negative values of x are defined differently than the y values for the positive values of x



Sometimes we are given a graph and need to write a piecewise description of the function it describes.



The piecewise function pictured above could be described as follows:

$$f(x) = \begin{cases} 6 & x < -6 \\ x + 7 & -6 \leq x < 2 \\ -x + 6 & x \geq 2 \end{cases}$$

Exercises 4.6

Sketch a graph for each of the piecewise functions described below.

$$1) \qquad f(x) = \begin{cases} 1 & x < 2 \\ 2 & x \geq 2 \end{cases}$$

2) $$f(x) = \begin{cases} 0 & x \leq 1 \\ x - 1 & x > 1 \end{cases}$$

3) $$f(x) = \begin{cases} 4 & x < 2 \\ x - 2 & x \geq 2 \end{cases}$$

4) $$f(x) = \begin{cases} 2 - x & x < -2 \\ 4 & x \geq -2 \end{cases}$$

5) $$f(x) = \begin{cases} x & x \leq 1 \\ x + 1 & x > 1 \end{cases}$$

6) $$f(x) = \begin{cases} 2x+3 & x < -2 \\ 3-x & x \geq -2 \end{cases}$$

7) $$f(x) = \begin{cases} -1 & x < -1 \\ 1 & -1 \leq x \leq 1 \\ -1 & x > 1 \end{cases}$$

8) $$f(x) = \begin{cases} -1 & x < -1 \\ x & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

9) $$f(x) = \begin{cases} x^2 & x < 0 \\ x+1 & x \geq 0 \end{cases}$$

10) $$f(x) = \begin{cases} x^2 + 2x & x \leq -1 \\ x & x > -1 \end{cases}$$

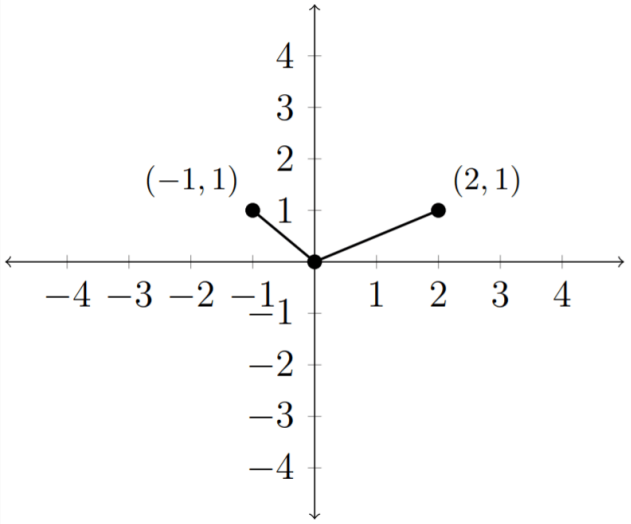
11) $$f(x) = \begin{cases} 2 & x \leq -1 \\ x^2 & x > -1 \end{cases}$$

12) $$f(x) = \begin{cases} 1 - x^2 & x \leq 2 \\ x & x > 2 \end{cases}$$

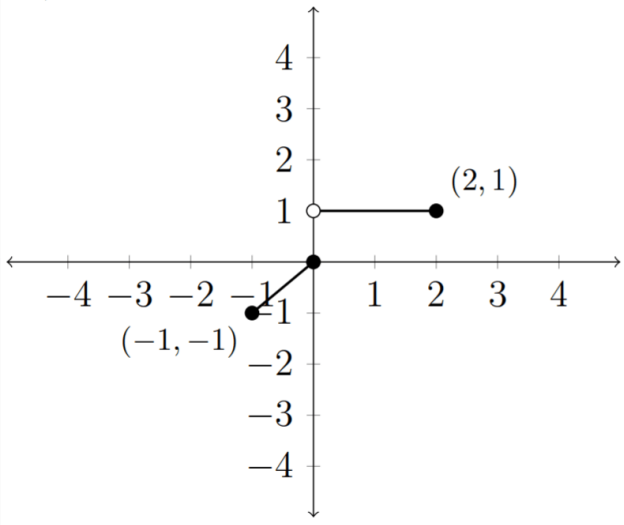
13) $$f(x) = \begin{cases} 4 & x < -2 \\ x^2 & -2 \leq x \leq 2 \\ -x + 6 & x > 2 \end{cases}$$

14) $$f(x) = \begin{cases} 3x & x < 0 \\ x + 1 & 0 \leq x \leq 2 \\ (x-2)^2 & x > 2 \end{cases}$$

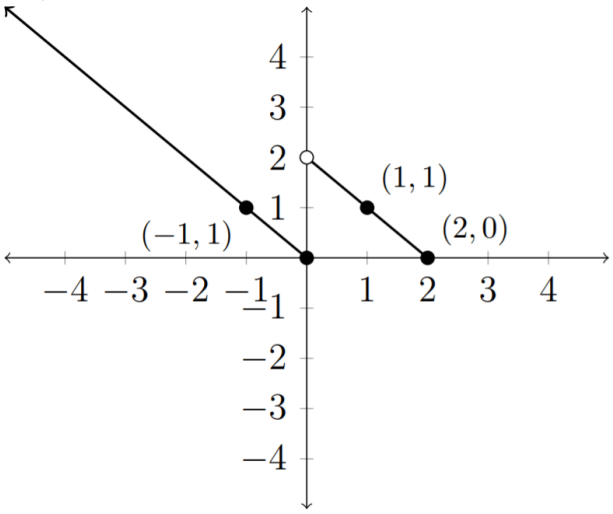
15)



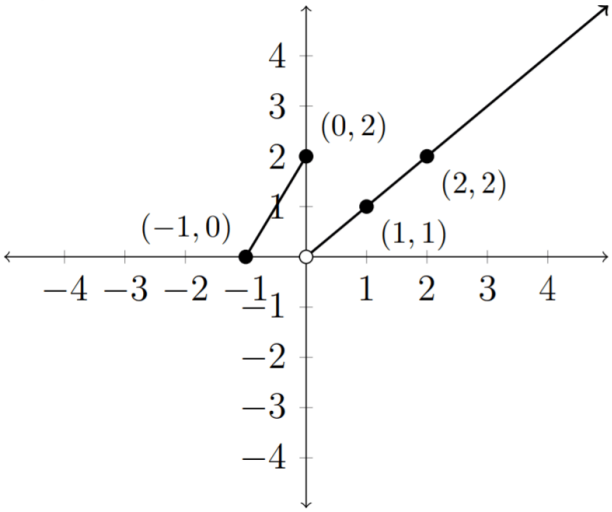
16)



17)



18)



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4.7: Composite functions

Similar to the way in which we used transformations to analyze the equation of a function, it is sometimes helpful to consider a given function as being several functions of the variable combined together.

For example, instead of thinking of the function $f(x) = (2x-7)^3$ as being a single function, we can think of it as being two functions:

$$\begin{gathered} g(x) = 2x - 7 \\ \text{and} \\ h(x) = x^3 \end{gathered} \tag{4.7.1}$$

Then $f(x)$ is the combination or "composition" of these two functions together. The first function multiplies the variable by 2, and subtracts 7 from the result. The second function takes this answer and raises it to the third power. The notation for the composition of functions is an open circle: ∘

In the example above we would say that the function $f(x) = (2x-7)^3$ is equivalent to the composition $h \circ g(x)$ or $h(g(x))$. The order of function composition is important. The function $g \circ h(x)$ would be equivalent to $g(h(x))$, which would be

$$\text{equal to } g\left(x^3\right) = 2\left(x^3\right) - 7 = 2x^3 - 7 \tag{4.7.2}$$

Exercises 4.7

Find $f \circ g(x)$ and $g \circ f(x)$ for each of the following problems.

1) $f(x) = x^2 \; g(x) = x - 1$
2) $f(x) = |x-3| \; g(x) = 2x + 3$

3) $f(x) = \frac{x}{x-2} \; g(x) = \frac{x+3}{x}$
4) \(\quad f(x)=x^{3}-1 \ g(x)=\frac{1}{x^{3}+1\)

5) $f(x) = \sqrt{x+1} \; g(x) = x^4 - 1$
6) \(\quad f(x)=2 x^{3}-1 \ g(x)=\sqrt[3]{\frac{x+1}{2}\)

Find functions $f(x)$ and $g(x)$ so that the given function $h(x) = f \circ g(x)$

7) $h(x) = (3x+1)^2$
8) $h(x) = \left(x^2 - 2x\right)^3$
9) $h(x) = \sqrt{1-4x}$
10) $h(x) = \sqrt[3]{x^2-1}$
11) $h(x) = \left(\frac{x+1}{x-1}\right)^2$
12) $h(x) = \left(\frac{1-2x}{1+2x}\right)^3$
13) $h(x) = \left(3x^2-1\right)^{-3}$
14) $h(x) = \left(1+\frac{1}{x}\right)^{-2}$
15) $h(x) = \sqrt{\frac{x}{x-1}}$
16) $h(x) = \sqrt[3]{\frac{x-1}{x}}$
17) $h(x) = \sqrt{\left(x^2-x-1\right)^3}$
18) $h(x) = \sqrt[3]{(1-x^4)^2}$
19) $h(x) = \frac{2}{\sqrt{4-x^2}}$
20) $h(x) = -\left(\frac{3}{x-1}\right)^5$

21) A spherical weather balloon is inflated so that the radius at time t is given by the equation:

$$r = f(t) = \frac{1}{2}t + 2 \tag{4.7.3}$$

Assume that r is in meters and t is in seconds, with $t = 0$ corresponding to the time the balloon begins to be inflated. If the

volume of a sphere is given by the formula:

$$v(r) = \frac{4}{3}\pi r^3 \tag{4.7.4}$$

Find $V(f(t))$ and use this to compute the time at which the volume of the balloon is $36\pi \mathrm{m}^3$

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4.8: Inverse Functions

An inverse function undoes the action of the original function. So the inverse of a function that squared a number would be a function that square rooted a number. In general, an inverse function will take a y value from the original function and return the x value that produced it.

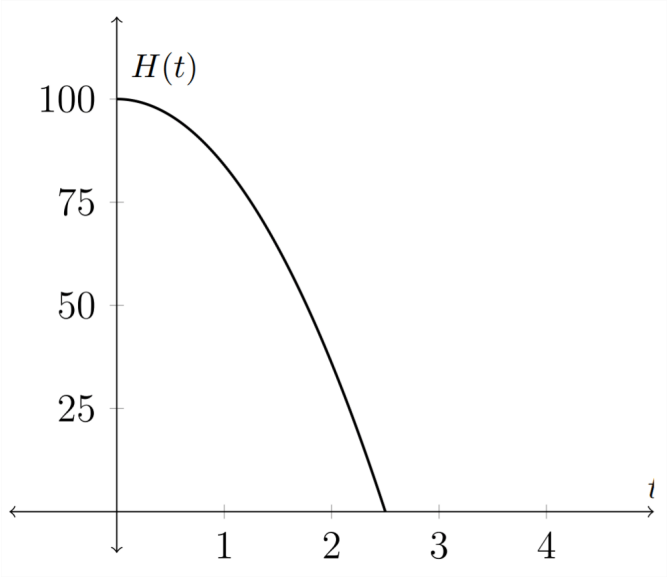
We can see this in an application. Given an object with little or no air resistance that is dropped from 100ft, the function that describes its height as a function of time would be:

$$H(t) = 100 - 16t^2 \tag{4.8.1}$$

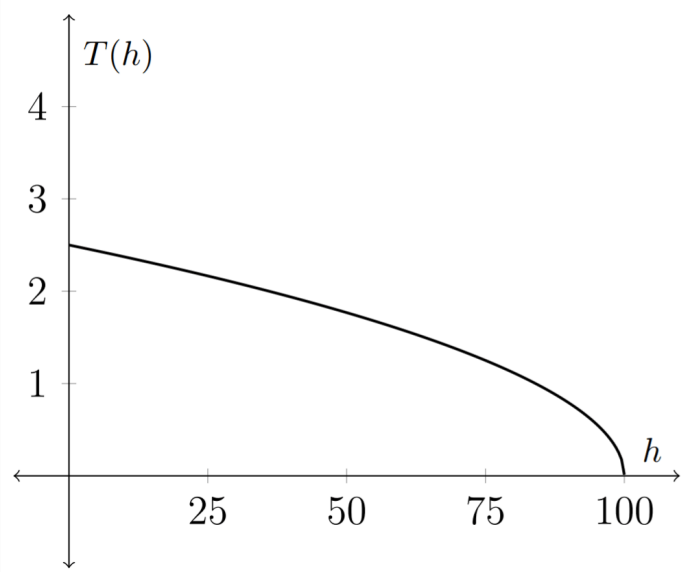
In this function, $H(t)$ is the height of the object at time t. If we wanted to turn this around so that it described the time for a given height, then we would want to isolate the t variable. In this example, the graph of the function would change in that the original independent variable - t, becomes the dependent variable in the inverse function.

$$\begin{aligned} h &= 100 - 16t^2 \\ 16t^2 &= 100 - h \\ t^2 &= \frac{100-h}{16} \\ t &= \sqrt{\frac{100-h}{16}} \\ T(h) &= \frac{\sqrt{100-h}}{4} \end{aligned} \tag{4.8.2}$$

Original function: $H(t) = 100 - 16t^2$



Inverse function: $T(h) = \frac{\sqrt{100-h}}{4}$



Notice that the graph of the function inverse is the original function reflected over the line $y = x$, because an inverse function interchanges the independent and dependent variables.

Finding a formula for an inverse function can be more confusing when we consider a standard function $y = f(x)$. In our standard notation, x is always considered to be the independent variable and y is always considered to be the dependent variable.

Notice in the example above that when we graphed the function and its inverse, the label on the x axis changed from t to h. In a standard function, the x axis will always be the x axis and the y axis will always be the y axis. To compensate for this, when we find a function inverse for a function stated in terms of x and y we generally interchange the x and y terms so that x remains the independent variable.
In our example, we had

$$H(t) = 100 - 16t^2 \tag{4.8.3}$$

and found the inverse to be

$$T(h) = \frac{\sqrt{100 - h}}{4} \tag{4.8.4}$$

If the original function had been stated in terms of x and y, then the process would have looked like this:

$$\begin{aligned} f(x) &= 100 - 16x^2 \\ y &= 100 - 16x^2 \\ 16x^2 &= 100 - y \\ x^2 &= \frac{100 - y}{16} \\ x &= \sqrt{\frac{100 - y}{16}} \\ x &= \frac{\sqrt{100 - y}}{4} \end{aligned} \tag{4.8.5}$$

Then we switch the x and y variables to keep the x as the independent variable:

$$y = f^{-1}(x) = \frac{\sqrt{100 - x}}{4} \tag{4.8.6}$$

So

$$f(x) = 100 - 16x^2 \tag{4.8.7}$$

and

$$f^{-1}(x) = \frac{\sqrt{100 - x}}{4} \tag{4.8.8}$$

Exercises 4.8

Given the function $f(x)$, find the inverse function $f^{-1}(x)$

1) $f(x) = 3x$
2) $f(x) = -4x$
3) $f(x) = 4x + 2$
4) $f(x) = 1 - 3x$
5) $f(x) = x^3 - 1$
6) $f(x) = x^3 + 1$
7) $f(x) = x^2 + 4 (x \geq 4)$
8) $f(x) = x^2 + 9 (x \geq 9)$
9) $f(x) = \frac{4}{x}$
10) $f(x) = -\frac{3}{x}$
11) $f(x) = \frac{1}{x-2}$
12) $f(x) = \frac{4}{x+2}$
13) $f(x) = \frac{2}{x+3}$
14) $f(x) = \frac{4}{2-x}$
15) $f(x) = \frac{3x}{x+2}$
16) $f(x) = -\frac{2x}{x-1}$
17) $f(x) = \frac{2x}{3x-1}$
18) $f(x) = -\frac{3x+1}{x}$
19) $f(x) = \frac{3x+4}{2x-3}$
20) $f(x) = \frac{2x-3}{x+4}$
21) $f(x) = \frac{2x+3}{x+2}$
22) $f(x) = -\frac{3x+4}{x-2}$

Find the inverse function for each of the following applications.

23) The volume of water left in a 1000 gallon tank that drains in 40 minutes is modeled by the equation:

$$V(t) = 1000 \left(1 - \frac{t}{40}\right)^2 \tag{4.8.9}$$

Find $T(v)$ - the function that tells you how long the water has been draining given a particular volume left in the tank. Time is measured in minutes and the volume is measured in gallons.

24) The speed of a vehicle in miles per hour that leaves skid marks d feet long is modeled by the equation:

$$R(d) = 2\sqrt{5d} \tag{4.8.10}$$

Find $D(r)$ - the function that tells you the stopping distance for a vehicle traveling r miles per hour.

25) The period of a pendulum of length ℓ can be expressed by the relationship:

$$T(\ell) = 2\pi\sqrt{\frac{\ell}{980}} \tag{4.8.11}$$

Find the function $L(t)$ that determines the length of a pendulum given its period. Here the time is measured in seconds and the length is measured in centimeters.

26) The volume of a sphere of radius r is given by the formula:

$$V(r) = \frac{4}{3}\pi r^3 \tag{4.8.12}$$

Find $R(v)$ - the function that determines the radius of a sphere given its volume.

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4.9: Optimization

One of the major applications of differential calculus is optimization. This is the process of finding maximum or minimum function values for a given relationship. There are four typical types of problems that we will examine in this section.
a) Analytic Optimization - these problems typically use the distance formula to determine the closest point to a particular curve.

b) Geometry/cost Optimization - these problems generally give a box or container of a particular shape and ask either to determine the cheapest manufacturing cost given a particular volume or to determine the greatest volume given a particular cost.

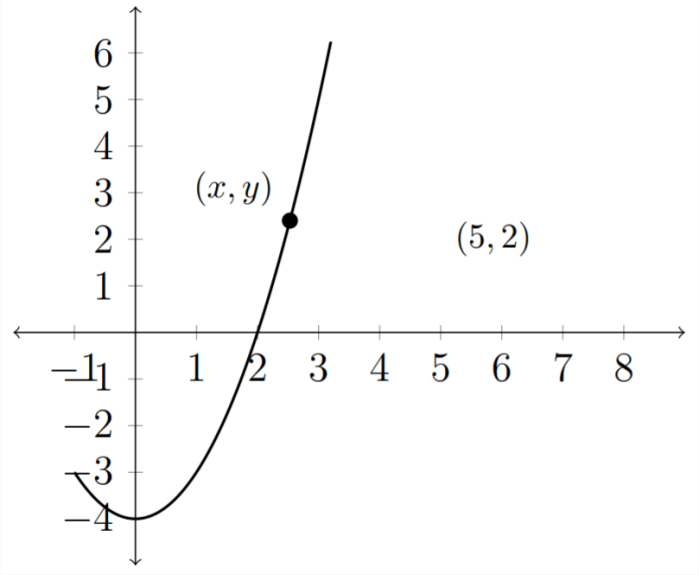
c) Distance Optimization - these problems generally use two objects travelling at right angles to each other and determine the maximum or minimum distance between the objects.

d) Distance/cost Optimization - these problems are usually focused on a situation in which two distances at right angles can be cut with a diagonal at a certain point to minimize cost or time.

Analytic Optimization

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{4.9.1}$$

1) Express the distance of a point (x, y) (in the first quadrant) on the graph of the parabola $y = x^2 - 4$ from the point (5,2) as a function of x
2) Use the graph of the distance function $d(x)$ from Part I to determine the point of the graph of the parabola $y = x^2 - 4$ that is closest to the point (5,2)
3) How far is this point from the point (5,2)?



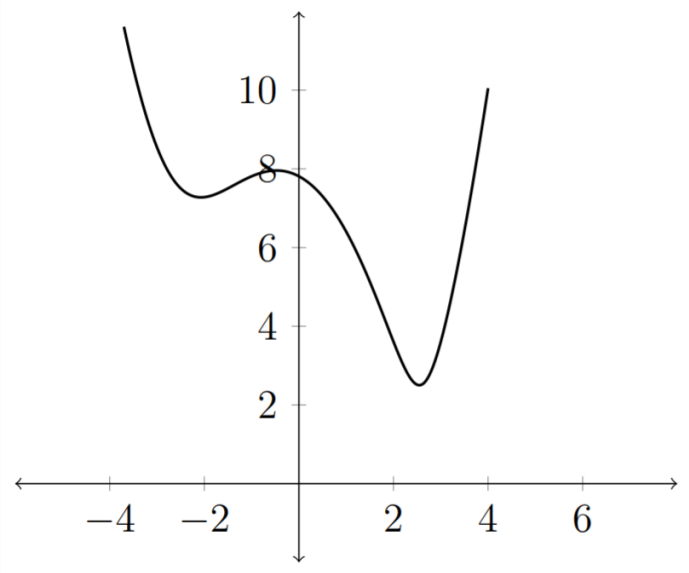
1) Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we can say that any point on the curve $y = x^2 - 4$ is a distance of:

$$d = \sqrt{(x-5)^2 + (y-2)^2} \tag{4.9.2}$$

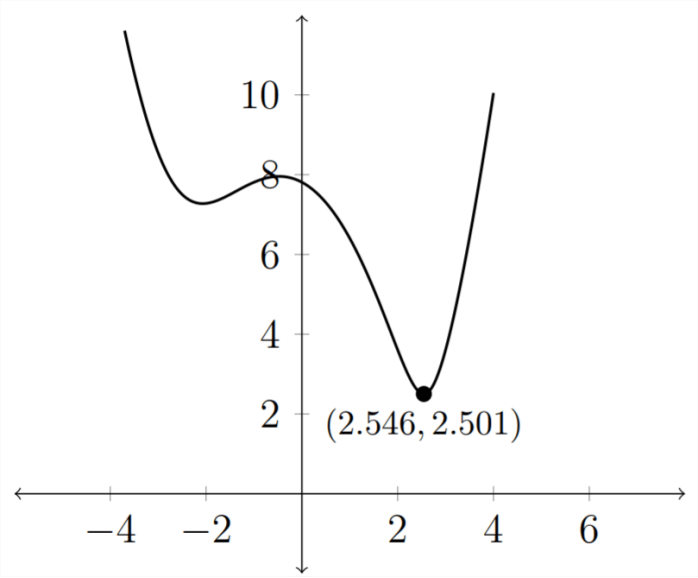
from the point $(5, 2)$. Because the question asks for this to be expressed as a function of x, we should replace the y variable with an equivalent expression involving x

$$d = \sqrt{(x-5)^2 + (x^2-4-2)^2}$$
$$\text{or} \tag{4.9.3}$$
$$d = \sqrt{(x-5)^2 + (x^2-6)^2}$$

2) The graph of this distance function is below:



Note that in this graph, the x axis represents the x values from the original graph, but the y axis in this graph is the distance of a point on the original curve from the point $(5, 2)$. We can see that this distance function has a clear minimum value that occurs between $x = 2$ and $x = 4$. We can use the graphing calculator to find this value - in Calculus we would use algebra to find this value.



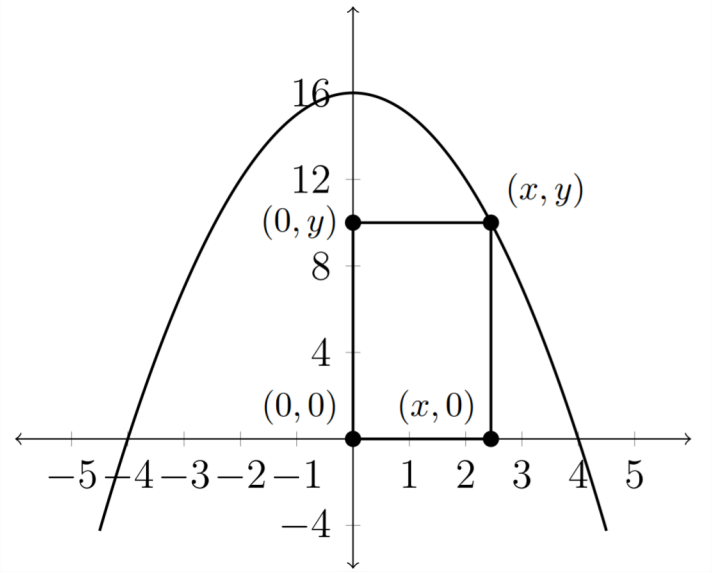
Both values indicated in the graph are approximate. The value $x \approx 2.546$ indicates the x value of the point on the original curve that is closest to the point (5,2) The $y \approx 2.501$ indicates the distance of the point from the point (5,2)
since the x value of the point on the original curve that is closest to the point (5,2) is about 2.546, that means that the y value is:

$$y = x^2 - 4 \approx 2.546^2 - 4 \approx 6.482 - 4 \approx 2.482 \tag{4.9.4}$$

So the answer for Part 2 is that (2.546,2.482) is closest point on the curve $y = x^2 - 4$ to the point (5,2)
3) We already determined the distance when we found the minimum point on the distance graph. The distance is about 2.501 units.

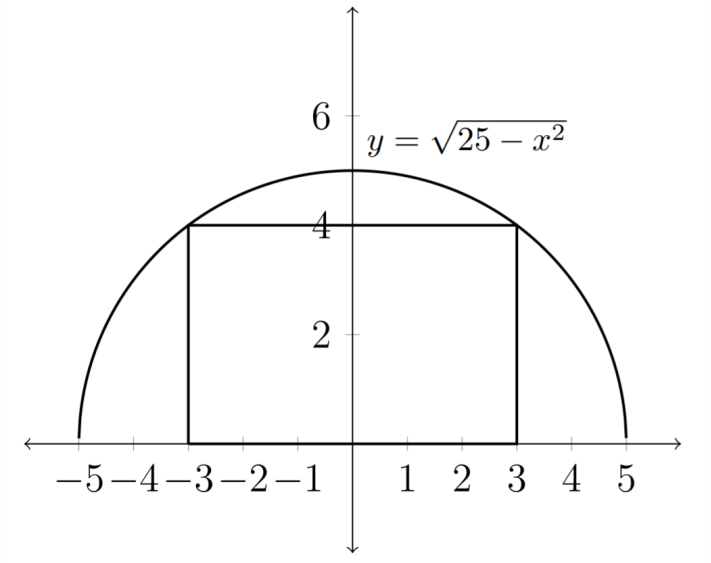
Exercises $4.9(a)$
1) Given the function $f(x) = 4 - x^2$, find x and y coordinates of the point on the curve that is closest to the point $(7, 3)$. How far away from the point (7,3) is the point you found?
2) Given the function $f(x) = 6 - x^2$, find x and y coordinates of the point on the curve that is closest to the point $(1, 1)$. How far away from the point (1,1) is the point you found?

3) Given the function $f(x) = x^2 + 2x - 5$, find x and y coordinates of the point
on the curve that is closest to the point $(-2, 5)$. How far away from the point (-2,5) is the point you found?
4) Given the function $f(x) = x^2 - 5x + 8$, find x and y coordinates of the point on the curve that is closest to the point $(-2, 2)$. How far away from the point (-2,2) is the point you found?
5) Given the function $f(x) = x^3 - 4x^2 + 2x + 1$, find x and y coordinates of the point on the curve that is closest to the point $(2, 1)$. How far away from the point
(2,1) is the point you found?
6) Given the function $f(x) = 2x^3 - 5x - 1$, find x and y coordinates of the point
on the curve that is closest to the point $(-2, 8)$. How far away from the point (-2,8) is the point you found?
7) A rectangle has one corner in the 1 st quadrant on the graph of $y = 16 - x^2$ one corner at the origin and the other two corners on the positive x axis and the positive y axis.



a) Express the area A of the rectangle as a function of x.
b) For what value of x is the area largest?
c) What is the maximum area?

8) A rectangle is inscribed in a semi-circle of radius 5 so that two corners are on the x axis and two corners are on the semi-circle.



The formula for the graph of the semi-circle is $y = \sqrt{25 - x^2}$
a) Express the area A of the rectangle as a function of x.
b) For what value of x is the area largest?

c) What is the maximum area?

Geometry/cost Optimization
A closed box with a square base is built from material that costs $\$1$ per ft^2 for the four sides and the bottom and $\$5$ per ft^2 for the top. What are the dimensions of the box of largest volume that can be constructed for $\$72$?

There are two issues we are working with in this problem - the volume of the box and the surface area of the box. The volume of the box is important because that's what we're looking to maximize. The surface area of the box is important because that's what will control the cost of the box - notice that the costs are given in terms of ft^2 or square feet which is related to the surface area.

The box has a square base, but the height is some other dimension - h. So the volume of the box will be length $*$ width $*$ height, but since the box has a square base the length and width will be the same - we'll call them x. From this we see that the volume can be expressed as:

$$V = x * x * h = x^2 h \tag{4.9.5}$$

For the cost of the box, we need to know the surface area. Each side of the box should be included - the base and the top have the same area (x^2) because they are both squares that are x units on each side. The four sides of the box are all rectangles that are x by h units. That means each one has an area of $x * h -$ so all four would be $4 * x * h$ or $4xh$
This makes the surface area of the box:

$$S = 2x^2 + 4xh \tag{4.9.6}$$

The cost of the box then will be:

$$C = x^2(\$1) + x^2(\$5) + 4xh(\$1) \tag{4.9.7}$$

Here we see the one x^2 is multiplied by $\$1$, because the bottom will cost $\$1$ per ft^2, but the other x^2 is multiplied by $\$5$, because the top costs $\$5$ per ft^2. The four sides : $4xh$ is also multiplied by $1.

So, our final formula for the cost is:

$$C = 6x^2 + 4xh \tag{4.9.8}$$

but we already know that we want to spend $\$72$ on the box, so we can say that:

$$72 = 6x^2 + 4xh \tag{4.9.9}$$

This means that:

$$\begin{gathered} \frac{72-6x^2}{4x} = h \\ \text{or} \\ \frac{18}{x} - 1.5x = h \end{gathered} \tag{4.9.10}$$

The reason why this is important is that it will allow us to express the volume of the box in terms of x. Remember that the volume was:

$$V = x^2 h \tag{4.9.11}$$

so now we can see that:

$$\begin{gathered} V = x^2\left(\frac{18}{x} - 1.5x\right) \\ \text{or} \\ V = 18x - 1.5x^3 \end{gathered} \tag{4.9.12}$$

The graph for this function is below:

We can see in the graph that the maximum volume given a cost of $\$72$ will be when the base of the box is 2 feet by 2 feet. The volume will be 24ft^3 and the height will be 6 feet. So the answer to the question is $2\text{ft} \times 2\text{ft} \times 6\text{ft}$

Exercises 4.9**(b)**

1) A rectangular storage container with an open top has a volume of 10m^3. The length of its base is twice the width. Material for the base costs $\$10$ per square meter. The sides require material that costs $\$6$ per square meter. Find the cost of materials for the cheapest such container.

2) A poster is to contain 108cm^2 of printed matter, with margins of 6cm each at the top and bottom of the page and 2cm on each side of the page. What is the minimum cost of the poster if it is to be made of material which costs $\$0.20$ per square centimeter?

3) If 1200cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

4) A box with a square base and open top must have a volume of $32,000\text{cm}^3$ Find the dimensions of the box that minimizes the amount of material used.

5) A Norman window has the shape of a semi-circle on top of a rectangle so that the diameter of the semi-circle is the same as the width of the rectangle. Given that the perimeter of the window should be 30ft, find the dimensions of the window that will admit the greatest amount of sunlight (maximize the area).

6) A rectangular box with no top is to be made from material costing $\$0.50$ per square foot. The width of the base is 5 ft and it will have a volume of 6ft^3. What are the dimensions of the box that minimizes the cost? What is the cost of this box?

Surface Area of a cylinder $= 2\pi r^2 + 2\pi rh$

Volume of a cylinder $= \pi r^2 h$

7) A cylindrical can with a top will be made from 200 in ^{2} of tin. What would be the height and radius of the can of maximum volume?

8) A cylindrical can with a top will have a volume of 500cm^3. If the sides and bottom cost $\$0.002$ per square cm and the top costs $\$0.0035$ per square cm, find the dimensions of the least expensive can that will have a volume of 500cm^3 How much does it cost to make this can?

Distance Optimization

One ship is 10 miles due east of a buoy and is sailing due west, towards the buoy at 12 mph. Another ship is 10 miles due south of the same buoy and sailing due north, also towards the buoy at 7 mph.

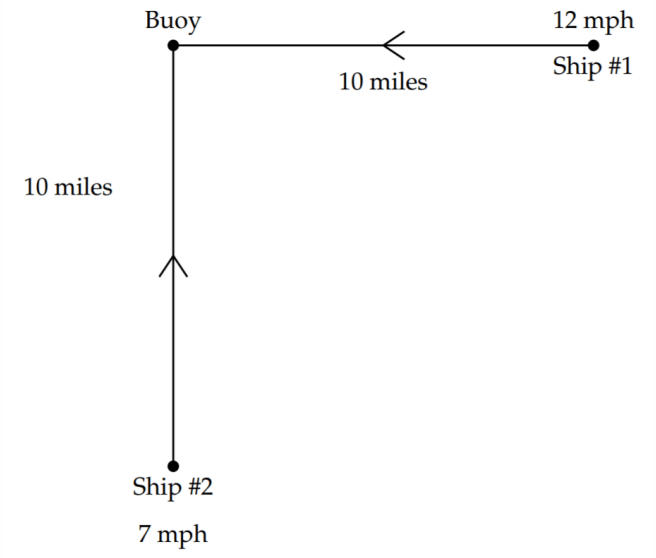
a) Write a function that represents the distance between the two ships in terms of t, the elapsed time in hours.

b) Graph the function and determine the value of t when the ships are closest together. How far apart are the ships at this time?

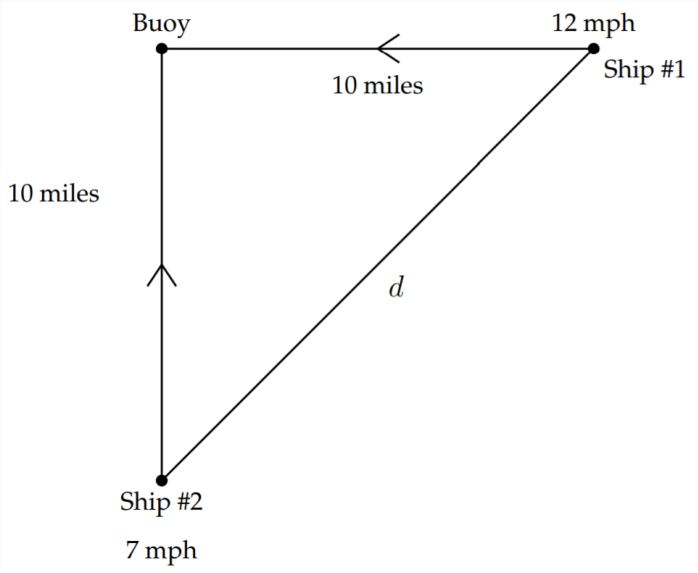
Round the value of t to the nearest 100 th of an hour.

Round the distance to the nearest tenth of a mile.

Here is a diagram that represents the problem:



The distance between the two ships is measured along the diagonal. Because this creates a right triangle, we can use the Pythagorean Theorem to represent the distance between the ships.



In this case the legs of the right triangle begin as 10 miles, but they get shorter as the ships travel closer to the buoy. For Ship #1, the distance decreases by 12 miles every hour - this means that the first ship will pass the buoy in less than one hour. The distance between Ship #1 and the buoy can be represented as $(10 - 12t)$ where t is the number of hours spent traveling. Similarly, the distance between Ship #2 and the buoy can be represented as $(10 - 7t)$

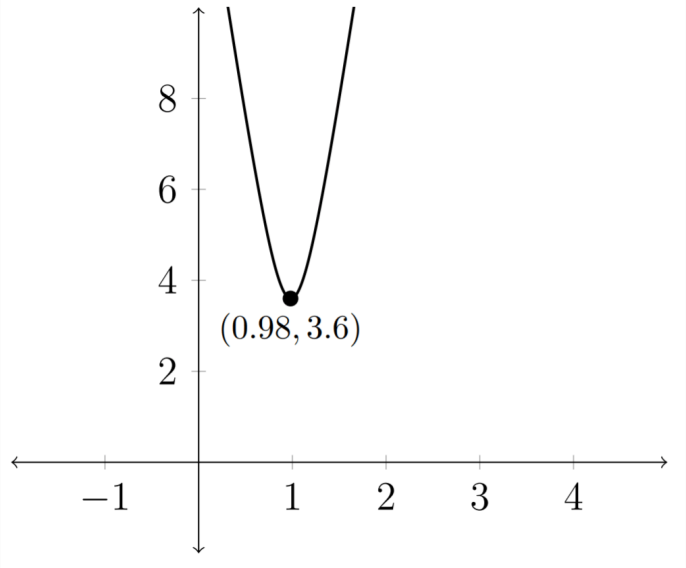
So, using the Pythagorean Theorem, we can represent the distance between the ships at any given time as:

$$d^2 = (10 - 7t)^2 + (10 - 12t)^2 \tag{4.9.13}$$

or

$$d = \sqrt{(10 - 7t)^2 + (10 - 12t)^2} \tag{4.9.14}$$

The graph of this distance function looks like this:



The minimum point on the distance graph indicates the time at which the ships are closest together. After traveling for 0.98 hours, the ships will be about 3.6 miles apart. They will then start moving farther apart.

Exercises $4.9(c)$

1) A runner starts running north from a given point running at 5 meters per second. At the same time, a second runner heads east from the same starting point, running at 8 meters per second. Represent the distance between the runners as a function of t, the

elapsed time. How long will it take for the runners to be 150 meters apart?

2) A runner starts running north from a given point running at 10 meters per second. At the same time, a second runner heads east from the same starting point, running at 7 meters per second. Represent the distance between the runners as a function of t, the elapsed time. How long will it take for the runners to be 200 meters apart?

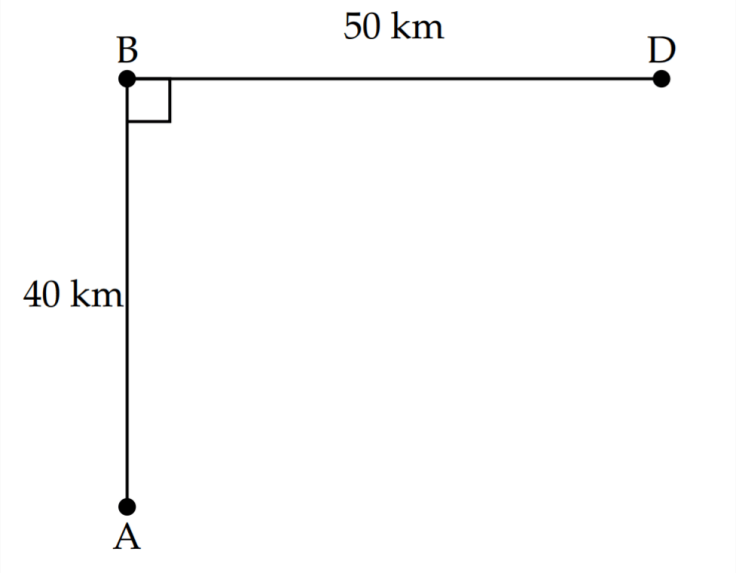
3) At 1pm a bicycle is 9 miles due north of an ice cream shop and is traveling south (towards the ice cream shop) at 15 mph. At the same time, another bike rider is 2 miles east of the ice cream shop and is traveling east (away from the ice cream shop) at 12 mph. Represent the distance between the bike riders at any given time after 1pm. At what time is the distance between the riders at a minimum? How far apart are they at this point?

4) A car leaves a four-way intersection at 9: 15 am, traveling south at 70 mph. At the same time, another car is 65 miles west of the intersection traveling east at 55 mph. Express the distance between the two cars as a function of the time since they left the intersection. At what time are the cars closest together? How far apart from each other are they at this time?

At Noon, a dune buggy is in the desert at the point A indicated below. Point A is 40km from a paved road, which covers the 50km between points B and D

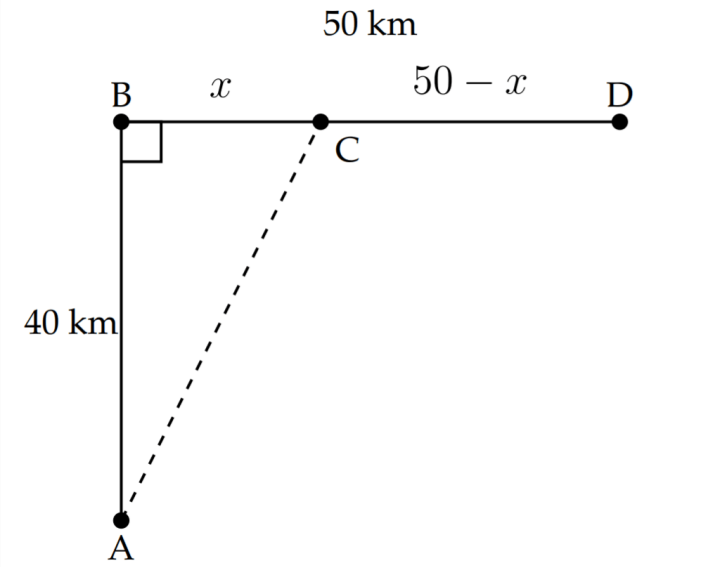
If the dune buggy can travel 45 kph in the desert and 75 kph on the road, where is the best place for the driver to leave the desert and turn onto the road in order to minimize the travel time?

What time does the driver arrive at Point D assuming they follow the path for minimal travel time?



One option is that the dune buggy can travel in a straight line from A to B (at 45 kph) and then travel on the paved road from B to D (at 75kph). This would take 1.5hrs
Another option is for the dune buggy to travel straight from A to D at 45 kph. The distance here is about 64km (found using the Pythagorean Theorem). That means the travel time would be about 1.423 hrs.

The optimization here says that, if the dune buggy were to travel part of the way in the desert (along a diagonal) and the rest of the way on the paved road, then there might be a minimal time that is shorter than the 1.423 hrs. it would take going straight from A to D



If we express the distance between B and C as x, then the distance between C and D will be $50 - x$. The distance travelled in the desert is the distance from A to C and this is found using the Pythagorean Theorem.

$$\overline{AC} = \sqrt{40^2 + x^2} = \sqrt{x^2 + 1600} \tag{4.9.15}$$

since $d = r * t$, then the time it takes travel a given distance will be $\frac{d}{r} = t$
In the case of this problem the time to travel from A to C is:

$$t_1 = \frac{\sqrt{x^2 + 1600}}{45} \tag{4.9.16}$$

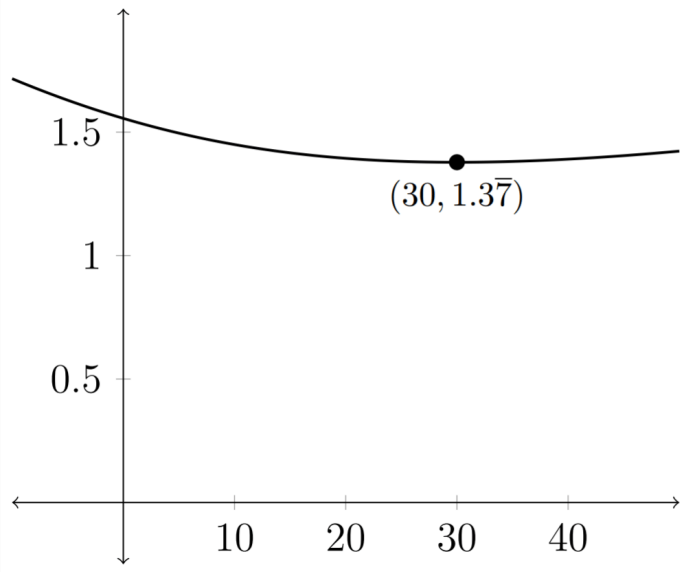
The time to travel from C to D is:

$$t_2 = \frac{50 - x}{75} \tag{4.9.17}$$

So, the total travel time will be:

$$t = t_1 + t_2 = \frac{x^2 + 1600}{45} + \frac{50 - x}{75} \tag{4.9.18}$$

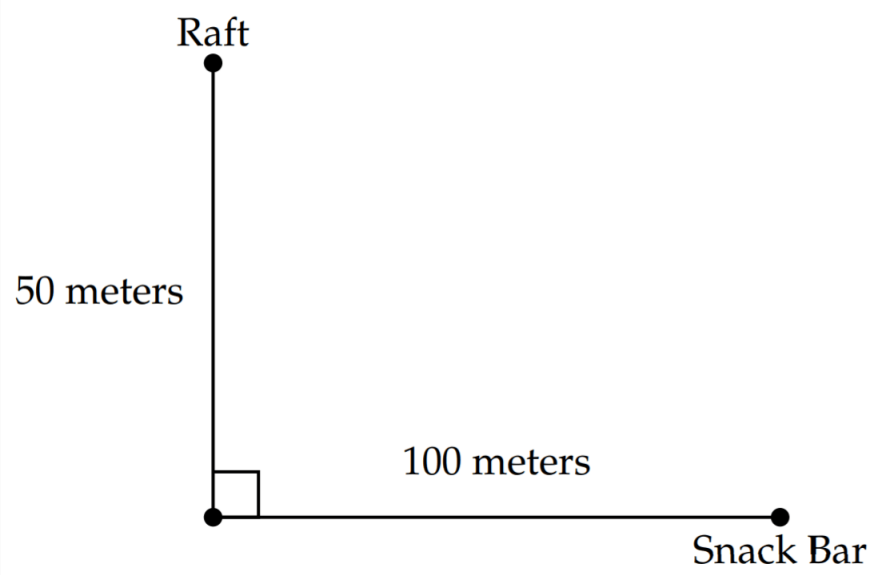
The graph of this function is below:



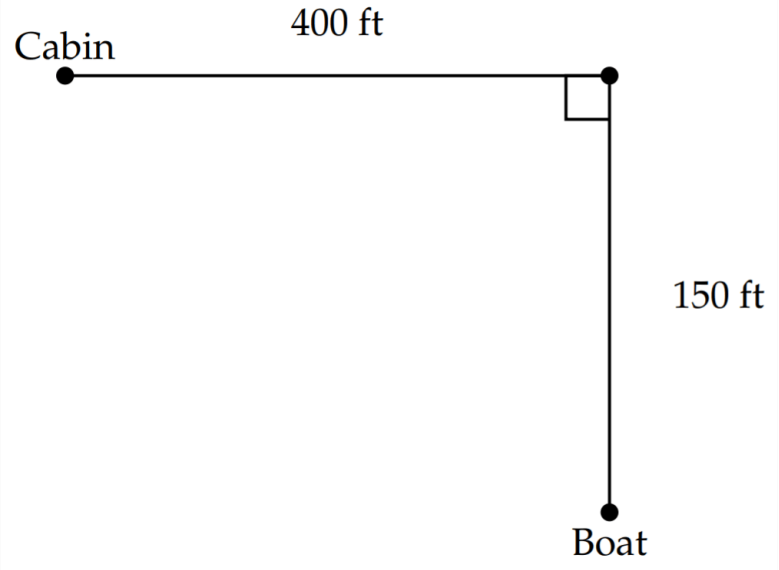
It is hard to see, but there is a minimum for the graph where $x = 30$ and $t = 1.3\overline{7}$

Exercises $4.9(d)$

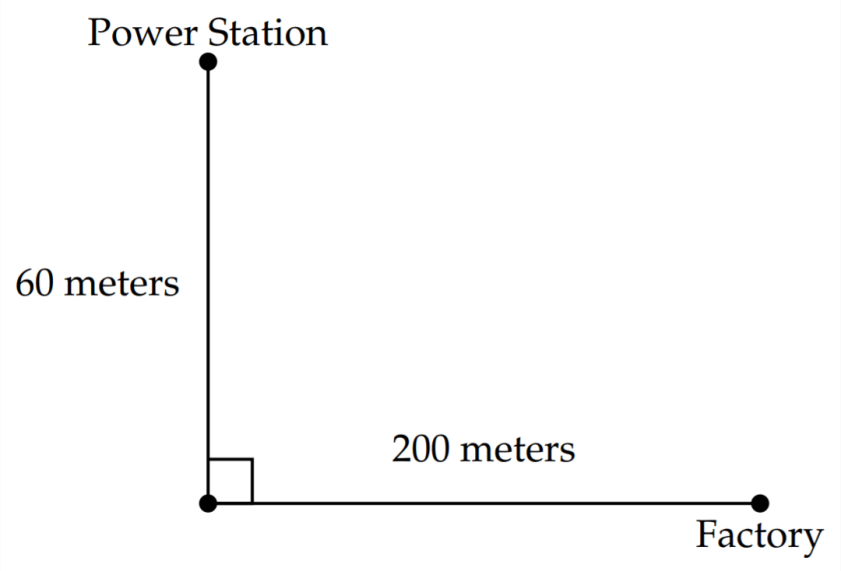
1) A woman wants to swim to shore from a raft that is 50 meters offshore and then run to the snack bar located 100 meters down the shoreline. The woman swims 1 meter per second and runs at 5 meters per second. Where on the shoreline should she swim to minimize her travel time?



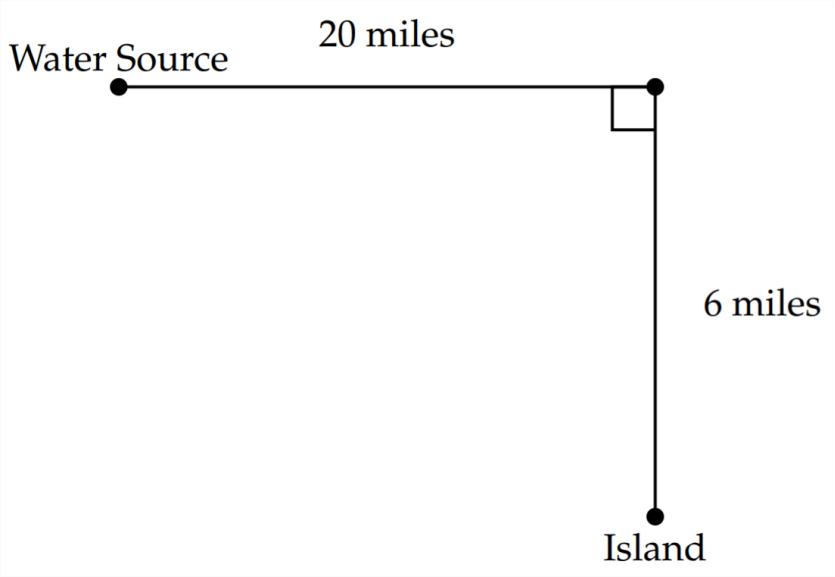
2) A man is in a boat on a lake 150ft from shore. He wants to return to his cabin. He is 400ft down the shoreline from the cabin. He plans to row to shore, leave the boat where he lands and then run the rest of the way to the cabin. If he rows 7 ft per second and runs 15 ft per second, where should he beach the rowboat to minimize the travel time?



3) A power station and a factory are located on opposite sides of a river that is
60 meters wide. The factory is 200 meters from the point on the shore directly opposite the power station. A power line must be laid between the power station and the factory. It costs $\$25$ per meter to run the cable in the river and $\$20$ per meter to run the cable on the land. How much of the cable should be laid in the river and how much should be laid on land to minimize the cost? Where should the cable come ashore? What is the minimum cost?



4) A freshwater pipeline is being constructed to an island that is 6 miles from the mainland. The water source is 20 miles down the coast from the point on the shore that is closest to the island. The cost of laying the pipeline along the shore is $\$8,000$ per mile while the cost of laying the pipeline in the water is $\$12,000$ per mile. How much of the pipeline should be laid in the water and how much should be laid on land to minimize the cost? Where should the pipeline leave the shore? What is the minimum cost?



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5: CONIC SECTIONS - CIRCLE AND PARABOLA

The conic sections as originally conceived in ancient Greece were "slices" of two cones tip-to-tip. Slicing this figure in different ways produces each of the four conic sections - Circle, Ellipse, Parabola and Hyperbola. This chapter will examine the Circle and the Parabola.

5.1: The Equation of the Circle

5.2: The Equation of the Parabola

5.3: Applications of the Parabola

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5.1: The Equation of the Circle

The equation for a circle is typically given as:

$$(x-h)^2+(y-k)^2=r^2 \tag{5.1.1}$$

In this equation, the point (h,k) represents the center of the circle and r represents the radius of the circle. This equation is derived from the distance formula. The definition of a circle is the locus (or collection) of points that are equidistant from a given point (the center of the circle).

So, taking this definition, we can say that any point (x,y) that is on the circle should be a distance of r from the center. Using the distance formula:

$$d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \tag{5.1.2}$$

Squaring both sides:

$$d^2=(x_1-x_2)^2+(y_1-y_2)^2 \tag{5.1.3}$$

In this situation, the distance d is the radius of the circle, r. This distance should be the same for all points on the circle. So any point on the edge of the circle, (x,y) should be a distance of r from the center of the circle, (h,k)

$$(x-h)^2+(y-k)^2=r^2 \tag{5.1.4}$$

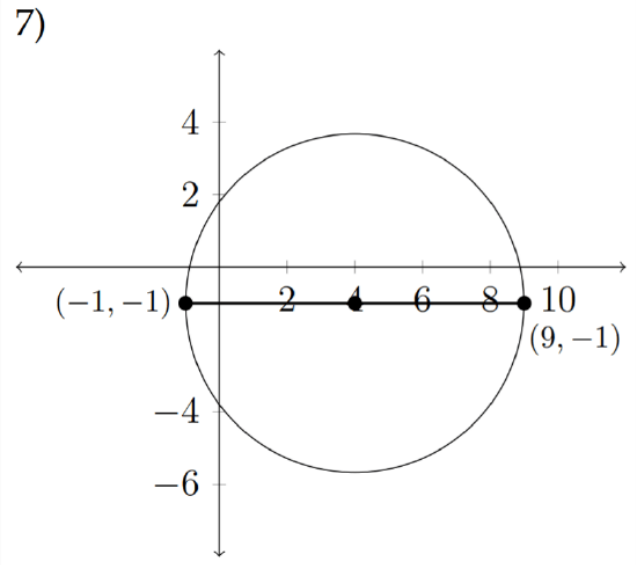
Exercises 5.1.1

Determine the center and radius for each circle and sketch the graph of the circle.

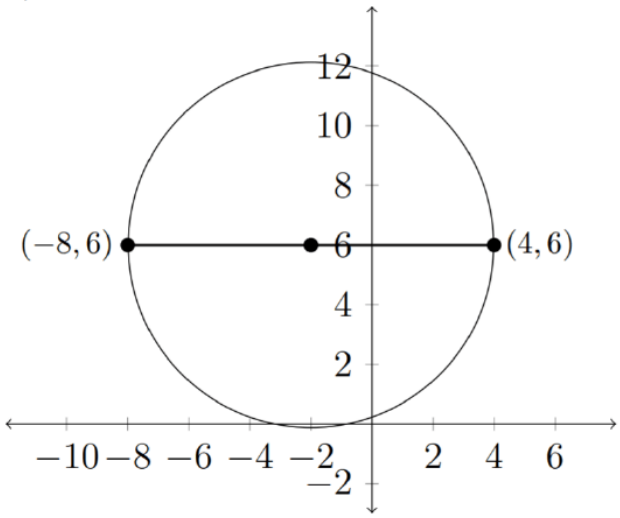
1) $(x-3)^2+(y+1)^2=9$
2) $(x+4)^2+(y+2)^2=16$
3) $(x-1)^2+(y-6)^2=20$
4) $(x-2)^2+(y+5)^2=27$
5) $(x+7)^2+(y-4)^2=33$
6) $(x-5)^2+(y-3)^2=50$

Determine the equation for each circle. Each line represents a diameter

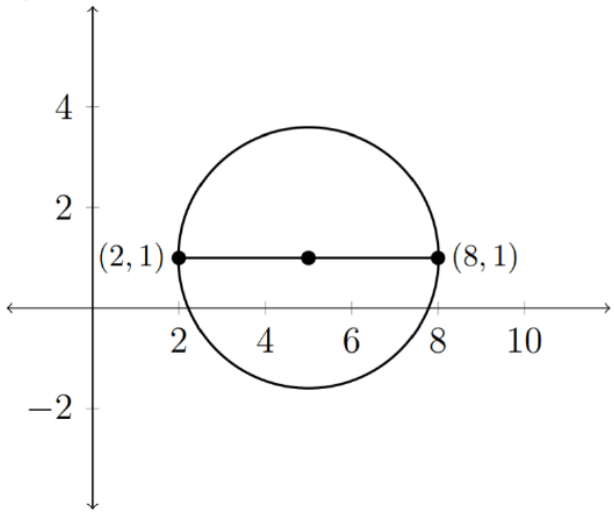
7)



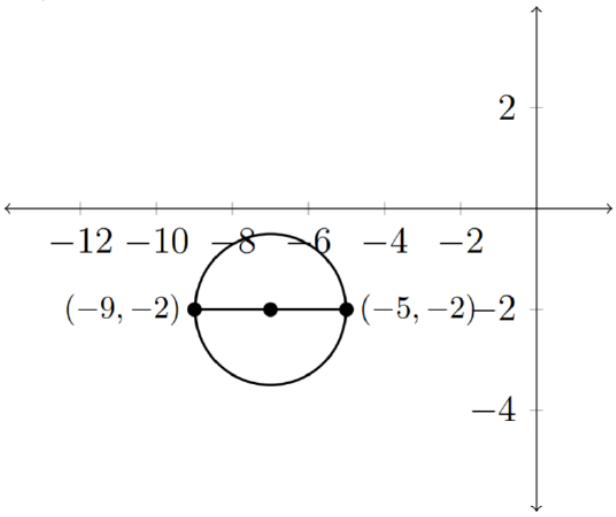
8)


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9)


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(8, 1)
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−2

10)


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−12 −10 −8 −6 −4 −2
(−9, −2)
(−5, −2)
−2
−4

Sometimes the equation for a circle is not given in the standard form. In this situation, you need to put the equation into standard form and then determine the center and radius. In order to put the equation into standard form you will need to complete the square. Completing the square is a mathematical technique that is often useful and is the basis for how the quadratic formula is derived.

Suppose that we are given the equation of a circle that is not in standard form:

$$x^2 + 6x + y^2 + 10y + 25 = 0 \tag{5.1.5}$$

We need to restate this relationship so that the center and radius can be easily determined from the equation. In order to do this, we need to complete the square for both the x and the y variables. There are a variety of methods for completing the square I will demonstrate one of these below.

Take the original equation and move any term that doesn't have a variable to the other side:

$$x^2 + 6x + y^2 + 10y = -25 \tag{5.1.6}$$

Then open a space after the $6x$ and the $10y$:

$$x^2 + 6x + y^2 + 10y = -25 \tag{5.1.7}$$

The idea is that we want trinomial expressions for both x and y that are perfect squares.
If we look at squaring binomial expressions, we can see that there is a pattern:

$$\begin{aligned}(x+3)^2 &= (x+3)(x+3) = x^2 + 6x + 9\\ (x+4)^2 &= (x+4)(x+4) = x^2 + 8x + 16\\ (x+5)^2 &= (x+5)(x+5) = x^2 + 10x + 25\\ (x+6)^2 &= (x+6)(x+6) = x^2 + 12x + 36\end{aligned} \tag{5.1.8}$$

We can see that if we have $x^2 + 6x$, then this would correspond to $(x+3)(x+3)$ or $(x+3)^2$. But there's a problem: $(x+3)^2$ is not equal to $x^2 + 6x$. It's equal to $x^2 + 6x + 9$. We can't just add a 9 to the $x^2 + 6x$, but we can add a 9 to both sides.

$$x^2 + 6x + 9 + y^2 + 10y = -25 + 9 \tag{5.1.9}$$

Similarly, to complete the square on the y, we see that $y^2 + 10y$ corresponds to $(x+5)(x+5)$ or $(x+5)^2$. Here, we would need to add 25 to both sides to create a perfect square.

$$\begin{aligned}x^2 + 6x + 9 + y^2 + 10y + 25 &= -25 + 9 + 25\\ x^2 + 6x + 9 + y^2 + 10y + 25 &= 9\\ (x+3)^2 + (y+5)^2 &= 9\end{aligned} \tag{5.1.10}$$

So, the center of the circle is (-3,-5) and the radius is 3
Sometimes it is not so obvious what the values of h and k should be in completing the square. Consider the equation below:

$$x^2 + 20x + y^2 + 30y + 15 = 0 \tag{5.1.11}$$

If we look back at the examples for squaring binomials, we can see the pattern that relates the coefficient of the linear term to the values for h and k

$$\begin{aligned}(x+3)^2 &= (x+3)(x+3) = x^2 + 6x + 9\\ (x+4)^2 &= (x+4)(x+4) = x^2 + 8x + 16\\ (x+5)^2 &= (x+5)(x+5) = x^2 + 10x + 25\\ (x+6)^2 &= (x+6)(x+6) = x^2 + 12x + 36\end{aligned} \tag{5.1.12}$$

Notice that the coefficient of the linear term is always double the value of the numeral in the parentheses and the constant term is always that number squared.

$$(x+n)^2 = (x+n)(x+n) = x^2 + 2nx + n^2 \tag{5.1.13}$$

So, given the problem above, $x^2 + 20x + y^2 + 30y + 15 = 0$, first we can move the
15 to the right hand side of the equation and then complete the squares on the x and y variables.

$$\begin{gathered} x^2+20x+y^2+30y=-15 \\ (x+?)^2+(y+?)^2 \\ x^2+20x+100+y^2+30y+225=-15+100+225 \\ (x+10)^2+(y+15)^2=310 \end{gathered} \tag{5.1.14}$$

We can now see that the center of the circle is at the point (-10,-15) and the radius of the circle is $\sqrt{310} \approx 17.6$

Exercises 5.1.2

Find the center and radius for each circle.
If the equation is not written in standard form, complete the square to rewrite the equation in standard form: $(x-h)^2+(y-k)^2=r^2$

1. $x^2+y^2=4$
2. 2) $x^2+(y-1)^2=1$
3. 3) $2(x-3)^2+2y^2=8$
4. 4) $3(x+1)^2+3(y-1)^2=6$
5. 5) $x^2+y^2-2x-4y-4=0$
6. 6) $x^2+y^2+4x+2y-20=0$
7. 7) $x^2+y^2+4x-4y-1=0$
8. 8) $x^2+y^2-6x+2y+9=0$
9. 9) $2x^2+2y^2+8x+7=0$
10. 10) $3x^2+3y^2-12y=0$
11. 11) $2x^2+2y^2-12x+8y-24=0$
12. 12) $2x^2+8x+2y^2=0$
13. 13) $x^2+y^2+15x=0$
14. 14) $x^2+y^2+x+y-\frac{1}{2}=0$
15. 15) $x^2+y^2-x+2y+1=0$
16. 16) $3x^2+3y^2-6x+3y=4$
17. 17) $2x^2+2y^2-10x-18y=1$
18. 18) $2x^2+2y^2-5x+y=0$
19. 19) $9x^2+9y^2+9x-27y+1=0$
20. 20) $2x^2+2y^2-7x+y=0$

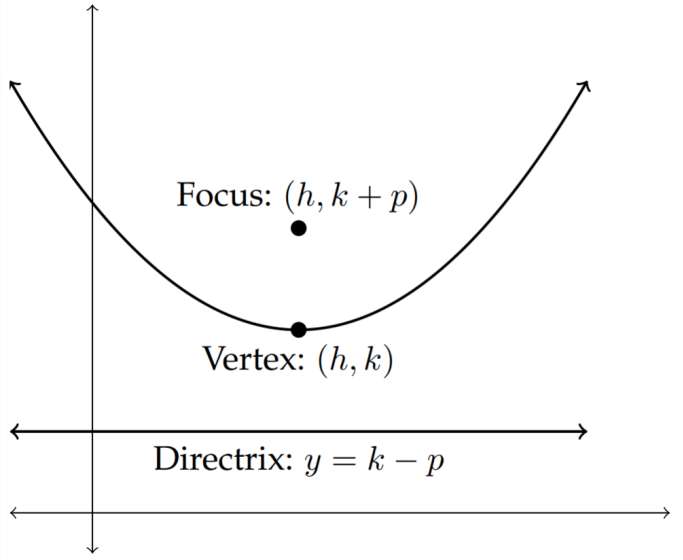
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5.2: The Equation of the Parabola

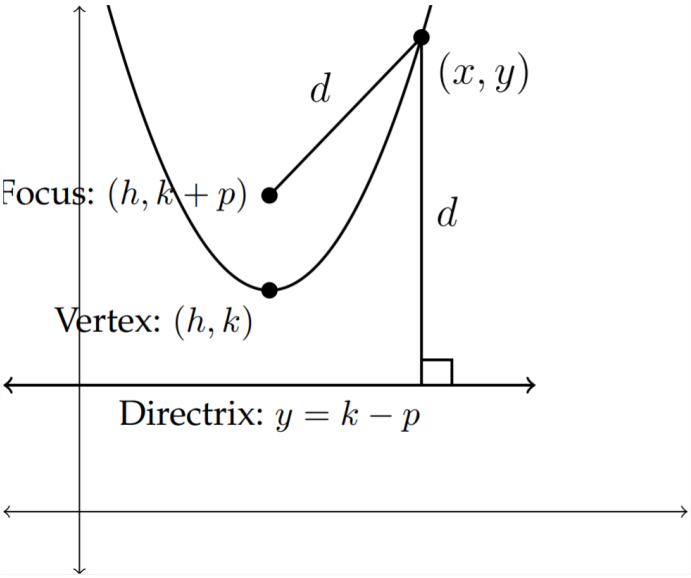
The equation of the parabola is often given in a number of different forms. One of the simplest of these forms is:

$$(x-h)^2 = 4p(y-k) \tag{5.2.1}$$

A parabola is defined as the locus (or collection) of points equidistant from a given point (the focus) and a given line (the directrix). Another important point is the vertex or turning point of the parabola. If the equation of a parabola is given in standard form then the vertex will be (h,k). The focus will be a distance of p units from the vertex within the curve of the parabola and the directrix will be a distance of p units from the vertex outside the curve of the parabola. This value (p) is called the focal distance.



Any point on the curve of the parabola is equidistant from the focus $(h, k+p)$ and the directrix $(h, k-p)$. Notice that the focus is a point and is identified with the coordinates of the point while the directrix is a line and is identified with the equation for that line.



We can derive the standard equation for a parabola using the distance formula.

In the picture above the two distances labeled " d'' should be the same distance. The vertical distance between the point (x,y) and the directrix $y = k-p$ is simply the difference between their y coordinates:

$$d = y - (k - p) \tag{5.2.2}$$

To find the distance between the point (x, y) and the focus $(h, k + p)$ we need to use the distance formula:

$$d = \sqrt{(x - h)^2 + (y - (k + p))^2} \tag{5.2.3}$$

Then we set the two distances equal to each other:

$$\sqrt{(x - h)^2 + (y - (k + p))^2} = y - (k - p) \tag{5.2.4}$$

Square both sides:

$$(x - h)^2 + (y - (k + p))^2 = (y - (k - p))^2 \tag{5.2.5}$$

We'll need to expand each side and collect like terms, but we'll leave the $(x - h)^2$ alone because it will appear in this form in the final equation.

$$(x - h)^2 + y^2 - 2(k + p)y + (k + p)^2 = y^2 - 2(k - p)y + (k - p)^2 \tag{5.2.6}$$

The y^2 terms cancel each other out:

$$(x - h)^2 - 2(k + p)y + (k + p)^2 = -2(k - p)y + (k - p)^2 \tag{5.2.7}$$

Now we'll expand the k, p and y terms:

$$(x - h)^2 - 2ky - 2py + k^2 + 2pk + p^2 = -2ky + 2py + k^2 - 2pk + p^2 \tag{5.2.8}$$

Here, the k^2 and p^2 terms cancel out, as do the $-2ky$ terms leaving:

$$(x - h)^2 - 2py + 2pk = 2py - 2pk \tag{5.2.9}$$

If we collect everything excep the $(x - h)^2$ on the right hand side, we'll have:

$$(x - h)^2 = 4py - 4pk \tag{5.2.10}$$

Factor out $4p$ and we have the standard equation for a parabola:

$$(x - h)^2 = 4p(y - k) \tag{5.2.11}$$

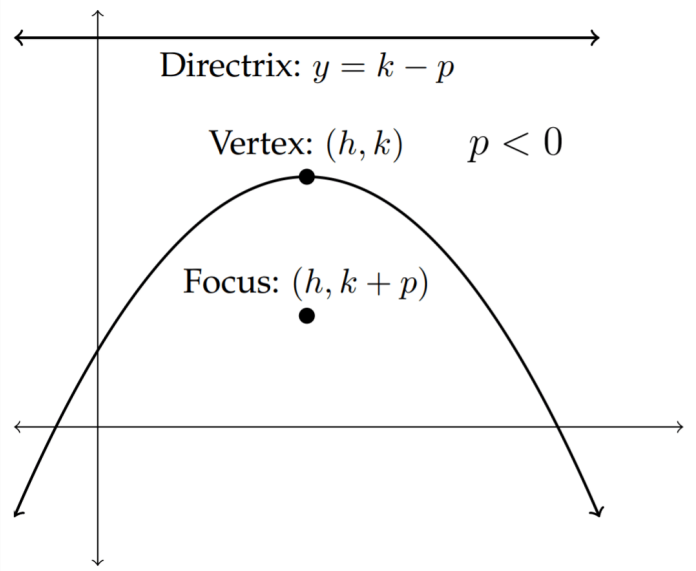
This equation will be different depending on the orientation of the parabola. An upward facing parabola will have this standard equation and both sides will have the same sign. For example, $(x - 5)^2 = 12(y - 1)$ is an upward facing parabola, as is $-(x - 5)^2 = -12(y - 1)$. You can see that this is the same equation, but has been multiplied by -1 on both sides.
A parabola in which one side is positive and one side negative, like

$$(x + 1)^2 = -8(y - 10) \tag{5.2.12}$$

is a downward facing parabola. This form might also appear as

$$-(x + 1)^2 = 8(y - 10) \tag{5.2.13}$$

The picture below illustrates this situation:

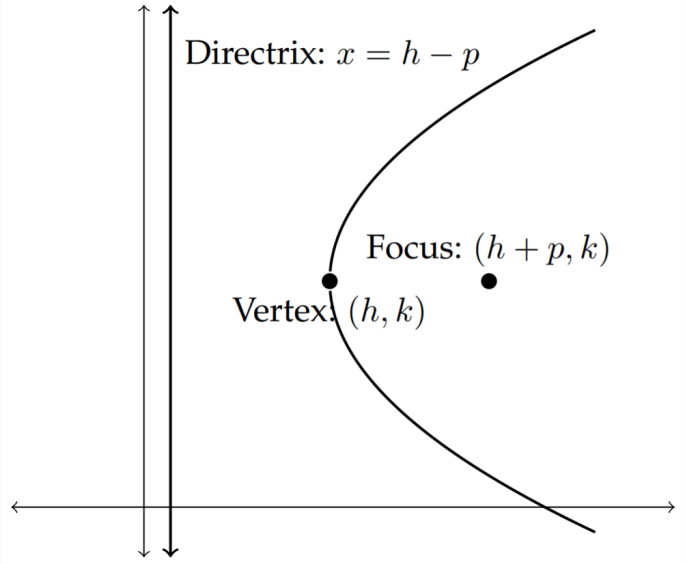


Notice that, in this case, the focus is below the vertex and the directrix is above it. Remember that the focus is always in the interior of the curve, while the directrix is always outside of the curve.

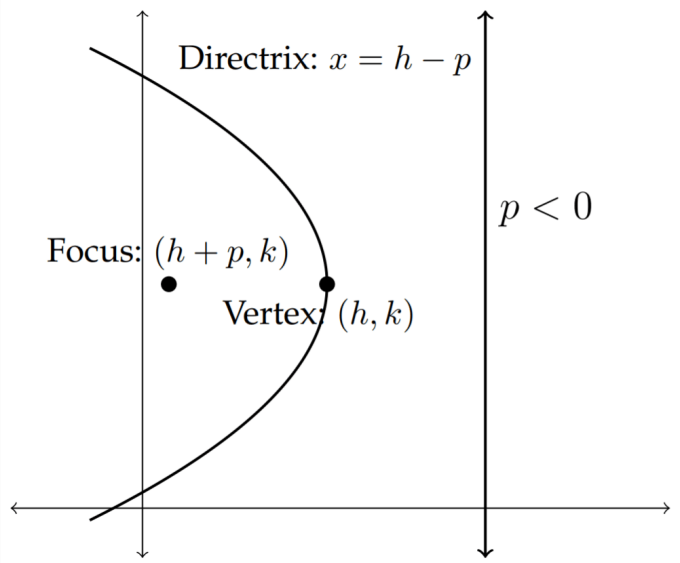
Parabolas can also open to the right or left. In these cases, it is the y variable that is squared in the standard equation:

$$(y - k)^2 = 4p(x - h) \tag{5.2.14}$$

In these situations, the focus is to the left or right of the vertex. If both sides of the equation have the same sign, such as $(y - 5)^2 = 12(x - 1)$ or $-(y - 5)^2 = -12(x - 1)$ then this would be a parabola opening to the right.

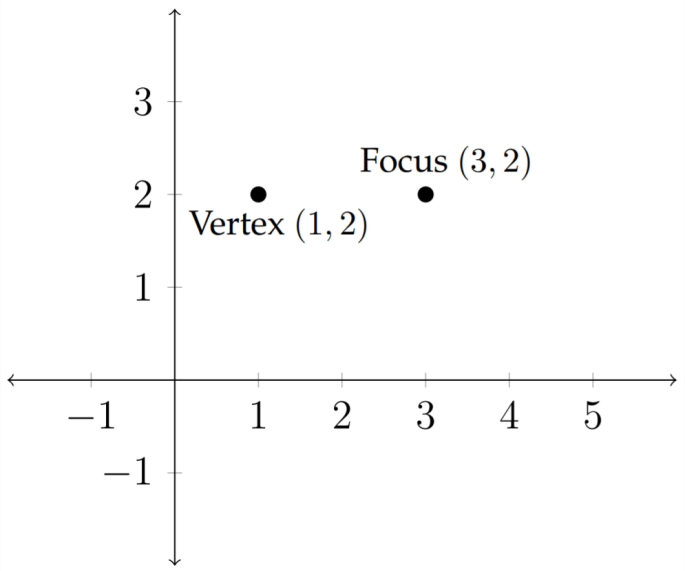


If the two sides have opposite signs, then the parabola will open to the left

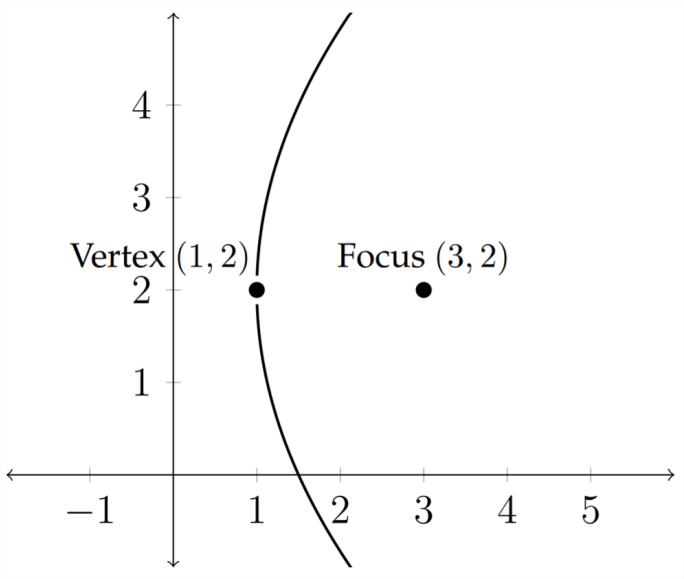


The orientation of the vertex, focus and directrix allows us to determine the equation of a parabola if we are given certain pieces of information about the vertex, focus and directrix. The vertex always falls halfway between the focus and directrix. The key pieces of information in determining the equation of a parabola are:
1) the vertex: this gives us the values for h and k for the equation.
2) the orientation: this allows us to determine the appropriate form of the equation
a) $(x-h)^2 = 4p(y-k)$
b) $(x-h)^2 = -4p(y-k)$
c) $(y-k)^2 = 4p(x-h)$
d) $(y-k)^2 = -4p(x-h)$

Given the following information determine an equation for the parabola described.
Focus at (3,2) Vertex at (1,2)
First draw a little sketch of the problem:



since the focus always falls within the interior of the parabola's curve, this parabola is facing to the right. Its directrix is the line $x = -1$



The equation for this parabola would be:

$$(y-2)^2 = 8(x-1) \tag{5.2.15}$$

Exercises $5.2(a)$
Determine the equation in standard form for each of the parabolas described below.
1) Focus: (2,5) & Vertex: (2,6)
2) Focus: (-4,3) & Vertex: (-3,3)
3) Focus: (1.5,0) & Vetrex: (0,0)
4) Focus: (-9,0) & Vertex: (-9,-0.5)
5) Focus: (5,1) & Directrix: x=12
6) Focus: (0,3) & Directrix: y=1
7) Focus: (1,-3) & Directrix: y=2
8) Focus: (-4,8) & Directrix: x=-6
9) Vertex: (1,3), axis of symmetry parallel to y -axis
contains the point (4,7.5)
10) Vertex: (4,2), axis of symmetry parallel to y -axis
contains the point (5,0)
11) Focus: (0,0) & Vertex: (0,-1.25)
12) Focus: (0,0) & Vertex: (0,2.5)
13) Focus: (2,7) & Directrix: x=0.5
14) Focus: (-3,0) & Directrix: x=-2
15) Vertex: (3,-2), axis of symmetry parallel to x -axis,
contains the point (7,3)
16) Vertex: (-6,-6), axis of symmetry parallel to y -axis,
contains the point (0,0)

If we are given the equation of a parabola and need to find the vertex, focus and directrix, it is often helpful to put the equation in standard form. This usually requires completing the square.
If we're given the equation:

$$x^2 + 6x - 4y + 1 = 0 \tag{5.2.16}$$

then we know that this will be either an upward or downward facing parabola. Let's move everything away from the x terms but leave a space to complete the
square:

$$x^2 + 6x = 4y - 1 \tag{5.2.17}$$

To complete the square on this, we need add 9 to both sides so that we can rewrite the left side as $(x+3)^2$

$$\begin{aligned} x^2+6x+9 &= 4y-1+9 \\ (x+3)^2 &= 4y+8 \end{aligned} \tag{5.2.18}$$

Then we can factor out the coefficient of the y variable (even if it doesn't factor evenly).

$$(x+3)^2 = 4(y+2) \tag{5.2.19}$$

So, this is an upward facing parabola with the vertex at the point (-3,-2) . To find the focus and directrix, we need to know the vlaue of p. since $4p=4$, then we know that $p=1$. This means that the focus will be 1 unit above the vertex at the point (-3,-1) and the directrix will be one unit below the vertex at the line y=-3.

$$\begin{aligned} \text{Vertex:} \quad & (-3,-2) \\ \text{Focus:} \quad & (-3,-1) \\ \text{Directrix:} \quad & (y=-3) \end{aligned} \tag{5.2.20}$$

Here's another example:
Express the equation in standard form and determine the vertex, focus and directrix of the following parabola.

$$y^2-2y-1+8x=0 \tag{5.2.21}$$

We know that this parabola will be either right or left facing.
First, let's move all the terms not containing y to the right-hand side but leave space to complete the square.

$$y^2-2y=-8x+1 \tag{5.2.22}$$

Add 1 to both sides to complete the square:

$$\begin{aligned} y^2-2y+1 &= -8x+1+1 \\ (y-1)^2 &= -8x+2 \end{aligned} \tag{5.2.23}$$

Factor out the coefficient of the x variable:

$$(y-1)^2 = -8(x-0.25) \tag{5.2.24}$$

So, this is a left-facing parabola with a vertex at the point $(0.25,1)$. To find the focal distance, we say that $4p=-8$, so $p=-2$ since it is left-facing, the focus will be a distance of 2 units to the left of the vertex at the point (-1.75,1) and the directrix will be a distance of 2 units to the right of the vertex at the line $x=2.25$

Exercises $5.2(b)$
Express each equation in standard form and determine the vertex, focus and directrix of each parabola.
Standard form:

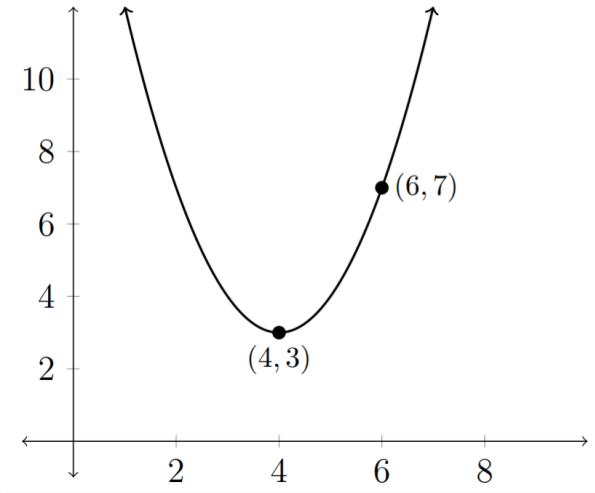
$$\begin{aligned} 4p(y-k) &= (x-h)^2 \\ 4p(x-h) &= (y-k)^2 \end{aligned} \tag{5.2.25}$$

1) $(x-2)^2 = 8(y+1)$
2) $(x+5)^2 = 12(y-3)$
3) $(y+1)^2 = 6(x-2)$
4) $(y+4)^2 = 10(x+1)$
5) $(x+3)^2 = -5(y+2)$
6) $(x-4)^2 = -7(y-6)$
7) $(y+8)^2 = -6(x+4)$
8) $(y-3)^2 = -9(x+4)$
9) $(x^2+8x+y+6=0$
10) $x^2+6x+y-3=0$
11) $y^2+6y+8x+1=0$

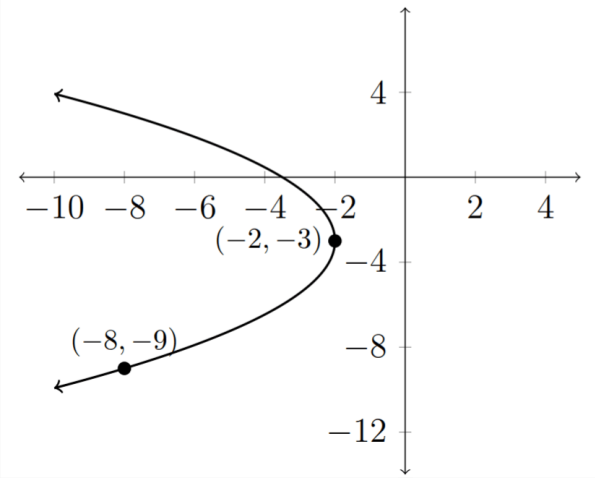
12) $y^2 + 8y - 4x + 8 = 0$
13) $x^2 + 4x - 3y + 7 = 0$
14) $x^2 + 2x - 6y - 11 = 0$
15) $y^2 + 6y - 4x + 4 = 0$
16) $y^2 - 4y + 3x + 9 = 0$
17) $x + y^2 - 3y + 1 = 0$
18) $10 + x + y^2 + 5y = 0$
19) $x + y^2 - 3y + 4 = 0$
20) $3x + y^2 + 8y + 4 = 0$
21) $x^2 + 3x + 3y - 1 = 0$
22) $x^2 + 5x - 4y - 1 = 0$
23) $x^2 - 8x - 4y + 3 = 0$
24) $6x - y^2 - 12y + 4 = 0$
25) $2x + 4y^2 + 8y - 5 = 0$
26) $4x^2 - 12x + 12y + 7 = 0$
27) $3x^2 - 6x - 9y + 4 = 0$
28) $2x - 3y^2 + 9y + 5 = 0$

Write an equation in standard form for each parabola pictured below.

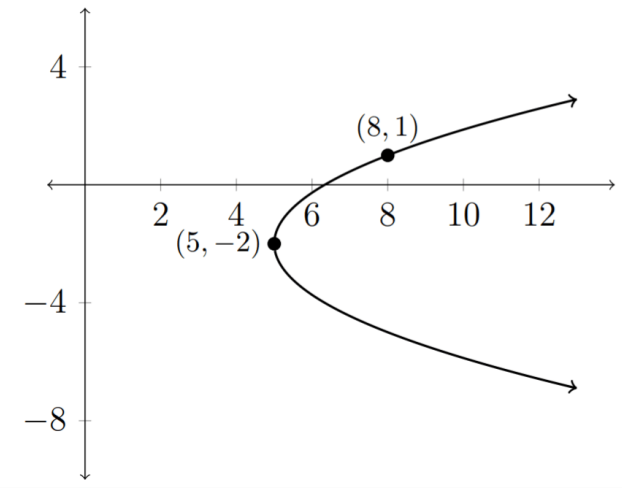
29)



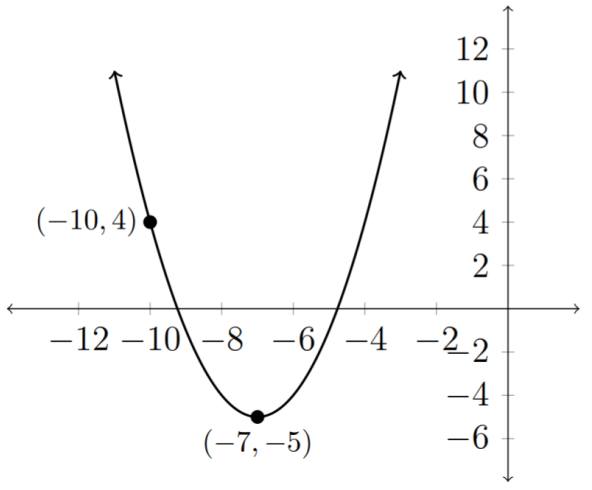
30)



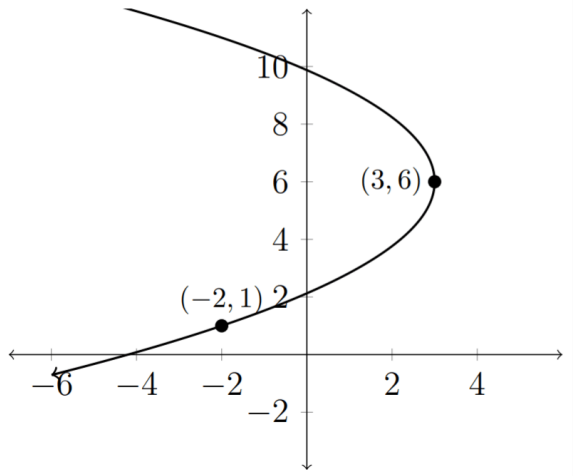
31)


4
2
4
6
8
10
12
(8, 1)
(5, −2)
−4
−8

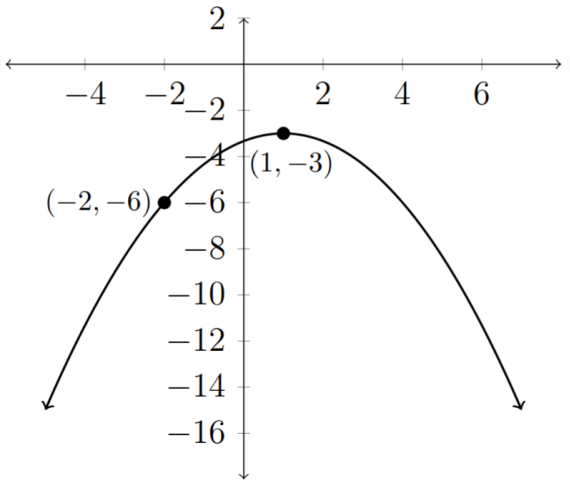
32)


12
10
8
6
4
2
(−10, 4)
−12
−10
−8
−6
−4
−2
−2
−4
−6
(−7, −5)

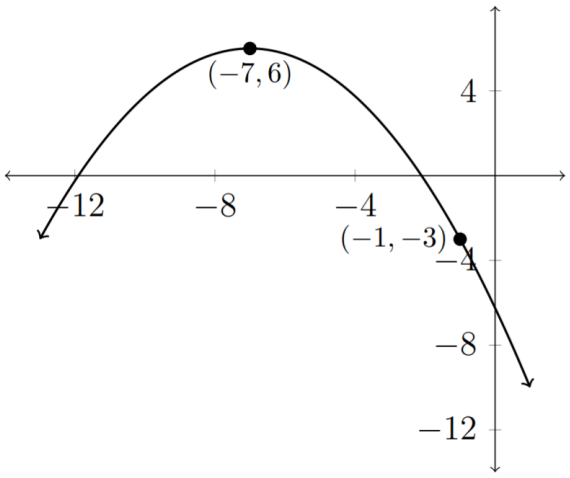
33)


10
8
6
4
2
(3, 6)
(−2, 1)
−6
−4
−2
2
4
−2

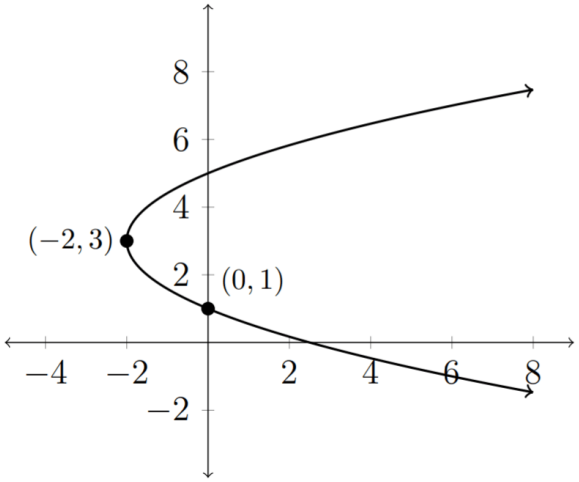
34)


2
−4
−2
−2
2
4
6
−4
(1, −3)
(−2, −6)
−6
−8
−10
−12
−14
−16

35)


(−7, 6)
4
−12
−8
−4
(−1, −3)
−4
−8
−12

36)


8
6
4
(−2, 3)
2
(0, 1)
−4
−2
2
4
6
8
−2

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5.3: Applications of the Parabola

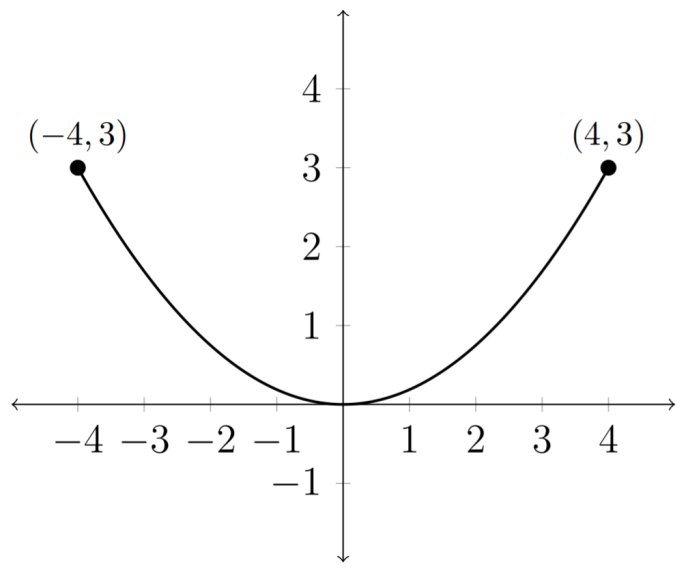
A parabola that is rotated around its axis of symmetry to create a three dimensional object is called a paraboloid. One of the special properties of a parabola is that any light (or other electromagnetic wave) striking the interior of the parabola is reflected to the focus. The proof of this is somewhat complex, but this property makes the paraboloid a very useful shape in application.

The paraboloid is used to make satellite dishes so that the signal from the satellite is reflected to the center of the dish. This strengthens the signal. Parabolic microphones are often used at sporting events so that noises on the field can be heard more clearly on the sidelines. Flashlights and headlights also use this property in reverse. With the light at the focus of the paraboloid, all the light is reflected straight ahead, thus concentrating the beam of light.

One last application which has become used more frequently over the last five or ten years is the use of the paraboloid in solar power. One use of these properties sets up a tower at the focus of the paraboloid with mirrors banked around the tower in the paraboloid shape. This focuses all of the sun's power on the tower in the center. Often, salt is used in the tower since it has very high melting point. The salt is melted by the reflected sunlight and flows into a steam turbine. This has been found to be more efficient than just using standard solar panels.

Example
If a satellite dish is 8 feet across and 3ft. deep, how far from the bottom of the dish should the receiver be placed so that it is at the focus of the paraboloid?
First let's consider what this looks like in two dimensions. Because we are setting up the graph of this parabola, we can choose to place the vertex at the origin. This makes things a little easier.
since the dish was a total of 8 feet across, we split this between the two sides of the graph, creating the points (4,3) and $(-4,3)$.
since the vertex for this parabola is at the origin, the standard equation is somewhat simplified.

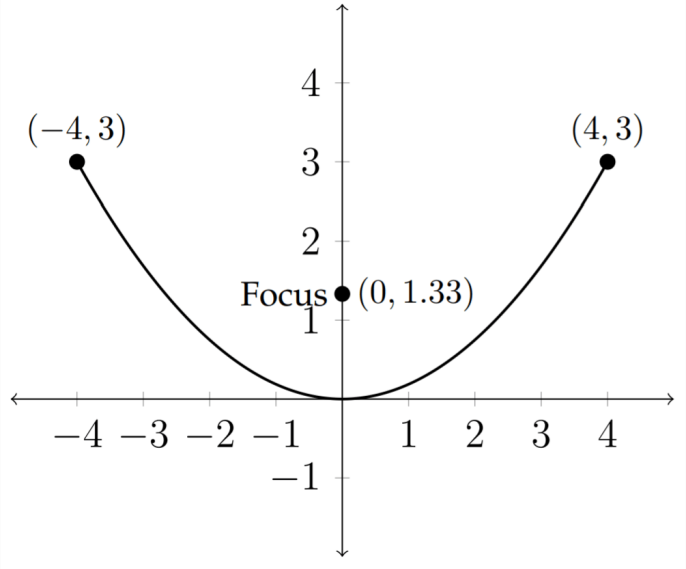
Here's what this cross-section looks like:



Taking the standard equation for an upward facing parabola: $4p(y-k)=(x-h)^2$ and using the point (0,0) for the vertex leaves us with $4p(y-0)=(x-0)^2$ or $4py=x^2$
The main thing we need to do is find the value of p for this situation. This will tell us where the receiver should be on the satellite dish. Plugging in the points on the graph that we know will allow us to solve for p

$$\begin{aligned} 4py &= x^2 \\ 4p(3) &= (4)^2 \\ 12p &= 16 \\ p &= \frac{4}{3} \end{aligned} \tag{5.3.1}$$

Thus the receiver should be $\frac{4}{3}$ feet or 1 foot 4 inches from the bottom of the dish.



Exercises 5.3

1) A satellite dish in the shape of a paraboloid is 10ft. across and 3 ft. deep. How far from the vertex at the bottom of the dish should the receiver be placed?

2) A satellite dish in the shape of a paraboloid is 6 ft. across and 2 ft. deep. How far from the vertex at the bottom of the dish should the receiver be placed?

3) The reflector in a flashlight in the shape of a paraboloid is 2 inches across and 1 inch deep. How far from the vertex at the bottom of the reflector should the light source be placed so that the light is reflected straight ahead?

4) The reflector in a flashlight in the shape of a paraboloid is 5 inches across and 3 inches deep. How far from the vertex at the bottom of the reflector should the light source be placed so that the light is reflected straight ahead?

5) A car headlight in the shape of a paraboloid has a bulb that is placed 1.25 inches from the vertex at the bottom of the headlight. If the headlight is 6 inches in diameter, how deep should the headlight be for the bulb to be at the focus of the paraboloid?

6) A spotlight in the shape of a paraboloid has a bulb that is placed 3 inches from the vertex at the bottom of the light housing. If the spotlight is 18 inches in diameter, how deep should the light be for the bulb to be at the focus of the paraboloid?

When a cable, chain or string hangs between two ends, it makes the shape of a catenary, which is generally described by an equation of the form:

$$f(x) = \frac{e^x + e^{-x}}{2} \tag{5.3.2}$$

However, when the cables of a suspension bridge support the weight of the roadbed on the bridge, the cables are pulled into the shape of a parabola.

7) The cables of a suspension bridge are in the shape of a parabola. The towers of the bridge that support the cable are 400ft. apart and 600ft. tall. If the cables touch the roadway at the center of the bridge mid-way between the towers, how high is the cable 75ft. from the center of the bridge?

8) The cables of a suspension bridge are in the shape of a parabola. The towers of the bridge that support the cable are 800ft. apart and 160ft. tall. If the cables touch the roadway at the center of the bridge mid-way between the towers, how high is the cable 120ft. from the center of the bridge?

9) A spotlight in the shape of a paraboloid has the light source 2.5ft. from the vertex at the bottom of the light. If the depth of the spotlight housing is 4ft, how wide should the face of the spotlight be?

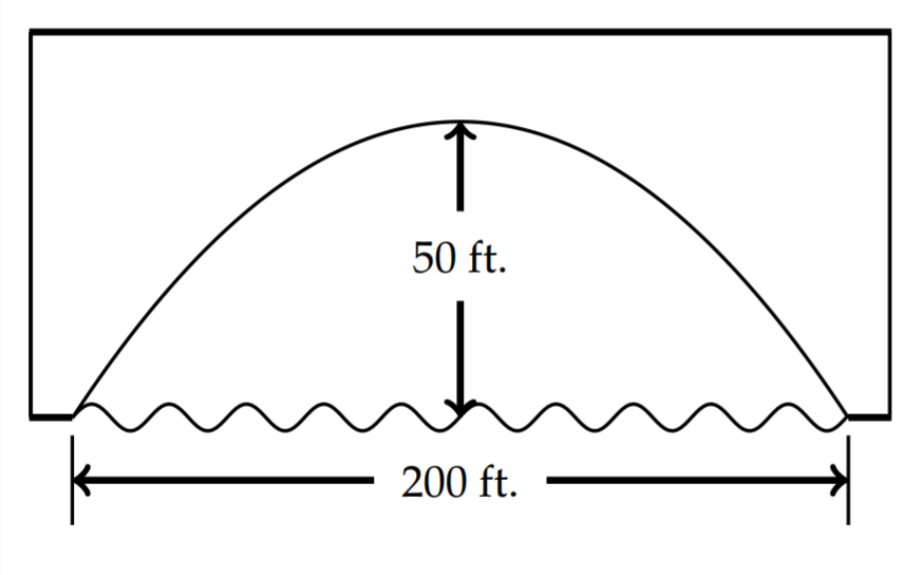
10) A flashlight reflector in the shape of a paraboloid has a light source that is 0.5 inches from the vertex at the bottom of the reflector. How deep should the reflector be if the face of the flashlight is 3 inches in diameter?

11) The parabolic mirror for the Mount Palomar observatory telescope is 200 inches in diameter and the mirror is 3.75 inches deep at the center. How far from the center of the mirror is the focal point where the light is reflected?

12) A reflecting telescope contains a mirror in the shape of a paraboloid. If the mirror is 5 inches across at the opening and 2 inches deep, how far from the vertex at the bottom of the mirror is the focal point?

13) The cables of a suspension bridge are in the shape of a parabola. The towers of the bridge that support the cable are 500ft. apart and 80ft. tall. If the cables are 15 ft. above the roadway at the center of the bridge mid-way between the towers, how high is

the cable 100ft. from the center of the bridge?
14) The cables of a suspension bridge are in the shape of a parabola. The towers of the bridge that support the cable are 2000ft. apart and 400ft. tall. If the cables are 25ft. above the roadway at the center of the bridge mid-way between the towers, how high is the cable 500ft. from the center of the bridge?
15) A flashlight reflector housing in the shape of a paraboloid is 4 inches deep and 4 inches across at its widest point. How far from the vertex at the bottom of the housing should the light source be placed?
16) A flashlight reflector housing in the shape of a paraboloid is 8 inches in diameter with the light source placed 1 inch from the vertex at the bottom of the housing. How deep should the housing be for the light source to be placed at the focal point?
17) A bridge is built in the shape of a parabolic arch (see figure below). If the arch is 50ft. above the water at the center and 200ft. wide at the water's surface, will a boat that is 35 ft. tall clear the arch 30 ft. from the center? If not, how much taller than the bridge is the boat? If so, how much clearance will there be?



18) A bridge is built in the shape of a parabolic arch. If the arch is 30 ft. above the water at the center and 150ft. wide at the water's surface, find the height of the arch above the water at distances of $10, 25, 40,$ and 50ft. from the center.

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6: SEQUENCES AND SERIES

6.1: Sequences
6.2: Arithmetic and Geometric Sequences
6.3: Series
6.4: Sum of a Series

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6.1: Sequences

A sequence of numbers in a one-to-one correspondence with the natural numbers $\{1, 2, 3, 4, ...\}$ can be defined in several ways. The terms of the sequence may simply be listed:

$$\{2, 4, 8, 16, 32, ...\} \tag{6.1.1}$$

A general expression for the sequence may be identified:

$$a_n = 2^n \tag{6.1.2}$$

In this situation, n is generally understood to be drawn from the ordered set of natural numbers. In addition, a sequence may be defined recursively. That is to say that each successive term will be defined in relation to the preceding term.
For the example that we're using above, a recursive definition would be as follows:

$$\begin{gathered} a_1 = 2 \\ a_n = 2 * a_{n-1} \\ \text{or} \\ a_{n+1} = 2 * a_n \end{gathered} \tag{6.1.3}$$

A sequence may be thought of as a function or relation in which the domain is restricted to positive whole numbers.

Examples
Find the first four terms of the given sequence and the 10 th term of the sequence.
1) $a_n = n^2 + 3$

$$a_1 = 4, a_2 = 7, a_3 = 12, a_4 = 19, a_{10} = 103 \tag{6.1.4}$$

2) $a_1 = 5$

$$\begin{gathered} a_n = a_{n-1} + 6 \\ a_1 = 5, a_2 = 11, a_3 = 17, a_4 = 23, a_{10} = 59 \end{gathered} \tag{6.1.5}$$

Finding a general or recursive definition for a sequence can be trickier than just writing out the terms. Common things to look for -

Is this an alternating sequence? That is, do the terms bounce back and forth between positive and negative values. If so, then you will need to include $(-1)^n$ or $(-1)^{n+1}$ in your general term.

$$\begin{gathered} \text{Example: } \{-1, 2, -3, 4, ...\} \\ a_n = (-1)^n(n) \end{gathered} \tag{6.1.6}$$

or

$$a_1 = -1 \tag{6.1.7}$$

$$a_n = (-1)\,(a_{n-1}) + (-1)^n \tag{6.1.8}$$

Is there a common difference between the terms? If so, then the sequence behaves much like a linear function and will have a form similar to $y = mx + b$, where m is the common difference.

$$\begin{gathered} \text{Example: } \{5, 8, 11, 14, ...\} \\ a_n = 3n + 2 \end{gathered} \tag{6.1.9}$$

or

$$\begin{aligned} a_1 &= 5 \\ a_n &= a_{n-1} + 3 \end{aligned} \tag{6.1.10}$$

Is there a common multiplier? If so, then this should be a power function where a particular base is being raised to the power of n.

$$\begin{aligned} &\text{Example: } \{3, 15, 75, 375, ...\} \\ &a_n = 3 * 5^{n-1} \\ &\text{ or} \\ &a_1 = 3 \\ &a_n = 5 * a_{n-1} \end{aligned} \tag{6.1.11}$$

Other patterns to look for are perfect squares and perfect cubes.

Exercises 6.1

Find the first four terms of the given sequence and the 10 th term of the sequence.

1) $a_n = 3n + 1$
2) $a_n = 4n - 12$
3) $a_n = -5n + 3$
4) $a_n = -2n + 7$
5) $a_n = 2n^2$
6) $a_n = 5n^2 - 1$
7) $a_n = (-1)^n(4n)$
8) $a_n = (-1)^{n+1}\left(\frac{1}{n}\right)$
9) $a_n = \frac{2^n}{3^{n-1}}$
10) $a_n = \frac{5^n}{2^{n+1}}$
11) $a_n = \frac{(-1)^{n-1}}{2n+5}$
12) $a_n = \frac{(-1)^n}{3n-2}$
13) $a_1 = -3 \text{and} a_n = a_{n-1} + 4$
14) $a_1 = 2 \text{and} a_n = a_{n-1} + 12$
15) $a_1 = 7 \text{and} a_n = 9 - a_{n-1}$
16) $a_1 = -5 \text{and} a_n = 17 - a_{n-1}$
17) $a_1 = 1 \text{and} a_n = n + a_{n-1}$
18) $a_1 = 4 \text{and} a_n = n - a_{n-1}$
19) $a_1 = \frac{1}{2} \text{and} a_n = \frac{(-1)^n}{a_{n-1}}$
20) $a_1 = \frac{2}{5} \text{and} a_n = \frac{(-1)^{n+1}}{2a_{n-1}}$

For each of the given sequences - find a general term a_n, and also find a recursive definition for the sequence.

21) $\{6, 7, 8, 9, 10, ...\}$
22) $\{9, 11, 13, 15, 17, ...\}$
23) $\{1, 4, 7, 10, 13, ...\}$
24) $\{-5, 4, 13, 22, 31, ...\}$
25) $\{-2, 6, -18, 54, ...\}$
26) $\{5, -10, 20, -40, 80, ...\}$
27) $\{1, -1, -3, -5, -7, ...\}$
28) $\{-8, -15, -22, -29, ...\}$
29) $\left\{\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, ...\right\}$
30) $\left\{\frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, ...\right\}$
31) $\left\{-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, ...\right\}$
32) $\left\{\frac{1}{2}, -\frac{2}{5}, \frac{3}{8}, -\frac{4}{11}, ...\right\}$
33) $\{5, -25, 125, -625, ...\}$
34) $\left\{1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, ...\right\}$
35) $\left\{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, ...\right\}$
36) $\left\{\frac{2}{3}, \frac{9}{4}, \frac{8}{27}, \frac{81}{16}, \frac{32}{243} ...\right\}$

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6.2: Arithmetic and Geometric Sequences

Two common types of mathematical sequences are arithmetic sequences and geometric sequences. An arithmetic sequence has a constant difference between each consecutive pair of terms. This is similar to the linear functions that have the form $y = mx + b$. A geometric sequence has a constant ratio between each pair of consecutive terms. This would create the effect of a constant multiplier.

Examples

Arithmetic Sequence:
$\{5, 11, 17, 23, 29, 35, ...\}$

Notice here the constant difference is 6. If we wanted to write a general term for this sequence, there are several approaches. One approach is to take the constant difference as the coefficient for the n term: $a_n = 6n + ?$ Then we just need to fill in the question mark with a value that matches the sequence. We could say for the sequence:
$\{5, 11, 17, 23, 29, 35, ...\}$

$a_n = 6n - 1$

There is also a formula which you can memorize that says that any arithmetic sequence with a constant difference d is expressed as:
$a_n = a_1 + (n - 1)d$

Notice that if we plug in the values from our example, we get the same answer as before:
$a_n = a_1 + (n - 1)d$

$a_1 = 5, d = 6$

So, $a_1 + (n - 1)d = 5 + (n - 1) * 6 = 5 + 6n - 6 = 6n - 1$

or $a_n = 6n - 1$

If the terms of an arithmetic sequence are getting smaller, then the constant difference is a negative number.
$\{24, 19, 14, 9, 4, -1, -6, ...\}$

$a_n = -5n + 29$

Geometric Sequence

In a geometric sequence there is always a constant multiplier. If the multiplier is greater than 1, then the terms will get larger. If the multiplier is less than 1, then the terms will get smaller.
$\{2, 6, 18, 54, 162, ...\}$

Notice in this sequence that there is a constant multiplier of 3. This means that 3 should be raised to the power of n in the general expression for the sequence. The fact that these are not multiples of 3 means that we must have a coefficient before the 3^n
$\{2, 6, 18, 54, 162, ...\}$

$a_n = 2 * 3^{n-1}$

If the terms are getting smaller, then the multiplier would be in the denominator:
$\{50, 10, 2, 0.4, 0.08, ...\}$

Notice here that each term is begin divided by 5 (or multiplied by $\frac{1}{5}$).
$\{50, 10, 2, 0.4, 0.08, ...\}$

$a_n = \frac{50}{5^{n-1}}$ or $a_n = \frac{250}{5^n}$ or $a_n = 50 * \left(\frac{1}{5}\right)^{n-1}$ and so on

Exercises 6.2

Determine whether each sequence is arithmetic, geometric or neither.

If it is arithmetic, determine the constant difference.

If it is geometric determine the constant ratio.

1) $\{18, 22, 26, 30, 34, ...\}$

2) $\{9, 19, 199, 1999, ...\}$

3) $\{8, 12, 18, 27, ...\}$

4) $\{15, 7, -1, -9, -17, ...\}$

5) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, ...\}$

6) $\{100, -50, 25, -12.5, ...\}$

7) $\{-8, 12, 32, 52, ...\}$

8) $\{1, 4, 9, 16, 25, ...\}$

9) $\{11, 101, 1001, 10001, ...\}$

10) $\{12, 15, 18, 21, 24, ...\}$

11) $\{80, 20, 5, 1.25, ...\}$

12) $\{5, 15, 45, 135, 405, ...\}$

13) $\{1, 3, 6, 10, 15, ...\}$

14) $\{2, 4, 6, 8, 10, ...\}$

15) $\{-1, -2, -4, -8, -16, ...\}$

16) $\{1, 1, 2, 3, 5, 8, 13, 21, ...\}$

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6.3: Series

Learning about mathematical sequences is usually a precursor to learning about mathematical series. A mathematical series is a sequence of numbers that is being added together. The importance of mathematical series cannot be understated. Many equations in the sciences cannot be solved by algebraic methods and must resort to series solutions. The notation for a mathematical series is typically the Greek capital letter sigma: Σ. The sigma notation is used as a short-hand method of representing a mathematical series with a particular form.

For example, if we are given the mathematical series:
$1+5+9+13+17+21$

This can be represented as follows:
$\sum_{k=0}^{5} 4k+1$

We could also express the same series as:
$\sum_{k=1}^{6} 4k-3$

Both expressions represent the terms being added together. This first example is an example of a finite series because it has a last term. Many mathematical series are infinite series. For example:
$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots$

is an example of an infinite series.

Working with infinite series can be quite useful, but also somewhat confusing. The behavior of an infinite series can be contradictory depending on how you analyze it.

Exercises 6.3
Write out each series in expanded notation.

1) $\sum_{k=1}^{10} 2k-5$

2) $\sum_{k=3}^{7} 6k-3$

3) $\sum_{k=2}^{9} (-1)^k \left(\frac{1}{k}\right)$

4) $\sum_{k=0}^{10} (-1)^{k+1}(k-4)$

5) $\sum_{k=0}^{4} \frac{k^2}{2}$

6) $\sum_{k=1}^{8} \frac{k}{3^k}$

Write each series using sigma notation.

7) $8+12+16+20+24+28$

8) $5+10+15+20+25+30$

9) $2+9+16+23+\cdots+65$

10) $5+8+11+14+\cdots+95$

11) $1+4+9+16+\cdots+256$

12) $1+8+27+64+\cdots+1331$

13) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$

14) $27-9+3-1+\frac{1}{3}-\frac{1}{9}$

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6.4: Sum of a Series

To find approximate solutions to problems in the sciences, it is often necessary to calculate the sum of a finite or infinite series. There are a variety of formulas that are used to accomplish this. Some of these formulas will be presented with proofs, but others will not. If you are interested in the proofs that are not included, please let me know.
General Formulas

Constant Series - notice that there is no k in the summation, the c is a constant that does not depend on the value of k
$\sum_{k=1}^{n} c = c + c + c + \cdots + c = n * c$

Sum of the first n integers:
$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Sum of the first n perfect squares:
$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of the first n perfect cubes:
$\sum_{k=1}^{n} k^3 = 1 + 8 + 27 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

The first formula should be obvious. The other three formulas are usually proved using mathematical induction, which we won't cover in this course. If you're interested in these proofs and how mathematical induction works, please let me know.

Formulas for the sum of arithmetic and geometric series:

Arithmetic Series: like an arithmetic sequence, an arithmetic series has a constant difference d. If we write out the terms of the series:
$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$

we can rewrite this in terms of the first term (a_1) and the constant difference d
$\sum_{k=1}^{n} a_k = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-1)d)$

This expression is equivalent to:
$\sum_{k=1}^{n} a_k = (a_1 + a_1 + a_1 + \cdots + a_1) + (d + 2d + 3d + \cdots (n-1)d)$
$\sum_{k=1}^{n} a_k = na_1 + d(1 + 2 + 3 + \cdots (n-1))$

Using the previous formula for the sum $1 + 2 + 3 + \cdots + (n-1)$ gives us:
$\sum_{k=1}^{n} a_k = na_1 + d\left(\frac{(n-1)n}{2}\right)$

This formula is often stated in various forms:
$\sum_{k=1}^{n} a_k = \frac{n}{2}(2a_1 + (n-1)d)$

or
$\sum_{k=1}^{n} a_k = \frac{n}{2}(a_1 + a_n)$
since $a_1 + (n-1)d = a_n$

Geometric Series:

Given a geometric series, whose first term is a and with a constant ratio of r $\sum_{k=1}^{n} a * r^{k-1}$, we can write out the terms of the series in a similar way that we did for the arithmetic series.
$\sum_{k=1}^{n} a * r^{k-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

The trick to finding a formula for the sum of this type of series is to multiply both sides of the previous equation by r
For simplicity's sake let's rename the sum of the series $\sum_{k=1}^{n} a * r^{k-1}$ as S_n

So,
$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

and
$r * S_n = r\left(a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}\right) = ar + ar^2 + ar^3 + \cdots + ar^n$

If we subtract these two equations, we will have:
$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$
$-\left(r * S_n = ar + ar^2 + ar^3 + \cdots + ar^n\right)$

Then we'll have:
$S_n - rS_n = a - ar^n$

Factor out S_n on the left-hand side:
$S_n(1 - r) = a - ar^n$

and divide on both sides to isolate S_n :
$\frac{S_n(1-r)}{(1-r)} = \frac{a-ar^n}{(1-r)}$
$S_n = \frac{a-ar^n}{1-r}$

So for a finite geometric series, we can use this formula to find the sum. This formula can also be used to help find the sum of an infinite geometric series, if the series converges. Typically this will be when the value of r is between -1 and 1. In other words, $|r| < 1$ or \(-1 $S_n = \frac{a}{1-r}$

Exercises 6.4
Find the sum for each of the following finite geometric series.
1) $\sum_{k=1}^{7} 3\left(\frac{1}{4}\right)^{k-1}$

2) $\sum_{k=1}^{7} 16\left(\frac{1}{3}\right)^{k-1}$

3) $\sum_{k=1}^{7} 3^k$

4) $\sum_{k=1}^{10} 2^{k-1}$

5) $\sum_{k=1}^{5} 4^{k-1}$

6) $\sum_{k=1}^{4} 6^{k-1}$

7) $\sum_{k=1}^{7} 2^k$

8) $\sum_{k=1}^{8} 3^k$

9) $\sum_{k=1}^{5} 2^{k+2}$

10) $\sum_{1}^{6} 3^{k-4}$

Determine whether each of the following geometric series has a sum. If it does, use the formula $S_n = \frac{a}{1-r}$ to find the sum.
11) $\sum_{k=1}^{\infty} 5 * \left(\frac{2}{3}\right)^{k-1}$

12) $\sum_{k=1}^{\infty} 12 * \left(\frac{1}{2}\right)^{k-1}$

13) $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k-1}$

14) $\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^{k-1}$

15) $\sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^{k+1}$

16) $\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^{k+1}$

17) $\sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^{k-1}$

18) $\sum_{k=1}^{\infty} \left(\frac{3}{2}\right)^{k-1}$

19) $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^{k+2}$

20) $\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^{k+4}$

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7: COMBINATORICS

7.1: The Fundamental Principle of Counting
7.2: Factorial Notation and Permutations
7.3: Permutations and Combinations
7.4: General Combinatorics Problems
7.5: Distinguishable Permutations
7.6: Probability

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7.1: The Fundamental Principle of Counting

Combinatorics is a major area of what is generally known as "Discrete Mathematics." The word discrete refers to quantities that are individual, separate or distinct. This creates a major division in mathematics between "continuous mathematics" and "discrete mathematics." The difference between these two areas is that continuous mathematics considers and uses all parts of the number line whole numbers, rational numbers (fractions), irrational numbers and so forth. Discrete mathematics typically uses only the whole numbers.

Rather than limiting the possibilities in mathematics, this restriction actually opens amazing new areas of consideration. The binary codes that computers use are generally controlled and kept (mostly) error free through the use of discrete mathematics. Computer security for the simplest (checking your on-line bank balance) and the most complex (high level classified data) digital information is handled through encryption that relies on the concepts of discrete mathematics.

Any type of application in the sciences that involves choices and possibilities often uses the concepts of combinatorics. Combinatorial chemistry explores the results when a series of different chemical groups are added to the same basic chemical structure to investigate the qualities of the resulting compound. In addition, combinatorics is very important to the study of probability. In order to calculate the probability of an event, it is often necessary to calculate how many
different ways something can happen.

The first major idea of combinatorics is the fundamental principle of counting. This is the idea that if two events occur in succession and there are m ways to do the first one and n ways to do the second (after the first has occurred), then there are $m * n$ ways to complete the two tasks in succession.

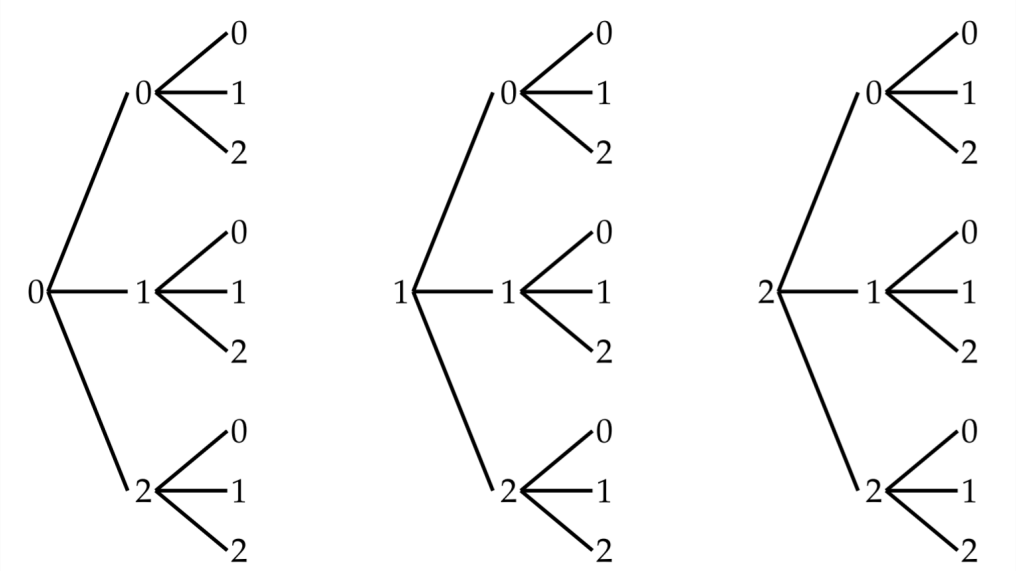
For example, in throwing two six-sided dice, there are 36 possibilities - six possibilities from the first die and six from the second. These possibilities are listed below:

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

If we were to throw a six-sided die and an eight sided-sided die, then there would be $6 * 8 = 48$ different possibilities.

Often the creation of a tree diagram can help us to visualize the possibilities. Suppose that a three-digit code using the numbers 0,1 and 2 is created so that repetition of the numbers is allowed. There would 3 possibilities for the first number, three for the second number and three for the third number meaning that there would be a total of $3 * 3 * 3 = 27$ different possible codes.

The tree diagram to illustrate this is shown below.



All of the possibilities are listed out and can be constructed from the tree diagram.

Here are some sample problems for using the fundamental principle of counting (also known as the multiplication principle).

Example:
An ice cream shop offers Vanilla, Chocolate, Strawberry, Boysenberry and Rocky Road ice cream. The ice cream comes with either a waffle cone, a sugar cone or a wafer cone and can be plain or with sprinkles. How many different ways are there to order a single scoop of ice cream?

With 5 different types of ice cream, 3 different cones and 2 choices for sprinkles, there would be $5 * 3 * 2 = 30$ different possibilities.

Example:
In many states the license plates for automobiles consist of three letters followed three numbers.

How many different possibilities are there:

- if repetition of letters and numbers is allowed?
- if repetition of letters and numbers is not allowed?
- There are six positions for letters and numbers that make up the license plate.

The first three are to be letters and the second three numbers.

There are 26 choices for the first three and 10 choices for the second three, so there are $26 * 26 * 26 * 10 * 10 * 10 = 17,576,000$ possible license plates if repetition is allowed.

If repetition is not allowed then we have to discount whichever letter or number is chosen for a particular position. So, for the first letter there are 26 possibilities, but then only 25 for the second letter and 24 for the third. Likewise, with the numbers there are 10 choices initially, then 9 choices and then 8

So, the solution without repetition would be $26 * 25 * 24 * 10 * 9 * 8 = 11,232,000$

4) A coin is flipped five times and the result each time is recorded. How many different possible outcomes are there?

5) A coin is flipped and a six-sided die is rolled and the results are recorded. If this is done three times, how many possible outcomes are there?

6) Two cards are chosen from a deck of 52 cards. If the first card is not replaced before the second card is chosen, how many ways are there to choose:

a) A spade first and a heart second?
b) Two spades?

7) A company has 3000 employees. They plan to implement an employee ID numbering system that would consist of a letter followed by two digits. Is it possible to give each employee a different ID code under this plan?

8) A baseball team has 7 pitchers and 3 catchers. How many different batteries (pitcher - catcher combinations) are possible?

9) A string of five letters is created using the letters $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and E. How many of these letter strings are possible if:
a) No conditions are imposed
b) Repetition of the letter A is not allowed
c) Each letter string must begin with C
d) B must be the middle letter
e) A, B and C must be the middle letters in any order with no repetition
For Part (e) please list all possibilities.

10) (\quad\) A combination lock is numbered from 0 to 30. Each combination consists of three numbers in succession. Successive numbers must be different, but the first and third can be the same. How many different combinations are possible?

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7.2: Factorial Notation and Permutations

In considering the number of possibilities of various events, particular scenarios typically emerge in different problems. One of these scenarios is the multiplication of consecutive whole numbers. For example, given the question of how many ways there are to seat a given number of people in a row of chairs, there will obviously not be repetition of the individuals. So, if we wanted to know how many different ways there are to seat 5 people in a row of five chairs, there would be 5 choices for the first seat, 4 choices for the second seat, 3 choices for the third seat and so on.

$$\underline{5} * \underline{4} * \underline{3} * \underline{2} * \underline{1} = 120 \text{ choices} \tag{7.2.1}$$

In these situations the 1 is sometimes omitted because it doesn't change the value of the answer. This process of multiplying consecutive decreasing whole numbers is called a "factorial." The notation for a factorial is an exclamation point. So the problem above could be answered: $5! = 120.$ By definition, $0! = 1.$ Although this may not seem logical intuitively, the definition is based on its application in permutation problems.

A "permutation" uses factorials for solving situations in which not all of the possibilities will be selected.

So, for example, if we wanted to know how many ways can first, second and third place finishes occur in a race with 7 contestants, there would be seven possibilities for first place, then six choices for second place, then five choices for third place.
So, there are $\underline{7} * \underline{6} * \underline{5} = 210$ possible ways to accomplish this.
The standard notation for this type of permutation is generally ${}_nP_r$ or $P(n, r)$
This notation represents the number of ways of allocating r distinct elements into separate positions from a group of n possibilities.

In the example above the expression $\underline{7} * \underline{6} * \underline{5}$ would be represented as ${}_7P_3$ or

$$P(7, 3) \tag{7.2.2}$$

The standard definition of this notation is:

$${}_nP_r = \frac{n!}{(n-r)!} \tag{7.2.3}$$

You can see that, in the example, we were interested in ${}_7P_3,$ which would be calculated as:

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7*6*5*4*3*2*1}{4*3*2*1} \tag{7.2.4}$$

The $4 * 3 * 2 * 1$ in the numerator and denominator cancel each other out, so we are just left with the expression we fouind intuitively:

$${}_7P_3 = 7 * 6 * 5 = 210 \tag{7.2.5}$$

Although the formal notation may seem cumbersome when compared to the intuitive solution, it is handy when working with more complex problems, problems that involve large numbers, or problems that involve variables.

Note that, in this example, the order of finishing the race is important. That is to say that the same three contestants might comprise different finish orders.
1st place: Alice 1st place: Bob 2nd place: Bob 2nd place: Charlie 3rd place: Charlie 3rd place: Alice
The two finishes listed above are distinct choices and are counted separately in the 210 possibilities. If we were only concerned with selecting 3 people from a group of $7,$ then the order of the people wouldn't be important - this is generally referred to a "combination" rather than a permutation and will be discussed in the next section.

Returning to the original example in this section - how many different ways are there to seat 5 people in a row of 5 chairs? If we use the standard definition of permutations, then this would be ${}_5P_5$

$$ {}_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{120}{1} = 120 \tag{7.2.6} $$

This is the reason why $0!$ is defined as 1

EXERCISES 7.2

1) $4 * 5!$
2) $3! * 4!$
3) $5! * 3!$
4) $\frac{8!}{6!}$
5) $\frac{10!}{7!}$
6) $\frac{9!*6!}{3!*7!}$
7) $\frac{12!*3!}{8!*6!}$
8) ${}_{10}P_4$
9) ${}_4P_3$
10) ${}_7P_5$
11) ${}_9P_2$
12) ${}_8P_4$
13) so P_3
14) n_1
15) ${}_{10}P_r$
16) List all the permutations of the letters $\{a, b, c\}$
17) List all the permutations of the letters $\{a, b, c\}$ taken two at a time.
18) How many permutations are there of the group of letters $\{a, b, c, d, e\}$?
19) How many permutations are there of the group of letters $\{a, b, c, d\}$?

List these permutations.

20) How many ways can a president, vice president and secretary be chosen from a group of 20 students?
21) How many ways can a president, vice president, secretary and treasurer be chosen from a group of 50 students?
22) How many ways can 5 boys and 5 girls be seated in a row containing ten seats:
    a) with no restrictions?
    b) if boys and girls must alternate seats?
23) How many ways can 5 boys and 4 girls be seated in a row containing nine seats:
    a) with no restrictions?
    b) if boys and girls must alternate seats?
24) How many ways can 6 people be seated if there are 10 chairs to choose from?
25) How many ways can 4 people be seated if there are 9 chairs to choose from?
26) How many ways can a group of 8 people be seated in a row of 8 seats if two people insist on sitting together?
27) How many ways can a group of 10 people be seated in a row of 10 seats if three people insist on sitting together?

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7.3: Permutations and Combinations

We saw in the last section that, when working with permutations, the order is always important. If we were choosing 3 people from a group of 7 to serve on a committee with no assigned roles, the nature of the problem would change.
For example, if we were choosing 3 people from a group of 7 to serve on a committee as president, vice-president and treasurer, the answer would be $_7P_3 = 210$ But - if we wanted to choose 3 people from a group of 7 with no assigned roles, then some of the choices in the permutation would be the same.
In a permutation:
1st place: Alice 1st place: Bob 2nd place: Bob 2nd place: Charlie 3rd place: Charlie 3rd place: Alice
the two choices listed above would be considered as being different and would be counted separately. In a "combination" in which the order of selection is not important and there are no assigned roles, we must compensate for these extra choices.

If we are choosing 3 people from a group of 7 to serve on a committee with no assigned roles then we should consider that any selection from a permutation that includes the same three people should only be counted once.
So, when we select the three people, we should consider how many different ways there are to group them and then remove those extra choices. In this example, we are choosing three people. Each group of three can be arranged in six different ways $3! = 3 * 2 = 6$, so each distinct group of three is counted six times.
In order to find the actual number of choices we take the number of possible permutations and divide by 6 to arrive at the actual answer:

$$_7C_3 = \frac{7P_3}{3!} = \frac{7!}{4! * 3!} \tag{7.3.1}$$

In a combination in which the order is not important and there are no assigned roles the number of possibilities is defined as:

$$_nC_r = \frac{n!}{(n-r)! * r!} \tag{7.3.2}$$

One way to remember the difference between a permutation and a combination is that on a combination pizza it doesn't make any difference whether the sausage goes on before the pepperoni or whether the onions are put on first-so in a combination, order is not important!

EXERCISES 7.3
Find the value of the following expressions.
1) $_{10}C_4$
2) $_8C_3$
3) $_{10}C_6$
4) $_8C_5$
5) $_{15}C_{12}$
6) $_{18}C_2$
7) $_nC_4$
8) $_9C_r$
9) How many three-topping pizzas can be made if there are twelve toppings to choose from?
10) How many bridge hands of 13 cards are possible from a deck of 52 cards?
11) How many poker hands of 5 cards are possible from a deck of 52 cards?
12) How many different bridge hands of 13 cards are possible if none of the cards is higher than 10 (i.e. no face cards)?
13) How many different poker hands of 5 cards are possible if none of the cards is higher than 8?
14) If a person has 10 different t-shirts, how many ways are there to choose 4 to take on a trip?
15) If a band has practiced 15 songs, how many ways are there for them to select 4 songs to play at a battle of the bands? How many different performances of four songs are possible?
16) Fifteen boys and 12 girls are on a camping trip. How many ways can a group of seven be selected to gather firewood:
    a) with no conditions
    b) the group contains four girls and three boys
    c) the group contains at least four girls
17) A class of 25 students is comprised of 15 girls and 10 boys. In how many ways can a committee of 8 students be selected if:
    a) there are no restrictions

b) no males are included on the committee
c) no females are included on the committee
d) the committee must have 5 boys and 3 girls

18) From a group of 12 male and 12 female tennis players, two men and two women will be chosen to compete in a men-vs-women doubles match. How many different matches are possible?

19) In a seventh-grade dance class, there are 20 girls and 17 boys.
a) How many ways can the students be paired off to create dance couples consisting of one boy and one girl?
b) How many ways are there to create a group of 17 boy/girl couples?
c) How many ways are there to create a group of 18 couples without restrictions?

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7.4: General Combinatorics Problems

As is often the case, practicing one type of problem at a time is helpful to master the techniques necessary to solve that type of problem. More difficult is deciding which type of problem you are working on and choosing which techniques to use to solve the problem. In this section, the problem types from the previous three sections are combined.

EXERCISES 4.4
SET I
1) How many strings of six lower case letters from the English alphabet contain
a) the letter a?
b) the letters a and b in consecutive positions with a preceding b, with all the letters distinct?
c) the letters a and b, where a is somewhere to the left of b in the string, with all the letters distinct?
2) Seven women and nine men are on the faculty in the mathematics department at a school.
a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?
b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?
3) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with 6 members if it must have the same number of men and women?
4) The English alphabet contains 21 consonants and 5 vowels. How many strings of 6 lower case letters of the English alphabet contain:
a) exactly one vowel?
b) exactly 2 vowels?
c) at least 1 vowel?
d) at least 2 vowels?
5) How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45,4 are selected from a block of 57, and the others are selected from the remaining 69 countries?
6) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with 6 members if it must have more women than men?
7) How many license plates consisting of three letters followed by three digits contain no letter or digit twice?
8) In the women's tennis tournament at Wimbledon, two finalists, A and B, are competing for the title, which will be awarded to the first player to win two sets. In how many different ways can the match be completed?
9) In the men's tennis tournament at Wimbledon, two finalists, A and B, are competing for the title, which will be awarded to the first player to win three sets. In how many different ways can the match be completed?
10) In how many different ways can a panel of 12 jurors and 2 alternate jurors be chosen from a group of 30 prospective jurors?

SET II
11) A class has 20 students, of which 12 are female and 8 are male. In how many ways can a committee of five students be picked from the class if:
a) No restrictions are placed on the choice of students
b) No males are included on the committee
c) The committee must have three female members and 2 male members
12) A school dance committee is to be chosen from a group of students consisting of six freshman, eight sophomores, twelve juniors and ten seniors. If the committee should consist of two freshman, three sophomores, four juniors and five seniors, how many ways can this be done?
13) A theater company consists of 22 actors - 10 men and 12 women. In the next play, the director needs to cast a leading man, leading lady, supporting male role, supporting female role and eight extras (three men and five women). In how many ways can this play be cast?
14) A hockey coach has 20 players of which twelve play forward, six play defense and two are goalies. In how many ways can the coach pick a lineup consisting of three forwards, two defense players and one goalie?
15) In how many ways can ten students be arranged in a row for a class picture if John and Jane want to stand next to each other and Ed and Sally also want to stand next to each other?
16) In how many ways can the students in the previous problem be arranged if Ed and Sally want to stand next to each other, but John and Jane refuse to stand next to each other?
17) In how many ways can four men and four women be seated in a row of eight seats if:
a) The first seat is occupied by a man

b) The first and last seats are occupied by women

SET III

18) The social security number of a person is a sequence of nine digits that are not necessarily distinct. How many social security numbers are possible?

19) A variable name in the BASIC programming language is either a letter of the alphabet or a letter followed by a digit. How many distinct variable names are there in the BASIC language?

20) a) How many even numbers are there between 0 and 100?

b) How many even numbers with distinct digits are there between 0 and $100?$

21) There are six characters - three letters of the English alphabet followed by three digits - which appear on the back panel of a particular brand of printer as an identification number. Find the number of possible identification numbers if

a) characters can be repeated

b) digits cannot repeat

c) letters cannot repeat

d) characters cannot repeat

22) A sequence of characters is called a palindrome if it reads the same forwards and backwards. For example $\mathrm{K98EE89K}$ is an eight character palindrome and $\mathrm{K98E89K}$ is a seven character palindrome. A MAN A PLAN A CANAL PANAMA is also a palindrome as are WAS IT A RAT I SAW, TACO CAT, TANGY GNAT, and NEVER ODD OR EVEN. Find the number of nine character palindromes that can be formed using the letters of the alphabet such that no letter appears more than twice in each one.

23) Find the number of ways to form a four-letter sequence using the letters A, B, C, D and E if:

a) repetitions are permitted

b) repetitions are not permitted

c) the sequence contains the letter A but repetitions are not permitted

d) the sequence contains the letter A and repetitions are permitted

24) There are 10 members A, B, C, D, E, F, G, H, I and J in a fund raising committee. The first task of the committee is to choose a chairperson, a secretary and a treasurer from this group. No individual can hold more than one office. How many ways can these three positions be filled if:

a) no one has any objection for holding any of these offices

b) Cneeds to be the chairperson

c) B does not want to be the chairperson

d) A is not willing to serve as chairperson or secretary

e) either I or J must be the treasurer

f) E or F or G must hold one of the three offices

25) Find the number of ways of picking each of the following from a standard deck of cards.

a) a king and a queen

b) a king or a queen

c) a king and a red card

d) a king or a red card

26) There are three bridges connecting two towns A and B. Between towns B and C there are four bridges. A salesperson needs to travel from A to C via B. Find:

a) the number of possible choices of bridges from A to C

b) the number of choices for a round trip from A to C and back to A

c) the number of choices for a round trip if no bridge may be crossed twice

27) A sequence of digits in which each digit is 0 or 1 is called binary number. Eight digit binary numbers are often referred to as "bytes."

a) How many bytes are possible?

b) How many bytes begin with 10 and end with $01?$

c) How many bytes begin with 10 but do not end with $01?$

d) How many bytes begin with 10 or end with $01?$

28) A group of 12 is to be seated in a row of chairs. How many ways can this be done if:

a) two people, A and B must be seated next to each other?

b) two people A and B must not be seated next to each other?

29) A variable in the FORTRAN language is a sequence that has at most six characters with the first character being a letter of the alphabet and the remaining characters (if any) being either letters or digits. How many distinct variable names are possible?

30) Four station wagons, five sedans and six vans are to be parked in a row of 15 spaces. Find the nubmer of ways to park the vehicles if:

a) the station wagons are parked at the beginning, then the sedans, then the vans
b) vehicles of the same type must be parked together

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7.5: Distinguishable Permutations

If there is a collection of 15 balls of various colors, then the number of permutations in lining the balls up in a row is ${}_{15}P_{15} = 15!$. If all of the balls were the same color there would only be one distinguishable permutation in lining them up in a row because the balls themselves would look the same no matter how they were arranged.

If 10 of the balls were yellow and the other 5 balls are all different colors, how many distinguishable permutations would there be?

No matter how the balls are arranged, because the 10 yellow balls are indistinguishable from each other, they could be interchanged without any perceptable change in the overall arrangement. As a result, the number of distinguishable permutations in this case would be $\frac{15!}{10!}$, since there are $10!$ rearrangements of the yellow balls for each fixed position of the other balls.

The general rule for this type of scenario is that, given n objects in which there are n_1 objects of one kind that are indistinguishable, n_2 objects of another kind that are indistinguishable and so on, then number of distinguishable permutations will be:

$$\begin{gathered} \frac{n!}{n_1!*n_2!*n_3!*\cdots*n_k!} \\ \text{with } n_1+n_2+n_3+\cdots+n_k=n \end{gathered} \tag{7.5.1}$$

Example
Find the number of ways of placing 12 balls in a row given that 5 are red, 3 are green and 4 are yellow.
This would be $\frac{12!}{5!3!4!} = \frac{12*11*10*9*8*7*6*5*4*3*2*1}{5*4*3*2*3*2*4*3*2}$

$$\begin{aligned} &= \frac{12*11*10*9*8*7*6}{3*2*4*3*2} \\ &= 27,720 \end{aligned} \tag{7.5.2}$$

Another way to think about this problem is to choose five of the twelve spaces in which to place the red balls - since the order of selection is not important, there are ${}_{12}C_5$ ways to do this. Then, from the remaining 7 spaces available, we need to choose three of them in which to place the green balls. There are ${}_7C_3$ ways to do that. The four yellow balls are then placed in the remaining four spaces.

The result of this process is that there are ${}_{12}C_5$ ways to choose the places for the red balls and ${}_7C_3$ ways to choose the places for the green balls, which results in:

$${}_{12}C_5 *_7 C_3 = \frac{12!}{5!7!} * \frac{7!}{3!4!} = \frac{12!}{5!3!4!} \tag{7.5.3}$$

This results in the same answer as when we approached the problem as a permutation. Considering the problem in this way helps us to solve problems which involve the assigning of tasks to groups of individuals.
Example
Fourteen construction workers are to be assigned to three different tasks. Six workers are needed for mixing cement, five for laying bricks and three for carrying the bricks to the brick layers. In how many different ways can the workers be assigned to these tasks?

This is also a problem of distinguishable permutation. Although the order of the workers is not important here, the result is the same:

$$\frac{14!}{6!5!3!} \tag{7.5.4}$$

Another way to think about problems of this type is that they are combination problems, since the order in which the workers are assigned is not important. In that case, we need to select six of the fourteen workers to mix cement, five to lay bricks and three to carry bricks.
To select six workers to mix cement: ${}_{14}C_6 = \frac{14!}{6!8!}$
To select five workers (from the remaining 8) to lay bricks: ${}_8C_5 = \frac{8!}{5!3!}$
To select three workers (from the reamining three) to carry bricks: 1

If there are $\frac{14!}{6!8!}$ ways to choose the cement crew and $\frac{8!}{5!3!}$ ways to choose the bricklayers from the remaining eight workers, then there will be:

$$\frac{14!}{6!8!} * \frac{8!}{5!3!} = \frac{14!}{6!5!3!} \tag{7.5.5}$$

ways to assign the workers to these tasks.

Exercises 7.5

Find the number of distinguishable permutations of the given letters.

1) $AAABBC$
2) $AAABBBCCC$
3) $AABCD$
4) $ABCDDDEE$

5) In how many ways can two blue marbles and four red marbles be arranged in a row?

6) In how many ways can five red balls, two white balls, and seven yellow balls be arranged in a row?

7) In how many different ways can four pennies, three nickels, two dimes and three quarters be arranged in a row?

8) In how many ways can the letters of the word ELEEMOSYNARY be arranged?

9) A man bought three vanilla ice-cream cones, two chocolate cones, four strawberry cones and five butterscotch cones for 14 children. In how many ways can he distribute the cones among the children.

10) When seven students take a trip, they find a hotel with three rooms available - a room for one person, a room for two poeple and a room for three people. In how many different ways can the students be assigned to these rooms? (one student will sleep in the car)

11) Eight workers are cleaning a large house. Five are needed to clean windows, two to clean carpets and one to clean the rest of the house. In how many different ways can these tasks be assigned to the eight workers?

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7.6: Probability

If a coin is tossed three times, what is the probability of getting exactly two heads? at least two heads? no heads?

Probability is defined as the number of ways the event in question can happen divided by the number of total possibilities. We can define the collection of all possible outcomes in an experiement with all outcomes equally likely as the sample space of the experiment. Then the number of ways the event in question can happen can be defined as $n(E)$ and size of the sample space can be defined as $n(S)$. In this notation, the probability of an event $P(E)$ is defined below:

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S} \tag{7.6.1}$$

The question at the opening of this section addresses a situation in which a coin is tossed three times. We can use a tree diagram to examine the sample space of this situation.
The possibilities in this experiment are:
$\{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}$
since there are eight possibilities, with 3 of them having exactly two heads, the probability is $\frac{3}{8} = 0.375$

The probability of getting at least two heads is $\frac{4}{8} = \frac{1}{2}$ or 0.5
The probability of getting no heads is $\frac{1}{8} = 0.125$

Example
A five-card poker hand is drawn from a standard deck of 52 cards. What is the probability that all five cards are hearts?

The number of ways to draw five cards from a deck of 52 is ${}_{52}C_5$. This is a combination since the order that the cards are drawn does not affect the outcome.
The number of ways of choosing 5 hearts from the 13 in the deck is 13C_5
So the probability of drawing five hearts from a deck of 52 cards is $\frac{13C_5}{52C_5} = \frac{1287}{2598960} \approx 0.000495$

Example
A bag contains 20 tennis balls, of which four are defective. If two balls are drawn at random from the bag, what is the probability that both are defective?

The sample space is the number of ways to choose two balls from the bag: ${}_{20}C_2$
since there are 4 defective balls, the number of ways to draw two of them is ${}_4C_2$
So, the probability that both balls will be defective is $\frac{4}{20C_2} = \frac{6}{190} \approx 0.0316$
We can also calculate the probability of drawing one defective ball:
$\frac{{}_4C_1 * {}_{16}C_1}{{}_{20}C_2} = \frac{64}{190} \approx 0.3368$

And the probability of drawing no defective balls:
$\frac{{}_{16}C_2}{{}_{20}C_2} = \frac{120}{190} \approx 0.6316$
Notice that $6 + 64 + 120 = 190$ and that $0.0316 + 0.3368 + 0.6316 = 1$ That is to say that the probabilities for all the possible events should add up to 1

This allows us to compute probabilities based on the probability that an event won't happen.

If the probability of an event is $P(E)$, then the probability that the event will not happen is $1 - P(E)$

Example
An urn contains 10 red balls and 15 green balls. If six of the balls are drawn at random, what is the probability that at least one of them is red?

We can calculate this probability by finding the probability that no red balls are drawn. The number of ways to draw six green balls is ${}_{15}C_6$. The number of ways to draw six balls from the urn is ${}_{25}C_6$. So the probability of drawing six green balls is:
$\frac{{}_{15}C_6}{{}_{25}C_6} = \frac{5005}{177{,}100} \approx 0.02826$
This means that the probability of drawing at least one red ball is $1 - 0.02826 = 0.97174$
We could also find this answer by adding up the possibilities of drawing one red ball, two red balls and so on, up to six red balls:

$\frac{{}_{10}C_1 * {}_{15}C_5}{{}_{25}C_6} + \frac{{}_{10}C_2 * {}_{15}C_4}{{}_{25}C_6} + \frac{{}_{10}C_3 * {}_{15}C_3}{{}_{25}C_6} + \frac{{}_{10}C_4 * {}_{15}C_2}{{}_{25}C_6} + \frac{{}_{10}C_5 * {}_{15}C_1}{{}_{25}C_6} + \frac{{}_{10}C_6}{{}_{25}C_6}$
$= 0.1696 + 0.3468 + 0.3083 + 0.1245 + 0.0213 + 0.0012 =$
0.9717

Example
Given 100 computer components, it is known that 10 of the 100 are defective. If someone were to choose 6 of these components at random, what is the probability that:
a) two of them are defective?
b) at least 1 of them is defective?
There are ${}_{10}C_2$ ways to choose 2 out of the 10 defective components and ${}_{90}C_4$ ways to choose 4 non-defective components. The size of the sample space (the number of ways to choose 6 components from the 100) is ${}_{100}C_6$
So, the probability for part (a) would be:
$\frac{{}_{10}C_2 * {}_{90}C_4}{{}_{100}C_6} \approx 0.096$
The probability in part (b) is most easily computed by finding the probability that no defective components are selected and then subtracting that value from 1
$1 - \frac{{}_{90}C_6}{{}_{100}C_6} \approx 1 - 0.5223 \approx 0.4777$

Exercises 7.6
SET I
1)
a) If a coin is tossed two times, describe the sample space.
b) Find the probability of getting exactly two heads.
c) Find the probability of getting at least one head.
d) Find the probability of getting exactly one head.
2)
a) If a coin is tossed and a single six-sided die is rolled, describe the sample space.
b) Find the probability of getting heads and an even number.
c) Find the probability of getting heads and a number greater than 4
d) Find the probability of getting tails and an odd number.
3) When rolling a single, six-sided die, find the probability of:
a) rolling a six
b) rolling an even number
c) rolling a number greater than 5
4) When rolling a single, six-sided die, find the probability of:
a) rolling a two or a three
b) rolling an odd number
c) rolling a number divisible by three
5) If a card is drawn randomly from a standard 52 card deck, find the probability of:
a) drawing a king
b) drawing a face card
c) drawing a card that is not a face card
6) If a card is drawn randomly from a standard 52 card deck, find the probability of:
a) drawing a heart
b) drawing a heart or a spade
c) drawing a card that is a heart, a diamond or a spade
7) A ball is drawn randomly from an urn that contains 8 balls - five red, two white and one yellow. Find the probability that:
a) a red ball is chosen
b) a yellow ball is not chosen
c) a green ball is chosen
8 , A ball is drawn randomly from an urn that contains 8 balls - five red, two white and one yellow. Find the probability that:
a) the ball drawn is neither yellow nor white
b) the ball drawn is either white, yellow, or red
c) the ball chosen is not white
9) A drawer contains 18 socks of which 6 are red, 4 are white and 8 are black.
a) If one sock is drawn at random from the drawer, what is the probability that it is red?
b) If one red sock is drawn with the first choice, what is the probability that the next sock drawn is also red?
A poker hand is drawn at random from a standard deck of 52 cards.
10) Find the probability of getting five hearts.

11) Find the probability of getting five cards of the same suit.
12) Find the probability of getting five face cards.
13) Find the probability of getting ace, king, queen, jack and ten of the same suit.
14) For problem #14 refer to the sample space in Section 4.1 (or create your own
If a pair of standard six-sided dice are rolled, what is the probability of:
a) rolling a 7
b) rolling a 9
c) rolling doubles (the same number on each die)
d) not rolling doubles
e) rolling 9 or higher

SET II
An unbiased coin is tossed 5 times. Find the probability that:
15) the coin lands heads five times
16) the coin lands heads exactly once
17) the coin lands heads at least once
18) the coin lands heads more than once
Two cards are drawn without replacement from a standard deck of 52 cards. Find the probability that:
19) a pair is drawn
20) a pair is not drawn
21) two black cards are drawn
22) two cards of the same suit are drawn
A jar contains three yellow balls and five red balls. If four balls are drawn at random (without replacement), find the probability that:
23) two of the balls are yellow and two are red
24) all of the balls are red
25) exactly three of the balls are red
26) two or three of the balls are yellow
Assume that the probability of a boy being born is the same as the probability of a
girl being born. Find the probability that a family with three children will have:
27) two boys and one girl
28) at least one girl
29) no boys
30) the two oldest children are girls
31) An exam consists of ten True-or-False questions. If a student guesses at every answer, what is the probability that he or she will answer exactly six questions correctly?
32) A law firm employs 14 lawyers, 8 of whom are partners in the firm. If a group of 3 lawyers is chosen at random to attend a conference, what is the probability that 3 partners will be selected?
33) In a lot of 24 computer components, there are four defective components. If two of the components are chosen at random, what is the probability that:
a) both of the components are defective?
b) at least one of the components is defective?
34) A barrel of 60 apples contains 4 rotten apples. If 3 apples are selected at random, what is the probability that 1 or more of the apples is rotten?
35) A shelf at the home improvement store contains 80 light bulbs of which 6 are defective. If a customer chooses 2 light bulbs at random, what is the probability that:
a) both are defective?
b) at least one is defective?
36) Some computer components are shipped in boxes of 24. Before they are shipped, the quality control inspector randomly chooses 8 components from each box. If any of the 8 components selected are defective, the box is not shipped. What is the probability that a lot containing exactly 2 defective components would be shipped anyway?
37) A business is preparing to choose 12 people to go on a business trip from a group of 100 employees, 60 of whom are women and 40 of whom are men. Suppose that Ed and Mary are both employees and that the 12 people for the trip will be chosen randomly.
a) What is the probability that Ed will be chosen?
b) What is the probability that Ed and Mary will both be chosen?
c) If an equal number of men and women are to go on the trip, what is the probability that Ed will be chosen?
d) If an equal number of men and women are to go on the trip, what is the probability that Ed and Mary will both be chosen?

38) A company has has 50 sales representatives on staff. There are three sales calls that must be made on the east side of town and five sales calls on the west side. If the sales reps are chosen at random for these eight sales calls:

a) What is the probability that an individual sales rep will be chosen for any of the eight sales calls?

b) What is the probability that two sales reps will be selected to make their sales calls on the same side of town.

39) A student studying for a test knows how to do 12 of the 20 problems from the study guide. If the test contains 10 problems chosen at random from the study guide, what is the probability that at least 8 of the problems on the test are problems the student knows how to do?

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8: RIGHT TRIANGLE TRIGONOMETRY

The precursors to what we study today as Trigonometry had their origin in ancient Mesopotamia, Greece and India. These cultures used the concepts of angles and lengths as an aid to understanding the movements of the heavenly bodies in the night sky. Ancient trigonometry typically used angles and triangles that were embedded in circles so that many of the calculations used were based on the lengths of chords within a circle. The relationships between the lengths of the chords and other lines drawn within a circle and the measure of the corresponding central angle represent the foundation of trigonometry - the relationship between angles and distances.

The earliest values for the sine function were calculated by Indian mathematicians in the 5th century. The cosine and tangent, as well as the cotangent, secant and cosecant were developed by Islamic mathematicians by the 11th century. European navigators used these ideas extensively to help calculate distances and direction during the Middle Ages. Modern European trigonometry as we understand it was then developed throughout the Renaissance (1450-1650) and Enlightenment (1650-1800).

8.1: Measuring Angles

8.2: The Trigonometric Ratios

8.3: Solving Triangles

8.4: Applications

8.5: More Applications

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8.1: Measuring Angles

Measuring Angles in Degrees

The two most common units for measuring angles are degrees and radians. Degrees are based on the ancient Mesopotamian assignment of 360° to a complete circle. This has its origin in the division of the horizon of the nighttime sky as the earth takes 365 days to travel around the sun. Because degrees were originally developed by the Mesopotamians, they are often also broken out into 60 unit measures of minutes and seconds. Sixty seconds make one minute and sixty minutes makes one degree.

60 seconds $= 1$ minute or $60'' = 1'$
60 minutes $= 1$ degree or $60' = 1^2$

Angles measured in degrees may also be expressed using decimal portions of a degree, for example:

$72.5^\circ = 72^\circ 30'$

Converting from decimal to DMS

Converting between degrees expressed with decimals and the degrees, minutes, seconds format (DMS) is relatively simple. If you're converting from degrees expressed with decimals to DMS, simply take the portion of the angle behind the decimal point and multiply by 60. In our previous example, we would take the .5 from 72.5° and multiply this by $60 : 0.5^*60 = 30$. So, the angle in DMS units would be $72^\circ 30'$

Examples

Convert 21.85° to DMS units

$0.85 * 60 = 51$

$\text{So}, 21.85^\circ = 21^\circ 51'$

Convert 143.27^{to} DMS units

$0.27 * 60 = 16.2$

So, $143.27^\circ = 143^\circ 16.2'$
In order to compute the number of seconds needed to express this angle in DMS units, we take the decimal portion of the minutes and multiply by 60:
$0.2 * 60 = 12$
So, $143.27^\circ = 143^\circ 16.2' = 143^\circ 16' 12''$

In the example above we ended with a whole number of seconds. If you don'tget a whole number for the seconds then you can leave the seconds with a decimal portion. For example, if you wanted to convert 22.847° to DMS units:

$22.847^\circ = 22^\circ 50.82' = 22^\circ 50' 49.2''$

Converting from DMS to decimal

To convert from DMS units to decimals, simply take the seconds portion and divide by 60 to make it a decimal:

$129^\circ 19' 30'' = 129^\circ 19.5'$

Then take the new minutes portion and divide it by 60

$\frac{19.5}{60} = 0.325$

This is the decimal portion of the angle

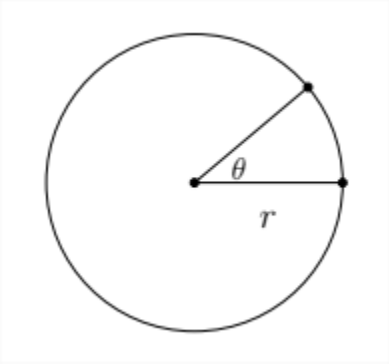
$129^\circ 19' 30'' = 129^\circ 19.5' = 129.325^\circ$

If you end up with repeating decimals in this process that's fine-just indicate the repeating portion with a bar.

Examples
Convert $42^\circ .27' 36''$ to decimal degrees
$\frac{36}{60} = 0.6$
$42^\circ \quad 27' 36'' = 42^\circ \quad 27.6'$
$\frac{27.6}{60} = 0.46$
$42^\circ \quad 27.6' = 42.46^\circ$
Convert $17^\circ 40' 18''$ to decimal degrees
$\frac{18}{60} = 0.3$
$17^\circ 40' 18'' = 17^\circ 40.3'$
$\frac{40.3}{60} = 0.671\overline{6}$
$17^\circ 40' 18'' = 17^\circ 40.3' = 17.671\overline{6}^\circ$

Measuring Angles in Radians

The other most commonly used method for measuring angles is radian measure. Radian measure is based on the central angle of a circle. A given central angle will trace out an arc of a particular length on the circle. The ratio of the arc length to the radius of the circle is the angle measure in radians. The benefit of radian measure is that it is based on a ratio of distances whereas degree measure is not. This allows radians to be used in calculus in situations in which degree measure would be inappropriate.



The length of the arc intersected by the central angle is the portion of the circumference swept out by the angle along the edge of the circle. The circumference of the circle would be $2\pi r$, so the length of the arc would be $\frac{\theta}{360^\circ} * 2\pi r$. The ratio of this arclength to the radius is $\frac{\frac{\theta}{1000}+2\pi r}{r}$ or

$$\frac{2\pi}{360^\circ} * \theta \tag{8.1.1}$$

or in reduced form

$$\frac{\pi}{180^\circ} * \theta \tag{8.1.2}$$

This assumes that the angle has been expressed in degrees to begin with. If an angle is expressed in radian measure, then to convert it into degrees, simply multiply by $\frac{180^\circ}{\pi}$

Examples - Degrees to Radians
Convert 60° to radians
$\frac{\pi}{180^\circ} * 60^\circ = \frac{\pi}{3}$
Convert 142° to radians

$\frac{\pi}{180^\circ} * 142^\circ = \frac{71\pi}{90}$ or $0.7\overline{8}\pi$
Examples - Radians to Degrees
Convert $\frac{\pi}{10}$ to degrees
$\frac{180^\circ}{\pi} * \frac{\pi}{10} = 18^\circ$
Convert $\frac{\pi}{2}$ to degrees
$\frac{180^\circ}{\pi} * \frac{\pi}{2} = 90^\circ$
Another way to convert radians to degrees is to simply replace the π with 180° :
$\frac{\pi}{10} = \frac{180^\circ}{10} = 18^\circ$
$\frac{\pi}{2} = \frac{180^\circ}{2} = 90^\circ$

Exercises 1.1 Convert each angle measure to decimal degrees.
$3.91^\circ 50'$
1. 27×40
4 34%

Unknown environment 'tabular'

10. $141^\circ 6' 9''$
11. $211^\circ 46' 48''$
12. $19^\circ 12' 18''$
Convert each angle measure to DMS notation. 13. 31.425°
14. 159.84°
15. 6.78°
16. 24.56°
19. 18.9°
17. 110.25°
18. 64.16°
19. 18.9°
22. 55.17°
20. 85.14°
21. 220.43°
23. 70.214°
24. 116.32°
Convert each angle measure from degrees to radians.
25. 30°
26. 120°
27. 45°
28. 225°
29. 60°
30. 150°
31. 90°
32. 270°
33. 15°
34. 36°
35. 12°
36. 104°

Convert each angle measure from radians to degrees.
37
38. $\frac{\pi}{5}$
39. $\frac{\pi}{3}$
40. $\frac{\pi}{6}$
4^7
42. $\frac{7\pi}{3}$
43. $\frac{5\pi}{2}$
44. $\frac{7\pi}{4}$
45. $\frac{5\pi}{6}$
46. $\frac{2\pi}{3}$

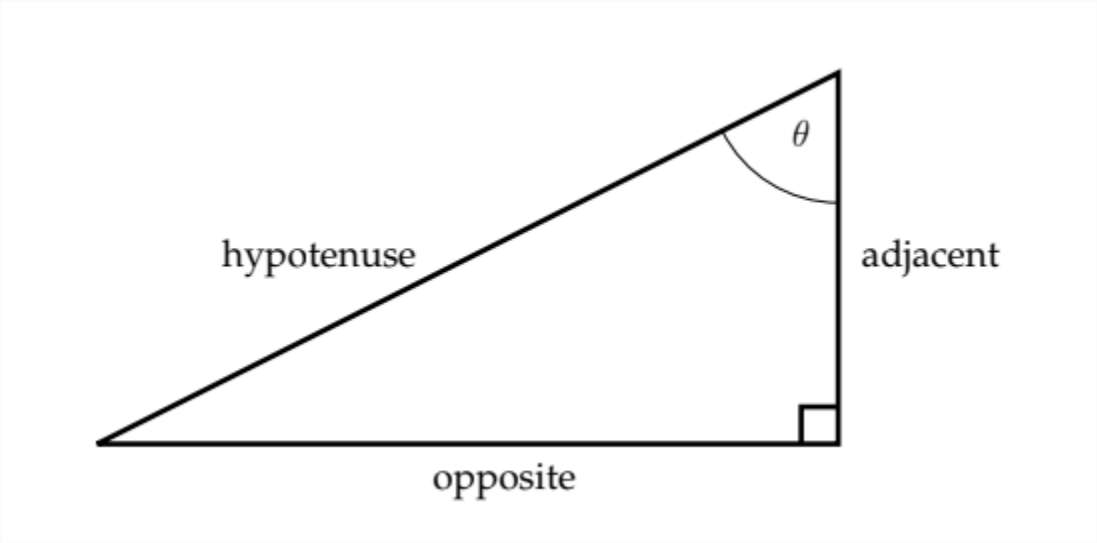
47. π
48. $\frac{7\pi}{2}$

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8.2: The Trigonometric Ratios

There are six common trigonometric ratios that relate the sides of a right triangle to the angles within the triangle. The three standard ratios are the sine, cosine and tangent. These are often abbreviated sin, cos and tan. The other three (cosecant, secant and cotangent) are the reciprocals of the sine, cosine and tangent and are often abbreviated csc, sec, and cot.

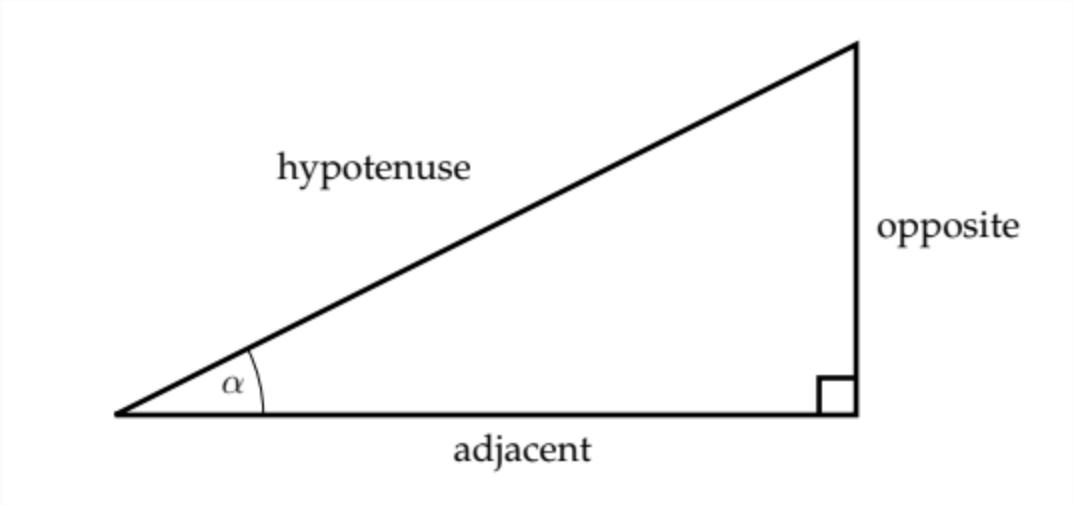


Given an angle situated in a right triangle, the sine function is defined as the ratio of the side opposite the angle to the hypotenuse, the cosine is defined as the ratio of the side adjacent to the angle to the hypotenuse and the tangent is defined as the ratio of the side opposite the angle to the side adjacent to the angle.

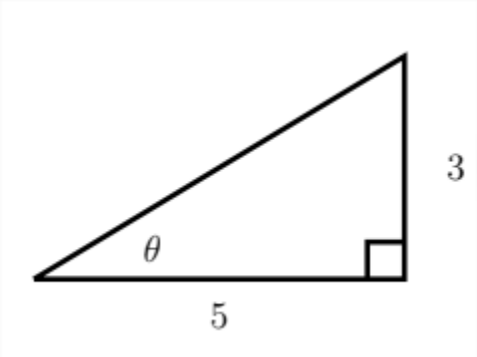
$$\begin{aligned} \sin\theta &= \frac{opp}{hyp} \\ \cos\theta &= \frac{adj}{hyp} \\ \tan\theta &= \frac{opp}{adj} \end{aligned} \tag{8.2.1}$$

A common mneumonic device to help remember these relationships is
-SOHCAHTOA-which identifies the sin as Opp over Hyp Cos as Adj over Hyp and the Tan as Opp over Adj.

An acute angle placed in the other position of a right triangle would have different oppposite and adjacent sides although the hypotenuse would remain the same.



Examples: Trigonometric Ratios
Find $\sin\theta, \cos\theta$ and $\tan\theta$ for the given angle θ



In order to find the sin and cos of the angle θ, we must first find the hypotenuse by using the Pythagorean Theorem $(a^2 + b^2 = c^2)$

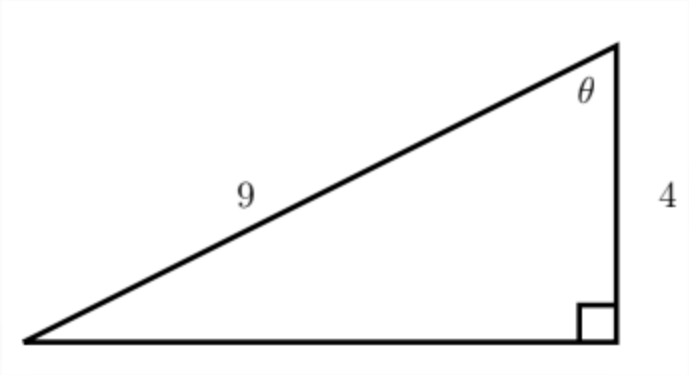
since we know the legs of the triangle, we can substitute these values for a and b in the Pythagorean Theorem:

$$\begin{aligned} 3^2 + 5^2 &= c^2 \\ 9 + 25 &= c^2 \\ 34 &= c^2 \\ \sqrt{34} &= c \end{aligned} \tag{8.2.2}$$

Now that we know the hypotenuse $(\sqrt{34})$, we can determine the sin, cos and tan for the angle θ

$$\begin{aligned} \sin\theta &= \frac{3}{\sqrt{34}} \\ \cos\theta &= \frac{5}{\sqrt{34}} \\ \tan\theta &= \frac{3}{5} \end{aligned} \tag{8.2.3}$$

Find $\sin\theta, \cos\theta$ and $\tan\theta$ for the given angle θ



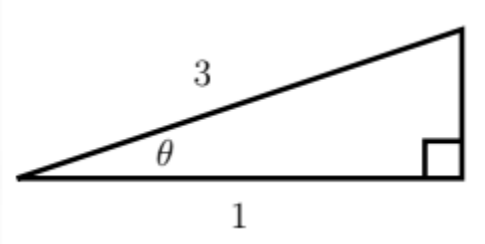
Again, in order to find the sin, cos and tan of the angle θ, we must find the missing side of the triangle by using the Pythagorean Theorem. since, in this case, we know the hypotenuse and one of the legs, the value of the hypotenuse must be substituted for c and the length of the leg we're given can be substituted for either
a or b

$$\begin{aligned} 4^2 + b^2 &= 9^2 \\ 16 + b^2 &= 81 \\ b^2 &= 65 \\ b &= \sqrt{65} \end{aligned} \tag{8.2.4}$$

Now that we know the length of the other leg of the triangle $(\sqrt{65})$, we can determine the sin, cos and tan for the angle θ

$$\begin{aligned} \sin\theta &= \frac{\sqrt{65}}{9} \\ \cos\theta &= \frac{4}{9} \\ \tan\theta &= \frac{\sqrt{65}}{4} \end{aligned} \tag{8.2.5}$$

In addition to the examples above, if we are given the value of one of the trigonometric ratios, we can find the value of the other two.
Example
Given that $\cos\theta = \frac{1}{3}$, find $\sin\theta$ and $\tan\theta$
Given the information about the cosine of the angle θ, we can create a triangle that will allow us to find $\sin\theta$ and $\tan\theta$



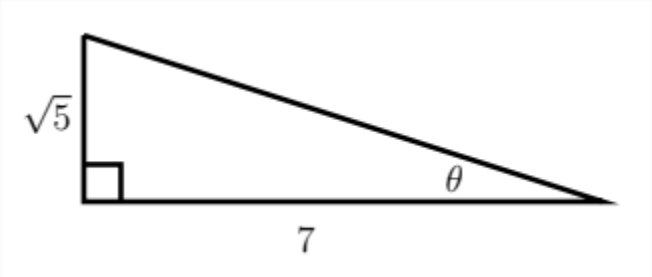
Using the Pythagorean Theorem, we can find the missing side of the triangle:

$$\begin{aligned} a^2 + 1^2 &= 3^2 \\ a^2 + 1 &= 9 \end{aligned} \tag{8.2.6}$$

$a^2 = 8$
$a = \sqrt{8} = 2\sqrt{2}$
Then $\sin\theta = \frac{\sqrt{8}}{3}$ and $\tan\theta = \frac{\sqrt{8}}{1} = \sqrt{8}$
You might say to yourself, "Wait a minute, just because the cosine of the angle θ is $\frac{1}{3}$, that doesn't necessarily mean that the sides of the triangle are 1 and 3, they could be 2 and 6, or 3 and 9 or any values n and $3n$. "

This is true, and if the sides are expressed as n and $3n$, then the missing side would be $n\sqrt{8}$, so that whenever we find a trigonometric ratio, the n' s will cancel out, so we just leave them out to begin with and call the sides 1 and 3

Example
Given that $\tan\theta = \frac{\sqrt{5}}{7}$, find $\sin\theta$ and $\cos\theta$
First we'll take the infomation about the tangent and use this to draw a triangle.



Then use the Pythagorean Theorem to find the missing side of the triangle:

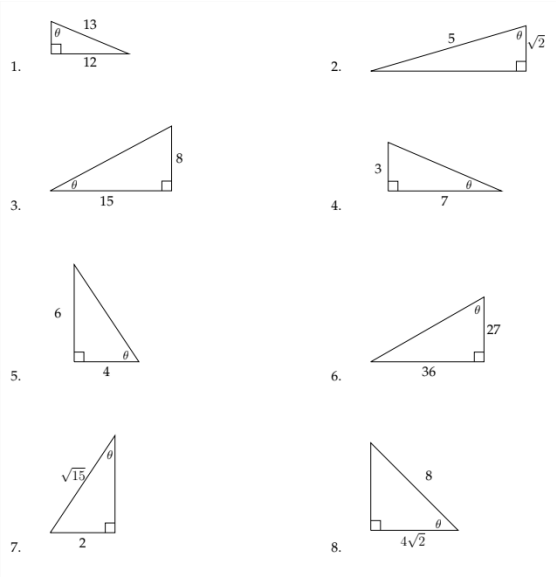
$$\begin{aligned} \sqrt{5}^2 + 7^2 &= c^2 \\ 5 + 49 &= c^2 \\ 54 &= c^2 \\ \sqrt{54} = 3\sqrt{6} &= c \end{aligned} \tag{8.2.7}$$

So then:

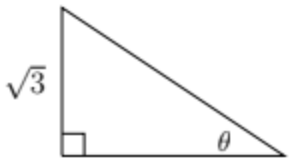
$$\sin\theta = \frac{\sqrt{5}}{\sqrt{54}} = \sqrt{\frac{5}{54}} \tag{8.2.8}$$

$$\cos\theta = \frac{7}{\sqrt{54}} = \frac{7}{3\sqrt{6}} \tag{8.2.9}$$

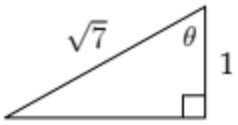
Exercises 1.2
Find $\sin\theta, \cos\theta$ and $\tan\theta$ for the given triangles.



9.



10.



Use the information given to find the other two trigonometric ratios.

11. $\tan\theta = \frac{1}{2}$
12. $\sin\theta = \frac{3}{4}$
13. $\cos\theta = \frac{3}{\sqrt{20}}$
14. $\tan\theta = 2$
15. $\sin\theta = \frac{5}{\sqrt{40}}$
16. $\sin\theta = \frac{7}{10}$
17. $\cos\theta = \frac{9}{40}$
18. $\tan\theta = \sqrt{3}$
19. $\cos\theta = \frac{1}{2}$
20. $\cos\theta = \frac{3}{7}$
21. $\sin\theta = \frac{\sqrt{5}}{7}$
22. $\tan\theta = 1.5$

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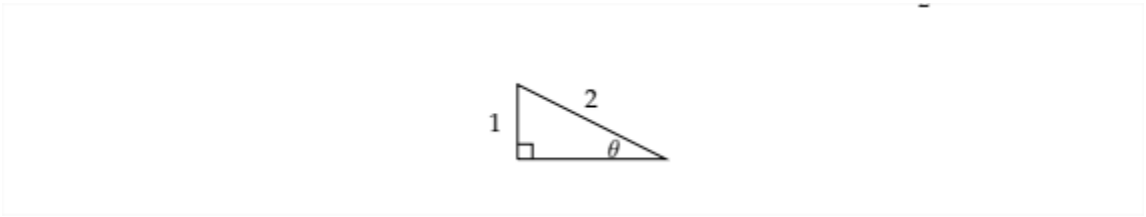
8.3: Solving Triangles

Using information about some of the sides and angles of a triangle in order to find the unknown sides and angles is called "solving the triangle." If two sides of a triangle are known, the Pythagorean Theorem can be used to find the third side. If one of the acute angles in a right triangle is known, the other angle will be its complement with their sum being 90°

Suppose that we have a right triangle in which we know the sides, but no angles. Another situation could involve knowing the angles but just one side. How could we solve for the missing measurements in these situations?

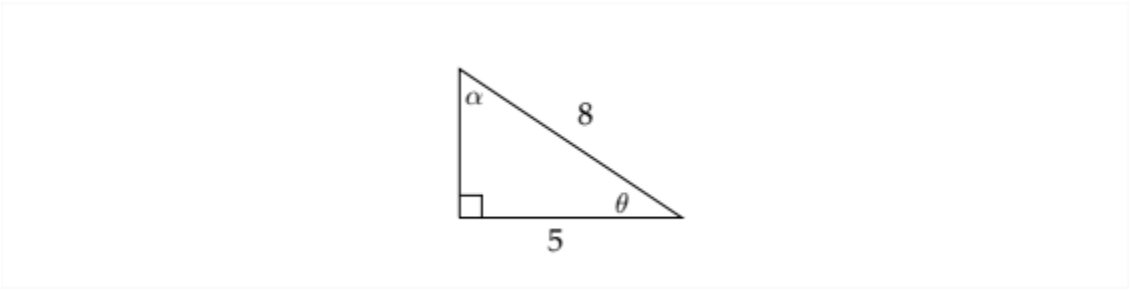
Solving problems like these uses precalculated values of the trigonometric ratios to match the lengths with the appropriate angles and vice versa. Up until the 1980's, these values were printed in tables that were included in the back of every textbook (along with tables of logarithms), but have recently been programmed into calculators using methods that are studied in Calculus.

Most calculators have a button or function designed to find the inverse sine, inverse cosine and inverse tangent $\left(\sin^{-1}, \cos^{-1}, \text{and } \tan^{-1}\right)$, these are the functions that tell you the measure of the angle that has a sine, cosine or tangent equal to a particular value.
For example, if we are given an angle θ and know that the $\sin\theta = \frac{1}{2}$



Then we can find $\sin^{-1}\left(\frac{1}{2}\right)$ on a calculator, which should return a value of 30°. If the calculator is in radian mode, it will return a value of ≈ 0.523598776. If you divided this number by π, you would get $0.1\overline{6}$, which means that $0.523598776 \approx \frac{\pi}{6}$ In this chapter we will work mainly in degrees. In Chapter 2, when we graph the trigonometric functions we will typically use radian measure.

Solve the triangle. Round side lengths to the nearest 100th and angles to the nearest 10th of a degree.



We can find the third side of the triangle by using the Pythagorean Theorem.

$$\begin{aligned} a^2 + 5^2 &= 8^2 \\ a^2 + 25 &= 64 \end{aligned} \tag{8.3.1}$$

$a^2 = 39$

$$a = \sqrt{39} \approx 6.24 \tag{8.3.2}$$

When solving problems of this type, I encourage people to use the most accurate values that are available in the problem. This way, there is less chance for rounding error to occur.

If we take the values for the sides that were given in the problem (5 and 8), then we can say that

$$\begin{aligned} \cos\theta &= \tfrac{5}{8} \\ \theta &= \cos^{-1}\left(\tfrac{5}{8}\right) \\ \theta &\approx 51.3^\circ \end{aligned} \tag{8.3.3}$$

Then α would be $90^\circ - 51.3^\circ = 38.7^\circ \approx \alpha$

Example 2
Solve the triangle. Round side lengths to the nearest 100th and angles to the nearest 10th of a degree.



First, we can find the other angle in the right triangle: $90^\circ - 68^\circ = 22^\circ$
Next, to find the sides, we choose a trigonometric ratio for which we know one of the sides. In this problem, we can use either the sine or the cosine.
$\sin 68^\circ = \frac{a}{5}$
Approximating $\sin 68^\circ$ on a calculator:
$0.9272 \approx \frac{a}{5}$

$$5 * 0.9272 \approx a \tag{8.3.4}$$

$4.6 \approx a$
When approximating a trigonometric value from the calculator, it is important to use at least 4 decimal places of accuracy. Again, this is to avoid rounding errors.
To solve for the remaining side we can either use the Pythagorean Theorem or use the method demonstrated above, but with the $\cos 68^\circ$

$\cos 68^\circ = \frac{b}{5}$
Approximating cos 68° on a calculator:

$$\begin{aligned} 0.3746 &\approx \tfrac{b}{5} \\ 5 * 0.3746 &\approx b \\ 1.9 &\approx b \end{aligned} \tag{8.3.5}$$

If we use the Pythagorean Theorem with two sides of the triangle to find the third, then we would say that:

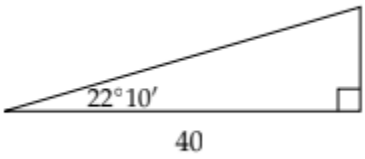
$$\begin{aligned} b^2 + 4.6^2 &= 5^2 \\ b^2 + 21.16 &= 25 \\ b^2 &= 3.84 \\ b &= \sqrt{3.84} \approx 1.959 \approx 2.0 \end{aligned} \tag{8.3.6}$$

The rounding error in this example comes from the fact that the first side we found was not exactly 4.6. If we wanted a more accurate answer that matches the answer we found using the cosine ratio, we just need more accuracy in the leg of the triangle we found.
Calculating $5 * \sin 68^\circ \approx 4.636$ should provide enough accuracy.

$$\begin{aligned} b^2 + 4.636^2 &= 5^2 \\ b^2 + 21.492496 &= 25 \\ b^2 &= 3.507504 \\ b &= \sqrt{3.507504} \approx 1.8728 \approx 1.9 \end{aligned} \tag{8.3.7}$$

Example 3

Solve the triangle. Round side lengths to the nearest 100th and angles to the nearest 10th of a degree.



If we convert the angle $22^\circ 10'$ to $22.1\overline{6}^\circ$, then the other acute angle in the right triangle is $90^\circ - 22.1\overline{6}^\circ = 67.8\overline{3}^\circ$ or $67^\circ 50'$

Finding the remaining sides requires the use of either the cosine or tangent function.

$$\cos 22.1\overrightarrow{6}^\circ = \frac{40}{c} \tag{8.3.8}$$

Approximating cos $22.1\overline{6}^\circ$ on a calculator:

$$0.9261 \approx \frac{40}{c} \tag{8.3.9}$$

Next we need to multiply on both sides by c :

$$c * 0.9261 \approx \frac{40}{c} * c \tag{8.3.10}$$

$0.9261c \approx 40$

Then divide on both sides by 0.9261:

$$\begin{aligned} \tfrac{0.9261c}{0.9261} &\approx \tfrac{40}{0.9261} \\ c &\approx 43.2 \end{aligned} \tag{8.3.11}$$

To find the other leg of the triangle, we can use the tangent ratio.

$$\tan 22.1\overline{6}^\circ = \frac{a}{40} \tag{8.3.12}$$

Approximating tan $22.1\overline{6}^\circ$ on a calculator:

$$0.4074 \approx \frac{a}{40} \tag{8.3.13}$$

Then, multiply on both sides by 40:

$$\begin{aligned} 40 * 0.4074 &\approx a \\ 16.3 &\approx a \end{aligned} \tag{8.3.14}$$

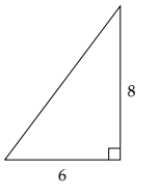
We can check this answer by the Pythagorean Theorem:

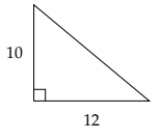
$$16.3^2 + 40^2 = 1865.69 \tag{8.3.15}$$

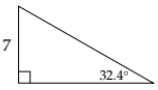
$$\sqrt{1865.69} \approx 43.2 \tag{8.3.16}$$

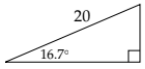
Exercises 1.3

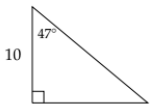
In each problem below, solve the triangle. Round side lengths to the nearest 100th and angle measures to the nearest 10th of a degree.

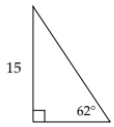
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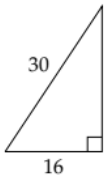
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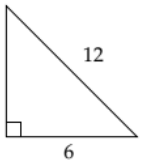
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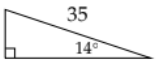
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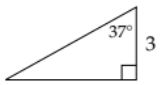
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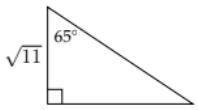
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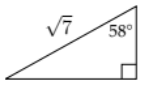
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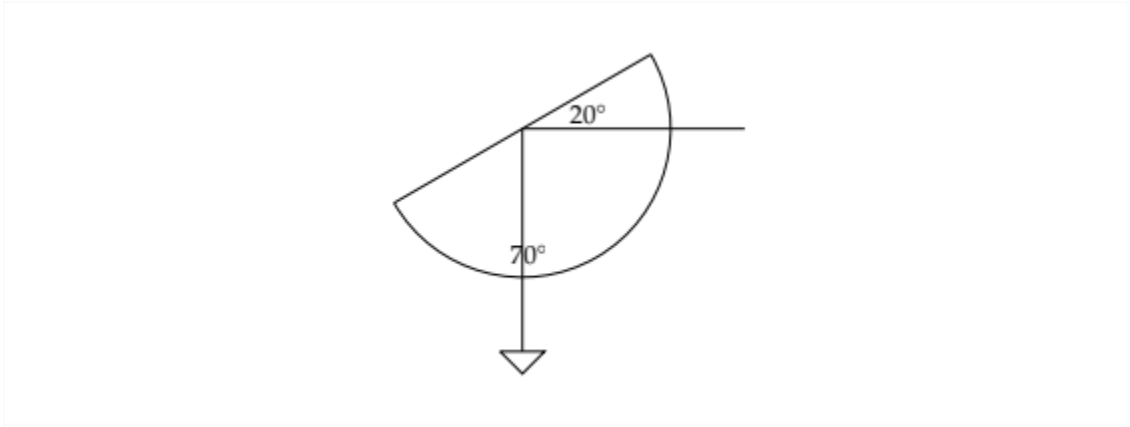
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8.4: Applications

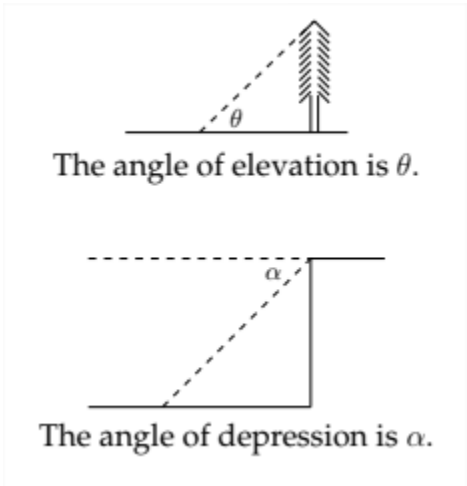
1.4 Applications
Trigonometry is often used for what is called "indirect measurement." This is a method of measuring inaccessible distances by using the relationships between lengths and angles within a triangle. Two simple examples of this process are measuring the height of a tall tree and measuring the distance across a body of water. In both cases, while it might be possible to measure the distance directly, it is often much easier to use indirect measurement.

In one example of indirect measurement, the angle of elevation of an object can be used to create a right triangle in which one angle and one side are known. The other sides of the triangle may then be solved for. In the problems in this text, the angle of elevation will typically be given in the problem. In order to actually measure the angle of elevation of an object, it is possible to use a simple protractor.

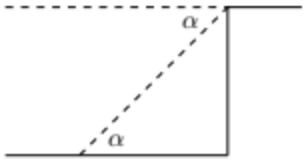
If you wanted to measure the height of a tall tree that sits on flat ground, you could use a specially modified protractor to do this. Modifying the protractor by tying a weight to the end of a string and tying the other end of the string through the hole in the protractor will help to measure the angle of elevation. Once the protractor is ready, hold it upside down and sight the top of the tree along the straight edge of the protractor. The weight hanging down will show the complement to the angle of elevation. In other words, if the angle of elevation is 20°, the string will mark out a measurement of 70° on the protractor.



Closely related to the concept of the angle of elevation is the angle of depression. This is the angle that is formed by looking down on something from above.



In a situation in which the angle of depression is measured, the angle of elevation and the angle of depression are alternate interior angles, which makes them equal.

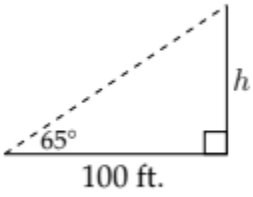


The angle of elevation equals the corrresponding angle of depression.

Example 1

Pacing off 100ft on flat ground from the base of a tree, a forester measures the angle of elevation to the top of the tree as 65°. What is the height of the tree?

The situation described in the problem creates a diagram like the one below:



since this is a right triangle, we can use an appropriate trigonometric ratio to find the height of the tree. In this case,

$$\tan 65^\circ = \frac{h}{100} \tag{8.4.1}$$

$100 * \tan 65^\circ = h$

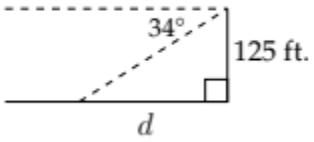
$214.45 \approx h$

So, the tree is about 214.45 feet tall.

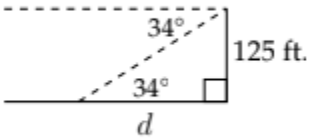
Example 2

From the top of a building 125 feet tall, the angle of depression of an intersection is 34°. How far from the base of the building is the intersection?

As in the previous example, it is often helpful to draw a diagram.



Again, the angle of elevation will be equal to the corresponding angle of depression, so we can use the triangle as seen below to solve the problem:



In this problem

$$\tan 34^\circ = \frac{125}{d} \tag{8.4.2}$$

Multiply on both sides by d

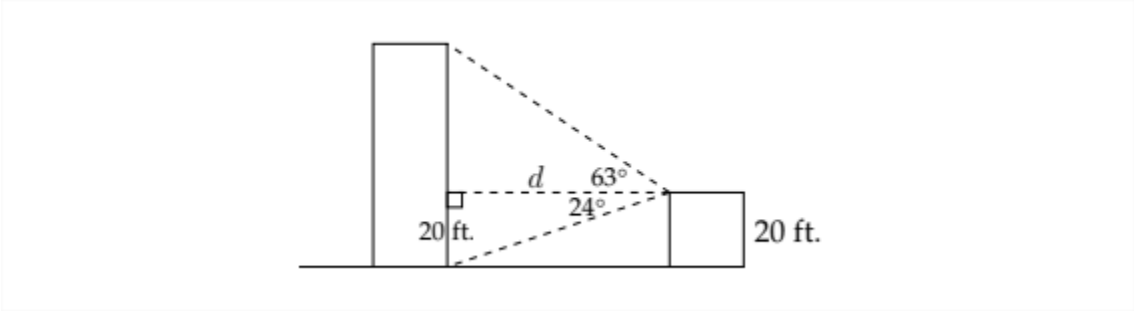
$$d * \tan 34^\circ = 125 \tag{8.4.3}$$

Then divide on both sides by $\tan 34^\circ$

$$\begin{gathered} \frac{d \tan 34^\circ}{\tan 34^\circ} = \frac{125}{\tan 34^\circ} \\ d \approx 185.32 \end{gathered} \tag{8.4.4}$$

Example 3
Sometimes a problem involves both an angle of elevation and an angle of depression.

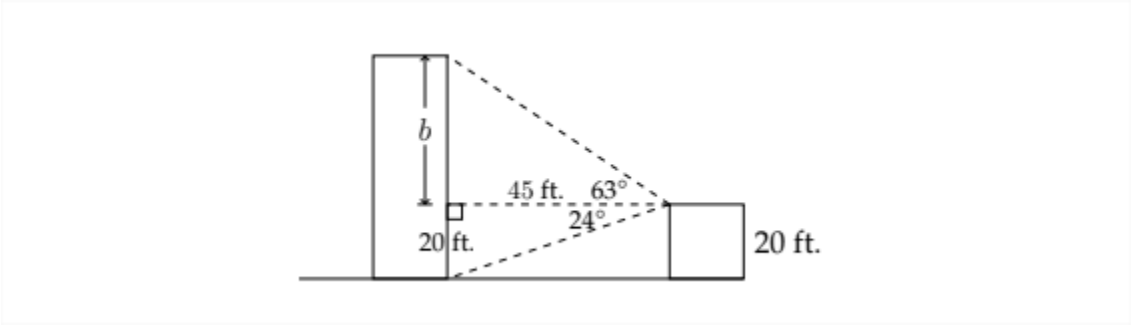
From the roof of a house 20 feet off the ground, the angle of elevation of the top of an apartment building is 63° and the angle of depression to the base of the building is 24°. How far away from the house is the apartment building? How tall is the apartment building?



If we work on the bottom triangle first, then we know that the height of the triangle is 20ft and the angle opposite this side is 24°. So, we can say that:

$$\begin{gathered} \tan 24^\circ = \frac{20}{d} \\ d = \frac{20}{\tan 24^\circ} \\ d \approx 45\text{ft} \end{gathered} \tag{8.4.5}$$

Now that we know that the apartment building is 45 feet away, we can use the upper triangle to determine the height ofthe building.

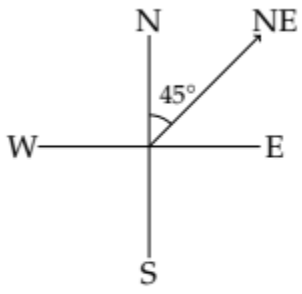


$$\begin{gathered} \tan 63^\circ = \frac{b}{45} \\ 45 * \tan 63^\circ = b \\ 88.3\text{ft.} \approx b \\ 20 + 88.3 = 108.3\text{ft} \end{gathered} \tag{8.4.6}$$

since the variable b only represents the part of the building that is in the second triangle, we need to add 20 feet to b to find the actual height of the building.

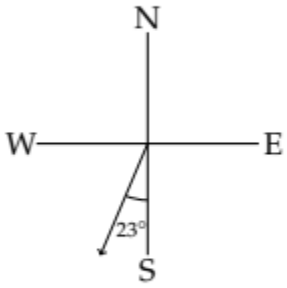
Problems on bearing and direction
Some applications of trigonometry involve ship navigation. One common method used to describe direction in this type of problem is what is known as bearing. Bearing describes a direction by the angle deviation from north or south. For example, the direction we typically describe as northeast is exactly halfway between north and east. The bearing for northeast would be $N45^\circ E$

, and is read as "Forty-five degrees east of north."

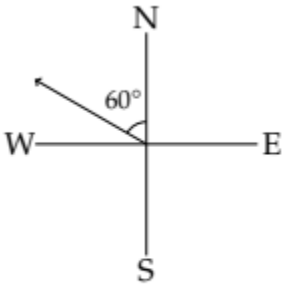


Forty-five degrees East of North: N45°E

Here are a few examples of what a bearing looks like in a N-S-E-W diagram.

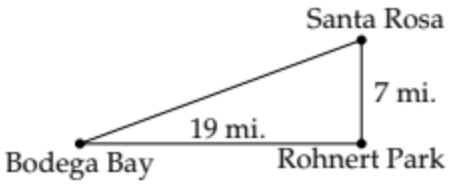


Twenty-three degrees West of South: S23°W

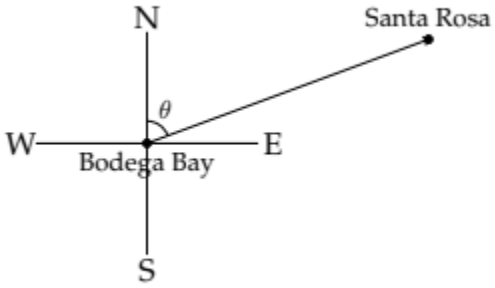


Sixty degrees West of North: N60°W

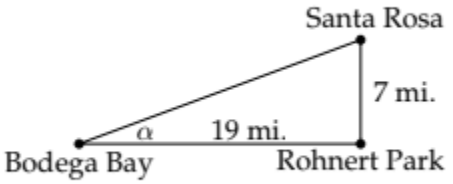
Example 4
Santa Rosa, California is 7 miles due north of Rohnert Park. Bodega Bay is 19 miles due west of Rohnert Park (as the crow flies). What is the bearing of Santa Rosa from Bodega Bay?
First, it would be helpful to draw a diagram to represent the situation:



To answer the question we'll need another diagram:



If we knew the angle θ, then we could conclude that the bearing of Santa Rosa from Bodega Bay is θ degrees East of North. From the previous diagram:



we can see that we can't find θ directly, but we can find the complement of θ

$$\tan \alpha = \frac{7}{19} \tag{8.4.7}$$

$\alpha \approx 20.2^\circ$

$$\text{Therefore, } \theta \approx 90^\circ - 20.2^\circ \approx 69.8^\circ \tag{8.4.8}$$

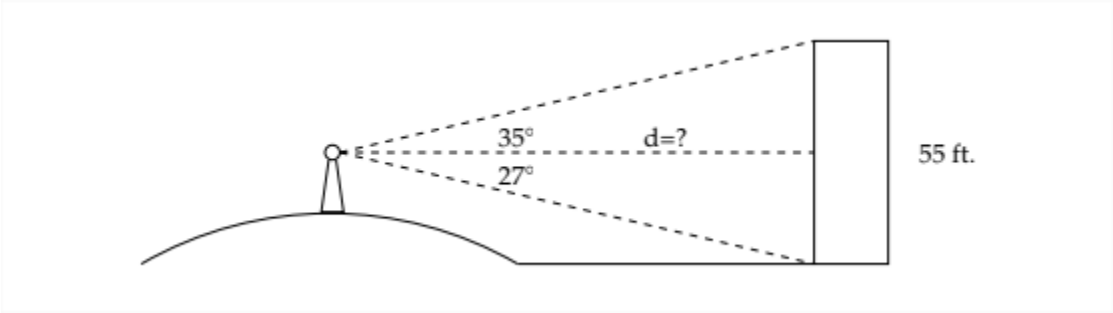
That means that the bearing of Santa Rosa from Bodega Bay is $N69.8^\circ E$, or 69.8° East of North.

Exercises 1.4

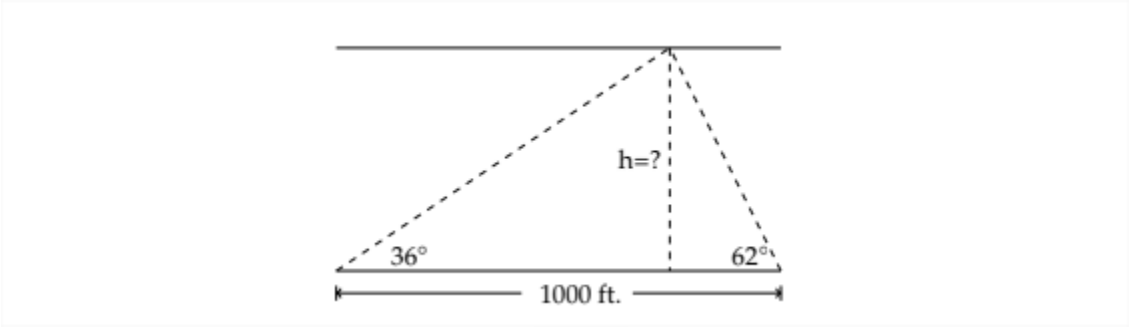
Round answers to the nearest 10th

1. From the top of a lighthouse 180 feet above sea level, the angle of depression to a ship in the ocean is 28°. How far is the ship from the base of the lighthouse?
2. A helicopter that is 700 feet in the air measures the angle of depression to a landing pad as 24°. How far is the landing pad from the point directly beneath the helicopter's current position?
3. An 88 foot tree casts a shadow that is 135 feet long. What is the angle of elevation of the sun?
4. A 275 foot guy wire is attached to the top of a communication tower. If the wire makes an angle of 53° with the ground, how tall is the tower?
5. A woman standing on a hill sees a building that she knows is 55 feet tall. The angle of depression to the bottom of the building

is 27° and the angle of elevation to the top of the building is 35°. Find the straight line distance from the woman to the building.



6. To measure the height of the cloud cover at an airport, a spotlight is shined upward at an angle of 62°. An observer 1000 feet away measures the angle of elevation to the spotlight to be 36°. Find the height of the cloud cover.



7. At ground level, a water tower is 430 feet from the base of a building. From one of the upper floors of the building, the angle of elevation to the top of the water tower is 15° and the angle of depression to the bottom of the water tower is 28°. How tall is the water tower? How high off the ground is the observer?
8. A small airplane is flying at the altitude of 7000 feet following a straight road directly beneath it. A car in front of the airplane is sighted with an angle of depression of 72° and a car behind the plane is sighted with an angle of depression of 48°. How far apart are the cars?
9. From a point on the floor, the angle of elevation to the top of a doorway is 43". The angle of elevation to the ceiling directly above the doorway is 56°. If the ceiling is 10 feet above the floor, how high is the doorway? How far in front of the doorway were the angles of elevation measured?
10. A man standing on the roof of a building 70 feet high looks at the building next door. The angle of depression to the roof of the building next door is 36° The angle of depression to the bottom of the building next door is 65°. How tall is the building next door?
11. A boat leaves the harbor and travels 30 miles in the direction of $N38^\circ W$ The boat turns 90° and then travels in the direction $S52^\circ W$ for 12 miles. At that time, how far is the boat from the harbor and what is the bearing of the boat from the harbor entrance?

12. A man walking in the desert travels 1.6 miles in the direction $S57^\circ E$. He then turns 90° and continues walking for 3.2 miles in the direction $N33^\circ E$. At that time, how far is he from his starting point and what is his bearing from the starting point?
13. Madras, Oregon is 26 miles due north of Redmond. Prineville is due east of Redmond and $S34^\circ 42' E$ from Madras. How far is Prineville from Redmond?
14. Raymond, Washington is 22 miles due south of Aberdeen. Montesano is due east of Aberdeen and $N26^\circ 34' E$ from Raymond. How far is Montesano from Raymond?
15. A boat travels on a course bearing of $S41^\circ 40' W$ for 84 miles. How far south and how far west is the boat from its starting point?
16. A boat travels on a course bearing of $N17^\circ 10' E$ for 10 miles. How far north and how far east is the boat from its starting point?

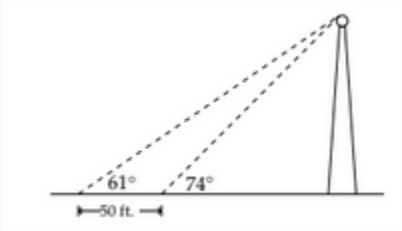
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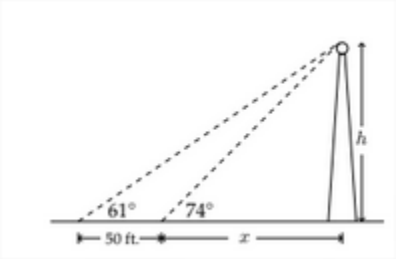
8.5: More Applications

1.5 More Applications

Sometimes solving problems involving right triangles requires the use of a system of equations. A common method for determining the height of an object whose base is inaccessible is that of measuring the angle of elevation from two different places in front of the object. If you measure the angle of elevation to the top of of a radio antenna as 74 , then walk back 50 feet and measure the angle of elevation to the top of the antenna as 61 ', then we would have something like the diagram below:



One of the first things we can do is introduce some labels for the unknown distances:



Then, we can say that:

$\tan 74^\circ = \frac{h}{x}$

$\tan 61^\circ = \frac{h}{x+50}$

To solve this system of equations, we'll set the first one equal to h :

$$\begin{aligned} \tan 74^\circ &= \tfrac{h}{x} \\ x * \tan 74^\circ &= h \end{aligned} \tag{8.5.1}$$

Then, substitute this into the second equation:

$$\tan 61^\circ = \frac{h}{x+30} \tag{8.5.2}$$

$$\tan 61^\circ = \frac{x \tan 74^\circ}{x+50} \tag{8.5.3}$$

Multiply on both sides by $x+50$:

$(x+50)\tan 61^\circ = \frac{x \tan 74^\circ}{x+30}(x+50)$

$(x+50)\tan 61^\circ = x \tan 74^\circ$

There are two options to solve this equation - we can hold on to the tangents as they are and solve for x in terms $\tan 74^\circ$ and $\tan 61^\circ$, or we can approximate $\tan 74^\circ$ and $\tan 61^\circ$ and generate an approximate value for x and h. First we'll approximate:

$(x+50)\tan 61^\circ = x \tan 74^\circ$

$$(x+50) * 1.804 \approx 3.4874x \tag{8.5.4}$$

$1.804x + 90.2024 \approx 3.4874x$

$90.2024 \approx 1.6834x$

$$53.58 \approx x \tag{8.5.5}$$

$x * \tan 74^\circ = h$

$$\begin{aligned} 53.58 * \tan 74^\circ &\approx h \\ 186.87 \text{ feet } &\approx h \end{aligned} \tag{8.5.6}$$

The other method is a little tricky algebraically:

$$(x+50)\tan 61^\circ = x\tan 74^\circ \tag{8.5.7}$$

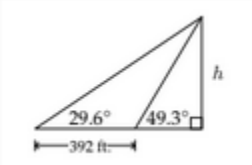
$$\begin{aligned} x\tan 61^\circ + 50\tan 61^\circ &= x\tan 74^\circ \\ 50\tan 61^\circ &= x\tan 74^\circ - x\tan 61^\circ \\ 50\tan 61^\circ &= x\left(\tan 74^\circ - \tan 61^\circ\right) \\ \tfrac{50\tan 61^\circ}{(\tan 74^\circ - \tan 61^\circ)} &= x \end{aligned} \tag{8.5.8}$$

At this point, you can approximate the value of x and solve for h, or express the value of h exactly as

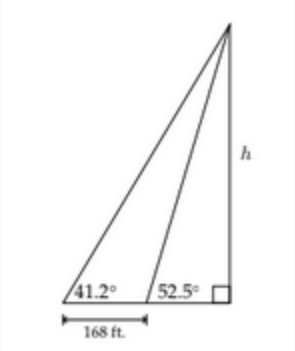
$$\tan 74^\circ * \frac{50\tan 61^\circ}{(\tan 74^\circ - \tan 61^\circ)} = h \tag{8.5.9}$$

Exercises 1.5

1. Find the indicated height h



2. Find the indicated height h



3. A small airplane flying at an altitude of 5300 feet sights two cars in front of the plane traveling on a road directly beneath it. The angle of depression to the nearest car is 62° and the angle of depression to the more distant car is 41°. How far apart are the cars?

4. A hot air balloon is flying above a straight road. In order to estimate their altitude, the people in the balloon measure the angles of depression to two consecutive mile markers on the same side of the balloon. The angle to the doser marker is 17° and the angle to the farther one is 13°. At what altitude is the balloon flying?

5. To estimate the height of a mountain, the angle of elevation from a spot on level ground to the top of the mountain is measured to be 32°. From a point

1000 feet closer to the mountain, the angle of elevation is measured to be 35° How high is the mountain above the ground from which the measurements were taken?

6. The angle of elevation from a point on the ground to the top of a pyramid is $35^\circ 30'$. The angle of elevation from a point 135 feet farther back to the top of the pyramid is $21^\circ 10$. What is the height of the pyramid?

7. An observer in a lighthouse 70 feet above sea level sights the angle of depression of an approaching ship to be $15^\circ 50'$. A few minutes later the angle of depression is sighted at $35^\circ 40'$. Find the distance traveled by the ship during that time.

8. To estimate the height of a tree, one forester stands due west of the tree and another forester stands due north of the tree. The two foresters are the same distance from the base of the tree and they are 45 feet from each other. If the angle of elevation for each forester is 40°, how tall is the tree?

9. A ship is anchored off of a long straight shoreline that runs east to west. From two observation points located 10 miles apart on the shoreline, the bearings of the ship from each observation point are $S35^\circ E$ and $S17^\circ W$. How far from shore is the ship?

10. From fire lookout Station Alpha the bearing of a forest fire is $N52^\circ E$. From lookout Station Beta, sited 6 miles due east of Station Alpha, the bearing is $N38^\circ$W How far is the fire from Station Alpha?

11. From a point 200 feet from the base of a church, the angle of elevation to the top of the steeple is 28°, while the angle of elevation to the bottom of the steeple is 20°. How high off the ground is the top of the steeple?

12. A television tower 75 feet tall is installed on the top of a building. From a point on the ground in front of the building, the angle of elevation to the top of the tower is 62° and the the angle of elevation to the bottom of the tower is 44°. How tall is the building?

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9: GRAPHING THE TRIGONOMETRIC FUNCTIONS

9.1: Trigonometric Functions of Non-Acute Angles

9.2: Graphing Trigonometric Functions

9.3: The Vertical Shift of a Trigonometric Function

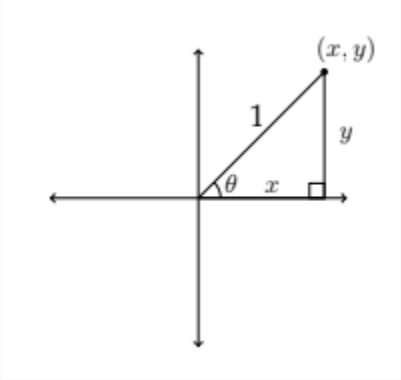
9.4: Phase Shift

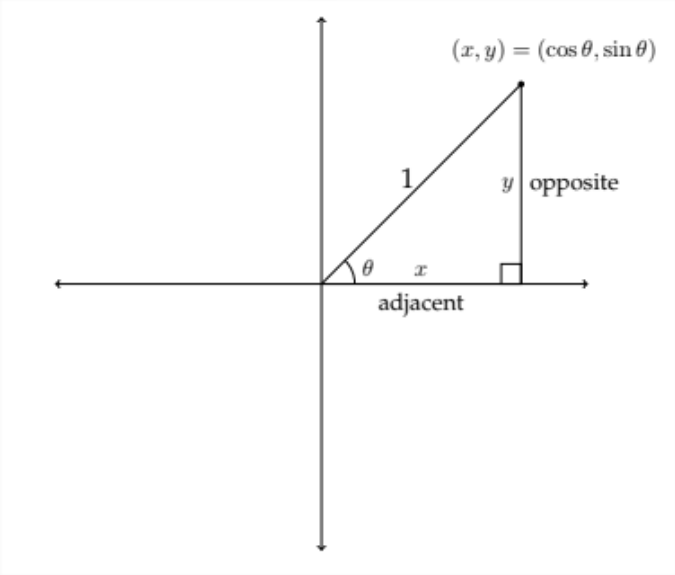
9.5: Combining the Transformations

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9.1: Trigonometric Functions of Non-Acute Angles

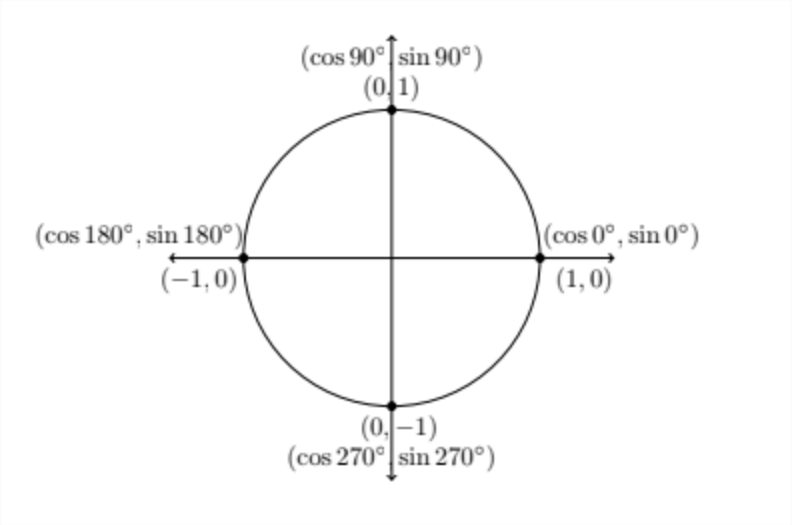
2.1 Trigonometric Functions of Non-Acute Angles
In Chapter 1, we learned about the trigonometric functions of positive acute angles that occur within right triangles. If we wish to extend the definition of the trigonometric functions, then we need to define how to determine the values for the sine and cosine of other angles. To do this, consider a right triangle drawn on the coordinate axes. The positive acute angle θ will be the angle created between the x -axis and the hypotenuse of the triangle. The lengths of the two legs of the triangle will be the x and y coordinates of a point in the first quadrant.





In the picture above we see a triangle in the first quadrant with a hypotenuse of
1. In this situation, the value of $\sin\theta = \frac{opp}{hyp} = \frac{y}{1} = y$, which is just the y -coordinate of the point at the top of the triangle. Correspondingly, the value of $\cos\theta = \frac{adj}{hyp} = \frac{x}{1} = x$, or the value of the x -coordinate of the same point.

This allows us to find the sine or cosine for what are known as the quadrantal angles - the angles that are multiples of 90°. If we look at the unit circle (the circle with a radius of 1), then we can see the values of the sine and cosine for these angles.

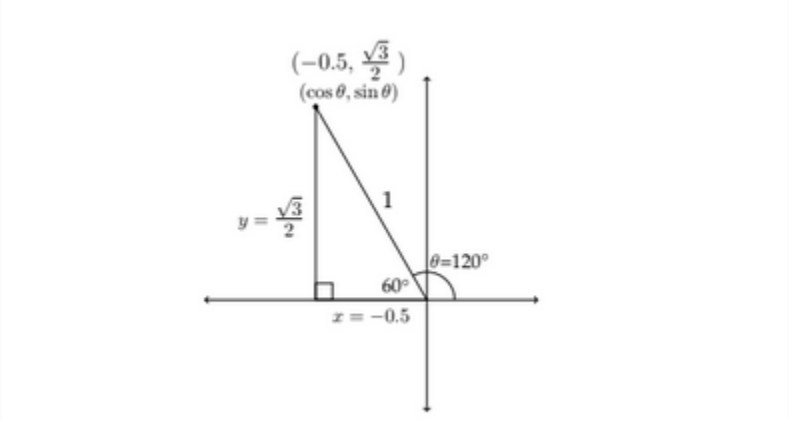


In the previous diagram, we see the values for the sine and cosine of the quadrantal angles:

$$\cos 0^\circ = 1 \quad \cos 90^\circ = 0 \quad \cos 180^\circ = -1 \quad \cos 270^\circ = 0 \tag{9.1.1}$$

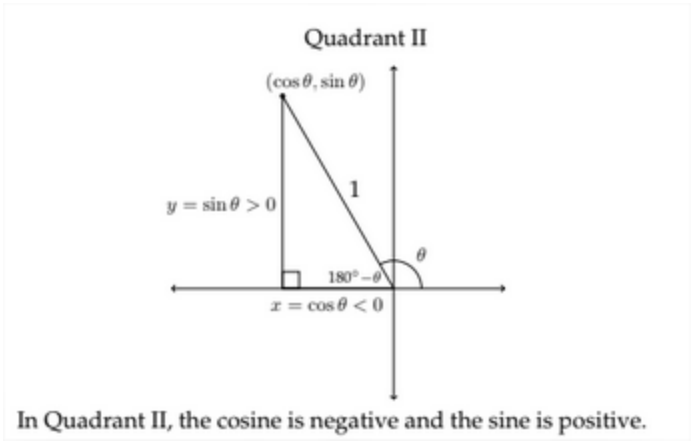
$\sin 0^\circ = 0 \quad \sin 90^\circ = 1 \quad \sin 180^\circ = 0 \quad \sin 270^\circ = -1$

If we take a radius of length 1 and rotate it counter-clockwise in the coordinate plane, the x and y coordinates of the point at the tip will correspond to the values of the cosine and sine of the angle that is created in the rotation. Let's look at an example in the second quadrant. If we rotate a line segment of length 1 by 120 ", it will terminate in Quadrant II.

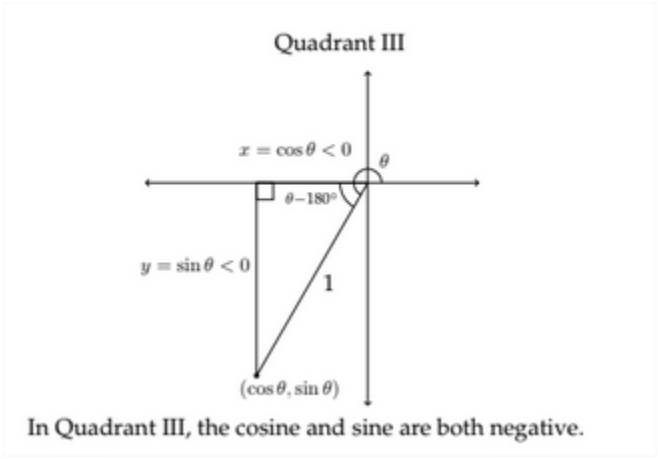


In the diagram above we notice several things. The radius of length 1 has been rotated by 120^{ento} Quadrant II. If we then drop a perpendicular line from the endpoint of the radius to the x -axis, we create a triangle in Quadrant II. Notice that the angle supplementary to 120° appears in the triangle and this allows us to find the lengths of the sides of the triangle and hence the values for the x and y coordinates of the point at the tip of the radius.

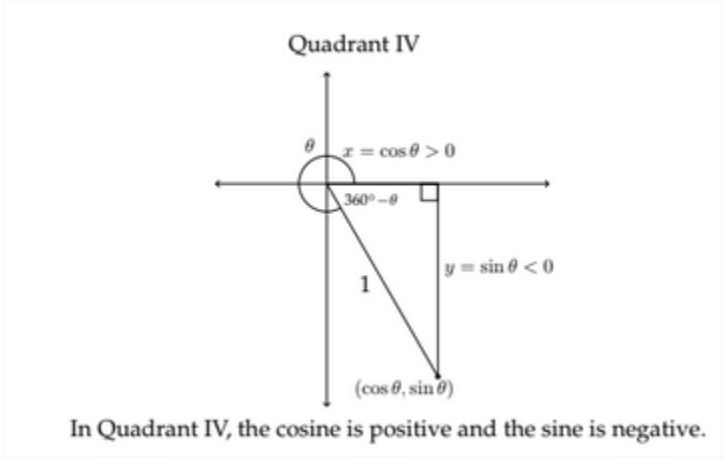
Whenever an angle greater than 90° is created on the coordinate axes, simply drop a perpendicular to the x -axis. The angle created is the reference angle. The values of the trigonometric functions of the angle of rotation and the reference angle will differ only in their sign $(+, -)$. On the next page are examples for Quadrants 1,
III, and IV.



In Quadrant II, the cosine is negative and the sine is positive.

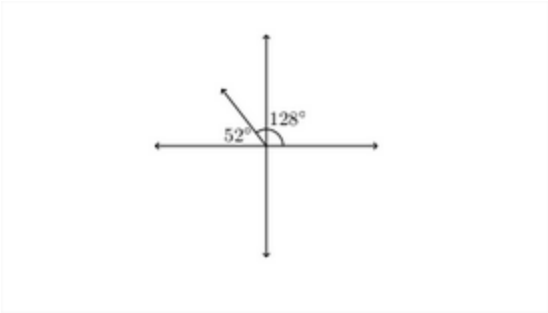


In Quadrant III, the cosine and sine are both negative.

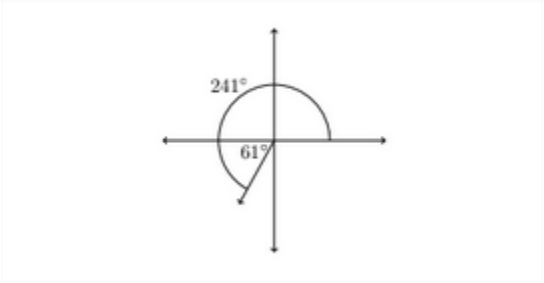


In Quadrant IV, the cosine is positive and the sine is negative.

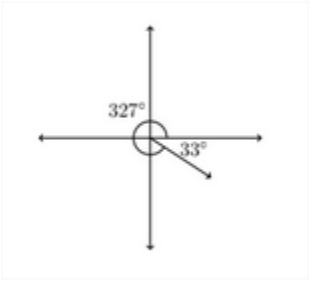
The process for finding reference angles depends on which quadrant the angle terminates in.
Examples
Find the reference angle for the following angles:
1.128°
2.41°
3.327°
1. An angle of 128° terminates in Quadrant II. To find the reference angle, we would subtract the angle from 180° : $180^\circ - 128^\circ = 52^\circ$



2. An angle of 241° terminates in Quadrant III. To find the reference angle, we would subtract 180° from the angle: $241^\circ - 180^\circ = 61^\circ$



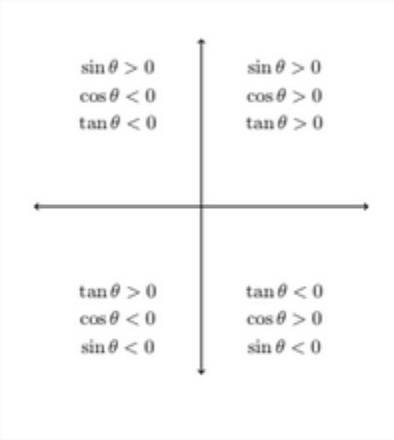
3. An angle of 327° terminates in Quadrant IV. To find the reference angle, we subtract the angle from 360° : $360^\circ - 327^\circ = 33^\circ$



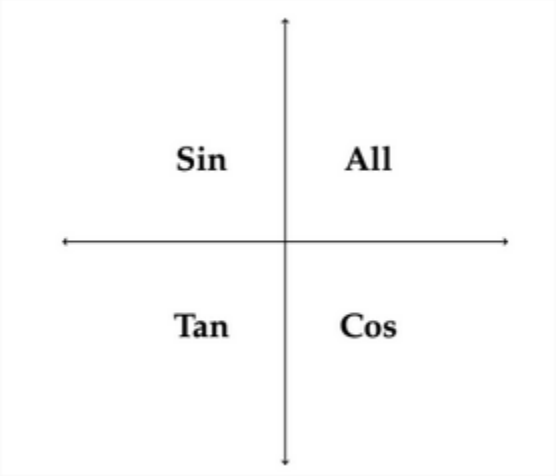
Once we know the reference angle, we can find the trigonometric functions for the original angle itself. In example 1, we had 128°, an angle in Quadrant II with a reference angle of 52°. Therefore, if we want to find the sine, cosine and tangent of 128°, then we should find the sine, cosine and tangent of 52° and apply the appropriate positive or negative sign.

Example 1 Quadrant II
In Quadrant II, x -coordinates are negative and y -coordinates are positive. This means that $\cos\theta < 0$ and $\sin\theta > 0$. The values for this process are given below:
$\sin 52^\circ \approx 0.7880$
$\sin 128^\circ \approx 0.7880$
$\cos 52^\circ \approx 0.6157 \quad \cos 128^\circ \approx -0.6157$
$\tan 52^\circ \approx 1.280 \quad \tan 128^\circ \approx -1.280^\circ$
Example 2 Quadrant III
In Quadrant III, x -coordinates are negative and y -coordinates are also negative. This means that $\cos\theta < 0$ and $\sin\theta < 0$. The values for this process are given below:
$\sin 61^\circ \approx 0.8746 \quad \sin 241^\circ \approx -0.8746$
$\cos 61^\circ \approx 0.4848$
$\cos 241^\circ \approx -0.4848$
$\tan 61^\circ \approx 1.8040 \quad \tan 241^\circ \approx 1.8040$
Example 3 Quadrant IV
In Quadrant IV, x -coordinates are positive and y -coordinates are negative. This means that $\cos\theta > 0$ and $\sin\theta < 0$. The values for this process are given below:
$\sin 33^\circ \approx 0.5446$
$\sin 327^\circ \approx -0.5446$
$\cos 33^\circ \approx 0.8387$
$\cos 327^\circ \approx 0.8387$

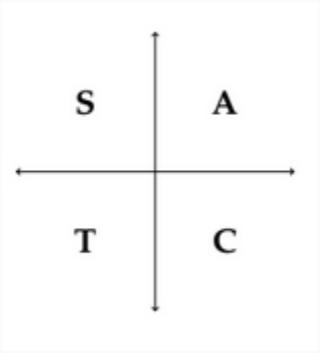
$\tan 33^\circ \approx 0.6494 \quad \tan 327^\circ \approx -0.6494$



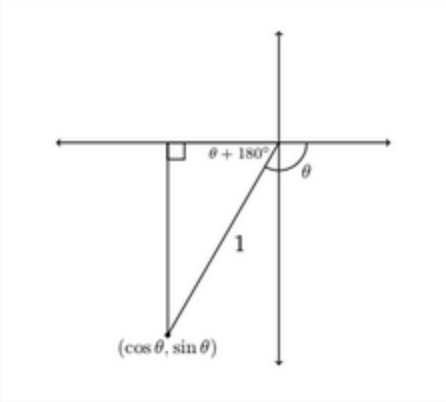
In Quadrant I, ALL the trigonometric functions are positive. In Quadrant II, the SIN function is positive (as well as the CSC).
In Quadrant III, the TAN function is positive (as well as the COT).
In Quadrant IV, the COS function is positive (as well as the SEC).



A common mneumonic device to remember these relationships is the phrase:
"All Students Take Calculus." This can help you remember which trigonometric functions are positive in each of the four quadrants.



Reference Angles for Negative Angles
Negatively measured angles rotate in a clockwise direction.

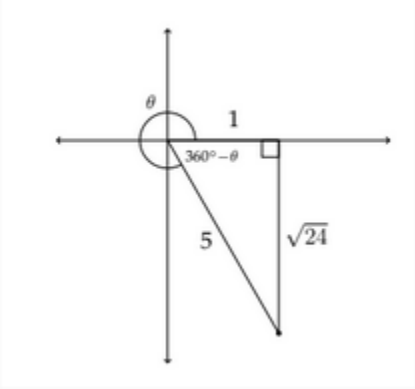


There are a variety of methods for finding the reference angle for a negatively valued angle. You can find a positive angle that is co-terminal with the negative angle and then find the reference angle for the positive angle. You can also drop a perpendicular to the x -axis to find the reference angle for the negative angle directly.

For example, the angle $-120°$ terminates in Quadrant III and is co-terminal with the positive angle $240°$. Either way, when you drop a perpendicular to the x -axis, you find that the reference angle is $60°$.

If you are given the value of one of the trigonometric functions of an angle θ and know which quadrant θ is located in, you can find the other trigonometric functions for that angle.
Example
Given θ in Quadrant IV with $\cos\theta = \frac{1}{5}$, find $\sin\theta$ and $\tan\theta$
If $\cos\theta = \frac{1}{5}$, then the adjacent side and the hypotenuse must be in a ratio of 15.

We can label these sides as 1 and 5 and then find the length of the third side in the triangle. This will allow us to find $\sin\theta$ and $\tan\theta$



Using the Pythagorean Theorem:
13
$25 - 10$
$x = 1$
we find that the side opposite the reference angle for θ is $\sqrt{24}$ or $2\sqrt{6}$. We can now find $\sin\theta$ and tan θ :

$$\sin\theta = \frac{\sqrt{24}}{5} \tag{9.1.2}$$

and

$$\tan\theta = \frac{\sqrt{24}}{1} = \sqrt{24} \tag{9.1.3}$$

In the problems in this section, the reciprocal functions secant, cosecant and cotangent are used. Remember that:

$$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{ady} \tag{9.1.4}$$

$\csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{adj}{opp}$

Exercises 2.1

Determine the quadrant in which the angle θ lies.

1. $\cos\theta > 0, \tan\theta > 0$
$\sin\theta < 0, \cos\theta > 0$
3. $\sec\theta > 0, \tan\theta < 0$
4. $\cot\theta > 0, \cos\theta < 0$
5. $\sin\theta > 0, \cos\theta < 0$
6. $\sin\theta > 0, \cot\theta > 0$
7. $\sin\theta < 0, \cos\theta < 0$
8. $\csc\theta > 0, \cot\theta < 0$

Determine which quadrant the given angle terminates in and find the reference angle for each.

9.195°
10.330°
11.120°
12.210°
13.135°
14.300°
15. -100°
16.225°
17. 315°
18. $\frac{5\pi}{4}$
19. $-\frac{2\pi}{3}$
20. $\frac{7\pi}{3}$
21. $\frac{11\pi}{4}$
22. $\frac{7\pi}{6}$
23. $\frac{11\pi}{6}$

Find $\sin\theta, \cos\theta$ and $\tan\theta$ in each problem.

24. $\sin\theta = -\frac{12}{13}, \theta$ in Quadrant IV
25. $\cos\theta = -\frac{4}{5}, \theta$ in Quadrant II
26. $\cos\theta = \frac{1}{4}, \theta$ in Quadrant I
27. $\tan\theta = \frac{3}{2}, \theta$ in Quadrant III
28. $\tan\theta = -\frac{4}{5}, \theta$ in Quadrant II
29. $\sin\theta = \frac{3}{8}, \theta$ in Quadrant II
30. $\sin\theta = -\frac{1}{3}, \theta$ in Quadrant III
31. $\tan\theta = 5, \theta$ in Quadrant I
32. $\sec\theta = -2, \tan\theta < 0$
33. $\cot\theta = \sqrt{3}, \cos\theta < 0$
34. $\tan\theta = -\frac{1}{3}, \sin\theta < 0$
35. $\csc\theta = \sqrt{2}, \cos\theta > 0$
36. $\cos\theta = -\frac{2}{5}, \tan\theta > 0$
37. $\sec\theta = 2, \sin\theta < 0$
38. $\sin\theta = \frac{1}{\sqrt{2}}, \cos\theta > 0$
39. $\sin\theta = -\frac{2}{3}, \cot\theta > 0$

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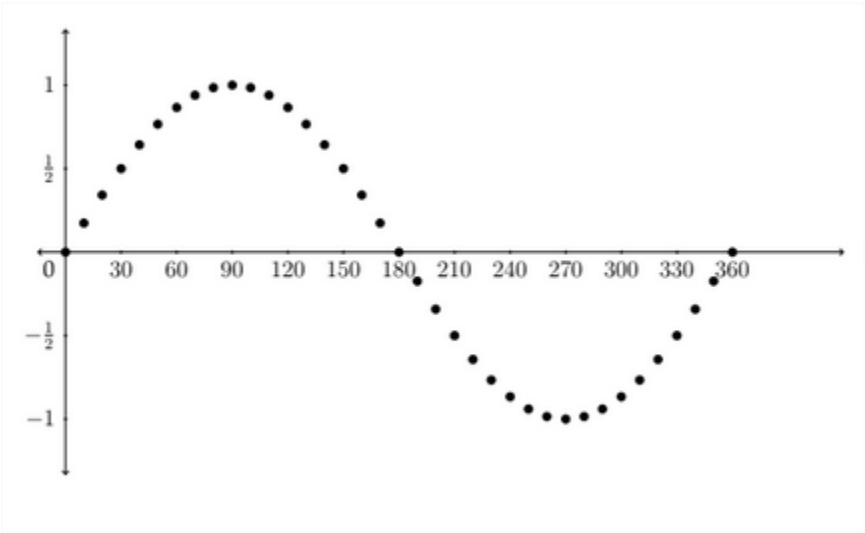
9.2: Graphing Trigonometric Functions

We have seen how to determine the values of trigonometric functions for angles terminating in Quadrants II, III, and IV. This allows us to make a graph of the values of the sine function for any angle. In the chart below, I have listed the values for the sine function for angles between 0° and 360°.

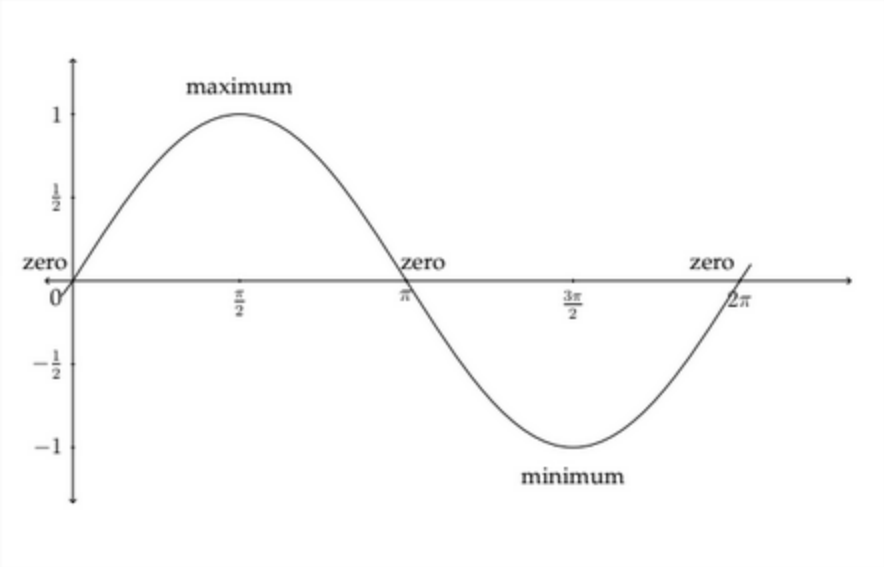
θ	$\sin\theta$		θ	$\sin\theta$
0°	$=0$		100°	≈ 0.9848
10°	≈ 0.1737		110°	≈ 0.9397
20°	≈ 0.3420		120°	≈ 0.8660
30°	$=0.5$		130°	≈ 0.7660
40°	≈ 0.6428		140°	≈ 0.6428
50°	≈ 0.7660		150°	$=0.5$
60°	≈ 0.8660		160°	≈ 0.3420
70°	≈ 0.9397		170°	≈ 0.1737
80°	≈ 0.9848		180°	$=0$
90°	$=1$			

θ	$\sin\theta$		θ	$\sin\theta$
180°	$=0$		280°	≈ -0.9848
190°	≈ -0.1737		290°	≈ -0.9397
200°	≈ -0.3420		300°	≈ -0.8660
210°	$=-0.5$		310°	≈ -0.7660
220°	≈ -0.6428		320°	≈ -0.6428
230°	≈ -0.7660		330°	$=-0.5$
240°	≈ -0.8660		340°	≈ -0.3420
250°	≈ -0.9397		350°	≈ -0.1737
260°	≈ -0.9848		360°	$=0$
270°	$=-1$			

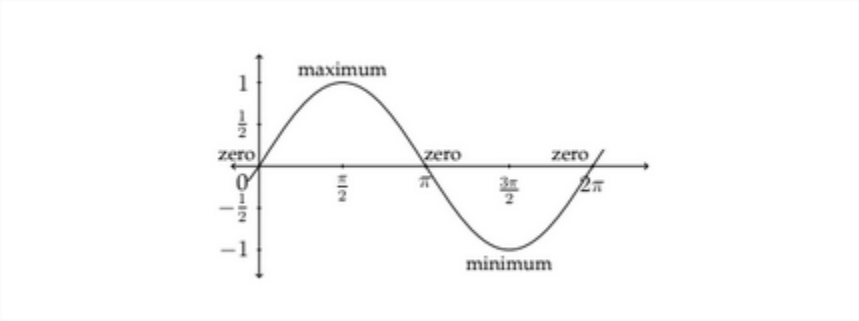
Below we see a graph of these points plotted on the coordinate axes.



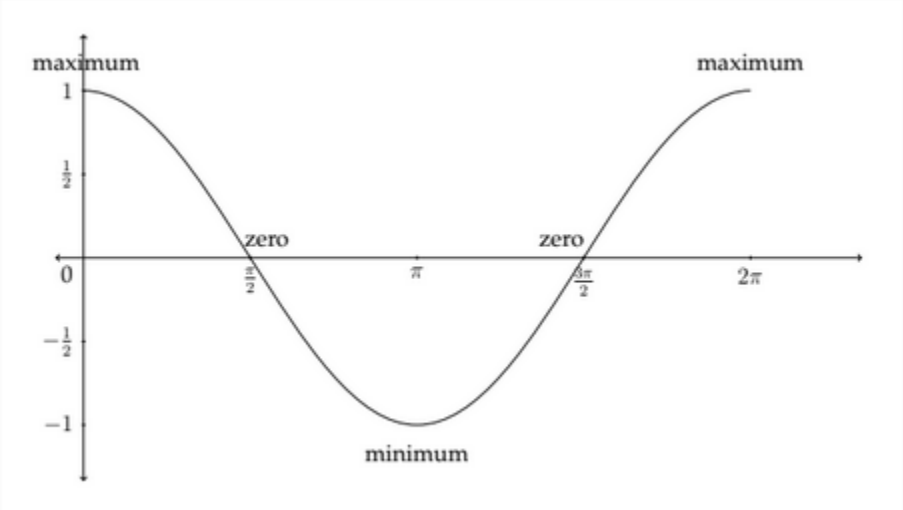
In graphing trigonometric functions, we typically use radian measure along the x -axis, so the graph would generally look like this:



The graph of the standard sine function begins at the zero point, then rises to the maximum value of 1 between 0 and $\frac{7}{3}$ radians. It then decreases back to 0 at
\pi radians before crossing over into the negative values and hitting its minimum value at $\frac{3\pi}{2}$ radians. It then goes back up to 0 at 2π radians before starting all over again.



The standard cosine graph behaves in a similar but slightly different way. We saw earlier that $\cos 0^\circ = 1$, so the cosine graph would start at the point $(0, 1)$, then gradually decrease to zero. A picture of the standard cosine graph would look like the figure below:
maximum



The sine and cosine graphs are sometimes referred to as a "sine wave" or "sinusoid" and can be very useful in modeling phenomena that occur in waves. Examples of this are the rise and fall of the tides; sound waves and music; electricity; and the length of day throughout the year. The standard sine and cosine graphs must be modified to fit a particular application so that

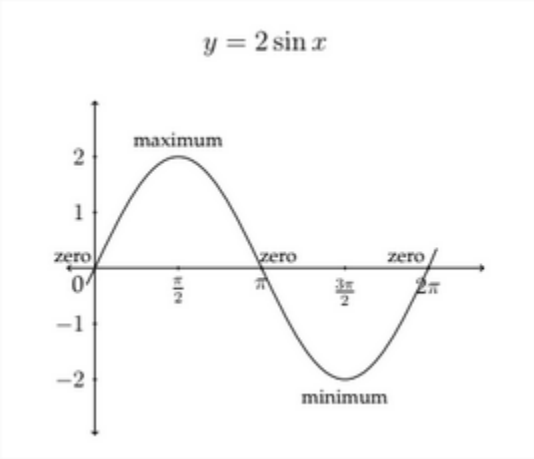
they will effectively model the situation. The ideas that we examine next will explain how to modify the sine and cosine graphs to fit a variety of different situations.

There are four aspects to the sine and cosine functions to take into consideration when making a graph. These are:
1) The Amplitude of the graph
2) The Period of the graph
3) The Vertical Shift of the graph
4) The Phase Shift of the graph

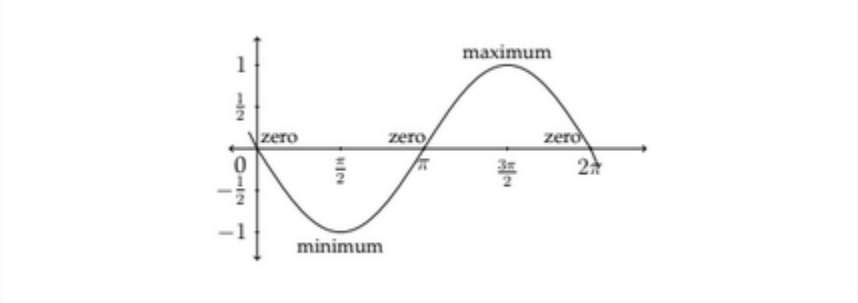
Amplitude

The amplitude of a sine or cosine function refers to the maximum and minimum values of the function. In the standard sine and cosine graphs, the maximum value is 1 and the minimum value is -1. The amplitude is one-half the difference between the maximum and minimum values. In the standard graphs the difference between the maximum and minimum is $1-(-1)=2$; one-half of this is 1 so the amplitude of the standard sine and cosine functions is 1.

The value of the amplitude is also the absolute value of the coefficient of the sine or cosine expression. In the standard graph, $y=\sin x$, the coefficient of the sine function is 1, so the amplitude is 1. In the function $y=2\sin x$, all the y values will be multiplied by 2 and the amplitude of the function will be 2. The graph for $y=2\sin x$ is shown on the next page.

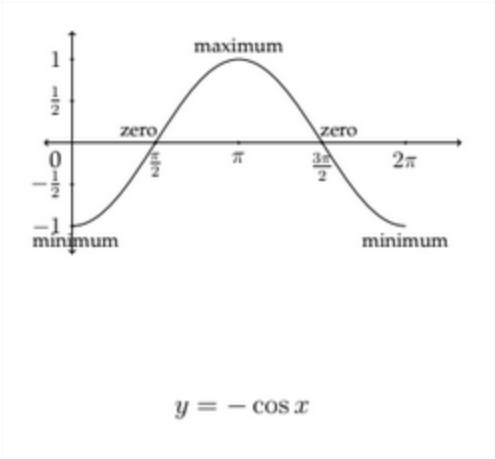


A negative value of the coefficient in front of a trigonometric function will not change the Amplitude of the function, but it will change the shape of the function. For example, the function:
has an amplitude of 1, but the graph will be different from the graph $y=\sin x$ All of the y -values of the function $y=-\sin x$ will have the opposite sign as the y -values of the function $y=\sin x$. The graph for $y=-\sin x$ appears below:



Notice that, because of the negation of the y -values, the graph begins at 0, as does the standard sine function, but the graph of $y=-\sin x$ first goes to a minimum value before crossing through 0 again up to the maximum value.

Likewise, the graph of $y=-\cos x$ begins at the minimum value before crossing through 0 and going to the maximum value, back through 0 and ending at the minimum value again.



$y = -\cos x$

Period

The period of the graph refers to how long it takes the graph to complete one full cycle of values. In the standard sine and cosine functions, the period is 2π radians. The function completes a single "wave" and returns to its starting place between 0 and 2π. A coefficient in front of the variable in a sine or cosine function will affect the period of the graph. In the general expression $y = A\sin Bx$, the value of A affects the amplitude of the function and the value of B affects the period of the function.

If we examine the table of values for the standard sine function, we can see how the coefficient of the x -variable will affect the period of the graph. Starting with the table from the standard sine function:

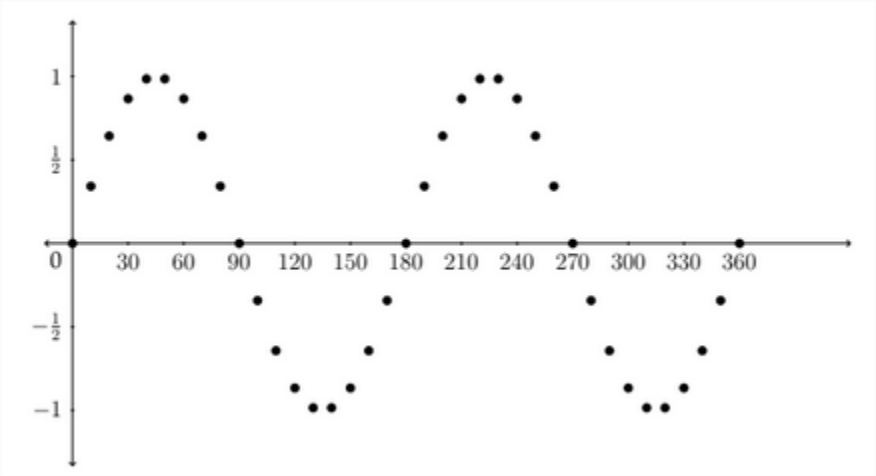
θ	$\sin\theta$		θ	$\sin\theta$
$0°$	$=0$		$100°$	≈ 0.9848
$10°$	≈ 0.1737		$110°$	≈ 0.9397
$20°$	≈ 0.3420		$120°$	≈ 0.8660
$30°$	≈ 0.5		$130°$	≈ 0.7660
$40°$	≈ 0.6428		$140°$	≈ 0.6428
$50°$	≈ 0.7660		$150°$	≈ 0.5
$60°$	≈ 0.8660		$160°$	≈ 0.3420
$70°$	≈ 0.9397		$170°$	≈ 0.1737
$80°$	≈ 0.9848		$180°$	$=0$
$90°$	$=1$			

θ	$\sin\theta$		θ	$\sin\theta$
$180°$	$=0$		$280°$	≈ -0.9848
$190°$	≈ -0.1737		$290°$	≈ -0.9397
$200°$	≈ -0.3420		$300°$	≈ -0.8660
$210°$	≈ -0.5		$310°$	≈ -0.7660
$220°$	≈ -0.6428		$320°$	≈ -0.6428
$230°$	≈ -0.7660		$330°$	$=-0.5$
$240°$	≈ -0.8660		$340°$	≈ -0.3420
$250°$	≈ -0.9397		$350°$	≈ -0.1737
$260°$	≈ -0.9848		$360°$	$=0$
$270°$	$=-1$			

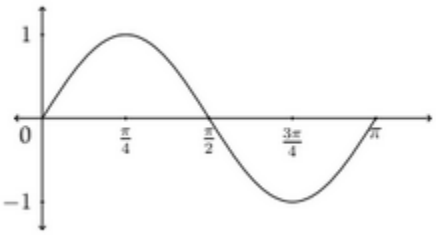
If we create a similar table for the function $y = \sin(2x)$, then we can see how this will affect the graph:

θ	2θ	$\sin(2\theta)$		θ	2θ	$\sin(2\theta)$
$0°$	$0°$	$=0$		$100°$	$200°$	≈ -0.3420
$10°$	$20°$	≈ 0.3420		$110°$	$220°$	≈ -0.6428
$20°$	$40°$	≈ 0.6428		$120°$	$240°$	≈ -0.8660
$30°$	$60°$	≈ 0.8660		$130°$	$260°$	≈ -0.9848
$40°$	$80°$	≈ 0.9848		$140°$	$280°$	≈ -0.9848
$50°$	$100°$	≈ 0.9848		$150°$	$300°$	≈ -0.8660
$60°$	$120°$	≈ 0.8660		$160°$	$320°$	≈ -0.6428
$70°$	$140°$	≈ 0.6428		$170°$	$340°$	≈ -0.3420
$80°$	$160°$	≈ 0.3420		$180°$	$360°$	$=0$
$90°$	$180°$	$=0$				

In the previous table we can see that the function $y = \sin(2x)$ completes one full cycle between 0 and π radians instead of the the standard 0 to 2π radians. The graph for these points is shown below. The coordinates for the x -values between π and 2π radians are shown as well.



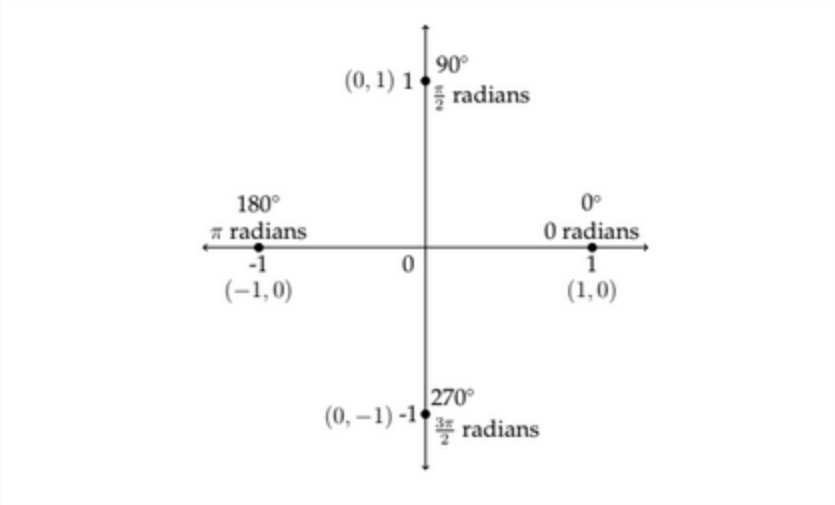
In this graph, you can see that there are two complete waves between 0 and 2π radians, or one complete wave between 0 and π radians. So, in a sine or cosine function of the form $y = A\sin Bx$, the amplitude will be $|A|$ and the period will be $\frac{2\pi}{b}$. The standard graph for one complete cycle of the function $y = \sin(2x)$ is shown below:



Notice that, because the period has been cut in half, the x -coordinates that correspond to the maximum, minimum, and zero y -coondinates are cut in half as

Example 1
Graph one full period of the function $y = -2\sin 3x$
The amplitude in this case is 2, but since the coefficient is negative, this sine graph will begin by first going to the minimum value. The period of the graph will be $\frac{2\pi}{B}$, or in this case $\frac{2\pi}{3}$ instead of 2π. To determine the x -values for the maximum, minimum and zero y -values, we should examine how these are determined for the standard sine curve.

The maximum, minimum and zero y-values for a standard sine curve occur at the quadrantal angles, that is to say, the angles that separate the four quadrants from each other. The quadrantal angles are 0 ' or 0 radians, 90° or $\frac{\pi}{2}$ radians, 180° or \pi radians, 270° or $\frac{3\pi}{2}$ radians and 360° or 2π radians. These x -values produce the "critical" y-values of the zero, maximum and minimum.



In the standard sine or cosine graph, the distance from each "critical value" of the

graph to the next is always a "jump" of $\frac{\pi}{2}$ along the x -axis. This is one-fourth of the period: $\frac{2\pi}{1} * \frac{1}{4} = \frac{\pi}{2}$. So, to determine the labels for the critical values of the graph along the x -axis, we should take the new period and multiply by $\frac{1}{4}$

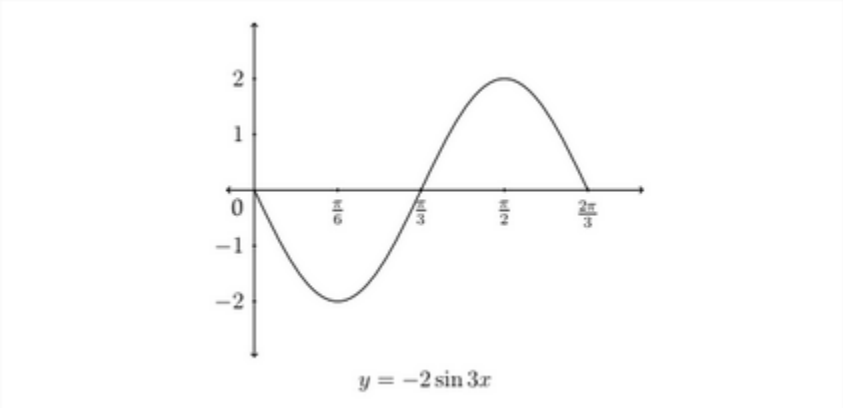
The function we are working with is $y = -2\sin 3x$, so to find the new period we calculated $\frac{2\pi}{B}$, which was $\frac{2\pi}{3}$. Then, in order to label the x -axis properly we should next take $\frac{2\pi}{3}$ and multiply by $\frac{1}{4}$

$$\frac{2\pi}{3} * \frac{1}{4} = \frac{2\pi}{12} = \frac{\pi}{6} \tag{9.2.1}$$

So, the critical values along the x -axis will be:

$$\frac{1\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \text{ and } \frac{4\pi}{6} \tag{9.2.2}$$

We want to express these in lowest terms, so we would label them as $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, and $\frac{2\pi}{3}$. The graph will start at zero, then (because the value of the coefficient A is negative) it will go down to a minimum value at $\frac{\pi}{6}$, back to zero at $\frac{\pi}{3}$, then up to the maximum at $\frac{\pi}{2}$ and back down to zero at $\frac{2\pi}{3}$ to complete one full period of the graph. The graph for this function is pictured below. Notice that the minimum y-value is -2 and the maximum y -value is 2 because $A = 2$

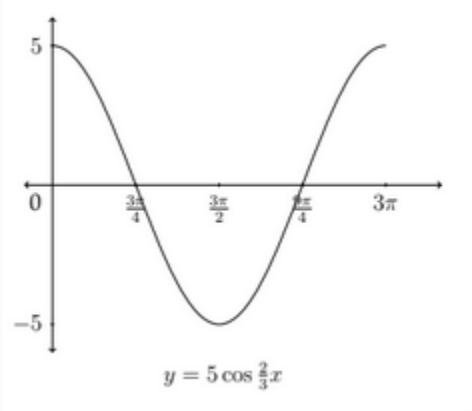


Let's look at an example using the cosine graph.

Example 2
Graph one full period of the function $y = 5\cos\frac{2}{3}x$
The amplitude of the function is 5 because $A = 5$, so the maximum y-value will be 5 and the minimum y -value will be -5. The period of the graph will be $\frac{2\pi}{B}$ which in this case is $\frac{2\pi}{7} = 2\pi * \frac{3}{2} = 3\pi$. So the period is 3 π. The critical values along the x -axis will start at 0 and be separated by "jumps" of $3\pi * \frac{1}{4} = \frac{3\pi}{4}$. So the critical values along the x -axis will be:

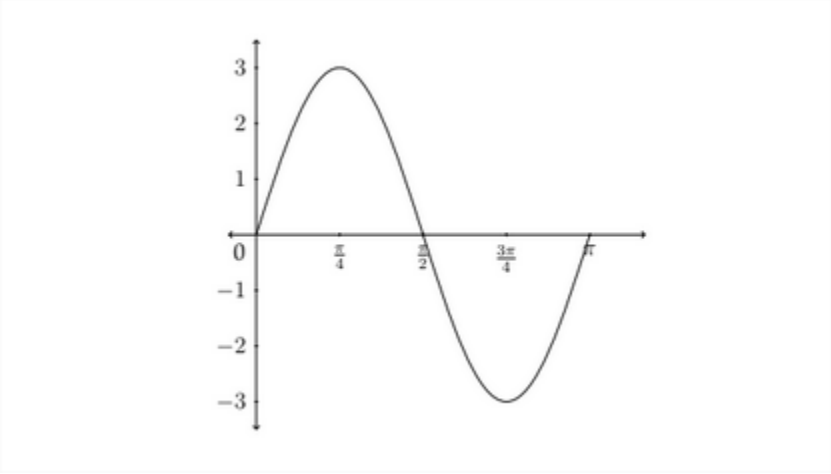
$$0, \frac{3\pi}{4}, \frac{6\pi}{4}, \frac{9\pi}{4}, \text{ and } \frac{12\pi}{4} \tag{9.2.3}$$

We want to express these in lowest terms so we would label them as $\frac{3\pi}{4}$, $\frac{3\pi}{2}$, $\frac{9\pi}{4}$ and 3π. The graph will start at the maximum y -value of 5 at $x = 0$, then it will go to zero at $x = \frac{3\pi}{4}$, down to the minimum y -value of -5 at $x = \frac{3\pi}{2}$, back through 0 at $x = \frac{9\pi}{4}$, and then up to the maximum y -value of 5 at $x = 3\pi$ to complete one full period of the graph. The graph of $y = 5\cos\frac{2}{3}x$. is shown below.



Determining an equation from a graph
Sometimes, you will be given a graph and asked to determine an equation which satisfies the conditions visible in the graph. So far, we have only discussed two of the possible transformations of a trigonometric function - the amplitude and period. Remember that in an equation of the form $y = A\sin Bx$ or $y = A\cos Bx$, the amplitude is $|A|$ and the period is $\frac{2\pi}{B}$. So, to write an equation for a trigonometric function, we need to determine the values of A and B.
Example 3

Deterimine an equation that satisfies the given graph.



First note that the maximum y -value for the graph is 3 and the minimum is -3 This means that the amplitude is 3. Next we see that there is one complete period of the function between 0 and π, this means that the period is π. From this information, we know that $A = 3$ and that the period for the graph is π. Since the period $P = \frac{2\pi}{B}$, then we know that $B = \frac{2\pi}{P}$. So, $B = \frac{2\pi}{\pi} = 2$

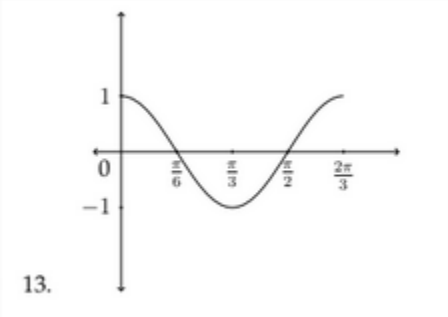
Lastly, this graph starts with a y -value of 5, which is the maximum y -value. It then goes to 0 and down to the minimum, back through 0 and then back to the maximum to form one complete wave. since this is the signature of the cosine function, the answer to this problem would be:

$$y = 5\cos\frac{8}{9}x \tag{9.2.4}$$

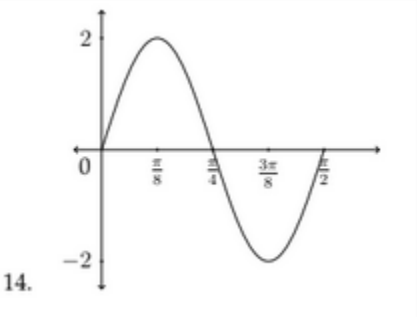
Exercises 2.2
Determine the amplitude and the period for each problem and graph one period of the function. Identify important points on the x and y axes.
1. $y = \cos 4x$
2. $y = -\sin 2x$

3. $y = 3\sin 3x$
4. $y = -2\cos 5x$
5. $y = 4\cos \frac{1}{2}x$
6. $y = 2\sin \frac{1}{3}x$
7. $y = -\frac{1}{2}\sin \frac{2}{3}x$
8. $y = -3\cos \frac{1}{5}x$
9. $y = -4\sin 6x$
10. $y = 3\sin 4x$
11. $y = 2\cos \frac{1}{2}x$
12. $y = 3\cos \frac{1}{3}x$

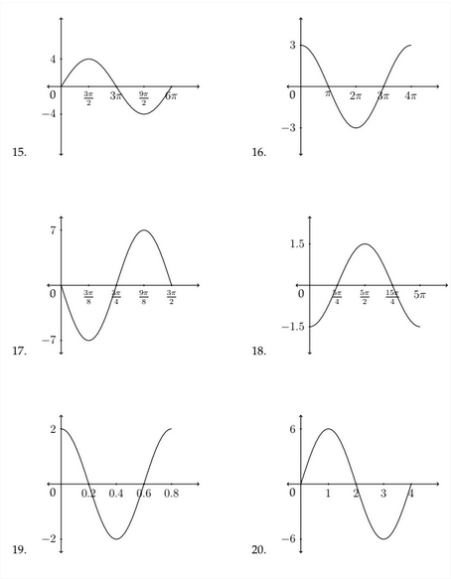
Determine an equation that satifies the given graph.



13.



14.


15.

16.

17.

18.

19.

20.

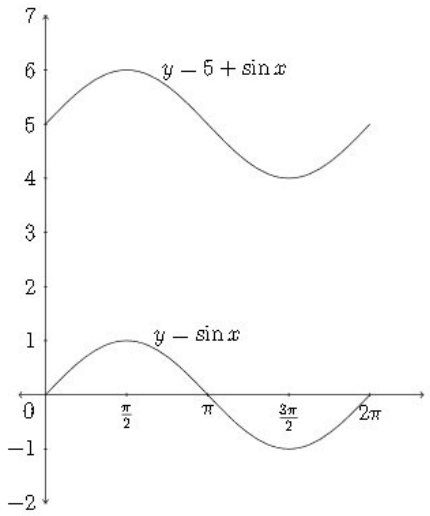
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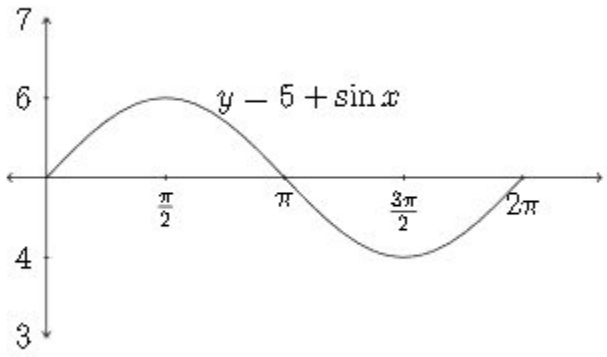
9.3: The Vertical Shift of a Trigonometric Function

If a constant is added or subtracted to a trigonometric function, this will affect the y -values of the function If we consider the function $y-5+\sin x$, then each of the standard y -values would have 5 added to if, which would shift the graph up 5 units.
The chart below considers just the quadrantal values for the sine function:

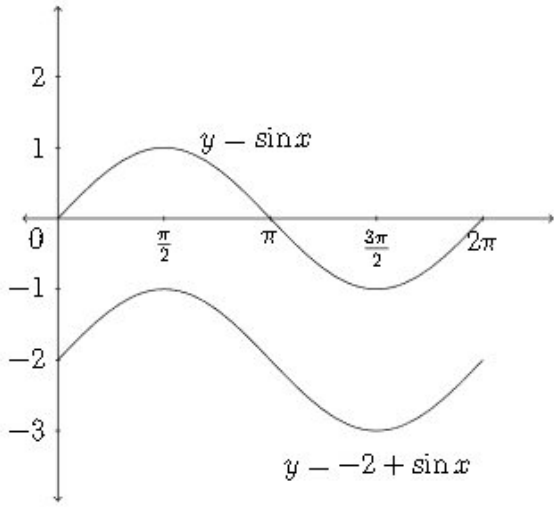
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Sometimes the x -axis is drawn through the line that is the new "zero" or "midline" for the function - in this case it would be $y-5$



Likewise, a negative constant would move the graph down, as each y-value would be less than the corresponding y -value in the standard sine function



In the previous examples the constant has been written in front of the sine function for clarity. Often the constant is written after the function:

$$\begin{gathered} y - \sin x + 5 \\ \text{or} \\ y - \sin x - 2 \end{gathered} \tag{9.3.1}$$

We have now examined three of the four transformations of trigonometric functions that are discussed in this chapter - amplitude, period and vertical shift. A general equation for a sinusoid that involves these three transformations would be:

$$y = A\sin(Bx) + D \tag{9.3.2}$$

Or

$$y = A\cos(Bx) + D \tag{9.3.3}$$

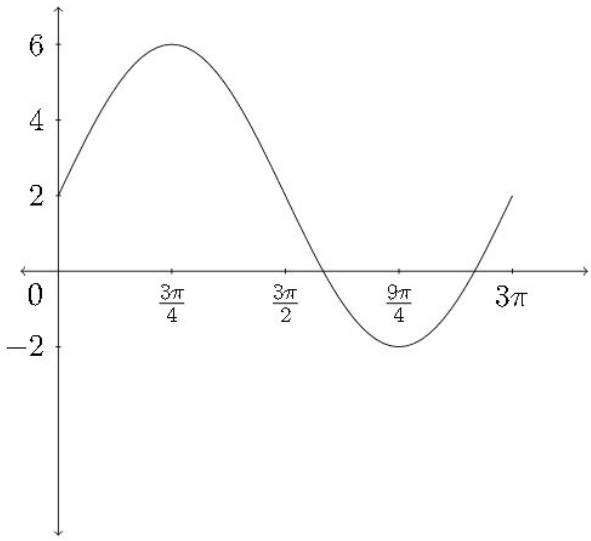
In determining an equation from a graph that involves a vertical shift, the value of A will be half the distance between the maximum and minimum values:

$$A = \frac{max - min}{2} \tag{9.3.4}$$

and the value of D will be the average of the maximum and minimum values:

$$D = \frac{max + min}{2} \tag{9.3.5}$$

Example
Determine an equation that satisfies the given graph.



In this graph, the maximum y -value is 6 and the minimum y -value is -2. The average of these two:

$$\frac{max+min}{2}=\frac{6+(-2)}{2}=\frac{4}{2}=2=D \tag{9.3.6}$$

is the value of D, the vertical shift.
The distance between 6 and -2 is $6-(-2)=8$. Half the distance between the $\max$ and $\min$ is 4, which is the value of A

$$\frac{max-min}{2}=\frac{6-(-2)}{2}=\frac{8}{2}=4=A \tag{9.3.7}$$

The graph completes one full cycle between 0 and 3π, so the period would be 3π and the value of B would be $B=\frac{2\pi}{P}=\frac{2\pi}{3\pi}=\frac{2}{3}$ $=B$. So a correct equation for the graph would be:

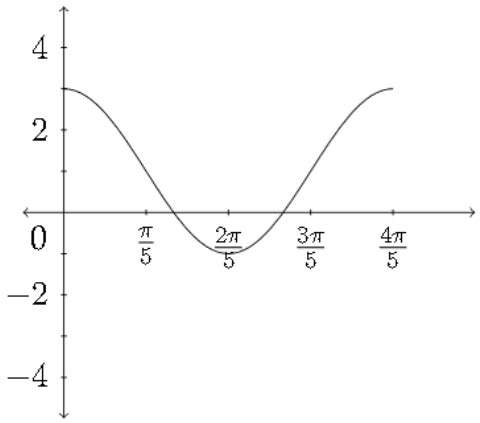
$$y=4\sin\frac{2}{3}x+2 \tag{9.3.8}$$

Exercises 2.3
Determine the Amplitude, Period and Vertical Shift for each function below and graph one period of the function. Identify the important points on the x and y axes.

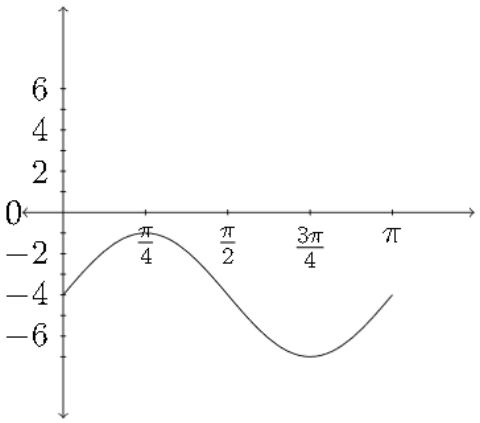
1. $y=\sin x+1$
2. $y=\cos x-1$
3. $y=2\cos x-\frac{1}{2}$
4. $y=5\sin x+4$
5. $y=-\sin\left(\frac{1}{4}x\right)+1$
6. $y=-\cos(2x)+7$
7. $y=\frac{1}{3}\sin(\pi x)-4$
8. $y=-\frac{1}{2}\cos(2\pi x)+2$
9. $y=5\cos\left(\frac{1}{2}x\right)+1$
10. $y=4\sin\left(\frac{1}{3}x\right)-1$
11. $y=3\cos x+2$
12. $y=2\sin x+3$
13. $y=2-4\cos(3x)$
14. $y=5-3\sin(2x)$

Determine an equation that satisfies the given graph.

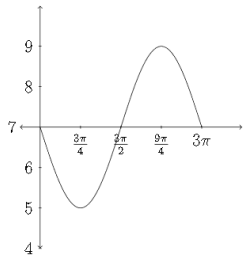
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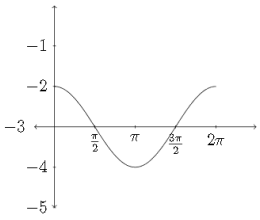
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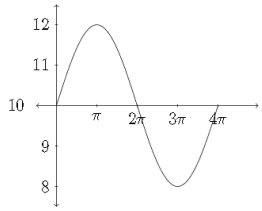
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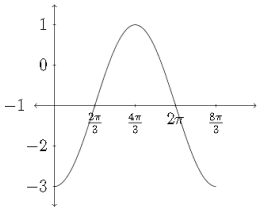
18.



19.



20.



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9.4: Phase Shift

The last form of transformation we will discuss in the graphing of trigonometric functions is the phase shift, or horizontal displacement. So far, we have considered the amplitude, period and vertical shift transformations of trigonometric functions. In the standard equation $y = A\sin(Bx) + D$, these corrrespond to the coefficients A, B and D. Notice that the amplitude and vertical shift coefficients $(A \text{ and } D)$, which affect the y -axis occur outside of the trigonometric function, whereas the coefficient that affects the period of the graph along the x -axis occurs within the sine function. This is true of the phase shift as well.
If we consider a general equation of:

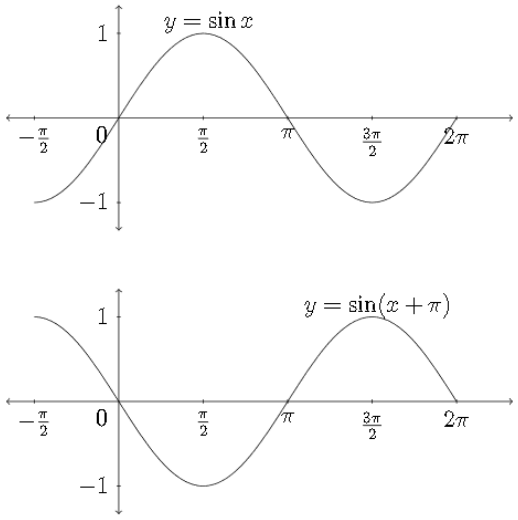
$$y = A\sin(Bx + C) + D \tag{9.4.1}$$

the constant C will affect the phase shift, or horizontal displacement of the function. Let's look at a simple example.

Graph at least one period of the given function: $y = \sin(x + \pi)$ Be sure to indicate important points along the x and y axes.
Let's examine this function by looking at a table of values.

x	$x + \pi$	$\sin(x + \pi)$
0	π	0
$\pi/2$	$3\pi/2$	-1
π	2π	0
$3\pi/2$	$5\pi/2$	1
2π	3π	0

Now let's look at a graph of $y = \sin(x + \pi)$ as compared to the standard graph of $y = \sin x$



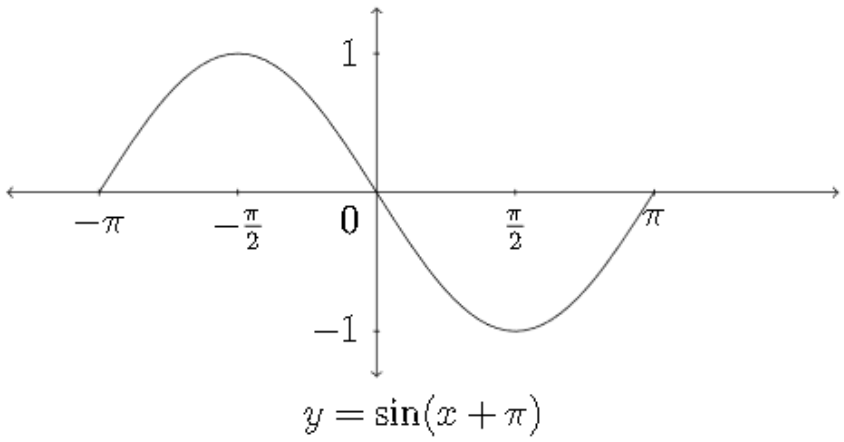
Notice that if we take the standard graph of $y = \sin x$ and drag it backwards along the x -axis a distance of π, we would have the graph of $y = \sin(x + \pi)$ That's because each x value is having π added to it, so to arrive at the x value that produces a particular y -value, we would need to subtract π. Here's an example:

$x+\pi$	$y=\sin(x+\pi)$
0	0
$\pi/2$	1
π	0
$3\pi/2$	-1
2π	0

In the table above we see the standard x and y values for the graph of the sine function. In the table below, we add a column that shows the value that x would need to be for $x+\pi$ to be the standard values:

x	$x+\pi$	$y=\sin(x+\pi)$
$-\pi$	0	0
$-\pi/2$	$\pi/2$	1
0	π	0
$\pi/2$	$3\pi/2$	-1
π	2π	0

Here's a graph of these values:



This is the same graph of $y=\sin(x+\pi)$ that we saw on the previous page, but anchored to different points on the x -axis. Either graph would be a correct response to a question asking for at least one period of the graph of $y=\sin(x+\pi)$
Let's look at another example:

Example 2
Graph at least one period of the given function: $y=\sin\left(x+\frac{\pi}{3}\right)$ Be sure to indicate important points along the x and y axes.
In this simplified example, we really have only one transformation to worry about - the phase shift. Notice that the amplitude, period and vertical shift have all been left out. When considering a sine or cosine graph that has a phase shift, a good way to start

the graph of the function is to determine the new starting point of the graph. In the previous example, we saw how the function $y = \sin(x + \pi)$

shifted the graph a distance of π to the left and made the new starting point of the sine curve $-\pi$

In graphing the standard sine curve we're generally interested in the quadrantal angles that produce the maximum, minimum and zero points of the graph. In graphing the function $y = \sin\left(x + \frac{\pi}{3}\right)$, we want to know which values of x will produce the quadrantal angles when we add $\frac{\pi}{3}$ to them.

So, to determine the new starting point we want to know the solution to the equation $x + \frac{\pi}{3} = 0$

$$\begin{aligned} x + \tfrac{\pi}{3} &= 0 \\ -\tfrac{\pi}{3} - \tfrac{\pi}{3} & \\ x &= -\tfrac{\pi}{3} \end{aligned} \tag{9.4.2}$$

This is the new starting point for the graph $y = \sin\left(x + \frac{\pi}{3}\right)$. Because this graph has a standard period, the "jump" between each of the quadrantal angles will be
$\frac{\pi}{2}$. To graph one period of a typical trigonometric function we'll need at least five quadrantal angle values. So, if our new starting point is $-\frac{\pi}{3}$, then the next critical value along the x -axis will be:

$$-\frac{\pi}{3} + \frac{\pi}{2} = -\frac{2\pi}{6} + \frac{3\pi}{6} = \frac{\pi}{6} \tag{9.4.3}$$

Then the subsequent critical values would be:

$$\begin{gathered} \tfrac{\pi}{6} + \tfrac{\pi}{2} = \tfrac{\pi}{6} + \tfrac{3\pi}{6} = \tfrac{4\pi}{6} = \tfrac{2\pi}{3} \\ \tfrac{4\pi}{6} + \tfrac{3\pi}{6} = \tfrac{7\pi}{6} \\ \tfrac{7\pi}{6} + \tfrac{3\pi}{6} = \tfrac{10\pi}{6} = \tfrac{5\pi}{3} \end{gathered} \tag{9.4.4}$$

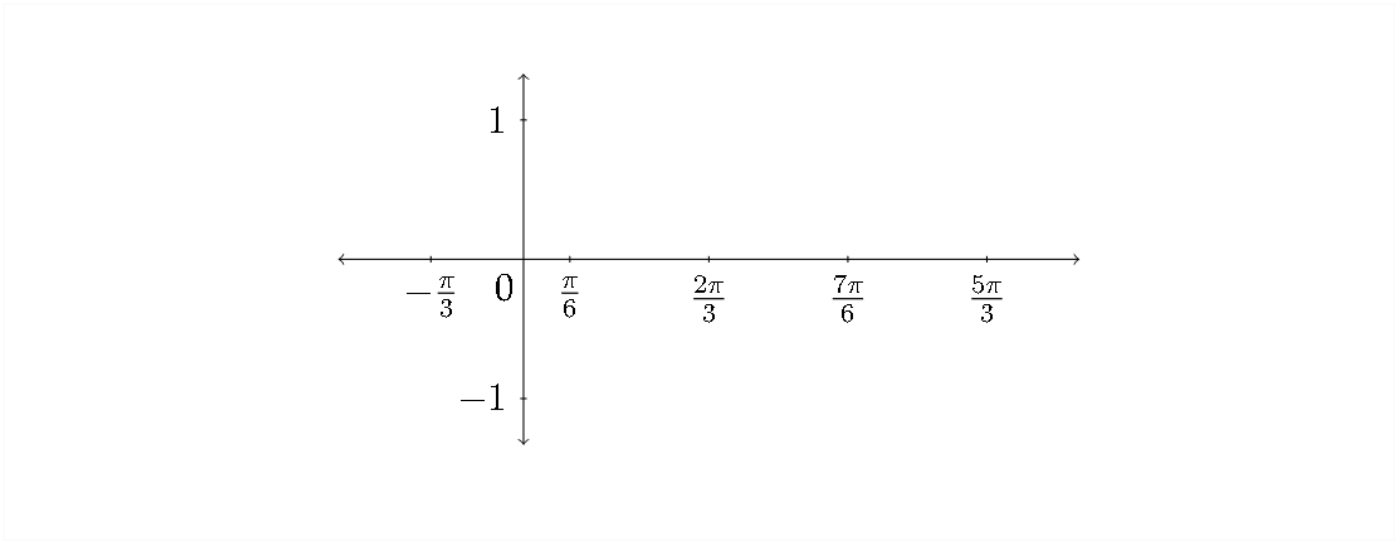
So the five critical values along the x -axis are:

$$-\frac{2\pi}{6}, \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6} \text{ and } \frac{10\pi}{6} \tag{9.4.5}$$

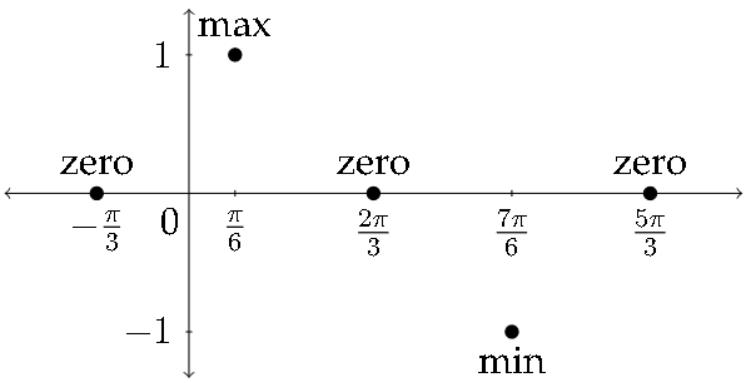
or, in reduced form:

$$-\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6} \text{ and } \frac{5\pi}{3} \tag{9.4.6}$$

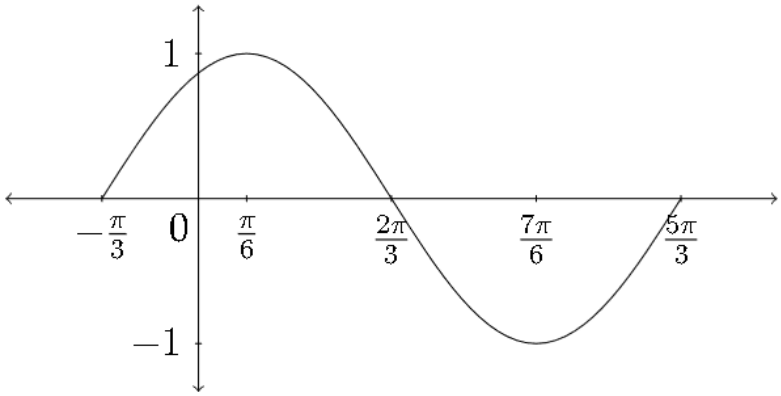
In order to graph the function, we would put these values along the x -axis and plot the standard quadrantal y -values to match up with them:



The y -values for the sine function start at zero, go up to the maximum, back down through zero to the minimum and then back to zero:



Connecting these points to make a sine curve produces the following graph:

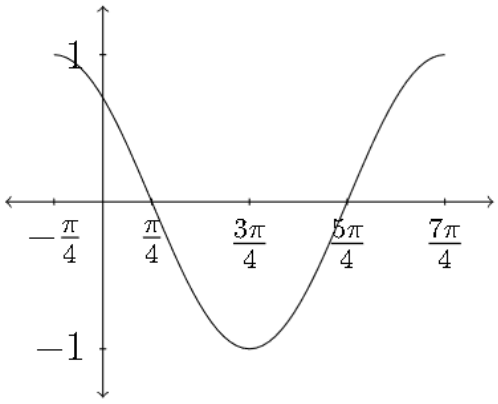


Exercises 2.4

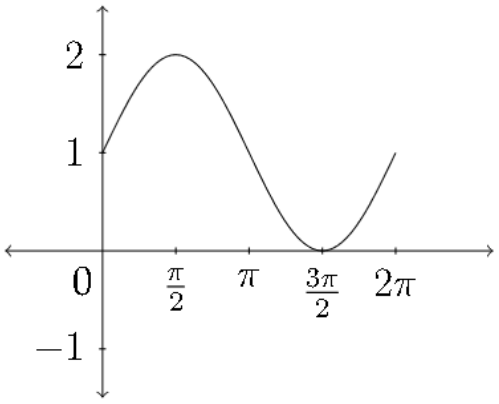
Match each function with the appropriate graph.

1. $y = \cos\left(x - \frac{\pi}{4}\right)$
2. $y = \sin\left(x + \frac{\pi}{4}\right)$
3. $y = \cos x - 1$
4. $y = \sin x + 1$
5. $y = \sin\left(x - \frac{\pi}{4}\right)$
6. $y = 1 - \cos x$
7. $y = \sin x - 1$
8. $y = \cos\left(x + \frac{\pi}{4}\right)$

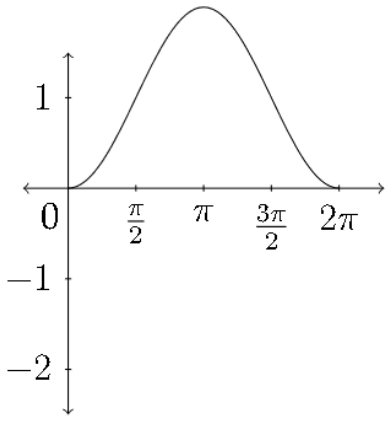
A.


1
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
$\frac{3\pi}{4}$
$\frac{5\pi}{4}$
$\frac{7\pi}{4}$
-1

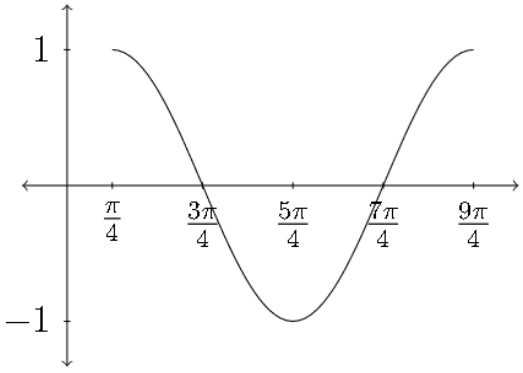
B.


2
1
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-1

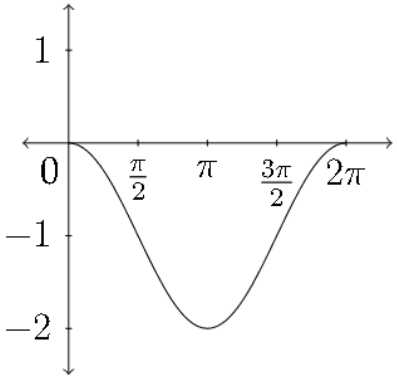
C.


1
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-1
-2

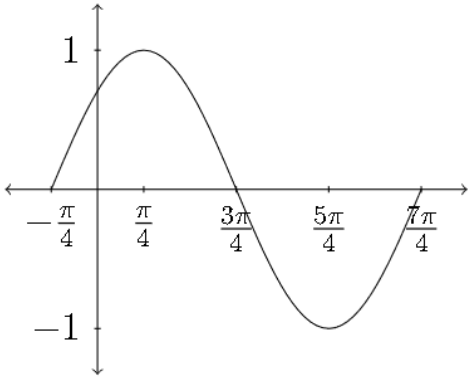
D.


1
$\frac{\pi}{4}$
$\frac{3\pi}{4}$
$\frac{5\pi}{4}$
$\frac{7\pi}{4}$
$\frac{9\pi}{4}$
-1

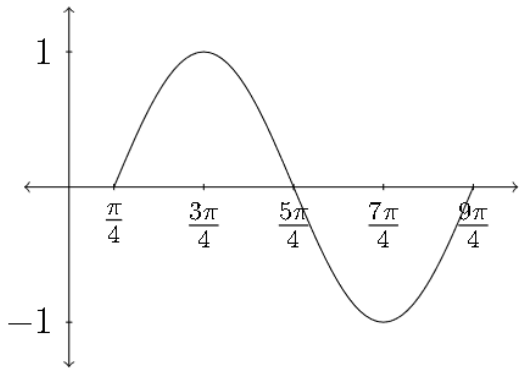
E.


1
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
-1
-2

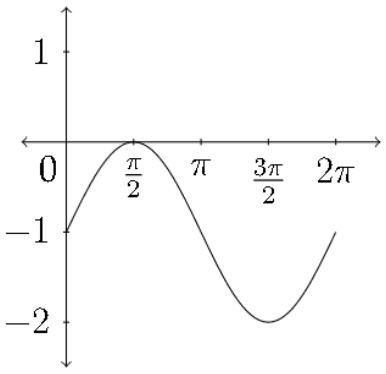
F.


1
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
$\frac{3\pi}{4}$
$\frac{5\pi}{4}$
$\frac{7\pi}{4}$
-1

G.



H.



Sketch at least one period for each function. Be sure to include the important values along the x and y axes.

9. $y = \sin\left(x + \frac{\pi}{6}\right)$
10. $y = \cos\left(x - \frac{\pi}{6}\right)$
11. $y = \cos\left(x - \frac{\pi}{3}\right)$
12. $y = \sin\left(x + \frac{\pi}{3}\right)$
13. $y = \sin\left(x - \frac{3\pi}{4}\right)$
14. $y = \cos\left(x + \frac{3\pi}{4}\right)$
15. $y = \cos\left(x + \frac{2\pi}{3}\right)$
16. $y = \sin\left(x - \frac{2\pi}{3}\right)$

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9.5: Combining the Transformations

In the previous sections, we have seen how the various transformations act on the trigonometric functions and we have worked with the first three (amplitude, period and vertical shift) in combination with each other. Combining the phase shift with the other transformations is tricky because of the way that the period and the phase shift interact with each other.
Now we have two standard equations for the sinusoid:

$$y = A\sin(Bx + C) + D \tag{9.5.1}$$

and

$$y = A\cos(Bx + C) + D \tag{9.5.2}$$

A and D, the amplitude and the vertical shift affect the y -axis, while B and C affect the x -axis.

y-axis	x-axis
Amplitude$= \lvert A \rvert$	Period$=\frac{2\pi}{B}$
Vertical Shift$= D$	Phase Shift$=-\frac{C}{B}$

Let's look at an example in which we need to combine a change in the period of the graph with a phase shift.

Example 1
Graph at least one period of the given function. Indicate the important values along the x and y axes.
$y = \cos(4x + \pi)$
The transformations in this example only affect the x -axis. The period of the function is $\frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$. So, the function will complete one full cycle over a distance of $\frac{\pi}{2}$ along the x -axis.

However, because of the phase shift, this graph will not start at 0 and end at $\frac{\pi}{2}$. We need to find the new starting point that is caused by the phase shift. So, we take what is called the "argument," or what it is we're finding the cosine of: $(4x + \pi)$ and set that equal to zero.

$$\begin{aligned} 4x + \pi &= 0 \\ 4x &= -\pi \\ x &= -\tfrac{\pi}{4} \end{aligned} \tag{9.5.3}$$

This is our new starting point. To identify the critical values along the x -axis, we'll need to determine how far each "jump ′′ would be given a period of $\frac{\pi}{2}$

$$\frac{\pi}{2} * \frac{1}{4} = \frac{\pi}{8} \tag{9.5.4}$$

So, each subsequent critical value along the x -axis will be a distance of $\frac{\pi}{8}$ from the previous one. If we start at our new starting point for this function $-\frac{\pi}{4}$, then if we add $\frac{\pi}{8}$ a total of 4 times, we will arrive at each of the five critical values for this function.

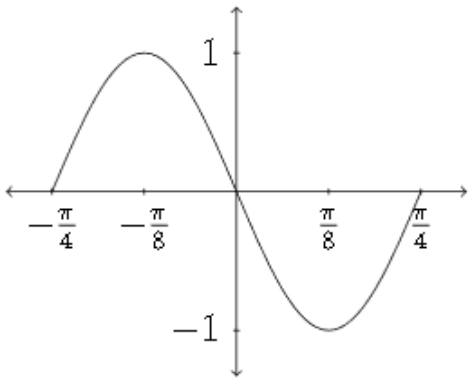
So, each subsequent critical value along the x -axis will be a distance of $\frac{\pi}{8}$ from the previous one. If we start at our new starting point for this function $-\frac{\pi}{4}$, then if we add $\frac{\pi}{8}$ a total of 4 times, we will arrive at each of the five critical values for this function

$$\begin{aligned} -\tfrac{\pi}{4} + \tfrac{\pi}{8} = -\tfrac{2\pi}{8} + \tfrac{\pi}{8} = -\tfrac{\pi}{8} \\ -\tfrac{\pi}{8} + \tfrac{\pi}{8} = 0 \\ 0 + \tfrac{\pi}{8} = \tfrac{\pi}{8} \\ \tfrac{\pi}{8} + \tfrac{\pi}{8} = \tfrac{2\pi}{8} = \tfrac{\pi}{4} \end{aligned} \tag{9.5.5}$$

So the critical values along the x -axis would be:

$$-\frac{\pi}{4}, -\frac{\pi}{8}, 0, \frac{\pi}{8}, \text{ and } \frac{\pi}{4} \tag{9.5.6}$$

Notice that the distance between the starting point $-\frac{\pi}{4}$ and the ending point $\frac{\pi}{4}$ is equal to the period we found at the beginning of the problem, which was $\frac{\pi}{2}$. Now let's graph the function:



since there were no changes to the y -axis, the amplitude for the function is 1 and the vertical shift is 0. Along the x -axis, we see a positive sine function that starts $\text{at}\left(-\frac{\pi}{4}, 0\right)$ then goes up to $\left(-\frac{\pi}{8}, 1\right)$, back down through (0,0) to $\left(\frac{\pi}{8}, -1\right)$ and back up to $\left(\frac{\pi}{4}, 0\right)$ to complete one full cycle of the graph.

Let's look at an example in which there are some changes to the y -axis as well as the x -axis.

Example 2
Graph at least one period of the given function. Be sure to identify critical values along the x and y axes.

$$y = -\frac{5}{2} + \cos(3x - \pi) \tag{9.5.7}$$

Remember which coefficients affect which axis in graphing:

y-axis	x-axis
Amplitude= $\lvert A \rvert$	Period= $\frac{2\pi}{B}$
Vertical Shift= D	Phase Shift= $-\frac{C}{B}$

In this example, the amplitude is 1, since there is no coefficient in front of the cosine function. The vertical shift is $-\frac{5}{2}$, which will shift the function down a distance of 2.5 on the y -axis. So, the mid-line or "zero" points of the graph will be at -2.5, the maximum y value will be -1.5 and the minimum y value will be -3.5

Along the x -axis, the period for the graph will be $\frac{2\pi}{B} = \frac{2\pi}{3}$, since the coefficient B in this problem is 3. To find the new starting point, we'll take the argument of the cosine function and set it equal to zero.

$$\begin{aligned} 3x - \pi &= 0 \\ 3x &= \pi \\ x = \pi * \tfrac{1}{3} &= \tfrac{\pi}{3} \end{aligned} \tag{9.5.8}$$

So, our new starting point will be at $\frac{\pi}{3}$. To determine the other critical values along the x -axis, we can find out how far each "jump" between the critical values would be. To do this, we take the period $\left(\frac{2\pi}{3}\right)$ and divide it by 4 (or multiply by $\frac{1}{4}$)

$$\frac{2\pi}{3} * \frac{1}{4} = \frac{2\pi}{12} = \frac{\pi}{6} \tag{9.5.9}$$

Now we can add this value to our new starting point four times to determine the other critical values along the x -axis.

$$\begin{aligned} \frac{\pi}{3}+\frac{\pi}{6}&=\frac{2\pi}{6}+\frac{\pi}{6}=\frac{3\pi}{6}=\frac{\pi}{2} \\ \frac{3\pi}{6}+\frac{\pi}{6}&=\frac{4\pi}{6}=\frac{2\pi}{3} \\ \frac{4\pi}{6}+\frac{\pi}{6}&=\frac{5\pi}{6} \\ \frac{5\pi}{6}+\frac{\pi}{6}&=\pi \end{aligned} \tag{9.5.10}$$

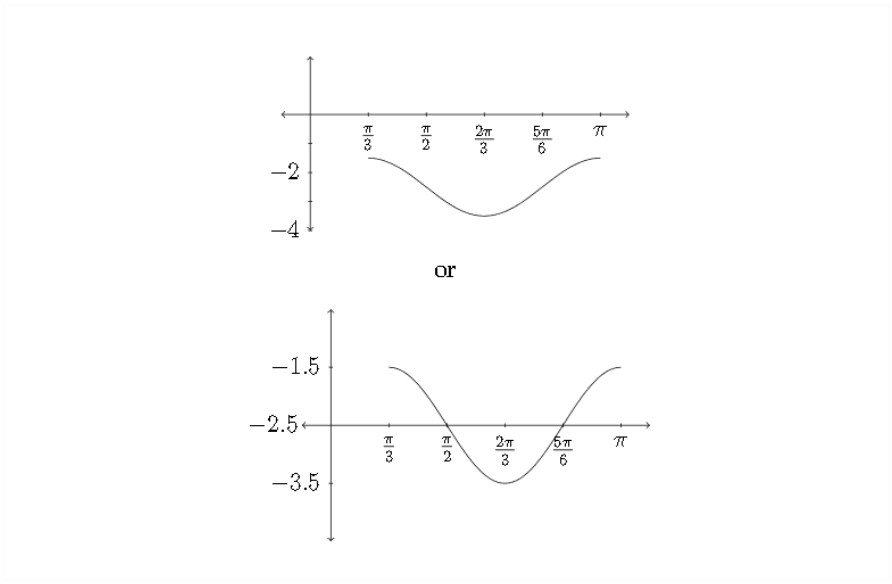
So the critical values along the x -axis would be:

$$\frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}, \text{ and } \frac{6\pi}{6} \tag{9.5.11}$$

or

$$\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \text{ and } \pi \tag{9.5.12}$$

Again, notice that the distance along the x -axis from the starting point to the ending point is the period: $\frac{2\pi}{3}$. Now let's graph the function:



Let's look at one more example.
Example 3
Sometimes the coefficient B appears factored out of the argument as it does in the problem below.

Graph at least one period of the given function. Be sure to include the critical values along the x and y axes.
$y=4\sin 2\left(x+\frac{\pi}{3}\right)-1$
First let's see how the x and y axes are affected by the transformations in this problem.

y-axis	x-axis
Amplitude$=\lvert A\rvert$	Period$=\frac{2\pi}{B}$
Vertical Shift$=D$	Phase Shift$=-\frac{C}{B}$

$$y=4\sin 2\left(x+\frac{\pi}{3}\right)-1 \tag{9.5.13}$$

The amplitude in this problem is 4 and the vertical shift is -1
The period for this graph is $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$. Notice that the value of B is 2 in this example, even though it's been factored out from the rest of the argument.

The new starting point for the graph is actually easier to find in problems of this type. If we take the argument as it is and set it equal to zero:

$$2\left(x + \frac{\pi}{3}\right) = 0 \tag{9.5.14}$$

we can divide through on both sides by 2 to cancel out the factor of B :

$$\begin{gathered} \frac{2(x+\frac{\pi}{3})}{2} = \frac{0}{2} \\ x + \frac{\pi}{3} = 0 \\ x = -\frac{\pi}{3} \end{gathered} \tag{9.5.15}$$

So, the new starting point for the function is $-\frac{\pi}{3}$
Now let's find the rest of the critical values along the x -axis. The period for this graph is π, so the "jump " between the critical values along the x -axis will be:

$$\pi * \frac{1}{4} = \frac{\pi}{4} \tag{9.5.16}$$

To find the rest of the critical values we'll need to add $\frac{\pi}{4}$ to the starting point of the graph $\left(-\frac{\pi}{3}\right)$ four times:

$$\begin{gathered} -\frac{\pi}{3} + \frac{\pi}{4} = -\frac{4\pi}{12} + \frac{3\pi}{12} = -\frac{\pi}{12} \\ -\frac{\pi}{12} + \frac{3\pi}{12} = \frac{2\pi}{12} = \frac{\pi}{6} \\ \frac{2\pi}{12} + \frac{3\pi}{12} = \frac{5\pi}{12} \\ \frac{5\pi}{12} + \frac{3\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3} \end{gathered} \tag{9.5.17}$$

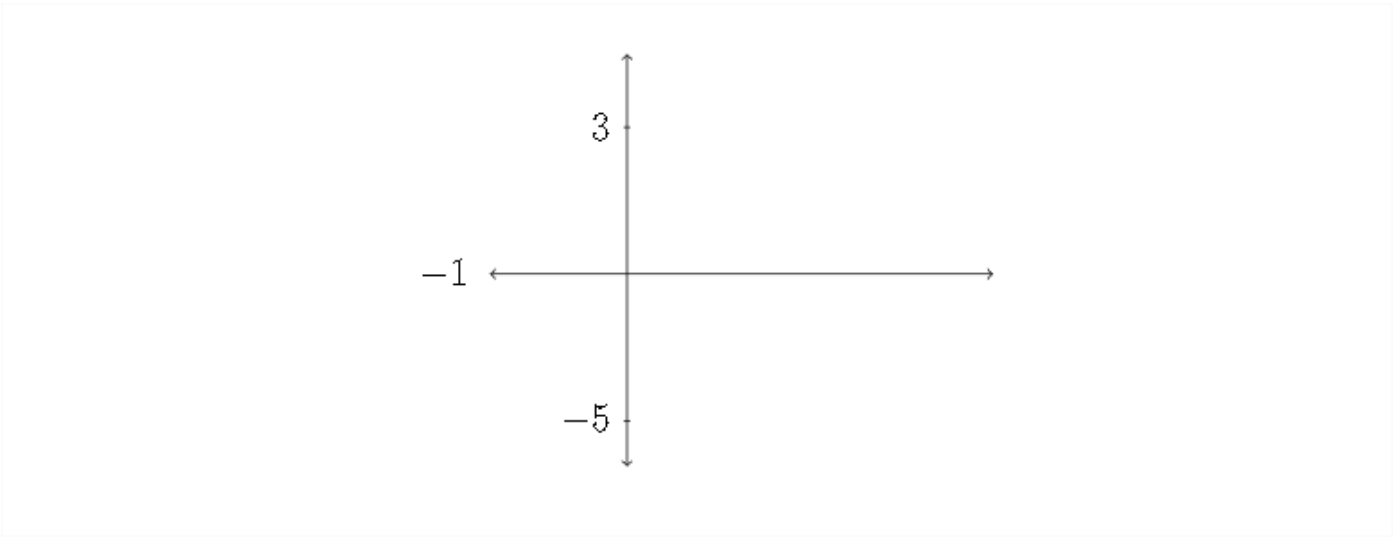
So the critical values along the x -axis would be:

$$-\frac{4\pi}{12}, -\frac{1\pi}{12}, \frac{2\pi}{12}, \frac{5\pi}{12}, \text{ and } \frac{8\pi}{12} \tag{9.5.18}$$

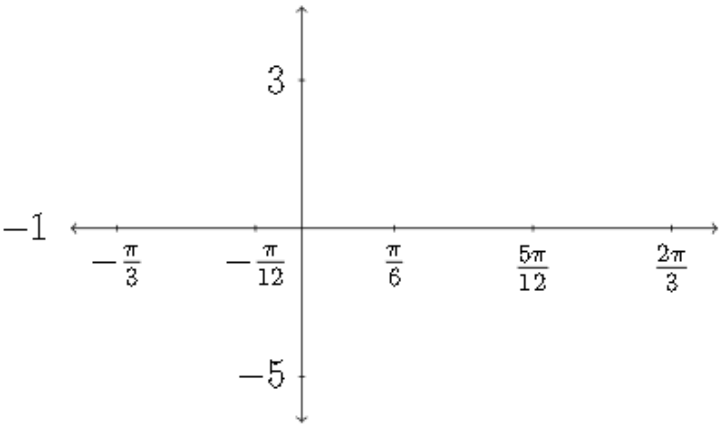
Or

$$-\frac{\pi}{3}, -\frac{\pi}{12}, \frac{\pi}{6}, \frac{5\pi}{12}, \text{ and } \frac{2\pi}{3} \tag{9.5.19}$$

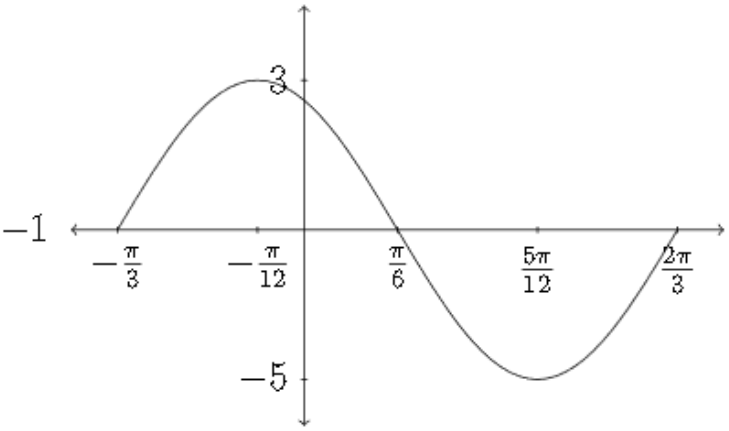
Now that we've addressed each of the four transformations let's use this information to draw the graph. First the y -axis - the amplitude is 4 and the vertical shift is -1:



Now, let's fill in the information for the x -axis. The critical values along the x -axis are $-\frac{\pi}{3}, -\frac{\pi}{12}, \frac{\pi}{6}, \frac{5\pi}{12}$, and $\frac{2\pi}{3}$



The function we're graphing is a positive sine function, so it will start at the "midline" or zero value (which in this case is -1), go up to the maximum, back through the mid-line to the minimum and back to the mid-line:



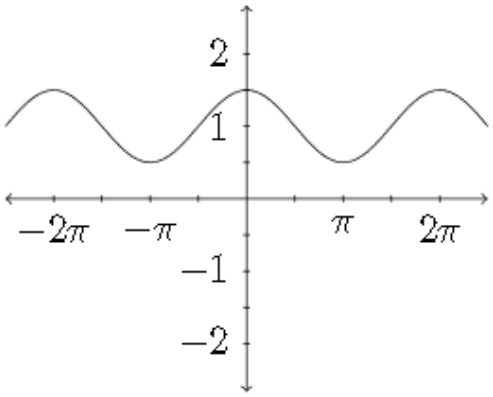
Determine the Amplitude, Period, Vertical Shift and Phase Shift for each function and graph at least one complete period. Be sure to identify the critical values along the x and y axes.

1. $y = \sin\left(x + \frac{\pi}{2}\right)$
2. $y = \sin(x - \pi)$
3. $y = 3\cos\left(x - \frac{\pi}{2}\right)$
4. $y = \frac{1}{2}\cos(x + \pi)$
5. $y = 3 + \cos\left(x - \frac{\pi}{4}\right)$
6. $y = -2 + \sin\left(x + \frac{\pi}{6}\right)$
7. $y = \sin(2x - \pi)$
8. $y = \sin\left(4x + \frac{\pi}{4}\right)$
9. $y = 2\cos\left(\frac{x}{2} + \pi\right)$
10. $y = -3\sin(6x - \pi)$
11. $y = -\frac{1}{3}\sin\left(2x + \frac{\pi}{4}\right)$
12. $y = \frac{1}{2}\cos\left(\frac{x}{2} - \pi\right)$
13. $y = 2\sin\left(2x - \frac{\pi}{3}\right) - 1$

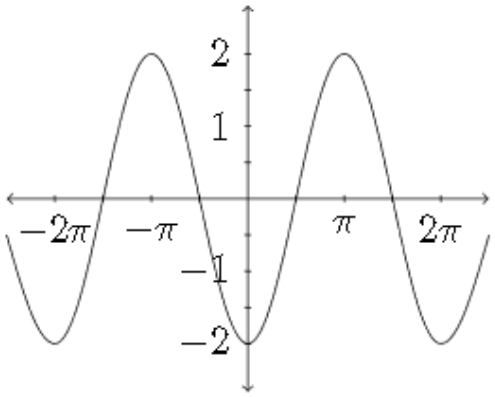
14. $y = 1 + 2\cos\left(3x + \frac{\pi}{2}\right)$
15. $y = 3\cos 2\left(x + \frac{\pi}{6}\right)$
16. $y = -4\sin 2\left(x + \frac{\pi}{2}\right)$
17. $y = \sin\frac{1}{2}\left(x + \frac{\pi}{4}\right)$
18. $y = 3 + 2\sin 3\left(x + \frac{\pi}{2}\right)$

In problems $19 - 22$, determine an equation for the function that is shown.

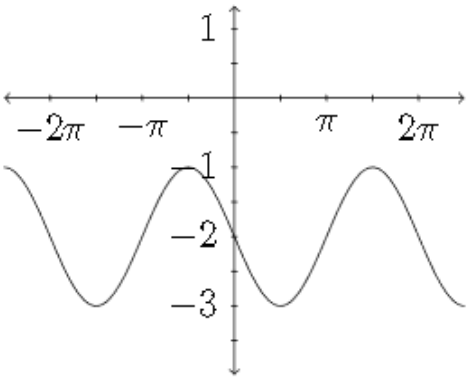
19.



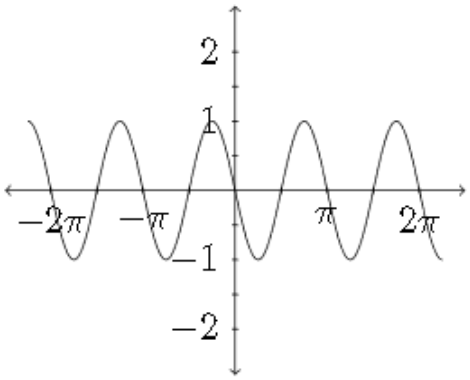
20.



21.

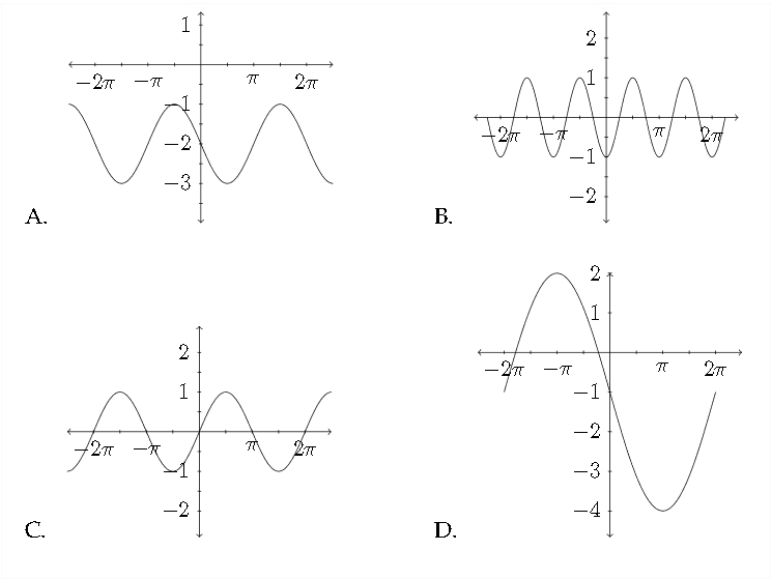


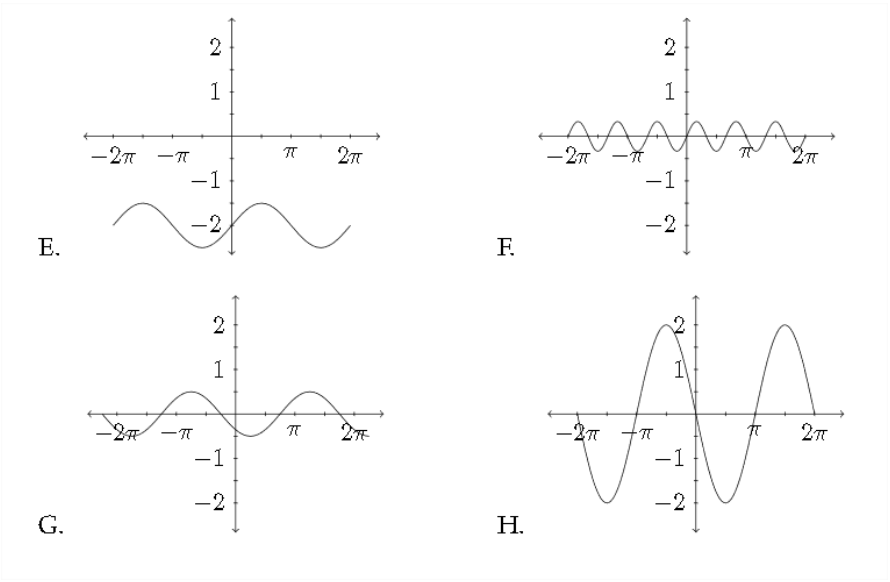
22.



Match the function to the appropriate graph

23. $y = -\cos 2x$
24. $y = \frac{1}{2}\sin x - 2$
25. $y = 2\cos\left(x + \frac{\pi}{2}\right)$
26. $y = -3\sin\frac{1}{2}x - 1$
27. $y = \sin(x - \pi) - 2$
28. $y = -\frac{1}{2}\cos\left(x - \frac{\pi}{4}\right)$
29. $y = \frac{1}{3}\sin 3x$
30. $y = \cos\left(x - \frac{\pi}{2}\right)$





Determine the Amplitude, Period, Vertical Shift and Phase Shift for each function and graph at least one complete period. Be sure to identify the critical values along the x and y axes.

31. $y = 2\cos\left(2x + \frac{\pi}{2}\right) - 1$
32. $y = -4\cos(3x - 2\pi)$
33. $y = \sin\left(2x - \frac{\pi}{4}\right)$
34. $y = -\sin(3x + \pi)$
35. $y = 3\cos\left(x + \frac{\pi}{3}\right) + 1$
36. $y = -2\sin\left(3x - \frac{\pi}{2}\right) + 4$
37. $y = -\frac{1}{2}\sin\left(x - \frac{\pi}{2}\right) - 2$
38. $y = 2 - \cos\left(2x - \frac{\pi}{3}\right)$

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10: TRIGONOMETRIC IDENTITIES AND EQUATIONS

Due to the nature of the trigonometric ratios, they have some interesting properties that make them useful in a number of mathematical problem-solving situations. One of the hallmarks of mathematical problem-solving is to change the appearance of the problem without changing its value. Trigonometric identities can be very helpful in changing the appearance of a problem.

The process of demonstrating the validity of a trigonometric identity involves changing one trigonometric expression into another, using a series of clearly defined steps. We'll look at a few examples briefly, but first, let's examine some of the fundamental trigonometric identities.

10.1: Reciprocal and Pythagorean Identities

10.2: Double-Angle Identities

10.3: Trigonometric Equations

10.4: More Trigonometric Equations

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10.1: Reciprocal and Pythagorean Identities

The two most basic types of trigonometric identities are the reciprocal identities and the Pythagorean identities. The reciprocal identities are simply definitions of the reciprocals of the three standard trigonometric ratios:

$$\sec\theta = \frac{1}{\cos\theta} \quad \csc\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{1}{\tan\theta} \tag{10.1.1}$$

Also, recall the definitions of the three standard trigonometric ratios (sine, cosine and tangent):

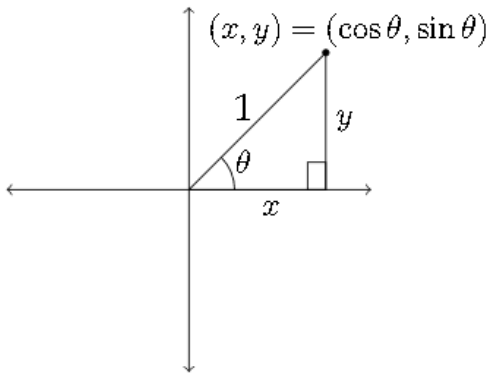
$$\begin{aligned} \sin\theta &= \tfrac{opp}{hyp} \\ \cos\theta &= \tfrac{adj}{hyp} \\ \tan\theta &= \tfrac{opp}{ady} \end{aligned} \tag{10.1.2}$$

If we look more closely at the relationships between the sine, cosine and tangent, we'll notice that $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$$\frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{opp}{hyp}\right)}{\left(\frac{adj}{hyp}\right)} = \frac{opp}{hyp} * \frac{hyp}{adj} = \frac{opp}{adj} = \tan\theta \tag{10.1.3}$$

Pythagorean Identities

The Pythagorean Identities are, of course, based on the Pythagorean Theorem. If we recall a diagram that was introduced in Chapter 2, we can build these identities from the relationships in the diagram:



Using the Pythagorean Theorem in this diagram, we see that $x^2 + y^2 = 1^2$, so $x^2 + y^2 = 1$. But, also remember that, in the unit circle, $x = \cos\theta$ and $y = \sin\theta$

Substituting this equality gives us the first Pythagorean Identity:

$$x^2 + y^2 = 1 \tag{10.1.4}$$

or

$$\cos^2\theta + \sin^2\theta = 1 \tag{10.1.5}$$

This identity is usually stated in the form:

$$\sin^2\theta + \cos^2\theta = 1 \tag{10.1.6}$$

If we take this identity and divide through on both sides by $\cos^2\theta$, this will result in the first of two additional Pythagorean Identities:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{10.1.7}$$

or

$$\tan^2\theta + 1 = \sec^2\theta \tag{10.1.8}$$

Dividing through by $\sin^2\theta$ gives us the second:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \tag{10.1.9}$$

or

$$1 + \cot^2\theta = \csc^2\theta \tag{10.1.10}$$

So, the three Pythagorean Identities we will be using are:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned} \tag{10.1.11}$$

These Pythagorean Identities are often stated in other terms, such as:

$$\begin{aligned} \sin^2\theta &= 1 - \cos^2\theta \\ \cos^2\theta &= 1 - \sin^2\theta \\ \tan^2\theta &= \sec^2\theta - 1 \\ \cot^2\theta &= \csc^2\theta - 1 \end{aligned} \tag{10.1.12}$$

At the beginning of this chapter, we discussed verifying trigonometric identities. Now that we have some basic identities to work with, let's use them to verify the equality of some more complicated statements. The process of verifying trigonometric identities involves changing one side of the given expression into the other side. since these are not really equations, we will not treat them the way we treat equations. That is to say, we won't add or subtract anything to both sides of the statement (or multiply or divide by anything on both sides either).

Another reason for not treating a trigonometric identity as an equation is that, in practice, this process typically involves just one side of the statement. In problem solving, mathematicians typically use trigonometric identities to change the appearance of a problem without changing its value. In this process, a trigonometric expression is changed into another trigonometric expression rather than showing that two trigonometric expressions are the same, which is what we will be doing.

Example 1
Verify the identity $(\sin\theta)(\cot\theta) = \cos\theta$
This is a very straightforward identity and it can solved by using one of the fundamental approaches to working with trigonometric identities. This is the approach of writing everything in terms of sines and cosines.

Beginning with the original statement:

$$(\sin\theta)(\cot\theta) = \cos\theta \tag{10.1.13}$$

Replace $\cot\theta$ with $\frac{\cos\theta}{\sin\theta}$

$$(\sin\theta)\frac{\cos\theta}{\sin\theta} = \cos\theta \tag{10.1.14}$$

Then canceling out the $\sin\theta$:

$$\cos\theta = \cos\theta \tag{10.1.15}$$

There are four fundamental approaches to verifying trigonometric identities:
1. write everything in terms of sines and cosines
2. make a common denominator and add fractions
3. split a fraction
4. factor and cancel
Not all of these can be used in every problem and some problems will use combinations of these strategies. Here is another example.

Example 2
Verify the identity $\tan\theta + \cot\theta = \sec\theta\csc\theta$

First we'll write everything in terms of sines and cosines:

$$\begin{aligned}\tan\theta+\cot\theta&=\sec\theta\csc\theta\\ \tfrac{\sin\theta}{\cos\theta}+\tfrac{\cos\theta}{\sin\theta}&=\tfrac{1}{\cos\theta}\cdot\tfrac{1}{\sin\theta}\end{aligned}\tag{10.1.16}$$

Next, on the left hand side, we can add the two fractions together by making a common denominator of $\cos\theta\sin\theta$

$$\begin{aligned}\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}&=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ \frac{\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\cos\theta}&=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ \frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}&=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}&=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ \frac{1}{\sin\theta\cos\theta}&=\frac{1}{\sin\theta\cos\theta}\end{aligned}\tag{10.1.17}$$

In this example, you can see that we have first written everything in terms of sines and cosines, then created common denominators and added the fractions on the left hand side together. After this is done, we can replace the expression $\sin^2\theta+\cos^2\theta$ with 1, since this is the fundamental Pythagorean Identity.

Example 3

Verify the identity $\frac{\tan\theta-\cot\theta}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta$

We'll begin this problem by splitting the fraction over the denominator. This can be helpful in problems in which there is no addition or subtraction in the denominator. The idea here is that since $\frac{a}{x}+\frac{b}{x}=\frac{a+b}{x}$, then we can reverse this process and say that $\frac{a+b}{x}=\frac{a}{x}+\frac{b}{x}$

In the problem above we'll say that:

$$\frac{\tan\theta-\cot\theta}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.18}$$

$$\frac{\tan\theta}{\sin\theta\cos\theta}-\frac{\cot\theta}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.19}$$

$$\frac{\frac{\sin\theta}{\cos\theta}}{\sin\theta\cos\theta}-\frac{\frac{\cos\theta}{\sin\theta}}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.20}$$

$$\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta\cos\theta}-\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.21}$$

$$\frac{\cancel{\sin\theta}}{\cos\theta}\cdot\frac{1}{\cancel{\sin\theta}\cos\theta}-\frac{\cancel{\cos\theta}}{\sin\theta}\cdot\frac{1}{\sin\theta\cancel{\cos\theta}}=\sec^2\theta-\csc^2\theta\tag{10.1.22}$$

$$\frac{1}{\cos^2\theta}-\frac{1}{\sin^2\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.23}$$

$$\sec^2\theta-\csc^2\theta=\sec^2\theta-\csc^2\theta\tag{10.1.24}$$

Example 4

Verify the identity $\frac{\tan^2\theta-\cos^2\theta}{1-\cos^2\theta}=\sec^2\theta-\cot^2\theta$

On the left-hand side, notice the expression $1-\cos^2\theta$ in the denominator. We can replace this with $\sin^2\theta$, which is a simpler expression. It is often helpful to have a simpler expression in the denominator rather than a more complicated expression.

$$\begin{aligned}\tfrac{\tan^2\theta-\cos^2\theta}{1-\cos^2\theta}&=\sec^2\theta-\cot^2\theta\\ \tfrac{\tan^2\theta-\cos^2\theta}{\sin^2\theta}&=\sec^2\theta-\cot^2\theta\end{aligned}\tag{10.1.25}$$

Next, we can split the fraction over the denominator of $\sin^2\theta$

$$\frac{\tan^2\theta-\cos^2\theta}{\sin^2\theta}=\sec^2\theta-\cot^2\theta\tag{10.1.26}$$

$$\frac{\tan^2\theta}{\sin^2\theta}-\frac{\cos^2\theta}{\sin^2\theta}=\sec^2\theta-\cot^2\theta\tag{10.1.27}$$

We can see on the left-hand side that the expression $\frac{\cos^2\theta}{\sin^2\theta}$ is equivalent to $\cot^2\theta$ but the first piece on the left-hand side needs to

be simplified a little more. We'll rewrite $\tan^2\theta$ as $\frac{\sin^2\theta}{\cos^2\theta}$ and then simplify the complex fraction.

$$\begin{aligned}\frac{\frac{\tan^2\theta}{\sin^2\theta}}{} - \frac{\frac{\cos^2\theta}{\sin^2\theta}}{} &= \sec^2\theta - \cot^2\theta \\ \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\sin^2\theta} - \cot^2\theta &= \sec^2\theta - \cot^2\theta \\ \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\sin^2\theta} - \cot^2\theta &= \sec^2\theta - \cot^2\theta \\ \frac{\cancel{\sin^2\theta}}{\cos^2\theta} \cdot \frac{1}{\cancel{\sin^2\theta}} - \cot^2\theta &= \sec^2\theta - \cot^2\theta\end{aligned} \tag{10.1.28}$$

After we cancel out the $\sin^2\theta$, we're almost done:

$$\begin{aligned}\frac{\cancel{\sin^2\theta}}{\cos^2\theta} \cdot \frac{1}{\cancel{\sin^2\theta}} - \cot^2\theta &= \sec^2\theta - \cot^2\theta \\ \frac{1}{\cos^2\theta} - \cot^2\theta &= \sec^2\theta - \cot^2\theta \\ \sec^2\theta - \cot^2\theta &= \sec^2\theta - \cot^2\theta\end{aligned} \tag{10.1.29}$$

The trigonometric identities we have discussed in this section are summarized below:

Pythagorean Identities	Reciprocal Identities
$\sin^2\theta + \cos^2\theta = 1$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\tan^2\theta + 1 = \sec^2\theta$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$1 + \cot^2\theta = \csc^2\theta$	$\sec\theta = \frac{1}{\cos\theta}$
	$\csc\theta = \frac{1}{\sin\theta}$

In the examples above and in the exercises, the form sin θ or $\cos\theta$ is typically used, however any letter may be used to represent the angle in question so long as it is the SAME letter in all expressions. For example, we can say that:

$$\sin^2\theta + \cos^2\theta = 1 \tag{10.1.30}$$

or we can say that

$$\sin^2 x + \cos^2 x = 1 \tag{10.1.31}$$

however:

$$\sin^2\theta + \cos^2 x \neq 1 \tag{10.1.32}$$

because θ and x could be different angles!

Exercises 3.1
In each problem verify the given trigonometric identify.

1. $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$

2. $\tan\theta(\cot\theta + \tan\theta) = \sec^2\theta$

3. $\tan\theta(\csc\theta + \cot\theta) - \sec\theta = 1$

4. $\cot\theta(\sec\theta+\tan\theta)-\csc\theta=1$

5. $\tan^2\theta\csc^2\theta-\tan^2\theta=1$

6. $\sin^2\theta\cot^2\theta+\sin^2\theta=1$

7. $\dfrac{\sin\theta\tan\theta+\sin\theta}{\tan\theta+\tan^2\theta}=\cos\theta$

8. $\dfrac{\cos\theta\cot\theta+\cos\theta}{\cot\theta+\cot^2\theta}=\sin\theta$

9. $\dfrac{(\sin\theta+\cos\theta)^2}{\cos\theta}-\sec\theta=2\sin\theta$

10. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2=2$

11. $\cos\theta(\tan\theta+\cot\theta)=\csc\theta$

12. $\sin\theta(\cot\theta+\tan\theta)=\sec\theta$

13. $\dfrac{\cos\theta}{\tan\theta}-\csc\theta=-\sin\theta$

14. $\dfrac{\sin\theta}{\cot\theta}-\sec\theta=-\cos\theta$

15. $\dfrac{\csc\theta}{\cos\theta}-\dfrac{\cos\theta}{\csc\theta}=\dfrac{\cot^2\theta+\sin^2\theta}{\cot\theta}$

16. $\dfrac{\sec\theta+\csc\theta}{\tan\theta+\cot\theta}=\sin\theta+\cos\theta$

17. $\dfrac{\sin\theta}{1+\sin\theta}-\dfrac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

18. $\dfrac{\cos\theta}{1+\cos\theta}-\dfrac{\cos\theta}{1-\cos\theta}=-2\cot^2\theta$

19. $\dfrac{\cot\theta}{1+\csc\theta}-\dfrac{\cot\theta}{1-\csc\theta}=2\sec\theta$

20. $\dfrac{\tan\theta}{1+\sec\theta}-\dfrac{\tan\theta}{1-\sec\theta}=2\csc\theta$

21. $\dfrac{\sec^2\theta}{1+\cot^2\theta}=\tan^2\theta$

22. $\dfrac{\csc^2\theta}{1+\tan^2\theta}=\cot^2\theta$

23. $\sec^4\theta-\sec^2\theta=\tan^4\theta+\tan^2\theta$

24. $\csc^4\theta-\csc^2\theta=\cot^4\theta+\cot^2\theta$

25. $1-\dfrac{\cos^2\theta}{1+\sin\theta}=\sin\theta$

26. $1-\dfrac{\sin^2\theta}{1+\cos\theta}=\cos\theta$

27. $\dfrac{\sec\theta}{\csc\theta}+\dfrac{\sin\theta}{\cos\theta}=2\tan\theta$

28. $\dfrac{1-\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{1-\sin\theta}=2\sec\theta$

29. $\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{1+\sin\theta}{\cos\theta}=2\sec\theta$

30. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=\sin^2\theta-\cos^2\theta$

31. $\dfrac{\sec\theta-\cos\theta}{\sec\theta+\cos\theta}=\dfrac{\sin^2\theta}{1+\cos^2\theta}$

32. $\dfrac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta}=\tan\theta\sec\theta$

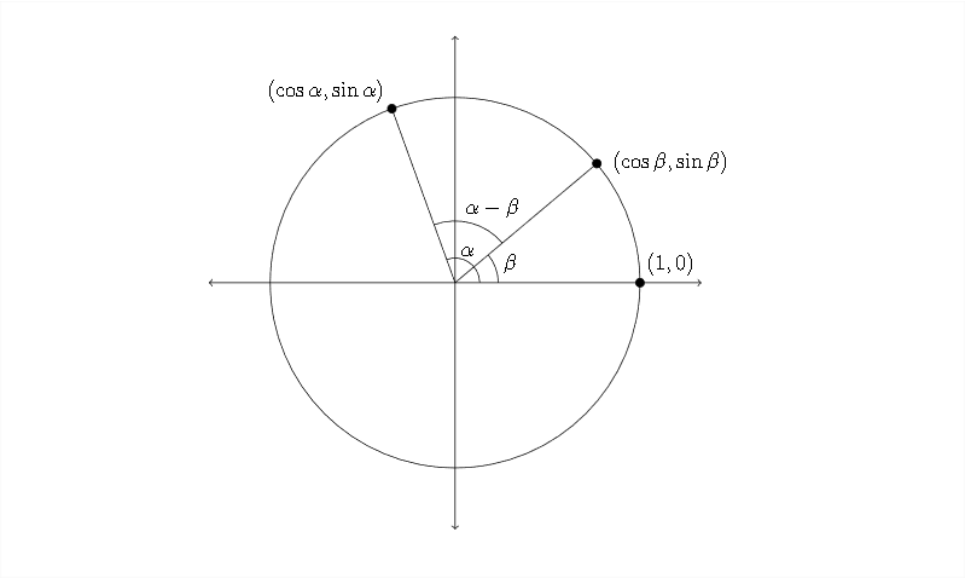
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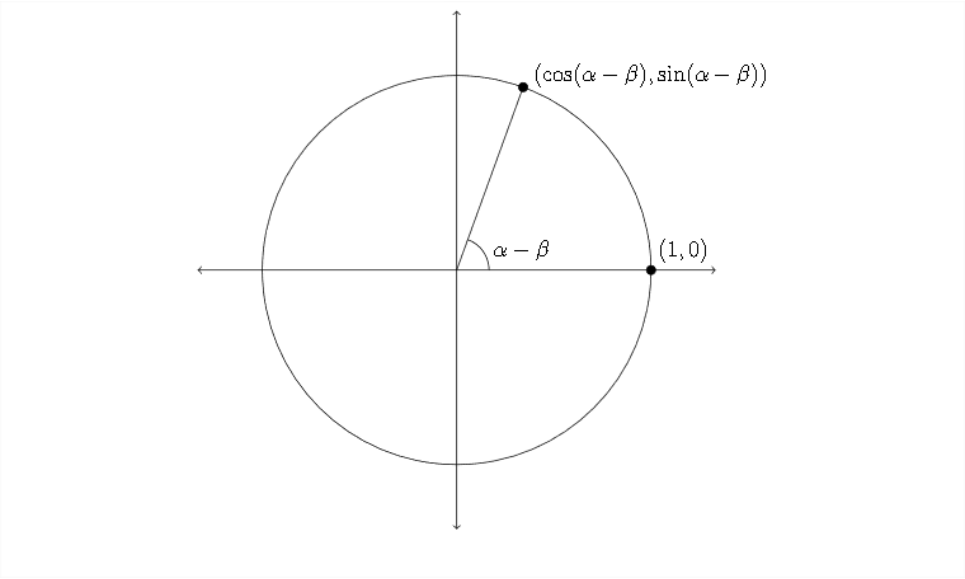
10.2: Double-Angle Identities

In this section we will include several new identities to the collection we established in the previous section. These new identities are called "Double-Angle Identities ’’ because they typically deal with relationships between trigonometric functions of a particular angle and functions of "two times" or double the original angle.

To establish the validity of these identities we need to use what are known as the Sum and Difference Identities. These are identities that deal with expressions such as $\sin(\alpha+\beta)$. First we will establish an expression that is equivalent to $\cos(\alpha-\beta)$
Let's start with the unit circle:

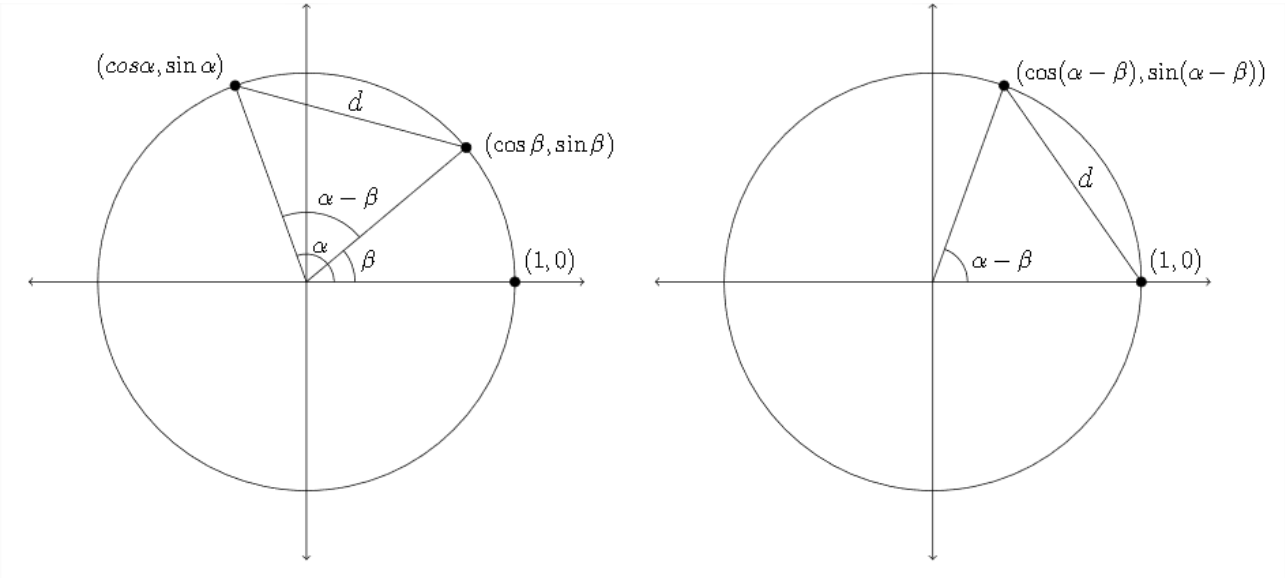


If we rotate everything in this picture clockwise so that the point labeled $(\cos\beta, \sin\beta)$ slides down to the point labeled $(1,0)$, then the angle of rotation in the diagram will be $\alpha-\beta$ and the corresponding point on the edge of the circle will be:
$(\cos(\alpha-\beta), \sin(\alpha-\beta))$
The diagram that represents this rotation is on the next page.



since the the second diagram is created by rotating the lines and points from the first diagram, the distance between the points

$(\cos\alpha, \sin\alpha)$ and $(\cos\beta, \sin\beta)$ in the first diagram is the same as the distance between $(\cos(\alpha-\beta), \sin(\alpha-\beta))$ and the point (1,0) in the second diagram.



In the diagram above the length of d in each picture is the same.

We can represent this distance d with the distance formula used to calculate the distance between two points in the coordinate plane:
The distance between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{10.2.1}$$

So, in the first diagram the distance d will be:

$$d = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \tag{10.2.2}$$

In the second diagram the distance d will be:

$$d = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2} \tag{10.2.3}$$

since these distances are the same, we can set them equal to each other:

$$\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2} \tag{10.2.4}$$

We'll square both sides to clear the radicals:

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = (\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2 \tag{10.2.5}$$

Next, we'll rewrite $(\sin(\alpha-\beta) - 0)^2$ as $\sin^2(\alpha-\beta)$

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = (\cos(\alpha-\beta) - 1)^2 + \sin^2(\alpha-\beta) \tag{10.2.6}$$

Now we'll work to simplify the expressions on the left-hand side of this equation.

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = (\cos(\alpha-\beta) - 1)^2 + \sin^2(\alpha-\beta) \tag{10.2.7}$$

First, each one needs to be squared:

$$\begin{aligned}(\cos\alpha - \cos\beta)^2 &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta \\ (\sin\alpha - \sin\beta)^2 &= \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta\end{aligned} \tag{10.2.8}$$

So, the left-hand side will now be:

$$\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta \tag{10.2.9}$$

If we rearrange this a little, it will simplify nicely:

$$\begin{aligned}&\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta\\&\sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta\end{aligned} \tag{10.2.10}$$

Notice the Pythagorean Identities at the front of this expression - these are each equal to 1:

$$\begin{aligned}\sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta\\1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta\\2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta\\2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta)\end{aligned} \tag{10.2.11}$$

Now that we've simplified the left-hand side, we'll simplify the right-hand side First we'll square the expression $(\cos(\alpha-\beta)-1)^2$

$$(\cos(\alpha-\beta)-1)^2 = \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 \tag{10.2.12}$$

So, the right-hand side is now:

$$\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta) \tag{10.2.13}$$

If we rearrange this expression, we'll again have a nice Pythagorean Identity:

$$\begin{aligned}\sin^2(\alpha-\beta) + \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1\\1 - 2\cos(\alpha-\beta) + 1\\2 - 2\cos(\alpha-\beta)\\2(1 - \cos(\alpha-\beta))\end{aligned} \tag{10.2.14}$$

So the left-hand side was equal to:

$$2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta) \tag{10.2.15}$$

And the right-hand side was equal to:

$$2(1 - \cos(\alpha-\beta)) \tag{10.2.16}$$

So, our original statement in simplified form is:

$$2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta) = 2(1 - \cos(\alpha-\beta)) \tag{10.2.17}$$

If we divide by 2 on both sides, we'll have:

$$1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta = 1 - \cos(\alpha-\beta) \tag{10.2.18}$$

then subtract 1

$$-\cos\alpha\cos\beta - \sin\alpha\sin\beta = -\cos(\alpha-\beta) \tag{10.2.19}$$

and multiply through by -1

$$\begin{aligned}\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)\\\text{So, } \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned} \tag{10.2.20}$$

This will help us to generate the double-angle formulas, but to do this, we don't want $\cos(\alpha-\beta)$, we want $\cos(\alpha+\beta)$ (you'll see why in a minute).

So, to change this around, we'll use identities for negative angles. Recall that in the fourth quadrant the sine function is negative and the cosine function is positive. For this reason, $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$
Now we can say that $\cos(2\theta) = \cos(\theta+\theta) = \cos(\theta-(-\theta))$. Going back to our identity for $\cos(\alpha-\beta)$, we can say that:

$$\begin{aligned}\cos(\theta-(-\theta)) &= \cos\theta\cos(-\theta) + \sin\theta\sin(-\theta)\\\cos(\theta-(-\theta)) &= \cos\theta\cos\theta + \sin\theta(-\sin\theta)\\\cos(\theta-(-\theta)) &= \cos\theta\cos\theta - \sin\theta\sin\theta\\\cos(\theta-(-\theta)) &= \cos^2\theta - \sin^2\theta\\\cos(\theta+\theta) &= \cos^2\theta - \sin^2\theta\\\cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned} \tag{10.2.21}$$

This is the double-angle identity for the cosine: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

This identity actually appears in any one of three forms because the Pythagorean Identities can be applied to this to change its appearance:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \cos(2\theta) &= 1 - \sin^2\theta - \sin^2\theta \\ \cos(2\theta) &= 1 - 2\sin^2\theta\end{aligned} \tag{10.2.22}$$

If we substitute for the $\sin^2\theta$ term:

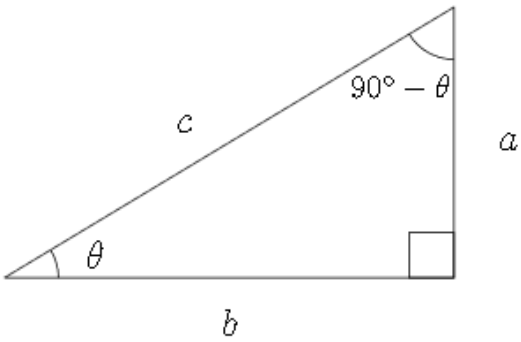
$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \cos(2\theta) &= \cos^2\theta - (1 - \cos^2\theta) \\ \cos(2\theta) &= \cos^2\theta - 1 + \cos^2\theta \\ \cos(2\theta) &= 2\cos^2\theta - 1\end{aligned} \tag{10.2.23}$$

So, the three forms of the cosine double angle identity are:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned} \tag{10.2.24}$$

The double-angle identity for the sine function uses what is known as the cofunction identity. Remember that, in a right triangle, the sine of one angle is the same as the cosine of its complement (which is the other acute angle). This is because the adjacent side for one angle is the opposite side for the other angle. The denominator in both cases is the hypotenuse, so the cofunctions of complementary angles are equal.

In the diagram below, we can see this more clearly:



In the diagram above note that:

$$\sin\theta = \frac{a}{c} = \cos\left(90^\circ - \theta\right) \tag{10.2.25}$$

So, if we want an identity for $\sin(\theta + \theta)$, we'll start with $\sin(\alpha + \beta)$ which is equivalent to $\cos\left(90^\circ - (\alpha + \beta)\right)$. We'll use a trick here and restate this as:

$$\begin{aligned}\sin(\alpha + \beta) &= \cos\left(90^\circ - (\alpha + \beta)\right) \\ &= \cos\left(90^\circ - \alpha - \beta\right) \\ &= \cos\left((90^\circ - \alpha) - \beta\right) \\ &= \cos\left(90^\circ - \alpha\right)\cos\beta + \sin\left(90^\circ - \alpha\right)\sin\beta \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned} \tag{10.2.26}$$

Now, we can use this to find an expression for $\sin 2\theta = \sin(\theta + \theta)$

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta\end{aligned} \tag{10.2.27}$$

Exercises 3.2

Here is a summary of all the identities we've worked with in this chapter:

Pythagorean Identities
$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Reciprocal Identities
$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\sec\theta = \frac{1}{\cos\theta}$
$\csc\theta = \frac{1}{\sin\theta}$

Double-Angle Identities
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\cos(2\theta) = 2\cos^2\theta - 1$
$\cos(2\theta) = 1 - 2\sin^2\theta$
$\sin(2\theta) = 2\sin\theta\cos\theta$

Working problems involving double-angle identities is very similar to the other identities we've worked with previously- you just have more identities to choose from!

Example
Verify the given identity: $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$
We have three possible identities to choose from for the left-hand side, so we'll wait on that for a moment while we simplify the right-hand side.

$$\begin{aligned}\frac{1-\tan^2 x}{1+\tan^2 x} &= \frac{1-\tan^2 x}{\sec^2 x}\\ &= \frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x}\\ &= \cos^2 x - \frac{\left(\frac{\sin^2 x}{\cos^2 x}\right)}{\left(\frac{1}{\cos^2 x}\right)}\\ &= \cos^2 x - \frac{\sin^2 x}{\cos^2 x}\cdot\frac{\cos^2 x}{1}\\ &= \cos^2 x - \frac{\sin^2 x}{\cancel{\cos^2 x}}\cdot\frac{\cancel{\cos^2 x}}{1}\\ &= \cos^2 x - \sin^2 x\end{aligned} \tag{10.2.28}$$

This is one of the identities for $\cos(2\theta)$ so we can stop and simply state $\cos(2x) = \cos^2 x - \sin^2 x$

In each problem verify the given trigonometric identity.
1. $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} = \tan(2x)$
2. $\sec(2x) = \frac{\sec^2 x}{2-\sec^2 x}$
3. $\sin(2x)\csc x = 2\cos x$
4. $\frac{2\cos x}{\sin(2x)} = \csc x$
5. $\frac{\cos(2x)}{\sin x} + \sin x = \frac{\cot x}{\sec x}$
6. $\frac{\sin x + \sin(2x)}{\sec x + 2} = \sin x\cos x$
7. $(\sin x + \cos x)^2 = 1 + \sin(2x)$
8. $\left(\sin^2 x - 1\right)^2 = \sin^4 x + \cos(2x)$
9. $2\cos x - \frac{\cos(2x)}{\cos x} = \sec x$
10. $\frac{1+\cos(2x)}{1-\cos(2x)} = \cot^2 x$
11. $\frac{\cos(2x)}{\sin^2 x} = \cot^2 x - 1$
12. $\frac{\cos(2x)}{\sin^2 x} = \csc^2 x - 2$
13. $\frac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$
14. $\sin 2x = \frac{2(\tan x - \cot x)}{\tan^2 x - \cot^2 x}$
15. $\frac{2\cos 2x}{\sin 2x} = \cot x - \tan x$
16. $\tan 2x = \frac{2}{\cot x - \tan x}$

17. $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$
18. $\tan x + \cot x = 2\csc(2x)$
19. $\cos(2x) = \frac{\cot^2 x - 1}{\cot^2 x + 1}$
20. $\sin(2x) = \frac{2\tan x}{1+\tan^2 x}$
21. $\frac{2\sin^2 x}{\sin(2x)} + \cot x = \sec x \csc x$
22. $\sec^2 x \cos(2x) = \sec^2 x - 2\tan^2 x$
23. $\frac{\cos(2x)}{\sin x} + \sin x = \csc x - \sin x$
24. $\frac{2\tan x - \sin(2x)}{2\sin^2 x} = \tan x$

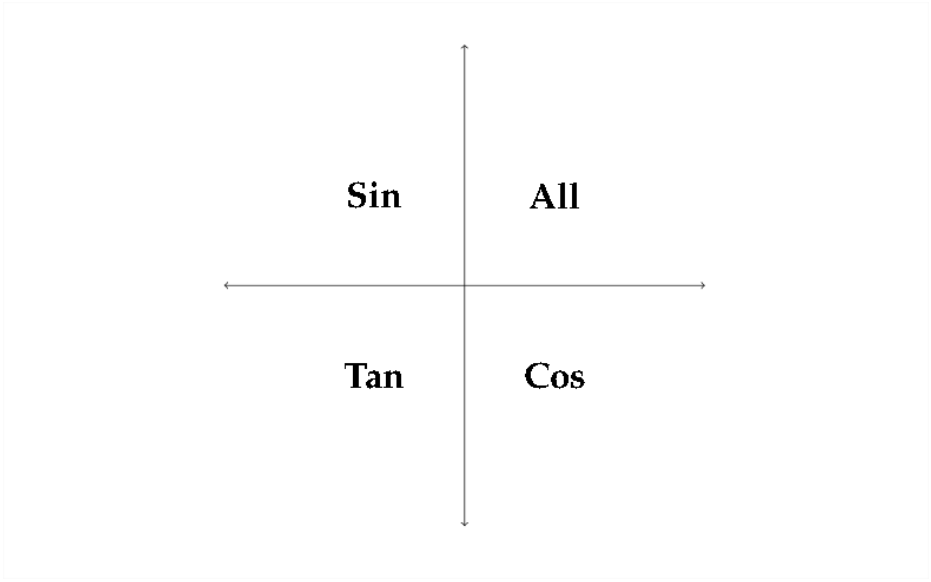
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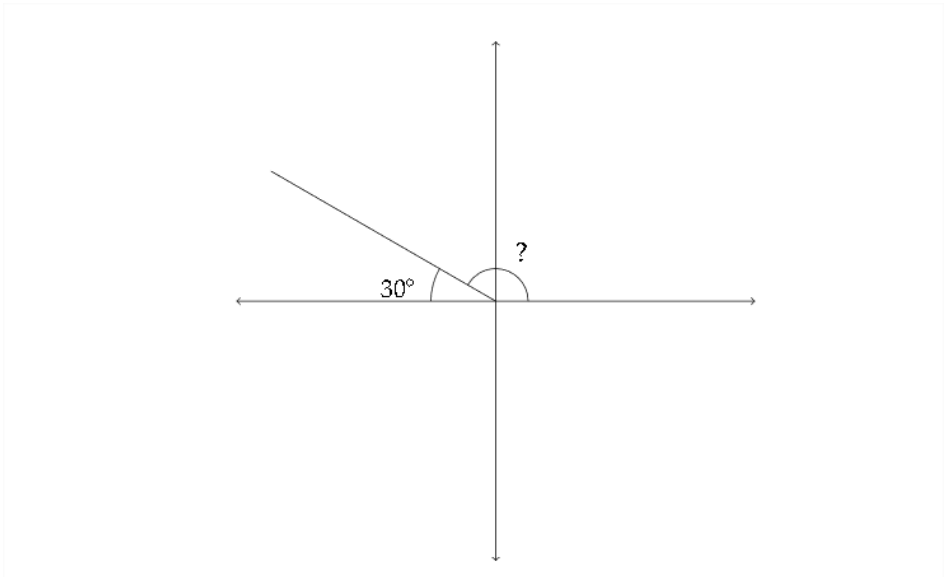
10.3: Trigonometric Equations

In the previous section on trigonometric identities we worked with equations that would be true for all values of a particular angle θ. These are sort of like the algebraic equations whose solution set is "all real numbers," like $2x + 10 = 2(x + 1) + 8$. In this section, we will solve trigonometric equations whose solution set involves only certain values for the angle in question. Because of the cyclical nature of the angles we're working with, there will often be an infinite number of solutions although not "all real numbers."

Example 1
Here's an example. Suppose that we consider the equation $\sin x = 0.5$. Whether we use technology, a table or reasoning to solve this equation, it's clear that one solution is 30°. However, remember from the beginning of Chapter 2 that the sine function is positive in Quadrant II. That means that a second quadrant angle with a reference angle of 30° also has a sine equal to 0.5. Recall the ASTC diagram from Chapter 2 :



So, the sine function is positive in Quadrants I and II. This means that in addition to a solution of 30°, there is another solution in Quadrant II. As mentioned above, this second quadrant solution has a reference angle of 30°



To find this angle, we simply subtract $180^\circ - 30^\circ = 150^\circ$
In Quadrant II, we subtract the reference angle from 180°
In Quadrant II, we add the reference angle to 180°

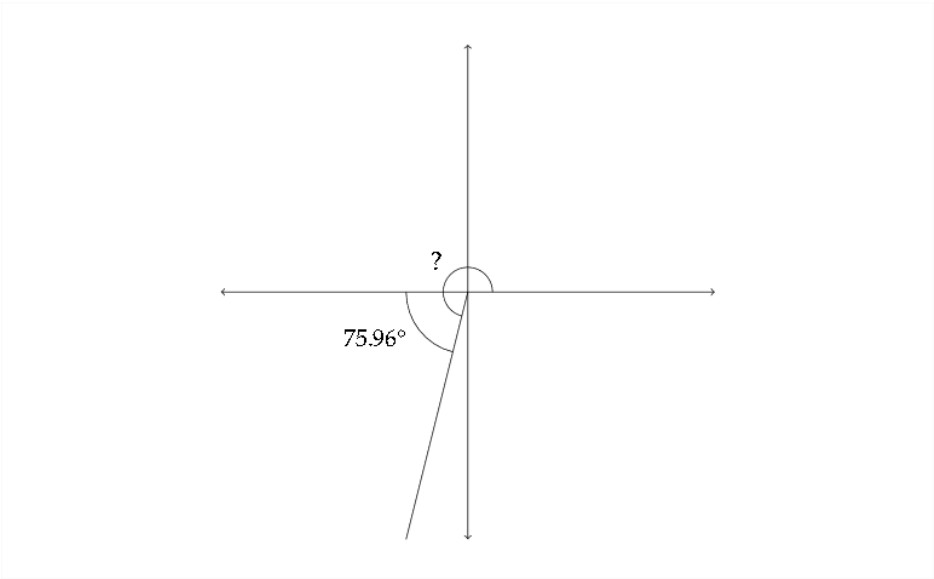
In Quadrant $\text{II}V$, we subtract the reference angle from 360°
So, the solutions to the equation $\sin x = 0.5$ between 0° and 360° are $x = 30^\circ, 150^\circ$ In this chapter we will consider mainly solutions with this restriction:

$$0^\circ \leq x < 360^\circ \tag{10.3.1}$$

The infinite solutions to this equation can be expressed as:

$$30^\circ + n \cdot 360^\circ \text{ and } 150^\circ + n \cdot 360^\circ \tag{10.3.2}$$

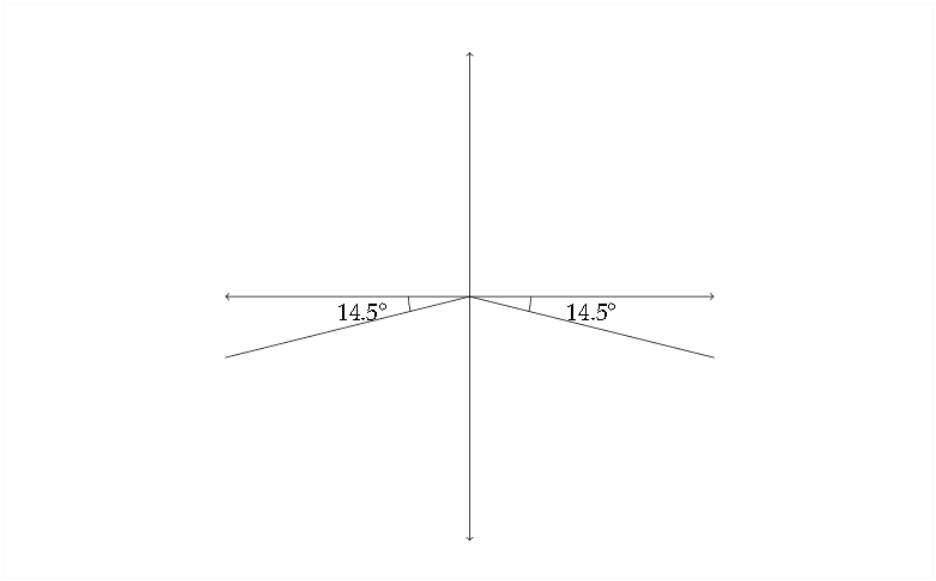
Let's look at another example:
Example 2
Find all solutions of the given equation for $0^\circ \leq x < 360^\circ$
$\tan x = 4$
Using a calculator to find $\tan^{-1}(4)$, we find that it returns an answer of $x \approx 75.96^\circ$ So this is the solution to the equation that lies in QuadrantI. The tangent function is also positive in Quadrant II, so we should also consider the third quadrant angle with a reference angle of 75.96°



In Quadrant II, we add the reference angle to 180°
$180^\circ + 75.96^\circ = 255.96^\circ$, so our solutions for this equation are $x \approx 75.96^\circ, 255.96^\circ$
Often, calculators are programmed to return an angle value that is not between $0^\circ \leq x < 360^\circ$

Example 3
Find all solutions of the given equation for $0^\circ \leq x < 360^\circ$
$\sin x = -0.25$
Solving this on a TI calculator would generally return a value of -14.5°. However, -14.5° is clearly not between 0° and 360°, so we need to use this information to find the solutions that are between 0° and 360°

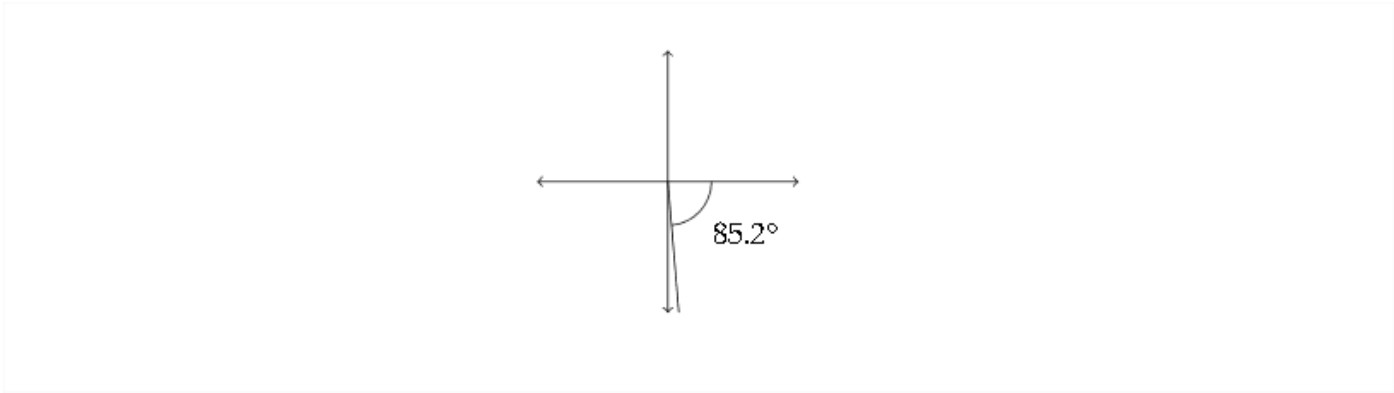
With the calculator returning a vlue of -14.5°, we know that the reference angle for all answers will be 14.5°. Knowing this, we can say that the sine is negative in Quadrants III and IV, so we'll need angles in those quadrants with reference angles of 14.5°



In Quadrant II we'll add 180° to the reference angle: $180^\circ + 14.5^\circ = 194.5^\circ$
In Quadrant II we'll subtract the reference angle from 360° : $360^\circ - 14.5^\circ = 345.5^\circ$
So, $x \approx 194.5^\circ, 345.5^\circ$

Some trigonometric equations have no real number solutions. The equation $\sin x = 2$ has no real number solutions. Recall that the sine ratio was originally defined as the ratio of the side opposite an angle to the hypotenuse. The hypotenuse is always the longest side in a right triangle so there is no way the sine function could be greater than 1 if we're working with real-valued angles. However, in the same way that complex numbers are used to solve equations like $x^2 = -7$, complex-valued angles can be used to solve equations such as $\sin x = 2$ We won't go into this here, however, there is a relatively straightforward way to solve these equations.
If you encounter an equation like $\cos x = 3$ and are solving for values of x $0^\circ \leq x < 360^\circ$, then the proper response is "no solution" or "no real solution." However, remember that the tangent function can take any value between $-\infty$ and ∞

Example 4
Solving an equation that includes a reciprocal trigonometric function simply involves the extra step of finding the reciprocal:
Find all solutions of the given equation for $0^\circ \leq x < 360^\circ$
$\sec x = 12$
The trick here is to restate the equation so that we can use the preprogrammed values from a calculator to find the solution.
If $\sec x = 12$ then $\cos x = \frac{1}{12}$. Finding $\cos^{-1}\left(\frac{1}{12}\right)$ gives a solution of $x \approx 85.2^\circ$
The cosine and the secant are both positive in Quadrant IIV, so we'll also want a fourth quadrant angle whose reference angle is 85.2°



In Quadrant II, we'll subtract the reference angle from 360°

$$360^\circ - 85.2^\circ \approx 274.8^\circ$$
$$x \approx 85.2^\circ, 274.8^\circ \tag{10.3.3}$$

Example 5
Solving a quadratic trigonometric equation often involves the use of the quadratic formula:
Find all solutions of the given equation for $0^\circ \leq x < 360^\circ$
$2\sin^2 x - \sin x - 2 = 0$
Using the quadratic formula we arrive at approximate values for $\sin x$ of $\sin x \approx -0.7808, 1.2808$
The solution $\sin x \approx 1.2808$ yields no real solutions, so we will focus on solving $\sin x \approx -0.7808$
Finding $\sin^{-1}(-0.7808)$ gives us an answer of $\approx -51.3^\circ$. This means our answers will lie in Quadrants III and IV with reference angles of 51.3°. In Quadrant III we'll say $180^\circ + 51.3^\circ \approx 231.3^\circ$. In Quadrant IV, we'll subtract the reference angle from 360° : $360^\circ - 51.3^\circ \approx 308.7^\circ$
So, $x \approx 231.3^\circ, 308.7^\circ$

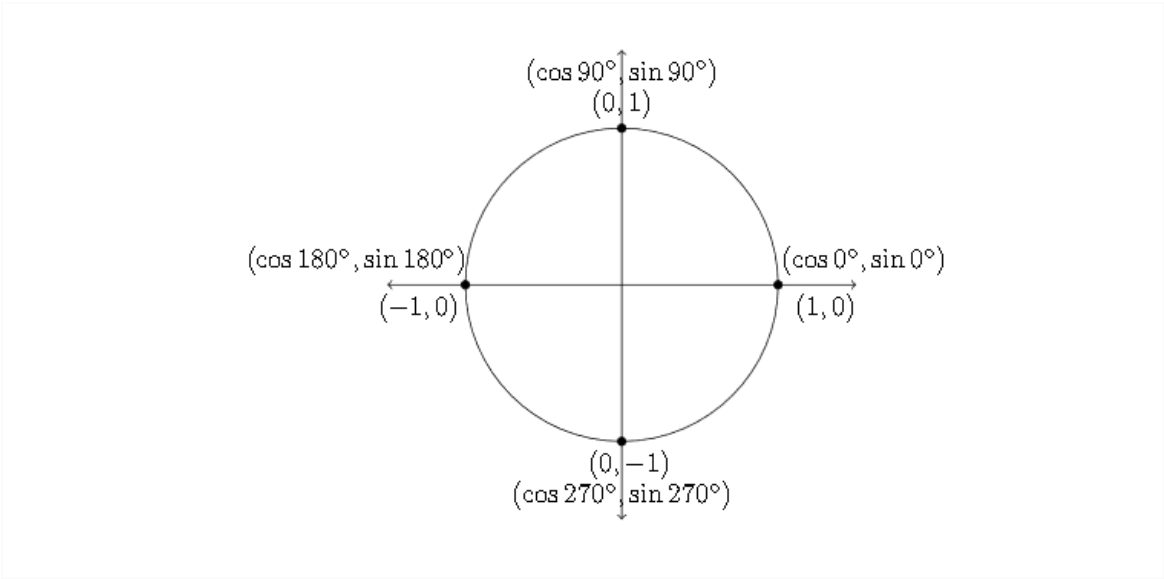
Exercises 3.3
Find all solutions for $0^\circ \leq x < 360^\circ$ Round all angle measures to the nearest 10th of a degree.
1. $\cos x - 0.75 = 0$
2. $\sin x + 0.432 = 0$
3. $3\sin x - 5 = 0$
4. $\sin x - 4 = 0$
5. $3\sec x + 8 = 0$
6. $4\csc x + 9 = 0$
7. $3 - 5\sin x = 4\sin x + 1$
8. $4\cos x - 5 = \cos x - 3$
9. $3\tan^2 x + 2\tan x = 0$
10. $4\cos^2 x - \cos x = 0$
11. $3\cos^2 x + 5\cos x - 2 = 0$
12. $2\cot^2 x - 7\cot x + 3 = 0$
13. $2\tan^2 x - \tan x - 10 = 0$
14. $2\sin^2 x + 5\sin x + 3 = 0$
15. $2\cos^2 x - 5\cos x - 5 = 0$
16. $3\sin^2 x - \sin x - 1 = 0$

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10.4: More Trigonometric Equations

When the solution to a trigonometric equation is one of the quadrantal angles $(0°, 90°, 180°, 270° \text{ and so on })$, then determining all the solutions between $0°$ and $360°$ can work a little differently. The calculator will return some of these values, but in some cases it may not. If we go back to the unit circle, we can see this more clearly:



In the diagram above we can see the sine and cosine for $0°, 90°, 180°$, and $270°$ since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then we can see that $\tan 0° = 0, \tan 90°$ is undefined, $\tan 180° =$
0 and $\tan 270°$ is also undefined.

The real issue with the quadrantal angles is finding $\sin^{-1}(0), \cos^{-1}(0)$ or $\tan^{-1}(0)$ The calculator returns values of:

$$\begin{aligned} \sin^{-1}(0) &= 0° \\ \cos^{-1}(0) &= 90° \\ \tan^{-1}(0) &= 0° \end{aligned} \tag{10.4.1}$$

In each case, there is another possibility than differs from the given angle by $180°$ so:

$$\begin{aligned} \sin^{-1}(0) &= 0°, 180° \\ \cos^{-1}(0) &= 90°, 270° \\ \tan^{-1}(0) &= 0°, 180° \end{aligned} \tag{10.4.2}$$

Let's look at how this is used in solving an equation:

Example 1
Solve the given equation for $0° \le x < 360°$

$$\tan^2 x - \tan x = 0 \tag{10.4.3}$$

We could use the quadratic formula to solve this, but we can also solve by factoring:

$$\begin{aligned} \tan^2 x - \tan x &= 0 \\ \tan x(\tan x - 1) &= 0 \\ \tan x = 0 \text{ or } \tan x &= 1 \end{aligned} \tag{10.4.4}$$

Using a calculator to find $\tan^{-1}(0)$ and $\tan^{-1}(1)$ returns values of $\tan^{-1}(0) = 0°$ and $\tan^{-1}(1) = 45°$. Once we know the reference angle for $\tan^{-1}(1)$, then we know that since the tangent is also positive in Quadrant III, the solutions here are $45°$ and $225°$. The calculator returns an answer of $0°$ for $\tan^{-1}(0)$, but we just saw that $\tan 180° = 0$ as well.
The answers for this equation are $x = 45°, 225°, 0°, 180°$

Another approach to solving trigonometric equations involves using Pythagorean Identities to make a substitution that so that the equation can be simply solved by the quadratic formula. Here's an example:

Example 2
Solve the given equation for $0^\circ \leq x < 360^\circ$

$$\sin^2\theta - 6\cos\theta = 4 \tag{10.4.5}$$

Notice that, unlike the problems we saw in the previous section, this equation involves both the sine and the cosine. To remedy this, we can replace the $\sin^2\theta$ term with the expression $1 - \cos^2\theta$

$$\begin{aligned} \sin^2\theta - 6\cos\theta &= 4 \\ 1 - \cos^2\theta - 6\cos\theta &= 4 \\ 0 &= \cos^2\theta + 6\cos\theta + 3 \end{aligned} \tag{10.4.6}$$

using the quadratic formula:

$$\cos\theta \approx -5.449, -0.5505 \tag{10.4.7}$$

since $\cos^{-1}(-5.449)$ is not a real-valued angle, we can focus on the other answer:
$\cos^{-1}(-0.5505) \approx 123.4^\circ$. since the cosine function is also negative in the third quadrant, we need to find the reference angle that will help us identify the third quadrant angle that is a solution for this equation:

$$180^\circ - 123.4^\circ = 56.6^\circ \tag{10.4.8}$$

So the reference angle is 56.6°

$$180^\circ + 56.6^\circ = 236.6^\circ \tag{10.4.9}$$

The solutions are $\theta \approx 123.4^\circ, 236.6^\circ$

Example 3
Solve the given equation for $0^\circ \leq x < 360^\circ$

$$2\cos^2\theta - \sin\theta = \sin^2\theta + 1 \tag{10.4.10}$$

First, we'll substitute $1 - \sin^2\theta$ for the $\cos^2\theta$

$$\begin{aligned} 2\cos^2\theta - \sin\theta &= \sin^2\theta + 1 \\ 2\left(1 - \sin^2\theta\right) - \sin\theta &= \sin^2\theta + 1 \\ 2 - 2\sin^2\theta - \sin\theta &= \sin^2\theta + 1 \\ 0 &= 3\sin^2\theta + \sin\theta - 1 \end{aligned} \tag{10.4.11}$$

Solving this with the quadratic formula gives us solutions of $\sin\theta \approx -0.7676, 0.43426$

$$\sin^{-1}(-0.7676) \approx -50.1^\circ \tag{10.4.12}$$

$$\sin^{-1}(0.43426) \approx 25.7^\circ \tag{10.4.13}$$

We'll work with the positive solution first. since the sine is also positive in Quadrant II, the other angle will be $180^\circ - 25.7^\circ = 154.3^\circ$

For the negative solution, we know that the sine is negative in Quadrants III and IV, so with a reference angle of 50.1°, in the third quadrant $180^\circ + 50.1^\circ = 230.1^\circ$ and in the fourth quadrant $360^\circ - 50.1^\circ = 309.9^\circ$
The solution set is $\theta \approx 25.7^\circ, 154.3^\circ, 230.1^\circ, 309.9^\circ$

Exercises 3.4
Solve the given equations for $0^\circ \leq x < 360^\circ$

1. $9\sin^2\theta - 6\sin\theta = 1$
2. $4\cos^2\theta + 4\cos\theta = 1$
3. $\sec^2\alpha - 2\sec\alpha - 3 = 0$
4. $\csc^2\beta + 4\csc\beta - 10 = 0$
5. $\csc^2 x + 4\csc x - 7 = 0$
6. $3\cot^2 x - 3\cot x - 1 = 0$
7. $2\sin^2 x = 1 - \cos x$
8. $\cos^2\alpha + 4 = 2\sin\alpha - 3$
9. $\cos^2\beta - 3\sin\beta + 2\sin^2\beta = 0$

10. $\sin^2\theta = 2\cos\theta + 3\cos^2\theta$
11. $\sec^2 x = 2\tan x + 4$
12. $3\tan^2 x = \sec x + 2$
13. $\cos\alpha + 1 = 2\cos 2\alpha$
14. $\cos 2x - 3\sin x - 2 = 0$
15. $\csc^2\theta = \cot\theta + 5$
16. $\csc\theta + 5 = 2\cot^2\theta + 2$

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11: THE LAW OF SINES AND THE LAW OF COSINES

Previously, we used the fundamental trigonometric relationships in right triangles to find unknown distances and angles. Unfortunately, in many problem solving situations, it is inconvenient to use right triangle relationships. Therefore, from the right triangle relationships, we can derive relationships that can be used in any triangle.

11.1: The Law of Sines

11.2: The Law of Sines - the Ambiguous Case

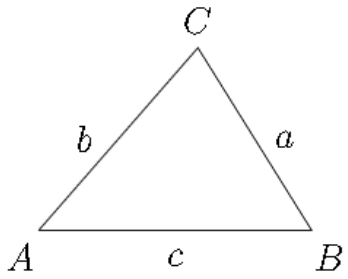
11.3: The Law of Cosines

11.4: Applications

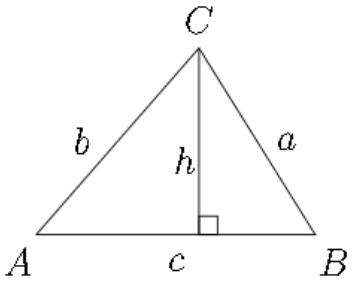
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11.1: The Law of Sines

The Law of sines is based on right triangle relationships that can be created with the height of a triangle. Often, in this type of a problem, the angles are labeled with capital letters and their corresponding sides are labeled with lower case letters.



If we drop a perpendicular to the base of the triangle from the vertex point at $\angle C$ this creates two right triangles with which we can make use of the right triangle trigonometry covered in Chapter 1. This perpendicular would be the height of the triangle.



The Law of sines is derived from this configuration and allows us to calculate the value of sides and angles in a triangle without a right angle, based on information about known sides and angles. Given the right triangles in the diagram above, we can see that:

$$\sin B = \frac{h}{a} \tag{11.1.1}$$

and

$$\sin A = \frac{h}{b} \tag{11.1.2}$$

Clearing the denominator in each fraction, we can see that:

$$a \sin B = h \tag{11.1.3}$$

and

$$\begin{gathered} b \sin A = h \\ \text{so} \\ a \sin B = b \sin A \end{gathered} \tag{11.1.4}$$

To put this in the form in which the Law of sines is normally stated, we can divide on both sides of the previous expression by ab :

$$\begin{aligned} a\sin B &= b\sin A \\ \frac{a\sin B}{ab} &= \frac{b\sin A}{ab} \\ \frac{\sin B}{b} &= \frac{\sin A}{a} \end{aligned} \tag{11.1.5}$$

A similar process will show that $\frac{\sin C}{c}$ is equivalent to $\frac{\sin B}{b}$ and $\frac{\sin A}{a}$. The diagram we derived this from used an acute triangle in which all the angles were less than
90°. The process to show that this is true for an obtuse triangle (which has one angle larger than 90°) is relatively simple and is left to the reader to discover or look up in another resource.
The Law of sines

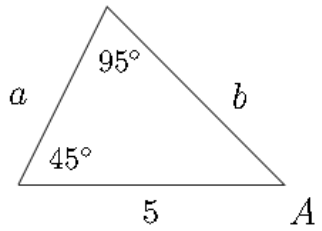
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{11.1.6}$$

Sometimes it is handy to set up a problem with the side lengths in the numerator:
The Law of sines

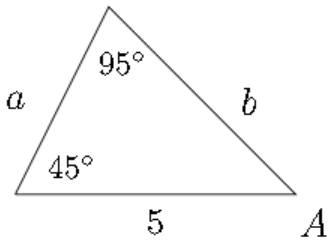
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{11.1.7}$$

\

Solve the triangle. Round side lengths to the nearest 100th .



In this problem we're given two angles and one side. It's important that the side we're given corresponds to one of the known angles, otherwise we wouldn't be able to use the Law of sines.



since we know two of the angles, then the third will just be $180^\circ - (45^\circ + 95^\circ) = 180^\circ - 140^\circ = 40^\circ = \angle A$. To find the lengths of the unknown sides, we'll use the Law of sines. We should start by choosing a side-angle pair for which we know both the side and the angle. In this case, we know that $\angle C = 95^\circ$ and side $c = 5$

$$\begin{aligned} \frac{c}{\sin C} &= \frac{b}{\sin B} \\ \frac{5}{\sin 95^\circ} &= \frac{b}{\sin 45^\circ} \end{aligned} \tag{11.1.8}$$

If we multiply on both sides by $\sin 45^\circ$, then

$$\sin 45^\circ * \frac{5}{\sin 95^\circ} = b \tag{11.1.9}$$

To arrive at an approximate value for $\sin 45^\circ * \frac{5}{\sin 95^\circ}$, we can say:

$$0.7071 * \frac{5}{0.9962} \approx b \tag{11.1.10}$$
$$3.55 \approx b$$

To find the length of side a, I would recommend that we use the exact side-angle pair that was given in the problem, rather than using the approximate value of side b that we just solved for.

This will make our value for side a more accurate:

$$\frac{c}{\sin C} = \frac{a}{\sin A} \tag{11.1.11}$$
$$\frac{5}{\sin 95^\circ} = \frac{a}{\sin 40^\circ}$$

Multiplying on both sides by $\sin 40^\circ$, then

$$\sin 40^\circ * \frac{5}{\sin 95^\circ} = a \tag{11.1.12}$$

To arrive at an approximate value for $\sin 40^\circ * \frac{5}{\sin 95^\circ}$, we can say:

$$0.6428 * \frac{5}{0.9962} \approx a \tag{11.1.13}$$
$$3.23 \approx a$$

$$\begin{aligned} \angle A = 40^\circ \quad & a \approx 3.23 \\ \angle B = 45^\circ \quad & b \approx 3.55 \\ \angle C = 95^\circ \quad & c = 5 \end{aligned} \tag{11.1.14}$$

Example 2
Some problems don't come with diagrams:
Solve the triangle if: $\angle A = 40^\circ, \quad \angle B = 20^\circ, \quad a = 2$
Round side lengths to the nearest 100th.
Just as in the previous example, we can begin by finding the measure of the third angle $\angle C$. This would be $180^\circ - (40^\circ + 20^\circ)$ $= 180^\circ - 60^\circ = 120^\circ = \angle C$
To find the missing sides, we should use the complete side-angle pair that is given in the problem: $\angle A = 40^\circ$ and $a = 2$

We can find side b first or side c first, it doesn't matter which:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{11.1.15}$$
$$\frac{2}{\sin 40^\circ} = \frac{b}{\sin 20^\circ}$$
$$\sin 20^\circ * \frac{2}{\sin 40^\circ} = b$$

Then,

$$0.3420 * \frac{2}{0.6428} \approx b \tag{11.1.16}$$
$$1.06 \approx b \tag{11.1.17}$$

For side c :

$$\frac{a}{\sin A} = \frac{c}{\sin C} \tag{11.1.18}$$
$$\frac{2}{\sin 40^\circ} = \frac{c}{\sin 120^\circ}$$
$$\sin 120^\circ * \frac{2}{\sin 40^\circ} = c$$

Then,

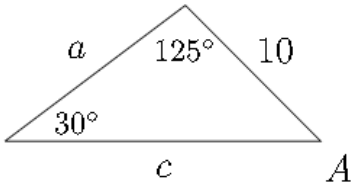
$$0.8660 * \frac{2}{0.6428} \approx c \tag{11.1.19}$$
$$2.69 \approx c \tag{11.1.20}$$

$$\begin{aligned} \angle A = 40^\circ \quad & a = 2 \\ \angle B = 20^\circ \quad & b \approx 1.06 \\ \angle C = 120^\circ \quad & c \approx 2.69 \end{aligned} \tag{11.1.21}$$

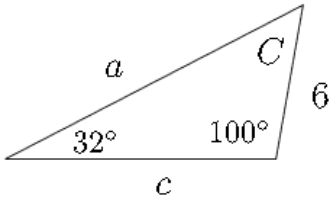
Exercises 4.1

In each problem below, solve the triangle. Round side lengths to the nearest 100th

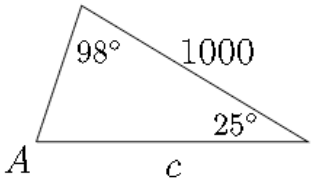
1.



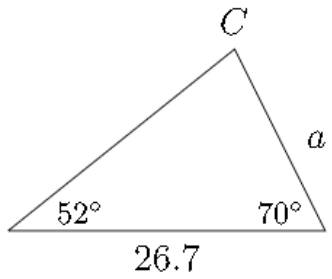
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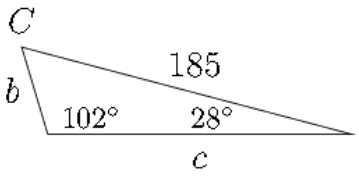
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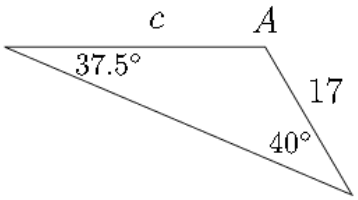
4.



5.



6.



7. $\angle A = 50^\circ, \quad \angle C = 27^\circ, \quad a = 3$
8. $\angle B = 70^\circ, \quad \angle C = 10^\circ, \quad b = 5$
9. $\angle A = 110^\circ, \quad \angle C = 30^\circ, \quad c = 3$
10. $\angle A = 50^\circ, \quad \angle B = 68^\circ, \quad a = 230$
11. $\angle A = 23^\circ, \quad \angle B = 110^\circ, \quad c = 50$
12. $\angle A = 22^\circ, \quad \angle B = 95^\circ, \quad a = 420$
13. $\angle B = 10^\circ, \quad \angle C = 100^\circ, \quad c = 11$
14. $\angle A = 30^\circ, \quad \angle C = 65^\circ, \quad b = 10$
15. $\angle A = 82^\circ, \quad \angle B = 65.4^\circ, \quad b = 36.5$
16. $\angle B = 28^\circ, \quad \angle C = 78^\circ, \quad c = 44$
17. $\angle A = 42^\circ, \quad \angle B = 61^\circ, \quad a = 12$
18. $\angle A = 42.5^\circ, \quad \angle B = 71.4^\circ, \quad a = 215$

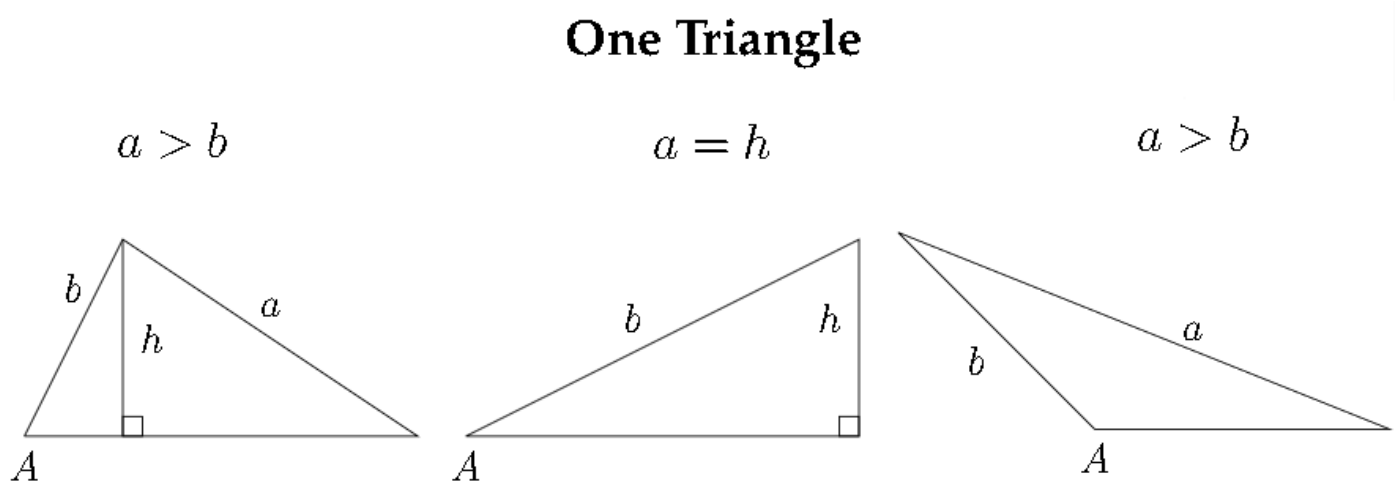
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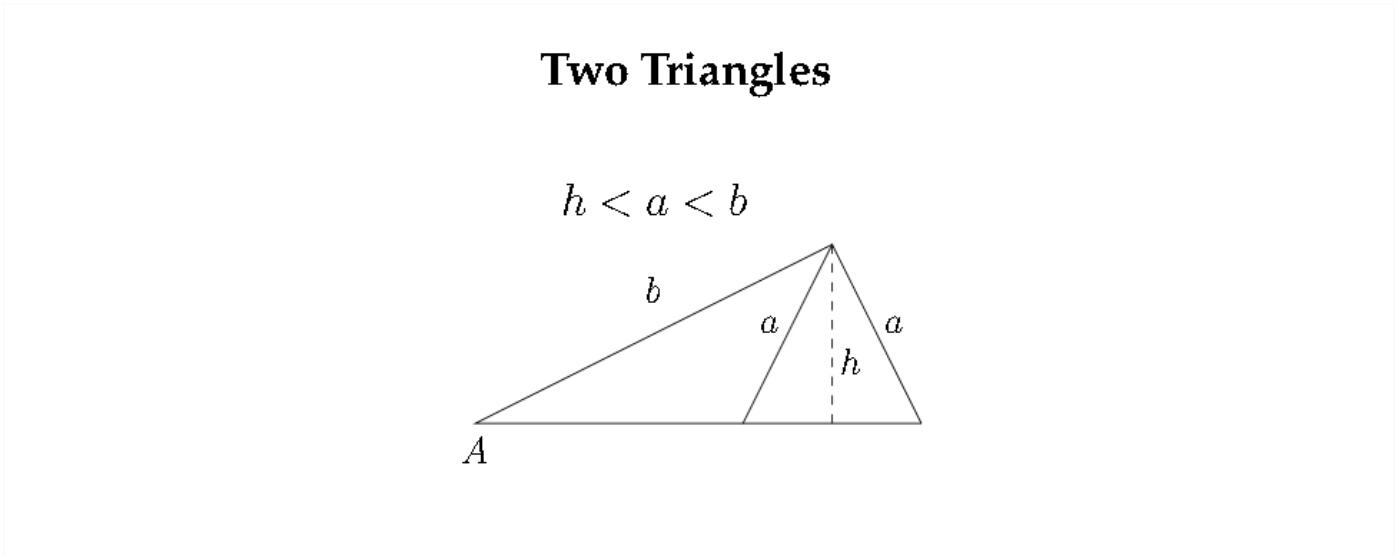
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11.2: The Law of Sines - the Ambiguous Case

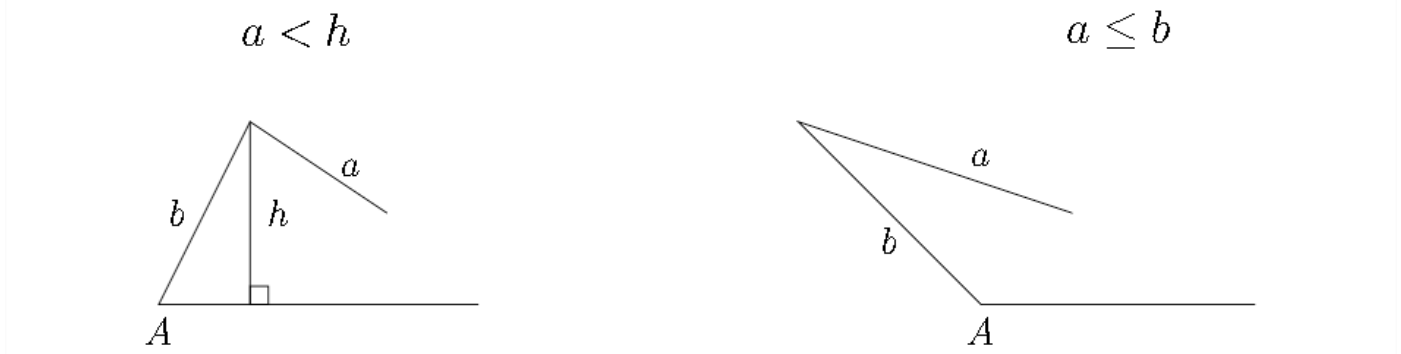
In all of the examples and problems in Section 4.1, notice that we were always given two angles and one side, although we could use the Law of sines if we were given one angle and two sides (as long as one of the sides corresponded to the given angle). This is because when we use the Law of sines to find an angle, an ambiguity can arise due to the sine function being positive in Quadrant I and Quadrant II.

We saw in Chapter 3 that multiple answers arise when we use the inverse trigonometric functions. For problems in which we use the Law of sines given one angle and two sides, there may be one possible triangle, two possible triangles or no possible triangles. There are six different scenarios related to the ambiguous case of the Law of sines: three result in one triangle, one results in two triangles and two result in no triangle.





No Triangle



We'll look at three examples: one for one triangle, one for two triangles and one for no triangles.

Example 11.2.1

Solve the triangle if: $\angle A = 112^\circ, \quad a = 45, \quad b = 24$
Round the angles and side lengths to the nearest 10th

Solution

Using the Law of sines, we can say that:

$$\begin{gathered} \frac{\sin 112^\circ}{45} = \frac{\sin B}{24} \\ \frac{0.9272}{45} \approx \frac{\sin B}{24} \\ 24 * \frac{0.9272}{45} \approx \sin B \\ 0.4945 \approx \sin B \end{gathered} \tag{11.2.1}$$

Then, we find $\sin^{-1}(0.4945) \approx 29.6^\circ$. Remember from Chapter 3 that there is a Quadrant II angle that has $\sin\theta \approx 0.4945$, with a reference angle of 29.6°.So, $\angle B$ could also be $\approx 150.4^\circ$. However, with $\angle A = 112^\circ$, there is no way that another angle of 150.4° would fit inside the same triangle. For this reason, we know then that $\angle B$ must be 29.6°

$$29.6^\circ \approx B \tag{11.2.2}$$

So now

$$\begin{gathered} \angle A = 112^\circ \\ \angle B \approx 29.6^\circ \\ \text{and } \angle C = 180^\circ - (112^\circ + 29.6^\circ) = 180^\circ - 141.6^\circ \approx 38.4^\circ \\ \angle C \approx 38.4^\circ \end{gathered} \tag{11.2.3}$$

We already know that $a = 45$ and $b = 24$. To find side c, I would recommend using the most exact values possible in the Law of sines calculation. This will provide the most accurate result in finding the length of side c

$$\begin{aligned}
\frac{45}{\sin 112^\circ} &= \frac{c}{\sin 38.4^\circ} \\
\frac{45}{0.9272} &\approx \frac{c}{0.6211} \\
0.6211 * \frac{45}{0.9272} &\approx c \\
30.1 &\approx c \\
\angle A = 112^\circ \qquad & a = 45 \\
\angle B \approx 29.6^\circ \qquad & b = 24 \\
\angle C \approx 38.4^\circ \qquad & c \approx 30.1
\end{aligned} \tag{11.2.4}$$

Example 11.2.2

Solve the triangle if: $\angle A = 38^\circ, \quad a = 40, \quad b = 52$
Round the angles and side lengths to the nearest 10th.

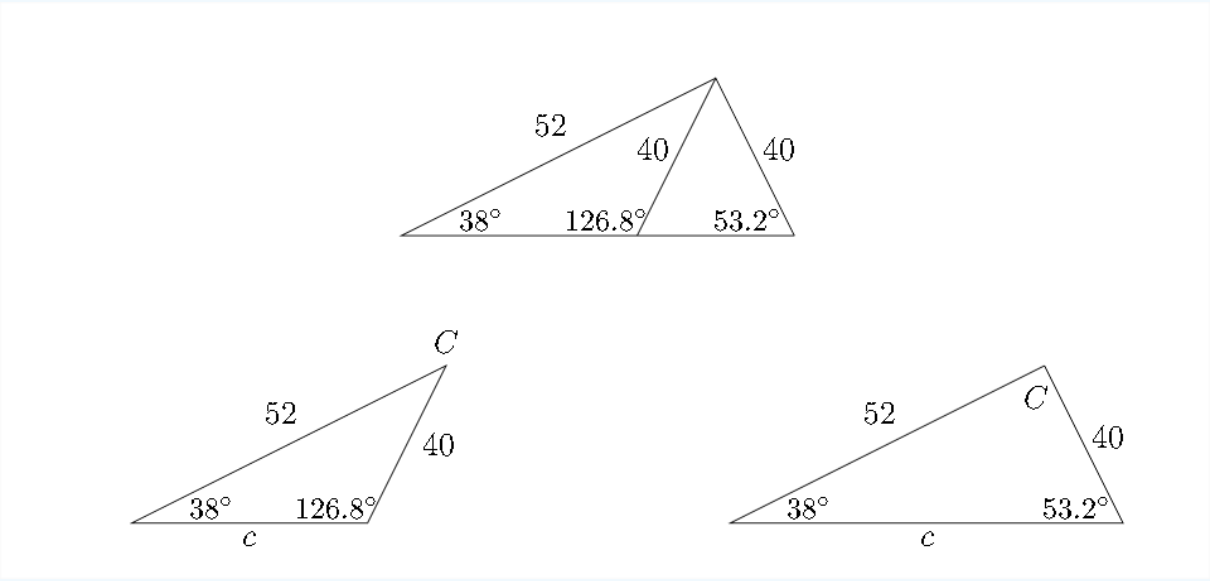
Solution

Using the Law of sines, we can say that:

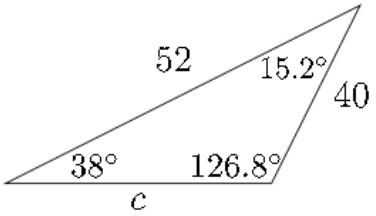
$$\begin{aligned}
\frac{\sin 38^\circ}{40} &= \frac{\sin B}{52} \\
\frac{0.6157}{40} &\approx \frac{\sin B}{52} \\
52 * \frac{0.6157}{40} &\approx \sin B \\
0.8004 &\approx \sin B
\end{aligned} \tag{11.2.5}$$

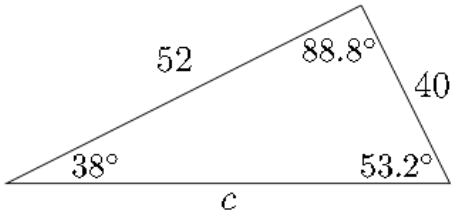
Just as in the previous example, we can find $\sin^{-1}(0.8004) \approx 53.2^\circ$. But again, there is a Quadrant II angle whose sine has the same value ≈ 0.8004. The angle 126.8° has a sine ≈ 0.8004 and a reference angle of 53.2°. With $\angle A = 38^\circ$, both of these angles $(53.2^\circ \text{ and } 126.8^\circ)$ could potentially fit in the triangle with angle A

If we go back to the diagrams we looked at earlier in this section, we can see how this would happen:



In the first possibility $\angle C$ would be $\approx 15.2^\circ$
In the second possibility $\angle C$ would be $\approx 88.8^\circ$





To find the two possible lengths for side c, we'll need to solve two Law of sines calculations, one with $\angle C \approx 15.2^\circ$ and one with the $\angle C \approx 88.8^\circ$

$$\begin{gathered} \frac{40}{\sin 38^\circ} = \frac{c}{\sin 15.2^\circ} \\ \frac{40}{0.6157} \approx \frac{c}{0.2622} \\ 0.2622 * \frac{40}{0.6157} \approx c \\ 17.0 \approx c \end{gathered} \tag{11.2.6}$$

With $\angle C \approx 88.8^\circ$:

$$\begin{gathered} \frac{40}{\sin 38^\circ} = \frac{c}{\sin 88.8^\circ} \\ \frac{40}{0.6157} \approx \frac{c}{0.9998} \\ 0.9998 * \frac{40}{0.6157} \approx c \\ 65.0 \approx c \end{gathered} \tag{11.2.7}$$

So, our two possible solutions would be:

$$\begin{array}{ll} \angle A = 38^\circ & a = 40 \\ \angle B \approx 126.8^\circ & b = 52 \\ \angle C \approx 15.2^\circ & c \approx 17.0 \end{array} \tag{11.2.8}$$

OR

$$\angle A = 38^\circ \quad a = 40 \tag{11.2.9}$$

$$\begin{array}{ll} \angle B \approx 53.2^\circ & b = 52 \\ \angle C \approx 88.8^\circ & c \approx 65.0 \end{array} \tag{11.2.10}$$

Example 11.2.3

Solve the triangle if:

$$\angle B = 73^\circ, \quad b = 51, \quad a = 92. \tag{11.2.11}$$

Round the angles and side lengths to the nearest 10th.

Solution

Using the Law of sines, we can say that:

$$\frac{\sin 73^\circ}{51} = \frac{\sin A}{92} \tag{11.2.12}$$

$$\begin{gathered} \frac{0.9563}{51} \approx \frac{\sin A}{92} \\ 92 * \frac{0.9563}{51} \approx \sin A \\ 1.7251 \approx \sin A \end{gathered} \tag{11.2.13}$$

As we saw previously, no real-valued angle has a sine greater than 1. Therefore, no triangle is possible.

Exercises

In each problem, solve the triangle. Round side lengths to the nearest 100th and angle measures to the nearest 10th.

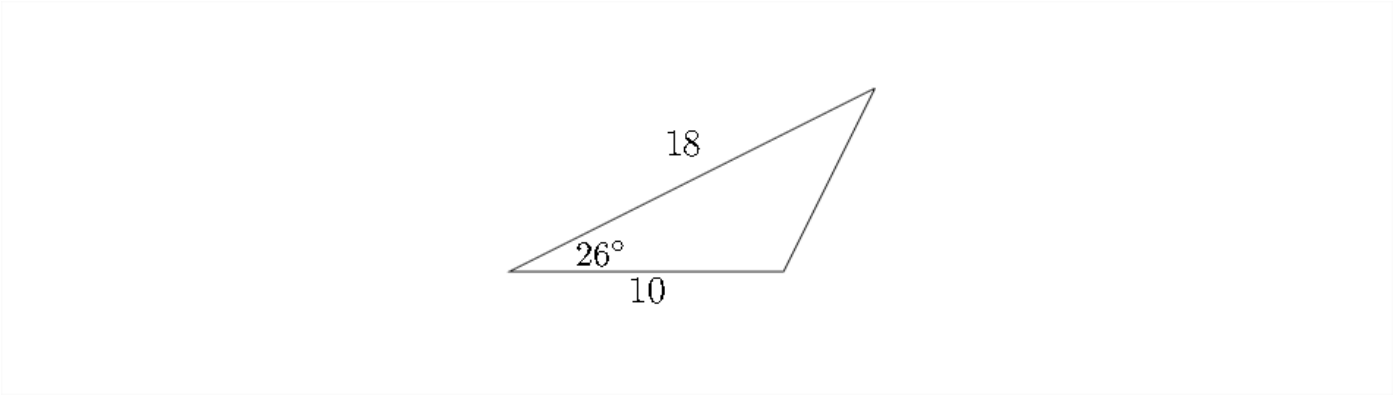
1. $\angle A = 50^\circ, \quad b = 20, \quad a = 32$
2. $\angle B = 40^\circ, \quad b = 4, \quad c = 3$
3. $\angle A = 43^\circ, \quad a = 23, \quad b = 29$
4. $\angle C = 20^\circ, \quad c = 43, \quad a = 55$
5. $\angle B = 62^\circ, \quad b = 4, \quad a = 5$
6. $\angle A = 75^\circ, \quad b = 8, \quad a = 3$
7. $\angle B = 24^\circ, \quad a = 17, \quad b = 8$
8. $\angle A = 40^\circ, \quad a = 4, \quad c = 5$
9. $\angle A = 108^\circ, \quad a = 12, \quad b = 7$
10. $\angle B = 117^\circ, \quad b = 19.6, \quad c = 10.5$
11. $\angle A = 42^\circ, \quad a = 18, \quad c = 11$
12. $\angle C = 27^\circ, \quad a = 42, \quad c = 37$
13. $\angle C = 125^\circ, \quad c = 2.7, \quad b = 5.2$
14. $\angle B = 115^\circ, \quad b = 68, \quad a = 92$
15. $\angle A = 43^\circ, \quad a = 31, \quad b = 37$
16. $\angle A = 28^\circ, \quad b = 3.5, \quad a = 4.3$
17. $\angle C = 132^\circ, \quad c = 22, \quad b = 16$
18. $\angle B = 114.2^\circ, \quad b = 87.2, \quad a = 12.1$
19. $\angle B = 52^\circ, \quad c = 82.7, \quad b = 70$
20. $\angle C = 65^\circ, \quad b = 7.6, \quad c = 7.1$

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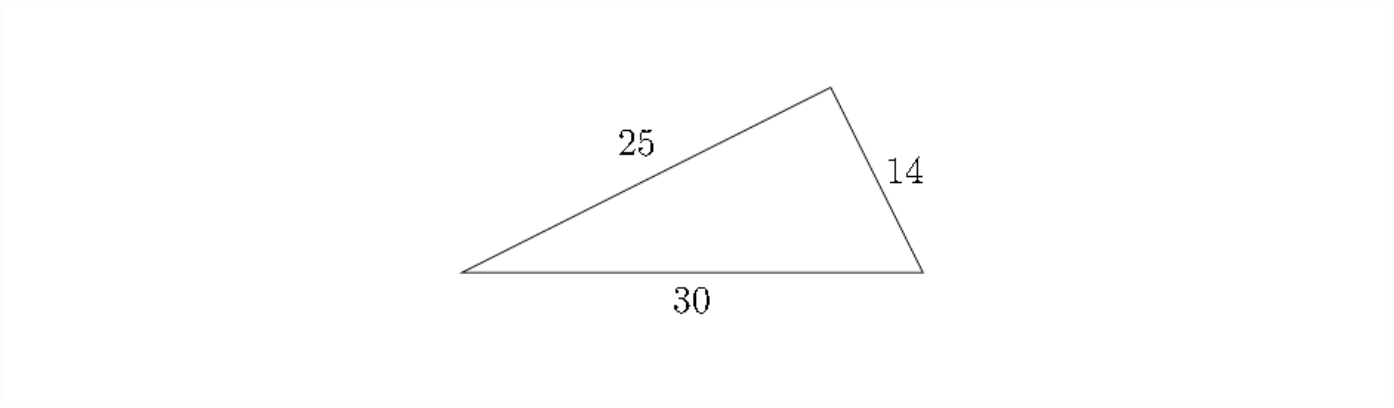
11.3: The Law of Cosines

There are situations in which the Law of sines cannot be used to solve a triangle. In the diagram below, we have information about two sides and the included angle:



The problem above lacks a complete angle-side pair which is necessary to set up the Law of sines calculation.

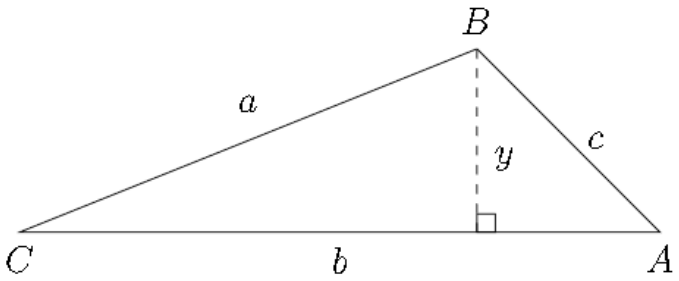
Another common situation involves a triangle in which all three sides are known but no angles are known:



Again, the lack of an angle-side pair would prevent us from setting up a Law of sines calculation.

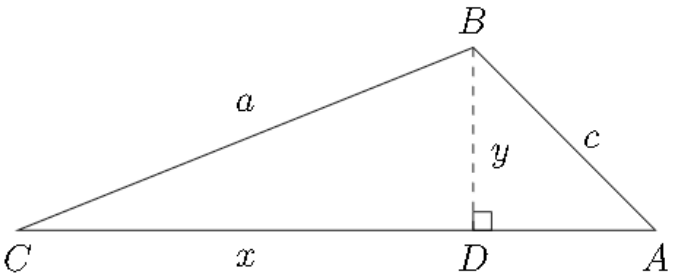
The Law of Cosines is one way to get around this difficulty. Using the Law of cosines is more complicated than using the Law of sines, however, as we have just seen, the Law of sines will not always be enough to solve a triangle.

To derive The Law of cosines, we begin with an arbitrary triangle, like the one seen on the next page:

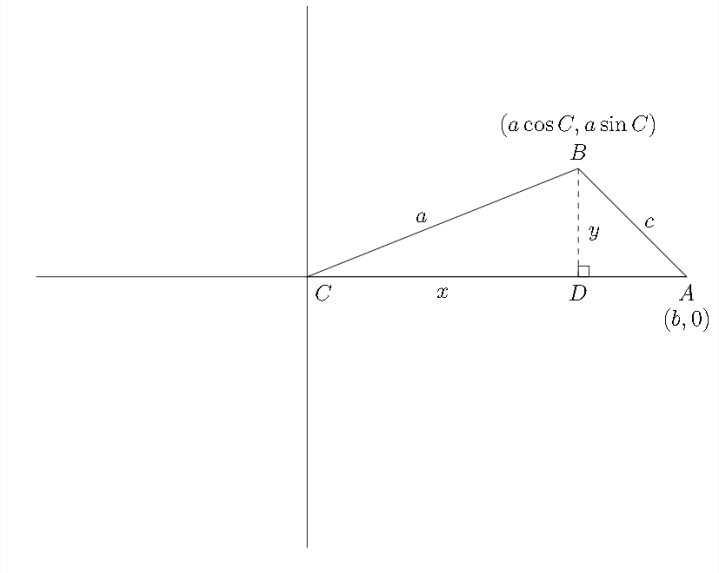


In this diagram we have taken the arbitrary triangle and created a perpendicular with length y. From this, we can say that $\sin C = \frac{y}{a}$ and that $a \sin C = y$

We can split side AC into two pieces AD and CD, as seen below, and label the distance CD as x



Then, we can say that $\cos C = \frac{x}{a}$ and that $a \cos C = x$
If we then put this triangle onto the coordinate axes with $\angle C$ at the origin (0,0) we can derive the Law of cosines. The coordinate of the vertex at $\angle B$ will be $(a \cos C, a \sin C)$, and the coordinates of the vertex at $\angle A$ will be $(b, 0)$



Using the distance formula, we can say that:

$$c = \sqrt{(a\cos C - b)^2 + (a\sin C - 0)^2} \tag{11.3.1}$$

Squaring both sides:

$$c^2 = (a\cos C - b)^2 + (a\sin C - 0)^2 \tag{11.3.2}$$

and

$$c^2 = a^2\cos^2 C - 2ab\cos C + b^2 + a^2\sin^2 C \tag{11.3.3}$$

or

$$c^2 = a^2\sin^2 C + a^2\cos^2 C + b^2 - 2ab\cos C \tag{11.3.4}$$

Factoring out the a^2 and replacing $\sin^2 C + \cos^2 C$ with 1 , we come out with one of the most common forms of the Law of cosines:

$$\begin{aligned} c^2 &= a^2\sin^2 C + a^2\cos^2 C + b^2 - 2ab\cos C \\ c^2 &= a^2\left(\sin^2 C + \cos^2 C\right) + b^2 - 2ab\cos C \\ c^2 &= a^2(1) + b^2 - 2ab\cos C \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned} \tag{11.3.5}$$

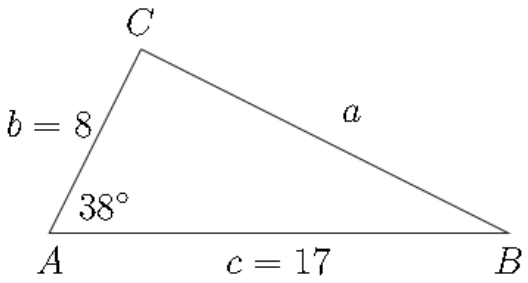
Any letter may be used to represent each of the sides, but the order that the letters are used in is very important. The side of the triangle isolated on the left hand side must correspond to the angle used on the right hand side.

The Law of cosines

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned} \tag{11.3.6}$$

We'll look at three examples- two in which two sides and the included angle are given and one in which the three sides of the triangle are given.

Example 1
Solve the triangle: $\angle A = 38^\circ, c = 17, b = 8$ Round angle measures and side lengths to the nearest 10th .



It's usually a good idea to see if you can use the Law of sines first, since it is easier to calculate. In this case we can't because we don't have a complete angle-side pair. So, using the Law of cosines to find side a :

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ a^2 &= 8^2 + 17^2 - 2 * 8 * 17 * \cos 38^\circ \\ a^2 &\approx 64 + 289 - 272 * 0.7880 \\ a^2 &\approx 353 - 214.336 \\ a^2 &\approx 138.664 \\ a &\approx 11.8 \end{aligned} \tag{11.3.7}$$

Once we know that $a \approx 11.8$ we can use this to find the other angles using the Law of sines. Because of the issue of the ambiguous case in using the Law of sines, it's often a good idea to find the angles that correspond to the two shortest sides in the triangle, because if there is an obtuse angle in the triangle it will have to correspond to the longest side. If we find the two smaller angles, we can be assured that they will both be acute and we can subtract from 180° to find the largest angle.

$$\begin{aligned} \tfrac{\sin 38^\circ}{11.8} &= \tfrac{\sin B}{8} \\ 8 \times \tfrac{0.61566}{11.8} &= \sin B \\ 0.4174 &\approx \sin B \\ 24.7^\circ &\approx B \end{aligned} \tag{11.3.8}$$

So with $\angle A = 38^\circ$ and $\angle B \approx 24.7^\circ$, then:

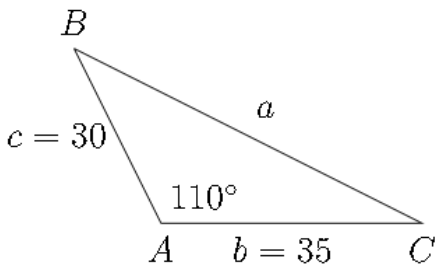
$$\angle C \approx 180^\circ - (38^\circ + 24.7^\circ) \approx 180^\circ - 62.7^\circ \approx 117.3^\circ \tag{11.3.9}$$

So, the angles and sides of the triangle would be:

$$\begin{aligned} \angle A &= 38^\circ \quad & a &\approx 11.8 \\ \angle B &\approx 24.7^\circ & b &= 8 \\ \angle C &\approx 117.3^\circ & c &= 17 \end{aligned} \tag{11.3.10}$$

In example 2, we'll look at a problem in which an obtuse angle is given.

Example 2
Solve the triangle: $\angle A = 110^\circ, c = 30, b = 35$ Round angle measures and side lengths to the nearest 10th .



The calculation for this problem is slightly different from the last one because the cosine of 110° will be negative:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ a^2 &= 35^2 + 30^2 - 2 * 35 * 30 * \cos 110^\circ \\ a^2 &\approx 1225 + 900 - 2100 * (-0.3420) \\ a^2 &\approx 2125 + 718.2 \\ a^2 &\approx 2843.2 \\ a &\approx 53.3 \end{aligned} \tag{11.3.11}$$

In this problem, since we were given an obtuse angle, then the other two angles must be acute and we don't have to worry about the ambiguous case in using the Law of sines.

$$\frac{\sin 110^\circ}{53.3} = \frac{\sin B}{35} \quad \frac{\sin 110^\circ}{53.3} = \frac{\sin C}{30} \tag{11.3.12}$$

$$35 * \frac{0.9397}{53.3} = \sin B \quad 30 * \frac{0.9397}{53.3} = \sin C \tag{11.3.13}$$

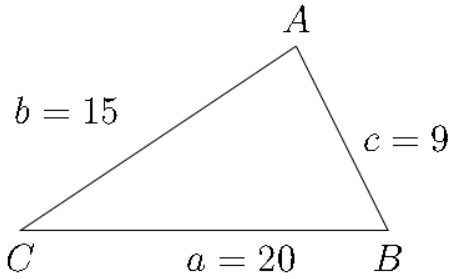
$$0.61706 \approx \sin B \quad 0.5289 \approx \sin C \tag{11.3.14}$$

$$38.1^\circ \approx B \quad 31.9^\circ \approx C \tag{11.3.15}$$

So the angles and sides of the triangle would be:

$$\begin{aligned} \angle A &= 110^\circ \quad a \approx 53.3 \\ \angle B &\approx 38.1^\circ \quad b = 35 \\ \angle C &\approx 31.9^\circ \quad c = 30 \end{aligned} \tag{11.3.16}$$

In example 3, we'll look at a problem in which three side lengths are given and we find an angle using the Law of cosines.
Example 3
Solve the triangle: $a = 20, \quad c = 9, \quad b = 15$ Round angle measures to the nearest 10th



It doesn't matter which angle we choose to solve, but whichever angle we choose must correspond to the side isolated on the left-hand side of the formula. If we want to solve for $\angle B$, we would say:

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ 15^2 &= 20^2 + 9^2 - 2 * 20 * 9 * \cos B \\ 225 &= 400 + 81 - 360 * \cos B \\ 225 &= 481 - 360\cos B \\ -256 &= -360\cos B \\ \tfrac{-256}{-360} &= \tfrac{-360\cos B}{-360} \\ 0.7\overline{1} &= \cos B \\ 44.7^\circ &\approx B \end{aligned} \tag{11.3.17}$$

Once we know the measure of $\angle B$, we'll use this to find the measure of $\angle C$, which corresponds to side c, the smallest side. Then we'll subtract to find the biggest angle.

$$\begin{aligned} \frac{\sin 44.7^\circ}{15} &= \frac{\sin C}{9} \\ 9 * \frac{0.7034}{15} &= \sin C \\ 0.42204 &\approx \sin C \\ 25.0^\circ &\approx C \end{aligned} \tag{11.3.18}$$

So, with $\angle B \approx 44.7^\circ$ and $\angle C \approx 25.0^\circ$, then:

$$\angle A \approx 180^\circ - (44.7^\circ + 25.0^\circ) \approx 180^\circ - 69.7^\circ \approx 110.3^\circ \tag{11.3.19}$$

So the angles and sides of the triangle would be:

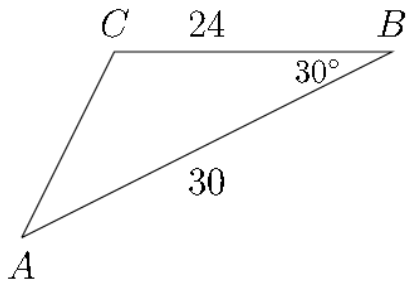
$$\begin{aligned} \angle A &= 110.3^\circ \quad & a &\approx 53.3 \\ \angle B &\approx 44.7^\circ \quad & b &= 35 \\ \angle C &\approx 25.0^\circ \quad & c &= 30 \end{aligned} \tag{11.3.20}$$

If we had used the Law of sines to find $\angle A$, the calculator would have returned the value of the reference angle for $\angle A$, rather than the angle that is actually in the triangle described in the problem!

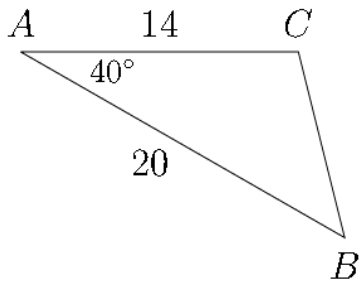
Exercises 4.3

In each problem, solve the triangle. Round side lengths to the nearest 100th and angle measures to the nearest 10th.

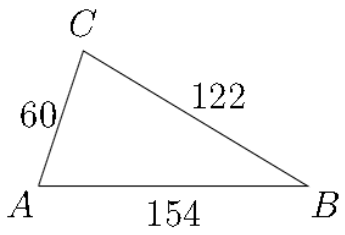
1.



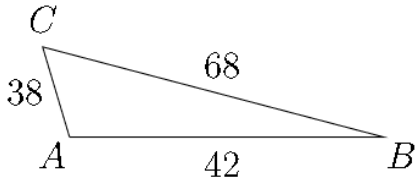
2.



3.



4.



5. $\angle A = 52^\circ, \quad c = 27, \quad b = 36$
6. $\angle B = 75^\circ, \quad a = 32, \quad c = 59$
7. $\angle B = 135^\circ, \quad a = 12, \quad c = 18$
8. $\angle C = 120^\circ, \quad b = 22, \quad a = 30$
9. $a = 21, \quad b = 26, \quad c = 23$
10. $a = 11, \quad b = 13, \quad c = 17$
11. $a = 25, \quad b = 32, \quad c = 40$
12. $a = 60, \quad b = 88, \quad c = 120$
13. $\angle A = 77.4^\circ, \quad b = 444, \quad c = 390$
14. $\angle B = 10^\circ, \quad a = 18, \quad c = 30$
15. $a = 112.7, \quad b = 96.5, \quad c = 130.2$
16. $a = 4.7, \quad b = 3.2, \quad c = 5.9$

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11.4: Applications

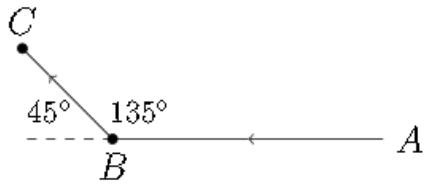
In the previous sections on applications, we saw situations in which right triangle trigonometry was used to find distances and angles. In this section, we will use the Law of sines and the Law of cosines to find distances and angles.

Example 11.4.1

A car travels along a straight road, heading west for 1 hour, then traveling on another straight road northwest for a half hour. If the speed of the car was a constant 50 mph how far is the car from its starting point?

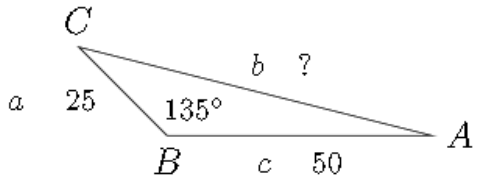
Solution

First, let's draw a diagram:



In the picture above, we know the angles 45° and 135° because of the direction the car was traveling. The direction northwest cuts exactly halfway between north and west creating a 45° angle. On the other side of this 45° angle is a 135° angle which is in the triangle we'll use to answer the question (triangle ABC).

The length of $\overline{AB}$ is 50 miles and the length of $\overline{BC}$ is 25 miles. This comes from the information about the speed and traveling time given in the problem. So the triangle we need to answer the question is pictured below:



We can use the Law of cosines to solve this problem:

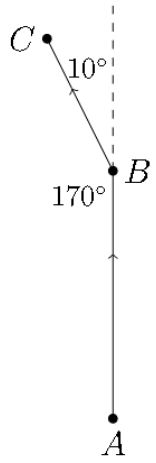
$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ b^2 &= 25^2 + 50^2 - 2 * 25 * 50 * \cos 135^\circ \\ b^2 &\approx 625 + 2500 - 2500 * (-0.7071) \\ b^2 &\approx 3125 + 1767.75 \\ b^2 &\approx 4892.75 \\ b &\approx 69.9 \text{miles} \end{aligned} \tag{11.4.1}$$

Example 11.4.2

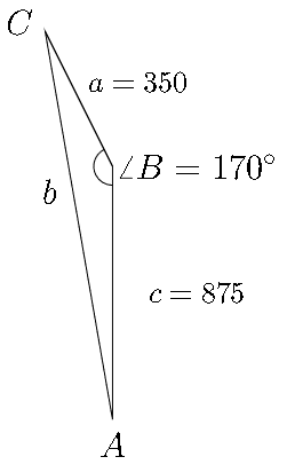
A pilot flies an airplane in a straight path for 2.5 hours and then makes a course correction, heading 10° to the left of the original course. The pilot then flies in this direction for 1 hour. If the speed of the plane is a constant 350mph, how far is the plane from its starting position?

Solution

Again, we'll start by making a diagram:



In this problem, we'll be working with triangle ABC, shown below. We can calculate the lengths of $\overline{AB}$ and $\overline{BC}$ from the information given in the problem and use this to calculate the length of $\overline{AC}$:

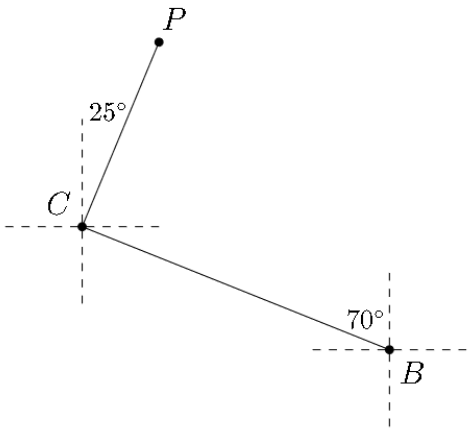


Using the Law of cosines:

$$
\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\cos B \\
b^2 &= 350^2 + 875^2 - 2 * 350 * 1050 * \cos 170^\circ \\
b^2 &\approx 122,500 + 765,625 - 735,000 * (-0.9848) \\
b^2 &\approx 888,125 + 723,828 \\
b^2 &\approx 1,611,953 \\
b &\approx 1270\text{miles}
\end{aligned} \tag{11.4.2}
$$

Example 11.4.3

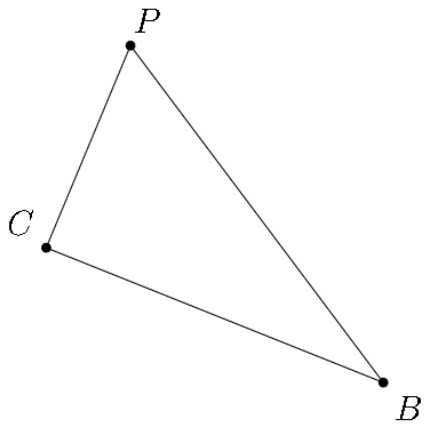
A pilot leaves the airport in Bend, headed towards Corvallis with the bearing $N70^\circ W$. He travels the 103 miles and makes a delivery before taking off and flying at a bearing of $N25^\circ E$ for 72 miles to arrive in Portland.



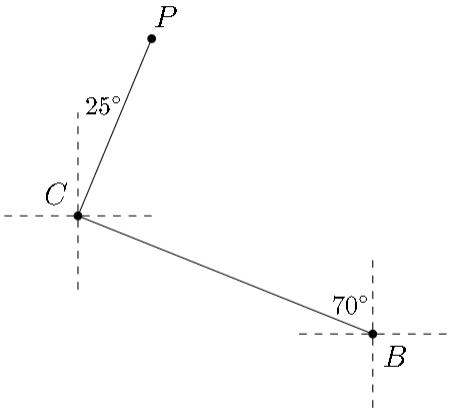
a) Based on this information, find the air distance between Portland and Bend.
b) Find the bearing from Portland to Bend.

Solution

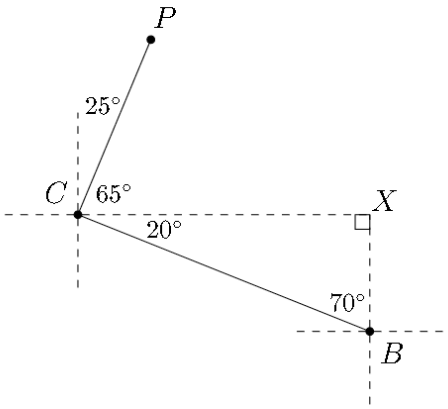
In this problem, a diagram has been given. We'll amend this to make it into a triangle:



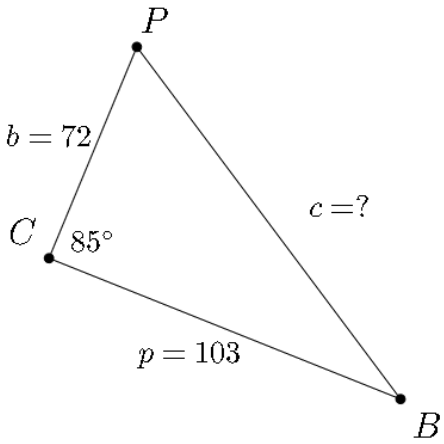
Filling the measures of the angles is tricky in this problem, so let's look at the original diagram again:



If we extend the dashed line east from Corvallis so that it meets the dashed line running north from Bend, we can create a triangle that shows us that the angle $\angle BCX = 20^\circ$. Also, notice that $\angle PCX = (90^\circ - 25^\circ) = 65^\circ$



This means that $\angle BCP = 85°$. We know from the problem that $\overline{BC} = 103$ and $\overline{CP} = 72$. We'll need to find the length of $\overline{BP}$ and the measure of $\angle CPB$ to answer the questions.



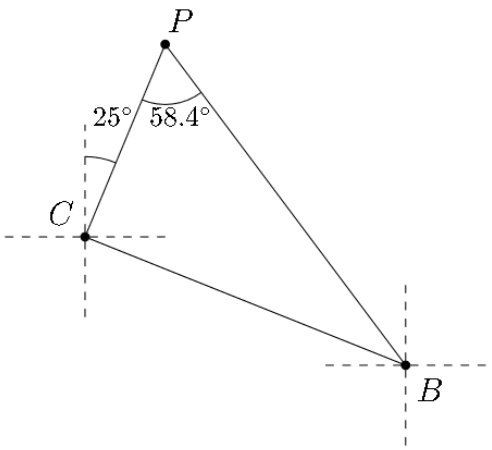
Now we're working with a triangle like the one pictured above, so we can use the Law of cosines to find the air distance from Portland to Bend:

$$\begin{aligned} c^2 &= b^2 + p^2 - 2bp\cos C \\ c^2 &= 72^2 + 103^2 - 2 * 72 * 103 * \cos 85° \\ c^2 &\approx 5184 + 10,609 - 14,832 * (0.087156) \\ c^2 &\approx 15,793 - 1292.7 \\ c^2 &\approx 14,500.3 \\ c &\approx 120.4 \text{ miles} \end{aligned} \tag{11.4.3}$$

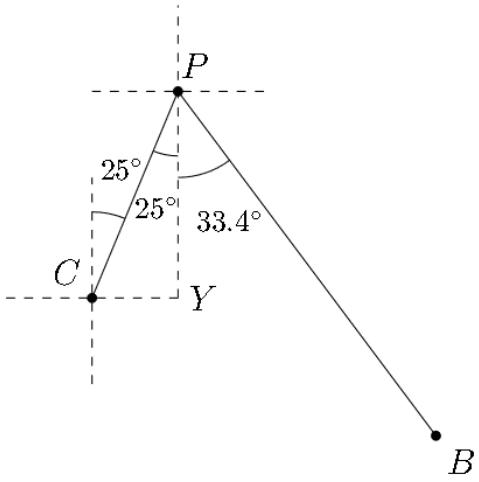
To find $\angle P$, we'll use the Law of sines:

$$\begin{aligned} \frac{\sin 85^\circ}{120.4} &= \frac{\sin P}{103} \\ 103 * \frac{\sin 85^\circ}{120.4} &= \sin P \\ 103 * \frac{0.9962}{120.4} &\approx \sin P \\ 0.85223 &\approx \sin P \\ 58.4^\circ &\approx P \end{aligned} \tag{11.4.4}$$

Now that we know the measure of $\angle P$, we can determine the bearing of Bend from Portland.

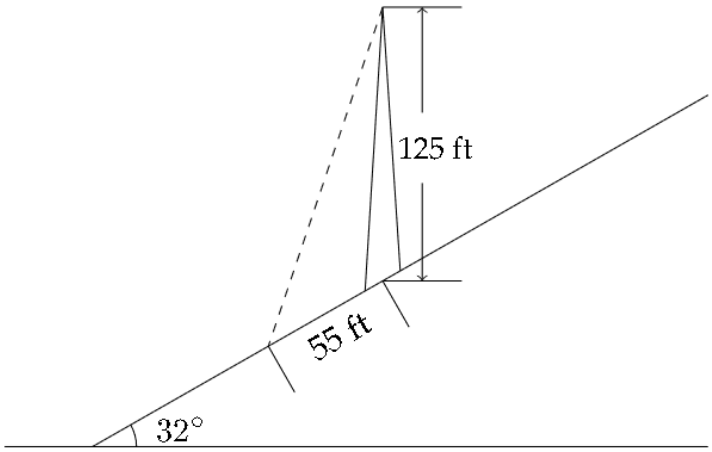


In the picture below notice that $\angle YPC = 25^\circ$. This means that the bearing from Portland to Bend will be east of south by the difference between $\angle P = 58.4^\circ$ and $\angle YPC = 25^\circ$. This makes the bearing of Bend from Portland equal to $S33.4^\circ E$



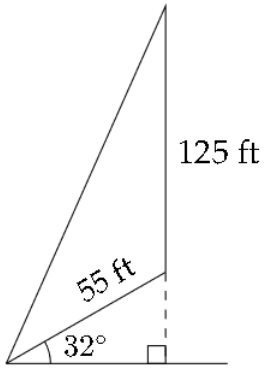
Example 11.4.4

A 125 foot tower is located on the side of a mountain that is inclined 32° to the horizontal. A guy wire is to be attached to the top of the tower and anchored at a point 55 feet downhill from the base of the tower. Find the shortest length of wire needed.



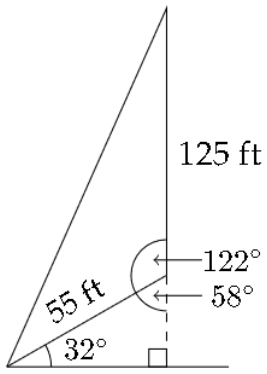
Solution

An important aspect in solving this problem is to identify a triangle in the problem which involves the unknown quantity we're being asked to find. If we're looking for the length of the guy wire, we can use a triangle that involves the wire, the distance from the wire to the center of the tower and the height of the tower:



The angle between the horizontal and the hill will stay 32° at any point on the hill. If we drop a perpendicular to the horizontal, we'll be able to find the angle included between the two given sides.

In the little right triangle, we know the 32 'angle. That means the other acute angle must be 58°, and the supplementary angle (which is in the triangle we're interested in) will be 122°

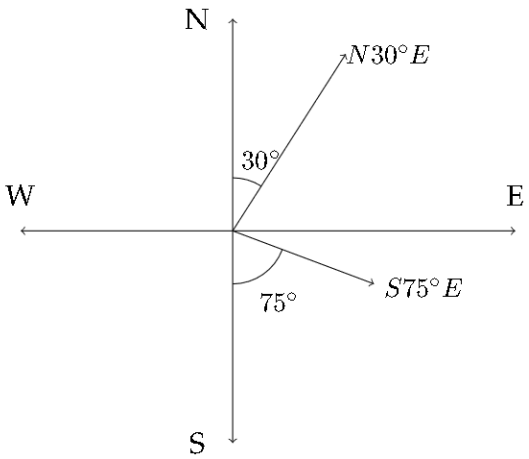


Now we can use the Law of cosines to find the length of the guy wire:

$$\begin{aligned} x^2 &= 125^2 + 55^2 - 2 * 125 * 55 * \cos 122^\circ \\ x^2 &= 15,625 + 3025 - 13,750 * \cos 122^\circ \\ x^2 &\approx 18,650 + 7286.39 \\ x^2 &\approx 25,936.39 \\ x &\approx 161 \text{ feet} \end{aligned} \tag{11.4.5}$$

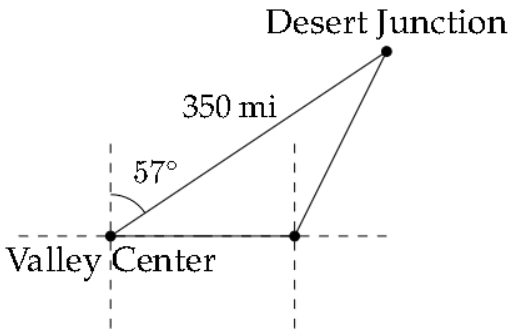
Exercises

1. Two straight roads diverge at an angle of 50°. Two cars leave the intersection at 1pm, one traveling 60mph and the other traveling 45mph. How far apart are the cars (as the crow flies) at $1:30\text{pm}$?
2. Two boats leave the same port at the same time. One travels at a speed of 40mph in the direction $N30^\circ E$ and the other travels at a speed of 28 mph in the direction $S75^\circ E$. How far apart are the two boats after one hour?



3. The airport in Desert Junction is 350 miles from the airport in Valley Center at a bearing of $N57^\circ E$. A pilot who wants to fly from Valley Center to Desert Junction mistakenly flies due east at 225 mph for 30 minutes before correcting the error. How far

is the plane from its destination when the pilot notices the error? What bearing should the plane use in order to arrive at Desert Junction?



4. An airplane leaves airport A and travels 520 miles to airport B at a bearing of $N35^\circ W$. The plane leaves airport B and travels to airport $C310$ miles away at a bearing of $S65^\circ W$ from airport B. Find the distance from airport A to airport C.

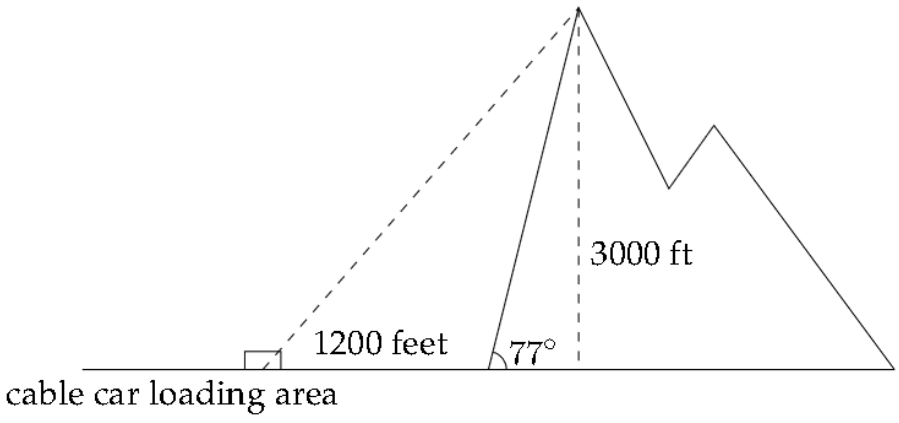
5. Two planes take off at the same time from an airport. The first plane flies at 300 mph at a bearing of $S45^\circ E$. The second plane is flying at a bearing of $S5^\circ W$ with a speed of 330 mph. How far apart are they after 3 hours?

6. Two planes leave an airport at the same time. Their speeds are 180mph and 110mph, and the angle between their flight paths is 43°. How far apart are they after 2.5 hours?

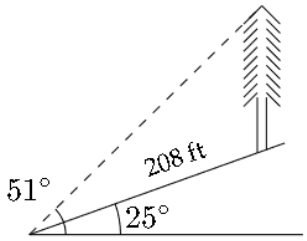
7. Two ships leave a harbor entrance at the same time. The first travels at a speed of 23 mph and the second travels at 17 mph. If the angle between the courses of the ships is 110°, how far apart are they after one hour?

8. A ship leaves the entrance to a harbor and travels 15 miles with a bearing $S10^\circ W$, then turns and travels 45 miles with a bearing of $N43^\circ W$. How far from the harbor entrance is the ship and what is the bearing of the ship from the harbor?

9. A steep mountain is inclined 77° to the horizontal and rises 3000 feet above the surrounding plain. A cable car is to be installed that will connect the plain to the top of the mountain. The distance from the foot of the mountain to the cable car entry loading area is 1200 feet (see diagram below). Find the shortest necessary length of the cable.



10. A tree on a hillside casts a shadow 208ft down the hill. If the angle of inclination of the hillside is 25° to the horizontal and the angle of elevation of the sun is 51°, find the height of the tree.



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12: FUNCTIONS AND GRAPHS

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

12.1: Prelude to Functions and Graphs

12.2: Review of Functions

12.2E: Exercises for Section 1.1

12.3: Basic Classes of Functions

12.3E: Exercises for Section 1.2

12.4: Trigonometric Functions

12.4E: Exercises for Section 1.3

12.5: Inverse Functions

12.5E: Exercises for Section 1.4

12.6: Exponential and Logarithmic Functions

12.6E: Exercises for Section 1.5

12.R: Chapter 1 Review Exercises

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12.1: Prelude to Functions and Graphs

In the past few years, major earthquakes have occurred in several countries around the world. In January 2010, an earthquake of magnitude 7.3 hit Haiti. A magnitude 9 earthquake shook northeastern Japan in March 2011. In April 2014, an 8.2-magnitude earthquake struck off the coast of northern Chile. What do these numbers mean? In particular, how does a magnitude 9 earthquake compare with an earthquake of magnitude 8.2? Or 7.3? Later in this chapter, we show how logarithmic functions are used to compare the relative intensity of two earthquakes based on the magnitude of each earthquake.



Figure 12.1.1: A portion of the San Andreas Fault in California. Major faults like this are the sites of most of the strongest earthquakes ever recorded. (credit: modification of work by Robb Hannawacker, NPS)

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

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12.2: Review of Functions

Learning Objectives

- Use functional notation to evaluate a function.
- Determine the domain and range of a function.
- Draw the graph of a function.
- Find the zeros of a function.
- Recognize a function from a table of values.
- Make new functions from two or more given functions.
- Describe the symmetry properties of a function.

In this section, we provide a formal definition of a function and examine several ways in which functions are represented—namely, through tables, formulas, and graphs. We study formal notation and terms related to functions. We also define composition of functions and symmetry properties. Most of this material will be a review for you, but it serves as a handy reference to remind you of some of the algebraic techniques useful for working with functions.

Functions

Given two sets A and B a set with elements that are ordered pairs (x, y) where x is an element of A and y is an element of B, is a relation from A to B. A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the ***input***; the element of the second set is called the ***output***. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. The cost of mailing a package is a function of the weight of the package. Since functions have so many uses, it is important to have precise definitions and terminology to study them.



Figure 12.2.1: A function can be visualized as an input/output device

Definition: Functions

A *function* f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the *domain* of the function. The set of outputs is called the *range* of the *function*.

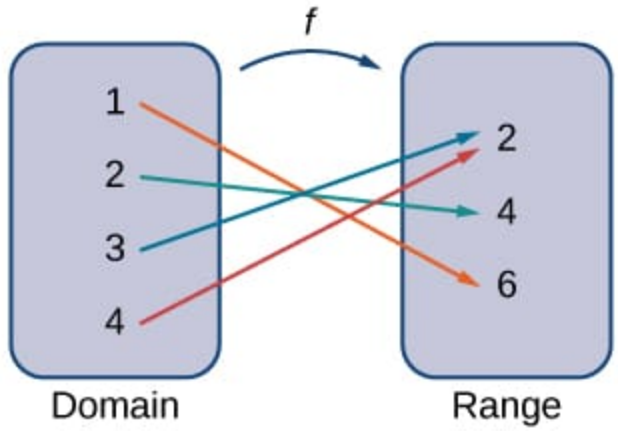


Figure 12.2.2: A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

For example, consider the function f, where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$.

Since every nonnegative real number has a real-value square root, every nonnegative number is an element of the range of this function. Since there is no real number with a square that is negative, the negative real numbers are not elements of the range. We conclude that the range is the set of nonnegative real numbers.

For a general function f with domain D, we often use x to denote the input and y to denote the output associated with x. When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x. Using function notation, we write $y = f(x)$, and we read this equation as "y equals f of x." For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using Figures 12.2.1 - 12.2.3.

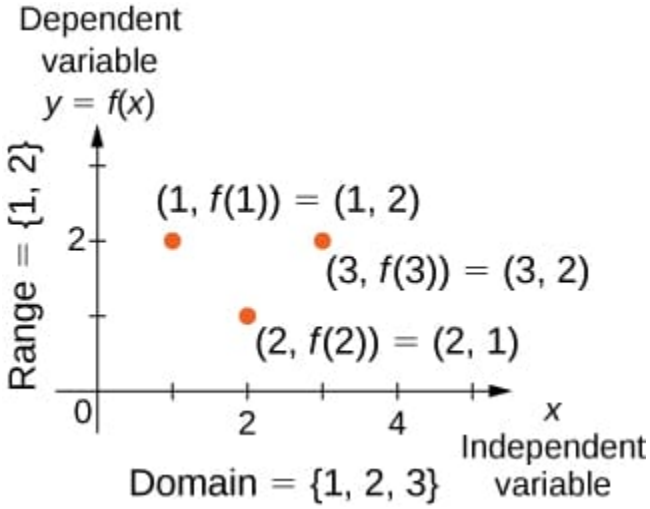


Figure 12.2.3: In this case, a graph of a function f has a domain of $\{1, 2, 3\}$ and a range of $\{1, 2\}$. The independent variable is x and the dependent variable is y.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The graph of a function is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In Figure 12.2.4, we plot a graph of this function.

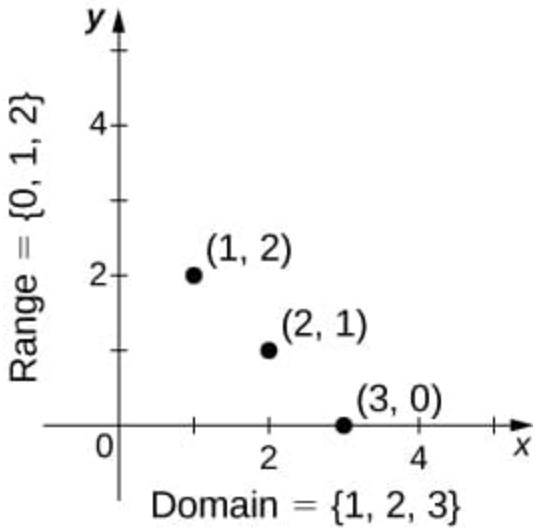


Figure 12.2.4: Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For example, since any real number can be squared, if no other domain is specified, we consider the domain of $f(x) = x^2$ to be the set of all real numbers. On the other hand, the square root function $f(x) = \sqrt{x}$ only gives a real output if x is nonnegative. Therefore, the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers, sometimes called the natural domain.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$, the domains are sets with an infinite number of elements. Clearly we cannot list all these elements. When describing a set with an infinite number of elements, it is often helpful to use set-builder or interval notation. When using set-builder notation to describe a subset of all real numbers, denoted R, we write

$$\{x \,|\, \text{x has some property}\}.$$

We read this as the set of real numbers x such that x has some property. For example, if we were interested in the set of real numbers that are greater than one but less than five, we could denote this set using set-builder notation by writing

\[\{x\,|\,1

A set such as this, which contains all numbers greater than a and less than b, can also be denoted using the *interval notation* (a, b). Therefore,

\[(1,5)=\{x\,|\,1

The numbers 1 and 5 are called the endpoints of this set. If we want to consider the set that includes the endpoints, we would denote this set by writing

$$[1, 5] = \{x \,|\, 1 \leq x \leq 5\}.$$

We can use similar notation if we want to include one of the endpoints, but not the other. To denote the set of nonnegative real numbers, we would use the set-builder notation

$$\{x \,|\, x \geq 0\}.$$

The smallest number in this set is zero, but this set does not have a largest number. Using interval notation, we would use the symbol ∞, which refers to positive infinity, and we would write the set as

$$[0, \infty) = \{x \,|\, x \geq 0\}.$$

It is important to note that ∞ is not a real number. It is used symbolically here to indicate that this set includes all real numbers greater than or equal to zero. Similarly, if we wanted to describe the set of all nonpositive numbers, we could write

$$(-\infty, 0] = \{x \mid x \leq 0\}.$$

Here, the notation $-\infty$ refers to negative infinity, and it indicates that we are including all numbers less than or equal to zero, no matter how small. The set

$$(-\infty, \infty) = \{x \mid \text{x is any real number}\}$$

refers to the set of all real numbers. Some functions are defined using different equations for different parts of their domain. These types of functions are known as piecewise-defined functions. For example, suppose we want to define a function f with a domain that is the set of all real numbers such that $f(x) = 3x + 1$ for $x \geq 2$ and $f(x) = x^2$ for $x < 2$. We denote this function by writing

$$f(x) = \begin{cases} 3x + 1, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$$

When evaluating this function for an input x, the equation to use depends on whether $x \geq 2$ or $x < 2$. For example, since $5 > 2$, we use the fact that $f(x) = 3x + 1$ for $x \geq 2$ and see that $f(5) = 3(5) + 1 = 16$. On the other hand, for $x = -1$, we use the fact that $f(x) = x^2$ for $x < 2$ and see that $f(-1) = 1$.

Example 12.2.1: Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate:

a. $f(-2)$
b. $f(\sqrt{2})$
c. $f(a + h)$

Solution

Substitute the given value for x in the formula for $f(x)$.

a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$
b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$
c. $f(a + h) = 3(a + h)^2 + 2(a + h) - 1 = 3(a^2 + 2ah + h^2) + 2a + 2h - 1 = 3a^2 + 6ah + 3h^2 + 2a + 2h - 1$

Exercise 12.2.1

For $f(x) = x^2 - 3x + 5$, evaluate $f(1)$ and $f(a + h)$.

Hint available at the end of this section.

Example 12.2.2: Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x - 4)^2 + 5$
b. $f(x) = \sqrt{3x + 2} - 1$
c. $f(x) = \dfrac{3}{x - 2}$

Solution a

Consider $f(x) = (x - 4)^2 + 5$.

i. Since $f(x) = (x - 4)^2 + 5$ is a real number for any real number x, the domain of f is the interval $(-\infty, \infty)$.

ii. Since $(x-4)^2 \geq 0$, we know $f(x) = (x-4)^2 + 5 \geq 5$. Therefore, the range must be a subset of $\{y \mid y \geq 5\}$. To show that every element in this set is in the range, we need to show that for a given y in that set, there is a real number x such that $f(x) = (x-4)^2 + 5 = y$. Solving this equation for x, we see that we need x such that

$$(x-4)^2 = y - 5.$$

This equation is satisfied as long as there exists a real number x such that

$$x - 4 = \pm\sqrt{y-5}$$

Since $y \geq 5$, the square root is well-defined. We conclude that for $x = 4 \pm \sqrt{y-5}$, $f(x) = y$, and therefore the range is $\{y \mid y \geq 5\}$.

Solution b

Consider $f(x) = \sqrt{3x+2} - 1$.

i. To find the domain of f, we need the expression $3x + 2 \geq 0$. Solving this inequality, we conclude that the domain is $\{x \mid x \geq -2/3\}$.

ii. To find the range of f, we note that since $\sqrt{3x+2} \geq 0$, $f(x) = \sqrt{3x+2} - 1 \geq -1$. Therefore, the range of f must be a subset of the set $\{y \mid y \geq -1\}$. To show that every element in this set is in the range of f, we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y$. Let $y \geq -1$. Then, $f(x) = y$ if and only if

$$\sqrt{3x+2} - 1 = y.$$

Solving this equation for x, we see that x must solve the equation

$$\sqrt{3x+2} = y + 1.$$

Since $y \geq -1$, such an x could exist. Squaring both sides of this equation, we have $3x + 2 = (y+1)^2$.
Therefore, we need

$$3x = (y+1)^2 - 2,$$

which implies

$$x = \frac{1}{3}(y+1)^2 - \frac{2}{3}.$$

We just need to verify that x is in the domain of f. Since the domain of f consists of all real numbers greater than or equal to $\frac{-2}{3}$, and

$$\frac{1}{3}(y+1)^2 - \frac{2}{3} \geq -\frac{2}{3},$$

there does exist an x in the domain of f. We conclude that the range of f is $\{y \mid y \geq -1\}$.

Solution c

Consider $f(x) = \dfrac{3}{x-2}$.

i. Since $3/(x-2)$ is defined when the denominator is nonzero, the domain is $\{x \mid x \neq 2\}$.

ii. To find the range of f, we need to find the values of y such that there exists a real number x in the domain with the property that

$$\frac{3}{x-2} = y.$$

Solving this equation for x, we find that

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$. Thus, the range is $\{y \mid y \neq 0\}$.

Exercise 12.2.2

Find the domain and range for $f(x) = \sqrt{4 - 2x} + 5$.

Hint available at the end of this section.

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

We can identify a function in each form, but we can also use them together. For instance, we can plot on a graph the values from a table or create a table from a formula.

Tables

Functions described using a **table of values** arise frequently in real-world applications. Consider the following simple example. We can describe temperature on a given day as a function of time of day. Suppose we record the temperature every hour for a 24-hour period starting at midnight. We let our input variable x be the time after midnight, measured in hours, and the output variable y be the temperature x hours after midnight, measured in degrees Fahrenheit. We record our data in Table 12.2.1.

Table 12.2.1: Temperature as a Function of Time of Day

Hour After Midnight	Temperature(°F)	Hour After Midnight	Temperature(°F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

We can see from the table that temperature is a function of time, and the temperature decreases, then increases, and then decreases again. However, we cannot get a clear picture of the behavior of the function without graphing it.

Graphs

Given a function f described by a table, we can provide a visual picture of the function in the form of a graph. Graphing the temperatures listed in Table 12.2.1 can give us a better idea of their fluctuation throughout the day. Figure 12.2.5 shows the plot of the temperature function.

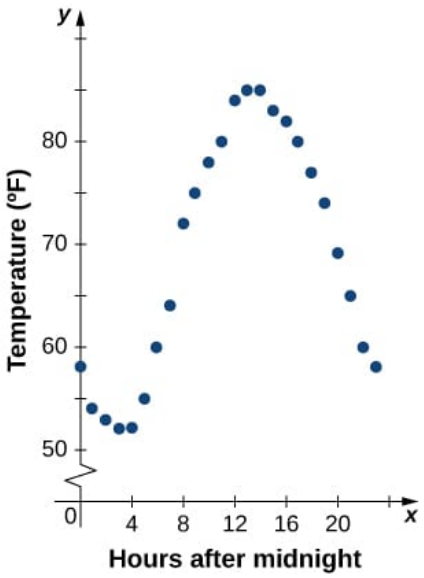


Figure 12.2.5: The graph of the data from Table 12.2.1 shows temperature as a function of time.

From the points plotted on the graph in Figure 12.2.5, we can visualize the general shape of the graph. It is often useful to connect the dots in the graph, which represent the data from the table. In this example, although we cannot make any definitive conclusion regarding what the temperature was at any time for which the temperature was not recorded, given the number of data points collected and the pattern in these points, it is reasonable to suspect that the temperatures at other times followed a similar pattern, as we can see in Figure 12.2.6.

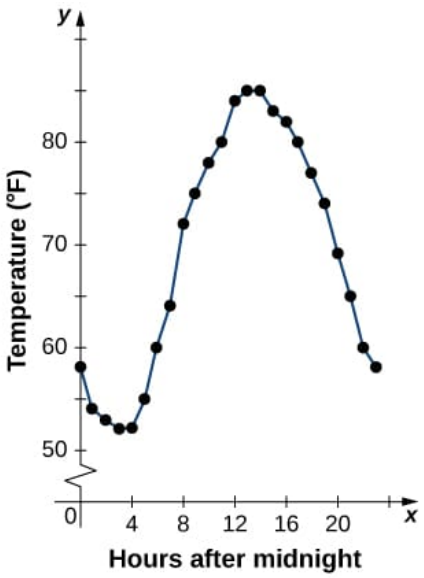


Figure 12.2.6: Connecting the dots in Figure 12.2.5 shows the general pattern of the data.

Algebraic Formulas

Sometimes we are not given the values of a function in table form, rather we are given the values in an explicit formula. Formulas arise in many applications. For example, the area of a circle of radius r is given by the formula $A(r) = \pi r^2$. When an object is thrown upward from the ground with an initial velocity v_0 ft/s, its height above the ground from the time it is thrown until it hits the ground is given by the formula $s(t) = -16t^2 + v_0 t$. When P dollars are invested in an account at an annual interest rate r compounded continuously, the amount of money after t years is given by the formula $A(t) = Pe^{rt}$. Algebraic formulas are important tools to calculate function values. Often we also represent these functions visually in graph form.

Given an algebraic formula for a function f, the graph of f is the set of points $(x, f(x))$, where x is in the domain of f and $f(x)$

is in the range. To graph a function given by a formula, it is helpful to begin by using the formula to create a table of inputs and outputs. If the domain of f consists of an infinite number of values, we cannot list all of them, but because listing some of the inputs and outputs can be very useful, it is often a good way to begin.

When creating a table of inputs and outputs, we typically check to determine whether zero is an output. Those values of x where $f(x) = 0$ are called the zeros of a function. For example, the zeros of $f(x) = x^2 - 4$ are $x = \pm 2$. The zeros determine where the graph of f intersects the x-axis, which gives us more information about the shape of the graph of the function. The graph of a function may never intersect the x-axis, or it may intersect multiple (or even infinitely many) times.

Another point of interest is the y -intercept, if it exists. The y-intercept is given by $(0, f(0))$.

Since a function has exactly one output for each input, the graph of a function can have, at most, one y-intercept. If $x = 0$ is in the domain of a function f, then f has exactly one y-intercept. If $x = 0$ is not in the domain of f, then f has no y-intercept. Similarly, for any real number c, if c is in the domain of f, there is exactly one output $f(c)$, and the line $x = c$ intersects the graph of f exactly once. On the other hand, if c is not in the domain of f, $f(c)$ is not defined and the line $x = c$ does not intersect the graph of f. This property is summarized in the vertical line test.

Vertical Line Test

Given a function f, every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

We can use this test to determine whether a set of plotted points represents the graph of a function (Figure 12.2.7).

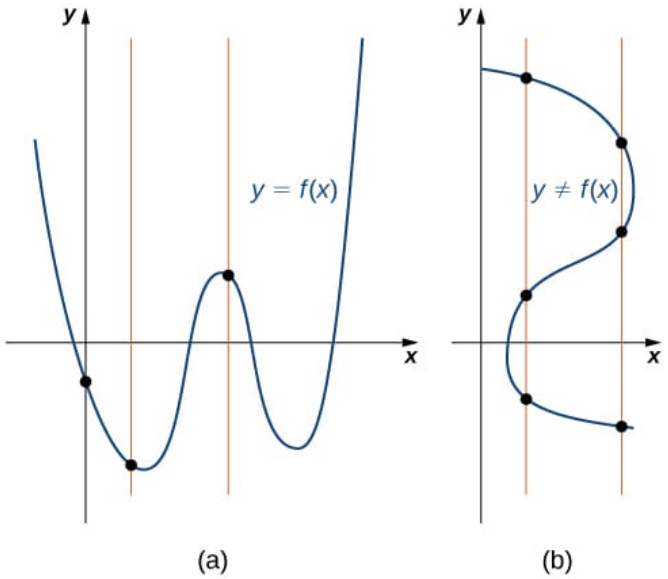


Figure 12.2.7: (a) The set of plotted points represents the graph of a function because every vertical line intersects the set of points, at most, once. (b) The set of plotted points does not represent the graph of a function because some vertical lines intersect the set of points more than once.

Example 12.2.3: Finding Zeros and y-Intercepts of a Function

Consider the function $f(x) = -4x + 2$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solutions

a. To find the zeros, solve $f(x) = -4x + 2 = 0$. We discover that f has one zero at $x = 1/2$.
b. The y-intercept is given by $(0, f(0)) = (0, 2)$.
c. Given that f is a linear function of the form $f(x) = mx + b$ that passes through the points $(1/2, 0)$ and $(0, 2)$, we can sketch the graph of f (Figure 12.2.8).

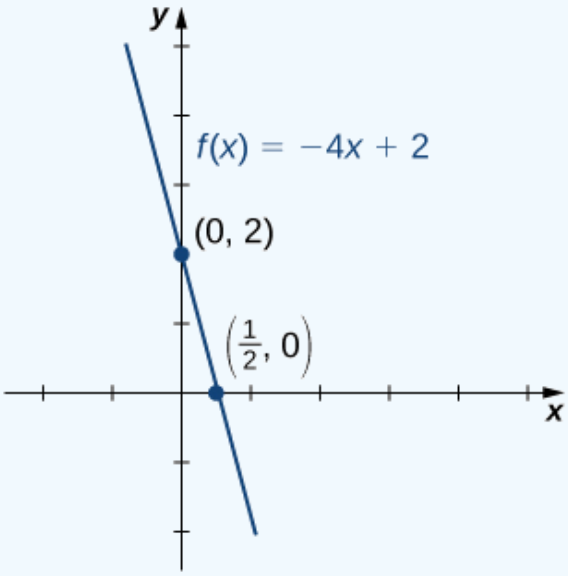


Figure 12.2.8: The function $f(x) = -4x + 2$ is a line with x-intercept $(1/2, 0)$ and y-intercept $(0, 2)$.

Example 12.2.4: Using Zeros and y-Intercepts to Sketch a Graph

Consider the function $f(x) = \sqrt{x + 3} + 1$.

a. Find all zeros of f.
b. Find the y-intercept (if any).
c. Sketch a graph of f.

Solution

a. To find the zeros, solve $\sqrt{x + 3} + 1 = 0$. This equation implies $\sqrt{x + 3} = -1$. Since $\sqrt{x + 3} \geq 0$ for all x, this equation has no solutions, and therefore f has no zeros.
b. The y-intercept is given by $(0, f(0)) = (0, \sqrt{3} + 1)$.
c. To graph this function, we make a table of values. Since we need $x + 3 \geq 0$, we need to choose values of $x \geq -3$. We choose values that make the square-root function easy to evaluate.

x	-3	-2	1
$f(x)$	1	2	3

Making use of the table and knowing that, since the function is a square root, the graph of f should be similar to the graph of $y = \sqrt{x}$, we sketch the graph (Figure 12.2.9).

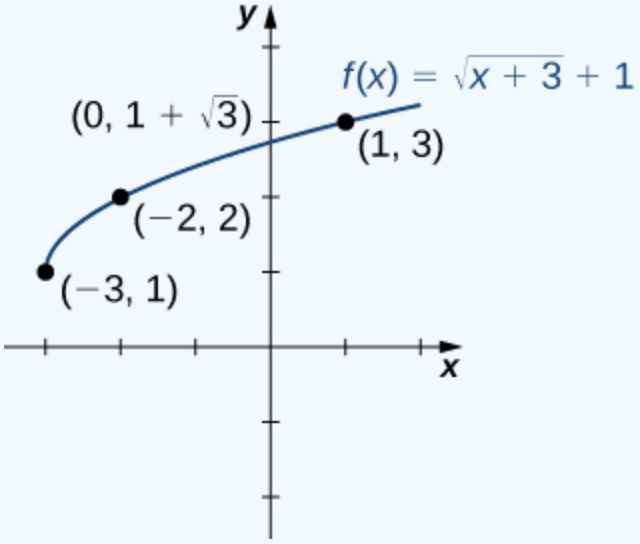


Figure 12.2.9: The graph of $f(x) = \sqrt{x+3} + 1$ has a y-intercept but no x-intercepts.

Exercise 12.2.4

Find the zeros of $f(x) = x^3 - 5x^2 + 6x$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 12.2.5: Finding the Height of a Free-Falling Object

If a ball is dropped from a height of 100 ft, its height s at time t is given by the function $s(t) = -16t^2 + 100$, where s is measured in feet and t is measured in seconds. The domain is restricted to the interval $[0, c]$, where $t = 0$ is the time when the ball is dropped and $t = c$ is the time when the ball hits the ground.

a. Create a table showing the height s(t) when $t = 0,\ 0.5,\ 1,\ 1.5,\ 2$, and 2.5. Using the data from the table, determine the domain for this function. That is, find the time c when the ball hits the ground.
b. Sketch a graph of s.

Solution a

t	0	0.5	1	1.5	2	2.5
$s(t)$	100	96	84	64	36	0

Since the ball hits the ground when $t = 2.5$, the domain of this function is the interval $[0, 2.5]$.

Solution b

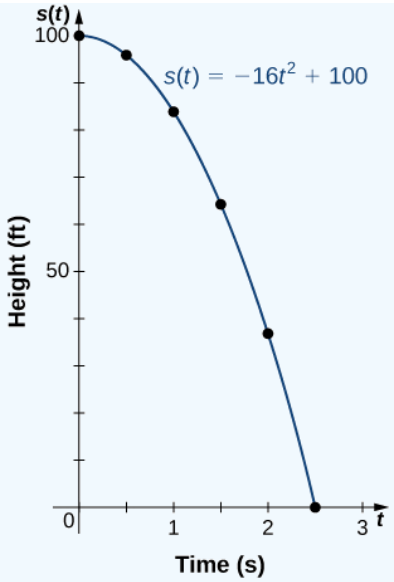


Figure $12.2.8$, the values of $f(x)$ are getting smaller as x is getting larger. A function with this property is said to be decreasing. On the other hand, for the function $f(x) = \sqrt{x+3} + 1$ graphed in Figure $12.2.9$, the values of $f(x)$ are getting larger as the values of x are getting larger. A function with this property is said to be increasing. It is important to note, however, that a function can be increasing on some interval or intervals and decreasing over a different interval or intervals. For example, using our temperature function plotted above, we can see that the function is decreasing on the interval $(0, 4)$, increasing on the interval $(4, 14)$, and then decreasing on the interval $(14, 23)$. We make the idea of a function increasing or decreasing over a particular interval more precise in the next definition.

Definition: Increasing and Decreasing on an Interval

We say that a function f is ***increasing on the interval*** I if for all $x_1,\ x_2 \in I$,

\[f(x_{1})≤f(x_{2}) \quad \text{when} \,x_{1}

We say f is strictly increasing on the interval I if for all $x_1, x_2 \in I$,

\[f(x_{1})

We say that a function f is ***decreasing on the interval*** I if for all $x_1, x_2 \in I$,

\[f(x_{1})≥f(x_{2}) \quad \text{if} \, x_{1}

We say that a function f is strictly decreasing on the interval I if for all $x_1, x_2 \in I$,

\[f(x_{1})>f(x_{2}) \quad \text{if} \, x_{1}

For example, the function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$ because $3x_1 < 3x_2$ whenever \(x_{1}–x^3_{2}\) whenever \(x_{1}

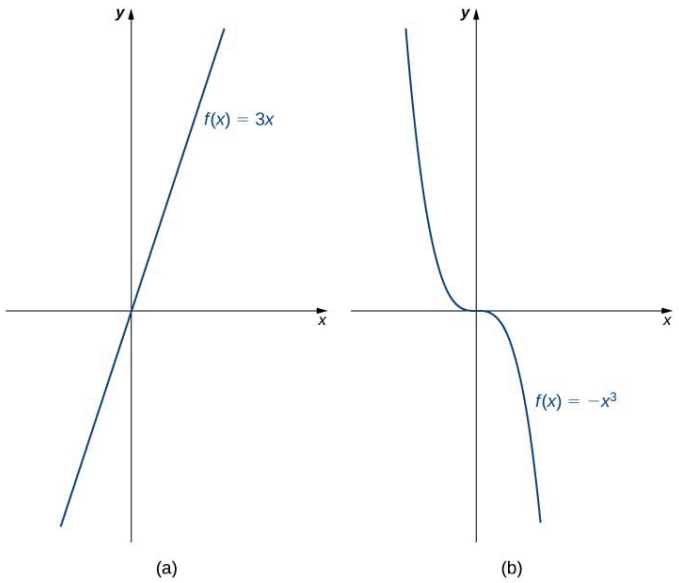


Figure 12.2.10: (a) The function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$. (b) The function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$.

Combining Functions

Now that we have reviewed the basic characteristics of functions, we can see what happens to these properties when we combine functions in different ways, using basic mathematical operations to create new functions. For example, if the cost for a company to manufacture x items is described by the function $C(x)$ and the revenue created by the sale of x items is described by the function $R(x)$, then the profit on the manufacture and sale of x items is defined as $P(x) = R(x) - C(x)$. Using the difference between two functions, we created a new function.

Alternatively, we can create a new function by composing two functions. For example, given the functions $f(x) = x^2$ and $g(x) = 3x + 1$, the composite function $f \circ g$ is defined such that

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (3x + 1)^2.$$

The composite function $g \circ f$ is defined such that

$$(g \circ f)(x) = g(f(x)) = 3f(x) + 1 = 3x^2 + 1.$$

Note that these two new functions are different from each other.

Combining Functions with Mathematical Operators

To combine functions using mathematical operators, we simply write the functions with the operator and simplify. Given two functions f and g, we can define four new functions:

$(f + g)(x) = f(x) + g(x)$	Sum
$(f - g)(x) = f(x) - g(x)$	Difference
$(f \cdot g)(x) = f(x)g(x)$	Product
$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ for $g(x) \neq 0$	Quotient

Example 12.2.6: Combining Functions Using Mathematical Operations

Given the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find each of the following functions and state its domain.

a. $(f+g)(x)$
b. $(f-g)(x)$
c. $(f \cdot g)(x)$
d. $\left(\frac{f}{g}\right)(x)$

Solution a

$$(f+g)(x) = (2x-3) + (x^2-1) = x^2 + 2x - 4.$$

The domain of this function is the interval $(-\infty, \infty)$.

Solution b

$$(f-g)(x) = (2x-3) - (x^2-1) = -x^2 + 2x - 2.$$

The domain of this function is the interval $(-\infty, \infty)$.

Solution c

$$(f \cdot g)(x) = (2x-3)(x^2-1) = 2x^3 - 3x^2 - 2x + 3.$$

The domain of this function is the interval $(-\infty, \infty)$.

Solution d

$$\left(\frac{f}{g}\right)(x) = \frac{2x-3}{x^2-1}.$$

The domain of this function is $\{x \mid x \neq \pm 1\}$.

Exercise 12.2.6

For $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, find $(f/g)(x)$ and state its domain.

Hint available at the end of this section.

Answer available at the end of this section.

Function Composition

When we compose functions, we take a function of a function. For example, suppose the temperature T on a given day is described as a function of time t (measured in hours after midnight) as in Table 12.2.1. Suppose the cost C, to heat or cool a building for 1 hour, can be described as a function of the temperature T. Combining these two functions, we can describe the cost of heating or cooling a building as a function of time by evaluating $C(T(t))$. We have defined a new function, denoted $C \circ T$, which is defined such that $(C \circ T)(t) = C(T(t))$ for all t in the domain of T. This new function is called a composite function. We note that since cost is a function of temperature and temperature is a function of time, it makes sense to define this new function $(C \circ T)(t)$. It does not make sense to consider $(T \circ C)(t)$, because temperature is not a function of cost.

Definition: Composite Functions

Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the composite function $(g \circ f)(x)$ is the function with domain A such that

$$(g \circ f)(x) = g(f(x))$$

A composite function $g \circ f$ can be viewed in two steps. First, the function f maps each input x in the domain of f to its output $f(x)$ in the range of f. Second, since the range of f is a subset of the domain of g, the output $f(x)$ is an element in the domain of g, and therefore it is mapped to an output $g(f(x))$ in the range of g. In Figure 12.2.11, we see a visual image of a composite function.

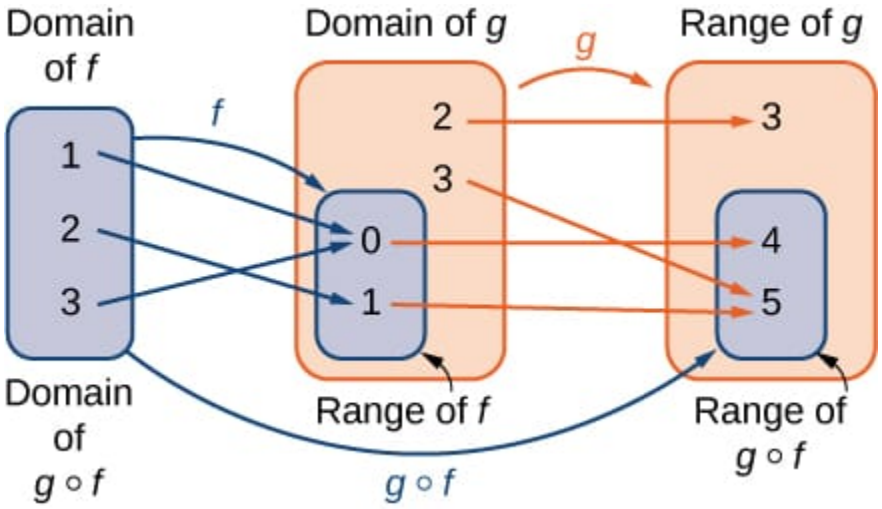


Figure 12.2.11: For the composite function $g \circ f$, we have $(g \circ f)(1) = 4$, $(g \circ f)(2) = 5$, and $(g \circ f)(3) = 4$.

Example 12.2.7: Compositions of Functions Defined by Formulas

Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$.

a. Find $(g \circ f)(x)$ and state its domain and range.
b. Evaluate $(g \circ f)(4)$, $(g \circ f)(-1/2)$.
c. Find $(f \circ g)(x)$ and state its domain and range.
d. Evaluate $(f \circ g)(4)$, $(f \circ g)(-1/2)$.

Solution a

We can find the formula for $(g \circ f)(x)$ in two different ways. We could write

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Alternatively, we could write

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^2 + 1}.$$

Since $x^2 + 1 \neq 0$ for all real numbers x, the domain of $(g \circ f)(x)$ is the set of all real numbers. Since $0 < 1/(x^2 + 1) \leq 1$, the range is, at most, the interval $(0, 1]$. To show that the range is this entire interval, we let $y = 1/(x^2 + 1)$ and solve this equation for x to show that for all y in the interval $(0, 1]$, there exists a real number x such that $y = 1/(x^2 + 1)$. Solving this equation for x, we see that $x^2 + 1 = 1/y$, which implies that

$$x = \pm\sqrt{\frac{1}{y} - 1}$$

If y is in the interval $(0, 1]$, the expression under the radical is nonnegative, and therefore there exists a real number x such that $1/(x^2 + 1) = y$. We conclude that the range of $g \circ f$ is the interval $(0, 1]$.

Solution b

$$(g \circ f)(4) = g(f(4)) = g(4^2 + 1) = g(17) = \frac{1}{17}$$

$$(g \circ f)(-\frac{1}{2}) = g\left(f\left(-\frac{1}{2}\right)\right) = g\left(\left(-\frac{1}{2}\right)^2 + 1\right) = g\left(\frac{5}{4}\right) = \frac{4}{5}$$

Solution c

We can find a formula for $(f \circ g)(x)$ in two ways. First, we could write

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 + 1.$$

Alternatively, we could write

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 + 1 = \left(\frac{1}{x}\right)^2 + 1.$$

The domain of $f \circ g$ is the set of all real numbers x such that $x \neq 0$. To find the range of f, we need to find all values y for which there exists a real number $x \neq 0$ such that

$$\left(\frac{1}{x}\right)^2 + 1 = y.$$

Solving this equation for x, we see that we need x to satisfy

$$\left(\frac{1}{x}\right)^2 = y - 1,$$

which simplifies to

$$\frac{1}{x} = \pm\sqrt{y-1}$$

Finally, we obtain

$$x = \pm\frac{1}{\sqrt{y-1}}.$$

Since $1/\sqrt{y-1}$ is a real number if and only if $y > 1$, the range of f is the set $\{y \mid y \geq 1\}$.

Solution d

$$(f \circ g)(4) = f(g(4)) = f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + 1 = \frac{17}{16}$$

$$(f \circ g)\left(-\frac{1}{2}\right) = f\left(g\left(-\frac{1}{2}\right)\right) = f(-2) = (-2)^2 + 1 = 5$$

In Example 12.2.7, we can see that $(f \circ g)(x) \neq (g \circ f)(x)$. This tells us, in general terms, that the order in which we compose functions matters.

Exercise 12.2.7

Let $f(x) = 2 - 5x$. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Solution

$$(f \circ g)(x) = 2 - 5\sqrt{x}.$$

Example 12.2.8: Composition of Functions Defined by Tables

Consider the functions f and g described by

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	4	2	4	-2	0	-2	4

x	-4	-2	0	2	4
$g(x)$	1	0	3	0	5

a. Evaluate $(g \circ f)(3)$,$(g \circ f)(0)$.
b. State the domain and range of $(g \circ f)(x)$.
c. Evaluate $(f \circ f)(3)$,$(f \circ f)(1)$.
d. State the domain and range of $(f \circ f)(x)$.

Solution a

$$(g \circ f)(3) = g(f(3)) = g(-2) = 0$$

$$(g \circ f)(0) = g(4) = 5$$

Solution b

The domain of $g \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $g \circ f$ is the set $\{0, 3, 5\}$.

Solution c

$$(f \circ f)(3) = f(f(3)) = f(-2) = 4$$

$$(f \circ f)(1) = f(f(1)) = f(-2) = 4$$

Solution d

The domain of $f \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $f \circ f$ is the set $\{0, 4\}$.

Example 12.2.9: Application Involving a Composite Function

A store is advertising a sale of 20% off all merchandise. Caroline has a coupon that entitles her to an additional 15% off any item, including sale merchandise. If Caroline decides to purchase an item with an original price of x dollars, how much will she end up paying if she applies her coupon to the sale price? Solve this problem by using a composite function.

Solution

Since the sale price is 20% off the original price, if an item is x dollars, its sale price is given by $f(x) = 0.80x$. Since the coupon entitles an individual to 15% off the price of any item, if an item is y dollars, the price, after applying the coupon, is given by g(y)=0.85y. Therefore, if the price is originally x dollars, its sale price will be $f(x) = 0.80x$ and then its final price after the coupon will be $g(f(x)) = 0.85(0.80x) = 0.68x$.

Exercise 12.2.9

If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Hint

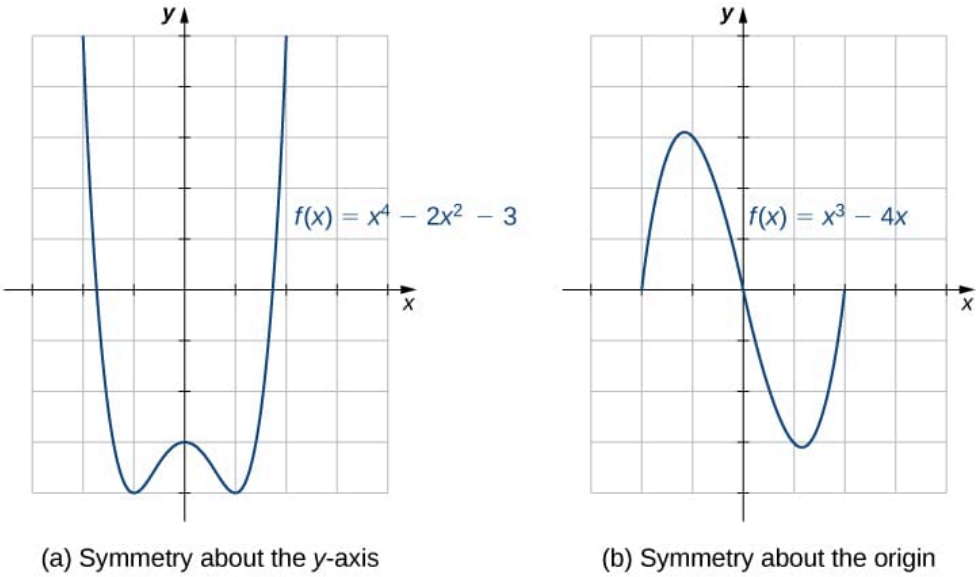
The sale price of an item with an original price of x dollars is $f(x) = 0.90x$. The coupon price for an item that is y dollars is $g(y) = 0.70y$.

Solution

$$(g \circ f)(x) = 0.63x$$

Symmetry of Functions

The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph. For example, consider the function $f(x) = x^4 - 2x^2 - 3$ shown in Figure $12.2.12a$. If we take the part of the curve that lies to the right of the y-axis and flip it over the y-axis, it lays exactly on top of the curve to the left of the y-axis. In this case, we say the function has **symmetry about the y-axis**. On the other hand, consider the function $f(x) = x^3 - 4x$ shown in Figure $12.2.12b$. If we take the graph and rotate it $180\,^\circ$ about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.



(a) Symmetry about the y-axis (b) Symmetry about the origin

Figure $12.2.12$: (a) A graph that is symmetric about the y-axis. (b) A graph that is symmetric about the origin.

If we are given the graph of a function, it is easy to see whether the graph has one of these symmetry properties. But without a graph, how can we determine algebraically whether a function f has symmetry? Looking at Figure $12.2.12a$ again, we see that since f is symmetric about the y-axis, if the point (x, y) is on the graph, the point $(-x, y)$ is on the graph. In other words, $f(-x) = f(x)$. If a function f has this property, we say f is an even function, which has symmetry about the y-axis. For example, $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

In contrast, looking at Figure $12.2.12b$ again, if a function f is symmetric about the origin, then whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph. In other words, $f(-x) = -f(x)$. If f has this property, we say f is an odd function, which has symmetry about the origin. For example, $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Definition: Even and Odd Functions

- If $f(x) = f(-x)$ for all x in the domain of f, then f is an *even* function. An even function is symmetric about the y-axis.

- If $f(-x) = -f(x)$ for all x in the domain of f, then f is an *odd* function. An odd function is symmetric about the origin.

Example 12.2.10: Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

a. $f(x) = -5x^4 + 7x^2 - 2$
b. $f(x) = 2x^5 - 4x + 5$
c. $f(x) = \frac{3x}{x^2+1}$

Solution

To determine whether a function is even or odd, we evaluate $f(-x)$ and compare it to $f(x)$ and $-f(x)$.

Solution a

$$f(-x) = -5(-x)^4 + 7(-x)^2 - 2 = -5x^4 + 7x^2 - 2 = f(x).$$

Therefore, f is even.

Solution b

$$f(-x) = 2(-x)^5 - 4(-x) + 5 = -2x^5 + 4x + 5.$$

Now, $f(-x) \neq f(x)$.

Furthermore, noting that $-f(x) = -2x^5 + 4x - 5$, we see that $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd.

Solution c

$$\begin{aligned} f(-x) &= 3(-x)/((-x)2 + 1) \\ &= -3x/(x^2 + 1) \\ &= -[3x/(x^2 + 1)] = -f(x). \end{aligned}$$

Therefore, f is odd.

Exercise 12.2.10

Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

Hint available at the end of this section.

Answer available at the end of this section.

One symmetric function that arises frequently is the **absolute value function**, written as $|x|$. The absolute value function is defined as

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Some students describe this function by stating that it "makes everything positive." By the definition of the absolute value function, we see that if $x < 0$, then $|x| = -x > 0$, and if $x > 0$, then $|x| = x > 0$. However, for $x = 0$, $|x| = 0$. Therefore, it is more accurate to say that for all nonzero inputs, the output is positive, but if $x = 0$, the output $|x| = 0$. We conclude that the range of the absolute value function is $\{y \mid y \geq 0\}$. In Figure 12.2.13, we see that the absolute value function is symmetric about the y-axis and is therefore an even function.

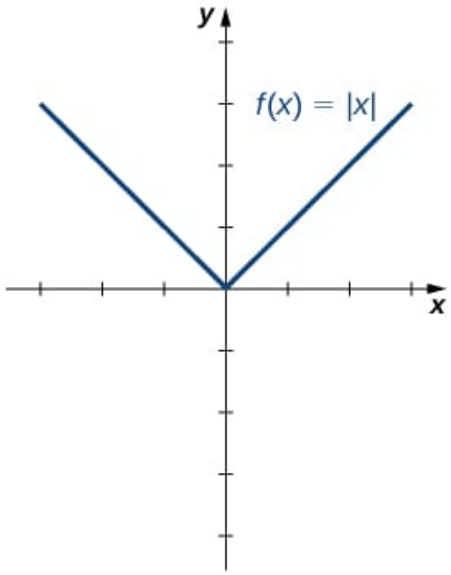


Figure 12.2.13: The graph of $f(x) = |x|$ is symmetric about the y-axis.

Example 12.2.11: Working with the Absolute Value Function

Find the domain and range of the function $f(x) = 2|x - 3| + 4$.

Solution

Since the absolute value function is defined for all real numbers, the domain of this function is $(-\infty, \infty)$. Since $|x - 3| \geq 0$ for all x, the function $f(x) = 2|x - 3| + 4 \geq 4$. Therefore, the range is, at most, the set $\{y \mid y \geq 4\}$. To see that the range is, in fact, this whole set, we need to show that for $y \geq 4$ there exists a real number x such that

$$2|x - 3| + 4 = y$$

A real number x satisfies this equation as long as

$$|x - 3| = \frac{1}{2}(y - 4)$$

Since $y \geq 4$, we know $y - 4 \geq 0$, and thus the right-hand side of the equation is nonnegative, so it is possible that there is a solution. Furthermore,

$$|x - 3| = \begin{cases} -(x - 3), & \text{if } x < 3 \\ x - 3, & \text{if } x \geq 3 \end{cases}$$

Therefore, we see there are two solutions:

$$x = \pm\tfrac{1}{2}(y - 4) + 3.$$

The range of this function is $\{y \mid y \geq 4\}$.

Exercise 12.2.11: Domain and Range

For the function $f(x) = |x + 2| - 4$, find the domain and range.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- A function is a mapping from a set of inputs to a set of outputs with exactly one output for each input.
- If no domain is stated for a function $y = f(x)$, the domain is considered to be the set of all real numbers x for which the function is defined.
- When sketching the graph of a function f, each vertical line may intersect the graph, at most, once.
- A function may have any number of zeros, but it has, at most, one y-intercept.
- To define the composition $g \circ f$, the range of f must be contained in the domain of g.
- Even functions are symmetric about the y-axis whereas odd functions are symmetric about the origin.

Key Equations

- **Composition of two functions**

$$(g \circ f)(x) = g\big(f(x)\big)$$

- **Absolute value function**

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Glossary

absolute value function
$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

composite function
given two functions f and g, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

decreasing on the interval I
a function decreasing on the interval I if, for all $x_1,\ x_2 \in I,\ \ f(x_1) \geq f(x_2)$ if \(x_1

dependent variable
the output variable for a function

domain
the set of inputs for a function

even function
a function is even if $f(-x) = f(x)$ for all x in the domain of f

function
a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

graph of a function
the set of points (x, y) such that x is in the domain of f and $y = f(x)$

increasing on the interval I
a function increasing on the interval I if for all $x_1,\ x_2 \in I,\ \ f(x_1) \leq f(x_2)$ if \(x_1

independent variable
the input variable for a function

odd function
a function is odd if $f(-x) = -f(x)$ for all x in the domain of f

range
the set of outputs for a function

symmetry about the origin
: the graph of a function f is symmetric about the origin if $(-x,-y)$ is on the graph of f whenever (x,y) is on the graph

symmetry about the y-axis
: the graph of a function f is symmetric about the y-axis if $(-x,y)$ is on the graph of f whenever (x,y) is on the graph

table of values
: a table containing a list of inputs and their corresponding outputs

vertical line test
: given the graph of a function, every vertical line intersects the graph, at most, once

zeros of a function
: when a real number x is a zero of a function f, $f(x)=0$

Answers

Exercise 12.2.1

Substitute 1 and $a+h$ for x in the formula for $f(x)$.

$$f(1)=3\)and\(f(a+h)=a^2+2ah+h^2-3a-3h+5$$

Exercise 12.2.2

Use $4-2x\geq 0$.

Domain = $\{x \mid x\leq 2\}$ and range = $\{y \mid y\geq 5\}$

Exercise 12.2.4

Factor the polynomial.

Exercise 12.2.4

$$x=0,2,3$$

Exercise 12.2.6

The new function $(f/g)(x)$ is a quotient of two functions. For what values of x is the denominator zero?

Exercise 12.2.6

$$\left(\frac{f}{g}\right)(x)=\frac{x^2+3}{2x-5}.\)Thedomainis\(\{x\,|\,x\neq\frac{5}{2}\}.$$

Exercise 12.2.10

Compare $f(-x)$ with $f(x)$ and $-f(x)$.

Exercise 12.2.10

$f(x)$ is odd.

Exercise 12.2.11**: Domain and Range**

$|x+2|\geq 0$ for all real numbers x.

Exercise 12.2.11**: Domain and Range**

Domain = $(-\infty,\infty)$, range = $\{y \mid y\geq -4\}$.

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12.2E: Exercises for Section 1.1

For exercises 1 - 6, (a) determine the domain and the range of each relation, and (b) state whether the relation is a function.

1)

x	y	x	y
-3	9	1	1
-2	4	2	4
-1	1	3	9
0	0		

Answer

a. Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, Range = $\{0, 1, 4, 9\}$

b. Yes, a function

2)

x	y	x	y
-3	-2	1	1
-2	-8	2	8
-1	-1	3	-2
0	0		

3)

x	y	x	y
1	-3	1	1
2	-2	2	2
3	-1	3	3
0	0		

Answer

a. Domain = $\{0, 1, 2, 3\}$, Range = $\{-3, -2, -1, 0, 1, 2, 3\}$

b. No, not a function

4)

x	y	x	y
1	1	5	1
2	1	6	1
3	1	7	1

x	y	x	y
4	1		

5)

x	y	x	y
3	3	15	1
5	2	21	2
8	1	33	3
10	0		

Answer

a. Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, Range = $\{0, 1, 2, 3\}$

b. Yes, a function

6)

x	y	x	y
-7	11	1	-2
-2	5	3	4
-2	1	6	11
0	-1		

For exercises 7 - 13, find the values for each function, if they exist, then simplify.

a. $f(0)$ b. $f(1)$ c. $f(3)$ d. $f(-x)$ e. $f(a)$ f. $f(a+h)$

7) $f(x) = 5x - 2$

Answer

a. -2 b. 3 c. 13 d. $-5x - 2$ e. $5a - 2$ f. $5a + 5h - 2$

8) $f(x) = 4x^2 - 3x + 1$

9) $f(x) = \dfrac{2}{x}$

Answer

a. Undefined b. 2 c. $\frac{2}{3}$ d. $-\dfrac{2}{x}$ e. $\dfrac{2}{a}$ f. $\dfrac{2}{a+h}$

10) $f(x) = |x - 7| + 8$

11) $f(x) = \sqrt{6x + 5}$

Answer

a. $\sqrt{5}$ b. $\sqrt{11}$ c. $\sqrt{23}$ d. $\sqrt{-6x + 5}$ e. $\sqrt{6a + 5}$ f. $\sqrt{6a + 6h + 5}$

12) $f(x) = \dfrac{x - 2}{3x + 7}$

13) $f(x) = 9$

Answer

a. 9 b. 9 c. 9 d. 9 e. 9 f. 9

For exercises 14 - 21, find the domain, range, and all zeros/intercepts, if any, of the functions.

14) $f(x) = \dfrac{x}{x^2 - 16}$

15) $g(x) = \sqrt{8x - 1}$

Answer

$x \geq \frac{1}{8}; \quad y \geq 0; \quad x = \frac{1}{8}$; no y-intercept

16) $h(x) = \dfrac{3}{x^2 + 4}$

17) $f(x) = -1 + \sqrt{x + 2}$

Answer

$x \geq -2; \quad y \geq -1; \quad x = -1; \quad y = -1 + \sqrt{2}$

18) $f(x) = 1x - \sqrt{9}$

19) $g(x) = \dfrac{3}{x - 4}$

Answer

$x \neq 4; \quad y \neq 0$; no x-intercept; $y = -\frac{3}{4}$

20) $f(x) = 4|x + 5|$

21) $g(x) = \sqrt{\dfrac{7}{x - 5}}$

Answer

$x > 5; \quad y > 0$; no intercepts

For exercises 22 - 27, set up a table to sketch the graph of each function using the following values: $x = -3, -2, -1, 0, 1, 2, 3.$

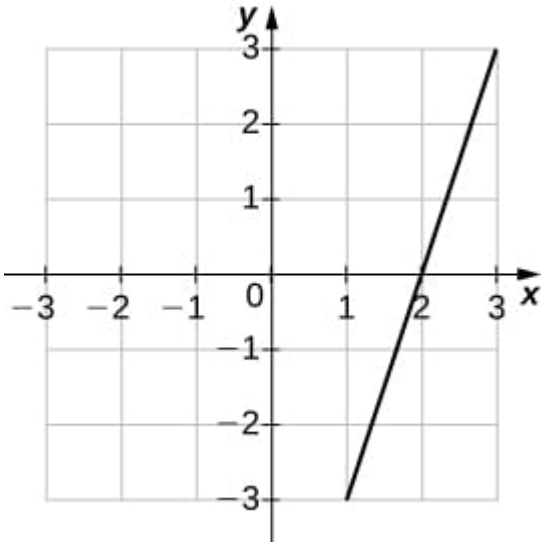
22) $f(x) = x^2 + 1$

x	y	x	y
-3	10	1	2
-2	5	2	5
-1	2	3	10
0	1		

23) $f(x) = 3x - 6$

x	y	x	y
-3	-15	1	-3
-2	-12	2	0
-1	-9	3	3
0	-6		

Answer



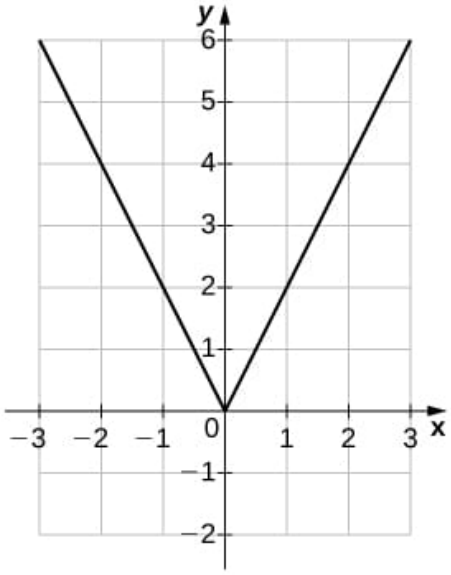
24) $f(x) = \frac{1}{2}x + 1$

x	y	x	y
-3	$-\frac{1}{2}$	1	$\frac{3}{2}$
-2	0	2	2
-1	$\frac{1}{2}$	3	$\frac{5}{2}$
0	1		

25) $f(x) = 2|x|$

x	y	x	y
-3	6	1	2
-2	4	2	4
-1	2	3	6
0	0		

Answer



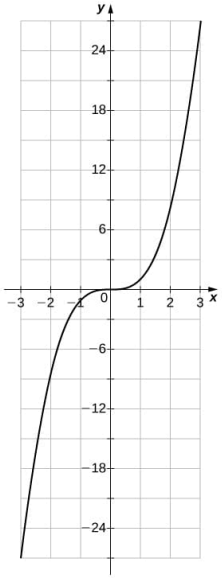
26) $f(x) = -x^2$

x	y	x	y
-3	-9	1	-1
-2	-4	2	-4
-1	-1	3	-9
0	0		

27) $f(x) = x^3$

x	y	x	y
-3	-27	1	1
-2	-8	2	8
-1	-1	3	27
0	0		

Answer



For exercises 28 - 35, use the vertical line test to determine whether each of the given graphs represents a function. Assume that a graph continues at both ends if it extends beyond the given grid. If the graph represents a function, then determine the following for each graph:

a. Domain and range

b. x -intercept, if any (estimate where necessary)

c. y-Intercept, if any (estimate where necessary)

d. The intervals for which the function is increasing

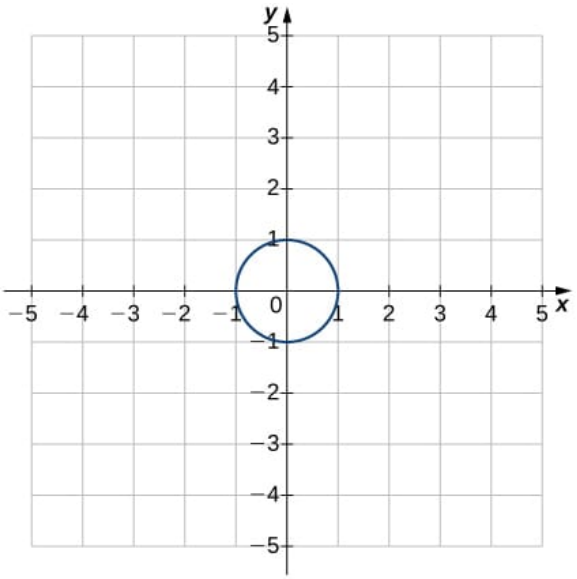
e. The intervals for which the function is decreasing

f. The intervals for which the function is constant

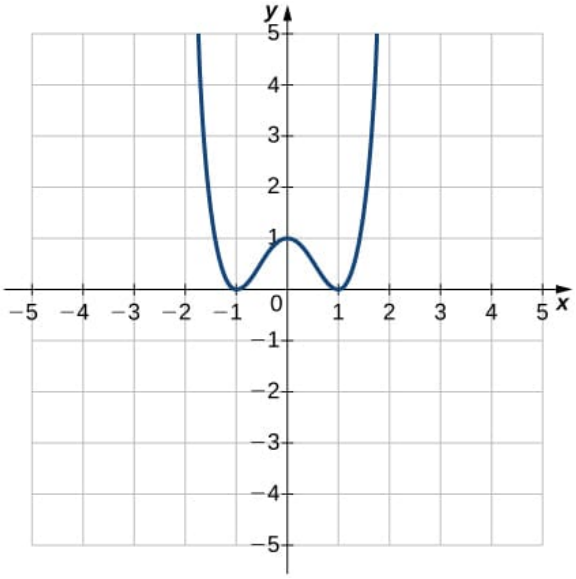
g. Symmetry about any axis and/or the origin

h. Whether the function is even, odd, or neither

28)



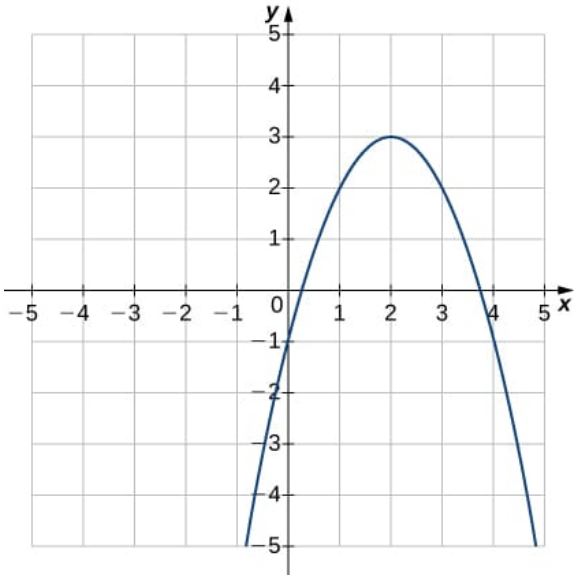
29)



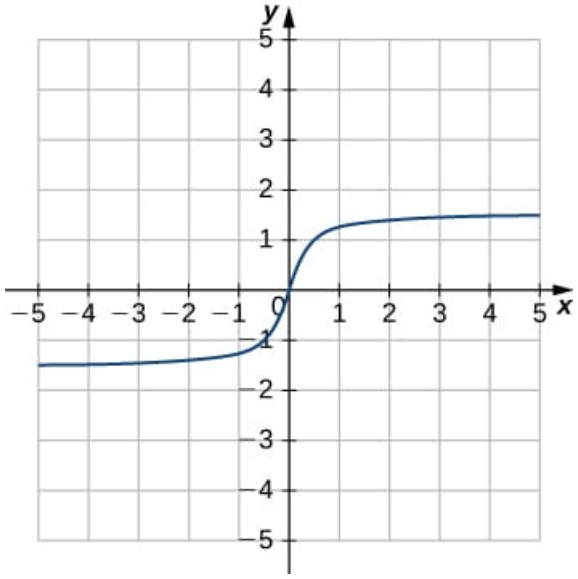
Answer

Function;
a. Domain: all real numbers, range: $y \geq 0$
b. $x = \pm 1$
c. $y = 1$
d. \(–1 e. \(–∞ f. Not constant
g. y-axis
h. Even

30)



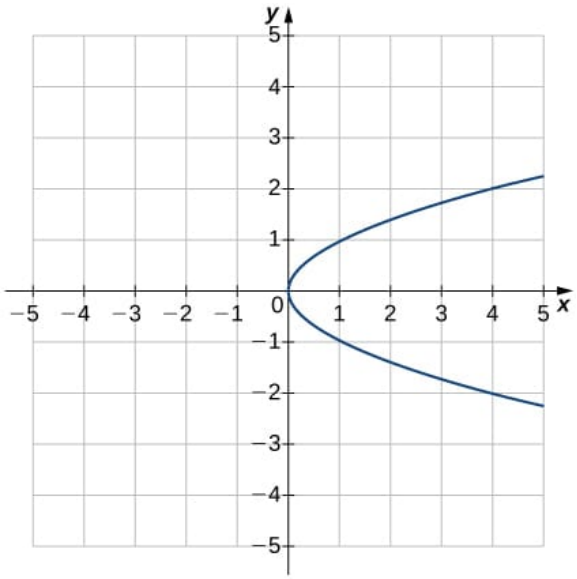
31)



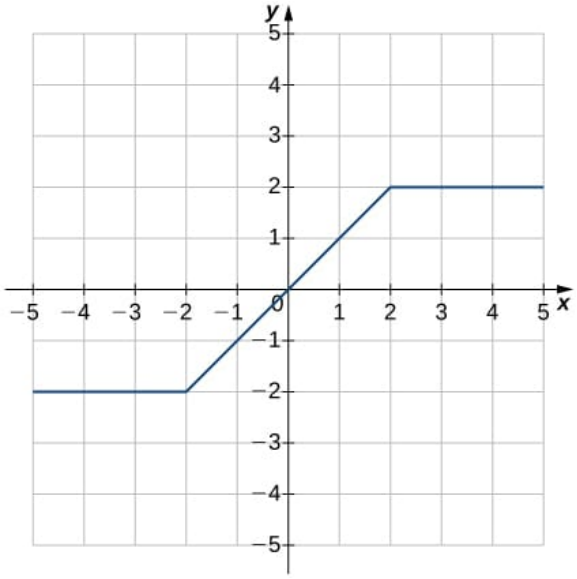
Answer

Function;
a. Domain: all real numbers, range: $-1.5 \leq y \leq 1.5$
b. $x = 0$
c. $y = 0$
d. all real numbers
e. None
f. Not constant
g. Origin
h. Odd

32)



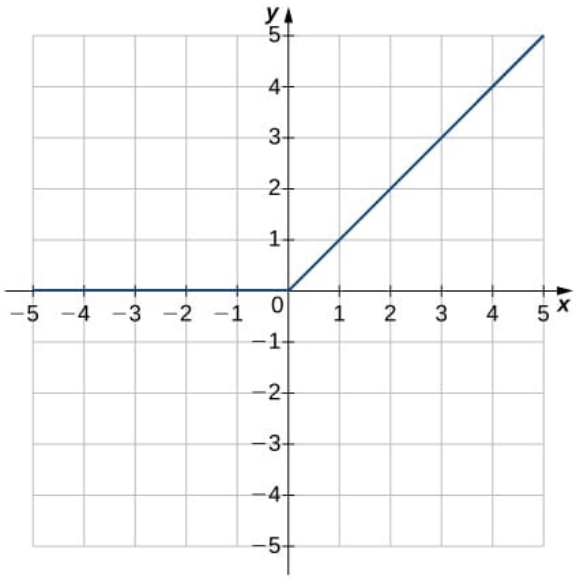
33)



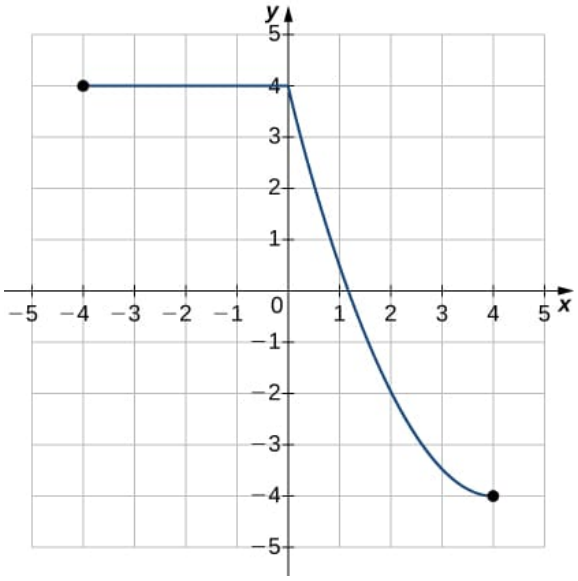
Answer

Function;
a. Domain: \(−∞ b. $x = 0$
c. $y = 0$
d. \(−2 e. Not decreasing
f. \(−∞ g. Origin
h. Odd

34)



35)



Answer

Function;
a. Domain: $-4 \leq x \leq 4$, range: $-4 \leq y \leq 4$
b. $x = 1.2$
c. $y = 4$
d. Not increasing
e. \(0 f. \(−4 g. No Symmetry
h. Neither

For exercises 36 - 41, for each pair of functions, find a. $f+g$ b. $f-g$ c. $f \cdot g$ d. f/g. Determine the domain of each of these new functions.

36) $f(x) = 3x + 4, \quad g(x) = x - 2$

37) $f(x) = x - 8, \quad g(x) = 5x^2$

Answer

a. $5x^2 + x - 8$; all real numbers
b. $-5x^2 + x - 8$; all real numbers
c. $5x^3 - 40x^2$; all real numbers
d. $\dfrac{x-8}{5x^2}; \quad x \neq 0$

38) $f(x) = 3x^2 + 4x + 1, \quad g(x) = x + 1$

39) $f(x) = 9 - x^2, \quad g(x) = x^2 - 2x - 3$

Answer

a. $-2x + 6$; all real numbers
b. $-2x^2 + 2x + 12$; all real numbers
c. $-x^4 + 2x^3 + 12x^2 - 18x - 27$; all real numbers
d. $-\dfrac{x+3}{x+1}; \quad x \neq -1, 3$

40) $f(x) = \sqrt{x}, \quad g(x) = x - 2$

41) $f(x) = 6 + \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

Answer

a. $6 + \dfrac{2}{x}; \quad x \neq 0$

b. $6; \quad x \neq 0$

c. $6x + \dfrac{1}{x^2}; \quad x \neq 0$

d. $6x + 1; \quad x \neq 0$

For exercises 42 - 48, for each pair of functions, find a. $(f \circ g)(x)$ and b. $(g \circ f)(x)$ Simplify the results. Find the domain of each of the results.

42) $f(x) = 3x, \quad g(x) = x + 5$

43) $f(x) = x + 4, \quad g(x) = 4x - 1$

Answer

a. $4x + 3$; all real numbers
b. $4x + 15$; all real numbers

44) $f(x) = 2x + 4, \quad g(x) = x^2 - 2$

45) $f(x) = x^2 + 7, \quad g(x) = x^2 - 3$

Answer

a. $x^4 - 6x^2 + 16$; all real numbers
b. $x^4 + 14x^2 + 46$; all real numbers

46) $f(x) = \sqrt{x}, \quad g(x) = x + 9$

47) $f(x) = \dfrac{3}{2x+1}, \quad g(x) = \dfrac{2}{x}$

Answer

a. $\dfrac{3x}{4+x}; \quad x \neq 0, -4$

b. $\frac{4x+2}{3}; \quad x \neq -\frac{1}{2}$

48) $f(x) = |x+1|, \quad g(x) = x^2 + x - 4$

49) The table below lists the NBA championship winners for the years 2001 to 2012.

Year	Winner
2001	LA Lakers
2002	LA Lakers
2003	Sam Antonio Spurs
2004	Detroit Pistons
2005	Sam Antonio Spurs
2006	Miami Heat
2007	Sam Antonio Spurs
2008	Boston Celtics
2009	LA Lakers
2010	LA Lakers
2011	Dallas Mavericks
2012	Miami Heat

a. Consider the relation in which the domain values are the years 2001 to 2012 and the range is the corresponding winner. Is this relation a function? Explain why or why not.

b. Consider the relation where the domain values are the winners and the range is the corresponding years. Is this relation a function? Explain why or why not.

Answer

a. Yes, because there is only one winner for each year.
b. No, because there are three teams that won more than once during the years 2001 to 2012.

50) [T] The area A of a square depends on the length of the side s.

a. Write a function $A(s)$ for the area of a square.

b. Find and interpret $A(6.5)$.

c. Find the exact and the two-significant-digit approximation to the length of the sides of a square with area 56 square units.

51) [T] The volume of a cube depends on the length of the sides s.

a. Write a function $V(s)$ for the area of a square.

b. Find and interpret $V(11.8)$.

Answer

a. $V(s) = s^3$
b. $V(11.8) \approx 1643$; a cube of side length 11.8 each has a volume of approximately 1643 cubic units.

52) **[T]** A rental car company rents cars for a flat fee of $20 and an hourly charge of $10.25. Therefore, the total cost C to rent a car is a function of the hours t the car is rented plus the flat fee.

a. Write the formula for the function that models this situation.

b. Find the total cost to rent a car for 2 days and 7 hours.

c. Determine how long the car was rented if the bill is $432.73.

53) [T] A vehicle has a 20-gal tank and gets 15 mpg. The number of miles N that can be driven depends on the amount of gas x in the tank.

a. Write a formula that models this situation.

b. Determine the number of miles the vehicle can travel on (i) a full tank of gas and (ii) 3/4 of a tank of gas.

c. Determine the domain and range of the function.

d. Determine how many times the driver had to stop for gas if she has driven a total of 578 mi.

Answer

a. $N(x) = 15x$
b. i. $N(20) = 15(20) = 300$; therefore, the vehicle can travel 300 mi on a full tank of gas.
ii. $N(15) = 225$; therefore, the vehicle can travel 225 mi on 3/4 of a tank of gas.
c. Domain: $0 \leq x \leq 20$; range: $[0, 300]$
d. The driver had to stop at least once, given that it takes approximately 39 gal of gas to drive a total of 578 mi.

54) [T] The volume V of a sphere depends on the length of its radius as $V = (4/3)\pi r^3$. Because Earth is not a perfect sphere, we can use the mean radius when measuring from the center to its surface. The mean radius is the average distance from the physical center to the surface, based on a large number of samples. Find the volume of Earth with mean radius 6.371×106 m.

55) [T] A certain bacterium grows in culture in a circular region. The radius of the circle, measured in centimeters, is given by $r(t) = 6 - \dfrac{5}{t^2+1}$, where t is time measured in hours since a circle of a 1-cm radius of the bacterium was put into the culture.

a. Express the area of the bacteria as a function of time.

b. Find the exact and approximate area of the bacterial culture in 3 hours.

c. Express the circumference of the bacteria as a function of time.

d. Find the exact and approximate circumference of the bacteria in 3 hours.

Answer

a. $A(t) = A(r(t)) = \pi \cdot (6 - \frac{5}{t^2+1})^2$
b. Exact: $\frac{121\pi}{4}$; approximately $95\ \text{cm}^2$
c. $C(t) = C(r(t)) = 2\pi(6 - \frac{5}{t^2+1})$
d. Exact: 11π; approximately 35 cm

56) [T] An American tourist visits Paris and must convert U.S. dollars to Euros, which can be done using the function $E(x) = 0.79x$, where x is the number of U.S. dollars and $E(x)$ is the equivalent number of Euros. Since conversion rates fluctuate, when the tourist returns to the United States 2 weeks later, the conversion from Euros to U.S. dollars is $D(x) = 1.245x$, where x is the number of Euros and $D(x)$ is the equivalent number of U.S. dollars.

a. Find the composite function that converts directly from U.S. dollars to U.S. dollars via Euros. Did this tourist lose value in the conversion process?

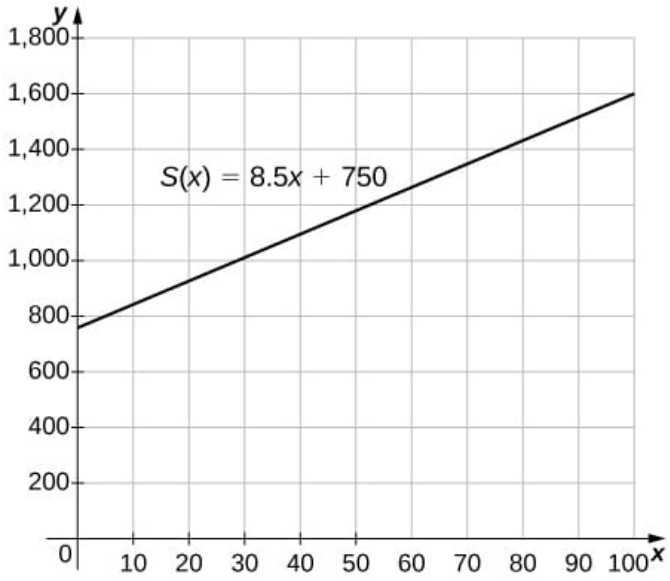
b. Use (a) to determine how many U.S. dollars the tourist would get back at the end of her trip if she converted an extra $200 when she arrived in Paris.

57) [T] The manager at a skateboard shop pays his workers a monthly salary S of $750 plus a commission of $8.50 for each skateboard they sell.

a. Write a function $y = S(x)$ that models a worker's monthly salary based on the number of skateboards x he or she sells.

b. Find the approximate monthly salary when a worker sells 25, 40, or 55 skateboards.

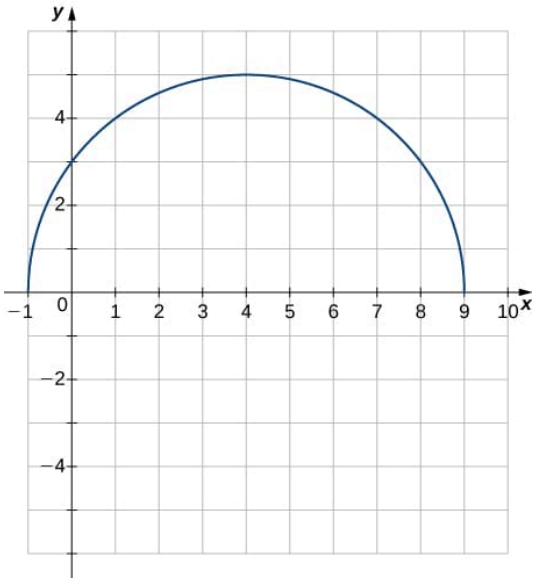
c. Use the INTERSECT feature on a graphing calculator to determine the number of skateboards that must be sold for a worker to earn a monthly income of $1400. (Hint: Find the intersection of the function and the line $y = 1400$.)



Answer

a. $S(x) = 8.5x + 750$ b. $962.50, $1090, $1217.50 c. 77 skateboards

58) [T] Use a graphing calculator to graph the half-circle $y = \sqrt{25 - (x-4)^2}$. Then, use the INTERCEPT feature to find the value of both the x- and y-intercepts.



Contributors

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12.3: Basic Classes of Functions

Learning Objectives

- Calculate the slope of a linear function and interpret its meaning.
- Recognize the degree of a polynomial.
- Find the roots of a quadratic polynomial.
- Describe the graphs of basic odd and even polynomial functions.
- Identify a rational function.
- Describe the graphs of power and root functions.
- Explain the difference between algebraic and transcendental functions.
- Graph a piecewise-defined function.
- Sketch the graph of a function that has been shifted, stretched, or reflected from its initial graph position.

We have studied the general characteristics of functions, so now let's examine some specific classes of functions. We begin by reviewing the basic properties of linear and quadratic functions, and then generalize to include higher-degree polynomials. By combining root functions with polynomials, we can define general algebraic functions and distinguish them from the transcendental functions we examine later in this chapter. We finish the section with examples of piecewise-defined functions and take a look at how to sketch the graph of a function that has been shifted, stretched, or reflected from its initial form.

Linear Functions and Slope

The easiest type of function to consider is a **linear function**. Linear functions have the form $f(x) = ax + b$, where a and b are constants. In Figure $12.3.1$, we see examples of linear functions when a is positive, negative, and zero. Note that if $a > 0$, the graph of the line rises as x increases. In other words, $f(x) = ax + b$ is increasing on $(-\infty, \infty)$. If $a < 0$, the graph of the line falls as x increases. In this case, $f(x) = ax + b$ is decreasing on $(-\infty, \infty)$. If $a = 0$, the line is horizontal.

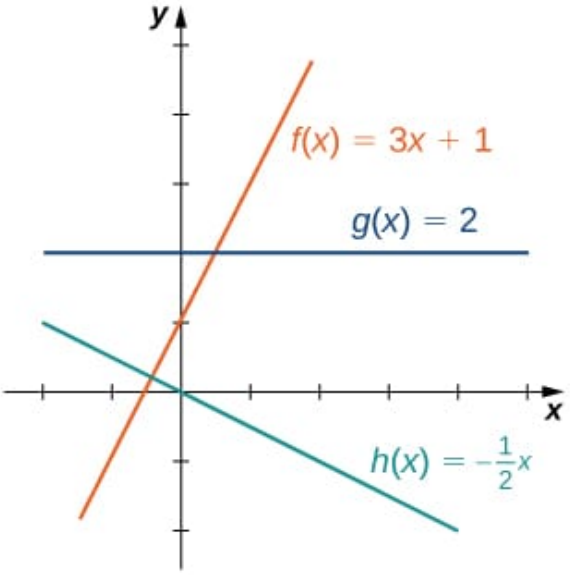


Figure $12.3.1$: These linear functions are increasing or decreasing on (∞, ∞) and one function is a horizontal line.

As suggested by Figure $12.3.1$, the graph of any linear function is a line. One of the distinguishing features of a line is its slope. The **slope** is the change in y for each unit change in x. The slope measures both the steepness and the direction of a line. If the slope is positive, the line points upward when moving from left to right. If the slope is negative, the line points downward when moving from left to right. If the slope is zero, the line is horizontal. To calculate the slope of a line, we need to determine the ratio of the change in y versus the change in x. To do so, we choose any two points (x_1, y_1) and (x_2, y_2) on the line and calculate

$\frac{y_2 - y_1}{x_2 - x_1}$. In Figure 12.3.2, we see this ratio is independent of the points chosen.

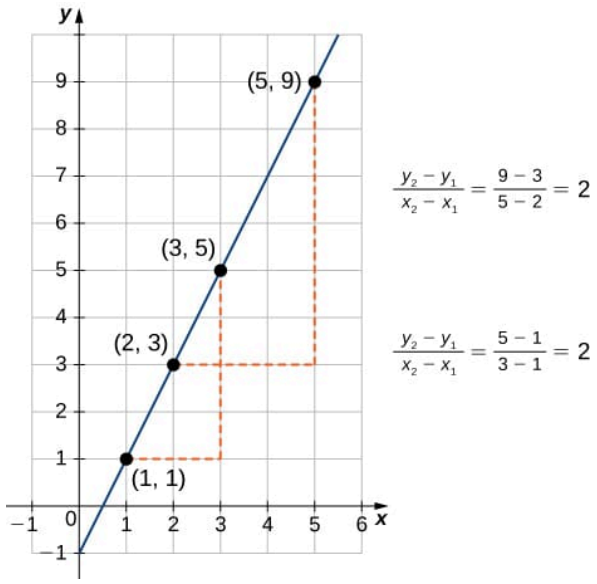


Figure 12.3.2: For any linear function, the slope $(y_2 - y_1)/(x_2 - x_1)$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.

Definition: Slope of a Linear Function

Consider line L passing through points (x_1, y_1) and (x_2, y_2). Let $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ denote the changes in y and x,respectively. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We now examine the relationship between slope and the formula for a linear function. Consider the linear function given by the formula $f(x) = ax + b$. As discussed earlier, we know the graph of a linear function is given by a line. We can use our definition of slope to calculate the slope of this line. As shown, we can determine the slope by calculating $(y_2 - y_1)/(x_2 - x_1)$ for any points (x_1, y_1) and (x_2, y_2) on the line. Evaluating the function f at $x = 0$, we see that $(0, b)$ is a point on this line. Evaluating this function at $x = 1$, we see that $(1, a + b)$ is also a point on this line. Therefore, the slope of this line is

$$\frac{(a + b) - b}{1 - 0} = a.$$

We have shown that the coefficient a is the slope of the line. We can conclude that the formula $f(x) = ax + b$ describes a line with slope a. Furthermore, because this line intersects the y-axis at the point $(0, b)$, we see that the y-intercept for this linear function is $(0, b)$. We conclude that the formula $f(x) = ax + b$ tells us the slope, a, and the y-intercept, $(0, b)$, for this line. Since we often use the symbol m to denote the slope of a line, we can write

$$\underbrace{f(x) = mx + b}_{\text{slope-intercept form}}$$

to denote the **slope-intercept form** of a linear function.

Sometimes it is convenient to express a linear function in different ways. For example, suppose the graph of a linear function passes through the point (x_1, y_1) and the slope of the line is m. Since any other point $(x, f(x))$ on the graph of f must satisfy the equation

$$m = \frac{f(x) - y_1}{x - x_1},$$

this linear function can be expressed by writing

$$\underbrace{f(x) - y_1 = m(x - x_1)}_{\text{point-slope equation}}.$$

We call this equation the **point-slope equation** for that linear function.

Since every nonvertical line is the graph of a linear function, the points on a nonvertical line can be described using the slope-intercept or point-slope equations. However, a vertical line does not represent the graph of a function and cannot be expressed in either of these forms. Instead, a vertical line is described by the equation $x = k$ for some constant k. Since neither the slope-intercept form nor the point-slope form allows for vertical lines, we use the notation

$$\underbrace{ax + by = c}_{\text{standard form}},$$

where a, b are both not zero, to denote the **standard form of a line.**

Definition: Point-Slope Equation, and the Slope-Intercept Form and Standard Form of the Equation of a Line

Consider a line passing through the point (x_1, y_1) with slope m. The equation

$$y - y_1 = m(x - x_1)$$

is the *point-slope equation* for that line.

Consider a line with slope m and y-intercept $(0, b)$. The equation

$$y = mx + b$$

is an equation for that line in *slope-intercept form*.

The *standard form of a line* is given by the equation

$$ax + by = c,$$

where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

Example 12.3.1: Finding the Slope and Equations of Lines

Consider the line passing through the points $(11, -4)$ and $(-4, 5)$, as shown in Figure 12.3.3.

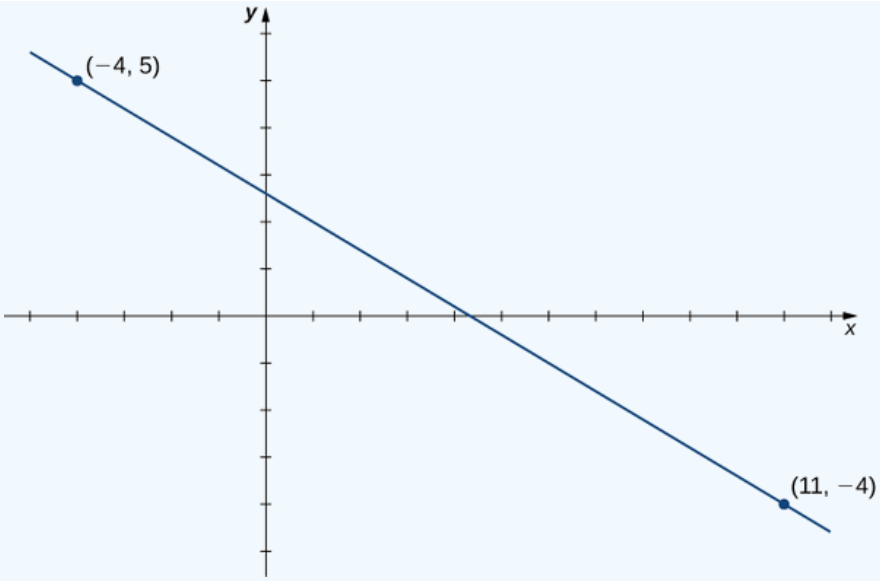


Figure 12.3.3: Finding the equation of a linear function with a graph that is a line between two given points.

1. Find the slope of the line.
2. Find an equation for this linear function in point-slope form.
3. Find an equation for this linear function in slope-intercept form.

Solution

1. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{-4 - 11} = -\frac{9}{15} = -\frac{3}{5}.$$

2. To find an equation for the linear function in point-slope form, use the slope $m = -3/5$ and choose any point on the line. If we choose the point $(11, -4)$, we get the equation

$$f(x) + 4 = -\frac{3}{5}(x - 11).$$

3. To find an equation for the linear function in slope-intercept form, solve the equation in part b. for $f(x)$. When we do this, we get the equation

$$f(x) = -\frac{3}{5}x + \frac{13}{5}.$$

Exercise 12.3.1

Consider the line passing through points $(-3, 2)$ and $(1, 4)$.

a. Find the slope of the line.
b. Find an equation of that line in point-slope form.
c. Find an equation of that line in slope-intercept form.

Hint available at the end of this section.

Answer a available at the end of this section.

Example 12.3.2:

Jessica leaves her house at 5:50 a.m. and goes for a 9-mile run. She returns to her house at 7:08 a.m. Answer the following questions, assuming Jessica runs at a constant pace.

a. Describe the distance D (in miles) Jessica runs as a linear function of her run time t (in minutes).
b. Sketch a graph of D.
c. Interpret the meaning of the slope.

Solution

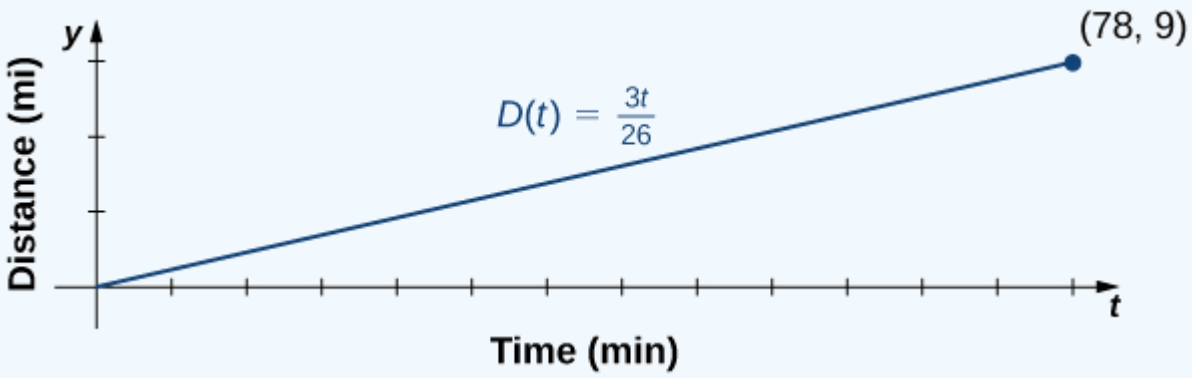
a. At time $t = 0$, Jessica is at her house, so $D(0) = 0$. At time $t = 78$ minutes, Jessica has finished running 9 mi, so $D(78) = 9$. The slope of the linear function is

$$m = \frac{9 - 0}{78 - 0} = \frac{3}{26}.$$

The y-intercept is $(0, 0)$, so the equation for this linear function is

$$D(t) = \frac{3}{26}t.$$

b. To graph D, use the fact that the graph passes through the origin and has slope $m = 3/26$.



c. The slope $m = 3/26 \approx 0.115$ describes the distance (in miles) Jessica runs per minute, or her average velocity.

Polynomials

A linear function is a special type of a more general class of functions: polynomials. A polynomial function is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

for some integer $n \geq 0$ and constants $a_n, a_{n-1}, \ldots, a_0$, where $a_n \neq 0$. In the case when $n = 0$, we allow for $a_0 = 0$; if $a_0 = 0$, the function $f(x) = 0$ is called the **zero function**. The value n is called the **degree** of the polynomial; the constant a_n is called the **leading coefficient**. A linear function of the form $f(x) = mx + b$ is a polynomial of degree 1 if $m \neq 0$ and degree 0 if $m = 0$. A polynomial of degree 0 is also called a **constant function**. A polynomial function of degree 2 is called a **quadratic function**. In particular, a quadratic function has the form

$$f(x) = ax^2 + bx + c,$$

where $a \neq 0$. A polynomial function of degree 3 is called a **cubic function**.

Power Functions

Some polynomial functions are power functions. A **power function** is any function of the form $f(x) = ax^b$, where a and b are any real numbers. The exponent in a power function can be any real number, but here we consider the case when the exponent is a positive integer. (We consider other cases later.) If the exponent is a positive integer, then $f(x) = ax^n$ is a polynomial. If n is even, then $f(x) = ax^n$ is an even function because $f(-x) = a(-x)^n = ax^n$ if n is even. If n is odd, then $f(x) = ax^n$ is an odd function because $f(-x) = a(-x)^n = -ax^n$ if n is odd (Figure 12.3.4).

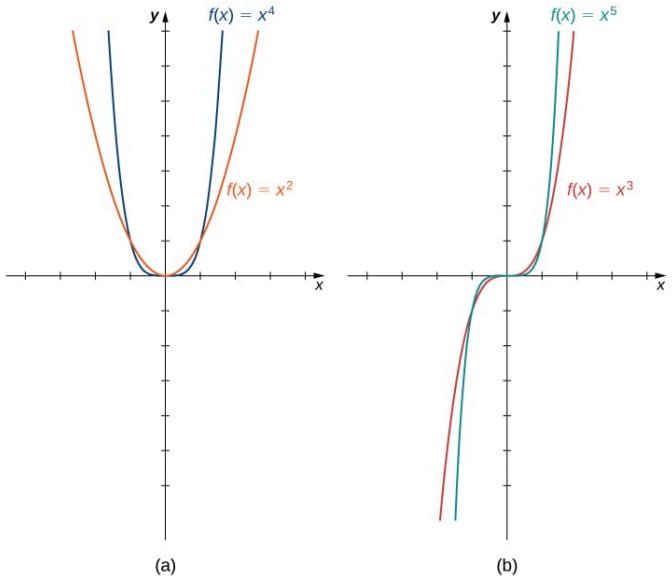


Figure 12.3.4: (a) For any even integer n,$f(x) = ax^n$ is an even function. (b) For any odd integer n,$f(x) = ax^n$ is an odd function.

Behavior at Infinity

To determine the behavior of a function f as the inputs approach infinity, we look at the values $f(x)$ as the inputs, x, become larger. For some functions, the values of $f(x)$ approach a finite number. For example, for the function $f(x) = 2 + 1/x$, the values $1/x$ become closer and closer to zero for all values of x as they get larger and larger. For this function, we say "$f(x)$ approaches two as x goes to infinity," and we write $f(x) \to 2$ as $x \to \infty$. The line $y = 2$ is a horizontal asymptote for the function $f(x) = 2 + 1/x$ because the graph of the function gets closer to the line as x gets larger.

For other functions, the values $f(x)$ may not approach a finite number but instead may become larger for all values of x as they get larger. In that case, we say "$f(x)$ approaches infinity as x approaches infinity," and we write $f(x) \to \infty$ as $x \to \infty$. For example, for the function $f(x) = 3x^2$, the outputs $f(x)$ become larger as the inputs x get larger. We can conclude that the function $f(x) = 3x^2$ approaches infinity as x approaches infinity, and we write $3x^2 \to \infty$ as $x \to \infty$. The behavior as $x \to -\infty$ and the meaning of $f(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$ can be defined similarly. We can describe what happens to the values of $f(x)$ as $x \to \infty$ and as $x \to -\infty$ as the **end behavior** of the function.

To understand the end behavior for polynomial functions, we can focus on quadratic and cubic functions. The behavior for higher-degree polynomials can be analyzed similarly. Consider a quadratic function $f(x) = ax^2 + bx + c$. If $a > 0$, the values $f(x) \to \infty$ as $x \to \pm\infty$. If $a < 0$, the values $f(x) \to -\infty$ as $x \to \pm\infty$. Since the graph of a quadratic function is a parabola, the parabola opens upward if $a > 0$.; the parabola opens downward if $a < 0$ (Figure 12.3.5a).

Now consider a cubic function $f(x) = ax^3 + bx^2 + cx + d$. If $a > 0$, then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. If $a < 0$, then $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. As we can see from both of these graphs, the leading term of the polynomial determines the end *behavior* (Figure 12.3.5b).

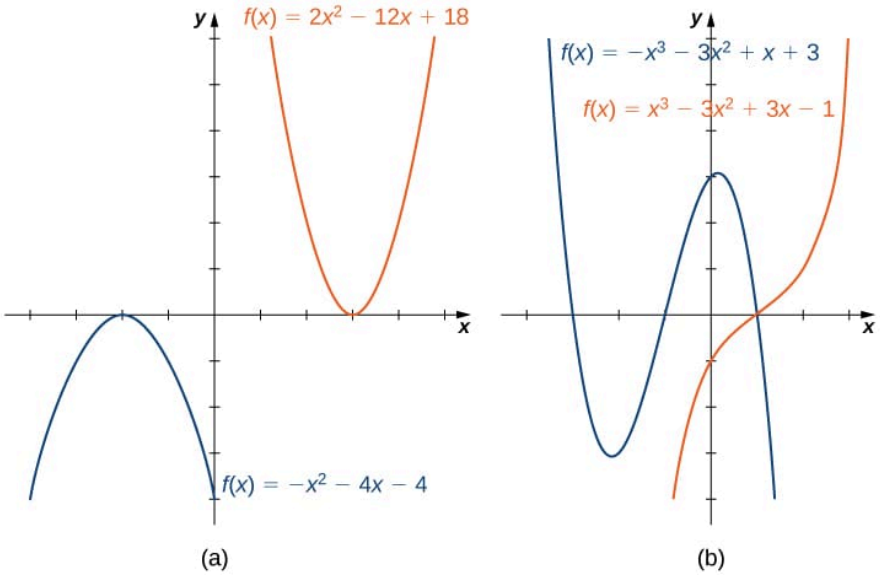


Figure 12.3.5: (a) For a quadratic function, if the leading coefficient $a > 0$,the parabola opens upward. If $a < 0$, the parabola opens downward. (b) For a cubic function f, if the leading coefficient $a > 0$, the values $f(x) \to \infty$ as $x \to \infty$ and the values $f(x) \to -\infty$ as $x \to -\infty$. If the leading coefficient $a < 0$, the opposite is true.

Zeros of Polynomial Functions

Another characteristic of the graph of a polynomial function is where it intersects the x-axis. To determine where a function f intersects the x-axis, we need to solve the equation $f(x) = 0$ for x. In the case of the linear function $f(x) = mx + b$, the x-intercept is given by solving the equation $mx + b = 0$. In this case, we see that the x-intercept is given by $(-b/m, 0)$. In the case of a quadratic function, finding the x-intercept(s) requires finding the zeros of a quadratic equation: $ax^2 + bx + c = 0$. In some cases, it is easy to factor the polynomial $ax^2 + bx + c$ to find the zeros. If not, we make use of the quadratic formula.

The Quadratic Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. The solutions of this equation are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{12.3.1}$$

If the discriminant $b^2 - 4ac > 0$, Equation 12.3.1 tells us there are two real numbers that satisfy the quadratic equation. If $b^2 - 4ac = 0$, this formula tells us there is only one solution, and it is a real number. If $b^2 - 4ac < 0$, no real numbers satisfy the quadratic equation.

In the case of higher-degree polynomials, it may be more complicated to determine where the graph intersects the x-axis. In some instances, it is possible to find the x-intercepts by factoring the polynomial to find its zeros. In other cases, it is impossible to calculate the exact values of the x-intercepts. However, as we see later in the text, in cases such as this, we can use analytical tools to approximate (to a very high degree) where the x-intercepts are located. Here we focus on the graphs of polynomials for which we can calculate their zeros explicitly.

Example 12.3.3: Graphing Polynomial Functions

For the following functions,

a. $f(x) = -2x^2 + 4x - 1$
b. $f(x) = x^3 - 3x^2 - 4x$

i. describe the behavior of $f(x)$ as $x \to \pm\infty$,
ii. find all zeros of f, and
iii. sketch a graph of f.

Solution

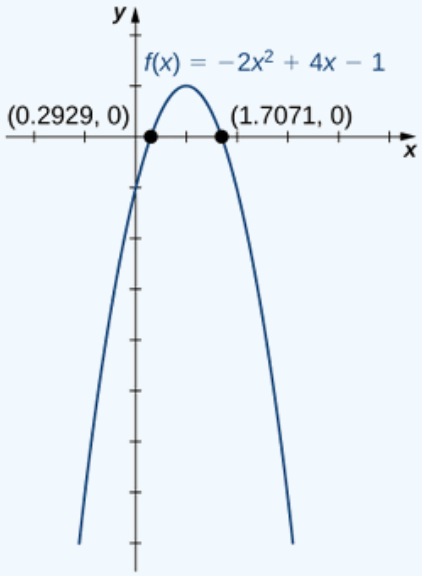
1. The function $f(x) = -2x^2 + 4x - 1$ is a quadratic function.

1. Because $a = -2 < 0$, as $x \to \pm\infty, f(x) \to -\infty$.

2. To find the zeros of f, use the quadratic formula. The zeros are

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-1)}}{2(-2)} = \frac{-4 \pm \sqrt{8}}{-4} = \frac{-4 \pm 2\sqrt{2}}{-4} = \frac{2 \pm \sqrt{2}}{2}.$$

3. To sketch the graph of f,use the information from your previous answers and combine it with the fact that the graph is a parabola opening downward.



2. The function $f(x) = x^3 - 3x^2 - 4x$ is a cubic function.

1. Because $a = 1 > 0$, as $x \to \infty$, $f(x) \to \infty$. As $x \to -\infty$, $f(x) \to -\infty$.

2. To find the zeros of f, we need to factor the polynomial. First, when we factor x out of all the terms, we find

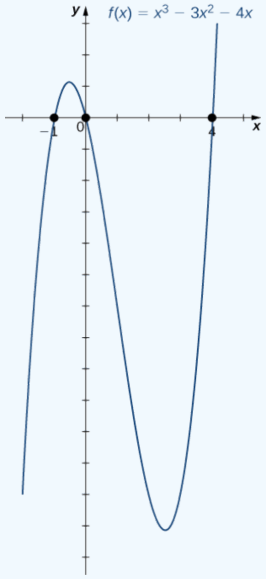
$$f(x) = x(x^2 - 3x - 4).$$

Then, when we factor the quadratic function $x^2 - 3x - 4$, we find

$$f(x) = x(x - 4)(x + 1).$$

Therefore, the zeros of f are $x = 0, 4, -1$.

3. Combining the results from parts i. and ii., draw a rough sketch of f.



Exercise 12.3.2

Consider the quadratic function $f(x) = 3x^2 - 6x + 2$. Find the zeros of f. Does the parabola open upward or downward?

Hint available at the end of this section.

Answer available at the end of this section.

Mathematical Models

A large variety of real-world situations can be described using **mathematical models**. A mathematical model is a method of simulating real-life situations with mathematical equations. Physicists, engineers, economists, and other researchers develop models by combining observation with quantitative data to develop equations, functions, graphs, and other mathematical tools to describe the behavior of various systems accurately. Models are useful because they help predict future outcomes. Examples of mathematical models include the study of population dynamics, investigations of weather patterns, and predictions of product sales.

As an example, let's consider a mathematical model that a company could use to describe its revenue for the sale of a particular item. The amount of revenue R a company receives for the sale of n items sold at a price of p dollars per item is described by the equation $R = p \cdot n$. The company is interested in how the sales change as the price of the item changes. Suppose the data in Table 12.3.1 show the number of units a company sells as a function of the price per item.

Table 12.3.1: Number of Units Sold n (in Thousands) as a Function of Price per Unit p (in Dollars)

p	6	8	10	12	14
n	19.4	18.5	16.2	13.8	12.2

In Figure 12.3.6, we see the graph the number of units sold (in thousands) as a function of price (in dollars). We note from the shape of the graph that the number of units sold is likely a linear function of price per item, and the data can be closely approximated by the linear function $n = -1.04p + 26$ for $0 \le p \le 25$, where n predicts the number of units sold in thousands. Using this linear function, the revenue (in thousands of dollars) can be estimated by the quadratic function

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p \text{ for } 0 \le p \le 25.$$

In Example 12.3.4, we use this quadratic function to predict the amount of revenue the company receives depending on the price

the company charges per item. Note that we cannot conclude definitively the actual number of units sold for values of p, for which no data are collected. However, given the other data values and the graph shown, it seems reasonable that the number of units sold (in thousands) if the price charged is p dollars may be close to the values predicted by the linear function $n = -1.04p + 26$.

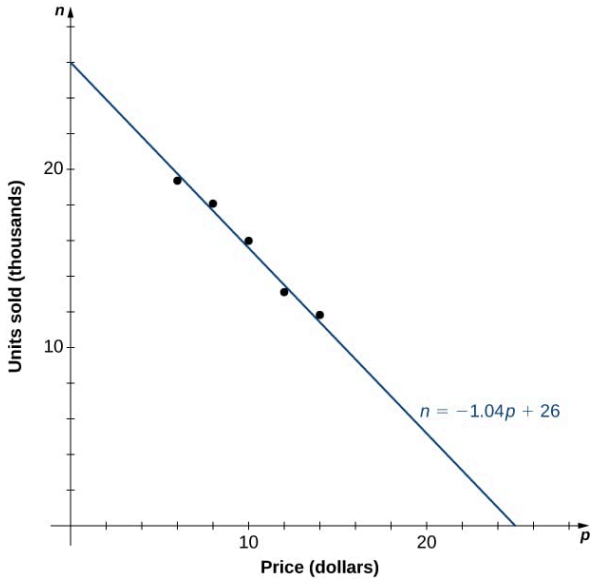


Figure 12.3.6: The data collected for the number of items sold as a function of price is roughly linear. We use the linear function $n = -1.04p + 26$ to estimate this function.

Example 12.3.4: Maximizing Revenue

A company is interested in predicting the amount of revenue it will receive depending on the price it charges for a particular item. Using the data from Table 12.3.1, the company arrives at the following quadratic function to model revenue R as a function of price per item p:

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \le p \le 25$.

a. Predict the revenue if the company sells the item at a price of $p = \$5$ and $p = \$17$.
b. Find the zeros of this function and interpret the meaning of the zeros.
c. Sketch a graph of R.
d. Use the graph to determine the value of p that maximizes revenue. Find the maximum revenue.

Solution

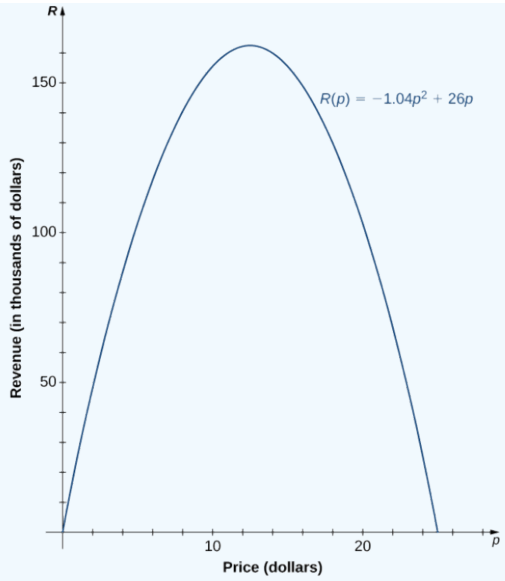
a. Evaluating the revenue function at $p = 5$ and $p = 17$, we can conclude that

$$R(5) = -1.04(5)^2 + 26(5) = 104, \text{ so revenue} = \$104,000;$$

$$R(17) = -1.04(17)^2 + 26(17) = 141.44, \text{ so revenue} = \$141,440.$$

b. The zeros of this function can be found by solving the equation $-1.04p^2 + 26p = 0$. When we factor the quadratic expression, we get $p(-1.04p + 26) = 0$. The solutions to this equation are given by $p = 0, 25$. For these values of p, the revenue is zero. When $p = \$0$, the revenue is zero because the company is giving away its merchandise for free. When $p = \$25$,the revenue is zero because the price is too high, and no one will buy any items.

c. Knowing the fact that the function is quadratic, we also know the graph is a parabola. Since the leading coefficient is negative, the parabola opens downward. One property of parabolas is that they are symmetric about the axis of symmetry, so since the zeros are at $p = 0$ and $p = 25$, the parabola must be symmetric about the line halfway between them, or $p = 12.5$.



d. The function is a parabola with zeros at $p = 0$ and $p = 25$, and it is symmetric about the line $p = 12.5$, so the maximum revenue occurs at a price of $p = \$12.50$ per item. At that price, the revenue is $R(p) = -1.04(12.5)^2 + 26(12.5) = \$162,500$.

Algebraic Functions

By allowing for quotients and fractional powers in polynomial functions, we create a larger class of functions. An **algebraic function** is one that involves addition, subtraction, multiplication, division, rational powers, and roots. Two types of algebraic functions are rational functions and root functions.

Just as rational numbers are quotients of integers, rational functions are quotients of polynomials. In particular, a **rational function** is any function of the form $f(x) = p(x)/q(x)$,where $p(x)$ and $q(x)$ are polynomials. For example,

$$f(x) = \frac{3x-1}{5x+2} \text{ and } g(x) = \frac{4}{x^2+1}$$

are rational functions. A **root function** is a power function of the form $f(x) = x^{1/n}$, where n is a positive integer greater than one. For example, $f(x) = x^{1/2} = \sqrt{x}$ is the square-root function and $g(x) = x^{1/3} = \sqrt[3]{x}$ is the cube-root function. By allowing for compositions of root functions and rational functions, we can create other algebraic functions. For example, $f(x) = \sqrt{4-x^2}$ is an algebraic function.

Example 12.3.5: Finding Domain and Range for Algebraic Functions

For each of the following functions, find the domain and range.

a. $f(x) = \dfrac{3x-1}{5x+2}$
b. $f(x) = \sqrt{4-x^2}$

Solution

1. It is not possible to divide by zero, so the domain is the set of real numbers x such that $x \neq -2/5$. To find the range, we need to find the values y for which there exists a real number x such that

$$y = \frac{3x-1}{5x+2}$$

When we multiply both sides of this equation by $5x + 2$, we see that x must satisfy the equation

$$5xy + 2y = 3x - 1.$$

From this equation, we can see that x must satisfy

$$2y + 1 = x(3 - 5y).$$

If y=$3/5$, this equation has no solution. On the other hand, as long as $y \neq 3/5$,

$$x = \frac{2y + 1}{3 - 5y}$$

satisfies this equation. We can conclude that the range of f is $\{y \mid y \neq 3/5\}$.

2. To find the domain of f, we need $4 - x^2 \geq 0$. When we factor, we write $4 - x^2 = (2 - x)(2 + x) \geq 0$. This inequality holds if and only if both terms are positive or both terms are negative. For both terms to be positive, we need to find x such that

$$2 - x \geq 0 \text{ and } 2 + x \geq 0.$$

These two inequalities reduce to $2 \geq x$ and $x \geq -2$. Therefore, the set $\{x \mid -2 \leq x \leq 2\}$ must be part of the domain. For both terms to be negative, we need

$$2 - x \leq 0 \text{ and } 2 + x \leq 0.$$

These two inequalities also reduce to $2 \leq x$ and $x \leq -2$. There are no values of x that satisfy both of these inequalities. Thus, we can conclude the domain of this function is $\{x \mid -2 \leq x \leq 2\}$.

If $-2 \leq x \leq 2$, then $0 \leq 4 - x^2 \leq 4$. Therefore, $0 \leq \sqrt{4 - x2} \leq 2$, and the range of f is $\{y \mid 0 \leq y \leq 2\}$.

Exercise 12.3.3

Find the domain and range for the function $f(x) = (5x + 2)/(2x - 1)$.

Hint available at the end of this section.

The root functions $f(x) = x^{1/n}$ have defining characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x) = x^{1/n}$ is the interval $[0, \infty)$. For all odd integers $n \geq 1$, the domain of $f(x) = x^{1/n}$ is the set of all real numbers. Since $x^{1/n} = (-x)^{1/n}$ for odd integers n,$f(x) = x^{1/n}$ is an odd function if$n$ is odd. See the graphs of root functions for different values of n in Figure 12.3.7.

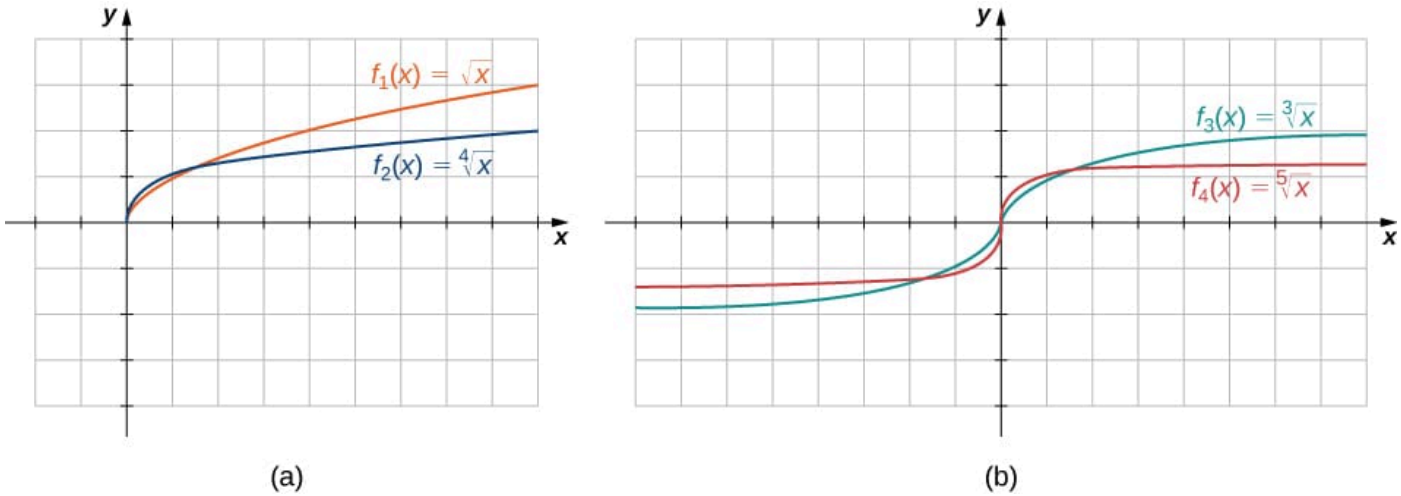


Figure 12.3.7: (a) If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$. (b) If n is odd, the domain of $f(x) = \sqrt[n]{x}$ is $(-\infty, \infty)$ and the function $f(x) = \sqrt[n]{x}$ is an odd function.

Example 12.3.6: Finding Domains for Algebraic Functions

For each of the following functions, determine the domain of the function.

a. $f(x) = \dfrac{3}{x^2 - 1}$
b. $f(x) = \dfrac{2x + 5}{3x^2 + 4}$
c. $f(x) = \sqrt{4 - 3x}$
d. $f(x) = \sqrt[3]{2x - 1}$

Solution

a. You cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $\{x \mid x \neq \pm 1\}$.
b. You need to determine the values of x for which the denominator is zero. Since $3x^2 + 4 \geq 4$ for all real numbers x, the denominator is never zero. Therefore, the domain is $(-\infty, \infty)$.
c. Since the square root of a negative number is not a real number, the domain is the set of values x for which $4 - 3x \geq 0$. Therefore, the domain is $\{x \mid x \leq 4/3\}$.
d. The cube root is defined for all real numbers, so the domain is the interval $(-\infty, \infty)$.

Exercise 12.3.4

Find the domain for each of the following functions: $f(x) = (5 - 2x)/(x^2 + 2)$ and $g(x) = \sqrt{5x - 1}$.

Hint available at the end of this section.

Answer available at the end of this section.

Transcendental Functions

Thus far, we have discussed algebraic functions. Some functions, however, cannot be described by basic algebraic operations. These functions are known as **transcendental functions** because they are said to "transcend," or go beyond, algebra. The most common transcendental functions are trigonometric, exponential, and logarithmic functions. A trigonometric function relates the ratios of two sides of a right triangle. They are $\sin x, \cos x, \tan x, \cot x, \sec x,$ and $\csc x$. (We discuss trigonometric functions later in the chapter.) An exponential function is a function of the form $f(x) = b^x$, where the base $b > 0,\ b \neq 1$. A logarithmic function is a function of the form $f(x) = \log_b(x)$ for some constant $b > 0,\ b \neq 1$, where $\log_b(x) = y$ if and only if $b^y = x$. (We also discuss exponential and logarithmic functions later in the chapter.)

Example 12.3.7: Classifying Algebraic and Transcendental Functions

Classify each of the following functions, a. through c., as algebraic or transcendental.

a. $f(x) = \dfrac{\sqrt{x^3 + 1}}{4x + 2}$
b. $f(x) = 2^{x^2}$
c. $f(x) = \sin(2x)$

Solution

a. Since this function involves basic algebraic operations only, it is an algebraic function.
b. This function cannot be written as a formula that involves only basic algebraic operations, so it is transcendental. (Note that algebraic functions can only have powers that are rational numbers.)

c. As in part b, this function cannot be written using a formula involving basic algebraic operations only; therefore, this function is transcendental.

Exercise 12.3.5:

Is $f(x) = x/2$ an algebraic or a transcendental function?

Answer available at the end of this section.

Piecewise-Defined Functions

Sometimes a function is defined by different formulas on different parts of its domain. A function with this property is known as a piecewise-defined function. The absolute value function is an example of a piecewise-defined function because the formula changes with the sign of x:

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}.$$

Other piecewise-defined functions may be represented by completely different formulas, depending on the part of the domain in which a point falls. To graph a piecewise-defined function, we graph each part of the function in its respective domain, on the same coordinate system. If the formula for a function is different for \(xa\), we need to pay special attention to what happens at $x = a$ when we graph the function. Sometimes the graph needs to include an open or closed circle to indicate the value of the function at $x = a$. We examine this in the next example.

Example 12.3.8: Graphing a Piecewise-Defined Function

Sketch a graph of the following piecewise-defined function:

$$f(x) = \begin{cases} x + 3, & \text{if } x < 1 \\ (x-2)^2, & \text{if } x \geq 1 \end{cases}$$

Solution

Graph the linear function $y = x + 3$ on the interval $(-\infty, 1)$ and graph the quadratic function $y = (x-2)^2$ on the interval $[1, \infty)$. Since the value of the function at $x = 1$ is given by the formula $f(x) = (x-2)^2$, we see that $f(1) = 1$. To indicate this on the graph, we draw a closed circle at the point $(1, 1)$. The value of the function is given by $f(x) = x + 3$ for all $x < 1$, but not at $x = 1$. To indicate this on the graph, we draw an open circle at $(1, 4)$.

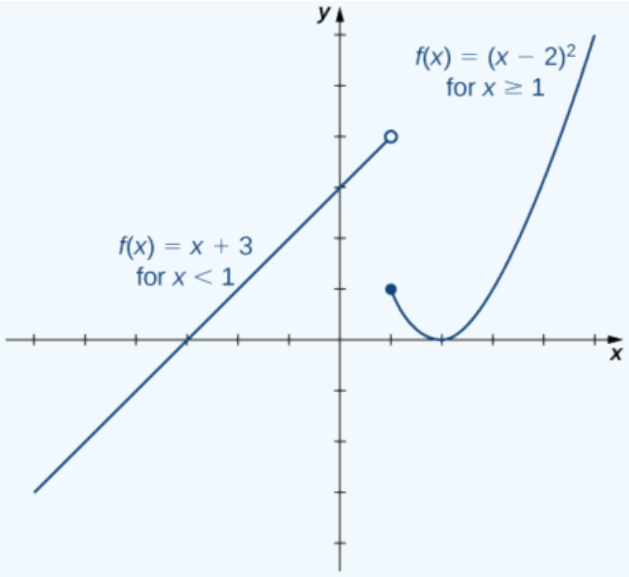
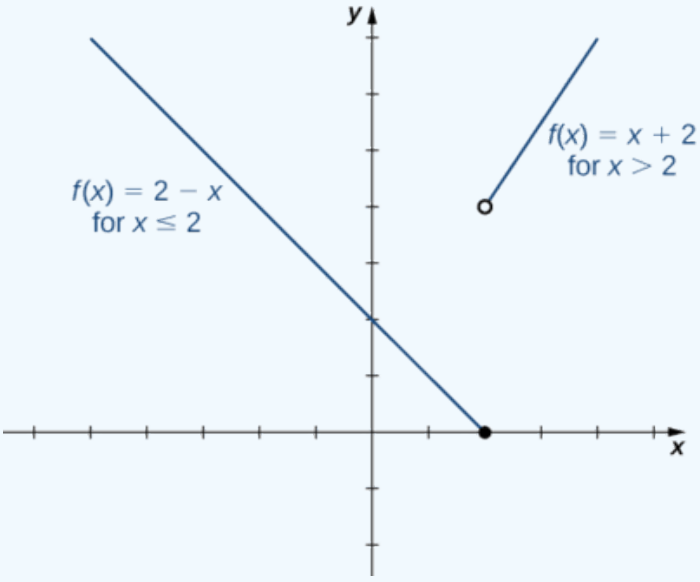


Figure 12.3.8: This piecewise-defined function is linear for $x < 1$ and quadratic for $x \geq 1$.

2) Sketch a graph of the function

$$f(x) = \begin{cases} 2 - x, & \text{if } x \leq 2 \\ x + 2, & \text{if } x > 2 \end{cases}.$$

Solution:



Example 12.3.9: Parking Fees Described by a Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged $10 for the first hour or any part of the first hour and an additional $2 for each hour or part thereof up to a maximum of $30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

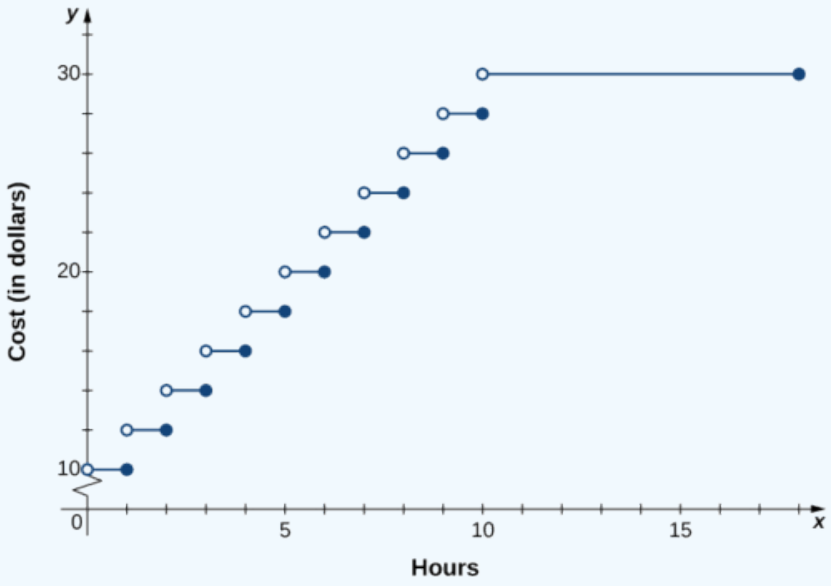
a. Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x.
b. Sketch a graph of this function $C(x)$.

Solution

1.Since the parking garage is open 18 hours each day, the domain for this function is \(\{x\,|\,0

$$C(x)=\begin{cases}10, & \text{for } 0$$

2.The graph of the function consists of several horizontal line segments.



Exercise 12.3.6

The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is $49¢$ for the first ounce and $21¢$ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for \(0

Hint available at the end of this section.

Answer available at the end of this section.

Transformations of Functions

We have seen several cases in which we have added, subtracted, or multiplied constants to form variations of simple functions. In the previous example, for instance, we subtracted 2 from the argument of the function $y = x^2$ to get the function $f(x) = (x - 2)^2$. This subtraction represents a shift of the function $y = x^2$ two units to the right. A shift, horizontally or vertically, is a type of **transformation of a function**. Other transformations include horizontal and vertical scalings, and reflections about the axes.

A vertical shift of a function occurs if we add or subtract the same constant to each output y. For $c > 0$, the graph of $f(x) + c$ is a shift of the graph of $f(x)$ up c units, whereas the graph of $f(x) - c$ is a shift of the graph of $f(x)$ down c units. For example, the graph of the function $f(x) = x^3 + 4$ is the graph of $y = x^3$ shifted up 4 units; the graph of the function $f(x) = x^3 - 4$ is the graph of $y = x^3$ shifted down 4 units (Figure $12.3.9$).

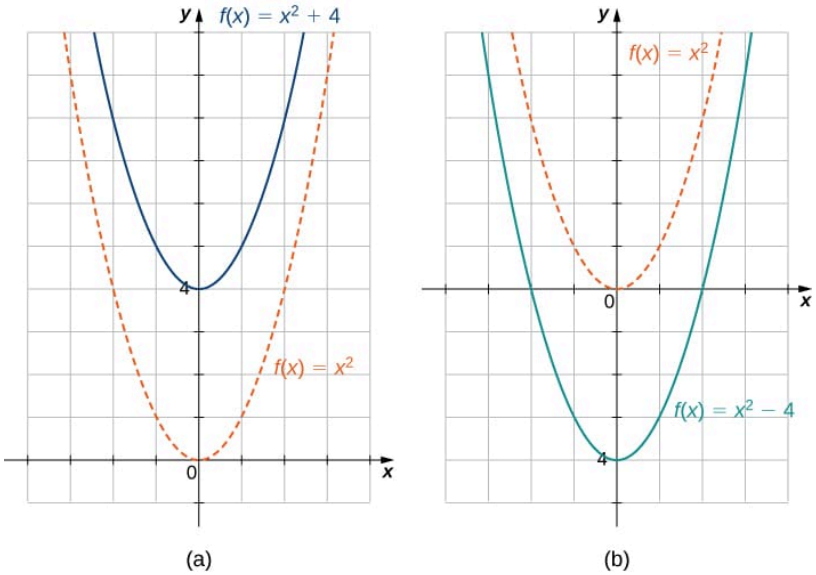


Figure 12.3.9: (a) For $c > 0$, the graph of $y = f(x) + c$ is a vertical shift up c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x) - c$ is a vertical shift down c units of the graph of $y = f(x)$.

A horizontal shift of a function occurs if we add or subtract the same constant to each input x. For $c > 0$, the graph of $f(x + c)$ is a shift of the graph of $f(x)$ to the left c units; the graph of $f(x - c)$ is a shift of the graph of $f(x)$ to the right c units. Why does the graph shift left when adding a constant and shift right when subtracting a constant? To answer this question, let's look at an example.

Consider the function $f(x) = |x + 3|$ and evaluate this function at $x - 3$. Since $f(x - 3) = |x|$ and \(x–3

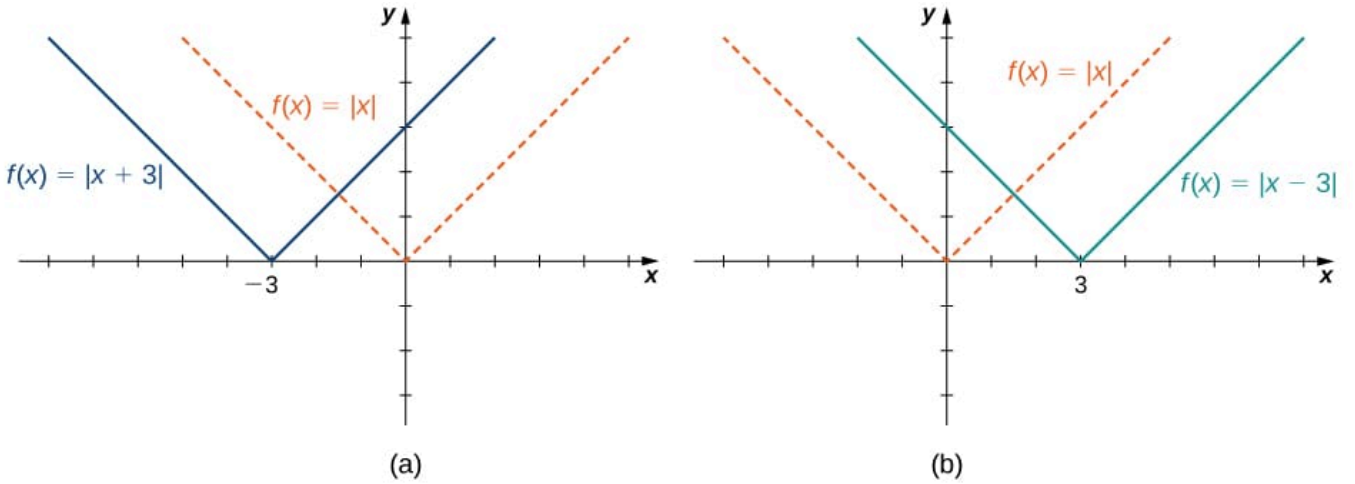


Figure 12.3.10: (a) For $c > 0$, the graph of $y = f(x + c)$ is a horizontal shift left c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x - c)$ is a horizontal shift right c units of the graph of $y = f(x)$.

A vertical scaling of a graph occurs if we multiply all outputs y of a function by the same positive constant. For $c > 0$, the graph of the function $cf(x)$ is the graph of $f(x)$ scaled vertically by a factor of c. If $c > 1$, the values of the outputs for the function $cf(x)$ are larger than the values of the outputs for the function $f(x)$; therefore, the graph has been stretched vertically. If \(0

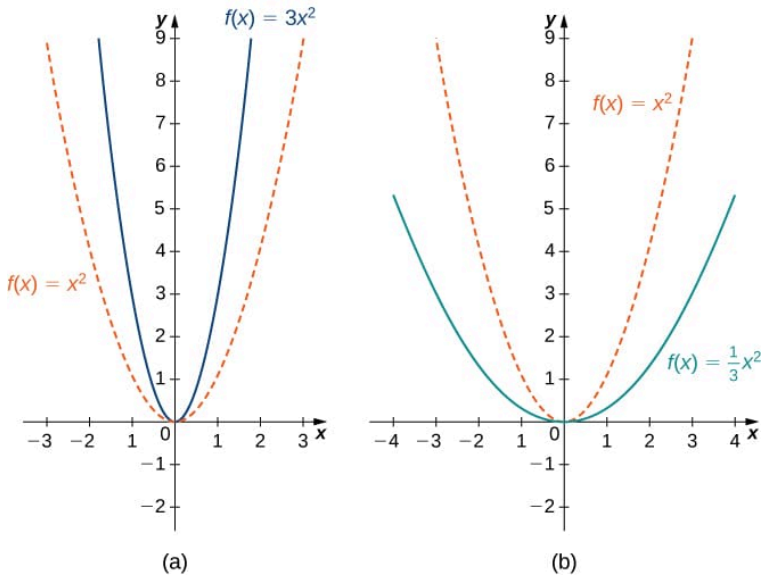


Figure 12.3.11: (a) If $c > 1$, the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$. (b) If \(0

The horizontal scaling of a function occurs if we multiply the inputs x by the same positive constant. For $c > 0$, the graph of the function $f(cx)$ is the graph of $f(x)$ scaled horizontally by a factor of c. If $c > 1$, the graph of $f(cx)$ is the graph of $f(x)$ compressed horizontally. If \(0

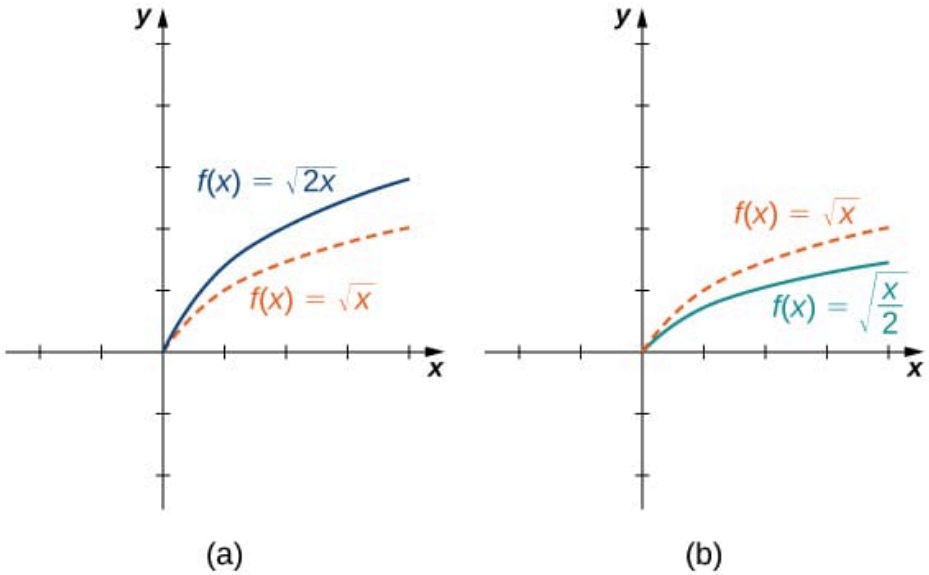


Figure 12.3.12: (a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$. (b) If \(0

We have explored what happens to the graph of a function f when we multiply f by a constant $c > 0$ to get a new function $cf(x)$. We have also discussed what happens to the graph of a function $f$when we multiply the independent variable x by $c > 0$ to get a new function $f(cx)$. However, we have not addressed what happens to the graph of the function if the constant c is negative. If we have a constant $c < 0$, we can write c as a positive number multiplied by -1; but, what kind of transformation do we get when we multiply the function or its argument by -1? When we multiply all the outputs by -1, we get a reflection about the x-axis. When we multiply all inputs by -1, we get a reflection about the y-axis. For example, the graph of $f(x) = -(x^3 + 1)$ is the graph of $y = (x^3 + 1)$ reflected about the x-axis. The graph of $f(x) = (-x)^3 + 1$ is the graph of $y = x^3 + 1$ reflected about the y-axis (Figure 12.3.13).

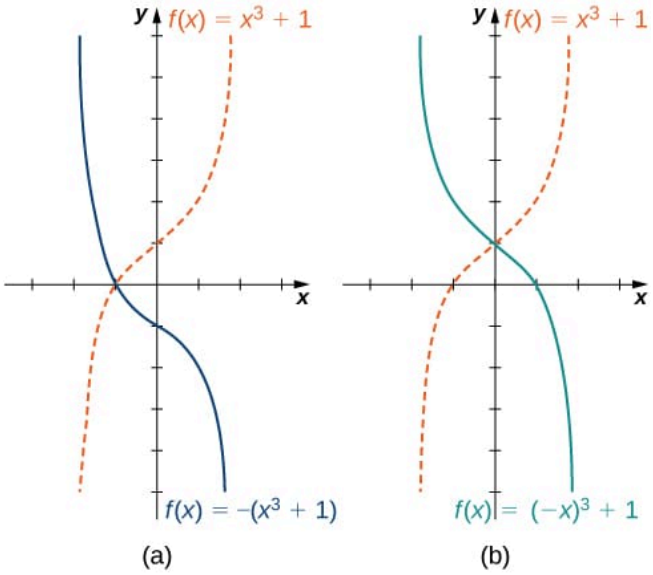


Figure 12.3.13: (a) The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis. (b) The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis.

If the graph of a function consists of more than one transformation of another graph, it is important to transform the graph in the correct order. Given a function $f(x)$, the graph of the related function $y = cf(a(x + b)) + d$ can be obtained from the graph of $y = f(x)$by performing the transformations in the following order.

- Horizontal shift of the graph of $y = f(x)$. If $b > 0$, shift left. If $b < 0$ shift right.
- Horizontal scaling of the graph of $y = f(x + b)$ by a factor of $|a|$. If $a < 0$, reflect the graph about the y-axis.
- Vertical scaling of the graph of $y = f(a(x + b))$ by a factor of $|c|$. If $c < 0$, reflect the graph about the x -axis.
- Vertical shift of the graph of $y = cf(a(x + b))$. If $d > 0$, shift up. If $d < 0$, shift down.

We can summarize the different transformations and their related effects on the graph of a function in the following table.

Transformation of $f(c > 0)$	Effect of the graph of f
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if \(0
$f(cx)$	Horizontal stretch if \(0 horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis

Example 12.3.10: Transforming a Function

For each of the following functions, a. and b., sketch a graph by using a sequence of transformations of a well-known function.

a. $f(x) = -|x + 2| - 3$
b. $f(x) = 3\sqrt{-x} + 1$

Solution

1.Starting with the graph of $y = |x|$, shift 2 units to the left, reflect about the x-axis, and then shift down 3 units.

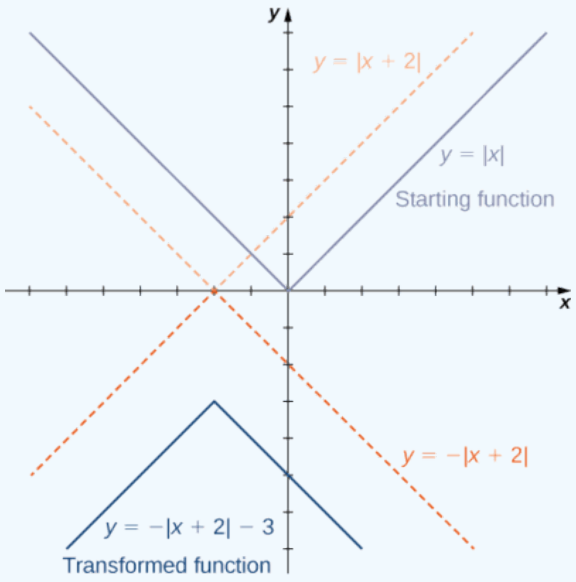


Figure 12.3.14: The function $f(x) = -|x + 2| - 3$ can be viewed as a sequence of three transformations of the function $y = |x|$.

2. Starting with the graph of $y = \sqrt{x}$, reflect about the y-axis, stretch the graph vertically by a factor of 3, and move up 1 unit.

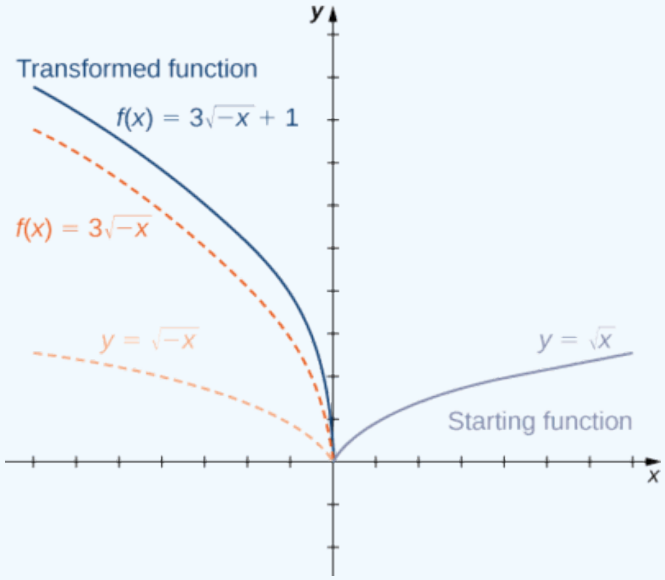


Figure 12.3.15: The function $f(x) = 3\sqrt{-x} + 1$can be viewed as a sequence of three transformations of the function $y = \sqrt{x}$.

Exercise 12.3.7

Describe how the function $f(x) = -(x+1)^2 - 4$ can be graphed using the graph of $y = x^2$ and a sequence of transformations

Answer available at the end of this section.

Key Concepts

- The power function $f(x) = x^n$ is an even function if n is even and $n \neq 0$, and it is an odd function if n is odd.
- The root function $f(x) = x^{1/n}$ has the domain $[0, \infty)$ if n is even and the domain $(-\infty, \infty)$ if n is odd. If n is odd, then $f(x) = x^{1/n}$ is an odd function.
- The domain of the rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomial functions, is the set of x such that $q(x) \neq 0$.
- Functions that involve the basic operations of addition, subtraction, multiplication, division, and powers are algebraic functions. All other functions are transcendental. Trigonometric, exponential, and logarithmic functions are examples of transcendental functions.
- A polynomial function f with degree $n \geq 1$ satisfies $f(x) \to \pm\infty$ as $x \to \pm\infty$. The sign of the output as $x \to \infty$ depends on the sign of the leading coefficient only and on whether n is even or odd.
- Vertical and horizontal shifts, vertical and horizontal scalings, and reflections about the x- and y-axes are examples of transformations of functions.

Key Equations

- **Point-slope equation of a line**

$$y - y_1 = m(x - x_1)$$

- **Slope-intercept form of a line**

$$y = mx + b$$

- **Standard form of a line**

$$ax + by = c$$

- **Polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Glossary

algebraic function
a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

cubic function
a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

degree
for a polynomial function, the value of the largest exponent of any term

linear function
a function that can be written in the form $f(x) = mx + b$

logarithmic function
a function of the form $f(x) = \log_b(x)$ for some base $b > 0,\ b \neq 1$ such that $y = \log_b(x)$ if and only if $b^y = x$

mathematical model
A method of simulating real-life situations with mathematical equations

piecewise-defined function
a function that is defined differently on different parts of its domain

point-slope equation
equation of a linear function indicating its slope and a point on the graph of the function

polynomial function
a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$

power function
a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

quadratic function
a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

rational function
a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

root function
a function of the form $f(x) = x^{1/n}$ for any integer $n \geq 2$

slope
the change in y for each unit change in x

slope-intercept form
equation of a linear function indicating its slope and y-intercept

transcendental function
a function that cannot be expressed by a combination of basic arithmetic operations

transformation of a function
a shift, scaling, or reflection of a function

Answers

Exercise 12.3.1
The slope $m = \Delta y/\Delta x$.

Exercise 12.3.1
$m = 1/2$.
The point-slope form is $y - 4 = \frac{1}{2}(x - 1)$.
The slope-intercept form is $y = \frac{1}{2}x + \frac{7}{2}$.

Exercise 12.3.2
Use the quadratic formula.

Exercise 12.3.2
The zeros are $x = 1 \pm \sqrt{3}/3$. The parabola opens upward.

Exercise 12.3.3
The denominator cannot be zero. Solve the equation $y = (5x + 2)/(2x - 1)$ for x to find the range.
The domain is the set of real numbers x such that $x \neq 1/2$. The range is the set $\{y \mid y \neq 5/2\}$.

Exercise 12.3.4
Determine the values of x when the expression in the denominator of f is nonzero, and find the values of x when the expression inside the radical of g is nonnegative.

Exercise 12.3.4
The domain of f is $(-\infty, \infty)$. The domain of g is $\{x \mid x \geq 1/5\}$.

Exercise 12.3.5**:**
Algebraic

Exercise 12.3.6
The piecewise-defined function is constant on the intervals $(0, 1],\ (1, 2],\ \ldots$.

Exercise 12.3.6
\[C(x)=\begin{cases}49, 0

Exercise 12.3.7

Shift the graph $y = x^2$ to the left 1 unit, reflect about the x-axis, then shift down 4 units.

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12.3E: Exercises for Section 1.2

In exercises 1 - 8, for each pair of points,

a. find the slope of the line passing through the points and

b. indicate whether the line is increasing, decreasing, horizontal, or vertical.

1) $(-2,4)$ and $(1,1)$

Answer

a. $m=-1$
b. Decreasing

2) $(-1,4)$ and $(3,-1)$

3) $(3,5)$ and $(-1,2)$

Answer

a. $m=3/4$
b. Increasing

4) $(6,4)$ and $(4,-3)$

5) $(2,3)$ and $(5,7)$

Answer

a. $m=4/3$
b. Increasing

6) $(1,9)$ and $(-8,5)$

7) $(2,4)$ and $(1,4)$

Answer

a. $m=0$
b. Horizontal

8) $(1,4)$ and $(1,0)$

In exercises 9 - 16, write the equation of the line satisfying the given conditions in slope-intercept form.

9) Slope $=-6$, passes through $(1,3)$

Answer

$y=-6x+9$

10) Slope $=3$, passes through $(-3,2)$

11) Slope $=\frac{1}{3}$, passes through $(0,4)$

Answer

$y=\frac{1}{3}x+4$

12) Slope $=\frac{2}{5}$, x-intercept $=8$

13) Passing through $(2,1)$ and $(-2,-1)$

Answer

$y = \frac{1}{2}x$

14) Passing through $(-3, 7)$ and $(1, 2)$

15) x-intercept $=5$ and y-intercept $=-3$

Answer

$y = \frac{3}{5}x - 3$

16) x-Intercept $=-6$ and y-intercept $=9$

In exercises 17 - 24, for each linear equation,

a. give the slope m and y-intercept b, if any, and

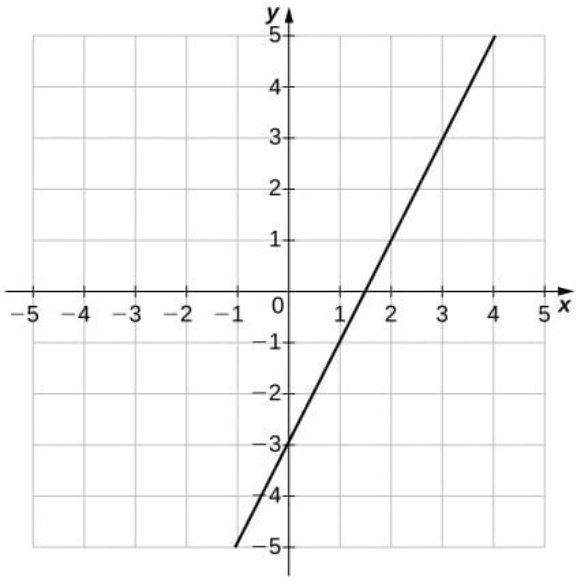
b. graph the line.

17) $y = 2x - 3$

Answer

a. $m = 2, \quad b = -3$

b.



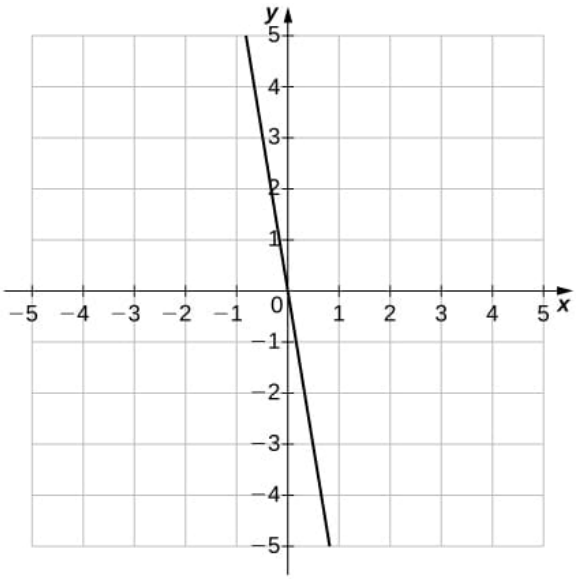
18) $y = -\frac{1}{7}x + 1$

19) $f(x) = -6x$

Answer

a. $m = -6, \quad b = 0$

b.



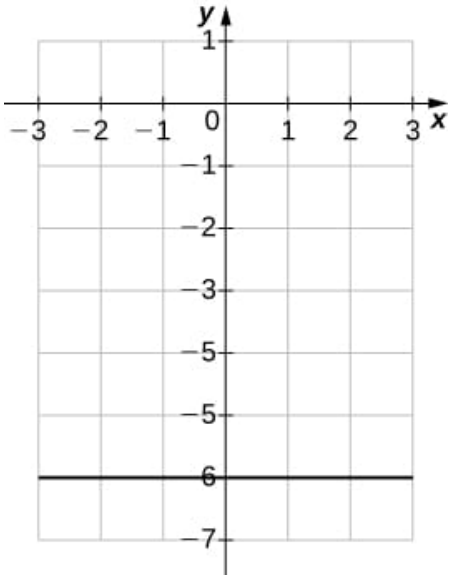
20) $f(x) = -5x + 4$

21) $4y + 24 = 0$

Answer

a. $m = 0, \quad b = -6$

b.



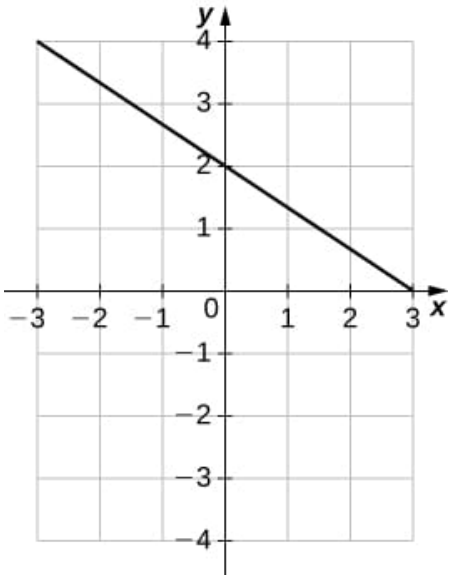
22) $8x - 4 = 0$

23) $2x + 3y = 6$

Answer

a. $m = -\frac{2}{3}, \quad b = 2$

b.



24) $6x - 5y + 15 = 0$

In exercises 25 - 29, for each polynomial,

a. find the degree;

b. find the zeros, if any;

c. find the y-intercept(s), if any;

d. use the leading coefficient to determine the graph's end behavior; and

e. determine algebraically whether the polynomial is even, odd, or neither.

25) $f(x) = 2x^2 - 3x - 5$

Answer

a. 2
b. $\frac{5}{2},\ -1$;
c. -5
d. Both ends rise
e. Neither

26) $f(x) = -3x^2 + 6x$

27) $f(x) = \frac{1}{2}x^2 - 1$

Answer

a. 2
b. $\pm\sqrt{2}$
c. -1
d. Both ends rise
e. Even

28) $f(x) = x^3 + 3x^2 - x - 3$

29) $f(x) = 3x - x^3$

Answer

a. 3

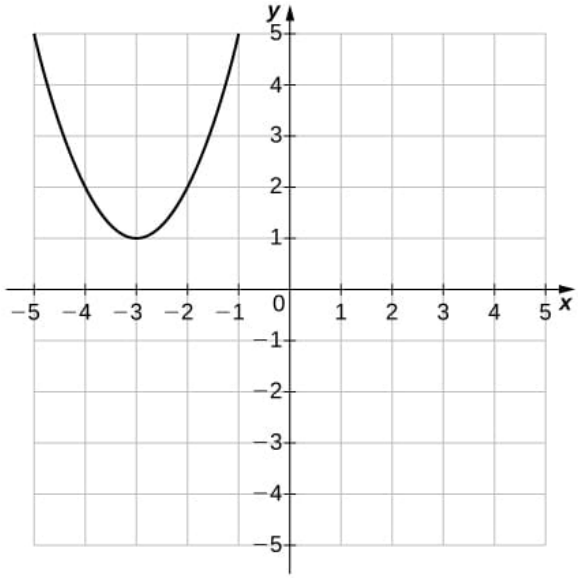
b. $0, \pm\sqrt{3}$
c. 0
d. Left end rises, right end falls
e. Odd

For exercises 30 - 31, use the graph of $f(x) = x^2$ to graph each transformed function g.

30) $g(x) = x^2 - 1$

31) $g(x) = (x + 3)^2 + 1$

Answer

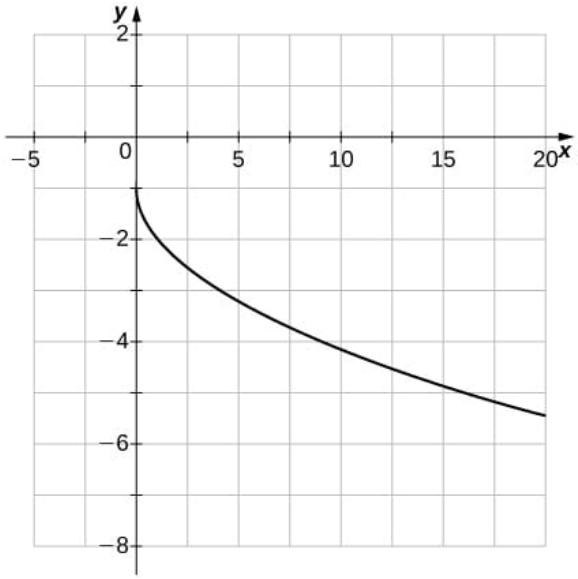


For exercises 32 - 33, use the graph of $f(x) = \sqrt{x}$ to graph each transformed function g.

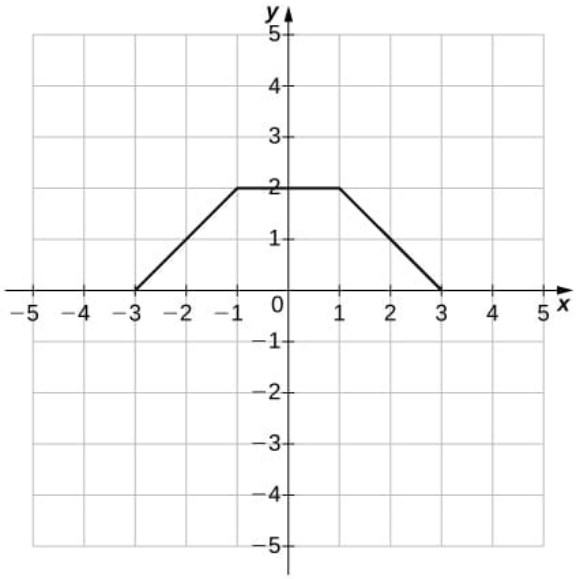
32) $g(x) = \sqrt{x + 2}$

33) $g(x) = -\sqrt{x} - 1$

Answer



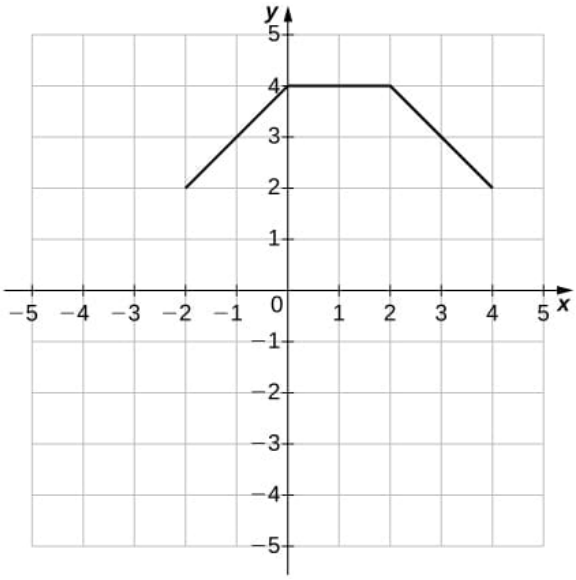
For exercises 34 - 35, use the graph of $y = f(x)$ to graph each transformed function g.



34) $g(x) = f(x) + 1$

35) $g(x) = f(x - 1) + 2$

Answer



In exercises 36 - 39, for each of the piecewise-defined functions,

a. evaluate at the given values of the independent variable, and

b. sketch the graph.

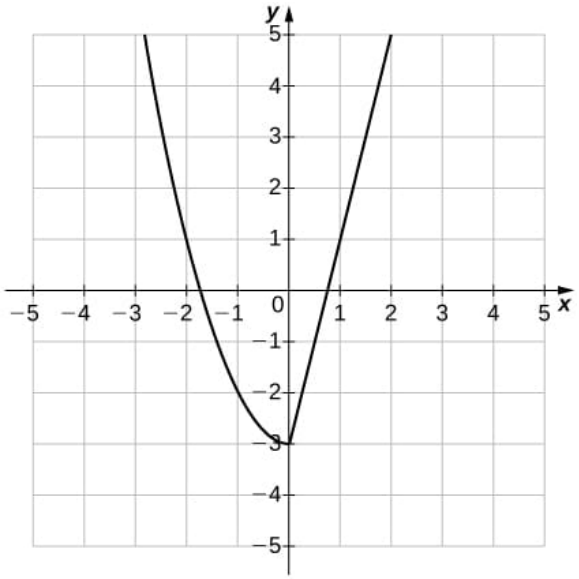
36) $f(x) = \begin{cases} 4x + 3, & \text{if } x \le 0 \\ -x + 1, & \text{if } x > 0 \end{cases}$; $f(-3)$; $f(0)$; $f(2)$

37) $f(x) = \begin{cases} x^2 - 3, & \text{if } x \le 0 \\ 4x - 3, & \text{if } x > 0 \end{cases}$; $f(-4)$; $f(0)$; $f(2)$

Answer

a. $f(-4) = 13, \quad f(0) = -3, \quad f(2) = 5$

b.



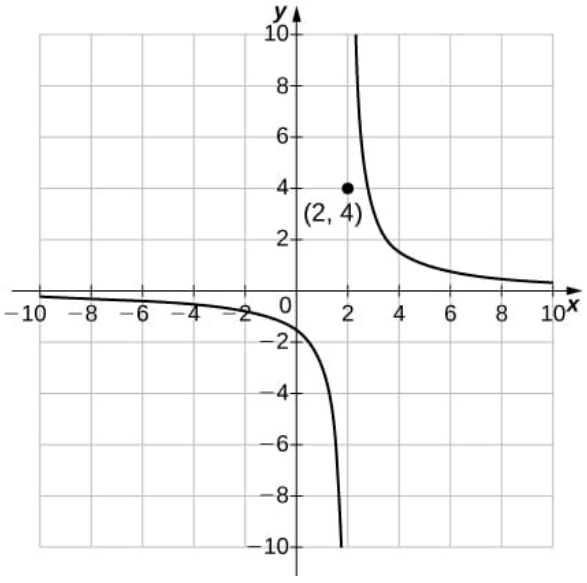
38) $h(x) = \begin{cases} x+1, & \text{if } x \le 5 \\ 4, & \text{if } x > 5 \end{cases}$; $\quad h(0); \;\; h(\pi); \;\; h(5)$

39) $g(x) = \begin{cases} \dfrac{3}{x-2}, & \text{if } x \neq 2 \\ 4, & \text{if } x = 2 \end{cases}$; $\quad g(0); \;\; g(-4); \;\; g(2)$

Answer

a. $g(0) = -\frac{3}{2}, \;\; g(-4) = -\frac{1}{2}, \;\; g(2) = 4$

b.



In exercises 40 - 44, determine whether the statement is *true or false*. Explain why.

40) $f(x) = \dfrac{4x+1}{7x-2}$ is a transcendental function.

41) $g(x) = \sqrt[3]{x}$ is an odd root function.

Answer

True; $n = 3$

42) A logarithmic function is an algebraic function.

43) A function of the form $f(x) = x^b$, where b is a real valued constant, is an exponential function.

Answer

False; $f(x) = x^b$, where b is a real-valued constant, is a power function

44) The domain of an even root function is all real numbers.

45) [T] A company purchases some computer equipment for $20,500. At the end of a 3-year period, the value of the equipment has decreased linearly to $12,301.

a. Find a function $y = V(t)$ that determines the value V of the equipment at the end of t years.

b. Find and interpret the meaning of the x- and y-intercepts for this situation.

c. What is the value of the equipment at the end of 5 years?

d. When will the value of the equipment be $3000?

Answer

a. $V(t) = -2733t + 20500$
b. $(0, 20, 500)$ means that the initial purchase price of the equipment is $20,500; $(7.5, 0)$ means that in 7.5 years the computer equipment has no value.
c. $6835
d. In approximately 6.4 years

46) [T] Total online shopping during the Christmas holidays has increased dramatically during the past 5 years. In 2012 $(t = 0)$,total online holiday sales were $42.3 billion, whereas in 2013 they were $48.1 billion.

a. Find a linear function S that estimates the total online holiday sales in the year t.

b. Interpret the slope of the graph of S.

c. Use part a. to predict the year when online shopping during Christmas will reach $60 billion.

47) [T] A family bakery makes cupcakes and sells them at local outdoor festivals. For a music festival, there is a fixed cost of $125 to set up a cupcake stand. The owner estimates that it costs $0.75 to make each cupcake. The owner is interested in determining the total cost C as a function of number of cupcakes made.

a. Find a linear function that relates cost C to x, the number of cupcakes made.

b. Find the cost to bake 160 cupcakes.

c. If the owner sells the cupcakes for $1.50 apiece, how many cupcakes does she need to sell to start making profit? (Hint: Use the INTERSECTION function on a calculator to find this number.)

Answer

a. $C = 0.75x + 125$
b. $245
c. 167 cupcakes

48) [T] A house purchased for $250,000 is expected to be worth twice its purchase price in 18 years.

a. Find a linear function that models the price P of the house versus the number of years t since the original purchase.

b. Interpret the slope of the graph of P.

c. Find the price of the house 15 years from when it was originally purchased.

49) [T] A car was purchased for $26,000. The value of the car depreciates by $1500 per year.

a. Find a linear function that models the value V of the car after t years.

b. Find and interpret $V(4)$.

Answer

a. $V(t) = -1500t + 26,000$
b. In 4 years, the value of the car is $20,000.

50) [T] A condominium in an upscale part of the city was purchased for $432,000. In 35 years it is worth $60,500. Find the rate of depreciation.

51) [T] The total cost C (in thousands of dollars) to produce a certain item is modeled by the function $C(x) = 10.50x + 28,500$, where x is the number of items produced. Determine the cost to produce 175 items.

Answer

$30,337,500

52) [T] A professor asks her class to report the amount of time t they spent writing two assignments. Most students report that it takes them about 45 minutes to type a four-page assignment and about 1.5 hours to type a nine-page assignment.

a. Find the linear function $y = N(t)$ that models this situation, where N is the number of pages typed and t is the time in minutes.

b. Use part a. to determine how many pages can be typed in 2 hours.

c. Use part a. to determine how long it takes to type a 20-page assignment.

53) [T] The output (as a percent of total capacity) of nuclear power plants in the United States can be modeled by the function $P(t) = 1.8576t + 68.052$, where t is time in years and $t = 0$ corresponds to the beginning of 2000. Use the model to predict the percentage output in 2015.

Answer

96% of the total capacity

54) [T] The admissions office at a public university estimates that 65% of the students offered admission to the class of 2019 will actually enroll.

a. Find the linear function $y = N(x)$, where N is the number of students that actually enroll and x is the number of all students offered admission to the class of 2019.

b. If the university wants the 2019 freshman class size to be 1350, determine how many students should be admitted.

Contributors

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12.4: Trigonometric Functions

Learning Objectives

- Convert angle measures between degrees and radians.
- Recognize the triangular and circular definitions of the basic trigonometric functions.
- Write the basic trigonometric identities.
- Identify the graphs and periods of the trigonometric functions.
- Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the unit circle, a circle with radius 1. The radian measure of an angle is defined as follows. Given an angle θ, let s be the length of the corresponding arc on the unit circle (Figure $12.4.1$). We say the angle corresponding to the arc of length 1 has radian measure 1.

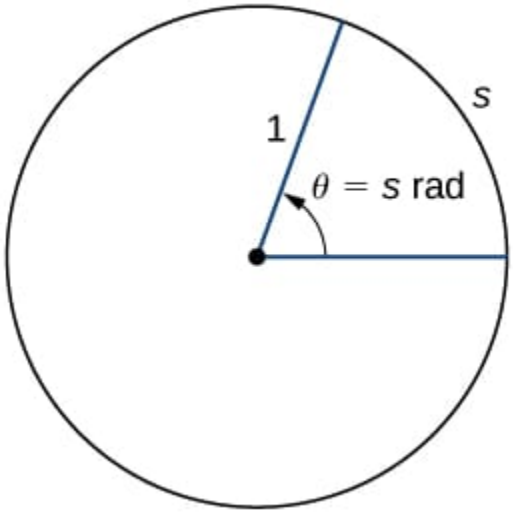


Figure $12.4.1$: The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of $360\,^\circ$ corresponds to the circumference of a circle, or an arc of length 2π, we conclude that an angle with a degree measure of $360\,^\circ$ has a radian measure of 2π. Similarly, we see that $180\,^\circ$ is equivalent to π radians. Table $12.4.1$ shows the relationship between common degree and radian values.

Table $12.4.1$: Common Angles Expressed in Degrees and Radians

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$
60	$\pi/3$	180	π

Degrees	Radians	Degrees	Radians
90	$\pi/2$		

Converting between Radians and Degrees

a. Express $225\,^\circ$ using radians.
b. Express $5\pi/3$ rad using degrees.

Solution

Use the fact that $180°$ is equivalent to π radians as a **conversion factor** (Table $12.4.1$):

$$1 = \frac{\pi \text{ rad}}{180°} = \frac{180°}{\pi \text{ rad}}.$$

a. $225\,^\circ = 225\,^\circ \cdot \left(\frac{\pi}{180°}\right) = \left(\frac{5\pi}{4}\right)$ rad
b. $\frac{5\pi}{3}$ rad = $\frac{5\pi}{3} \cdot \frac{180°}{\pi}$=$300°$

Exercise $12.4.1$

a. Express $210\,^\circ$ using radians.
b. Express $11\pi/6$ rad using degrees.

Hint available at the end of this section.

Answer available at the end of this section.

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle—not only on a unit circle—or to find an angle given a point on a circle. They also define the relationship between the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x-axis and with a terminal side that is the line segment OP. An angle in this position is said to be in standard position (Figure $12.4.2$). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y.

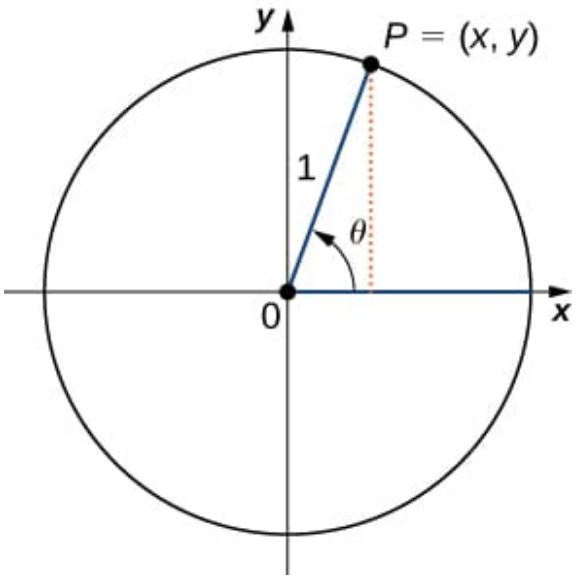


Figure 12.4.2: The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y.

Definition: Trigonometric functions

Let $P = (x, y)$ be a point on the unit circle centered at the origin O. Let θ be an angle with an initial side along the positive x -axis and a terminal side given by the line segment OP. The trigonometric functions are then defined as

$\sin\theta = y$	$\csc\theta = \dfrac{1}{y}$
$\cos\theta = x$	$\sec\theta = \dfrac{1}{x}$
$\tan\theta = \dfrac{y}{x}$	$\cot\theta = \dfrac{x}{y}$

If $x = 0, \sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ, the coordinates x and y satisfy

$$\cos\theta = \frac{x}{r} \tag{12.4.1}$$

$$x = r\cos\theta \tag{12.4.2}$$

and

$$\sin\theta = \frac{y}{r} \tag{12.4.3}$$

$$y = r\sin\theta. \tag{12.4.4}$$

The values of the other trigonometric functions can be expressed in terms of x, y, and r (Figure 12.4.3).

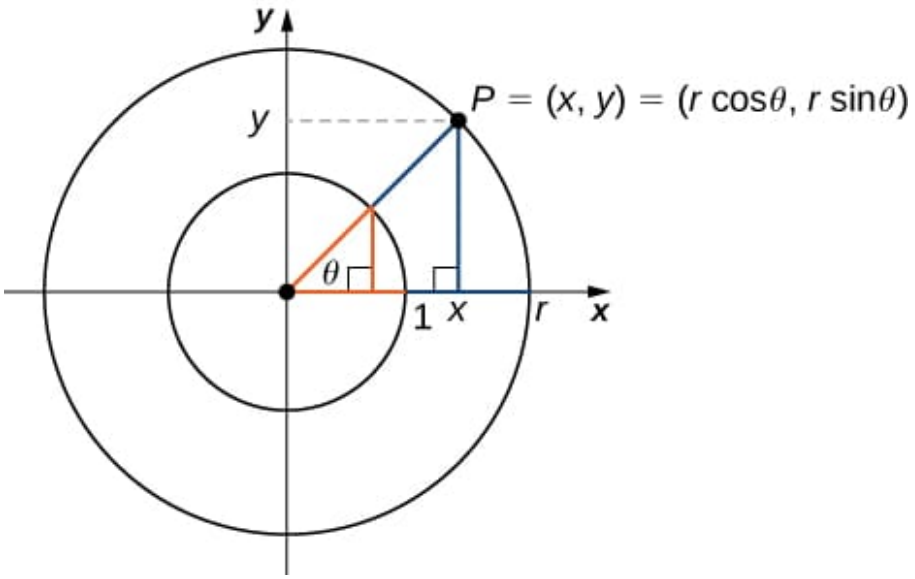


Figure 12.4.3: For a point $P = (x, y)$ on a circle of radius r, the coordinates x and y satisfy $x = r\cos\theta$ and $y = r\sin\theta$.

Table 12.4.2 shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin\theta$ and $\cos\theta$.

Table 12.4.2: Values of $\sin\theta$ and $\cos\theta$ at Major Angles θ in the First Quadrant

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Example 12.4.2: Evaluating Trigonometric Functions

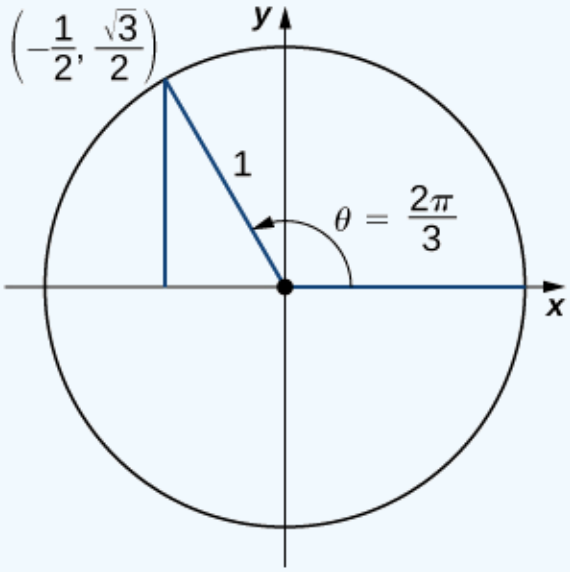
Evaluate each of the following expressions.

a. $\sin\left(\frac{2\pi}{3}\right)$
b. $\cos\left(-\frac{5\pi}{6}\right)$
c. $\tan\left(\frac{15\pi}{4}\right)$

Solution:

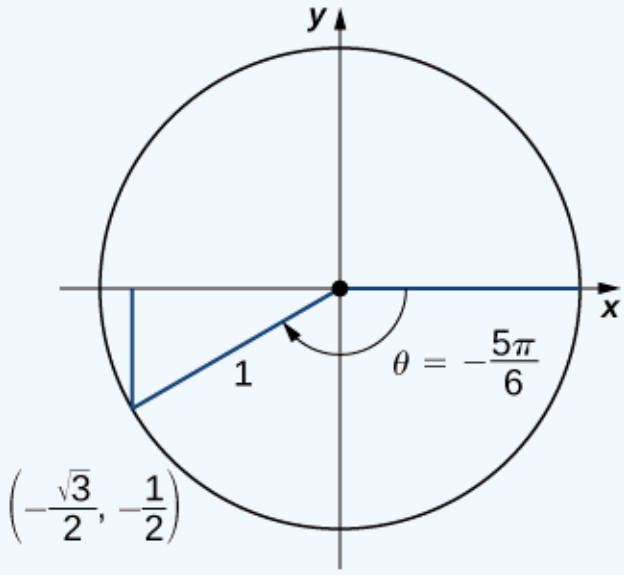
a) On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore,

$$\sin\left(\frac{2\pi}{3}\right) = y = \left(\frac{\sqrt{3}}{2}\right).$$



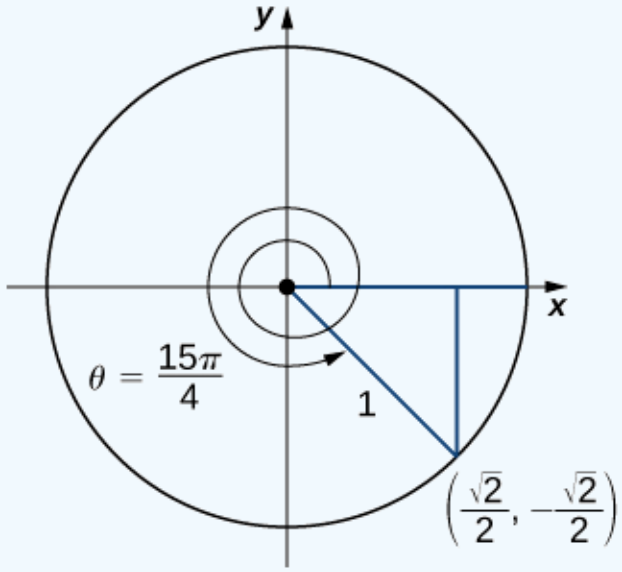
b) An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore,

$$\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}.$$



c) An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, we can conclude that

$$\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1.$$



Exercise 12.4.2

Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

Hint available at the end of this section.

Answer available at the end of this section.

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H, as shown in Figure 12.4.4, we see that A, H, and O satisfy the following relationships with θ:

$\sin\theta = \frac{O}{H}$	$\csc\theta = \frac{H}{O}$
$\cos\theta = \frac{A}{H}$	$\sec\theta = \frac{H}{A}$
$\tan\theta = \frac{O}{A}$	$\cot\theta = \frac{A}{O}$

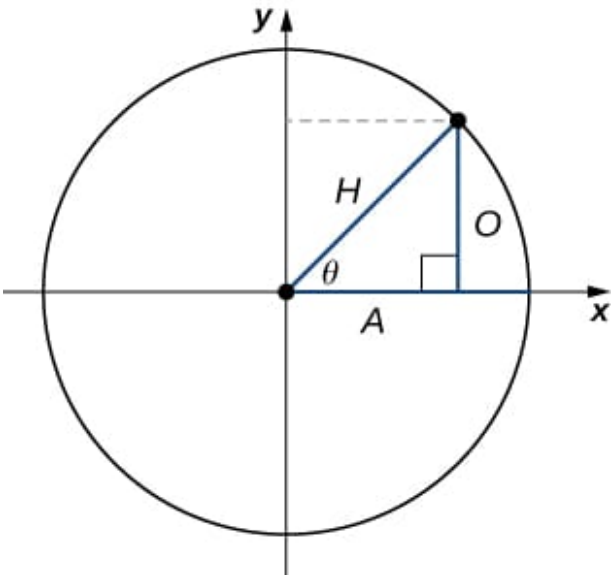


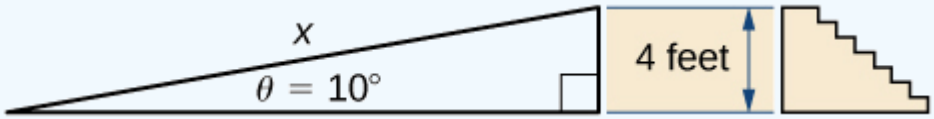
Figure 12.4.4: By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ.

Example 12.4.3: Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be $10°$, how long does the ramp need to be?

Solution

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10\ °) = 4/x$. Solving this equation for x, we see that $x = 4/\sin(10\ °) \approx 23.035$ ft.



Exercise 12.4.3

A house painter wants to lean a 20-ft ladder against a house. If the angle between the base of the ladder and the ground is to be $60°$, how far from the house should she place the base of the ladder?

Hint available at the end of this section.

Answer available at the end of this section.

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed next.

Trigonometric Identities

Reciprocal identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{12.4.5}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{12.4.6}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{12.4.7}$$

Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{12.4.8}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \tag{12.4.9}$$

$$= 1 - 2\sin^2\theta \tag{12.4.10}$$

$$= \cos^2\theta - \sin^2\theta \tag{12.4.11}$$

Example 12.4.4: Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos(2\theta) = \cos\theta$
b. $\sin(2\theta) = \tan\theta$

Solution

a) Using the double-angle formula for $\cos(2\theta)$, we see that θ is a solution of

$$1 + \cos(2\theta) = \cos\theta$$

if and only if

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots,$$

and $\cos\theta = 1/2$ when

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \ldots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \ldots,$$

we conclude that the set of solutions to this equation is

$$\theta = \frac{\pi}{2} + n\pi, \ \theta = \frac{\pi}{3} + 2n\pi$$

and

$$\theta = -\frac{\pi}{3} + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots.$$

b) Using the double-angle formula for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions. Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \ldots$. After checking for extraneous solutions, the set of solutions to the equation is

$$\theta = n\pi$$

and

$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}$$

with $n = 0, \pm 1, \pm 2, \ldots$.

Exercise 12.4.4

Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 12.4.5: Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution:

We start with the Pythagorean identity (Equation 12.4.5)

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

$$\tan^2\theta + 1 = \sec^2\theta.$$

Exercise 12.4.5

Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Answer available at the end of this section.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle . Since the angle θ and $\theta + 2\pi$ correspond to the same point P, the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions**. The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f. The sine, cosine, secant, and cosecant functions have a period of 2π. Since the tangent and cotangent functions repeat on an interval of length π, their period is π (Figure 12.4.5).

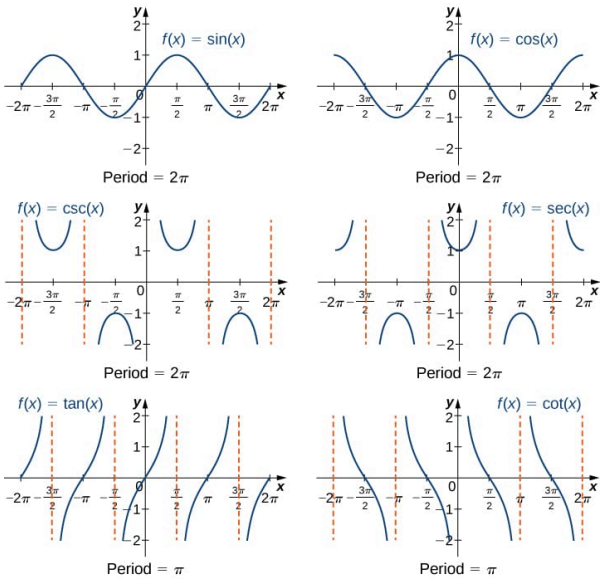


Figure 12.4.5: The six trigonometric functions are periodic.

Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

$$f(x) = A\sin(B(x-\alpha)) + C.$$

In Figure 12.4.6, the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the "amplitude of f." The constant C causes a vertical shift.

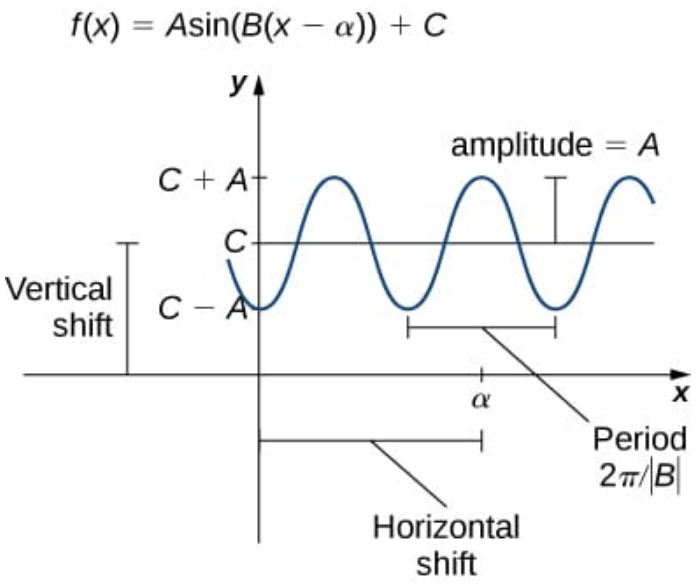


Figure 12.4.6: A graph of a general sine function.

Notice in Figure 12.4.6 that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write

$$\cos x = \sin(x + \pi/2).$$

Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the

shortest day of the year with 8.3 hours. It can be shown that the function

$$h(t) = 3.7 \sin\left(\frac{2\pi}{365}(x - 80.5)\right) + 12$$

is a model for the number of hours of daylight h as a function of day of the year t (Figure 12.4.7).

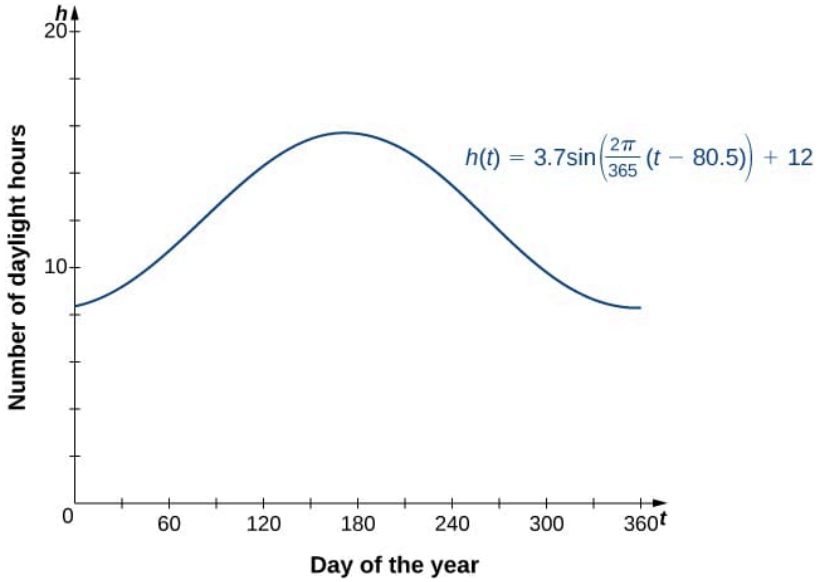


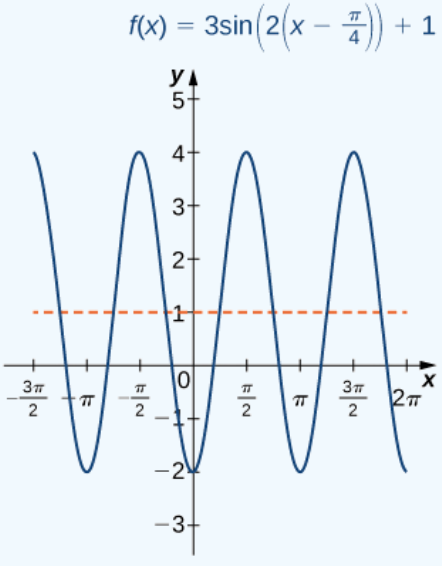
Figure 12.4.7: The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

Example 12.4.6: Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3\sin(2(x - \frac{\pi}{4})) + 1$.

Solution

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π.



Exercise 12.4.6

Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of $180°$ has a radian measure of π rad.
- For acute angles θ,the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ.
- For a general angle θ, let (x, y) be a point on a circle of radius r corresponding to this angle θ. The trigonometric functions can be written as ratios involving x, y, and r.
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π. The tangent and cotangent functions have period π.

Key Equations

- **Generalized sine function**

$$f(x) = A\sin(B(x - \alpha)) + C$$

Glossary

periodic function
: a function is periodic if it has a repeating pattern as the values of x move from left to right

radians
: for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

trigonometric functions
: functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity
: an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

Answers

Exercise 12.4.1

π radians is equal to 180°

Exercise 12.4.1

a. $7\pi/6$
b. 330°

Exercise 12.4.2

Look at angles on the unit circle.

Exercise 12.4.2

$$\cos(3\pi/4) = -\sqrt{2}/2$$
$$\sin(-\pi/6) = -1/2$$

Exercise 12.4.3

Draw a right triangle with hypotenuse 20.

Exercise 12.4.3

10 ft

Exercise 12.4.4

Use the double-angle formula for cosine (Equation 12.4.8).

Exercise 12.4.4

$\theta = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

for $n = 0, \pm 1, \pm 2, \ldots$.

Exercise 12.4.5

Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

Exercise 12.4.6

The graph of f can be sketched using the graph of $y = \sin(x)$ and a sequence of three transformations.

Exercise 12.4.6

To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function f will have a period of $\pi/2$ and an amplitude of 3.

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12.4E: Exercises for Section 1.3

In exercises 1 - 5, convert each angle in degrees to radians. Write the answer as a multiple of π.

1) $240\ ^\circ$

Answer

$\frac{4\pi}{3}$ rad

2) $15\ ^\circ$

3) $60\ ^\circ$

Answer

$\frac{\pi}{3}$ rad

4) $-225\ ^\circ$

5) $330\ ^\circ$

Answer

$\frac{11\pi}{6}$ rad

In exercises 6 - 10, convert each angle in radians to degrees.

6) $\frac{\pi}{2}$ rad

7) $\frac{7\pi}{6}$ rad

Answer

$210\ ^\circ$

8) $\frac{11\pi}{2}$ rad

9) -3π rad

Answer

$-540\ ^\circ$

10) $\frac{5\pi}{12}$ rad

In exercises 11 - 16, evaluate the functional values.

11) $\cos\frac{4\pi}{3}$

Answer

$\cos\frac{4\pi}{3} = -0.5$

12) $\tan\frac{19\pi}{4}$

13) $\sin\left(-\frac{3\pi}{4}\right)$

Answer

$\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

14) $\sec\left(-\frac{\pi}{6}\right)$

15) $\sin\left(-\frac{\pi}{12}\right)$

Answer

$$\sin\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

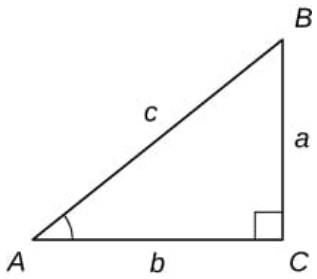
16) $\cos\left(-\frac{5\pi}{12}\right)$

In exercises 17 - 22, consider triangle ABC, a right triangle with a right angle at C.

a. Find the missing side of the triangle.

b. Find the six trigonometric function values for the angle at A.

Where necessary, round to one decimal place.



17) $a = 4,\ \ c = 7$

Answer

a. $b = 5.7$
b. $\sin A = \frac{4}{7},\ \ \cos A = \frac{5.7}{7},\ \ \tan A = \frac{4}{5.7},\ \ \csc A = \frac{7}{4},\ \ \sec A = \frac{7}{5.7},\ \ \cot A = \frac{5.7}{4}$

18) $a = 21,\ \ c = 29$

19) $a = 85.3,\ \ b = 125.5$

Answer

a. $c = 151.7$
b. $\sin A = 0.5623,\ \ \cos A = 0.8273,\ \ \tan A = 0.6797,\ \ \csc A = 1.778,\ \ \sec A = 1.209,\ \ \cot A = 1.471$

20) $b = 40,\ \ c = 41$

21) $a = 84,\ \ b = 13$

Answer

a. $c = 85$
b. $\sin A = \frac{84}{85},\ \ \cos A = \frac{13}{85},\ \ \tan A = \frac{84}{13},\ \ \csc A = \frac{85}{84},\ \ \sec A = \frac{85}{13},\ \ \cot A = \frac{13}{84}$

22) $b = 28,\ \ c = 35$

In exercises 23 - 26, P is a point on the unit circle.

a. Find the (exact) missing coordinate value of each point and

b. find the values of the six trigonometric functions for the angle θ with a terminal side that passes through point P.

Rationalize denominators.

23) $P\left(\frac{7}{25}, y\right), \quad y > 0$

Answer

a. $y = \frac{24}{25}$
b. $\sin\theta = \frac{24}{25}, \ \cos\theta = \frac{7}{25}, \ \tan\theta = \frac{24}{7}, \ \csc\theta = \frac{25}{24}, \ \sec\theta = \frac{25}{7}, \ \cot\theta = \frac{7}{24}$

24) $P\left(-\frac{15}{17}, y\right), \quad y > 0$

25) $P\left(x, \frac{\sqrt{7}}{3}\right), \quad x > 0$

Answer

a. $x = -\frac{\sqrt{2}}{3}$
b. $\sin\theta = \frac{\sqrt{7}}{3}, \ \cos\theta = -\frac{\sqrt{2}}{3}, \ \tan\theta = -\frac{\sqrt{14}}{2}, \ \csc\theta = \frac{3\sqrt{7}}{7}, \ \sec\theta = -\frac{3\sqrt{2}}{2}, \ \cot\theta = -\frac{\sqrt{14}}{7}$

26) $P\left(x, -\frac{\sqrt{15}}{4}\right), \quad y > 0$

In exercises 27 - 34, simplify each expression by writing it in terms of sines and cosines, then simplify. The final answer does not have to be in terms of sine and cosine only.

27) $\tan^2 x + \sin x \csc x$

Answer

$\sec^2 x$

28) $\sec x \sin x \cot x$

29) $\frac{\tan^2 x}{\sec^2 x}$

Answer

$\sin^2 x$

30) $\sec x - \cos x$

31) $(1 + \tan\theta)^2 - 2\tan\theta$

Answer

$\sec^2\theta$

32) $(\sin x)(\csc x - \sin x)$

33) $\frac{\cos t}{\sin t} + \frac{\sin t}{1 + \cos t}$

Answer

$\frac{1}{\sin t} = \csc t$

34) $\frac{1 + \tan^2\alpha}{1 + \cot^2\alpha}$

In exercises 35 - 42, verify that each equation is an identity.

35) $\frac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

36) $\frac{\sec^2\theta}{\tan\theta} = \sec\theta\csc\theta$

37) $\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = 1$

38) $\dfrac{\sin x}{\cos x + 1} + \dfrac{\cos x - 1}{\sin x} = 0$

39) $\cot \gamma + \tan \gamma = \sec \gamma \csc \gamma$

40) $\sin^2 \beta + \tan^2 \beta + \cos^2 \beta = \sec^2 \beta$

41) $\dfrac{1}{1 - \sin \alpha} + \dfrac{1}{1 + \sin \alpha} = 2 \sec^2 \alpha$

42) $\dfrac{\tan \theta - \cot \theta}{\sin \theta \cos \theta} = \sec^2 \theta - \csc^2 \theta$

In exercises 43 - 50, solve the trigonometric equations on the interval $0 \leq \theta < 2\pi$.

43) $2 \sin \theta - 1 = 0$

Answer

$\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

44) $1 + \cos \theta = \frac{1}{2}$

45) $2 \tan^2 \theta = 2$

Answer

$\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

46) $4 \sin^2 \theta - 2 = 0$

47) $\sqrt{3} \cot \theta + 1 = 0$

Answer

$\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

48) $3 \sec \theta - 2\sqrt{3} = 0$

49) $2 \cos \theta \sin \theta = \sin \theta$

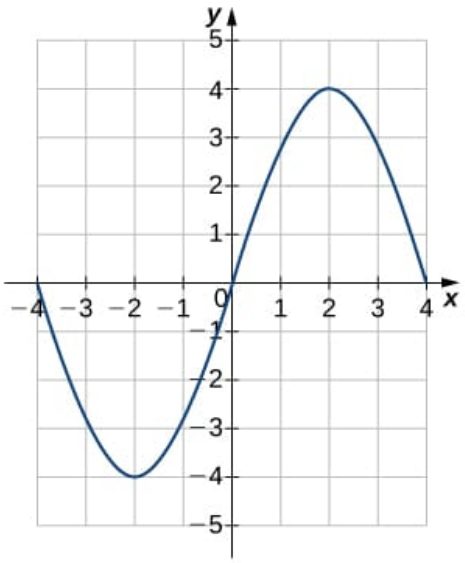
Answer

$\left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$

50) $\csc^2 \theta + 2 \csc \theta + 1 = 0$

In exercises 51 - 54, each graph is of the form $y = A \sin Bx$ or $y = A \cos Bx$, where $B > 0$. Write the equation of the graph.

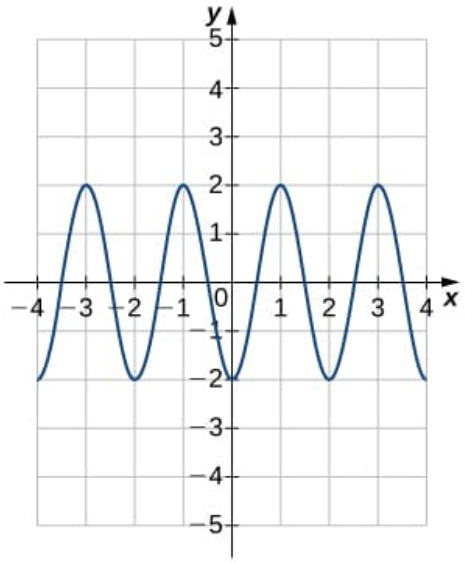
51)



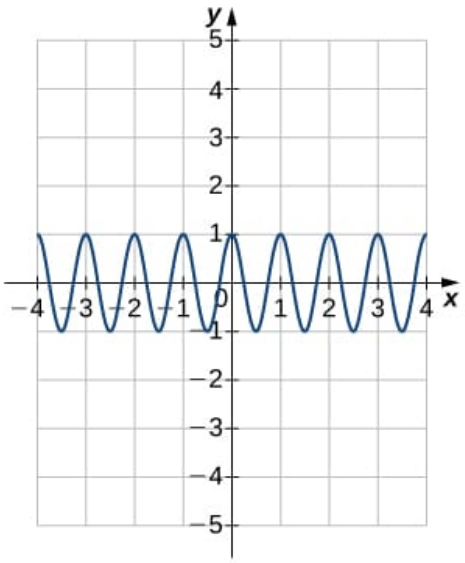
Answer

$y = 4\sin\left(\frac{\pi}{4}x\right)$

52)



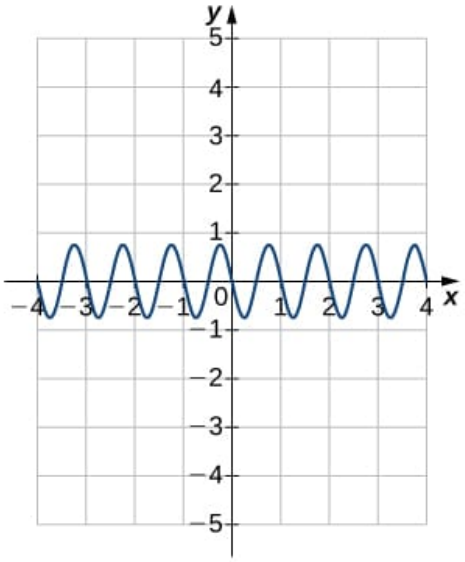
53)



Answer

$y = \cos 2\pi x$

54)



In exercises 55 - 60, find

a. the amplitude,

b. the period, and

c. the phase shift with direction for each function.

55) $y = \sin\left(x - \frac{\pi}{4}\right)$

Answer

a. 1
b. 2π
c. $\frac{\pi}{4}$ units to the right

56) $y = 3\cos(2x + 3)$

57) $y = -\frac{1}{2}\sin\left(\frac{1}{4}x\right)$

Answer

a. $\frac{1}{2}$
b. 8π
c. No phase shift

58) $y = 2\cos\left(x - \frac{\pi}{3}\right)$

59) $y = -3\sin(\pi x + 2)$

Answer

a. 3
b. 2
c. $\frac{2}{\pi}$ units to the left

60) $y = 4\cos\left(2x - \frac{\pi}{2}\right)$

61) [T] The diameter of a wheel rolling on the ground is 40 in. If the wheel rotates through an angle of $120°$, how many inches does it move? Approximate to the nearest whole inch.

Answer

Approximately 42 in.

62) [T] Find the length of the arc intercepted by central angle θ in a circle of radius r. Round to the nearest hundredth.

a. $r = 12.8$ cm, $\theta = \frac{5\pi}{6}$ rad b. $r = 4.378$ cm, $\theta = \frac{7\pi}{6}$ rad c. $r = 0.964$ cm, $\theta = 50°$ d. $r = 8.55$ cm, $\theta = 325°$

63) [T] As a point P moves around a circle, the measure of the angle changes. The measure of how fast the angle is changing is called angular speed, ω, and is given by $\omega = \theta/t$, where θ is in radians and t is time. Find the angular speed for the given data. Round to the nearest thousandth.

a. $\theta = \frac{7\pi}{4}$ rad, $t = 10$ sec b. $\theta = \frac{3\pi}{5}$ rad, $t = 8$ sec c. $\theta = \frac{2\pi}{9}$ rad, $t = 1$ min d. $\theta = 23.76$ rad, $t = 14$ min

Answer

a. 0.550 rad/sec
b. 0.236 rad/sec
c. 0.698 rad/min
d. 1.697 rad/min

64) [T] A total of $250,000\ \text{m}^2$ of land is needed to build a nuclear power plant. Suppose it is decided that the area on which the power plant is to be built should be circular.

a)Find the radius of the circular land area.

b)If the land area is to form a $45°$ sector of a circle instead of a whole circle, find the length of the curved side.

65) [T] The area of an isosceles triangle with equal sides of length x is $\frac{1}{2}x^2\sin\theta$,

where θ is the angle formed by the two sides. Find the area of an isosceles triangle with equal sides of length 8 in. and angle $\theta = \frac{5\pi}{12}$ rad.

Answer

$\approx 30.9\ \text{in}^2$

66) [T] A particle travels in a circular path at a constant angular speed ω. The angular speed is modeled by the function $\omega = 9|\cos(\pi t - \pi/12)|$. Determine the angular speed at $t = 9$ sec.

67) [T] An alternating current for outlets in a home has voltage given by the function $V(t) = 150\cos 368t$,

where V is the voltage in volts at time t in seconds.

a) Find the period of the function and interpret its meaning.

b) Determine the number of periods that occur when 1 sec has passed.

Answer

a. $\frac{\pi}{184}$; the voltage repeats every $\frac{\pi}{184}$ sec
b. Approximately 59 periods

68) [T] The number of hours of daylight in a northeast city is modeled by the function

$$N(t) = 12 + 3\sin\left[\frac{2\pi}{365}(t - 79)\right],$$

where t is the number of days after January 1.

a) Find the amplitude and period.

b) Determine the number of hours of daylight on the longest day of the year.

c) Determine the number of hours of daylight on the shortest day of the year.

d) Determine the number of hours of daylight 90 days after January 1.

e) Sketch the graph of the function for one period starting on January 1.

69) [T] Suppose that $T = 50 + 10\sin\left[\frac{\pi}{12}(t - 8)\right]$ is a mathematical model of the temperature (in degrees Fahrenheit) at t hours after midnight on a certain day of the week.

a) Determine the amplitude and period.

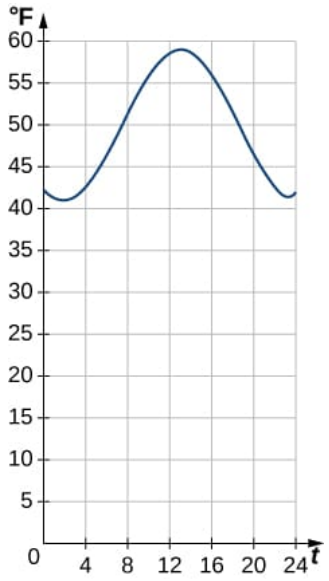
b) Find the temperature 7 hours after midnight.

c) At what time does $T = 60°$?

d) Sketch the graph of T over $0 \le t \le 24$.

Answer

a. Amplitude = 10; Period=24
b. 47.4°F
c. 14 hours later, or 2 p.m.
d.



70) [T] The function $H(t) = 8\sin\left(\frac{\pi}{6}t\right)$ models the height H (in feet) of the tide t hours after midnight. Assume that $t = 0$ is midnight.

a) Find the amplitude and period.

b) Graph the function over one period.

c) What is the height of the tide at 4:30 a.m.?

Contributors

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12.5: Inverse Functions

Learning Objectives

- Determine the conditions for when a function has an inverse.
- Use the horizontal line test to recognize when a function is one-to-one.
- Find the inverse of a given function.
- Draw the graph of an inverse function.
- Evaluate inverse trigonometric functions.

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function and study the relationship between the graph of a function and the graph of its inverse. Then we apply these ideas to define and discuss properties of the inverse trigonometric functions.

Existence of an Inverse Function

We begin with an example. Given a function f and an output $y = f(x)$, we are often interested in finding what value or values x were mapped to y by f. For example, consider the function $f(x) = x^3 + 4$. Since any output $y = x^3 + 4$, we can solve this equation for x to find that the input is $x = \sqrt[3]{y-4}$. This equation defines x as a function of y. Denoting this function as f^{-1}, and writing $x = f^{-1}(y) = \sqrt[3]{y-4}$, we see that for any x in the domain of f, $f^{-1}f(x)) = f^{-1}(x^3 + 4) = x$. Thus, this new function, f^{-1}, "undid" what the original function f did. A function with this property is called the inverse function of the original function.

Definition: Inverse Functions

Given a function f with domain D and range R, its *inverse function* (if it exists) is the function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if and only if $f(x) = y$. In other words, for a function f and its inverse f^{-1},

$$f^{-1}(f(x)) = x$$

for all x in D and

$$f(f^{-1}(y)) = y$$

for all y in R.

Note that f^{-1} is read as "f inverse." Here, the -1 is not used as an exponent so

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Figure $12.5.1$shows the relationship between the domain and range of f and the domain and range of f^{-1}.

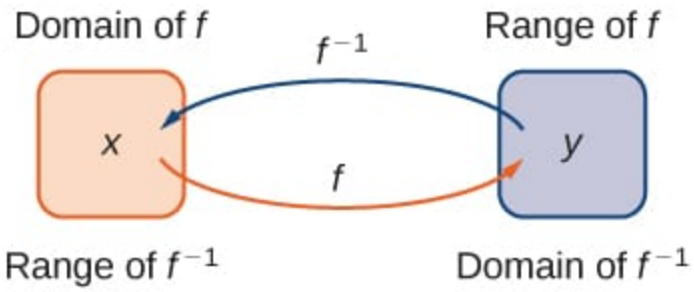


Figure 12.5.1: Given a function f and its inverse f^{-1}, $f^{-1}(y) = x$ if and only if $f(x) = y$. The range of f becomes the domain of f^{-1} and the domain of f becomes the range of f^{-1}.

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $y = x^2$ for x, we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe x as a function of y because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a different output. A function that sends each input to a different output is called a one-to-one function.

Definition: One-to-One functions

We say a function f is a one-to-one function if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the xy-plane, according to the **horizontal line test**, it cannot intersect the graph more than once. We note that the horizontal line test is different from the vertical line test. The vertical line test determines whether a graph is the graph of a function. The horizontal line test determines whether a function is one-to-one (Figure 12.5.2).

Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.

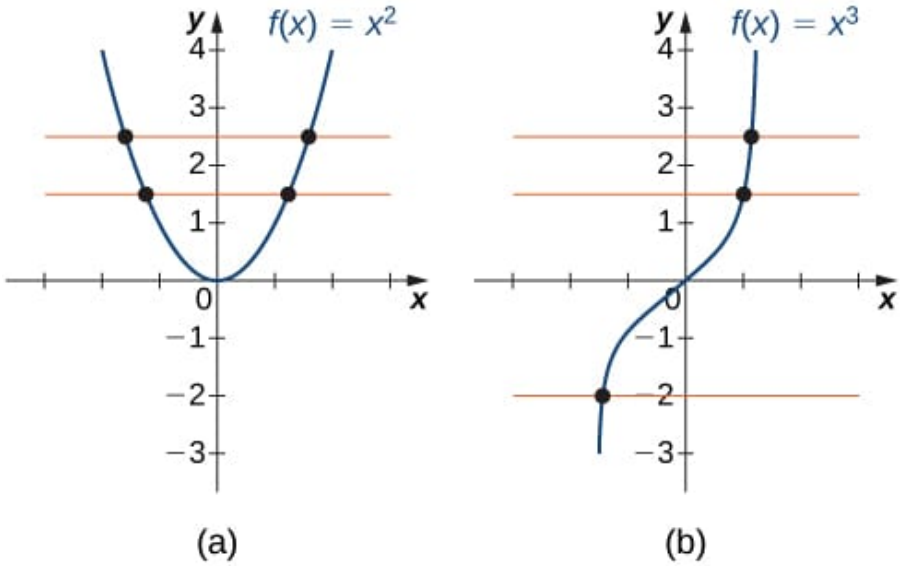
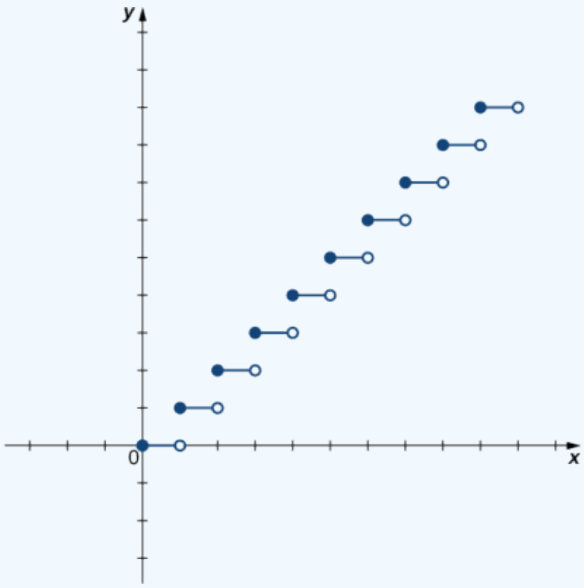


Figure 12.5.2: (a) The function $f(x) = x^2$ is not one-to-one because it fails the horizontal line test. (b) The function $f(x) = x^3$ is one-to-one because it passes the horizontal line test.

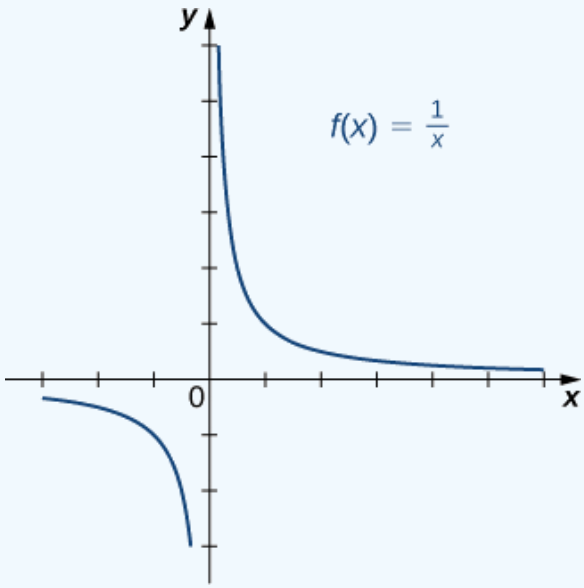
Example 12.5.1: Determining Whether a Function Is One-to-One

For each of the following functions, use the horizontal line test to determine whether it is one-to-one.

a)

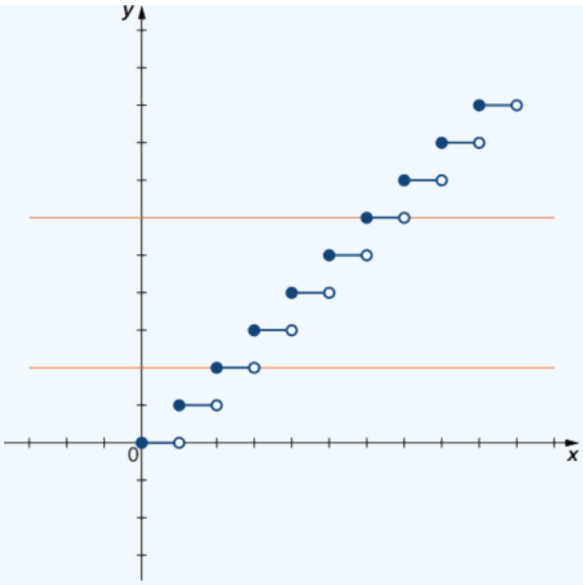


b)

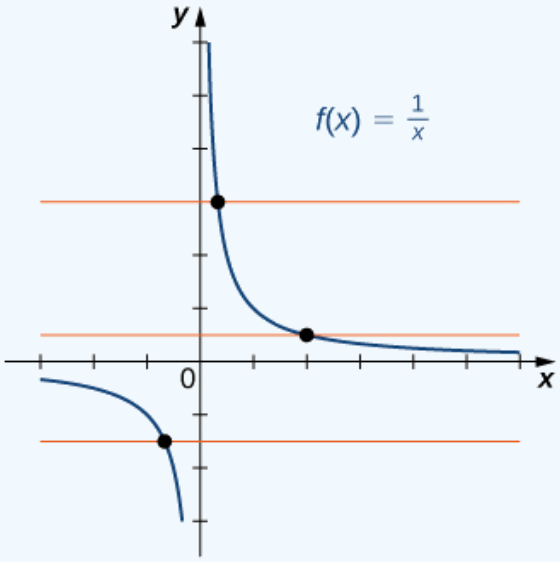


Solution

a) Since the horizontal line $y = n$ for any integer $n \geq 0$ intersects the graph more than once, this function is not one-to-one.

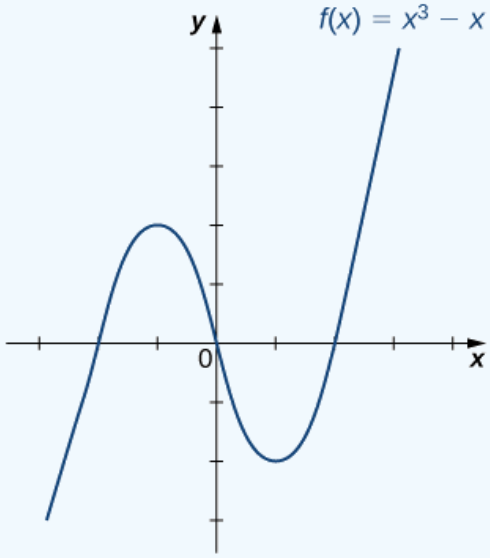


b) Since every horizontal line intersects the graph once (at most), this function is one-to-one.



Exercise 12.5.1

Is the function f graphed in the following image one-to-one?



Solution available at the end of this section.

Answer available at the end of this section.

Finding a Function's Inverse

We can now consider one-to-one functions and show how to find their inverses. Recall that a function maps elements in the domain of f to elements in the range of f. The inverse function maps each element from the range of f back to its corresponding element from the domain of f. Therefore, to find the inverse function of a one-to-one function f, given any y in the range of f, we need to determine which x in the domain of f satisfies $f(x) = y$. Since f is one-to-one, there is exactly one such value x. We can find that value x by solving the equation $f(x) = y$ for x. Doing so, we are able to write x as a function of y where the domain of this function is the range of f and the range of this new function is the domain of f. Consequently, this function is the inverse of f , and we write $x = f^{-1}(y)$. Since we typically use the variable x to denote the independent variable and y to denote the dependent variable, we often interchange the roles of x and y, and write $y = f^{-1}(x)$. Representing the inverse function in this way is also helpful later when we graph a function f and its inverse f^{-1} on the same axes.

Problem-Solving Strategy: Finding an Inverse Function

1. Solve the equation $y = f(x)$ for x.
2. Interchange the variables x and y and write $y = f^{-1}(x)$.

Example 12.5.2: Finding an Inverse Function

Find the inverse for the function $f(x) = 3x - 4$. State the domain and range of the inverse function. Verify that $f^{-1}(f(x)) = x$.

Solution

Follow the steps outlined in the strategy.

Step 1. If $y = 3x - 4$, then $3x = y + 4$ and $x = \frac{1}{3}y + \frac{4}{3}$.

Step 2. Rewrite as $y = \frac{1}{3}x + \frac{4}{3}$ and let $y = f^{-1}(x)$.Therefore, $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$.

Since the domain of f is $(-\infty, \infty)$, the range of f^{-1} is $(-\infty, \infty)$. Since the range of f is $(-\infty, \infty)$, the domain of f^{-1} is $(-\infty, \infty)$.

You can verify that $f^{-1}(f(x)) = x$ by writing

$$f^{-1}(f(x)) = f^{-1}(3x-4) = \tfrac{1}{3}(3x-4) + \tfrac{4}{3} = x - \tfrac{4}{3} + \tfrac{4}{3} = x.$$

Note that for $f^{-1}(x)$ to be the inverse of $f(x)$, both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all x in the domain of the inside function.

Exercise 12.5.2

Find the inverse of the function $f(x) = 3x/(x-2)$. State the domain and range of the inverse function.

Hint available at the end of this section.

Answer available at the end of this section.

Graphing Inverse Functions

Let's consider the relationship between the graph of a function f and the graph of its inverse. Consider the graph of f shown in Figure 12.5.3 and a point (a, b) on the graph. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, when we graph f^{-1}, the point (b, a) is on the graph. As a result, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

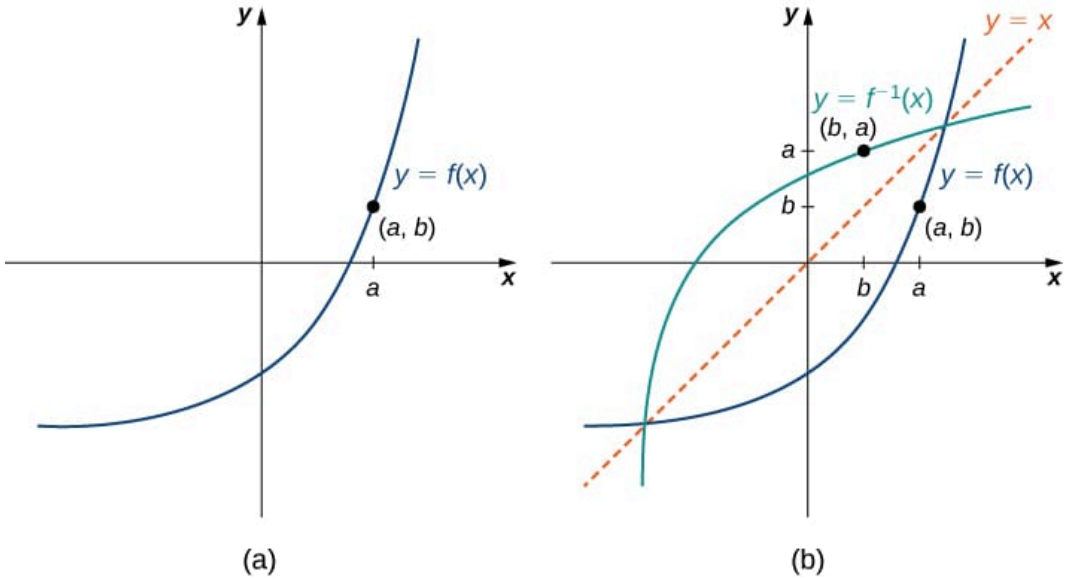
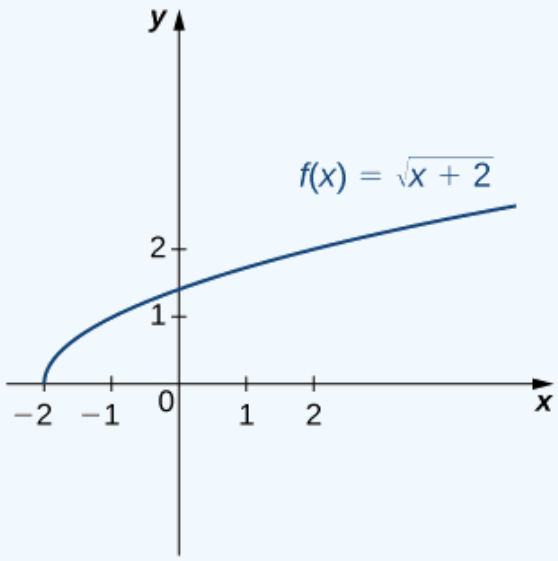


Figure 12.5.3: (a) The graph of this function f shows point (a, b) on the graph of f. (b) Since (a, b) is on the graph of f, the point (b, a) is on the graph of f^{-1}. The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

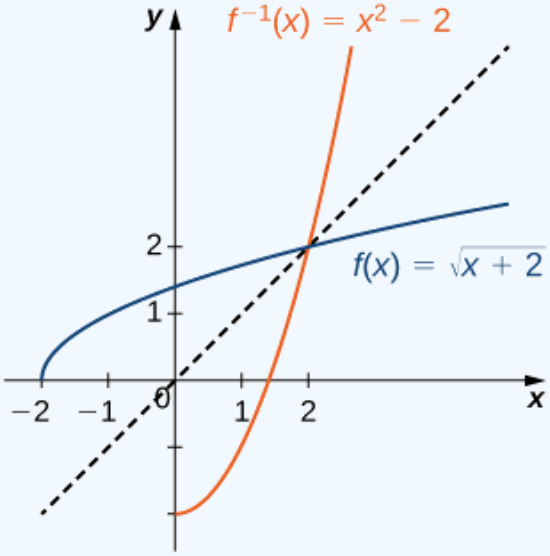
Example 12.5.3: Sketching Graphs of Inverse Functions

For the graph of f in the following image, sketch a graph of f^{-1} by sketching the line $y = x$ and using symmetry. Identify the domain and range of f^{-1}.



Solution

Reflect the graph about the line $y = x$. The domain of f^{-1} is $[0, \infty)$. The range of f^{-1} is $[-2, \infty)$. By using the preceding strategy for finding inverse functions, we can verify that the inverse function is $f^{-1}(x) = x^2 - 2$, as shown in the graph.



Exercise 12.5.3

Sketch the graph of $f(x) = 2x + 3$ and the graph of its inverse using the symmetry property of inverse functions.

Hint available at the end of this section.

Answer available at the end of this section.

Restricting Domains

As we have seen, $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of f such that the function is one-to-one. This subset is called a **restricted domain**. By restricting the domain of f, we can define a new function g such that the domain of g is the restricted domain of f and $g(x) = f(x)$ for all x in the domain of g. Then we can define an inverse function for g on that domain. For example, since $f(x) = x^2$ is one-to-one on the interval $[0, \infty)$, we can define a new function g such that the domain of g is $[0, \infty)$ and $g(x) = x^2$ for all x in its domain. Since g is a one-to-one function, it has an inverse function, given by the formula $g^{-1}(x) = \sqrt{x}$. On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could also define a new function h such that the domain of h is $(-\infty, 0]$ and $h(x) = x^2$ for all x in the domain of h. Then h is a one-to-one function and must also have an inverse. Its inverse is given by the formula $h^{-1}(x) = -\sqrt{x}$ (Figure 12.5.4).

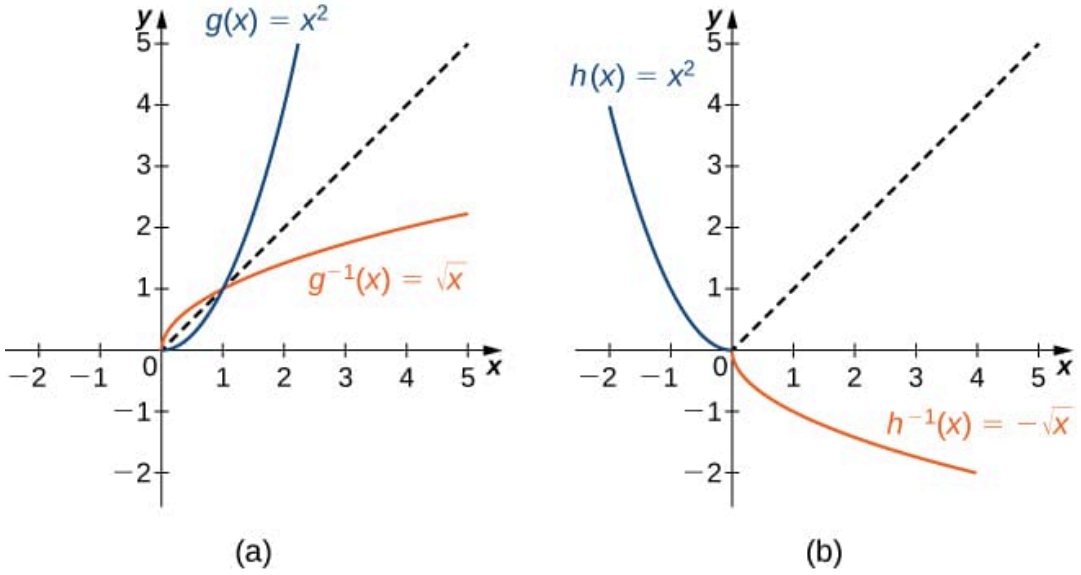


Figure 12.5.4: (a) For $g(x) = x^2$ restricted to $[0, \infty)$, $g^{-1}(x) = \sqrt{x}$. (b) For $h(x) = x^2$ restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x}$.

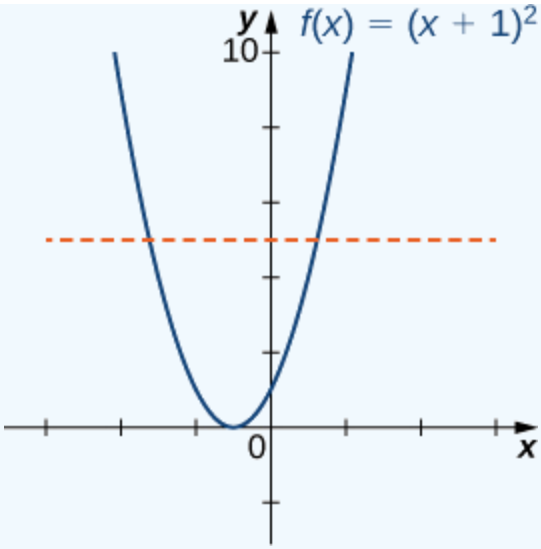
Example 12.5.4: Restricting the Domain

Consider the function $f(x) = (x + 1)^2$.

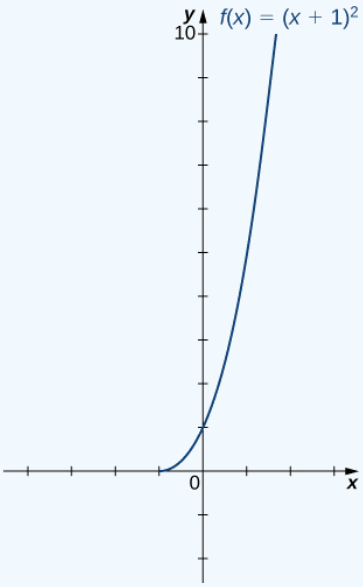
a. Sketch the graph of f and use the horizontal line test to show that f is not one-to-one.
b. Show that f is one-to-one on the restricted domain $[-1, \infty)$. Determine the domain and range for the inverse of f on this restricted domain and find a formula for f^{-1}.

Solution

a) The graph of f is the graph of $y = x^2$ shifted left 1 unit. Since there exists a horizontal line intersecting the graph more than once, f is not one-to-one.



b) On the interval $[-1, \infty)$, f is one-to-one.



The domain and range of f^{-1} are given by the range and domain of f, respectively. Therefore, the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[-1, \infty)$. To find a formula for f^{-1}, solve the equation $y = (x + 1)^2$ for x. If $y = (x + 1)^2$, then $x = -1 \pm \sqrt{y}$. Since we are restricting the domain to the interval where $x \geq -1$, we need $\pm\sqrt{y} \geq 0$. Therefore, $x = -1 + \sqrt{y}$. Interchanging x and y, we write $y = -1 + \sqrt{x}$ and conclude that $f^{-1}(x) = -1 + \sqrt{x}$.

Exercise 12.5.4

Consider $f(x) = 1/x^2$ restricted to the domain $(-\infty, 0)$. Verify that f is one-to-one on this domain. Determine the domain and range of the inverse of f and find a formula for f^{-1}.

Hint available at the end of this section.

Answer available at the end of this section.

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, and therefore they are not one-to-one. However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse. Consider the sine function. The sine function is one-to-one on an infinite number of intervals, but the standard convention is to restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By doing so, we define the inverse sine function on the domain $[-1, 1]$ such that for any x in the interval $[-1, 1]$, the inverse sine function tells us which angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfies $\sin\theta = x$. Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us which angle in a certain interval has a specified trigonometric value.

Definition: inverse trigonometric functions

The inverse sine function, denoted $\sin^{-1}$ or $\arcsin$, and the inverse cosine function, denoted $\cos^{-1}$ or $\arccos$, are defined on the domain $D = \{x| -1 \leq x \leq 1\}$ as follows:

$$\sin^{-1}(x) = y$$

- if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$;

$$\cos^{-1}(x) = y$$

- if and only if $\cos(y) = x$ and $0 \leq y \leq \pi$.

The inverse tangent function, denoted $\tan^{-1}$ or $\arctan$, and inverse cotangent function, denoted $\cot^{-1}$ or arccot, are defined on the domain \(D=\{x|–∞

$$\tan^{-1}(x) = y$$

- if and only if $\tan(y) = x$ and \(–\frac{π}{2}

$$\cot^{-1}(x) = y$$

- if and only if $\cot(y) = x$ and \(0

The inverse cosecant function, denoted $\csc^{-1}$ or arccsc, and inverse secant function, denoted $\sec^{-1}$ or arcsec, are defined on the domain $D = \{x \mid |x| \geq 1\}$ as follows:

$$\csc^{-1}(x) = y$$

- if and only if $\csc(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2},\ y \neq 0$;

$$\sec^{-1}(x) = y$$

- if and only if $\sec(y) = x$ and$0 \leq y \leq \pi,\ y \neq \pi/2$.

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line $y = x$ (Figure $12.5.5$).

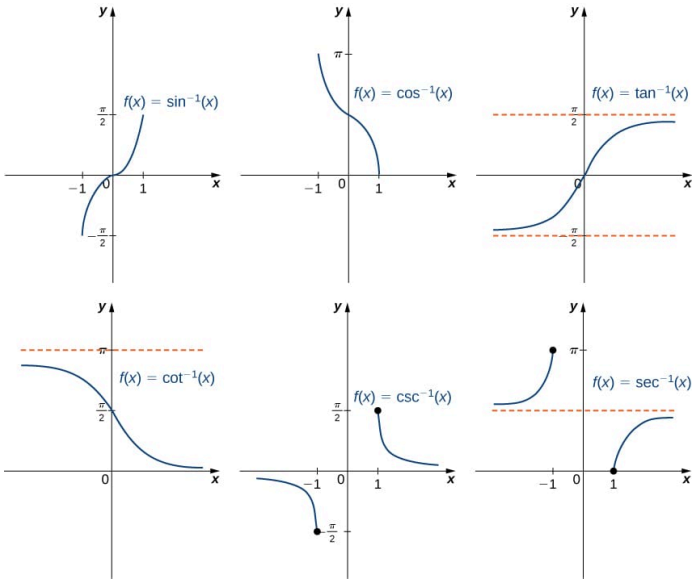


Figure 12.5.5: The graph of each of the inverse trigonometric functions is a reflection about the line $y = x$ of the corresponding restricted trigonometric function.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$, we need to find an angle θ such that $\cos\theta = \frac{1}{2}$. Clearly, many angles have this property. However, given the definition of $\cos^{-1}$, we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi]$. We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi))$.

For the first one, we simplify as follows:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

The inverse function is supposed to "undo" the original function, so why isn't $\sin^{-1}(\sin(\pi)) = \pi$? Recalling our definition of inverse functions, a function f and its inverse f^{-1} satisfy the conditions $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f, so what happened here? The issue is that the inverse sine function, $\sin^{-1}$, is the inverse of the restricted sine function defined on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is true that $\sin^{-1}(\sin x) = x$. However, for values of x outside this interval, the equation does not hold, even though $\sin^{-1}(\sin x)$ is defined for all real numbers x.

What about $\sin(\sin^{-1} y)$? Does that have a similar issue? The answer is no. Since the domain of $\sin^{-1}$ is the interval $[-1, 1]$, we conclude that $\sin\left(\sin^{-1} y\right) = y$ if $-1 \le y \le 1$ and the expression is not defined for other values of y. To summarize,

$$\sin(\sin^{-1} y) = y \text{ if } -1 \le y \le 1$$

and

$$\sin^{-1}(\sin x) = x \text{ if } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}.$$

Similarly, for the cosine function,

$$\cos(\cos^{-1} y) = y \text{ if } -1 \le y \le 1$$

and

$$\cos^{-1}(\cos x) = x \text{ if } 0 \leq x \leq \pi.$$

Similar properties hold for the other trigonometric *functions and their inverses*.

Example 12.5.5: Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluate each of the following expressions.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\tan\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$
d. $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

Solution

a. Evaluating $\sin^{-1}(-\sqrt{3}/2)$ is equivalent to finding the angle θ such that $\sin\theta = -\sqrt{3}/2$ and $-\pi/2 \leq \theta \leq \pi/2$. The angle $\theta = -\pi/3$ satisfies these two conditions. Therefore, $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$.
b. First we use the fact that $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Then $\tan(-\pi/6) = -1/\sqrt{3}$. Therefore, $\tan(\tan^{-1}(-1/\sqrt{3})) = -1/\sqrt{3}$.
c. To evaluate $\cos^{-1}(\cos(5\pi/4))$,first use the fact that $\cos(5\pi/4) = -\sqrt{2}/2$. Then we need to find the angle θ such that $\cos(\theta) = -\sqrt{2}/2$ and $0 \leq \theta \leq \pi$. Since $3\pi/4$ satisfies both these conditions, we have $\cos^{-1}(\cos(5\pi/4)) = \cos^{-1}(-\sqrt{2}/2)) = 3\pi/4$.
d. Since $\cos(2\pi/3) = -1/2$, we need to evaluate $\sin^{-1}(-1/2)$. That is, we need to find the angle θ such that $\sin(\theta) = -1/2$ and $-\pi/2 \leq \theta \leq \pi/2$. Since $-\pi/6$ satisfies both these conditions, we can conclude that $\sin^{-1}(\cos(2\pi/3)) = \sin^{-1}(-1/2) = -\pi/6$.

The Maximum Value of a Function

In many areas of science, engineering, and mathematics, it is useful to know the maximum value a function can obtain, even if we don't know its exact value at a given instant. For instance, if we have a function describing the strength of a roof beam, we would want to know the maximum weight the beam can support without breaking. If we have a function that describes the speed of a train, we would want to know its maximum speed before it jumps off the rails. Safe design often depends on knowing maximum values.

This project describes a simple example of a function with a maximum value that depends on two equation coefficients. We will see that maximum values can depend on several factors other than the independent variable x.

1. Consider the graph in Figure 12.5.6 of the function $y = \sin x + \cos x$. Describe its overall shape. Is it periodic? How do you know?

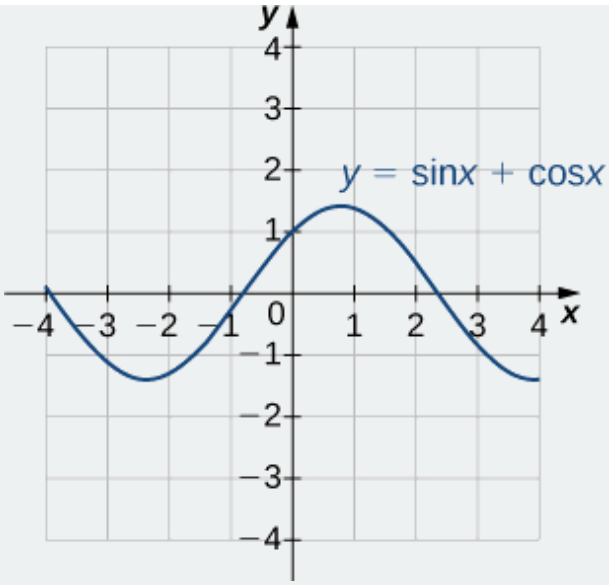


Figure 12.5.6: The graph of $y = \sin x + \cos x$.

Using a graphing calculator or other graphing device, estimate the x- and y-values of the maximum point for the graph (the first such point where $x > 0$). It may be helpful to express the x-value as a multiple of π.

2. Now consider other graphs of the form $y = A\sin x + B\cos x$ for various values of A and B. Sketch the graph when $A = 2$ and $B = 1$, and find the x- and y-values for the maximum point. (Remember to express the x-value as a multiple of π, if possible.) Has it moved?

3. Repeat for $A = 1,\ \ B = 2$. Is there any relationship to what you found in part (2)?

4. Complete the following table, adding a few choices of your own for A and B :

A	B	x	y	A	B	x	y
0	1			3	4		
1	0			4	3		
1	1			$\sqrt{3}$	1		
1	2			1	$\sqrt{3}$		
2	1			12	5		
2	2			5	12		

5. Try to figure out the formula for the y-values.

6. The formula for the x-values is a little harder. The most helpful points from the table are $(1, 1),\ \ (1, \sqrt{3}),\ \ (\sqrt{3}, 1)$. (Hint: Consider inverse trigonometric functions.)

7. If you found formulas for parts (5) and (6), show that they work together. That is, substitute the x-value formula you found into $y = A\sin x + B\cos x$ and simplify it to arrive at the y-value formula you found.

Key Concepts

- For a function to have an inverse, the function must be one-to-one. Given the graph of a function, we can determine whether the function is one-to-one by using the horizontal line test.
- If a function is not one-to-one, we can restrict the domain to a smaller domain where the function is one-to-one and then define the inverse of the function on the smaller domain.

- For a function f and its inverse f^{-1}, $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f.
- Since the trigonometric functions are periodic, we need to restrict their domains to define the inverse trigonometric functions.
- The graph of a function f and its inverse f^{-1} are symmetric about the line $y = x$.

Key Equations

- **Inverse function**

  $f^{-1}(f(x)) = x$ for all x in D, and $f(f^{-1}(y)) = y$ for all y in R.

Glossary

horizontal line test
: a function f is one-to-one if and only if every horizontal line intersects the graph of f, at most, once

inverse function
: for a function f, the inverse function f^{-1} satisfies $f^{-1}(y) = x$ if $f(x) = y$

inverse trigonometric functions
: the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

one-to-one function
: a function f is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

restricted domain
: a subset of the domain of a function f

Answers

Exercise 12.5.1

Use the horizontal line test.

Exercise 12.5.1

No

Exercise 12.5.2

Use the Problem-Solving Strategy for finding inverse functions.

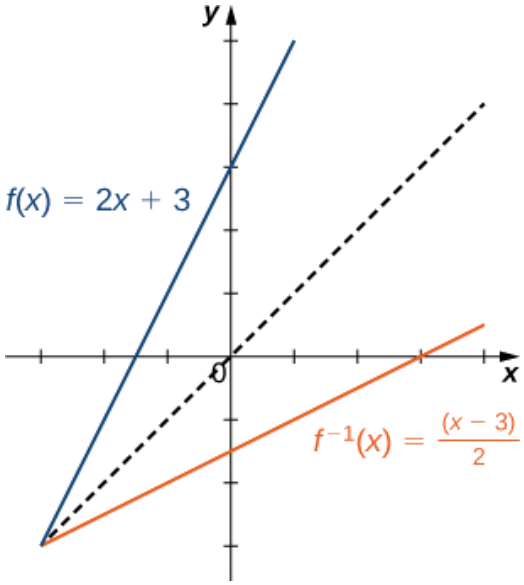
Exercise 12.5.2

$f^{-1}(x) = \dfrac{2x}{x-3}$. The domain of f^{-1} is $\{x \mid x \neq 3\}$. The range of f^{-1} is $\{y \mid y \neq 2\}$.

Exercise 12.5.3

The graphs are symmetric about the line $y = x$

Exercise 12.5.3



Exercise 12.5.4

The domain and range of f^{-1} is given by the range and domain of f, respectively. To find f^{-1}, solve $y = 1/x^2$ for x.

Exercise 12.5.4

The domain of f^{-1} is $(0, \infty)$. The range of f^{-1} is $(-\infty, 0)$. The inverse function is given by the formula $f^{-1}(x) = -1/\sqrt{x}$.

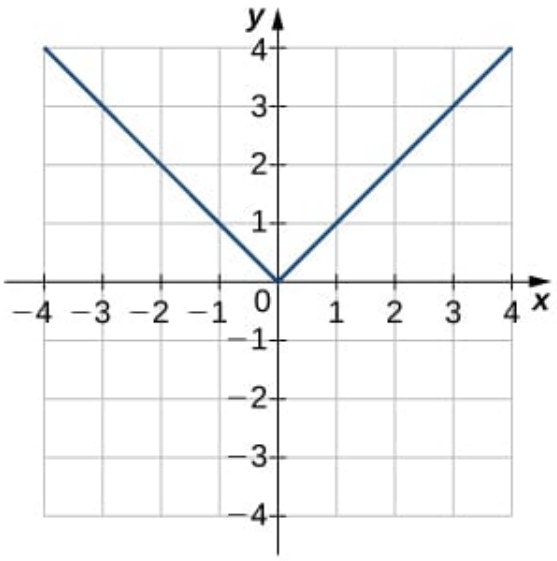
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12.5E: Exercises for Section 1.4

In exercises 1 - 6, use the horizontal line test to determine whether each of the given graphs is one-to-one.

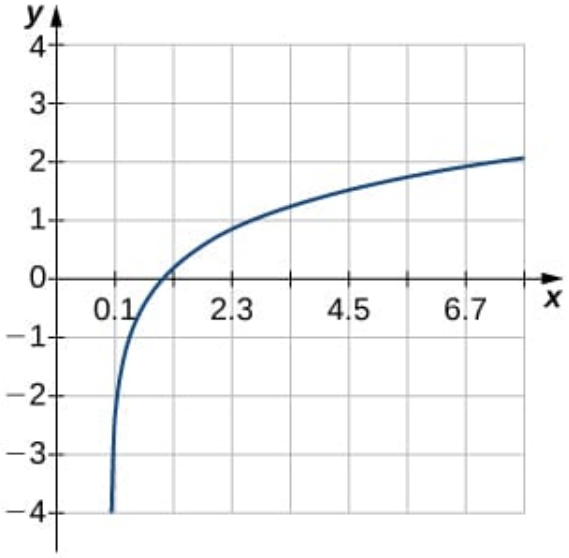
1)



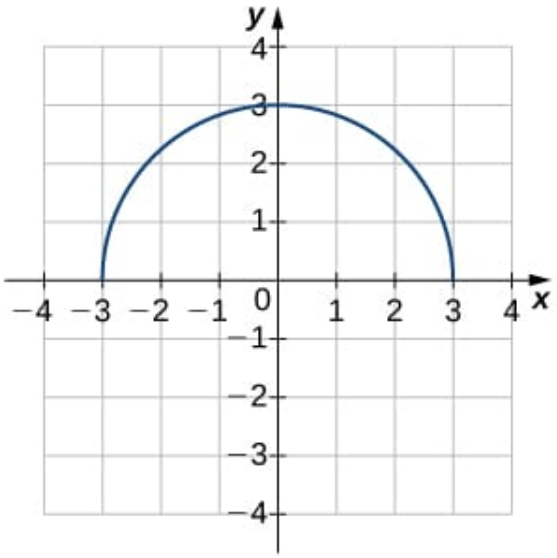
Answer

Not one-to-one

2)



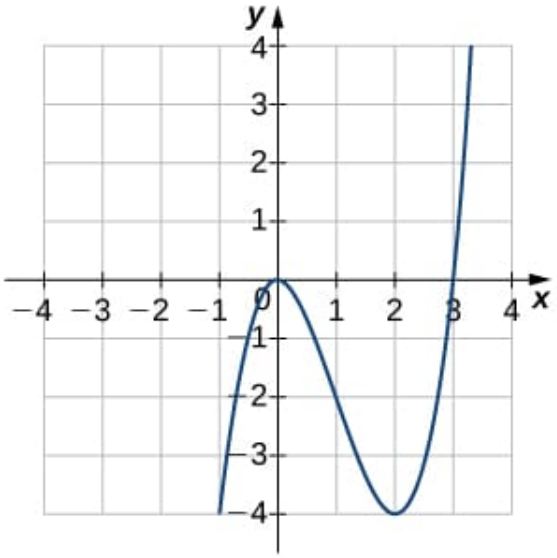
3)



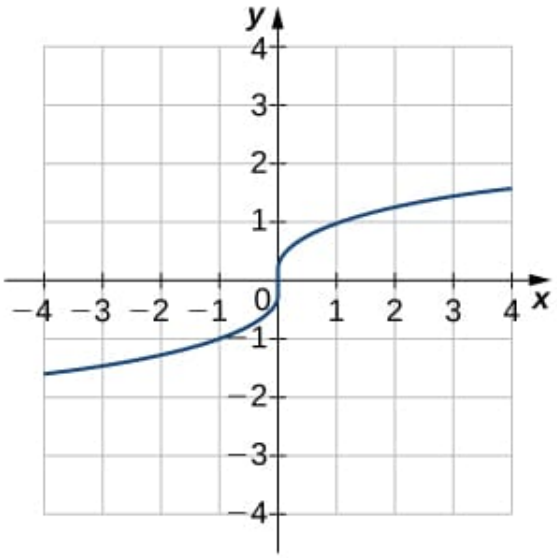
Answer

Not one-to-one

4)



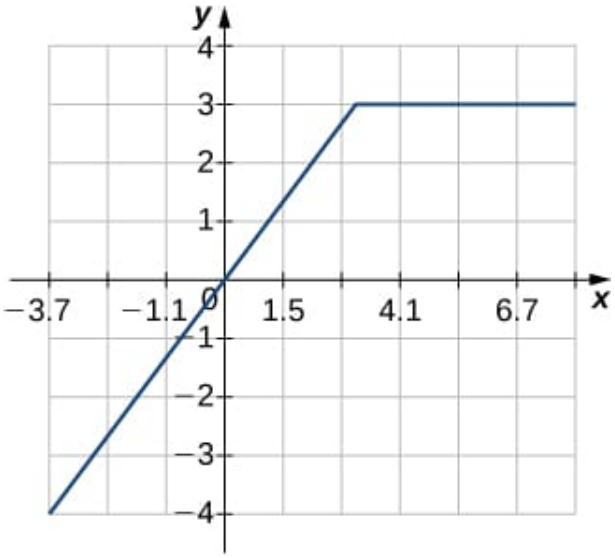
5)



Answer

One-to-one

6)



In exercises 7 - 12,

a. find the inverse function, and

b. find the domain and range of the inverse function.

7) $f(x) = x^2 - 4, \quad x \geq 0$

Answer

a. $f^{-1}(x) = \sqrt{x+4}$

b. Domain: $x \geq -4$, Range: $y \geq 0$

8) $f(x) = \sqrt[3]{x-4}$

9) $f(x) = x^3 + 1$

Answer

a. $f^{-1}(x) = \sqrt[3]{x-1}$
b. Domain: all real numbers, Range: all real numbers

10) $f(x) = (x-1)^2, \quad x \leq 1$

11) $f(x) = \sqrt{x-1}$

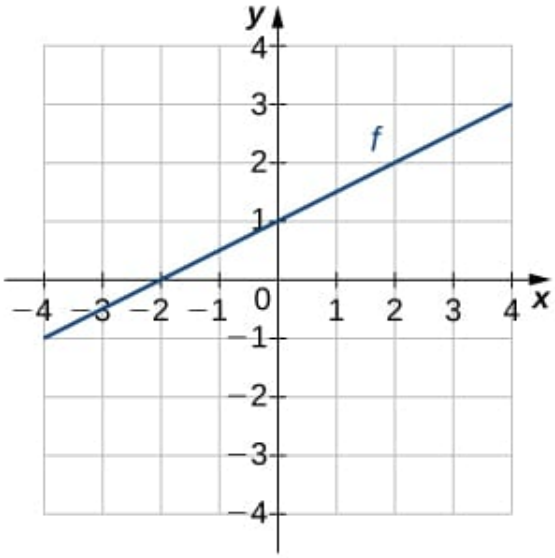
Answer

a. $f^{-1}(x) = x^2 + 1$,
b. Domain: $x \geq 0$, Range: $y \geq 1$

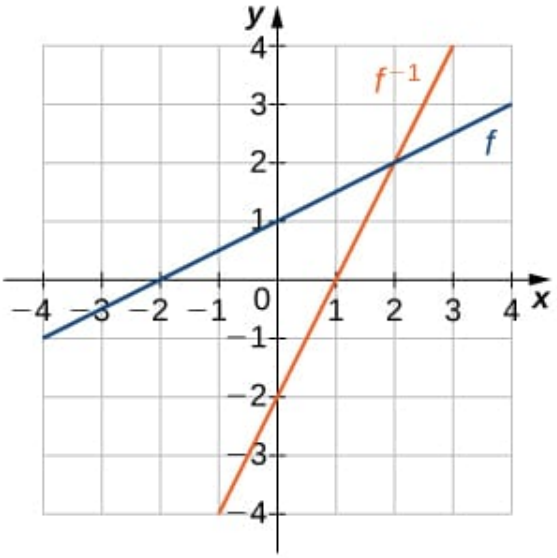
12) $f(x) = \dfrac{1}{x+2}$

In exercises 13 - 16, use the graph of f to sketch the graph of its inverse function.

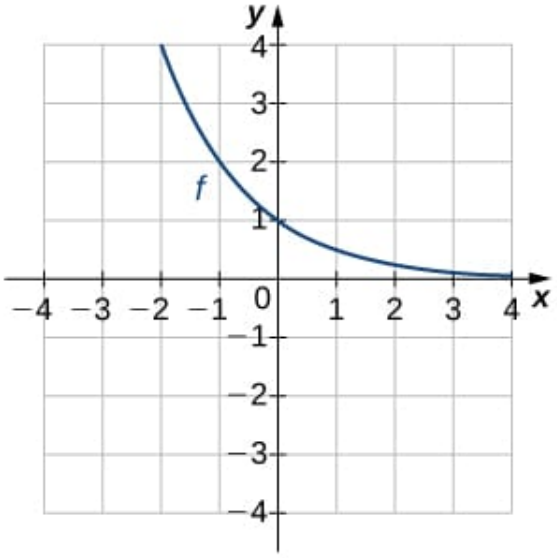
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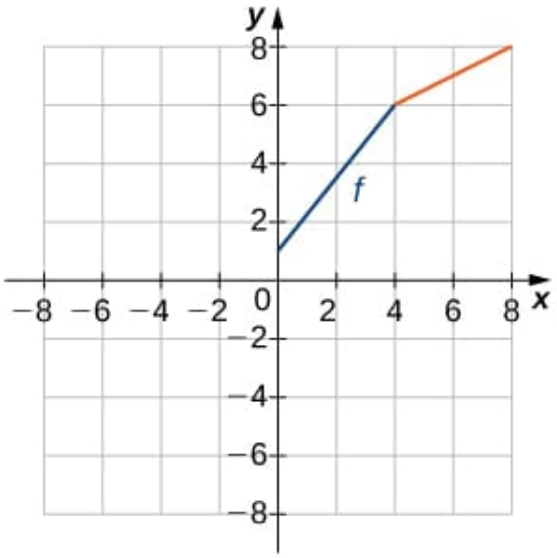
Answer



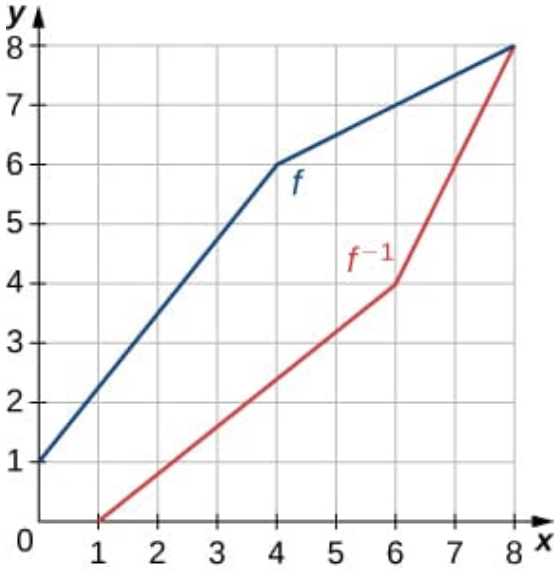
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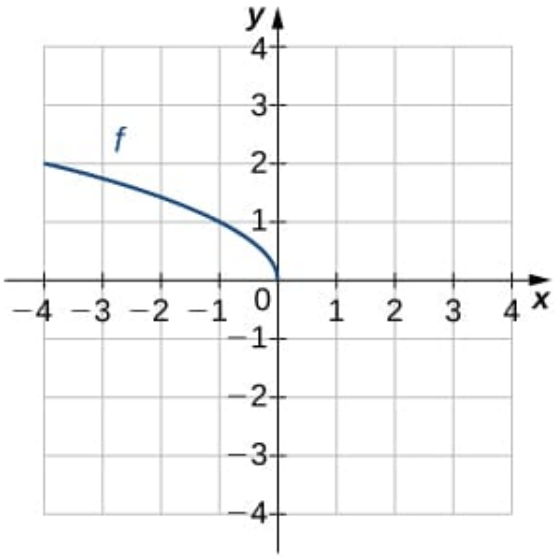
15)



Answer



16)



In exercises 17 - 24, use composition to determine which pairs of functions are inverses.

17) $f(x) = 8x, \quad g(x) = \dfrac{x}{8}$

Answer

These are inverses.

18) $f(x) = 8x + 3, \quad g(x) = \dfrac{x - 3}{8}$

19) $f(x) = 5x - 7, \quad g(x) = \dfrac{x + 5}{7}$

Answer

These are not inverses.

20) $f(x) = \frac{2}{3}x + 2, \quad g(x) = \frac{3}{2}x + 3$

21) $f(x) = \dfrac{1}{x - 1}, \ x \neq 1, \quad g(x) = \dfrac{1}{x} + 1, \ x \neq 0$

Answer

These are inverses.

22) $f(x) = x^3 + 1, \quad g(x) = (x - 1)^{1/3}$

23) $f(x) = x^2 + 2x + 1, \ x \geq -1, \quad g(x) = -1 + \sqrt{x}, \ x \geq 0$

Answer

These are inverses.

24) $f(x) = \sqrt{4 - x^2}, \ 0 \leq x \leq 2, \quad g(x) = \sqrt{4 - x^2}, \ 0 \leq x \leq 2$

In exercises 25 - 33, evaluate the functions. Give the exact value.

25) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

Answer

$\frac{\pi}{6}$

26) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

27) $\cot^{-1}(1)$

Answer

$\frac{\pi}{4}$

28) $\sin^{-1}(-1)$

29) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

$\frac{\pi}{6}$

30) $\cos\left(\tan^{-1}(\sqrt{3})\right)$

31) $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Answer

$\frac{\sqrt{2}}{2}$

32) $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

33) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Answer

$-\frac{\pi}{6}$

34) The function $C = T(F) = (5/9)(F - 32)$ converts degrees Fahrenheit to degrees Celsius.

a) Find the inverse function $F = T^{-1}(C)$

b) What is the inverse function used for?

35) [T] The velocity V (in centimeters per second) of blood in an artery at a distance x cm from the center of the artery can be modeled by the function $V = f(x) = 500(0.04 - x^2)$ for $0 \leq x \leq 0.2$.

a) Find $x = f^{-1}(V)$.

b) Interpret what the inverse function is used for.

c) Find the distance from the center of an artery with a velocity of 15 cm/sec, 10 cm/sec, and 5 cm/sec.

Answer

a. $x = f^{-1}(V) = \sqrt{0.04 - \frac{V}{500}}$
b. The inverse function determines the distance from the center of the artery at which blood is flowing with velocity V.
c. 0.1 cm; 0.14 cm; 0.17 cm

36) A function that converts dress sizes in the United States to those in Europe is given by $D(x) = 2x + 24$.

a) Find the European dress sizes that correspond to sizes 6, 8, 10, and 12 in the United States.

b) Find the function that converts European dress sizes to U.S. dress sizes.

c) Use part b. to find the dress sizes in the United States that correspond to 46, 52, 62, and 70.

37) [T] The cost to remove a toxin from a lake is modeled by the function $C(p) = \dfrac{75p}{85-p}$, where C is the cost (in thousands of dollars) and p is the amount of toxin in a small lake (measured in parts per billion [ppb]). This model is valid only when the amount of toxin is less than 85 ppb.

a) Find the cost to remove 25 ppb, 40 ppb, and 50 ppb of the toxin from the lake.

b) Find the inverse function.

c) Use part b. to determine how much of the toxin is removed for $50,000.

Answer

a. $31,250, $66,667, $107,143
b. $p = \dfrac{85C}{C+75}$
c. 34 ppb

38) [T] A race car is accelerating at a velocity given by $v(t) = \frac{25}{4}t + 54,$

where v is the velocity (in feet per second) at time t.

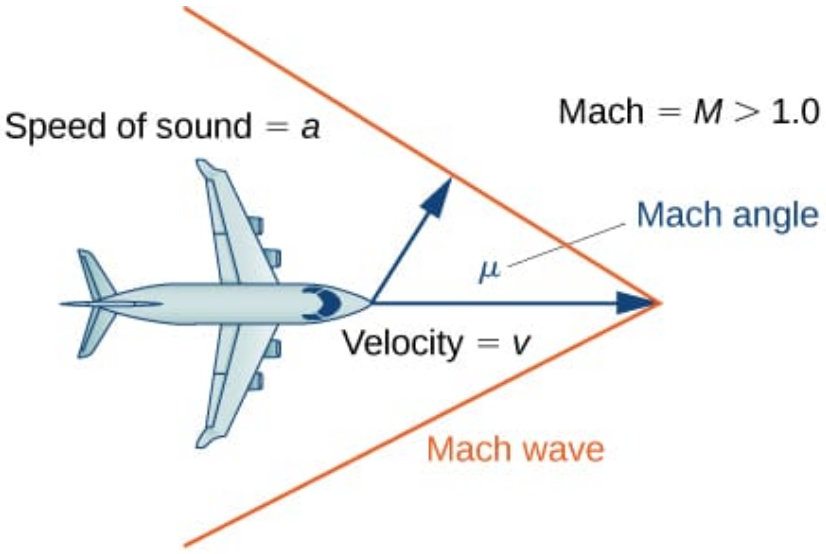
a) Find the velocity of the car at 10 sec.

b) Find the inverse function.

c) Use part b. to determine how long it takes for the car to reach a speed of 150 ft/sec.

39) [T] An airplane's Mach number M is the ratio of its speed to the speed of sound. When a plane is flying at a constant altitude, then its Mach angle is given by $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$.

Find the Mach angle (to the nearest degree) for the following Mach numbers.



a. $\mu = 1.4$

b. $\mu = 2.8$

c. $\mu = 4.3$

Answer

a. $\sim 92\,^\circ$ b. $\sim 42\,^\circ$ c. $\sim 27\,^\circ$

40) [T] Using $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$, find the Mach number M for the following angles.

a. $\mu = \frac{\pi}{6}$

b. $\mu = \frac{2\pi}{7}$

c. $\mu = \frac{3\pi}{8}$

41) [T] The temperature (in degrees Celsius) of a city in the northern United States can be modeled by the function

$$T(x) = 5 + 18\sin\left[\tfrac{\pi}{6}(x - 4.6)\right],$$

where x is time in months and $x = 1.00$ corresponds to January 1. Determine the month and day when the temperature is $21\ ^{\circ}C$.

Answer

$x \approx 6.69,\ 8.51$; so, the temperature occurs on June 21 and August 15

42) [T] The depth (in feet) of water at a dock changes with the rise and fall of tides. It is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Determine the first time after midnight when the depth is 11.75 ft.

43) [T] An object moving in simple harmonic motion is modeled by the function $s(t) = -6\cos\left(\dfrac{\pi t}{2}\right)$, where s is measured in inches and t is measured in seconds. Determine the first time when the distance moved is 4.5 in.

Answer

~ 1.5 sec

44) [T] A local art gallery has a portrait 3 ft in height that is hung 2.5 ft above the eye level of an average person. The viewing angle θ can be modeled by the function $\theta = \tan^{-1}\frac{5.5}{x} - \tan^{-1}\frac{2.5}{x}$, where x is the distance (in feet) from the portrait. Find the viewing angle when a person is 4 ft from the portrait.

45) [T] Use a calculator to evaluate $\tan^{-1}(\tan(2.1))$ and $\cos^{-1}(\cos(2.1))$. Explain the results of each.

Answer

$\tan^{-1}(\tan(2.1)) \approx -1.0416$; the expression does not equal 2.1 since $2.1 > 1.57 = \frac{\pi}{2}$—in other words, it is not in the restricted domain of $\tan x$. $\cos^{-1}(\cos(2.1)) = 2.1$, since 2.1 is in the restricted domain of $\cos x$.

46) [T] Use a calculator to evaluate $\sin(\sin^{-1}(-2))$ and $\tan(\tan^{-1}(-2))$. Explain the results of each.

Contributors

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12.6: Exponential and Logarithmic Functions

Learning Objectives

- Identify the form of an exponential function.
- Explain the difference between the graphs of x^b and b^x.
- Recognize the significance of the number e.
- Identify the form of a logarithmic function.
- Explain the relationship between exponential and logarithmic functions.
- Describe how to calculate a logarithm to a different base.
- Identify the hyperbolic functions, their graphs, and basic identities.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e. We also define hyperbolic and inverse hyperbolic functions, which involve combinations of exponential and logarithmic functions. (Note that we present alternative definitions of exponential and logarithmic functions in the chapter Applications of Integrations, and prove that the functions have the same properties with either definition.)

Exponential Functions

Exponential functions arise in many applications. One common example is **population growth.** For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

$$P(1) = P_0 + 0.02P_0 = P_0(1 + 0.02) = P_0(1.02).$$

Its population after 2 years is

$$P(2) = P(1) + 0.02P(1) = P(1)(1.02) = P_0(1.02)^2.$$

In general, its population after t years is

$$P(t) = P_0(1.02)^t,$$

which is an exponential function. More generally, any function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$, is an exponential function with **base** b and **exponent** x. Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is not an exponential function but a power function.

To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$. In Table $12.6.1$, we see that both 2^x and x^2 approach infinity as $x \to \infty$. Eventually, however, 2^x becomes larger than x^2 and grows more rapidly as $x \to \infty$. In the opposite direction, as $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$. The line $y = 0$ is a horizontal asymptote for $y = 2^x$.

Table $12.6.1$

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

In Figure $12.6.1$, we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.

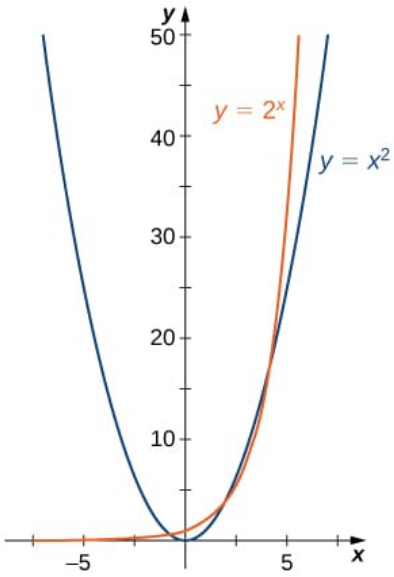


Figure 12.6.1: Both 2^x and x^2 approach infinity as $x \to \infty$, but 2^x grows more rapidly than x^2. As $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$.

Evaluating Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y, and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number, then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = \left(\sqrt{9}\right)^3 = 27$. However, how is b^x defined if x is an irrational number? For example, what do we mean by $2^{\sqrt{2}}$? This is too complex a question for us to answer fully right now; however, we can make an approximation.

Table 12.6.2: Values of 2^x for a List of Rational Numbers Approximating $\sqrt{2}$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
2^x	2.639	2.65737	2.66475	2.665119	2.665138	2.665143

In Table 12.6.2, we list some rational numbers approaching $\sqrt{2}$, and the values of 2^x for each rational number x are presented as well. We claim that if we choose rational numbers x getting closer and closer to $\sqrt{2}$, the values of 2^x get closer and closer to some number L. We define that number L to be $2^{\sqrt{2}}$.

Example 12.6.1: Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting $t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Solution

The number of bacteria after 6 hours is given by

$$n(6) = 1000 \cdot 2^{6/4} \approx 2828 \,\text{bacteria}.$$

The number of bacteria after 10 hours is given by

$$n(10) = 1000 \cdot 2^{10/4} \approx 5657 \,\text{bacteria}.$$

The number of bacteria after 24 hours is given by $n(24) = 1000 \cdot 2^6 = 64,000$ bacteria.

Exercise 12.6.1

Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Answer available at the end of this section.

Graphing Exponential Functions

For any base $b > 0$, $b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x, we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \to \infty$ as $x \to \infty$, whereas $b^x \to 0$ as $x \to -\infty$. On the other hand, if \(0

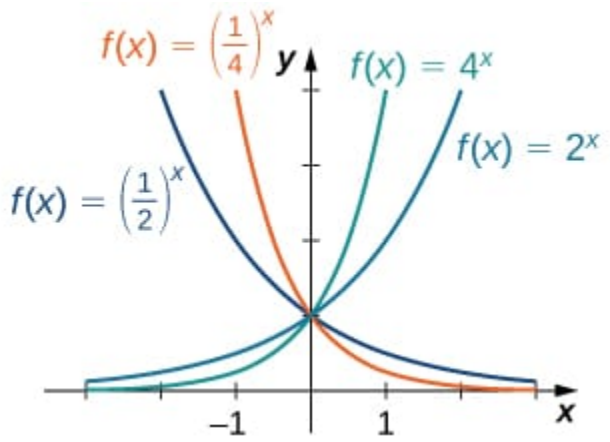


Figure 12.6.2: If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If \(0

Note that exponential functions satisfy the general laws of exponents. To remind you of these laws, we state them as rules.

Laws of Exponents

For any constants $a > 0$, $b > 0$, and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Example 12.6.2: Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

a. $\dfrac{(2x^{2/3})^3}{(4x^{-1/3})^2}$

b. $\dfrac{(x^3y^{-1})^2}{(xy^2)^{-2}}$

Soution

a. We can simplify as follows:

$$\frac{(2x^{2/3})^3}{(4x^{-1/3})^2} = \frac{2^3(x^{2/3})^3}{4^2(x^{-1/3})^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{(x^3y^{-1})^2}{(xy^2)^{-2}} = \frac{(x^3)^2(y^{-1})^2}{x^{-2}(y^2)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

Exercise 12.6.2

Use the laws of exponents to simplify $\frac{6x^{-3}y^2}{12x^{-4}y^5}$.

Hint available at the end of this section.

Answer available at the end of this section.

The Number e

A special type of exponential function appears frequently in real-world applications. To describe it, consider the following example of exponential growth, which arises from **compounding interest** in a savings account. Suppose a person invests P dollars in a savings account with an annual interest rate r, compounded annually. The amount of money after 1 year is

$$A(1) = P + rP = P(1 + r).$$

The amount of money after 2 years is

$$A(2) = A(1) + rA(1) = P(1 + r) + rP(1 + r) = P(1 + r)^2.$$

More generally, the amount after t years is

$$A(t) = P(1 + r)^t.$$

If the money is compounded 2 times per year, the amount of money after half a year is

$$A\left(\frac{1}{2}\right) = P + \left(\frac{r}{2}\right)P = P\left(1 + \left(\frac{r}{2}\right)\right).$$

The amount of money after 1 year is

$$A(1) = A\left(\frac{1}{2}\right) + \left(\frac{r}{2}\right)A\left(\frac{1}{2}\right) = P\left(1 + \frac{r}{2}\right) + \frac{r}{2}\left(\left(P(1 + \frac{r}{2}\right)\right) = P\left(1 + \frac{r}{2}\right)^2.$$

After t years, the amount of money in the account is

$$A(t) = P\left(1 + \frac{r}{2}\right)^{2t}.$$

More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

What happens as $n \to \infty$? To answer this question, we let $m = n/r$ and write

$$\left(1+\frac{r}{n}\right)^{nt}=\left(1+\frac{1}{m}\right)^{mrt},$$

and examine the behavior of $(1+1/m)^m$ as $m\to\infty$, using a table of values (Table 12.6.3).

Table 12.6.3: Values of $\left(1+\frac{1}{m}\right)^m$ *as* $m\to\infty$

m	10	100	1000	10,000	100,000	1,000,000
$\left(1+\frac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Looking at this table, it appears that $(1+1/m)^m$ is approaching a number between 2.7 and 2.8 as $m\to\infty$. In fact, $(1+1/m)^m$ does approach some number as $m\to\infty$. We call this number e. To six decimal places of accuracy,

$$e\approx 2.718282.$$

Leonhard Euler

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r, compounded continuously, then $A(t)=Pe^{rt}$. This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x)=e^x$ the **natural exponential function**. Not only is this function interesting because of the definition of the number e, but also, as discussed next, its graph has an important property.

Since $e>1$, we know $f(x)=e^x$ is increasing on $(-\infty,\infty)$. In Figure 12.6.3, we show a graph of $f(x)=e^x$ along with a tangent line to the graph of f at $x=0$. We give a precise definition of tangent line in the next chapter; but, informally, we say a tangent line to a graph of f at $x=a$ is a line that passes through the point $(a,f(a))$ and has the same "slope" as f at that point . The function $f(x)=e^x$ is the only exponential function b^x with tangent line at $x=0$ that has a slope of 1. As we see later in the text, having this property makes the natural exponential function the most simple exponential function to use in many instances.

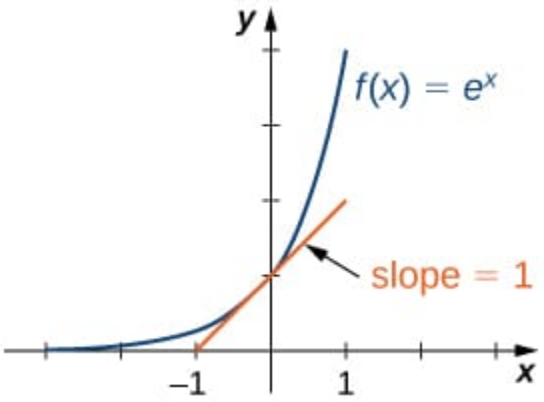


Figure 12.6.3: The graph of $f(x)=e^x$ has a tangent line with slope 1 at $x=0$.

Example 12.6.3: Compounding Interest

Suppose $\$500$ is invested in an account at an annual interest rate of $r=5.5\%$, compounded continuously.

a. Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t. Find a formula for $A(t)$.

b. Find the amount of money in the account after 10 years and after 20 years.

Solution

a. If P dollars are invested in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.

b. After 10 years, the amount of money in the account is

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

Exercise 12.6.3

If $\$750$ is invested in an account at an annual interest rate of 4%, compounded continuously, find a formula for the amount of money in the account after t years. Find the amount of money after 30 years.

Hint available at the end of this section.

Answer available at the end of this section.

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as the pH scale in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is *one-to-one*, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the logarithmic function with base b. For any $b > 0,\ b \neq 1$, the logarithmic function with base b, denoted $\log_b$, has domain $(0, \infty)$ and range $(-\infty, \infty)$,and satisfies

$$\log_b(x) = y$$

if and only if $b^y = x$.

For example,

$$\log_2(8) = 3$$

since $2^3 = 8$,

$$\log_{10}\left(\frac{1}{100}\right) = -2$$

since $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$,

$$\log_b(1) = 0$$

since $b^0 = 1$ for any base $b > 0$.

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

$$\log_b(b^x) = x$$

and

$$b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

$$\ln(e) = \log_e(e) = 1$$
$$\ln(e^3) = \log_e(e^3) = 3$$
$$\ln(1) = \log_e(1) = 0.$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ (Figure $12.6.4$).

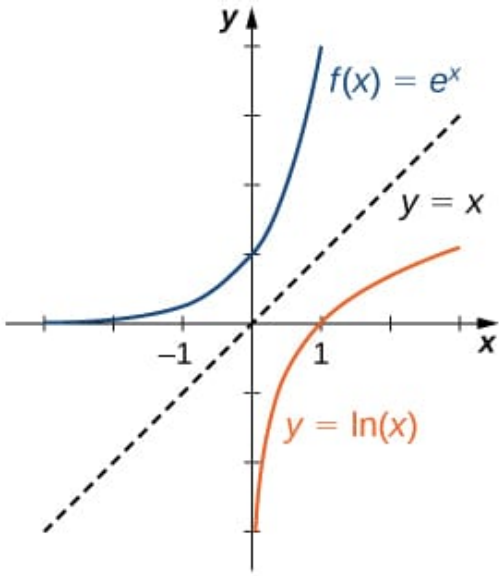


Figure $12.6.4$: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

In general, for any base $b > 0$, $b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ (Figure $12.6.5$).

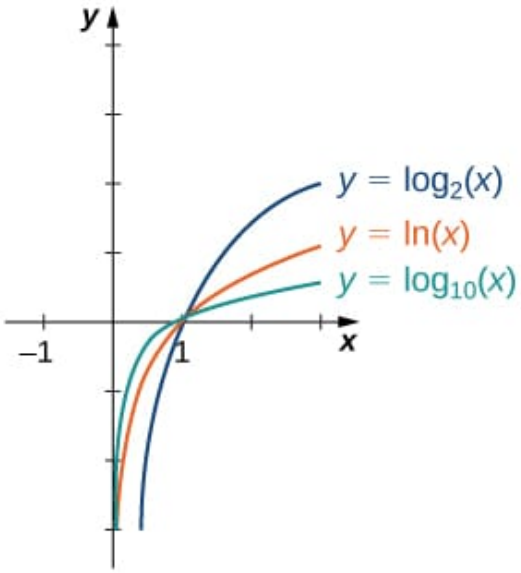


Figure 12.6.5: Graphs of $y = \log_b(x)$ are depicted for $b = 2,\ e,\ 10$.

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Properties of Logarithms

If $a,\ b,\ c > 0,\ b \neq 1$, and r is any real number, then

- **Product property**

$$\log_b(ac) = \log_b(a) + \log_b(c) \tag{12.6.1}$$

- **Quotient property**

$$\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c) \tag{12.6.2}$$

- **Power property**

$$\log_b(a^r) = r\log_b(a) \tag{12.6.3}$$

Example 12.6.4: Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$
b. $e^x + 6e^{-x} = 5$

Solution

a. Applying the natural logarithm function to both sides of the equation, we have

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

$$x \ln 5 = \ln 2.$$

Therefore,

$$x = \frac{\ln 2}{\ln 5}.$$

b. Multiplying both sides of the equation by e^x,we arrive at the equation

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Exercise 12.6.4

Solve

$$e^{2x}/(3 + e^{2x}) = 1/2.$$

Hint available at the end of this section.

Answer available at the end of this section.

Example 12.6.5: Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\frac{1}{x}\right) = 4$
b. $\log_{10}\sqrt{x} + \log_{10} x = 2$
c. $\ln(2x) - 3\ln(x^2) = 0$

Solution

a. By the definition of the natural logarithm function,

$$\ln\left(\frac{1}{x}\right) = 4$$

- if and only if $e^4 = \frac{1}{x}$.

Therefore, the solution is $x = 1/e^4$.

b. Using the product (Equation 12.6.1) and power (Equation 12.6.3) properties of logarithmic functions, rewrite the left-hand side of the equation as

$$\begin{aligned}\log_{10}\sqrt{x}+\log_{10}x &= \log_{10}x\sqrt{x}\\ &= \log_{10}x^{3/2}\\ &= \frac{3}{2}\log_{10}x.\end{aligned}$$

Therefore, the equation can be rewritten as

$$\frac{3}{2}\log_{10}x = 2$$

or

$$\log_{10}x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

c. Using the power property (Equation 12.6.3) of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln(x^6) = 0$.

Using the quotient property (Equation 12.6.2), this becomes

$$\ln\left(\frac{2}{x^5}\right) = 0$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

Exercise 12.6.5

Solve $\ln(x^3) - 4\ln(x) = 1$.

Hint available at the end of this section.

Answer available at the end of this section.

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are $\log_{10}$ or $\log$, called the **common logarithm**, or $\ln$, which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base b. If you need to use a calculator to evaluate an expression with a different base, you can apply the change-of-base formulas first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Rule: Change-of-Base Formulas

Let $a > 0,\ b > 0$, and $a \neq 1,\ b \neq 1$.

1. $a^x = b^{x\log_b a}$ for any real number x.

   If $b = e$, this equation reduces to $a^x = e^{x\log_e a} = e^{x\ln a}$.

2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0$.

   If $b = e$, this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}$.

Proof

For the first change-of-base formula, we begin by making use of the power property of logarithmic functions. We know that for any base $b > 0,\ b \neq 1$, $\log_b(a^x) = x \log_b a$. Therefore,

$$b^{\log_b(a^x)} = b^{x \log_b a}.$$

In addition, we know that b^x and $\log_b(x)$ are inverse functions. Therefore,

$$b^{\log_b(a^x)} = a^x.$$

Combining these last two equalities, we conclude that $a^x = b^{x \log_b a}$.

To prove the second property, we show that

$$(\log_b a) \cdot (\log_a x) = \log_b x.$$

Let $u = \log_b a, v = \log_a x$, and $w = \log_b x$. We will show that $u \cdot v = w$. By the definition of logarithmic functions, we know that $b^u = a,\ a^v = x$, and $b^w = x$. From the previous equations, we see that

$$b^{uv} = (b^u)^v = a^v = x = b^w.$$

Therefore, $b^{uv} = b^w$. Since exponential functions are one-to-one, we can conclude that $u \cdot v = w$.

□

Example 12.6.6: Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earlier.

Solution

Use the second equation with $a = 3$ and $b = e$: $\log_3 7 = \dfrac{\ln 7}{\ln 3} \approx 1.77124$.

Exercise 12.6.6

Use the change-of-base formula and a calculating utility to evaluate $\log_4 6$.

Hint available at the end of this section.

Example 12.6.7: The Richter Scale for Earthquakes

In 1935, Charles Richter developed a scale (now known as the Richter scale) to measure the magnitude of an **earthquake**. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake?



Figure 12.6.6: (credit: modification of work by Robb Hannawacker, NPS)

A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

$$R_1 - R_2 = \log_{10}\left(\frac{A1}{A2}\right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

$$8 - 7 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore,

$$\log_{10}\left(\frac{A1}{A2}\right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2, we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

$$\log_{10}\left(\frac{A1}{A2}\right) = 8 - 6 = 2.$$

Therefore, $A_1 = 100A_2$.That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

Exercise 12.6.7

Compare the relative severity of a magnitude 8.4 earthquake with a magnitude 7.4 earthquake.

Hint available at the end of this section.

Hyperbolic Functions

The hyperbolic functions are defined in terms of certain combinations of e^x and e^{-x}. These functions arise naturally in various engineering and physics applications, including the study of water waves and vibrations of elastic membranes. Another common

use for a hyperbolic function is the representation of a hanging chain or cable, also known as a *catenary* (Figure 12.6.7). If we introduce a coordinate system so that the low point of the chain lies along the y-axis, we can describe the height of the chain in terms of a hyperbolic function. First, we define the **hyperbolic functions.**



Figure 12.6.7:The shape of a strand of silk in a spider's web can be described in terms of a hyperbolic function. The same shape applies to a chain or cable hanging from two supports with only its own weight. (credit: "Mtpaley", Wikimedia Commons)

Definitions: hyperbolic functions

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The name $\cosh$ rhymes with "gosh," whereas the name $\sinh$ is pronounced "cinch." $\tanh$, sech, csch, and $\coth$ are pronounced "tanch," "seech," "coseech," and "cotanch," respectively.

Using the definition of $\cosh(x)$ and principles of physics, it can be shown that the height of a hanging chain, such as the one in Figure 12.6.8, can be described by the function $h(x) = a\cosh(x/a) + c$ for certain constants a and c.

But why are these functions called hyperbolic functions? To answer this question, consider the quantity $\cosh^2 t - \sinh^2 t$. Using the definition of $\cosh$ and $\sinh$, we see that

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1.$$

This identity is the analog of the trigonometric identity $\cos^2 t + \sin^2 t = 1$. Here, given a value t, the point $(x, y) = (\cosh t,\ \sinh t)$ lies on the unit hyperbola $x^2 - y^2 = 1$ (Figure 12.6.8).

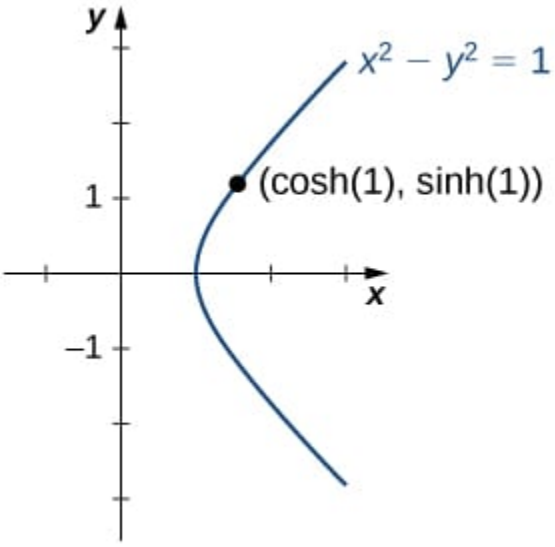


Figure 12.6.8: The unit hyperbola $\cosh^2 t - \sinh^2 t = 1$.

Graphs of Hyperbolic Functions

To graph $\cosh x$ and $\sinh x$, we make use of the fact that both functions approach $(1/2)e^x$ as $x \to \infty$, since $e^{-x} \to 0$ as $x \to \infty$. As $x \to -\infty, \cosh x$ approaches $1/2e^{-x}$, whereas $\sinh x$ approaches $-1/2e^{-x}$. Therefore, using the graphs of $1/2e^x, 1/2e^{-x}$, and $-1/2e^{-x}$ as guides, we graph $\cosh x$ and $\sinh x$. To graph $\tanh x$, we use the fact that $\tanh(0) = 0$, $-1 < \tanh(x) < 1$ for all x, $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$ as $x \to -\infty$. The graphs of the other three hyperbolic functions can be sketched using the graphs of $\cosh x$, $\sinh x$, and $\tanh x$ (Figure 12.6.9).

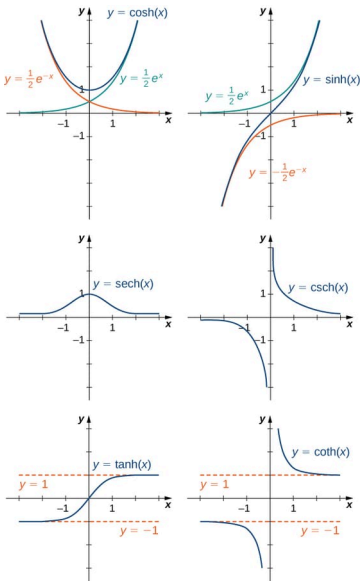


Figure 12.6.9: The hyperbolic functions involve combinations of e^x and e^{-x}.

Identities Involving Hyperbolic Functions

The identity $\cosh^2 t - \sinh^2 t = 1$, shown in Figure 12.6.8, is one of several identities involving the hyperbolic functions, some of which are listed next. The first four properties follow easily from the definitions of hyperbolic sine and hyperbolic cosine. Except for some differences in signs, most of these properties are analogous to identities for trigonometric functions.

Identities Involving Hyperbolic Functions

1. $\cosh(-x) = \cosh x$
2. $\sinh(-x) = -\sinh x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$
5. $\cosh^2 x - \sinh^2 x = 1$
6. $1 - \tanh^2 x = \operatorname{sech}^2 x$
7. $\coth^2 x - 1 = \operatorname{csch}^2 x$
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Example 12.6.8: Evaluating Hyperbolic Functions

a. Simplify $\sinh(5 \ln x)$.
b. If $\sinh x = 3/4$, find the values of the remaining five hyperbolic functions.

Solution:

a. Using the definition of the $\sinh$ function, we write

$$\sinh(5 \ln x) = \frac{e^{5 \ln x} - e^{-5 \ln x}}{2} = \frac{e^{\ln(x^5)} - e^{\ln(x^{-5})}}{2} = \frac{x^5 - x^{-5}}{2}.$$

b. Using the identity $\cosh^2 x - \sinh^2 x = 1$,we see that

$$\cosh^2 x = 1 + \left(\tfrac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x, we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5, \operatorname{csch} x = 4/3, \operatorname{sech} x = 4/5$, and $\coth x = 5/3$.

Exercise 12.6.8

Simplify $\cosh(2 \ln x)$.

Hint available at the end of this section.

Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh x$ and $\operatorname{sech} x$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the **inverse hyperbolic functions**. Since the hyperbolic functions themselves involve exponential functions, the inverse hyperbolic functions involve logarithmic functions.

Definitions: Inverse Hyperbolic Functions

$$\sinh^{-1} x = \operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \cosh^{-1} x = \operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \operatorname{arccot} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \operatorname{arcsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \operatorname{arccsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Let's look at how to derive the first equation. The others follow similarly. Suppose $y = \sinh^{-1} x$. Then, $x = \sinh y$ and, by the definition of the hyperbolic sine function, $x = \dfrac{e^y - e^{-y}}{2}$. Therefore,

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation, with the solution

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Since $e^y > 0$,the only solution is the one with the positive sign. Applying the natural logarithm to both sides of the equation, we conclude that

$$y = \ln(x + \sqrt{x^2+1}).$$

Example 12.6.9: Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

$$\sinh^{-1}(2)$$

$$\tanh^{-1}(1/4)$$

Solution:

$$\sinh^{-1}(2) = \ln(2 + \sqrt{2^2 + 1}) = \ln(2 + \sqrt{5}) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \frac{1}{2}\ln\left(\frac{1 + 1/4}{1 - 1/4}\right) = \frac{1}{2}\ln\left(\frac{5/4}{3/4}\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right) \approx 0.2554$$

Exercise 12.6.9

Evaluate $\tanh^{-1}(1/2)$.

Hint available at the end of this section.

Key Concepts

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if \(0
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a, we can make a change of base to convert this function to any base $b > 0$, $b \neq 1$. We typically convert to base e.
- The hyperbolic functions involve combinations of the exponential functions e^x and e^{-x}. As a result, the inverse hyperbolic functions involve the natural logarithm.

Glossary

base
: the number b in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

exponent
: the value x in the expression b^x

hyperbolic functions
: the functions denoted $\sinh$, $\cosh$, $\tanh$, csch, sech, and $\coth$, which involve certain combinations of e^x and e^{-x}

inverse hyperbolic functions
: the inverses of the hyperbolic functions where $\cosh$ and sech are restricted to the domain $[0, \infty)$;each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

natural exponential function
: the function $f(x) = e^x$

natural logarithm
: the function $\ln x = \log_e x$

number e
: as m gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be e; the value of e is approximately 2.718282

Answers

Exercise 12.6.1

$f(4) = 900$

$f(10) = 24,300.$

Exercise 12.6.2

$x^a/x^b = x^{a-b}$

Exercise 12.6.2

$x/(2y^3)$

Exercise 12.6.3

$A(t) = Pe^{rt}$

Exercise 12.6.3

$A(t) = 750e^{0.04t}$. After 30 years, there will be approximately $\$2,490.09$.

Exercise 12.6.4

First solve the equation for e^{2x}

Exercise 12.6.4

$x = \dfrac{\ln 3}{2}$.

Exercise 12.6.5

First use the power property, then use the product property of logarithms.

Exercise 12.6.5

$x = \dfrac{1}{e}$

Exercise 12.6.6

Use the change of base to rewrite this expression in terms of expressions involving the natural logarithm function.

$\log_4 6 = \dfrac{\ln 6}{\ln 4} \approx 1.29248$

Exercise 12.6.7

$R_1 - R_2 = \log_{10}(A1/A2)$.

The magnitude 8.4 earthquake is roughly 10 times as severe as the magnitude 7.4 earthquake.

Exercise 12.6.8

Use the definition of the $\cosh$ function and the power property of logarithm functions.

$(x^2 + x^{-2})/2$

Exercise 12.6.9

Use the definition of $\tanh^{-1} x$ and simplify.

$\dfrac{1}{2}\ln(3) \approx 0.5493$.

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12.6E: Exercises for Section 1.5

In exercises 1 - 4, evaluate the given exponential functions as indicated, accurate to two significant digits after the decimal.

1) $f(x) = 5^x$

a. $x = 3$

b. $x = \frac{1}{2}$

c. $x = \sqrt{2}$

Answer

a. 125
b. 2.24
c. 9.74

2) $f(x) = (0.3)^x$

a. $x = -1$

b. $x = 4$

c. $x = -1.5$

3) $f(x) = 10^x$

a. $x = -2$

b. $x = 4$

c. $x = \frac{5}{3}$

Answer

a. 0.01
b. $10,000$
c. 46.42

4) $f(x) = e^x$

a. $x = 2$

b. $x = -3.2$

c. $x = \pi$

In exercises 5 - 10, match the exponential equation to the correct graph.

a. $y = 4^{-x}$

b. $y = 3^{x-1}$

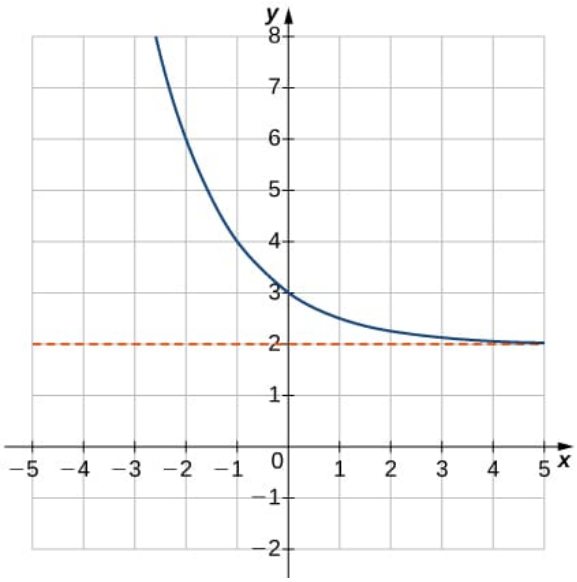
c. $y = 2^{x+1}$

d. $y = \left(\frac{1}{2}\right)^x + 2$

e. $y = -3^{-x}$

f. $y = 1 - 5^x$

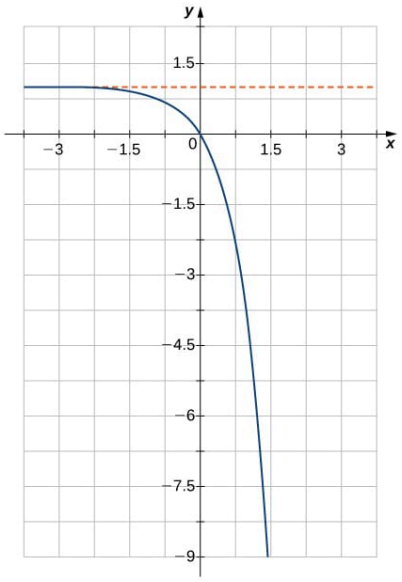
5)



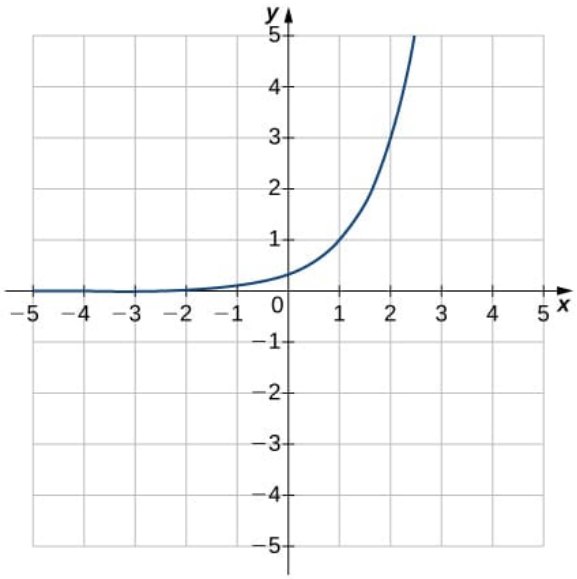
Answer

d

6)



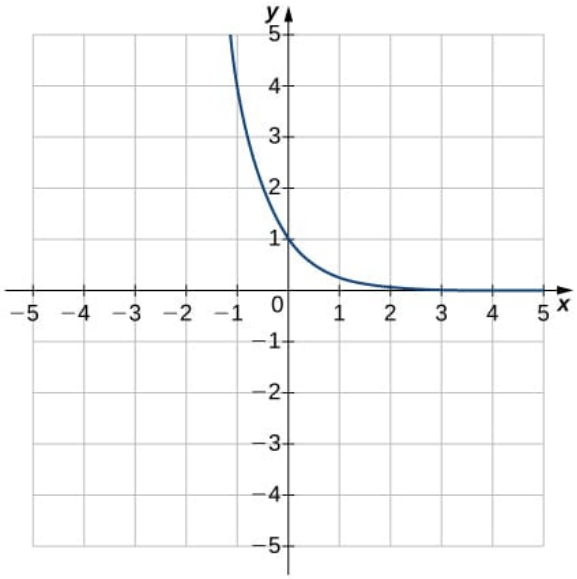
7)



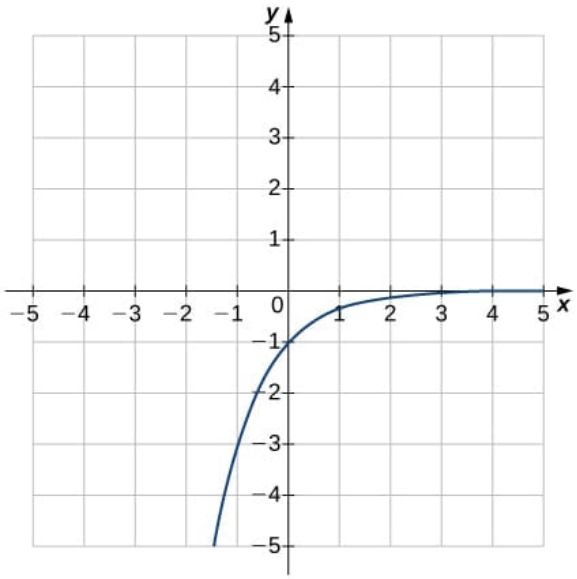
Answer

b

8)



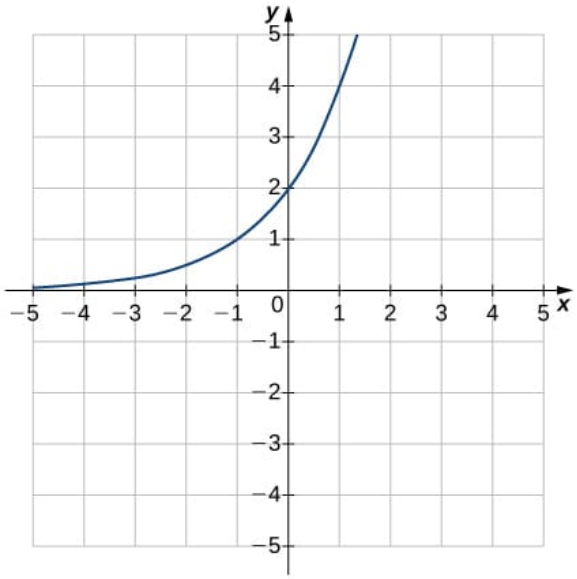
9)



Answer

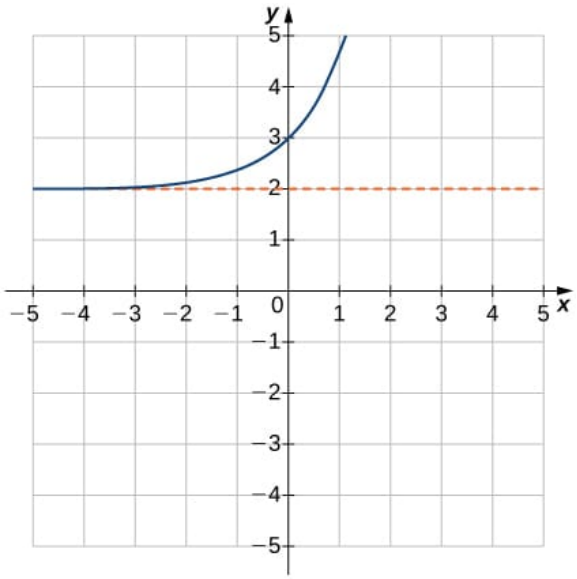
e

10)



In exercises 11 - 17, sketch the graph of the exponential function. Determine the domain, range, and horizontal asymptote.

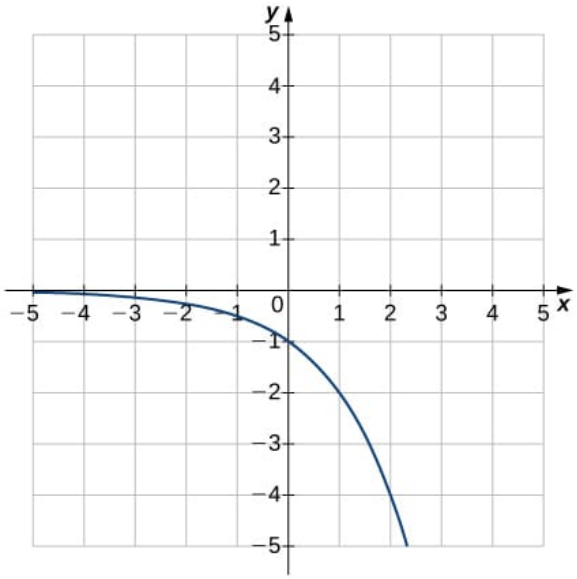
11) $f(x) = e^x + 2$



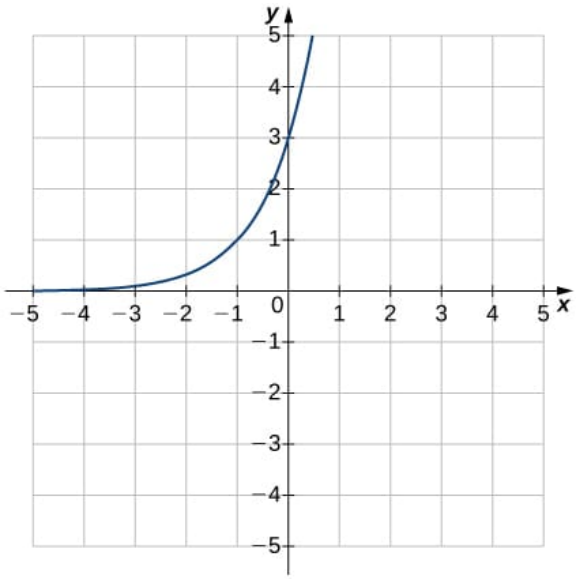
Answer

Domain: all real numbers, Range: $(2, \infty)$, $y = 2$

12) $f(x) = -2^x$



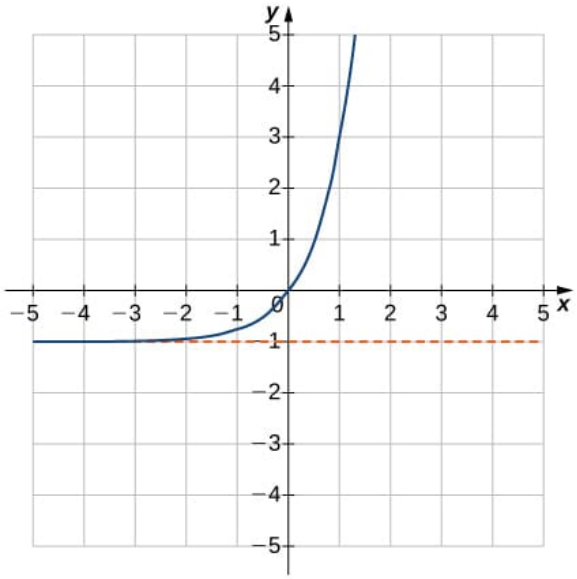
13) $f(x) = 3^{x+1}$



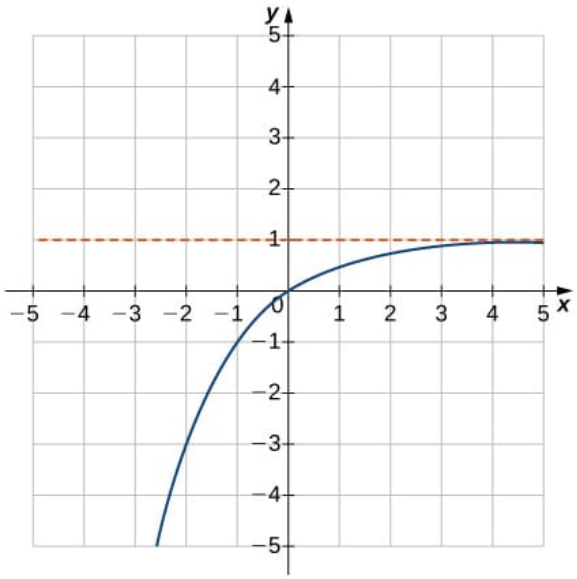
Answer

Domain: all real numbers, Range: $(0, \infty), \quad y = 0$

14) $f(x) = 4^x - 1$



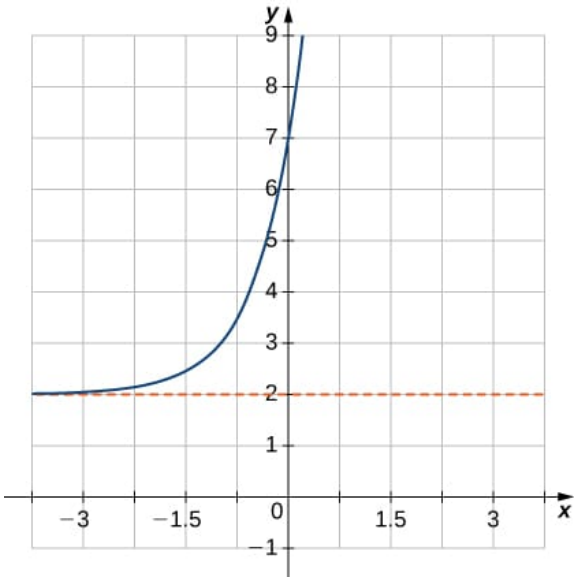
15) $f(x) = 1 - 2^{-x}$



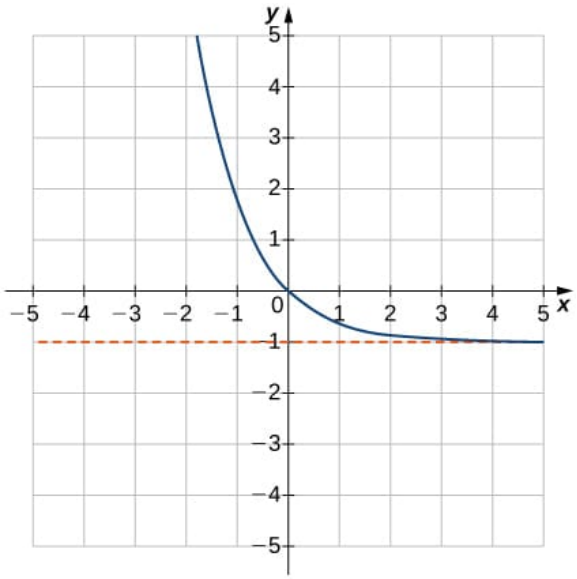
Answer

Domain: all real numbers, Range: $(-\infty, 1), \quad y = 1$

16) $f(x) = 5^{x+1} + 2$



17) $f(x) = e^{-x} - 1$



Answer

Domain: all real numbers, Range: $(-1, \infty)$, $y = -1$

In exercises 18 - 25, write the equation in equivalent exponential form.

18) $\log_3 81 = 4$

19) $\log_8 2 = \frac{1}{3}$

Answer

$8^{1/3} = 2$

20) $\log_5 1 = 0$

21) $\log_5 25 = 2$

Answer

$5^2 = 25$

22) $\log 0.1 = -1$

23) $\ln\left(\frac{1}{e^3}\right) = -3$

Answer

$e^{-3} = \frac{1}{e^3}$

24) $\log_9 3 = 0.5$

25) $\ln 1 = 0$

Answer

$e^0 = 1$

In exercises 26 - 35, write the equation in equivalent logarithmic form.

26) $2^3 = 8$

27) $4^{-2} = \frac{1}{16}$

Answer

$\log_4\left(\frac{1}{16}\right) = -2$

28) $10^2 = 100$

29) $9^0 = 1$

Answer

$\log_9 1 = 0$

30) $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

31) $\sqrt[3]{64} = 4$

Answer

$\log_{64} 4 = \frac{1}{3}$

32) $e^x = y$

33) $9^y = 150$

Answer

$\log_9 150 = y$

34) $b^3 = 45$

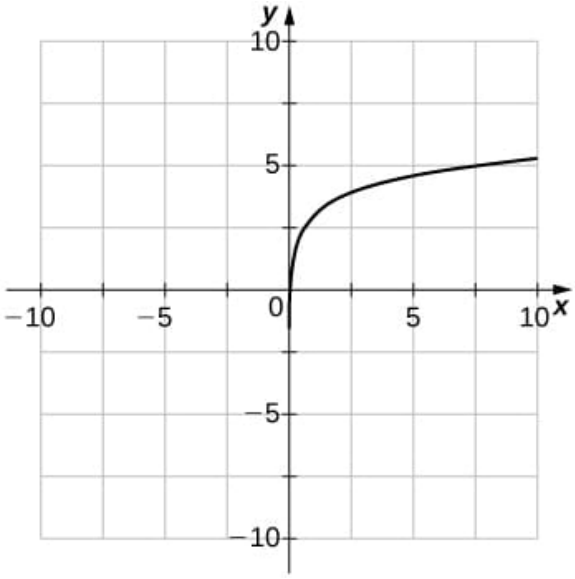
35) $4^{-3/2} = 0.125$

Answer

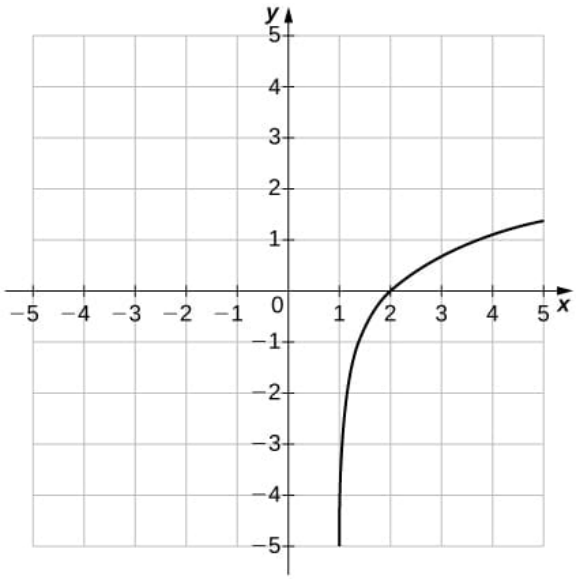
$\log_4 0.125 = -\frac{3}{2}$

In exercises 36 - 41, sketch the graph of the logarithmic function. Determine the domain, range, and vertical asymptote.

36) $f(x) = 3 + \ln x$



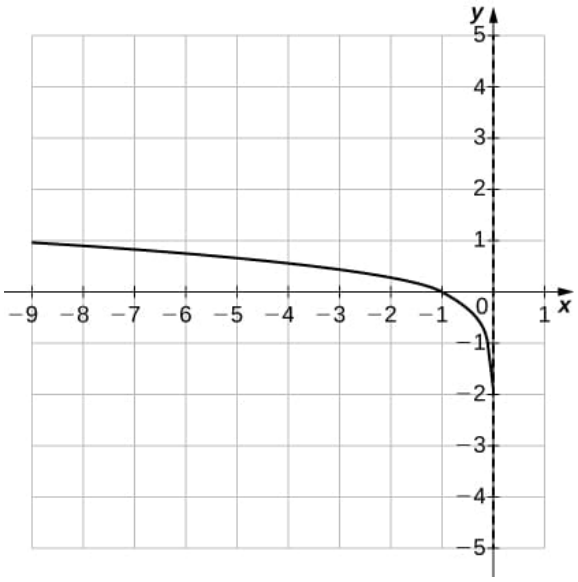
37) $f(x) = \ln(x - 1)$



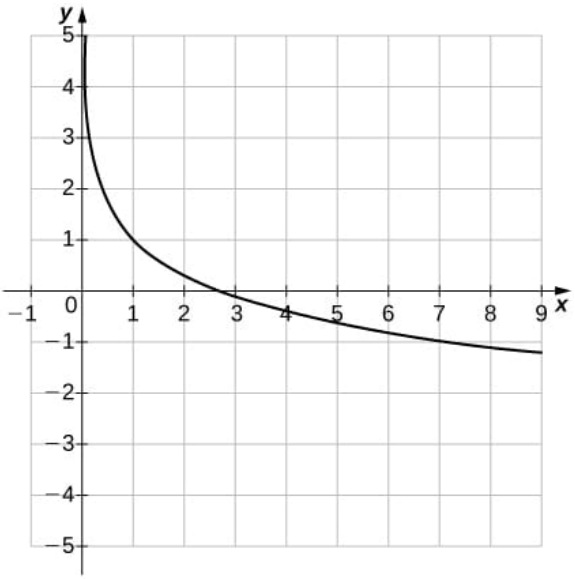
Answer

Domain: $(1, \infty)$, Range: $(-\infty, \infty)$, $x = 1$

38) $f(x) = \ln(-x)$



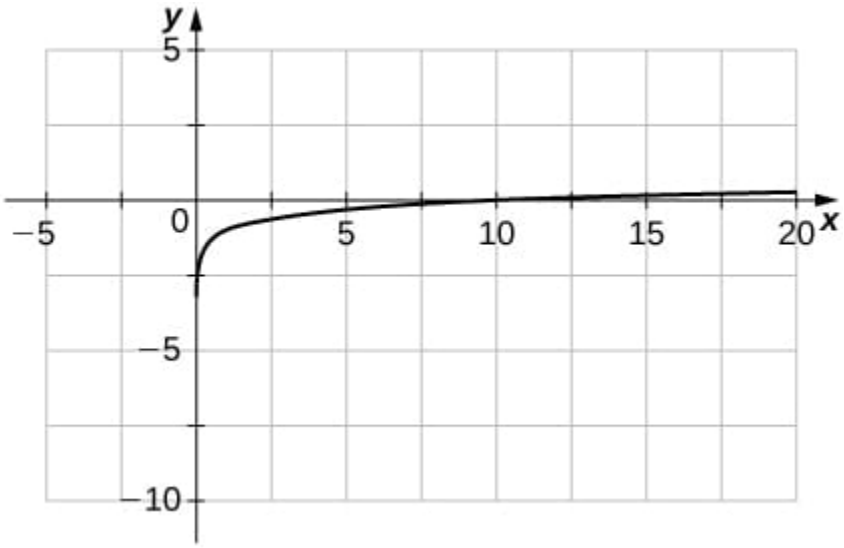
39) $f(x) = 1 - \ln x$



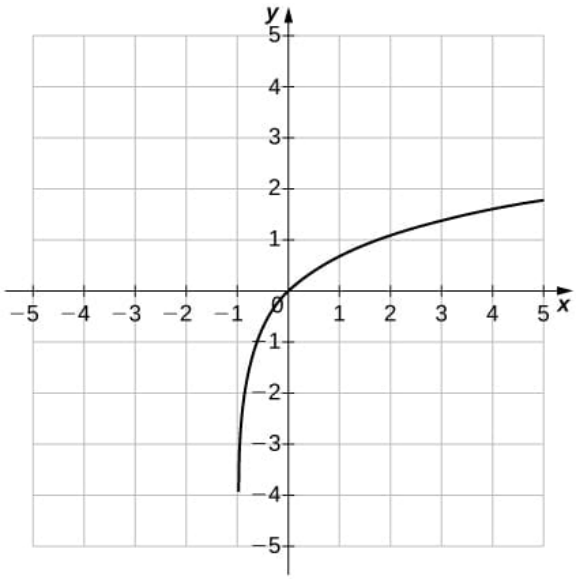
Answer

Domain: $(0, \infty)$, Range: $(-\infty, \infty)$, $x = 0$

40) $f(x) = \log x - 1$



41) $f(x) = \ln(x + 1)$



Answer

Domain: $(-1, \infty)$, Range: $(-\infty, \infty)$, $x = -1$

In exercises 42 - 47, use properties of logarithms to write the expressions as a sum, difference, and/or product of logarithms.

42) $\log x^4 y$

43) $\log_3 \frac{9a^3}{b}$

Answer

$2 + 3\log_3 a - \log_3 b$

44) $\ln a\sqrt[3]{b}$

45) $\log_5 \sqrt{125xy^3}$

Answer

$\frac{3}{2} + \frac{1}{2}\log_5 x + \frac{3}{2}\log_5 y$

46) $\log_4 \frac{\sqrt[3]{xy}}{64}$

47) $\ln\left(\frac{6}{\sqrt{e^3}}\right)$

Answer

$-\frac{3}{2} + \ln 6$

In exercises 48 - 55, solve the exponential equation exactly.

48) $5^x = 125$

49) $e^{3x} - 15 = 0$

Answer

$\frac{\ln 15}{3}$

50) $8^x = 4$

51) $4^{x+1} - 32 = 0$

Answer

$\frac{3}{2}$

52) $3^{x/14} = \frac{1}{10}$

53) $10^x = 7.21$

Answer

$\log 7.21$

54) $4 \cdot 2^{3x} - 20 = 0$

55) $7^{3x-2} = 11$

Answer

$\frac{2}{3} + \frac{\log 11}{3 \log 7}$

In exercises 56 - 63, solve the logarithmic equation exactly, if possible.

56) $\log_3 x = 0$

57) $\log_5 x = -2$

Answer

$x = \frac{1}{25}$

58) $\log_4(x + 5) = 0$

59) $\log(2x - 7) = 0$

Answer

$x = 4$

60) $\ln \sqrt{x + 3} = 2$

61) $\log_6(x + 9) + \log_6 x = 2$

Answer

$x = 3$

62) $\log_4(x + 2) - \log_4(x - 1) = 0$

63) $\ln x + \ln(x - 2) = \ln 4$

Answer

$1 + \sqrt{5}$

In exercises 64 - 69, use the change-of-base formula and either base 10 or base e to evaluate the given expressions. Answer in exact form and in approximate form, rounding to four decimal places.

64) $\log_5 47$

65) $\log_7 82$

Answer

$\frac{\log 82}{\log 7} \approx 2.2646$

66) $\log_6 103$

67) $\log_{0.5} 211$

Answer

$$\frac{\log 211}{\log 0.5} \approx -7.7211$$

68) $\log_2 \pi$

69) $\log_{0.2} 0.452$

Answer

$$\frac{\log 0.452}{\log 0.2} \approx 0.4934$$

70) Rewrite the following expressions in terms of exponentials and simplify.

a. $2\cosh(\ln x)$ b. $\cosh 4x + \sinh 4x$ c. $\cosh 2x - \sinh 2x$ d. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

71) [T] The number of bacteria N in a culture after t days can be modeled by the function $N(t) = 1300 \cdot (2)^{t/4}$. Find the number of bacteria present after 15 days.

Answer

$\sim 17,491$

72) [T] The demand D (in millions of barrels) for oil in an oil-rich country is given by the function $D(p) = 150 \cdot (2.7)^{-0.25p}$, where p is the price (in dollars) of a barrel of oil. Find the amount of oil demanded (to the nearest million barrels) when the price is between $15 and $20.

73) [T] The amount A of a $100,000 investment paying continuously and compounded for t years is given by $A(t) = 100,000 \cdot e^{0.055t}$. Find the amount A accumulated in 5 years.

Answer

Approximately $131,653 is accumulated in 5 years.

74) [T] An investment is compounded monthly, quarterly, or yearly and is given by the function $A = P\left(1 + \frac{j}{n}\right)^{nt}$, where A is the value of the investment at time t, P is the initial principle that was invested, j is the annual interest rate, and n is the number of time the interest is compounded per year. Given a yearly interest rate of 3.5% and an initial principle of $100,000, find the amount A accumulated in 5 years for interest that is compounded a. daily, b., monthly, c. quarterly, and d. yearly.

75) [T] The concentration of hydrogen ions in a substance is denoted by $[H+]$, measured in moles per liter. The pH of a substance is defined by the logarithmic function $pH = -\log[H+]$. This function is used to measure the acidity of a substance. The pH of water is 7. A substance with a pH less than 7 is an acid, whereas one that has a pH of more than 7 is a base.

a. Find the pH of the following substances. Round answers to one digit.

b. Determine whether the substance is an acid or a base.

i. Eggs: $[H+] = 1.6 \times 10^{-8}$ mol/L

ii. Beer: $[H+] = 3.16 \times 10^{-3}$ mol/L

iii. Tomato Juice: $[H+] = 7.94 \times 10^{-5}$ mol/L

Answer

i. a. pH = 8 b. Base
ii. a. pH = 3 b. Acid
iii. a. pH = 4 b. Acid

76) [T] Iodine-131 is a radioactive substance that decays according to the function $Q(t) = Q_0 \cdot e^{-0.08664t}$, where Q_0 is the initial quantity of a sample of the substance and t is in days. Determine how long it takes (to the nearest day) for 95% of a quantity to decay.

77) [T] According to the World Bank, at the end of 2013 $(t = 0)$ the U.S. population was 316 million and was increasing according to the following model:

$$P(t) = 316e^{0.0074t},$$

where P is measured in millions of people and t is measured in years after 2013.

a. Based on this model, what will be the population of the United States in 2020?

b. Determine when the U.S. population will be twice what it is in 2013.

Answer

a. ~ 333 million
b. 94 years from 2013, or in 2107

78) [T] The amount A accumulated after 1000 dollars is invested for t years at an interest rate of 4% is modeled by the function $A(t) = 1000(1.04)^t$.

a. Find the amount accumulated after 5 years and 10 years.

b. Determine how long it takes for the original investment to triple.

79) [T] A bacterial colony grown in a lab is known to double in number in 12 hours. Suppose, initially, there are 1000 bacteria present.

a. Use the exponential function $Q = Q_0e^{kt}$ to determine the value k, which is the growth rate of the bacteria. Round to four decimal places.

b. Determine approximately how long it takes for $200,000$ bacteria to grow.

Answer

a. $k \approx 0.0578$
b. ≈ 92 hours

80) [T] The rabbit population on a game reserve doubles every 6 months. Suppose there were 120 rabbits initially.

a. Use the exponential function $P = P_0a^t$ to determine the growth rate constant a. Round to four decimal places.

b. Use the function in part a. to determine approximately how long it takes for the rabbit population to reach 3500.

81) [T] The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time, in Japan, an earthquake with magnitude 4.9 caused only minor damage. Approximately how much more energy was released by the San Francisco earthquake than by the Japanese earthquake?

Answer

The San Francisco earthquake had $10^{3.4}$ or ~ 2512 times more energy than the Japan earthquake.

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12.R: Chapter 1 Review Exercises

True or False? Justify your answer with a proof or a counterexample.

1) A function is always one-to-one.

2) $f \circ g = g \circ f$, assuming f and g are functions.

Answer

False

3) A relation that passes the horizontal and vertical line tests is a one-to-one function.

4) A relation passing the horizontal line test is a function.

Answer

False

State the domain and range of the given functions:

$f = x^2 + 2x - 3$, $g = \ln(x - 5)$, $h = \dfrac{1}{x + 4}$

5) h

6) g

Answer

Domain: $x > 5$, Range: all real numbers

7) $h \circ f$

8) $g \circ f$

Answer

Domain: $x > 2$ and $x < -4$, Range: all real numbers

Find the degree, y-intercept, and zeros for the following polynomial functions.

9) $f(x) = 2x^2 + 9x - 5$

10) $f(x) = x^3 + 2x^2 - 2x$

Answer

Degree of 3, y-intercept: $(0, 0)$, Zeros: $0,\ \sqrt{3} - 1,\ -1 - \sqrt{3}$

Simplify the following trigonometric expressions.

11) $\dfrac{\tan^2 x}{\sec^2 x} + \cos^2 x$

12) $\cos^2 x - \sin^2 x$

Answer

$\cos(2x)$

Solve the following trigonometric equations on the interval $\theta = [-2\pi, 2\pi]$ exactly.

13) $6\cos 2x - 3 = 0$

14) $\sec^2 x - 2\sec x + 1 = 0$

Answer

$0, \pm 2\pi$

Solve the following logarithmic equations.

15) $5^x = 16$

16) $\log_2(x+4) = 3$

Answer

4

Are the following functions one-to-one over their domain of existence? Does the function have an inverse? If so, find the inverse $f^{-1}(x)$ of the function. Justify your answer.

17) $f(x) = x^2 + 2x + 1$

18) $f(x) = \dfrac{1}{x}$

Answer

One-to-one; yes, the function has an inverse; inverse: $f^{-1}(x) = \dfrac{1}{x}$

For the following problems, determine the largest domain on which the function is one-to-one and find the inverse on that domain.

19) $f(x) = \sqrt{9-x}$

20) $f(x) = x^2 + 3x + 4$

Answer

$x \geq -\frac{3}{2}, \quad f^{-1}(x) = -\frac{3}{2} + \frac{1}{2}\sqrt{4x-7}$

21) A car is racing along a circular track with diameter of 1 mi. A trainer standing in the center of the circle marks his progress every 5 sec. After 5 sec, the trainer has to turn 55° to keep up with the car. How fast is the car traveling?

For the following problems, consider a restaurant owner who wants to sell T-shirts advertising his brand. He recalls that there is a fixed cost and variable cost, although he does not remember the values. He does know that the T-shirt printing company charges $440 for 20 shirts and $1000 for 100 shirts.

22) a. Find the equation $C = f(x)$ that describes the total cost as a function of number of shirts and

b. determine how many shirts he must sell to break even if he sells the shirts for $10 each.

Answer

a. $C(x) = 300 + 7x$
b. 100 shirts

23) a. Find the inverse function $x = f^{-1}(C)$ and describe the meaning of this function.

b. Determine how many shirts the owner can buy if he has $8000 to spend.

For the following problems, consider the population of Ocean City, New Jersey, which is cyclical by season.

24) The population can be modeled by $P(t) = 82.5 - 67.5\cos[(\pi/6)t]$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). During a year, in what intervals is the population less than 20,000? During what intervals is the

population more than 140,000?

Answer

The population is less than 20,000 from December 8 through January 23 and more than 140,000 from May 29 through August 2

25) In reality, the overall population is most likely increasing or decreasing throughout each year. Let's reformulate the model as $P(t) = 82.5 - 67.5\cos[(\pi/6)t] + t$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). When is the first time the population reaches 200,000?

For the following problems, consider radioactive dating. A human skeleton is found in an archeological dig. Carbon dating is implemented to determine how old the skeleton is by using the equation $y = e^{rt}$, where y is the percentage of radiocarbon still present in the material, t is the number of years passed, and $r = -0.0001210$ is the decay rate of radiocarbon.

26) If the skeleton is expected to be 2000 years old, what percentage of radiocarbon should be present?

Answer

78.51%

27) Find the inverse of the carbon-dating equation. What does it mean? If there is 25% radiocarbon, how old is the skeleton?

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13: LIMITS

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

13.1: Prelude to Limits

13.2: A Preview of Calculus

13.2E: Exercises for Section 2.1

13.3: The Limit of a Function

13.3E: Exercises for Section 2.2

13.4: The Limit Laws

13.4E: Exercises for Section 2.3

13.5: Continuity

13.5E: Exercises for Section 2.4

13.6: The Precise Definition of a Limit

13.6E: Exercises for Section 2.5

13.R: Chapter 2 Review Exercises

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13.1: Prelude to Limits

Science fiction writers often imagine spaceships that can travel to far-off planets in distant galaxies. However, back in 1905, Albert Einstein showed that a limit exists to how fast any object can travel. The problem is that the faster an object moves, the more mass it attains (in the form of energy), according to the equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in the chapter)

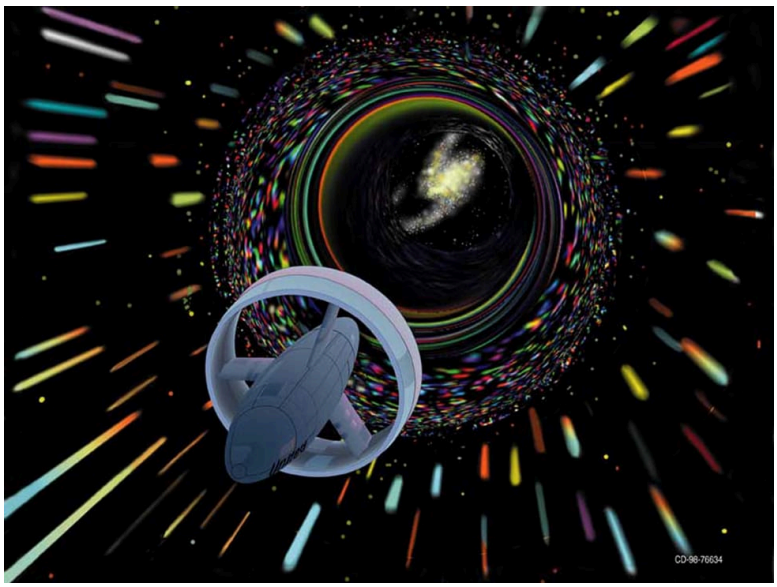


Figure 13.1.1:The vision of human exploration by the National Aeronautics and Space Administration (NASA) to distant parts of the universe illustrates the idea of space travel at high speeds. But, is there a limit to how fast a spacecraft can go? (credit: NASA)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

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13.2: A Preview of Calculus

Learning Objectives

- Describe the tangent problem and how it led to the idea of a derivative.
- Explain how the idea of a limit is involved in solving the tangent problem.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.
- Describe the area problem and how it was solved by the integral.
- Explain how the idea of a limit is involved in solving the area problem.
- Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in Figure $13.2.1$.

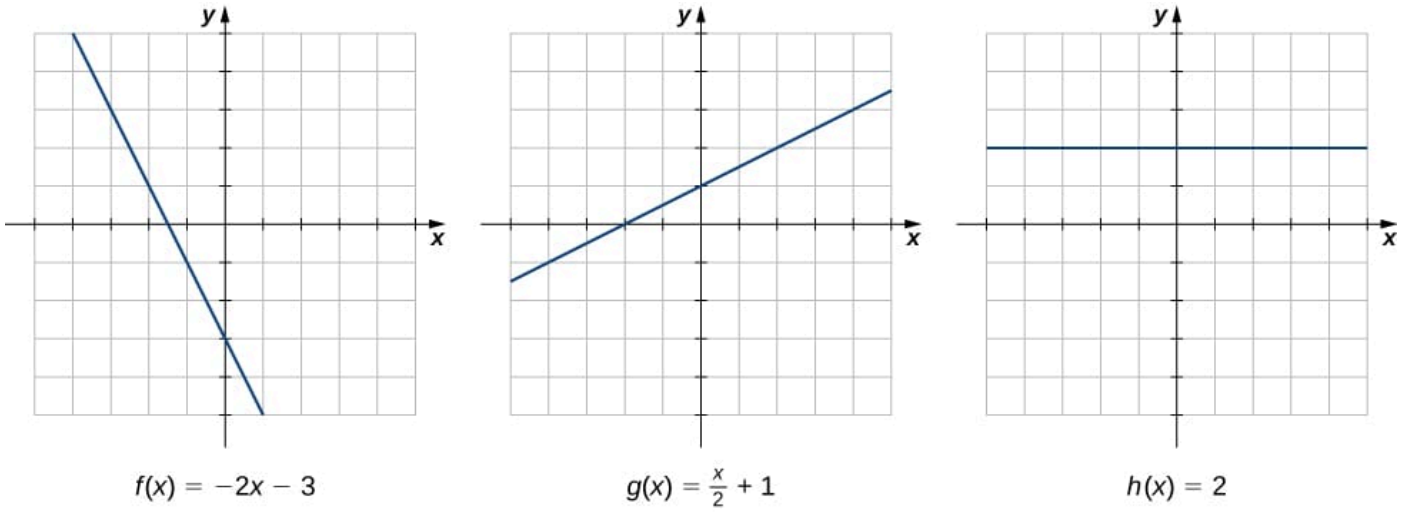


Figure $13.2.1$: The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This **rate of change** is determined by the slope (-2) of the line. Similarly, the slope of $1/2$ in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of $1/2$ unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (Figure $13.2.2$). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

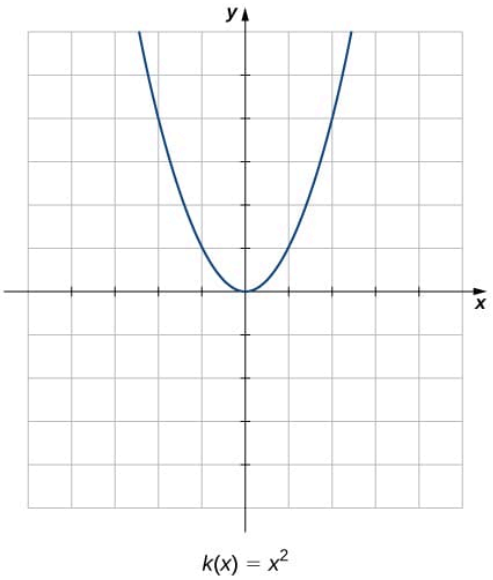


Figure 13.2.2: The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a secant line. Figure 13.2.3 shows a *secant line* to a function $f(x)$ at a point $(a, f(a))$.

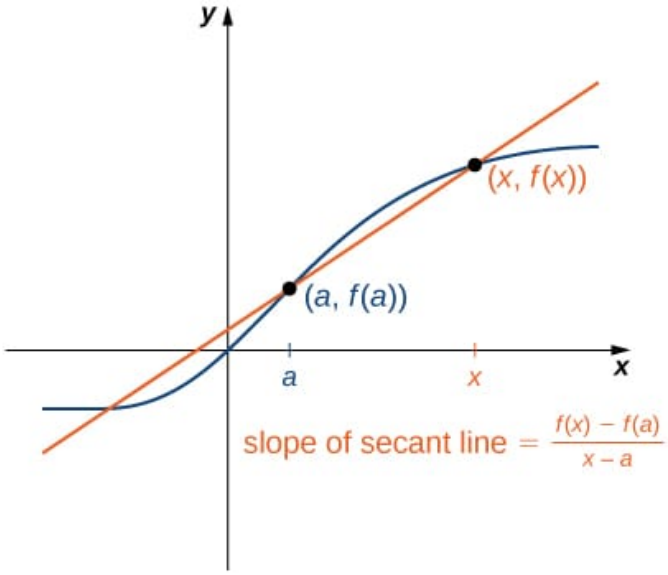


Figure 13.2.3: The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition: Secant Line

The *secant* to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{sec} = \frac{f(x) - f(a)}{x - a}. \tag{13.2.1}$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a. As we see in

Figure 13.2.4, if x is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.

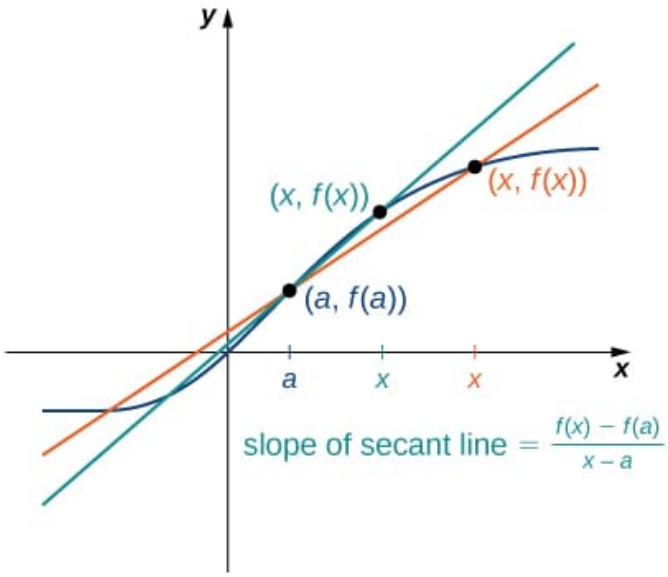


Figure 13.2.4: As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a (Figure 13.2.5). The slope of the tangent line to the graph at a measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

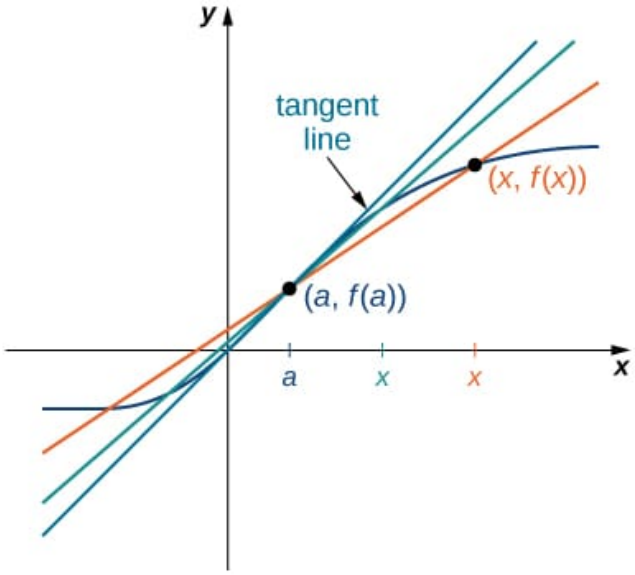


Figure 13.2.5: Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Example 13.2.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example 13.2.1: Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2,4)$
b. $\left(\frac{3}{2},\frac{9}{4}\right)$

Solution:

Use the formula for the slope of a secant line (Equation 13.2.1).

a. $m_{sec}=\frac{4-1}{2-1}=3$

b. $m_{sec}=\frac{\frac{9}{4}-1}{\frac{3}{2}-1}=\frac{5}{2}=2.5$

The point in part b. is closer to the point $(1,1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (Figure 13.2.6).

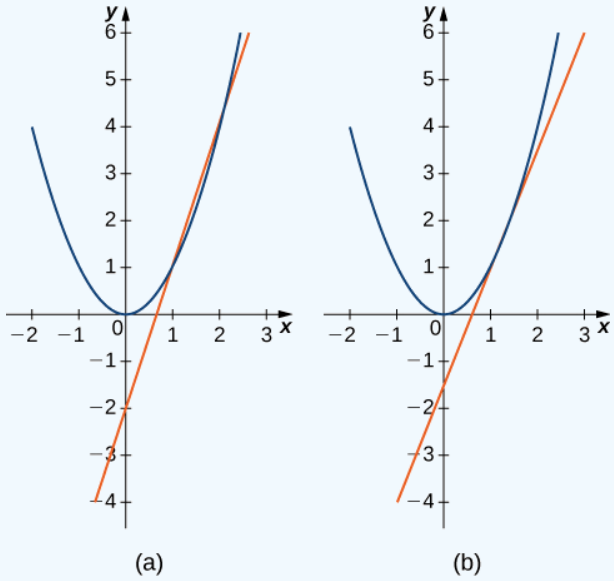


Figure 13.2.6: The secant lines to $f(x)=x^2$ at $(1,1)$ through (a) $(2,4)$ and (b) $(\frac{3}{2},\frac{9}{4})$ provide successively closer approximations to the tangent line to $f(x)=x^2$ at $(1,1)$.

Exercise 13.2.1

Estimate the slope of the tangent line (rate of change) to $f(x)=x^2$ at $x=1$ by finding slopes of secant lines through $(1,1)$ and the point $(\frac{5}{4},\frac{25}{16})$ on the graph of $f(x)=x^2$.

Answer available at the end of this section.

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time t. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t=a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the *average velocity* of an object over a time period to be the change in its position divided by the length of the time period.

Definition: Average Velocity

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The *average velocity* of the object over a time interval $[a, t]$ where \(a

$$v_{ave} = \frac{s(t) - s(a)}{t - a}. \tag{13.2.2}$$

As t is chosen closer to a, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point a, we let the x-values approach a in the slope of the secant line. Similarly, to find the instantaneous velocity at time a, we let the t-values approach a in the average velocity. This process of letting x or t approach a in an expression is called taking a **limit**. Thus, we may define the *instantaneous velocity* as follows.

Definition: Instantaneous Velocity

For a position function $s(t)$, the *instantaneous velocity* at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a, provided such a value exists.

Example $13.2.2$ illustrates this concept of limits and average velocity.

Example $13.2.2$: Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$
b. $[0.5, 0.51]$

Solution

Substitute the data into Equation $13.2.2$ for the definition of average velocity.

a. $$v_{ave} = \frac{s(0.49) - s(0.5)}{0.49 - 0.5} = -15.84$$

b. $$v_{ave} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.016$$

The instantaneous velocity is somewhere between –15.84 and –16.16 ft/sec. A good guess might be –16 ft/sec.

Exercise $13.2.2$

An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

Hint available at the end of this section.

Answer available at the end of this section.

The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may

be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the x-axis over an interval (Figure 13.2.7)?

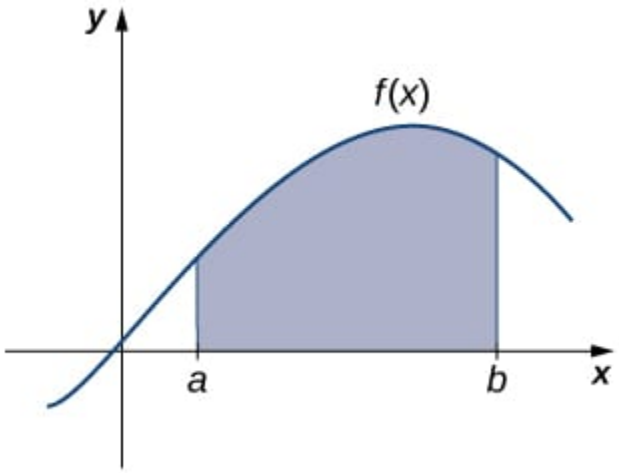


Figure 13.2.7: The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles (Figure 13.2.8).

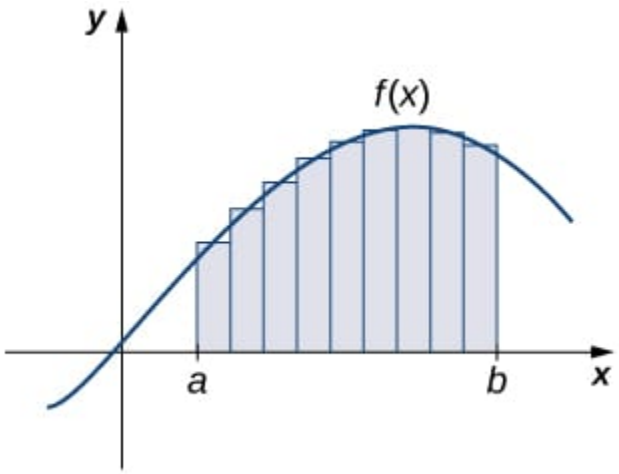


Figure 13.2.8: The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the x-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

Example 13.2.3: Estimation Using Rectangles

Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 13.2.9.

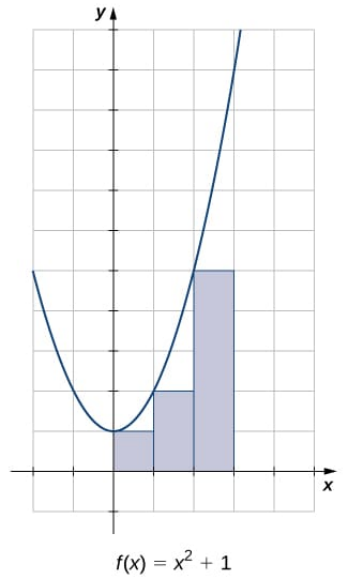


Figure 13.2.9: The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Solution

The areas of the three rectangles are 1 unit^2, 2 unit^2, and 5 unit^2. Using these rectangles, our area estimate is 8 unit^2.

Exercise 13.2.3

Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in Figure 13.2.10.

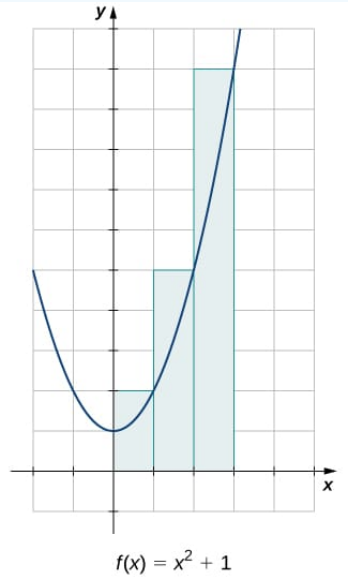


Figure 13.2.10: The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Hint available at the end of this section.

Answer available at the end of this section.

Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in Figure 13.2.11. These are only a few of the types of questions that can be asked and answered using *multivariable calculus*. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.

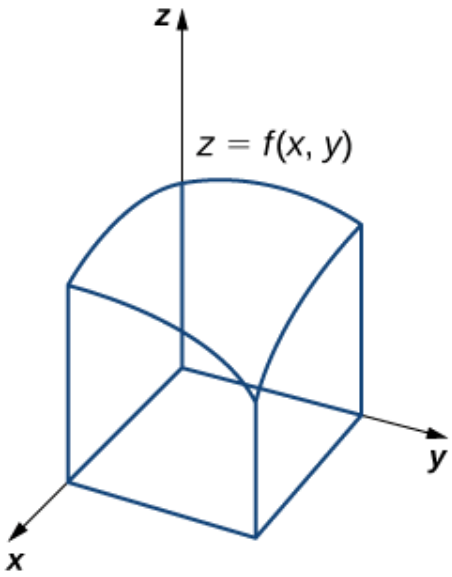


Figure 13.2.11: We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

Key Concepts

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the x-axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

Key Equations

- **Slope of a Secant Line**

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

- **Average Velocity over Interval [a,t]**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

Glossary

average velocity
: the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if \(ta\)), with a position given by $s(t)$, that is $v_{ave} = \dfrac{s(t) - s(a)}{t - a}$

differential calculus
: the field of calculus concerned with the study of derivatives and their applications

instantaneous velocity
: The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of t move closer to a, provided such a value exists

integral calculus
: the study of integrals and their applications

limit
: the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

multivariable calculus
: the study of the calculus of functions of two or more variables

secant
: A secant line to a function $f(x)$ at a is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{sec} = \dfrac{f(x) - f(a)}{x - a}$

tangent
: A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the rate of change of the function at a

Answers

Exercise 13.2.1

2.25

Exercise 13.2.2

Use Equation 13.2.2 with $v_{ave} = \dfrac{s(2.001) - s(2)}{2.001 - 2}$.

Exercise 13.2.2

12.006001

Exercise 13.2.3

Use Example 13.2.3 as a guide

Exercise 13.2.3

17 unit^2

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13.2E: Exercises for Section 2.1

For exercises 1 - 3 , points $P(1,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^2+1$.

1) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

Answer

a. 2.21000000
b. 2.02010000
c. 2.00200010
d. 2.00020000
e. (1.10000000, 2.21000000)
f. (1.01000000, 2.02010000)
g. (1.00100000, 2.00200010)
h. (1.00010000, 2.00020000)
i. 2.10000000
j. 2.01000000
k. 2.00100000
l. 2.00010000

2) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x=1$.

3) Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

Answer

$y=2x$

For the exercises 4-6, points $P(1,1)$ and $Q(x,y)$ are on the graph of the function $f(x)=x^3$.

4) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.2

5) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x=1$.

Answer

3

6) Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

For the exercises 7 - 9, points $P(4,2)$ and $Q(x,y)$ are on the graph of the function $f(x)=\sqrt{x}$.

7) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
4.1	a.	e.	i.
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

Answer

a. 2.0248457
b. 2.0024984
c. 2.0002500
d. 2.0000250
e. (4.1000000,2.0248457)
f. (4.0100000,2.0024984)
g. (4.0010000,2.0002500)
h. (4.00010000,2.0000250)
i. 0.24845673
j. 0.24984395
k. 0.24998438
l. 0.24999844

8) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x=4$.

9) Use the value in the preceding exercise to find the equation of the tangent line at point P.

Answer

$y=\frac{x}{4}+1$

For exercises 10 - 12, points $P(1.5,0)$ and $Q(\phi,y)$ are on the graph of the function $f(\phi)=\cos(\pi\phi)$.

10) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(\phi,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(\phi,y)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

11) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $\phi=1.5$.

Answer

π

12) Use the value in the preceding exercise to find the equation of the tangent line at point P.

For exercises 13 - 15, points $P(-1,-1)$ and $Q(x,y)$ are on the graph of the function $f(x)=\frac{1}{x}$.

13) [T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x,y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x,y)$	m_{sec}
-1.05	a.	e.	i.
-1.01	b.	f.	j.
-1.005	c.	g.	k.
-1.001	d.	h.	l.

Answer

a. −0.95238095
b. −0.99009901
c. −0.99502488
d. −0.99900100
e. (−1;.0500000,−0;.95238095)
f. (−1;.0100000,−0;.9909901)
g. (−1;.0050000,−0;.99502488)
h. (1.0010000,−0;.99900100)
i. −0.95238095
j. −0.99009901
k. −0.99502488
l. −0.99900100

14) Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x=-1$.

15) Use the value in the preceding exercise to find the equation of the tangent line at point P.

Answer

$y=-x-2$

For exercises 16 - 17, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t)=200-4.9t^2$, where position s is measured in meters and time t is measured in seconds. Round your answer to eight significant digits.

16) [T] Compute the average velocity of the ball over the given time intervals.

a. [4.99,5]

b. [5,5.01]

c. [4.999,5]

d. [5,5.001]

17) Use the preceding exercise to guess the instantaneous velocity of the ball at $t=5$ sec.

Answer

-49 m/sec (velocity of the ball is 49 m/sec downward)

For exercises 18 - 19, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t)=15t-4.9t^2$.

18) [T] Compute the average velocity of the stone over the given time intervals.

a. [1,1.05]

b. [1,1.01]

c. [1,1.005]

d. [1,1.001]

19) Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

Answer

5.2 m/sec

For exercises 20 - 21, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where t is measured in seconds.

20) [T] Compute the average velocity of the rocket over the given time intervals.

a. [9,9.01]

b. [8.99,9]

c. [9,9.001]

d. [8.999,9]

21) Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

Answer

-9.8 m/sec

For exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where d is the position in meters and t is the time elapsed, measured in seconds.

22) [T] Compute the average velocity of the runner over the given time intervals.

a. [1.95,2.05]

b. [1.995,2.005]

c. [1.9995,2.0005]

d. [2,2.00001]

23) Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

Answer

6 m/sec

For exercises 24 - 25, consider the function $f(x) = |x|$.

24) Sketch the graph of f over the interval $[-1, 2]$ and shade the region above the x-axis.

25) Use the preceding exercise to find the exact value of the area between the x-axis and the graph of f over the interval $[-1, 2]$ using rectangles. For the rectangles, use the square units, and approximate both above and below the lines. Use geometry to find the exact answer.

Answer

Under, 1 $unit^2$; over: 4 $unit^2$.
The exact area of the two triangles is $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = 2.5 units^2$.

For exercises 26 - 27, consider the function $f(x) = \sqrt{1 - x^2}$. (Hint: This is the upper half of a circle of radius 1 positioned at $(0, 0)$.)

26) Sketch the graph of f over the interval $[-1, 1]$.

27) Use the preceding exercise to find the exact area between the x-axis and the graph of f over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

Answer

Under, 0.96 units^2; over, 1.92 units^2).
The exact area of the semicircle with radius 1 is $\frac{\pi(1)^2}{2} = \frac{\pi}{2} \text{ units}^2$

For exercises 28 - 29, consider the function $f(x) = -x^2 + 1$.

28) Sketch the graph of f over the interval $[-1, 1]$.

29) Approximate the area of the region between the x-axis and the graph of f over the interval $[-1, 1]$.

Answer

Approximately $1.3333333 \text{ units}^2$

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13.3: The Limit of a Function

Learning Objectives

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

- $f(x) = \dfrac{x^2 - 4}{x - 2}$,
- $g(x) = \dfrac{|x - 2|}{x - 2}$, and
- $h(x) = \dfrac{1}{(x - 2)^2}$,

which are shown in Figure $13.3.1$. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

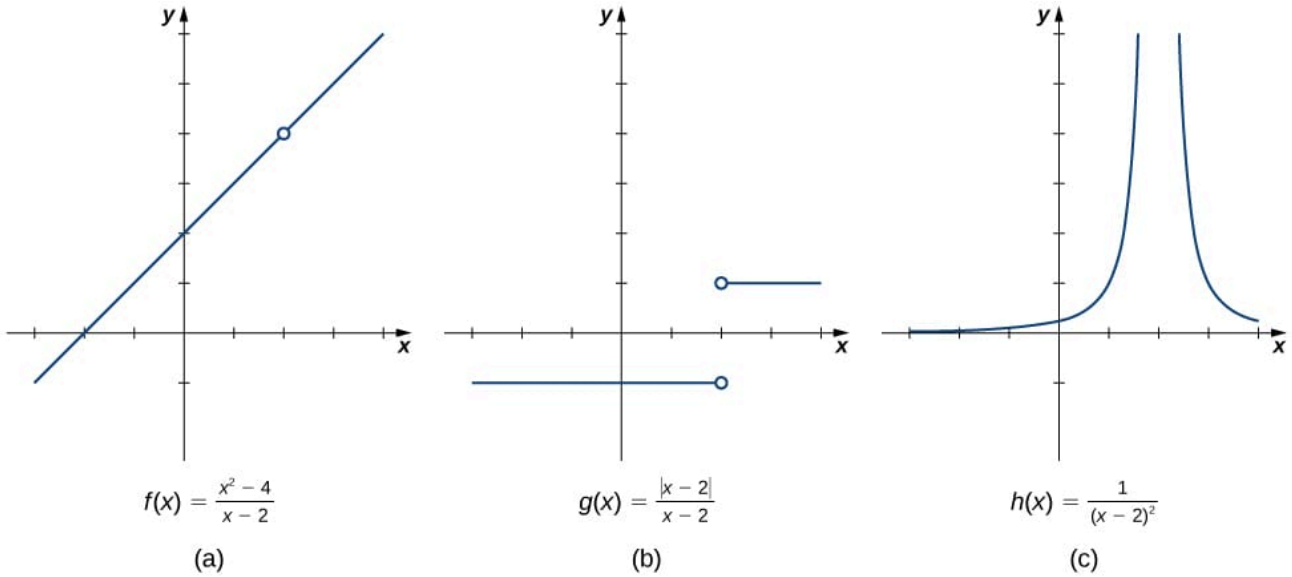


Figure $13.3.1$: These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in Figure $13.3.1$. As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4$$

From this very brief informal look at one limit, let's start to develop an *intuitive definition of the limit*. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition (Intuitive): Limit

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of $x (\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinctly, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \tag{13.3.1}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table $13.3.1$ demonstrates what your tables might look like.

Table $13.3.1$

x	$f(x)$	x	$f(x)$
$a - 0.1$	$f(a - 0.1)$	$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$	$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$	$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$	$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.		Use additional values as necessary.	

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a - 0.1)$, $f(a - 0.01)$, $f(a - 0.001)$, $f(a - 0.0001)$, and so on, and $f(a + 0.1)$, $f(a + 0.01)$, $f(a + 0.001)$, $f(a + 0.0001)$, and so on. (Note: Although we have chosen the x-values $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$, $a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)

3. If both columns approach a common y-value L, we state $\lim_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x-values near a are in our window. We can use the trace feature to move along the graph of the function and watch the y-value readout as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in Examples $13.3.1A$ and $13.3.1B$.

Example $13.3.1A$: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x \to 0} \dfrac{\sin x}{x}$ using a table of functional values.

Solution

We have calculated the values of $f(x) = \dfrac{\sin x}{x}$ for the values of x listed in Table $13.3.2$.

Table $13.3.2$

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
-0.1	0.998334166468	0.1	0.998334166468
-0.01	0.999983333417	0.01	0.999983333417
-0.001	0.999999833333	0.001	0.999999833333
-0.0001	0.999999998333	0.0001	0.999999998333

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\dfrac{\sin x}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$. A calculator-or computer-generated graph of $f(x) = \dfrac{\sin x}{x}$ would be similar to that shown in Figure $13.3.2$, and it confirms our estimate.

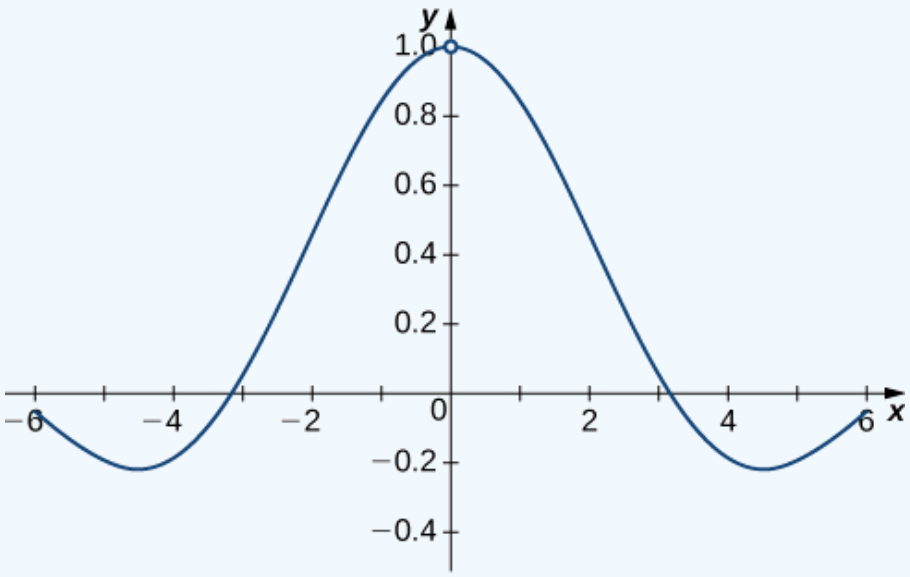


Figure $13.3.2$: The graph of $f(x) = (\sin x)/x$ confirms the estimate from Table $13.3.2$.

Example 13.3.1B: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Solution

As before, we use a table—in this case, Table 13.3.3—to list the values of the function for the given values of x.

Table 13.3.3

x	$\frac{\sqrt{x}-2}{x-4}$	x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869	4.1	0.248456731317
3.99	0.25015644562	4.01	0.24984394501
3.999	0.250015627	4.001	0.249984377
3.9999	0.250001563	4.0001	0.249998438
3.99999	0.25000016	4.00001	0.24999984

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}=0.25$. We confirm this estimate using the graph of $f(x)=\dfrac{\sqrt{x}-2}{x-4}$ shown in Figure 13.3.3.

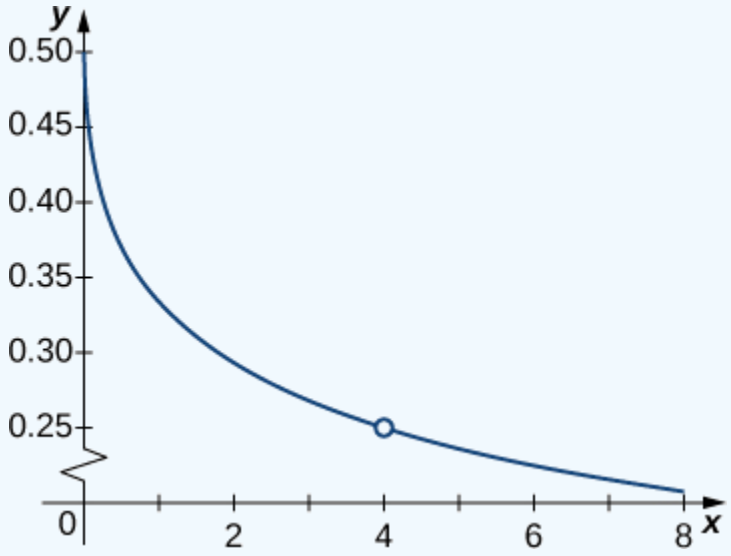


Figure 13.3.3: The graph of $\frac{\sqrt{x}-2}{x-4}$ confirms the estimate from Table 13.3.3.

Exercise 13.3.1

Estimate $\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$ using a table of functional values. Use a graph to confirm your estimate.

Hint available at the end of this section.

At this point, we see from Examples 13.3.1A and 13.3.1B that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example 13.3.2, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

Example 13.3.2: Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure 13.3.4, evaluate $\lim_{x \to -1} g(x)$.

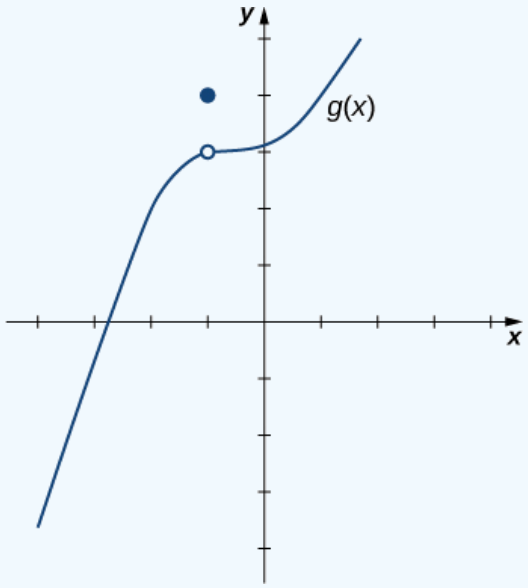


Figure 13.3.4: The graph of $g(x)$ includes one value not on a smooth curve.

Solution:

Despite the fact that $g(-1) = 4$, as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on Example 13.3.2, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

Exercise 13.3.2

Use the graph of $h(x)$ in Figure 13.3.5 to evaluate $\lim_{x \to 2} h(x)$, if possible.

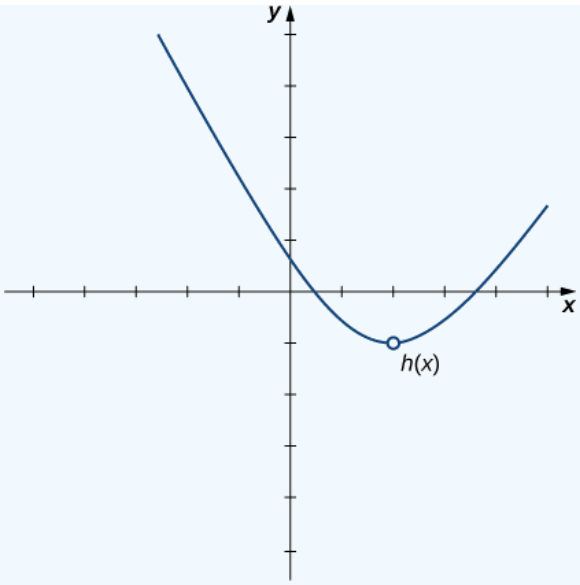


Figure 13.3.5:

Hint available at the end of this section.

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Two Important Limits

Let a be a real number and c be a constant.

i. $\lim\limits_{x\to a} x = a$
ii. $\lim\limits_{x\to a} c = c$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim\limits_{x\to a} x = a$.
ii. For the second limit, consider Table 13.3.4.

Table 13.3.4

x	$f(x) = c$	x	$f(x) = c$
$a - 0.1$	c	$a + 0.1$	c
$a - 0.01$	c	$a + 0.01$	c
$a - 0.001$	c	$a + 0.001$	c
$a - 0.0001$	c	$a + 0.0001$	c

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim\limits_{x\to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 13.3.3: Evaluating a Limit That Fails to Exist

Evaluate $\lim_{x\to 0} \sin(1/x)$ using a table of values.

Solution

Table 13.3.5 lists values for the function $\sin(1/x)$ for the given values of x.

Table 13.3.5

x	$\sin(1/x)$	x	$\sin(1/x)$
-0.1	0.544021110889	0.1	−0.544021110889
-0.01	0.50636564111	0.01	−0.50636564111
-0.001	−0.8268795405312	0.001	0.8268795405312
-0.0001	0.305614388888	0.0001	−0.305614388888
-0.00001	−0.035748797987	0.00001	0.035748797987
-0.000001	0.349993504187	0.000001	−0.349993504187

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim_{x\to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate "does not exist" as DNE. Thus, we would write $\lim_{x\to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in Figure 13.3.6 and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between -1 and 1 as x approaches 0.

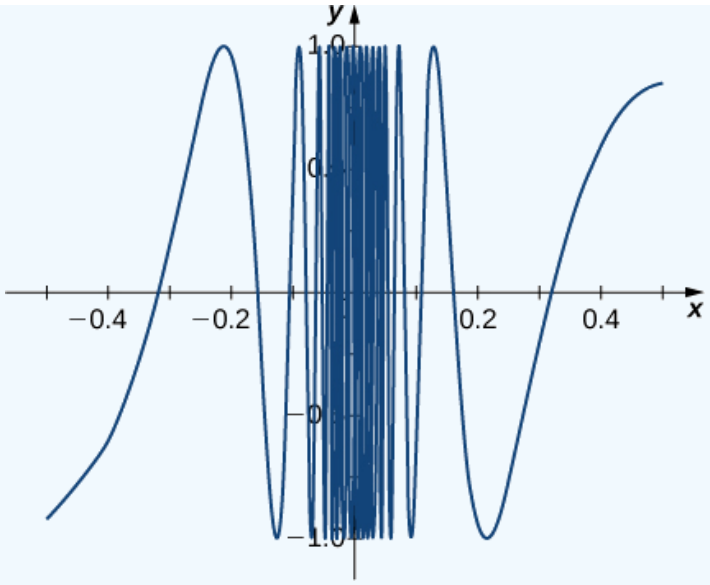


Figure 13.3.6: The graph of $f(x) = \sin(1/x)$ oscillates rapidly between -1 and 1 as x approaches 0.

Exercise 13.3.3

Use a table of functional values to evaluate $\lim_{x \to 2} \dfrac{|\, x^2 - 4 \,|}{x - 2}$, if possible.

Hint available at the end of this section.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2|/(x - 2)$ introduced at the beginning of the section (see Figure 13.3.1(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or the negative side of 2), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches -1. Mathematically, we say that the limit as x approaches 2 from the left is -1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or from the positive side), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Definition: One-sided Limits

We define two types of one-sided limits.

Limit from the left:

Let $f(x)$ be a function defined at all values in an open interval of the form (z, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where \(x

$$\lim_{x \to a^-} f(x) = L.$$

Limit from the right:

Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

Example 13.3.4: Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again. Observe in Table 13.3.6 that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

Table 13.3.6

x	$f(x) = x + 1$	x	$f(x) = x^2 - 4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001
1.99999	2.99999	2.00001	0.0000400001

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. Figure 13.3.7 shows a graph of $f(x)$ and reinforces our conclusion about these limits.

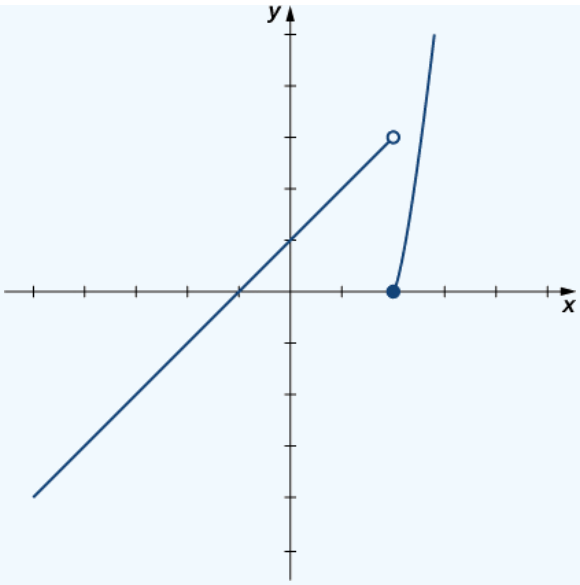


Figure 13.3.7: The graph of $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$ has a break at $x = 2$.

Exercise 13.3.4

Use a table of functional values to estimate the following limits, if possible.

a. $\lim\limits_{x \to 2^-} \dfrac{|\, x^2 - 4 \,|}{x - 2}$

b. $\lim\limits_{x \to 2^+} \dfrac{|\, x^2 - 4 \,|}{x - 2}$

Hint available at the end of this section.

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in Note.

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x \to a} f(x) = L$$

if and only if $\lim\limits_{x \to a^-} f(x) = L$ and $\lim\limits_{x \to a^+} f(x) = L$.

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see Figure 13.3.1(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger

and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x\to 2} h(x) = +\infty.$$

More generally, we define *infinite limits* as follows:

Definitions: Infinite Limits

We define three types of *infinite limits*.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where \(x

ii. If the values of $f(x)$ decrease without bound as the values of x (where \(x

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x\to a^+} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x\to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x\to a} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x\to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x\to a} f(x) = +\infty$ or $\lim_{x\to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x\to a} f(x) = +\infty$, we always write $\lim_{x\to a} f(x) = +\infty$ rather than $\lim_{x\to a} f(x)$ DNE.

Example 13.3.5: Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x\to 0^-} \frac{1}{x}$
b. $\lim_{x\to 0^+} \frac{1}{x}$
c. $\lim_{x\to 0} \frac{1}{x}$

Solution

Begin by constructing a table of functional values.

Table 13.3.7

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10,000	0.0001	10,000
-0.00001	-100,000	0.00001	100,000
-0.000001	-1,000,000	0.000001	1,000,000

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x\to 0} \frac{1}{x} \quad \text{DNE.}$$

The graph of $f(x) = 1/x$ in Figure 13.3.8 confirms these conclusions.

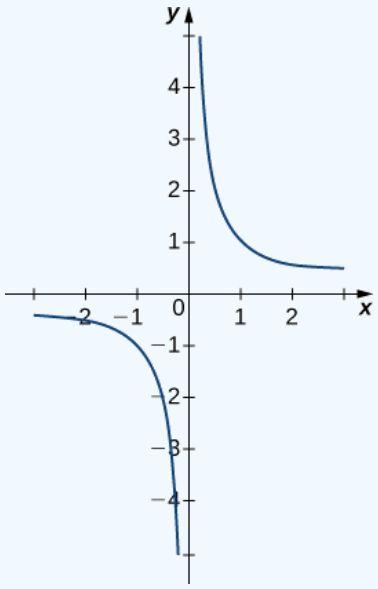


Figure 13.3.8: The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

Exercise 13.3.5

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim_{x\to 0^-} \frac{1}{x^2}$
b. $\lim_{x\to 0^+} \frac{1}{x^2}$
c. $\lim_{x\to 0} \frac{1}{x^2}$

Hint available at the end of this section.

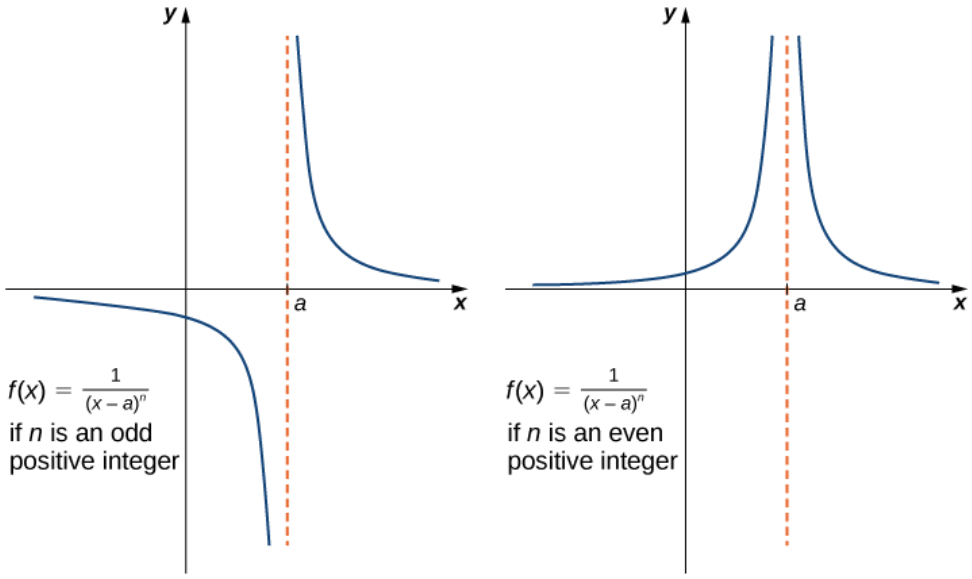


Figure 13.3.9: The function $f(x) = 1/(x-a)^n$ has infinite limits at a.

Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x\to a} \frac{1}{(x-a)^n} = +\infty. \tag{13.3.2}$$

If n is a positive odd integer, then

$$\lim_{x\to a^+} \frac{1}{(x-a)^n} = +\infty \tag{13.3.3}$$

and

$$\lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty. \tag{13.3.4}$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a vertical asymptote of the graph. We formally define a *vertical asymptote* as follows:

Definition: Vertical Asymptotes

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a vertical asymptote of $f(x)$.

$$\lim_{x\to a^-} f(x) = +\infty$$

$$\lim_{x\to a^-} f(x) = -\infty$$

$$\lim_{x\to a^+} f(x) = +\infty$$

$$\lim_{x\to a^+} f(x) = -\infty$$

$$\lim_{x\to a} f(x) = +\infty$$

$$\lim_{x\to a} f(x) = -\infty$$

Example 13.3.6: Finding a Vertical Asymptote

Evaluate each of the following limits using Equations 13.3.2, 13.3.3, and 13.3.4 above. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim_{x\to -3^-} \dfrac{1}{(x+3)^4}$
b. $\lim_{x\to -3^+} \dfrac{1}{(x+3)^4}$
c. $\lim_{x\to -3} \dfrac{1}{(x+3)^4}$

Solution

We can use the above equations directly.

a. $\lim_{x\to -3^-} \dfrac{1}{(x+3)^4} = +\infty$
b. $\lim_{x\to -3^+} \dfrac{1}{(x+3)^4} = +\infty$
c. $\lim_{x\to -3} \dfrac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

Exercise 13.3.6

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \dfrac{1}{(x-2)^3}$.

a. $\lim_{x\to 2^-} \dfrac{1}{(x-2)^3}$
b. $\lim_{x\to 2^+} \dfrac{1}{(x-2)^3}$
c. $\lim_{x\to 2} \dfrac{1}{(x-2)^3}$

Answer a available at the end of this section.

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 13.3.7: Behavior of a Function at Different Points

Use the graph of $f(x)$ in Figure 13.3.10 to determine each of the following values:

a. $\lim_{x\to -4^-} f(x)$; $\lim_{x\to -4^+} f(x)$; $\lim_{x\to -4} f(x)$; $f(-4)$

b. $\lim_{x\to -2^-} f(x)$; $\lim_{x\to -2^+} f(x)$; $\lim_{x\to -2} f(x)$; $f(-2)$
c. $\lim_{x\to 1^-} f(x)$; $\lim_{x\to 1^+} f(x)$; $\lim_{x\to 1} f(x)$; $f(1)$
d. $\lim_{x\to 3^-} f(x)$; $\lim_{x\to 3^+} f(x)$; $\lim_{x\to 3} f(x)$; $f(3)$

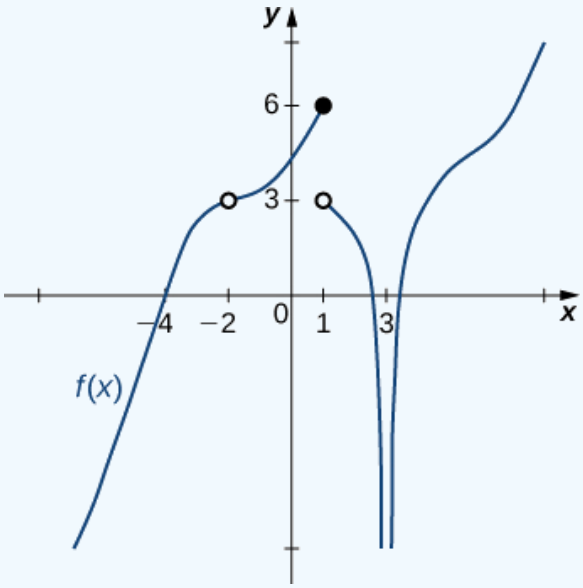


Figure 13.3.10: The graph shows $f(x)$.

Solution

Using the definitions above and the graph for reference, we arrive at the following values:

a. $\lim_{x\to -4^-} f(x) = 0$; $\lim_{x\to -4^+} f(x) = 0$; $\lim_{x\to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim_{x\to -2^-} f(x) = 3$; $\lim_{x\to -2^+} f(x) = 3$; $\lim_{x\to -2} f(x) = 3$; $f(-2)$ is undefined
c. $\lim_{x\to 1^-} f(x) = 6$; $\lim_{x\to 1^+} f(x) = 3$; $\lim_{x\to 1} f(x)$ DNE; $f(1) = 6$
d. $\lim_{x\to 3^-} f(x) = -\infty$; $\lim_{x\to 3^+} f(x) = -\infty$; $\lim_{x\to 3} f(x) = -\infty$; $f(3)$ is undefined

Exercise 13.3.7

Evaluate $\lim_{x\to 1} f(x)$ for $f(x)$ shown here:

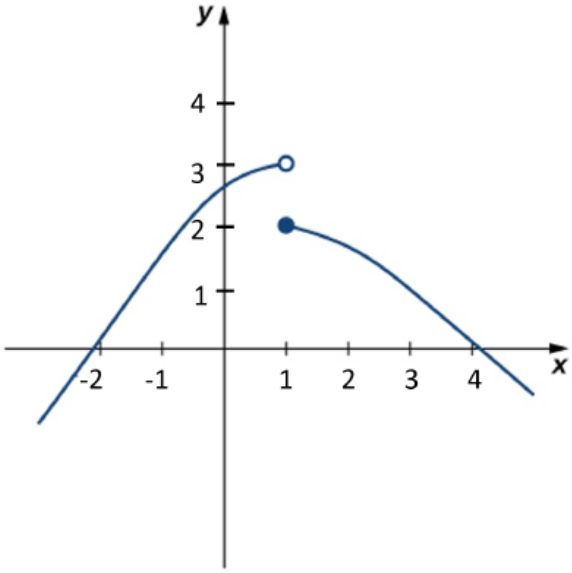


Figure 13.3.11. The graph of a piecewise function f.

Hint available at the end of this section.

Example 13.3.8: Einstein's Equation

In the Chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

what is the value of this bound?

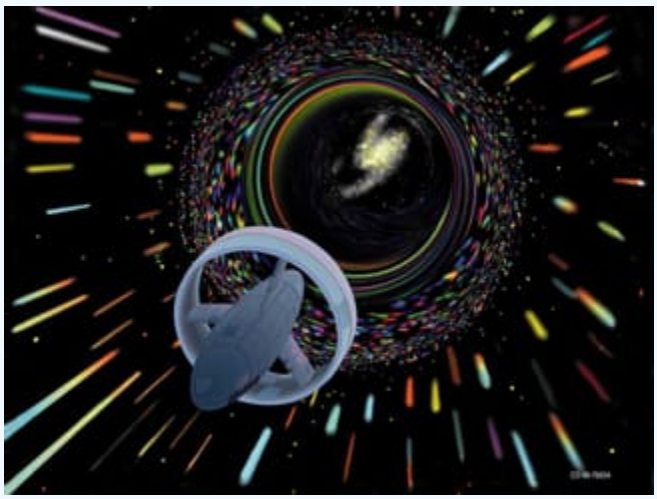


Figure 13.3.12. (Credit:NASA)

Solution

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c (Figure 13.3.13).

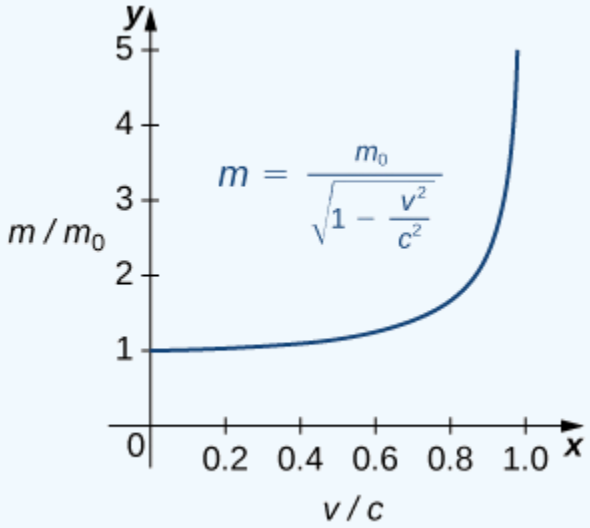


Figure 13.3.13: This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

Table 13.3.8

v/c	$\sqrt{1-\frac{v^2}{c^2}}$	m/m_o
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.7

Thus, according to Table 13.3.8:, if an object with mass 100 kg is traveling at 0.9999c, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.

Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

Key Equations

- **Intuitive Definition of the Limit**

$$\lim_{x \to a} f(x) = L$$

- **Two Important Limits**

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

- **One-Sided Limits**

$$\lim_{x\to a^-} f(x) = L \qquad \lim_{x\to a^+} f(x) = L$$

- **Infinite Limits from the Left**

$$\lim_{x\to a^-} f(x) = +\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

$$\lim_{x\to a^+} f(x) = +\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

$$\lim_{x\to a} f(x) = +\infty\text{: } \lim_{x\to a^-} f(x) = +\infty \text{ and } \lim_{x\to a^+} f(x) = +\infty$$

$$\lim_{x\to a} f(x) = -\infty\text{: } \lim_{x\to a^-} f(x) = -\infty \text{ and } \lim_{x\to a^+} f(x) = -\infty$$

Glossary

infinite limit
A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

intuitive definition of the limit
If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

one-sided limit
A one-sided limit of a function is a limit taken from either the left or the right

vertical asymptote
A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

---

Answers

Exercise 13.3.1

Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

$$\lim_{x\to 1} \frac{\frac{1}{x} - 1}{x - 1} = -1$$

Exercise 13.3.2

What y-value does the function approach as the x-values approach 2?

$\lim_{x\to 2} h(x) = -1.$

Exercise 13.3.3

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

$\lim_{x\to 2} \frac{|\, x^2 - 4\,|}{x - 2}$ does not exist.

Exercise 13.3.4

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 to estimate $\lim_{x\to 2^-} \frac{|\, x^2 - 4\,|}{x - 2}$.

Use x-values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim_{x\to 2^+} \frac{|\, x^2 - 4\,|}{x - 2}$.

(These tables are available from a previous Checkpoint problem.)

a. $\lim_{x\to 2^-} \frac{|\, x^2 - 4\,|}{x - 2} = -4$

$\lim_{x\to 2^+} \frac{|\, x^2 - 4\,|}{x - 2} = 4$

Exercise 13.3.5

Follow the procedures from Example 13.3.5.

a. $\lim_{x\to 0^-} \frac{1}{x^2} = +\infty$;

b. $\lim_{x \to 0^+} \frac{1}{x^2} = +\infty$;

c. $\lim_{x \to 0} \frac{1}{x^2} = +\infty$

It is useful to point out that functions of the form $f(x) = 1/(x-a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (Figure 13.3.9). These limits are summarized in the above definitions.

Exercise 13.3.6

$\lim_{x \to 2^-} \frac{1}{(x-2)^3} = -\infty$

$\lim_{x \to 2^+} \frac{1}{(x-2)^3} = +\infty$

$\lim_{x \to 2} \frac{1}{(x-2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x-2)^3$.

Exercise 13.3.7

Compare the limit from the right with the limit from the left.

$\lim_{x \to 1} f(x)$ does not exist

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13.3E: Exercises for Section 2.2

Intuitive Definition of Limits

For exercises 1 - 2, consider the function $f(x) = \dfrac{x^2 - 1}{|x - 1|}$.

1) [T] Complete the following table for the function. Round your solutions to four decimal places.

x	$f(x)$	x	$f(x)$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

2) What do your results in the preceding exercise indicate about the two-sided limit $\lim\limits_{x \to 1} f(x)$? Explain your response.

Answer

$\lim\limits_{x \to 1} f(x)$ does not exist because $\lim\limits_{x \to 1^-} f(x) = -2 \neq \lim\limits_{x \to 1^+} f(x) = 2$.

For exercises 3 - 5, consider the function $f(x) = (1 + x)^{1/x}$.

3) [T] Make a table showing the values of f for $x = -0.01,\ -0.001,\ -0.0001,\ -0.00001$ and for $x = 0.01,\ 0.001,\ 0.0001,\ 0.00001$. Round your solutions to five decimal places.

x	$f(x)$	x	$f(x)$
-0.01	a.	0.01	e.
-0.001	b.	0.001	f.
-0.0001	c.	0.0001	g.
-0.00001	d.	0.00001	h.

4) What does the table of values in the preceding exercise indicate about the function $f(x) = (1 + x)^{1/x}$?

Answer

$\lim\limits_{x \to 0} (1 + x)^{1/x} \approx 2.7183$.

5) To which mathematical constant do the values in the preceding exercise appear to be approaching? This is the actual limit here.

In exercises 6 - 8, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.

6) [T] $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$; $\pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm .0001$

x	$\frac{\sin 2x}{x}$	x	$\frac{\sin 2x}{x}$
-0.1	a.	0.1	e.

x	$\frac{\sin 2x}{x}$	x	$\frac{\sin 2x}{x}$
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Answer

a. 1.98669331; b. 1.99986667; c. 1.99999867; d. 1.99999999; e. 1.98669331; f. 1.99986667; g. 1.99999867; h. 1.99999999;

$\lim_{x\to 0}\frac{\sin 2x}{x}=2$

7) [T] $\lim_{x\to 0}\frac{\sin 3x}{x}\ \pm 0.1,\ \pm 0.01,\ \pm 0.001,\ \pm 0.0001$

x	$\frac{\sin 3x}{x}$	x	$\frac{\sin 3x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

8) Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x\to 0}\frac{\sin ax}{x}$ for a, a positive real value.

Answer

$\lim_{x\to 0}\frac{\sin ax}{x}=a$

[T] In exercises 9 - 14, set up a table of values to find the indicated limit. Round to eight significant digits.

9) $\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}$

x	$\frac{x^2-4}{x^2+x-6}$	x	$\frac{x^2-4}{x^2+x-6}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

10) $\lim_{x\to 1}(1-2x)$

x	$1-2x$	x	$1-2x$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

Answer

a. −0.80000000; b. −0.98000000; c. −0.99800000; d. −0.99980000; e. −1.2000000; f. −1.0200000; g. −1.0020000; h. −1.0002000;

$\lim_{x\to 1}(1-2x)=-1$

11) $\lim_{x\to 0} \dfrac{5}{1-e^{1/x}}$

x	$\frac{5}{1-e^{1/x}}$	x	$\frac{5}{1-e^{1/x}}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

12) $\lim_{z\to 0} \dfrac{z-1}{z^2(z+3)}$

z	$\frac{z-1}{z^2(z+3)}$	z	$\frac{z-1}{z^2(z+3)}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Answer

a. −37.931034; b. −3377.9264; c. −333,777.93; d. −33,337,778; e. −29.032258; f. −3289.0365; g. −332,889.04; h. −33,328,889

$\lim_{x\to 0} \dfrac{z-1}{z^2(z+3)} = -\infty$

13) $\lim_{t\to 0^+} \dfrac{\cos t}{t}$

t	$\frac{\cos t}{t}$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

14) $\lim_{x\to 2} \dfrac{1-\frac{2}{x}}{x^2-4}$

x	$\frac{1-\frac{2}{x}}{x^2-4}$	x	$\frac{1-\frac{2}{x}}{x^2-4}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

Answer

a. 0.13495277; b. 0.12594300; c. 0.12509381; d. 0.12500938; e. 0.11614402; f. 0.12406794; g. 0.12490631; h. 0.12499063;

$\therefore \lim_{x\to 2} \dfrac{1-\frac{2}{x}}{x^2-4} = 0.1250 = \dfrac{1}{8}$

[T] In exercises 15 - 16, set up a table of values and round to eight significant digits. Based on the table of values,

make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

15) $\lim_{\theta \to 0} \sin\left(\frac{\pi}{\theta}\right)$

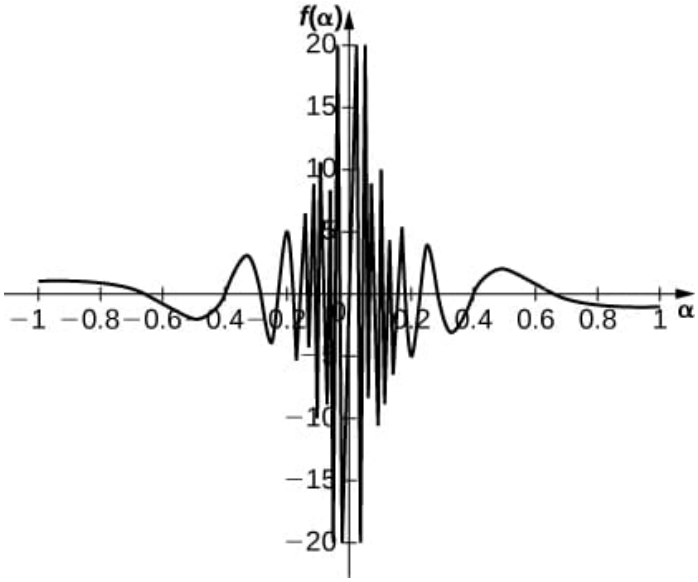
θ	$\sin\left(\frac{\pi}{\theta}\right)$	θ	$\sin\left(\frac{\pi}{\theta}\right)$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

16) $\lim_{\alpha \to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$

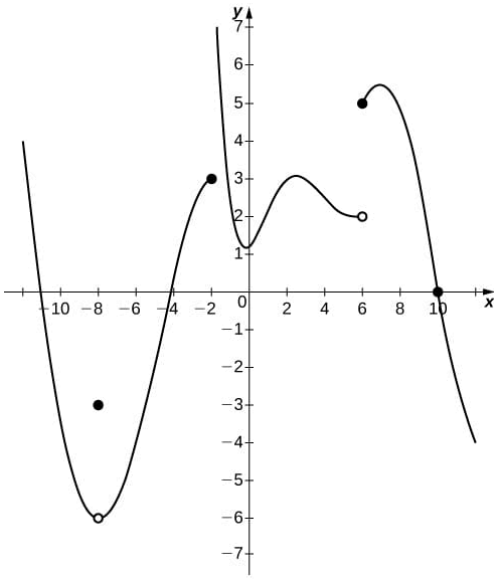
a	$\frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

Answer

a. 10.00000; b. 100.00000; c. 1000.0000; d. 10,000.000;
Guess: $\lim_{\alpha \to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right) = \infty$;
Actual: DNE , since the graph shows the function oscillates wildly between values approaching positive infinity and values approaching negative infinity, as the value of α approaches 0 from the positive side.



In exercises 17 - 20, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.



17) $\lim_{x \to 10} f(x) = 0$

18) $\lim_{x \to -2^+} f(x) = 3$

Answer

False; $\lim_{x \to -2^+} f(x) = +\infty$

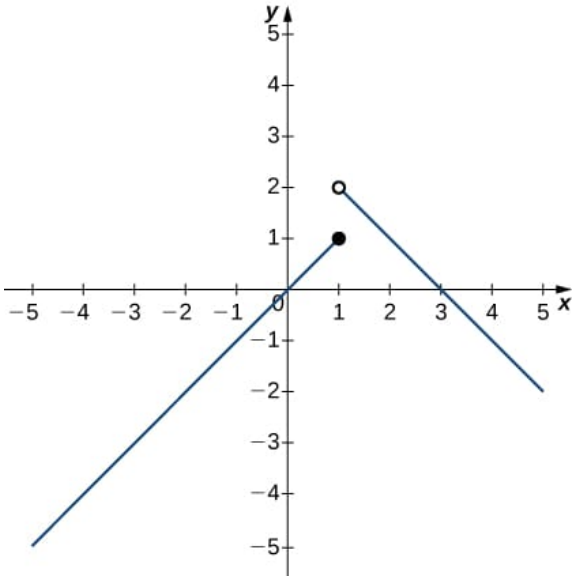
19) $\lim_{x \to -8} f(x) = f(-8)$

20) $\lim_{x \to 6} f(x) = 5$

Answer

False; $\lim_{x \to 6} f(x)$ DNE since $\lim_{x \to 6^-} f(x) = 2$ and $\lim_{x \to 6^+} f(x) = 5$.

In exercises 21 - 25, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



21) $\lim_{x \to 1^-} f(x)$

22) $\lim_{x \to 1^+} f(x)$

Answer

2

23) $\lim_{x \to 1} f(x)$

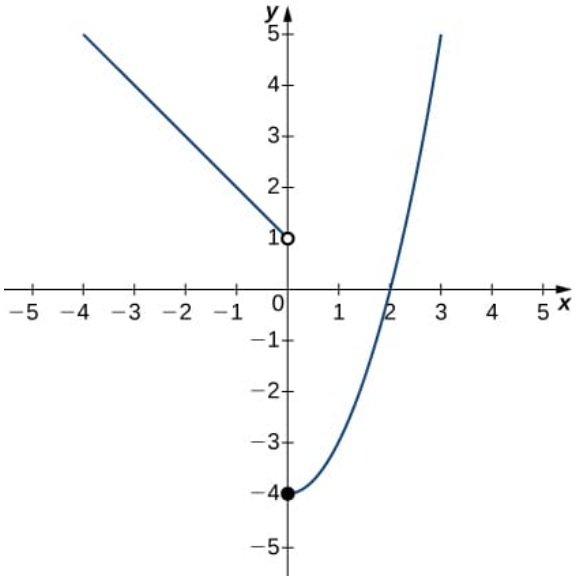
24) $\lim_{x \to 2} f(x)$

Answer

1

25) $f(1)$

In exercises 26 - 29, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



26) $\lim_{x \to 0^-} f(x)$

Answer

1

27) $\lim_{x \to 0^+} f(x)$

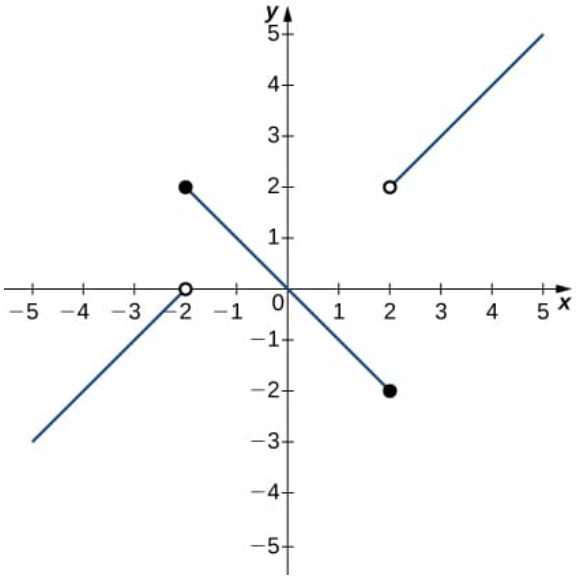
28) $\lim_{x \to 0} f(x)$

Answer

DNE

29) $\lim_{x \to 2} f(x)$

In exercises 30 - 35, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



30) $\lim_{x \to -2^-} f(x)$

Answer

0

31) $\lim_{x \to -2^+} f(x)$

32) $\lim_{x \to -2} f(x)$

Answer

DNE

33) $\lim_{x \to 2^-} f(x)$

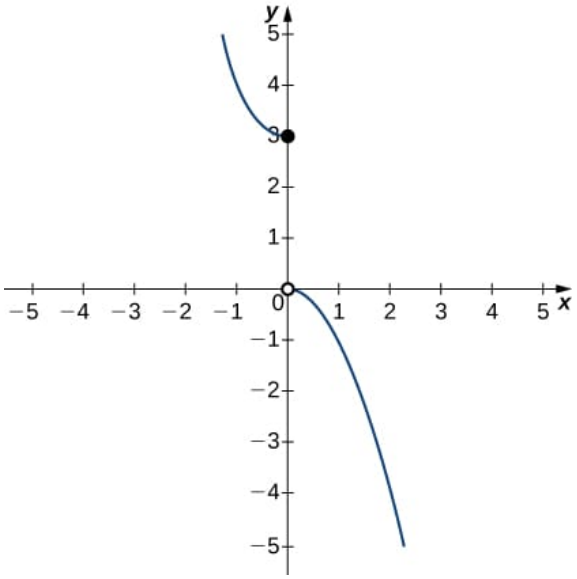
34) $\lim_{x \to 2^+} f(x)$

Answer

2

35) $\lim_{x \to 2} f(x)$

In exercises 36 - 38, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.



36) $\lim_{x \to 0^-} g(x)$

Answer

3

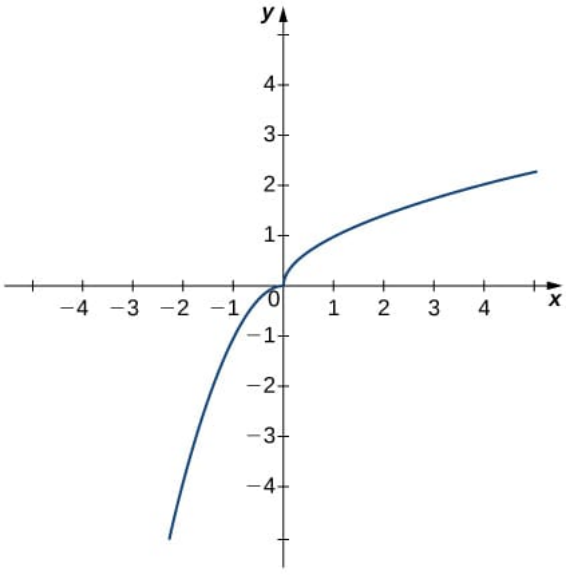
37) $\lim_{x \to 0^+} g(x)$

38) $\lim_{x \to 0} g(x)$

Answer

DNE

In exercises 39 - 41, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.



39) $\lim_{x \to 0^-} h(x)$

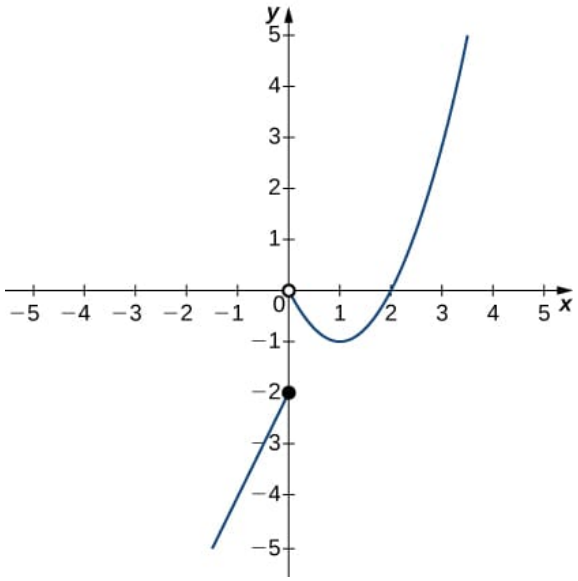
40) $\lim_{x \to 0^+} h(x)$

Answer

0

41) $\lim_{x \to 0} h(x)$

In exercises 42 - 46, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



42) $\lim_{x \to 0^-} f(x)$

Answer

-2

43) $\lim_{x \to 0^+} f(x)$

44) $\lim_{x \to 0} f(x)$

Answer

DNE

45) $\lim_{x \to 1} f(x)$

46) $\lim_{x \to 2} f(x)$

Answer

0

Infinite Limits

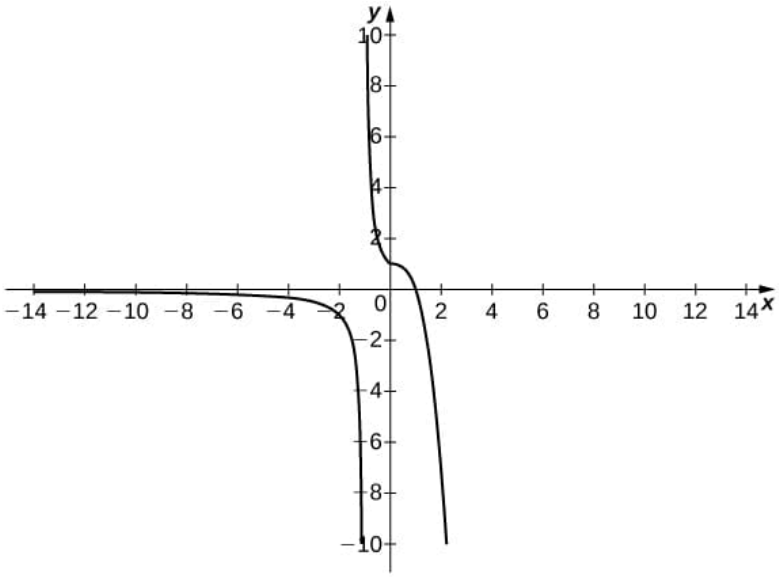
In exercises 47 - 51, sketch the graph of a function with the given properties.

47) $\lim_{x \to 2} f(x) = 1, \quad \lim_{x \to 4^-} f(x) = 3, \quad \lim_{x \to 4^+} f(x) = 6, \quad x = 4$ is not defined.

48) $\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to-1^-} f(x) = -\infty, \quad \lim_{x\to-1^+} f(x) = \infty, \quad \lim_{x\to 0} f(x) = f(0), \quad f(0) = 1, \quad \lim_{x\to\infty} f(x) = -\infty$

Answer

Answers may vary

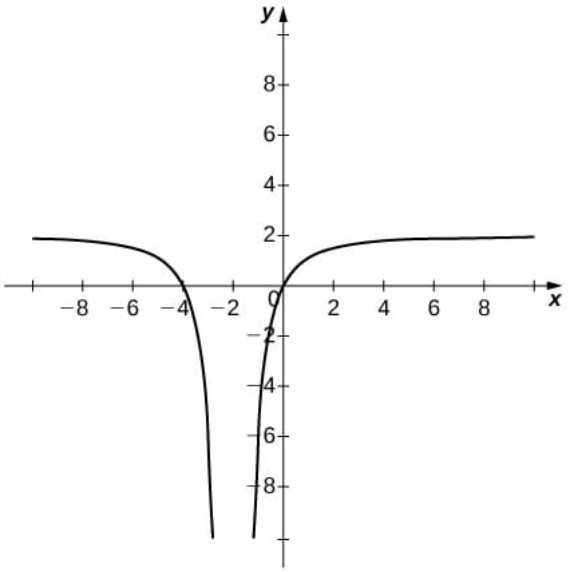


49) $\lim_{x\to-\infty} f(x) = 2, \quad \lim_{x\to 3^-} f(x) = -\infty, \quad \lim_{x\to 3^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 2, \quad f(0) = -\frac{1}{3}$

50) $\lim_{x\to-\infty} f(x) = 2, \quad \lim_{x\to-2} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = 2, \quad f(0) = 0$

Answer

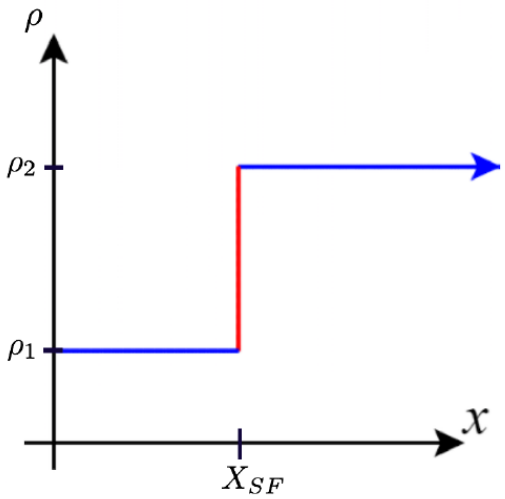
Answer may vary



51) $\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to-1^-} f(x) = \infty, \quad \lim_{x\to-1^+} f(x) = -\infty, \quad f(0) = -1, \quad \lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 0$

52) Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a

shock wave with respect to distance, x, is shown here. We are mainly interested in the location of the front of the shock, labeled X_{SF} in the diagram.



a. Evaluate $\lim_{x \to X_{SF}^{+}} \rho(x)$.

b. Evaluate $\lim_{x \to X_{SF}^{-}} \rho(x)$.

c. Evaluate $\lim_{x \to X_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

Answer

a. ρ_2 b. ρ_1 c. DNE unless $\rho_1 = \rho_2$. As you approach X_{SF} from the right, you are in the high-density area of the shock. When you approach from the left, you have not experienced the "shock" yet and are at a lower density.

53) A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where x is the position in meters of the runner and t is time in seconds. What is $\lim_{t \to 2} x(t)$? What does it mean physically?

$t(sec)$	$x(m)$
1.75	4.5
1.95	6.1
1.99	6.42
2.01	6.58
2.05	6.9
2.25	8.5

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13.4: The Limit Laws

Learning Objectives

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.
- Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated previously and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Theorem 13.4.1: Basic Limit Results

For any real number a and any constant c,

I. $\lim\limits_{x \to a} x = a$
II. $\lim\limits_{x \to a} c = c$

Example 13.4.1: Evaluating a Basic Limit

Evaluate each of the following limits using "Basic Limit Results."

a. $\lim\limits_{x \to 2} x$
b. $\lim\limits_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim\limits_{x \to 2} x = 2$.
b. The limit of a constant is that constant: $\lim\limits_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Theorem 13.4.2: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum law for limits**:

$$\lim_{x\to a}(f(x)+g(x))=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)=L+M$$

- **Difference law for limits**:

$$\lim_{x\to a}(f(x)-g(x))=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=L-M$$

- **Constant multiple law for limits**:

$$\lim_{x\to a}cf(x)=c\cdot\lim_{x\to a}f(x)=cL$$

- **Product law for limits**:

$$\lim_{x\to a}(f(x)\cdot g(x))=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=L\cdot M$$

- **Quotient law for limits**:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}=\frac{L}{M}$$

for $M\neq 0$.

- **Power law for limits**:

$$\lim_{x\to a}\big(f(x)\big)^n=\Big(\lim_{x\to a}f(x)\Big)^n=L^n$$

for every positive integer n.

- **Root law for limits**:

$$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}f(x)}=\sqrt[n]{L}$$

for all L if n is odd and for $L\geq 0$ if n is even.

We now practice applying these limit laws to evaluate a limit.

Example 13.4.2A: Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate

$$\lim_{x\to -3}(4x+2).$$

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned}\lim_{x\to -3}(4x+2)&=\lim_{x\to -3}4x+\lim_{x\to -3}2 && \text{Apply the sum law.}\\ &=4\cdot\lim_{x\to -3}x+\lim_{x\to -3}2 && \text{Apply the constant multiple law.}\\ &=4\cdot(-3)+2=-10. && \text{Apply the basic limit results and simplify.}\end{aligned}$$

Example 13.4.2B: Using Limit Laws Repeatedly

Use the limit laws to evaluate

$$\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}.$$

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4} &= \frac{\lim_{x\to 2}(2x^2-3x+1)}{\lim_{x\to 2}(x^3+4)} && \text{Apply the quotient law, make sure that } (2)^3+4\neq 0.\\
&= \frac{2\cdot\lim_{x\to 2}x^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\lim_{x\to 2}x^3+\lim_{x\to 2}4} && \text{Apply the sum law and constant multiple law.}\\
&= \frac{2\cdot\left(\lim_{x\to 2}x\right)^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\left(\lim_{x\to 2}x\right)^3+\lim_{x\to 2}4} && \text{Apply the power law.}\\
&= \frac{2(4)-3(2)+1}{(2)^3+4}=\frac{1}{4}. && \text{Apply the basic limit laws and simplify.}
\end{aligned}$$

Exercise 13.4.2

Use the limit laws to evaluate $\lim_{x\to 6}(2x-1)\sqrt{x+4}$. In each step, indicate the limit law applied.

Hint available at the end of this section.

Answer available at the end of this section.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x\to a}f(x)=f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Theorem 13.4.3: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x\to a}p(x)=p(a)$$

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$$

when $q(a)\neq 0$.

To see that this theorem holds, consider the polynomial

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)\\ &= c_n\left(\lim_{x\to a} x\right)^n + c_{n-1}\left(\lim_{x\to a} x\right)^{n-1} + \cdots + c_1\left(\lim_{x\to a} x\right) + \lim_{x\to a} c_0\\ &= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0\\ &= p(a)\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$,

then

$$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Example 13.4.3: Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x\to 3}\dfrac{2x^2-3x+1}{5x+4}$.

Solution

Since 3 is in the domain of the rational function $f(x) = \dfrac{2x^2-3x+1}{5x+4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x\to 3}\frac{2x^2-3x+1}{5x+4} = \frac{10}{19}.$$

Exercise 13.4.3

Evaluate $\lim_{x\to -2}(3x^3 - 2x + 7)$.

Hint available at the end of this section.

Answer available at the end of this section.

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x\to a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a$, $f(x) = g(x)$ over some open interval containing a, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

To understand this idea better, consider the limit $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$.

The function

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in Figure 13.4.1.

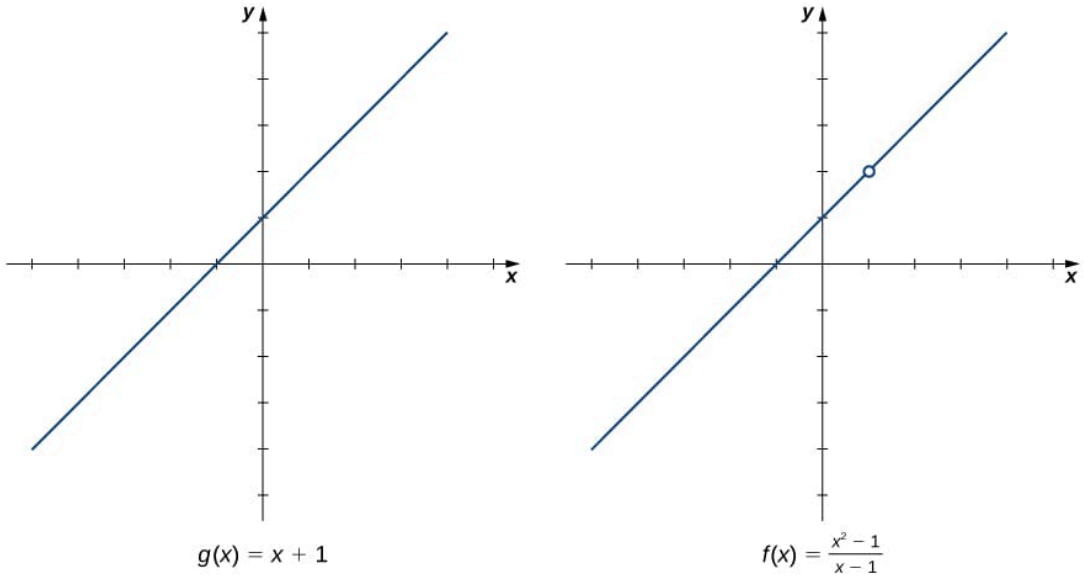


Figure 13.4.1: The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} (x+1) = 2.$$

The limit has the form $\lim_{x\to a} f(x)/g(x)$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form $0/0$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. Example 13.4.4 illustrates the factor-and-cancel technique; Example 13.4.5 shows multiplying by a conjugate. In Example 13.4.6, we look at simplifying a complex fraction.

Example 13.4.4: Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x\to 3} \dfrac{x^2-3x}{2x^2-5x-3}$.

Solution

Step 1. The function $f(x) = \dfrac{x^2-3x}{2x^2-5x-3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}=\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x\neq 3$, $\dfrac{x^2-3x}{2x^2-5x-3}=\dfrac{x}{2x+1}$. Therefore,

$$\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}=\lim_{x\to 3}\frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x\to 3}\frac{x}{2x+1}=\frac{3}{7}.$$

Exercise 13.4.4

Evaluate $\lim_{x\to -3}\dfrac{x^2+4x+3}{x^2-9}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 13.4.5: Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim_{x\to -1}\dfrac{\sqrt{x+2}-1}{x+1}$.

Solution

Step 1. $\dfrac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at –1. Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}=\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$=\lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)}.$$

Step 3. Then we cancel:

$$=\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}=\frac{1}{2}.$$

Exercise 13.4.5

Evaluate $\lim_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 13.4.6: Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim_{x\to 1}\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$.

Solution

Step 1. $\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}=\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}\cdot\frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$=\lim_{x\to 1}\frac{2-(x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$=\lim_{x\to 1}\frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out –1 from the numerator:

$$=\lim_{x\to 1}\frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$=\lim_{x\to 1}\frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x\to 1}\frac{-1}{2(x+1)}=-\frac{1}{4}.$$

Exercise 13.4.6

Evaluate $\lim_{x\to -3}\dfrac{\dfrac{1}{x+2}+1}{x+3}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 13.4.7 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 13.4.7: Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$.

Solution:

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\frac{1}{x}+\frac{5}{x(x-5)}=\frac{x-5+5}{x(x-5)}=\frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\lim_{x\to 0}\frac{x}{x(x-5)}=\lim_{x\to 0}\frac{1}{x-5}=-\frac{1}{5}.$$

Exercise 13.4.7

Evaluate $\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right)$.

Hint available at the end of this section.

Answer available at the end of this section.

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim_{x\to a^-}h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b,a); for a limit of the form $\lim_{x\to a^+}h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a,c). Example 13.4.8A illustrates this point.

Example 13.4.8A: Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim_{x\to 3^-}\sqrt{x-3}$
b. $\lim_{x\to 3^+}\sqrt{x-3}$

Solution

Figure 13.4.2 illustrates the function $f(x)=\sqrt{x-3}$ and aids in our understanding of these limits.

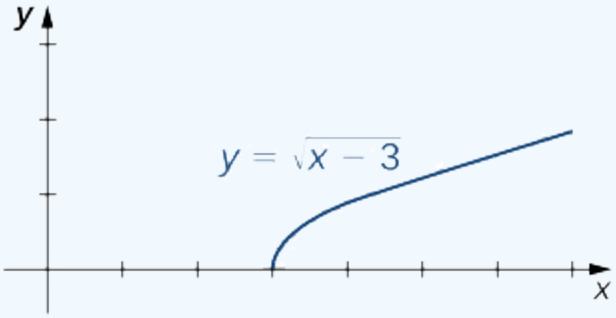


Figure 13.4.2: The graph shows the function $f(x) = \sqrt{x-3}$.

a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim\limits_{x \to 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim\limits_{x \to 3^-} \sqrt{x-3}$ does not exist.

b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim\limits_{x \to 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim\limits_{x \to 3^+} \sqrt{x-3} = 0$.

In Example 13.4.8B we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example 13.4.8B: Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x-3, & \text{if } x < 2 \\ (x-3)^2, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim\limits_{x \to 2^-} f(x)$
b. $\lim\limits_{x \to 2^+} f(x)$
c. $\lim\limits_{x \to 2} f(x)$

Solution

Figure 13.4.3 illustrates the function $f(x)$ and aids in our understanding of these limits.

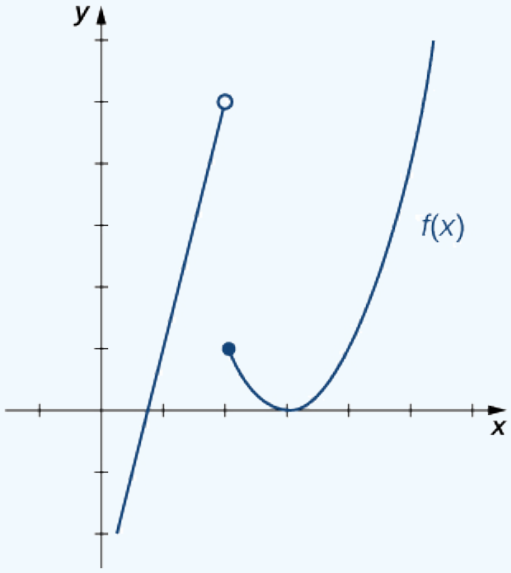


Figure 13.4.3: This graph shows a function $f(x)$.

a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - 3) = 5$$

b. Since $f(x) = (x - 3)^2$for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (x - 3)^2 = 1.$$

c. Since $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 1$, we conclude that $\lim_{x \to 2} f(x)$ does not exist.

Exercise 13.4.8

Graph $f(x) = \begin{cases} -x - 2, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ x^3, & \text{if } x > -1 \end{cases}$ and evaluate $\lim_{x \to -1^-} f(x)$.

Hint available at the end of this section.

Answer available at the end of this section.

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at a.

Example 13.4.9: Evaluating a Limit of the Form $K/0,\ K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \to 2^-} \frac{x - 3}{x^2 - 2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches -1; and the denominator approaches 0. Consequently, the magnitude of $\frac{x - 3}{x(x - 2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x\to 2^-}\frac{x-3}{x^2-2x}=\lim_{x\to 2^-}\frac{x-3}{x(x-2)}$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$=\lim_{x\to 2^-}\frac{x-3}{x}\cdot\frac{1}{x-2}$$

Step 3. Using the Limit Laws, we can write:

$$=\left(\lim_{x\to 2^-}\frac{x-3}{x}\right)\cdot\left(\lim_{x\to 2^-}\frac{1}{x-2}\right).$$

Step 4. $\lim_{x\to 2^-}\frac{x-3}{x}=-\frac{1}{2}$ and $\lim_{x\to 2^-}\frac{1}{x-2}=-\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x\to 2^-}\frac{x-3}{x^2-2x}=+\infty.$$

Exercise 13.4.9

Evaluate $\lim_{x\to 1}\frac{x+2}{(x-1)^2}$.

Solution available at the end of this section.

Answer available at the end of this section.

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem,** proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by "squeezing" a function, with a limit at a point a that is unknown, between two functions having a common known limit at a. Figure 13.4.4 illustrates this idea.

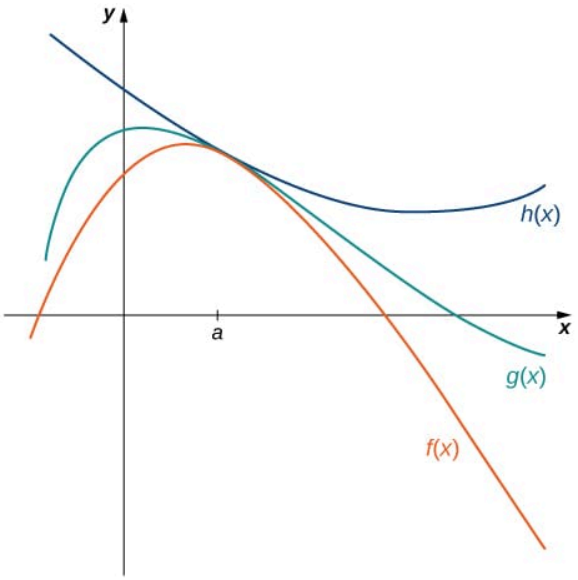


Figure 13.4.4: The Squeeze Theorem applies when $f(x)\le g(x)\le h(x)$ and $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)$.

Theorem 13.4.4: The Squeeze Theorem

Let $f(x), g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Example 13.4.10: Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \to 0} x \cos x$.

Solution

Because $-1 \leq \cos x \leq 1$ for all x, we have $-x \leq x \cos x \leq x$ for $x \geq 0$ and $-x \geq x \cos x \geq x$ for $x \leq 0$ (if x is negative the direction of the inequalities changes when we multiply). Since $\lim_{x \to 0}(-x) = 0 = \lim_{x \to 0} x$, from the squeeze theorem, we obtain $\lim_{x \to 0} x \cos x = 0$. The graphs of $f(x) = -x, \ \ g(x) = x \cos x$, and $h(x) = x$ are shown in Figure 13.4.5.

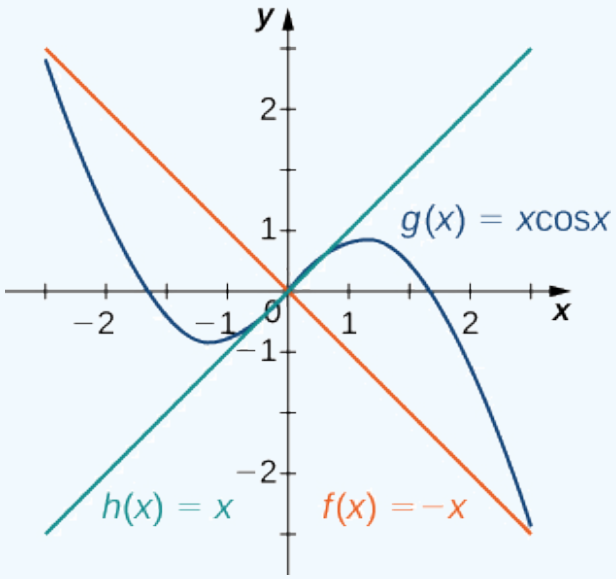


Figure 13.4.5: The graphs of $f(x), \ \ g(x)$, and $h(x)$ are shown around the point $x = 0$.

Exercise 13.4.10

Use the squeeze theorem to evaluate $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

Hint available at the end of this section.

Answer available at the end of this section.

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta \to 0} \sin \theta$. Consider the unit circle shown in Figure 13.4.6. In the figure, we see that $\sin \theta$ is the y-coordinate on the unit circle and

it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \dfrac{\pi}{2}$, we have $0 < \sin\theta < \theta$.

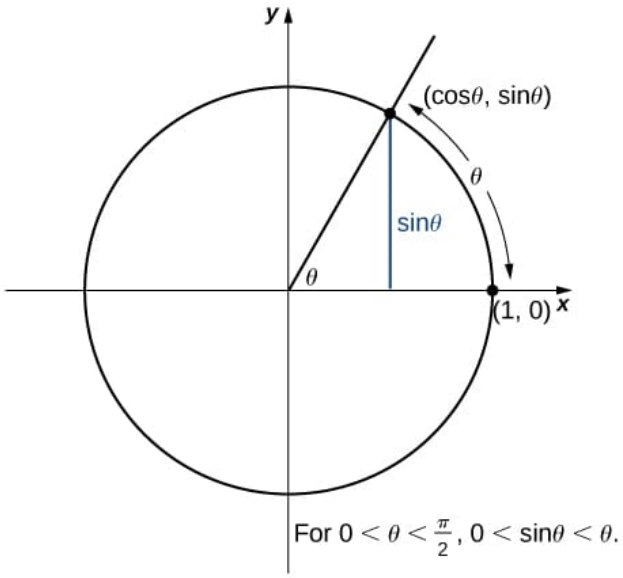


Figure 13.4.6: The sine function is shown as a line on the unit circle.

Because $\lim\limits_{\theta\to 0^+} 0 = 0$ and $\lim\limits_{\theta\to 0^+} \theta = 0$, by using the squeeze theorem we conclude that

$$\lim_{\theta\to 0^+} \sin\theta = 0.$$

To see that $\lim\limits_{\theta\to 0^-} \sin\theta = 0$ as well, observe that for $-\dfrac{\pi}{2} < \theta < 0, 0 < -\theta < \dfrac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta$. Consequently, $0 < -\sin\theta < -\theta$. It follows that $0 > \sin\theta > \theta$. An application of the squeeze theorem produces the desired limit. Thus, since $\lim\limits_{\theta\to 0^+} \sin\theta = 0$ and $\lim\limits_{\theta\to 0^-} \sin\theta = 0$,

$$\lim_{\theta\to 0} \sin\theta = 0$$

Next, using the identity $\cos\theta = \sqrt{1-\sin^2\theta}$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, we see that

$$\lim_{\theta\to 0} \cos\theta = \lim_{\theta\to 0} \sqrt{1-\sin^2\theta} = 1.$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$. To evaluate this limit, we use the unit circle in Figure 13.4.7. Notice that this figure adds one additional triangle to Figure 13.4.6. We see that the length of the side opposite angle θ in this new triangle is $\tan\theta$. Thus, we see that for $0 < \theta < \dfrac{\pi}{2}$, we have $\sin\theta < \theta < \tan\theta$.

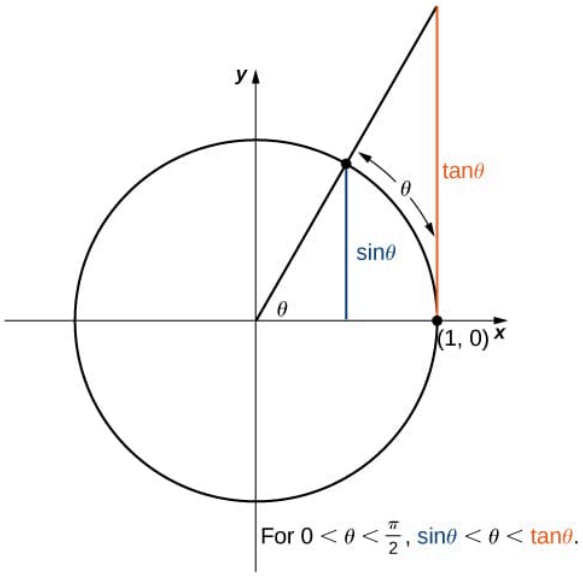


Figure 13.4.7: The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin\theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Equivalently, we have

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim\limits_{\theta\to 0^+} 1 = 1 = \lim\limits_{\theta\to 0^+} \cos\theta$, we conclude that $\lim\limits_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} = 1$, by the squeeze theorem. By applying a manipulation similar to that used in demonstrating that $\lim\limits_{\theta\to 0^-} \sin\theta = 0$, we can show that $\lim\limits_{\theta\to 0^-} \dfrac{\sin\theta}{\theta} = 1$. Thus,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

In Example 13.4.11, we use this limit to establish $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta} = 0$. This limit also proves useful in later chapters.

Example 13.4.11: Evaluating an Important Trigonometric Limit

Evaluate $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}$.

Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\\ &= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\\ &= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}\\ &= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\cdot\left(\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\right)\\ &= 1\cdot\frac{0}{2}=0.\end{aligned}$$

Therefore,

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0.$$

Exercise 13.4.11

Evaluate $\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}$.

Hint available at the end of this section.

Answer available at the end of this section.

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician *Archimedes* (ca. 287–212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height h and the base b of the isosceles triangle in Figure $13.4.8$ in terms of θ and r.

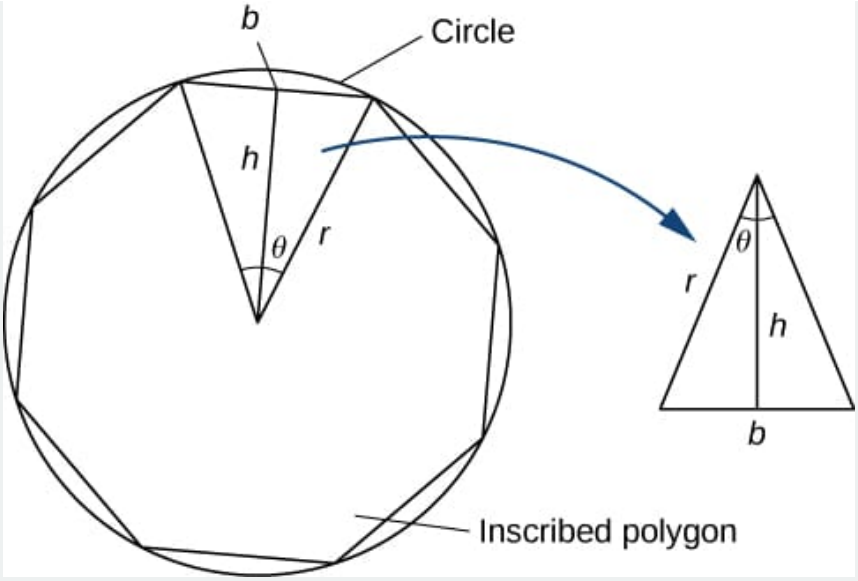


Figure 13.4.8

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r.

(Substitute $\frac{1}{2}\sin\theta$ for $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ in your expression.)

3. If an n-sided regular polygon is inscribed in a circle of radius r, find a relationship between θ and n. Solve this for n. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)

4. Find an expression for the area of the n-sided polygon in terms of r and θ.

5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (Hint: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$).

The technique of estimating areas of regions by using polygons is revisited in Introduction to Integration.

Key Concepts

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions,

$$\lim_{x\to a} f(x) = f(a).$$

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

Key Equations

- **Basic Limit Results**

$$\lim_{x\to a} x = a \qquad \lim_{x\to a} c = c$$

- **Important Limits**

$$\lim_{\theta\to 0}\sin\theta = 0$$

$$\lim_{\theta \to 0} \cos \theta = 1$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$$

Glossary

constant multiple law for limits
the limit law

$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$$

difference law for limits
the limit law

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

limit laws
the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let c be a constant

power law for limits
the limit law

$$\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n = L^n$$

for every positive integer n

product law for limits
the limit law

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

quotient law for limits
the limit law $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for M≠0

root law for limits
the limit law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

squeeze theorem
states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where L is a real number, then $\lim_{x \to a} g(x) = L$

sum law for limits
The limit law $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Answers

Exercise 13.4.2
Begin by applying the product law.

Exercise 13.4.2
$11\sqrt{10}$

Exercise 13.4.3
Use LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS as reference

Exercise 13.4.3
–13

Exercise 13.4.4

Follow the steps in the Problem-Solving Strategy

Exercise 13.4.4

$\frac{1}{3}$

Exercise 13.4.5

Follow the steps in the Problem-Solving Strategy

Exercise 13.4.5

$\frac{1}{4}$

Exercise 13.4.6

Follow the steps in the Problem-Solving Strategy

Exercise 13.4.6

–1

Exercise 13.4.7

Use the same technique as Example 13.4.7. Don't forget to factor $x^2 - 2x - 3$ before getting a common denominator.

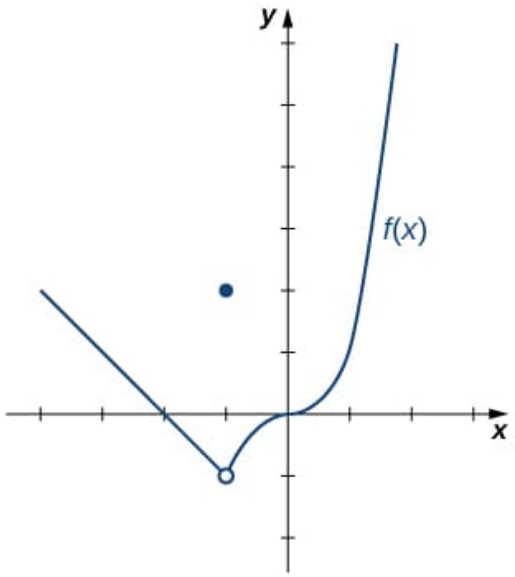
Exercise 13.4.7

$\frac{1}{4}$

Exercise 13.4.8

Use the method in Example $13.4.8B$ to evaluate the limit.

Exercise 13.4.8



$\lim_{x \to -1^-} f(x) = -1$

Exercise 13.4.9

Use the methods from Example 13.4.9.

Exercise 13.4.9

$+\infty$

Exercise 13.4.10

Use the fact that $-x^2 \leq x^2 \sin(1/x) \leq x^2$ to help you find two functions such that $x^2 \sin(1/x)$ is squeezed between them.

Exercise 13.4.10

0

Exercise 13.4.11

Multiply numerator and denominator by $1 + \cos\theta$.

Exercise 13.4.11

0

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13.4E: Exercises for Section 2.3

In exercises 1 - 4, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

1) $\lim_{x\to 0}\ (4x^2-2x+3)$

Answer

Use constant multiple law and difference law:

$$\lim_{x\to 0}\ (4x^2-2x+3)=4\lim_{x\to 0}x^2-2\lim_{x\to 0}x+\lim_{x\to 0}3=0+0+3=3$$

2) $\lim_{x\to 1}\dfrac{x^3+3x^2+5}{4-7x}$

3) $\lim_{x\to -2}\sqrt{x^2-6x+3}$

Answer

Use root law: $\lim_{x\to -2}\sqrt{x^2-6x+3}=\sqrt{\lim_{x\to -2}(x^2-6x+3)}=\sqrt{19}$

4) $\lim_{x\to -1}\ (9x+1)^2$

In exercises 5 - 10, use direct substitution to evaluate the limit of each continuous function.

5) $\lim_{x\to 7}x^2$

Answer

$$\lim_{x\to 7}x^2\ =\ 49$$

6) $\lim_{x\to -2}(4x^2-1)$

7) $\lim_{x\to 0}\dfrac{1}{1+\sin x}$

Answer

$$\lim_{x\to 0}\dfrac{1}{1+\sin x}\ =\ 1$$

8) $\lim_{x\to 2}e^{2x-x^2}$

9) $\lim_{x\to 1}\dfrac{2-7x}{x+6}$

Answer

$$\lim_{x\to 1}\dfrac{2-7x}{x+6}\ =\ -\dfrac{5}{7}$$

10) $\lim_{x\to 3}\ln e^{3x}$

In exercises 11 - 20, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit analytically.

11) $\lim_{x\to 4}\dfrac{x^2-16}{x-4}$

Answer

When $x = 4$, $\dfrac{x^2-16}{x-4} = \dfrac{16-16}{4-4} = \dfrac{0}{0}$;

then, $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4} = \lim\limits_{x\to 4} \dfrac{(x+4)(x-4)}{x-4} = \lim\limits_{x\to 4}(x+4) = 4+4 = 8$

12) $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-2x}$

13) $\lim\limits_{x\to 6} \dfrac{3x-18}{2x-12}$

Answer

When $x = 6$, $\dfrac{3x-18}{2x-12} = \dfrac{18-18}{12-12} = \dfrac{0}{0}$;

then, $\lim\limits_{x\to 6} \dfrac{3x-18}{2x-12} = \lim\limits_{x\to 6} \dfrac{3(x-6)}{2(x-6)} = \lim\limits_{x\to 6} \dfrac{3}{2} = \dfrac{3}{2}$

14) $\lim\limits_{h\to 0} \dfrac{(1+h)^2-1}{h}$

15) $\lim\limits_{t\to 9} \dfrac{t-9}{\sqrt{t}-3}$

Answer

When $t = 9$, $\dfrac{t-9}{\sqrt{t}-3} = \dfrac{9-9}{3-3} = \dfrac{0}{0}$;

then, $\lim\limits_{t\to 9} \dfrac{t-9}{\sqrt{t}-3} = \lim\limits_{t\to 9} \dfrac{t-9}{\sqrt{t}-3}\dfrac{\sqrt{t}+3}{\sqrt{t}+3} = \lim\limits_{t\to 9} \dfrac{(t-9)(\sqrt{t}+3)}{t-9} = \lim\limits_{t\to 9}(\sqrt{t}+3) = \sqrt{9}+3 = 6$

16) $\lim\limits_{h\to 0} \dfrac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h}$, where a is a real-valued constant

17) $\lim\limits_{\theta\to \pi} \dfrac{\sin\theta}{\tan\theta}$

Answer

When $\theta = \pi$, $\dfrac{\sin\theta}{\tan\theta} = \dfrac{\sin\pi}{\tan\pi} = \dfrac{0}{0}$;

then, $\lim\limits_{\theta\to \pi} \dfrac{\sin\theta}{\tan\theta} = \lim\limits_{\theta\to \pi} \dfrac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} = \lim\limits_{\theta\to \pi} \cos\theta = \cos\pi = -1$

18) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x^2-1}$

19) $\lim\limits_{x\to 1/2} \dfrac{2x^2+3x-2}{2x-1}$

Answer

When $x = 1/2$, $\dfrac{2x^2+3x-2}{2x-1} = \dfrac{\frac{1}{2}+\frac{3}{2}-2}{1-1} = \dfrac{0}{0}$;

then, $\lim\limits_{x\to 1/2} \dfrac{2x^2+3x-2}{2x-1} = \lim\limits_{x\to 1/2} \dfrac{(2x-1)(x+2)}{2x-1} = \lim\limits_{x\to 1/2}(x+2) = \dfrac{1}{2}+2 = \dfrac{5}{2}$

20) $\lim\limits_{x\to -3} \dfrac{\sqrt{x+4}-1}{x+3}$

In exercises 21 - 24, use direct substitution to obtain an undefined expression. Then, use the method used in Example 9 of this section to simplify the function and determine the limit.

21) $\lim_{x\to -2^-} \dfrac{2x^2+7x-4}{x^2+x-2}$

Answer

$-\infty$

22) $\lim_{x\to -2^+} \dfrac{2x^2+7x-4}{x^2+x-2}$

23) $\lim_{x\to 1^-} \dfrac{2x^2+7x-4}{x^2+x-2}$

Answer

$-\infty$

24) $\lim_{x\to 1^+} \dfrac{2x^2+7x-4}{x^2+x-2}$

In exercises 25 - 32, assume that $\lim_{x\to 6} f(x) = 4$, $\lim_{x\to 6} g(x) = 9$, and $\lim_{x\to 6} h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.

25) $\lim_{x\to 6} 2f(x)g(x)$

Answer

$$\lim_{x\to 6} 2f(x)g(x) = 2\left(\lim_{x\to 6} f(x)\right)\left(\lim_{x\to 6} g(x)\right) = 2(4)(9) = 72$$

26) $\lim_{x\to 6} \dfrac{g(x)-1}{f(x)}$

27) $\lim_{x\to 6} \left(f(x) + \dfrac{1}{3}g(x)\right)$

Answer

$$\lim_{x\to 6} \left(f(x) + \frac{1}{3}g(x)\right) = \lim_{x\to 6} f(x) + \frac{1}{3}\lim_{x\to 6} g(x) = 4 + \frac{1}{3}(9) = 7$$

28) $\lim_{x\to 6} \dfrac{\left(h(x)\right)^3}{2}$

29) $\lim_{x\to 6} \sqrt{g(x)-f(x)}$

Answer

$$\lim_{x\to 6} \sqrt{g(x)-f(x)} = \sqrt{\lim_{x\to 6} g(x) - \lim_{x\to 6} f(x)} = \sqrt{9-4} = \sqrt{5}$$

30) $\lim_{x\to 6} x \cdot h(x)$

31) $\lim_{x\to 6}[(x+1)\cdot f(x)]$

Answer

$$\lim_{x\to 6}[(x+1)f(x)] = \left(\lim_{x\to 6}(x+1)\right)\left(\lim_{x\to 6} f(x)\right) = 7(4) = 28$$

32) $\lim_{x\to 6}(f(x)\cdot g(x) - h(x))$

[T] In exercises 33 - 35, use a calculator to draw the graph of each piecewise-defined function and study the graph to

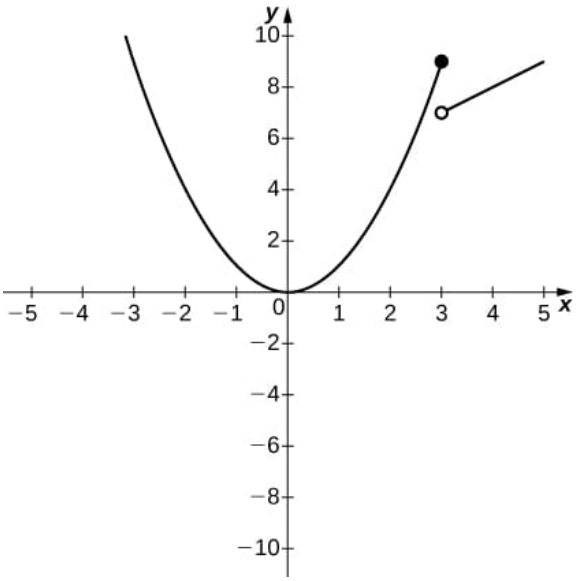
evaluate the given limits.

33) $f(x) = \begin{cases} x^2, & x \leq 3 \\ x+4, & x > 3 \end{cases}$

a. $\lim_{x \to 3^-} f(x)$

b. $\lim_{x \to 3^+} f(x)$

Answer



a. 9; b.7

34) $g(x) = \begin{cases} x^3 - 1, & x \leq 0 \\ 1, & x > 0 \end{cases}$

a. $\lim_{x \to 0^-} g(x)$

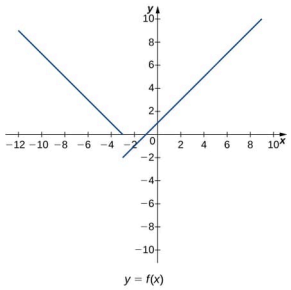
b. $\lim_{x \to 0^+} g(x)$

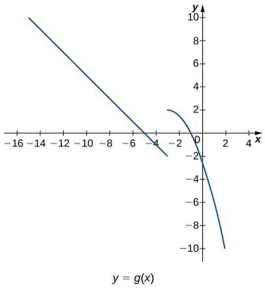
35) $h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \geq 2 \end{cases}$

a. $\lim_{x \to 2^-} h(x)$

b. $\lim_{x \to 2^+} h(x)$

In exercises 36 - 43, use the following graphs and the limit laws to evaluate each limit.





36) $\lim_{x\to -3^+}(f(x)+g(x))$

37) $\lim_{x\to -3^-}(f(x)-3g(x))$

Answer

$$\lim_{x\to -3^-}(f(x)-3g(x))=\lim_{x\to -3^-}f(x)-3\lim_{x\to -3^-}g(x)=0+6=6$$

38) $\lim_{x\to 0}\dfrac{f(x)g(x)}{3}$

39) $\lim_{x\to -5}\dfrac{2+g(x)}{f(x)}$

Answer

$$\lim_{x\to -5}\frac{2+g(x)}{f(x)}=\frac{2+\left(\lim_{x\to -5}g(x)\right)}{\lim_{x\to -5}f(x)}=\frac{2+0}{2}=1$$

40) $\lim_{x\to 1}(f(x))^2$

41) $\lim_{x\to 1}\sqrt[3]{f(x)-g(x)}$

Answer

$$\lim_{x\to 1}\sqrt[3]{f(x)-g(x)}=\sqrt[3]{\lim_{x\to 1}f(x)-\lim_{x\to 1}g(x)}=\sqrt[3]{2+5}=\sqrt[3]{7}$$

42) $\lim_{x\to -7}(x\cdot g(x))$

43) $\lim_{x\to -9}[x\cdot f(x)+2\cdot g(x)]$

Answer

$$\lim_{x\to -9}(xf(x)+2g(x))=\left(\lim_{x\to -9}x\right)\left(\lim_{x\to -9}f(x)\right)+2\lim_{x\to -9}g(x)=(-9)(6)+2(4)=-46$$

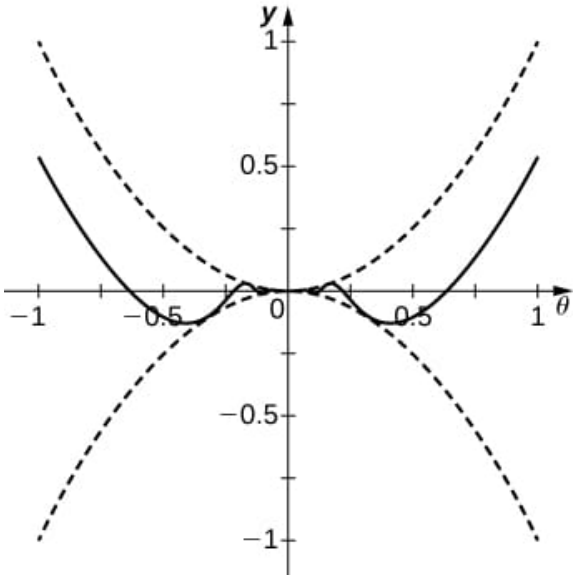
For exercises 44 - 46, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f(x)$, $g(x)$, and $h(x)$ when possible.

44) [T] True or False? If $2x - 1 \leq g(x) \leq x^2 - 2x + 3$, then $\lim_{x \to 2} g(x) = 0$.

45) [T] $\lim_{\theta \to 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$

Answer

The limit is zero.



46) $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrrational} \end{cases}$

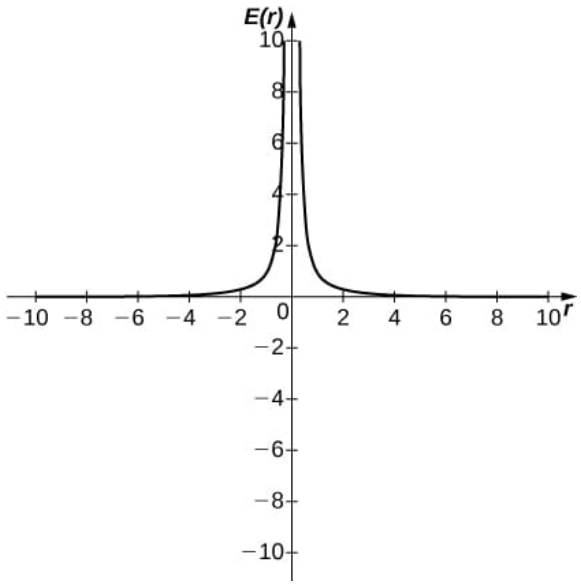
47) [T] In physics, the magnitude of an electric field generated by a point charge at a distance r in vacuum is governed by Coulomb's law: $E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$, where E represents the magnitude of the electric field, q is the charge of the particle, r is the distance between the particle and where the strength of the field is measured, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant: $8.988 \times 109N \cdot m^2/C^2$.

a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}$.

b. Evaluate $\lim_{r \to 0^+} E(r)$. What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

Answer

a.



b. ∞. The magnitude of the electric field as you approach the particle q becomes infinite. It does not make physical sense to evaluate negative distance.

48) [T] The density of an object is given by its mass divided by its volume: $\rho = m/V$.

a. Use a calculator to plot the volume as a function of density $(V = m/\rho)$, assuming you are examining something of mass 8 kg $(m = 8)$.

b. Evaluate $\lim_{x \to 0^+} V(\rho)$ and explain the physical meaning.

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13.5: Continuity

Learning Objectives

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- State the theorem for limits of composite functions.
- Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity* at a point where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure $13.5.1$. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

i. $f(a)$ is defined

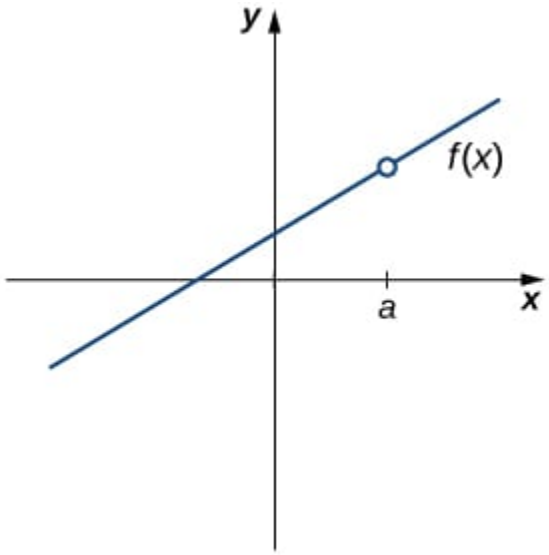


Figure $13.5.1$: The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in Figure $13.5.2$, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim\limits_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

ii. $\lim_{x \to a} f(x)$ **exists**

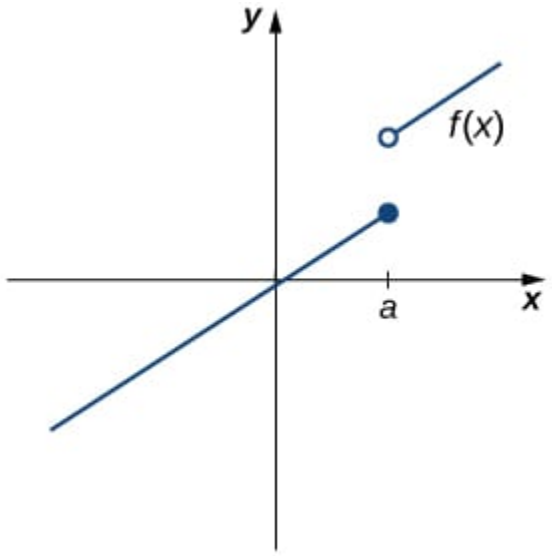


Figure 13.5.2: The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x)$ does not exist.

However, as we see in Figure 13.5.3, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

iii. $\lim_{x \to a} f(x) = f(a)$

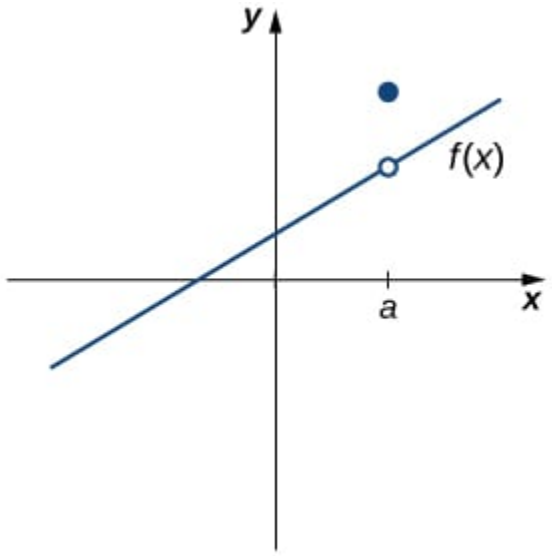


Figure 13.5.3: The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition: Continuous at a Point

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f(a)$ is defined
ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example $13.5.1A$: Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure $13.5.4$.

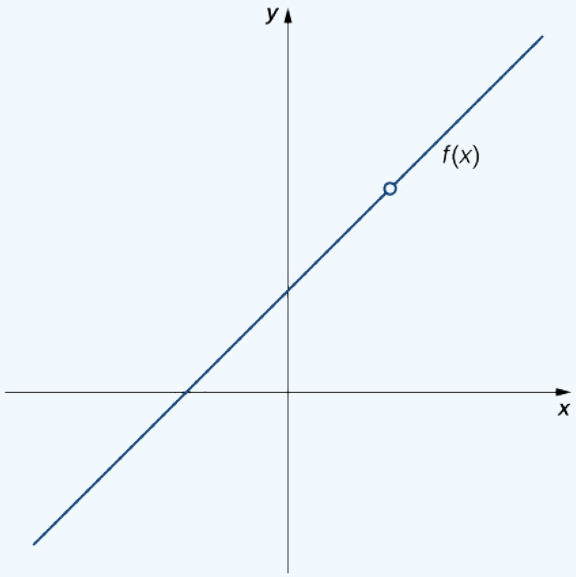


Figure $13.5.4$: The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example $13.5.1B$: Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4, & \text{if } x \leq 3 \\ 4x - 8, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim_{x \to 3} f(x)$. To do this, we must compute $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$:

$$\lim_{x \to 3^-} f(x) = -(3^2) + 4 = -5$$

and

$$\lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim_{x \to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in Figure $13.5.5$.

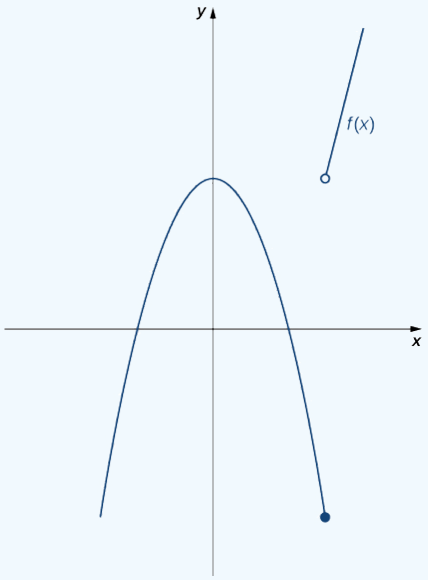


Figure $13.5.5$: The function $f(x)$ is not continuous at 3 because $\lim_{x \to 3} f(x)$ does not exist.

Example $13.5.1C$: Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1$$

Next,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x\to 0} f(x)$. We see that

$$f(0) = 1 = \lim_{x\to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

Exercise 13.5.1

Using the definition, determine whether the function $f(x) = \begin{cases} 2x+1, & \text{if } x < 1 \\ 2, & \text{if } x = 1 \\ -x+4, & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Hint available at the end of this section.

Answer available at the end of this section.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 13.5.1: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x\to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply Theorem 13.5.1 to determine the points at which a given rational function is continuous.

Example 13.5.2:Continuity of a Rational Function

For what values of *x* is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

Exercise 13.5.2

For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Hint available at the end of this section.

Answer available at the end of this section.

Types of Discontinuities

As we have seen in Example $13.5.1A$ and Example $13.5.1B$, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure $13.5.6$ illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

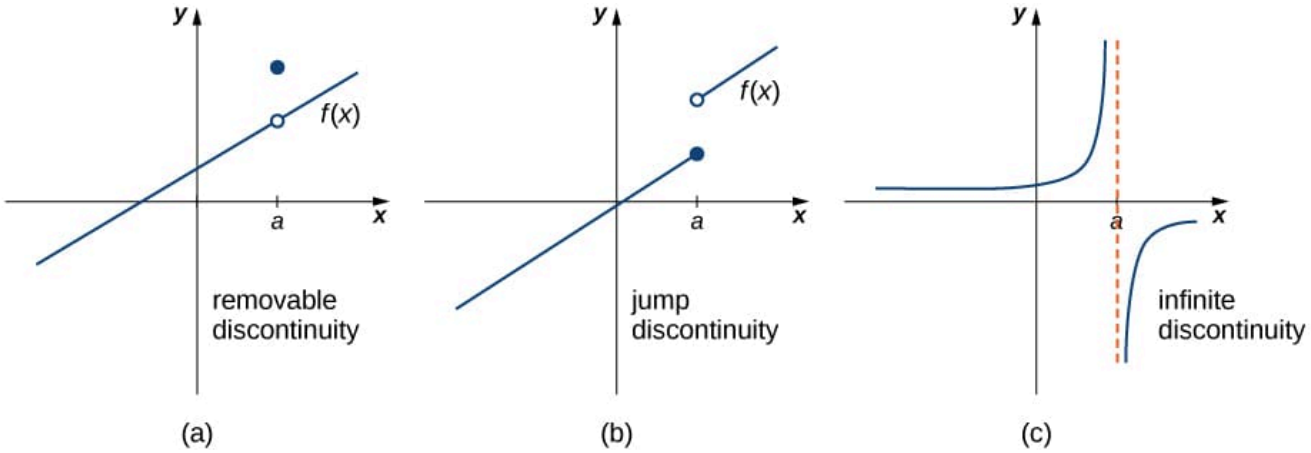


Figure $13.5.6$: Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim\limits_{x\to a} f(x)$ exists. (Note: When we state that $\lim\limits_{x\to a} f(x)$ exists, we mean that $\lim\limits_{x\to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$. (Note: When we state that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

Example $13.5.3$: Classifying a Discontinuity

In Example $13.5.1A$, we showed that $f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim\limits_{x\to 2} f(x)$:

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} (x+2) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x\to 2} f(x)$ exists, f has a removable discontinuity at $x=2$.

Example 13.5.4: Classifying a Discontinuity

In Example $13.5.1B$, we showed that $f(x) = \begin{cases} -x^2+4, & \text{if } x \le 3 \\ 4x-8, & \text{if } x > 3 \end{cases}$ is discontinuous at $x=3$. Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that f is discontinuous at 3 because $\lim_{x\to 3} f(x)$ does not exist. However, since $\lim_{x\to 3^-} f(x) = -5$ and $\lim_{x\to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example 13.5.5: Classifying a Discontinuity

Determine whether $f(x) = \dfrac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1. To determine the type of discontinuity, we must determine the limit at -1. We see that $\lim_{x\to -1^-} \dfrac{x+2}{x+1} = -\infty$ and $\lim_{x\to -1^+} \dfrac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1.

Exercise 13.5.3

For $f(x) = \begin{cases} x^2, & \text{if } x \ne 1 \\ 3, & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Hint available at the end of this section.

Answer available at the end of this section.

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Definition: Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim_{x \to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example 13.5.6: Continuity on an Interval

State the interval(s) over which the function $f(x) = \dfrac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \dfrac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty, -2), (-2, 0)$, and $(0, +\infty)$.

Example 13.5.7: Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4 - x^2}$ is continuous.

Solution

From the limit laws, we know that $\lim_{x \to a} \sqrt{4 - x^2} = \sqrt{4 - a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ exists and $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

Exercise 13.5.4

State the interval(s) over which the function $f(x) = \sqrt{x + 3}$ is continuous.

Hint available at the end of this section.

Answer available at the end of this section.

Theorem 13.5.2 allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Theorem 13.5.2: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then

$$\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big) = f(L). \tag{13.5.1}$$

Before we move on to Example 13.5.8, recall that earlier, in the section on limit laws, we showed $\lim_{x\to 0}\cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In Example 13.5.8, we see how to combine this result with the composite function theorem.

Example 13.5.8: Limit of a Composite Cosine Function

Evaluate $\lim_{x\to \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

Solution

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim_{x\to\pi/2}\left(x - \frac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

$$\lim_{x\to\pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x\to\pi/2}\left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

Exercise 13.5.4:

Evaluate $\lim_{x\to\pi} \sin(x - \pi)$.

Hint available at the end of this section.

Answer available at the end of this section.

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem 13.5.3: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x\to a}\cos x = \cos a$ for all values of a.

$$\begin{aligned}
\lim_{x\to a}\cos x &= \lim_{x\to a}\cos((x-a)+a) && \text{Rewrite } x = x - a + a.\\
&= \lim_{x\to a}(\cos(x-a)\cos a - \sin(x-a)\sin a) && \text{Apply the identity for the cosine of the sum of two angles.}\\
&= \cos(\lim_{x\to a}(x-a))\cos a - \sin(\lim_{x\to a}(x-a))\sin a && \text{Since } \lim_{x\to a}(x-a) = 0, \text{ and } \sin x \text{ and } \cos x \text{ are continuous at } 0.\\
&= \cos(0)\cos a - \sin(0)\sin a && \text{Evaluate } \cos(0) \text{ and } \sin(0) \text{ and simplify.}\\
&= 1\cdot\cos a - 0\cdot\sin a = \cos a.
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem.**

The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in Figure 13.5.7.

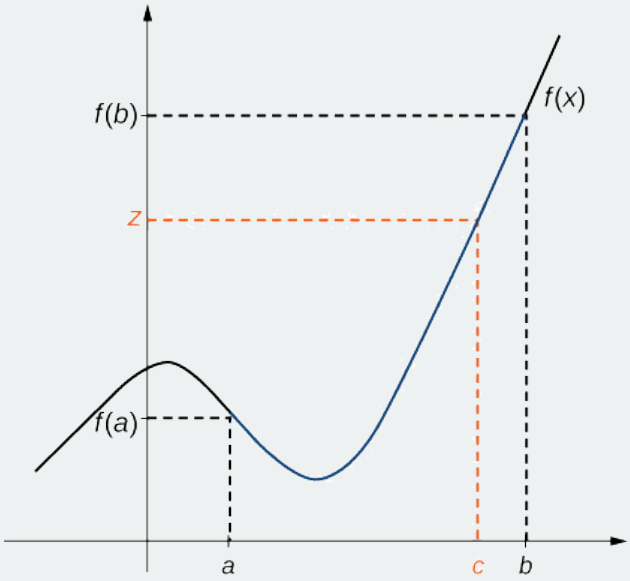


Figure 13.5.7: There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

Example 13.5.9: Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f(\tfrac{\pi}{2}) = \tfrac{\pi}{2} - \cos\tfrac{\pi}{2} = \tfrac{\pi}{2} > 0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example 13.5.10: When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0,2]$, $f(0)>0$ and $f(2)>0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0,2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x)=(x-1)^2$. It satisfies $f(0)=1>0, f(2)=1>0$, and $f(1)=0$.

Example 13.5.11: When Can You Apply the Intermediate Value Theorem?

For $f(x)=1/x, f(-1)=-1<0$ and $f(1)=1>0$. Can we conclude that $f(x)$ has a zero in the interval $[-1,1]$?

Solution

No. The function is not continuous over $[-1,1]$. The Intermediate Value Theorem does not apply here.

Exercise 13.5.5

Show that $f(x)=x^3-x^2-3x+1$ has a zero over the interval $[0,1]$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f(x)$ is continuous at L and $\lim\limits_{x\to a} g(x)=L$, then $\lim\limits_{x\to a} f\big(g(x)\big)=f\big(\lim\limits_{x\to a} g(x)\big)=f(L)$.
- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

Glossary

continuity at a point
A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim\limits_{x\to a} f(x)$ exists, and (3) $\lim x\to a f(x)=f(a)$

continuity from the left
A function is continuous from the left at b if $\lim\limits_{x\to b^-} f(x)=f(b)$

continuity from the right
A function is continuous from the right at a if $\lim\limits_{x\to a^+} f(x)=f(a)$

continuity over an interval
a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval

if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at a and from the left at b

discontinuity at a point
A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

infinite discontinuity
An infinite discontinuity occurs at a point a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

Intermediate Value Theorem
Let f be continuous over a closed bounded interval $[a, b]$ if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

jump discontinuity
A jump discontinuity occurs at a point a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

removable discontinuity
A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim_{x \to a} f(x)$ exists

Answers

Exercise 13.5.1
Check each condition of the definition.

Exercise 13.5.1
f is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x \to 1} f(x)$.

Exercise 13.5.2
Use the Continuity of Polynomials and Rational Functions stated above.

Exercise 13.5.2
$f(x)$ is continuous at every real number.

Exercise 13.5.3
Consider the definitions of the various kinds of discontinuity stated above. If the function is discontinuous at 1, look at $\lim_{x \to 1} f(x)$

Exercise 13.5.3
Discontinuous at 1; removable

Exercise 13.5.4
Use Example $13.5.7$ as a guide.

Exercise 13.5.4
$[-3, +\infty)$

Exercise 13.5.4**:**
$f(x) = \sin x$ is continuous at 0. Use Example $13.5.8$ as a guide.

Exercise 13.5.4**:**
0

Exercise 13.5.5
Find $f(0)$ and $f(1)$. Apply the Intermediate Value Theorem.

Exercise 13.5.5
$f(0) = 1 > 0, \quad f(1) = -2 < 0; \quad f(x)$ is continuous over $[0, 1]$. It must have a zero on this interval.

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13.5E: Exercises for Section 2.4

For exercises 1 - 8, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as *jump*, *removable*, *infinite*, or *other*.

1) $f(x) = \dfrac{1}{\sqrt{x}}$

Answer

The function is defined for all x in the interval $(0, \infty)$.

2) $f(x) = \dfrac{2}{x^2 + 1}$

3) $f(x) = \dfrac{x}{x^2 - x}$

Answer

Removable discontinuity at $x = 0$; infinite discontinuity at $x = 1$.

4) $g(t) = t^{-1} + 1$

5) $f(x) = \dfrac{5}{e^x - 2}$

Answer

Infinite discontinuity at $x = \ln 2$

6) $f(x) = \dfrac{|x - 2|}{x - 2}$

7) $H(x) = \tan 2x$

Answer

Infinite discontinuities at $x = \dfrac{(2k + 1)\pi}{4}$, for $k = 0, \ \pm 1, \ \pm 2, \ \pm 3, \ \ldots$

8) $f(t) = \dfrac{t + 3}{t^2 + 5t + 6}$

For exercises 9 - 14, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

9) $\dfrac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$

Answer

No. It is a removable discontinuity.

10) $h(\theta) = \dfrac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$

11) $g(u) = \begin{cases} \dfrac{6u^2 + u - 2}{2u - 1}, & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2}, & \text{if } u = \frac{1}{2} \end{cases}$, at $u = \frac{1}{2}$

Answer

Yes. It is continuous.

12) $f(y) = \dfrac{\sin(\pi y)}{\tan(\pi y)}$, at $y = 1$

13) $f(x) = \begin{cases} x^2 - e^x, & \text{if } x < 0 \\ x - 1, & \text{if } x \geq 0 \end{cases}$, at $x = 0$

Answer

Yes. It is continuous.

14) $f(x) = \begin{cases} x\sin(x), & \text{if } x \leq \pi \\ x\tan(x), & \text{if } x > \pi \end{cases}$, at $x = \pi$

In exercises 15 - 19, find the value(s) of k that makes each function continuous over the given interval.

15) \(f(x)=\begin{cases}3x+2, & \text{if }x

Answer

$k = -5$

16) $f(\theta) = \begin{cases} \sin\theta, & \text{if } 0 \leq \theta < \frac{\pi}{2} \\ \cos(\theta + k), & \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$

17) $f(x) = \begin{cases} \dfrac{x^2 + 3x + 2}{x + 2}, & \text{if } x \neq -2 \\ k, & \text{if } x = -2 \end{cases}$

Answer

$k = -1$

18) $f(x) = \begin{cases} e^{kx}, & \text{if } 0 \leq x < 4 \\ x + 3, & \text{if } 4 \leq x \leq 8 \end{cases}$

19) \(f(x)=\begin{cases}\sqrt{kx}, & \text{if }0≤x≤3\x+1, & \text{if }3

Answer

$k = \frac{16}{3}$

In exercises 20 - 21, use the Intermediate Value Theorem (IVT).

20) Let $h(x) = \begin{cases} 3x^2 - 4, & \text{if } x \leq 2 \\ 5 + 4x, & \text{if } x > 2 \end{cases}$ Over the interval $[0, 4]$, there is no value of x such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

21) A particle moving along a line for time t has a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value c for \(2

Answer

Since both s and $y = t$ are continuous everywhere, then $h(t) = s(t) - t$ is continuous everywhere and, in particular, it is continuous over the closed interval $[2, 5]$. Also, $h(2) = 3 > 0$ and $h(5) = -3 < 0$. Therefore, by the IVT, there is a value $x = c$ such that $h(c) = 0$.

22) [T] Use the statement “The cosine of t is equal to t cubed."

a. Write a mathematical equation of the statement.

b. Prove that the equation in part a. has at least one real solution.

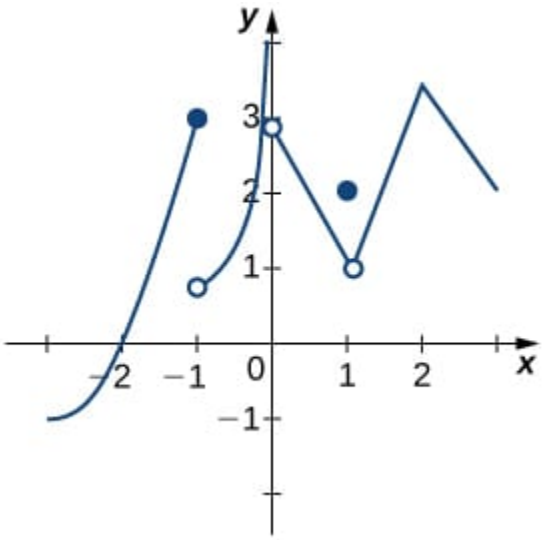
c. Use a calculator to find an interval of length 0.01 that contains a solution.

23) Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

Answer

The function $f(x) = 2^x - x^3$ is continuous over the interval $[1.25, 1.375]$ and has opposite signs at the endpoints.

24) Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.

b. For each value in part a., state why the formal definition of continuity does not apply.

c. Classify each discontinuity as either jump, removable, or infinite.

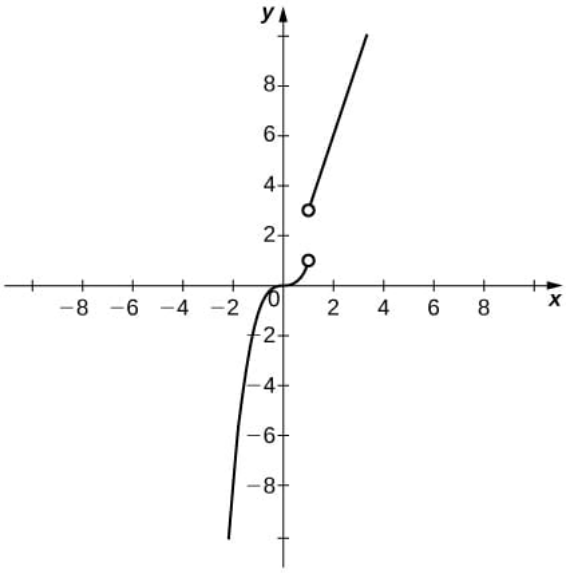
25) Let $f(x) = \begin{cases} 3x, & \text{if } x > 1 \\ x^3, & \text{if } x < 1 \end{cases}$.

a. Sketch the graph of f.

b. Is it possible to find a value k such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.

Answer

a.



b. It is not possible to redefine $f(1)$ since the discontinuity is a jump discontinuity.

26) Let $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$ for $x \neq -1, 1$.

a. Sketch the graph of f.

b. Is it possible to find values k_1 and k_2 such that $f(-1) = k$ and $f(1) = k_2$, and that makes $f(x)$ continuous for all real numbers? Briefly explain.

27) Sketch the graph of the function $y = f(x)$ with properties i. through vii.

i. The domain of f is $(-\infty, +\infty)$.

ii. f has an infinite discontinuity at $x = -6$.

iii. $f(-6) = 3$

iv. $\lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^+} f(x) = 2$

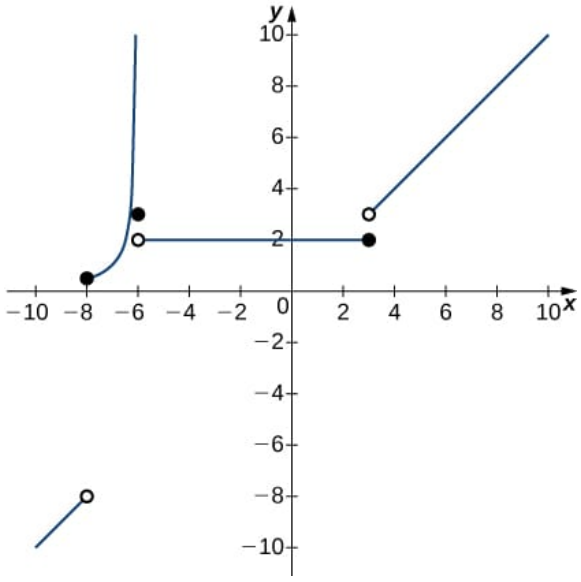
v. $f(-3) = 3$

vi. f is left continuous but not right continuous at $x = 3$.

vii. $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$

Answer

Answers may vary; see the following example:



28) Sketch the graph of the function $y = f(x)$ with properties i. through iv.

i. The domain of f is $[0, 5]$.

ii. $\lim\limits_{x \to 1^+} f(x)$ and $\lim\limits_{x \to 1^-} f(x)$ exist and are equal.

iii. $f(x)$ is left continuous but not continuous at $x = 2$, and right continuous but not continuous at $x = 3$.

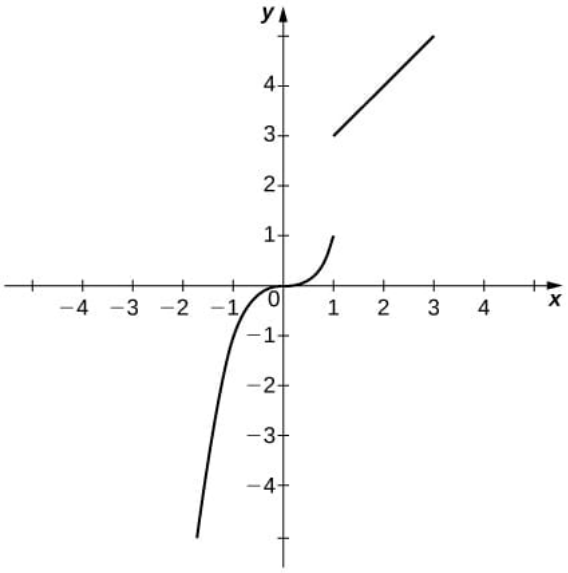
iv. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and the following limits hold: $\lim\limits_{x \to 3^-} f(x) = -\infty$ and $\lim\limits_{x \to 3^+} f(x) = 2$.

In exercises 29 - 30, suppose $y = f(x)$ is defined for all x. For each description, sketch a graph with the indicated property.

29) Discontinuous at $x = 1$ with $\lim_{x \to -1} f(x) = -1$ and $\lim_{x \to 2} f(x) = 4$

Answer

Answers may vary; see the following example:



30) Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \to 0} f(x) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

31) $f(t) = \dfrac{2}{e^t - e^{-t}}$ is continuous everywhere.

Answer

False. It is continuous over $(-\infty, 0) \cup (0, \infty)$.

32) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal, then f cannot be discontinuous at $x = a$.

33) If a function is not continuous at a point, then it is not defined at that point.

Answer

False. Consider $f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0 \end{cases}$.

34) According to the IVT, $\cos x - \sin x - x = 2$ has a solution over the interval $[-1, 1]$.

35) If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b]$.

Answer

False. Consider $f(x) = \cos(x)$ on $[-\pi, 2\pi]$.

36) The function $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$ is continuous over the interval $[0, 3]$.

37) If $f(x)$ is continuous everywhere and $f(a), f(b) > 0$, then there is no root of $f(x)$ in the interval $[a, b]$.

Answer

False. The IVT does not work in reverse! Consider $(x-1)^2$ over the interval $[-2, 2]$.

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \frac{|q_1 q_2|}{r^2}$, where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and r is the distance between the two particles.

38) To simplify the calculation of a model with many interacting particles, after some threshold value $r = R$, we approximate F as zero.

a. Explain the physical reasoning behind this assumption.

b. What is the force equation?

c. Evaluate the force F using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} **coulombs (C)**, and the Coulomb constant $k_e = 8.988 \times 10^9 Nm^2/C^2$ are 1 m apart. Also, assume $R < 1$ m. How much inaccuracy does our approximation generate? Is our approximation reasonable?

d. Is there any finite value of R for which this system remains continuous at R?

39) Instead of making the force 0 at R, we let the force be $10 - 20$ for $r \geq R$. Assume two protons, which have a magnitude of charge $1.6022 \times 10^{-19}\ C$, and the Coulomb constant $k_e = 8.988 \times 10^9\ \ Nm^2/C^2$. Is there a value R that can make this system continuous? If so, find it.

Answer

$R = 0.0001519$ m

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2$, where m is the mass of the rocket, d is the distance of the rocket from the center of Earth, and k is a constant.

40) [T] Determine the value and units of k given that the mass of the rocket on Earth is 3 million kg. (Hint: The distance from the center of Earth to its surface is 6378 km.)

41) [T] After a certain distance D has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by \(F(d)=\begin{cases}–\dfrac{mk}{d^2}, & \text{if }d

Answer

$D = 63.78$ km

42) As the rocket travels away from Earth's surface, there is a distance D where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as \(F(d)=\begin{cases} –\dfrac{m_1k}{d^2}, & \text{if }d

In Exercises 43 - 44, prove each function is continuous everywhere.

43) $f(\theta) = \sin\theta$

Answer

For all values of a, $f(a)$ is defined, $\lim_{\theta \to a} f(\theta)$ exists, and $\lim_{\theta \to a} f(\theta) = f(a)$. Therefore, $f(\theta)$ is continuous everywhere.

44) $g(x) = |x|$

45) Where is $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$ continuous?

Answer

Nowhere

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13.6: The Precise Definition of a Limit

Learning Objectives

- Describe the epsilon-delta definition of a limit.
- Apply the epsilon-delta definition to find the limit of a function.
- Describe the epsilon-delta definitions of one-sided limits and infinite limits.
- Use the epsilon-delta definition to prove the limit laws.

By now you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a very strong intuitive sense of what the limit of a function means and how you can find it. In this section, we convert this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is quite possibly one of the most challenging definitions you will encounter early in your study of calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points a and b on a number line is given by $|a-b|$.

- The statement $|f(x)-L|<\varepsilon$ may be interpreted as: The distance between $f(x)$ and L is less than ε.
- The statement $0<|x-a|<\delta$ may be interpreted as: $x\neq a$ and the distance between x and a is less than δ.

It is also important to look at the following equivalences for absolute value:

- The statement $|f(x)-L|<\varepsilon$ is equivalent to the statement \(L–ε
- The statement $0<|x-a|<\delta$ is equivalent to the statement \(a–δ

With these clarifications, we can state the formal **epsilon-delta definition of the limit**.

Definition: Finite Limits (Formal)

Let $f(x)$ be defined for all $x\neq a$ over an open interval containing a. Let L be a real number. Then

$$\lim_{x\to a} f(x)=L$$

if, for every $\varepsilon>0$, there exists a $\delta>0$, such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.

This definition may seem rather complex from a mathematical point of view, but it becomes easier to understand if we break it down phrase by phrase. The statement itself involves something called a **universal quantifier** (for every $\varepsilon>0$), an **existential quantifier** (there exists a $\delta>0$), and, last, a **conditional statement** (if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$). Let's take a look at Table $13.6.1$, which breaks down the definition and translates each part.

Table $13.6.1$

Definition	Translation
1. For every $\varepsilon>0$,	1. For every positive distance ε from L,
2. there exists a $\delta>0$,	2. There is a positive distance δ from a,
3. such that	3. such that

Definition	Translation
4. if $0 < \lvert x - a \rvert < \delta$, then $\lvert f(x) - L \rvert < \varepsilon$.	4. if x is closer than δ to a and $x \neq a$, then $f(x)$ is closer than ε to L.

We can get a better handle on this definition by looking at the definition geometrically. Figure $13.6.1$ shows possible values of δ for various choices of $\varepsilon > 0$ for a given function $f(x)$, a number a, and a limit L at a. Notice that as we choose smaller values of ε (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a, then the value of $f(x)$ is within ε of the limit L.

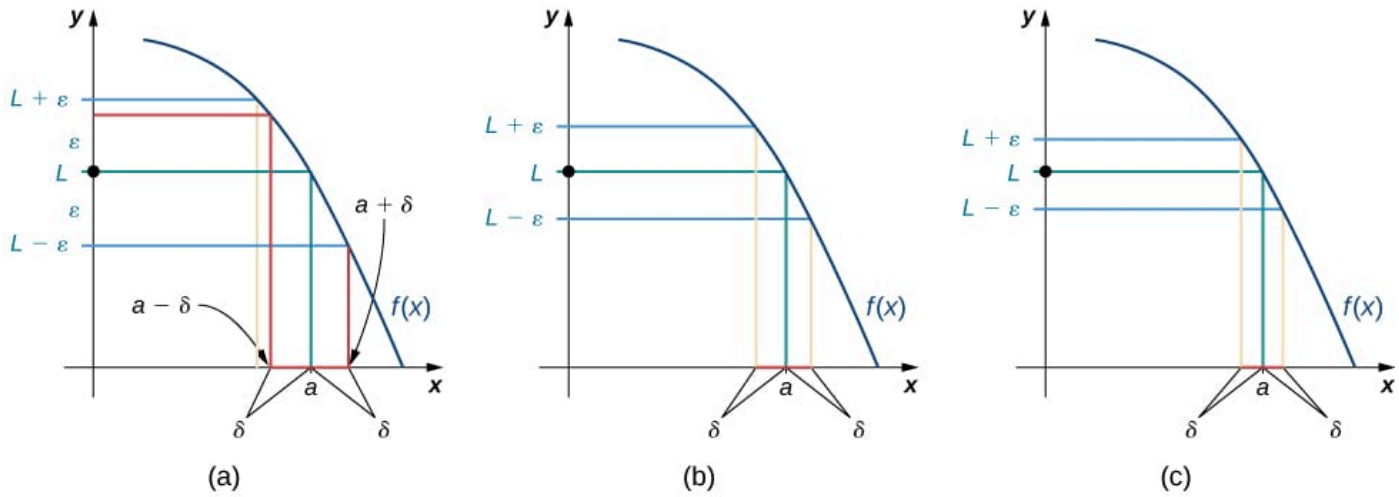


Figure $13.6.1$: These graphs show possible values of δ, given successively smaller choices of ε.

Note

Visit the following applet to experiment with finding values of δ for selected values of ε:

- The epsilon-delta definition of limit

Example $13.6.1$ shows how you can use this definition to prove a statement about the limit of a specific function at a specified value.

Example $13.6.1$: Proving a Statement about the Limit of a Specific Function

Prove that $\lim\limits_{x \to 1} (2x + 1) = 3$.

Solution

Let $\varepsilon > 0$.

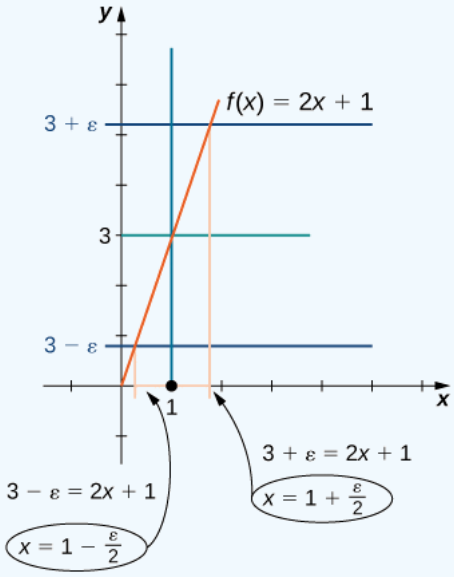
The first part of the definition begins "For every $\varepsilon > 0$."This means we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.

Choose $\delta = \frac{\varepsilon}{2}$.

The definition continues with "there exists a $\delta > 0$. " The phrase "there exists" in a mathematical statement is always a signal for a scavenger hunt. In other words, we must go and find δ. So, where exactly did $\delta = \varepsilon/2$ come from? There are two basic approaches to tracking down δ. One method is purely algebraic and the other is geometric.

We begin by tackling the problem from an algebraic point of view. Since ultimately we want $|(2x+1)-3|<\varepsilon$, we begin by manipulating this expression: $|(2x+1)-3|<\varepsilon$ is equivalent to $|2x-2|<\varepsilon$, which in turn is equivalent to $|2||x-1|<\varepsilon$. Last, this is equivalent to $|x-1|<\varepsilon/2$. Thus, it would seem that $\delta=\varepsilon/2$ is appropriate.

We may also find δ through geometric methods. Figure $13.6.2$ demonstrates how this is done.



δ is the length of the smaller of the two distances marked in brown.

$\delta = \min\{1+\frac{\varepsilon}{2}-1,\ 1-(1-\frac{\varepsilon}{2})\}$

$= \min\{\frac{\varepsilon}{2},\frac{\varepsilon}{2}\}$

$= \frac{\varepsilon}{2}$

Figure $13.6.2$: This graph shows how we find δ geometrically.

Assume $0<|x-1|<\delta$. When δ has been chosen, our goal is to show that if $0<|x-1|<\delta$, then $|(2x+1)-3|<\varepsilon$. To prove any statement of the form "If this, then that," we begin by assuming "this" and trying to get "that."

Thus,

$|(2x+1)-3|=|2x-2|$ **property of absolute value**

$=|2(x-1)|$

$=|2||x-1|\ |2|=2$

$=2|x-1|$

$<2\cdot\delta$ **here's where we use the assumption that** $0<|x-1|<\delta$

$=2\cdot\frac{\varepsilon}{2}=\varepsilon$ **here's where we use our choice of** $\delta=\varepsilon/2$

Analysis

In this part of the proof, we started with $|(2x+1)-3|$ and used our assumption $0<|x-1|<\delta$ in a key part of the chain of inequalities to get $|(2x+1)-3|$ to be less than ε. We could just as easily have manipulated the assumed inequality $0<|x-1|<\delta$ to arrive at $|(2x+1)-3|<\varepsilon$ as follows:

$0<|x-1|<\delta \Rightarrow |x-1|<\delta$

\(⇒–δ

\(⇒–\frac{ε}{2}

$\Rightarrow -\varepsilon<2x-2<\varepsilon$

$\Rightarrow |2x-2|<\varepsilon$

$$\Rightarrow |(2x+1)-3| < \varepsilon.$$

Therefore, $\lim\limits_{x\to 1}\ (2x+1) = 3$. (Having completed the proof, we state what we have accomplished.)

After removing all the remarks, here is a final version of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon/2$.

Assume $0 < |x-1| < \delta$.

Thus,

$$\begin{aligned} |(2x+1)-3| &= |2x-2| \\ &= |2(x-1)| \\ &= |2||x-1| \\ &= 2|x-1| \\ &< 2\cdot\delta \\ &= 2\cdot\frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{x\to 1}\ (2x+1) = 3$.

The following Problem-Solving Strategy summarizes the type of proof we worked out in Example $13.6.1$.

Problem-Solving Strategy: Proving That $\lim\limits_{x\to a} f(x) = L$ for a Specific Function $f(x)$

1. Let's begin the proof with the following statement: Let $\varepsilon > 0$.
2. Next, we need to obtain a value for δ. After we have obtained this value, we make the following statement, filling in the blank with our choice of δ: Choose $\delta =$_____.
3. The next statement in the proof should be (at this point, we fill in our given value for a): Assume $0 < |x-a| < \delta$.
4. Next, based on this assumption, we need to show that $|f(x)-L| < \varepsilon$, where $f(x)$ and L are our function $f(x)$ and our limit L. At some point, we need to use $0 < |x-a| < \delta$.
5. We conclude our proof with the statement: Therefore, $\lim\limits_{x\to a} f(x) = L$.

Example $13.6.2$: Proving a Statement about a Limit

Complete the proof that $\lim\limits_{x\to -1}\ (4x+1) = -3$ by filling in the blanks.

Let ____.

Choose $\delta =$_____.

Assume $0 < |x-$_____$| < \delta$.

Thus, |_____−_____|=_____ε.

Solution

We begin by filling in the blanks where the choices are specified by the definition. Thus, we have

Let $\varepsilon > 0$.

Choose δ=_____.

Assume $0 < |x - (-1)| < \delta$. (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta$_____$\varepsilon$.

Focusing on the final line of the proof, we see that we should choose $\delta = \frac{\varepsilon}{4}$.

We now complete the final write-up of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \frac{\varepsilon}{4}$.

Assume $0 < |x - (-1)| < \delta$ (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta = 4(\varepsilon/4) = \varepsilon$.

Exercise 13.6.1

Complete the proof that $\lim_{x\to 2} (3x - 2) = 4$ by filling in the blanks.

Let _____.

Choose δ =_____.

Assume $0 < |x -$___$| <$___.

Thus,

|_____–___|=_____ε.

Therefore, $\lim_{x\to 2} (3x - 2) = 4$.

Hint available at the end of this section.

Answer available at the end of this section.

In Examples 13.6.1 and 13.6.2, the proofs were fairly straightforward, since the functions with which we were working were linear. In Example 13.6.3, we see how to modify the proof to accommodate a nonlinear function.

Example 13.6.3: Proving a Statement about the Limit of a Specific Function (Geometric Approach)

Prove that $\lim_{x\to 2} x^2 = 4$.

Solution

1. Let $\varepsilon > 0$. The first part of the definition begins “For every $\varepsilon > 0$,” so we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating “Let $\varepsilon > 0$,” we signal our intent to do so.

2. Without loss of generality, assume $\varepsilon \leq 4$. Two questions present themselves: Why do we want $\varepsilon \leq 4$ and why is it okay to make this assumption? In answer to the first question: Later on, in the process of solving for δ, we will discover that δ involves the quantity $\sqrt{4-\varepsilon}$. Consequently, we need $\varepsilon \leq 4$. In answer to the second question: If we can find $\delta > 0$ that "works" for $\varepsilon \leq 4$, then it will "work" for any $\varepsilon > 4$ as well. Keep in mind that, although it is always okay to put an upper bound on ε, it is never okay to put a lower bound (other than zero) on ε.

3. Choose $\delta = \min\{2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2\}$. Figure 13.6.3 shows how we made this choice of δ.

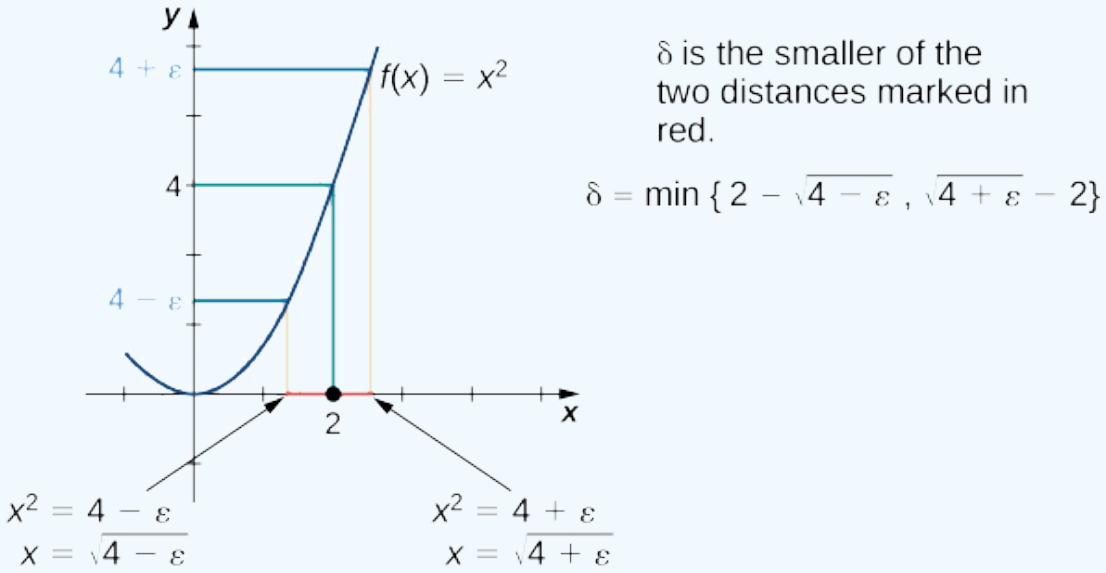


Figure 13.6.3: This graph shows how we find δ geometrically for a given ε for the proof in Example 13.6.3.

4. We must show: If $0 < |x-2| < \delta$, then $|x^2-4| < \varepsilon$, so we must begin by assuming

$$0 < |x-2| < \delta.$$

We don't really need $0 < |x-2|$ (in other words, $x \neq 2$) for this proof. Since $0 < |x-2| < \delta \Rightarrow |x-2| < \delta$, it is okay to drop $0 < |x-2|$.

$$|x-2| < \delta.$$

Hence,

\(–δ

Recall that $\delta = \min\{2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2\}$. Thus, $\delta \geq 2-\sqrt{4-\varepsilon}$ and consequently $-(2-\sqrt{4-\varepsilon}) \leq -\delta$. We also use $\delta \leq \sqrt{4+\varepsilon}-2$ here. We might ask at this point: Why did we substitute $2-\sqrt{4-\varepsilon}$ for δ on the left-hand side of the inequality and $\sqrt{4+\varepsilon}-2$ on the right-hand side of the inequality? If we look at Figure 13.6.3, we see that $2-\sqrt{4-\varepsilon}$ corresponds to the distance on the left of 2 on the x-axis and $\sqrt{4+\varepsilon}-2$ corresponds to the distance on the right. Thus,

\(–(2–\sqrt{4–ε})≤–δ

We simplify the expression on the left:

\(–2+\sqrt{4–ε}

Then, we add 2 to all parts of the inequality:

\(\sqrt{4–ε}

We square all parts of the inequality. It is okay to do so, since all parts of the inequality are positive:

\(4−ε

We subtract 4 from all parts of the inequality:

\(−ε

Last,

$$|x^2 - 4| < \varepsilon.$$

5. Therefore,

$$\lim_{x \to 2} x^2 = 4.$$

Exercise 13.6.2

Find δ corresponding to $\varepsilon > 0$ for a proof that $\lim_{x \to 9} \sqrt{x} = 3$.

Hint available at the end of this section.

Answer available at the end of this section.

The geometric approach to proving that the limit of a function takes on a specific value works quite well for some functions. Also, the insight into the formal definition of the limit that this method provides is invaluable. However, we may also approach limit proofs from a purely algebraic point of view. In many cases, an algebraic approach may not only provide us with additional insight into the definition, it may prove to be simpler as well. Furthermore, an algebraic approach is the primary tool used in proofs of statements about limits. For Example $13.6.4$, we take on a purely algebraic approach.

Example 13.6.4:Proving a Statement about the Limit of a Specific Function (Algebraic Approach)

Prove that $\lim_{x \to -1} (x^2 - 2x + 3) = 6$.

Solution

Let's use our outline from the Problem-Solving Strategy:

1. Let $\varepsilon > 0$.

2. Choose $\delta = \min\{1, \varepsilon/5\}$. This choice of δ may appear odd at first glance, but it was obtained by taking a look at our ultimate desired inequality: $|\ (x^2 - 2x + 3) - 6\ | < \varepsilon$. This inequality is equivalent to $|x + 1| \cdot |x - 3| < \varepsilon$. At this point, the temptation simply to choose $\delta = \frac{\varepsilon}{x-3}$ is very strong. Unfortunately, our choice of δ must depend on ε only and no other variable. If we can replace $|x - 3|$ by a numerical value, our problem can be resolved. This is the place where assuming $\delta \leq 1$ comes into play. The choice of $\delta \leq 1$ here is arbitrary. We could have just as easily used any other positive number. In some proofs, greater care in this choice may be necessary. Now, since $\delta \leq 1$ and $|x + 1| < \delta \leq 1$, we are able to show that $|x - 3| < 5$. Consequently, $|x + 1| \cdot |x - 3| < |x + 1| \cdot 5$. At this point we realize that we also need $\delta \leq \varepsilon/5$. Thus, we choose $\delta = \min\{1, \varepsilon/5\}$.

3. Assume $0 < |x + 1| < \delta$. Thus,

$$|x + 1| < 1 \text{ and } |x + 1| < \frac{\varepsilon}{5}.$$

Since $|x + 1| < 1$, we may conclude that \(−1

$$\left|(x^2 - 2x + 3) - 6\right| = |x + 1| \cdot |x - 3| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon.$$

Therefore,

$$\lim_{x\to -1}\,(x^2-2x+3)=6.$$

Exercise 13.6.3

Complete the proof that $\lim_{x\to 1} x^2 = 1$.

Let $\varepsilon > 0$; choose $\delta = \min\{1, \varepsilon/3\}$; assume $0 < |x-1| < \delta$.

Since $|x-1| < 1$, we may conclude that \(–1

Hint available at the end of this section.

Answer available at the end of this section.

You will find that, in general, the more complex a function, the more likely it is that the algebraic approach is the easiest to apply. The algebraic approach is also more useful in proving statements about limits.

Proving Limit Laws

We now demonstrate how to use the epsilon-delta definition of a limit to construct a rigorous proof of one of the limit laws. The **triangle inequality** is used at a key point of the proof, so we first review this key property of absolute value.

Definition: The Triangle Inequality

The **triangle inequality** states that if a and b are any real numbers, then $|a+b| \le |a| + |b|$.

Proof

We prove the following limit law: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}\,(f(x)+g(x)) = L+M$.

Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|f(x)-L| < \varepsilon/2$.

Choose $\delta_2 > 0$ so that if $0 < |x-a| < \delta_2$, then $|g(x)-M| < \varepsilon/2$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

Assume $0 < |x-a| < \delta$.

Thus,

$$0 < |x-a| < \delta_1 \text{ and } 0 < |x-a| < \delta_2.$$

Hence,

$$\begin{aligned} |(f(x)+g(x))-(L+M)| &= |(f(x)-L)+(g(x)-M)| \\ &\le |f(x)-L| + |g(x)-M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}.$$

□

We now explore what it means for a limit not to exist. The limit $\lim_{x\to a} f(x)$ does not exist if there is no real number L for which $\lim_{x\to a} f(x) = L$. Thus, for all real numbers L, $\lim_{x\to a} f(x) \neq L$. To understand what this means, we look at each part of the definition of $\lim_{x\to a} f(x) = L$ together with its opposite. A translation of the definition is given in Table 13.6.2.

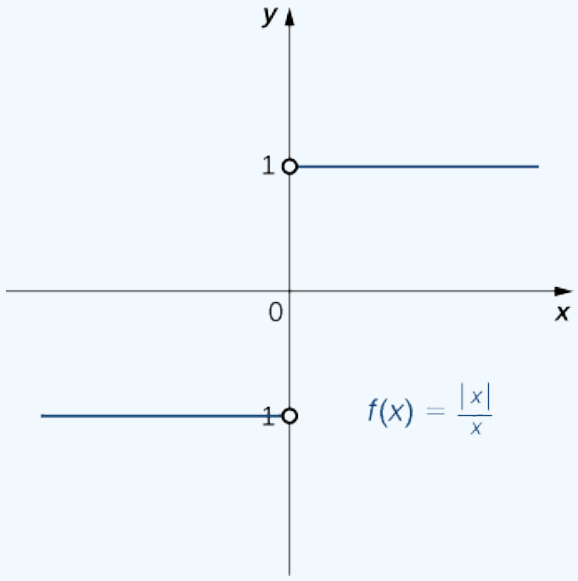
Table 13.6.2. Translation of the Definition of $\lim_{x\to a} f(x) = L$ and its Opposite

Definition	Opposite
1. For every $\varepsilon > 0$,	1. There exists $\varepsilon > 0$ so that
2. there exists a $\delta > 0$, so that	2. for every $\delta > 0$,
3. if $0 < \lvert x - a\rvert < \delta$, then $\lvert f(x) - L\rvert < \varepsilon$.	3. There is an x satisfying $0 < \lvert x - a\rvert < \delta$ so that $\lvert f(x) - L\rvert \geq \varepsilon$.

Finally, we may state what it means for a limit not to exist. The limit $\lim_{x\to a} f(x)$ does not exist if for every real number L, there exists a real number $\varepsilon > 0$ so that for all $\delta > 0$, there is an x satisfying $0 < \lvert x - a\rvert < \delta$, so that $\lvert f(x) - L\rvert \geq \varepsilon$. Let's apply this in Example 13.6.5 to show that a limit does not exist.

Example 13.6.5: Showing That a Limit Does Not Exist

Show that $\lim_{x\to 0} \dfrac{|x|}{x}$ does not exist. The graph of $f(x) = |x|/x$ is shown here:



Solution

Suppose that L is a candidate for a limit. Choose $\varepsilon = 1/2$.

Let $\delta > 0$. Either $L \geq 0$ or $L < 0$. If $L \geq 0$, then let $x = -\delta/2$.

Thus,

$$|x - 0| = |-\tfrac{\delta}{2} - 0| = \tfrac{\delta}{2} < \delta$$

and

$$\left|\frac{|-\frac{\delta}{2}|}{-\frac{\delta}{2}} - L\right| = |-1-L| = L+1 \geq 1 > \tfrac{1}{2} = \varepsilon.$$

On the other hand, if $L < 0$, then let $x = \delta/2$. Thus,

$$|x-0| = |\ \tfrac{\delta}{2} - 0\ | = \tfrac{\delta}{2} < \delta$$

and

$$\left|\frac{|\frac{\delta}{2}|}{\frac{\delta}{2}} - L\right| = |1-L| = |L|+1 \geq 1 > \tfrac{1}{2} = \varepsilon.$$

Thus, for any value of L, $\lim_{x\to 0}\dfrac{|x|}{x} \neq L$.

One-Sided Limits

Just as we first gained an intuitive understanding of limits and then moved on to a more rigorous definition of a limit, we now revisit one-sided limits. To do this, we modify the epsilon-delta definition of a limit to give formal epsilon-delta definitions for limits from the right and left at a point. These definitions only require slight modifications from the definition of the limit. In the definition of the limit from the right, the inequality \(0

Definition: One-Sided Limits (Formal)

Limit from the Right: Let $f(x)$ be defined over an open interval of the form (a,b) where \(a

$$\lim_{x\to a^+} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if \(0

Limit from the Left: Let $f(x)$ be defined over an open interval of the form (b,c) where \(b

$$\lim_{x\to c^-} f(x) = L$$

if for every $\varepsilon > 0$,there exists a $\delta > 0$ such that if \(−δ

Example 13.6.6: Proving a Statement about a Limit From the Right

Prove that

$$\lim_{x\to 4^+} \sqrt{x-4} = 0.$$

Solution

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon^2$. Since we ultimately want $|\ \sqrt{x-4} - 0\ | < \varepsilon$, we manipulate this inequality to get $\sqrt{x-4} < \varepsilon$ or, equivalently, \(0

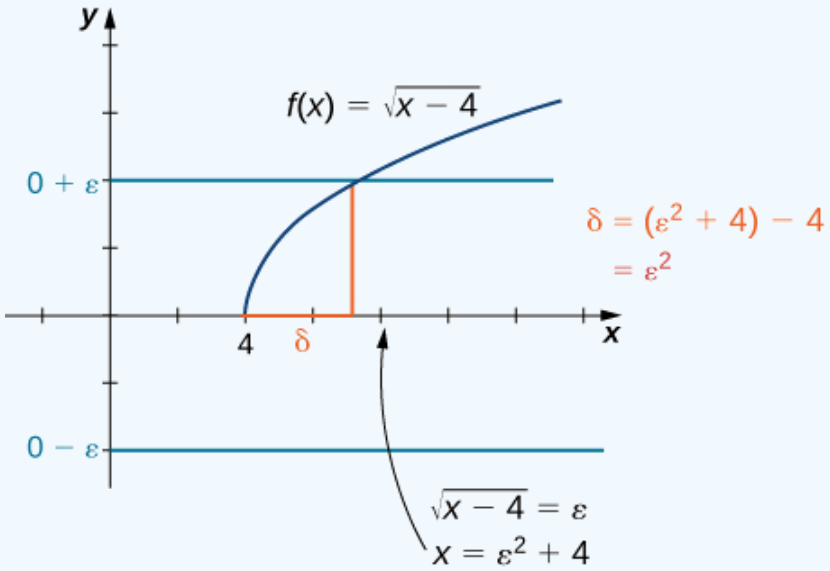


Figure 13.6.4: This graph shows how we find δ for the proof in Example 13.6.6.

Assume \(0

Exercise 13.6.4

Find δ corresponding to ε for a proof that $\lim_{x\to 1^-} \sqrt{1-x} = 0$.

Hint available at the end of this section.

Answer available at the end of this section.

Infinite Limits

We conclude the process of converting our intuitive ideas of various types of limits to rigorous formal definitions by pursuing a formal definition of infinite limits. To have $\lim_{x\to a} f(x) = +\infty$, we want the values of the function $f(x)$ to get larger and larger as x approaches a. Instead of the requirement that $|f(x) - L| < \varepsilon$ for arbitrarily small ε when $0 < |x - a| < \delta$ for small enough δ, we want $f(x) > M$ for arbitrarily large positive M when $0 < |x - a| < \delta$ for small enough δ. Figure 13.6.5 illustrates this idea by showing the value of δ for successively larger values of M.

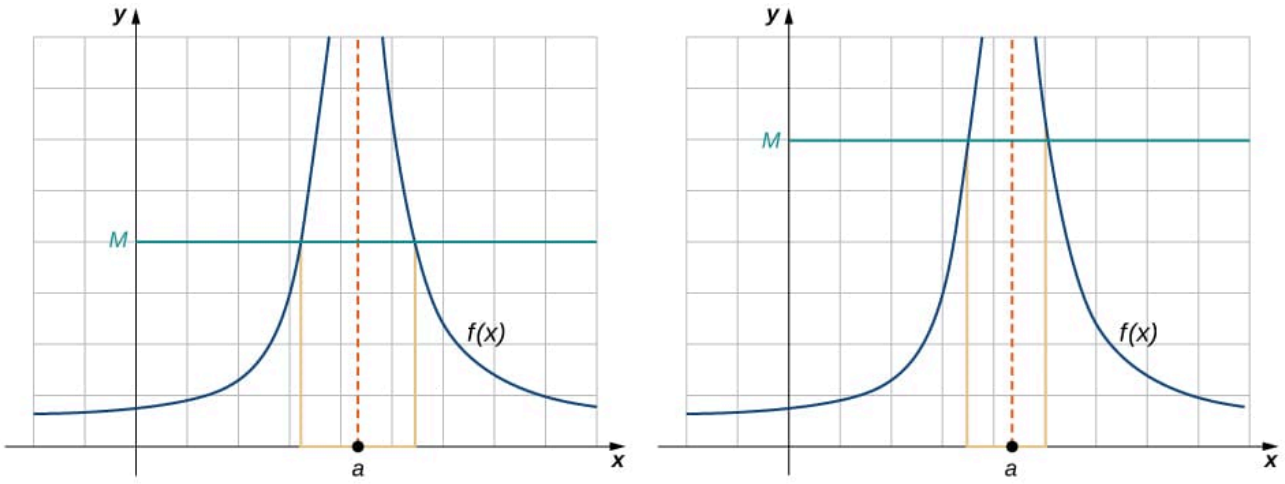


Figure 13.6.5**:** These graphs plot values of δ for M to show that $\lim_{x\to a} f(x) = +\infty$.

Definition: Infinite Limits (Formal)

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have an infinite limit

$$\lim_{x\to a} f(x) = +\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$.

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have a negative infinite limit

$$\lim_{x\to a} f(x) = -\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < -M$.

Example 13.6.7: Proving a Statement about an Infinite Limit

Prove that $\lim_{x\to 3} \dfrac{1}{(x-3)^2} = \infty$.

Solution

We use a very similar approach to our previous Problem-Solving Strategy. We first find an appropriate $\delta > 0$. Then we write our proof.

Step 1: First we find an appropriate $\delta > 0$.

1. Let M be any real number such that $M > 0$.

2. Let $f(x) = \dfrac{1}{(x-3)^2} > M$. Then we solve for the expression $x - 3$.

Multiplying both sides of the inequality by the positive quantity $(x-3)^2$ and dividing both sides by the positive quantity M gives us:

$$\frac{1}{M} > (x-3)^2$$

Taking the square root of both sides, we have,

$$\sqrt{\frac{1}{M}} > |x-3|. \qquad \left(\text{Remember that } \sqrt{x^2} = |x|.\right)$$

Rewriting this statement gives us, $0 < |x-3| < \sqrt{\frac{1}{M}}$. From this we choose $\delta = \sqrt{\frac{1}{M}}$.

Step 2: Now we write a proof.

3. Let $\delta = \sqrt{\frac{1}{M}}$ and assume $0 < |x-3| < \delta = \sqrt{\frac{1}{M}}$.

Thus,

$$|x-3| < \sqrt{\frac{1}{M}}.$$

Squaring both sides gives us,

$$(x-3)^2 < \frac{1}{M}.$$

Taking the reciprocal of both sides (and remembering that this will reverse the direction of the inequality),

$$\frac{1}{(x-3)^2} > M.$$

Therefore, we have proven that

$$\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty.$$

A very similar proof will be needed for a limit that is equal to $-\infty$.

Note that a one-sided limit approach will often need to be taken with this type of limit. For example, to prove: $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

Key Concepts

- The intuitive notion of a limit may be converted into a rigorous mathematical definition known as the epsilon-delta definition of the limit.
- The epsilon-delta definition may be used to prove statements about limits.
- The epsilon-delta definition of a limit may be modified to define one-sided limits.
- A similar definition of an infinite limit can be used to prove statements about infinite limits.

Glossary

epsilon-delta definition of the limit
$\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

triangle inequality
If a and b are any real numbers, then $|a+b| \le |a| + |b|$

formal definition of an infinite limit
$\lim_{x\to a} f(x) = \infty$ if for every $M > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $f(x) > M$

$\lim_{x \to a} f(x) = -\infty$ if for every $M > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < -M$

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College), added Example $13.6.7$ and entries for infinite limits under Key Concepts and the Glossary.

Answers

Exercise $13.6.1$

Follow the outline in the Problem-Solving Strategy that we worked out in full in Example $13.6.2$.

Exercise $13.6.1$

Let $\varepsilon > 0$; choose $\delta = \frac{\varepsilon}{3}$; assume $0 < |x - 2| < \delta$.

Thus, $|(3x - 2) - 4| = |3x - 6| = |3| \cdot |x - 2| < 3 \cdot \delta = 3 \cdot (\varepsilon/3) = \varepsilon$.

Therefore, $\lim_{x \to 2}(3x - 2) = 4$.

Exercise $13.6.2$

Draw a graph similar to the one in Example $13.6.3$.

Exercise $13.6.2$

Choose $\delta = \min\{9 - (3 - \varepsilon)^2,\ \ (3 + \varepsilon)^2 - 9\}$.

Exercise $13.6.3$

Use Example $13.6.4$ as a guide.

Exercise $13.6.3$

$| x^2 - 1 | = |x - 1| \cdot |x + 1| < \varepsilon/3 \cdot 3 = \varepsilon$

Exercise $13.6.4$

Sketch the graph and use Example $13.6.6$ as a solving guide.

Exercise $13.6.4$

$\delta = \varepsilon^2$

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13.6E: Exercises for Section 2.5

In exercises 1 - 4, write the appropriate $\varepsilon - \delta$ definition for each of the given statements.

1) $\lim\limits_{x \to a} f(x) = N$

2) $\lim\limits_{t \to b} g(t) = M$

Answer

For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |t - b| < \delta$, then $|g(t) - M| < \varepsilon$

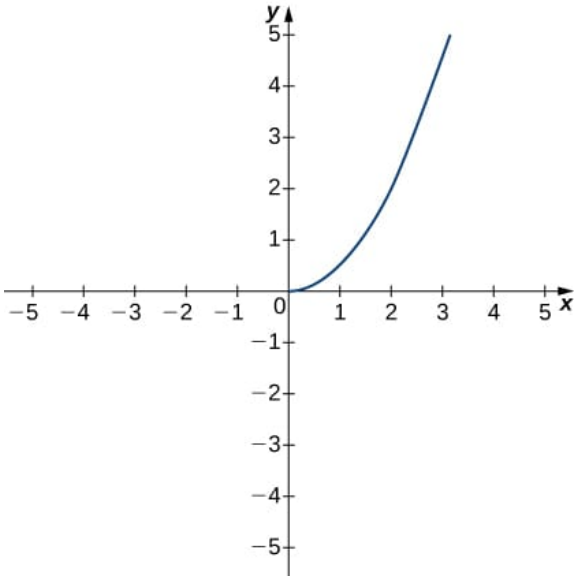
3) $\lim\limits_{x \to c} h(x) = L$

4) $\lim\limits_{x \to a} \varphi(x) = A$

Answer

For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - a| < \delta$, then $|\varphi(x) - A| < \varepsilon$

The following graph of the function f satisfies $\lim\limits_{x \to 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



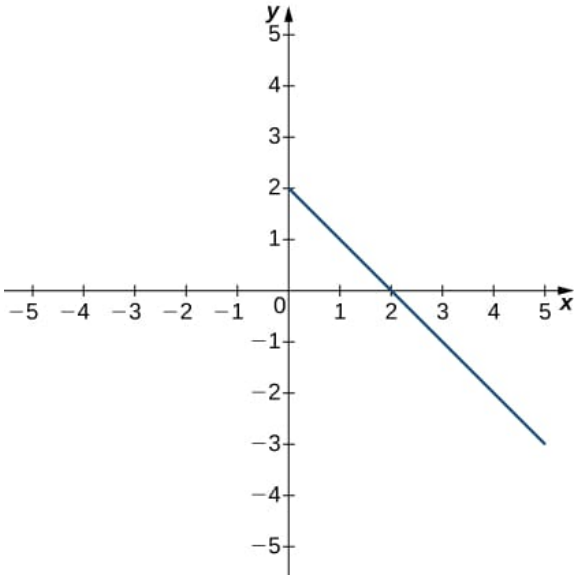
5) If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

6) If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 0.5$.

Answer

$\delta \leq 0.25$

The following graph of the function f satisfies $\lim\limits_{x \to 3} f(x) = -1$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



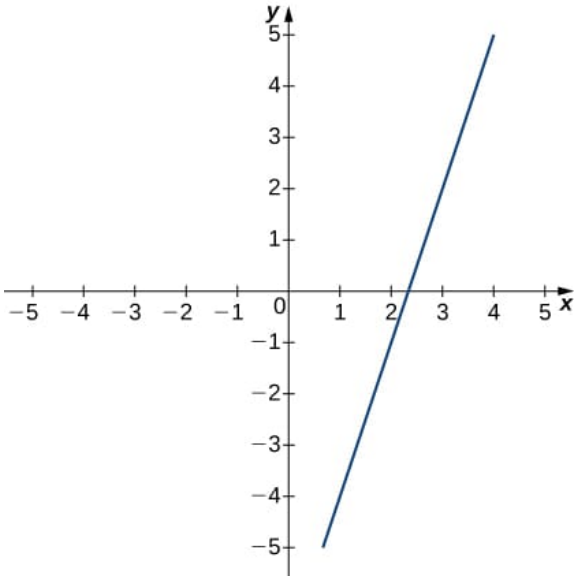
7) If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 1$.

8) If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 2$.

Answer

$\delta \leq 2$

The following graph of the function f satisfies $\lim_{x \to 3} f(x) = 2$. In the following exercises, for each value of ε, find a value of $\delta > 0$ such that the precise definition of limit holds true.



9) $\varepsilon = 1.5$

10) $\varepsilon = 3$

Answer

$\delta \leq 1$

[T] In exercises 11 - 12, use a graphing calculator to find a number δ such that the statements hold true.

11) $\left|\sin(2x) - \frac{1}{2}\right| < 0.1$, whenever $\left|x - \frac{\pi}{12}\right| < \delta$

12) $\left|\sqrt{x-4} - 2\right| < 0.1$, whenever $|x - 8| < \delta$

Answer

$\delta < 0.3900$

In exercises 13 - 17, use the precise definition of limit to prove the given limits.

13) $\lim\limits_{x \to 2} (5x + 8) = 18$

14) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = 6$

Answer

Let $\delta = \varepsilon$. If $0 < |x - 3| < \varepsilon$, then $\left|\dfrac{x^2 - 9}{x - 3} - 6\right| = \left|\dfrac{(x+3)(x-3)}{x-3} - 6\right| = |x + 3 - 6| = |x - 3| < \varepsilon$.

15) $\lim\limits_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2} = 5$

16) $\lim\limits_{x \to 0} x^4 = 0$

Answer

Let $\delta = \sqrt[4]{\varepsilon}$. If $0 < |x| < \sqrt[4]{\varepsilon}$, then $\left|x^4 - 0\right| = x^4 < \varepsilon$.

17) $\lim\limits_{x \to 2} (x^2 + 2x) = 8$

In exercises 18 - 20, use the precise definition of limit to prove the given one-sided limits.

18) $\lim\limits_{x \to 5^-} \sqrt{5 - x} = 0$

Answer

Let $\delta = \varepsilon^2$. If $-\varepsilon^2 < x - 5 < 0$, we can multiply through by -1 to get $0 < 5 - x < \varepsilon^2$.
Then $\left|\sqrt{5-x} - 0\right| = \sqrt{5-x} < \sqrt{\varepsilon^2} = \varepsilon$.

19) $\lim\limits_{x \to 0^+} f(x) = -2$, where $f(x) = \begin{cases} 8x - 3, & \text{if } x < 0 \\ 4x - 2, & \text{if } x \geq 0 \end{cases}$.

20) $\lim\limits_{x \to 1^-} f(x) = 3$, where $f(x) = \begin{cases} 5x - 2, & \text{if } x < 1 \\ 7x - 1, & \text{if } x \geq 1 \end{cases}$.

Answer

Let $\delta = \varepsilon/5$. If $-\varepsilon/5 < x - 1 < 0$, we can multiply through by -1 to get $0 < 1 - x < \varepsilon/5$.
Then $|f(x) - 3| = |5x - 2 - 3| = |5x - 5| = 5(1 - x)$, since $x < 1$ here.
And $5(1 - x) < 5(\varepsilon/5) = \varepsilon$.

In exercises 21 - 23, use the precise definition of limit to prove the given infinite limits.

21) $\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$

22) $\lim\limits_{x \to -1} \dfrac{3}{(x+1)^2} = \infty$

Answer

Let $\delta = \sqrt{\frac{3}{N}}$. If $0 < |x + 1| < \sqrt{\frac{3}{N}}$, then $f(x) = \frac{3}{(x+1)^2} > N$.

23) $\lim\limits_{x \to 2} -\dfrac{1}{(x-2)^2} = -\infty$

24) An engineer is using a machine to cut a flat square of Aerogel of area $144\ \text{cm}^2$. If there is a maximum error tolerance in the area of $8\ \text{cm}^2$, how accurately must the engineer cut on the side, assuming all sides have the same length? How do these numbers relate to δ, ε, a, and L?

Answer

$0.033\ \text{cm},\ \varepsilon = 8,\ \delta = 0.33,\ a = 12,\ L = 144$

25) Use the precise definition of limit to prove that the following limit does not exist: $\lim\limits_{x\to 1} \dfrac{|x-1|}{x-1}$.

26) Using precise definitions of limits, prove that $\lim\limits_{x\to 0} f(x)$ does not exist, given that $f(x)$ is the ceiling function. (Hint: Try any $\delta < 1$.)

Answer

Answers may very.

27) Using precise definitions of limits, prove that $\lim\limits_{x\to 0} f(x)$ does not exist: $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Think about how you can always choose a rational number \(0

28) Using precise definitions of limits, determine $\lim\limits_{x\to 0} f(x)$ for $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Break into two cases, x rational and x irrational.)

Answer

0

29) Using the function from the previous exercise, use the precise definition of limits to show that $\lim\limits_{x\to a} f(x)$ does not exist for $a \neq 0$

For exercises 30 - 32, suppose that $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$ both exist. Use the precise definition of limits to prove the following limit laws:

30) $\lim\limits_{x\to a}(f(x) - g(x)) = L - M$

Answer

$f(x) - g(x) = f(x) + (-1)g(x)$

31) $\lim\limits_{x\to a}[cf(x)] = cL$ for any real constant c (Hint: Consider two cases: $c = 0$ and $c \neq 0$.)

32) $\lim\limits_{x\to a}[f(x)g(x)] = LM$. (Hint: $|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|$.)

Answer

Answers may vary.

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13.R: Chapter 2 Review Exercises

***True or False*. In exercises 1 - 4, justify your answer with a proof or a counterexample.**

1) A function has to be continuous at $x = a$ if the $\lim_{x \to a} f(x)$ exists.

2) You can use the quotient rule to evaluate $\lim_{x \to 0} \dfrac{\sin x}{x}$.

Answer

False, since we cannot have $\lim_{x \to 0} x = 0$ in the denominator.

3) If there is a vertical asymptote at $x = a$ for the function $f(x)$, then f is undefined at the point $x = a$.

4) If $\lim_{x \to a} f(x)$ does not exist, then f is undefined at the point $x = a$.

Answer

False. A jump discontinuity is possible.

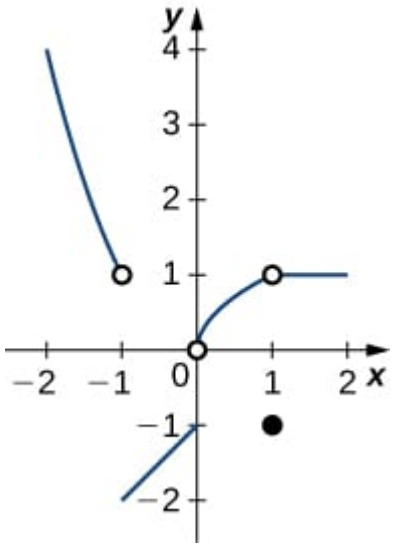
5) Using the graph, find each limit or explain why the limit does not exist.

a. $\lim_{x \to -1} f(x)$

b. $\lim_{x \to 1} f(x)$

c. $\lim_{x \to 0^+} f(x)$

d. $\lim_{x \to 2} f(x)$



In exercises 6 - 15, evaluate the limit algebraically or explain why the limit does not exist.

6) $\lim_{x \to 2} \dfrac{2x^2 - 3x - 2}{x - 2}$

Answer

5

7) $\lim_{x\to 0} 3x^2 - 2x + 4$

8) $\lim_{x\to 3} \dfrac{x^3 - 2x^2 - 1}{3x - 2}$

Answer

$8/7$

9) $\lim_{x\to \pi/2} \dfrac{\cot x}{\cos x}$

10) $\lim_{x\to -5} \dfrac{x^2 + 25}{x + 5}$

Answer

DNE

11) $\lim_{x\to 2} \dfrac{3x^2 - 2x - 8}{x^2 - 4}$

12) $\lim_{x\to 1} \dfrac{x^2 - 1}{x^3 - 1}$

Answer

$2/3$

13) $\lim_{x\to 1} \dfrac{x^2 - 1}{\sqrt{x} - 1}$

14) $\lim_{x\to 4} \dfrac{4 - x}{\sqrt{x} - 2}$

Answer

-4

15) $\lim_{x\to 4} \dfrac{1}{\sqrt{x} - 2}$

In exercises 16 - 17, use the squeeze theorem to prove the limit.

16) $\lim_{x\to 0} x^2 \cos(2\pi x) = 0$

Answer

Since $-1 \leq \cos(2\pi x) \leq 1$, then $-x^2 \leq x^2 \cos(2\pi x) \leq x^2$. Since $\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} -x^2$, it follows that $\lim_{x\to 0} x^2 \cos(2\pi x) = 0$.

17) $\lim_{x\to 0} x^3 \sin\left(\dfrac{\pi}{x}\right) = 0$

18) Determine the domain such that the function $f(x) = \sqrt{x-2} + xe^x$ is continuous over its domain.

Answer

$[2, \infty]$

In exercises 19 - 20, determine the value of c such that the function remains continuous. Draw your resulting function to ensure it is continuous.

19) $f(x) = \begin{cases} x^2 + 1, & \text{if } x > c \\ 2^x, & \text{if } x \leq c \end{cases}$

20) $f(x) = \begin{cases} \sqrt{x+1}, & \text{if } x > -1 \\ x^2 + c, & \text{if } x \leq -1 \end{cases}$

In exercises 21 - 22, use the precise definition of limit to prove the limit.

21) $\lim\limits_{x \to 1} (8x + 16) = 24$

22) $\lim\limits_{x \to 0} x^3 = 0$

Answer

$\delta = \sqrt[3]{\varepsilon}$

23) A ball is thrown into the air and the vertical position is given by $x(t) = -4.9t^2 + 25t + 5$. Use the Intermediate Value Theorem to show that the ball must land on the ground sometime between 5 sec and 6 sec after the throw.

24) A particle moving along a line has a displacement according to the function $x(t) = t^2 - 2t + 4$, where x is measured in meters and t is measured in seconds. Find the average velocity over the time period $t = [0, 2]$.

Answer

0 m/sec

25) From the previous exercises, estimate the instantaneous velocity at $t = 2$ by checking the average velocity within $t = 0.01$ sec.

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14: DERIVATIVES

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

14.1: Prelude to Derivatives

14.2: Defining the Derivative

14.2E: Exercises for Section 3.1

14.3: The Derivative as a Function

14.3E: Exercises for Section 3.2

14.4: Differentiation Rules

14.4E: Exercises for Section 3.3

14.5: Derivatives as Rates of Change

14.5E: Exercises for Section 3.4

14.6: Derivatives of Trigonometric Functions

14.6E: Exercises for Section 3.5

14.7: The Chain Rule

14.7E: Exercises for Section 3.6

14.8: Derivatives of Inverse Functions

14.8E: Exercises for Section 3.7

14.9: Implicit Differentiation

14.9E: Exercises for Section 3.8

14.10: Derivatives of Exponential and Logarithmic Functions

14.10E: Exercises for Section 3.9

14.R: Chapter 3 Review Exercises

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14.1: Prelude to Derivatives

The Hennessey Venom GT is one of the fastest cars in the world. In 2014, it reached a record-setting speed of 270.49 mph. It can go from 0 to 200 mph in 14.51 seconds. The techniques in this chapter can be used to calculate the acceleration the Venom achieves in this feat.)



Figure 14.1.1: The Hennessey Venom GT can go from 0 to 200 mph in 14.51 seconds. (credit: modification of work by Codex41, Flickr)

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

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14.2: Defining the Derivative

Learning Objectives

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac **Newton (1643–1727)** and the German Gottfried **Leibniz (1646–1716)**, whose images appear in Figure 14.2.1. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

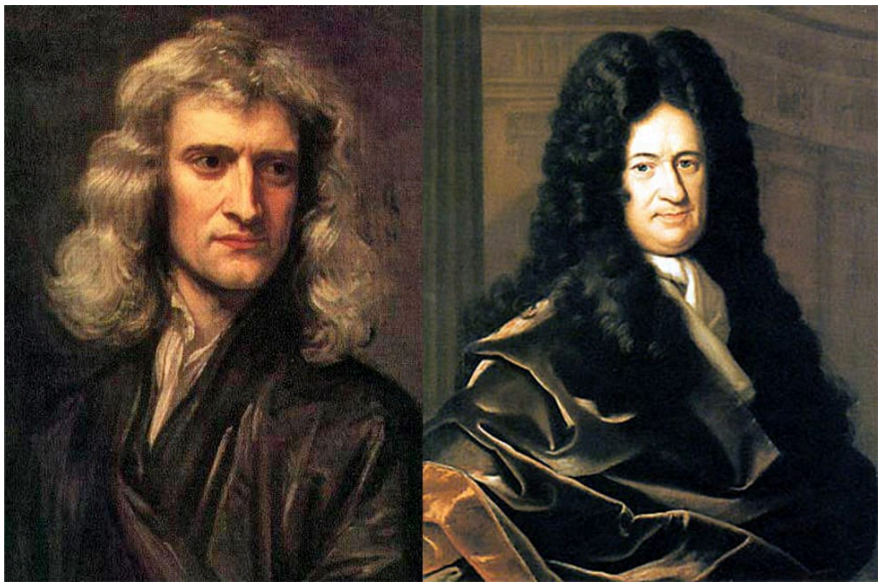


Figure 14.2.1: Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in Figure 14.2.2. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where

h is a value close to a. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a+h, f(a+h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{sec} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

Definition: Difference Quotient

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x)-f(a)}{x-a}$$

is a *difference quotient*.

Also, if $h \neq 0$ is chosen so that $a+h$ is in I, then

$$Q = \frac{f(a+h)-f(a)}{h}$$

is a difference quotient with increment h.

These two expressions for calculating the slope of a secant line are illustrated in Figure 14.2.2. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

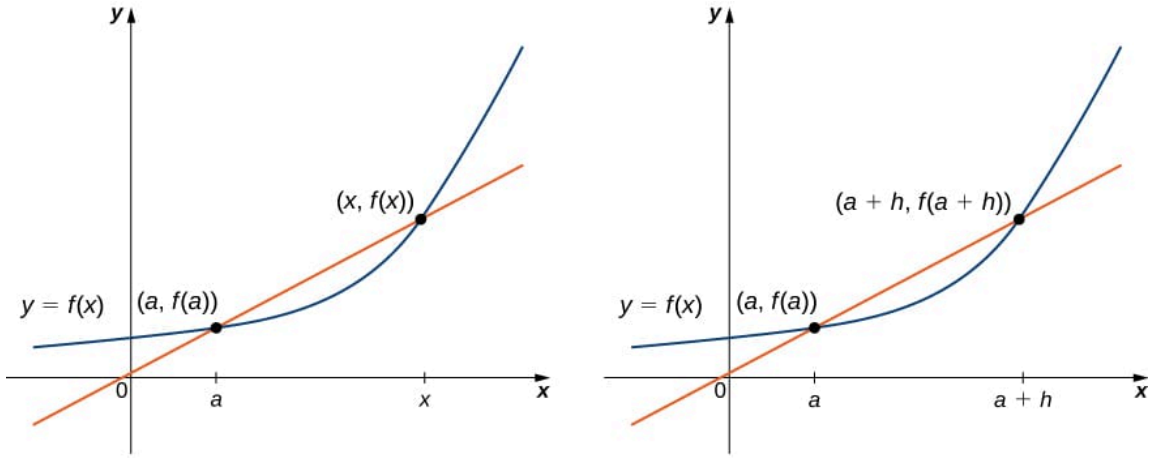


Figure 14.2.2: We can calculate the slope of a secant line in either of two ways.

In Figure 14.2.3a we see that, as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a. Furthermore, the secant lines themselves approach the tangent line to the function at a, which represents the limit of the secant lines. Similarly, Figure 14.2.3b shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a, as shown in Figure 14.2.3c.

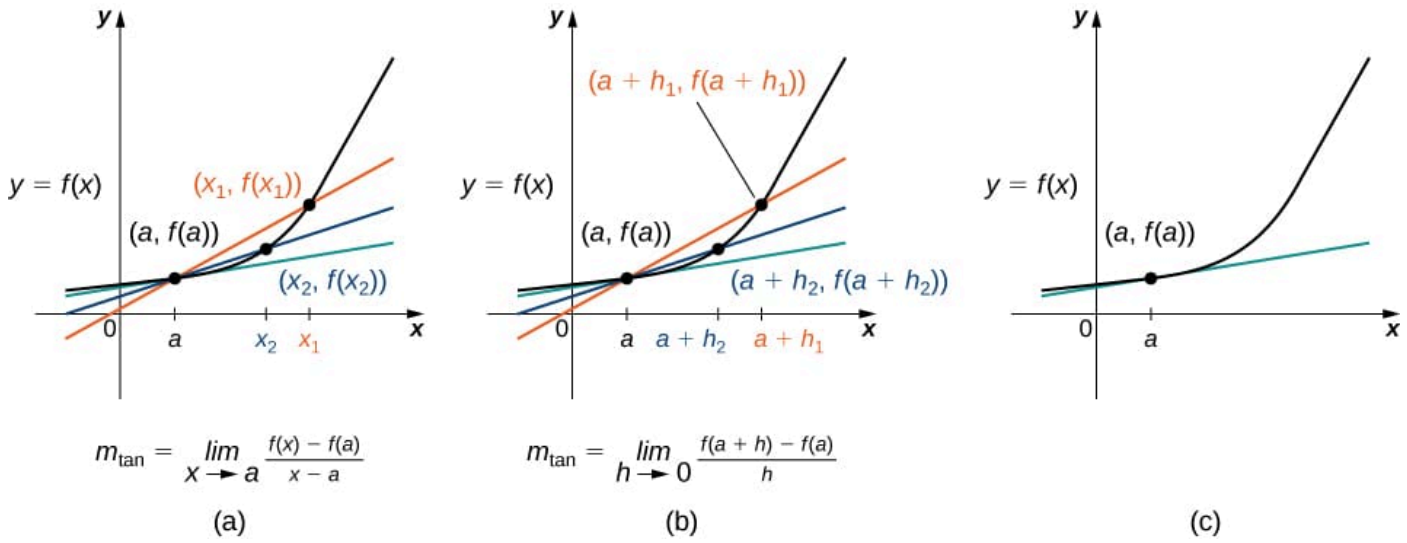


Figure 14.2.3: The secant lines approach the tangent line (shown in green) as the second point approaches the first.

In Figure 14.2.4 we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be **locally linear** in the immediate vicinity of $x = 1$.

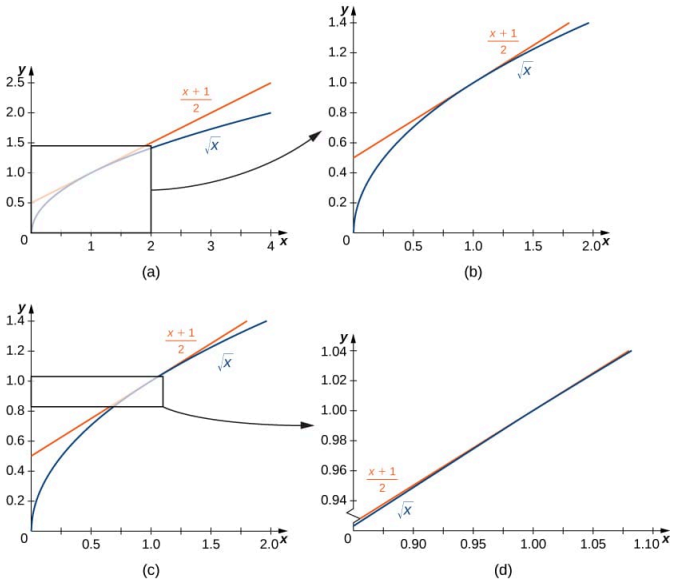


Figure 14.2.4: For values of x close to 1, the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Definition: Tangent Line

Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{14.2.1}$$

provided this limit exists.

Equivalently, we may define the *tangent line* to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{14.2.2}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example 14.2.1: Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

First find the slope of the tangent line. In this example, use Equation 14.2.1.

$$\begin{aligned} m_{tan} &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} && \text{Apply the definition.} \\ &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} && \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9 \\ &= \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3}(x+3) = 6 && \text{Factor the numerator to evaluate the limit.} \end{aligned}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in Figure 14.2.5.

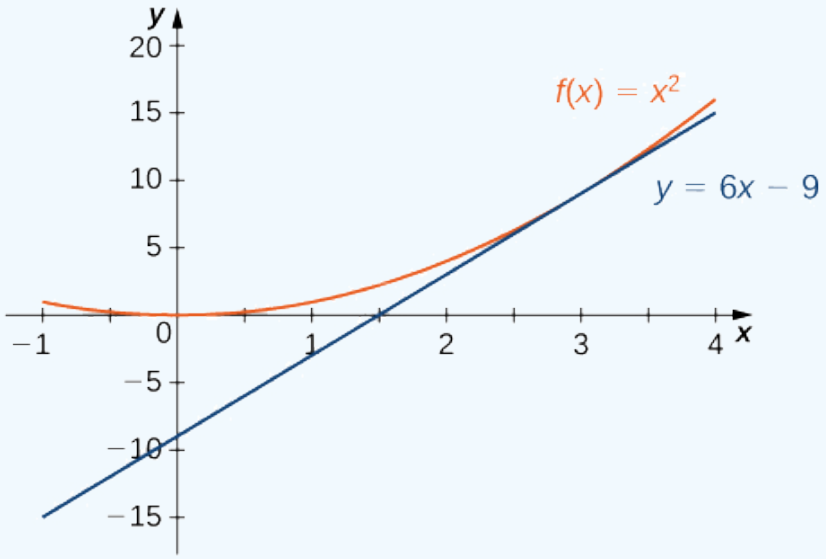


Figure 14.2.5: The tangent line to $f(x)$ at $x = 3$.

Example 14.2.2: The Slope of a Tangent Line Revisited

Use Equation 14.2.2 to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

The steps are very similar to Example 14.2.1. See Equation 14.2.2 for the definition.

$$
\begin{aligned}
m_{tan} &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} && \text{Apply the definition.} \\
&= \lim_{h\to 0} \frac{(3+h)^2 - 9}{h} && \text{Substitute } f(3+h) = (3+h)^2 \text{ and } f(3) = 9 \\
&= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h} && \text{Expand and simplify to evaluate the limit.} \\
&= \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6
\end{aligned}
$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example 14.2.3: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution

We can use Equation 14.2.1, but as we have seen, the results are the same if we use Equation 14.2.2.

Misplaced &

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in Figure 14.2.6.

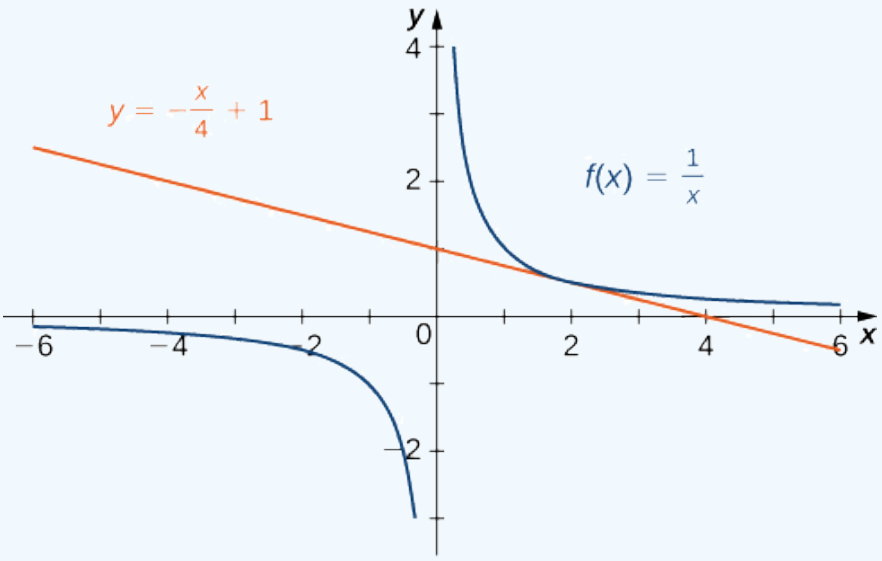


Figure 14.2.6:The line is tangent to $f(x)$ at $x = 2$.

Exercise 14.2.1

Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Hint available at the end of this section.

Answer available at the end of this section.

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative.** The process of finding a derivative is called **differentiation**.

Definition: Derivative

Let $f(x)$ be a function defined in an open interval containing a. The **derivative** of the function $f(x)$ at a, denoted by $f\,'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{14.2.3}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{14.2.4}$$

Example 14.2.4: Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f\,'(3)$ using Equation 14.2.3.

Solution

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2 - 9}{x - 3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f\,'(3) = 6$.

Exercise 14.2.2

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 14.2.4.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.2.6: Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 14.2.3.

Solution

Substitute the given function and value directly into the equation.

$$
\begin{aligned}
f'(x) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition.}\\
&= \lim_{x\to 2} \frac{(3x^2 - 4x + 1) - 5}{x - 2} && \text{Substitute } f(x) = 3x^2 - 4x + 1 \text{ and } f(2) = 5.\\
&= \lim_{x\to 2} \frac{(x - 2)(3x + 2)}{x - 2} && \text{Simplify and factor the numerator.}\\
&= \lim_{x\to 2} (3x + 2) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

Example 14.2.7: Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using Equation 14.2.4.

Solution

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in Example 14.2.6.

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2 + h) - f(2)}{h} && \text{Apply the definition.}\\
&= \lim_{h\to 0} \frac{(3(2 + h)^2 - 4(2 + h) + 1) - 5}{h} && \text{Substitute } f(2) = 5 \text{ and } f(2 + h) = 3(2 + h)^2 - 4(2 + h) + 1.\\
&= \lim_{h\to 0} \frac{3(4 + 4h + h^2) - 8 - 4h + 1 - 5}{h} && \text{Expand the numerator.}\\
&= \lim_{h\to 0} \frac{12 + 12h + 3h^2 - 12 - 4h}{h} && \text{Distribute and begin simplifying the numerator.}\\
&= \lim_{h\to 0} \frac{3h^2 + 8h}{h} && \text{Finish simplifying the numerator.}\\
&= \lim_{h\to 0} \frac{h(3h + 8)}{h} && \text{Factor the numerator.}\\
&= \lim_{h\to 0} (3h + 8) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

The results are the same whether we use Equation 14.2.3 or Equation 14.2.4.

Exercise 14.2.4

For $f(x) = x^2 + 3x + 2$, find $f\,'(1)$.

Hint available at the end of this section.

Answer available at the end of this section.

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the **average velocity** of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if \(t

$$v_{ave} = \frac{s(t) - s(a)}{t - a}. \tag{14.2.5}$$

As the values of t approach a, the values of v_{ave} approach the value we call the **instantaneous velocity** at a. That is, instantaneous velocity at a, denoted $v(a)$, is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \tag{14.2.6}$$

To better understand the relationship between average velocity and instantaneous velocity, see Figure 14.2.7. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t]$.

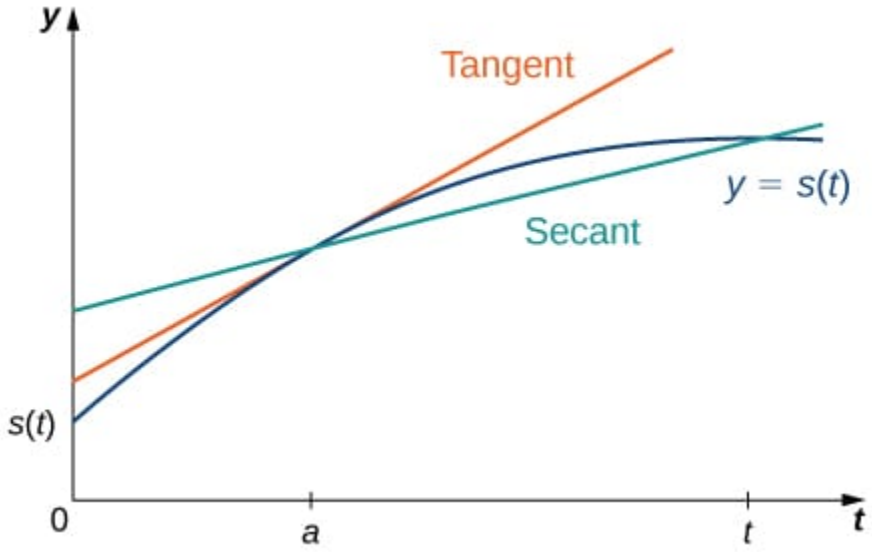


Figure 14.2.7: The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use Equation 14.2.6 to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using Equation 14.2.5.

Example 14.2.8: Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ (Figure 14.2.8). Use a table of values to estimate $v(0)$. Check the estimate by using Equation 14.2.6.

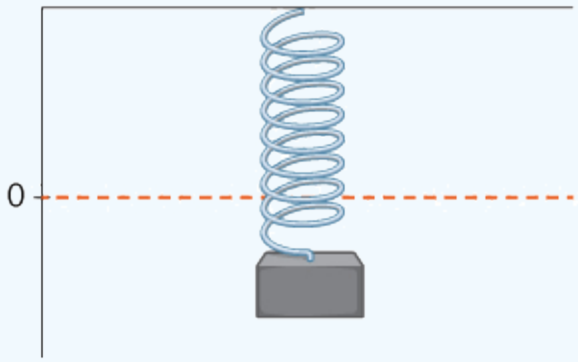


Figure 14.2.8: A lead weight suspended from a spring in vertical oscillatory motion.

Solution

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of t approaching 0, as shown in Table 14.2.2.

Table 14.2.2: Average velocities using values of t approaching 0

t	$\frac{\sin t - \sin 0}{t-0} = \frac{\sin t}{t}$
−0.1	0.998334166
−0.01	0.9999833333
−0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is 0.998334166, the average velocity over the time interval $[-0.01, 0]$ is 0.9999833333, and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1$.

By using Equation 14.2.6, we can see that

$$v(0) = s'(0) = \lim_{t \to 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1$.

Exercise 14.2.5

A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64, \quad 0 \leq t \leq 2$. Find its instantaneous velocity 1 second after it is dropped, using Equation 14.2.6.

Hint available at the end of this section.

Answer available at the end of this section.

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Definition: Instantaneous Rate of Change

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example 14.2.9: Chapter Opener: Estimating Rate of Change of Velocity

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its **acceleration**) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?



Figure 14.2.9: (credit: modification of work by Codex41, Flickr)

Solution: First observe that 60 mph = 88 ft/s, 100 mph ≈146.67 ft/s, 200 mph ≈293.33 ft/s, and 229.9 mph ≈337.19 ft/s. We can summarize the information in a table.

Table 14.2.3: $v(t)$ at different values of t

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

Now compute the average acceleration of the car in feet per second on intervals of the form $[t, 19.96]$ as t approaches 19.96, as shown in the following table.

Average acceleration

t	$\frac{v(t)-v(19.96)}{t-19.96}=\frac{v(t)-337.19}{t-19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph (337.19 ft/s).

Example 14.2.10: Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70\,°F$ at 9 p.m., reaches a low of $60\,°$ during the night, and rises back to $70\,°$ by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10$, where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution

Since midnight is 3 hours past 9 p.m., we want to compute $T\,'(3)$. Refer to Equation 14.2.3.

$$\begin{aligned}
T'(3) &= \lim_{t\to 3}\frac{T(t)-T(3)}{t-3} && \text{Apply the definition.}\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+70-61.6}{t-3} && \text{Substitute } T(t)=0.4t^2-4t+70 \text{ and } T(3)=61.6.\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+8.4}{t-3} && \text{Simplify.}\\
&= \lim_{t\to 3}\frac{0.4(t-3)(t-7)}{t-3} &&\\
&= \lim_{t\to 3}0.4(t-7) && \text{Cancel.}\\
&= -1.6 && \text{Evaluate the limit.}
\end{aligned}$$

The instantaneous rate of change of the temperature at midnight is $-1.6\,°F$ per hour.

Example 14.2.11: Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10,000$ dollars. Find the rate of change of profit when $10,000$ games are produced. Should the toy company increase or decrease production?

Solution

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x)$, where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p = -0.01x + 400$ per game,

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,

$$P(x) = -0.01x^2 + 300x - 10,000.$$

Therefore, evaluating the rate of change of profit gives

$$\begin{aligned}
P'(10000) &= \lim_{x\to 10000}\frac{P(x)-P(10000)}{x-10000}\\
&= \lim_{x\to 10000}\frac{-0.01x^2+300x-10000-1990000}{x-10000}\\
&= \lim_{x\to 10000}\frac{-0.01x^2+300x-2000000}{x-10000}\\
&= 100
\end{aligned}.$$

Since the rate of change of profit $P'(10,000) > 0$ and $P(10,000) > 0$, the company should increase production.

Exercise 14.2.6

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10$. The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25)$, the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h.
- The derivative of a function $f(x)$ at a value a is found using either of the definitions for the slope of the tangent line.
- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time t is the derivative of the position $s(t)$ at time t. Average velocity is given by

  $$v_{ave} = \frac{s(t) - s(a)}{t - a}.$$

  Instantaneous velocity is given by

  $$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

- We may estimate a derivative by using a table of values.

Key Equations

- **Difference quotient**

  $$Q = \frac{f(x) - f(a)}{x - a}$$

- **Difference quotient with increment h**

  $$Q = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

- **Slope of tangent line**

  $$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  $$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Derivative of f(x) at a**

  $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Average velocity**

$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$

- **Instantaneous velocity**

$$v(a) = s\,'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Glossary

derivative
: the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

difference quotient
: of a function $f(x)$ at a is given by

$$\frac{f(a+h) - f(a)}{h} \text{ or } \frac{f(x) - f(a)}{x - a}$$

differentiation
: the process of taking a derivative

instantaneous rate of change
: the rate of change of a function at any point along the function a, also called $f\,'(a)$, or the derivative of the function at a

Answers

Exercise 14.2.1
Use either Equation 14.2.1 or Equation 14.2.2. Multiply the numerator and the denominator by a conjugate.

Exercise 14.2.1
$\frac{1}{4}$

Exercise 14.2.2
Evaluate $\frac{(x+h)^2 - x^2}{h}$ at $h = -0.1,\ -0.01,\ -0.001,\ 0.001,\ 0.01,\ 0.1$

Exercise 14.2.2
6

Exercise 14.2.4
Use either Equation 14.2.3, Equation 14.2.4, or try both.

Exercise 14.2.4
$f\,'(1) = 5$

Exercise 14.2.5
$v(t) = s\,'(t)$. Follow the earlier examples of the derivative using Equation 14.2.6.

Exercise 14.2.5
–32 ft/s

Exercise 14.2.6
Use Example 14.2.11 for a guide.

Exercise 14.2.6
$P\,'(3.25) = 20 > 0$; raise prices

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14.2E: Exercises for Section 3.1

For exercises 1 - 10, use the equation $m_{\text{sec}} = \dfrac{f(x) - f(a)}{x - a}$ to find the slope of the secant line between the values x_1 and x_2 for each function $y = f(x)$.

1) $f(x) = 4x + 7; \quad x_1 = 2, \quad x_2 = 5$

Answer

$m_{\text{sec}} = 4$

2) $f(x) = 8x - 3; \quad x_1 = -1, \quad x_2 = 3$

3) $f(x) = x^2 + 2x + 1; \quad x_1 = 3, \quad x_2 = 3.5$

Answer

$m_{\text{sec}} = 8.5$

4) $f(x) = -x^2 + x + 2; \quad x_1 = 0.5, \quad x_2 = 1.5$

5) $f(x) = \dfrac{4}{3x - 1}; \quad x_1 = 1, \quad x_2 = 3$

Answer

$m_{\text{sec}} = -\frac{3}{4}$

6) $f(x) = \dfrac{x - 7}{2x + 1}; \quad x_1 = -2, \quad x_2 = 0$

7) $f(x) = \sqrt{x}; \quad x_1 = 1, \quad x_2 = 16$

Answer

$m_{\text{sec}} = 0.2$

8) $f(x) = \sqrt{x - 9}; \quad x_1 = 10, \quad x_2 = 13$

9) $f(x) = x^{1/3} + 1; \quad x_1 = 0, \quad x_2 = 8$

Answer

$m_{\text{sec}} = 0.25$

10) $f(x) = 6x^{2/3} + 2x^{1/3}; \quad x_1 = 1, \quad x_2 = 27$

For the functions in exercises 11 - 20,

a. use the equation $m_{\text{tan}} = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$ to find the slope of the tangent line $m_{\text{tan}} = f\,'(a)$, and

b. find the equation of the tangent line to f at $x = a$.

11) $f(x) = 3 - 4x, \quad a = 2$

Answer

a. $m_{\text{tan}} = -4$
b. $y = -4x + 3$

12) $f(x) = \dfrac{x}{5} + 6, \quad a = -1$

13) $f(x) = x^2 + x, \quad a = 1$

Answer

a. $m_{\tan} = 3$
b. $y = 3x - 1$

14) $f(x) = 1 - x - x^2, \quad a = 0$

15) $f(x) = \dfrac{7}{x}, \quad a = 3$

Answer

a. $m_{\tan} = \frac{-7}{9}$
b. $y = \frac{-7}{9}x + \frac{14}{3}$

16) $f(x) = \sqrt{x+8}, \quad a = 1$

17) $f(x) = 2 - 3x^2, \quad a = -2$

Answer

a. $m_{\tan} = 12$
b. $y = 12x + 14$

18) $f(x) = \dfrac{-3}{x-1}, \quad a = 4$

19) $f(x) = \dfrac{2}{x+3}, \quad a = -4$

Answer

a. $m_{\tan} = -2$
b. $y = -2x - 10$

20) $f(x) = \dfrac{3}{x^2}, \quad a = 3$

For the functions $y = f(x)$ in exercises 21 - 30, find $f\,'(a)$ using the equation $f\,'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

21) $f(x) = 5x + 4, \quad a = -1$

Answer

$f'(-1) = 5$

22) $f(x) = -7x + 1, \quad a = 3$

23) $f(x) = x^2 + 9x, \quad a = 2$

Answer

$f'(2) = 13$

24) $f(x) = 3x^2 - x + 2, \quad a = 1$

25) $f(x) = \sqrt{x}, \quad a = 4$

Answer

$f'(4) = \frac{1}{4}$

26) $f(x) = \sqrt{x-2}, \quad a = 6$

27) $f(x) = \dfrac{1}{x}, \quad a = 2$

Answer

$f'(2) = -\frac{1}{4}$

28) $f(x) = \dfrac{1}{x-3}, \quad a = -1$

29) $f(x) = \dfrac{1}{x^3}, \quad a = 1$

Answer

$f'(1) = -3$

30) $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 4$

For the following exercises, given the function $y = f(x)$,

a. find the slope of the secant line PQ for each point $Q(x, f(x))$ with x value given in the table.

b. Use the answers from a. to estimate the value of the slope of the tangent line at P.

c. Use the answer from b. to find the equation of the tangent line to f at point P.

31) [T] $f(x) = x^2 + 3x + 4, \quad P(1, 8)$ (Round to 6 decimal places.)

x	Slope m_{PQ}	x	Slope m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

Answer

$a.(i)5.100000, (ii)5.010000, (iii)5.001000, (iv)5.000100, (v)5.000010, (vi)5.000001, (vii)4.900000, (viii)4.990000, (ix)4.999000, (x)4.999900, (xi)4.999990, (x)4.999999$

b. $m_{\tan} = 5$

c. $y = 5x + 3$

32) [T] $f(x) = \dfrac{x+1}{x^2-1}, \quad P(0, -1)$

x	Slope m_{PQ}	x	Slope m_{PQ}
0.1	(i)	−0.1	(vii)
0.01	(ii)	−0.01	(viii)
0.001	(iii)	−0.001	(ix)
0.0001	(iv)	−0.0001	(x)
0.00001	(v)	−0.00001	(xi)
0.000001	(vi)	−0.000001	(xii)

33) [T] $f(x) = 10e^{0.5x}, \quad P(0, 10)$ (Round to 4 decimal places.)

x	Slope m_{PQ}

−0.1	(i)
−0.01	(ii)
−0.001	(iii)
−0.0001	(iv)
−0.00001	(v)
−0.000001	(vi)

Answer

a. $(i) 4.8771,\ (ii) 4.9875,\ (iii) 4.9988,\ (iv) 4.9999,\ (v) 4.9999,\ (vi) 4.9999$
b. $m_{\tan} = 5$
c. $y = 5x + 10$

34) [T] $f(x) = \tan(x), \quad P(\pi, 0)$

x	Slope m_{PQ}
3.1	(i)
3.14	(ii)
3.141	(iii)
3.1415	(iv)
3.14159	(v)
3.141592	(vi)

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

b. the average velocity between $t = 2$ and $t = 2 + h$, where $(i)\ h = 0.1,\ (ii)\ h = 0.01,\ (iii)\ h = 0.001$, and $(iv)\ h = 0.0001$; and

c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

35) $s(t) = \frac{1}{3}t + 5$

Answer

a. $\frac{1}{3}$;
b. $(i)\ \frac{1}{3}$ m/s, $(ii)\ \frac{1}{3}$ m/s, $(iii)\ \frac{1}{3}$ m/s, $(iv)\ \frac{1}{3}$ m/s;
c. $\frac{1}{3}$ m/s

36) $s(t) = t^2 - 2t$

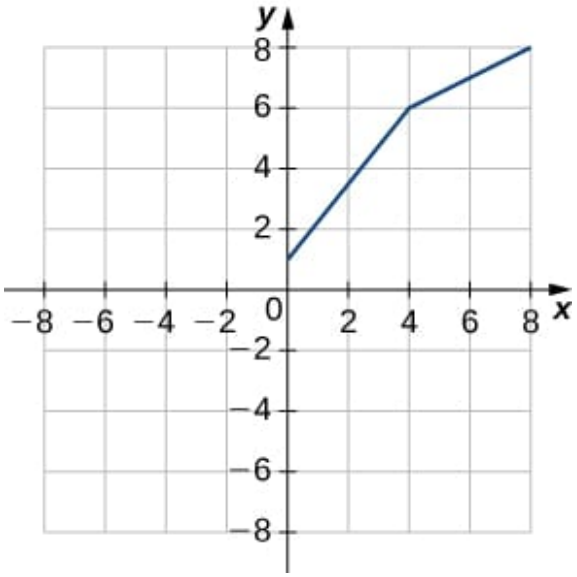
37) $s(t) = 2t^3 + 3$

Answer

a. $2(h^2 + 6h + 12)$;
b. $(i)\ 25.22$ m/s, $(ii)\ 24.12$ m/s, $(iii)\ 24.01$ m/s, $(iv)\ 24$ m/s;
c. 24 m/s

38) $s(t) = \dfrac{16}{t^2} - \dfrac{4}{t}$

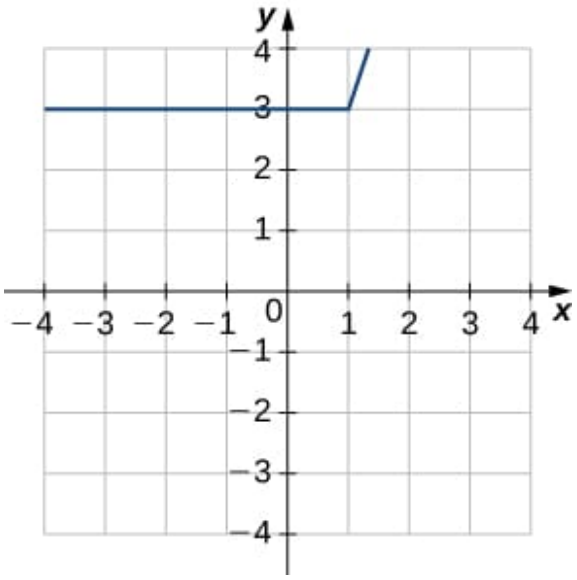
39) Use the following graph to evaluate a. $f\,'(1)$ and b. $f\,'(6)$.



Answer

a. 1.25; b. 0.5

40) Use the following graph to evaluate a. $f\,'(-3)$ and b. $f\,'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.

41) $f(x) = x^{1/3}, \quad x = 0$

Answer

$$\lim_{x \to 0^-} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \to 0^-} \frac{1}{x^{2/3}} = \infty$$

42) $f(x) = x^{2/3}, \quad x = 0$

43) $f(x) = \begin{cases} 1, & \text{if } x < 1 \\ x, & \text{if } x \geq 1 \end{cases}, \quad x = 1$

Answer

$$\lim_{x\to 1^-}\frac{1-1}{x-1}=0\neq 1=\lim_{x\to 1^+}\frac{x-1}{x-1}$$

44) $f(x)=\dfrac{|x|}{x},\quad x=0$

45) [T] The position in feet of a race car along a straight track after t seconds is modeled by the function $s(t)=8t^2-\frac{1}{16}t^3$.

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:

i. $[4, 4.1]$

ii. $[4, 4.01]$

iii. $[4, 4.001]$

iv. $[4, 4.0001]$

b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t=4$ seconds.

Answer

a. $(i)61.7244ft/s,\ \ (ii)61.0725ft/s,\ \ (iii)61.0072ft/s,\ \ (iv)61.0007ft/s$
b. At 4 seconds the race car is traveling at a rate/velocity of 61 ft/s.

46) [T] The distance in feet that a ball rolls down an incline is modeled by the function $s(t)=14t^2$,

where t is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:

i. [5, 5.1]

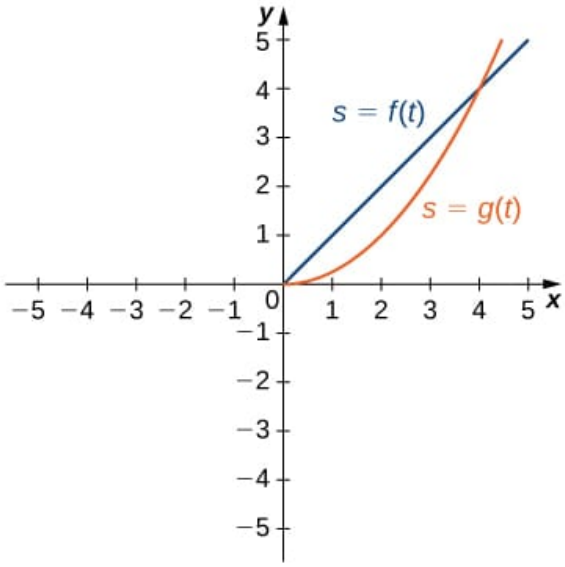
ii. [5, 5.01]

iii. [5, 5.001]

iv. [5, 5.0001]

b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t=5$ seconds.

47) Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s=f(t)$ and $s=g(t)$, where s is measured in feet and t is measured in seconds.



a. Which vehicle has traveled farther at $t = 2$ seconds?

b. What is the approximate velocity of each vehicle at $t = 3$ seconds?

c. Which vehicle is traveling faster at $t = 4$ seconds?

d. What is true about the positions of the vehicles at $t = 4$ seconds?

Answer

a. The vehicle represented by $f(t)$, because it has traveled 2 feet, whereas $g(t)$ has traveled 1 foot.
b. The velocity of $f(t)$ is constant at 1 ft/s, while the velocity of $g(t)$ is approximately 2 ft/s.
c. The vehicle represented by $g(t)$, with a velocity of approximately 4 ft/s.
d. Both have traveled 4 feet in 4 seconds.

48) [T] The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300$.

a. Calculate the average cost per jar over the following intervals:

i. [100, 100.1]

ii. [100, 100.01]

iii. [100, 100.001]

iv. [100, 100.0001]

b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

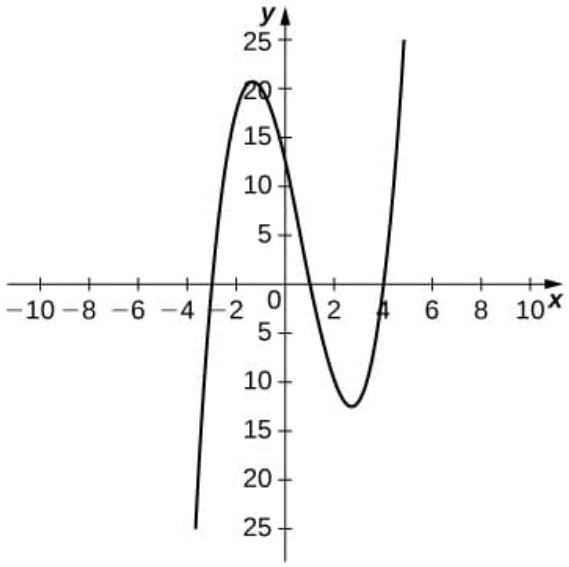
49) [T] For the function $f(x) = x^3 - 2x^2 - 11x + 12$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{tan} = f\,'(a) = 0$.

Answer

a.



b. $a \approx -1.361, \ \ 2.694$

50) [T] For the function $f(x) = \dfrac{x}{1+x^2}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\tan} = f\,'(a) = 0$.

51) Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for $N(x)$.

b. What is $N(100)$? Explain the physical meaning.

c. What is $N\,'(100)$? Explain the physical meaning.

Answer

a. $N(x) = \dfrac{x}{30}$
b. ~ 3.3 gallons. When the vehicle travels 100 miles, it has used 3.3 gallons of gas.
c. $\frac{1}{30}$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled 100 miles.

52) [T] For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the $nDeriv$ function, which numerically finds the derivative, on a graphing calculator to estimate $f\,'(-2), \ \ f\,'(-0.5), \ \ f\,'(1.7)$, and $f\,'(2.718)$.

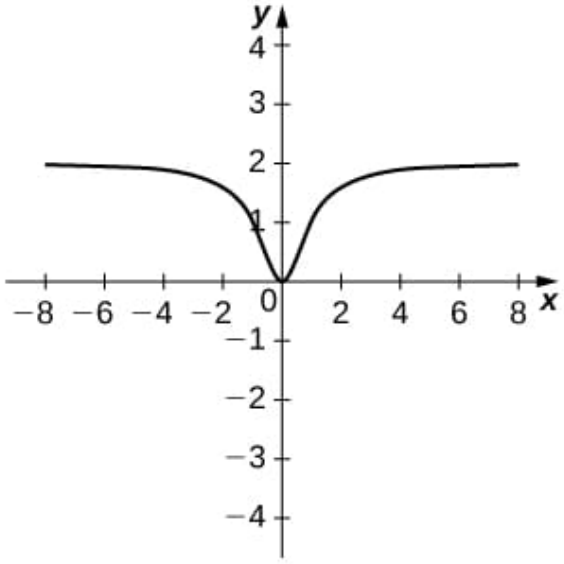
53) [T] For the function $f(x) = \dfrac{x^2}{x^2+1}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.

b. Use the $nDeriv$ function on a graphing calculator to find $f\,'(-4), \ \ f\,'(-2), \ \ f\,'(2)$, and $f\,'(4)$.

Answer

a.



b. $-0.028, -0.16, 0.16, 0.028$

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14.3: The Derivative as a Function

Learning Objectives

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions for when a function does not have a derivative.
- Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition: Derivative Function

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{14.3.1}$$

A function $f(x)$ is said to be **differentiable at** a if $f'(a)$ exists. More generally, a function is said to be **differentiable on** S if it is differentiable at every point in an open set S, and a differentiable function is one in which $f'(x)$ exists on its domain.

In the next few examples we use Equation 14.3.1 to find the derivative of a function.

Example 14.3.1: Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution

Start directly with the definition of the derivative function.

Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$	
$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$	Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ without distributing in the denominator.

$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$	Multiply the numerators and simplify.
$= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}$	Cancel the h.
$= \frac{1}{2\sqrt{x}}$	Evaluate the limit

Example 14.3.2: Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution

Follow the same procedure here, but without having to multiply by the conjugate.

Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim_{h\to 0} \frac{((x+h)^2 - 2(x+h)) - (x^2 - 2x)}{h}$	
$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$	Expand $(x+h)^2 - 2(x+h)$.
$= \lim_{h\to 0} \frac{2xh - 2h + h^2}{h}$	Simplify
$= \lim_{h\to 0} \frac{h(2x - 2 + h)}{h}$	Factor out h from the numerator
$= \lim_{h\to 0} (2x - 2 + h)$	Cancel the common factor of h
$= 2x - 2$	Evaluate the limit

Exercise 14.3.1

Find the derivative of $f(x) = x^2$.

Hint available at the end of this section.

Answer available at the end of this section.

We use a variety of different notations to express the derivative of a function. In Example 14.3.2 we showed that if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$. If we had expressed this function in the form $y = x^2 - 2x$, we could have expressed the derivative as $y' = 2x - 2$ or $\frac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\frac{d}{dx}(x^2 - 2x) = 2x - 2$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of $f(x)$:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}\big(f(x)\big).$$

In place of $f'(a)$ we may also use $\left.\frac{dy}{dx}\right|_{x=a}$. Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (Figure 14.3.1). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

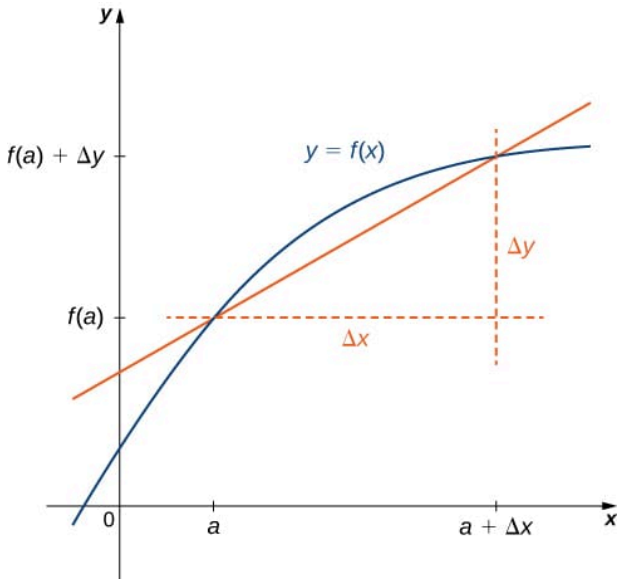


Figure 14.3.1: The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In Example 14.3.1, we found that for $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$. If we graph these functions on the same axes, as in Figure 14.3.2, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

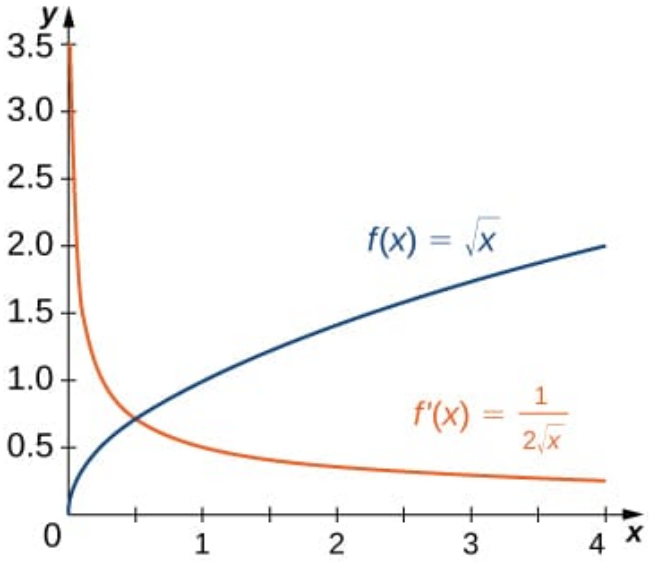


Figure 14.3.2: The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In Example 14.3.2, we found that for $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$. The graphs of these functions are shown in Figure 14.3.3. Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x, $f'(x) < 0$. For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

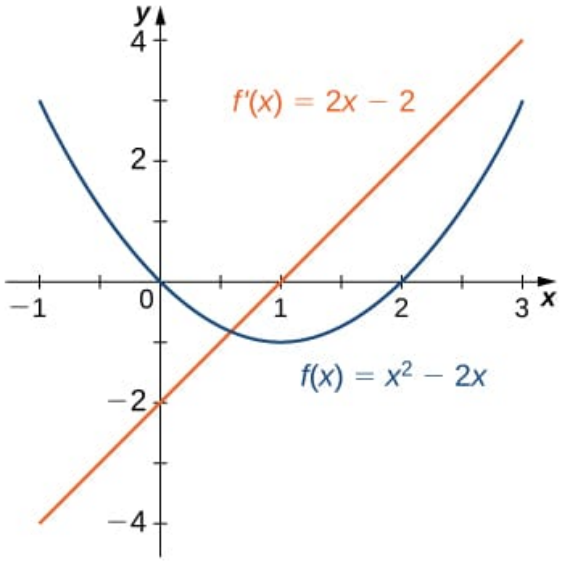
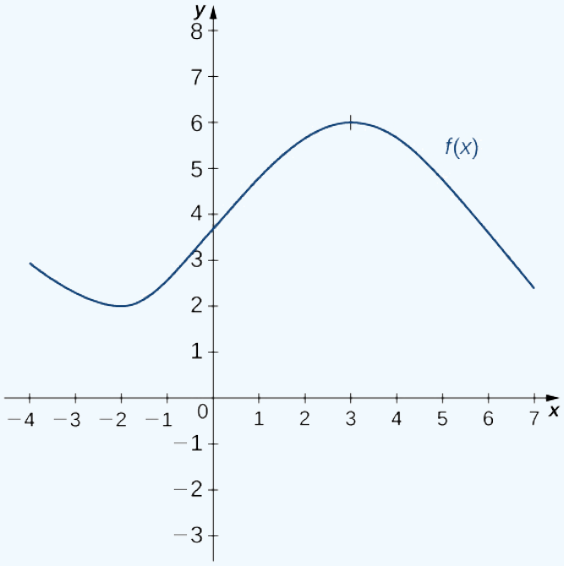


Figure 14.3.3: The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent

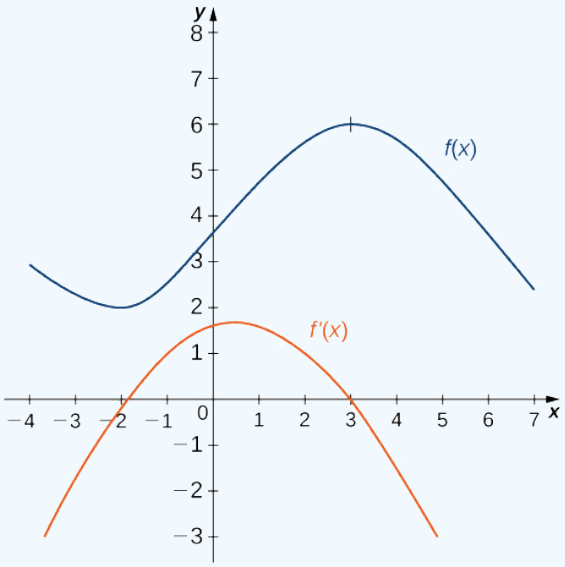
Example 14.3.3: Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3, and $f'(-2) = 0$ and $f'(3) = 0$.



Exercise 14.3.2

Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x-axis?

Hint available at the end of this section.

Answer available at the end of this section.

Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there; however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Proof

If $f(x)$ is differentiable at a, then $f'(a)$ exists and, if we let $h = x - a$, we have $x = a + h$, and as $h = x - a \to 0$, we can see that $x \to a$.

Then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

can be rewritten as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x\to a} f(x) = f(a)$. Thus,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} \big(f(x) - f(a) + f(a)\big) \\ &= \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a}\cdot(x-a) + f(a)\right) \qquad \text{Multiply and divide } (f(x)-f(a)) \text{ by } x-a. \\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot\Big(\lim_{x\to a}(x-a)\Big) + \lim_{x\to a} f(a) \\ &= f'(a)\cdot 0 + f(a) \\ &= f(a).\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x\to a} f(x) = f(a)$, we conclude that f is continuous at a.

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x) = |x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x) = |x|$,

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x|-|0|}{x-0} = \lim_{x\to 0}\frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x\to 0^-}\frac{|x|}{x} = -1 \text{ and } \lim_{x\to 0^+}\frac{|x|}{x} = 1.$$

See Figure 14.3.4.

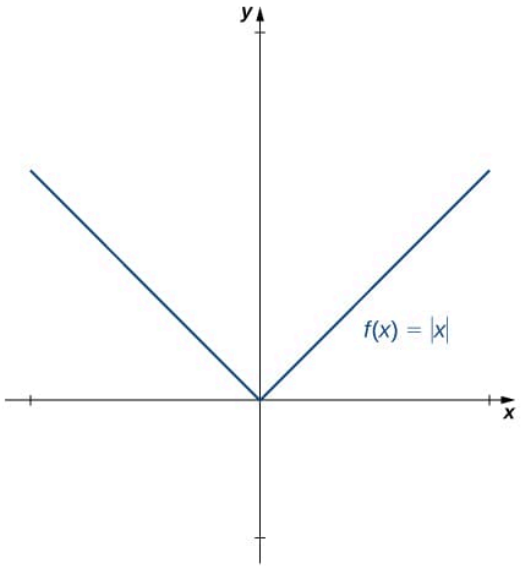


Figure 14.3.4: The function $f(x) = |x|$ is continuous at 0 but is not differentiable at 0.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x) = \sqrt[3]{x}$:

$$f'(0) = \lim_{x\to 0}\frac{\sqrt[3]{x}-0}{x-0} = \lim_{x\to 0}\frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x) = \sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at

0 (Figure $14.3.5$).

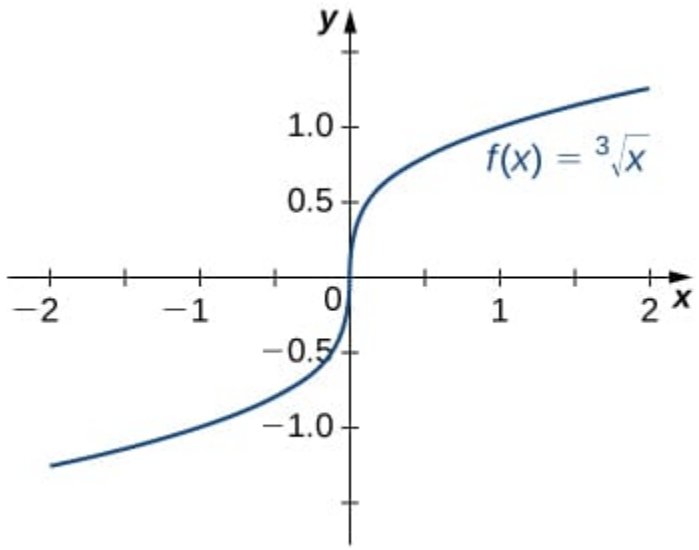


Figure $14.3.5$: The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0$. It is continuous at 0 but is not differentiable at 0.

The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at 0.

We see that

$$f'(0) = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero (Figure $14.3.6$).

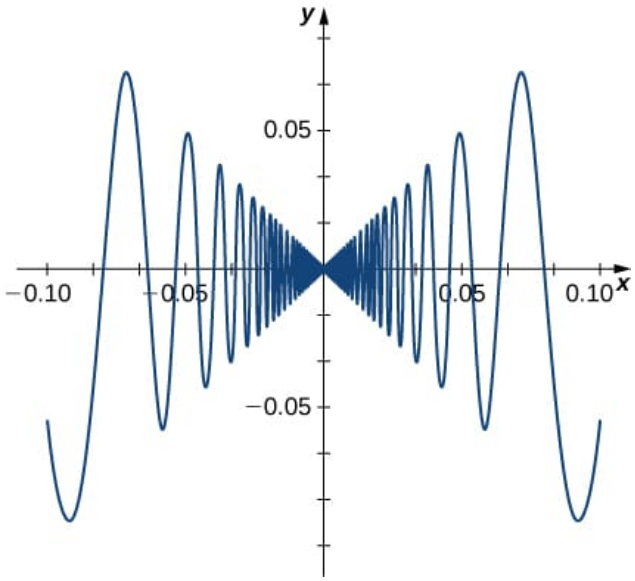


Figure $14.3.6$: The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right

were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be "smooth" at that point.

3. As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.
4. As we saw with $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example 14.3.4: A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (Figure 14.3.7). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c, & \text{if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2}, & \text{if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

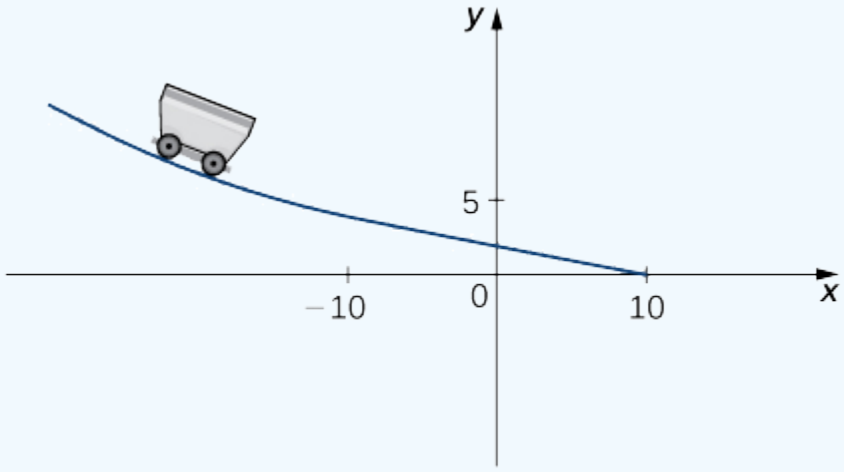


Figure 14.3.7: For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x = -10$, $\lim\limits_{x \to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x \to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x \to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\lim_{x\to -10^-}\frac{f(x)-f(-10)}{x+10}=\lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+c-5}{x+10}$$

$$=\lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+(10b-5)-5}{x+10} \quad \text{Substitute } c=10b-5.$$

$$=\lim_{x\to -10^-}\frac{x^2-100+10bx+100b}{10(x+10)}$$

$$=\lim_{x\to -10^-}\frac{(x+10)(x-10+10b)}{10(x+10)} \quad \text{Factor by grouping}$$

$$=b-2$$

We also have

$$\lim_{x\to -10^+}\frac{f(x)-f(-10)}{x+10}=\lim_{x\to -10^+}\frac{-\frac{1}{4}x+\frac{5}{2}-5}{x+10}$$

$$=\lim_{x\to -10^+}\frac{-(x+10)}{4(x+10)}$$

$$=-\frac{1}{4}$$

This gives us $b-2=-\frac{1}{4}$. Thus $b=\frac{7}{4}$ and $c=10(\frac{7}{4})-5=\frac{25}{2}$.

Exercise 14.3.3

Find values of a and b that make $f(x)=\begin{cases}ax+b, & \text{if } x<3\\ x^2, & \text{if } x\geq 3\end{cases}$ both continuous and differentiable at 3.

Hint available at the end of this section.

Answer available at the end of this section.

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y=f(x)$ can be expressed in any of the following forms:

$$f''(x),\ f'''(x),\ f^{(4)}(x),\ \ldots\ ,\ f^{(n)}(x)$$

$$y''(x),\ y'''(x),\ y^{(4)}(x),\ \ldots\ ,\ y^{(n)}(x)$$

$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dy^3},\ \frac{d^4y}{dy^4},\ \ldots\ ,\ \frac{d^ny}{dy^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly.

Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}$.

Example 14.3.5: Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution

First find $f'(x)$.

Substitute $f(x) = 2x^2 - 3x + 1$ and $f(x+h) = 2(x+h)^2 - 3(x+h) + 1$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$f'(x)$ $= \lim_{h \to 0} \frac{(2(x+h)^2 - 3(x+h) + 1) - (2x^2 - 3x + 1)}{h}$	
$= \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h}$	Simplify the numerator.
$= \lim_{h \to 0} (4x + 2h - 3)$	Factor out the h in the numerator and cancel with the h in the denominator.
$= 4x - 3$	Take the limit.

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.

$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$	Use $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ with $f'(x)$ in place of $f(x)$.
$= \lim_{h \to 0} \frac{(4(x+h) - 3) - (4x - 3)}{h}$	Substitute $f'(x+h) = 4(x+h) - 3$ and $f'(x) = 4x - 3$.
$= \lim_{h \to 0} 4$	Simplify.
$= 4$	Take the limit.

Exercise 14.3.4

Find $f''(x)$ for $f(x) = x^2$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.3.6: Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

Solution

Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

$$\begin{aligned} s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - (3t^2 - 4t + 1)}{h} \\ &= 6t - 4. \end{aligned}$$

Next,

$$\begin{aligned} s''(t) &= \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{h} \\ &= \lim_{h \to 0} \frac{6(t+h) - 4 - (6t-4)}{h} \\ &= 6. \end{aligned}$$

Thus, $a = 6 \text{ m/s}^2$.

Exercise 14.3.5

For $s(t) = t^3$, find $a(t)$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The derivative of a function $f(x)$ is the function whose value at x is $f'(x)$.
- The graph of a derivative of a function $f(x)$ is related to the graph of $f(x)$. Where $f(x)$ has a tangent line with positive slope, $f'(x) > 0$. Where $f(x)$ has a tangent line with negative slope, $f'(x) < 0$. Where $f(x)$ has a horizontal tangent line, $f'(x) = 0$.
- If a function is differentiable at a point, then it is continuous at that point. A function is not differentiable at a point if it is not continuous at the point, if it has a vertical tangent line at the point, or if the graph has a sharp corner or cusp.
- Higher-order derivatives are derivatives of derivatives, from the second derivative to the nth derivative.

Key Equations

- **The derivative function**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Glossary

derivative function
gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

differentiable at a
a function for which $f'(a)$ exists is differentiable at a

differentiable on S
a function for which $f'(x)$ exists for each x in the open set S is differentiable on S

differentiable function
a function for which $f'(x)$ exists is a differentiable function

higher-order derivative
a derivative of a derivative, from the second derivative to the nth derivative, is called a higher-order derivative

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Paul Seeburger (Monroe Community College) added explanation of the alternative definition of the derivative used in the

proof of that differentiability implies continuity.

Answers

Exercise 14.3.1

Use Equation 14.3.1 and follow the example.

Exercise 14.3.1

$f'(x) = 2x$

Exercise 14.3.2

The graph of $f'(x)$ is positive where $f(x)$ is increasing.

Exercise 14.3.2

$(0, +\infty)$

Exercise 14.3.3

Use Example 14.3.4 as a guide.

Exercise 14.3.3

$a = 6$ and $b = -9$

Exercise 14.3.4

We found $f'(x) = 2x$ in a previous checkpoint. Use Equation 14.3.1 to find the derivative of $f'(x)$

Exercise 14.3.4

$f''(x) = 2$

Exercise 14.3.5

Use Example 14.3.6 as a guide.

Exercise 14.3.5

$a(t) = 6t$

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14.3E: Exercises for Section 3.2

For exercises 1 - 10, use the definition of a derivative to find $f\,'(x)$.

1) $f(x) = 6$

2) $f(x) = 2 - 3x$

Answer

$$f'(x) = -3$$

3) $f(x) = \dfrac{2x}{7} + 1$

4) $f(x) = 4x^2$

Answer

$$f'(x) = 8x$$

5) $f(x) = 5x - x^2$

6) $f(x) = \sqrt{2x}$

Answer

$$f'(x) = \frac{1}{\sqrt{2x}}$$

7) $f(x) = \sqrt{x - 6}$

8) $f(x) = \dfrac{9}{x}$

Answer

$$f'(x) = \frac{-9}{x^2}$$

9) $f(x) = x + \dfrac{1}{x}$

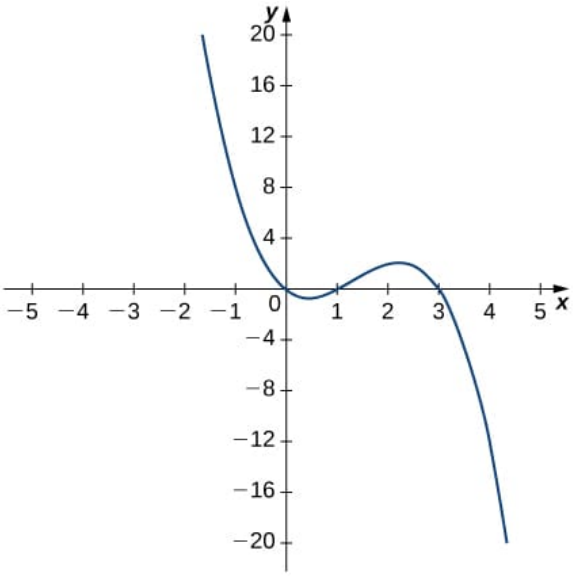
10) $f(x) = \dfrac{1}{\sqrt{x}}$

Answer

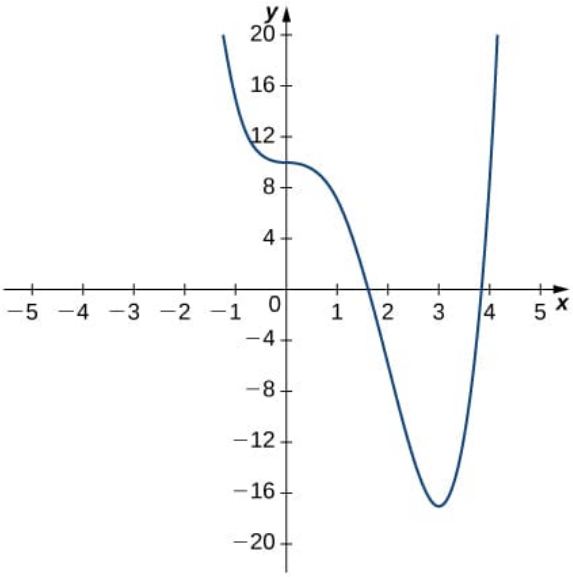
$$f'(x) = \frac{-1}{2x^{3/2}}$$

For the exercises 11 - 14, use the graph of $y = f(x)$ to sketch the graph of its derivative $f\,'(x)$.

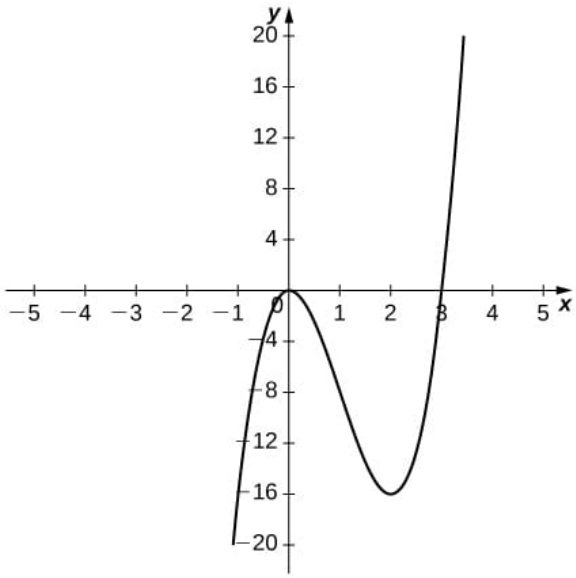
11)


y
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16
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8
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−5 −4 −3 −2 −1 0 1 2 3 4 5 x
−4
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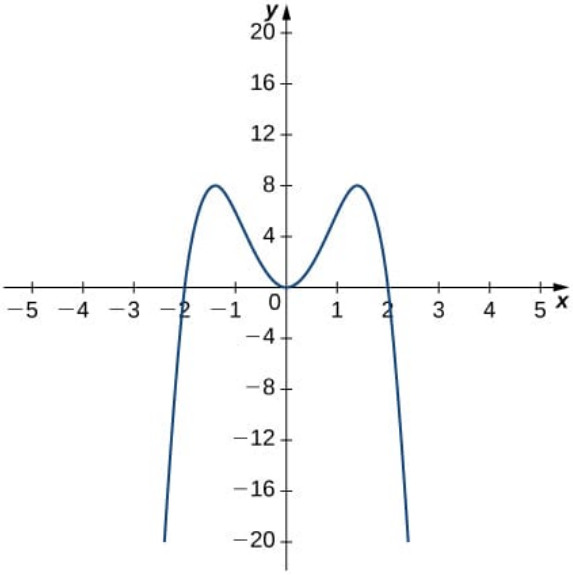
12)


y
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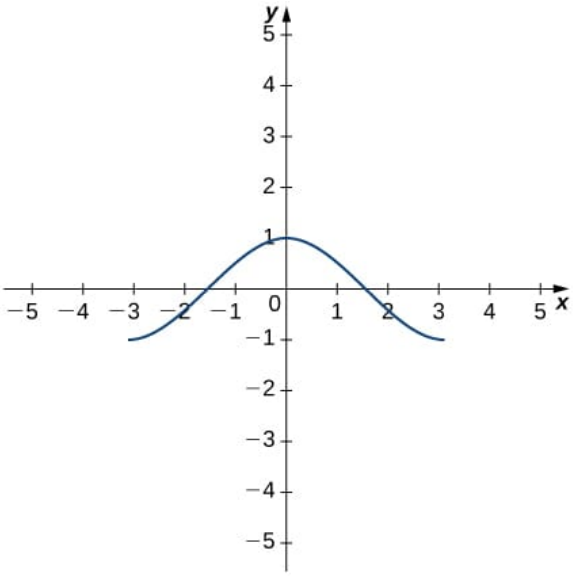
Answer


y
20
16
12
8
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−5 −4 −3 −2 −1 0 1 2 3 4 5 x
−4
−8
−12
−16
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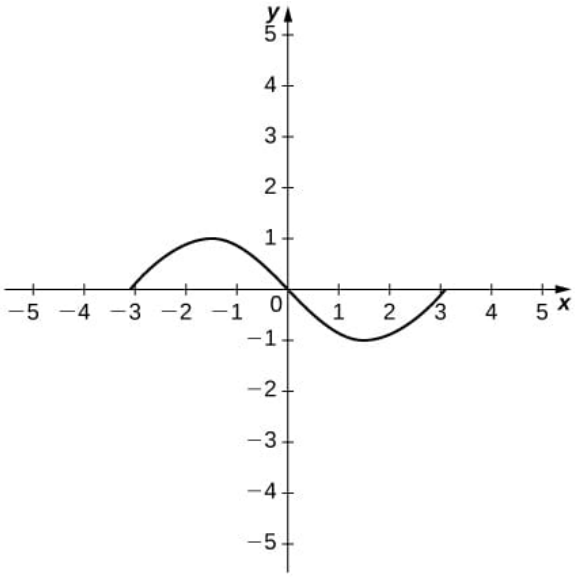
13)


y
20
16
12
8
4
−5 −4 −3 −2 −1 0 1 2 3 4 5 x
−4
−8
−12
−16
−20

14)



Answer



For exercises 15 - 20, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a.

15) $\lim\limits_{h \to 0} \dfrac{(1+h)^{2/3} - 1}{h}$

16) $\lim\limits_{h \to 0} \dfrac{[3(2+h)^2 + 2] - 14}{h}$

Answer

$f(x) = 3x^2 + 2, \quad a = 2$

17) $\lim\limits_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$

18) $\lim\limits_{h \to 0} \dfrac{(2+h)^4 - 16}{h}$

Answer

$$f(x) = x^4, \quad a = 2$$

19) $\lim_{h \to 0} \dfrac{[2(3+h)^2 - (3+h)] - 15}{h}$

20) $\lim_{h \to 0} \dfrac{e^h - 1}{h}$

Answer

$$f(x) = e^x, \quad a = 0$$

For the functions in exercises 21 - 24,

a. sketch the graph and

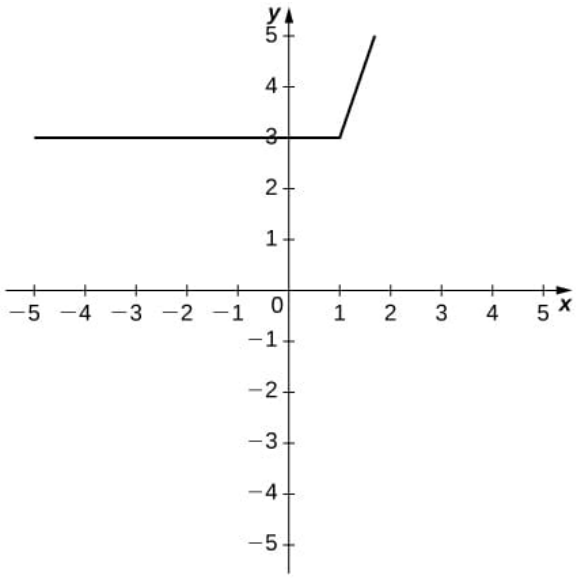
b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

21) $f(x) = \begin{cases} 2\sqrt{x}, & \text{if } 0 \le x \le 1 \\ 3x - 1, & \text{if } x > 1 \end{cases}$

22) $f(x) = \begin{cases} 3, & \text{if } x < 1 \\ 3x, & \text{if } x \ge 1 \end{cases}$

Answer

a.



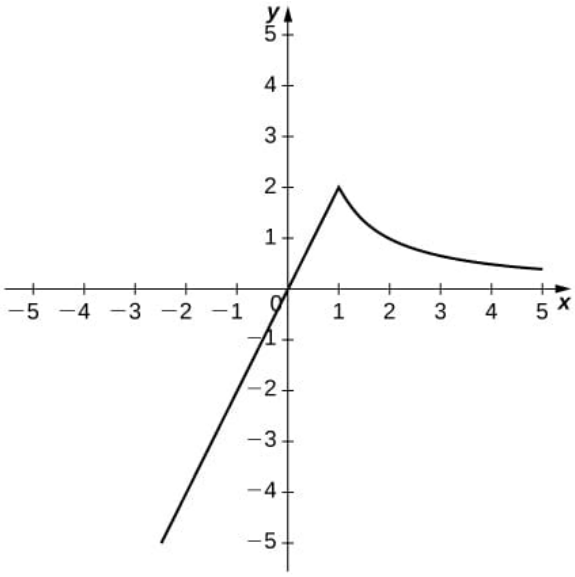
b. $\lim_{h \to 1^-} \dfrac{3-3}{h} \ne \lim_{h \to 1^+} \dfrac{3h}{h}$

23) $f(x) = \begin{cases} -x^2 + 2, & \text{if } x \le 1 \\ x, & \text{if } x > 1 \end{cases}$

24) $f(x) = \begin{cases} 2x, & \text{if } x \le 1 \\ \frac{2}{x}, & \text{if } x > 1 \end{cases}$

Answer

a.



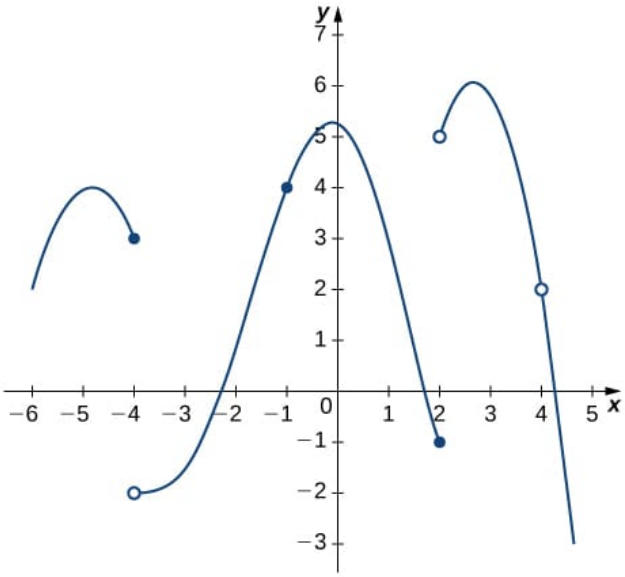
b. $\lim_{h\to 1^-} \frac{2h}{h} \neq \lim_{h\to 1^+} \frac{\frac{2}{x+h} - 2x}{h}$.

For the graphs in exercises 25 - 26,

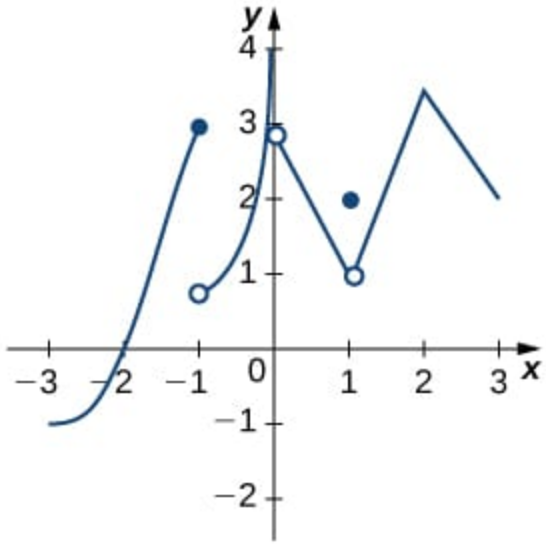
a. determine for which values of $x = a$ the $\lim_{x\to a} f(x)$ exists but f is not continuous at $x = a$, and

b. determine for which values of $x = a$ the function is continuous but not differentiable at $x = a$.

25)



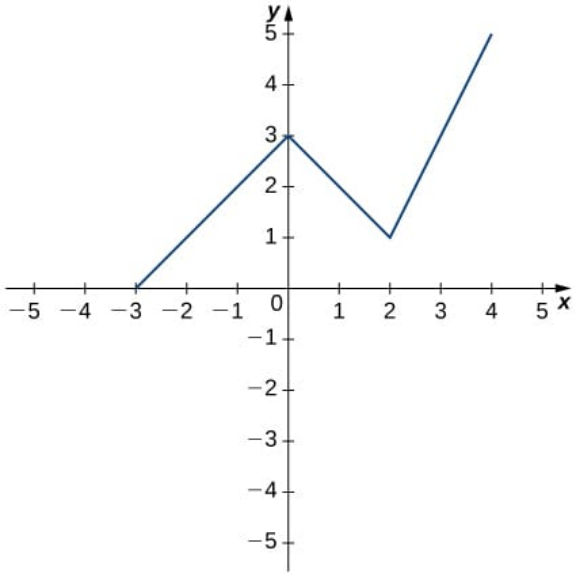
26)



Answer

a. $x = 1$,
b. $x = 2$

27) Use the graph to evaluate a. $f\,'(-0.5)$, b. $f\,'(0)$, c. $f\,'(1)$, d. $f\,'(2)$, and e. $f\,'(3)$, if it exists.



For the functions in exercises 28 - 30, use $f''(x) = \lim_{h \to 0} \dfrac{f'(x+h) - f'(x)}{h}$ **to find** $f''(x)$.

28) $f(x) = 2 - 3x$

Answer

$f''(x) = 0$

29) $f(x) = 4x^2$

30) $f(x) = x + \dfrac{1}{x}$

Answer

$$f''(x) = \frac{2}{x^3}$$

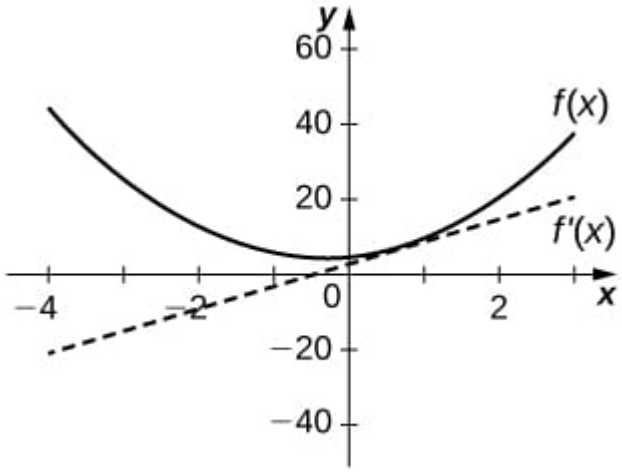
For exercises 31 - 36, use a calculator to graph $f(x)$. Determine the function $f'(x)$, then use a calculator to graph $f'(x)$.

31) [T] $f(x) = -\frac{5}{x}$

32) [T] $f(x) = 3x^2 + 2x + 4.$

Answer

$$f'(x) = 6x + 2$$

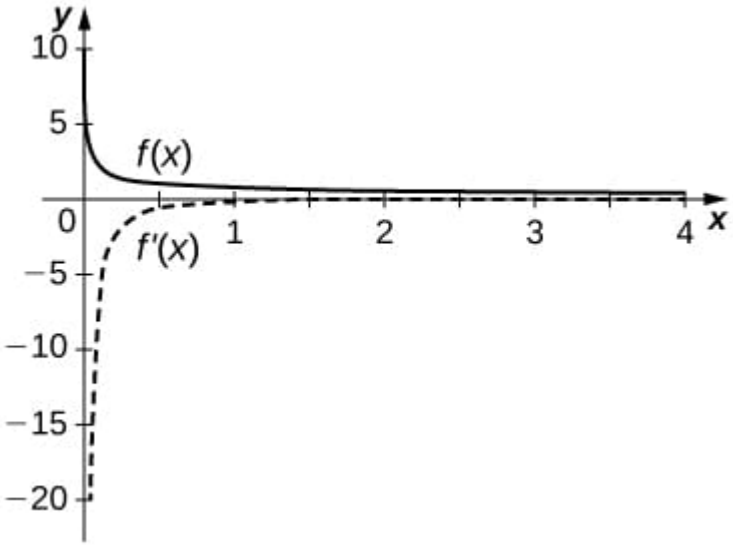


33) [T] $f(x) = \sqrt{x} + 3x$

34) [T] $f(x) = \frac{1}{\sqrt{2x}}$

Answer

$$f'(x) = -\frac{1}{(2x)^{3/2}}$$

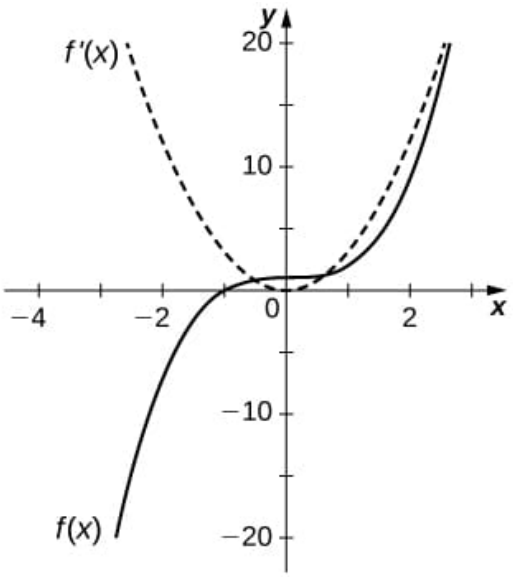


35) [T] $f(x) = 1 + x + \dfrac{1}{x}$

36) [T] $f(x) = x^3 + 1$

Answer

$$f\,'(x) = 3x^2$$



For exercises 37 - 42, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\dfrac{f(x+h) - f(x)}{h}$

b. $f\,'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

37) $P(x)$ denotes the population of a city at time x in years.

38) $C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

Answer

a. Average rate at which customers spent on concessions in thousands per customer.

b. Rate (in thousands per customer) at which x customers spent money on concessions in thousands per customer.

39) $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios

40) $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

Answer

a. Average grade received on the test with an average study time between two values.

b. Rate (in percentage points per hour) at which the grade on the test increased or decreased for a given average study time of x hours.

41) $B(x)$denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since 1990.

42) $p(x)$ denotes atmospheric pressure at an altitude of x feet.

Answer

a. Average change of atmospheric pressure between two different altitudes.

b. Rate (torr per foot) at which atmospheric pressure is increasing or decreasing at x feet.

43) Sketch the graph of a function $y = f(x)$ with all of the following properties:

a. $f\,'(x) > 0$ for $-2 \leq x < 1$

b. $f\,'(2) = 0$

c. $f\,'(x) > 0$ for $x > 2$

d. $f(2) = 2$ and $f(0) = 1$

e. $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$

f. $f\,'(1)$ does not exist.

44) Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y = T(x)$.

a. Give a physical interpretation, with units, of $T\,'(x)$.

b. If we know that $T\,'(1000) = -0.1$, explain the physical meaning.

Answer

a. The rate (in degrees per foot) at which temperature is increasing or decreasing for a given height x.

b. The rate of change of temperature as altitude changes at 1000 feet is -0.1 degrees per foot.

45) Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.

a. What does $\dfrac{P(b) - P(a)}{b - a}$ for \(0

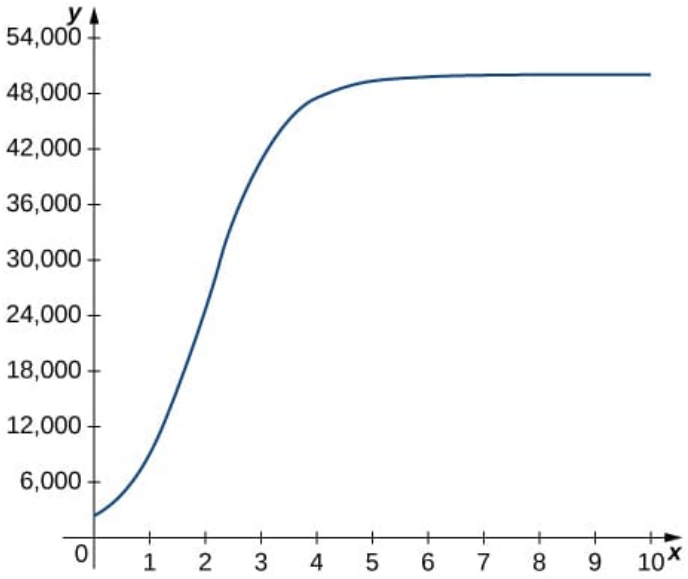
b. What does $P\,'(x)$ measure, and what are the units?

c. Suppose that $P\,'(30) = 5$, what is the approximate change in profit if the number of items sold increases from 30 to 31?

46) The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of 50,000 citizens.

a. Describe what $N\,'(t)$ represents and how it behaves as t increases.

b. What does the derivative tell us about how this town is affected by the flu outbreak?



Answer

a. The rate at which the number of people who have come down with the flu is changing t weeks after the initial outbreak.
b. The rate is increasing sharply up to the third week, at which point it slows down and then becomes constant.

For exercises 47 - 52, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	Height (meters)
0	0
1	2
2	4
3	13
4	25
5	32

47) What is the physical meaning of $h\,'(t)$? What are the units?

48) [T] Construct a table of values for $h\,'(t)$ and graph both $h(t)$ and $h\,'(t)$ on the same graph. (Hint: for interior points, estimate both the left limit and right limit and average them.)

Answer

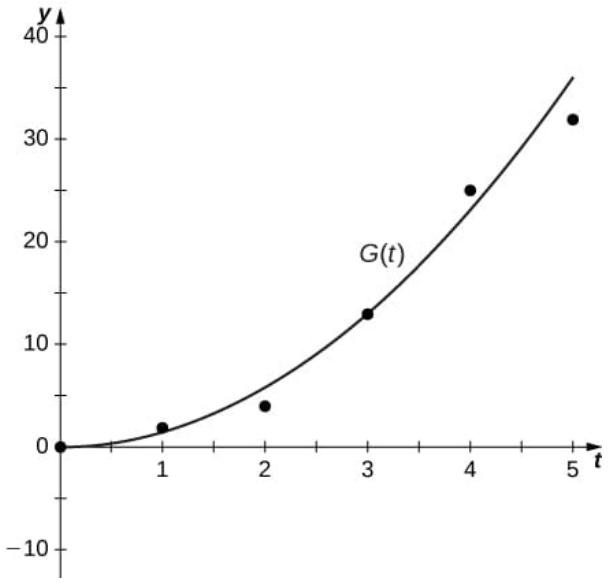
Time (seconds)	$h\,'(t)$ (m/s)
0	2
1	2
2	5.5
3	10.5
4	9.5
5	7

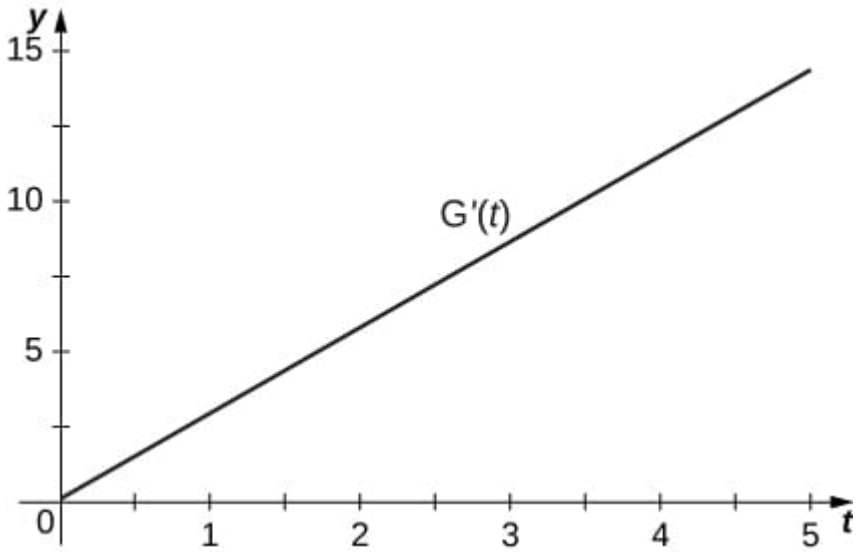
49) [T] The best linear fit to the data is given by $H(t) = 7.229t - 4.905$, where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t)$. Graph $H(t$ with the given data and, on a separate coordinate plane, graph $H'(t)$.

50) [T] The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429$, where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t)$. Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t)$.

Answer

$$G'(t) = 2.858t + 0.0857$$





51) [T] The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683$, where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t)$. Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t)$. Does the linear, quadratic, or cubic function fit the data best?

52) Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t)$, $G''(t)$ and $F''(t)$ are. What are the physical meanings of $H''(t)$, $G''(t)$ and $F''(t)$, and what are their units?

Answer

$H''(t) = 0$, $G''(t) = 2.858$ and $f''(t) = 1.222t + 5.912$ represent the acceleration of the rocket, with units of meters per second squared $(\mathrm{m/s^2})$.

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14.4: Differentiation Rules

Learning Objectives

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule for finding the derivative of a product of functions.
- Use the quotient rule for finding the derivative of a quotient of functions.
- Extend the power rule to functions with negative exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit.

The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated.

In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x + h) = c$, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

The Constant Rule

Let c be a constant. If $f(x) = c$, then $f\,'(x) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 14.4.1: Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution

This is just a one-step application of the rule: $f\,'(8) = 0$.

Exercise 14.4.1

Find the derivative of $g(x) = -3$.

Hint available at the end of this section.

Answer available at the end of this section.

The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right) = 2x \quad \text{and} \quad \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}\left(x^n\right)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, pay special attention to the portion of the expression in boldface, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example 14.4.2: Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution:

$\frac{d}{dx}\left(x^3\right) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$	
$= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$	Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2h$. All other terms contain powers of h that are two or greater
$= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h}$	In this step the x^3 terms have been cancelled, leaving only terms containing h.
$= \lim_{h\to 0}\frac{h(3x^2 + 3xh + h^2)}{h}$	Factor out the common factor of h.

$= \lim_{h \to 0}(3x^2 + 3xh + h^2)$	After cancelling the common factor of h,the only term not containing h is $3x^2$.
$= 3x^2$	Let h go to 0.

Exercise 14.4.2

Find $\frac{d}{dx}\left(x^4\right)$.

Hint available at the end of this section.

Answer available at the end of this section.

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

The Power Rule

Let n be a positive integer. If $f(x) = x^n$,then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1.}$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

Since

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,

$$f'(x) = \lim_{h\to 0}(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1})$$

$$= nx^{n-1}.$$

□

Example 14.4.3: Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

Solution

Using the power rule with $n = 10$, we obtain

$$f'(x) = 10x^{10-1} = 10x^9.$$

Exercise 14.4.3

Find the derivative of $f(x) = x^7$.

Hint available at the end of this section.

Answer available at the end of this section.

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big);$$

that is,

$$\text{for } s(x) = f(x) + g(x), \quad s'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$\text{for } d(x) = f(x) - g(x), \quad d'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx}\big(f(x)\big);$$

that is,

$$\text{for } m(x) = kf(x), \quad m'(x) = kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $s(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}.$$

By substituting $s(x+h) = f(x+h) + g(x+h)$ and $s(x) = f(x) + g(x)$, we obtain

$$s'(x) = \lim_{h \to 0} \frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h}.$$

Rearranging and regrouping the terms, we have

$$s'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).$$

□

Example 14.4.4: Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

Solution

We use the power rule directly:

$$g'(x) = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f\,'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of $f(x)$. This relationship is illustrated in Figure 14.4.1.

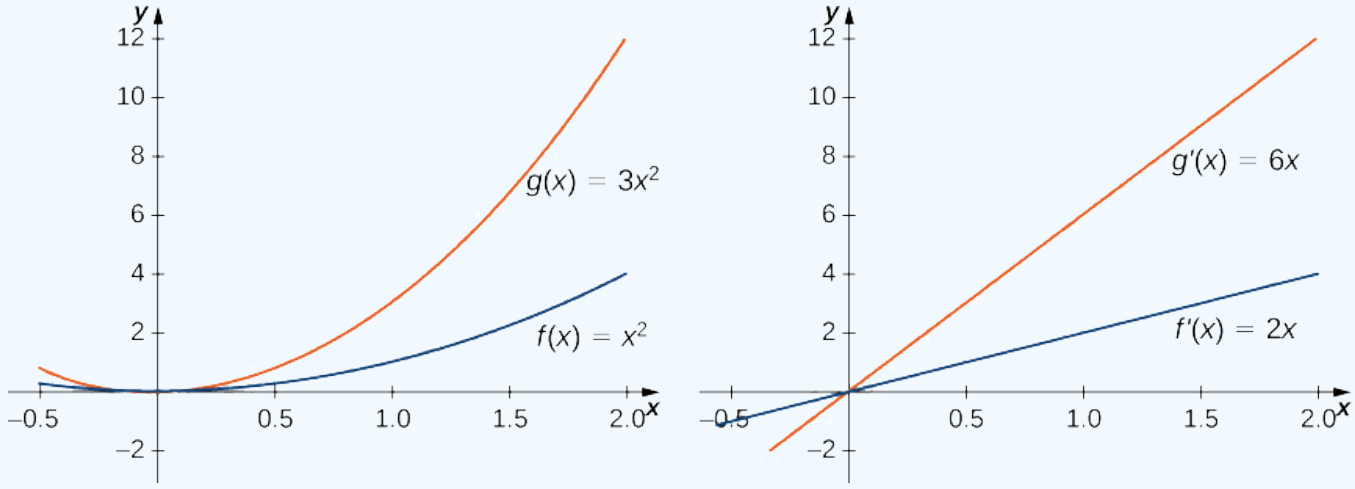


Figure 14.4.1: The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 14.4.5: Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^5 + 7\right) \\ &= \frac{d}{dx}(2x^5) + \frac{d}{dx}(7) && \text{Apply the sum rule.} \\ &= 2\frac{d}{dx}(x^5) + \frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\ &= 2(5x^4) + 0 && \text{Apply the power rule and the constant rule.} \\ &= 10x^4 && \text{Simplify.} \end{aligned}$$

Exercise 14.4.4

Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.4.6: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f\,'(1)$, we must first find $f\,'(x)$. Using the definition of a derivative, we have

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f\,'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y = -2x + 5.$$

Exercise 14.4.5

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

Hint available at the end of this section.

Answer available at the end of this section.

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f\,'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

$$\text{if } p(x) = f(x)g(x), \quad \text{then } p'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \to 0} g(x + h) = g(x)$.

By applying the limit definition of the derivative to $p(x) = f(x)g(x)$, we obtain

$$p'(x) = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x + h)$ in the numerator, we have

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$p'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}.$$

Rearranging, we obtain

$$\begin{aligned} p'(x) &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x) \right) \\ &= \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right) \cdot \left(\lim_{h \to 0} g(x+h) \right) + \left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right) \cdot f(x) \end{aligned}$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$p'(x) = f'(x)g(x) + g'(x)f(x).$$

□

Example 14.4.7: Applying the Product Rule to Constant Functions

For $p(x) = f(x)g(x)$, use the product rule to find $p\,'(2)$ if $f(2) = 3$, $f\,'(2) = -4$, $g(2) = 1$, and $g\,'(2) = 6$.

Solution

Since $p(x) = f(x)g(x)$, $p\,'(x) = f\,'(x)g(x) + g\,'(x)f(x)$, and hence

$p\,'(2) = f\,'(2)g(2) + g\,'(2)f(2) = (-4)(1) + (6)(3) = 14.$

Example 14.4.8: Applying the Product Rule to Binomials

For $p(x) = (x^2 + 2)(3x^3 - 5x)$, find $p\,'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f\,'(x) = 2x$ and $g\,'(x) = 9x^2 - 5$. Thus,

$p\,'(x) = f\,'(x)g(x) + g\,'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2).$

Simplifying, we have

$$p'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $p(x) = 3x^5 + x^3 - 10x$ and, consequently, $p\,'(x) = 15x^4 + 3x^2 - 10$.

Exercise 14.4.6

Use the product rule to obtain the derivative of $p(x) = 2x^5(4x^2 + x)$.

Hint available at the end of this section.

Answer available at the end of this section.

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}(x^2) = 2x, \text{ not } \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{\big(g(x)\big)^2}.$$

That is, if

$$q(x) = \frac{f(x)}{g(x)}$$

then

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}.$$

The proof of the quotient rule is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example 14.4.9: Applying the Quotient Rule

Use the quotient rule to find the derivative of $q(x) = \dfrac{5x^2}{4x+3}$.

Solution

Let $f(x) = 5x^2$ and $g(x) = 4x + 3$. Thus, $f\,'(x) = 10x$ and $g\,'(x) = 4$.

Substituting into the quotient rule, we have

$$q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2}.$$

Simplifying, we obtain

$$q'(x) = \frac{20x^2 + 30x}{(4x+3)^2}$$

Exercise 14.4.7

Find the derivative of $h(x) = \dfrac{3x+1}{4x-3}$.

Hint available at the end of this section.

Answer available at the end of this section.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer n,$x^{-n} = \dfrac{1}{x^n}$, we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f\,'(x) = 0$ and $g\,'(x) = nx^{n-1}$. Thus,

$$\frac{d}{dx}(x^{-n}) = \frac{0(x^n) - 1(nx^{n-1})}{(x^n)^2}.$$

Simplifying, we see that

$$\begin{aligned}\frac{d}{dx}(x^{-n}) &= \frac{-nx^{n-1}}{x^{2n}} \\ &= -nx^{(n-1)-2n} \\ &= -nx^{-n-1}.\end{aligned}$$

Finally, observe that since $k = -n$, by substituting we have

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

□

Example 14.4.10: Using the Extended Power Rule

Find $\dfrac{d}{dx}(x^{-4})$.

Solution

By applying the extended power rule with $k = -4$, we obtain

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}.$$

Example 14.4.11: Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find $f(x) = \frac{6}{x^2}$.

Solution

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

$$\begin{aligned} f'(x) = \frac{d}{dx}\left(\frac{6}{x^2}\right) = \frac{d}{dx}\left(6x^{-2}\right) & \quad \text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}. \\ = 6\frac{d}{dx}\left(x^{-2}\right) & \quad \text{Apply the constant multiple rule.} \\ = 6(-2x^{-3}) & \quad \text{Use the extended power rule to differentiate } x^{-2}. \\ = -12x^{-3} & \quad \text{Simplify.} \end{aligned}$$

Exercise 14.4.8

Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended power rule.

Hint available at the end of this section.

Answer available at the end of this section.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 14.4.12: Combining Differentiation Rules

For $k(x) = 3h(x) + x^2g(x)$, find $k'(x)$.

Solution: Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.

$k'(x) = \frac{d}{dx}\left(3h(x) + x^2g(x)\right) = \frac{d}{dx}\left(3h(x)\right) + \frac{d}{dx}\left(x^2g(x)\right)$	Apply the sum rule.
$= 3\frac{d}{dx}\left(h(x)\right) + \left(\frac{d}{dx}(x^2)g(x) + \frac{d}{dx}(g(x))x^2\right)$	Apply the constant multiple rule to differentiate $3h(x)$ and the product rule to differentiate $x^2g(x)$.
$= 3h'(x) + 2xg(x) + g'(x)x^2$	

Example 14.4.13: Extending the Product Rule

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x), g(x), h(x)$, and their derivatives.

Solution

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x) = (f(x)g(x)) \cdot h(x)$. Thus,

$$\begin{aligned} k'(x) &= \frac{d}{dx}\big(f(x)g(x)\big) \cdot h(x) + \frac{d}{dx}\big(h(x)\big) \cdot \big(f(x)g(x)\big). && \text{Apply the product rule to the product of } f(x)g(x) \text{ and } h(x). \\ &= \big(f'(x)g(x) + g'(x)f(x)\big)h(x) + h'(x)f(x)g(x) && \text{Apply the product rule to } f(x)g(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). && \text{Simplify.} \end{aligned}$$

Example 14.4.14: Combining the Quotient Rule and the Product Rule

For $h(x) = \dfrac{2x^3k(x)}{3x+2}$, find $h\,'(x)$.

Solution

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}(2x^3k(x)) \cdot (3x+2) - \frac{d}{dx}(3x+2) \cdot (2x^3k(x))}{(3x+2)^2} && \text{Apply the quotient rule.} \\ &= \frac{(6x^2k(x) + k'(x) \cdot 2x^3)(3x+2) - 3(2x^3k(x))}{(3x+2)^2} && \text{Apply the product rule to find } \frac{d}{dx}(2x^3k(x)). \text{ Use } \frac{d}{dx}(3x+2) = 3. \\ &= \frac{-6x^3k(x) + 18x^3k(x) + 12x^2k(x) + 6x^4k'(x) + 4x^3k'(x)}{(3x+2)^2} && \text{Simplify} \end{aligned}$$

Exercise 14.4.9

Find $\dfrac{d}{dx}(3f(x) - 2g(x))$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.4.15: Determining Where a Function Has a Horizontal Tangent

Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

Solution

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f\,'(x) = 0$.

Since $f\,'(x) = 3x^2 - 14x + 8 = (3x-2)(x-4)$,

we must solve $(3x-2)(x-4) = 0$. Thus we see that the function has horizontal tangent lines at $x = \dfrac{2}{3}$ and $x = 4$ as shown in the following graph.

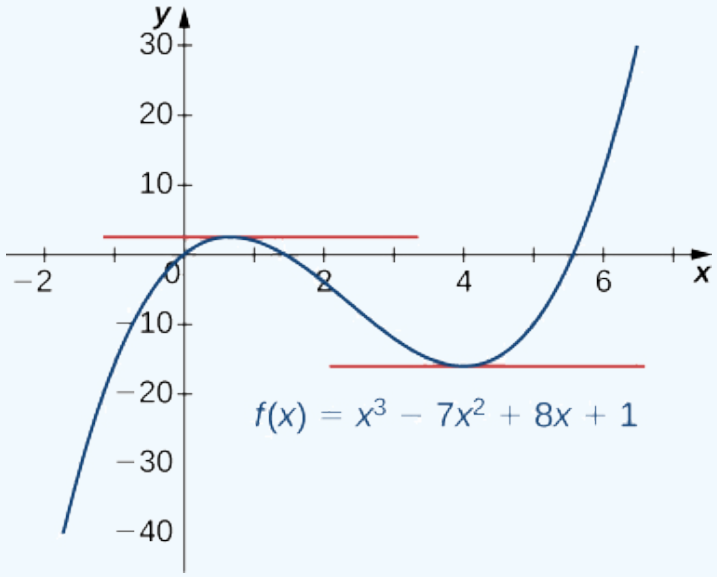


Figure 14.4.2: This function has horizontal tangent lines at $x = 2/3$ and $x = 4$.

Example 14.4.16: Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t) = \dfrac{t}{t^2+1}$. What is the initial velocity of the object?

Solution

Since the initial velocity is $v(0) = s\,'(0)$, begin by finding $s\,'(t)$ by applying the quotient rule:

$$s\,'(t) = \frac{1(t^2+1) - 2t(t)}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}.$$

After evaluating, we see that $v(0) = 1$.

Exercise 14.4.10

Find the values of x for which the line tangent to the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

Hint available at the end of this section.

Answer available at the end of this section.

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure 14.4.3).



Figure 14.4.3: The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (Figure 14.4.4). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

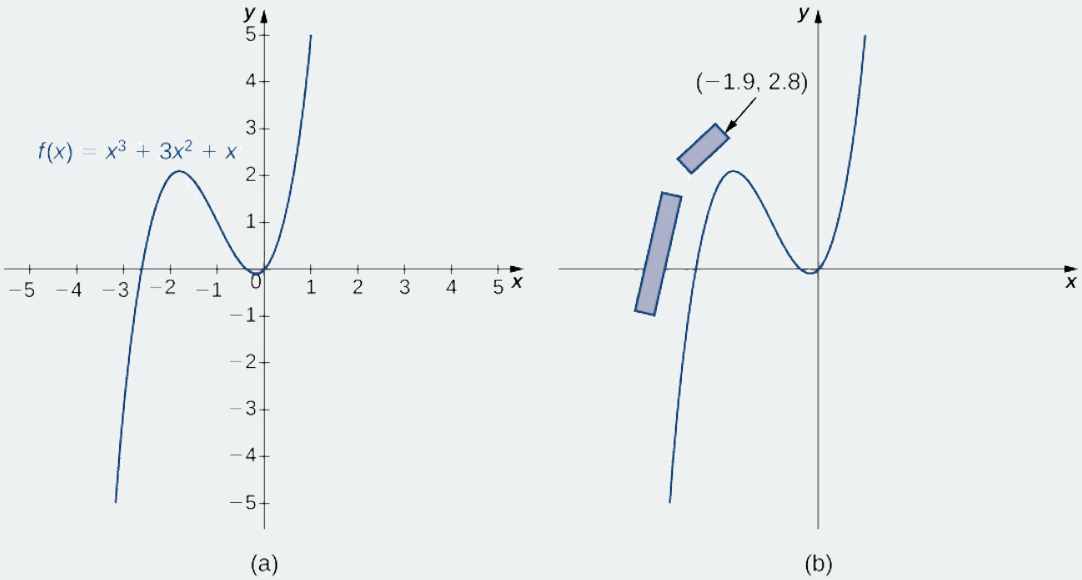


Figure 14.4.4: (a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the x-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?

4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Key Concepts

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

Glossary

constant multiple rule
: the derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big) = cf\,'(x)$

constant rule
: the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where c is a constant

difference rule
: the derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) - g(x)\big) = f\,'(x) - g\,'(x)$

power rule
: the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}\,(x^n) = nx^{n-1}$

product rule
: the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}\big(f(x)g(x)\big) = f\,'(x)g(x) + g\,'(x)f(x)$

quotient rule
: the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2}$

sum rule
: the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) + g(x)\big) = f\,'(x) + g\,'(x)$

Answers

Exercise 14.4.1

Use the preceding example as a guide

Exercise 14.4.1

0

Exercise 14.4.2

Use $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ and follow the procedure outlined in the preceding example.

Exercise 14.4.2

$\frac{d}{dx}(x^4) = 4x^3$

Exercise 14.4.3

Use the power rule with $n = 7$.

Exercise 14.4.3

$f\,'(x) = 7x^6$

Exercise 14.4.4

Use the preceding example as a guide.

Exercise 14.4.4

$f\,'(x) = 6x^2 - 12x$.

Exercise 14.4.5

Use the preceding example as a guide.

Exercise 14.4.5

$y = 12x - 23$

Exercise 14.4.6

Set $f(x) = 2x^5$ and $g(x) = 4x^2 + x$ and use the preceding example as a guide.

Exercise 14.4.6

$p\,'(x) = 10x^4(4x^2 + x) + (8x + 1)(2x^5) = 56x^6 + 12x^5$.

Exercise 14.4.7

Apply the quotient rule with $f(x) = 3x + 1$ and $g(x) = 4x - 3$.

Exercise 14.4.7

$h\,'(x) = -\frac{13}{(4x-3)^2}$.

Exercise 14.4.8

Rewrite $g(x) = \frac{1}{x^7} = x^{-7}$. Use the extended power rule with $k = -7$.

Exercise 14.4.8

$g\,'(x) = -7x^{-8}$.

Exercise 14.4.9

Apply the difference rule and the constant multiple rule.

Exercise 14.4.9

$3f\,'(x) - 2g\,'(x)$.

Exercise 14.4.10

Solve $f\,'(x) = 2$.

Exercise 14.4.10

$\frac{5}{8}$

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14.4E: Exercises for Section 3.3

In exercises 1 - 12, find $f'(x)$ for each function.

1) $f(x) = x^7 + 10$

2) $f(x) = 5x^3 - x + 1$

Answer

$$f'(x) = 15x^2 - 1$$

3) $f(x) = 4x^2 - 7x$

4) $f(x) = 8x^4 + 9x^2 - 1$

Answer

$$f'(x) = 32x^3 + 18x$$

5) $f(x) = x^4 + 2x$

6) $f(x) = 3x\left(18x^4 + \dfrac{13}{x+1}\right)$

Answer

$$f'(x) = 270x^4 + \frac{39}{(x+1)^2}$$

7) $f(x) = (x+2)(2x^2 - 3)$

8) $f(x) = x^2\left(\dfrac{2}{x^2} + \dfrac{5}{x^3}\right)$

Answer

$$f'(x) = \frac{-5}{x^2}$$

9) $f(x) = \dfrac{x^3 + 2x^2 - 4}{3}$

10) $f(x) = \dfrac{4x^3 - 2x + 1}{x^2}$

Answer

$$f'(x) = \frac{4x^4 + 2x^2 - 2x}{x^4}$$

11) $f(x) = \dfrac{x^2 + 4}{x^2 - 4}$

12) $f(x) = \dfrac{x + 9}{x^2 - 7x + 1}$

Answer

$$f'(x) = \frac{-x^2 - 18x + 64}{(x^2 - 7x + 1)^2}$$

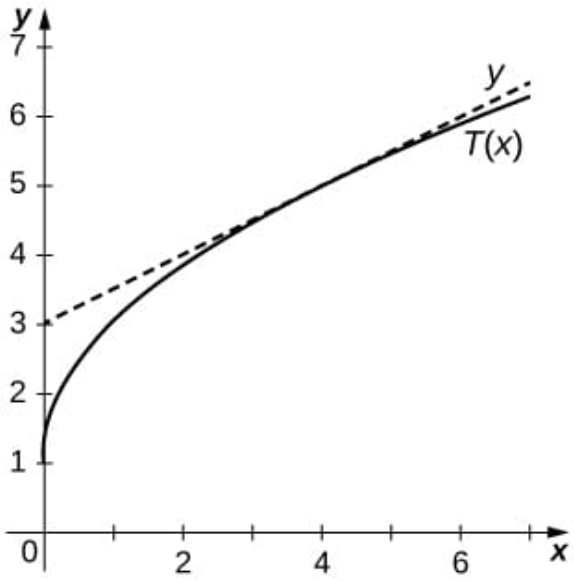
In exercises 13 - 16, find the equation of the tangent line $T(x)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

13) [T] $y = 3x^2 + 4x + 1$ at $(0, 1)$

14) [T] $y = 2\sqrt{x} + 1$ at $(4, 5)$

Answer

$T(x) = \frac{1}{2}x + 3$

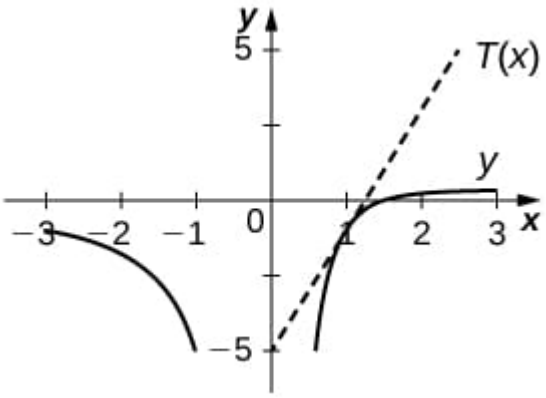


15) [T] $y = \dfrac{2x}{x-1}$ at $(-1, 1)$

16) [T] $y = \dfrac{2}{x} - \dfrac{3}{x^2}$ at $(1, -1)$

Answer

$T(x) = 4x - 5$



In exercises 17 - 20, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x. Find the derivative of each of the functions $h(x)$.

17) $h(x) = 4f(x) + \dfrac{g(x)}{7}$

18) $h(x) = x^3 f(x)$

Answer

$h'(x) = 3x^2 f(x) + x^3 f\,'(x)$

19) $h(x) = \dfrac{f(x)g(x)}{2}$

20) $h(x) = \dfrac{3f(x)}{g(x)+2}$

Answer

$$h'(x) = \frac{3f'(x)(g(x)+2) - 3f(x)g'(x)}{(g(x)+2)^2}$$

For exercises 21 - 24, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	−2	0
$g(x)$	2	3	−4	6
$f\,'(x)$	−1	7	8	−3
$g\,'(x)$	4	1	2	9

21) Find $h\,'(1)$ if $h(x) = xf(x) + 4g(x)$.

22) Find $h\,'(2)$ if $h(x) = \dfrac{f(x)}{g(x)}$.

Answer

$h'(2) = \frac{16}{9}$

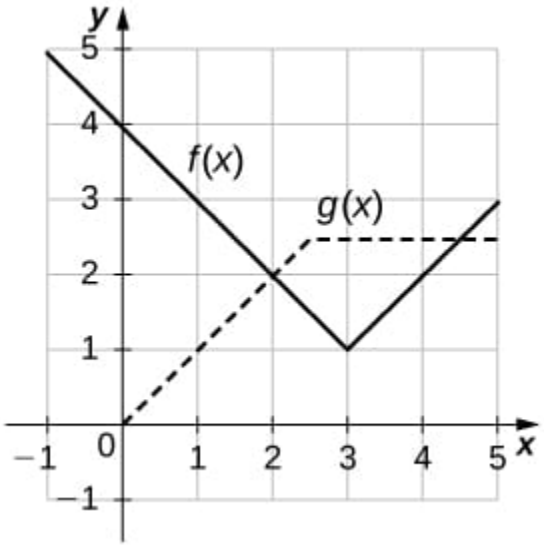
23) Find $h\,'(3)$ if $h(x) = 2x + f(x)g(x)$.

24) Find $h\,'(4)$ if $h(x) = \dfrac{1}{x} + \dfrac{g(x)}{f(x)}$.

Answer

$h'(4)$ is undefined.

In exercises 25 - 27, use the following figure to find the indicated derivatives, if they exist.



25) Let $h(x) = f(x) + g(x)$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

26) Let $h(x) = f(x)g(x)$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

Answer

a. $h'(1) = 2$,
b. $h'(3)$ does not exist,
c. $h'(4) = 2.5$

27) Let $h(x) = \dfrac{f(x)}{g(x)}$. Find

a) $h'(1)$,

b) $h'(3)$, and

c) $h'(4)$.

In exercises 28 - 31,

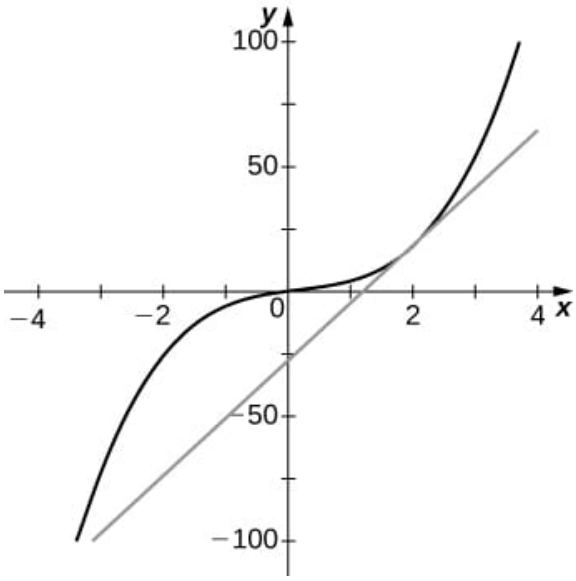
a) evaluate $f'(a)$, and

b) graph the function $f(x)$ and the tangent line at $x = a$.

28) [T] $f(x) = 2x^3 + 3x - x^2, \quad a = 2$

Answer

a. 23
b. $y = 23x - 28$

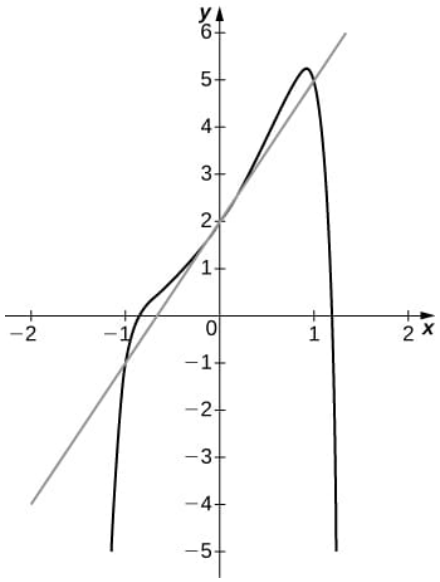


29) [T] $f(x) = \dfrac{1}{x} - x^2, \quad a = 1$

30) [T] $f(x) = x^2 - x^{12} + 3x + 2, \quad a = 0$

Answer

a. 3
b. $y = 3x + 2$



31) [T] $f(x) = \dfrac{1}{x} - x^{2/3}, \quad a = -1$

32) Find the equation of the tangent line to the graph of $f(x) = 2x^3 + 4x^2 - 5x - 3$ at $x = -1$.

Answer

$y = -7x - 3$

33) Find the equation of the tangent line to the graph of $f(x) = x^2 + \dfrac{4}{x} - 10$ at $x = 8$.

34) Find the equation of the tangent line to the graph of $f(x) = (3x - x^2)(3 - x - x^2)$ at $x = 1$.

Answer

$y = -5x + 7$

35) Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x-intercept of $(6, 0)$.

36) Find the equation of the line passing through the point $P(3, 3)$ and tangent to the graph of $f(x) = \dfrac{6}{x - 1}$.

Answer

$y = -\frac{3}{2}x + \frac{15}{2}$

37) Determine all points on the graph of $f(x) = x^3 + x^2 - x - 1$ for which the slope of the tangent line is

a. horizontal

b. –1.

38) Find a quadratic polynomial such that $f(1) = 5, \quad f\,'(1) = 3$ and $f''(1) = -6$.

Answer

$y = -3x^2 + 9x - 1$

39) A car driving along a freeway with traffic has traveled $s(t) = t^3 - 6t^2 + 9t$ meters in t seconds.

a. Determine the time in seconds when the velocity of the car is 0.

b. Determine the acceleration of the car when the velocity is 0.

40) [T] A herring swimming along a straight line has traveled $s(t) = \dfrac{t^2}{t^2 + 2}$ feet in t seconds. Determine the velocity of the herring when it has traveled 3 seconds.

Answer

$\frac{12}{121}$ or 0.0992 ft/s

41) The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P(t) = \dfrac{8t + 3}{0.2t^2 + 1}$, where t is measured in years.

a. Determine the initial flounder population.

b. Determine $P'(10)$ and briefly interpret the result.

42) [T] The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t) = \dfrac{2t^2 + t}{t^3 + 50}$, where C is measured in milligrams per liter of blood.

a. Find the rate of change of $C(t)$.

b. Determine the rate of change for $t = 8, 12, 24$,and 36.

c. Briefly describe what seems to be occurring as the number of hours increases.

Answer

a. $\dfrac{-2t^4 - 2t^3 + 200t + 50}{(t^3 + 50)^2}$

b. -0.02395 mg/L-hr, -0.01344 mg/L-hr, -0.003566 mg/L-hr, -0.001579 mg/L-hr

c. The rate at which the concentration of drug in the bloodstream decreases is slowing to 0 as time increases.

43) A book publisher has a cost function given by $C(x) = \dfrac{x^3 + 2x + 3}{x^2}$, where x is the number of copies of a book in thousands and C is the cost, per book, measured in dollars. Evaluate $C'(2)$and explain its meaning.

44) [T] According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \dfrac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

a. Suppose that $G, m_1,$ and m_2 are constants. Find the rate of change of force F with respect to distance d.

b. Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2$, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

Answer

a. $F'(d) = \dfrac{-2Gm_1m_2}{d_3}$

b. -1.33×10^{-7} N/m

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14.5: Derivatives as Rates of Change

Learning Objectives

- Determine a new value of a quantity from the old value and the amount of change.
- Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
- Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
- Predict the future population from the present value and the population growth rate.
- Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h)-f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

$$\frac{f(a+h)-f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

For small enough values of h, $f\,'(a)\approx\frac{f(a+h)-f(a)}{h}$. We can then solve for $f(a+h)$ to get the amount of change formula:

$$f(a+h)\approx f(a)+f'(a)h. \tag{14.5.1}$$

We can use this formula if we know only $f(a)$and $f\,'(a)$ and wish to estimate the value of $f(a+h)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in Figure $14.5.1$, we are approximating $f(a+h)$ by the y coordinate at a+h on the line tangent to $f(x)$ at $x=a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f\,'(a)$.

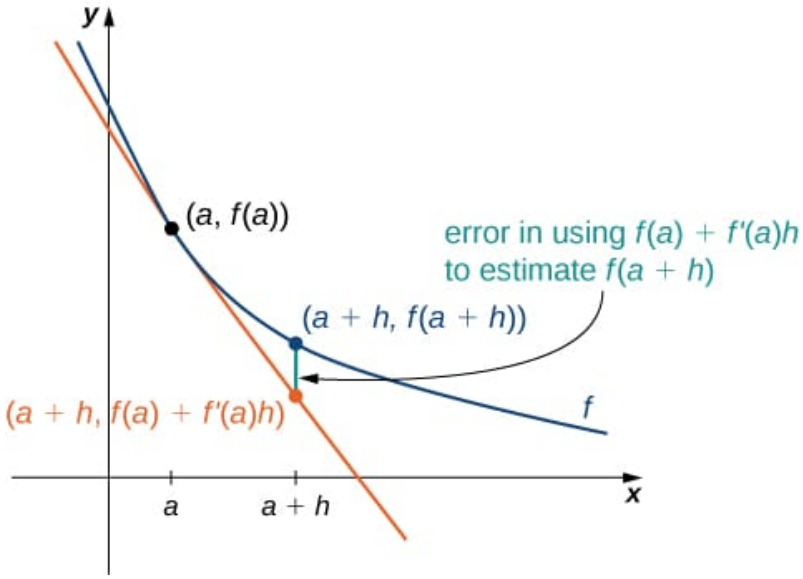


Figure 14.5.1: The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h) \approx f(a) + f\,'(a)h$.

Example 14.5.1: Estimating the Value of a Function

If $f(3) = 2$ and $f\,'(3) = 5$, estimate $f(3.2)$.

Solution

Begin by finding h. We have $h = 3.2 - 3 = 0.2$. Thus,

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f\,'(3) = 2 + 0.2(5) = 3.$$

Exercise 14.5.1

Given $f(10) = -5$ and $f\,'(10) = 6$, estimate $f(10.1)$.

Hint available at the end of this section.

Answer available at the end of this section.

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

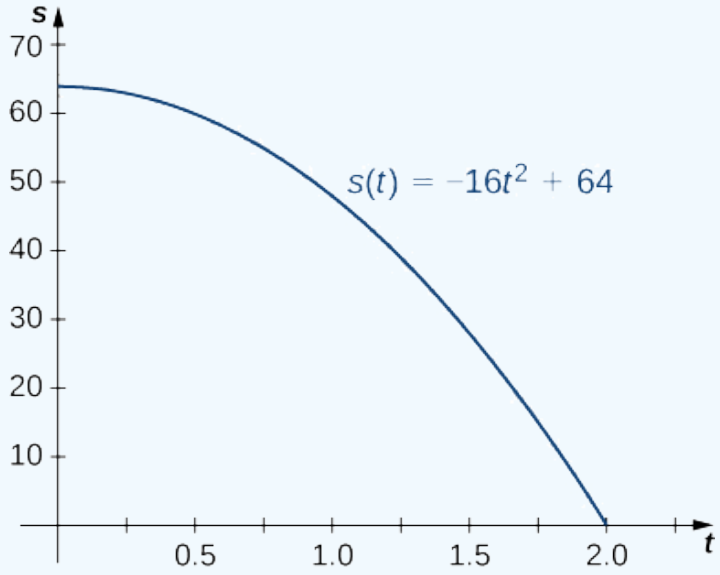
Definition

Let $s(t)$ be a function giving the position of an object at time t.

- The velocity of the object at time t is given by $v(t) = s\,'(t)$.
- The speed of the object at time t is given by $|v(t)|$.
- The acceleration of the object at t is given by $a(t) = v\,'(t) = s''(t)$.

Example 14.5.2: Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



a. What is the instantaneous velocity of the ball when it hits the ground?
b. What is the average velocity during its fall?

Solution

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0$. Solving $-16t^2 + 64 = 0$, we get $t = 2$, so it takes 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2)$. Since $v(t) = s'(t) = -32t$, we obtain $v(t) = -64$ ft/s.
b. The average velocity of the ball during its fall is

$$v_{ave} = \frac{s(2)-s(0)}{2-0} = \frac{0-64}{2} = -32 \text{ ft/s.}$$

Example 14.5.3: Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?
b. Is the particle speeding up or slowing down at time $t = 1$?

Solution

Begin by finding $v(t)$ and $a(t)$.

$v(t) = s'(t) = 3t^2 - 4$ and $a(t) = v'(t) = s''(t) = 6t$.

Evaluating these functions at $t = 1$, we obtain $v(1) = -1$ and $a(1) = 6$.

a. Because $v(1) < 0$, the particle is moving from right to left.

b. Because $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example 14.5.4: Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4, \quad t \geq 0.$

a. Find $v(t)$.
b. At what time(s) is the particle at rest?
c. On what time intervals is the particle moving from left to right? From right to left?
d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution

a. The velocity is the derivative of the position function:

$$v(t) = s\,'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0$, so set $3t^2 - 18t + 24 = 0$. Factoring the left-hand side of the equation produces $3(t - 2)(t - 4) = 0$. Solving, we find that the particle is at rest at $t = 2$ and $t = 4$.

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0$. Figure 14.5.2 gives the analysis of the sign of $v(t)$ for $t \geq 0$, but it does not represent the axis along which the particle is moving.



Figure 14.5.2:The sign of $v(t)$ determines the direction of the particle.

- Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (4, +\infty)$, the particle is moving from left to right on these intervals.
- Since $3t^2 - 18t + 24 < 0$ on $(2, 4)$, the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4)$. We have $s(0) = 4$, $s(2) = 24$, and $s(4) = 20$. This means that the particle begins on the coordinate axis at 4 and changes direction at 24 and 20 on the coordinate axis. The path of the particle is shown on a coordinate axis in Figure 14.5.3.

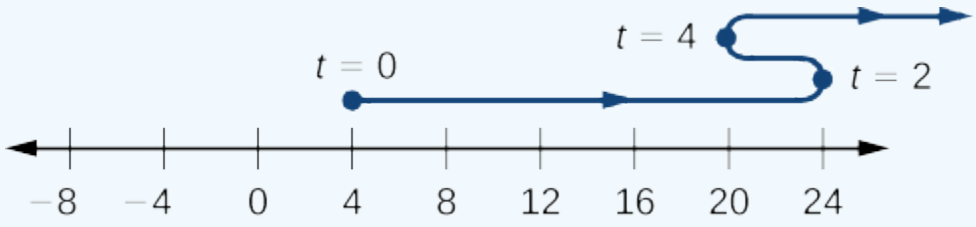


Figure 14.5.3: The path of the particle can be determined by analyzing $v(t)$.

Exercise 14.5.2

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Hint available at the end of this section.

Answer available at the end of this section.

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Definition

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

Example 14.5.5: Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution

Let $P(t)$ be the population (in thousands) t years from now. Thus, we know that $P(0) = 10$ and based on the information, we anticipate $P(5) = 30$. Now estimate $P'(0)$, the current growth rate, using

$$P'(0) \approx \frac{P(5)-P(0)}{5-0} = \frac{30-10}{5} = 4.$$

By applying Equation 14.5.1 to $P(t)$, we can estimate the population 2 years from now by writing

$$P(2) \approx P(0) + (2)P'(0) \approx 10 + 2(4) = 18;$$

thus, in 2 years the population will be 18,000.

Exercise 14.5.3

The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3,000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Hint available at the end of this section.

Answer available at the end of this section.

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Definition

- If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.
- If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C\,'(x) \approx C(x+1) - C(x)$. Consequently, $C\,'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R\,'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P\,'(x)$ approximates the profit obtained by producing and selling one additional item.

Example 14.5.6: Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x, 0 \leq x \leq 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \quad \text{for } 0 \leq x \leq 300.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution

First, find the marginal revenue function: $MR(x) = R\,'(x) = -0.06x + 9$.

Next, use $R\,'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R\,'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

Exercise 14.5.4

Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- Using $f(a+h) \approx f(a) + f\,'(a)h$, it is possible to estimate $f(a+h)$ given $f\,'(a)$ and $f(a)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.
- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

Glossary

acceleration
: is the rate of change of the velocity, that is, the derivative of velocity

amount of change
: the amount of a function $f(x)$ over an interval $[x, x+h] is f(x+h)-f(x)$

average rate of change
: is a function $f(x)$ over an interval $[x, x+h]$ is $\frac{f(x+h)-f(a)}{b-a}$

marginal cost
: is the derivative of the cost function, or the approximate cost of producing one more item

marginal revenue
: is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

marginal profit
: is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

population growth rate
: is the derivative of the population with respect to time

speed
: is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

Answers

Exercise 14.5.1
Use the same process as in the preceding example.

Exercise 14.5.1
-4.4

Exercise 14.5.2
Find $v(3)$ and look at the sign.

Exercise 14.5.2
left to right

Exercise 14.5.3
Use $P(3) \approx P(0) + 3P'(0)$

Exercise 14.5.3
3,300

Exercise 14.5.4
Use $P'(100)$ to approximate $P(101) - P(100)$.

Exercise 14.5.4
$2

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14.5E: Exercises for Section 3.4

In exercises 1 - 3, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.

b. Determine the time intervals when the object is slowing down or speeding up.

1) $s(t) = 2t^3 - 3t^2 - 12t + 8$

2) $s(t) = 2t^3 - 15t^2 + 36t - 10$

Answer

a. $v(t) = 6t^2 - 30t + 36, \quad a(t) = 12t - 30$;
b. speeds up for $(2, 2.5) \cup (3, \infty)$, slows down for $(0, 2) \cup (2.5, 3)$

3) $s(t) = \dfrac{t}{1 + t^2}$

4) A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.

a. Find the velocity of the rocket 3 seconds after being fired.

b. Find the acceleration of the rocket 3 seconds after being fired.

Answer

a. $464 \ \text{ft/s}^2$
b. $-32 \ \text{ft/s}^2$

5) A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

a. Determine how long it takes for the ball to hit the ground.

b. Determine the velocity of the ball when it hits the ground.

6) The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.

a. Determine the velocity of the car when $s(t) = 0$.

b. Determine the velocity of the car when $s(t) = 14$.

Answer

a. 5 ft/s
b. 9 ft/s

7) The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.

a. Determine the velocity of the bird at $t = 1$ sec.

b. Determine the acceleration of the bird at $t = 1$ sec.

c. Determine the acceleration of the bird when the velocity equals 0.

8) A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building.

The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

a. Find the velocity of the potato after 0.5 s and 5.75 s.

b. Find the speed of the potato at 0.5 s and 5.75 s.

c. Determine when the potato reaches its maximum height.

d. Find the acceleration of the potato at 0.5 s and 1.5 s.

e. Determine how long the potato is in the air.

f. Determine the velocity of the potato upon hitting the ground.

Answer

a. 84 ft/s, –84 ft/s
b. 84 ft/s
c. $\frac{25}{8}$ s
d. -32 ft/s^2 in both cases
e. $\frac{1}{8}(25 + \sqrt{965})$ s
f. $-4\sqrt{965}$ ft/s

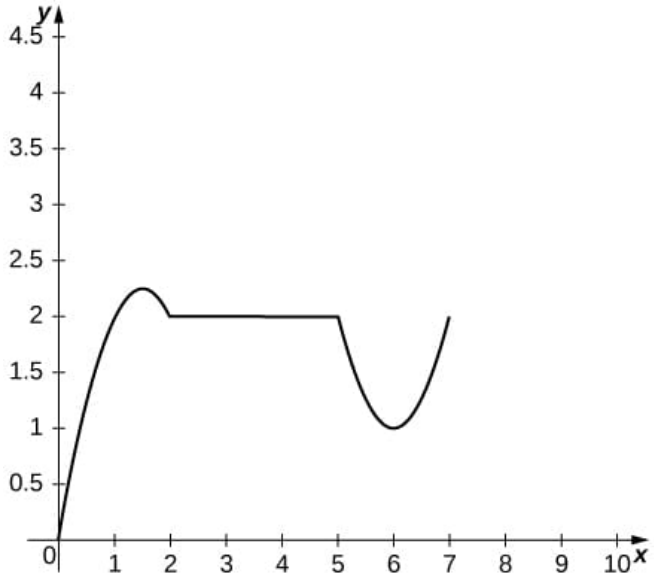
9) The position function $s(t) = t^3 - 8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

a. Determine the direction the train is traveling when $s(t) = 0$.

b. Determine the direction the train is traveling when $a(t) = 0$.

c. Determine the time intervals when the train is slowing down or speeding up.

10) The following graph shows the position $y = s(t)$ of an object moving along a straight line.



a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.

b. Sketch the graph of the velocity function.

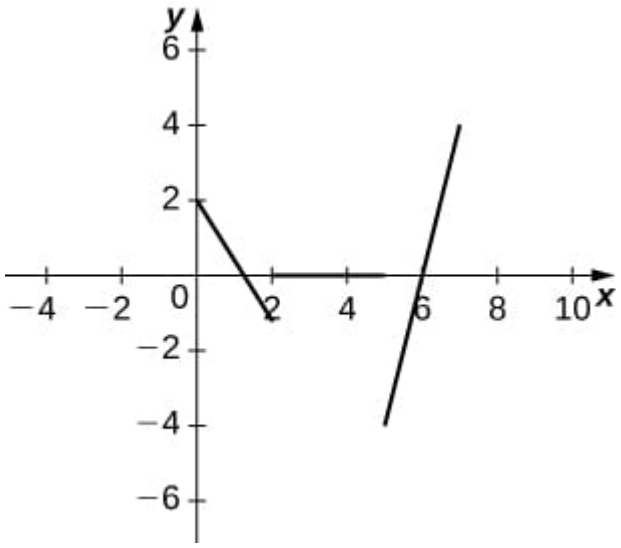
c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.

d. Determine the time intervals when the object is speeding up or slowing down.

Answer

a. Velocity is positive on $(0, 1.5) \cup (6, 7)$, negative on $(1.5, 2) \cup (5, 6)$, and zero on $(2, 5)$.

b.



c. Acceleration is positive on $(5, 7)$, negative on $(0, 2)$, and zero on $(2, 5)$.
d. The object is speeding up on $(6, 7) \cup (1.5, 2)$ and slowing down on $(0, 1.5) \cup (5, 6)$.

11) The cost function, in dollars, of a company that manufactures food processors is given by $C(x) = 200 + \frac{7}{x} + \frac{x}{27}$, where x is the number of food processors manufactured.

a. Find the marginal cost function.

b. Find the marginal cost of manufacturing 12 food processors.

c. Find the actual cost of manufacturing the thirteenth food processor.

12) The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p = 10 - 0.001x$.

a. Find the revenue function $R(x)$

b. Find the marginal revenue function.

c. Find the marginal revenue at $x = 2000$ and 5000.

Answer

a. $R(x) = 10x - 0.001x^2$
b. $R'(x) = 10 - 0.002x$
c. \$6 per item, \$0 per item

13) [T] A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.

b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

14) [T] In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.

Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75,000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

a. Find the marginal cost function.

b. Find the revenue and marginal revenue functions.

c. Find $R\,'(1000)$ and $R\,'(4000)$. Interpret the results.

d. Find the profit and marginal profit functions.

e. Find $P\,'(1000)$ and $P\,'(4000)$. Interpret the results.

Answer

a. $C\,'(x) = 65$
b. $R(x) = 143x - 0.03x^2, R\,'(x) = 143 - 0.06x$
c. $R\,'(1000) = 83, \quad R\,'(4000) = -97$. At a production level of 1000 cordless drills, revenue is increasing at a rate of \$83 per drill; at a production level of 4000 cordless drills, revenue is decreasing at a rate of \$97 per drill.
d. $P(x) = -0.03x^2 + 78x - 75000, \quad P\,'(x) = -0.06x + 78$
e. $P\,'(1000) = 18, \quad P\,'(4000) = -162$. At a production level of 1000 cordless drills, profit is increasing at a rate of \$18 per drill; at a production level of 4000 cordless drills, profit is decreasing at a rate of \$162 per drill.

15) A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

a. Find the rate of change function $P\,'(t)$ of the population function.

b. Find $P\,'(1), \ P\,'(2), \ P\,'(3)$, and $P\,'(4)$. Interpret what the results mean for the town.

c. Find $P''(1), \ P''(2), \ P''(3)$, and $P''(4)$. Interpret what the results mean for the town's population.

16) [T] A culture of bacteria grows in number according to the function $N(t) = 3000(1 + \frac{4t}{t^2 + 100})$, where t is measured in hours.

a. Find the rate of change of the number of bacteria.

b. Find $N\,'(0), \ N\,'(10), \ N\,'(20)$, and $N\,'(30)$.

c. Interpret the results in (b).

d. Find $N''(0), \ N''(10), \ N''(20)$, and $N''(30)$. Interpret what the answers imply about the bacteria population growth.

Answer

a. $N\,'(t) = 3000\left(\frac{-4t^2 + 400}{(t^2 + 100)^2}\right)$
b. $120, 0, -14.4, -9.6$
c. The bacteria population increases from time 0 to 10 hours; afterwards, the bacteria population decreases.
d. $0, -6, 0.384, 0.432$. The rate at which the bacteria is increasing is decreasing during the first 10 hours. Afterwards, the bacteria population is decreasing at a decreasing rate.

17) The centripetal force of an object of mass m is given by $F(r) = \frac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.

b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

Year Since 1800	Population (millions)
1	0.8975
11	1.040
21	1.264
31	1.516
41	1.661
51	2.000
61	2.634
71	3.272
81	3.911
91	4.422

Population of LondonSource: http://en.Wikipedia.org/wiki/Demographics_of_London

18) [T]

a. Using a calculator or a computer program, find the best-fit linear function to measure the population.

b. Find the derivative of the equation in a. and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

Answer

a. $P(t) = 0.03983 + 0.4280$
b. $P'(t) = 0.03983$. The population is increasing.
c. $P''(t) = 0$. The rate at which the population is increasing is constant.

19) [T]

a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.

b. Find the derivative of the equation and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut's position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	Position (m)
0	0
1	–1
2	–2
3	–5
4	–7
5	–14

20) [T]

a. Using a calculator or computer program, find the best-fit quadratic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

Answer

a. $p(t) = -0.6071x^2 + 0.4357x - 0.3571$
b. $p\,'(t) = -1.214x + 0.4357$. This is the velocity of the sensor.
c. $p''(t) = -1.214$. This is the acceleration of the sensor; it is a constant acceleration downward.

21) [T]

a. Using a calculator or computer program, find the best-fit cubic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

d. Using the result from c. explain why a cubic function is not a good choice for this problem.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

22) [T] The **Holling type I equation** is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.

a. Graph the Holling type I equation, given $a = 0.5$.

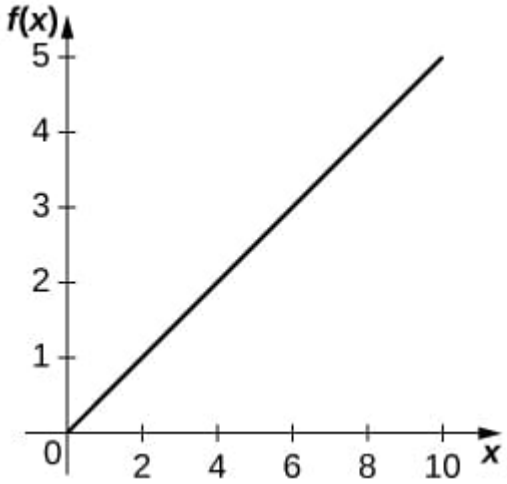
b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.

c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.

d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

Answer

a.



b. $f\,'(x) = a$. The more increase in prey, the more growth for predators.
c. $f''(x) = 0$. As the amount of prey increases, the rate at which the predator population growth increases is

constant.
d. This equation assumes that if there is more prey, the predator is able to increase consumption linearly. This assumption is unphysical because we would expect there to be some saturation point at which there is too much prey for the predator to consume adequately.

23) [T] The Holling type II equation is described by $f(x) = \frac{ax}{n+x}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type I and II equations?

b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.

c. Show that $f(n) = \frac{1}{2}a$ and interpret the meaning of the parameter n.

d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

24) [T] The Holling type III equation is described by $f(x) = \frac{ax^2}{n^2+x^2}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type II and III equations?

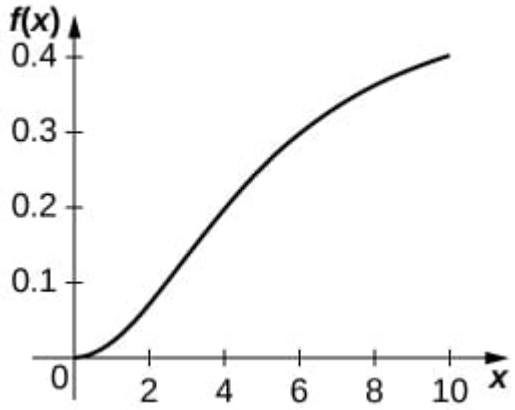
b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.

c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).

d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

Answer

a.



b. $f'(x) = \frac{2axn^2}{(n^2+x^2)^2}$. When the amount of prey increases, the predator growth increases.
c. $f''(x) = \frac{2an^2(n^2-3x^2)}{(n^2+x^2)^3}$. When the amount of prey is extremely small, the rate at which predator growth is increasing is increasing, but when the amount of prey reaches above a certain threshold, the rate at which predator growth is increasing begins to decrease.
d. At lower levels of prey, the prey is more easily able to avoid detection by the predator, so fewer prey individuals are consumed, resulting in less predator growth.

25) [T] The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	Population of lynx (hundreds)
20	10
5	15
65	55
95	60

Snowshoe Hare and Lynx PopulationsSource: http://www.biotopics.co.uk/newgcse/predatorprey.html.

a. Graph the data points and determine which Holling-type function fits the data best.

b. Using the meanings of the parameters a and n, determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.

c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

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14.6: Derivatives of Trigonometric Functions

Learning Objectives

- Find the derivatives of the sine and cosine function.
- Find the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

We see that by using $h = 0.01$,

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting

$$D(x) = \frac{\sin(x + 0.01) - \sin x}{0.01}$$

and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ (Figure $14.6.1$).

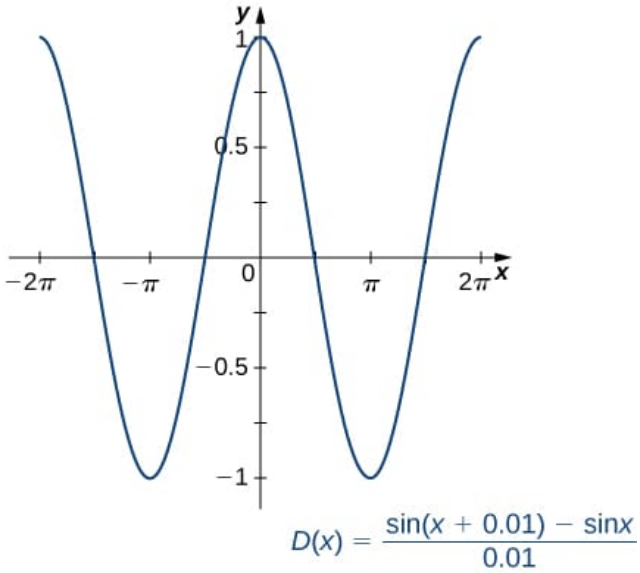


Figure 14.6.1: The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

The Derivatives of $\sin x$ and $\cos x$

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits:

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \text{ and } \lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{\sin h}{h}$ and $y = \frac{\cos h - 1}{h}$ are shown in Figure 14.6.2.

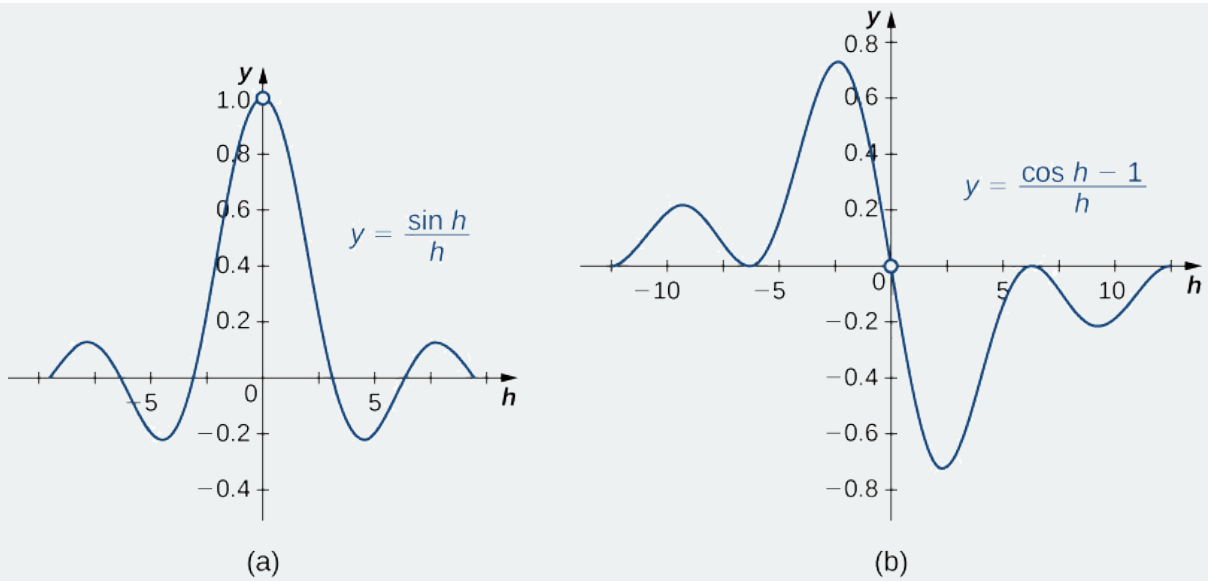


Figure 14.6.2: These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Apply the definition of the derivative.}\\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Use trig identity for the sine of the sum of two angles.}\\
&= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right) && \text{Regroup.}\\
&= \lim_{h\to 0} \left(\sin x \left(\frac{\cos h - 1}{h}\right) + (\cos x)\left(\frac{\sin h}{h}\right)\right) && \text{Factor out } \sin x \text{ and } \cos x\\
&= (\sin x) \lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right) + (\cos x) \lim_{h\to 0} \left(\frac{\sin h}{h}\right) && \text{Factor } \sin x \text{ and } \cos x \text{ out of limits.}\\
&= (\sin x)(0) + (\cos x)(1) && \text{Apply trig limit formulas.}\\
&= \cos x && \text{Simplify.}
\end{aligned}$$

□

Figure 14.6.3 shows the relationship between the graph of $f(x) = \sin x$ and its derivative $f\,'(x) = \cos x$. Notice that at the points where $f(x) = \sin x$ has a horizontal tangent, its derivative $f\,'(x) = \cos x$ takes on the value zero. We also see that where f$(x) = \sin x$ is increasing, $f\,'(x) = \cos x > 0$ and where $f(x) = \sin x$ is decreasing, $f\,'(x) = \cos x < 0$.

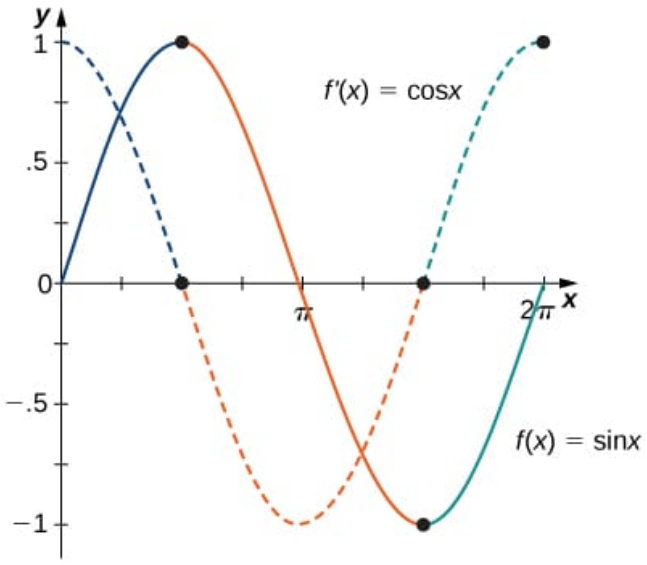


Figure 14.6.3: Where $f(x)$ has a maximum or a minimum, $f'(x) = 0$ that is, $f'(x) = 0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs

Example 14.6.1: Differentiating a Function Containing $\sin x$

Find the derivative of $f(x) = 5x^3 \sin x$.

Solution

Using the product rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}(5x^3) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 \\ &= 15x^2 \cdot \sin x + \cos x \cdot 5x^3. \end{aligned}$$

After simplifying, we obtain

$$f'(x) = 15x^2 \sin x + 5x^3 \cos x.$$

Exercise 14.6.1

Find the derivative of $f(x) = \sin x \cos x$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.6.2: Finding the Derivative of a Function Containing cos x

Find the derivative of $g(x) = \dfrac{\cos x}{4x^2}$.

Solution

By applying the quotient rule, we have

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{(4x^2)^2}.$$

Simplifying, we obtain

$$g'(x) = \frac{-4x^2 \sin x - 8x \cos x}{16x^4} = \frac{-x \sin x - 2 \cos x}{4x^3}.$$

Exercise 14.6.2

Find the derivative of $f(x) = \dfrac{x}{\cos x}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.6.3: An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 \sin t - t$ for $0 \leq t \leq 2\pi$. At what times is the particle at rest?

Solution

To determine when the particle is at rest, set $s\,'(t) = v(t) = 0$. Begin by finding $s\,'(t)$. We obtain

$$s'(t) = 2 \cos t - 1,$$

so we must solve

$$2 \cos t - 1 = 0 \text{ for } 0 \leq t \leq 2\pi.$$

The solutions to this equation are $t = \dfrac{\pi}{3}$ and $t = \dfrac{5\pi}{3}$. Thus the particle is at rest at times $t = \dfrac{\pi}{3}$ and $t = \dfrac{5\pi}{3}$.

Exercise 14.6.3

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2 \cos t$ for $0 \leq t \leq 2\pi$. At what times is the particle at rest?

Hint available at the end of this section.

Answer available at the end of this section.

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

Example 14.6.4: The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

Solution

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain

$$f\,'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1$, by the Pythagorean theorem, we now have

$$f'(x) = \frac{1}{\cos^2 x}$$

Finally, use the identity $\sec x = \dfrac{1}{\cos x}$ to obtain

$$f\,'(x) = \sec^2 x.$$

Exercise 14.6.4

Find the derivative of $f(x) = \cot x$.

Hint available at the end of this section.

Answer available at the end of this section.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{14.6.1}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{14.6.2}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{14.6.3}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x. \tag{14.6.4}$$

Example 14.6.5: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

$$f\left(\tfrac{\pi}{4}\right) = \cot\tfrac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $\left(\frac{\pi}{4}, 1\right)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

$$f\,'(x) = -\csc^2 x \text{ and } f\,'\left(\tfrac{\pi}{4}\right) = -\csc^2\left(\tfrac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

$$y - 1 = -2\left(x - \tfrac{\pi}{4}\right)$$

or equivalently,

$$y = -2x + 1 + \tfrac{\pi}{2}.$$

Example 14.6.6: Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x\tan x$.

Solution

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find

$$f\,'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x\tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x\cot x$, and by applying the product rule to the second term we obtain

$$\frac{d}{dx}(x\tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

$$f\,'(x) = -\csc x\cot x + \tan x + x\sec^2 x.$$

Exercise 14.6.5

Find the derivative of $f(x) = 2\tan x - 3\cot x$.

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 14.6.6

Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \dfrac{\pi}{6}$.

Hint available at the end of this section.

Answer available at the end of this section.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order

derivative of $\sin x$ and $\cos x$.

Example 14.6.7: Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution

Each step in the chain is straightforward:

$$\begin{aligned} y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x \end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of $\sin x$ equals $\sin x$, so

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \ldots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$

$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \ldots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.$$

Exercise 14.6.7

For $y = \cos x$, find $\frac{d^4y}{dx^4}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.6.8: Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

Exercise 14.6.8

For $y = \sin x$, find $\dfrac{d^{59}}{dx^{59}}(\sin x)$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.6.9: An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution

First find $v(t) = s\,'(t)$

$$v(t) = s'(t) = -\cos t.$$

Thus,

$$v\left(\tfrac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Next, find $a(t) = v\,'(t)$. Thus, $a(t) = v\,'(t) = \sin t$ and we have

$$a\left(\tfrac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Since $v\left(\tfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2} < 0$ and $a\left(\tfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is traveling. Consequently, the particle is slowing down.

Exercise 14.6.9

A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\tfrac{5\pi}{6}\right)$ and $a\left(\tfrac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- We can find the derivatives of $\sin x$ and $\cos x$ by using the definition of derivative and the limit formulas found earlier. The results are

  $$\frac{d}{dx}\big(\sin x\big) = \cos x \quad \text{and} \quad \frac{d}{dx}\big(\cos x\big) = -\sin x.$$

- With these two formulas, we can determine the derivatives of all six basic trigonometric functions.

Key Equations

- **Derivative of sine function**

$$\frac{d}{dx}(\sin x) = \cos x$$

- **Derivative of cosine function**

$$\frac{d}{dx}(\cos x) = -\sin x$$

- **Derivative of tangent function**

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

- **Derivative of cotangent function**

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

- **Derivative of secant function**

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

- **Derivative of cosecant function**

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Answers

Exercise 14.6.1

Don't forget to use the product rule.

Exercise 14.6.1

$$f'(x) = \cos^2 x - \sin^2 x$$

Exercise 14.6.2

Use the quotient rule.

Exercise 14.6.2

$f'(x) = \dfrac{\cos x + x \sin x}{\cos^2 x}$

Exercise 14.6.3

Use the previous example as a guide.

Exercise 14.6.3

$t = \dfrac{\pi}{3}, \quad t = \dfrac{2\pi}{3}$

Exercise 14.6.4

Rewrite $\cot x$ as $\dfrac{\cos x}{\sin x}$ and use the quotient rule.

Exercise 14.6.4

$f\,'(x) = -\csc^2 x$

Exercise 14.6.5

Use the rule for differentiating a constant multiple and the rule for differentiating a difference of two functions.

Exercise 14.6.5

$f\,'(x) = 2\sec^2 x + 3\csc^2 x$

Exercise 14.6.6

Evaluate the derivative at $x = \dfrac{\pi}{6}$.

Exercise 14.6.6

$\dfrac{4}{3}$

Exercise 14.6.7

See the previous example.

Exercise 14.6.7

$\cos x$

Exercise 14.6.8

$\frac{d^{59}}{dx^{59}}(\sin x) = \frac{d^{4 \cdot 14+3}}{dx^{4 \cdot 14+3}}(\sin x)$

Exercise 14.6.8

$-\cos x$

Exercise 14.6.9

Use Example 14.6.9 as a guide.

Exercise 14.6.9

$v\left(\frac{5\pi}{6}\right) = -\sqrt{3} < 0$ and $a\left(\frac{5\pi}{6}\right) = -1 < 0$. The block is speeding up.

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14.6E: Exercises for Section 3.5

In exercises 1 - 10, find $\frac{dy}{dx}$ for the given functions.

1) $y = x^2 - \sec x + 1$

Answer

$$\frac{dy}{dx} = 2x - \sec x \tan x$$

2) $y = 3 \csc x + \frac{5}{x}$

3) $y = x^2 \cot x$

Answer

$$\frac{dy}{dx} = 2x \cot x - x^2 \csc^2 x$$

4) $y = x - x^3 \sin x$

5) $y = \frac{\sec x}{x}$

Answer

$$\frac{dy}{dx} = \frac{x \sec x \tan x - \sec x}{x^2}$$

6) $y = \sin x \tan x$

7) $y = (x + \cos x)(1 - \sin x)$

Answer

$$\frac{dy}{dx} = (1 - \sin x)(1 - \sin x) - \cos x(x + \cos x)$$

8) $y = \frac{\tan x}{1 - \sec x}$

9) $y = \frac{1 - \cot x}{1 + \cot x}$

Answer

$$\frac{dy}{dx} = \frac{2 \csc^2 x}{(1 + \cot x)^2}$$

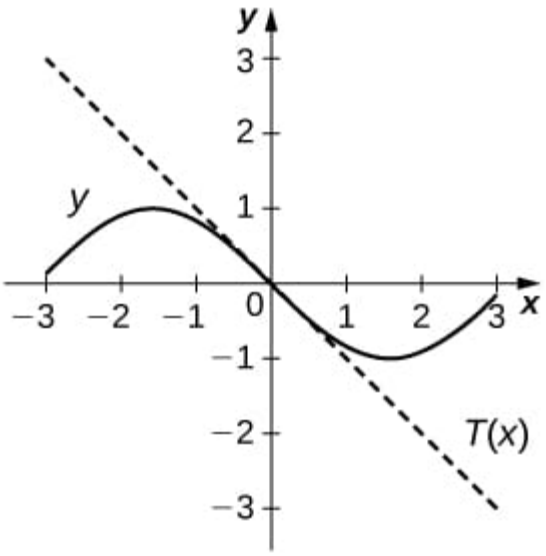
10) $y = (\cos x)(1 + \csc x)$

In exercises 11 - 16, find the equation of the tangent line to each of the given functions at the indicated values of x. Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

11) [T] $f(x) = -\sin x, \quad x = 0$

Answer

$$y = -x$$

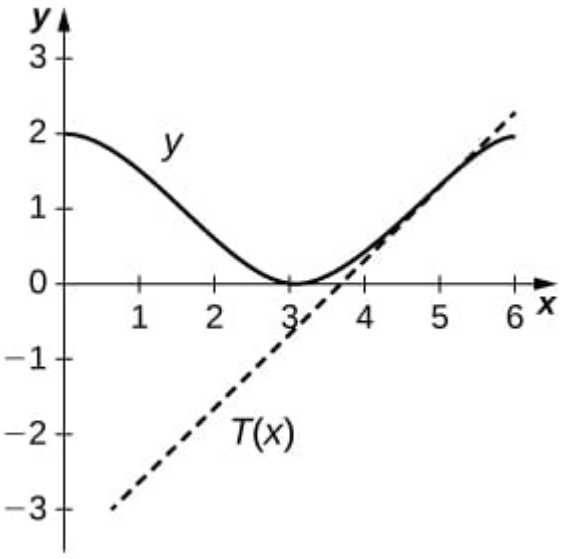


12) [T] $f(x) = \csc x, \quad x = \frac{\pi}{2}$

13) [T] $f(x) = 1 + \cos x, \quad x = \frac{3\pi}{2}$

Answer

$y = x + \frac{2-3\pi}{2}$

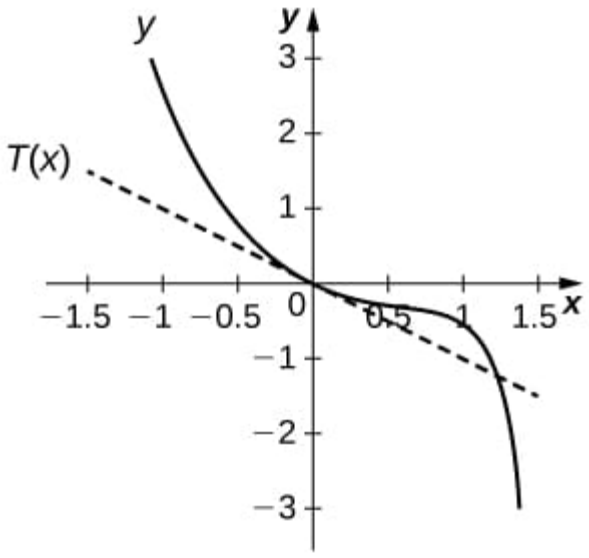


14) [T] $f(x) = \sec x, \quad x = \frac{\pi}{4}$

15) [T] $f(x) = x^2 - \tan x, \quad x = 0$

Answer

$y = -x$



16) [T] $f(x) = 5\cot x, \quad x = \frac{\pi}{4}$

In exercises 17 - 22, find $\frac{d^2y}{dx^2}$ for the given functions.

17) $y = x\sin x - \cos x$

Answer

$$\frac{d^2y}{dx^2} = 3\cos x - x\sin x$$

18) $y = \sin x \cos x$

19) $y = x - \frac{1}{2}\sin x$

Answer

$$\frac{d^2y}{dx^2} = \frac{1}{2}\sin x$$

20) $y = \dfrac{1}{x} + \tan x$

21) $y = 2\csc x$

Answer

$$\frac{d^2y}{dx^2} = 2\csc(x)\left(\csc^2(x) + \cot^2(x)\right)$$

22) $y = \sec^2 x$

23) Find all x values on the graph of $f(x) = -3\sin x \cos x$ where the tangent line is horizontal.

Answer

$x = \dfrac{(2n+1)\pi}{4}$,where n is an integer

24) Find all x values on the graph of $f(x) = x - 2\cos x$ for \(0

25) Let $f(x) = \cot x$. Determine the points on the graph of f for \(0

Answer

$\left(\frac{\pi}{4}, 1\right), \quad \left(\frac{3\pi}{4}, -1\right), \quad \left(\frac{5\pi}{4}, 1\right), \quad \left(\frac{7\pi}{4}, -1\right)$

26) [T] A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6\cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

27) Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a\cos t + b\sin t$. Find the constants a and b such that when the velocity is 3 cm/s, $s = 0$ and $t = 0$.

Answer

$a = 0, \quad b = 3$

28) After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t) = -5\cos t$ cm at t seconds after the jump.

a. Sketch one period of the position function for $t \geq 0$.

b. Find the velocity function.

c. Sketch one period of the velocity function for $t \geq 0$.

d. Determine the times when the velocity is 0 over one period.

e. Find the acceleration function.

f. Sketch one period of the acceleration function for $t \geq 0$.

29) The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y = 10 + 5\sin x$ where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find y' and determine the intervals where the number of burgers being sold is increasing.

Answer

$y' = 5\cos(x)$, increasing on $\left(0, \frac{\pi}{2}\right), \quad \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, and $\left(\frac{7\pi}{2}, 12\right)$

30) [T] The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y(t) = 0.5 + 0.3\cos t$, where t is months since January. Find y' and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For exercises 31 - 33, use the quotient rule to derive the given equations.

31) $\frac{d}{dx}(\cot x) = -\csc^2 x$

32) $\frac{d}{dx}(\sec x) = \sec x \tan x$

33) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

34) Use the definition of derivative and the identity $\cos(x + h) = \cos x \cos h - \sin x \sin h$ to prove that $\frac{d}{dx}(\cos x) = -\sin x$.

For exercises 35 - 39, find the requested higher-order derivative for the given functions.

35) $\frac{d^3y}{dx^3}$ of $y = 3\cos x$

Answer

$\frac{d^3y}{dx^3} = 3\sin x$

36) $\frac{d^2y}{dx^2}$ of $y = 3\sin x + x^2\cos x$

37) $\frac{d^4y}{dx^4}$ of $y = 5\cos x$

Answer

$$\frac{d^4y}{dx^4} = 5\cos x$$

38) $\frac{d^2y}{dx^2}$ of $y = \sec x + \cot x$

39) $\frac{d^3y}{dx^3}$ of $y = x^{10} - \sec x$

Answer

$$\frac{d^3y}{dx^3} = 720x^7 - 5\tan(x)\sec^3(x) - \tan^3(x)\sec(x)$$

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14.7: The Chain Rule

Learning Objectives

- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
- Recognize the chain rule for a composition of three or more functions.
- Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions ($x^n, \sin x, \cos x$, etc.) as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin(x^3)$ or $k(x) = \sqrt{3x^2 + 1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the *chain rule*, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin(x^3)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x. Consequently, we want to know how $\sin(x^3)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin(x^3)$. This chain reaction gives us hints as to what is involved in computing the derivative of $\sin(x^3)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin(x^3)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u = x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin(x^3)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin(x^3)$.

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x - a}$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a$. That is,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}(x^3)\Big|_{x=a} = 3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^3$ and observing that as $x \to a, u \to a^3$:

$$\begin{aligned} \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} &= \lim_{u \to a^3} \frac{\sin u - \sin(a^3)}{u - a^3} \\ &= \frac{d}{du}(\sin u)\Big|_{u=a^3} \\ &= \cos(a^3) \end{aligned}.$$

Thus, $h'(a) = \cos(a^3) \cdot 3a^2$.

In other words, if $h(x) = \sin(x^3)$, then $h'(x) = \cos(x^3) \cdot 3x^2$. Thus, if we think of $h(x) = \sin(x^3)$ as the composition $(f \circ g)(x) = f\big(g(x)\big)$ where $f(x) = \sin x$ and $g(x) = x^3$, then the derivative of $h(x) = \sin(x^3)$ is the product of the derivative of $g(x) = x^3$ and the derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3$. At this point, we anticipate that for $h(x) = \sin\big(g(x)\big)$, it is quite likely that $h'(x) = \cos\big(g(x)\big)g'(x)$. As we determined above, this is the case for $h(x) = \sin(x^3)$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f\big(g(x)\big)$$

is given by

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f\big(g(x)\big)$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'\big(g(x)\big)$.
3. Find $g'(x)$.
4. Write $h'(x) = f'\big(g(x)\big) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = \big(g(x)\big)^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = \big(g(x)\big)^n$ as $f\big(g(x)\big)$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'\big(g(x)\big) = n\big(g(x)\big)^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$$

Rule: Power Rule for Composition of Functions (General Power Rule)

For all values of x for which the derivative is defined, if

$$h(x) = \big(g(x)\big)^n,$$

Then

$$h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x). \tag{14.7.1}$$

Example 14.7.1: Using the Chain and Power Rules

Find the derivative of $h(x) = \dfrac{1}{(3x^2+1)^2}$.

Solution

First, rewrite $h(x) = \dfrac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$.

Applying the power rule with $g(x) = 3x^2 + 1$, we have

$$h'(x) = -2(3x^2+1)^{-3} \cdot 6x.$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2+1)^3}$$

Exercise 14.7.1

Find the derivative of $h(x) = (2x^3 + 2x - 1)^4$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.7.2: Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example 14.7.3: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \dfrac{1}{(3x-5)^2}$ at $x = 2$.

Solution

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \dfrac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x-5)^{-2}$ and apply the power rule to obtain

$$h'(x) = -2(3x-5)^{-3}(3) = -6(3x-5)^{-3}.$$

By substituting, we have $h'(2) = -6(3(2)-5)^{-3} = -6$.

Therefore, the line has equation $y - 1 = -6(x-2)$. Rewriting, the equation of the line is $y = -6x + 13$.

Exercise 14.7.2

Find the equation of the line tangent to the graph of $f(x) = (x^2-2)^3$ at $x = -2$.

Hint available at the end of this section.

Answer available at the end of this section.

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

Example 14.7.4: Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos\big(g(x)\big)$.

Solution

Think of $h(x) = \cos\big(g(x)\big)$ as $f\big(g(x)\big)$ where $f(x) = \cos x$. Since $f'(x) = -\sin x$, we have $f'\big(g(x)\big) = -\sin\big(g(x)\big)$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'\big(g(x)\big)\cdot g'(x) && \text{Apply the chain rule.} \\ &= -\sin\big(g(x)\big)\cdot g'(x) && \text{Substitute } f'\big(g(x)\big) = -\sin\big(g(x)\big). \end{aligned}$$

Thus, the derivative of $h(x) = \cos\big(g(x)\big)$ is given by $h'(x) = -\sin\big(g(x)\big)\cdot g'(x)$.

In the following example we apply the rule that we have just derived.

Example 14.7.5: Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos(5x^2)$.

Solution

Let $g(x) = 5x^2$. Then $g'(x) = 10x$. Using the result from the previous example,

$$h'(x) = -\sin(5x^2)\cdot 10x = -10x\sin(5x^2)$$

Example 14.7.6: Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec(4x^5 + 2x)$.

Solution

Apply the chain rule to $h(x) = \sec\big(g(x)\big)$ to obtain

$$h'(x) = \sec(g(x))\tan\big(g(x)\big) \cdot g'(x).$$

In this problem, $g(x) = 4x^5 + 2x$, so we have $g'(x) = 20x^4 + 2$. Therefore, we obtain

$$h'(x) = \sec(4x^5 + 2x)\tan(4x^5 + 2x)(20x^4 + 2) = (20x^4 + 2)\sec(4x^5 + 2x)\tan(4x^5 + 2x).$$

Exercise 14.7.3

Find the derivative of $h(x) = \sin(7x + 2)$.

Hint available at the end of this section.

Answer available at the end of this section.

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in the examples above. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$\frac{d}{dx}\Big(\sin(g(x))\Big) = \cos(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\sin u\Big) = \cos u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\cos(g(x))\Big) = -\sin(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\cos u\Big) = -\sin u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\tan(g(x))\Big) = \sec^2(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\tan u\Big) = \sec^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\cot(g(x))\Big) = -\csc^2(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\cot u\Big) = -\csc^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\sec(g(x))\Big) = \sec(g(x))\tan(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\sec\ u\Big) = \sec\ u \tan u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\csc(g(x))\Big) = -\csc(g(x))\cot(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\csc\ u\Big) = -\csc\ u \cot u \cdot \frac{du}{dx}.$

Example 14.7.7: Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x + 1)^5(3x - 2)^7$.

Solution

First apply the product rule, then apply the chain rule to each term of the product.

$$
\begin{aligned}
h'(x) &= \frac{d}{dx}\Big((2x+1)^5\Big)\cdot(3x-2)^7 + \frac{d}{dx}\Big((3x-2)^7\Big)\cdot(2x+1)^5 && \text{Apply the product rule.}\\
&= 5(2x+1)^4\cdot 2\cdot(3x-2)^7 + 7(3x-2)^6\cdot 3\cdot(2x+1)^5 && \text{Apply the chain rule.}\\
&= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.}\\
&= (2x+1)^4(3x-2)^6(10(3x-2)+21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6\\
&= (2x+1)^4(3x-2)^6(72x+1) && \text{Simplify.}
\end{aligned}
$$

Exercise 14.7.4

Find the derivative of $h(x) = \dfrac{x}{(2x+3)^3}$.

Hint available at the end of this section.

Answer available at the end of this section.

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

$$k(x) = h\Big(f\big(g(x)\big)\Big).$$

Then, applying the chain rule once we obtain

$$k'(x) = \frac{d}{dx}\Big(h\big(f\big(g(x)\big)\big)\Big) = h'\Big(f\big(g(x)\big)\Big)\cdot\frac{d}{dx}\Big(f\big(g(x)\big)\Big).$$

Applying the chain rule again, we obtain

$$k'(x) = h'\Big(f\big(g(x)\big)\Big)\cdot f'\big(g(x)\big)\cdot g'(x).$$

Rule: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h\Big(f\big(g(x)\big)\Big),$$

then

$$k'(x) = h'\Big(f\big(g(x)\big)\Big)\cdot f'\big(g(x)\big)\cdot g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example 14.7.8: Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4(7x^2 + 1)$.

Solution

First, rewrite $k(x)$ as

$$k(x) = \Big(\cos(7x^2 + 1)\Big)^4.$$

Then apply the chain rule several times.

$$\begin{aligned}
k'(x) &= 4(\cos(7x^2+1))^3 \cdot \frac{d}{dx}\Big(\cos(7x^2+1)\Big) && \text{Apply the chain rule.}\\
&= 4(\cos(7x^2+1))^3(-\sin(7x^2+1)) \cdot \frac{d}{dx}\Big(7x^2+1\Big) && \text{Apply the chain rule.}\\
&= 4(\cos(7x^2+1))^3(-\sin(7x^2+1))(14x) && \text{Apply the chain rule.}\\
&= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{Simplify}
\end{aligned}$$

Exercise 14.7.5

Find the derivative of $h(x) = \sin^6(x^3)$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.7.9: Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

Solution

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s(t)$. Thus,

$$v(t) = s'(t) = 2\cos(2t) - 3\sin(3t).$$

Proof of Chain Rule

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a) = \lim_{x\to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{x - a}.$$

Rewriting, we obtain

$$h'(a) = \lim_{x\to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

Although it is clear that

$$\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=g'(a),$$

it is not obvious that

$$\lim_{x\to a}\frac{f\big(g(x)\big)-f\big(g(a)\big)}{g(x)-g(a)}=f'\big(g(a)\big).$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x\to a}g(x)=g(a).$$

Next, make the substitution $y=g(x)$and $b=g(a)$ and use change of variables in the limit to obtain

$$\lim_{x\to a}\frac{f\big(g(x)\big)-f\big(g(a)\big)}{g(x)-g(a)}=\lim_{y\to b}\frac{f(y)-f(b)}{y-b}=f'(b)=f'\big(g(a)\big).$$

Finally,

$$h'(a)=\lim_{x\to a}\frac{f\big(g(x)\big)-f\big(g(a)\big)}{g(x)-g(a)}\cdot\frac{g(x)-g(a)}{x-a}=f'\big(g(a)\big)\cdot g'(a).$$

□

Example 14.7.10: Using the Chain Rule with Functional Values

Let $h(x)=f\big(g(x)\big)$. If $g(1)=4, g'(1)=3$, and $f'(4)=7$, find $h'(1)$.

Solution

Use the chain rule, then substitute.

$$\begin{aligned} h'(1) &= f'\big(g(1)\big)\cdot g'(1) && \text{Apply the chain rule.}\\ &= f'(4)\cdot 3 && \text{Substitute } g(1)=4 \text{ and } g'(1)=3.\\ &= 7\cdot 3 && \text{Substitute } f'(4)=7.\\ &= 21 && \text{Simplify.}\end{aligned}$$

Exercise 14.7.6

Given $h(x)=f(g(x))$. If $g(2)=-3, g'(2)=4$, and $f'(-3)=7$, find $h'(2)$.

Hint available at the end of this section.

Answer available at the end of this section.

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x)=f(g(x))$, let $u=g(x)$ and $y=h(x)=f(u)$. Thus,

$$h'(x)=\frac{dy}{dx}$$

$$f'(g(x)) = f'(u) = \frac{dy}{du}$$

and

$$g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'\big(g(x)\big) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 14.7.11: Taking a Derivative Using Leibniz's Notation I

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\
&= 5u^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = \frac{2}{(3x+2)^2}. \\
&= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } u = \frac{x}{3x+2}. \\
&= \frac{10x^4}{(3x+2)^6} && \text{Simplify.}
\end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example 14.7.12: Taking a Derivative Using Leibniz's Notation II

Find the derivative of $y = \tan(4x^2 - 3x + 1)$.

Solution

First, let $u = 4x^2 - 3x + 1$. Then $y = \tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:

$$\frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= \sec^2 u \cdot (8x - 3) && \text{Use } \frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u. \\ &= \sec^2(4x^2 - 3x + 1) \cdot (8x - 3) && \text{Substitute } u = 4x^2 - 3x + 1. \end{aligned}$$

Exercise 14.7.7

Use Leibniz's notation to find the derivative of $y = \cos(x^3)$. Make sure that the final answer is expressed entirely in terms of the variable x.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h(x) = f\big(g(x)\big)$,

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

  In Leibniz's notation this rule takes the form

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:

$$\text{If } h(x) = \big(g(x)\big)^n, \text{ then } h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x).$$

- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h(x) = f\Big(g\big(k(x)\big)\Big)$, then $h'(x) = f'\Big(g\big(k(x)\big)\Big) \cdot g'\big(k(x)\big) \cdot k'(x)$.

Key Equations

- **The chain rule**

  $h'(x) = f'\big(g(x)\big) \cdot g'(x)$

- **The power rule for functions**

  $h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$

Glossary

chain rule
: the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

Answers

Exercise 14.7.1

Use the General Power Rule (Equation 14.7.1) with $g(x) = 2x^3 + 2x - 1$.

Exercise 14.7.1

$h'(x) = 4(2x^3 + 2x - 1)^3(6x^2 + 2) = 8(3x^2 + 1)(2x^3 + 2x - 1)^3$

Exercise 14.7.2

Use the preceding example as a guide.

Exercise 14.7.2

$y = -48x - 88$

Exercise 14.7.3

Apply the chain rule to $h(x) = \sin\big(g(x)\big)$ first and then use $g(x) = 7x + 2$.

Exercise 14.7.3

$h'(x) = 7\cos(7x + 2)$

Exercise 14.7.4

Start out by applying the quotient rule. Remember to use the chain rule to differentiate the denominator.

Exercise 14.7.4

$h'(x) = \dfrac{3 - 4x}{(2x + 3)^4}$

Exercise 14.7.5

Rewrite $h(x) = \sin^6(x^3) = \big(\sin(x^3)\big)^6$ and use Example 14.7.8 as a guide.

Exercise 14.7.5

$h'(x) = 18x^2 \sin^5(x^3)\cos(x^3)$

Exercise 14.7.6

Follow Example 14.7.10.

Exercise 14.7.6

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Exercise 14.7.7

Let $u = x^3$.

Exercise 14.7.7

$\dfrac{dy}{dx} = -3x^2 \sin(x^3).$

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14.7E: Exercises for Section 3.6

In exercises 1 - 6, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

1) $y = 3u - 6, \quad u = 2x^2$

2) $y = 6u^3, \quad u = 7x - 4$

Answer

$$\frac{dy}{dx} = 18u^2 \cdot 7 = 18(7x-4)^2 \cdot 7 = 126(7x-4)^2$$

3) $y = \sin u, \quad u = 5x - 1$

4) $y = \cos u, \quad u = -\frac{x}{8}$

Answer

$$\frac{dy}{dx} = -\sin u \cdot \left(-\tfrac{1}{8}\right) = \tfrac{1}{8}\sin(-\tfrac{x}{8})$$

5) $y = \tan u, \quad u = 9x + 2$

6) $y = \sqrt{4u+3}, \quad u = x^2 - 6x$

Answer

$$\frac{dy}{dx} = \frac{8x-24}{2\sqrt{4u+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and

b. find $\frac{dy}{dx}$ as a function of x.

7) $y = (3x-2)^6$

8) $y = (3x^2+1)^3$

Answer

a. $f(u) = u^3, \quad u = 3x^2 + 1;$

b. $\frac{dy}{dx} = 18x(3x^2+1)^2$

9) $y = \sin^5(x)$

10) $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

Answer

a. $f(u) = u^7, \quad u = \frac{x}{7} + \frac{7}{x};$

b. $\frac{dy}{dx} = 7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \cdot \left(\frac{1}{7} - \frac{7}{x^2}\right)$

11) $y = \tan(\sec x)$

12) $y = \csc(\pi x + 1)$

Answer

a. $f(u) = \csc u, \quad u = \pi x + 1;$

b. $\dfrac{dy}{dx} = -\pi \csc(\pi x + 1) \cdot \cot(\pi x + 1)$

13) $y = \cot^2 x$

14) $y = -6 \sin^{-3} x$

Answer

a. $f(u) = -6u^{-3}, \quad u = \sin x;$

b. $\dfrac{dy}{dx} = 18 \sin^{-4} x \cdot \cos x$

In exercises 15 - 24, find $\dfrac{dy}{dx}$ for each function.

15) $y = (3x^2 + 3x - 1)^4$

16) $y = (5 - 2x)^{-2}$

Answer

$$\frac{dy}{dx} = \frac{4}{(5 - 2x)^3}$$

17) $y = \cos^3(\pi x)$

18) $y = (2x^3 - x^2 + 6x + 1)^3$

Answer

$$\frac{dy}{dx} = 6(2x^3 - x^2 + 6x + 1)^2 \cdot (3x^2 - x + 3)$$

19) $y = \dfrac{1}{\sin^2(x)}$

20) $y = \Big(\tan x + \sin x\Big)^{-3}$

Answer

$$\frac{dy}{dx} = -3\Big(\tan x + \sin x\Big)^{-4} \cdot (\sec^2 x + \cos x)$$

21) $y = x^2 \cos^4 x$

22) $y = \sin(\cos 7x)$

Answer

$$\frac{dy}{dx} = -7 \cos(\cos 7x) \cdot \sin 7x$$

23) $y = \sqrt{6 + \sec \pi x^2}$

24) $y = \cot^3(4x + 1)$

Answer

$$\frac{dy}{dx} = -12 \cot^2(4x + 1) \cdot \csc^2(4x + 1)$$

25) Let $y = \big[f(x)\big]^3$ and suppose that $f\,'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1$. Find $f(1)$.

26) Let $y = \left(f(x) + 5x^2\right)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1$. Find $f\,'(-1)$

Answer

$f\,'(-1) = 10\frac{3}{4}$

27) Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f\,'(4)$.

28) [T] Find the equation of the tangent line to $y = -\sin(\frac{x}{2})$ at the origin. Use a calculator to graph the function and the tangent line together.

Answer

$y = -\frac{1}{2}x$

29) [T] Find the equation of the tangent line to $y = \left(3x + \frac{1}{x}\right)^2$ at the point $(1, 16)$. Use a calculator to graph the function and the tangent line together.

30) Find the x -coordinates at which the tangent line to $y = \left(x - \frac{6}{x}\right)^8$ is horizontal.

Answer

$x = \pm\sqrt{6}$

31) [T] Find an equation of the line that is normal to $g(\theta) = \sin^2(\pi\theta)$ at the point $\left(\frac{1}{4}, \frac{1}{2}\right)$. Use a calculator to graph the function and the normal line together.

For exercises 32 - 39, use the information in the following table to find $h\,'(a)$ at the given value for a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	–2	3	0
2	4	4	1	–1
3	3	–3	2	3

32) $h(x) = f\big(g(x)\big); \quad a = 0$

Answer

$h'(0) = 10$

33) $h(x) = g\big(f(x)\big); \quad a = 0$

34) $h(x) = \left(x^4 + g(x)\right)^{-2}; \quad a = 1$

Answer

$h'(1) = -\frac{1}{8}$

35) $h(x) = \left(\dfrac{f(x)}{g(x)}\right)^2; \quad a = 3$

36) $h(x) = f\big(x + f(x)\big); \quad a = 1$

Answer

$h'(1) = -4$

37) $h(x) = \left(1 + g(x)\right)^3; \quad a = 2$

38) $h(x) = g\big(2 + f(x^2)\big); \quad a = 1$

Answer

$h'(1) = -12$

39) $h(x) = f\big(g(\sin x)\big); \quad a = 0$

40) [T] The position function of a freight train is given by $s(t) = 100(t+1)^{-2}$, with s in meters and t in seconds. At time $t = 6$ s, find the train's

a. velocity and

b. acceleration.

c. Considering your results in parts a. and b., is the train speeding up or slowing down?

Answer

a. $v(6) = -\frac{200}{343}$ m/s,

b. $a(6) = \frac{600}{2401}\ \text{m/s}^2$,

c. The train is slowing down since velocity and acceleration have opposite signs.

41) [T] A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches:

$$s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right).$$

a. Determine the position of the spring at $t = 1.5$ s.

b. Find the velocity of the spring at $t = 1.5$ s.

42) [T] The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

a. Find the marginal cost $C'(x)$.

b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.

c. Use your result in part b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

Answer

a. $C'(x) = 0.0003x^2 - 0.04x + 3$

b. $\frac{dC}{dt} = 100 \cdot (0.0003x^2 - 0.04x + 3) = 100 \cdot (0.0003(1600 + 100t)^2 - 0.04(1600 + 100t) + 3) = 300t^2 + 9200t + 70700$

c. Approximately $90,300 per week

43) [T] The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r = 2 - \frac{100}{(t+7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.

b. Use your result in part a. to find the rate at which the area is expanding at $t = 4$ s.

44) [T] The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r = \frac{1}{(t+1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.

b. Use your result in part a. to find the rate at which the volume is changing at $t = 1$ min.

Answer

a. $\frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3} = -\frac{8\pi\left(\frac{1}{(t+1)^2} - \frac{1}{12}\right)^2}{(t+1)^3}$

b. The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft^3/min

45) [T] The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10\cos\left[\frac{\pi}{12}(x-2)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

46) [T] The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Answer

2.3 ft/hr

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14.8: Derivatives of Inverse Functions

Learning Objectives

- Calculate the derivative of an inverse function.
- Recognize the derivatives of the standard inverse trigonometric functions.

In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. Figure $14.8.1$ shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $(a,\ f^{-1}(a))$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of

$$\left(f^{-1}\right)'(a) = \frac{p}{q}.$$

This point corresponds to a point $(f^{-1}(a),\ a)$ on the graph of $f(x)$ having a tangent line with a slope of

$$f'\left(f^{-1}(a)\right) = \frac{q}{p}.$$

Thus, if $f^{-1}(x)$ is differentiable at a, then it must be the case that

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

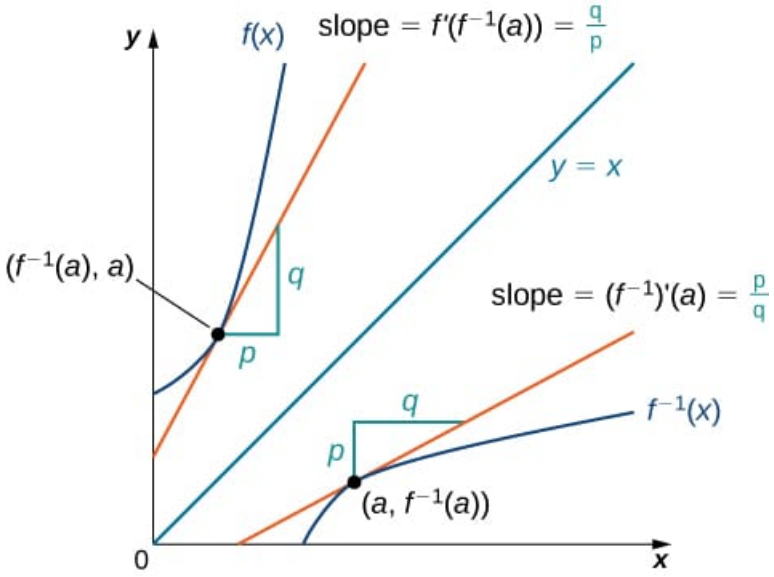


Figure $14.8.1$:The tangent lines of a function and its inverse are related; so, too, are the derivatives of these functions.

We may also derive the formula for the derivative of the inverse by first recalling that $x = f\left(f^{-1}(x)\right)$. Then by differentiating both sides of this equation (using the chain rule on the right), we obtain

$$1 = f'\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x).$$

Solving for $\left(f^{-1}\right)'(x)$, we obtain

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

We summarize this result in the following theorem.

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}. \tag{14.8.1}$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'\left(g(x)\right)}. \tag{14.8.2}$$

Example 14.8.1: Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \dfrac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution

The inverse of $g(x) = \dfrac{x+2}{x}$ is $f(x) = \dfrac{2}{x-1}$.

We will use Equation 14.8.2 and begin by finding $f'(x)$. Thus,

$$f'(x) = \frac{-2}{(x-1)^2}$$

and

$$f'\left(g(x)\right) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\dfrac{x+2}{x} - 1\right)^2} = -\frac{x^2}{2}.$$

Finally,

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

$$g'(x) = -\frac{2}{x^2}.$$

Exercise 14.8.1

Use the inverse function theorem to find the derivative of $g(x) = \dfrac{1}{x+2}$. Compare the result obtained by differentiating $g(x)$ directly.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.8.2: Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

Solution

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g\,'(x) = \dfrac{1}{f'\big(g(x)\big)}$, begin by finding $f\,'(x)$. Thus,

$$f'(x) = 3x^2$$

and

$$f'\big(g(x)\big) = 3\big(\sqrt[3]{x}\big)^2 = 3x^{2/3}$$

Finally,

$$g'(x) = \frac{1}{3x^{2/3}}.$$

If we were to differentiate $g(x)$ directly, using the power rule, we would first rewrite $g(x) = \sqrt[3]{x}$ as a power of x to get,

$$g(x) = x^{1/3}$$

Then we would differentiate using the power rule to obtain

$$g'(x) = \tfrac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}.$$

Exercise 14.8.2

Find the derivative of $g(x) = \sqrt[5]{x}$ by applying the inverse function theorem.

Hint available at the end of this section.

Answer available at the end of this section.

From the previous example, we see that we can use the inverse function theorem to extend the power rule to exponents of the form $\dfrac{1}{n}$, where n is a positive integer. This extension will ultimately allow us to differentiate x^q, where q is any rational number.

Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}.$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.$$

Proof

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g\,'(x) = \dfrac{1}{f'\left(g(x)\right)}$, begin by finding $f\,'(x)$. Thus,

$$f\,'(x) = nx^{n-1} \text{ and } f\,'\left(g(x)\right) = n\left(x^{1/n}\right)^{n-1} = nx^{(n-1)/n}.$$

Finally,

$$g\,'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{(1-n)/n} = \frac{1}{n}x^{(1/n)-1}.$$

To differentiate $x^{m/n}$ we must rewrite it as $(x^{1/n})^m$ and apply the chain rule. Thus,

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{d}{dx}\left(\left(x^{1/n}\right)^m\right) = m\left(x^{1/n}\right)^{m-1} \cdot \frac{1}{n}x^{(1/n)-1} = \frac{m}{n}x^{(m/n)-1}.$$

□

Example 14.8.3: Applying the Power Rule to a Rational Power

Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

Solution

First find $\dfrac{dy}{dx}$ and evaluate it at $x = 8$. Since

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$$

and

$$\left.\frac{dy}{dx}\right|_{x=8} = \frac{1}{3}$$

the slope of the tangent line to the graph at $x = 8$ is $\frac{1}{3}$.

Substituting $x = 8$ into the original function, we obtain $y = 4$. Thus, the tangent line passes through the point $(8, 4)$. Substituting into the point-slope formula for a line, we obtain the tangent line

$$y = \tfrac{1}{3}x + \tfrac{4}{3}.$$

Exercise 14.8.3

Find the derivative of $s(t) = \sqrt{2t+1}$.

Hint available at the end of this section.

Answer available at the end of this section.

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Example 14.8.4A: Derivative of the Inverse Sine Function

Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1} x$.

Solution

Since for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin x$ is the inverse of $g(x) = \sin^{-1} x$, begin by finding $f\,'(x)$. Since

$$f'(x) = \cos x$$

and

$$f'\big(g(x)\big) = \cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}$$

we see that

$$g'(x) = \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{\sqrt{1 - x^2}}$$

Analysis

To see that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$, consider the following argument. Set $\sin^{-1} x = \theta$. In this case, $\sin\theta = x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We begin by considering the case where $0 < \theta < \frac{\pi}{2}$. Since θ is an acute angle, we may construct a right triangle having acute angle θ, a hypotenuse of length 1 and the side opposite angle θ having length x. From the Pythagorean theorem, the side adjacent to angle θ has length $\sqrt{1 - x^2}$. This triangle is shown in Figure 14.8.2 Using the triangle, we see that $\cos(\sin^{-1} x) = \cos\theta = \sqrt{1 - x^2}$.

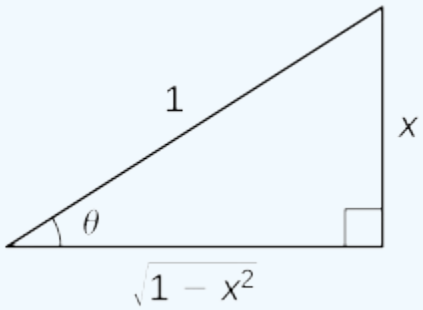


Figure 14.8.2: Using a right triangle having acute angle θ, a hypotenuse of length 1, and the side opposite angle θ having length x, we can see that $\cos(\sin^{-1} x) = \cos\theta = \sqrt{1 - x^2}$.

In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence

$$\cos\left(\sin^{-1} x\right) = \cos\theta = \cos(-\theta) = \sqrt{1 - x^2}.$$

Now if $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}, x = 1$ or $x = -1$, and since in either case $\cos\theta = 0$ and $\sqrt{1 - x^2} = 0$, we have

$$\cos\left(\sin^{-1}x\right) = \cos\theta = \sqrt{1-x^2}.$$

Consequently, in all cases,

$$\cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}.$$

Example 14.8.4B: Applying the Chain Rule to the Inverse Sine Function

Apply the chain rule to the formula derived in Example 14.8.4A to find the derivative of $h(x) = \sin^{-1}\left(g(x)\right)$ and use this result to find the derivative of $h(x) = \sin^{-1}(2x^3)$.

Solution

Applying the chain rule to $h(x) = \sin^{-1}\left(g(x)\right)$, we have

$$h\,'(x) = \frac{1}{\sqrt{1-\left(g(x)\right)^2}}g\,'(x).$$

Now let $g(x) = 2x^3$, so $g\,'(x) = 6x^2$. Substituting into the previous result, we obtain

$$\begin{aligned} h'(x) &= \frac{1}{\sqrt{1-4x^6}}\cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}} \end{aligned}$$

Exercise 14.8.4

Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1}x$.

Hint available at the end of this section.

Answer available at the end of this section.

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} \tag{14.8.3}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}} \tag{14.8.4}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2} \tag{14.8.5}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+x^2} \tag{14.8.6}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}} \tag{14.8.7}$$

$$\frac{d}{dx}\left(\csc^{-1}x\right) = \frac{-1}{|x|\sqrt{x^2-1}} \tag{14.8.8}$$

Example 14.8.5A: Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}(x^2)$.

Solution

Let $g(x) = x^2$, so $g\,'(x) = 2x$. Substituting into Equation 14.8.5, we obtain

$$f\,'(x) = \frac{1}{1 + (x^2)^2} \cdot (2x).$$

Simplifying, we have

$$f\,'(x) = \frac{2x}{1 + x^4}.$$

Example 14.8.5B: Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h(x) = x^2 \sin^{-1} x$.

Solution

By applying the product rule, we have

$$h\,'(x) = 2x \sin^{-1} x + \frac{1}{\sqrt{1 - x^2}} \cdot x^2$$

Exercise 14.8.5

Find the derivative of $h(x) = \cos^{-1}(3x - 1)$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.8.6: Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution

Begin by differentiating $s(t)$ in order to find $v(t)$.Thus,

$$v(t) = s\,'(t) = \frac{1}{1 + \left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

$$v(t) = -\frac{1}{t^2 + 1}.$$

Thus, $v(1) = -\frac{1}{2}$.

Exercise 14.8.6

Find the equation of the line tangent to the graph of $f(x) = \sin^{-1} x$ at $x = 0$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The inverse function theorem allows us to compute derivatives of inverse functions without using the limit definition of the derivative.
- We can use the inverse function theorem to develop differentiation formulas for the inverse trigonometric functions.

Key Equations

- **Inverse function theorem**

  $(f^{-1})'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$ whenever $f'\left(f^{-1}(x)\right) \neq 0$ and $f(x)$ is differentiable.

- **Power rule with rational exponents**

  $\dfrac{d}{dx}\left(x^{m/n}\right) = \dfrac{m}{n}x^{(m/n)-1}.$

- **Derivative of inverse sine function**

  $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$

- **Derivative of inverse cosine function**

  $\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$

Derivative of inverse tangent function

$\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$

Derivative of inverse cotangent function

$\dfrac{d}{dx}\left(\cot^{-1} x\right) = \dfrac{-1}{1+x^2}$

Derivative of inverse secant function

$\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$

Derivative of inverse cosecant function

$\dfrac{d}{dx}\left(\csc^{-1} x\right) = \dfrac{-1}{|x|\sqrt{x^2-1}}$

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Paul Seeburger (Monroe Community College) added the second half of Example $14.8.2$.

Answers

Exercise $14.8.1$

Use the preceding example as a guide.

Exercise $14.8.1$

$g\,'(x) = -\dfrac{1}{(x+2)^2}$

Exercise $14.8.2$

$g(x)$ is the inverse of $f(x) = x^5$.

Exercise $14.8.2$

$g(x) = \frac{1}{5}x^{-4/5}$

Exercise $14.8.3$

Use the chain rule.

Exercise $14.8.3$

$s\,'(t) = (2t+1)^{-1/2}$

Exercise $14.8.4$

The inverse of $g(x)$ is $f(x) = \tan x$. Use Example $14.8.4A$ as a guide.

Exercise $14.8.4$

$g\,'(x) = \dfrac{1}{1+x^2}$

Exercise $14.8.5$

Use Equation $14.8.4$. with $g(x) = 3x - 1$

Exercise $14.8.5$

$h\,'(x) = \dfrac{-3}{\sqrt{6x - 9x^2}}$

Exercise $14.8.6$

$f\,'(0)$ is the slope of the tangent line.

Exercise $14.8.6$

$y = x$

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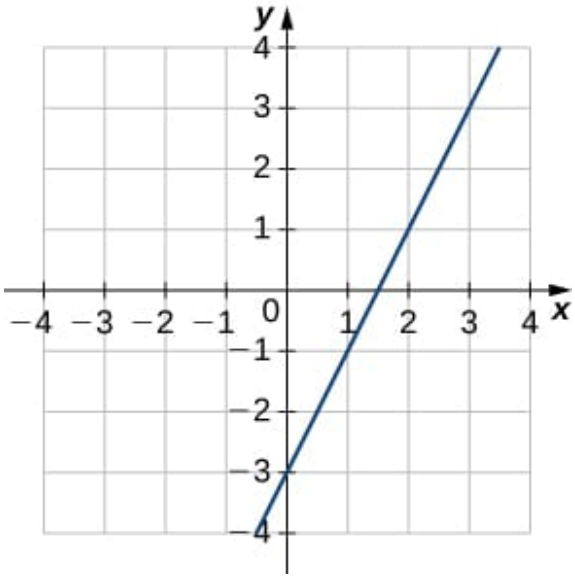
14.8E: Exercises for Section 3.7

In exercises 1 - 4, use the graph of $y = f(x)$ to

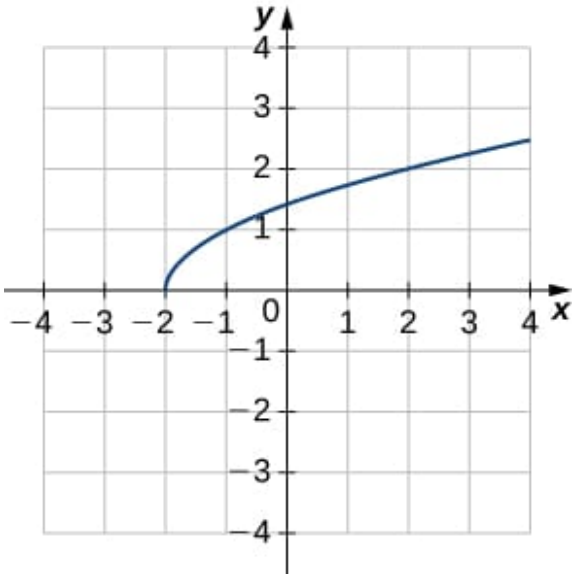
a. sketch the graph of $y = f^{-1}(x)$, and

b. use part a. to estimate $\left(f^{-1}\right)'(1)$.

1)

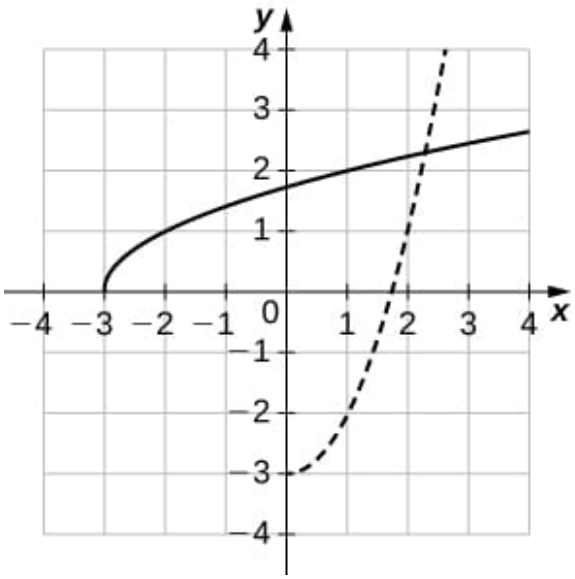


2)



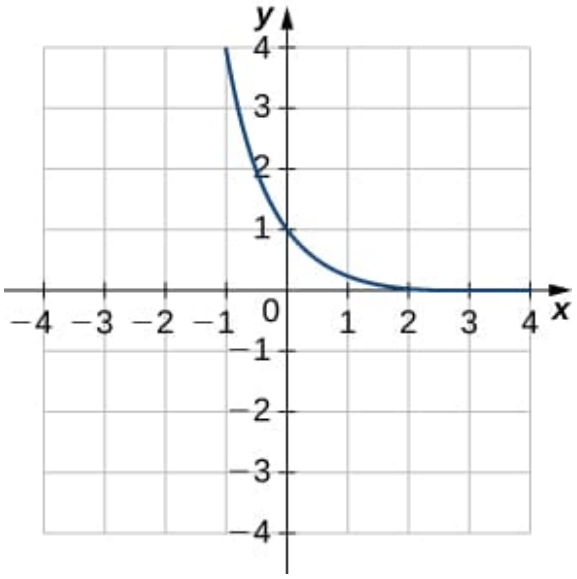
Answer

a.

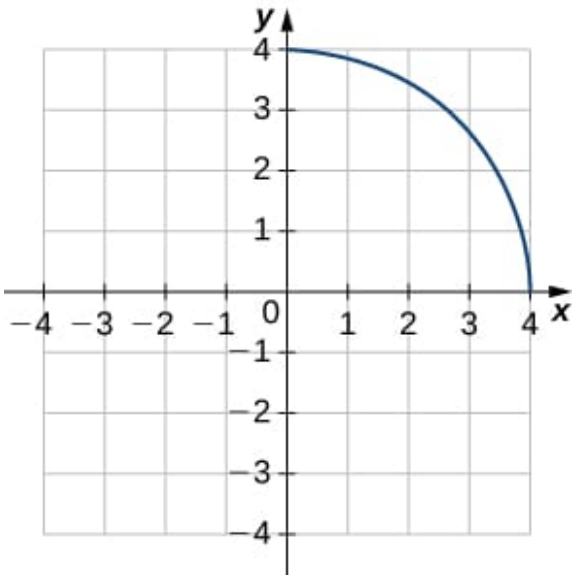


b. $(f^{-1})'(1) \approx 2$

3)

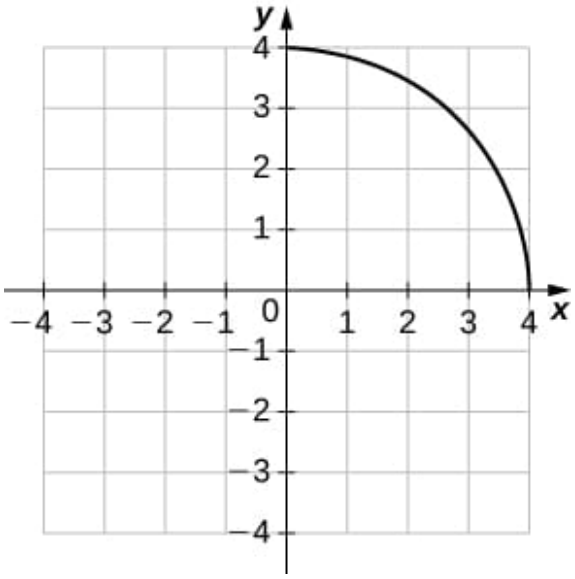


4)



Answer

a.



b. $(f^{-1})'(1) \approx -1/\sqrt{3}$

For exercises 5 - 8, use the given function $y = f(x)$ to find

a. $\frac{df}{dx}$ at $x = a$ and

b. $x = f^{-1}(y)$.

c. Then use part b. to find $\frac{df^{-1}}{dy}$ at $y = f(a)$.

5) $f(x) = 6x - 1, \quad x = -2$

6) $f(x) = 2x^3 - 3, \quad x = 1$

Answer

a. $\dfrac{df}{dx} = 6$

b. $x = f^{-1}(y) = \left(\dfrac{y+3}{2}\right)^{1/3}$

c. $\dfrac{df^{-1}}{dy} = \frac{1}{6}$

7) $f(x) = 9 - x^2, \quad 0 \le x \le 3, x = 2$

8) $f(x) = \sin x, \quad x = 0$

Answer

a. $\dfrac{df}{dx} = 1$

b. $x = f^{-1}(y) = \sin^{-1} y$

c. $\dfrac{df^{-1}}{dy} = 1$

For each function in exercises 9 - 14, find $\left(f^{-1}\right)'(a)$.

9) $f(x) = x^2 + 3x + 2, \quad x \ge -1, \quad a = 2$

10) $f(x) = x^3 + 2x + 3, \quad a = 0$

Answer

$\left(f^{-1}\right)'(1) = \frac{1}{5}$

11) $f(x) = x + \sqrt{x}, \quad a = 2$

12) $f(x) = x - \frac{2}{x}, \quad x < 0, \quad a = 1$

Answer

$\left(f^{-1}\right)'(1) = \frac{1}{3}$

13) $f(x) = x + \sin x, \quad a = 0$

14) $f(x) = \tan x + 3x^2, \quad a = 0$

Answer

$\left(f^{-1}\right)'(0) = 1$

For each function $y = f(x)$, given in exercises 15 - 19,

a. find the slope of the tangent line to its inverse function f^{-1} at the indicated point P, and

b. find the equation of the tangent line to the graph of f^{-1} at the indicated point.

15) $f(x) = \dfrac{4}{1+x^2}, \quad P(2,1)$

16) $f(x) = \sqrt{x-4}, \quad P(2,8)$

Answer

a. 4

b. $y = 4x$

17) $f(x) = (x^3+1)^4, \quad P(16,1)$

18) $f(x) = -x^3 - x + 2, \quad P(-8,2)$

Answer

a. $-\frac{1}{96}$
b. $y=-\frac{1}{13}x+\frac{18}{13}$

19) $f(x)=x^5+3x^3-4x-8, \quad P(-8,1)$

In exercises 20 - 29, find $\frac{dy}{dx}$ for the given function.

20) $y=\sin^{-1}(x^2)$

Answer

$$\frac{dy}{dx}=\frac{2x}{\sqrt{1-x^4}}$$

21) $y=\cos^{-1}\left(\sqrt{x}\right)$

22) $y=\sec^{-1}\left(\frac{1}{x}\right)$

Answer

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}$$

23) $y=\sqrt{\csc^{-1}x}$

24) $y=(1+\tan^{-1}x)^3$

Answer

$$\frac{dy}{dx}=\frac{3(1+\tan^{-1}x)^2}{1+x^2}$$

25) $y=\cos^{-1}(2x)\cdot\sin^{-1}(2x)$

26) $y=\dfrac{1}{\tan^{-1}(x)}$

Answer

$$\frac{dy}{dx}=\frac{-1}{(1+x^2)(\tan^{-1}x)^2}$$

27) $y=\sec^{-1}(-x)$

28) $y=\cot^{-1}\sqrt{4-x^2}$

Answer

$$\frac{dy}{dx}=\frac{x}{(5-x^2)\sqrt{4-x^2}}$$

29) $y=x\cdot\csc^{-1}x$

In exercises 30 - 35, use the given values to find $\left(f^{-1}\right)'(a)$.

30) $f(\pi)=0, f'(\pi)=-1, a=0$

Answer

$$\left(f^{-1}\right)'(0)=-1$$

31) $f(6)=2, \quad f'(6)=\frac{1}{3}, \quad a=2$

32) $f(\frac{1}{3})=-8, \quad f'(\frac{1}{3})=2, \quad a=-8$

Answer

$\left(f^{-1}\right)'(-8) = \frac{1}{2}$

33) $f(\sqrt{3}) = \frac{1}{2}, f'(\sqrt{3}) = \frac{2}{3}, a = \frac{1}{2}$

34) $f(1) = -3, \quad f'(1) = 10, \quad a = -3$

Answer

$\left(f^{-1}\right)'(-3) = \frac{1}{10}$

35) $f(1) = 0, \quad f'(1) = -2, \quad a = 0$

36) [T] The position of a moving hockey puck after t seconds is $s(t) = tan^{-1}t$ where s is in meters.

a. Find the velocity of the hockey puck at any time t.

b. Find the acceleration of the puck at any time t.

c. Evaluate parts a. and b. for $t = 2, \quad 4$,and 6 seconds.

d. What conclusion can be drawn from the results in c.?

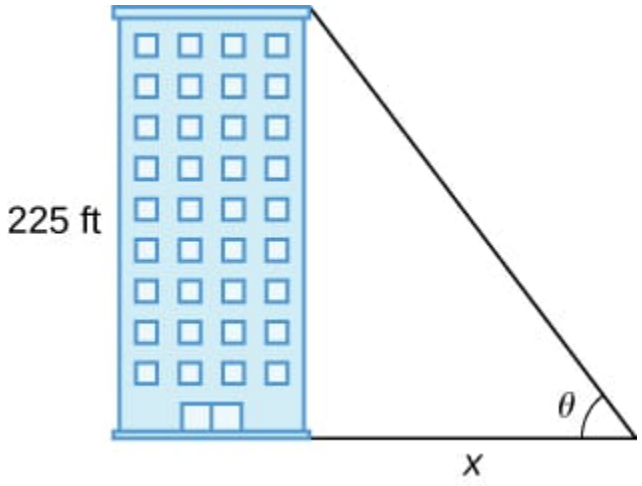
Answer

a. $v(t) = \dfrac{1}{1+t^2}$
b. $a(t) = \dfrac{-2t}{(1+t^2)^2}$
c. (a) $0.2, \quad 0.06, \quad 0.03$; (b) $-0.16, \quad -0.028, \quad -0.0088$

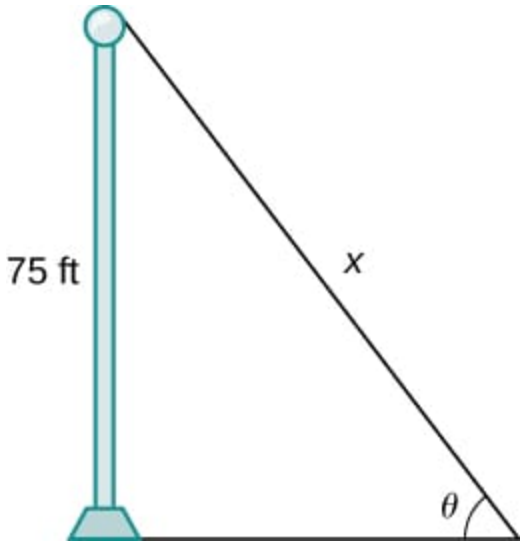
d. The hockey puck is decelerating/slowing down at 2, 4, and 6 seconds.

Solution:

37) [T] A building that is 225 feet tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 272$ feet.



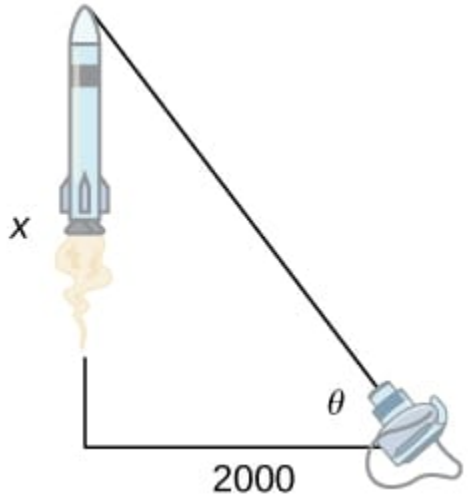
38) [T] A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



Answer

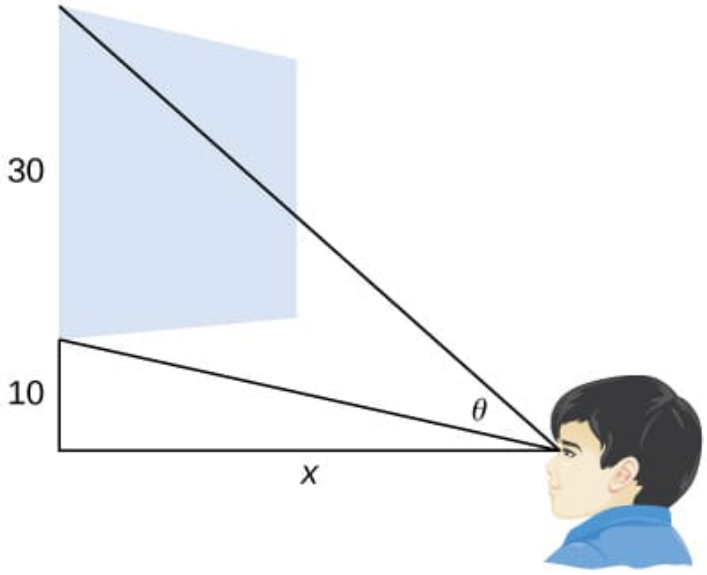
-0.0168 radians per foot

39) [T] A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by $\theta = \tan^{-1}\left(\frac{x}{2000}\right)$, where x is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.



40) [T] A local movie theater with a 30-foot-high screen that is 10 feet above a person's eye level when seated has a viewing angle θ (in radians) given by $\theta = \cot^{-1}\frac{x}{40} - \cot^{-1}\frac{x}{10}$,

where x is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.



a. Find $\frac{d\theta}{dx}$.

b. Evaluate $\frac{d\theta}{dx}$ for $x = 5,\ 10,\ 15$, and 20.

c. Interpret the results in part b.

d. Evaluate $\frac{d\theta}{dx}$ for $x = 25,\ 30,\ 35$, and 40.

e. Interpret the results in part d. At what distance x should the person stand to maximize his or her viewing angle?

Answer

a. $\frac{d\theta}{dx} = \frac{10}{100 + x^2} - \frac{40}{1600 + x^2}$

b. $\frac{18}{325},\ \frac{9}{340},\ \frac{42}{4745},\ 0$

c. As a person moves farther away from the screen, the viewing angle is increasing, which implies that as he or she moves farther away, his or her screen vision is widening. d. $-\frac{54}{12905},\ -\frac{3}{500},\ -\frac{198}{29945},\ -\frac{9}{1360}$

e. As the person moves beyond 20 feet from the screen, the viewing angle is decreasing. The optimal distance the person should stand for maximizing the viewing angle is 20 feet.

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14.9: Implicit Differentiation

Learning Objectives

- Find the derivative of a complicated function by using implicit differentiation.
- Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, we say that y is an explicit function of x. For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x. On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y implicitly in terms of x. For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

Implicit differentiation allows us to find slopes of tangents to curves that are clearly not functions (they fail the vertical line test). We are using the idea that portions of y are functions that satisfy the given equation, but that y is not actually a function of x.

In general, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

$$y = \sqrt{25 - x^2}$$

and

$$y = \begin{cases} \sqrt{25 - x^2}, & \text{if } -5 \leq x < 0 \\ -\sqrt{25 - x^2}, & \text{if } 0 \leq x \leq 5 \end{cases}$$

which are illustrated in Figure $14.9.1$, are just two of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.

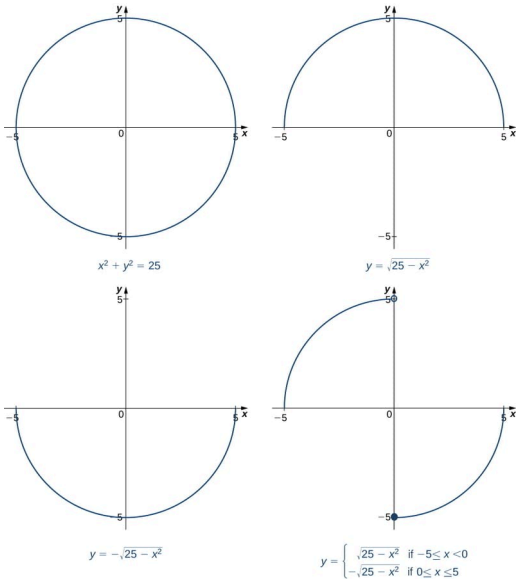


Figure 14.9.1:The equation $x^2 + y^2 = 25$ defines many functions implicitly.

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$, we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3$. On the other hand, if we want the slope of the tangent line at the point $(3, -4)$, we could use the derivative of $y = -\sqrt{25 - x^2}$. However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly. The process of finding $\dfrac{dy}{dx}$ using implicit differentiation is described in the following problem-solving strategy.

Problem-Solving Strategy: Implicit Differentiation

To perform *implicit differentiation* on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas
$$\frac{d}{dx}(\sin x) = \cos x$$
and
$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$
because we must use the chain rule to differentiate $\sin y$ with respect to x.
2. Rewrite the equation so that all terms containing dy/dx are on the left and all terms that do not contain dy/dx are on the right.
3. Factor out dy/dx on the left.
4. Solve for dy/dx by dividing both sides of the equation by an appropriate algebraic expression.

Example 14.9.1: Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

Solution

Follow the steps in the problem-solving strategy.

$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$	Step 1. Differentiate both sides of the equation.
$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=0$	Step 1.1. Use the sum rule on the left. On the right $\frac{d}{dx}(25)=0$.
$2x+2y\frac{dy}{dx}=0$	Step 1.2. Take the derivatives, so $\frac{d}{dx}(x^2)=2x$ and $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$.
$2y\frac{dy}{dx}=-2x$	Step 2. Keep the terms with $\frac{dy}{dx}$ on the left. Move the remaining terms to the right.
$\frac{dy}{dx}=-\frac{x}{y}$	Step 4. Divide both sides of the equation by $2y$. (Step 3 does not apply in this case.)

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example 14.9.2: Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3\sin y+y=4x+3$, find $\frac{dy}{dx}$.

Solution

$\frac{d}{dx}(x^3\sin y+y)=\frac{d}{dx}(4x+3)$	Step 1: Differentiate both sides of the equation.
$\frac{d}{dx}(x^3\sin y)+\frac{d}{dx}(y)=4$	Step 1.1: Apply the sum rule on the left. On the right, $\frac{d}{dx}(4x+3)=4$.
$\left(\frac{d}{dx}(x^3)\cdot\sin y+\frac{d}{dx}(\sin y)\cdot x^3\right)+\frac{dy}{dx}=4$	Step 1.2: Use the product rule to find $\frac{d}{dx}(x^3\sin y)$. Observe that $\frac{d}{dx}(y)=\frac{dy}{dx}$.
$3x^2\sin y+(\cos y\frac{dy}{dx})\cdot x^3+\frac{dy}{dx}=4$	Step 1.3: We know $\frac{d}{dx}(x^3)=3x^2$. Use the chain rule to obtain $\frac{d}{dx}(\sin y)=\cos y\frac{dy}{dx}$.
$x^3\cos y\frac{dy}{dx}+\frac{dy}{dx}=4-3x^2\sin y$	Step 2: Keep all terms containing $\frac{dy}{dx}$ on the left. Move all other terms to the right.
$\frac{dy}{dx}(x^3\cos y+1)=4-3x^2\sin y$	Step 3: Factor out $\frac{dy}{dx}$ on the left.
$\frac{dy}{dx}=\frac{4-3x^2\sin y}{x^3\cos y+1}$	Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by $x^3\cos y+1$.

Example 14.9.3: Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2y}{dx^2}$ if $x^2+y^2=25$.

Solution

In Example 14.9.1, we showed that $\frac{dy}{dx}=-\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(-\frac{x}{y}\right)$$ Differentiate both sides of $\frac{dy}{dx} = -\frac{x}{y}$.

$$= -\frac{\left(1 \cdot y - x\frac{dy}{dx}\right)}{y^2}$$ Use the quotient rule to find $\frac{d}{dy}\left(-\frac{x}{y}\right)$.

$$= \frac{-y + x\frac{dy}{dx}}{y^2}$$ Simplify.

$$= \frac{-y + x\left(-\frac{x}{y}\right)}{y^2}$$ Substitute $\frac{dy}{dx} = -\frac{x}{y}$.

$$= \frac{-y^2 - x^2}{y^3}$$ Simplify.

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2 + y^2 = 25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$.

Exercise 14.9.1

Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5 + \tan y = y^2 + 5x$.

Hint available at the end of this section.

Answer available at the end of this section.

Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Example 14.9.4: Finding a Tangent Line to a Circle

Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

Although we could find this equation without using implicit differentiation, using that method makes it much easier. In Example 14.9.1, we found $\frac{dy}{dx} = -\frac{x}{y}$.

The slope of the tangent line is found by substituting $(3, -4)$ into this expression. Consequently, the slope of the tangent line is $\frac{dy}{dx}\Big|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Using the point $(3, -4)$ and the slope $\frac{3}{4}$ in the point-slope equation of the line, we obtain the equation $y = \frac{3}{4}x - \frac{25}{4}$ (Figure 14.9.2).

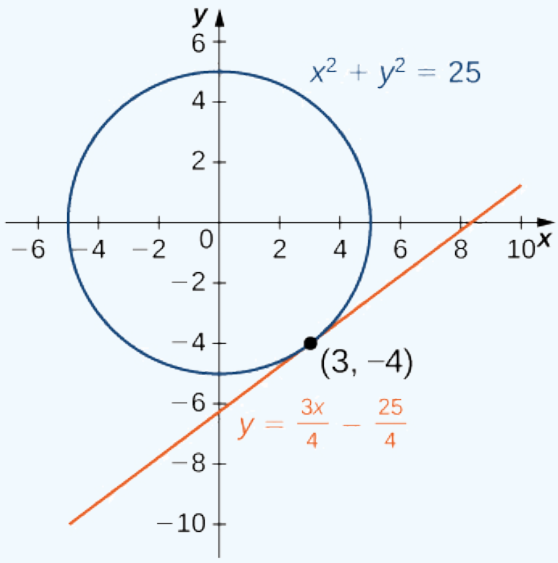


Figure 14.9.2: The line $y = \frac{3}{4}x - \frac{25}{4}$ is tangent to $x^2 + y^2 = 25$ at the point (3, –4).

Example 14.9.5: Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ (Figure 14.9.3). This curve is known as the folium (or leaf) of Descartes.

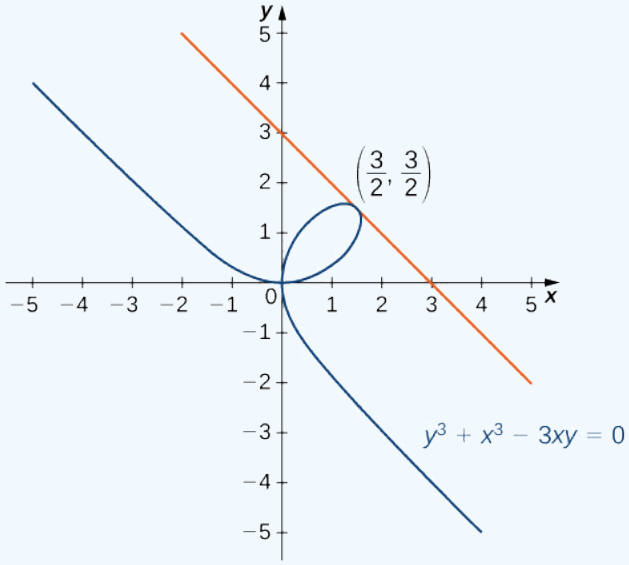


Figure 14.9.3: Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution

Begin by finding $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(y^3 + x^3 - 3xy\right) = \frac{d}{dx}\left(0\right)$$

$$3y^2\frac{dy}{dx} + 3x^2 - \left(3y + 3x\frac{dy}{dx}\right) = 0$$

$$3y^2\frac{dy}{dx}+3x^2-3y-3x\frac{dy}{dx}=0$$

$$(3y^2-3x)\frac{dy}{dx}=3y-3x^2$$

$$\frac{dy}{dx}=\frac{3y-3x^2}{3y^2-3x}.$$

Next, substitute $\left(\frac{3}{2},\frac{3}{2}\right)$ into $\frac{dy}{dx}=\frac{3y-3x^2}{3y^2-3x}$ to find the slope of the tangent line:

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2},\frac{3}{2}\right)}=-1.$$

Finally, substitute into the point-slope equation of the line to obtain

$$y=-x+3.$$

Example 14.9.6: Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2+25y^2=100$. The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x-axis toward $(0,0)$. If the rocket fires a missile when it is located at $\left(3,\frac{8}{5}\right)$, where will it intersect the x -axis?

Solution

To solve this problem, we must determine where the line tangent to the graph of

$4x^2+25y^2=100$ at $\left(3,\frac{8}{5}\right)$ intersects the x-axis. Begin by finding $\frac{dy}{dx}$ implicitly.

Differentiating, we have

$$8x+50y\frac{dy}{dx}=0.$$

Solving for $\frac{dy}{dx}$,

we have

$$\frac{dy}{dx}=-\frac{4x}{25y}.$$

The slope of the tangent line is $\left.\frac{dy}{dx}\right|_{\left(3,\frac{8}{5}\right)}=-\frac{3}{10}$. The equation of the tangent line is $y=-\frac{3}{10}x+\frac{5}{2}$. To determine where the line intersects the x-axis, solve $0=-\frac{3}{10}x+\frac{5}{2}$. The solution is $x=\frac{25}{3}$. The missile intersects the x-axis at the point $\left(\frac{25}{3},0\right)$.

Exercise 14.9.2

Find the equation of the line tangent to the hyperbola $x^2-y^2=16$ at the point $(5,3)$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- We use implicit differentiation to find derivatives of implicitly defined functions (functions defined by equations).
- By using implicit differentiation, we can find the equation of a tangent line to the graph of a curve.

Glossary

implicit differentiation

is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable y as a function) and solving for $\frac{dy}{dx}$

Answers

Exercise 14.9.1

Follow the problem solving strategy, remembering to apply the chain rule to differentiate $\tan y$ and y^2.

Exercise 14.9.1

$$\frac{dy}{dx} = \frac{5 - 20x^4}{\sec^2 y - 2y}$$

Exercise 14.9.2

$\frac{dy}{dx} = \frac{x}{y}$

Exercise 14.9.2

$y = \frac{5}{3}x - \frac{16}{3}$

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14.9E: Exercises for Section 3.8

In exercises 1 - 10, use implicit differentiation to find $\frac{dy}{dx}$.

1) $x^2 - y^2 = 4$

2) $6x^2 + 3y^2 = 12$

Answer

$$\frac{dy}{dx} = \frac{-2x}{y}$$

3) $x^2y = y - 7$

4) $3x^3 + 9xy^2 = 5x^3$

Answer

$$\frac{dy}{dx} = \frac{x}{3y} - \frac{y}{2x}$$

5) $xy - \cos(xy) = 1$

6) $y\sqrt{x+4} = xy + 8$

Answer

$$\frac{dy}{dx} = \frac{y - \dfrac{y}{2\sqrt{x+4}}}{\sqrt{x+4} - x}$$

7) $-xy - 2 = \frac{x}{7}$

8) $y\sin(xy) = y^2 + 2$

Answer

$$\frac{dy}{dx} = \frac{y^2\cos(xy)}{2y - \sin(xy) - xy\cos(xy)}$$

9) $(xy)^2 + 3x = y^2$

10) $x^3y + xy^3 = -8$

Answer

$$\frac{dy}{dx} = \frac{-3x^2y - y^3}{x^3 + 3xy^2}$$

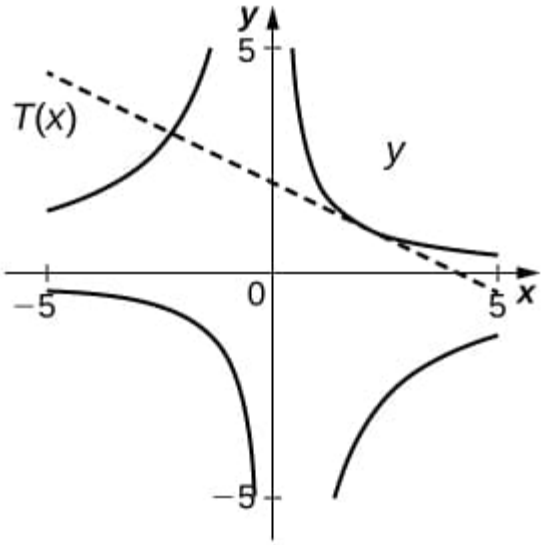
For exercises 11 - 16, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

11) [T] $x^4y - xy^3 = -2, \quad (-1, -1)$

12) [T] $x^2y^2 + 5xy = 14, \quad (2, 1)$

Answer

$$y = -\tfrac{1}{2}x + 2$$

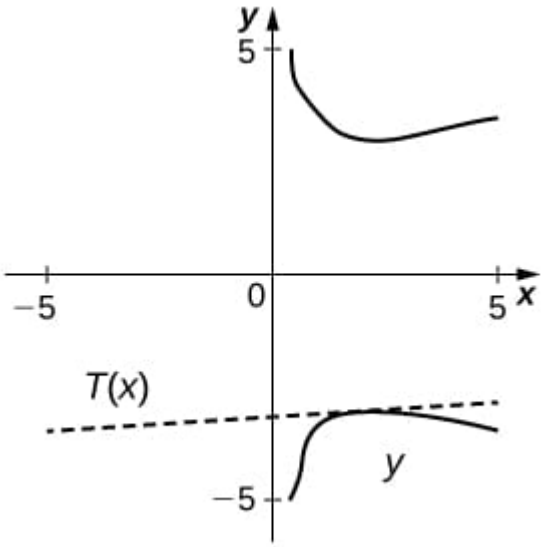


13) [T] $\tan(xy) = y, \quad (\frac{\pi}{4}, 1)$

14) [T] $xy^2 + \sin(\pi y) - 2x^2 = 10, \quad (2, -3)$

Answer

$y = \frac{1}{\pi+12}x - \frac{3\pi+38}{\pi+12}$

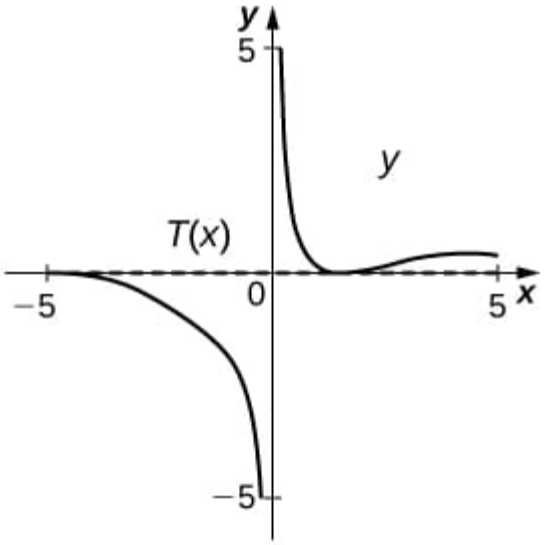


15) [T] $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y, \quad (1, 2)$

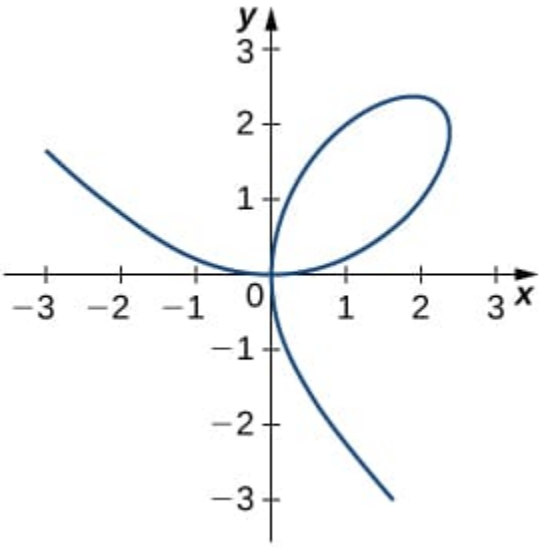
16) [T] $xy + \sin(x) = 1, \quad (\frac{\pi}{2}, 0)$

Answer

$y = 0$



17) [T] The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



a. Find the equation of the tangent line at the point $(2, 1)$. Graph the tangent line along with the folium.

b. Find the equation of the normal line to the tangent line in a. at the point $(2, 1)$.

18) For the equation $x^2 + 2xy - 3y^2 = 0$,

a. Find the equation of the normal to the tangent line at the point $(1, 1)$.

b. At what other point does the normal line in a. intersect the graph of the equation?

Answer

a. $y = -x + 2$
b. $(3, -1)$

19) Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

20) For the equation $x^2 + xy + y^2 = 7$,

a. Find the x-intercept(s).

b.Find the slope of the tangent line(s) at the x-intercept(s).

c. What does the value(s) in part b. indicate about the tangent line(s)?

Answer

a. $\left(\pm\sqrt{7}, 0\right)$
b. -2
c. They are parallel since the slope is the same at both intercepts.

21) Find the equation of the tangent line to the graph of the equation $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{6}$ at the point $\left(0, \frac{1}{2}\right)$.

22) Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

Answer

$y = -x + 1$

23) Find y' and y'' for $x^2 + 6xy - 2y^2 = 3$.

24) [T] The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4} = 3240$.

a. Find $\frac{dy}{dx}$ and evaluate at the point $(81, 16)$.

b. Interpret the result of a.

Answer

a. $\frac{dy}{dx} = -0.5926$
b. When \$81 is spent on labor and \$16 is spent on capital, the amount spent on capital is decreasing by \$0.5926 per \$1 spent on labor.

25) [T] The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $30x^{1/3}y^{2/3} = 360$.

(Both $x$and y are measured in thousands of dollars.)

a. Find $\frac{dy}{dx}$ and evaluate at the point $(27, 8)$.

b. Interpret the result of part a.

26) The volume of a right circular cone of radius x and height y is given by $V = \frac{1}{3}\pi x^2 y$. Suppose that the volume of the cone is $85\pi \text{ cm}^3$. Find $\frac{dy}{dx}$ when $x = 4$ and $y = 16$.

Answer

$\frac{dy}{dx} = -8$

For exercises 27 - 28, consider a closed rectangular box with a square base with side x and height y.

27) Find an equation for the surface area of the rectangular box, $S(x, y)$.

28) If the surface area of the rectangular box is 78 square feet, find $\frac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.

Answer

$\frac{dy}{dx} = -2.67$

In exercises 29 - 31, use implicit differentiation to determine y'. Does the answer agree with the formulas we have previously determined?

29) $x = \sin y$

30) $x = \cos y$

Answer

$$y' = -\frac{1}{\sqrt{1-x^2}}$$

31) $x = \tan y$

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14.10: Derivatives of Exponential and Logarithmic Functions

Learning Objectives

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in Introduction to Functions and Graphs, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x,\ b > 0$, is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n, where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1, b^{-n} = \dfrac{1}{b^n}$, for a positive integer n, and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t. These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the continuity of $B(x) = b^x, b > 0$, we may interpret b^r as $\lim\limits_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4, \quad 4^{3.1} < 4^\pi < 4^{3.2}, \quad 4^{3.14} < 4^\pi < 4^{3.15},$$

$$4^{3.141} < 4^\pi < 4^{3.142}, \quad 4^{3.1415} < 4^\pi < 4^{3.1416}, \quad \ldots.$$

As we see in the following table, $4^\pi \approx 77.88$.

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Approximating a Value of 4^π

We also assume that for $B(x) = b^x,\ b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in Introduction to Functions and Graphs. Figure $14.10.1$ provides graphs of the functions $y = 2^x,\ y = 3^x,\ y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function.**

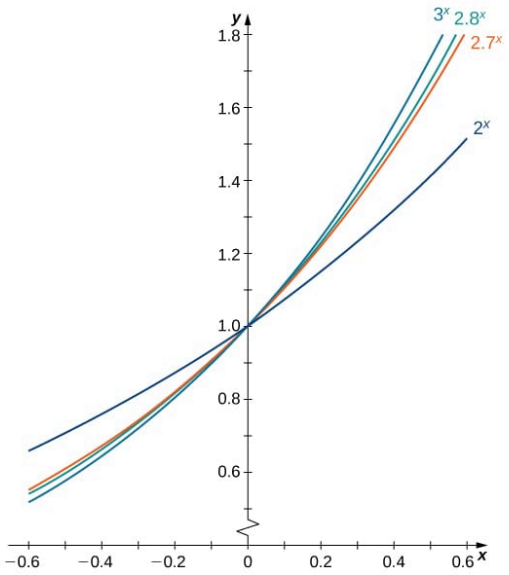


Figure $14.10.1$: The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$

to obtain the estimate

\[\frac{b^{−0.00001}−1}{−0.00001}

See the following table.

Table : Estimating the value of e

b	\(\frac{b^{−0.00001}−1}{−0.00001}	b	\(\frac{b^{−0.00001}−1}{−0.00001}
2	\(0.693145	2.7183	\(1.000002
2.7	\(0.993247	2.719	\(1.000259
2.71	\(0.996944	2.72	\(1.000627
2.718	\(0.999891	2.8	\(1.029614
2.7182	\(0.999965	3	\(1.098606

The evidence from the table suggests that \(2.7182

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in Figure $14.10.2$. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.

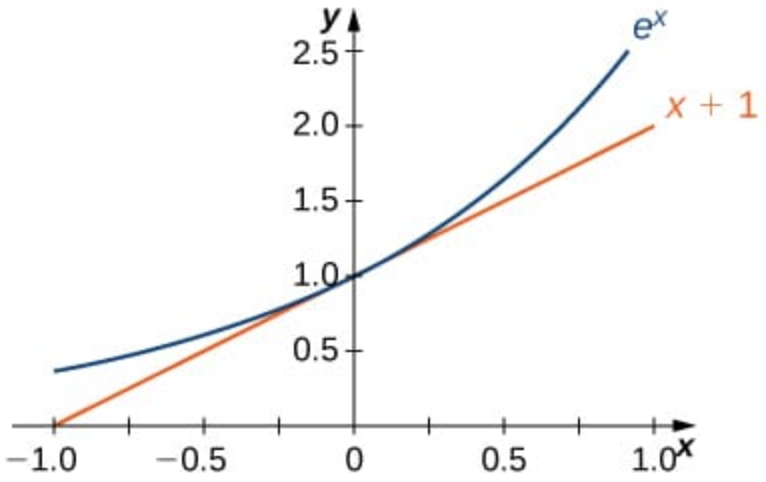


Figure 14.10.2: The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x,\ b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h\to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h\to 0} \frac{b^h - 1}{h}$$

Turning to $B'(x)$, we obtain the following.

$$\begin{aligned}
B'(x) &= \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.}\\
&= \lim_{h\to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h.\\
&= \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} && \text{Factor out } b^x.\\
&= b^x \lim_{h\to 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.}\\
&= b^x B'(0) && \text{Use } B'(0) = \lim_{h\to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h\to 0} \frac{b^h - 1}{h}.
\end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0).$$

For $E(x) = e^x,\ E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x,\ b > 0$, will be derived later.)

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$

Example 14.10.1: Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution:

Using the derivative formula and the chain rule,

$$f'(x) = e^{\tan(2x)} \frac{d}{dx}\Big(\tan(2x)\Big) = e^{\tan(2x)} \sec^2(2x) \cdot 2$$

Example 14.10.2: Combining Differentiation Rules

Find the derivative of $y = \dfrac{e^{x^2}}{x}$.

Solution

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

$$\begin{aligned} y' &= \frac{(e^{x^2} \cdot 2)x \cdot x - 1 \cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\ &= \frac{e^{x^2}(2x^2 - 1)}{x^2} && \text{Simplify.} \end{aligned}$$

Exercise 14.10.1

Find the derivative of $h(x) = xe^{2x}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.10.3: Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Exercise 14.10.2

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Hint available at the end of this section.

Answer available at the end of this section.

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Definition: The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$\frac{dy}{dx} = \frac{1}{x}.$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g\,'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)}g'(x).$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$

is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

$$h'(x) = \frac{1}{g(x)}g'(x). \tag{14.10.1}$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in Figure $14.10.3$.

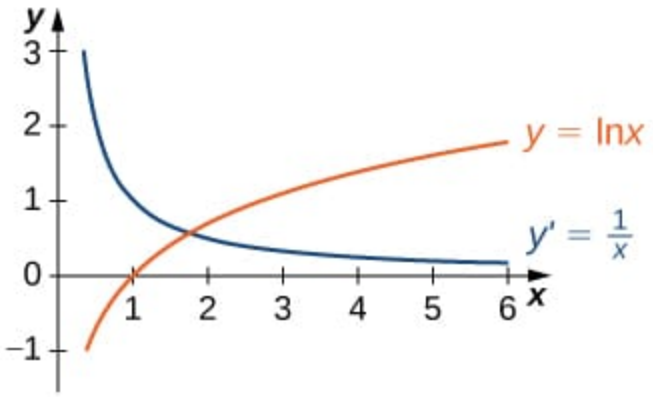


Figure 14.10.3: The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$

Example 14.10.4: Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution

Use Equation 14.10.1 directly.

$$f'(x) = \frac{1}{x^3 + 3x - 4} \cdot (3x^2 + 3) \qquad \text{Use } g(x) = x^3 + 3x - 4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x).$$

$$= \frac{3x^2 + 3}{x^3 + 3x - 4} \qquad \text{Rewrite.}$$

Example 14.10.5: Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right)$.

Solution

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

$$f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right) = 2 \ln x + \ln(\sin x) - \ln(2x + 1) \qquad \text{Apply properties of logarithms.}$$

$$f'(x) = \frac{2}{x} + \cot x - \frac{2}{2x + 1} \qquad \text{Apply sum rule and } h'(x) = \frac{1}{g(x)} g'(x).$$

Exercise 14.10.3

Differentiate: $f(x) = \ln(3x + 2)^5$.

Hint available at the end of this section.

Answer available at the end of this section.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ and $y = b^x$ for $b > 0,\ b \neq 1$.

Derivatives of General Exponential and Logarithmic Functions

Let $b > 0, b \neq 1$, and let $g(x)$ be a differentiable function.

i. If $y = \log_b x$, then

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,

$$h'(x) = \frac{g'(x)}{g(x) \ln b}. \tag{14.10.2}$$

ii. If $y = b^x$, then

$$\frac{dy}{dx} = b^x \ln b.$$

More generally, if $h(x) = b^{g(x)}$, then

$$h'(x) = b^{g(x)} g'(x) \ln b \tag{14.10.3}$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in Equation 14.10.2 now follows from the chain rule.

If $y = b^x$. then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y}\frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative (Equation 14.10.3) follows from the chain rule.

□

Example 14.10.6: Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x + 2}$.

Solution

Use the quotient rule and Note.

$$h'(x) = \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} \qquad \text{Apply the quotient rule.}$$

$$= \frac{2 \cdot 3^x \ln 3}{(3x + 2)^2} \qquad \text{Simplify.}$$

Example 14.10.7: Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1$. Using Equation 14.10.2, we see that

$$\frac{dy}{dx} = \frac{3}{(3x + 1)\ln 2}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

$$\frac{dy}{dx}\Big|_{x=1} = \frac{3}{4\ln 2} = \frac{3}{\ln 16}.$$

Exercise 14.10.4

Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Hint available at the end of this section.

Answer available at the end of this section.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 14.10.8: Using Logarithmic Differentiation

Find the derivative of $y = (2x^4 + 1)^{\tan x}$.

Solution

Use logarithmic differentiation to find this derivative.

$\ln y = \ln (2x^4+1)^{\tan x}$	Step 1. Take the natural logarithm of both sides.
$\ln y = \tan x \ln(2x^4+1)$	Step 2. Expand using properties of logarithms.
$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(2x^4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x$	Step 3. Differentiate both sides. Use the product rule on the right.
$\frac{dy}{dx} = y \cdot (\sec^2 x \ln(2x^4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x)$	Step 4. Multiply by y on both sides.
$\frac{dy}{dx} = (2x^4+1)^{\tan x}(\sec^2 x \ln(2x^4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x)$	Step 5. Substitute $y = (2x^4+1)^{\tan x}$.

Example 14.10.9: Extending the Power Rule

Find the derivative of $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$\ln y = \ln \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$	Step 1. Take the natural logarithm of both sides.
$\ln y = \ln x + \frac{1}{2}\ln(2x+1) - x \ln e - 3 \ln \sin x$	Step 2. Expand using properties of logarithms.
$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos x}{\sin x}$	Step 3. Differentiate both sides.
$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$	Step 4. Multiply by y on both sides.
$\frac{dy}{dx} = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$	Step 5. Substitute $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Exercise 14.10.5

Use logarithmic differentiation to find the derivative of $y = x^x$.

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 14.10.6

Find the derivative of $y = (\tan x)^\pi$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- On the basis of the assumption that the exponential function $y = b^x,\ b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \dfrac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.

- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

Key Equations

- **Derivative of the natural exponential function**

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g\,'(x)$$

- **Derivative of the natural logarithmic function**

$$\frac{d}{dx}\left(\ln g(x)\right) = \frac{1}{g(x)}g\,'(x)$$

- **Derivative of the general exponential function**

$$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}g\,'(x)\ln b$$

- **Derivative of the general logarithmic function**

$$\frac{d}{dx}\left(\log_b g(x)\right) = \frac{g'(x)}{g(x)\ln b}$$

Glossary

logarithmic differentiation
: is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

Answers

Exercise 14.10.1
Don't forget to use the product rule.

Exercise 14.10.1
$h\,'(x) = e^{2x} + 2xe^{2x}$

Exercise 14.10.2
Find $A\,'(4)$.

Exercise 14.10.2
996

Exercise 14.10.3
Use a property of logarithms to simplify before taking the derivative.

Exercise 14.10.3
$f\,'(x) = \dfrac{15}{3x+2}$

Exercise 14.10.4
Evaluate the derivative at $x = 2$.

Exercise 14.10.4
$9\ln(3)$

Exercise 14.10.5
Follow the problem solving strategy.

Exercise 14.10.5
Solution: $\dfrac{dy}{dx} = x^x(1+\ln x)$

Exercise 14.10.6
Use the power rule (since the exponent π is a constant) and the chain rule.

Exercise 14.10.6
$y\,' = \pi(\tan x)^{\pi-1}\sec^2 x$

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14.10E: Exercises for Section 3.9

In exercises 1 - 15, find $f'(x)$ for each function.

1) $f(x) = x^2 e^x$

Answer

$$f'(x) = 2xe^x + x^2 e^x$$

2) $f(x) = \dfrac{e^{-x}}{x}$

3) $f(x) = e^{x^3 \ln x}$

Answer

$$f'(x) = e^{x^3 \ln x}\left(3x^2 \ln x + x^2\right)$$

4) $f(x) = \sqrt{e^{2x} + 2x}$

5) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Answer

$$f'(x) = \frac{4}{(e^x + e^{-x})^2}$$

6) $f(x) = \dfrac{10^x}{\ln 10}$

7) $f(x) = 2^{4x} + 4x^2$

Answer

$$f'(x) = 2^{4x+2} \cdot \ln 2 + 8x$$

8) $f(x) = 3^{\sin 3x}$

9) $f(x) = x^\pi \cdot \pi^x$

Answer

$$f'(x) = \pi x^{\pi-1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$$

10) $f(x) = \ln(4x^3 + x)$

11) $f(x) = \ln \sqrt{5x - 7}$

Answer

$$f'(x) = \frac{5}{2(5x - 7)}$$

12) $f(x) = x^2 \ln 9x$

13) $f(x) = \log(\sec x)$

Answer

$$f'(x) = \frac{\tan x}{\ln 10}$$

14) $f(x) = \log_7 (6x^4 + 3)^5$

15) $f(x) = 2^x \cdot \log_3 7^{x^2-4}$

Answer

$$f'(x) = 2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4} + 2^x \cdot \frac{2x \ln 7}{\ln 3}$$

For exercises 16 - 23, use logarithmic differentiation to find $\frac{dy}{dx}$.

16) $y = x^{\sqrt{x}}$

17) $y = (\sin 2x)^{4x}$

Answer

$$\frac{dy}{dx} = (\sin 2x)^{4x}\left[4 \cdot \ln(\sin 2x) + 8x \cdot \cot 2x\right]$$

18) $y = (\ln x)^{\ln x}$

19) $y = x^{\log_2 x}$

Answer

$$\frac{dy}{dx} = x^{\log_2 x} \cdot \frac{2 \ln x}{x \ln 2}$$

20) $y = (x^2 - 1)^{\ln x}$

21) $y = x^{\cot x}$

Answer

$$\frac{dy}{dx} = x^{\cot x} \cdot \left[-\csc^2 x \cdot \ln x + \frac{\cot x}{x}\right]$$

22) $y = \dfrac{x + 11}{\sqrt[3]{x^2 - 4}}$

23) $y = x^{-1/2}(x^2 + 3)^{2/3}(3x - 4)^4$

Answer

$$\frac{dy}{dx} = x^{-1/2}(x^2 + 3)^{2/3}(3x - 4)^4 \cdot \left[\frac{-1}{2x} + \frac{4x}{3(x^2 + 3)} + \frac{12}{3x - 4}\right]$$

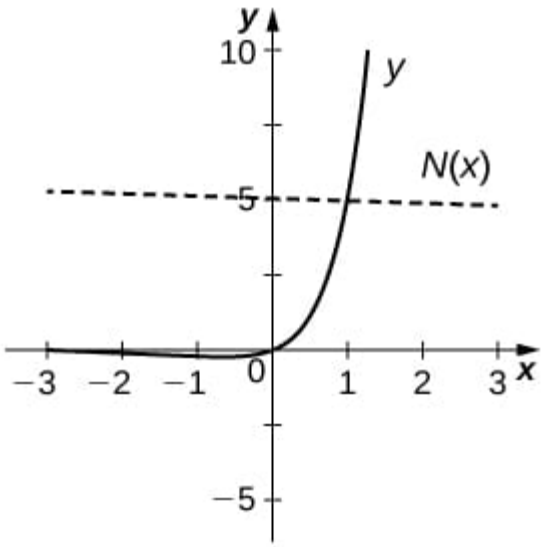
24) [T] Find an equation of the tangent line to the graph of $f(x) = 4xe^{(x^2-1)}$ at the point where

$x = -1$. Graph both the function and the tangent line.

25) [T] Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

Answer

$$y = \tfrac{-1}{5+5\ln 5}x + \left(5 + \tfrac{1}{5+5\ln 5}\right)$$



26) [T] Find the equation of the tangent line to the graph of $x^3 - x \ln y + y^3 = 2x + 5$ at the point where $x = 2$. (Hint: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

27) Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.

b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Answer

a. $x = e \approx 2.718$
b. $y' > 0$ for $(0, e)$ and $y' < 0$ for (e, ∞).

28) The formula $I(t) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\pi/2$	(ii)
π	(iii)
$3\pi/2$	(vi)
2π	(v)
2π	(vi)
3π	(vii)

b. Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

29) [T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.

b. Use part a. to determine the rate at which the population is increasing in t years.

c. Use part b. to determine the rate at which the population is increasing in 10 years

Answer

a. $P = 500,000(1.05)^t$ individuals
b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year
c. $39,737$ individuals per year

30)[T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.

b. Use a. to determine the rate at which the substance is decaying in t hours.

c. Use b. to determine the rate of decay at $t = 4$ hours.

31) [T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1964 is modeled by the function $N(t) = 5.3e^{0.093t^2 - 0.87t}, (0 \leq t \leq 4)$, where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.

b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in the United States.

Answer

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States.
b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

32) [T] The relative rate of change of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}$ One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a, b, and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.

b. Use part a. to find the relative rate of change of a population in $x = 20$ months when $a = 204,\ \ b = 0.0198$, and $c = 0.15$.

c. Briefly interpret what the result of part b. means.

For exercises 33 - 36, use the population of New York City from 1790 to 1860, given in the following table.

Year since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over TimeSource: http://en.Wikipedia.org/wiki/Largest..._United_States

_by_population_by_decade

33) [T] Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Answer

$p = 35741(1.045)^t$

34) [T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

35) [T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Answer

Year since 1790	P''
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

36) [T] Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?

b. Estimate the population in 2010. Was the prediction correct from part a.?

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14.R: Chapter 3 Review Exercises

Chapter Review Exercises

True or False? Justify the answer with a proof or a counterexample.

1) Every function has a derivative.

Answer

False

2) A continuous function has a continuous derivative.

3) A continuous function has a derivative.

Answer

False

4) If a function is differentiable, it is continuous.

In exercises 5 and 6, use the limit definition of the derivative to exactly evaluate the derivative.

5) $f(x) = \sqrt{x+4}$

Answer

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

6) $f(x) = \dfrac{3}{x}$

In exercises 7 - 15, find the derivatives of the given functions.

7) $f(x) = 3x^3 - \dfrac{4}{x^2}$

Answer

$f'(x) = 9x^2 + \frac{8}{x^3}$

9) $f(x) = (4 - x^2)^3$

10) $f(x) = e^{\sin x}$

Answer

$$f'(x) = e^{\sin x}\cos x$$

11) $f(x) = \ln(x+2)$

12) $f(x) = x^2\cos x + x\tan x$

Answer

$$f'(x) = x\sec^2 x + 2x\cos x + \tan x - x^2\sin x$$

13) $f(x) = \sqrt{3x^2+2}$

14) $f(x) = \dfrac{x}{4}\sin^{-1}(x)$

Answer

$$f'(x) = \frac{1}{4}\left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1} x\right)$$

15) $x^2 y = (y+2) + xy \sin x$

In exercises 16 - 18, find the indicated derivatives of various orders.

16) First derivative of $y = x(\ln x)\cos x$

Answer

$$\frac{dy}{dx} = \cos x \cdot (\ln x + 1) - x(\ln x)\sin x$$

17) Third derivative of $y = (3x+2)^2$

18) Second derivative of $y = 4^x + x^2 \sin x$

Answer

$$\frac{d^2y}{dx^2} = 4^x (\ln 4)^2 + 2\sin x + 4x\cos x - x^2 \sin x$$

In exercises 19 and 20, find the equation of the tangent line to the following equations at the specified point.

19) $y = \cos^{-1}(x) + x$ at $x = 0$

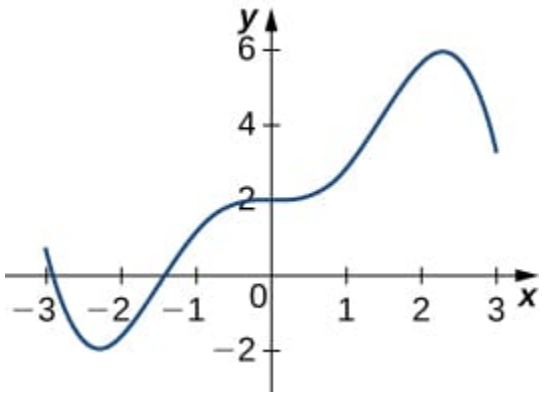
20) $y = x + e^x - \frac{1}{x}$ at $x = 1$

Answer

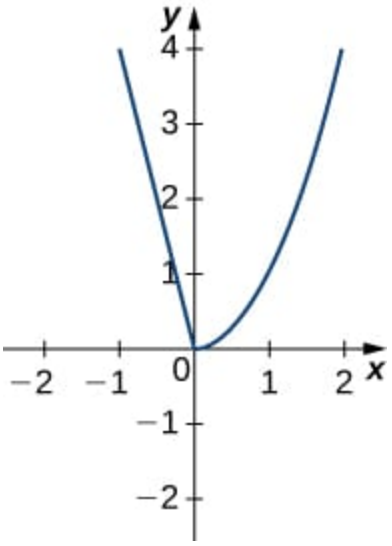
$$y = (2+e)x - 2$$

In exercises 21 and 22, draw the derivative of the functions with the given graphs.

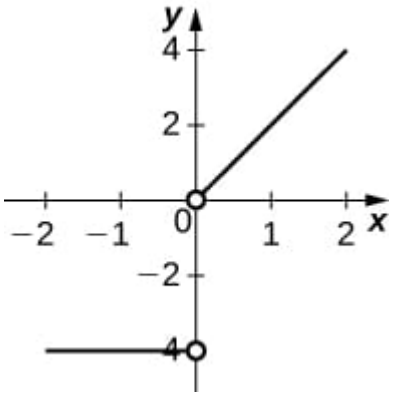
21)



22)



Answer



Questions 22 and 23 concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9\cos(\frac{\pi}{6}t)$, where t is measured in hours after midnight, and the height is measured in feet.

22) Find and graph the derivative. What is the physical meaning?

23) Find $w'(3)$. What is the physical meaning of this value?

Answer

$w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

Questions 24 and 25 consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175

73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource: news.nationalgeographic.com/n..._timeline.html.

24) Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

25) Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Answer

-7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

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15: APPLICATIONS OF DERIVATIVES

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

15.1: Prelude to Applications of Derivatives

15.2: Related Rates

15.2E: Exercises for Section 4.1

15.3: Linear Approximations and Differentials

15.3E: Exercises for Section 4.2

15.4: Maxima and Minima

15.4E: Exercises for Section 4.3

15.5: The Mean Value Theorem

15.5E: Exercises for Section 4.4

15.6: Derivatives and the Shape of a Graph

15.6E: Exercises for Section 4.5

15.7: Limits at Infinity and Asymptotes

15.7E: Exercises for Section 4.6

15.8: Applied Optimization Problems

15.8E: Exercises for Section 4.7

15.9: L'Hôpital's Rule

15.9E: Exercises for Section 4.8

15.10: Newton's Method

15.10E: Exercises for Section 4.9

15.11: Antiderivatives

15.11E: Exercises for Section 4.10

15.R: Chapter 4 Review Exercises

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15.1: Prelude to Applications of Derivatives

A rocket is being launched from the ground and cameras are recording the event. A video camera is located on the ground a certain distance from the launch pad. At what rate should the angle of inclination (the angle the camera makes with the ground) change to allow the camera to record the flight of the rocket as it heads upward?



Figure 15.1.1: As a rocket is being launched, at what rate should the angle of a video camera change to continue viewing the rocket? (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

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15.2: Related Rates

Learning Objectives

- Express changing quantities in terms of derivatives.
- Find relationships among the derivatives in a given problem.
- Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 15.2.1: Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$ (Figure 15.2.1). How fast is the radius increasing when the radius is 3 cm?

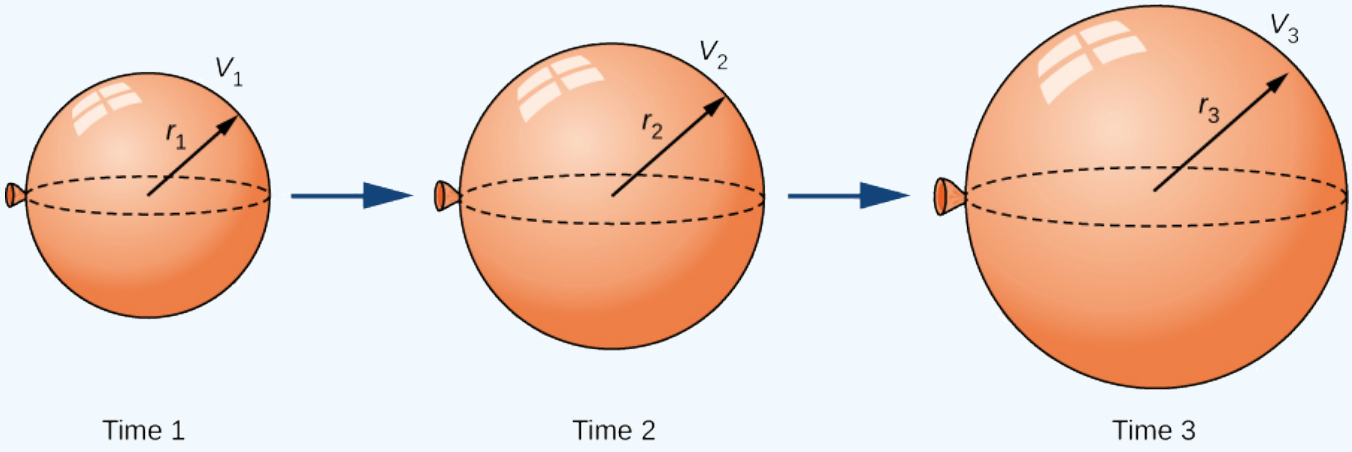


Figure 15.2.1: As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution

The volume of a sphere of radius r centimeters is

$$V = \tfrac{4}{3}\pi r^3\ \text{cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \tfrac{4}{3}\pi\big[r(t)\big]^3\text{cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi\big[r(t)\big]^2 r'(t).$$

The balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$, so $V'(t) = 2\ \text{cm}^3/\text{sec}$. Therefore,

$$2\ \text{cm}^3/\text{sec} = \Big(4\pi\big[r(t)\big]^2\ \text{cm}^2\Big) \cdot \Big(r'(t)\ \ \text{cm/s}\Big),$$

which implies

$$r'(t) = \frac{1}{2\pi\big[r(t)\big]^2}\ \text{cm/sec}.$$

When the radius $r = 3$ cm,

$$r'(t) = \frac{1}{18\pi}\ \text{cm/sec}.$$

Exercise 15.2.1

What is the instantaneous rate of change of the radius when $r = 6$ cm?

Hint available at the end of this section.

Answer available at the end of this section.

Before looking at other examples, let's outline the problem-solving strategy we will be using to solve related-rates problems.

Problem-Solving Strategy: Solving a Related-Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change

Note that when solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4. We examine this potential error in the following example.

Examples of the Process

Let's now implement the strategy just described to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the plane and the rate at which the distance between the plane and a person on the ground is changing.

Example 15.2.2: An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Solution

Step 1. Draw a picture, introducing variables to represent the different quantities involved.

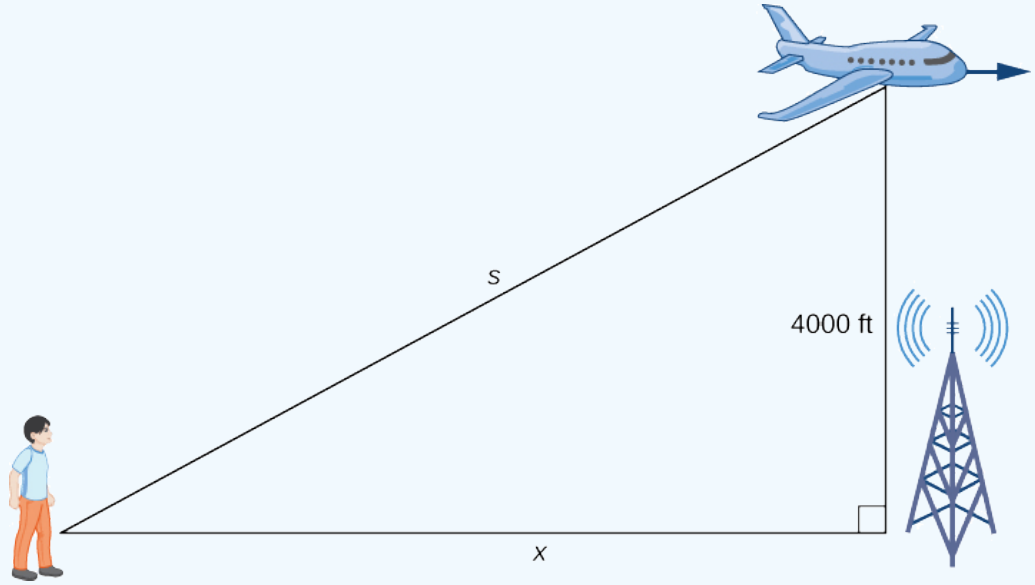


Figure 15.2.2: An airplane is flying at a constant height of 4000 ft. The distance between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables s and x, respectively.

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the height of the plane because it remains at a constant elevation of 4000 ft. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Step 2. Since x denotes the horizontal distance between the man and the point on the ground below the plane, dx/dt represents the speed of the plane. We are told the speed of the plane is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find ds/dt when $x = 3000$ ft.

Step 3. From the figure, we can use the **Pythagorean theorem** to write an equation relating x and s:

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Step 4. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

$$x\frac{dx}{dt} = s\frac{ds}{dt}.$$

Step 5. Find the rate at which the distance between the man and the plane is increasing when the plane is directly over the radio tower. That is, find $\frac{ds}{dt}$ when $x = 3000$ ft. Since the speed of the plane is 600 ft/sec, we know that $\frac{dx}{dt} = 600$ ft/sec. We are not given an explicit value for s; however, since we are trying to find $\frac{ds}{dt}$ when $x = 3000$ ft, we can use the Pythagorean theorem to determine the distance s when $x = 3000$ ft and the height is 4000 ft. Solving the equation

$$3000^2 + 4000^2 = s^2$$

for s, we have $s = 5000$ ft at the time of interest. Using these values, we conclude that ds/dt

is a solution of the equation

$$(3000)(600) = (5000) \cdot \frac{ds}{dt}.$$

Therefore,

$$\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360 \text{ ft/sec}.$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in step 3, we related the variable quantities $x(t)$ and $s(t)$ by the equation

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

$$3000^2 + 4000^2 = [s(t)]^2.$$

After differentiating, our equation would become

$$0 = s(t)\frac{ds}{dt}.$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0$.

Exercise 15.2.2

What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

Hint available at the end of this section.

Answer available at the end of this section.

We now return to the problem involving the rocket launch from the beginning of the chapter.

Example 15.2.3: Chapter Opener - A Rocket Launch

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec.



Figure 15.2.3: (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

Solution

Step 1. Draw a picture introducing the variables.

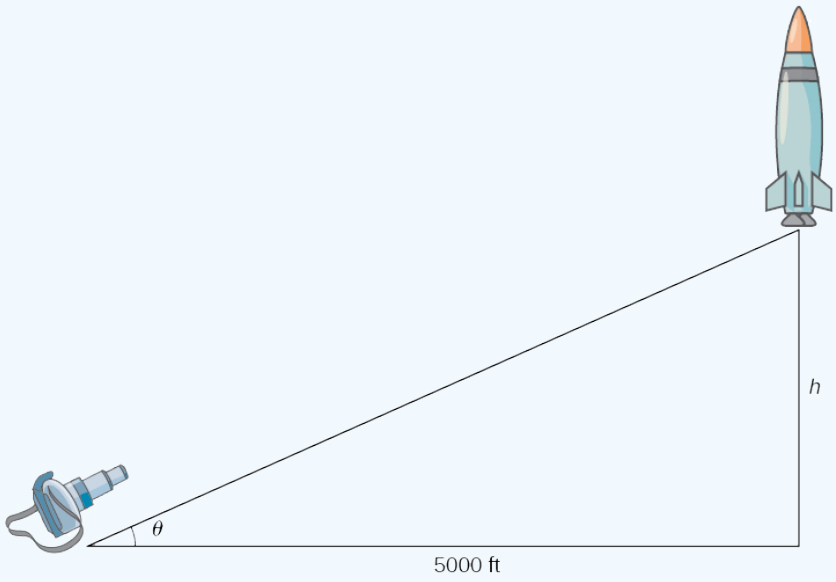


Figure 15.2.4: A camera is positioned 5000 ft from the launch pad of the rocket. The height of the rocket and the angle of the camera are changing with respect to time. We denote those quantities with the variables h and θ,respectively.

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Step 2. We are trying to find the rate of change in the angle of the camera with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600$ ft/sec.

Step 3. Now we need to find an equation relating the two quantities that are changing with respect to time: h and θ. How can we create such an equation? Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan\theta$ is the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

$$\tan\theta = \frac{h}{5000}.$$

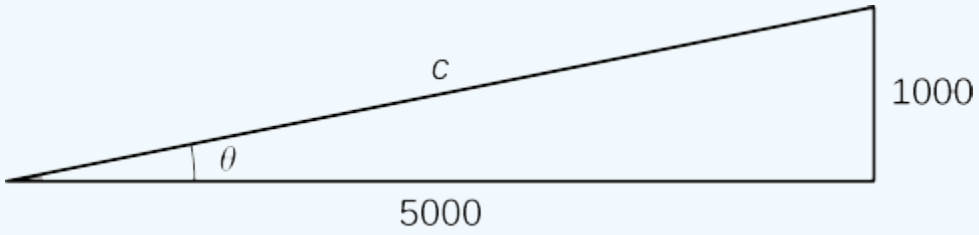
This gives us the equation

$$h = 5000\tan\theta.$$

Step 4. Differentiating this equation with respect to time t, we obtain

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

Step 5. We want to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At this time, we know that $\frac{dh}{dt} = 600$ ft/sec. We need to determine $\sec^2\theta$. Recall that $\sec\theta$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is 5000 ft. To determine the length of the hypotenuse, we use the Pythagorean theorem, where the length of one leg is 5000 ft, the length of the other leg is $h = 1000$ ft, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

$$c = 1000\sqrt{26}\text{ ft}.$$

Therefore, when $h = 1000$, we have

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Recall from step 4 that the equation relating $\frac{d\theta}{dt}$ to our known values is

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

When $h = 1000$ ft, we know that $\frac{dh}{dt} = 600$ ft/sec and $\sec^2\theta = \frac{26}{25}$. Substituting these values into the previous equation, we arrive at the equation

$$600 = 5000\left(\tfrac{26}{25}\right)\frac{d\theta}{dt}.$$

Therefore, $\frac{d\theta}{dt} = \frac{3}{26}$ rad/sec.

Exercise 15.2.3

What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Hint available at the end of this section.

Answer available at the end of this section.

In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example 15.2.4: Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\ \text{ft}^3/\text{sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}$ ft?

Solution

Step 1: Draw a picture introducing the variables.

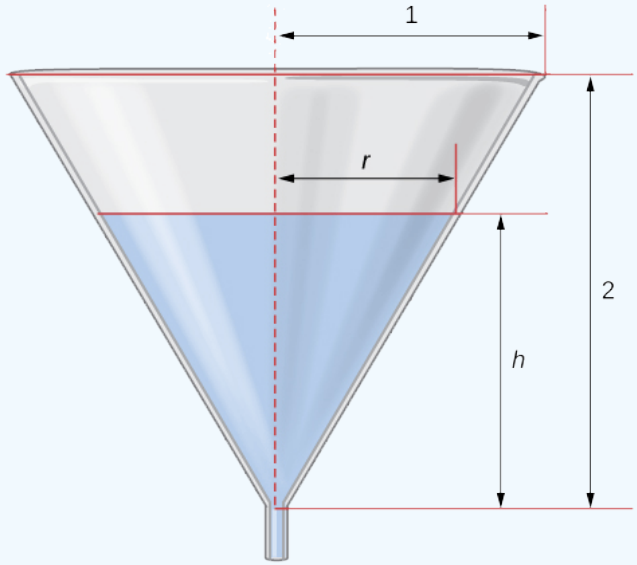


Figure 15.2.5: Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03$ ft/sec.

Step 3: The volume of water in the cone is

$$V = \tfrac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \tfrac{1}{3}\pi\left(\tfrac{h}{2}\right)^2 h = \tfrac{\pi}{12}h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\ \text{ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03\ \text{ft}^3/\text{sec}$. Therefore,

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2 \frac{dh}{dt},$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\,\text{ft/sec}.$$

Exercise 15.2.4

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- To solve a related rates problem, first draw a picture that illustrates the relationship between the two or more related quantities that are changing with respect to time.
- In terms of the quantities, state the information given and the rate to be found.
- Find an equation relating the quantities.
- Use differentiation, applying the chain rule as necessary, to find an equation that relates the rates.
- Be sure not to substitute a variable quantity for one of the variables until after finding an equation relating the rates.

Glossary

related rates
are rates of change associated with two or more related quantities that are changing over time

Answers

Exercise 15.2.1
$\frac{dr}{dt} = \frac{1}{2\pi r^2}$

Exercise 15.2.1
$\frac{1}{72\pi}$ cm/sec, or approximately 0.0044 cm/sec

Exercise 15.2.2
$\frac{ds}{dt} = 300$ ft/sec

Exercise 15.2.2
500 ft/sec

Exercise 15.2.3
Find $\frac{d\theta}{dt}$ when $h = 2000$ ft. At that time, $\frac{dh}{dt} = 500$ ft/sec.

Exercise 15.2.3
$\frac{1}{10}$ rad/sec

Exercise 15.2.4
We need to find $\frac{dh}{dt}$ when $h = \frac{1}{4}$.

Exercise 15.2.4

-0.61 ft/sec

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15.2E: Exercises for Section 4.1

In exercises 1 - 3, find the quantities for the given equation.

1) Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$.

Answer

$$\frac{dy}{dt} = 8$$

2) Find $\frac{dx}{dt}$ at $x = -2$ and $y = 2x^2 + 1$ if $\frac{dy}{dt} = -1$.

3) Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3$.

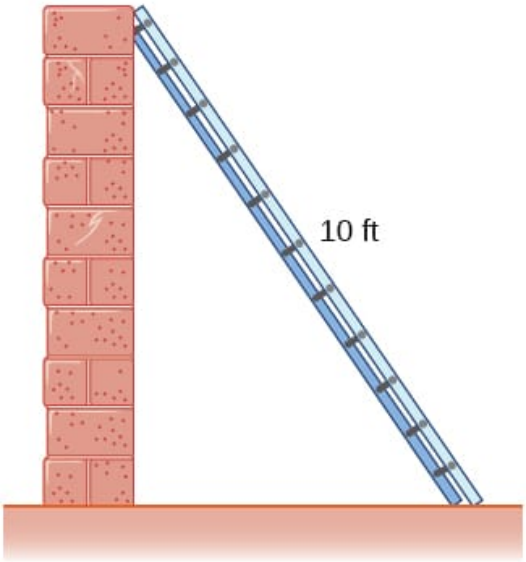
Answer

$$\frac{dz}{dt} = \frac{13}{\sqrt{10}}$$

In exercises 4 - 15, sketch the situation if necessary and used related rates to solve for the quantities.

4) [T] If two electrical resistors are connected in parallel, the total resistance (measured in ohms, denoted by the Greek capital letter omega, Ω) is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is increasing at a rate of $0.5\Omega/\text{min}$ and R_2 decreases at a rate of $1.1\Omega/\text{min}$, at what rate does the total resistance change when $R_1 = 20\Omega$ and $R_2 = 50\Omega/\text{min}$?

5) A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?

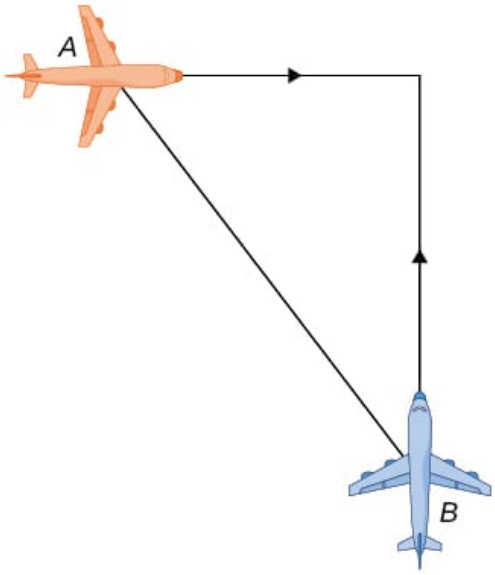


Answer

$2\sqrt{3}$ ft/sec

6) A 25-ft ladder is leaning against a wall. If we push the ladder toward the wall at a rate of 1 ft/sec, and the bottom of the ladder is initially 20 ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?

7) Two airplanes are flying in the air at the same height: airplane A is flying east at 250 mi/h and airplane B is flying north at 300 mi/h. If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing?



Answer

The distance is decreasing at 390 mi/h.

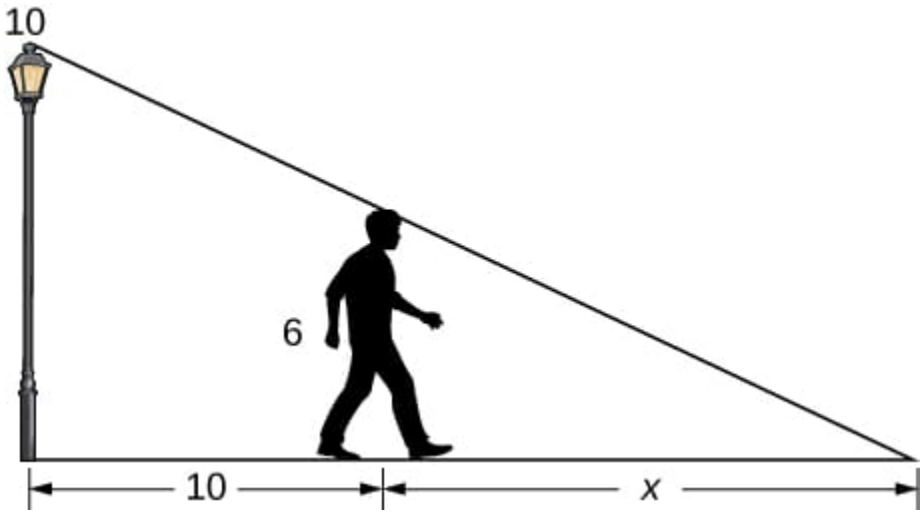
8) You and a friend are riding your bikes to a restaurant that you think is east; your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16 mph east and your friend riding 12 mph north. After you traveled 4 mi, at what rate is the distance between you changing?

9) Two buses are driving along parallel freeways that are 5 mi apart, one heading east and the other heading west. Assuming that each bus drives a constant 55 mph, find the rate at which the distance between the buses is changing when they are 13 mi apart (as the crow flies), heading toward each other.

Answer

The distance between them shrinks at a rate of $\frac{1320}{13} \approx 101.5$ mph.

10) A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



11) Using the previous problem, what is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

Answer

$\frac{9}{2}$ ft/sec

12) A 5-ft-tall person walks toward a wall at a rate of 2 ft/sec. A spotlight is located on the ground 40 ft from the wall. How fast does the height of the person's shadow on the wall change when the person is 10 ft from the wall?

13) Using the previous problem, what is the rate at which the shadow changes when the person is 10 ft from the wall, if the person is walking away from the wall at a rate of 2 ft/sec?

Answer

It grows at a rate $\frac{4}{9}$ ft/sec

14) A helicopter starting on the ground is rising directly into the air at a rate of 25 ft/sec. You are running on the ground starting directly under the helicopter at a rate of 10 ft/sec. Find the rate of change of the distance between the helicopter and yourself after 5 sec.

15) Using the previous problem, assuming the helicopter is again rising at a rate of 25 ft/sec and you are running on the ground starting directly under the helicopter at a rate of 10 ft/sec, what is the rate at which the distance between you and the helicopter is changing when the helicopter has risen to a height of 60 ft in the air, assuming that, initially, it was 30 ft above you?

Answer

The distance is increasing at $\frac{135\sqrt{26}}{26}$ ft/sec

In exercises 16 - 24, draw and label diagrams to help solve the related-rates problems.

16) The side of a cube increases at a rate of $\frac{1}{2}$ m/sec. Find the rate at which the volume of the cube increases when the side of the cube is 4 m.

17) The volume of a cube decreases at a rate of $10\ \mathrm{m}^3$/sec. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

Answer

$-\frac{5}{6}$ m/sec

18) The radius of a circle increases at a rate of 2 m/sec. Find the rate at which the area of the circle increases when the radius is 5 m.

19) The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

Answer

$240\pi\ \mathrm{m}^2/\mathrm{sec}$

20) The radius of a sphere increases at a rate of 1 m/sec. Find the rate at which the volume increases when the radius is 20 m.

21) The radius of a sphere is increasing at a rate of 9 cm/sec. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

Answer

$\frac{1}{2\sqrt{\pi}}$ cm

22) The base of a triangle is shrinking at a rate of 1 cm/min and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when the height is 22 cm and the base is 10 cm.

23) A triangle has two constant sides of length 3 ft and 5 ft. The angle between these two sides is increasing at a rate of 0.1 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the two sides is $\pi/6$.

Answer

The area is increasing at a rate $\frac{3\sqrt{3}}{8}\ \mathrm{ft}^2/\mathrm{sec}$.

24) A triangle has a height that is increasing at a rate of 2 cm/sec and its area is increasing at a rate of $4\ \text{cm}^2/\text{sec}$. Find the rate at which the base of the triangle is changing when the height of the triangle is 4 cm and the area is $20\ \text{cm}^2$.

In exercises 25 - 27, consider an inverted right cone that is leaking water. (Inverted means the cone's point is facing down, like a funnel.) The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.

25) How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of $10\ \text{ft}^3/\text{min}$?

Answer

The depth of the water decreases at $\frac{128}{125\pi}$ ft/min.

26) Find the rate at which the surface area of the water changes when the water is 10 ft high if the cone leaks water at a rate of $10\ \text{ft}^3/\text{min}$.

27) If the water level is decreasing at a rate of 3 in/min when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.

Answer

The volume is decreasing at a rate of $\frac{25\pi}{16}\ \text{ft}^3/\text{min}$.

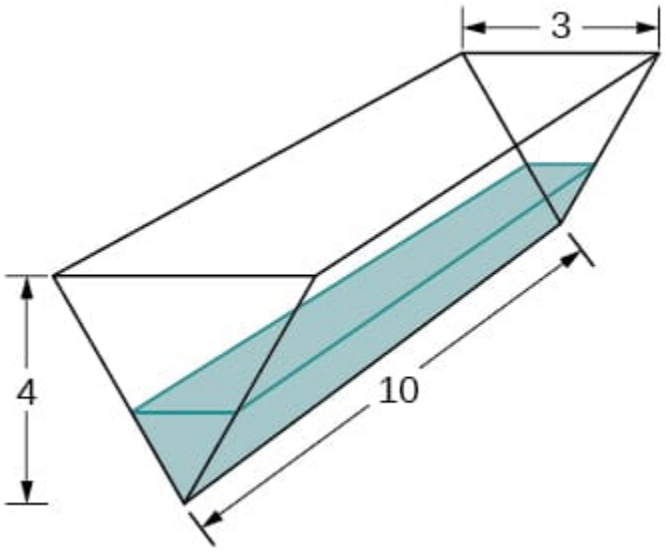
28) A vertical cylinder is leaking water at a rate of $1\ \text{ft}^3/\text{sec}$. If the cylinder has a height of 10 ft and a radius of 1 ft, at what rate is the height of the water changing when the height is 6 ft?

29) A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is 10 cm/min when the height is 1 m.

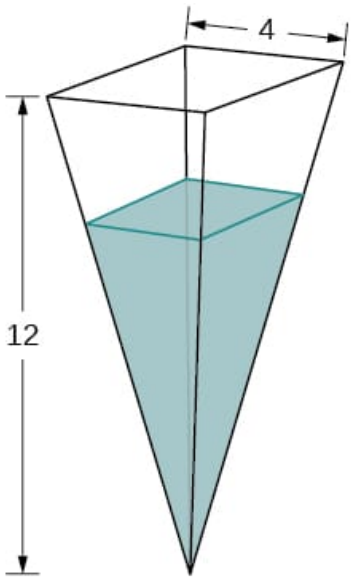
Answer

The water flows out at rate $\frac{2\pi}{5}\ \text{m}^3/\text{min}$.

30) A trough has ends shaped like isosceles triangles, with width 3 m and height 4 m, and the trough is 10 m long. Water is being pumped into the trough at a rate of $5\ \text{m}^3/\text{min}$. At what rate does the height of the water change when the water is 1 m deep?



31) A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}\ \text{m}^3$/sec?



Answer

$\frac{3}{2}$ m/sec

For exercises 32 - 34, consider a pool shaped like the bottom half of a sphere, that is being filled at a rate of $25\ \text{ft}^3$ /min. The radius of the pool is 10 ft.

32) Find the rate at which the depth of the water is changing when the water has a depth of 5 ft.

33) Find the rate at which the depth of the water is changing when the water has a depth of 1 ft.

Answer

$\frac{25}{19\pi}$ ft/min

34) If the height is increasing at a rate of 1 in/sec when the depth of the water is 2 ft, find the rate at which water is being pumped in.

35) Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of $10\ \text{ft}^3/\text{min}$. The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.

Answer

$\frac{2}{45\pi}$ ft/min

36) Using a similar setup from the preceding problem, find the rate at which the gravel is being unloaded if the pile is 5 ft high and the height is increasing at a rate of 4 in/min.

In exercises 37 - 41, draw the situations and solve the related-rate problems.

37) You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

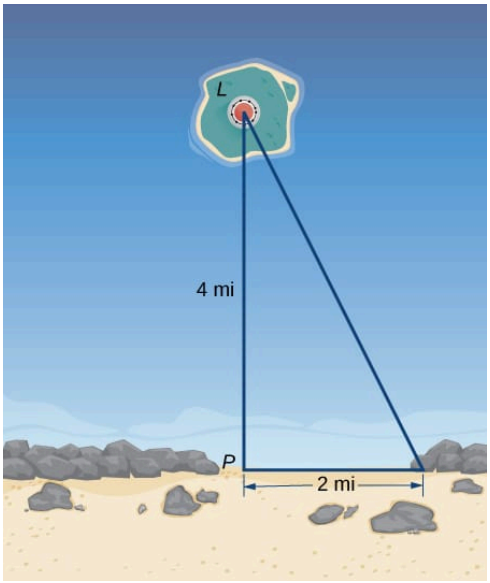
Answer

The angle decreases at $\frac{400}{1681}$ rad/sec.

38) You stand 40 ft from a bottle rocket on the ground and watch as it takes off vertically into the air at a rate of 20 ft/sec. Find the rate at which the angle of elevation changes when the rocket is 30 ft in the air.

39) A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach (see the following image). If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away

from the closest point on the beach?



Answer

100π mi/min

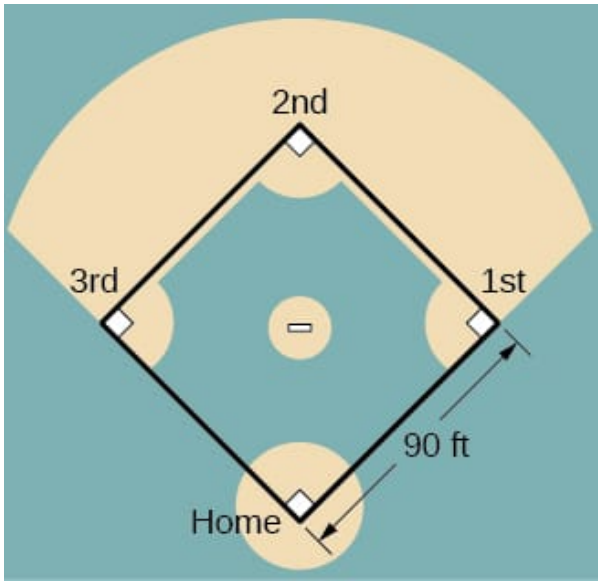
40) Using the same setup as the previous problem, determine at what rate the beam of light moves across the beach 1 mi away from the closest point on the beach.

41) You are walking to a bus stop at a right-angle corner. You move north at a rate of 2 m/sec and are 20 m south of the intersection. The bus travels west at a rate of 10 m/sec away from the intersection – you have missed the bus! What is the rate at which the angle between you and the bus is changing when you are 20 m south of the intersection and the bus is 10 m west of the intersection?

Answer

The angle is changing at a rate of $\frac{11}{25}$ rad/sec.

In exercises 42 - 45, refer to the figure of baseball diamond, which has sides of 90 ft.



42) [T] A batter hits a ball toward third base at 75 ft/sec and runs toward first base at a rate of 24 ft/sec. At what rate does the

distance between the ball and the batter change when 2 sec have passed?

43) [T] A batter hits a ball toward second base at 80 ft/sec and runs toward first base at a rate of 30 ft/sec. At what rate does the distance between the ball and the batter change when the runner has covered one-third of the distance to first base? (Hint: Recall the law of cosines.)

Answer

The distance is increasing at a rate of 62.50 ft/sec.

44) [T] A batter hits the ball and runs toward first base at a speed of 22 ft/sec. At what rate does the distance between the runner and second base change when the runner has run 30 ft?

45) [T] Runners start at first and second base. When the baseball is hit, the runner at first base runs at a speed of 18 ft/sec toward second base and the runner at second base runs at a speed of 20 ft/sec toward third base. How fast is the distance between runners changing 1 sec after the ball is hit?

Answer

The distance is decreasing at a rate of 11.99 ft/sec.

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15.3: Linear Approximations and Differentials

Learning Objectives

- Describe the linear approximation to a function at a point.
- Write the linearization of a given function.
- Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
- Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials Introduction to Power Series and Functions.

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a$. Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

Figure $15.3.1a$ shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of $f(2.1)$ is given by

$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ (Figure $15.3.1b$). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the y-value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$.

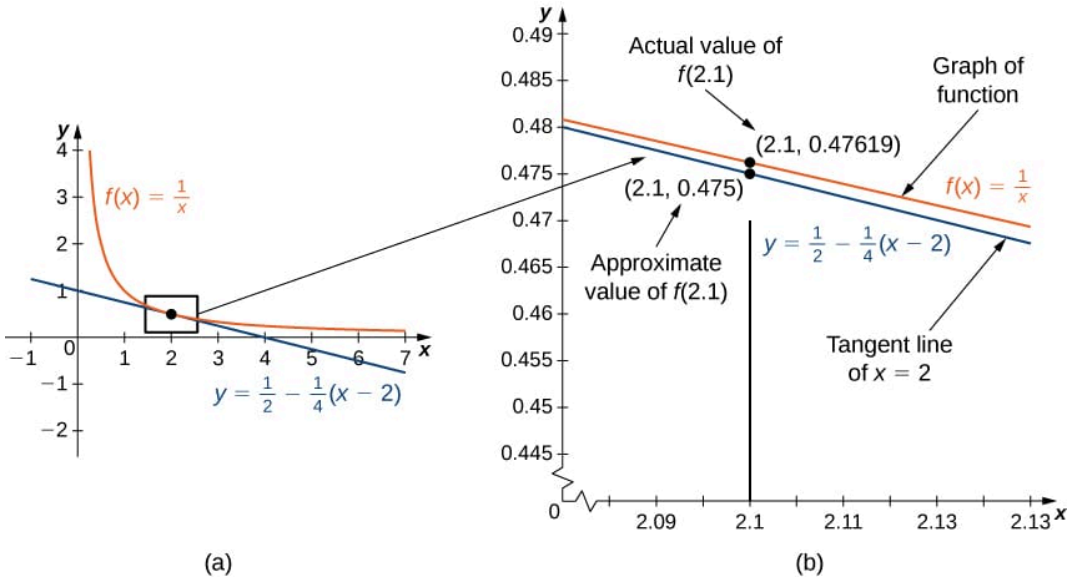


Figure 15.3.1: (a) The tangent line to $f(x) = 1/x$ at $x = 2$ provides a good approximation to f for x near 2. (b) At $x = 2.1$, the value of y on the tangent line to $f(x) = 1/x$ is 0.475. The actual value of $f(2.1)$ is $1/2.1$, which is approximately 0.47619.

In general, for a differentiable function f, the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near a. Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \tag{15.3.1}$$

the **linear approximation,** or **tangent line approximation**, of f at $x = a$. This function L is also known as the **linearization** of f at $x = a$.

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9$.

Example 15.3.1: Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}$.

Solution

Since we are looking for the linear approximation at $x = 9$, using Equation 15.3.1 we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9)$.

$$f(x) = \sqrt{x} \Rightarrow f(9) = \sqrt{9} = 3$$

$$f'(x) = \tfrac{1}{2\sqrt{x}} \Rightarrow f'(9) = \tfrac{1}{2\sqrt{9}} = \tfrac{1}{6}$$

Therefore, the linear approximation is given by Figure 15.3.2.

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

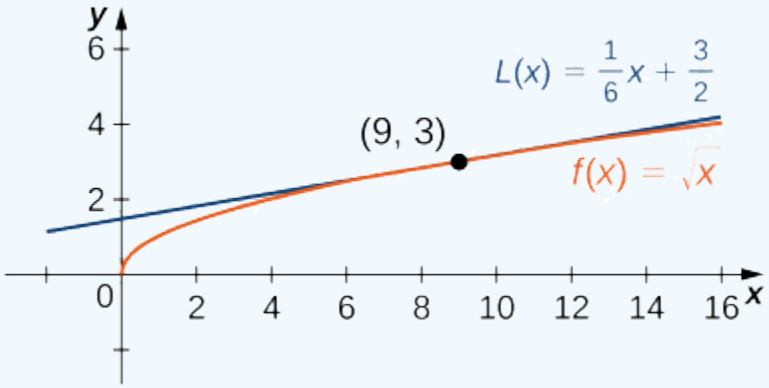


Figure 15.3.2: The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate $\sqrt{x}$, at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can just push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

Exercise 15.3.1

Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.3.2: Linear Approximation of $\sin x$

Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62\,°)$.

Solution

First we note that since $\frac{\pi}{3}$ rad is equivalent to $60\,°$, using the linear approximation at $x = \pi/3$ seems reasonable. The linear approximation is given by

$$L(x) = f(\tfrac{\pi}{3}) + f'(\tfrac{\pi}{3})(x - \tfrac{\pi}{3}).$$

We see that

$$f(x) = \sin x \Rightarrow f(\tfrac{\pi}{3}) = \sin(\tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2}$$

$$f'(x) = \cos x \Rightarrow f'(\tfrac{\pi}{3}) = \cos(\tfrac{\pi}{3}) = \tfrac{1}{2}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by Figure 15.3.3.

$$L(x) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(x - \tfrac{\pi}{3})$$

To estimate $\sin(62\,^\circ)$ using L, we must first convert $62\,^\circ$ to radians. We have $62\,^\circ = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62\,^\circ)$ is given by

$$\sin(62\,^\circ) = f(\tfrac{62\pi}{180}) \approx L(\tfrac{62\pi}{180}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(\tfrac{62\pi}{180} - \tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(\tfrac{2\pi}{180}) = \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{180} \approx 0.88348.$$

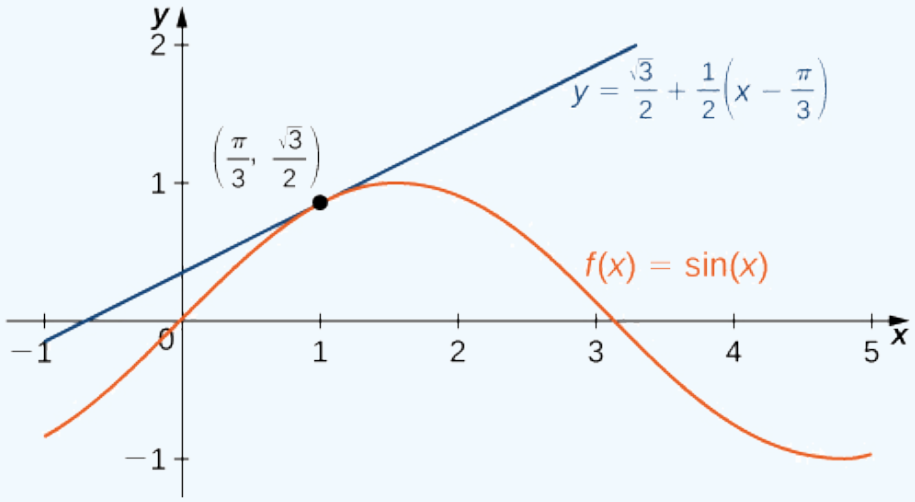


Figure 15.3.3: The linear approximation to $f(x) = \sin x$ at $x = \pi/3$ provides an approximation to $\sin x$ for x near $\pi/3$.

Exercise 15.3.2

Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Hint available at the end of this section.

Answer available at the end of this section.

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1 + x)^n$ at $x = 0$, which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m + x)^n$ to estimate roots and powers near a different number m.

Example 15.3.3: Approximating Roots and Powers

Find the linear approximation of $f(x) = (1 + x)^n$ at $x = 0$. Use this approximation to estimate $(1.01)^3$.

Solution

The linear approximation at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

$$f(x) = (1 + x)^n \Rightarrow f(0) = 1$$

$$f'(x) = n(1 + x)^{n-1} \Rightarrow f'(0) = n,$$

the linear approximation is given by Figure 15.3.4*a*.

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3$. We conclude that

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$

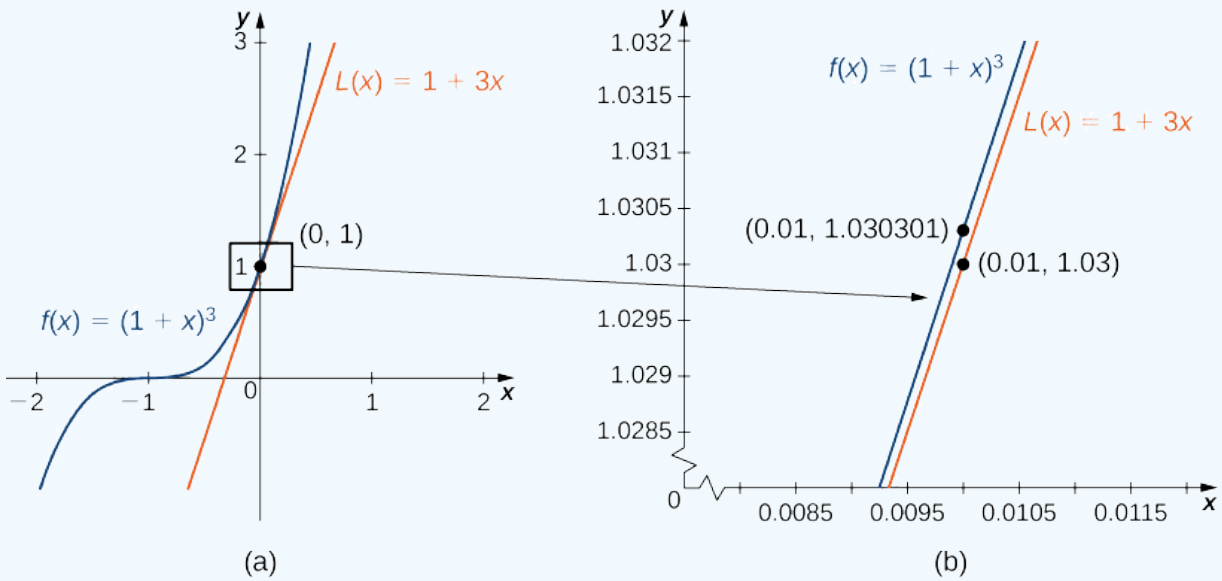


Figure 15.3.4: (a) The linear approximation of $f(x)$ at $x = 0$ is $L(x)$. (b) The actual value of 1.01^3 is 1.030301. The linear approximation of $f(x)$ at $x = 0$ estimates 1.01^3 to be 1.03.

Exercise 15.3.3

Find the linear approximation of $f(x) = (1 + x)^4$ at $x = 0$ without using the result from the preceding example.

Hint available at the end of this section.

Answer available at the end of this section.

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: **differentials**. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x. Although we used the expressions dy and dx in this notation, they did not have meaning on their own. Here we see a meaning to the expressions dy and dx. Suppose $y = f(x)$ is a differentiable function. Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)\,dx. \tag{15.3.2}$$

It is important to notice that dy is a function of both x and dx. The expressions dy and dx are called differentials. We can divide both sides of Equation 15.3.2 by dx, which yields

$$\frac{dy}{dx} = f'(x). \tag{15.3.3}$$

This is the familiar expression we have used to denote a derivative. Equation 15.3.3 is known as the **differential form** of Equation 15.3.2.

Example 15.3.4: Computing Differentials

For each of the following functions, find dy and evaluate when $x = 3$ and $dx = 0.1$.

a. $y = x^2 + 2x$
b. $y = \cos x$

Solution

The key step is calculating the derivative. When we have that, we can obtain dy directly.

a. Since $f(x) = x^2 + 2x$, we know $f'(x) = 2x + 2$, and therefore

$$dy = (2x + 2)\ dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x, f'(x) = -\sin(x)$. This gives us

$$dy = -\sin x\ \ dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = -\sin(3)(0.1) = -0.1\sin(3).$$

Exercise 15.3.4

For $y = e^{x^2}$, find dy.

Hint available at the end of this section.

Answer available at the end of this section.

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx$, then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near $a, f(x)$ can be approximated by the linear approximation (Equation 15.3.1)

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

$$f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\, dx.$$

In other words, the actual change in the function f if x increases from a to $a + dx$ is approximately the difference between $L(a + dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)\ dx$. In summary,

$$\Delta y = f(a+dx) - f(a) \approx L(a+dx) - f(a) = f'(a)\,dx = dy.$$

Therefore, we can use the differential $dy = f'(a)\ dx$ to approximate the change in y if x increases from $x = a$ to $x = a + dx$. We can see this in the following graph.

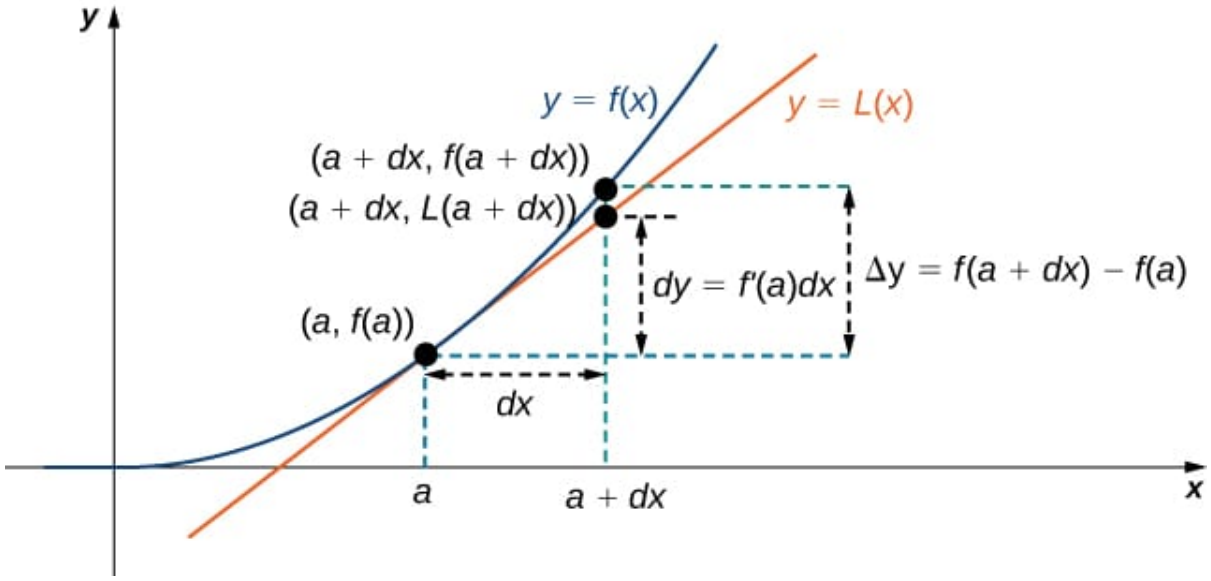


Figure 15.3.5: The differential $dy = f'(a)\ dx$ is used to approximate the actual change in y if x increases from a to $a + dx$.

We now take a look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note the calculation with differentials is much simpler than calculating actual values of functions and the result is very close to what we would obtain with the more exact calculation.

Example 15.3.5: Approximating Change with Differentials

Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$ if $dx = 0.1$.

Solution

The actual change in y if x changes from $x = 3$ to $x = 3.1$ is given by

$$\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [3^2 + 2(3)] = 0.81.$$

The approximate change in y is given by $dy = f'(3)\ dx$. Since $f'(x) = 2x + 2$, we have

$$dy = f'(3)\ dx = (2(3) + 2)(0.1) = 0.8.$$

Exercise 15.3.5

For $y = x^2 + 2x$, find Δy and dy at $x = 3$ if $dx = 0.2$.

Hint available at the end of this section.

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the measurement error is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is a differentiable function at a,the propagated error is

$$\Delta y \approx dy = f'(a)\, dx.$$

Unfortunately, we do not know the exact value a. However, we can use the measured value $a + dx$, and estimate

$$\Delta y \approx dy \approx f'(a + dx)\, dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example 15.3.6: Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

a. Use differentials to estimate the error in the computed volume of the cube.
b. Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solution

a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \leq dx \leq 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2 dx.$$

Using the measured side length of 5 cm, we can estimate that

$$-3(5)^2(0.1) \leq dV \leq 3(5)^2(0.1).$$

Therefore,

$$-7.5 \leq dV \leq 7.5.$$

b. If the side length is actually 4.9 cm, then the volume of the cube is

$$V(4.9) = (4.9)^3 = 117.649 \text{cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

$$V(5.1) = (5.1)^3 = 132.651 \text{cm}^3.$$

Therefore, the actual volume of the cube is between 117.649 and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125$. Therefore, the error in the computed volume is

$$117.649 - 125 \leq \Delta V \leq 132.651 - 125.$$

That is,

$$-7.351 \leq \Delta V \leq 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

Exercise 15.3.6

Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

Hint available at the end of this section.

Answer available at the end of this section.

The measurement error $dx\ (=\Delta x)$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q}$, where q is the actual value of the quantity. The **percentage error** is the relative error expressed as a percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in. but the relative error is $\frac{1}{62} = 0.016$, or 1.6%. By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8} = \frac{1}{32}$, or 3.1%. Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example 15.3.7: Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution: If the measurement of the radius is accurate to within ± 80, we have

$$-80 \leq dr \leq 80.$$

Since the volume of a sphere is given by $V = (\frac{4}{3})\pi r^3$, we have

$$dV = 4\pi r^2 dr.$$

Using the measured radius of 4000 mi, we can estimate

$$-4\pi(4000)^2(80) \leq dV \leq 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\dfrac{dV}{V}$. Since we do not know the exact value of the volume V, use the measured radius $r = 4000$ mi to estimate V. We obtain $V \approx (\frac{4}{3})\pi(4000)^3$. Therefore the relative error satisfies

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \leq \frac{dV}{V} \leq \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

$$-0.06 \leq \frac{dV}{V} \leq 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

Exercise 15.3.7

Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- A differentiable function $y = f(x)$ can be approximated at a by the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

- For a function $y = f(x)$, if x changes from a to $a + dx$, then

$$dy = f'(x)\ dx$$

is an approximation for the change in y. The actual change in y is

$$\Delta y = f(a + dx) - f(a).$$

- A measurement error dx can lead to an error in a calculated quantity $f(x)$. The error in the calculated quantity is known as the propagated error. The propagated error can be estimated by

$$dy \approx f'(x)\ dx.$$

- To estimate the relative error of a particular quantity q, we estimate $\frac{\Delta q}{q}$.

Key Equations

- **Linear approximation**

$L(x) = f(a) + f'(a)(x - a)$

- **A differential**

$dy = f'(x)\ dx$

Glossary

differential
the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)\ dx$

differential form
given a differentiable function $y = f'(x)$, the equation $dy = f'(x)\ dx$ is the differential form of the derivative of y with respect to x

linear approximation
the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

percentage error
the relative error expressed as a percentage

propagated error
the error that results in a calculated quantity $f(x)$ resulting from a measurement error dx

relative error
given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

tangent line approximation (linearization)
since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

Answers

Exercise 15.3.1
$L(x) = f(a) + f'(a)(x - a)$

Exercise 15.3.1
$L(x) = 2 + \frac{1}{12}(x - 8)$; 2.00833

Exercise 15.3.2
$L(x) = f(a) + f'(a)(x - a)$

Exercise 15.3.2
$L(x) = -x + \frac{\pi}{2}$

Exercise 15.3.3
$f'(x) = 4(1 + x)^3$

Exercise 15.3.3
$L(x) = 1 + 4x$

Exercise 15.3.4
$dy = f'(x)\ dx$

Exercise 15.3.4
$dy = 2xe^{x^2}dx$

Exercise 15.3.5
$dy = f'(3)\ dx,\ \ \Delta y = f(3.2) - f(3)$
$dy = 1.6,\ \ \Delta y = 1.64$

Exercise 15.3.6
$dV = 3x^2dx$

Exercise 15.3.6
The volume measurement is accurate to within $21.6\ \text{cm}^3$.

Exercise 15.3.7
Use the fact that $dV = 4\pi r^2 dr$ to find dV/V.

Exercise 15.3.7
7.6%

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15.3E: Exercises for Section 4.2

1) What is the linear approximation for any generic linear function $y = mx + b$?

2) Determine the necessary conditions such that the linear approximation function is constant. Use a graph to prove your result.

Answer

$f'(a) = 0$

3) Explain why the linear approximation becomes less accurate as you increase the distance between x and a. Use a graph to prove your argument.

4) When is the linear approximation exact?

Answer

The linear approximation exact when $y = f(x)$ is linear or constant.

In exercises 5 - 10, find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$ for the function.

5) [T] $f(x) = x + x^4, \quad a = 0$

6) [T] $f(x) = \dfrac{1}{x}, \quad a = 2$

Answer

$L(x) = \frac{1}{2} - \frac{1}{4}(x - 2)$

7) [T] $f(x) = \tan x, \quad a = \frac{\pi}{4}$

8) [T] $f(x) = \sin x, \quad a = \frac{\pi}{2}$

Answer

$L(x) = 1$

9) [T] $f(x) = x \sin x, \quad a = 2\pi$

10) [T] $f(x) = \sin^2 x, \quad a = 0$

Answer

$L(x) = 0$

In exercises 11 - 16, compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x - a)$. Check your answer using a calculator.

11) [T] $(2.001)^6$

12) [T] $\sin(0.02)$

Answer

$\sin(0.02) \approx 0.02$

13) [T] $\cos(0.03)$

14) [T] $(15.99)^{1/4}$

Answer

$(15.99)^{1/4} \approx 1.9996875$

15) [T] $\frac{1}{0.98}$

16) [T] $\sin(3.14)$

Answer

$\sin(3.14) \approx 0.001593$

In exercises 17 - 22, determine the appropriate $f(x)$ and a, and evaluate $L(x) = f(a) + f\,'(a)(x - a)$. Calculate the numerical error in the linear approximations that follow.

17) $(1.01)^3$

18) $\cos(0.01)$

Answer

$\cos(0.01) \approx L(0.01) = f(0) + f'(0)(0 - 0.01) = 1$; error, 0.00005

19) $(\sin(0.01))^2$

20) $(1.01)^{-3}$

Answer

$(1.01)^{-3} \approx L(1.01) = f(1) + f'(1)(1.01 - 1) = 0.97$; error, 0.0006

21) $\left(1 + \frac{1}{10}\right)^{10}$

22) $\sqrt{8.99}$

Answer

$\sqrt{8.99} \approx L(8.99) = f(9) + f'(9)(8.99 - 9) = 3 - \frac{1}{600}$; error, 4.632×10^{-7}

In exercises 23 - 26, find the differential of the function.

23) $y = 3x^4 + x^2 - 2x + 1$

24) $y = x \cos x$

Answer

$dy = (\cos x - x \sin x)\ dx$

25) $y = \sqrt{1 + x}$

26) $y = \dfrac{x^2 + 2}{x - 1}$

Answer

$dy = (\dfrac{x^2 - 2x - 2}{(x - 1)^2})dx$

In exercises 27 - 32, find the differential and evaluate for the given x and dx.

27) $y = 3x^2 - x + 6,\ \ x = 2,\ \ dx = 0.1$

28) $y = \dfrac{1}{x + 1},\ \ x = 1,\ \ dx = 0.25$

Answer

$dy = -\dfrac{1}{(x + 1)^2}\,dx, \quad dy = -\frac{1}{16}$

29) $y = \tan x,\ \ x = 0,\ \ dx = \frac{\pi}{10}$

30) $y = \dfrac{3x^2+2}{\sqrt{x+1}},\ \ x=0,\ \ dx=0.1$

Answer

$$dy = \frac{9x^2+12x-2}{2(x+1)^{3/2}}dx, \quad dy=-0.1$$

31) $y = \dfrac{\sin(2x)}{x},\ \ x=\pi,\ \ dx=0.25$

32) $y = x^3+2x+\dfrac{1}{x},\ \ x=1,\ \ dx=0.05$

Answer

$$dy = \left(3x^2+2-\frac{1}{x^2}\right)dx, \quad dy=0.2$$

In exercises 33 - 38, find the change in volume dV or in surface area dA.

33) dV if the sides of a cube change from 10 to 10.1.

34) dA if the sides of a cube change from x to $x+dx$.

Answer

$$dA = 12x\ dx$$

35) dA if the radius of a sphere changes from r by dr.

36) dV if the radius of a sphere changes from r by dr.

Answer

$$dV = 4\pi r^2 dr$$

37) dV if a circular cylinder with $r=2$ changes height from 3 cm to $3.05cm$.

38) dV if a circular cylinder of height 3 changes from $r=2$ to $r=1.9$ cm.

Answer

$$dV = -1.2\pi\ \text{cm}^3$$

In exercises 39 - 41, use differentials to estimate the maximum and relative error when computing the surface area or volume.

39) A spherical golf ball is measured to have a radius of 5 mm, with a possible measurement error of 0.1 mm. What is the possible change in volume?

40) A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

Answer

$$-100\ \text{ft}^3$$

41) An ice cream cone has height 4 in. and radius 1 in. If the cone is 0.1 in. thick, what is the difference between the volume of the cone, including the shell, and the volume of the ice cream you can fit inside the shell?

In exercises 42 - 44, confirm the approximations by using the linear approximation at $x=0$.

42) $\sqrt{1-x} \approx 1-\frac{1}{2}x$

43) $\dfrac{1}{\sqrt{1-x^2}} \approx 1$

44) $\sqrt{c^2+x^2} \approx c$

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15.4: Maxima and Minima

Learning Objectives

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance, because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty, f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (Figure $15.4.1$).

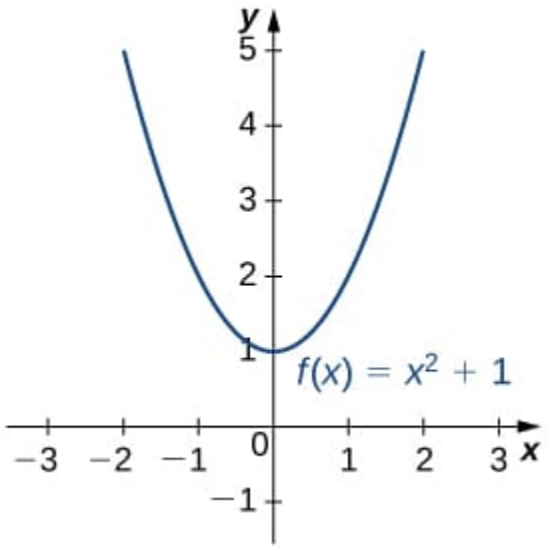


Figure $15.4.1$: The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition: Absolute Extrema

Let f be a function defined over an interval I and let $c \in I$. We say f has an *absolute maximum* on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an *absolute minimum* on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has *an absolute extremum* on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2+1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in Figure $15.4.2$(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure $15.4.2$ shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

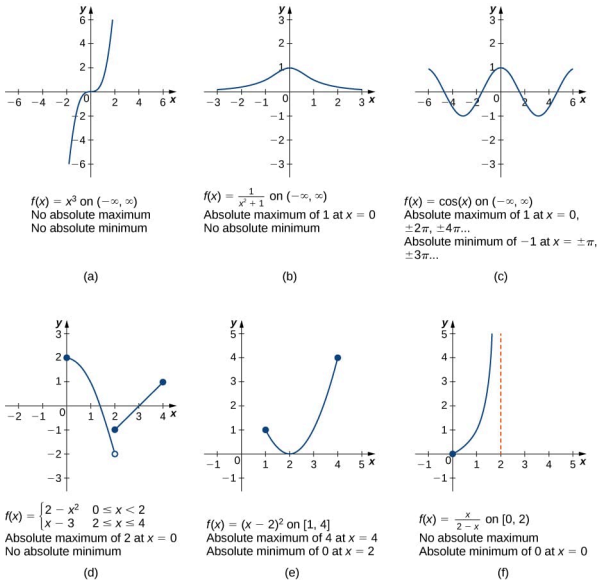


Figure $15.4.2$: Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem $15.4.1$: Extreme Value Theorem

If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in Figure $15.4.2$ (d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in Figure $15.4.3$. The graph can be described as two mountains with a valley in the middle. The

absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.

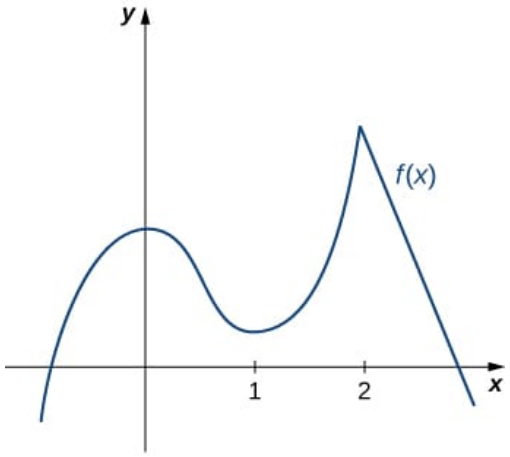


Figure 15.4.3: This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition: Local Extrema

A function f has a *local maximum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a *local minimum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a *local extremum* at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure 15.4.3 again. The local extrema occur at $x = 0, x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition: Critical Points

Let c be an interior point in the domain of f. We say that c is a *critical point* of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as **Fermat's theorem.**

Theorem 15.4.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c,} \tag{15.4.1}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c, but $x > c$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$. From Equation 15.4.1 we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In Figure 15.4.4, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

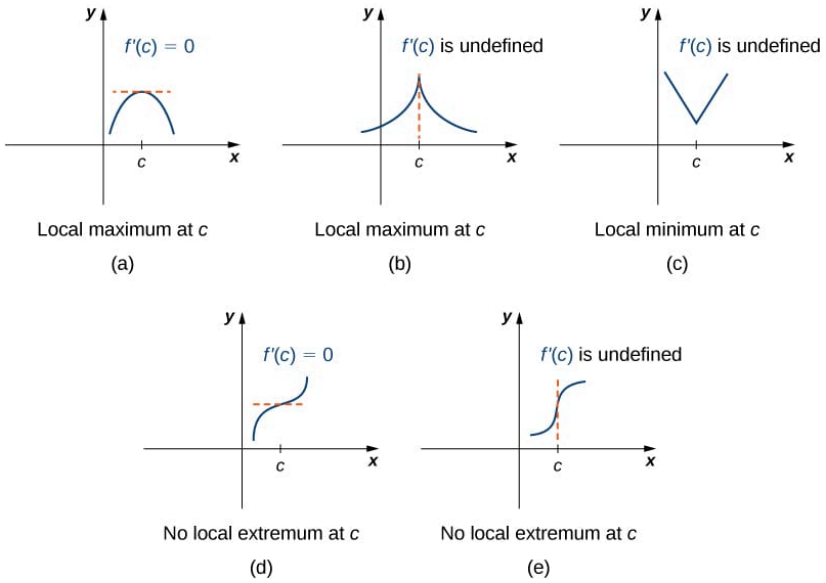


Figure 15.4.4: (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 15.4.1: Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$
b. $f(x) = (x^2 - 1)^3$
c. $f(x) = \frac{4x}{1+x^2}$

Solution

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in Figure 15.4.5, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

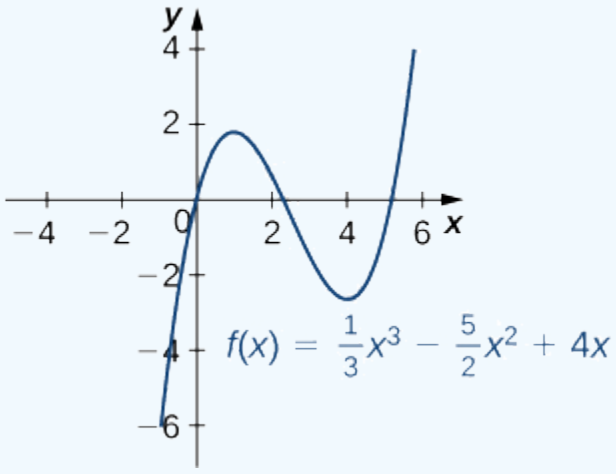


Figure 15.4.5: This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in Figure $15.4.6$, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.

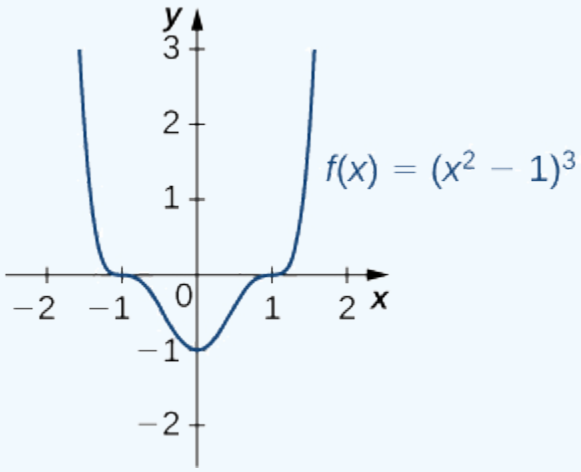


Figure $15.4.6$: This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the quotient rule, we see that the derivative is

$$f'(x) = \frac{4(1+x^2) - 4x(2x)}{(1+x^2)^2} = \frac{4 - 4x^2}{(1+x^2)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in Figure $15.4.7$, we see that f has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I , then f has a local extremum at c.)

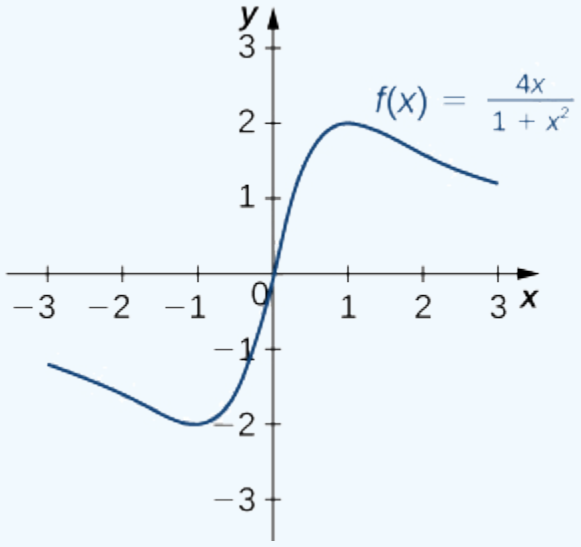


Figure $15.4.7$: This function has an absolute maximum and an absolute minimum.

Exercise 15.4.1

Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Hint available at the end of this section.

Answer available at the end of this section.

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in Figure 15.4.2, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by Fermat's Theorem, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 15.4.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From "Location of Absolute Extrema," the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 15.4.2: Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f(\frac{3}{2})$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f(\frac{3}{2})$ and find

$$f\left(\tfrac{3}{2}\right) = \tfrac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval $[1, 3]$ is -2, and it occurs at $x = 3$ as shown in Figure $15.4.8$.

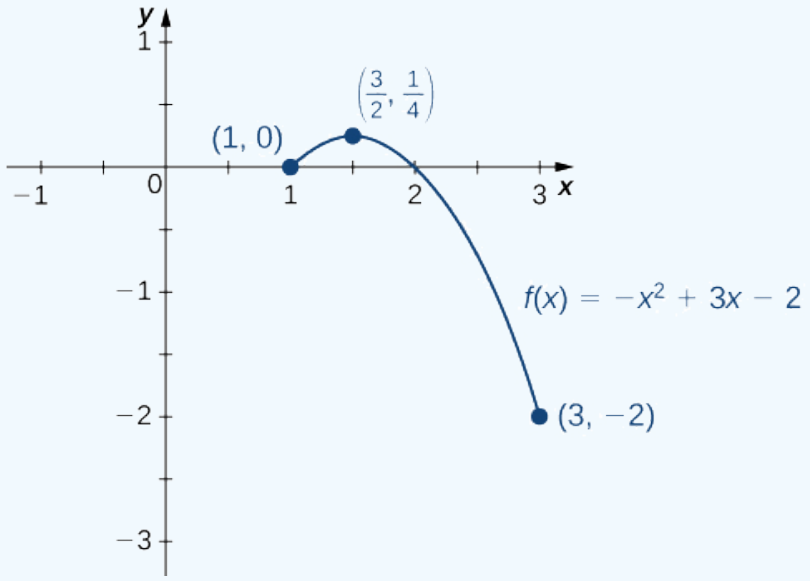


Figure $15.4.8$: This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3\,(2)^{2/3} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \tfrac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval $[0, 2]$ is zero, and it occurs at $x = 0$. The absolute minimum is -2, and it occurs at $x = 1$ as shown in Figure 15.4.9.

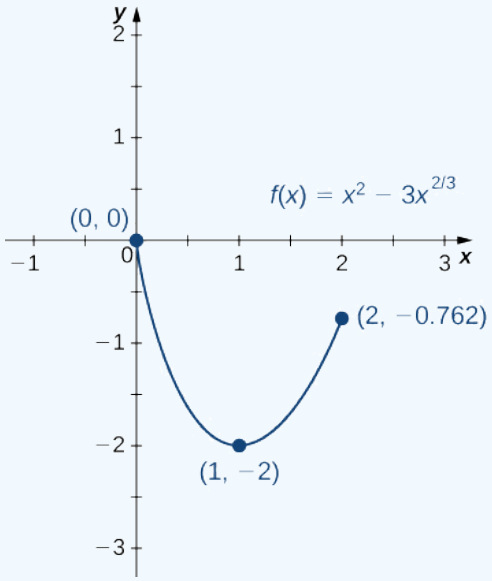


Figure 15.4.9: This function has an absolute maximum at an endpoint of the interval.

Exercise 15.4.2

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Hint available at the end of this section.

Answer available at the end of this section.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

Glossary

absolute extremum
: if f has an absolute maximum or absolute minimum at c, we say f has an absolute extremum at c

absolute maximum
: if $f(c) \geq f(x)$ for all x in the domain of f, we say f has an absolute maximum at c

absolute minimum
: if $f(c) \leq f(x)$ for all x in the domain of f, we say f has an absolute minimum at c

critical point
: if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

extreme value theorem
: if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem
: if f has a local extremum at c, then c is a critical point of f

local extremum
: if f has a local maximum or local minimum at c, we say f has a local extremum at c

local maximum
: if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum
: if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

Answers

Exercise 15.4.1

Calculate $f'(x)$.

Exercise 15.4.1

$x = \frac{-2}{3}, x = 1$

Exercise 15.4.2

Look for critical points. Evaluate f at all critical points and at the endpoints.

Exercise 15.4.2

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

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15.4E: Exercises for Section 4.3

1) In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $m = -\frac{b}{2a}$. Prove this formula using calculus.

2) If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

Answer

On a closed interval, the endpoints often lie above or below any local (relative) extrema. Answers may vary for the graph.

3) If you are examining a function over an interval (a, b), for a and b finite, is it possible not to have an absolute maximum or absolute minimum?

4) When you are checking for critical points to locate the extrema of a function f, explain why you also need to determine points where $f'(x)$ is undefined. Draw a graph to support your explanation.

Answer

Points on the graph of f where there is a corner, a cusp, or a jump discontinuity or removable discontinuity can easily be absolute (or local) extrema of the function. Answers may vary for the graph.

5) Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

6) Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain why or why not using graphical arguments.

Answer

No; answers will vary

7) Let m be the number of local minima and M be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

8) Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

Answer

Since the absolute maximum is the function (output) value rather than the x value, the answer is no; answers will vary

9) Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

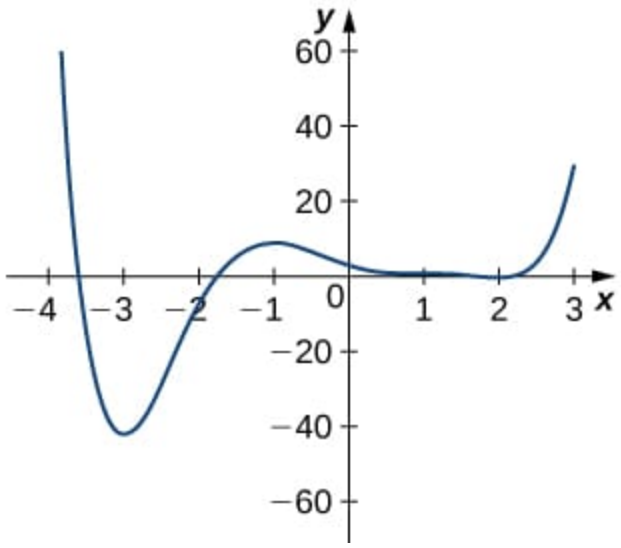
10) [T] Graph the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?

Answer

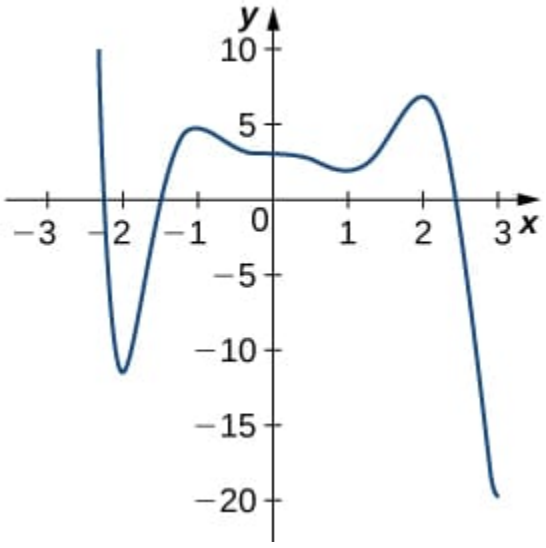
When $a = 0$

In exercises 11 - 14, determine where the local and absolute maxima and minima occur on the graph given. Assume domains are closed intervals unless otherwise specified.

11)



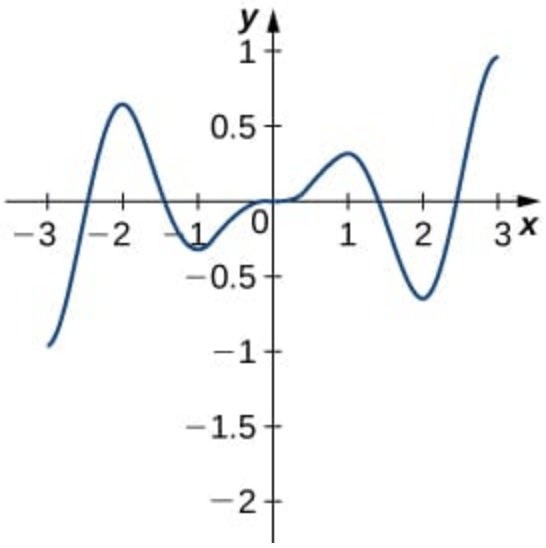
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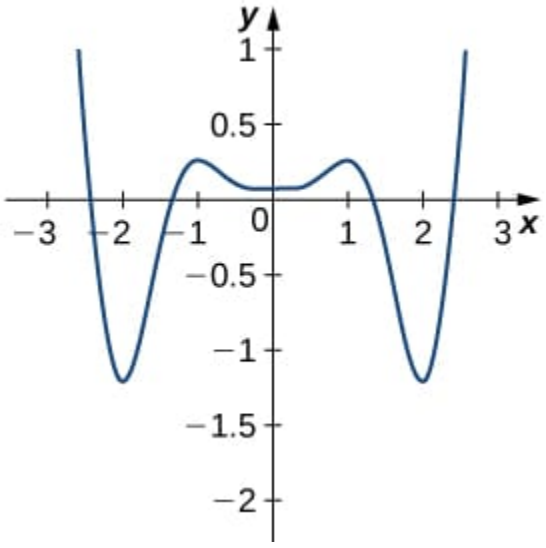
Answer

Absolute minimum at 3; Absolute maximum at −2.2; local minima at −2, 1; local maxima at −1, 2

13)



14)



Answer

Absolute minima at –2, 2; absolute maxima at –2.5, 2.5; local minimum at 0; local maxima at –1, 1

For exercises 15 - 18, draw graphs of $f(x)$, which is continuous, over the interval $[-4, 4]$ with the following properties:

15) Absolute maximum at $x = 2$ and absolute minima at $x = \pm 3$

16) Absolute minimum at $x = 1$ and absolute maximum at $x = 2$

Answer

Answers may vary.

17) Absolute maximum at $x = 4$, absolute minimum at $x = -1$, local maximum at $x = -2$, and a critical point that is not a maximum or minimum at $x = 2$

18) Absolute maxima at $x = 2$ and $x = -3$, local minimum at $x = 1$, and absolute minimum at $x = 4$

Answer

Answers may vary.

In exercises 19 - 28, find the critical points in the domains of the given functions.

19) $y = 4x^3 - 3x$

20) $y = 4\sqrt{x} - x^2$

Answer

$x = 1$

21) $y = \dfrac{1}{x-1}$

22) $y = \ln(x-2)$

Answer

None

23) $y = \tan(x)$

24) $y = \sqrt{4-x^2}$

Answer

$x = 0$

25) $y = x^{3/2} - 3x^{5/2}$

26) $y = \dfrac{x^2-1}{x^2+2x-3}$

Answer

None

27) $y = \sin^2(x)$

28) $y = x + \dfrac{1}{x}$

Answer

$x = -1$ and $x = 1$

In exercises 29 - 39, find the local and/or absolute maxima for the functions over the specified domain.

29) $f(x) = x^2 + 3$ over $[-1, 4]$

30) $y = x^2 + \dfrac{2}{x}$ over $[1, 4]$

Answer

Absolute maximum: $x = 4, y = \frac{33}{2}$; absolute minimum: $x = 1, y = 3$

31) $y = (x - x^2)^2$ over $[-1, 1]$

32) $y = \dfrac{1}{x - x^2}$ over $[0, 1]$

Answer

Absolute minimum: $x = \frac{1}{2}, y = 4$

33) $y = \sqrt{9-x}$ over $[1, 9]$

34) $y = x + \sin(x)$ over $[0, 2\pi]$

Answer

Absolute maximum: $x = 2\pi, y = 2\pi$; absolute minimum: $x = 0, y = 0$

35) $y = \dfrac{x}{1+x}$ over $[0, 100]$

36) $y = |x+1| + |x-1|$ over $[-3, 2]$

Answer

Absolute maximum: $x = -3, y = 6$; absolute minimum: $-1 \le x \le 1, y = 2$

37) $y = \sqrt{x} - \sqrt{x^3}$ over $[0, 4]$

38) $y = \sin x + \cos x$ over $[0, 2\pi]$

Answer

Absolute maximum: $x = \frac{\pi}{4}, y = \sqrt{2}$; absolute minimum: $x = \frac{5\pi}{4}, y = -\sqrt{2}$

39) $y = 4\sin\theta - 3\cos\theta$ over $[0, 2\pi]$

In exercises 40 - 45, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

40) $y = x^2 + 4x + 5$

Answer

Absolute minimum: $x = -2, y = 1$

41) $y = x^3 - 12x$

42) $y = 3x^4 + 8x^3 - 18x^2$

Answer

Absolute minimum: $x = -3, y = -135$; local maximum: $x = 0, y = 0$; local minimum: $x = 1, y = -7$

43) $y = x^3(1-x)^6$

44) $y = \dfrac{x^2 + x + 6}{x - 1}$

Answer

Local maximum: $x = 1 - 2\sqrt{2}, y = 3 - 4\sqrt{2}$; local minimum: $x = 1 + 2\sqrt{2}, y = 3 + 4\sqrt{2}$

45) $y = \dfrac{x^2 - 1}{x - 1}$

In exercises 46 - 50, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

46) [T] $y = 3x\sqrt{1 - x^2}$

Answer

Absolute maximum: $x = \frac{\sqrt{2}}{2}, y = \frac{3}{2}$; absolute minimum: $x = -\frac{\sqrt{2}}{2}, y = -\frac{3}{2}$

47) [T] $y = x + \sin(x)$

48) [T] $y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$

Answer

Local maximum: $x = -2, y = 59$; local minimum: $x = 1, y = -130$

49) [T] $y = \dfrac{x^3 + 6x^2 - x - 30}{x - 2}$

50) [T] $y = \dfrac{\sqrt{4-x^2}}{\sqrt{4+x^2}}$

Answer

Absolute maximum: $x = 0, y = 1$; absolute minimum: $x = -2, 2, y = 0$

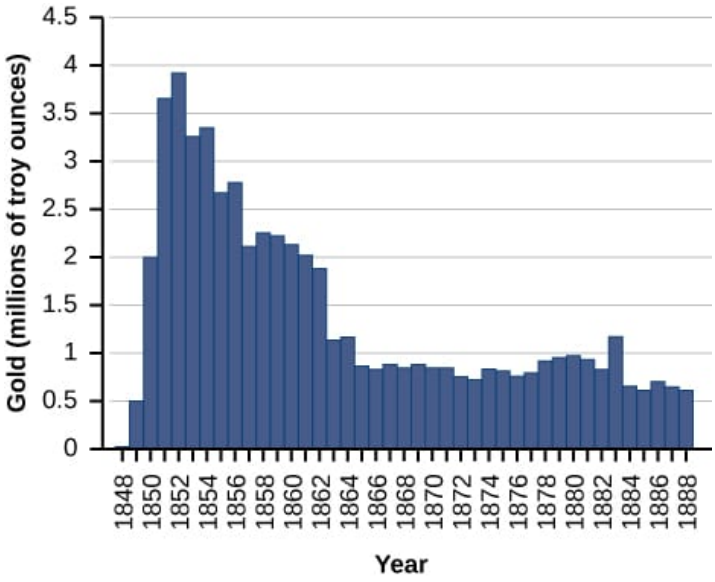
51) A company that produces cell phones has a cost function of $C = x^2 - 1200x + 36,400$, where C is cost in dollars and x is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

52) A ball is thrown into the air and its position is given by $h(t) = -4.9t^2 + 60t + 5m$. Find the height at which the ball stops ascending. How long after it is thrown does this happen?

Answer

$h = \frac{9245}{49}$ m, $t = \frac{300}{49}$ s

For exercises 53-54, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G(t) = \dfrac{(25t)}{(t^2+16)}$, where t is the number of years since the rush began $(0 \le t \le 40)$ and G is ounces of gold produced (in millions). A summary of the data is shown in the following figure.



53) Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

54) Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Answer

The global minimum was in 1848, when no gold was produced.

In exercises 55 & 56, find the critical points, maxima, and minima for the given piecewise functions.

55) \(y= \begin{cases} x^2−4x, & \text{if }0≤x≤1\x^2−4, & \text{if }1

56) $y = \begin{cases} x^2 + 1, & \text{if } x \le 1 \\ x^2 - 4x + 5, & \text{if } x > 1 \end{cases}$

Answer

Absolute minima: $x = 0, x = 2, y = 1$; local maximum at $x = 1, y = 2$

In exercises 57 - 58, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

57) $y = ax^2 + bx + c$, given that $a > 0$

58) $y = (x - 1)^a$, given that $a > 1$

Answer

No maxima/minima if a is odd, minimum at $x = 1$ if a is even

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added answers for exercises 2 and 4.

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15.5: The Mean Value Theorem

Learning Objectives

- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

The *Mean Value Theorem* is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure $15.5.1$ illustrates this theorem.

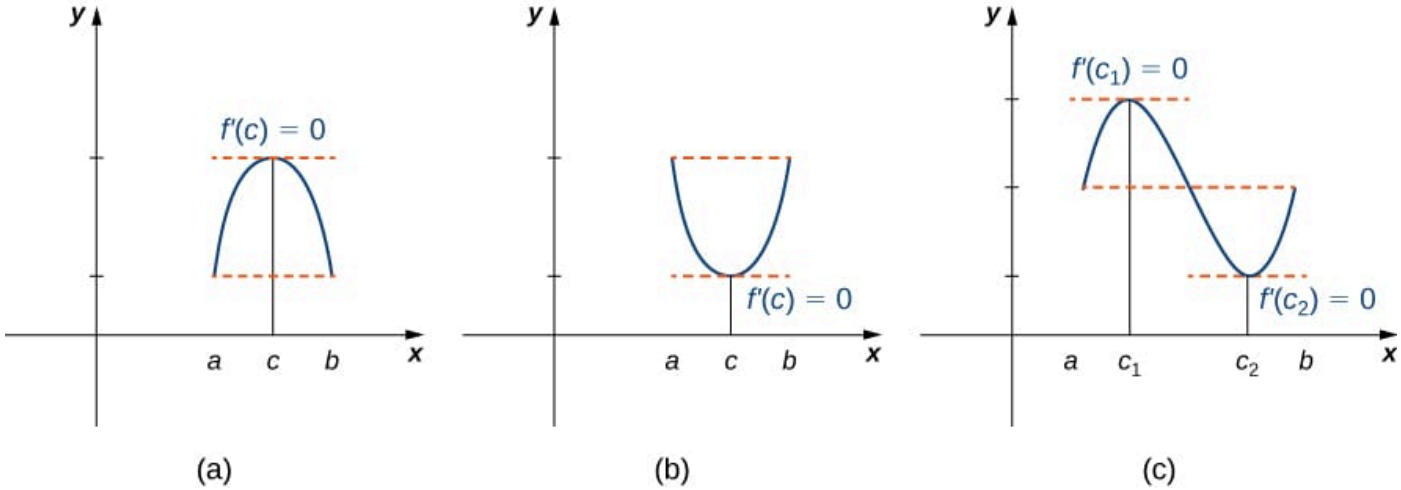


Figure $15.5.1$: If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.
2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that \(f(x)

Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that \(f(x)

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

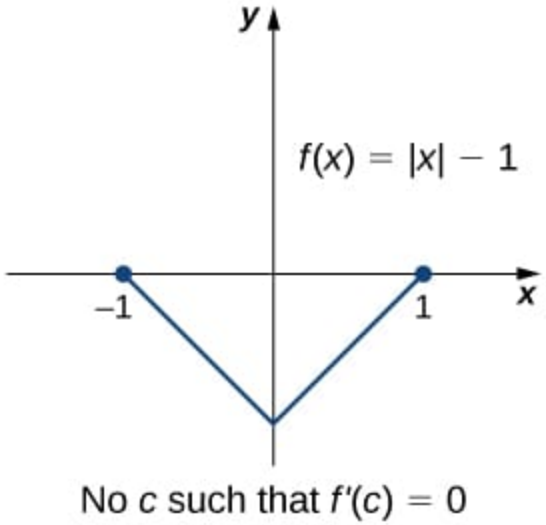


Figure 15.5.2: Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0$.

Example 15.5.1: Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$
b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

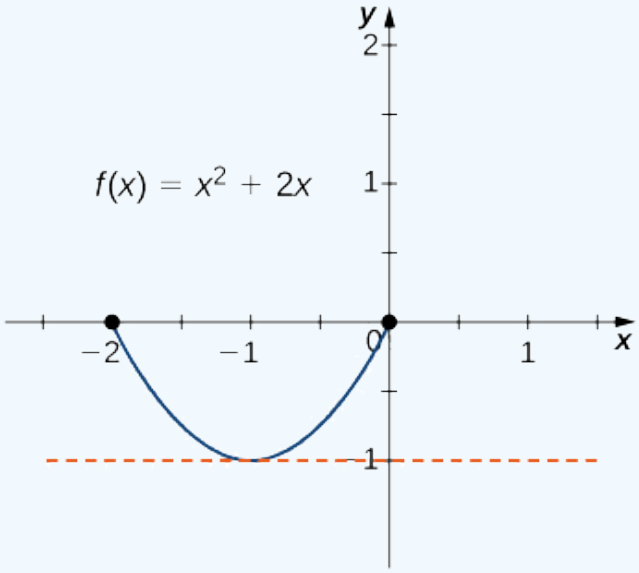


Figure 15.5.3: This function is continuous and differentiable over [−2,0], $f'(c) = 0$ when $c = -1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.

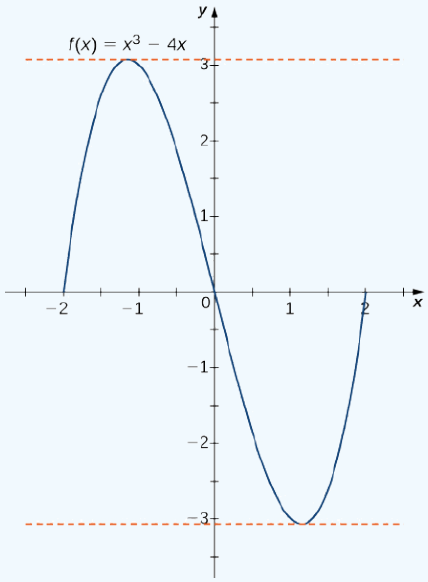


Figure 15.5.4: For this polynomial over $[-2, 2]$, $f'(c) = 0$ at $x = \pm 2/\sqrt{3}$.

Exercise 15.5.1

Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

Hint available at the end of this section.

Answer available at the end of this section.

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f that are zero at the endpoints. The Mean Value Theorem generalizes Rolle's theorem by considering functions that are not necessarily zero at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (Figure $15.5.5$). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

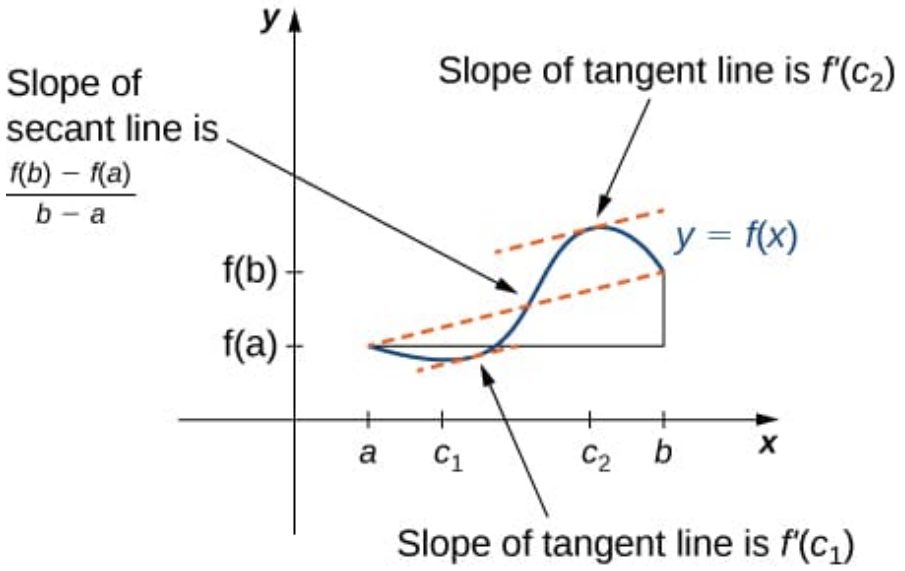


Figure $15.5.5$: The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

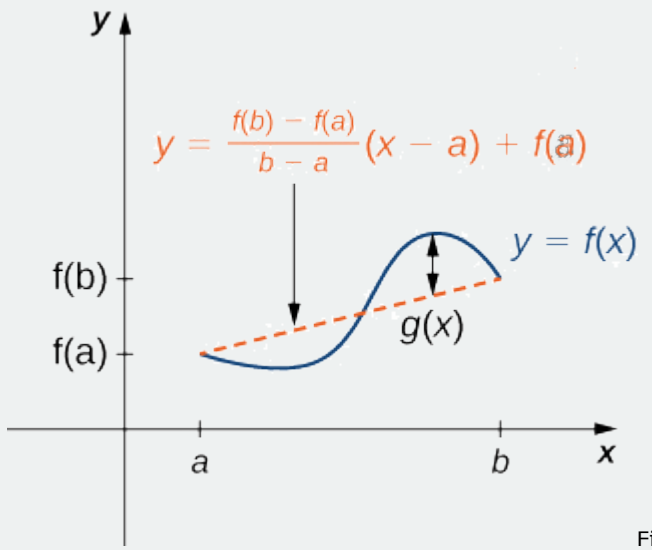


Figure 15.5.6: The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 15.5.2: Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (Figure 15.5.7). To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9)-f(0)}{9-0}=\frac{\sqrt{9}-\sqrt{0}}{9-0}=\frac{3}{9}=\frac{1}{3}.$$

We want to find c such that $f'(c)=\frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}}=\frac{1}{3}.$$

Solving this equation for c, we obtain $c=\frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

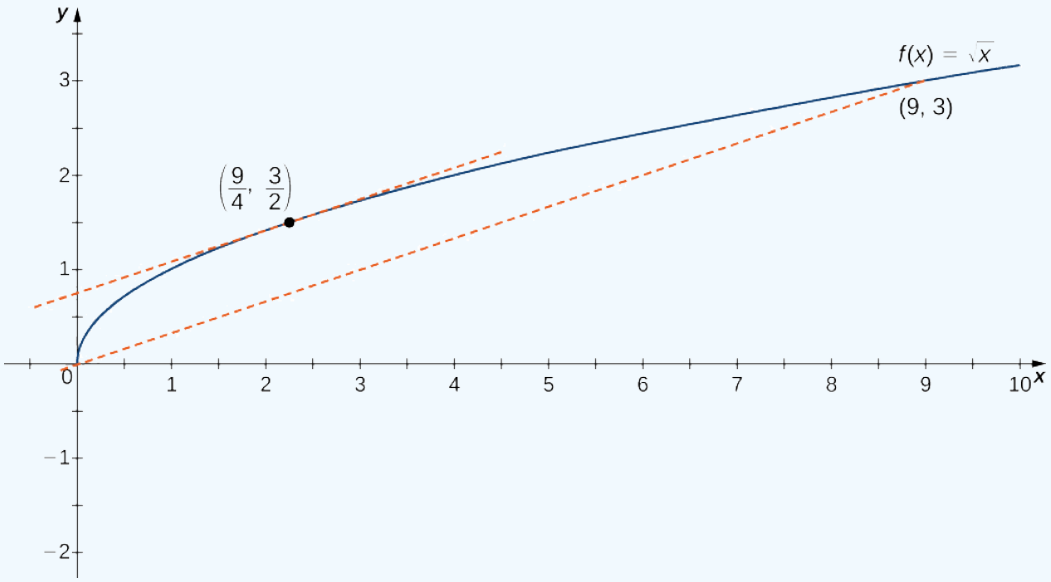


Figure 15.5.7: The slope of the tangent line at $c=9/4$ is the same as the slope of the line segment connecting (0,0) and (9,3).

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0\le t\le 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c\in(0,1)$, the speed of the car was exactly

$$v(c)=s'(c)=\frac{s(1)-s(0)}{1-0}=45\,\text{mph}.$$

Example 15.5.3: Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t)=-16t^2+100$.

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solution

a. When the rock hits the ground, its position is $s(t)=0$. Solving the equation $-16t^2+100=0$ for t, we find that $t=\pm\frac{5}{2}sec.$ Since we are only considering $t\ge 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.

b. The average velocity is given by

$$v_{avg}=\frac{s(5/2)-s(0)}{5/2-0}=\frac{0-100}{5/2}=-40\,\text{ft/sec}.$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t) = s\,'(t) = v_{avg} = -40$ ft/sec. Since $s(t)$ is continuous over the interval $[0, 5/2]$ and differentiable over the interval $(0, 5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c \in (0, 5/2)$ such that

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$, we find that $s\,'(t) = -32t$. Therefore, the equation reduces to $s\,'(c) = -32c = -40$. Solving this equation for c, we have $c = \frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.

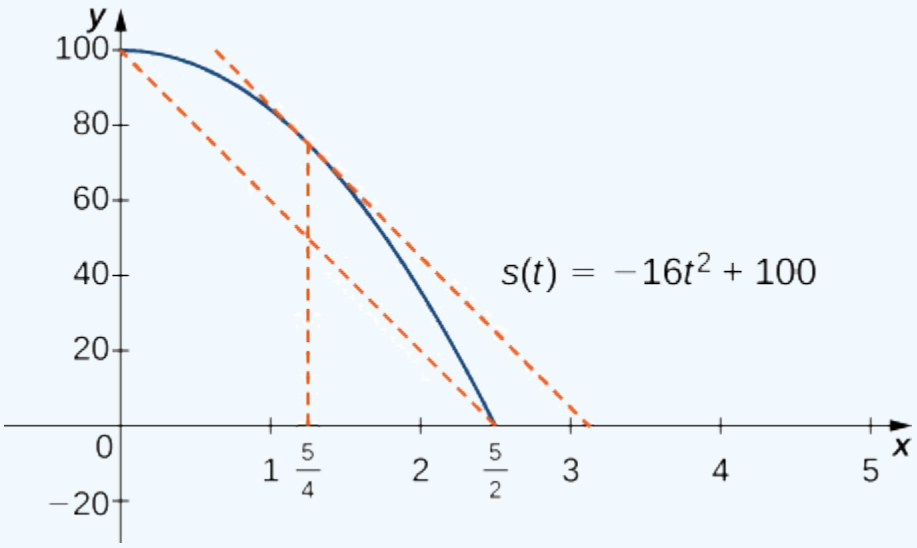


Figure 15.5.8: At time $t = 5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

Exercise 15.5.2

Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Hint available at the end of this section.

Answer available at the end of this section.

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f\,'(x) = 0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f\,'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof

Since f is differentiable over I, f must be continuous over I. Suppose $f(x)$ is not constant for all x in I. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that \(a

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f\,'(c) \neq 0$, which contradicts the assumption that $f\,'(x) = 0$ for all $x \in I$.

□

From "Corollary 1: Functions with a Derivative of Zero," it follows that if two functions have the same derivative, they differ by, at most, a constant.

Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f\,'(x) = g\,'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof

Let $h(x) = f(x) - g(x)$. Then, $h\,'(x) = f\,'(x) - g\,'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if \(f(x_1)f(x_2)\) whenever \(x_1

This fact is important because it means that for a given function f, if there exists a function F such that $F\,'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C. We discuss this result in more detail later in the chapter.

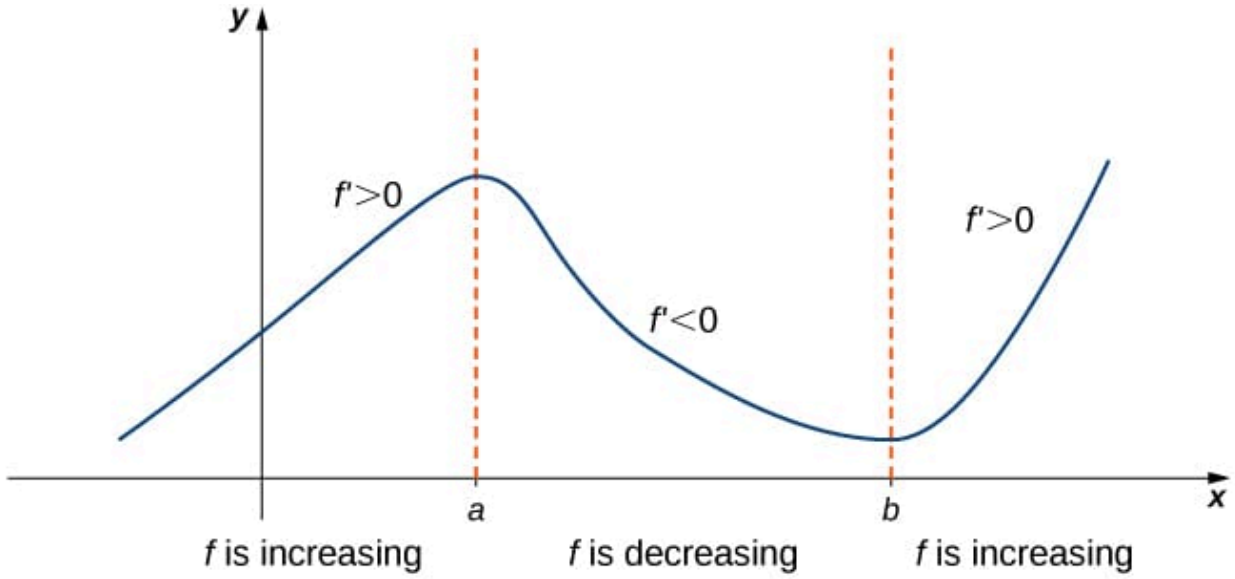


Figure 15.5.9: If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.

Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.
ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I. Then there exist a and b in I such that \(a

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, \(a0.\) We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I.

□

Key Concepts

- If f is continuous over $[a, b]$ and differentiable over (a, b) and $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. This is Rolle's theorem.
- If f is continuous over $[a, b]$ and differentiable over (a, b), then there exists a point $c \in (a, b)$ such that

  $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  This is the Mean Value Theorem.
- If $f'(x) = 0$ over an interval I, then f is constant over I.

- If two differentiable functions f and g satisfy $f\,'(x) = g\,'(x)$ over I, then $f(x) = g(x) + C$ for some constant C.
- If $f\,'(x) > 0$ over an interval I, then f is increasing over I. If $f\,'(x) < 0$ over I, then f is decreasing over I.

Glossary

mean value theorem

if f is continuous over $[a, b]$ and differentiable over (a, b), then there exists $c \in (a, b)$ such that $f\,'(c) = \frac{f(b)-f(a)}{b-a}$

rolle's theorem

if f is continuous over $[a, b]$ and differentiable over (a, b), and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f\,'(c) = 0$

Answers

Exercise $15.5.1$

Find all values c, where $f'(c) = 0$.

Exercise $15.5.1$

$c = 2$

Exercise $15.5.2$

First, determine how long it takes for the ball to hit the ground. Then, find the average velocity of the ball from the time it is dropped until it hits the ground.

Exercise $15.5.2$

$\frac{5}{2\sqrt{2}}$ sec

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15.5E: Exercises for Section 4.4

1)Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

2) Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

Answer

One example is $f(x) = |x| + 3, -2 \leq x \leq 2$

3) When are Rolle's theorem and the Mean Value Theorem equivalent?

4) If you have a function with a discontinuity, is it still possible to have $f\,'(c)(b-a) = f(b) - f(a)$? Draw such an example or prove why not.

Answer

Yes, but the Mean Value Theorem still does not apply

In exercises 5 - 9, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

5) $y = \sin(\pi x)$

6) $y = \dfrac{1}{x^3}$

Answer

$(-\infty, 0), (0, \infty)$

7) $y = \sqrt{4 - x^2}$

8) $y = \sqrt{x^2 - 4}$

Answer

$(-\infty, -2), (2, \infty)$

9) $y = \ln(3x - 5)$

In exercises 10 - 13, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f\,'(c)(b-a) = f(b) - f(a)$.

10) [T] $y = 3x^3 + 2x + 1$ over $[-1, 1]$

Answer

2 points

11) [T] $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

12) [T] $y = x^2 \cos(\pi x)$ over $[-2, 2]$

Answer

5 points

13) [T] $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

In exercises 14 - 19, use the Mean Value Theorem and find all points \(0such that $f(2) - f(0) = f\,'(c)(2-0)$.

14) $f(x) = x^3$

Answer

$c = \frac{2\sqrt{3}}{3}$

15) $f(x) = \sin(\pi x)$

16) $f(x) = \cos(2\pi x)$

Answer

$c = \frac{1}{2}, 1, \frac{3}{2}$

17) $f(x) = 1 + x + x^2$

18) $f(x) = (x-1)^{10}$

Answer

$c = 1$

19) $f(x) = (x-1)^9$

In exercises 20 - 23, show there is no c such that $f(1) - f(-1) = f\,'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

20) $f(x) = \left|x - \frac{1}{2}\right|$

Answer

Not differentiable

21) $f(x) = \dfrac{1}{x^2}$

22) $f(x) = \sqrt{|x|}$

Answer

Not differentiable

23) $f(x) = \lfloor x \rfloor$ (Hint: This is called the floor function and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

In exercises 24 - 34, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

24) $y = e^x$ over $[0, 1]$

Answer

Yes

25) $y = \ln(2x + 3)$ over $[-\frac{3}{2}, 0]$

26) $f(x) = \tan(2\pi x)$ over $[0, 2]$

Answer

The Mean Value Theorem does not apply since the function is discontinuous at $x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$.

27) $y = \sqrt{9 - x^2}$ over $[-3, 3]$

28) $y = \dfrac{1}{|x+1|}$ over $[0, 3]$

Answer

Yes

29) $y = x^3 + 2x + 1$ over $[0, 6]$

30) $y = \dfrac{x^2 + 3x + 2}{x}$ over $[-1, 1]$

Answer

The Mean Value Theorem does not apply; discontinuous at $x = 0$.

31) $y = \dfrac{x}{\sin(\pi x) + 1}$ over $[0, 1]$

32) $y = \ln(x + 1)$ over $[0, e - 1]$

Answer

Yes

33) $y = x \sin(\pi x)$ over $[0, 2]$

34) $y = 5 + |x|$ over $[-1, 1]$

Answer

The Mean Value Theorem does not apply; not differentiable at $x = 0$.

For exercises 35 - 37, consider the roots of each equation.

35) Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root. What is it?

36) Find the conditions for exactly one root (double root) for the equation $y = x^2 + bx + c$

Answer

$b = \pm 2\sqrt{c}$

37) Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

In exercises 38 - 42, use a calculator to graph the function over the interval $[a, b]$ and graph the secant line from a to b. Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of c, if possible, or write the final equation and use a calculator to estimate to four digits.

38) [T] $y = \tan(\pi x)$ over $\left[-\frac{1}{4}, \frac{1}{4}\right]$

Answer

$c \approx \pm 0.1533$
$c = \pm \frac{1}{\pi} \cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$

39) [T] $y = \dfrac{1}{\sqrt{x + 1}}$ over $[0, 3]$

40) [T] $y = |x^2 + 2x - 4|$ over $[-4, 0]$

Answer

The Mean Value Theorem does not apply.

41) [T] $y = x + \dfrac{1}{x}$ over $\left[\frac{1}{2}, 4\right]$

42) [T] $y = \sqrt{x + 1} + \dfrac{1}{x^2}$ over $[3, 8]$

Answer

$$\frac{1}{2\sqrt{c + 1}} - \frac{2}{c^3} = \frac{521}{2880}$$
$c \approx 3.133, 5.867$

43) At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at

10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

44) Two cars drive from one stoplight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

Answer

Yes

45) Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

46) Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

Answer

It is constant.

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15.6: Derivatives and the Shape of a Graph

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point c, then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x)=x^3$ has a critical point at $x=0$ since $f'(x)=3x^2$ is zero at $x=0$, but f does not have a local extremum at $x=0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval I, then the function is decreasing over I as shown in the following figure.

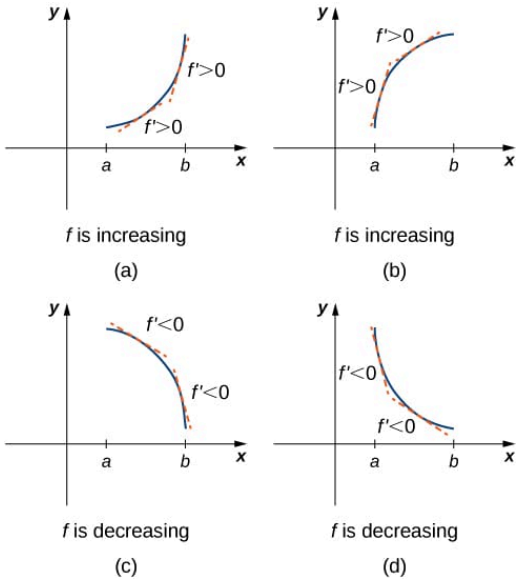


Figure $\PageIndex{1}$: Both functions are increasing over the interval (a,b). At each point x, the derivative $f'(x)>0$. Both functions are decreasing over the interval (a,b). At each point x, the derivative $f'(x)<0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f' can change sign as x increases through c is if $f'(c)=0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c)=0$ or $f'(c)$ is undefined. Consequently, to locate

local extrema for a function \(f\), we look for points \(c\) in the domain of \(f\) such that \(f'(c)=0\) or \(f'(c)\) is undefined. Recall that such points are called critical points of \(f\).

Note that \(f\) need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure \(\PageIndex{2}\), we show that if a continuous function \(f\) has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if \(f\) has a local extremum at a critical point, then the sign of \(f'\) switches as \(x\) increases through that point.

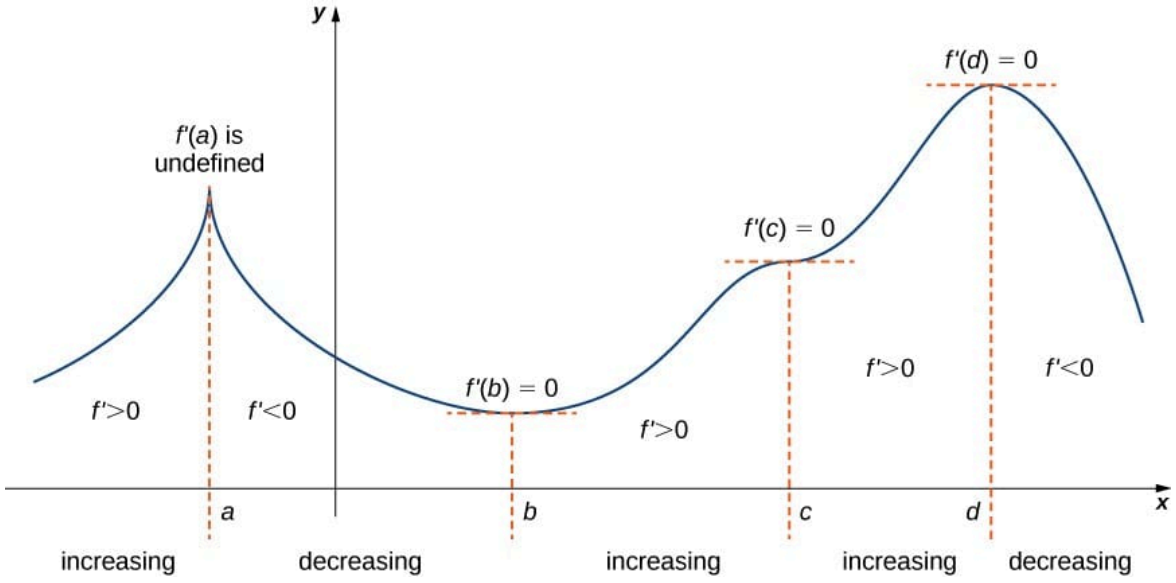


Figure \(\PageIndex{2}\): The function \(f\) has four critical points: \(a,b,c\),and \(d\). The function \(f\) has local maxima at \(a\) and \(d\), and a local minimum at \(b\). The function \(f\) does not have a local extremum at \(c\). The sign of \(f'\) changes at all local extrema.

Using Figure \(\PageIndex{2}\), we summarize the main results regarding local extrema.

- If a continuous function \(f\) has a local extremum, it must occur at a critical point \(c\).
- The function has a local extremum at the critical point \(c\) if and only if the derivative \(f'\) switches sign as \(x\) increases through \(c\).
- Therefore, to test whether a function has a local extremum at a critical point \(c\), we must determine the sign of \(f'(x)\) to the left and right of \(c\).

This result is known as the **first derivative test.**

First Derivative Test

Suppose that \(f\) is a continuous function over an interval \(I\) containing a critical point \(c\). If \(f\) is differentiable over \(I\), except possibly at point \(c\), then \(f(c)\) satisfies one of the following descriptions:

i. If \(f'\) changes sign from positive when \(xc\), then \(f(c)\) is a local maximum of \(f\).
ii. If \(f'\) changes sign from negative when \(xc\), then \(f(c)\) is a local minimum of \(f\).
iii. If \(f'\) has the same sign for \(xc\), then \(f(c)\) is neither a local maximum nor a local minimum of \(f\)

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example \(\PageIndex{1}\): Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for \(f(x)=x^3−3x^2−9x−1.\) Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is $f'(x)=3x^2-6x-9.$ To find the critical points, we need to find where $f'(x)=0.$ Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2-2x-3)=3(x-3)(x+1)=0. \nonumber$$

Therefore, the critical points are $x=3,-1.$ Now divide the interval $(-\infty,\infty)$ into the smaller intervals $(-\infty,-1),(-1,3)$ and $(3,\infty).$

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty,-1),(-1,3)$ and $(3,\infty)$ and determine the sign of f' at each of these points. For example, let's choose $x=-2$, $x=0$, and $x=4$ as test points.

Table: $\PageIndex{1}$: First Derivative Test for $f(x)=x^3-3x^2-9x-1.$

Interval	Test Point	Sign of $f'(x)=3(x-3)(x+1)$ at Test Point	Conclusion
$(-\infty,-1)$	$x=-2$	$(+)(-)(-)=+$	f is increasing.
$(-1,3)$	$x=0$	$(+)(-)(+)=-$	f is decreasing.
$(3,\infty)$	$x=4$	$(+)(+)(+)=+$	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through -1, f has a local maximum at $x=-1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x=3$. These analytical results agree with the following graph.

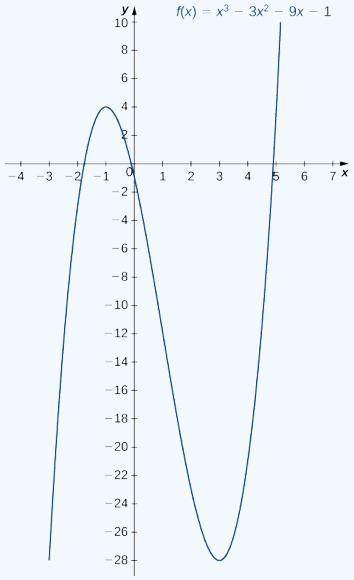


Figure $\PageIndex{3}$: The function f has a maximum at $x=-1$ and a minimum at $x=3$

Exercise $\PageIndex{1}$

Use the first derivative test to locate all local extrema for $f(x)=-x^3+\frac{3}{2}x^2+18x.$

Hint available at the end of this section.

Answer available at the end of this section.

Example \(\PageIndex{2}\): Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for \(f(x)=5x^{1/3}−x^{5/3}.\) Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

$$f'(x)=\frac{5}{3}x^{−2/3}−\frac{5}{3}x^{2/3}=\frac{5}{3x^{2/3}}−\frac{5x^{2/3}}{3}=\frac{5−5x^{4/3}}{3x^{2/3}}=\frac{5(1−x^{4/3})}{3x^{2/3}}.$$

The derivative \(f'(x)=0\) when \(1−x^{4/3}=0.\) Therefore, \(f'(x)=0\) at \(x=±1\). The derivative \(f'(x)\) is undefined at \(x=0.\) Therefore, we have three critical points: \(x=0\), \(x=1\), and \(x=−1\). Consequently, divide the interval \((−∞,∞)\) into the smaller intervals \((−∞,−1),\,(−1,0),\,(0,1)\), and \((1,∞)\).

Step 2: Since \(f'\) is continuous over each subinterval, it suffices to choose a test point \(x\) in each of the intervals from step 1 and determine the sign of \(f'\) at each of these points. The points \(x=−2,\,x=−\frac{1}{2},\,x=\frac{1}{2}\), and \(x=2\) are test points for these intervals.

Table: \(\PageIndex{2}\): First Derivative Test for \(f(x)=5x^{1/3}−x^{5/3}.\)

Interval	Test Point	Sign of \(f'(x)=\frac{5(1−x^{4/3})}{3x^{2/3}}\) at Test Point	Conclusion
\((−∞,−1)\)	\(x=−2\)	\(\frac{(+)(−)}{+}=−\)	\(f\) is decreasing.
\((−1,0)\)	\(x=−\frac{1}{2}\)	\(\frac{(+)(+)}{+}=+\)	\(f\) is increasing.
\((0,1)\)	\(x=\frac{1}{2}\)	\(\frac{(+)(+)}{+}=+\)	\(f\) is increasing.
\((1,∞)\)	\(x=2\)	\(\frac{(+)(−)}{+}=−\)	\(f\) is decreasing.

Step 3: Since \(f\) is decreasing over the interval \((−∞,−1)\) and increasing over the interval \((−1,0)\), \(f\) has a local minimum at \(x=−1\). Since \(f\) is increasing over the interval \((−1,0)\) and the interval \((0,1)\), \(f\) does not have a local extremum at \(x=0\). Since \(f\) is increasing over the interval \((0,1)\) and decreasing over the interval \((1,∞)\), \(f\) has a local maximum at \(x=1\). The analytical results agree with the following graph.

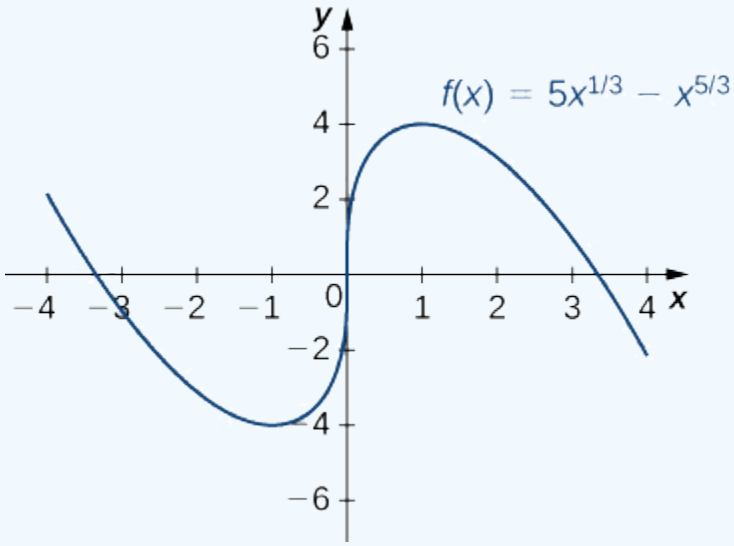


Figure \(\PageIndex{4}\): The function \(f\) has a local minimum at \(x=−1\) and a local maximum at \(x=1\)

Exercise \(\PageIndex{2}\)

Use the first derivative test to find all local extrema for \(f(x)=\dfrac{3}{x−1}\).

Hint available at the end of this section.

Answer available at the end of this section.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure \(\PageIndex{5a}\) shows a function \(f\) with a graph that curves upward. As \(x\) increases, the slope of the tangent line increases. Thus, since the derivative increases as \(x\) increases, \(f'\) is an increasing function. We say this function \(f\) is concave up. Figure \(\PageIndex{5b}\) shows a function \(f\) that curves downward. As \(x\) increases, the slope of the tangent line decreases. Since the derivative decreases as \(x\) increases, \(f'\) is a decreasing function. We say this function \(f\) is concave down.

Definition: concavity test

Let \(f\) be a function that is differentiable over an open interval \(I\). If \(f'\) is increasing over \(I\), we say \(f\) is **concave up** over \(I\). If \(f'\) is decreasing over \(I\), we say \(f\) is **concave down** over \(I\).

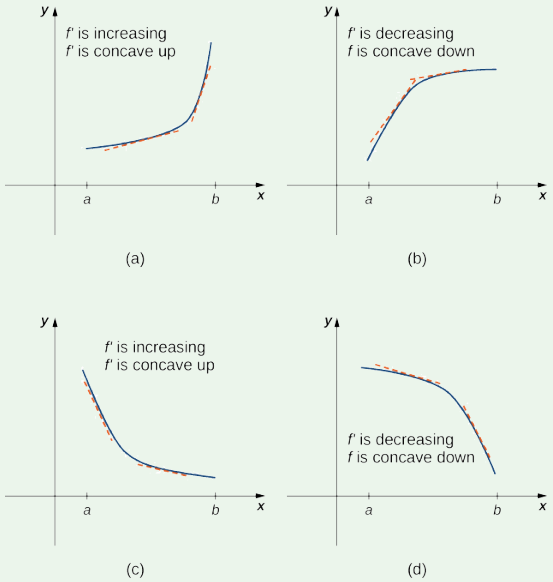


Figure \(\PageIndex{5}\): (a), (c) Since \(f'\) is increasing over the interval \((a,b)\), we say \(f\) is concave up over \((a,b)\). (b), (d)\) Since \(f'\) is decreasing over the interval \((a,b)\), we say \(f\) is concave down over \((a,b).\)

In general, without having the graph of a function \(f\), how can we determine its concavity? By definition, a function \(f\) is concave up if \(f'\) is increasing. From Corollary \(3\), we know that if \(f'\) is a differentiable function, then \(f'\) is increasing if its derivative \(f''(x)>0\). Therefore, a function \(f\) that is twice differentiable is concave up when \(f''(x)>0\). Similarly, a function \(f\) is concave down if \(f'\) is decreasing. We know that a differentiable function \(f'\) is decreasing if its derivative \(f''(x)<0\). Therefore, a twice-differentiable function \(f\) is concave down when \(f''(x)<0\). Applying this logic is known as the *concavity test*.

Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x)>0$ for all $x\in I$, then f is concave up over I
ii. If $f''(x)<0$ for all $x\in I,$ then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure \PageIndex{6}). However, a continuous function can switch concavity only at a point x if $f''(x)=0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x)=0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x)=0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a,f(a))$ is an **inflection point** of f.

Definition: inflection point

If f is continuous at a and f changes concavity at a, the point $(a, \,f(a))$ is an *inflection point* of f.

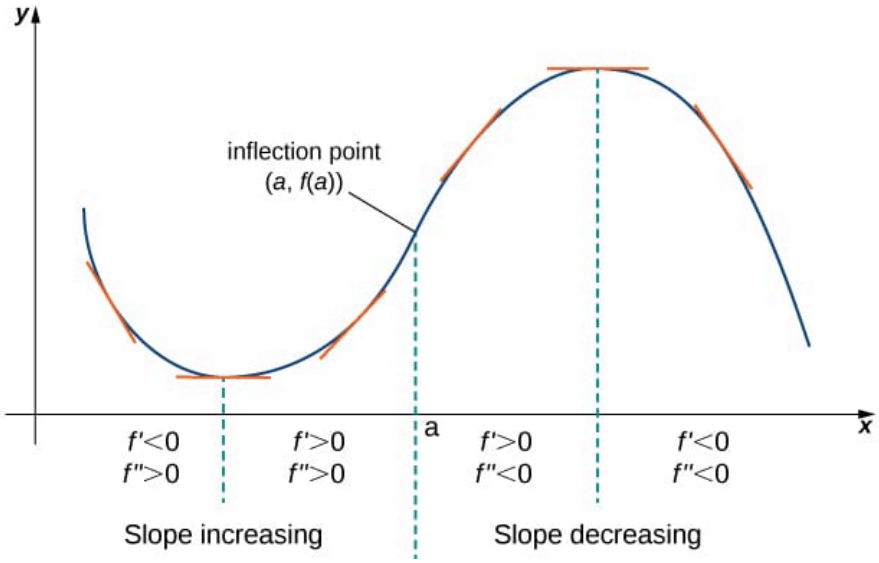


Figure \PageIndex{6}: Since $f''(x)>0$ for xa, the function f is concave down over the interval (a,∞). The point $(a,f(a))$ is an inflection point of f.

Example \PageIndex{3}: Testing for Concavity

For the function $f(x)=x^3−6x^2+9x+30,$ determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x).$ The first derivative is $f'(x)=3x^2−12x+9,$ so the second derivative is $f''(x)=6x−12.$ If the function changes concavity, it occurs either when $f''(x)=0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x)=0$. Solving the equation $6x−12=0$, we see that $x=2$ is the only place where f could change concavity. We now test points over the intervals $(−\infty,2)$ and $(2,\infty)$ to determine the concavity of f. The points $x=0$ and $x=3$ are test points for these intervals.

Table: \(\PageIndex{3}\): Test for Concavity for \(f(x)=x^3−6x^2+9x+30.\)

Interval	Test Point	Sign of \(f''(x)=6x−12\) at Test Point	Conclusion
\((−∞,2)\)	\(x=0\)	−	\(f\) is concave down
\((2,∞)\)	\(x=3\)	+	\(f\) is concave up

We conclude that \(f\) is concave down over the interval \((−∞,2)\) and concave up over the interval \((2,∞)\). Since \(f\) changes concavity at \(x=2\), the point \((2,f(2))=(2,32)\) is an inflection point. Figure \(\PageIndex{7}\) confirms the analytical results.

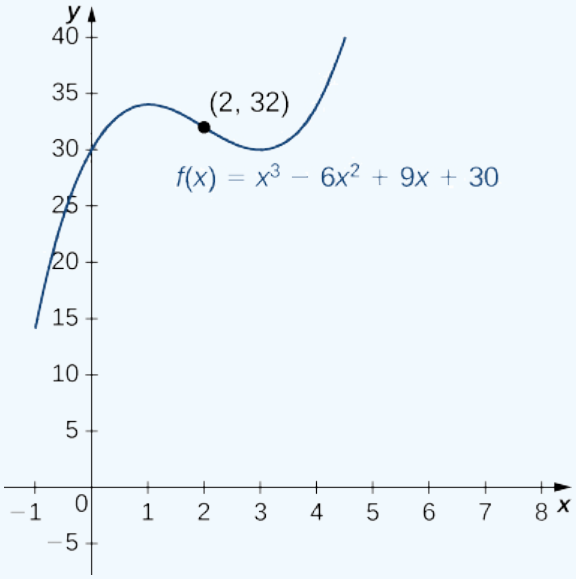


Figure \(\PageIndex{7}\): The given function has a point of inflection at \((2,32)\) where the graph changes concavity.

Exercise \(\PageIndex{3}\)

For \(f(x)=−x^3+\frac{3}{2}x^2+18x\), find all intervals where \(f\) is concave up and all intervals where \(f\) is concave down.

Hint available at the end of this section.

Answer available at the end of this section.

We now summarize, in Table \(\PageIndex{4}\), the information that the first and second derivatives of a function \(f\) provide about the graph of \(f\), and illustrate this information in Figure \(\PageIndex{8}\).

Table: \(\PageIndex{4}\): What Derivatives Tell Us about Graphs

Sign of \(f'\)	Sign of \(f''\)	Is \(f\) increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

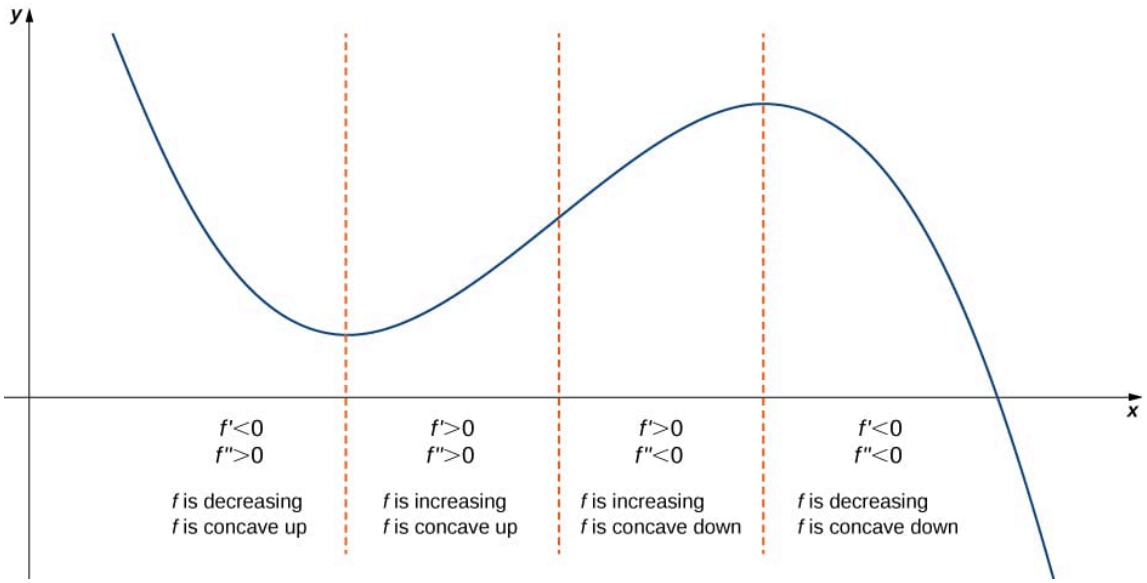


Figure \(\PageIndex{8}\):Consider a twice-differentiable function \(f\) over an open interval \(I\). If \(f'(x)>0\) for all \(x∈I\), the function is increasing over \(I\). If \(f'(x)<0\) for all \(x∈I\), the function is decreasing over \(I\). If \(f''(x)>0\) for all \(x∈I\), the function is concave up. If \(f''(x)<0\) for all \(x∈I\), the function is concave down on \(I\).

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extremum, it must occur at a critical point. However, a function need not have a local extremum at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let \(f\) be a twice-differentiable function such that \(f'(a)=0\) and \(f''\) is continuous over an open interval \(I\) containing \(a\). Suppose \(f''(a)<0\). Since \(f''\) is continuous over \(I, f''(x)<0\) for all \(x∈I\) (Figure \(\PageIndex{9}\)). Then, by Corollary \(3\), \(f'\) is a decreasing function over \(I\). Since \(f'(a)=0\), we conclude that for all \(x∈I, \,f'(x)>0\) if \(xa\). Therefore, by the first derivative test, \(f\) has a local maximum at \(x=a\).

On the other hand, suppose there exists a point \(b\) such that \(f'(b)=0\) but \(f''(b)>0\). Since \(f''\) is continuous over an open interval \(I\) containing \(b\), then \(f''(x)>0\) for all \(x∈I\) (Figure \(\PageIndex{9}\)). Then, by Corollary \(3\), \(f'\) is an increasing function over \(I\). Since \(f'(b)=0\), we conclude that for all \(x∈I\), \(f'(x)<0\) if \(x0\) if \(x>b\). Therefore, by the first derivative test, \(f\) has a local minimum at \(x=b.\)

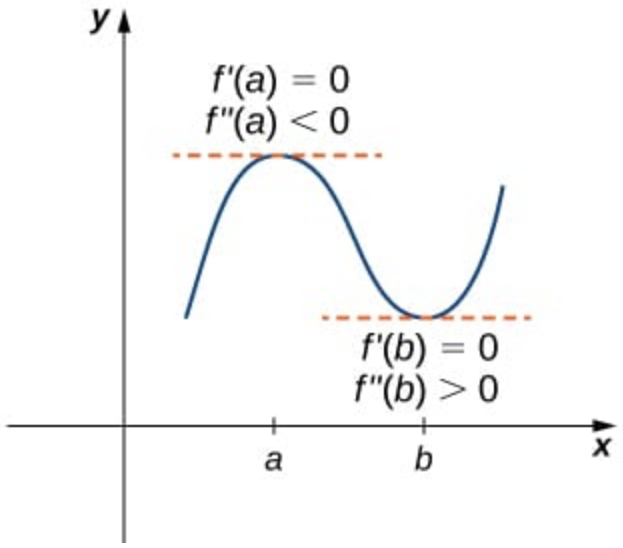


Figure \(\PageIndex{9}\): Consider a twice-differentiable function \(f\) such that \(f''\) is continuous. Since \(f'(a)=0\) and \(f''(a)<0\), there is an interval \(I\) containing \(a\) such that for all \(x\) in \(I\), \(f\) is increasing if \(xa\). As a result, \(f\) has a local maximum at \(x=a\). Since \(f'(b)=0\) and \(f''(b)>0\), there is an interval \(I\) containing \(b\) such that for all \(x\) in \(I\), \(f\) is decreasing if \(xb\). As a result, \(f\) has a local minimum at \(x=b\).

Second Derivative Test

Suppose \(f'(c)=0\) and \(f''\) is continuous over an interval containing \(c\).

i. If \(f''(c)>0\), then \(f\) has a local minimum at \(c\).
ii. If \(f''(c)<0\), then \(f\) has a local maximum at \(c\).
iii. If \(f''(c)=0,\) then the test is inconclusive.

Note that for case iii. when \(f''(c)=0\), then \(f\) may have a local maximum, local minimum, or neither at \(c\). For example, the functions \(f(x)=x^3, \; f(x)=x^4,\) and \(f(x)=−x^4\) all have critical points at \(x=0\). In each case, the second derivative is zero at \(x=0\). However, the function \(f(x)=x^4\) has a local minimum at \(x=0\) whereas the function \(f(x)=−x^4\) has a local maximum at \(x=0\), and the function \(f(x)=x^3\) does not have a local extremum at \(x=0\).

Let's now look at how to use the second derivative test to determine whether \(f\) has a local maximum or local minimum at a critical point \(c\) where \(f'(c)=0.\)

Example \(\PageIndex{4}\): Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for \(f(x)=x^5−5x^3.\)

Solution

To apply the second derivative test, we first need to find critical points \(c\) where \(f'(c)=0\). The derivative is \(f'(x)=5x^4−15x^2\). Therefore, \(f'(x)=5x^4−15x^2=5x^2(x^2−3)=0\) when \(x=0,\,±\sqrt{3}\).

To determine whether \(f\) has a local extremum at any of these points, we need to evaluate the sign of \(f''\) at these points. The second derivative is

\(f''(x)=20x^3−30x=10x(2x^2−3).\)

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether \(f\) has a local maximum or local minimum at any of these points.

Table: \(\PageIndex{5}\): Second Derivative Test for \(f(x)=x^5−5x^3.\)

\(x\)	\(f''(x)\)	Conclusion
\(−\sqrt{3}\)	\(−30\sqrt{3}\)	Local maximum
\(0\)	\(0\)	Second derivative test is inconclusive
\(\sqrt{3}\)	\(30\sqrt{3}\)	Local minimum

By the second derivative test, we conclude that \(f\) has a local maximum at \(x=−\sqrt{3}\) and \(f\) has a local minimum at \(x=\sqrt{3}\). The second derivative test is inconclusive at \(x=0\). To determine whether \(f\) has a local extrema at \(x=0,\) we apply the first derivative test. To evaluate the sign of \(f'(x)=5x^2(x^2−3)\) for \(x∈(−\sqrt{3},0)\) and \(x∈(0,\sqrt{3})\), let \(x=−1\) and \(x=1\) be the two test points. Since \(f'(−1)<0\) and \(f'(1)<0\), we conclude that \(f\) is decreasing on both intervals and, therefore, \(f\) does not have a local extrema at \(x=0\) as shown in the following graph.

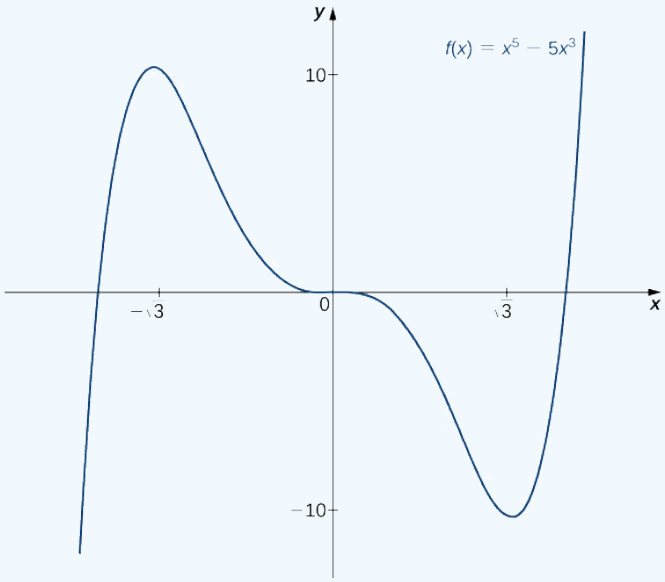


Figure \(\PageIndex{10}\):The function \(f\) has a local maximum at \(x=−\sqrt{3}\) and a local minimum at \(x=\sqrt{3}\)

Exercise \(\PageIndex{4}\)

Consider the function \(f(x)=x^3−(\frac{3}{2})x^2−18x\). The points \(c=3,\,−2\) satisfy \(f'(c)=0\). Use the second derivative test to determine whether \(f\) has a local maximum or local minimum at those points.

Hint available at the end of this section.

Answer available at the end of this section.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as \(x→±∞.\) At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Key Concepts

- If \(c\) is a critical point of \(f\) and \(f'(x)>0\) for \(xc\), then \(f\) has a local maximum at \(c\).
- If \(c\) is a critical point of \(f\) and \(f'(x)<0\) for \(x0\) for \(x>c,\) then \(f\) has a local minimum at \(c\).
- If \(f''(x)>0\) over an interval \(I\), then \(f\) is concave up over \(I\).
- If \(f''(x)<0\) over an interval \(I\), then \(f\) is concave down over \(I\).
- If \(f'(c)=0\) and \(f''(c)>0\), then \(f\) has a local minimum at \(c\).
- If \(f'(c)=0\) and \(f''(c)<0\), then \(f\) has a local maximum at \(c\).

- If $f'(c)=0$ and $f''(c)=0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

Glossary

concave down
: if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up
: if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity
: the upward or downward curve of the graph of a function

concavity test
: suppose f is twice differentiable over an interval I; if $f''>0$ over I, then f is concave up over I; if $f''<$ over I, then f is concave down over I

first derivative test
: let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

inflection point
: if f is continuous at c and f changes concavity at c, the point $(c,f(c))$ is an inflection point of f

second derivative test
: suppose $f'(c)=0$ and f'' is continuous over an interval containing c; if $f''(c)>0$, then f has a local minimum at c; if $f''(c)<0$, then f has a local maximum at c; if $f''(c)=0$, then the test is inconclusive

Answers

Exercise $\PageIndex{1}$
Find all critical points of f and determine the signs of $f'(x)$ over particular intervals determined by the critical points.

Exercise $\PageIndex{1}$
f has a local minimum at -2 and a local maximum at 3.

Exercise $\PageIndex{2}$
The only critical point of f is $x=1.$

Exercise $\PageIndex{2}$
f has no local extrema because f' does not change sign at $x=1$.

Exercise $\PageIndex{3}$
Find where $f''(x)=0$

Exercise $\PageIndex{3}$
f is concave up over the interval $(-\infty,\frac{1}{2})$ and concave down over the interval $(\frac{1}{2},\infty)$

Exercise $\PageIndex{4}$
$f''(x)=6x-3$

Exercise $\PageIndex{4}$
f has a local maximum at -2 and a local minimum at 3.

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15.6E: Exercises for Section 4.5

1) If c is a critical point of $f(x)$, when is there no local maximum or minimum at c? Explain.

2) For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

Answer

It is not a local maximum/minimum because f' does not change sign

3) For the function $y = x^3$, is $x = 0$ an inflection point?

4) Is it possible for a point c to be both an inflection point and a local extremum of a twice differentiable function?

Answer

No

5) Why do you need continuity for the first derivative test? Come up with an example.

6) Explain whether a concave-down function has to cross $y = 0$ for some value of x.

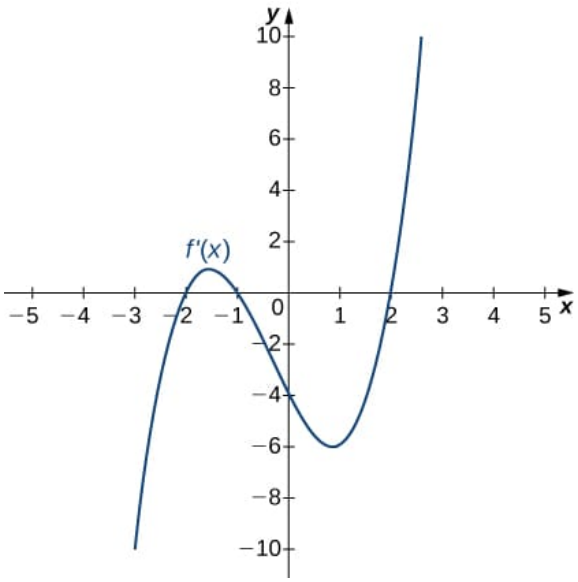
Answer

False; for example, $y = \sqrt{x}$.

7) Explain whether a polynomial of degree 2 can have an inflection point.

In exercises 8 - 12, analyze the graphs of f', then list all intervals where f is increasing or decreasing.

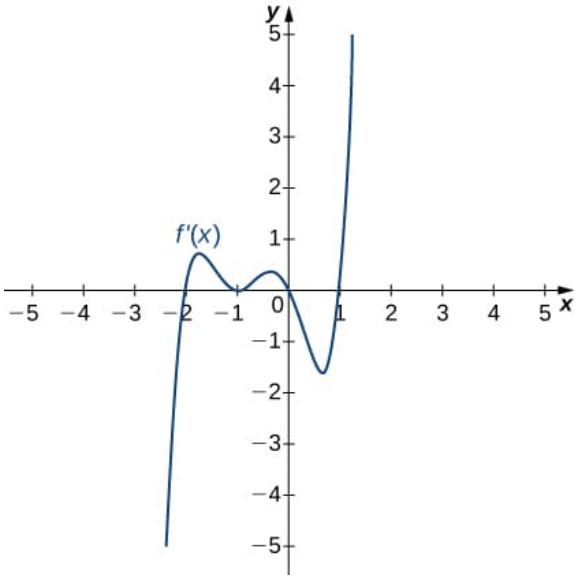
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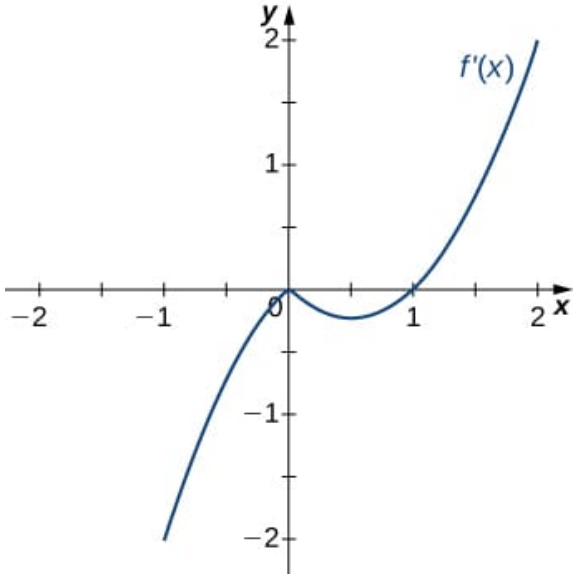
Answer

Increasing for \(−22\);
Decreasing for $x < -2$ and \(−1

9)



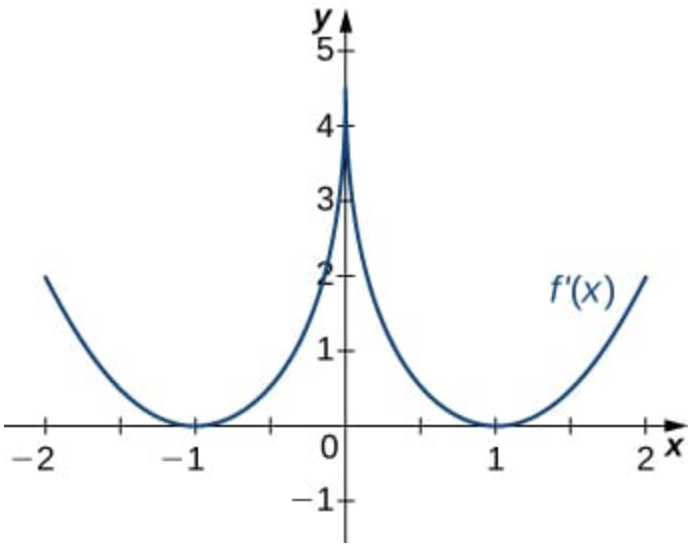
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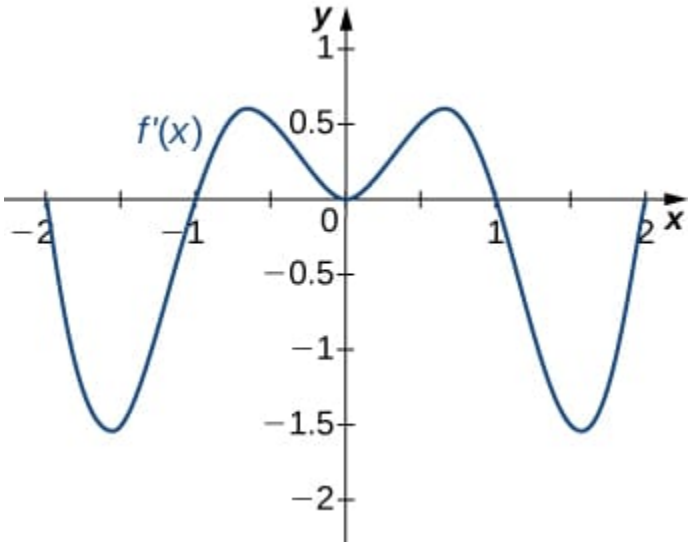
Answer

Decreasing for $x < 1$,
Increasing for $x > 1$

11)



12)



Answer

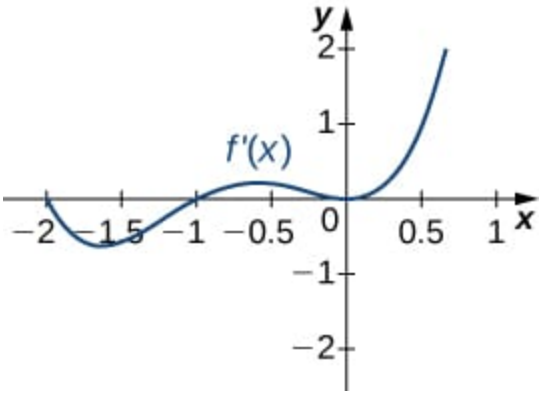
Decreasing for \(–2 Increasing for \(–12\)

In exercises 13 - 17, analyze the graphs of f', then list all intervals where

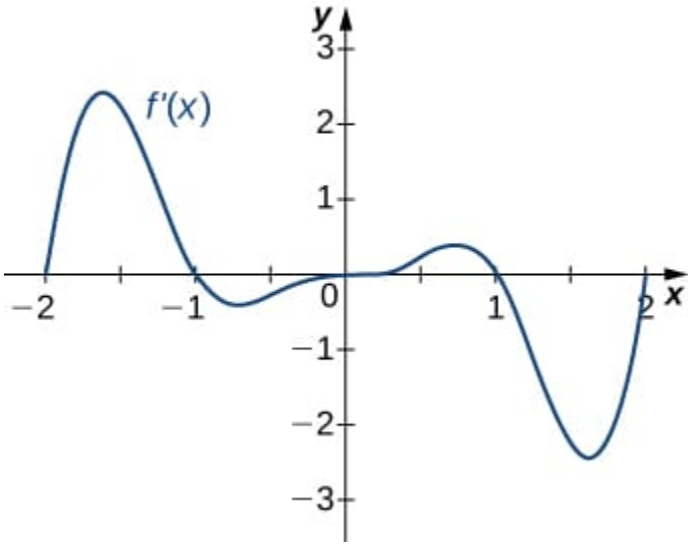
a. f is increasing and decreasing and

b. the minima and maxima are located.

13)



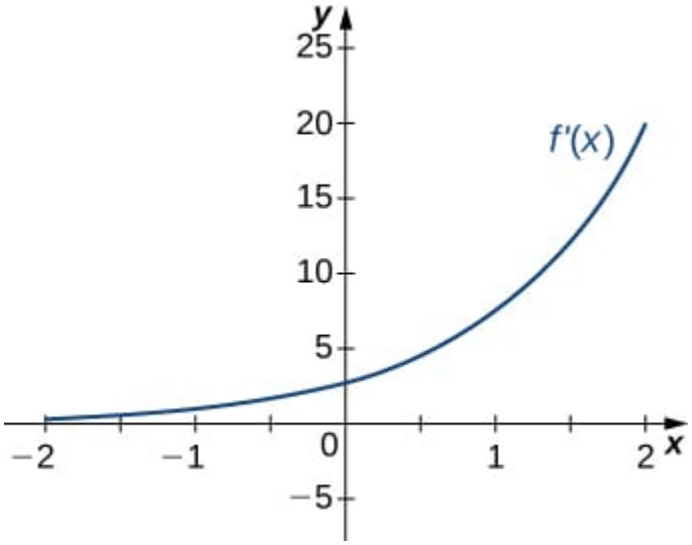
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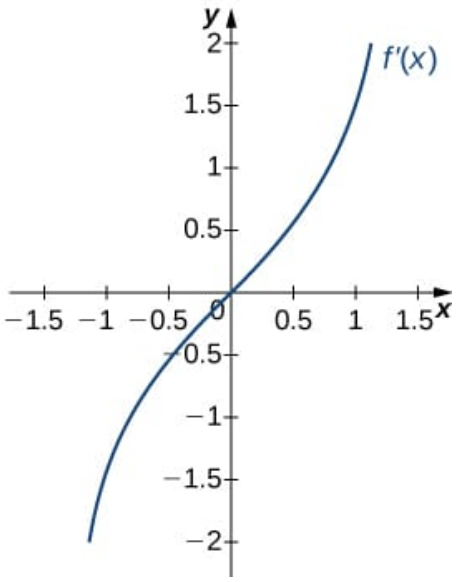
Answer

a. Increasing over \(−22\), Decreasing over \(x<−2, \;−1 b. Maxima at $x = -1$ and $x = 1$, Minima at $x = -2$ and $x = 0$ and $x = 2$

15)



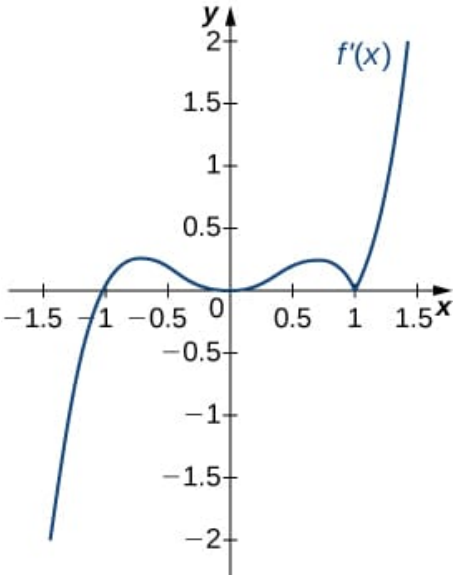
16)



Answer

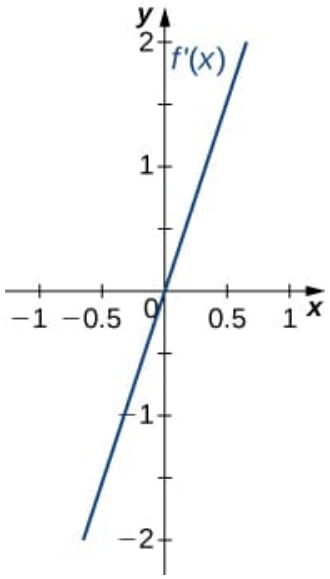
a. Increasing over $x > 0$, Decreasing over $x < 0$;
b. Minimum at $x = 0$

17)



In exercises 18 - 22, analyze the graphs of f', then list all inflection points and intervals f that are concave up and concave down.

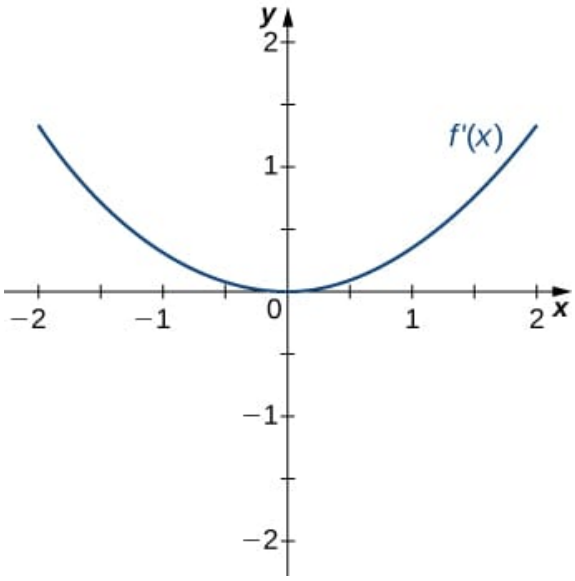
18)



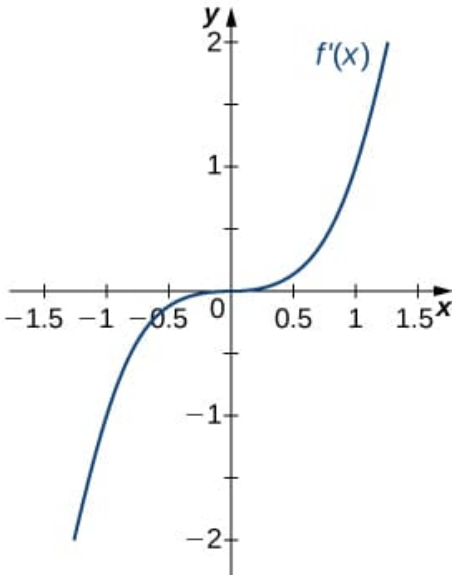
Answer

Concave up for all x,
No inflection points

19)



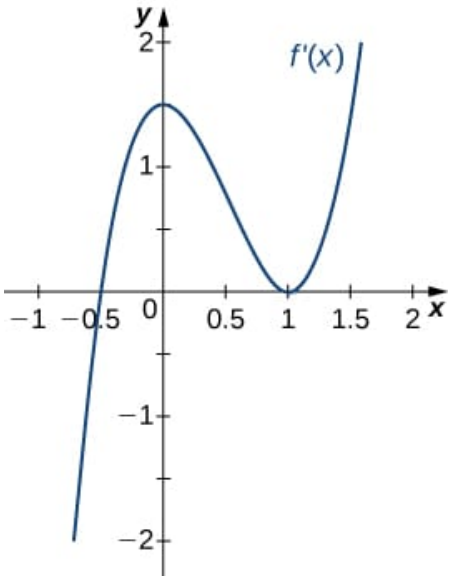
20)



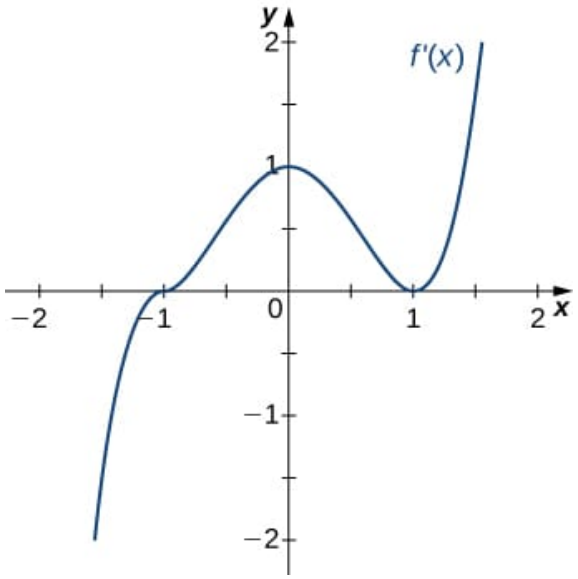
Answer

Concave up for all x,
No inflection points

21)



22)



Answer

Concave up for $x < 0$ and $x > 1$,
Concave down for \(0 Inflection points at $x = 0$ and $x = 1$

For exercises 23 - 27, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

23) $f(x) > 0,\ \ f'(x) > 0$ over $x > 1,\ \ -3 < x < 0,\ \ f'(x) = 0$ over="" $0 < x < 1$<="" p=""> </x<0,\;f'(x)=0\)>

24) $f'(x) > 0$ over $x > 2,\ \ -3 < x < -1,\ \ f'(x) < 0$ over="" $-1 < x < 2,\ \ f''(x) < 0$="" for="" all="" x<="" p=""> </x<–1,\;f'(x)<0\)>

Answer

Answers will vary

25) $f''(x) < 0$ over \(–1<x<1,\;f''(x)>0,\;–3 </x<1,\;f''(x)>

26) There is a local maximum at $x = 2$, local minimum at $x = 1$, and the graph is neither concave up nor concave down.

Answer

Answers will vary

27) There are local maxima at $x = \pm 1$, the function is concave up for all x, and the function remains positive for all x.

For the following exercises, determine

a. intervals where f is increasing or decreasing and

b. local minima and maxima of f.

28) $f(x) = \sin x + \sin^3 x$ over \(–π

Answer

a. Increasing over \(–\frac{π}{2}\frac{π}{2}\)

b. Local maximum at $x = \frac{\pi}{2}$; local minimum at $x = -\frac{\pi}{2}$

29) $f(x) = x^2 + \cos x$

For exercise 30, determine

a. intervals where f is concave up or concave down, and

b. the inflection points of f.

30) $f(x) = x^3 - 4x^2 + x + 2$

Answer

a. Concave up for $x > \frac{4}{3}$, concave down for $x < \frac{4}{3}$

b. Inflection point at $x = \frac{4}{3}$

For exercises 31 - 37, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f.

31) $f(x) = x^2 - 6x$

32) $f(x) = x^3 - 6x^2$

Answer

a. Increasing over $x < 0$ and $x > 4$, decreasing over \(0 b. Maximum at $x = 0$, minimum at $x = 4$
c. Concave up for $x > 2$, concave down for $x < 2$
d. Inflection point at $x = 2$

33) $f(x) = x^4 - 6x^3$

34) $f(x) = x^{11} - 6x^{10}$

Answer

a. Increasing over $x < 0$ and $x > \frac{60}{11}$, decreasing over \(0 b. Maximum at $x = 0$, minimum at $x = \frac{60}{11}$
c. Concave down for $x < \frac{54}{11}$, concave up for $x > \frac{54}{11}$
d. Inflection point at $x = \frac{54}{11}$

35) $f(x) = x + x^2 - x^3$

36) $f(x) = x^2 + x + 1$

Answer

a. Increasing over $x > -\frac{1}{2}$, decreasing over $x < -\frac{1}{2}$
b. Minimum at $x = -\frac{1}{2}$
c. Concave up for all x
d. No inflection points

37) $f(x) = x^3 + x^4$

For exercises 38 - 47, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

38) [T] $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$

Answer

a. Increases over \(–\frac{1}{4}\frac{3}{4}\) and $x < -\frac{1}{4}$
b. Minimum at $x = -\frac{1}{4}$, maximum at $x = \frac{3}{4}$
c. Concave up for \(–\frac{3}{4}\frac{1}{4}\)
d. Inflection points at $x = -\frac{3}{4}, \quad x = \frac{1}{4}$

39) [T] $f(x) = x + \sin(2x)$ over $x = [-\frac{\pi}{2}, \frac{\pi}{2}]$

40) [T] $f(x) = \sin x + \tan x$ over $(-\frac{\pi}{2}, \frac{\pi}{2})$

Answer

a. Increasing for all x
b. No local minimum or maximum
c. Concave up for $x > 0$, concave down for $x < 0$
d. Inflection point at $x = 0$

41) [T] $f(x) = (x-2)^2(x-4)^2$

42) [T] $f(x) = \dfrac{1}{1-x}, \quad x \neq 1$

Answer

a. Increasing for all x where defined
b. No local minima or maxima
c. Concave up for $x < 1$; concave down for $x > 1$
d. No inflection points in domain

43) [T] $f(x) = \dfrac{\sin x}{x}$ over $x = [-2\pi, 0) \cup (0, 2\pi]$

44) $f(x) = \sin(x)e^x$ over $x = [-\pi, \pi]$

Answer

a. Increasing over \(–\frac{π}{4}\frac{3π}{4},\;x<–\frac{π}{4}\)
b. Minimum at $x = -\frac{\pi}{4}$, maximum at $x = \frac{3\pi}{4}$
c. Concave up for \(–\frac{π}{2}\frac{π}{2}\)
d. Inflection points at $x = \pm\frac{\pi}{2}$

45) $f(x) = \ln x\sqrt{x}, \quad x > 0$

46) $f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}, \quad x > 0$

Answer

a. Increasing over $x > 4$, decreasing over \(0 b. Minimum at $x = 4$
c. Concave up for \(08\sqrt[3]{2}\)
d. Inflection point at $x = 8\sqrt[3]{2}$

47) $f(x) = \dfrac{e^x}{x}, \quad x \neq 0$

In exercises 48 - 52, interpret the sentences in terms of f, f', and f''.

48) The population is growing more slowly. Here f is the population.

Answer

$f > 0, \quad f' > 0, \quad f'' < 0$

49) A bike accelerates faster, but a car goes faster. Here $f =$ Bike's position minus Car's position.

50) The airplane lands smoothly. Here f is the plane's altitude.

Answer

$f > 0, \quad f' < 0, \quad f'' > 0$

51) Stock prices are at their peak. Here $f$is the stock price.

52) The economy is picking up speed. Here f is a measure of the economy, such as GDP.

Answer

$f > 0, \quad f' > 0, \quad f'' > 0$

For exercises 53 - 57, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0$ and $f'(3) = 0$.

Determine whether the following statements are true or false. Justify your answer.

53) $f(x) = 0$ for some $1 \leq x \leq 3$.

54) $f''(x) = 0$ for some $1 \leq x \leq 3$.

Answer

True, by the Mean Value Theorem

55) There is no absolute maximum at $x = 3$.

56) If $f(x)$ has three roots, then it has 1 inflection point.

Answer

True, examine derivative

57) If $f(x)$ has one inflection point, then it has three real roots.

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15.7: Limits at Infinity and Asymptotes

Learning Objectives

- Calculate the limit of a function as x increases or decreases without bound.
- Recognize a horizontal asymptote on the graph of a function.
- Estimate the end behavior of a function as x increases or decreases without bound.
- Recognize an oblique asymptote on the graph of a function.
- Analyze a function and its derivatives to draw its graph.

We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$. In this section, we define limits at infinity and show how these limits affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function f.

We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity. Back in Introduction to Functions and Graphs, we looked at vertical asymptotes; in this section we deal with horizontal and oblique asymptotes.

Limits at Infinity and Horizontal Asymptotes

Recall that $\lim_{x\to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a. We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in Figure $15.7.1$ and numerically in Table $15.7.1$, as the values of x get larger, the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim_{x\to\infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approach 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim_{x\to-\infty} f(x) = 2$.

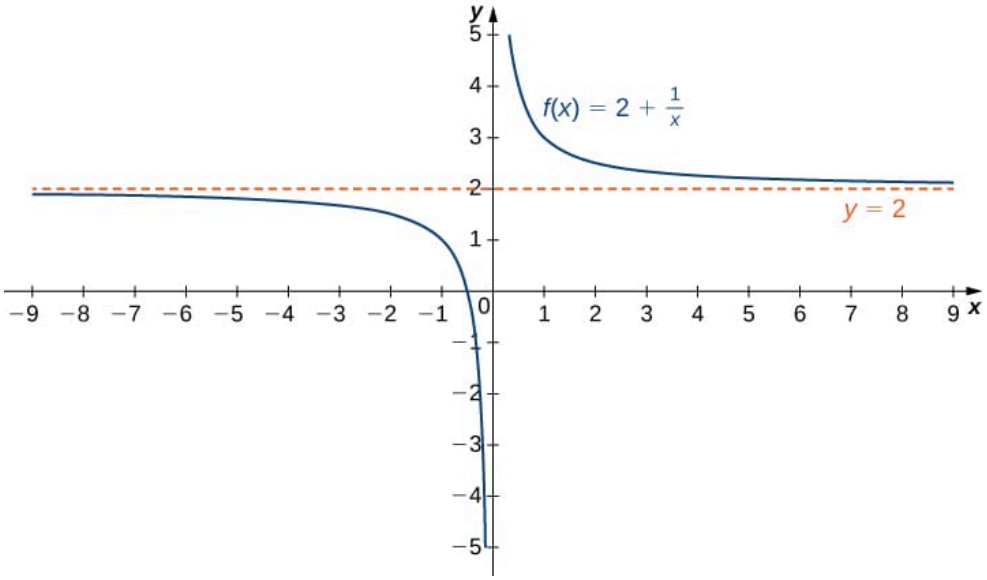


Figure $15.7.1$:The function approaches the asymptote $y = 2$ as x approaches $\pm\infty$.

Table $15.7.1$: *Values of a function* f *as* $x \to \pm\infty$

x	10	100	1,000	10,000
$2 + \frac{1}{x}$	2.1	2.01	2.001	2.0001

x	−10	−100	−1000	−10,000
$2+\frac{1}{x}$	1.9	1.99	1.999	1.9999

More generally, for any function f, we say the limit as $x \to \infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as x is sufficiently large. In that case, we write $\lim_{x\to\infty} f(x) = L$. Similarly, we say the limit as $x \to -\infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim_{x\to-\infty} f(x) = L$. We now look at the definition of a function having a limit at infinity.

Definition: Limit at Infinity (Informal)

If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write

$$\lim_{x\to\infty} f(x) = L.$$

If the values of $f(x)$ becomes arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function f has a limit at negative infinity and write

$$\lim_{x\to-\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value L as $x \to \infty$ or $x \to -\infty$, the graph of f approaches the line $y = L$. In that case, the line $y = L$ is a horizontal asymptote of f (Figure 15.7.2). For example, for the function $f(x) = \frac{1}{x}$, since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \frac{1}{x}$.

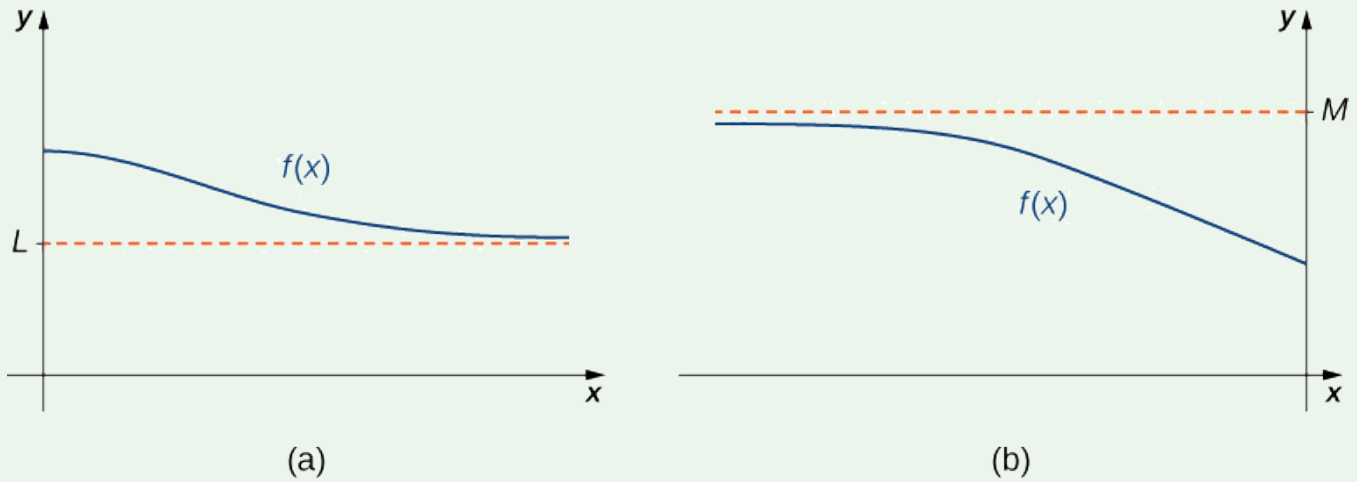


Figure 15.7.2: (a) As $x \to \infty$, the values of f are getting arbitrarily close to L. The line $y = L$ is a horizontal asymptote of f. (b) As $x \to -\infty$, the values of f are getting arbitrarily close to M. The line $y = M$ is a horizontal asymptote of f.

Definition: Horizontal Asymptote

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a *horizontal asymptote* of f.

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as x approaches the vertical asymptote. However, a function may cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function $f(x) = \frac{\cos x}{x} + 1$ shown in Figure 15.7.3 intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.

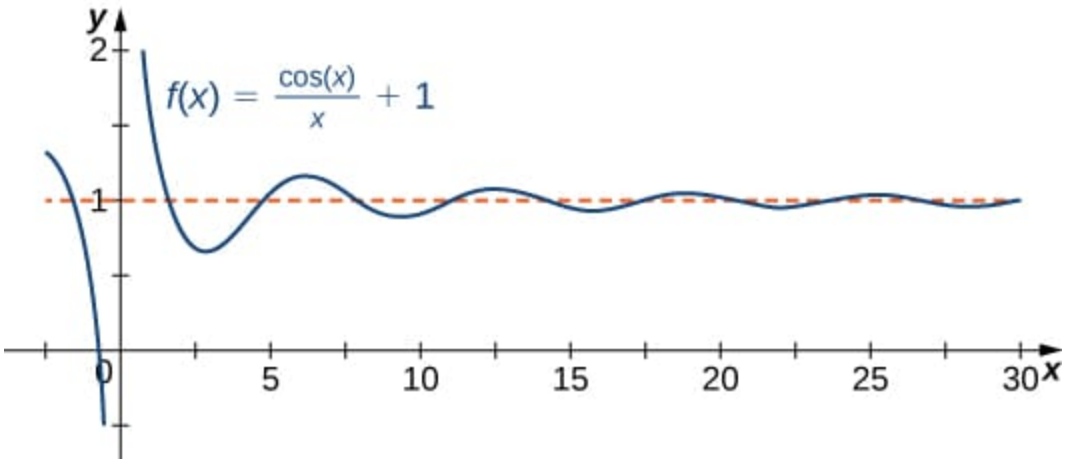


Figure 15.7.3: The graph of $f(x) = (\cos x)/x + 1$ crosses its horizontal asymptote $y = 1$ an infinite number of times.

The algebraic limit laws and squeeze theorem we introduced in Introduction to Limits also apply to limits at infinity. We illustrate how to use these laws to compute several limits at infinity.

Example 15.7.1: Computing Limits at Infinity

For each of the following functions f, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Determine the horizontal asymptote(s) for f.

a. $f(x) = 5 - \dfrac{2}{x^2}$
b. $f(x) = \dfrac{\sin x}{x}$
c. $f(x) = \tan^{-1}(x)$

Solution

a. Using the algebraic limit laws, we have

$$\lim_{x\to\infty}\left(5 - \frac{2}{x^2}\right) = \lim_{x\to\infty} 5 - 2\left(\lim_{x\to\infty}\frac{1}{x}\right)\cdot\left(\lim_{x\to\infty}\frac{1}{x}\right) = 5 - 2\cdot 0 = 5.$$

Similarly, $\lim_{x\to-\infty} f(x) = 5$. Therefore, $f(x) = 5 - \dfrac{2}{x^2}$ has a horizontal asymptote of $y = 5$ and f approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

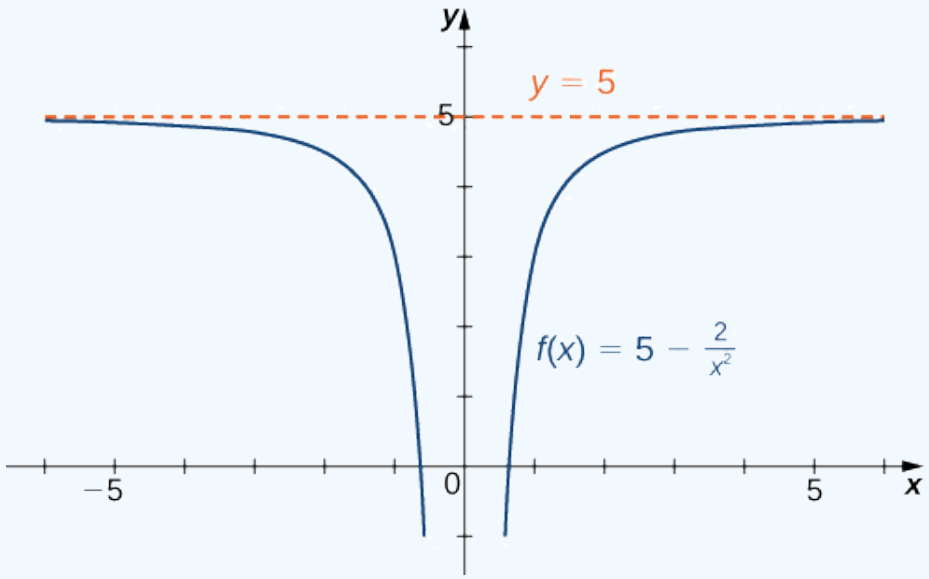


Figure 15.7.4: This function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since $-1 \leq \sin x \leq 1$ for all x, we have

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

for all $x \neq 0$. Also, since

$$\lim_{x\to\infty} \frac{-1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x},$$

we can apply the squeeze theorem to conclude that

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

Similarly,

$$\lim_{x\to-\infty} \frac{\sin x}{x} = 0.$$

Thus, $f(x) = \dfrac{\sin x}{x}$ has a horizontal asymptote of $y = 0$ and $f(x)$ approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

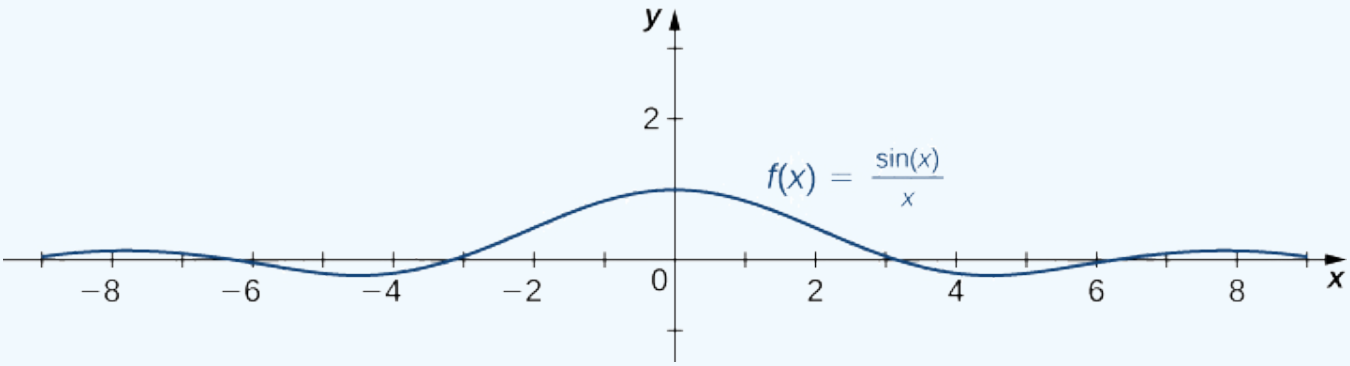


Figure 15.7.5: This function crosses its horizontal asymptote multiple times.

c. To evaluate $\lim_{x\to\infty} \tan^{-1}(x)$ and $\lim_{x\to-\infty} \tan^{-1}(x)$, we first consider the graph of $y = \tan(x)$ over the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ as shown in the following graph.

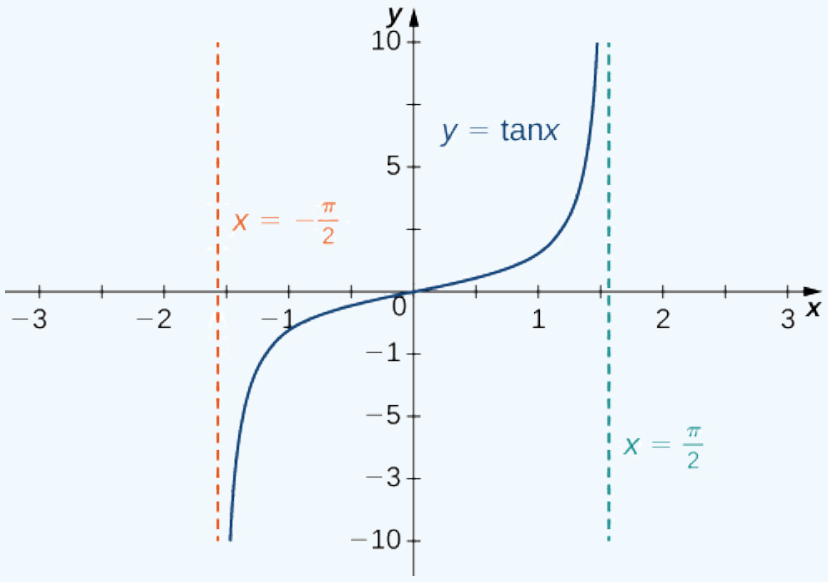


Figure 15.7.6: The graph of $y = \tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}$

Since

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty,$$

it follows that

$$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

Similarly, since

$$\lim_{x\to-\frac{\pi}{2}^+} \tan x = -\infty,$$

it follows that

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

As a result, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes of $f(x) = \tan^{-1}(x)$ as shown in the following graph.

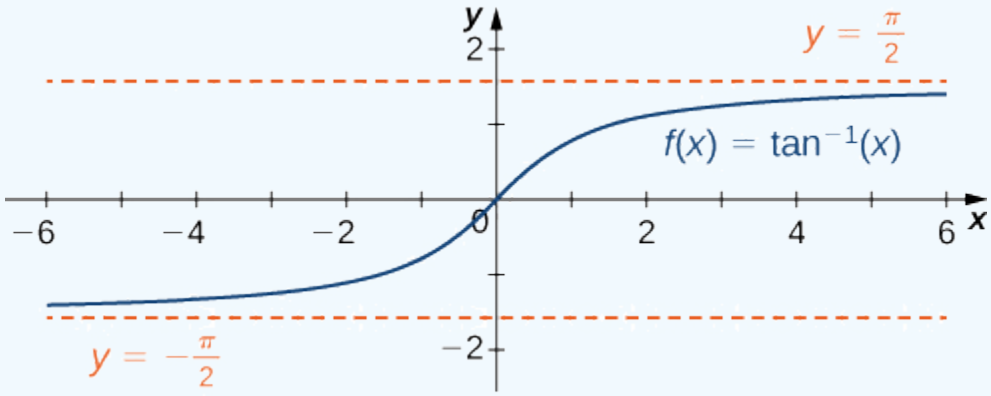


Figure 15.7.7: This function has two horizontal asymptotes.

Exercise 15.7.1

Evaluate $\lim_{x\to-\infty}\left(3+\frac{4}{x}\right)$ and $\lim_{x\to\infty}\left(3+\frac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x)=3+\frac{4}{x}$, if any.

Hint available at the end of this section.

Answer available at the end of this section.

Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x\to\infty$(or as $x\to-\infty$). In this case, we write $\lim_{x\to\infty} f(x)=\infty$ (or $\lim_{x\to-\infty} f(x)=\infty$). On the other hand, if the values of f are negative but become arbitrarily large in magnitude as $x\to\infty$ (or as $x\to-\infty$), we write $\lim_{x\to\infty} f(x)=-\infty$ (or $\lim_{x\to-\infty} f(x)=-\infty$).

For example, consider the function $f(x)=x^3$. As seen in Table 15.7.2 and Figure 15.7.8, as $x\to\infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim_{x\to\infty} x^3=\infty$. On the other hand, as $x\to-\infty$, the values of $f(x)=x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim_{x\to-\infty} x^3=-\infty$.

Table 15.7.2: Values of a power function as $x\to\pm\infty$

x	10	20	50	100	1000
x^3	1000	8000	125,000	1,000,000	1,000,000,000
x	−10	−20	−50	−100	−1000
x^3	−1000	−8000	−125,000	−1,000,000	−1,000,000,000

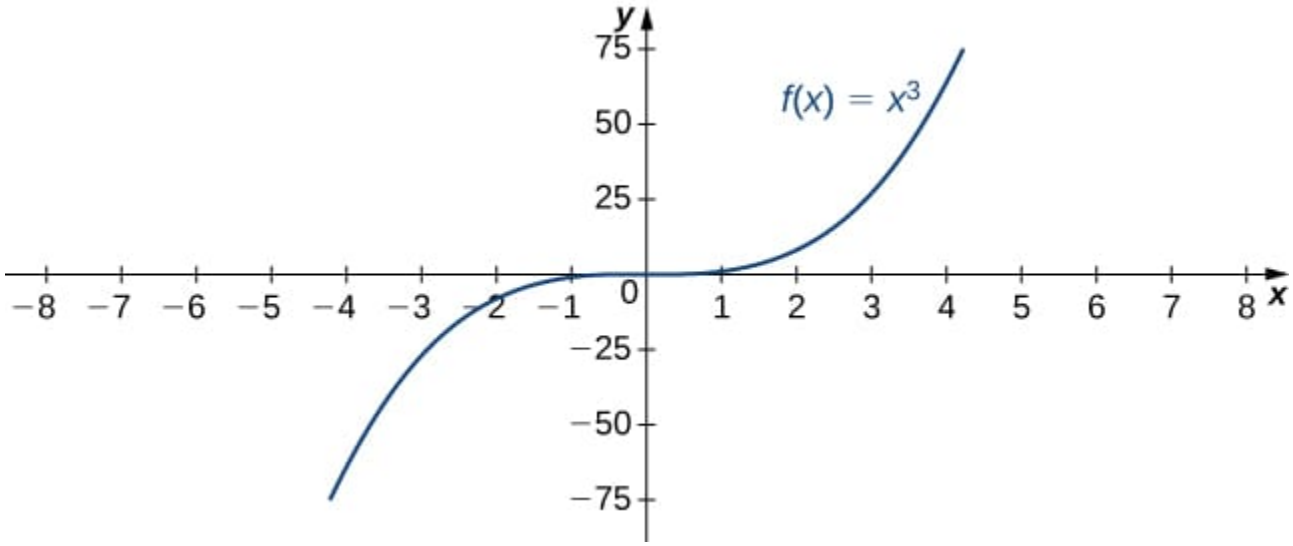


Figure 15.7.8: For this function, the functional values approach ±infinity as $x\to\pm\infty$.

Definition: Infinite Limit at Infinity (Informal)

We say a function f has an infinite limit at infinity and write

$$\lim_{x\to\infty} f(x)=\infty.$$

if $f(x)$ becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

Formal Definitions

Earlier, we used the terms arbitrarily close, arbitrarily large, and sufficiently large to define limits at infinity informally. Although these terms provide accurate descriptions of limits at infinity, they are not precise mathematically. Here are more formal definitions of limits at infinity. We then look at how to use these definitions to prove results involving limits at infinity.

Definition: Limit at Infinity (Formal)

We say a function f has a *limit at infinity*, if there exists a real number L such that for all $\varepsilon > 0$, there exists $N > 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x > N$. in that case, we write

$$\lim_{x\to\infty} f(x) = L$$

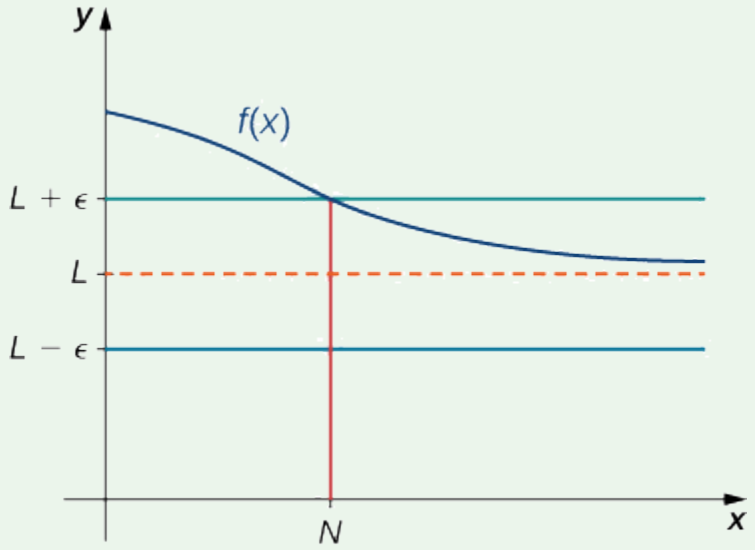


Figure 15.7.9: For a function with a limit at infinity, for all $x > N, |f(x) - L| < \varepsilon$.

Earlier in this section, we used graphical evidence in Figure 15.7.1 and numerical evidence in Table 15.7.1 to conclude that $\lim_{x\to\infty}\left(2 + \frac{1}{x}\right) = 2$. Here we use the formal definition of limit at infinity to prove this result rigorously.

Example 15.7.2:

Use the formal definition of limit at infinity to prove that $\lim_{x\to\infty}\left(2 + \frac{1}{x}\right) = 2$.

Solution

Let $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. Therefore, for all $x > N$, we have

$$\left|2+\frac{1}{x}-2\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{N} = \varepsilon$$

Exercise 15.7.2

Use the formal definition of limit at infinity to prove that $\lim_{x\to\infty}\left(3-\frac{1}{x^2}\right)=3$.

Hint available at the end of this section.

Answer available at the end of this section.

We now turn our attention to a more precise definition for an infinite limit at infinity.

Definition: Infinite Limit at Infinity (Formal)

We say a function f has an *infinite limit at infinity* and write

$$\lim_{x\to\infty} f(x) = \infty$$

if for all $M > 0$, there exists an $N > 0$ such that

$$f(x) > M$$

for all $x > N$ (see Figure 15.7.10).

We say a function has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty$$

if for all $M < 0$, there exists an $N > 0$ such that

\(f(x)

for all $x > N$.

Similarly we can define limits as $x \to -\infty$.

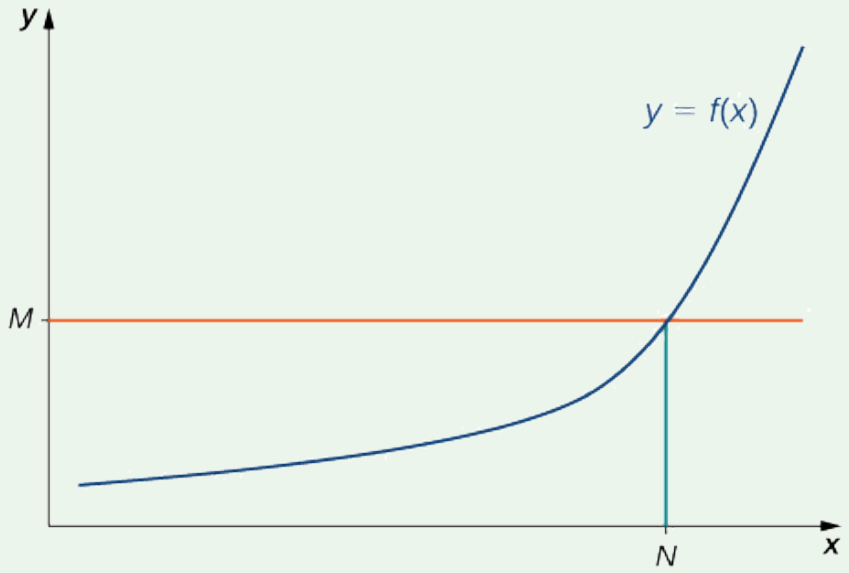


Figure 15.7.10: For a function with an infinite limit at infinity, for all $x > N$, $f(x) > M$.

Earlier, we used graphical evidence (Figure 15.7.8) and numerical evidence (Table 15.7.2) to conclude that $\lim_{x\to\infty} x^3 = \infty$. Here we use the formal definition of infinite limit at infinity to prove that result.

Example 15.7.3

Use the formal definition of infinite limit at infinity to prove that $\lim_{x\to\infty} x^3 = \infty$.

Solution

Let $M > 0$. Let $N = \sqrt[3]{M}$. Then, for all $x > N$, we have

$$x^3 > N^3 = (\sqrt[3]{M})^3 = M.$$

Therefore, $\lim_{x\to\infty} x^3 = \infty$.

Exercise 15.7.3

Use the formal definition of infinite limit at infinity to prove that $\lim_{x\to\infty} 3x^2 = \infty$.

Hint available at the end of this section.

Answer available at the end of this section.

End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where n is a positive integer. From Figure 15.7.11 and Figure 15.7.12, we see that

$$\lim_{x\to\infty} x^n = \infty;\ n = 1, 2, 3, \ldots$$

and

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty, & n = 2, 4, 6, \ldots \\ -\infty, & n = 1, 3, 5, \ldots . \end{cases}$$

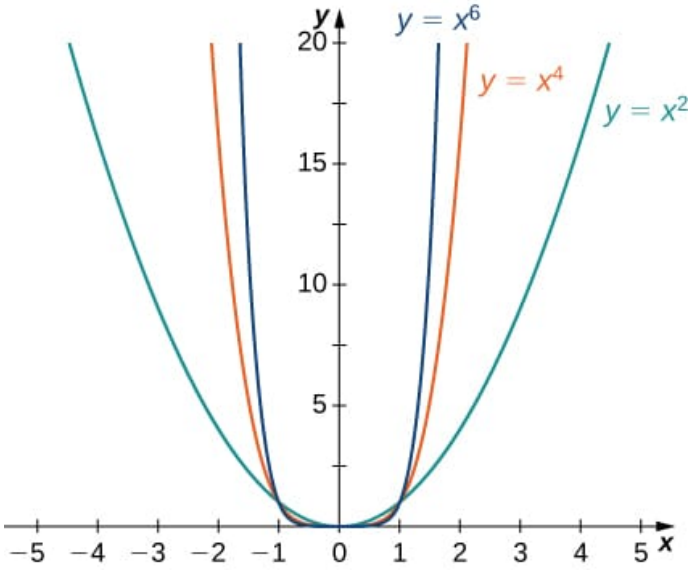


Figure 15.7.11: For power functions with an even power n, $\lim_{x\to\infty} x^n = \infty = \lim_{x\to-\infty} x^n$.

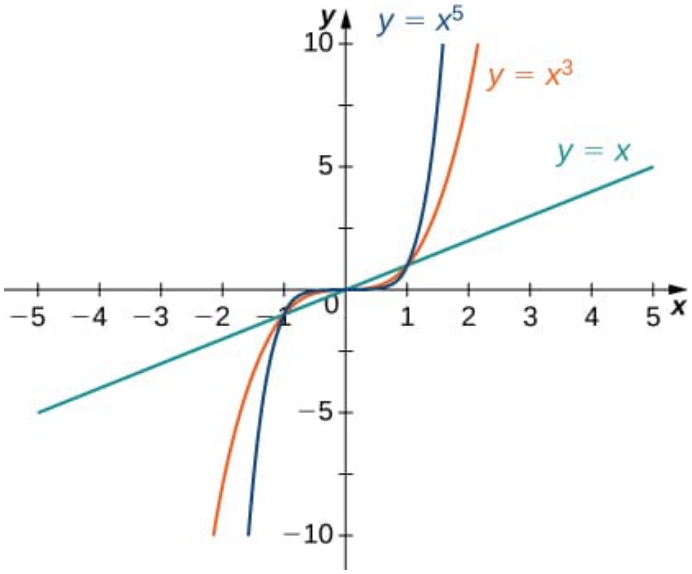


Figure 15.7.12: For power functions with an odd power n, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$.

Using these facts, it is not difficult to evaluate $\lim_{x\to\infty} cx^n$ and $\lim_{x\to-\infty} cx^n$, where c is any constant and n is a positive integer. If $c > 0$, the graph of $y = cx^n$ is a vertical stretch or compression of $y = x^n$, and therefore

$$\lim_{x\to\infty} cx^n = \lim_{x\to\infty} x^n \text{ and } \lim_{x\to-\infty} cx^n = \lim_{x\to-\infty} x^n \text{ if } c > 0.$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the x-axis, and therefore

$$\lim_{x\to\infty} cx^n = -\lim_{x\to\infty} x^n \text{ and } \lim_{x\to-\infty} cx^n = -\lim_{x\to-\infty} x^n \text{ if } c < 0.$$

If $c = 0, y = cx^n = 0$, in which case $\lim_{x\to\infty} cx^n = 0 = \lim_{x\to-\infty} cx^n$.

Example 15.7.4: Limits at Infinity for Power Functions

For each function f, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

a. $f(x) = -5x^3$
b. $f(x) = 2x^4$

Solution

a. Since the coefficient of x^3 is -5, the graph of $f(x) = -5x^3$ involves a vertical stretch and reflection of the graph of $y = x^3$ about the x-axis. Therefore, $\lim_{x\to\infty}(-5x^3) = -\infty$ and $\lim_{x\to-\infty}(-5x^3) = \infty$.
b. Since the coefficient of x^4 is 2, the graph of $f(x) = 2x^4$ is a vertical stretch of the graph of $y = x^4$. Therefore, $\lim_{x\to\infty} 2x^4 = \infty$ and $\lim_{x\to-\infty} 2x^4 = \infty$.

Exercise 15.7.4

Let $f(x) = -3x^4$. Find $\lim_{x\to\infty} f(x)$.

Hint available at the end of this section.

Answer available at the end of this section.

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x\to\pm\infty} f(x)$ for any polynomial function f. Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a^1 x + a^0$$

of degree $n \geq 1$ so that $a_n \neq 0$.

Factoring, we see that

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \ldots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right).$$

As $x \to \pm\infty$, all the terms inside the parentheses approach zero except the first term. We conclude that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n.$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$ as shown in Figure 15.7.13 and Table 15.7.3.

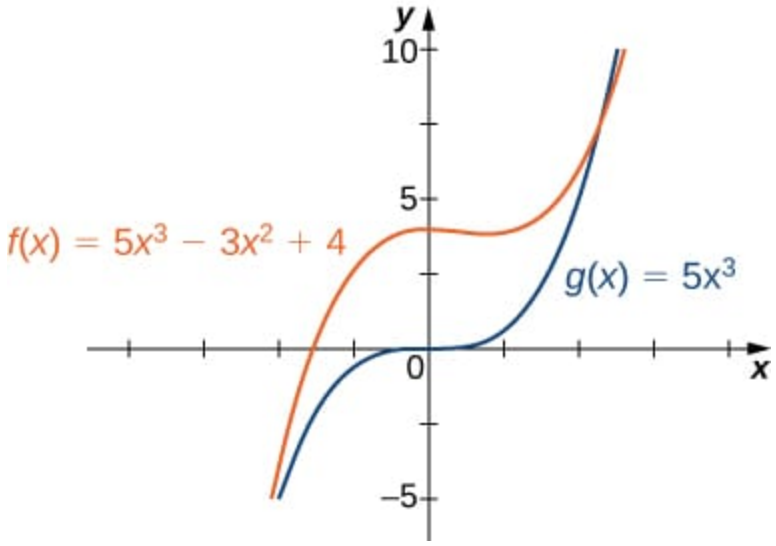


Figure 15.7.13: The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

Table 15.7.3: A polynomial's end behavior is determined by the term with the largest exponent

x	10	100	1000
$f(x) = 5x^3 - 3x^2 + 4$	4704	4,970,004	4,997,000,004
$g(x) = 5x^3$	5000	5,000,000	5,000,000,000
x	−10	−100	−000
$f(x) = 5x^3 - 3x^2 + 4$	−5296	−5,029,996	−5,002,999,996
$g(x) = 5x^3$	−5000	−5,000,000	−5,000,000,000

End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In Example 15.7.5, we show that the limits at infinity of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ depend on the relationship between the degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x.

Example 15.7.5: Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \to \infty$ and $x \to -\infty$. Then, use this information to describe the end behavior of the function.

a. $f(x) = \dfrac{3x-1}{2x+5}$ (Note: The degree of the numerator and the denominator are the same.)

b. $f(x) = \dfrac{3x^2+2x}{4x^3-5x+7}$ (Note: The degree of numerator is less than the degree of the denominator.)

c. $f(x) = \dfrac{3x^2+4x}{x+2}$ (Note: The degree of the numerator is greater than the degree of the denominator.)

Solution

a. The highest power of x in the denominator is x. Therefore, dividing the numerator and denominator by x and applying the algebraic limit laws, we see that

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{3x-1}{2x+5} &= \lim_{x\to\pm\infty}\frac{3-1/x}{2+5/x}\\ &= \frac{\lim_{x\to\pm\infty}(3-1/x)}{\lim_{x\to\pm\infty}(2+5/x)}\\ &= \frac{\lim_{x\to\pm\infty}3-\lim_{x\to\pm\infty}1/x}{\lim_{x\to\pm\infty}2+\lim_{x\to\pm\infty}5/x}\\ &= \frac{3-0}{2+0}=\frac{3}{2}.\end{aligned}$$

Since $\lim_{x\to\pm\infty} f(x)=\frac{3}{2}$, we know that $y=\frac{3}{2}$ is a horizontal asymptote for this function as shown in the following graph.

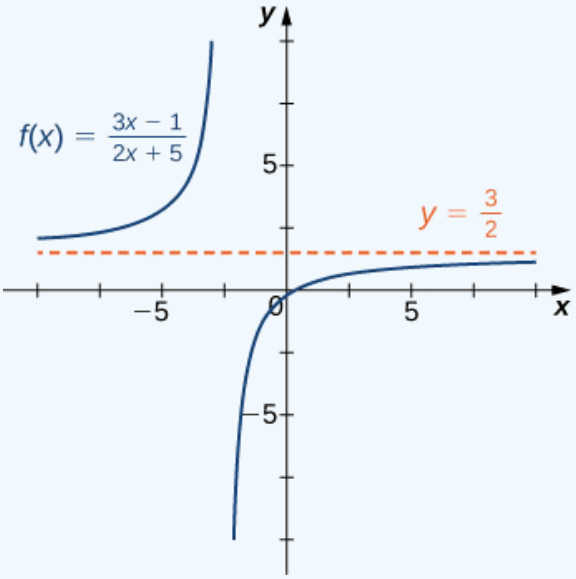


Figure 15.7.14: The graph of this rational function approaches a horizontal asymptote as $x\to\pm\infty$.

b. Since the largest power of x appearing in the denominator is x^3, divide the numerator and denominator by x^3. After doing so and applying algebraic limit laws, we obtain

$$\lim_{x\to\pm\infty}\frac{3x^2+2x}{4x^3-5x+7}=\lim_{x\to\pm\infty}\frac{3/x+2/x^2}{4-5/x^2+7/x^3}=\frac{3\cdot 0+2\cdot 0}{4-5\cdot 0+7\cdot 0}=\frac{0}{4}=0.$$

Therefore f has a horizontal asymptote of $y=0$ as shown in the following graph.

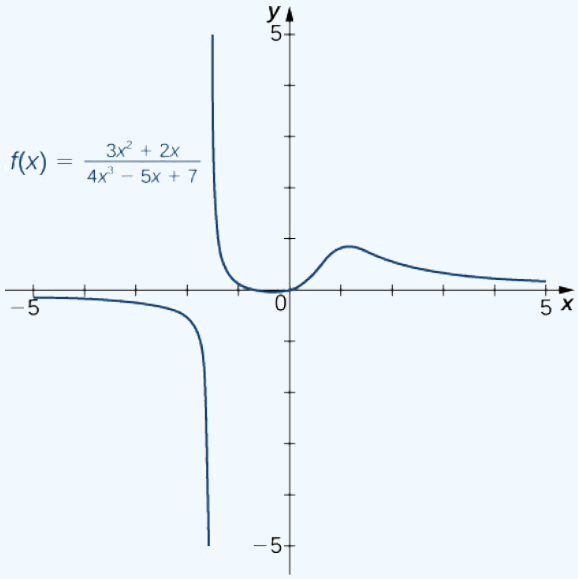


Figure 15.7.15: The graph of this rational function approaches the horizontal asymptote $y = 0$ as $x \to \pm\infty$.

c. Dividing the numerator and denominator by x, we have

$$\lim_{x\to\pm\infty} \frac{3x^2+4x}{x+2} = \lim_{x\to\pm\infty} \frac{3x+4}{1+2/x}.$$

As $x \to \pm\infty$, the denominator approaches 1. As $x \to \infty$, the numerator approaches $+\infty$. As $x \to -\infty$, the numerator approaches $-\infty$. Therefore $\lim_{x\to\infty} f(x) = \infty$, whereas $\lim_{x\to-\infty} f(x) = -\infty$ as shown in the following figure.

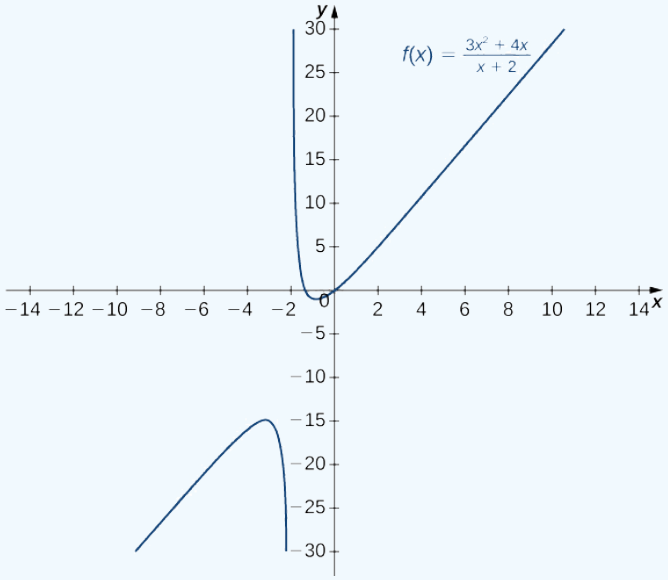


Figure 15.7.16: As $x \to \infty$, the values $f(x) \to \infty$. As $x \to -\infty$, the values $f(x) \to -\infty$.

Exercise 15.7.5

Evaluate $\lim_{x\to\pm\infty} \frac{3x^2+2x-1}{5x^2-4x+7}$ and use these limits to determine the end behavior of $f(x) = \frac{3x^2+2x-1}{5x^2-4x+7}$.

Hint available at the end of this section.

Answer available at the end of this section.

Before proceeding, consider the graph of $f(x) = \dfrac{3x^2 + 4x}{x + 2}$ shown in Figure 15.7.16. As $x \to \infty$ and $x \to -\infty$, the graph of f appears almost linear. Although f is certainly not a linear function, we now investigate why the graph of f seems to be approaching a linear function. First, using long division of polynomials, we can write

$$f(x) = \frac{3x^2 + 4x}{x + 2} = 3x - 2 + \frac{4}{x + 2}.$$

Since $\dfrac{4}{x + 2} \to 0$ as $x \to \pm\infty$, we conclude that

$$\lim_{x \to \pm\infty} (f(x) - (3x - 2)) = \lim_{x \to \pm\infty} \frac{4}{x + 2} = 0.$$

Therefore, the graph of f approaches the line $y = 3x - 2$ as $x \to \pm\infty$. This line is known as an **oblique asymptote** for f (Figure 15.7.17).

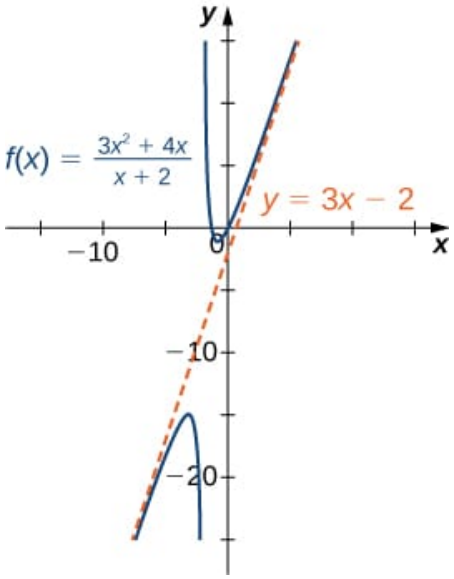


Figure 15.7.17: The graph of the rational function $f(x) = (3x^2 + 4x)/(x + 2)$ approaches the oblique asymptote $y = 3x - 2$ as $x \to \pm\infty$.

We can summarize the results of Example 15.7.5 to make the following conclusion regarding end behavior for rational functions. Consider a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0},$$

where $a_n \neq 0$ and $b_m \neq 0$.

1. If the degree of the numerator is the same as the degree of the denominator $(n = m)$, then f has a horizontal asymptote of $y = a_n/b_m$ as $x \to \pm\infty$.
2. If the degree of the numerator is less than the degree of the denominator \((n
3. If the degree of the numerator is greater than the degree of the denominator $(n > m)$, then f does not have a horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

   $$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

   where the degree of $r(x)$ is less than the degree of $q(x)$. As a result, $\lim_{x \to \pm\infty} r(x)/q(x) = 0$. Therefore, the values of $[f(x) - g(x)]$ approach zero as $x \to \pm\infty$. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ (i.e., $n = m + 1$), the function $g(x)$

is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

Example 15.7.6: Determining End Behavior for a Function Involving a Radical

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3x-2}{\sqrt{4x^2+5}}$ and describe the end behavior of f.

Solution

Let's use the same strategy as we did for rational functions: divide the numerator and denominator by a power of x. To determine the appropriate power of x, consider the expression $\sqrt{4x^2+5}$ in the denominator. Since

$$\sqrt{4x^2+5} \approx \sqrt{4x^2} = 2|x|$$

for large values of x in effect x appears just to the first power in the denominator. Therefore, we divide the numerator and denominator by $|x|$. Then, using the fact that $|x| = x$ for $x > 0$, $|x| = -x$ for $x < 0$, and $|x| = \sqrt{x^2}$ for all x, we calculate the limits as follows:

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to\infty} \frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}} \\
&= \lim_{x\to\infty} \frac{(1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}} \\
&= \lim_{x\to\infty} \frac{3-2/x}{\sqrt{4+5/x^2}} = \frac{3}{\sqrt{4}} = \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to-\infty} \frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}} \\
&= \lim_{x\to-\infty} \frac{(-1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}} \\
&= \lim_{x\to-\infty} \frac{-3+2/x}{\sqrt{4+5/x^2}} = \frac{-3}{\sqrt{4}} = \frac{-3}{2}.
\end{aligned}$$

Therefore, $f(x)$ approaches the horizontal asymptote $y = \frac{3}{2}$ as $x \to \infty$ and the horizontal asymptote $y = -\frac{3}{2}$ as $x \to -\infty$ as shown in the following graph.

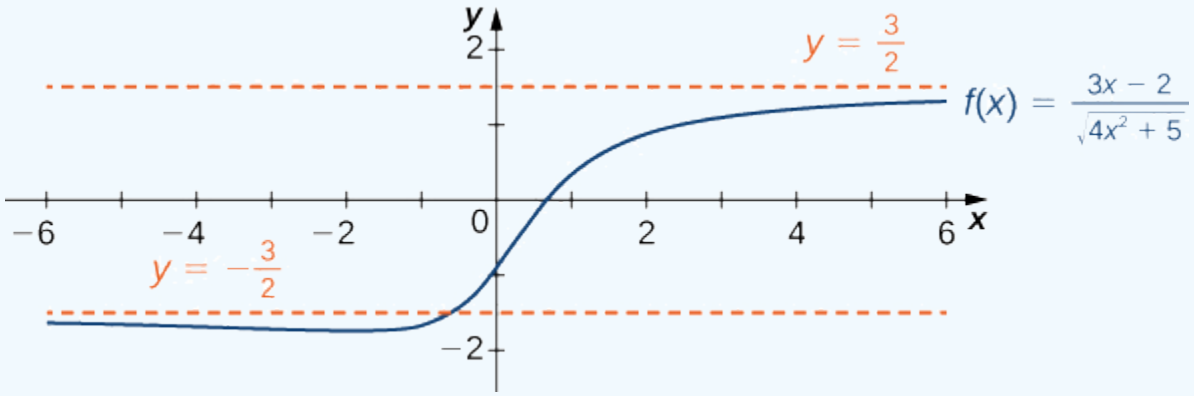


Figure 15.7.18: This function has two horizontal asymptotes and it crosses one of the asymptotes.

Exercise 15.7.6

Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{3x^2+4}}{x+6}$.

Hint available at the end of this section.

Answer available at the end of this section.

Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as $x \to \pm\infty$. For example, $\sin x$ oscillates between 1 and –1 (Figure 15.7.19). The tangent function x has an infinite number of vertical asymptotes as $x \to \pm\infty$; therefore, it does not approach a finite limit nor does it approach $\pm\infty$ as $x \to \pm\infty$ as shown in Figure 15.7.20.

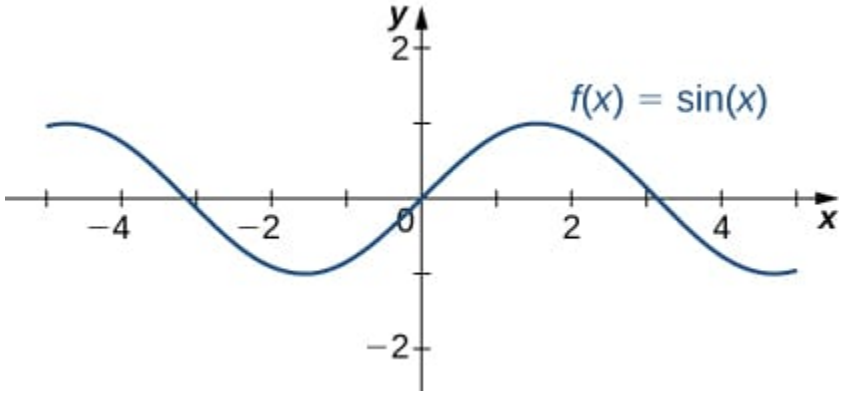


Figure 15.7.19: The function $f(x) = \sin x$ oscillates between 1 and -1 as $x \to \pm\infty$

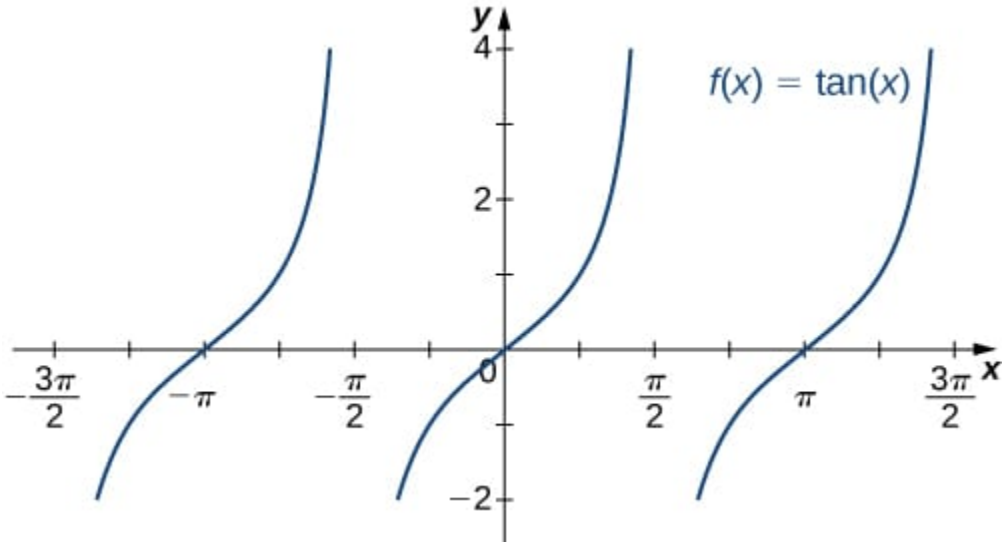


Figure 15.7.20: The function $f(x) = \tan x$ does not approach a limit and does not approach $\pm\infty$ as $x \to \pm\infty$

Recall that for any base $b > 0,\ \ b \neq 1$, the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1,\ \ y = b^x$ is increasing over $(-\infty, \infty)$. If \(01\). Therefore, $f(x) = e^x$ is increasing on $(-\infty, \infty)$ and the range is $(0, \infty)$. The exponential function $f(x) = e^x$ approaches ∞ as $x \to \infty$ and approaches 0 as $x \to -\infty$ as shown in Table 15.7.4 and Figure 15.7.21 .

Table 15.7.4: End behavior of the natural exponential function

x	−5	−2	0	2	5
e^x	0.00674	0.135	1	7.389	148.413

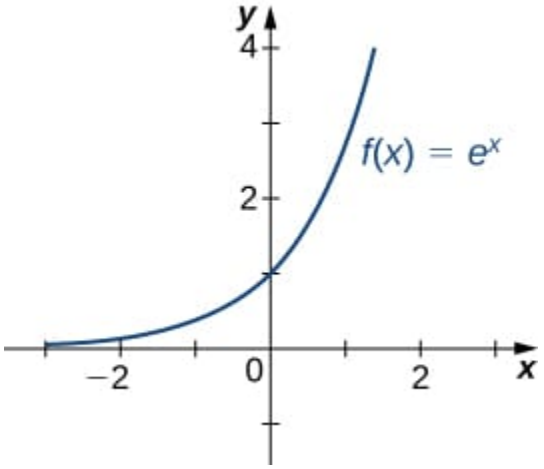


Figure 15.7.21: The exponential function approaches zero as $x \to -\infty$ and approaches ∞ as $x \to \infty$.

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \to -\infty$ as $x \to 0^+$ and $\ln(x) \to \infty$ as $x \to \infty$ as shown in Figure 15.7.22 and Table 15.7.5.

Table 15.7.5: End behavior of the natural logarithm function

x	0.01	0.1	1	10	100
$\ln(x)$	−4.605	−2.303	0	2.303	4.605

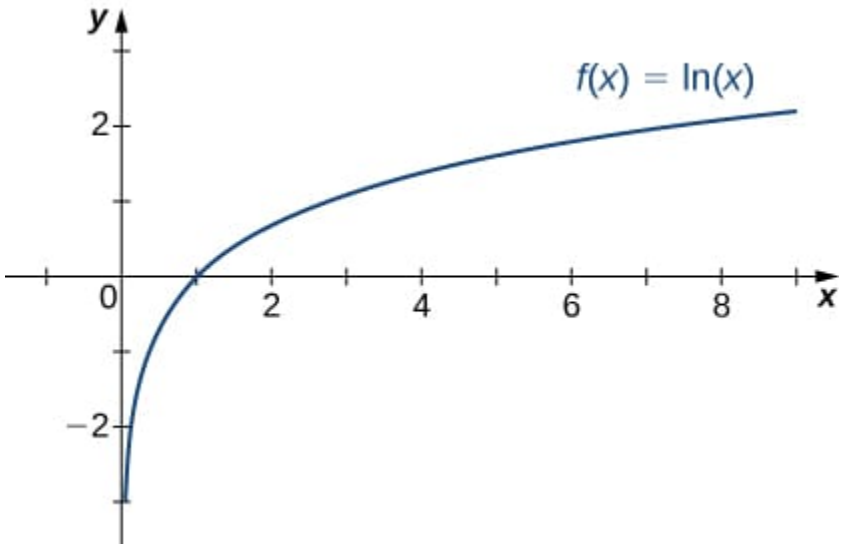


Figure 15.7.22: The natural logarithm function approaches ∞ as $x \to \infty$.

Example 15.7.7: Determining End Behavior for a Transcendental Function

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{2 + 3e^x}{7 - 5e^x}$ and describe the end behavior of f.

Solution

To find the limit as $x \to \infty$, divide the numerator and denominator by e^x:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2+3e^x}{7-5e^x}$$
$$= \lim_{x\to\infty} \frac{(2/e^x)+3}{(7/e^x)-5.}$$

As shown in Figure 15.7.21, $e^x \to \infty$ as $x \to \infty$. Therefore,

$$\lim_{x\to\infty} \frac{2}{e^x} = 0 = \lim_{x\to\infty} \frac{7}{e^x}.$$

We conclude that $\lim_{x\to\infty} f(x) = -\frac{3}{5}$, and the graph of f approaches the horizontal asymptote $y = -\frac{3}{5}$ as $x \to \infty$. To find the limit as $x \to -\infty$, use the fact that $e^x \to 0$ as $x \to -\infty$ to conclude that $\lim_{x\to-\infty} f(x) = \frac{2}{7}$, and therefore the graph of $f(x)$ approaches the horizontal asymptote $y = \frac{2}{7}$ as $x \to -\infty$.

Exercise 15.7.7

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \dfrac{3e^x - 4}{5e^x + 2}$.

Hint available at the end of this section.

Answer available at the end of this section.

Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

Problem-Solving Strategy: Drawing the Graph of a Function

Given a function f, use the following steps to sketch a graph of f:

1. Determine the domain of the function.
2. Locate the x- and y-intercepts.
3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number L, then $y = L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x) = \dfrac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as
$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x),}$$
where the degree of $r(x)$ is less than the degree of $q(x)$. The values of $f(x)$ approach the values of $g(x)$ as $x \to \pm\infty$. If $g(x)$ is a linear function, it is known as an oblique asymptote.
4. Determine whether f has any vertical asymptotes.
5. Calculate $f\,'$. Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
6. Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Example 15.7.8: Sketching a Graph of a Polynomial

Sketch a graph of $f(x) = (x-1)^2(x+2)$.

Solution

Step 1: Since f is a polynomial, the domain is the set of all real numbers.

Step 2: When $x = 0$, $f(x) = 2$. Therefore, the y-intercept is $(0, 2)$. To find the x-intercepts, we need to solve the equation $(x-1)^2(x+2) = 0$, which gives us the x-intercepts $(1, 0)$ and $(-2, 0)$

Step 3: We need to evaluate the end behavior of f. As $x \to \infty$, $(x-1)^2 \to \infty$ and $(x+2) \to \infty$. Therefore, $\lim\limits_{x\to\infty} f(x) = \infty$.

As $x \to -\infty$, $(x-1)^2 \to \infty$ and $(x+2) \to -\infty$. Therefore, $\lim\limits_{x\to-\infty} f(x) = -\infty$.

To get even more information about the end behavior of f, we can multiply the factors of f. When doing so, we see that

$$f(x) = (x-1)^2(x+2) = x^3 - 3x + 2.$$

Since the leading term of f is x^3, we conclude that f behaves like $y = x^3$ as $x \to \pm\infty$.

Step 4: Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5: The first derivative of f is

$$f'(x) = 3x^2 - 3.$$

Therefore, f has two critical points: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2, x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f'(x)$ at each of these test points, as shown in the following table.

Interval	Test point	Sign of Derivative $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$	Conclusion
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f decreasing
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f is increasing

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6: The second derivative of f is

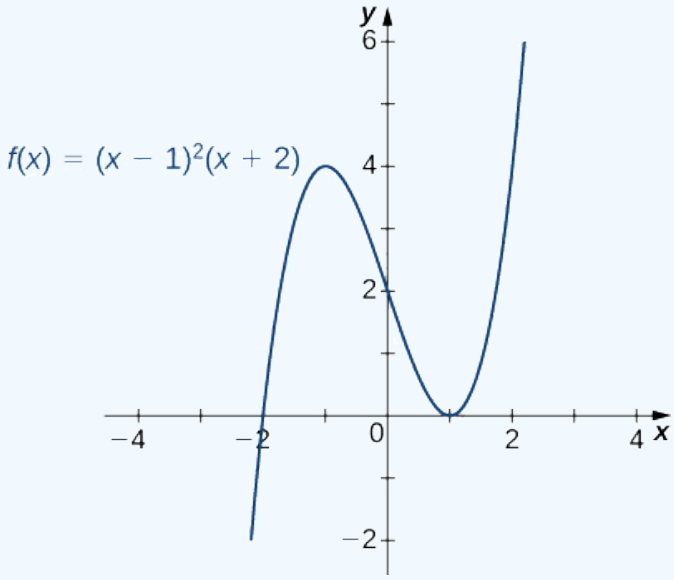
$$f''(x) = 6x.$$

The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	Test Point	Sign of $f''(x) = 6x$	Conclusion
$(-\infty, 0)$	$x = -1$	$-$	f is concave down.
$(0, \infty)$	$x = 1$	$+$	f is concave up.

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f'(x) = 0$ at those points—combined with the fact that f'' changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f.

Combining this information, we arrive at the graph of $f(x) = (x-1)^2(x+2)$ shown in the following graph.



Exercise 15.7.8

Sketch a graph of $f(x) = (x-1)^3(x+2)$.

Hint available at the end of this section.

Example 15.7.9: Sketching a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2}{1-x^2}$.

Solution

Step 1: The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2: Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\dfrac{x^2}{1-x^2} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since $f$is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2.We obtain

$$\lim_{x \to \pm\infty} \frac{x^2}{1-x^2} = \lim_{x \to \pm\infty} \frac{1}{\frac{1}{x^2} - 1} = -1.$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \to \infty$ and $x \to -\infty$.

Step 4: To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f, evaluate $\lim_{x \to 1} f(x)$ and $\lim_{x \to -1} f(x)$. By looking at each one-sided limit as $x \to 1$, we see that

$$\lim_{x \to 1^+} \frac{x^2}{1 - x^2} = -\infty \text{ and } \lim_{x \to 1^-} \frac{x^2}{1 - x^2} = \infty.$$

In addition, by looking at each one-sided limit as $x \to -1$, we find that

$$\lim_{x \to -1^+} \frac{x^2}{1 - x^2} = \infty \text{ and } \lim_{x \to -1^-} \frac{x^2}{1 - x^2} = -\infty.$$

Step 5: Calculate the first derivative:

$$f\,'(x) = \frac{(1 - x^2)(2x) - x^2(-2x)}{\left(1 - x^2\right)^2} = \frac{2x}{\left(1 - x^2\right)^2}.$$

Critical points occur at points x where $f\,'(x) = 0$ or $f\,'(x)$ is undefined. We see that $f\,'(x) = 0$ when $x = 0$. The derivative $f\,'$ is not undefined at any point in the domain of f. However, $x = \pm 1$ are not in the domain of f. Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty, \infty)$ into four smaller intervals: $(-\infty, -1), (-1, 0), (0, 1)$, and $(1, \infty)$, and choose a test point in each interval to determine the sign of $f\,'(x)$ in each of these intervals. The values $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are good choices for test points as shown in the following table.

Interval	Test point	Sign of $f\,'(x) = \frac{2x}{(1-x^2)^2}$	Conclusion
$(-\infty, -1)$	$x = -2$	$-/+ = -$	f is decreasing.
$(-1, 0)$	$x = -1/2$	$-/+ = -$	f is decreasing.
$(0, 1)$	$x = 1/2$	$+/+ = +$	f is increasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

From this analysis, we conclude that f has a local minimum at $x = 0$ but no local maximum.

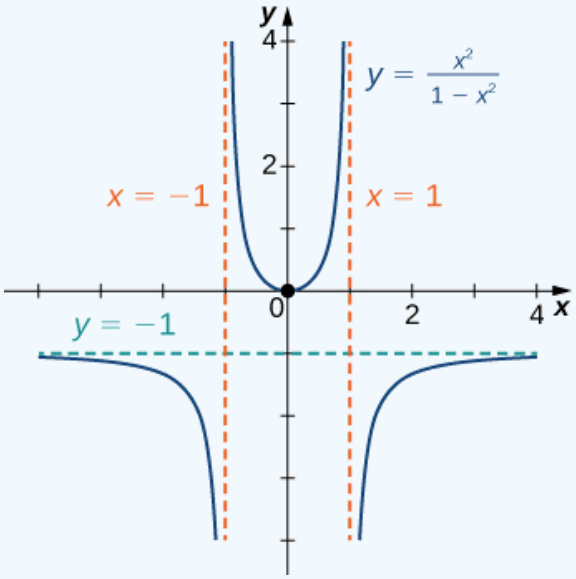
Step 6: Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(1 - x^2)^2(2) - 2x(2(1 - x^2)(-2x))}{(1 - x^2)^4} \\ &= \frac{(1 - x^2)[2(1 - x^2) + 8x^2]}{\left(1 - x^2\right)^4} \\ &= \frac{2(1 - x^2) + 8x^2}{\left(1 - x^2\right)^3} \\ &= \frac{6x^2 + 2}{\left(1 - x^2\right)^3}. \end{aligned}$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x) = 0$ or $f''(x)$ is undefined. Since the numerator $6x^2 + 2 \neq 0$ for any x, $f''(x)$ is never zero. Furthermore, f'' is not undefined for any x in the domain of f. However, as discussed earlier, $x = \pm 1$ are not in the domain of f. Therefore, to determine the concavity of f, we divide the interval $(-\infty, \infty)$ into the three smaller intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. The values $x = -2$, $x = 0$, and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{6x^2+2}{(1-x^2)^3}$	Conclusion
$(-\infty, -1)$	$x = -2$	$+/-=-$	f is concave down.
$(-1, 1)$	$x = 0$	$+/+=+$	f is concave up
$(1, \infty)$	$x = 2$	$+/-=-$	f is concave down.

Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x = -1$ and $x = 1$, there are no inflection points at either of these places because f is not continuous at $x = -1$ or $x = 1$.



Exercise 15.7.9

Sketch a graph of $f(x) = \dfrac{3x+5}{8+4x}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.7.10: Sketching a Rational Function with an Oblique Asymptote

Sketch the graph of $f(x) = \dfrac{x^2}{x-1}$

Solution

Step 1: The domain of f is the set of all real numbers x except $x = 1$.

Step 2: Find the intercepts. We can see that when $x = 0,\ \ f(x) = 0$, so $(0, 0)$ is the only intercept.

Step 3: Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

Since $\frac{1}{x-1} \to 0$ as $x \to \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \to \pm\infty$. The line $y = x + 1$ is an oblique asymptote for f.

Step 4: To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \to 1$, we find

$$\lim_{x\to 1^+} \frac{x^2}{x-1} = \infty \text{ and } \lim_{x\to 1^-} \frac{x^2}{x-1} = -\infty.$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5: Calculate the first derivative:

$$f\,'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

We have $f\,'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical points. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0), (0, 1), (1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f\,'(x)$ in each of these smaller intervals. For example, let $x = -1, x = \frac{1}{2}, x = \frac{3}{2}$, and $x = 3$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{x^2-2x}{(x-1)^2}$	Conclusion
$(-\infty, 0)$	$x = -1$	(−)(−)/+=+	f is increasing.
$(0, 1)$	$x = 1/2$	(+)(−)/+=−	f is decreasing.
$(1, 2)$	$x = 3/2$	(+)(−)/+=−	f is decreasing.
$(2, \infty)$	$x = 3$	(+)(+)/+=+	f is increasing.

From this table, we see that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. The value of f at the local maximum is $f(0) = 0$ and the value of f at the local minimum is $f(2) = 4$. Therefore, $(0, 0)$ and $(2, 4)$ are important points on the graph.

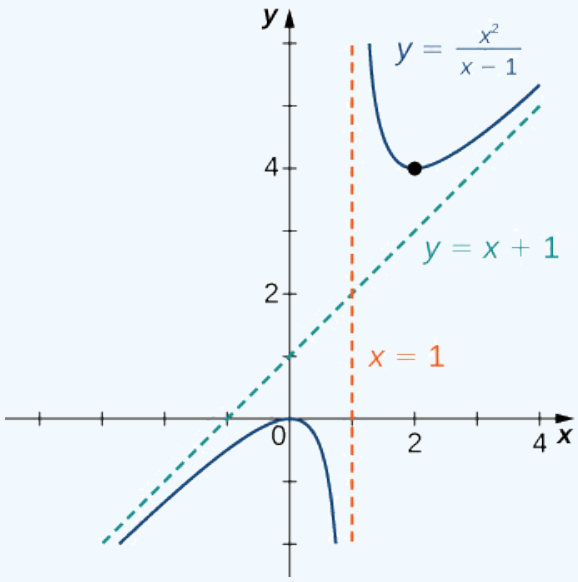
Step 6. Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{(x-1)^2(2x-2) - 2(x-1)(x^2-2x)}{(x-1)^4} \\ &= \frac{2(x-1)[(x-1)^2 - (x^2-2x)]}{(x-1)^4} \\ &= \frac{2[x^2-2x+1-x^2+2x]}{(x-1)^3} \\ &= \frac{2}{(x-1)^3}. \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x = 1$, to check concavity we just divide the interval $(-\infty, \infty)$ into the two smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x = 0$ and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{2}{(x-1)^3}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/-=-$	f is concave down.
$(1, \infty)$	$x = 2$	$+/+=+$	f is concave up

From the information gathered, we arrive at the following graph for f.



Exercise 15.7.10

Find the oblique asymptote for $f(x) = \dfrac{3x^3 - 2x + 1}{2x^2 - 4}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.7.11: Sketching the Graph of a Function with a Cusp

Sketch a graph of $f(x) = (x-1)^{2/3}$

Solution

Step 1: Since the cube-root function is defined for all real numbers x and $(x-1)^{2/3} = (\sqrt[3]{x-1})^2$, the domain of f is all real numbers.

Step 2: To find the y-intercept, evaluate $f(0)$. Since $f(0) = 1$, the y-intercept is $(0, 1)$. To find the x-intercept, solve $(x-1)^{2/3} = 0$. The solution of this equation is $x = 1$, so the x-intercept is $(1, 0)$.

Step 3: Since $\lim\limits_{x \to \pm\infty} (x-1)^{2/3} = \infty$, the function continues to grow without bound as $x \to \infty$ and $x \to -\infty$.

Step 4: The function has no vertical asymptotes.

Step 5: To determine where f is increasing or decreasing, calculate $f\,'$. We find

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

This function is not zero anywhere, but it is undefined when $x = 1$. Therefore, the only critical point is $x = 1$. Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points in each of these intervals to determine the sign of $f\,'(x)$ in each of these smaller intervals. Let $x = 0$ and $x = 2$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{2}{3(x-1)^{1/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/-=-$	f is decreasing
$(1, \infty)$	$x = 2$	$+/+=+$	f is increasing

We conclude that f has a local minimum at $x = 1$. Evaluating f at $x = 1$, we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim_{x \to 1} f'(x)$. Looking at the one-sided limits, we have

$$\lim_{x \to 1^+} \frac{2}{3(x-1)^{1/3}} = \infty \text{ and } \lim_{x \to 1^-} \frac{2}{3(x-1)^{1/3}} = -\infty.$$

Therefore, f has a cusp at $x = 1$.

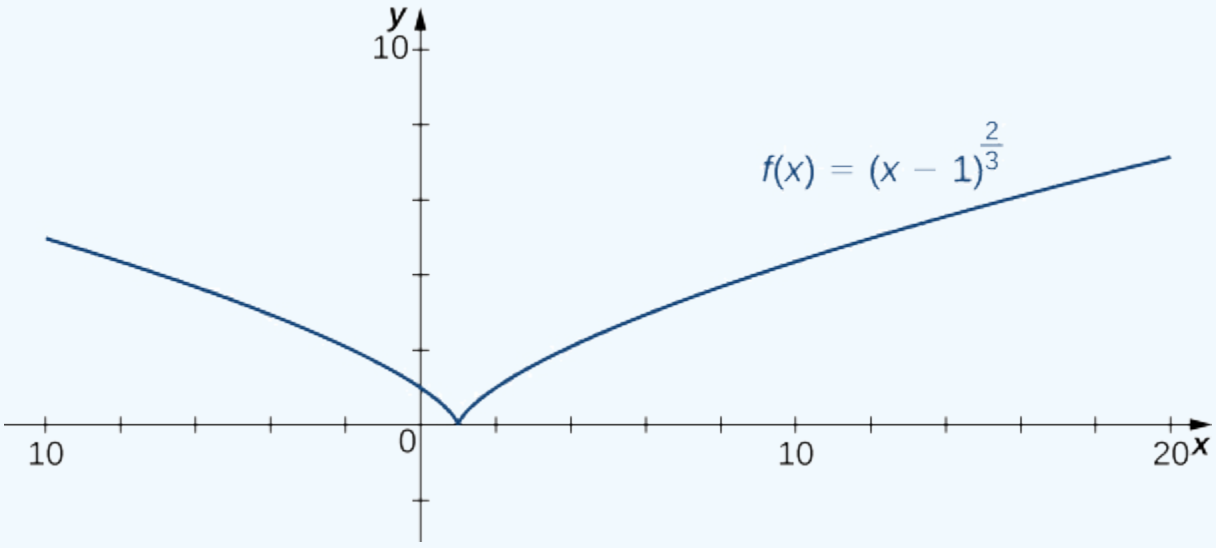
Step 6: To determine concavity, we calculate the second derivative of f:

$$f''(x) = -\frac{2}{9}(x-1)^{-4/3} = \frac{-2}{9(x-1)^{4/3}}.$$

We find that $f''(x)$ is defined for all x, but is undefined when $x = 1$. Therefore, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points to evaluate the sign of $f''(x)$ in each of these intervals. As we did earlier, let $x = 0$ and $x = 2$ be test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{-2}{9(x-1)^{4/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$-/+=-$	f is concave down
$(1, \infty)$	$x = 2$	$-/+=-$	f is concave down

From this table, we conclude that f is concave down everywhere. Combining all of this information, we arrive at the following graph for f.



Exercise 15.7.11

Consider the function $f(x) = 5 - x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The limit of $f(x)$ is L as $x \to \infty$ (or as $x \to -\infty$) if the values $f(x)$ become arbitrarily close to L as x becomes sufficiently large.
- The limit of $f(x)$ is ∞ as $x \to \infty$ if $f(x)$ becomes arbitrarily large as x becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x \to \infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large as x becomes sufficiently large. We can define the limit of $f(x)$ as x approaches $-\infty$ similarly.
- For a polynomial function $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, where $a_n \neq 0$, the end behavior is determined by the leading term $a_n x^n$. If $n \neq 0, p(x)$ approaches ∞ or $-\infty$at each end.
- For a rational function $f(x) = \dfrac{p(x)}{q(x),}$ the end behavior is determined by the relationship between the degree of p and the degree of q. If the degree of p is less than the degree of q, the line $y = 0$ is a horizontal asymptote for f. If the degree of p is equal to the degree of q, then the line $y = \dfrac{a_n}{b_n}$ is a horizontal asymptote, where a_n and b_n are the leading coefficients of p and q, respectively. If the degree of p is greater than the degree of q, then f approaches ∞ or $-\infty$ at each end.

Glossary

end behavior
: the behavior of a function as $x \to \infty$ and $x \to -\infty$

horizontal asymptote
: if $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f

infinite limit at infinity
: a function that becomes arbitrarily large as x becomes large

limit at infinity
: a function that approaches a limit value L as x becomes large

oblique asymptote
: the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or$x \to -\infty$

Answers

Exercise 15.7.1

$\lim\limits_{x\to\pm\infty} \dfrac{1}{x} = 0$

Exercise 15.7.1

Both limits are 3. The line $y = 3$ is a horizontal asymptote.

Exercise 15.7.2

Let $N = \frac{1}{\sqrt{\varepsilon}}$.

Exercise 15.7.2

Let $\varepsilon > 0$. Let $N = \frac{1}{\sqrt{\varepsilon}}$. Therefore, for all $x > N$, we have

$$\left|3 - \frac{1}{x^2} - 3\right| = \frac{1}{x^2} < \frac{1}{N^2} = \varepsilon$$

Therefore, $\lim\limits_{x\to\infty}(3 - 1/x^2) = 3$.

Exercise 15.7.3

Let $N = \sqrt{\frac{M}{3}}$.

Exercise 15.7.3

Let $M > 0$. Let $N = \sqrt{\frac{M}{3}}$. Then, for all $x > N$, we have

$3x^2 > 3N^2 = 3\left(\sqrt{\frac{M}{3}}\right)^2 = \frac{3M}{3} = M$

Exercise 15.7.4

The coefficient -3 is negative.

Exercise 15.7.4

$-\infty$

Exercise 15.7.5

Divide the numerator and denominator by x^2.

Exercise 15.7.5

$\frac{3}{5}$

Exercise 15.7.6

Divide the numerator and denominator by x.

Exercise 15.7.6

$\sqrt{3}$

Exercise 15.7.7

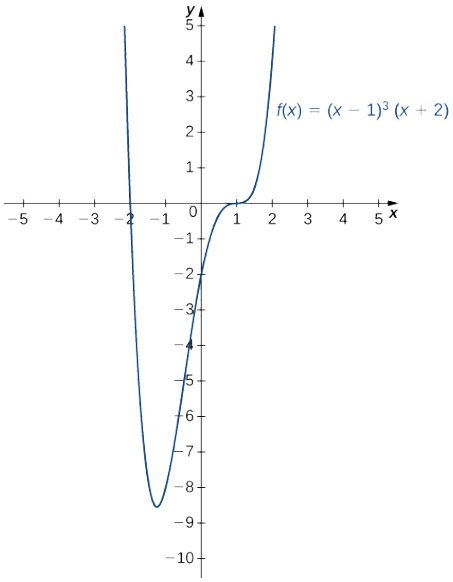
$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

Exercise 15.7.7

$\lim_{x\to\infty} f(x) = \frac{3}{5}, \quad \lim_{x\to-\infty} f(x) = -2$

Exercise 15.7.8

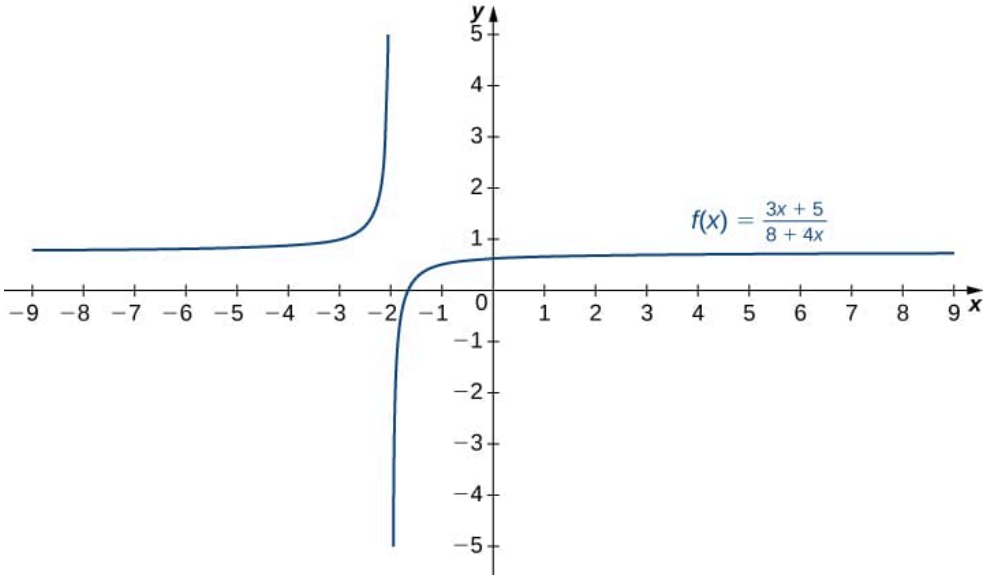
f is a fourth-degree polynomial.



Exercise 15.7.9

A line $y = L$ is a horizontal asymptote of f if the limit as $x \to \infty$ or the limit as $x \to -\infty$ of $f(x)$ is L. A line $x = a$ is a vertical asymptote if at least one of the one-sided limits of f as $x \to a$ is ∞ or $-\infty$.

Exercise 15.7.9



Exercise 15.7.10

Use long division of polynomials.

Exercise 15.7.10

$y = \frac{3}{2}x$

Exercise 15.7.11

A function f has a cusp at a point a if $f(a)$ exists, $f'(a)$ is undefined, one of the one-sided limits as $x \to a$ of $f'(x)$ is $+\infty$, and the other one-sided limit is $-\infty$.

Exercise 15.7.11

The function f has a cusp at $(0, 5)$, since $\lim\limits_{x \to 0^-} f'(x) = \infty$ and $\lim\limits_{x \to 0^+} f'(x) = -\infty$. For end behavior, $\lim\limits_{x \to \pm\infty} f(x) = -\infty$.

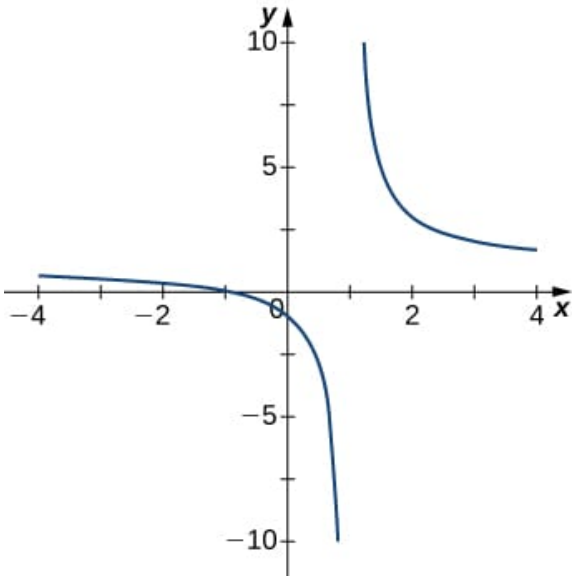
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15.7E: Exercises for Section 4.6

For exercises 1 - 5, examine the graphs. Identify where the vertical asymptotes are located.

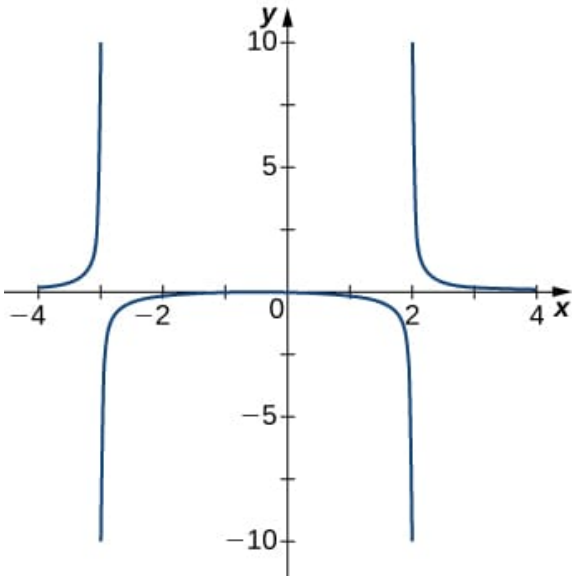
1)



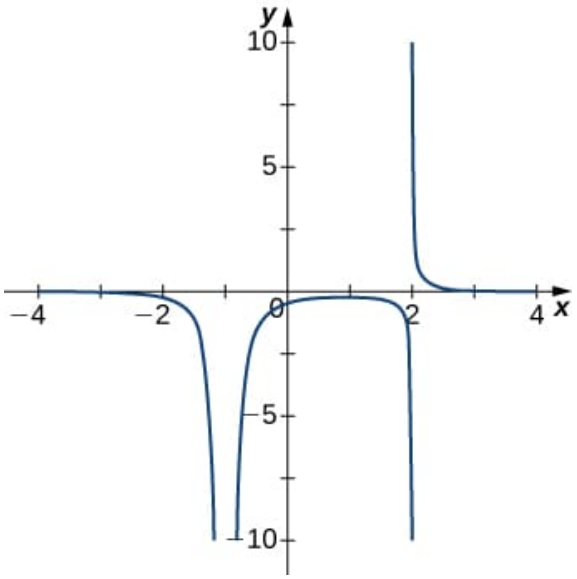
Answer

$x = 1$

2)



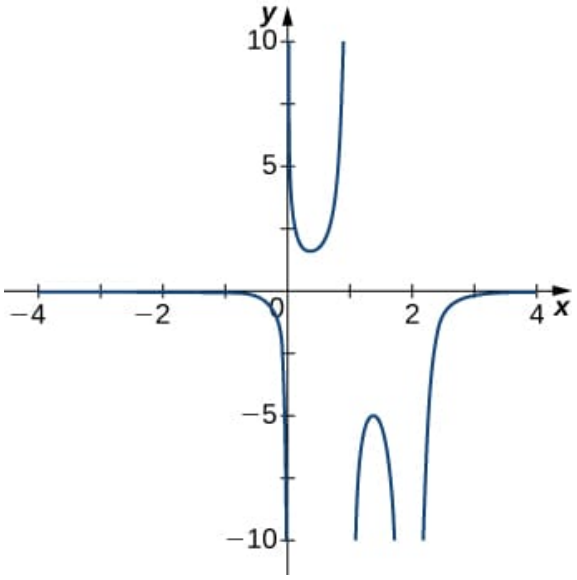
3)



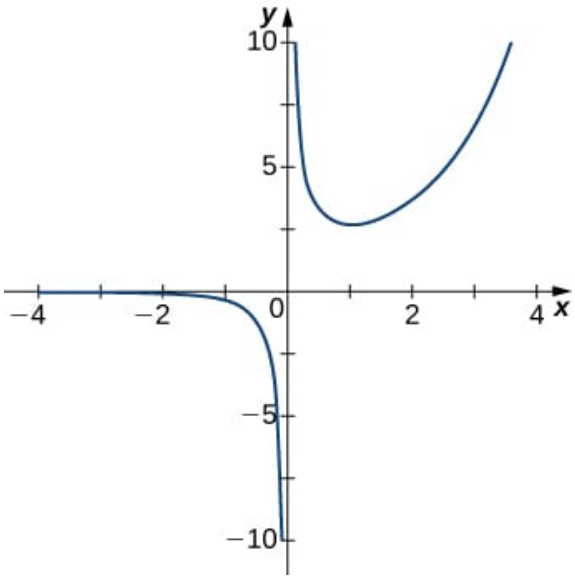
Answer

$x = -1, \;\; x = 2$

4)



5)



Answer

$x = 0$

For the functions $f(x)$ in exercises 6 - 10, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

6) $f(x) = \dfrac{x+1}{x^2+5x+4}, \quad a = -1$

7) $f(x) = \dfrac{x}{x-2}, \quad a = 2$

Answer

Yes, there is a vertical asymptote at $x = 2$.

8) $f(x) = (x+2)^{3/2}, \quad a = -2$

9) $f(x) = (x-1)^{-1/3}, \quad a = 1$

Answer

Yes, there is vertical asymptote at $x = 1$.

10) $f(x) = 1 + x^{-2/5}, \quad a = 1$

In exercises 11 - 20, evaluate the limit.

11) $\lim\limits_{x\to\infty} \dfrac{1}{3x+6}$

Answer

$\lim\limits_{x\to\infty} \dfrac{1}{3x+6} = 0$

12) $\lim\limits_{x\to\infty} \dfrac{2x-5}{4x}$

13) $\lim\limits_{x\to\infty} \dfrac{x^2-2x+5}{x+2}$

Answer

$\lim\limits_{x\to\infty} \dfrac{x^2-2x+5}{x+2} = \infty$

14) $\lim_{x\to-\infty}\dfrac{3x^3-2x}{x^2+2x+8}$

15) $\lim_{x\to-\infty}\dfrac{x^4-4x^3+1}{2-2x^2-7x^4}$

Answer

$$\lim_{x\to-\infty}\frac{x^4-4x^3+1}{2-2x^2-7x^4}=-\frac{1}{7}$$

16) $\lim_{x\to\infty}\dfrac{3x}{\sqrt{x^2+1}}$

17) $\lim_{x\to-\infty}\dfrac{\sqrt{4x^2-1}}{x+2}$

Answer

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2-1}}{x+2}=-2$$

18) $\lim_{x\to\infty}\dfrac{4x}{\sqrt{x^2-1}}$

19) $\lim_{x\to-\infty}\dfrac{4x}{\sqrt{x^2-1}}$

Answer

$$\lim_{x\to-\infty}\frac{4x}{\sqrt{x^2-1}}=-4$$

20) $\lim_{x\to\infty}\dfrac{2\sqrt{x}}{x-\sqrt{x}+1}$

For exercises 21 - 25, find the horizontal and vertical asymptotes.

21) $f(x)=x-\dfrac{9}{x}$

Answer

Horizontal: none,
Vertical: $x=0$

22) $f(x)=\dfrac{1}{1-x^2}$

23) $f(x)=\dfrac{x^3}{4-x^2}$

Answer

Horizontal: none,
Vertical: $x=\pm2$

24) $f(x)=\dfrac{x^2+3}{x^2+1}$

25) $f(x)=\sin(x)\sin(2x)$

Answer

Horizontal: none,
Vertical: none

26) $f(x)=\cos x+\cos(3x)+\cos(5x)$

27) $f(x) = \dfrac{x \sin(x)}{x^2 - 1}$

Answer

Horizontal: $y = 0$,
Vertical: $x = \pm 1$

28) $f(x) = \dfrac{x}{\sin(x)}$

29) $f(x) = \dfrac{1}{x^3 + x^2}$

Answer

Horizontal: $y = 0$,
Vertical: $x = 0$ and $x = -1$

30) $f(x) = \dfrac{1}{x - 1} - 2x$

31) $f(x) = \dfrac{x^3 + 1}{x^3 - 1}$

Answer

Horizontal: $y = 1$,
Vertical: $x = 1$

32) $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

33) $f(x) = x - \sin x$

Answer

Horizontal: none,
Vertical: none

34) $f(x) = \dfrac{1}{x} - \sqrt{x}$

For exercises 35 - 38, construct a function $f(x)$ that has the given asymptotes.

35) $x = 1$ and $y = 2$

Answer

Answers will vary, for example: $y = \dfrac{2x}{x - 1}$

36) $x = 1$ and $y = 0$

37) $y = 4, \quad x = -1$

Answer

Answers will vary, for example: $y = \dfrac{4x}{x + 1}$

38) $x = 0$

In exercises 39 - 43, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

39) [T] $f(x) = \dfrac{1}{x + 10}$

Answer

$\lim_{x\to\infty} \dfrac{1}{x+10} = 0$ so f has a horizontal asymptote of $y = 0$.

40) [T] $f(x) = \dfrac{x+1}{x^2+7x+6}$

41) [T] $\lim_{x\to-\infty} x^2 + 10x + 25$

Answer

$\lim_{x\to-\infty} x^2 + 10x + 25 = \infty$

42) [T] $\lim_{x\to-\infty} \dfrac{x+2}{x^2+7x+6}$

43) [T] $\lim_{x\to\infty} \dfrac{3x+2}{x+5}$

Answer

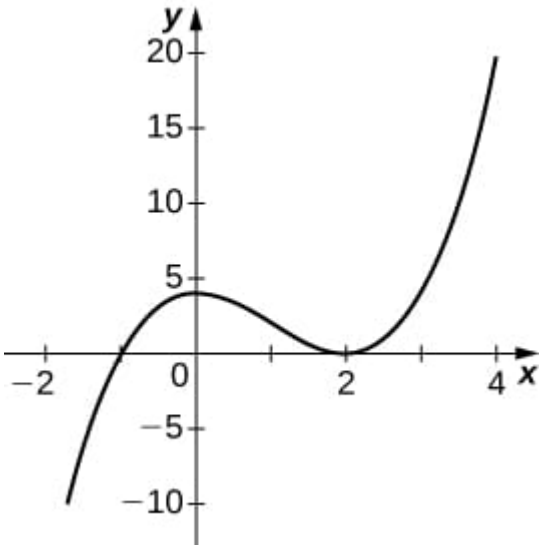
$\lim_{x\to\infty} \dfrac{3x+2}{x+5} = 3$ so this function has a horizontal asymptote of $y = 3$.

In exercises 44 - 55, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

44) $y = 3x^2 + 2x + 4$

45) $y = x^3 - 3x^2 + 4$

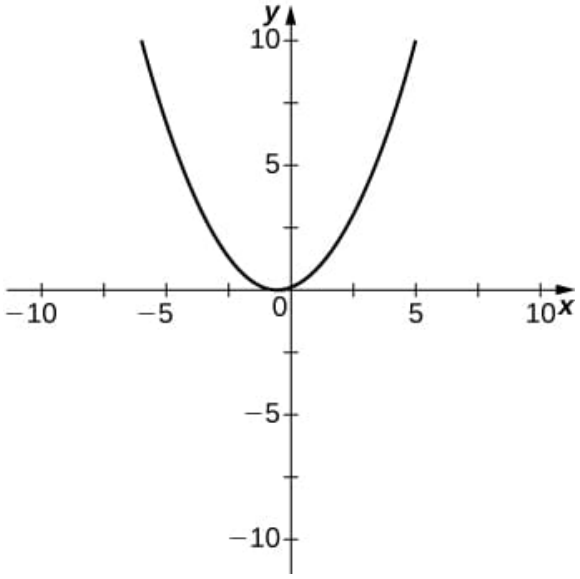
Answer



46) $y = \dfrac{2x+1}{x^2+6x+5}$

47) $y = \dfrac{x^3+4x^2+3x}{3x+9}$

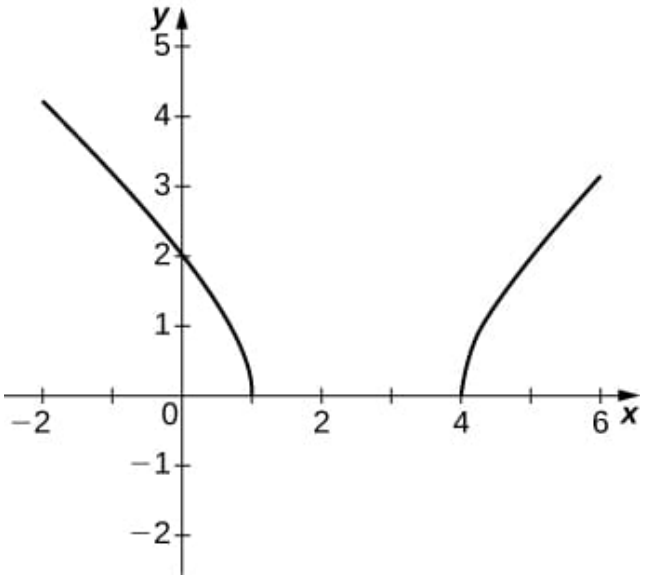
Answer



48) $y = \dfrac{x^2 + x - 2}{x^2 - 3x - 4}$

49) $y = \sqrt{x^2 - 5x + 4}$

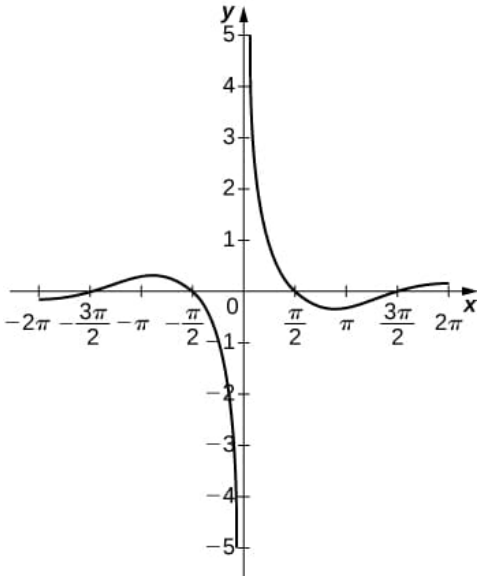
Answer



50) $y = 2x\sqrt{16 - x^2}$

51) $y = \dfrac{\cos x}{x}$, on $x = [-2\pi, 2\pi]$

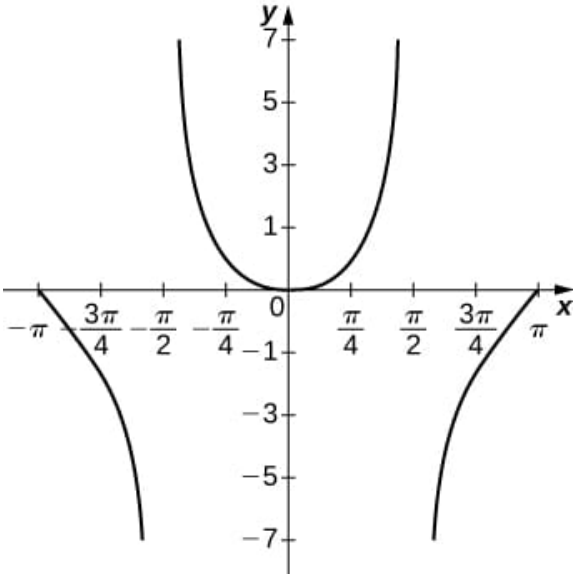
Answer



52) $y = e^x - x^3$

53) $y = x \tan x, \quad x = [-\pi, \pi]$

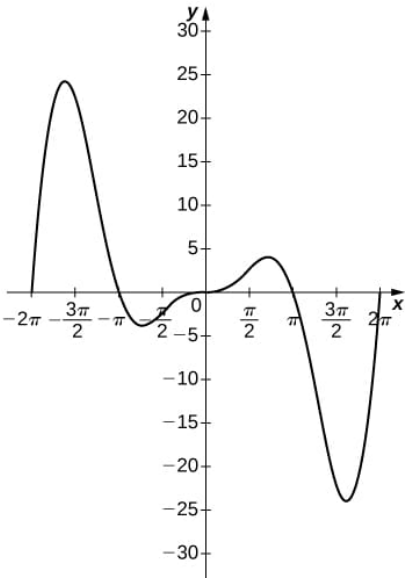
Answer



54) $y = x \ln(x), \quad x > 0$

55) $y = x^2 \sin(x), \quad x = [-2\pi, 2\pi]$

Answer



56) For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $y = 2$ then the polynomials $P(x)$ and $Q(x)$ must have what relation?

57) For $f(x) = \dfrac{P(x)}{Q(x)}$ to have an asymptote at $x = 0$, then the polynomials $P(x)$ and $Q(x)$. must have what relation?

Answer

$Q(x)$. must have have x^{k+1} as a factor, where $P(x)$ has x^k as a factor.

58) If $f\,'(x)$ has asymptotes at $y = 3$ and $x = 1$, then $f(x)$ has what asymptotes?

59) Both $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{1}{(x-1)^2}$ have asymptotes at $x = 1$ and $y = 0$. What is the most obvious difference between these two functions?

Answer

$$\lim_{x \to 1^-} f(x) = -\infty \text{ and } \lim_{x \to 1^-} g(x) = \infty$$

60) True or false: Every ratio of polynomials has vertical asymptotes.

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15.8: Applied Optimization Problems

Learning Objectives

- Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example $15.8.1$, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example $15.8.1$: Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure $15.8.1$). Given $100\ \text{ft}$ of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

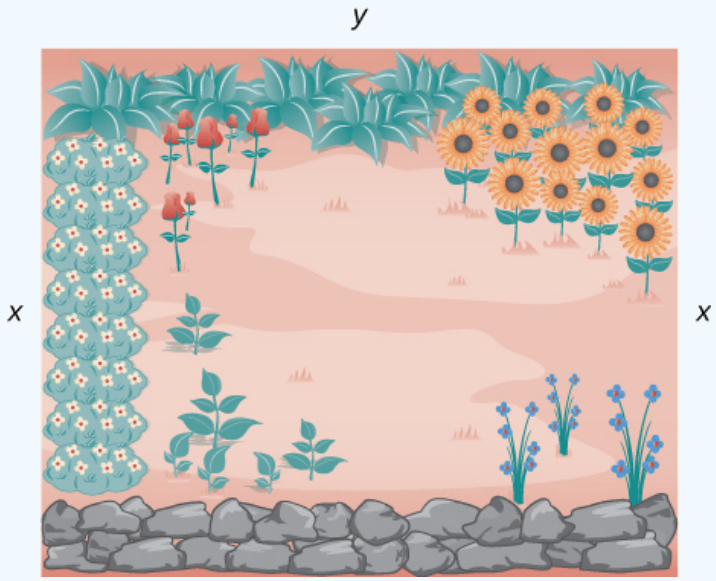


Figure $15.8.1$: We want to determine the measurements x and y that will create a garden with a maximum area using $100\ \text{ft}$ of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From Figure $15.8.1$, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden.

Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (Figure $15.8.2$). We conclude that the maximum area must occur when $x = 25$.

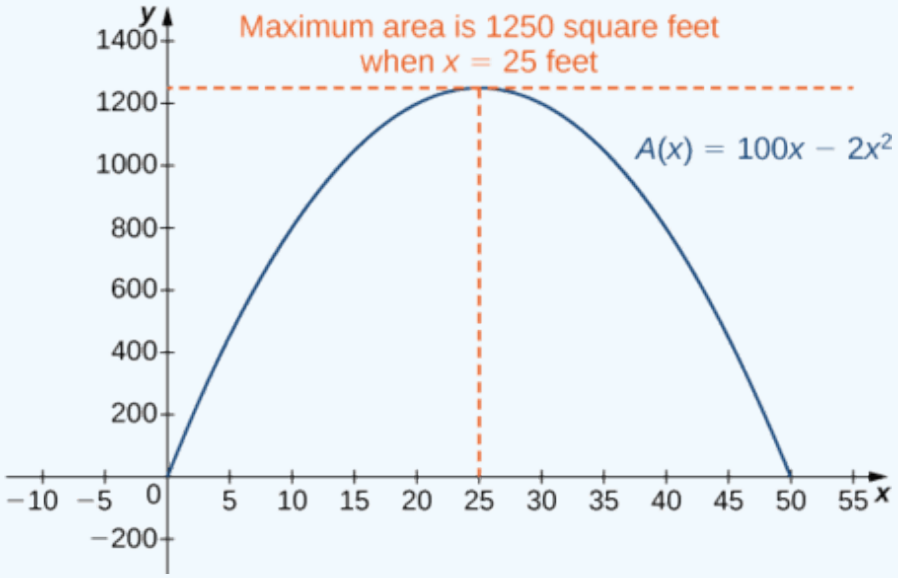


Figure $15.8.2$: To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50$ ft. The area of this garden is 1250 ft^2.

Exercise 15.8.1

Determine the maximum area if we want to make the same rectangular garden as in Figure 15.8.2, but we have 200 ft of fencing.

Hint available at the end of this section.

Answer available at the end of this section.

Now let's look at a general strategy for solving optimization problems similar to Example 15.8.1.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 15.8.2: Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (Figure 15.8.3). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

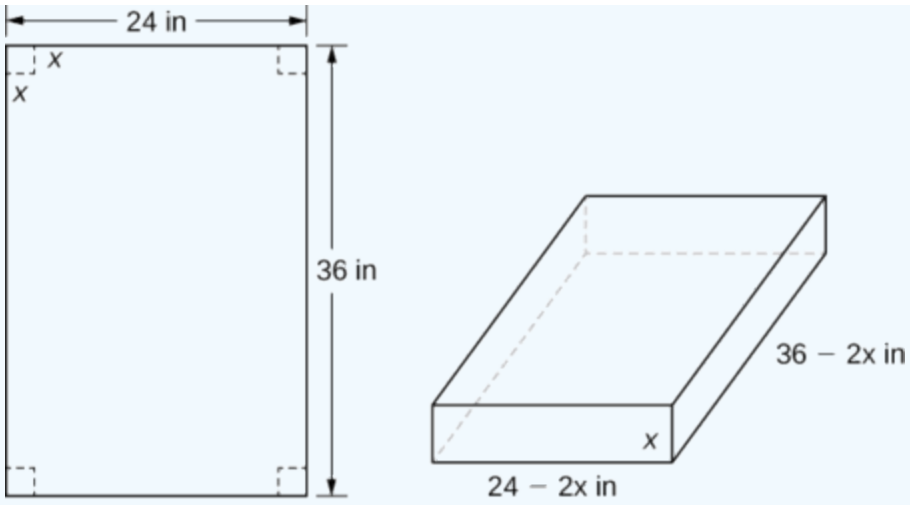


Figure 15.8.3: A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is

$$V = L \cdot W \cdot H,$$

where $L,\ W$,and H are the length, width, and height, respectively.

Step 4: From Figure 15.8.3, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$\begin{aligned} V(x) &= (36 - 2x)(24 - 2x)x \\ &= 4x^3 - 120x^2 + 864x \end{aligned}.$$

Step 5: To determine the domain of consideration, let's examine Figure 15.8.3. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for \(0

$$V\,'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$
\begin{aligned}
x &= \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} \\
&= \frac{20 \pm \sqrt{112}}{2} \\
&= \frac{20 \pm 4\sqrt{7}}{2} \\
&= 10 \pm 2\sqrt{7}
\end{aligned}
.
$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is

$$V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825\,\text{in}^3.$$

as shown in the following graph.

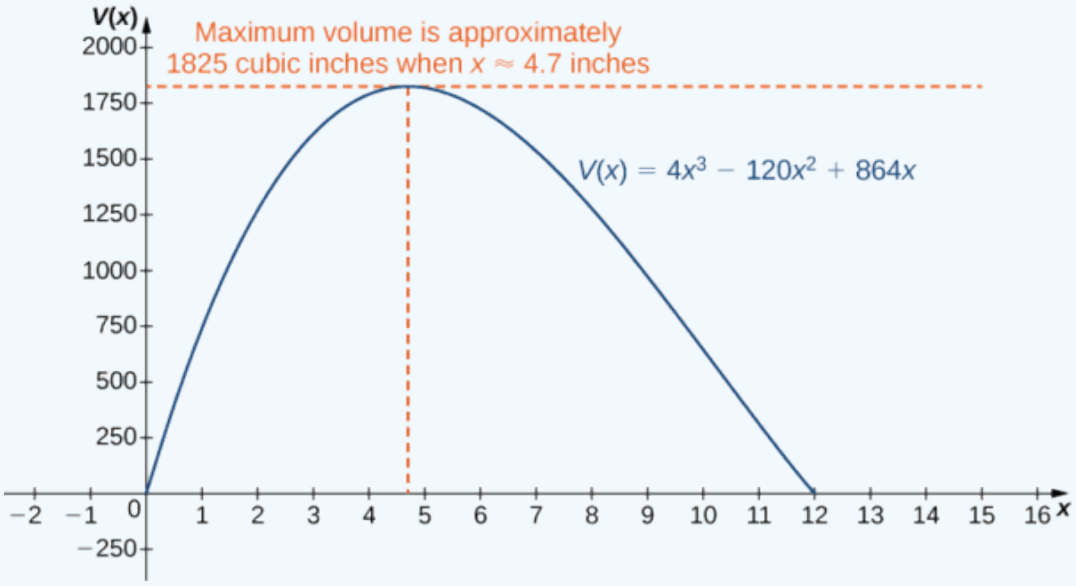


Figure 15.8.4: Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Exercise 15.8.2

Suppose the dimensions of the cardboard in Example 15.8.2 are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.8.3: Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (Figure $15.8.5$). Let T be the time it takes to get from the cabin to the island.

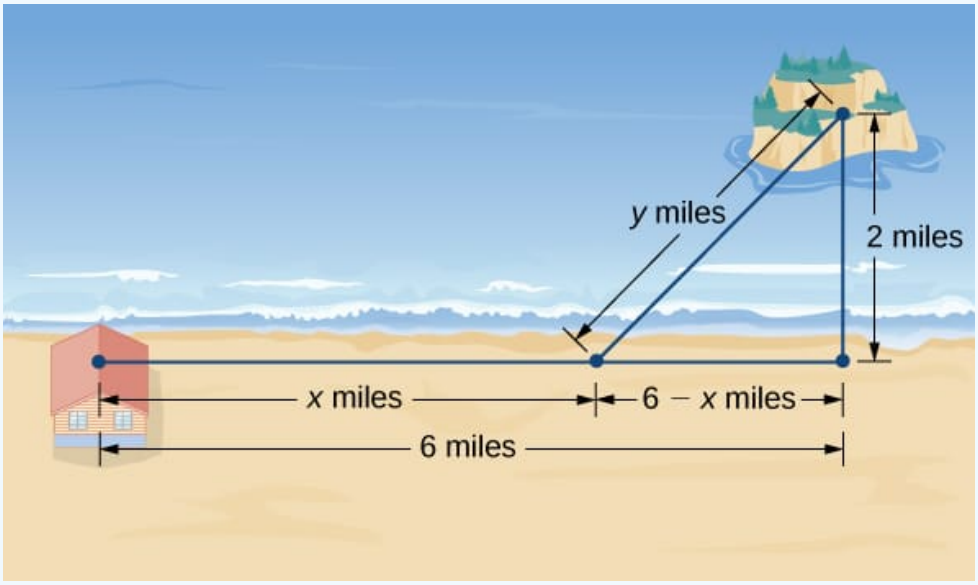


Figure $15.8.5$: How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time $(D = R \times T)$, the time spent running is

$$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{swimming} = \frac{D_{swimming}}{R_{swimming}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From Figure $15.8.5$, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem, $2^2 + (6 - x)^2 = y^2$, and we obtain $y = \sqrt{(6 - x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6 - x)^2 + 4}}{3}.$$

Step 5: From Figure $15.8.5$, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$\begin{aligned} T'(x) &= \frac{1}{8} - \frac{1}{2}\frac{[(6 - x)^2 + 4]^{-1/2}}{3} \cdot 2(6 - x) \\ &= \frac{1}{8} - \frac{(6 - x)}{3\sqrt{(6 - x)^2 + 4}} \end{aligned}$$

If $T'(x) = 0$,, then

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2+4}} \tag{15.8.1}$$

Therefore,

$$3\sqrt{(6-x)^2+4} = 8(6-x). \tag{15.8.2}$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9[(6-x)^2+4] = 64(6-x)^2,$$

which implies

$$55(6-x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x-6)^2 = \frac{36}{55}.$$

[Note that since we are squaring, $(x-6)^2 = (6-x)^2$.]

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of Equation 15.8.2 to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies Equation 15.8.1. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas

$$T(6 - 6/\sqrt{55}) \approx 1.368\,\text{h}.$$

Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

Exercise 15.8.3

Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Hint available at the end of this section.

Answer available at the end of this section.

In business, companies are interested in **maximizing revenue.** In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

Example 15.8.4: Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge $\$50$ per day or less, they will rent all their cars. If they charge $\$200$ per day or more, they will not rent any cars. Assuming the owners plan to charge customers between $\$50$ per day and $\$200$ per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$\begin{aligned} R(p) &= n \times p \\ &= (1000 - 5p)p \\ &= -5p^2 + 1000p. \end{aligned}$$

Step 5: Since the owners plan to charge between $\$50$ per car per day and $\$200$ per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$. When $p = 100, R(100) = \$50,000$. When $p = 50, R(p) = \$37,500$. When $p = 200, R(p) = \$0$.

Therefore, the absolute maximum occurs at $p = \$100$. The car rental company should charge $\$100$ per day per car to maximize revenue as shown in the following figure.

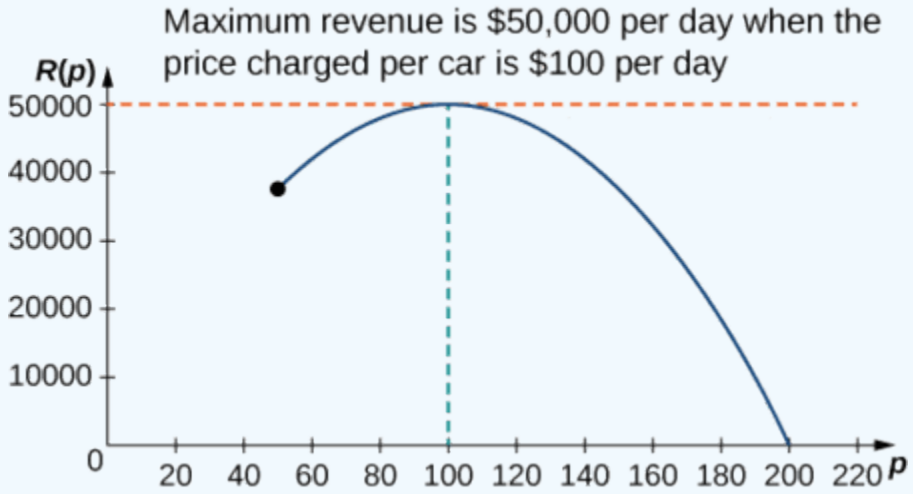


Figure 15.8.6: To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

Exercise 15.8.4

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.8.5: Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

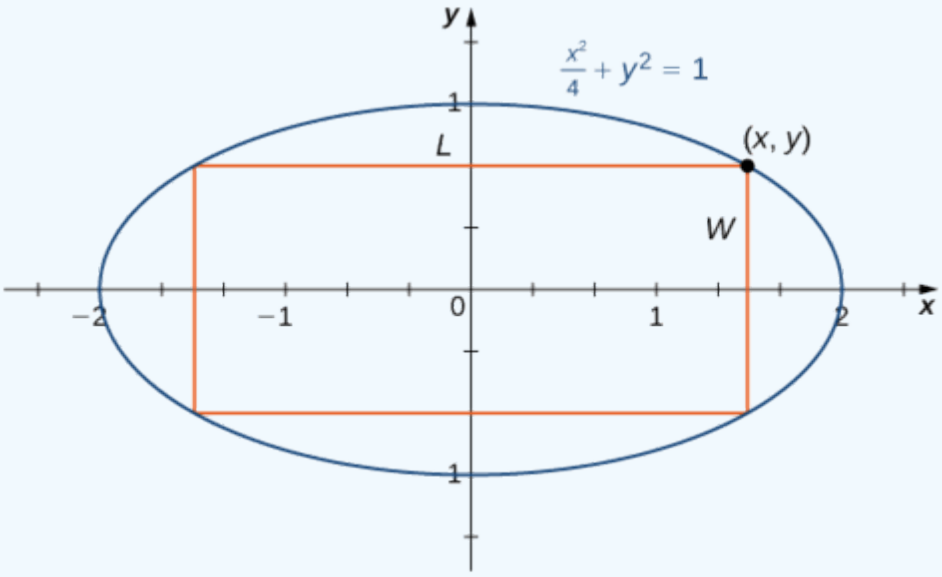


Figure 15.8.7: We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in Figure 15.8.7. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

Step 5: From Figure 15.8.7, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy \(0

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0,2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x=0$ and $x=2$, $A(x)=0$. For \(00\).

Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4-x^2}+2x\cdot\frac{1}{2\sqrt{4-x^2}}(-2x) \\ &= 2\sqrt{4-x^2}-\frac{2x^2}{\sqrt{4-x^2}} \\ &= \frac{8-4x^2}{\sqrt{4-x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x)=0$. We can see that if x is a solution of

$$\frac{8-4x^2}{\sqrt{4-x^2}}=0, \tag{15.8.3}$$

then x must satisfy

$$8-4x^2=0.$$

Therefore, $x^2=2$. Thus, $x=\pm\sqrt{2}$ are the possible solutions of Equation 15.8.3. Since we are considering x over the interval $[0,2]$, $x=\sqrt{2}$ is a possibility for a critical point, but $x=-\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of Equation 15.8.3. Since $x=\sqrt{2}$ is a solution of Equation 15.8.3, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0,2]$.

Therefore, $A(x)$ must have an absolute maximum at the critical point $x=\sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x=\sqrt{2}$ then

$$y=\sqrt{1-\frac{(\sqrt{2})^2}{4}}=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L=2x=2\sqrt{2}$ and $W=2y=\dfrac{2}{\sqrt{2}}=\sqrt{2}$. The area of this rectangle is $A=LW=(2\sqrt{2})(\sqrt{2})=4$.

Exercise 15.8.5

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2+y^2=1$. What is the domain of consideration?

Hint available at the end of this section.

Answer available at the end of this section.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For

example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example 15.8.6: Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of $216\ \text{in}^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

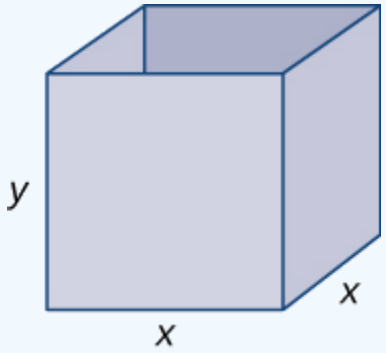


Figure 15.8.8: We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is x^2y and the volume is given as $216\ \text{in}^3$, the constraint equation is

$$x^2y = 216.$$

Solving the constraint equation for y, we have $y = \dfrac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \dfrac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is

the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+,\ S(x) \to \infty$. Also, as $x \to \infty,\ S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S\,'(x) = 0$ when $2x = \dfrac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see Figure 15.8.9).

When $x = 6\sqrt[3]{2}$, $y = \dfrac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

$$S(6\sqrt[3]{2}) = \frac{864}{6\sqrt[3]{2}} + (6\sqrt[3]{2})^2 = 108\sqrt[3]{4}\,\text{in}^2$$

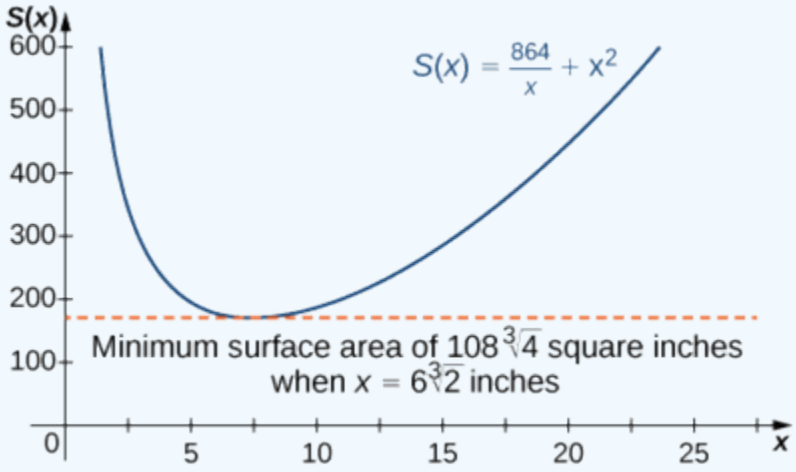


Figure 15.8.9: We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

Exercise 15.8.6

Consider the same open-top box, which is to have volume $216\ \text{in}^3$. Suppose the cost of the material for the base is $20¢/\text{in}^2$ and the cost of the material for the sides is $30¢/\text{in}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.
- Look for critical points to locate local extrema.

Glossary

optimization problems
: problems that are solved by finding the maximum or minimum value of a function

Answers

Exercise 15.8.1

We need to maximize the function $A(x) = 200x - 2x^2$ over the interval $[0, 100]$.

Exercise 15.8.1

The maximum area is $5000\ \text{ft}^2$.

Exercise 15.8.2

The volume of the box is $L \cdot W \cdot H$.

Exercise 15.8.2

$V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

Exercise 15.8.3

The time $T = T_{running} + T_{swimming}$.

Exercise 15.8.3

$$T(x) = \frac{x}{6} + \frac{\sqrt{(15 - x)^2 + 1}}{2.5}$$

Exercise 15.8.4

$R(p) = n \times p$, where n is the number of cars rented and p is the price charged per car.

Exercise 15.8.4

The company should charge $\$75$ per car per day.

Exercise 15.8.5

If (x, y) is the vertex of the square that lies in the first quadrant, then the area of the square is $A = (2x)(2y) = 4xy$.

Exercise 15.8.5

$A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

Exercise 15.8.6

If the cost of one of the sides is $30\text{¢}/\text{in}^2$, the cost of that side is $0.30xy$ dollars.

Exercise 15.8.6

$c(x) = \dfrac{259.2}{x} + 0.2x^2$ dollars

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15.8E: Exercises for Section 4.7

For exercises 1 - 4, answer by proof, counterexample, or explanation.

1) When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

Answer

The critical points can be the minima, maxima, or neither.

2) Why do you need to check the endpoints for optimization problems?

3) True or False. For every continuous nonlinear function, you can find the value x that maximizes the function.

Answer

False; $y = -x^2$ has a minimum only

4) True or False. For every continuous non-constant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

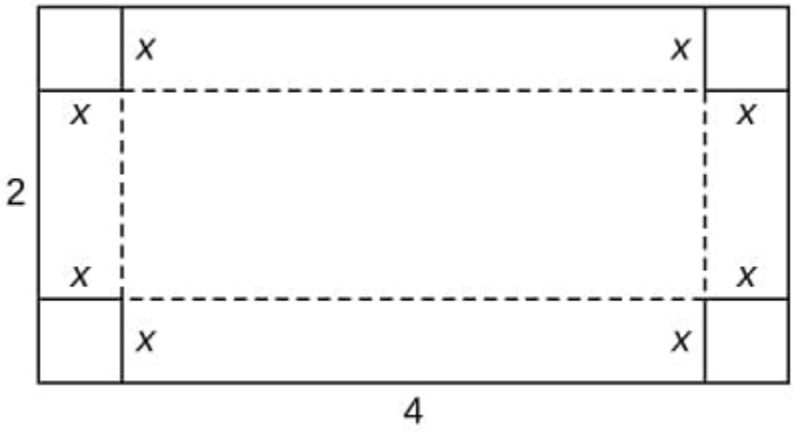
In exercises 5 - 8, set up and evaluate each optimization problem.

5) To carry a suitcase on an airplane, the $\text{length} + \text{width} + \text{height}$ of the box must be less than or equal to 62 in. Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \frac{1}{2}h\right)^2$. What height allows you to have the largest volume?

Answer

$h = \frac{62}{3}$ in.

6) You are constructing a cardboard box with the dimensions 2 m by 4 m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



7) Find the positive integer that minimizes the sum of the number and its reciprocal.

Answer

1

8) Find two positive integers such that their sum is 10, and minimize and maximize the sum of their squares.

In exercises 9 - 11, consider the construction of a pen to enclose an area.

9) You have 400 ft of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the

area?

Answer

100 ft by 100 ft

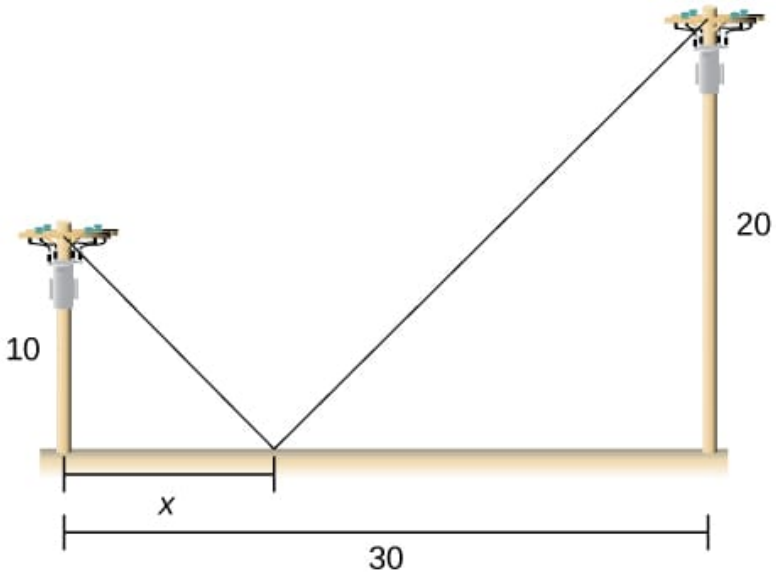
10) You have 800 ft of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

11) You need to construct a fence around an area of 1600 ft^2. What are the dimensions of the rectangular pen to minimize the amount of material needed?

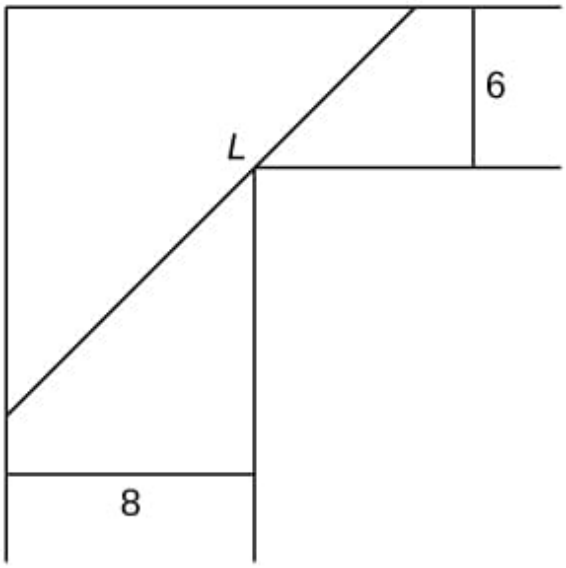
Answer

40 ft by 40 ft

12) Two poles are connected by a wire that is also connected to the ground. The first pole is 20 ft tall and the second pole is 10 ft tall. There is a distance of 30 ft between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



13) [T] You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



Answer

19.73 ft

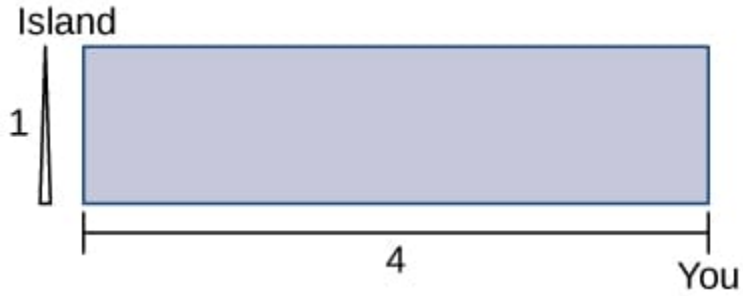
14) A patient's pulse measures $70 \text{ bpm}, \ 80 \text{ bpm}$, then 120 bpm. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2+(x-80)^2+(x-120)^2$? What value minimizes it?

15) In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

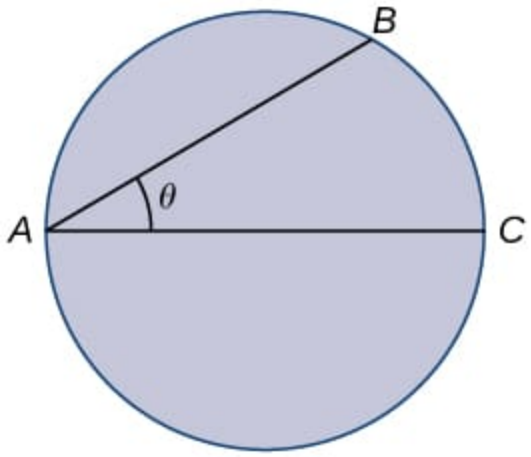
Answer

84 bpm

16) You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For exercises 17 - 19, consider a lifeguard at a circular pool with diameter 40 m. He must reach someone who is drowning on the exact opposite side of the pool, at position C. The lifeguard swims with a speed v and runs around the pool at speed $w=3v$.



17) Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, θ.

Answer

$$T(\theta) = \frac{40\theta}{3v} + \frac{40\cos\theta}{v}$$

18) Find at what angle θ the lifeguard should swim to reach the drowning person in the least amount of time.

19) A truck uses gas as $g(v) = av + \dfrac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

Answer

$$v = \sqrt{\frac{b}{a}}$$

For exercises 20 - 21, consider a limousine that gets $m(v) = \frac{120-2v}{5}$ mi/gal at speed v, the chauffeur costs $\$15/\text{h}$, and gas is $\$3.50/\text{gal}$.

20) Find the cost per mile at speed v.

21) Find the cheapest driving speed.

Answer

approximately 34.02 mph

For exercises 22 - 24, consider a pizzeria that sell pizzas for a revenue of $R(x) = ax$ and costs $C(x) = b + cx + dx^2$, where x represents the number of pizzas.

22) Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

23) Assume that $R(x) = 10x$ and $C(x) = 2x + x^2$. How many pizzas sold maximizes the profit?

Answer

Selling 4 pizzas will maximize the profit.

24) Assume that $R(x) = 15x$, and $C(x) = 60 + 3x + \frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For exercises 25 - 26, consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x.

25) Choose x to maximize the sum of their areas.

Answer

$x = 0$

26) Choose x to minimize the sum of their areas.

For exercises 27 - 30, consider two nonnegative numbers x and y such that $x + y = 10$. Maximize and minimize the quantities.

27) xy

Answer

Maximal: $x = 5,\ y = 5$;
Minimal: $x = 0,\ y = 10$ and $y = 0,\ x = 10$

28 x^2y^2

29) $y - \dfrac{1}{x}$

Answer

Maximal: $x = 1,\ y = 9$;
Minimal: none

30) $x^2 - y$

In exercises 31 - 36, draw the given optimization problem and solve.

31) Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

Answer

$V = \frac{4\pi}{3\sqrt{3}}$

32) Find the volume of the largest right cone that fits in a sphere of radius 1.

33) Find the area of the largest rectangle that fits into the triangle with sides $x = 0,\ y = 0$ and $\dfrac{x}{4} + \dfrac{y}{6} = 1$.

Answer

$A = 6$

34) Find the largest volume of a cylinder that fits into a cone that has base radius R and height h.

35) Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

Answer

$r = 2,\ h = 4$

36) Find the dimensions of a right cone with surface area $S = 4\pi$ that has the largest volume.

For exercises 37 - 40, consider the points on the graphs of the given equations. Use a calculator to graph the functions.

37) [T] Where is the line $y = 5 - 2x$ closest to the origin?

Answer

$(2, 1)$

38) [T] Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

39) [T] Where is the parabola $y = x^2$ closest to point $(2, 0)$?

Answer

$(0.8351, 0.6974)$

40) [T] Where is the parabola $y = x^2$ closest to point $(0, 3)$?

In exercises 41 - 45, set up, but do not evaluate, each optimization problem.

41) A window is composed of a semicircle placed on top of a rectangle. If you have 20 ft of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use r to represent the radius of the semicircle.



Answer

$A = 20r - 2r^2 - \frac{1}{2}\pi r^2$

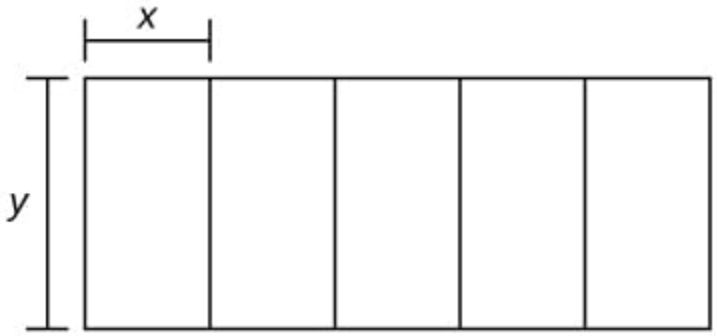
42) You have a garden row of 20 watermelon plants that produce an average of 30 watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

43) You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume 4 ft^3. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

Answer

$C(x) = 5x^2 + \dfrac{32}{x}$

44) You are building five identical pens adjacent to each other with a total area of 1000 m^2, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?



45) You are the manager of an apartment complex with 50 units. When you set rent at $\$800/\text{month}$, all apartments are rented. As you increase rent by $\$25/\text{month}$, one fewer apartment is rented. Maintenance costs run $\$50/\text{month}$ for each occupied unit. What is the rent that maximizes the total amount of profit?

Answer

$$P(x) = (50 - x)(800 + 25x - 50)$$

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15.9: L'Hôpital's Rule

Learning Objectives

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule,** uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4.$$

For $\lim_{x\to 0}\frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more importantly, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and

g such that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and such that $g\,'(a) \neq 0$ For x near a,we can write

$$f(x) \approx f(a) + f'(a)(x-a)$$

and

$$g(x) \approx g(a) + g'(a)(x-a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}.$$

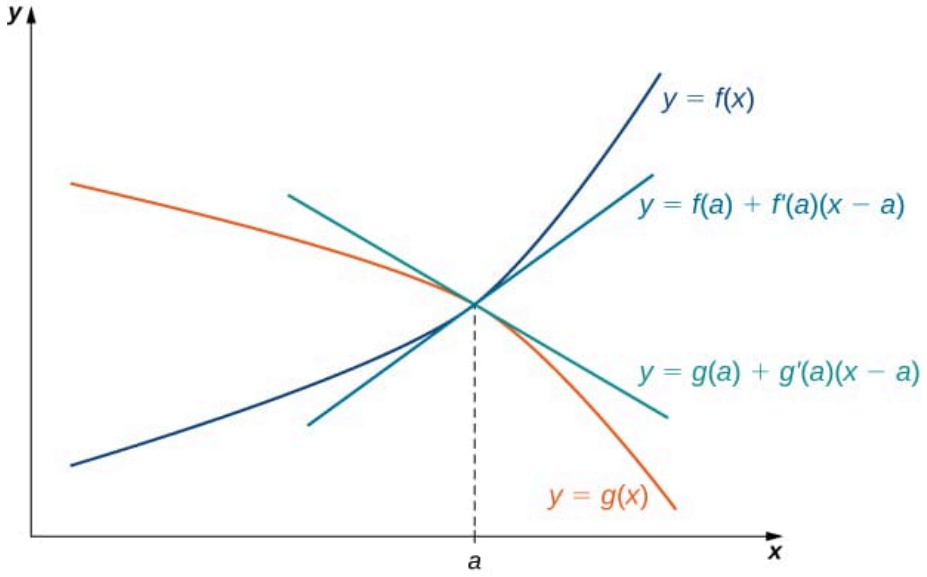


Figure 15.9.1: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x\to a} f(x) = 0$. Similarly, $g(a) = \lim_{x\to a} g(x) = 0$. If we also assume that $f\,'$ and $g\,'$ are continuous at $x = a$, then $f\,'(a) = \lim_{x\to a} f\,'(x)$ and $g\,'(a) = \lim_{x\to a} g\,'(x)$. Using these ideas, we conclude that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that $f\,'$ and $g\,'$ are continuous at a and $g\,'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{Since } f(a)=0=g(a)\\
&= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} && \text{Multiply numerator and denominator by } \frac{1}{x-a}\\
&= \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} && \text{The limit of a quotient is the quotient of the limits.}\\
&= \frac{f'(a)}{g'(a)} && \text{By the definition of the derivative}\\
&= \frac{\lim\limits_{x\to a} f'(x)}{\lim\limits_{x\to a} g'(x)} && \text{By the continuity of } f' \text{ and } g'\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)}. && \text{The limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 15.9.1: Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x\to 0}\dfrac{1-\cos x}{x}$
b. $\lim_{x\to 1}\dfrac{\sin(\pi x)}{\ln x}$
c. $\lim_{x\to\infty}\dfrac{e^{1/x}-1}{1/x}$
d. $\lim_{x\to 0}\dfrac{\sin x - x}{x^2}$

Solution

a. Since the numerator $1-\cos x \to 0$ and the denominator $x \to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}\left(1-\cos x\right)}{\frac{d}{dx}\left(x\right)} = \lim_{x\to 0}\frac{\sin x}{1} = \frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0} 1} = \frac{0}{1} = 0.$$

b. As $x \to 1$, the numerator $\sin(\pi x) \to 0$ and the denominator $\ln(x) \to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 1}\frac{\sin(\pi x)}{\ln x}=\lim_{x\to 1}\frac{\pi\cos(\pi x)}{1/x}$$
$$=\lim_{x\to 1}(\pi x)\cos(\pi x)$$
$$=(\pi\cdot 1)(-1)=-\pi.$$

c. As $x\to\infty$, the numerator $e^{1/x}-1\to 0$ and the denominator $\frac{1}{x}\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to\infty}\frac{e^{1/x}-1}{\frac{1}{x}}=\lim_{x\to\infty}\frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}e^{1/x}=e^0=1.$$

d. As $x\to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 0}\frac{\sin x-x}{x^2}=\lim_{x\to 0}\frac{\cos x-1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x\to 0$, we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x\to 0}\frac{\cos x-1}{2x}=\lim_{x\to 0}\frac{-\sin x}{2}=0.$$

Therefore, we conclude that

$$\lim_{x\to 0}\frac{\sin x-x}{x^2}=0.$$

Exercise 15.9.1

Evaluate $\lim_{x\to 0}\frac{x}{\tan x}$.

Hint available at the end of this section.

Answer available at the end of this section.

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$. Limits of this form are classified as indeterminate forms of type ∞/∞. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞ is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a}f(x)=\infty$ (or $-\infty$) and $\lim_{x\to a}g(x)=\infty$ (or $-\infty$). Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a=\infty$ or $-\infty$, or the limit is one-sided.

Example 15.9.2: Applying L'Hôpital's Rule (∞/∞) Case

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x\to\infty}\dfrac{3x+5}{2x+1}$

b. $\lim_{x\to 0^+}\dfrac{\ln x}{\cot x}$

Solution

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x\to\infty}(3x+5)=\infty$ and $\lim_{x\to\infty}(2x+1)=\infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3}{2}=\frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+5/x}{2+1/x}=\frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x\to 0^+}\ln x=-\infty$ and $\lim_{x\to 0^+}\cot x=\infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x}=\lim_{x\to 0^+}\frac{1/x}{-\csc^2 x}=\lim_{x\to 0^+}\frac{1}{-x\csc^2 x}.$$

Now as $x\to 0^+,\csc^2 x\to\infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x\to 0^+}\frac{1}{-x\csc^2 x}=\lim_{x\to 0^+}\frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to 0^+}\sin^2 x=0$ and $\lim_{x\to 0^+}-x=0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x\to 0^+}\frac{\sin^2 x}{-x}=\lim_{x\to 0^+}\frac{2\sin x\cos x}{-1}=\frac{0}{-1}=0.$$

We conclude that

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x}=0.$$

Exercise 15.9.2

Evaluate $\lim_{x\to\infty}\dfrac{\ln x}{5x}$.

Hint available at the end of this section.

Answer available at the end of this section.

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\dfrac{f(x)}{g(x)}$, it is essential that the limit of $\dfrac{f(x)}{g(x)}$ be of the form $\dfrac{0}{0}$ or ∞/∞. Consider the following example.

Example 15.9.3: When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x\to 1}\dfrac{x^2+5}{3x+4}$.

Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5)=2x$$

and

$$\frac{d}{dx}(3x+4)=3.$$

At which point we would conclude erroneously that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\lim_{x\to 1}\frac{2x}{3}=\frac{2}{3}.$$

However, since $\lim_{x\to 1}(x^2+5)=6$ and $\lim_{x\to 1}(3x+4)=7$, we actually have

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}.$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\dfrac{d}{dx}(x^2+5)}{\dfrac{d}{dx}(3x+4)}.$$

Exercise 15.9.3

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x\to 0^+}\dfrac{\cos x}{x}$. Evaluate $\lim_{x\to 0^+}\dfrac{\cos x}{x}$ by other means.

Hint available at the end of this section.

Answer available at the end of this section.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\dfrac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0\cdot\infty, \infty-\infty, 1^\infty, \infty^0$, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\dfrac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim_{x \to a}(f(x) \cdot g(x))$, where $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form that arises in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x \to \infty$. For example, let n be a positive integer and consider

$$f(x) = \frac{1}{(x^n + 1)} \text{ and } g(x) = 3x^2.$$

As $x \to \infty, f(x) \to 0$ and $g(x) \to \infty$. However, the limit as $x \to \infty$ of $f(x)g(x) = \dfrac{3x^2}{(x^n + 1)}$ varies, depending on n. If $n = 2$, then $\lim_{x \to \infty} f(x)g(x) = 3$. If $n = 1$, then $\lim_{x \to \infty} f(x)g(x) = \infty$. If $n = 3$, then $\lim_{x \to \infty} f(x)g(x) = 0$. Here we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 15.9.4: Indeterminate Form of Type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution

First, rewrite the function $x \ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x \ln x = \frac{\ln x}{1/x}$$

we see that $\ln x \to -\infty$ as $x \to 0^+$ and $\dfrac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{\frac{d}{dx}\left(\ln x\right)}{\frac{d}{dx}\left(1/x\right)} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

We conclude that

$$\lim_{x \to 0^+} x \ln x = 0.$$

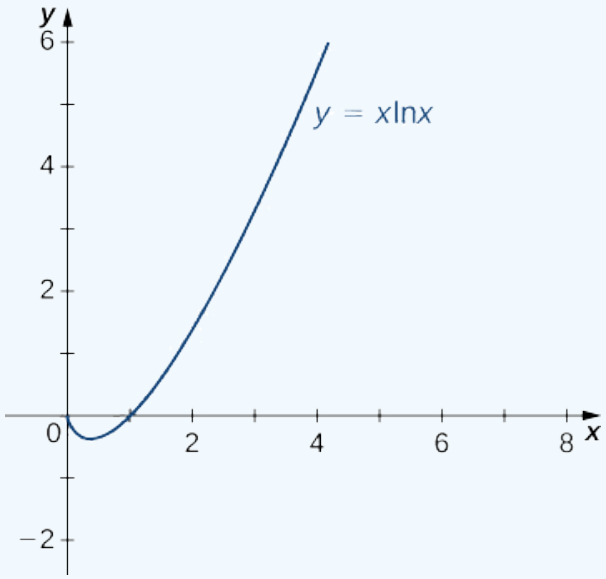


Figure 15.9.2: Finding the limit at $x = 0$ of the function $f(x) = x \ln x$.

Exercise 15.9.4

Evaluate $\lim_{x \to 0} x \cot x$.

Hint available at the end of this section.

Answer available at the end of this section.

Indeterminate Form of Type ∞-∞

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty, f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim_{x \to \infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 15.9.5: Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right)$.

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $(\sec^2 x) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = \infty.$$

Exercise 15.9.5

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

Hint available at the end of this section.

Answer available at the end of this section.

Indeterminate Forms Involving Exponents

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions $0^0, \infty^0$, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln(f(x)^{g(x)}) = g(x) \ln(f(x)).$$

Therefore,

$$\lim_{x\to a}[\ln(y)] = \lim_{x\to a}[g(x)\ln(f(x))].$$

Since $\lim_{x\to a} f(x) = \infty$, we know that $\lim_{x\to a}\ln(f(x)) = \infty$. Therefore, $\lim_{x\to a} g(x)\ln(f(x))$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x\to a} g(x)\ln(f(x)) = L$, where L may be ∞ or $-\infty$. Then

$$\lim_{x\to a}[\ln(y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to a} y\right) = L,$$

which gives us

$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^L.$$

Example 15.9.6: Indeterminate Form of Type ∞^0

Evaluate $\lim_{x\to\infty} x^{1/x}$.

Solution

Let $y = x^{1/x}$.Then,

$$\ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x\to\infty}\dfrac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0.$$

Therefore, $\lim_{x\to\infty}\ln y = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty} y\right) = 0,$$

which leads to

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} y = e^{\ln\left(\lim_{x\to\infty} y\right)} = e^0 = 1.$$

Hence,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

Exercise 15.9.6

Evaluate $\lim_{x\to\infty} x^{1/\ln(x)}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.9.7: Indeterminate Form of Type 0^0

Evaluate $\lim\limits_{x\to 0^+} x^{\sin x}$.

Solution

Let

$$y = x^{\sin x}.$$

Therefore,

$$\ln y = \ln(x^{\sin x}) = \sin x \ln x.$$

We now evaluate $\lim\limits_{x\to 0^+} \sin x \ln x$. Since $\lim\limits_{x\to 0^+} \sin x = 0$ and $\lim\limits_{x\to 0^+} \ln x = -\infty$, we have the indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x\to 0^+} \frac{\cos x}{-1/(x(\ln x)^2)} = \lim_{x\to 0^+} (-x(\ln x)^2 \cos x).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x,}$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x\to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x)\right] = \left(\lim_{x\to 0^+} \frac{\sin x}{x}\right) \cdot \left(\lim_{x\to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

We conclude that $\lim\limits_{x\to 0^+} \ln y = 0$. Therefore, $\ln\left(\lim\limits_{x\to 0^+} y\right) = 0$ and we have

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x\to 0^+} x^{\sin x} = 1.$$

Exercise 15.9.7

Evaluate $\lim_{x\to 0^+} x^x$.

Hint available at the end of this section.

Answer available at the end of this section.

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as Table 15.9.1 shows, the values of x^3 are growing much faster than the values of x^2.

Table 15.9.1: Comparing the Growth Rates of x^2 and x^3

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

In fact,

$$\lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty.$$

or, equivalently

$$\lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in Table 15.9.2. In fact,

$$\lim_{x\to\infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

Table 15.9.2: Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 15.9.8: Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

a. $f(x) = x^2$ and $g(x) = e^x$
b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution

a. Since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

Since $\lim_{x\to\infty} 2x = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See Figure 15.9.3 and Table 15.9.3)

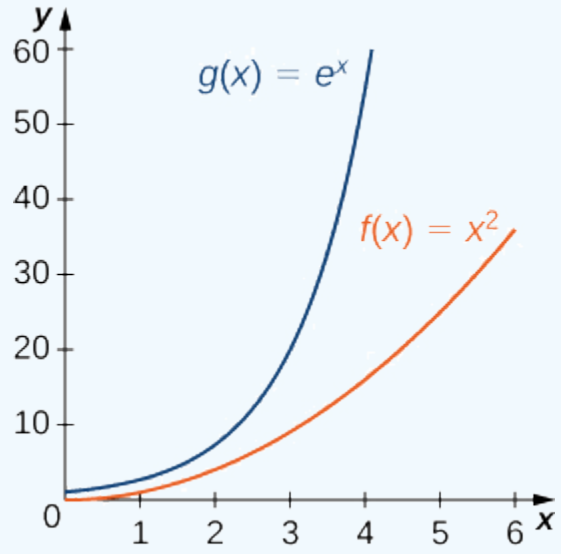


Figure 15.9.3: An exponential function grows at a faster rate than a power function.

Table 15.9.3: Growth rates of a power function and an exponential function.

x	5	10	15	20
x^2	25	100	225	400

e^x	148	22,026	3,269,017	485,165,195

b. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$. We obtain

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see Figure $15.9.4$ and Table $15.9.4$).

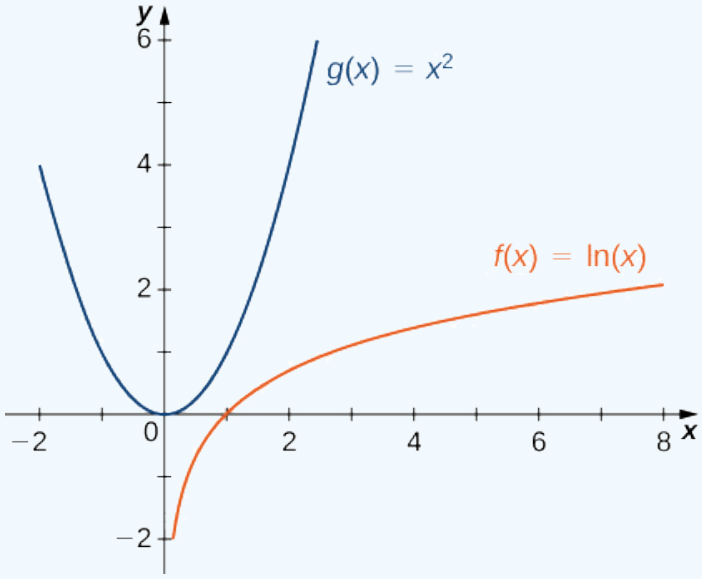


Figure $15.9.4$: A power function grows at a faster rate than a logarithmic function.

Table $15.9.4$: Growth rates of a power function and a logarithmic function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

Exercise 15.9.8

Compare the growth rates of x^{100} and 2^x.

Hint available at the end of this section.

Answer available at the end of this section.

Using the same ideas as in Example $15.9.8a$. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure $15.9.5$ and Table $15.9.5$, we compare e^x with x^3 and x^4 as $x \to \infty$.

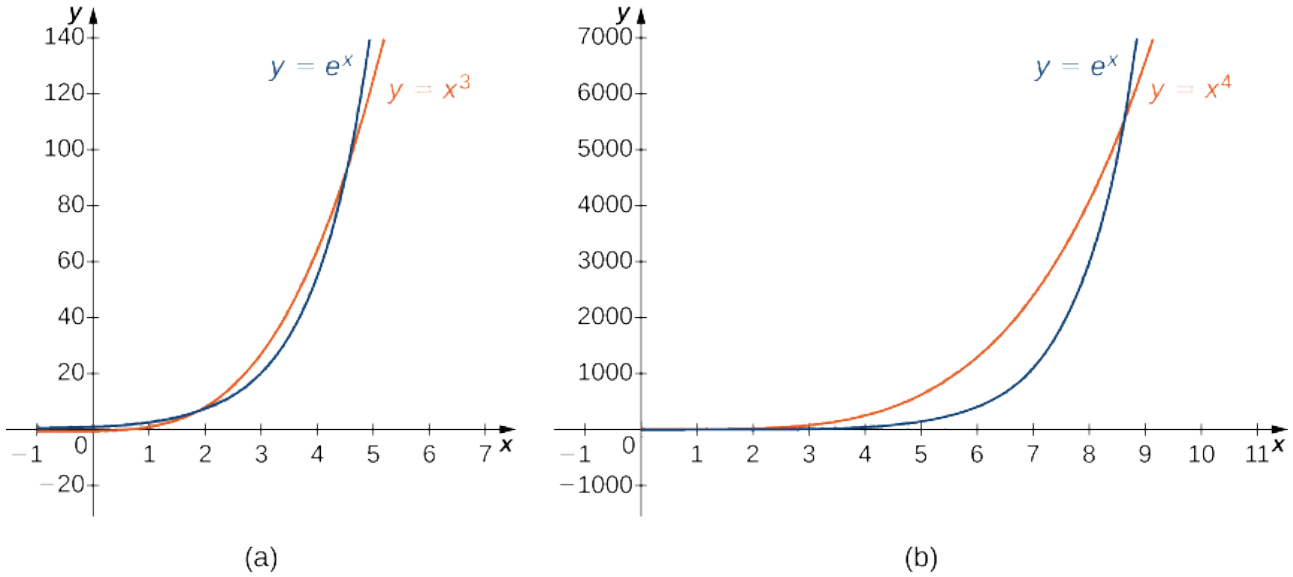


Figure 15.9.5: The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

Table 15.9.5: An exponential function grows at a faster rate than any power function

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In Figure 15.9.6 and Table 15.9.6, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

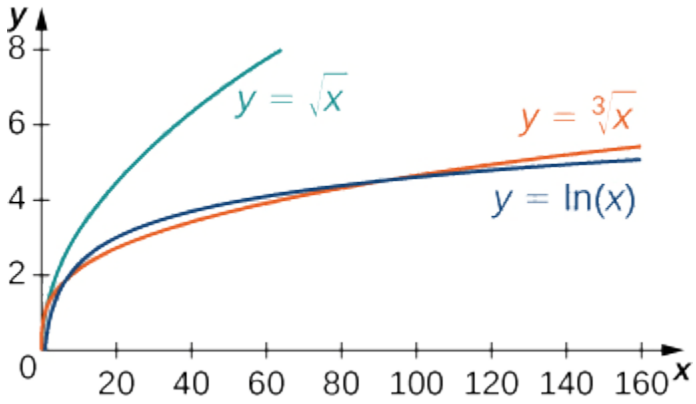


Figure 15.9.6: The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

Table 15.9.6: A logarithmic function grows at a slower rate than any root function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Key Concepts

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.
- The exponential function e^x grows faster than any power function $x^p, p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function $x^p, p > 0$.

Glossary

indeterminate forms

When evaluating a limit, the forms $\frac{0}{0}, \infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

L'Hôpital's rule

If f and g are differentiable functions over an interval a, except possibly at a, and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$.

Answers

Exercise 15.9.1

$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

Exercise 15.9.1

1

Exercise 15.9.2

$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$

Exercise 15.9.2

0

Exercise 15.9.3

Determine the limits of the numerator and denominator separately.

Exercise 15.9.3

$\lim_{x \to 0^+} \cos x = 1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞.

Exercise 15.9.4

Write $x \cot x = \frac{x \cos x}{\sin x}$

Exercise 15.9.4

1

Exercise 15.9.5

Rewrite the difference of fractions as a single fraction.

Exercise 15.9.5

0

Exercise 15.9.6

Let $y = x^{1/\ln(x)}$ and apply the natural logarithm to both sides of the equation.

Exercise 15.9.6

e

Exercise 15.9.7

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Exercise 15.9.7

1

Exercise 15.9.8

Apply L'Hôpital's rule to $x^{100}/2^x$.

Exercise 15.9.8

The function 2^x grows faster than x^{100}.

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15.9E: Exercises for Section 4.8

In exercises 1 - 6, evaluate the limit.

1) Evaluate the limit $\lim_{x\to\infty}\dfrac{e^x}{x}$.

2) Evaluate the limit $\lim_{x\to\infty}\dfrac{e^x}{x^k}$.

Answer

$$\lim_{x\to\infty}\frac{e^x}{x^k} \quad = \quad \infty$$

3) Evaluate the limit $\lim_{x\to\infty}\dfrac{\ln x}{x^k}$.

4) Evaluate the limit $\lim_{x\to a}\dfrac{x-a}{x^2-a^2}$.

Answer

$$\lim_{x\to a}\frac{x-a}{x^2-a^2} \quad = \quad \frac{1}{2a}$$

5. Evaluate the limit $\lim_{x\to a}\dfrac{x-a}{x^3-a^3}$.

6. Evaluate the limit $\lim_{x\to a}\dfrac{x-a}{x^n-a^n}$.

Answer

$$\lim_{x\to a}\frac{x-a}{x^n-a^n} \quad = \quad \frac{1}{na^{n-1}}$$

In exercises 7 - 11, determine whether you can apply L'Hôpital's rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply L'Hôpital's rule.

7) $\lim_{x\to 0^+} x^2\ln x$

8) $\lim_{x\to\infty} x^{1/x}$

Answer

Cannot apply directly; use logarithms

9) $\lim_{x\to 0} x^{2/x}$

10) $\lim_{x\to 0}\dfrac{x^2}{1/x}$

Answer

Cannot apply directly; rewrite as $\lim_{x\to 0} x^3$

11) $\lim_{x\to\infty}\dfrac{e^x}{x}$

In exercises 12 - 40, evaluate the limits with either L'Hôpital's rule or previously learned methods.

12) $\lim_{x\to 3}\dfrac{x^2-9}{x-3}$

Answer

$$\lim_{x\to 3}\frac{x^2-9}{x-3} \quad = \quad 6$$

13) $\lim_{x\to 3}\frac{x^2-9}{x+3}$

14) $\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x}$

Answer

$$\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x} \quad = \quad -2$$

15) $\lim_{x\to \pi/2}\frac{\cos x}{\frac{\pi}{2}-x}$

16) $\lim_{x\to \pi}\frac{x-\pi}{\sin x}$

Answer

$$\lim_{x\to \pi}\frac{x-\pi}{\sin x} \quad = \quad -1$$

17) $\lim_{x\to 1}\frac{x-1}{\sin x}$

18) $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$

Answer

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x} \quad = \quad n$$

19) $\lim_{x\to 0}\frac{(1+x)^n-1-nx}{x^2}$

20) $\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}$

Answer

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3} \quad = \quad -\frac{1}{2}$$

21) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

22) $\lim_{x\to 0}\frac{e^x-x-1}{x^2}$

Answer

$$\lim_{x\to 0}\frac{e^x-x-1}{x^2} \quad = \quad \frac{1}{2}$$

23) $\lim_{x\to 0}\frac{\tan x}{\sqrt{x}}$

24) $\lim_{x\to 1}\frac{x-1}{\ln x}$

Answer

$$\lim_{x\to 1}\frac{x-1}{\ln x} \quad = \quad 1$$

25) $\lim_{x\to 0}\ (x+1)^{1/x}$

26) $\lim_{x\to 1}\dfrac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

Answer

$$\lim_{x\to 1}\frac{\sqrt{x}-\sqrt[3]{x}}{x-1} \quad = \quad \frac{1}{6}$$

27) $\lim_{x\to 0^+} x^{2x}$

28) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Answer

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) \quad = \quad 1$$

29) $\lim_{x\to 0}\dfrac{\sin x - x}{x^2}$

30) $\lim_{x\to 0^+} x\ln\left(x^4\right)$

Answer

$$\lim_{x\to 0^+} x\ln\left(x^4\right) \quad = \quad 0$$

31) $\lim_{x\to\infty}(x-e^x)$

32) $\lim_{x\to\infty} x^2e^{-x}$

Answer

$$\lim_{x\to\infty} x^2e^{-x} \quad = \quad 0$$

33) $\lim_{x\to 0}\dfrac{3^x-2^x}{x}$

34) $\lim_{x\to 0}\dfrac{1+1/x}{1-1/x}$

Answer

$$\lim_{x\to 0}\frac{1+1/x}{1-1/x} \quad = \quad -1$$

35) $\lim_{x\to\pi/4}(1-\tan x)\cot x$

36) $\lim_{x\to\infty} xe^{1/x}$

Answer

$$\lim_{x\to\infty} xe^{1/x} \quad = \quad \infty$$

37) $\lim_{x\to 0} x^{1/\cos x}$

38) $\lim_{x\to 0^+} x^{1/x}$

Answer

$$\lim_{x\to 0^+} x^{1/x} \quad = \quad 0$$

39) $\lim_{x\to 0}\left(1-\dfrac{1}{x}\right)^x$

40) $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$

Answer

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}$$

For exercises 41 - 50, use a calculator to graph the function and estimate the value of the limit, then use L'Hôpital's rule to find the limit directly.

41) [T] $\lim_{x\to 0}\frac{e^x-1}{x}$

42) [T] $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Answer

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

43) [T] $\lim_{x\to 1}\frac{x-1}{1-\cos(\pi x)}$

44) [T] $\lim_{x\to 1}\frac{e^{x-1}-1}{x-1}$

Answer

$$\lim_{x\to 1}\frac{e^{x-1}-1}{x-1} = 1$$

45) [T] $\lim_{x\to 1}\frac{(x-1)^2}{\ln x}$

46) [T] $\lim_{x\to \pi}\frac{1+\cos x}{\sin x}$

Answer

$$\lim_{x\to \pi}\frac{1+\cos x}{\sin x} = 0$$

47) [T] $\lim_{x\to 0}\left(\csc x-\frac{1}{x}\right)$

48) [T] $\lim_{x\to 0^+}\tan\left(x^x\right)$

Answer

$$\lim_{x\to 0^+}\tan\left(x^x\right) = \tan 1$$

49) [T] $\lim_{x\to 0^+}\frac{\ln x}{\sin x}$

50) [T] $\lim_{x\to 0}\frac{e^x-e^{-x}}{x}$

Answer

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x} = 2$$

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15.10: Newton's Method

Learning Objectives

- Describe the steps of Newton's method.
- Explain what an iterative process means.
- Recognize when Newton's method does not work.
- Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0$. For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the **zeroes of functions**. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x) = 0$. If f is the first-degree polynomial $f(x) = ax + b$, then the solution of $f(x) = 0$ is given by the formula $x = -\frac{b}{a}$. If f is the second-degree polynomial $f(x) = ax^2 + bx + c$, the solutions of $f(x) = 0$ can be found by using the quadratic formula. However, for polynomials of degree **3** or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree **5** or greater, it is known that no such formulas exist. For example, consider the function

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0$. Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x) - x = 0$. No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x) = 0$. By sketching a graph of f, we can estimate a root of $f(x) = 0$. Let's call this estimate x_0. We then draw the tangent line to f at x_0. If $f\,'(x_0) \neq 0$, this tangent line intersects the x-axis at some point $(x_1, 0)$. Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at x_1. If $f\,'(x_1) \neq 0$, this tangent line also intersects the x-axis, producing another approximation, x_2. We continue in this way, deriving a list of approximations: $x_0,\ x_1,\ x_2,\ \ldots$. Typically, the numbers $x_0,\ x_1,\ x_2,\ \ldots$ quickly approach an actual root x^*, as shown in the following figure.

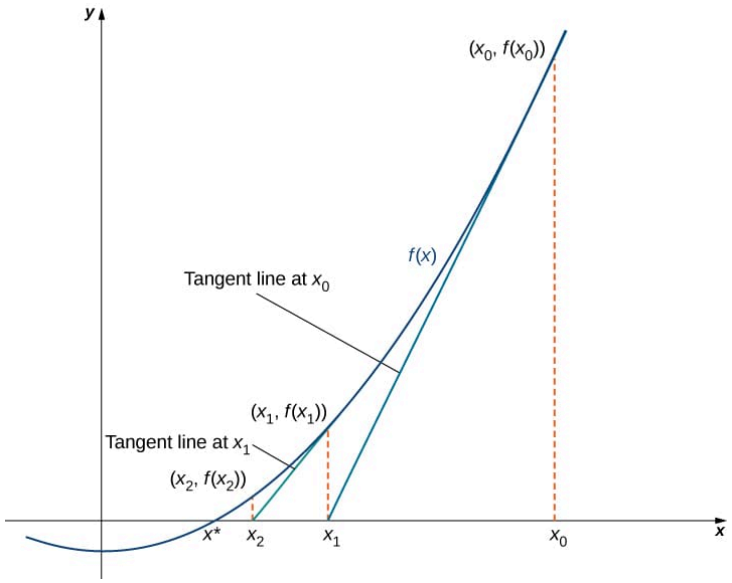


Figure 15.10.1:The approximations $x_0,\ x_1,\ x_2,\ \ldots$ approach the actual root x^*. The approximations are derived by looking at tangent lines to the graph of f.

Now let's look at how to calculate the approximations $x_0,\ x_1,\ x_2,\ \ldots$. If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0, x_n$ satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{15.10.1}$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example 15.10.1: Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find $x_1,\ x_2,\ x_3,\ x_4$, and x_5.

Solution

From Figure 15.10.2, we see that f has one root over the interval $[1, 2]$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use Equation 15.10.1. Since $f(x) = x^3 - 3x + 1$, the derivative is $f'(x) = 3x^2 - 3$. Using Equation 15.10.1 with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, x_2, we use Equation 15.10.1 with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

- $x_1 \approx 1.666666667$
- $x_2 \approx 1.548611111$
- $x_3 \approx 1.532390162$
- $x_4 \approx 1.532088989$
- $x_5 \approx 1.532088886$
- $x_6 \approx 1.532088886.$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

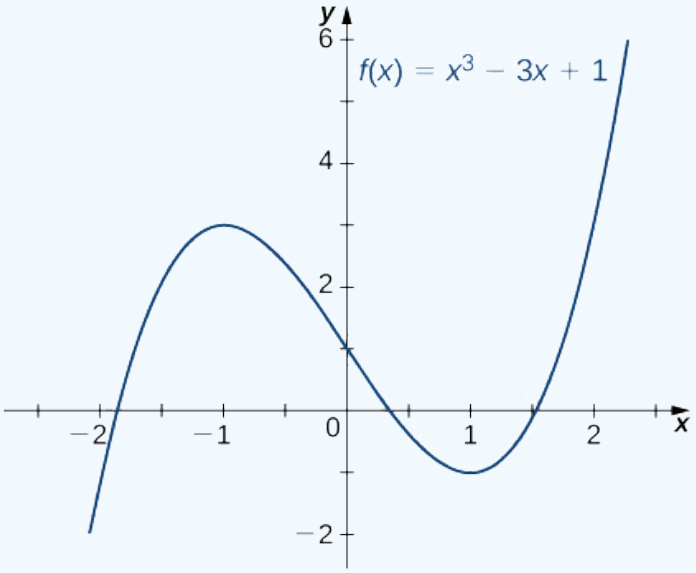


Figure 15.10.2: The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

Exercise 15.10.1

Letting $x_0 = 0$, let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2.

Hint available at the end of this section.

Answer available at the end of this section.

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

Example 15.10.2: Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (Figure 15.10.3). Let $f(x) = x^2 - 2$, let $x_0 = 2$, and calculate $x_1,\ x_2,\ x_3,\ x_4,\ x_5$. (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2}$, the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$).

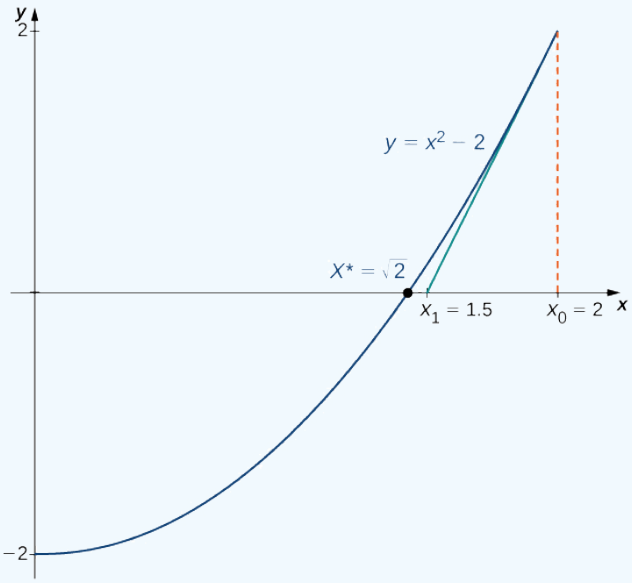


Figure 15.10.3: We can use Newton's method to find $\sqrt{2}$.

Solution

For $f(x) = x^2 - 2,\quad f\,'(x) = 2x$. From Equation 15.10.1, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$x_1 = \tfrac{1}{2}\left(x_0 + \tfrac{2}{x_0}\right) = \tfrac{1}{2}\left(2 + \tfrac{2}{2}\right) = 1.5$$

$$x_2 = \tfrac{1}{2}\left(x_1 + \tfrac{2}{x_1}\right) = \tfrac{1}{2}\left(1.5 + \tfrac{2}{1.5}\right) \approx 1.416666667.$$

Continuing in this way, we find that

$$x_1 = 1.5$$

$$x_2 \approx 1.416666667$$

$$x_3 \approx 1.414215686$$

$$x_4 \approx 1.414213562$$

$$x_5 \approx 1.414213562.$$

Since we obtained the same value for x_4 and x_5, it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562$.

Exercise 15.10.2

Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2.

Hint available at the end of this section.

Answer available at the end of this section.

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$, we can rewrite Equation 15.10.1 as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an iterative process. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0,\ x_1,\ x_2,\ \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see Figure 15.10.4). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In Example 15.10.3, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

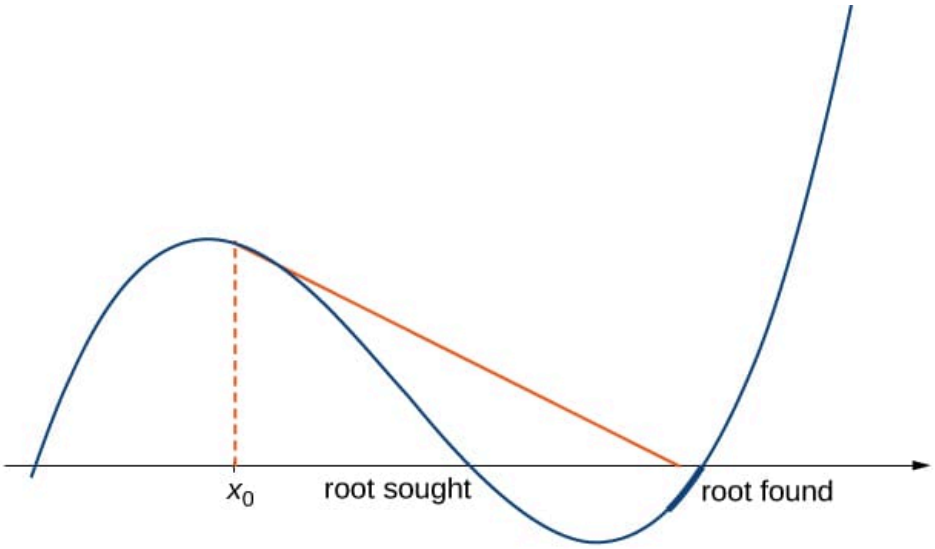


Figure 15.10.4: If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example 15.10.3: When Newton's Method Fails

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence $x_1,\ x_2,\ \ldots$ fails to approach a root of f.

Solution

For $f(x) = x^3 - 2x + 2$, the derivative is $f\,'(x) = 3x^2 - 2$.Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0,\ x_1,\ x_2,\ \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (Figure 15.10.5). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

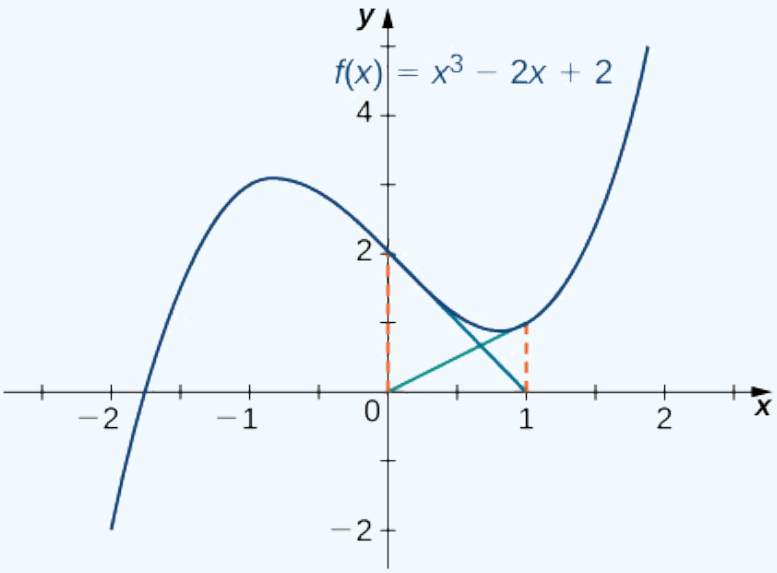


Figure 15.10.5: The approximations continue to alternate between 0 and 1 and never approach the root of f.

Exercise 15.10.3

For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2.

Hint available at the end of this section.

Answer available at the end of this section.

From Example 15.10.3, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative

process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative process that creates a list of numbers $x_0,\ x_1,\ x_2,\ \ldots,\ x_n,\ \ldots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In Example 15.10.4, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example 15.10.4: Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1$, let $x_n = F(x_{n-1})$. Find the values $x_1,\ x_2,\ x_3,\ x_4,\ x_5$. Make a conjecture about what happens to this list of numbers $x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ \ldots$ as $n \to \infty$. If the list of numbers $x_1,\ x_2,\ x_3,\ \ldots$ approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution

If $x_0 = 0$, then

- $x_1 = \frac{1}{2}(0) + 4 = 4$
- $x_2 = \frac{1}{2}(4) + 4 = 6$
- $x_3 = \frac{1}{2}(6) + 4 = 7$
- $x_4 = \frac{1}{2}(7) + 4 = 7.5$
- $x_5 = \frac{1}{2}(7.5) + 4 = 7.75$
- $x_6 = \frac{1}{2}(7.75) + 4 = 7.875$
- $x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$
- $x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$
- $x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375.$

From this list, we conjecture that the values x_n approach 8.

Figure 15.10.6 provides a graphical argument that the values approach 8 as $n \to \infty$. Starting at the point (x_0, x_0), we draw a vertical line to the point $(x_0, F(x_0))$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2. Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1))$. The output $F(x_1)$ becomes x_2. Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0,\ x_1,\ x_2,\ \ldots$ approaches $x^* = 8$ as $n \to \infty$.

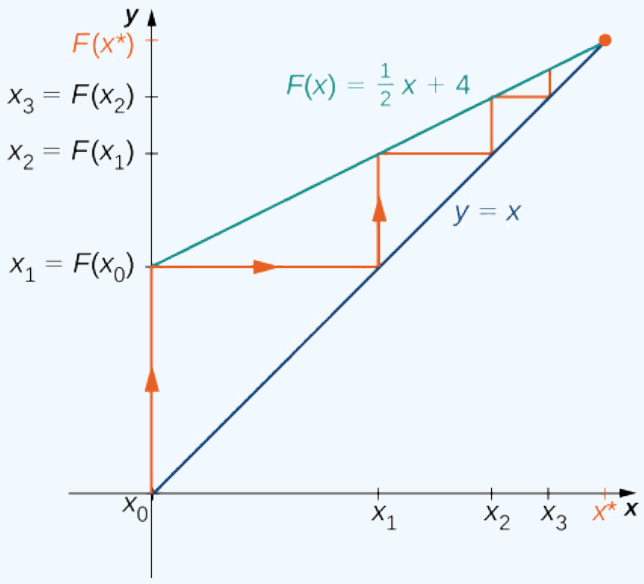


Figure 15.10.6: This iterative process approaches the value $x^* = 8$.

Exercise 15.10.4

Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find $x_1,\ x_2,\ x_3,\ x_4,\ x_5$. Make a conjecture about what happens to the list of numbers $x_1,\ x_2,\ x_3,\ \ldots\ x_n,\ \ldots$ as $n \to \infty$.

Hint available at the end of this section.

Answer available at the end of this section.

Iterative Processes and Chaos

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in Example 15.10.3 that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-cycle. Iterative processes can converge to cycles with various periodicities, such as 2–cycles, 4–cycles (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call chaos. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a complete exploration of **chaos** is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the **Mandelbrot set** (see Figure 15.10.7), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

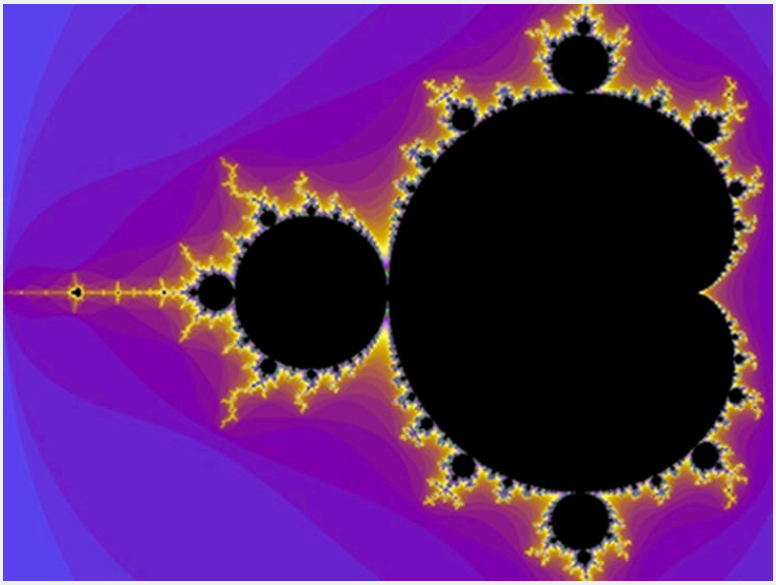


Figure 15.10.7: The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

In this project we use the logistic map

$$f(x) = rx(1 - x)$$

where $x \in [0, 1]$ and $r > 0$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of r, the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a cobweb diagram. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1), then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. Figure $15.10.8$ shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.

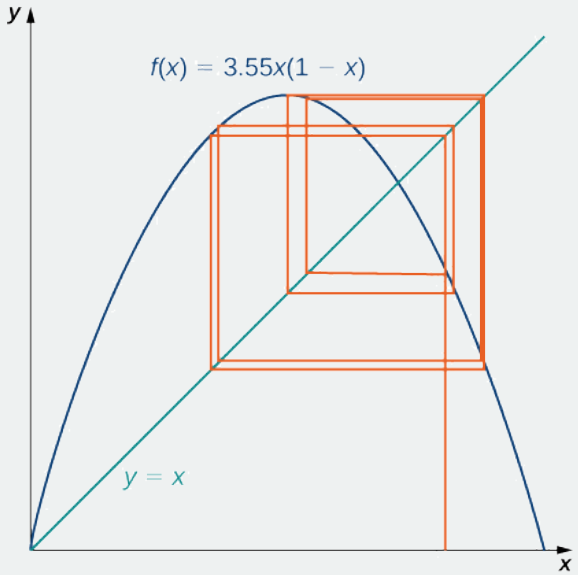


Figure $15.10.8$: A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8-cycle.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, 2–cycle, 4–cycle.)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?

Key Concepts

- Newton's method approximates roots of $f(x) = 0$ by starting with an initial approximation x_0, then uses tangent lines to the graph of f to create a sequence of approximations $x_1,\ x_2,\ x_3,\ \ldots$.
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers $x_0,\ x_1,\ x_2,\ \ldots$ does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers $x_0,\ x_1,\ x_2,\ \ldots$ is generated by defining an initial number x_0 and defining the subsequent numbers by the equation $x_n = F(x_{n-1})$ for some function F is an iterative process. Newton's method is an example of an iterative process, where the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$ for a given function f.

Glossary

iterative process

process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

Newton's method

method for approximating roots of $f(x) = 0$; using an initial guess x_0; each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

Answers

Exercise 15.10.1

Use Equation 15.10.1.

Exercise 15.10.1

$x_1 \approx 0.33333333$
$x_2 \approx 0.347222222$

Exercise 15.10.2

For $f(x) = x^2 - 3$, Equation 15.10.1 reduces to $x_n = \frac{x_{n-1}}{2} + \frac{3}{2x_{n-1}}$.

Exercise 15.10.2

$x_1 = 2$
$x_2 = 1.75$

Exercise 15.10.3

Use Equation 15.10.1.

Exercise 15.10.3

$x_1 \approx -1.842105263$
$x_2 \approx -1.772826920$

Exercise 15.10.4

Consider the point where the lines $y = x$ and $y = F(x)$ intersect.

Exercise 15.10.4

$x_1 = 6,\ \ x_2 = 8,\ \ x_3 = \frac{26}{3},\ \ x_4 = \frac{80}{9},\ \ x_5 = \frac{242}{27};\ \ x^* = 9$

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15.10E: Exercises for Section 4.9

In exercises 1 - 5, write Newton's formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.

1) $f(x) = x^2 + 1$

2) $f(x) = x^3 + 2x + 1$

Answer

$$F(x_n) = x_n - \frac{x_n^3 + 2x_n + 1}{3x_n^2 + 2}$$

3) $f(x) = \sin x$

4) $f(x) = e^x$

Answer

$$F(x_n) = x_n - \frac{e^{x_n}}{e^{x_n}}$$

5) $f(x) = x^3 + 3xe^x$

In exercises 6 - 8, solve $f(x) = 0$ using the iteration $x_{n+1} = x_{n-c}f(x_n)$, which differs slightly from Newton's method. Find a c that works and a c that fails to converge, with the exception of $c = 0$.

6) $f(x) = x^2 - 4$, with $x_0 = 0$

Answer

$|c| > 0.5$ fails, $|c| \leq 0.5$ works

7) $f(x) = x^2 - 4x + 3$, with $x_0 = 2$

8) What is the value of "c" for Newton's method?

Answer

$$c = \frac{1}{f'(x_n)}$$

In exercises 9 - 16, compute x_1 and x_2 using the specified iterative method.

Start at

a. $x_0 = 0.6$ **and**

b. $x_0 = 2$.

9) $x_{n+1} = x_n^2 - \frac{1}{2}$

10) $x_{n+1} = 2x_n\,(1 - x_n)$

Answer

a. $x_1 = \frac{12}{25}, \quad x_2 = \frac{312}{625}$;
b. $x_1 = -4, \quad x_2 = -40$

11) $x_{n+1} = \sqrt{x_n}$

12) $x_{n+1} = \frac{1}{\sqrt{x_n}}$

Answer

a. $x_1 = 1.291, \quad x_2 = 0.8801;$
b. $x_1 = 0.7071, \quad x_2 = 1.189$

13) $x_{n+1} = 3x_n(1 - x_n)$

14) $x_{n+1} = x_n^2 + x_{n-2}$

Answer

a. $x_1 = -\frac{26}{25}, \quad x_2 = -\frac{1224}{625};$
b. $x_1 = 4, \quad x_2 = 18$

15) $x_{n+1} = \frac{1}{2}x_n - 1$

16) $x_{n+1} = |x_n|$

Answer

a. $x_1 = \frac{6}{10}, \quad x_2 = \frac{6}{10};$
b. $x_1 = 2, \quad x_2 = 2$

In exercises 17 - 26, solve to four decimal places using Newton's method and a computer or calculator. Choose any initial guess x_0 that is not the exact root.

17) $x^2 - 10 = 0$

18) $x^4 - 100 = 0$

Answer

3.1623 or -3.1623

19) $x^2 - x = 0$

20) $x^3 - x = 0$

Answer

0, -1 or 1

21) $x + 5\cos x = 0$

22) $x + \tan x = 0$, choose $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Answer

0

23) $\dfrac{1}{1-x} = 2$

24) $1 + x + x^2 + x^3 + x^4 = 2$

Answer

0.5188 or -1.2906

25) $x^3 + (x+1)^3 = 10^3$

26) $x = \sin^2(x)$

Answer

0

In exercises 27 - 30, use Newton's method to find the fixed points of the function where $f(x) = x$; round to three decimals.

27) $\sin x$

28) $\tan x$ on $x = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Answer

4.493

29) $e^x - 2$

30) $\ln(x) + 2$

Answer

$0.159, \ \ 3.146$

Newton's method can be used to find maxima and minima of functions in addition to the roots. In this case apply Newton's method to the derivative function $f'(x)$ to find its roots, instead of the original function. In exercises 31 - 32, consider the formulation of the method.

31) To find candidates for maxima and minima, we need to find the critical points $f'(x) = 0$. Show that to solve for the critical points of a function $f(x)$, Newton's method is given by $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$.

32) What additional restrictions are necessary on the function f?

Answer

We need f to be twice continuously differentiable.

In exercises 33 - 40, use Newton's method to find the location of the local minima and/or maxima of the following functions; round to three decimals.

33) Minimum of $f(x) = x^2 + 2x + 4$

34) Minimum of $f(x) = 3x^3 + 2x^2 - 16$

Answer

$x = 0$

35) Minimum of $f(x) = x^2 e^x$

36) Maximum of $f(x) = x + \dfrac{1}{x}$

Answer

$x = -1$

37) Maximum of $f(x) = x^3 + 10x^2 + 15x - 2$

38) Maximum of $f(x) = \dfrac{\sqrt{x} - \sqrt[3]{x}}{x}$

Answer

$x = 5.619$

39) Minimum of $f(x) = x^2 \sin x$, closest non-zero minimum to $x = 0$

40) Minimum of $f(x) = x^4 + x^3 + 3x^2 + 12x + 6$

Answer

$x = -1.326$

In exercises 41 - 44, use the specified method to solve the equation. If it does not work, explain why it does not work.

41) Newton's method, $x^2 + 2 = 0$

42) Newton's method, $0 = e^x$

Answer

There is no solution to the equation.

43) Newton's method, $0 = 1 + x^2$ starting at $x_0 = 0$

44) Solving $x_{n+1} = -x_n^3$ starting at $x_0 = -1$

Answer

It enters a cycle.

In exercises 45 - 48, use the *secant method*, an alternative iterative method to Newton's method. The formula is given by

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

45) a root to $0 = x^2 - x - 3$ accurate to three decimal places.

46) Find a root to $0 = \sin x + 3x$ accurate to four decimal places.

Answer

0

47) Find a root to $0 = e^x - 2$ accurate to four decimal places.

48) Find a root to $\ln(x + 2) = \frac{1}{2}$ accurate to four decimal places.

Answer

-0.3513

49) Why would you use the secant method over Newton's method? What are the necessary restrictions on f?

In exercises 50 - 54, use both Newton's method and the secant method to calculate a root for the following equations. Use a calculator or computer to calculate how many iterations of each are needed to reach within three decimal places of the exact answer. For the secant method, use the first guess from Newton's method.

50) $f(x) = x^2 + 2x + 1, \quad x_0 = 1$

Answer

Newton: 11 iterations, secant: 16 iterations

51) $f(x) = x^2, \quad x_0 = 1$

52) $f(x) = \sin x, \quad x_0 = 1$

Answer

Newton: three iterations, secant: six iterations

53) $f(x) = e^x - 1, \quad x_0 = 2$

54) $f(x) = x^3 + 2x + 4, \quad x_0 = 0$

Answer

Newton: five iterations, secant: eight iterations

In exercises 55 - 56, consider Kepler's equation regarding planetary orbits, $M = E - \varepsilon \sin(E)$, where M is the mean

anomaly, E is eccentric anomaly, and ε measures eccentricity.

55) Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{\pi}{3}$ and the eccentricity of the orbit $\varepsilon = 0.25$; round to three decimals.

56) Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{3\pi}{2}$ and the eccentricity of the orbit $\varepsilon = 0.8$; round to three decimals.

Answer

$E = 4.071$

In exercises 57 - 58, consider a bank investment. The initial investment is $\$10,000$. After 25 years, the investment has tripled to $\$30,000$.

57) Use Newton's method to determine the interest rate if the interest was compounded annually.

58) Use Newton's method to determine the interest rate if the interest was compounded continuously.

Answer

4.394

59) The cost for printing a book can be given by the equation $C(x) = 1000 + 12x + \frac{1}{2}x^{2/3}$. Use Newton's method to find the break-even point if the printer sells each book for $\$20$.

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15.11: Antiderivatives

Learning Objectives

- Find the general antiderivative of a given function.
- Explain the terms and notation used for an indefinite integral.
- State the power rule for integrals.
- Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in Derivatives of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s\,'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$ —that is, $a(t) = v\,'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v\,'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s\,'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (Introduction to Techniques of Integration).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F with derivative f, we call F an antiderivative of f.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F\,'(x) = 2x$.

Are there any other antiderivatives of f?

Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives.

Are there any others that are not of the form $x^2 + C$ for some constant C?

The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F\,'(x) = G\,'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

Theorem 15.11.1: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x)+C$ is also an antiderivative of f over I;
II. if G is an antiderivative of f over I, there is a constant C for which $G(x)=F(x)+C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x)+C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 15.11.1: Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x)=3x^2$
b. $f(x)=\dfrac{1}{x}$
c. $f(x)=\cos x$
d. $f(x)=e^x$

Solution:

a. Because

$$\frac{d}{dx}\left(x^3\right)=3x^2$$

then $F(x)=x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form x^3+C for some constant C, and every function of the form x^3+C is an antiderivative of $3x^2$.

b. Let $f(x)=\ln|x|$.

For $x>0,\quad f(x)=\ln|x|=\ln(x)$ and

$$\frac{d}{dx}\Big(\ln x\Big)=\frac{1}{x}.$$

For $x<0,\quad f(x)=\ln|x|=\ln(-x)$ and

$$\frac{d}{dx}\Big(\ln(-x)\Big)=-\frac{1}{-x}=\frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}\Big(\ln|x|\Big)=\frac{1}{x}.$$

Thus, $F(x)=\ln|x|$ is an antiderivative of $\dfrac{1}{x}$. Therefore, every antiderivative of $\dfrac{1}{x}$ is of the form $\ln|x|+C$ for some constant C and every function of the form $\ln|x|+C$ is an antiderivative of $\dfrac{1}{x}$.

c. We have

$$\frac{d}{dx}\Big(\sin x\Big)=\cos x,$$

so $F(x)=\sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x+C$ for some constant C and every function of the form $\sin x+C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}\left(e^x\right) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

Exercise 15.11.1

Find all antiderivatives of $f(x) = \sin x$.

Hint available at the end of this section.

Answer available at the end of this section.

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f\,'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x) + C$ is the most general antiderivative of f and write

$$\int f(x)\,dx = F(x) + C.$$

The symbol $\int$ is called an integral sign, and $\int f(x)\ dx$ is called the indefinite integral of f.

Definition: Indefinite Integrals

Given a function f, the indefinite integral of f, denoted

$$\int f(x)\,dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C.$$

The expression $f(x)$ is called the integrand and the variable x is the variable of integration.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as integrating f.

For a function f and an antiderivative F, the functions $F(x) + C$, where C is any real number, is often referred to as the family of antiderivatives of f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form $x^2 + C$, we write

$$\int 2x\,dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the family of antiderivatives of $2x$. Figure 15.11.1 shows a graph of this family of antiderivatives.

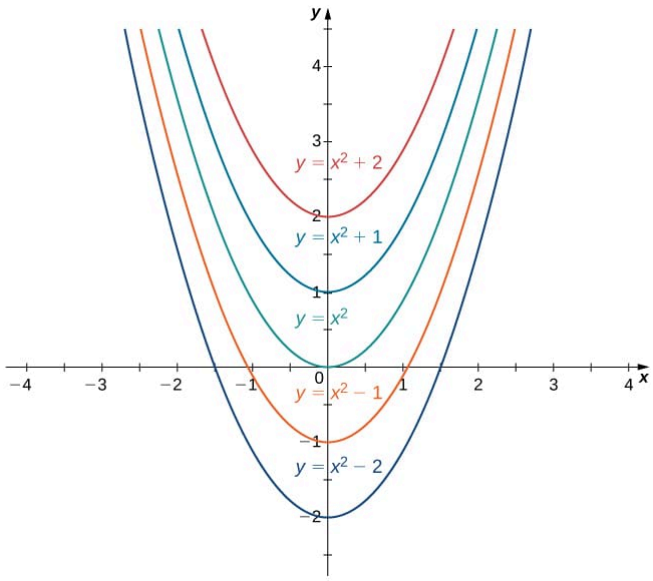


Figure 15.11.1: The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n.$$

This fact is known as the power rule for integrals.

Power Rule for Integrals

For $n \neq -1$,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in Appendix B.

Table 15.11.1: Integration Formulas

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}\left(k\right) = 0$	$\int k \, dx = \int kx^0 \, dx = kx + C$
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}\left(\ln\lvert x\rvert\right) = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\left(e^x\right) = e^x$	$\int e^x \, dx = e^x + C$
$\frac{d}{dx}\left(\sin x\right) = \cos x$	$\int \cos x \, dx = \sin x + C$

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}\left(\cos x\right) = -\sin x$	$\int \sin x \; dx = -\cos x + C$
$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$	$\int \sec^2 x \; dx = \tan x + C$
$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$	$\int \csc x \cot x \; dx = -\csc x + C$
$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$	$\int \sec x \tan x \; dx = \sec x + C$
$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$	$\int \csc^2 x \; dx = -\cot x + C$
$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \; dx = \tan^{-1} x + C$
$\frac{d}{dx}\left(\sec^{-1} \lvert x\rvert\right) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \; dx = \sec^{-1} \lvert x\rvert + C$

From the definition of indefinite integral of f, we know

$$\int f(x)\, dx = F(x) + C$$

if and only if F is an antiderivative of f.

Therefore, when claiming that

$$\int f(x)\, dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F\,'(x) = f(x)$.

Example 15.11.2: Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x) \; dx = F(x) + C$. Verify that each statement is correct by showing that $F\,'(x) = f(x)$.

a. $\int \left(x + e^x\right) \; dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x \; dx = xe^x - e^x + C$

Solution:

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x,$$

the statement

$$\int \left(x + e^x\right) dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}\left(xe^x - e^x + C\right) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x\,dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2e^x}{2}\right) = xe^x + \frac{x^2e^x}{2} \neq xe^x.$$

In general, the product of antiderivatives is not an antiderivative of a product.

Exercise 15.11.2

Verify that $\int x\cos x\quad dx = x\sin x + \cos x + C.$

Hint available at the end of this section.

Answer available at the end of this section.

In Table 15.11.1, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g$. In Example 15.11.2a we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\frac{x^2}{2} + e^x$—that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g, respectively, then

$$\frac{d}{dx}\Big(F(x) + G(x)\Big) = F\,'(x) + G\,'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

$$\int \Big(f(x) + g(x)\Big)\,dx = F(x) + G(x) + C.$$

Similarly,

$$\int \Big(f(x) - g(x)\Big)\,dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since

$$\frac{d}{dx}\Big(kF(x)\Big) = k\frac{d}{dx}\Big(F(x)\Big) = kF'(x)$$

for any real number k, we conclude that

$$\int kf(x)\,dx = kF(x) + C.$$

These properties are summarized next.

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int \big(f(x) \pm g(x)\big)\,dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)\,dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated. (See Example $15.11.2b$ for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in Introduction to Integration. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 15.11.3: Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int \left(5x^3 - 7x^2 + 3x + 4\right)\,dx$

b. $\int \dfrac{x^2 + 4\sqrt[3]{x}}{x}\,dx$

c. $\int \dfrac{4}{1+x^2}\,dx$

d. $\int \tan x \cos x\,dx$

Solution:

a. Using Properties of Indefinite Integrals, we can integrate each of the four terms in the integrand separately. We obtain

$$\int \left(5x^3 - 7x^2 + 3x + 4\right)\,dx = \int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx.$$

From the second part of Properties of Indefinite Integrals, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx = 5\int x^3\,dx - 7\int x^2\,dx + 3\int x\,dx + 4\int 1\,dx.$$

Using the power rule for integrals, we conclude that

$$\int \left(5x^3 - 7x^2 + 3x + 4\right)\,dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x}.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\begin{aligned}\int \left(x + \frac{4}{x^{2/3}}\right)\, dx &= \int x\, dx + 4\int x^{-2/3}\, dx \\ &= \frac{1}{2}x^2 + 4\frac{1}{\left(\frac{-2}{3}\right)+1}x^{(-2/3)+1} + C \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C.\end{aligned}$$

c. Using Properties of Indefinite Integrals, write the integral as

$$4\int \frac{1}{1+x^2}\, dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\dfrac{1}{1+x^2}$ to conclude that

$$\int \frac{4}{1+x^2}\, dx = 4\tan^{-1}(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x\, dx = \int \sin x\, dx = -\cos x + C.$$

Exercise 15.11.3

Evaluate $\int \left(4x^3 - 5x^2 + x - 7\right)\, dx$.

Hint available at the end of this section.

Answer available at the end of this section.

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A differential equation is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \tag{15.11.1}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation 15.11.1 are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2\,dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition

$$y(x_0) = y_0$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

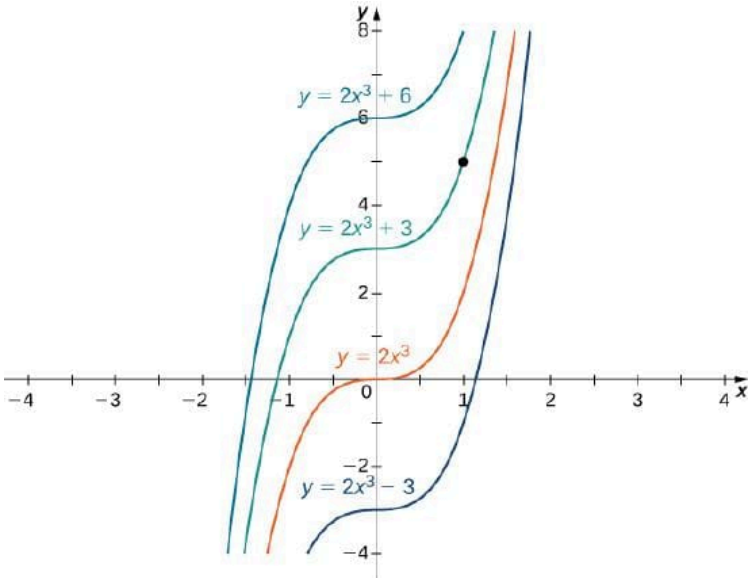


Figure 15.11.2: Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example 15.11.4: Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, \quad y(0) = 5.$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x$, then

$$y = \int \sin(x)\,dx = -\cos x + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that $-\cos x + C = 5$. Therefore,

$$C = 5 + \cos(0) = 6.$$

The solution of the initial-value problem is $y = -\cos x + 6$.

Exercise 15.11.4

Solve the initial value problem $\dfrac{dy}{dx} = 3x^{-2}, \quad y(1) = 2.$

Hint available at the end of this section.

Answer available at the end of this section.

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 15.11.5:

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t) = -15 \text{ ft/sec}^2.$$

The acceleration is the derivative of the velocity,

$$v\,'(t) = -15.$$

Therefore, we have an initial-value problem to solve:

$$v\,'(t) = -15, \quad v(0) = 88.$$

Integrating, we find that

$$v(t) = -15t + C.$$

Since $v(0) = 88, C = 88$. Thus, the velocity function is

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0$, we obtain $t = \frac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

$$s\,'(t) = -15t + 88, \quad s(0) = 0.$$

Integrating, we have

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0$, the constant is $C = 0$. Therefore, the position function is

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

Exercise 15.11.5

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C.
- Solving the initial-value problem

  $$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

  requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

Glossary

antiderivative
: a function F such that $F\,'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral
: the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)\ dx$ to denote the indefinite integral of f

initial value problem
: a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

Answers

Exercise 15.11.1

What function has a derivative of $\sin x$?

Exercise 15.11.1

$F(x) = -\cos x + C$

Exercise 15.11.2

Calculate $\frac{d}{dx}\left(x \sin x + \cos x + C\right)$.

Exercise 15.11.2

$\frac{d}{dx}\left(x \sin x + \cos x + C\right) = \sin x + x \cos x - \sin x = x \cos x$

Exercise 15.11.3

Integrate each term in the integrand separately, making use of the power rule.

Exercise 15.11.3

$\int \left(4x^3 - 5x^2 + x - 7\right)\, dx = \quad x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$

Exercise 15.11.4

Find all antiderivatives of $f(x) = 3x^{-2}$.

Exercise 15.11.4

$y = -\frac{3}{x} + 5$

Exercise 15.11.5

$v(t) = -15t + 44$.

Exercise 15.11.5

2.93 sec, 64.5 ft

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15.11E: Exercises for Section 4.10

In exercises 1 - 20, find the antiderivative $F(x)$ of each function $f(x)$.

1) $f(x) = \dfrac{1}{x^2} + x$

2) $f(x) = e^x - 3x^2 + \sin x$

Answer

$$F(x) = e^x - x^3 - \cos x + C$$

3) $f(x) = e^x + 3x - x^2$

4) $f(x) = x - 1 + 4\sin(2x)$

Answer

$$F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C$$

5) $f(x) = 5x^4 + 4x^5$

6) $f(x) = x + 12x^2$

Answer

$$F(x) = \tfrac{1}{2}x^2 + 4x^3 + C$$

7) $f(x) = \dfrac{1}{\sqrt{x}}$

8) $f(x) = \left(\sqrt{x}\right)^3$

Answer

$$F(x) = \tfrac{2}{5}\left(\sqrt{x}\right)^5 + C$$

9) $f(x) = x^{1/3} + \left(2x\right)^{1/3}$

10) $f(x) = \dfrac{x^{1/3}}{x^{2/3}}$

Answer

$$F(x) = \tfrac{3}{2}x^{2/3} + C$$

11) $f(x) = 2\sin(x) + \sin(2x)$

12) $f(x) = \sec^2 x + 1$

Answer

$$F(x) = x + \tan x + C$$

13) $f(x) = \sin x \cos x$

14) $f(x) = \sin^2(x)\cos(x)$

Answer

$$F(x) = \tfrac{1}{3}\sin^3(x) + C$$

15) $f(x) = 0$

16) $f(x) = \frac{1}{2}\csc^2 x + \frac{1}{x^2}$

Answer

$$F(x) = -\frac{1}{2}\cot x - \frac{1}{x} + C$$

17) $f(x) = \csc x \cot x + 3x$

18) $f(x) = 4\csc x \cot x - \sec x \tan x$

Answer

$$F(x) = -\sec x - 4\csc x + C$$

19) $f(x) = 8(\sec x)\big(\sec x - 4\tan x\big)$

20) $f(x) = \frac{1}{2}e^{-4x} + \sin x$

Answer

$$F(x) = -\frac{1}{8}e^{-4x} - \cos x + C$$

For exercises 21 - 29, evaluate the integral.

21) $\int (-1)\ dx$

22) $\int \sin x\ dx$

Answer

$$\int \sin x\ dx = -\cos x + C$$

23) $\int \left(4x + \sqrt{x}\right)\ dx$

24) $\int \frac{3x^2 + 2}{x^2}\ dx$

Answer

$$\int \frac{3x^2 + 2}{x^2}\ dx = 3x - \frac{2}{x} + C$$

25) $\int \left(\sec x \tan x + 4x\right)\ dx$

26) $\int \left(4\sqrt{x} + \sqrt[4]{x}\right)\ dx$

Answer

$$\int \left(4\sqrt{x} + \sqrt[4]{x}\right)\ dx = \frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

27) $\int \left(x^{-1/3} - x^{2/3}\right)\ dx$

28) $\int \frac{14x^3 + 2x + 1}{x^3}\ dx$

Answer

$$\int \frac{14x^3 + 2x + 1}{x^3}\ dx = 14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

29) $\int \left(e^x + e^{-x}\right)\, dx$

In exercises 30 - 34, solve the initial value problem.

30) $f\,'(x) = x^{-3}, \quad f(1) = 1$

Answer

$$f(x) = -\frac{1}{2x^2} + \frac{3}{2}$$

31) $f\,'(x) = \sqrt{x} + x^2, \quad f(0) = 2$

32) $f\,'(x) = \cos x + \sec^2(x), \quad f(\frac{\pi}{4}) = 2 + \frac{\sqrt{2}}{2}$

Answer

$$f(x) = \sin x + \tan x + 1$$

33) $f\,'(x) = x^3 - 8x^2 + 16x + 1, \quad f(0) = 0$

34) $f\,'(x) = \dfrac{2}{x^2} - \dfrac{x^2}{2}, \quad f(1) = 0$

Answer

$$f(x) = -\tfrac{1}{6}x^3 - \frac{2}{x} + \frac{13}{6}$$

In exercises 35 - 39, find two possible functions f given the second- or third-order derivatives

35) $f''(x) = x^2 + 2$

36) $f''(x) = e^{-x}$

Answer

Answers may vary; one possible answer is $f(x) = e^{-x}$

37) $f''(x) = 1 + x$

38) $f'''(x) = \cos x$

Answer

Answers may vary; one possible answer is $f(x) = -\sin x$

39) $f'''(x) = 8e^{-2x} - \sin x$

40) A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of $10\ \text{ft/sec}^2$. How long before the car stops?

Answer

5.867 sec

41) In the preceding problem, calculate how far the car travels in the time it takes to stop.

42) You are merging onto the freeway, accelerating at a constant rate of $12\ \text{ft/sec}^2$. How long does it take you to reach merging speed at 60 mph?

Answer

7.333 sec

43) Based on the previous problem, how far does the car travel to reach merging speed?

44) A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant

deceleration, find the value of deceleration that accomplishes this.

Answer

13.75 ft/sec^2

45) A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

In exercises 46 - 51, find the antiderivative of the function, assuming $F(0) = 0$.

46) [T] $f(x) = x^2 + 2$

Answer

$F(x) = \frac{1}{3}x^3 + 2x$

47) [T] $f(x) = 4x - \sqrt{x}$

48) [T] $f(x) = \sin x + 2x$

Answer

$F(x) = x^2 - \cos x + 1$

49) [T] $f(x) = e^x$

50) [T] $f(x) = \dfrac{1}{(x+1)^2}$

Answer

$F(x) = -\dfrac{1}{x+1} + 1$

51) [T] $f(x) = e^{-2x} + 3x^2$

In exercises 52 - 55, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

52) If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Answer

True

53) If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

54) If $f(x)$ is the antiderivative of $v(x)$, then $f(x) + 1$ is the antiderivative of $v(x) + 1$.

Answer

False

55) If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

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15.R: Chapter 4 Review Exercises

True or False? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.

1) If $f(-1) = -6$ and $f(1) = 2$, then there exists at least one point $x \in [-1, 1]$ such that $f\,'(x) = 4$.

Answer

True, by Mean Value Theorem

2) If $f\,'(c) = 0$, there is a maximum or minimum at $x = c$.

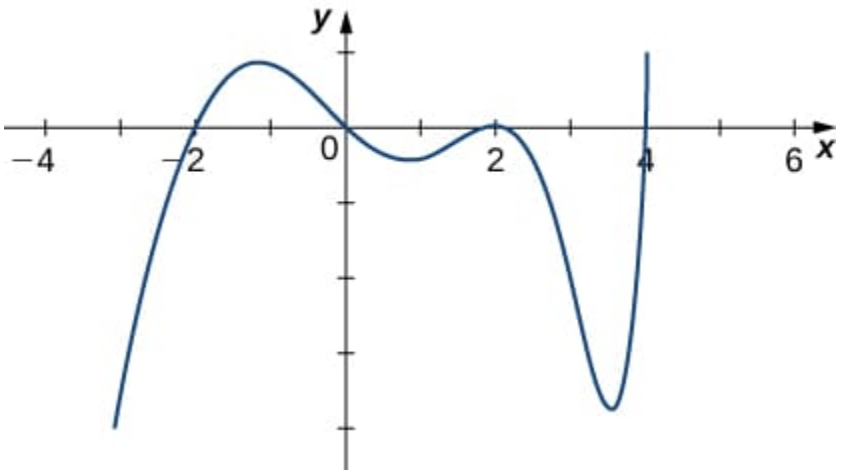
3) There is a function such that $f(x) < 0, f\,'(x) > 0$, and $f''(x) < 0$. (A graphical "proof" is acceptable for this answer.)

Answer

True

4) There is a function such that there is both an inflection point and a critical point for some value $x = a$.

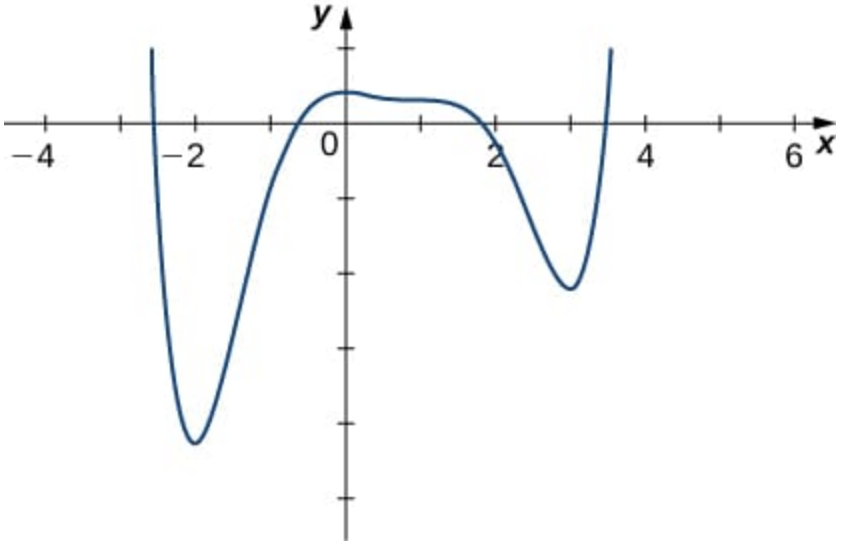
5) Given the graph of $f\,'$, determine where f is increasing or decreasing.



Answer

Increasing: $(-2, 0) \cup (4, \infty)$, decreasing: $(-\infty, -2) \cup (0, 4)$

6) The graph of f is given below. Draw $f\,'$.



7) Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

Answer

$L(x) = \frac{17}{16} + \frac{1}{2}(1 + 4\pi)\left(x - \frac{1}{4}\right)$

8) Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Find the critical points and the local and absolute extrema of the following functions on the given interval.

9) $f(x) = x + \sin^2(x)$ over $[0, \pi]$

Answer

Critical point: $x = \frac{3\pi}{4}$,
Absolute minimum: 0 when $x = 0$,
Absolute maximum: π when $x = \pi$

Solution:

10) $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

11) $x(t) = 3t^4 - 8t^3 - 18t^2$

Answer

Increasing: $(-1, 0) \cup (3, \infty)$,
Decreasing: $(-\infty, -1) \cup (0, 3)$,
Concave up: $\left(-\infty, \frac{1}{3}\left(2 - \sqrt{13}\right)\right) \cup \left(\frac{1}{3}\left(2 + \sqrt{13}\right), \infty\right)$,
Concave down: $\left(\frac{1}{3}\left(2 - \sqrt{13}\right), \frac{1}{3}\left(2 + \sqrt{13}\right)\right)$

12) $y = x + \sin(\pi x)$

13) $g(x) = x - \sqrt{x}$

Answer

Increasing: $\left(\frac{1}{4}, \infty\right)$,
Decreasing: $\left(0, \frac{1}{4}\right)$,
Concave up: $(0, \infty)$,
Concave down: nowhere

14) $f(\theta) = \sin(3\theta)$

Evaluate the following limits.

15) $\lim\limits_{x\to\infty} \dfrac{3x\sqrt{x^2+1}}{\sqrt{x^4-1}}$

Answer

3

16) $\lim\limits_{x\to\infty} \cos\left(\dfrac{1}{x}\right)$

17) $\lim\limits_{x\to 1} \dfrac{x-1}{\sin(\pi x)}$

Answer

$-\frac{1}{\pi}$

18) $\lim\limits_{x\to\infty} (3x)^{1/x}$

Use Newton's method to find the first two iterations, given the starting point.

19) $y = x^3 + 1, \quad x_0 = 0.5$

Answer

$x_1 = -1, \;\; x_2 = -1$

20) $\dfrac{1}{x+1} = \dfrac{1}{2}, \quad x_0 = 0$

Find the antiderivatives $F(x)$ of the following functions.

21) $g(x) = \sqrt{x} - \dfrac{1}{x^2}$

Answer

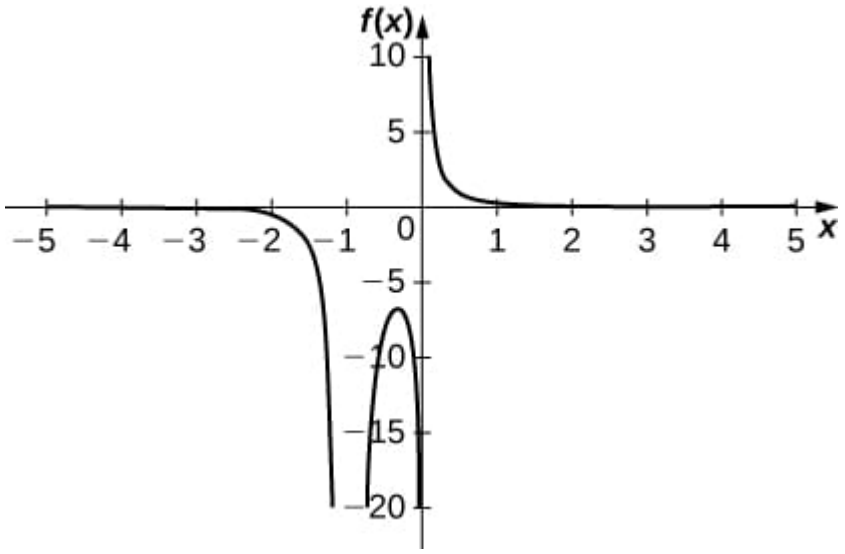
$F(x) = \dfrac{2x^{3/2}}{3} + \dfrac{1}{x} + C$

22) $f(x) = 2x + 6\cos x, \quad F(\pi) = \pi^2 + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

23) $y = \dfrac{1}{x(x+1)^2}$

Answer



Inflection points: none;
Critical points: $x = -\frac{1}{3}$;
Zeros: none;
Vertical asymptotes: $x = -1, \;\; x = 0$;
Horizontal asymptote: $y = 0$

24) $y = x - \sqrt{4 - x^2}$

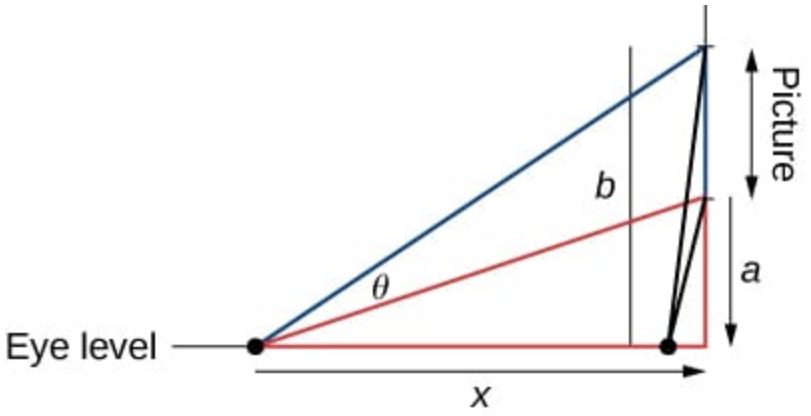
25) A car is being compacted into a rectangular solid. The volume is decreasing at a rate of $2 \text{ m}^3/\text{sec}$. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec, find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m.

Answer

The height is decreasing at a rate of 0.125 m/sec

26) A rocket is launched into space; its kinetic energy is given by $K(t) = \frac{1}{2}m(t)v(t)^2$, where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of $15 \text{ m}/\text{sec}^2$ and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J.

27) The famous **Regiomontanus' problem** for angle maximization was proposed during the 15th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ?



Answer

$x = \sqrt{ab}$ feet

28) An airline sells tickets from Tokyo to Detroit for $\$1200$. There are 500 seats available and a typical flight books 350 seats. For every $\$10$ decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?

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16: THE DERIVATIVE

16.1: Prelude to the Derivative
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16.3: The Derivative
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16.E: The Derivative (Exercises)

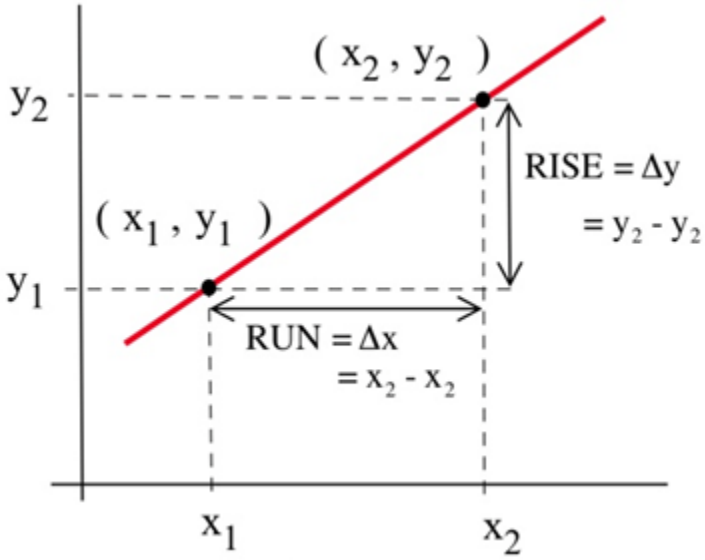
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16.1: Prelude to the Derivative

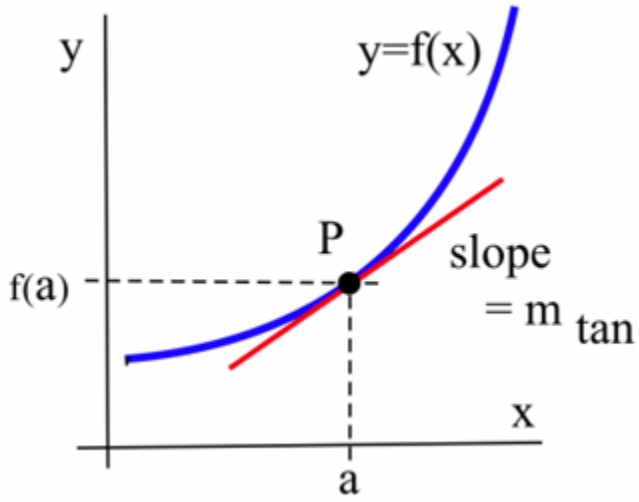
Precalculus Idea: Slope and Rate of Change

The slope of a line measures how fast a line rises or falls as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. If the line represents the distance traveled over time, for example, then its slope represents the velocity. In the figure, you can remind yourself of how we calculate slope using two points on the line:



$$m = \text{Slope from } P \text{ to } Q = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We would like to be able to get that same sort of information (how fast the curve rises or falls, velocity from distance) even if the graph is not a straight line. But what happens if we try to find the slope of a curve, as in the figure below?

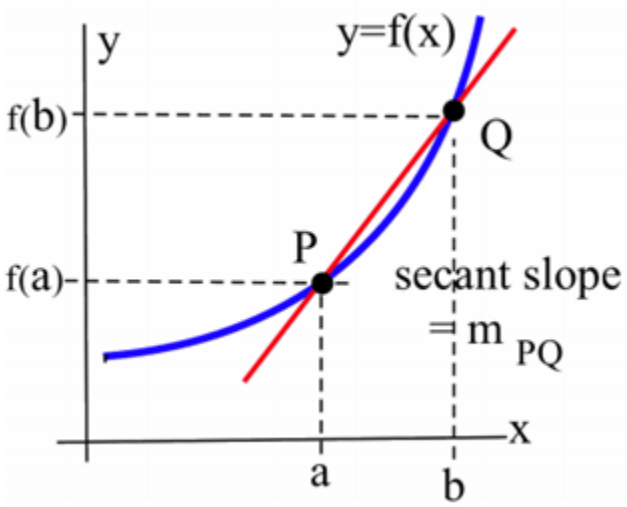


We need two points in order to determine the slope of a line. How can we find a slope of a curve, at just one point? The answer, as suggested in the figure, is to find the slope of the tangent line to the curve at that point. Most of us have an intuitive idea of what a tangent line is. Unfortunately, “tangent line” is hard to define precisely.

Definition: Secant Line

A **secant line** is a line between two points on a curve.

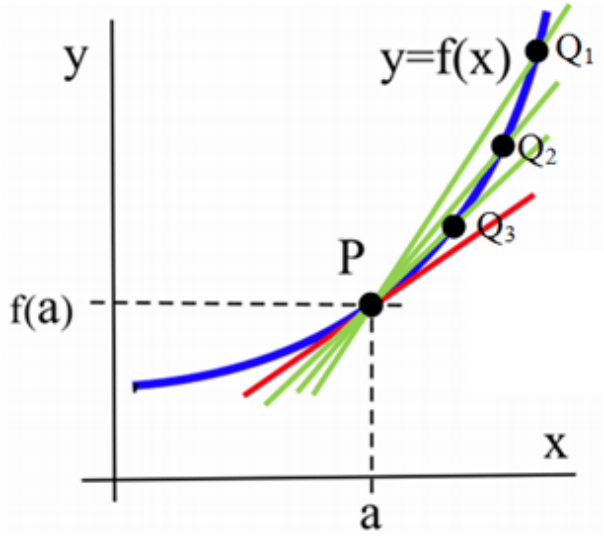
See the image below:



Can't-quite-do-it-yet Definition: Tangent Line

A **tangent line** is a line at one point on a curve... that does its best to be the curve at that point?

As you may be able to see in the image below, the closer the point Q is to the point P, the closer the secant slope gets to the tangent slope. This will be key to finding the tangent slope, but first we need to more carefully define the idea of "getting closer to."



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16.2: Limits and Continuity

Limit

In the last section, we saw that as the interval over which we calculated got smaller, the secant slopes approached the tangent slope. The **limit** gives us better language with which to discuss the idea of “approaches.”

The limit of a function describes the behavior of the function when the variable is near, **but does not equal**, a specified number (see the figure below).

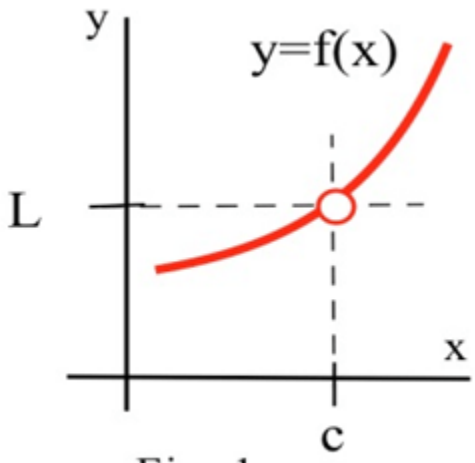


Fig. 1

Limit

If the values of $f(x)$ get closer and closer, as close as we want, to one number L as we take values of x very close to (but not equal to) a number c, then we say "**the limit of** $f(x)$ **as** x **approaches** c **is** L" and we write

$$\lim_{x \to c} f(x) = \mathbf{L}.$$

The symbol "$\to$" means "approaches" or, less formally, "gets very close to".

(This definition of the limit isn't stated as formally as it could be, but it is sufficient for our purposes in this course.)

Note:

- $\mathbf{f(c)}$ is a single number that describes the behavior (value) of $f(x)$ **AT** the point $x = c$.
- $\lim\limits_{x \to c} f(x)$ is a single number that describes the behavior of $f(x)$ **NEAR, *BUT NOT AT***, the point $x = c$.

If we have a graph of the function near $x = c$, then it is usually easy to determine $\lim\limits_{x \to c} f(x)$.

Example 16.2.1

Use the graph of $y = f(x)$ in the figure below to determine the following limits:

1. $\lim\limits_{x \to 1} f(x)$
2. $\lim\limits_{x \to 2} f(x)$

3. $\lim_{x\to 3} f(x)$
4. $\lim_{x\to 4} f(x)$

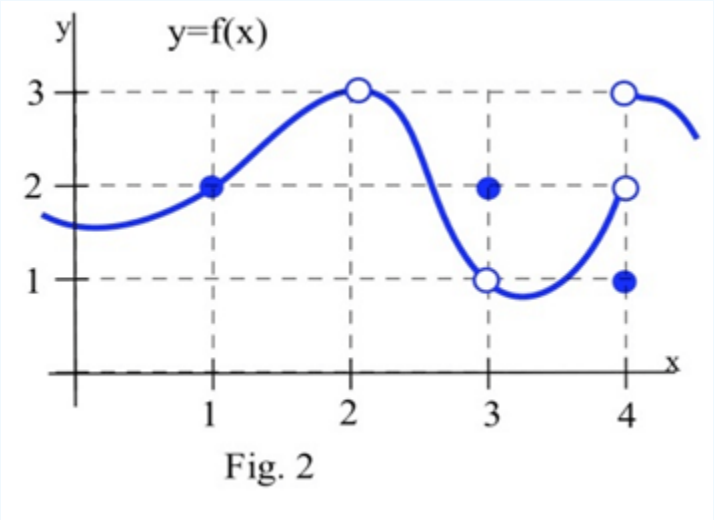


Fig. 2

Solution

1. $$\lim_{x\to 1} f(x) = 2$$
   When x is very close to 1, the values of $f(x)$ are very close to $y = 2$. In this example, it happens that $f(1) = 2$, but that is **irrelevant** for the limit. The only thing that matters is what happens for x *close to* 1 but $x \neq 1$.
2. $f(2)$ is undefined, but we only care about the behavior of $f(x)$ for x *close to* 2 but *not equal to* 2. When x is close to 2, the values of $f(x)$ are close to 3. If we restrict x close enough to 2, the values of y will be as close to 3 as we want, so $\lim_{x\to 2} f(x) = 3$.
3. When x is close to 3 (or "as x approaches the value 3"), the values of $f(x)$ are close to 1 (or "approach the value 1"), so $\lim_{x\to 3} f(x) = 1$. For this limit it is completely irrelevant that $f(3) = 2$, We only care about what happens to $f(x)$ for x close to and not equal to 3.
4. This one is harder and we need to be careful. When x is close to 4 and slightly **less than** 4 (x is just to the left of 4 on the x-axis), then the values of $f(x)$ are close to 2. But if x is close to 4 and slightly **larger than** 4 then the values of $f(x)$ are close to 3. If we only know that x is very close to 4, then we cannot say whether $y = f(x)$ will be close to 2 or close to 3 – it depends on whether x is on the right or the left side of 4. In this situation, the $f(x)$ values are not close to a *single* number so we say $\lim_{x\to 4} f(x)$ **does not exist**. It is irrelevant that $f(4) = 1$. The limit, as x approaches 4, would still be undefined if $f(4)$ was 3 or 2 or anything else.

We can also explore limits using tables and using algebra.

Example 16.2.2

Find $\lim_{x\to 1} \dfrac{2x^2 - x - 1}{x - 1}$.

Solution

You might try to evaluate $f(x) = \frac{2x^2-x-1}{x-1}$ at $x = 1$, but $f(x)$ is not defined at $x = 1$. It is tempting, **but wrong**, to conclude that this function does not have a limit as x approaches 1.

Using tables: Trying some "test" values for x which get closer and closer to 1 from both the left and the right, we get

x	$f(x)$
0.9	2.82
0.9998	2.9996
0.999994	2.999988
0.9999999	2.9999998
$\to 1$	$\to 3$

x	$f(x)$
1.1	3.2
1.003	3.006
1.0001	3.0002
1.000007	3.000014
$\to 1$	$\to 3$

The function f is not defined at $x = 1$, but when x is **close to** 1, the values of $f(x)$ are getting very close to 3. We can get $f(x)$ as close to 3 as we want by taking x very close to 1 so

$$\lim_{x\to 1} \frac{2x^2 - x - 1}{x - 1} = 3.$$

Using algebra: We could have found the same result by noting that

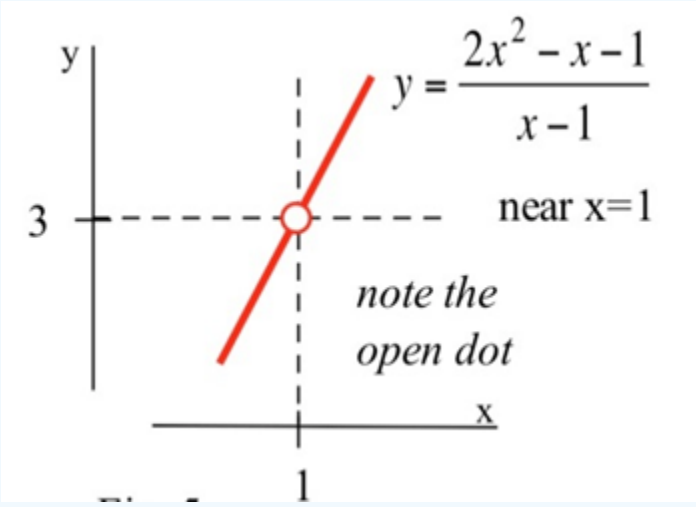
$$f(x) = \frac{2x^2 - x - 1}{x - 1} = \frac{(2x + 1)(x - 1)}{(x - 1)} = 2x + 1$$

as long as $x \neq 1$. (If $x \neq 1$, then $x–1 \neq 0$ so it is valid to divide the numerator and denominator by the factor $x–1$.) The "$x \to 1$" part of the limit means that x is close to 1 but **not equal to 1**, so our division step is valid and

$$\lim_{x\to 1} \frac{2x^2 - x - 1}{x - 1} = \lim_{x\to 1} 2x + 1 = 3,$$

which is our answer.

Using a graph: We can graph $y = f(x) = \frac{2x^2 - x - 1}{x - 1}$ for x close to 1:



Notice that whenever x is close to 1, the values of $y = f(x)$ are close to 3. Since f is not defined at $x = 1$, the graph has a hole above $x = 1$, but we only care about what $f(x)$ is doing for x close to but **not equal to** 1.

Example 16.2.3

Find $\lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3}$.

Solution

Notice that this function is not defined at $x = 3$. We can find the limit using algebra. Giving the two terms in the numerator a common denominator, we can simplify:

$$\frac{\frac{1}{x} - \frac{1}{3}}{x-3} = \frac{\frac{1}{x}\cdot\frac{3}{3} - \frac{1}{3}\cdot\frac{x}{x}}{x-3} = \frac{\frac{3}{3x} - \frac{x}{3x}}{x-3} = \frac{\frac{3-x}{3x}}{x-3}$$

Remember that dividing a fraction is the same as multiplying by the reciprocal, so

$$\frac{\frac{3-x}{3x}}{x-3} = \frac{\frac{3-x}{3x}}{\frac{x-3}{1}}$$

is equivalent to

$$\frac{3-x}{3x}\cdot\frac{1}{x-3}$$

To simplify further, we need to factor a negative 1 out of the numerator. Then we can cancel the term $(x-3)$
$\frac{-1(x-3)}{3x}\cdot\frac{1}{x-3} = \frac{-1}{3x}$ as long as $x \neq 3$

Now we can evaluate the limit using this simplified form.

$$\lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{-1}{3x} = -\frac{1}{9}$$

One Sided Limits

Sometimes, what happens to us at a place depends on the direction we use to approach that place. If we approach Niagara Falls from the upstream side, then we will be 182 feet higher and have different worries than if we approach from the downstream side. Similarly, the values of a function near a point may depend on the direction we use to approach that point.

Left and Right Limits

The **left limit** of $f(x)$ as x approaches c is L if the values of $f(x)$ get as close to L as we want when x is very close to and **left of** c (i.e., $x < \mathrm{c}$). We write

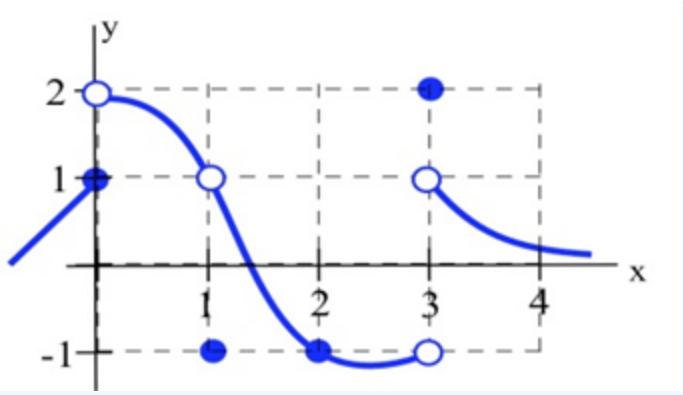
$$\lim_{x\to c^-} f(x) = L.$$

The **right limit** of $f(x)$ as x approaches c, written with $\mathrm{x} \to \mathrm{c}^+$, is L if the values of $f(x)$ get as close to L as we want when x is very close to and **right of** c (i.e., $x > \mathrm{c}$). We write

$$\lim_{x\to c^+} f(x) = L.$$

Example 16.2.4

Evaluate the one sided limits of the function $f(x)$ graphed below at $x = 0$ and $x = 1$.



Solution

As x approach 0 from the *left*, the value of the function is getting closer to 1, so $\lim_{x\to 0^-} f(x) = 1.$

As x approaches 0 from the *right*, the value of the function is getting closer to 2, so $\lim_{x\to 0^+} f(x) = 2.$

Notice that since the limit from the left and limit from the right are different, the general limit, $\lim_{x\to 0} f(x)$, does not exit.

At x approaches 1 from either direction, the value of the function is approaching 1, so

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 1.$$

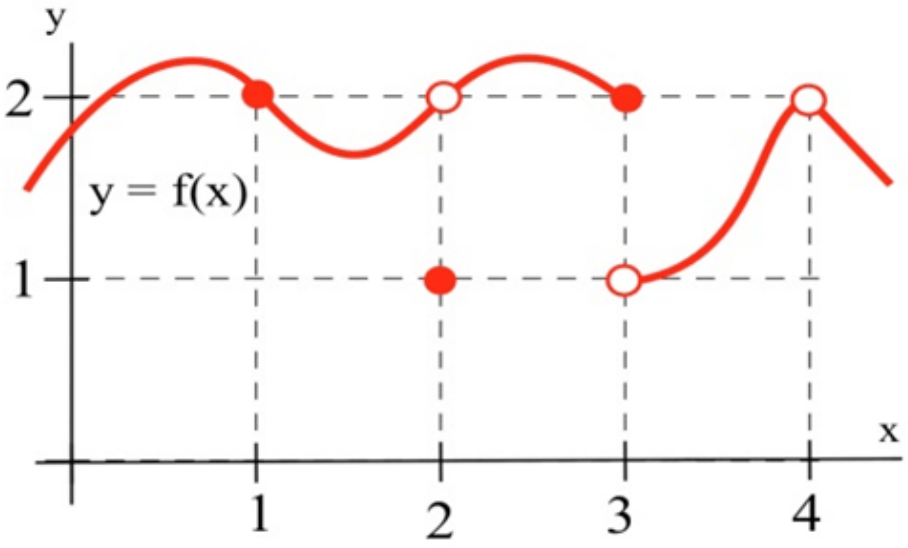
Continuity

A function that is "friendly" and doesn't have any breaks or jumps in it is called **continuous.** More formally,

Continuity at a Point

A function $\mathbf{f}$ **is continuous at** $\mathbf{x = a}$ if and only if $\lim_{x\to a} \mathbf{f(x) = f(a)}$.

The graph below illustrates some of the different ways a function can behave at and near a point, and the table contains some numerical information about the function and its behavior.



a	$f(a)$	$\lim_{x\to a} f(x)$
1	2	2
2	1	2
3	2	Does not exist (DNE)
4	Undefined	2

Based on the information in the table, we can conclude that f is continuous at 1 since $\lim_{x\to 1} f(x) = 2 = f(1)$.

We can also conclude from the information in the table that f is not continuous at 2 or 3 or 4, because $\lim_{x\to 2} f(x) \neq f(2)$, $\lim_{x\to 3} f(x) \neq f(3)$, and $\lim_{x\to 4} f(x) \neq f(4)$.

The behaviors at $x = 2$ and $x = 4$ exhibit a **hole** in the graph, sometimes called a **removable discontinuity**, since the graph could be made continuous by changing the value of a single point. The behavior at $x = 3$ is called a **jump discontinuity**, since the graph jumps between two values.

So which functions are continuous? It turns out pretty much every function you've studied is continuous where it is defined: **polynomial, radical, rational, exponential, and logarithmic functions are all continuous where they are defined**. Moreover, **any combination of continuous functions is also continuous**.

This is helpful, because the definition of continuity says that for a continuous function, $\lim_{x\to a} f(x) = f(a)$. That means for a continuous function, we can find the limit by **direct substitution** (evaluating the function) if the function is continuous at a.

Example 16.2.5

Evaluate using continuity, if possible:

1. $\lim_{x\to 2} x^3 - 4x$
2. $\lim_{x\to 2} \dfrac{x-4}{x+3}$
3. $\lim_{x\to 2} \dfrac{x-4}{x-2}$

Solution

1. The given function is polynomial, and is defined for all values of x, so we can find the limit by direct substitution:
$$\lim_{x\to 2} x^3 - 4x = 2^3 - 4(2) = 0.$$
2. The given function is rational. It is not defined at $x = -3$, but we are taking the limit as x approaches 2, and the function is defined at that point, so we can use direct substitution:
$$\lim_{x\to 2} \frac{x-4}{x+3} = \frac{2-4}{2+3} = -\frac{2}{5}.$$
3. This function is not defined at $x = 2$, and so is not continuous at $x = 2$. We cannot use direct substitution.

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16.3: The Derivative

Definition of the Derivative

When working with linear functions, we could find the slope of a line to determine the rate at which the function is changing. For an arbitrary function, we can determine the **average rate of change** of the function. This is the slope of the secant line through those two points on the graph.

Average Rate of Change

Average Rate of Change = $\frac{\Delta \text{output}}{\Delta \text{input}} = \frac{f(b) - f(a)}{b - a}$

= **the slope of the secant line through the two points** $(a, f(a))$ **and** $(b, f(b))$

Example 16.3.1

Suppose the total cost of producing x items is given by $TC(x) = 200 + 30x - 0.1x^2$. Determine the average rate of change of cost when increasing production from 25 units to 100 units.

Solution

The cost when producing 25 units is $TC(25) = 200 + 30(25) - 0.1(25)^2 = \$887.50$

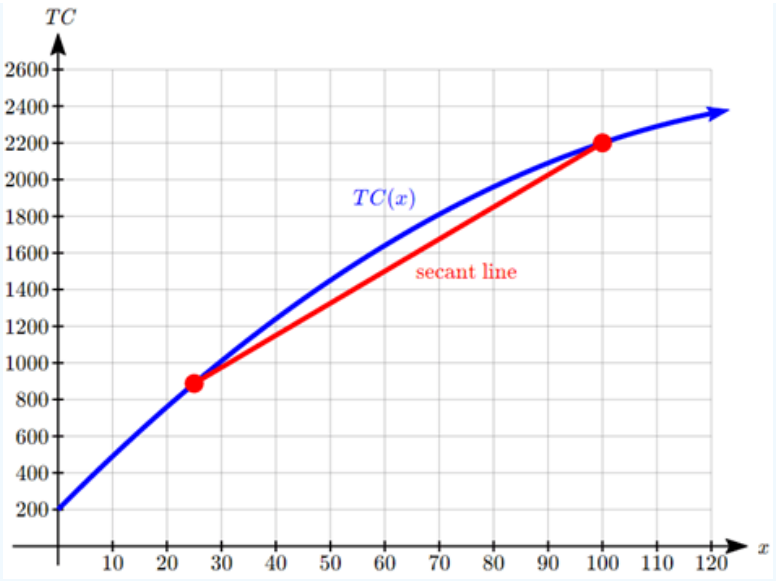
The cost when producing 100 units is $TC(100) = 200 + 30(100) - 0.1(100)^2 = \$2200$

The cost increased by \$2200-\$887.50 = \$1312.50 while production increased by 100 – 25 = 75 items. The average rate of change is:

$$\frac{TC(100) - TC(25)}{100 - 25} = \frac{2200 - 887.50}{100 - 25} = \frac{1312.50}{75} = 17.50 \text{ dollars per unit}$$

This tells us that on average the cost increases by \$17.50 for each unit produced.

The average rate of change is the slope of the secant line between (25, 887.50) and (100, 2200)



Thinking about the last example, suppose instead we asked the question "How fast are costs increasing when production is 25 units?" Notice this is a different kind of question. The question in the example asked for the rate of change over an interval, as production increased from one value to another. This question is again asking for a rate of change, but an **instantaneous rate of change**, at a particular moment.

We don't yet have a way to calculated rate of change except over an interval. In the next example we will explore a couple ways to estimate the instantaneous rate of change.

Example 16.3.2

Suppose the total cost of producing x items is given by $TC(x) = 200 + 30x - 0.1x^2$. Estimate the instantaneous rate of change when 25 items are being produced.

Solution

We can see in the previous graph that the secant slope on the interval $25 \le x \le 100$ is not a particularly good estimate of the instantaneous rate of change since the cost function seems steeper at $x = 25$ then over the secant slope. We could improve the estimate by choosing a smaller interval.

Approach 1: Using smaller intervals

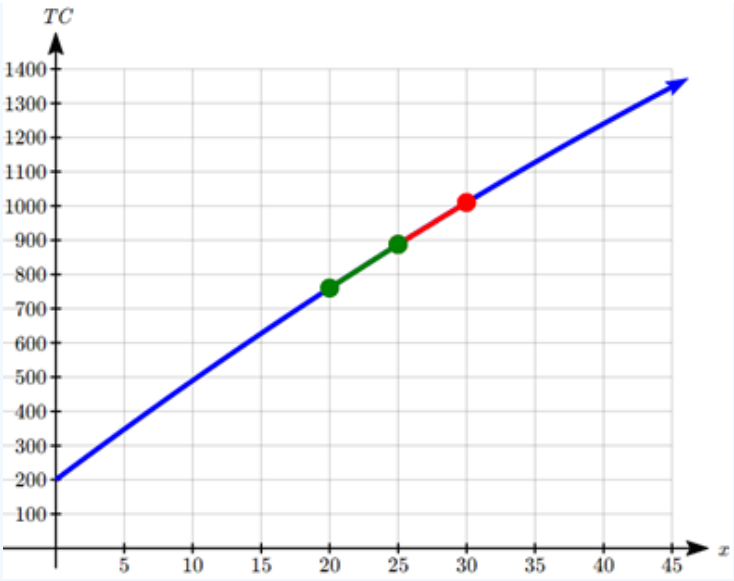
Using the interval $25 \le x \le 30$, the average rate of change would be:

$$\frac{TC(30) - TC(25)}{30 - 25} = \frac{1010 - 887.50}{30 - 25} = \frac{122.50}{5} = 24.50 \text{ dollars per unit}$$

Using an interval on the other side, $20 \le x \le 25$, the average rate of change would be:

$$\frac{TC(25) - TC(20)}{25 - 20} = \frac{887.50 - 760}{25 - 20} = \frac{127.5}{5} = 25.50 \text{ dollars per unit}$$

Visually, we can see both these secant lines seem to approximate the function pretty well.

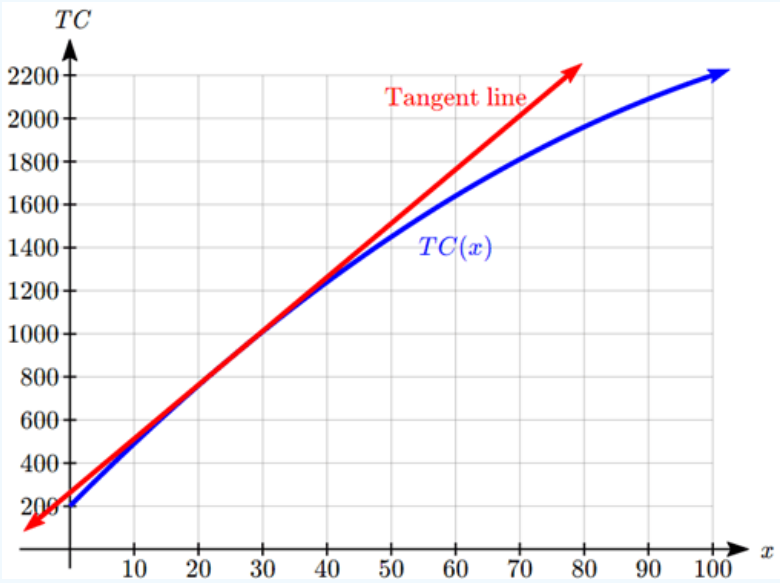


We would expect the instantaneous rate of change to be somewhere between these two values. Averaging them, we get an estimate of $25 per unit for the instantaneous rate of change.

Approach 2: Estimating from the graph.

This approach is more commonly used when we only have the graph of a function, and don't have a formula to evaluate, but we will illustrate it here using the same function.

The instantaneous rate of change is the slope of the tangent line, which is the line that just touches the graph at the point of interest, and has the same rate of change (slope) as the function does at the point. Given a graph, we can sketch in a **tangent line** by holding a ruler up to the graph, and drawing in a line that touches the graph when $x = 25$ and has the same slope.

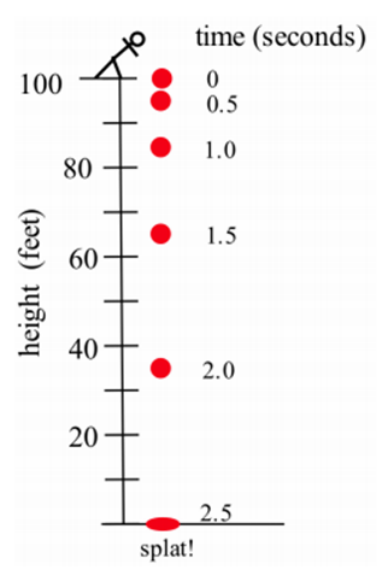


To estimate the instantaneous rate of change, we can now calculate the slope of this drawn-in line. Looking at the graph, the tangent line appears to pass approximately through (30, 1000) and (70, 2000) which would give a slope of $\frac{2000-1000}{70-30} = \frac{1000}{40} = 25$ dollars per unit.

Next we will explore the same ideas using a function defined in a table, and in another context.

Example 16.3.3

Suppose we drop a tomato from the top of a 100 foot building and time its fall.



Time (sec)	Height (ft)
0.0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0

1. How long did it take for the tomato to drop 100 feet?
2. How far did the tomato fall during the first second?
3. How far did the tomato fall during the last second?
4. How far did the tomato fall between $t = 0.5$ and $t = 1$?
5. What was the average velocity of the tomato during its fall?
6. What was the average velocity between $t = 1$ and $t = 2$ seconds?
7. How fast was the tomato falling 1 second after it was dropped?

Solution

Some of these questions are pretty easy to answer, while some are more complex.

1. From the table, it took 2.5 seconds for the tomato to drop 100 feet
2. During the first second, the tomato fell 100 – 84 = 16 feet
3. During the last second, the tomato fell 64 – 0 = 64 feet
4. Between $t = 0.5$ and $t = 1$ the tomato fell 96 – 84 = 12 feet
5. Velocity is similar to speed, and is a rate of change. We can calculate **average velocity** the same way we did average rate of change earlier.

$$\text{Average velocity} = \frac{\text{distance fallen}}{\text{total time}} = \frac{\Delta\text{position}}{\Delta\text{time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40 \text{ ft/s}$$

6. Between $t = 1$ and $t = 2$ seconds,

$$\text{Average velocity} = \frac{\Delta\text{position}}{\Delta\text{time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48 \text{ ft/s}$$

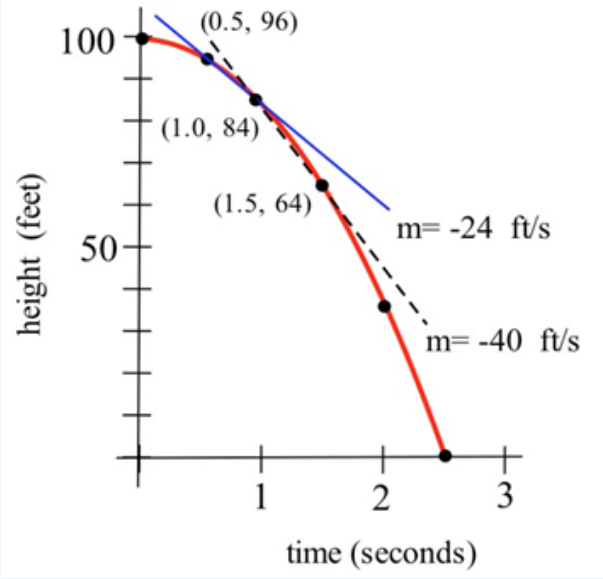
7. This question is significantly different from the previous two questions about average velocity. Here we want the **instantaneous velocity,** the velocity at an instant in time. Unfortunately the tomato is not equipped with a speedometer so we will have to give an approximate answer. Like in our Approach 1 in the previous example, we will estimate it using secant slopes.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, -40 ft/s. But the tomato fell slowly at the beginning and rapidly near the end so the "-40 ft/s" estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at $t = 1$ by calculating the average velocities over a short time interval near $t = 1$. The average velocity between $t = 0.5$ and $t = 1$ is $\frac{-12 \text{ feet}}{0.5 \text{ s}} = -24 \text{ ft/s}$, and the average velocity between $t = 1$ and $t = 1.5$ is $\frac{-20 \text{ feet}}{0.5 \text{ s}} = -40 \text{ ft/s}$ so we can be reasonably sure that the instantaneous velocity is between -24 ft/s and -40 ft/s. The average, –32 ft/s, would be a good estimate for the instantaneous velocity.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity.

The average velocity over a time interval is $\frac{\Delta\text{position}}{\Delta\text{time}}$, which is the slope of the **secant line** through two points on the graph of height versus time. The instantaneous velocity at a particular time and height is the slope of the **tangent line** to the graph at the point given by that time and height.



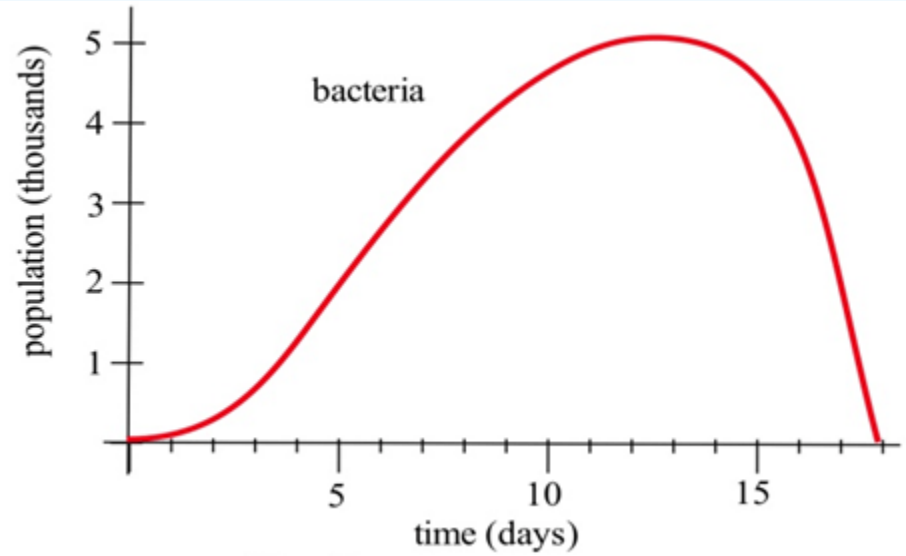
Average vs Instantaneous Velocity

Average velocity = $\frac{\Delta \text{position}}{\Delta \text{time}}$ = slope of the secant line through 2 points.

Instantaneous velocity = slope of the line tangent to the graph.

Example 16.3.4

Suppose we set up a machine to count the number of bacteria growing on a Petri plate. At first there are few bacteria so the population grows slowly. Then there are more bacteria to divide so the population grows more quickly. Later, there are more bacteria and less room and nutrients available for the expanding population, so the population grows slowly again. Finally, the bacteria have used up most of the nutrients, and the population declines as bacteria die.



Use the population graph to estimate the answer to the questions below.

1. What is the bacteria population at time $t = 3$ days?
2. What is the population increment from $t = 3$ to $t = 10$ days?
3. What is the **rate** of population growth from $t = 3$ to $t = 10$ days?
4. What is the **rate** of population growth on the third day, at $t = 3$?

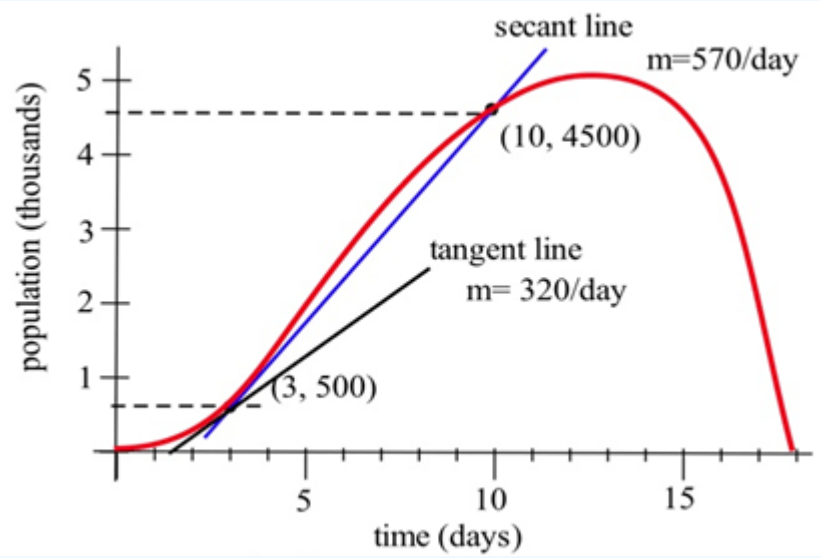
Solution

1. From the graph, at $t = 3$, the population is about 0.5 thousand, or 500 bacteria.
2. At $t = 10$, the population is about 4.5 thousand, so the increment is about 4000 bacteria.
3. The rate of growth from $t = 3$ to $t = 10$ is the average rate of change in population during that time:

$$\begin{aligned} \text{average change in population} &= \frac{\text{change in population}}{\text{change in time}} \\ &= \frac{\Delta \text{population}}{\Delta \text{time}} \\ &= \frac{4000 \text{ bacteria}}{7 \text{ days}} \\ &\approx 570 \text{ bacteria/day}. \end{aligned}$$

This is the slope of the secant line through the two points (3, 500) and (10, 4500).

4. This question is asking for the **instantaneous** rate of population change, the slope of the line which is **tangent** to the population curve at (3, 500). If we sketch a line approximately tangent to the curve at (3, 500) and pick two points near the ends of the tangent line segment, we can estimate that instantaneous rate of population growth is approximately 320 bacteria/day.



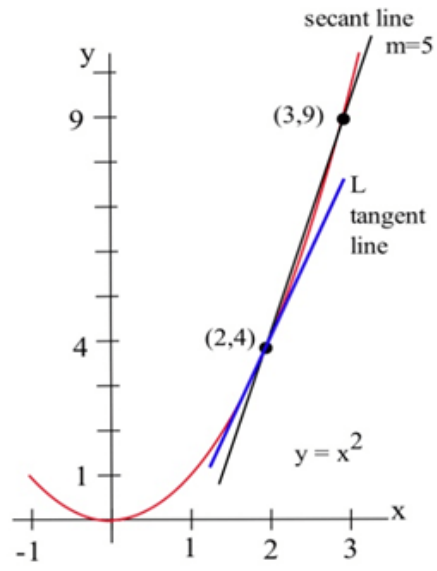
In the previous examples, we noticed that as the interval got smaller and smaller, the secant line got closer to the tangent line and its slope got closer to the slope of the tangent line. That's good news – we know how to find the slope of a secant line.

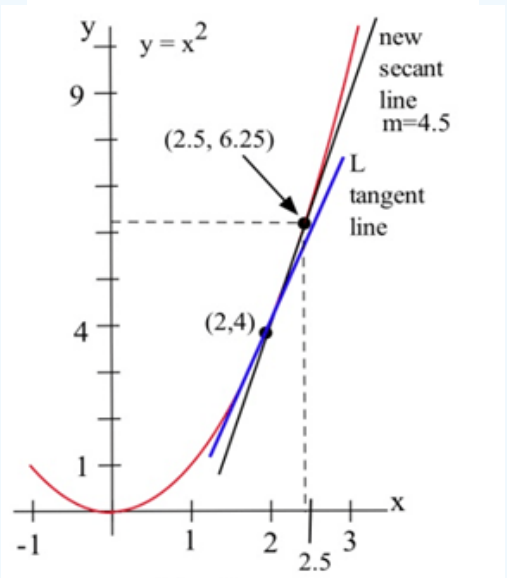
Let's explore further this idea of finding the tangent slope based on the secant slope.

Example 16.3.5

Find the slope of the line L in the graph below which is tangent to $f(x) = x^2$ at the point (2,4).

We could estimate the slope of L from the graph, but we won't. Instead, we will use the idea that secant lines over tiny intervals approximate the tangent line.





We can see that the line through (2,4) and (3,9) on the graph of f is an approximation of the slope of the tangent line, and we can calculate that slope exactly: $m = \frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5$. But $m = 5$ is only an estimate of the slope of the tangent line and not a very good estimate. It's too big. We can get a better estimate by picking a second point on the graph of f which is closer to (2,4). The point (2,4) is fixed and it must be one of the points we use.

From the second figure, we can see that the slope of the line through the points (2,4) and (2.5,6.25) is a better approximation of the slope of the tangent line at (2,4): $m = \frac{\Delta y}{\Delta x} = \frac{6.25-4}{2.5-2} = \frac{2.25}{0.5} = 4.5$, a better estimate, but still an approximation. We can continue picking points closer and closer to (2,4) on the graph of f, and then calculating the slopes of the lines through each of these points and the point (2,4):

Points to the left of (2,4)

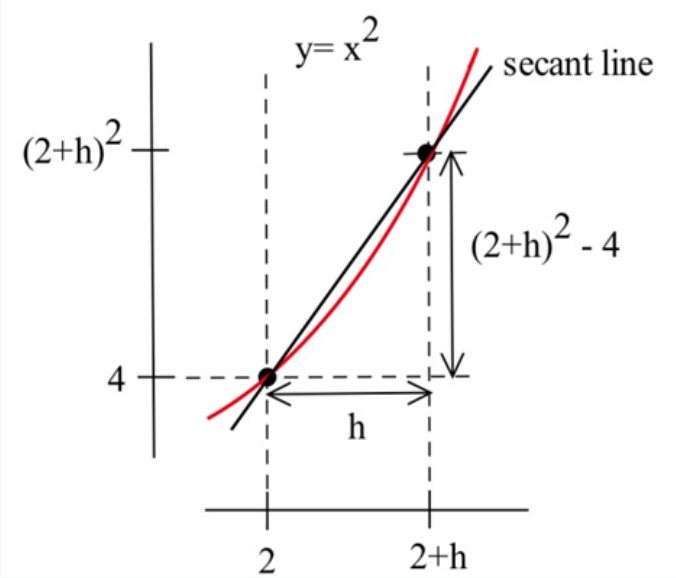
x	$y = x^2$	Slope of line through (x, y) and (2,4).
1.5	2.25	3.5
1.9	3.61	3.9
1.99	3.9601	3.99

Points to the right of (2,4)

x	$y = x^2$	Slope of line through (x, y) and (2,4).
3	9	5
2.5	6.25	4.5
2.01	4.0401	4.01

The only thing special about the x–values we picked is that they are numbers which are close, and very close, to $x = 2$. Someone else might have picked other nearby values for x. As the points we pick get closer and closer to the point (2,4) on the graph of $y = x^2$, the slopes of the lines through the points and (2,4) are better approximations of the slope of the tangent line, and these slopes are getting closer and closer to 4.

We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near (2,4). Define $x = 2 + h$ so h is the increment from 2 to x. If h is small, then $x = 2 + h$ is close to 2 and the point $(2 + h, f(2 + h)) = (2 + h, (2 + h)^2)$ is close to (2,4). The slope m of the line through the points (2,4) and $(2 + h, (2 + h)^2)$ is a good approximation of the slope of the tangent line at the point (2,4):



We can calculate the secant slope for any value of h:

$$m = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{(4 + 4h + h^2) - 4}{h} = \frac{4h + h^2}{h} = \frac{h(4+h)}{h} = 4 + h$$

The value $m = 4 + h$ is the slope of the secant line through the two points (2,4) and $(2 + h, (2 + h)^2)$. As h gets smaller and smaller, this slope approaches the slope of the tangent line to the graph of f at (2,4).

More formally, we could write:

$$\text{Slope of the tangent line} = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} (4 + h).$$

We can easily evaluate this limit using direct substitution, finding that as the interval h shrinks towards 0, the secant slope approaches the tangent slope, 4.

Use the applet below to explore this. You can drag the base point on the graph to explore the behavior at different locations on the graph. Once setting the base point, use the slider to see how the secant lines approach the tangent line as h approaches zero.

Finding tangent slopes and finding the instantaneous rate of change are the same problem. In each problem we wanted to know how rapidly something was **changing at an instant in time**, and the answer turned out to be finding the **slope of a tangent line**, which we *approximated* with the **slope of a secant line**. This idea is the key to defining the slope of a curve.

The Derivative

We can view the derivative in different ways. Here are a three of them:

- The **derivative** of a function f at a point $(x, f(x))$ is the instantaneous rate of change.
- The **derivative** is the slope of the tangent line to the graph of f at the point $(x, f(x))$.
- The **derivative** is the slope of the curve $f(x)$ at the point $(x, f(x))$.

A function is called **differentiable** at $(x, f(x))$ if its derivative exists at $(x, f(x))$.

Notation for the Derivative

The **derivative of** $y = f(x)$ **with respect to** x is written as

$$f'(x)$$

(read aloud as "f prime of x"), or

$$y'$$

(read aloud as "why prime") or

$$\frac{dy}{dx}$$

(read aloud as "dee why dee ex"), or

$$\frac{df}{dx}.$$

The notation that resembles a fraction is called **Leibniz notation**. It displays not only the name of the function (f or y), but also the name of the variable (in this case, x). It looks like a fraction because the derivative is a slope. In fact, this is simply $\frac{\Delta y}{\Delta x}$ written in Roman letters instead of Greek letters.

Verb Forms

We **find the derivative** of a function, or **take the derivative** of a function, or **differentiate** a function.

We use an adaptation of the $\frac{df}{dx}$ notation to mean "find the derivative of $f(x)$:"

$$\frac{d}{dx}[f(x)] = \frac{df}{dx}.$$

[The book uses parentheses instead of brackets–both are acceptable forms of the notation.]

Formal Algebraic Definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Practical Definition

The derivative can be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval. The tinier the interval, the closer this is to the true instantaneous rate of change, slope of the tangent line, or slope of the curve.

Looking Ahead

We will have methods for computing exact values of derivatives from formulas soon. If the function is given to you as a table or graph, you will still need to approximate this way.

This is the foundation for the rest of this chapter. It's remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative $f'(x)$) will lead to so many important ideas and applications.

Example 16.3.6

Find the slope of the tangent line to $f(x) = \frac{1}{x}$ at $x = 3$.

Solution

The slope of the tangent line is the value of the derivative $f'(3)$. $f(3) = \frac{1}{3}$ and $f(3+h) = \frac{1}{3+h}$, so, using the formal limit definition of the derivative,

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}.$$

We can simplify by giving the fractions a common denominator:

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} &= \lim_{h\to 0} \frac{\frac{1}{3+h}\cdot\frac{3}{3} - \frac{1}{3}\cdot\frac{3+h}{3+h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3}{9+3h} - \frac{3+h}{9+3h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3-(3+h)}{9+3h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3-3-h}{9+3h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{9+3h}}{h} \\
&= \lim_{h\to 0} \frac{-h}{9+3h}\cdot\frac{1}{h} \\
&= \lim_{h\to 0} \frac{-1}{9+3h}
\end{aligned}$$

and the evaluate using direct substitution:

$$\lim_{h\to 0} \frac{-1}{9+3h} = \frac{-1}{9+3(0)} = -\frac{1}{9}.$$

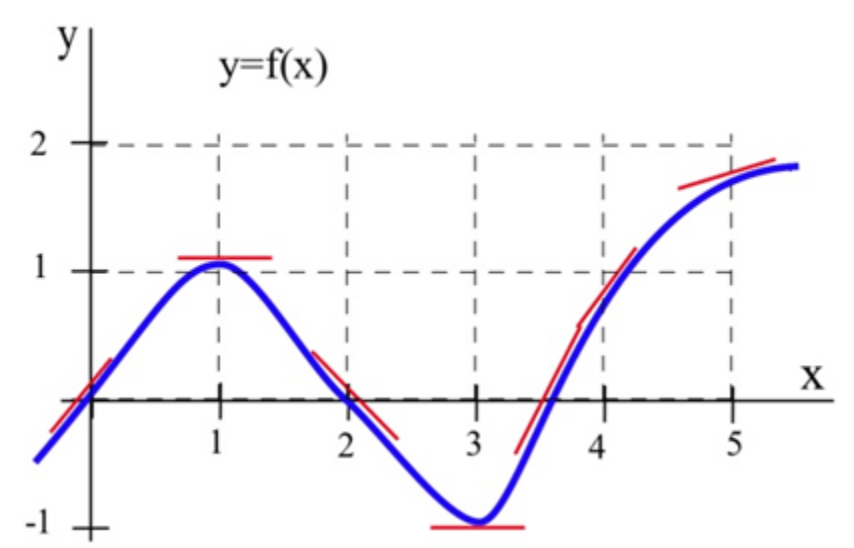
Thus, the slope of the tangent line to $f(x) = \frac{1}{x}$ at $x = 3$ is $-\frac{1}{9}$.

The Derivative as a Function

We now know how to find (or at least approximate) the derivative of a function for any x-value; this means we can think of the derivative as a function, too. The inputs are the same x's; the output is the value of the derivative at that x value.

Example 16.3.7

Below is the graph of a function $y = f(x)$. We can use the information in the graph to fill in a table showing values of $f'(x)$:



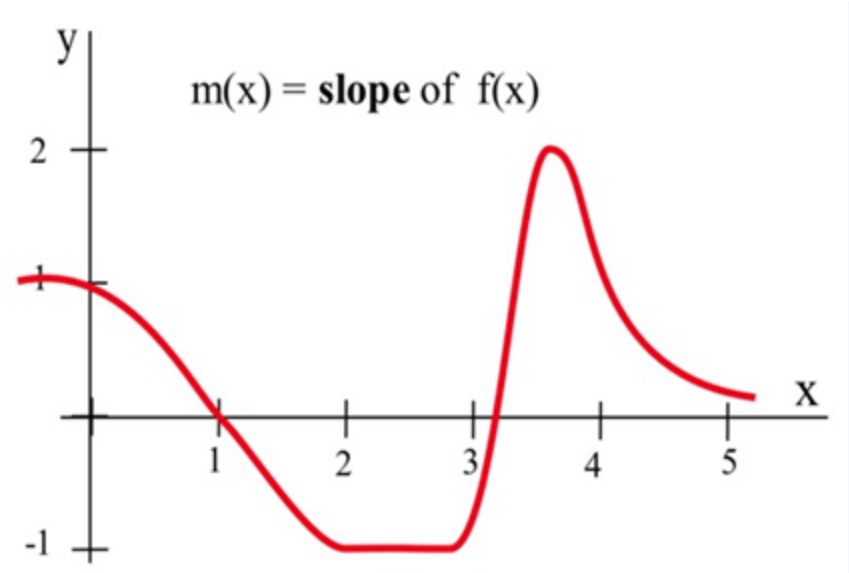
At various values of x, draw your best guess at the tangent line and measure its slope. You might have to extend your lines so you can read some points. In general, your estimate of the slope will be better if you choose points that are easy to read and far away from each other. Here are estimates for a few values of x (parts of the tangent lines used are shown above in the graph):

x	$y = f(x)$	$f'(x) =$ the estimated *slope* of the tangent line to the curve at the point (x, y).
0	0	1
1	1	0
2	0	-1
3	-1	0
3.5	0	2

We can estimate the values of $f'(x)$ at some non-integer values of x, too: $f'(0.5) \approx 0.5$ and $f'(1.3) \approx -0.3$.

We can even think about entire intervals. For example, if $0 < x < 1$, then $f(x)$ is increasing, all the slopes are positive, and so $f'(x)$ is positive.

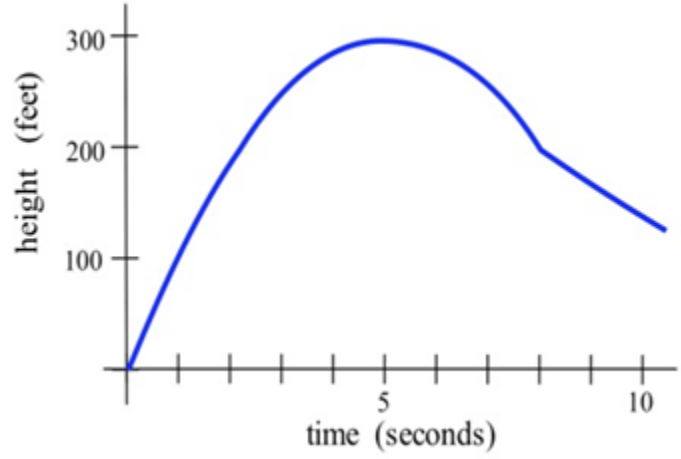
The values of $f'(x)$ definitely depend on the values of x, and $f'(x)$ is a function of x. We can use the results in the table to help sketch the graph of $f'(x)$.



To get a better feel for this, explore the applet below. The top graph is the graph of the original function $g(x)$. The bottom graph shows the slopes of $g(x)$, so is a graph of the derivative, $g'(x)$. Drag the point *a* and notice how the slope of the tangent line corresponds to the value of the derivative $g'(x)$.

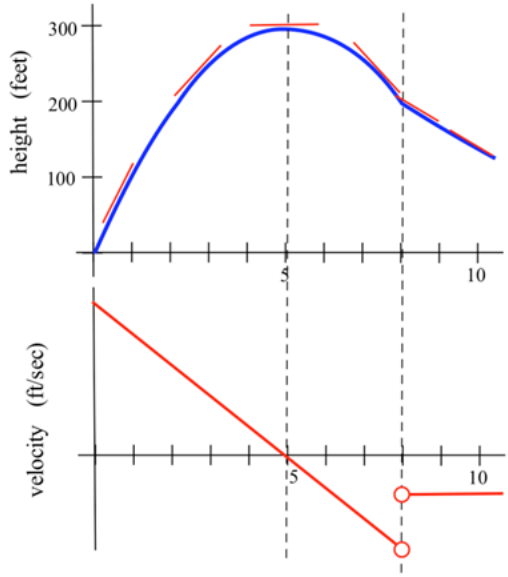
Example 16.3.8

Shown is the graph of the height $h(t)$ of a rocket at time t.



Sketch the graph of the **velocity** of the rocket at time t. (Velocity is the **derivative** of the height function, so it is the **slope of the tangent** to the graph of position or height.)

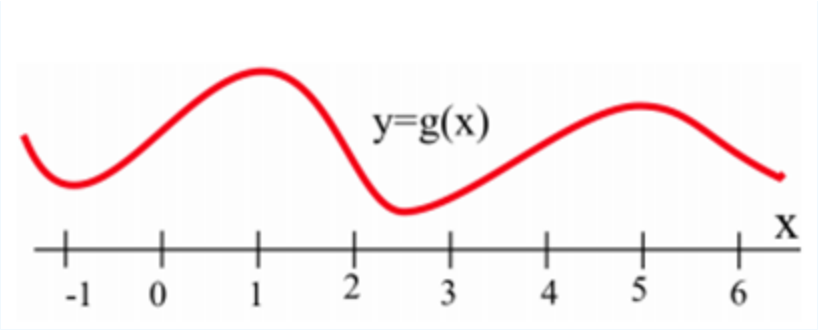
We can estimate the slope of the function at several points. The lower graph below shows the velocity of the rocket. This is $v(t) = h'(t)$.



In some applications, we need to know where the graph of a function $f(x)$ has horizontal tangent lines (slopes = 0).

Example 16.3.9

Below is the graph of $y = g(x)$. At what values of x does the graph of $g(x)$ have horizontal tangent lines?



The tangent lines to the graph of $g(x)$ are horizontal (slope = 0) when $x \approx -1, 1, 2.5, \text{ and } 5$.

We can also find derivative functions algebraically using limits.

Example 16.3.10

Find $\frac{d}{dx}(2x^2 - 4x - 1)$.

Setting up the derivative using a limit,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We will start by simplifying $f(x+h)$ by expanding:

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 4(x+h) - 1 \\ &= 2(x^2 + 2xh + h^2) - 4(x+h) - 1 \\ &= 2x^2 + 4xh + 2h^2 - 4x - 4h - 1 \end{aligned}$$

Now finding the limit:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(2x^2 + 4xh + 2h^2 - 4x - 4h - 1) - (2x^2 - 4x - 1)}{h} \\ &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 4x - 4h - 1 - 2x^2 + 4x + 1}{h} \qquad \text{(Substitute in the formulas.)} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 - 4h}{h} \qquad \text{(Now simplify.)} \\ &= \lim_{h \to 0} \frac{h(4x + 2h - 4)}{h} \qquad \text{(Factor out the } h\text{, then cancel.)} \\ &= \lim_{h \to 0} (4x + 2h - 4) \end{aligned}$$

We can find the limit of this expression by direct substitution:

$$f'(x) = \lim_{h \to 0} (4x + 2h - 4) = 4x - 4$$

Notice that the derivative depends on x, and that this formula will tell us the slope of the tangent line to f at any value x. For example, if we wanted to know the tangent slope of f at $x = 3$, we would simply evaluate: $f'(3) = 4(3) - 4 = 8$.

A formula for the derivative function is very powerful, but as you can see, calculating the derivative using the limit definition is **very** time consuming. In the next section, we will identify some patterns that will allow us to start building a set of rules for finding derivatives without needing the limit definition.

Interpreting the Derivative

So far we have emphasized the derivative as the slope of the line tangent to a graph. That interpretation is very visual and useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are used in a wide variety of fields and applications, and some of these fields use other interpretations. The following are a few interpretations of the derivative that are commonly used.

General

Rate of Change: $f'(x)$ is the **rate of change** of the function at x. If the units for x are years and the units for $f(x)$ are people, then the units for $\frac{df}{dx}$ are $\frac{\text{people}}{\text{year}}$, a rate of change in population.

Graphical

Slope: $f'(x)$ is the **slope of the line tangent to the graph of f at the point** $(x, f(x))$.

Physical

Velocity: If $f(x)$ is the position of an object at time x, then $f'(x)$ is the **velocity** of the object at time x. If the units for x are hours and $f(x)$ is distance measured in miles, then the units for $\mathbf{f'(x) = \frac{df}{dx}}$ are $\frac{\text{miles}}{\text{hour}}$, miles per hour, which is a measure of velocity.

Acceleration: If $f(x)$ is the velocity of an object at time x, then $f'(x)$ is the **acceleration** of the object at time x. If the units are for x are hours and $f(x)$ has the units $\frac{\text{miles}}{\text{hour}}$, then the units for the acceleration $\mathbf{f'(x) = \frac{df}{dx}}$ are $\frac{\text{miles/hour}}{\text{hour}} = \frac{\text{miles}}{\text{hour}^2}$, miles per hour per hour.

Business

Marginal Cost, Marginal Revenue, and Marginal Profit: We'll explore these terms in more depth later in the section. Basically, the marginal cost is approximately the *additional* cost of making one more object once we have already made x objects. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, the cost per bicycle.

In business contexts, the word "**marginal**" usually means the derivative or rate of change of some quantity.

One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and even the notations of calculus are still useful.

Example 16.3.11

Suppose the demand curve for widgets was given by $D(p) = \frac{1}{p}$, where D is the quantity of widgets, in thousands, at a price of p dollars. Interpret the derivative of D at $p =$\$3.

Note that we calculated $D'(3)$ earlier to be $D'(3) = -\frac{1}{9} \approx -0.111$.

Since D has units “thousands of widgets” and the units for p is dollars of price, the units for D' will be $\frac{\text{thousands of widgets}}{\text{dollar of price}}$. In other words, it shows how the demand will change as the price increases.

Specifically, $D'(3) \approx -0.111$ tells us that when the price is \$3, the demand will *decrease* by about 0.111 thousand items for every dollar the price increases.

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16.4: Power and Sum Rules for Derivatives

In the next few sections, we'll get the derivative rules that will let us find formulas for derivatives when our function comes to us as a formula. This is a very algebraic section, and you should get lots of practice. When you tell someone you have studied calculus, this is the one skill they will expect you to have.

Building Blocks

These are the simplest rules – rules for the basic functions. We won't prove these rules; we'll just use them. But first, let's look at a few so that we can see they make sense.

Example 16.4.1

Find the derivative of $y = f(x) = mx + b$.

Solution

This is a linear function, so its graph is its own tangent line! The slope of the tangent line, the derivative, is the slope of the line:

$$f'(x) = m$$

Rule

The derivative of a linear function is its slope.

Example 16.4.2

Find the derivative of $f(x) = 135$.

Solution

Think about this one graphically, too. The graph of $f(x)$ is a horizontal line. So its slope is zero:

$$f'(x) = 0$$

Rule

The derivative of a constant is zero.

Example 16.4.3

Find the derivative of $f(x) = x^2$.

Solution

Recall the formal definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Using our function $f(x) = x^2$, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$.

Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x \end{aligned}$$

From all that, we find that $f'(x) = 2x$.

Luckily, there is a handy rule we use to skip using the limit:

Power Rule

The derivative of $f(x) = x^n$ is

$$f'(x) = nx^{n-1}.$$

Example 16.4.4

Find the derivative of $g(x) = 4x^3$.

Solution

Using the power rule, we know that if $f(x) = x^3$, then $f'(x) = 3x^2$. Notice that g is 4 times the function f. Think about what this change means to the graph of g – it's now 4 times as tall as the graph of f. If we find the slope of a secant line, it will be $\frac{\Delta g}{\Delta x} = \frac{4\Delta f}{\Delta x} = 4\frac{\Delta f}{\Delta x}$; each slope will be 4 times the slope of the secant line on the f graph. This property will hold for the slopes of tangent lines, too:

$$\frac{d}{dx}\left(4x^3\right) = 4\frac{d}{dx}\left(x^3\right) = 4 \cdot 3x^2 = 12x^2.$$

Rule

Constants come along for the ride, i.e., $\frac{d}{dx}(kf) = kf'$.

Here are all the basic rules in one place.

Derivative Rules: Building Blocks

In what follows, f and g are differentiable functions of x.

Constant Multiple Rule

$$\frac{d}{dx}(kf) = kf'$$

Sum and Difference Rule

$$\frac{d}{dx}(f \pm g) = f' \pm g'$$

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Special cases:

$$\frac{d}{dx}(k) = 0 \quad (\text{Because } k = kx^0.)$$
$$\frac{d}{dx}(x) = 1 \quad (\text{Because } x = x^1.)$$

Exponential Functions

$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(a^x) = \ln(a)\, a^x$$

Natural Logarithm

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

The sum, difference, and constant multiple rule combined with the power rule allow us to easily find the derivative of any polynomial.

Example 16.4.5

Find the derivative of $p(x) = 17x^{10} + 13x^8 - 1.8x + 1003$.

Solution

$$\begin{aligned}\frac{d}{dx}\left(17x^{10} + 13x^8 - 1.8x + 1003\right) &= \frac{d}{dx}\left(17x^{10}\right) + \frac{d}{dx}\left(13x^8\right) - \frac{d}{dx}(1.8x) + \frac{d}{dx}(1003) \\ &= 17\frac{d}{dx}\left(x^{10}\right) + 13\frac{d}{dx}\left(x^8\right) - 1.8\frac{d}{dx}(x) + \frac{d}{dx}(1003) \\ &= 17\left(10x^9\right) + 13\left(8x^7\right) - 1.8\,(1) + 0 \\ &= 170x^9 + 104x^7 - 1.8\end{aligned}$$

You don't have to show every single step. Do be careful when you're first working with the rules, but pretty soon you'll be able to just write down the derivative directly:

Example 16.4.6

Find $\frac{d}{dx}(17x^2 - 33x + 12)$.

Solution

Writing out the rules, we'd write

$$\frac{d}{dx}\left(17x^2 - 33x + 12\right) = 17(2x) - 33(1) + 0 = 34x - 33.$$

Once you're familiar with the rules, you can, in your head, multiply the 2 times the 17 and the 33 times 1, and just write

$$\frac{d}{dx}\left(17x^2-33x+12\right)=34x-33.$$

The power rule works even if the power is negative or a fraction. In order to apply it, first translate all roots and basic rational expressions into exponents:

Example 16.4.7

Find the derivative of $y=3\sqrt{t}-\frac{4}{t^4}+5e^t$.

Solution

The first step is translate into exponents:

$$y=3\sqrt{t}-\frac{4}{t^4}+5e^t=3t^{1/2}-4t^{-4}+5e^t$$

Now you can take the derivative:

$$\begin{aligned}\frac{d}{dt}\left(3t^{1/2}-4t^{-4}+5e^t\right)&=3\left(\frac{1}{2}t^{-1/2}\right)-4\left(-4t^{-5}\right)+5\left(e^t\right)\\&=\frac{3}{2}t^{-1/2}+16t^{-5}+5e^t\end{aligned}$$

If there is a reason to, you can rewrite the answer with radicals and positive exponents:

$$y'=\frac{3}{2}t^{-1/2}+16t^{-5}+5e^t=\frac{3}{2\sqrt{t}}+\frac{16}{t^5}+5e^t$$

Be careful when finding the derivatives with negative exponents.

We can immediately apply these rules to solve the problem we started the chapter with - finding a tangent line.

Example 16.4.8

Find the equation of the line tangent to $g(t)=10-t^2$ when $t=2$.

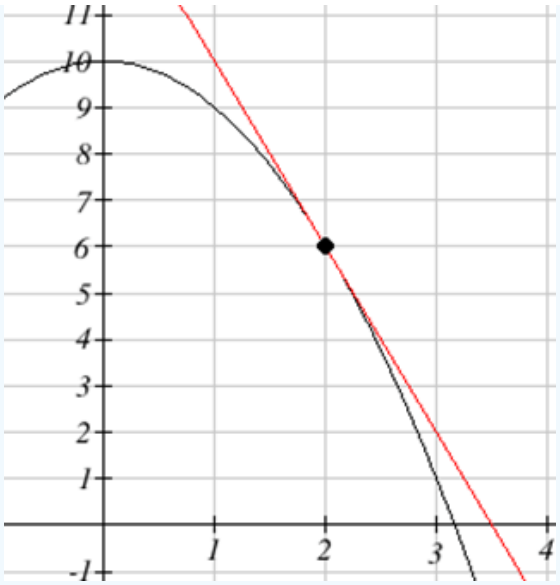
Solution

The slope of the tangent line is the value of the derivative. We can compute $g'(t)=-2t$. To find the slope of the tangent line when $t=2$, evaluate the derivative at that point. The slope of the tangent line is -4.

To find the equation of the tangent line, we also need a point on the tangent line. Since the tangent line touches the original function at $t=2$, we can find the point by evaluating the original function: $g(2)=10-2^2=6$. The tangent line must pass through the point (2, 6).

Using the point-slope equation of a line, the tangent line will have equation $y-6=-4(t-2)$. Simplifying to slope-intercept form, the equation is $y=-4t+14$.

Graphing, we can verify this line is indeed tangent to the curve:



We can also use these rules to help us find the derivatives we need to interpret the behavior of a function.

Example 16.4.9

In a memory experiment, a researcher asks the subject to memorize as many words from a list as possible in 10 seconds. Recall is tested, then the subject is given 10 more seconds to study, and so on. Suppose the number of words remembered after t seconds of studying could be modeled by $W(t) = 4t^{2/5}$. Find and interpret $W'(20)$.

Solution

$W'(t) = 4 \cdot \frac{2}{5}t^{-3/5} = \frac{8}{5}t^{-3/5}$, so $W'(20) = \frac{8}{5}(20)^{-3/5} \approx 0.2652$.

Since W is measured in words, and t is in seconds, W' has units words per second. $W'(20) \approx 0.2652$ means that after 20 seconds of studying, the subject is learning about 0.27 more words for each additional second of studying.

Business and Economics Terms

Next we will delve more deeply into some business applications. To do that, we first need to review some terminology.

Suppose you are producing and selling some item. The profit you make is the amount of money you take in minus what you have to pay to produce the items. Both of these quantities depend on how many you make and sell. (So we have functions here.) Here is a list of definitions for some of the terminology, together with their meaning in algebraic terms and in graphical terms.

Cost

Your **cost** is the money you have to spend to produce your items.

Fixed Cost

The **Fixed Cost** (**FC**) is the amount of money you have to spend regardless of how many items you produce. FC can include things like rent, purchase costs of machinery, and salaries for office staff. You have to pay the fixed costs even if you don't produce anything.

Total Variable Cost

The **Total Variable Cost** (**TVC**) for q items is the amount of money you spend to actually produce them. TVC includes things like the materials you use, the electricity to run the machinery, gasoline for your delivery vans, maybe the wages of your production workers. These costs will vary according to how many items you produce.

Total Cost

The **Total Cost** (**TC**, or sometimes just **C**) for q items is the total cost of producing them. It's the sum of the fixed cost and the total variable cost for producing q items.

Average Cost

The **Average Cost** (**AC**) for q items is the total cost divided by q, or

$$AC(q) = \frac{TC}{q}$$

You can also talk about the average fixed cost, $\frac{FC}{q}$, or the average variable cost, $\frac{TVC}{q}$.

Marginal Cost

The **Marginal Cost** (**MC**) at q items is the cost of producing the next item. Really, it's

$$MC(q) = TC(q+1) - TC(q).$$

In many cases, though, it's easier to approximate this difference using calculus (see Example 1 below). And some sources define the marginal cost directly as the derivative,

$$MC(q) = TC'(q).$$

In this course, we will use both of these definitions as if they were interchangeable.

The units on marginal cost is cost per item.

For the purposes of this course, if a question asks for marginal cost, revenue, profit, etc., compute it using the derivative if possible, unless specifically told otherwise.

Why is it okay that there are two definitions for Marginal Cost (and Marginal Revenue, and Marginal Profit)?

We have been using slopes of secant lines over tiny intervals to approximate derivatives. In this example, we'll turn that around – we'll use the derivative to approximate the slope of the secant line.

Notice that the "cost of the next item" definition is actually the slope of a secant line, over an interval of 1 unit:

$$MC(q) = C(q+1) - 1 = \frac{C(q+1) - 1}{1}.$$

So this is approximately the same as the derivative of the cost function at q:

$$MC(q) = C'(q).$$

In practice, these two numbers are so close that there's no practical reason to make a distinction. For our purposes, the marginal cost **is** the derivative **is** the cost of the next item.

Example 16.4.10

The table shows the total cost (TC) of producing q items.

Items, q	TC
0	$20,000
100	$35,000
200	$45,000
300	$53,000

1. What is the fixed cost?
2. When 200 items are made, what is the total variable cost? The average variable cost?
3. When 200 items are made, estimate the marginal cost.

Solution

1. The fixed cost is $20,000, the cost even when no items are made.
2. When 200 items are made, the total cost is $45,000. Subtracting the fixed cost, the total variable cost is $45,000 - $20,000 = $25,000.

   The average variable cost is the total variable cost divided by the number of items, so we would divide the $25,000 total variable cost by the 200 items made. $25,000/200 = $125. On average, each item had a variable cost of $125.

3. We need to estimate the value of the derivative, or the slope of the tangent line at $q = 200$. Finding the secant line from $q = 100$ to $q = 200$ gives a slope of

$$\frac{45,000 - 35,000}{200 - 100} = 100.$$

   Finding the secant line from $q = 200$ to $q = 300$ gives a slope of

$$\frac{53,000 - 45,000}{300 - 200} = 80.$$

   We could estimate the tangent slope by averaging these secant slopes, giving us an estimate of $90/item.

   This tells us that after 200 items have been made, it will cost about $90 to make one more item.

Example 16.4.11

The cost to produce x items is $C(x) = \sqrt{x}$ hundred dollars.

1. What is the cost for producing 100 items? 101 items? What is cost of the 101st item?
2. Calculate $C'(x)$ and evaluate C' at $x = 100$. How does $C'(100)$ compare with the last answer in Part a?

Solution

1. $C(100) = 10$ hundred dollars = $1000 and $C(101) = 10.0499$ hundred dollars = $1004.99, so it costs $4.99 for that 101st item. Using this definition, the marginal cost is $4.99.
2. $C'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, so $C'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$ hundred dollars = $5.00.

Note how close these answers are! This shows (again) why it's OK that we use both definitions for marginal cost.

Demand

Demand is the functional relationship between the price p and the quantity q that can be sold (that is demanded). Depending on your situation, you might think of p as a function of q, or of q as a function of p

Revenue

Your **revenue** is the amount of money you actually take in from selling your products.

Total Revenue

The **Total Revenue** (**TR**, or just **R**) for q items is the total amount of money you take in for selling q items. Total Revenue is price multiplied by quantity,

$$TR = p \cdot q.$$

Average Revenue

The **Average Revenue** (**AR**) for q items is the total revenue divided by q, or

$$\frac{TR}{q}.$$

Marginal Revenue

The Marginal Revenue (MR) at q items is the revenue from producing the next item,

$$MR(q) = TR(q+1) - TR(q).$$

Just as with marginal cost, we will use both this definition and the derivative definition:

$$MR(q) = TR'(q).$$

Profit

Your **profit** is what's left over from total revenue after costs have been subtracted.

The **Profit** (**P**) for q items is

$$TR(q) - TC(q),$$

the difference between total revenue and total costs.

The **average profit** for q items is

$$\frac{P}{q}.$$

The **marginal profit** at q items is

$$P(q+1) – P(q),$$

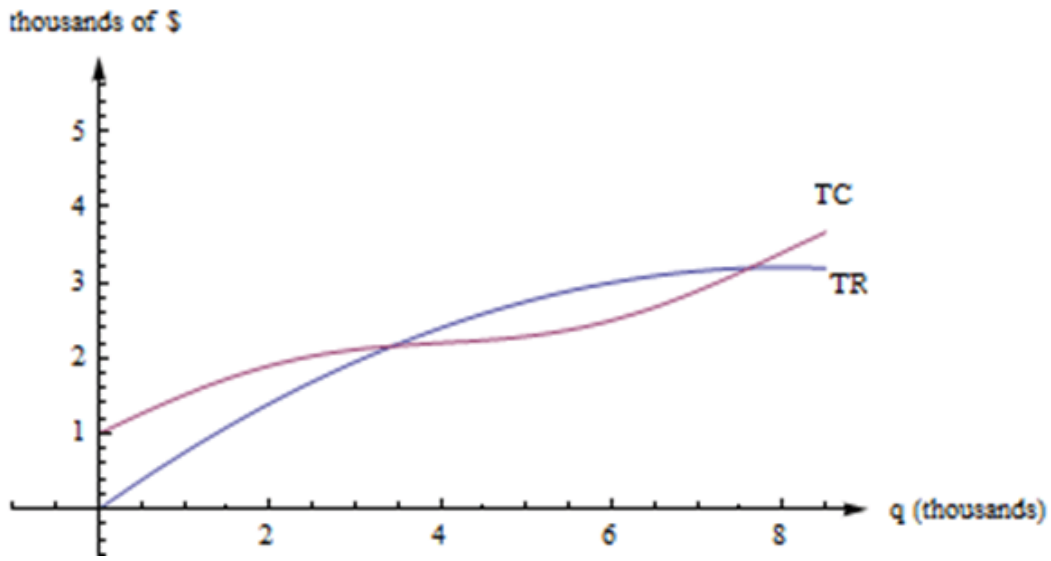
or

$$P'(q)$$

Graphical Interpretations of the Basic Business Math Terms

Illustration

Here are the graphs of TR and TC for producing and selling a certain item. The horizontal axis is the number of items, in thousands. The vertical axis is the number of dollars, also in thousands.



First, notice how to find the fixed cost and variable cost from the graph here. **FC is the y-intercept of the TC graph.** ($FC = TC(0)$.) The graph of TVC would have the same shape as the graph of TC, shifted down. ($TVC = TC - FC$.)

$MC(q) = TC(q+1) - TC(q)$, but that's impossible to read on this graph. How could you distinguish between TC(4022) and TC(4023)? On this graph, that interval is too small to see, and our best guess at the secant line is actually the tangent line to the TC curve at that point. (This is the reason we want to have the derivative definition handy.)

$MC(q)$ is the slope of the tangent line to the TC curve at $(q, TC(q))$.

$MR(q)$ is the slope of the tangent line to the TR curve at $(q, TR(q))$.

Profit is the distance between the TR and TC curve. If you experiment with a clear ruler, you'll see that the biggest profit occurs exactly when the tangent lines to the TR and TC curves are parallel. This is the rule **"profit is maximized when $MR = MC$"** which we'll explore later in the chapter.

Example 16.4.12

The demand, D, for a product at a price of p dollars is given by $D(p) = 200 - 0.2p^2$. Find the marginal revenue when the price is $10.

Solution

First we need to form a revenue equation. Since Revenue = Price$\times$Quantity, and the demand equation shows the quantity of product that can be sold, we have

$$R(p) = D(p) \cdot p = \left(200 - 0.2p^2\right) p = 200p - 0.2p^3.$$

Now we can find marginal revenue by finding the derivative:

$$R'(p) = 200(1) - 0.2(3p^2) = 200 - 0.6p^2$$

At a price of \$10, $R'(10) = 200 - 0.6(10)^2 = 140$.

Notice the units for R' are $\frac{\text{dollars of Revenue}}{\text{dollar of price}}$, so $R'(10) = 140$ means that when the price is \$10, the revenue will increase by \$140 for each dollar that the price was increased.

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16.5: Product and Quotient Rules

The basic rules will let us tackle simple functions. But what happens if we need the derivative of a combination of these functions?

Example 16.5.1

Find the derivative of $h(x) = \left(4x^3 - 11\right)(x + 3)$

This function is not a simple sum or difference of polynomials. It's a product of polynomials. We can simply multiply it out to find its derivative:

$$\begin{aligned} h(x) &= \left(4x^3 - 11\right)(x + 3) \\ &= 4x^4 - 11x + 12x^3 - 33 \\ h'(x) &= 16x^3 - 11 + 36x^2 \end{aligned}$$

Now suppose we wanted to find the derivative of

$$f(x) = \left(4x^5 + x^3 - 1.5x^2 - 11\right)\left(x^7 - 7.25x^5 + 120x + 3\right)$$

This function is not a simple sum or difference of polynomials. It's a product of polynomials. We could 'simply' multiply it out to find its derivative as before – who wants to volunteer? Nobody?

We'll need a rule for finding the derivative of a product so we don't have to multiply everything out.

It would be great if we can just take the derivatives of the factors and multiply them, but unfortunately that won't give the right answer. To see that, consider finding derivative of $h(x) = \left(4x^3 - 11\right)(x + 3)$. We already worked out the derivative, it is $h'(x) = 16x^3 - 11 + 36x^2$. What if we try differentiating the factors and multiplying them? We'd get $h'(x) = \left(12x^2\right)(1) = 12x^2$, which is radically different from the correct answer.

The rules for finding derivatives of products and quotients are a little complicated, but they save us the much more complicated algebra we might face if we were to try to multiply things out. They also let us deal with products where the factors are not polynomials. We can use these rules, together with the basic rules, to find derivatives of many complicated looking functions.

Derivative Rules: Product and Quotient Rules

In what follows, f and g are differentiable functions of x.

Product Rule

$\frac{d}{dx}(f \cdot g) = f' \cdot g + f \cdot g'$

The derivative of the first factor times the second left alone, plus the first left alone times the derivative of the second.

The product rule can extend to a product of several functions; the pattern continues – take the derivative of each factor in turn, multiplied by all the other factors left alone, and add them up:

$$\frac{d}{dx}(f \cdot g \cdot h) = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f' \cdot g - f \cdot g'}{g^2}$$

The numerator of the result resembles the product rule, but there is a minus instead of a plus; the minus sign goes with the g'. The denominator is simply the square of the original denominator – no derivatives there.

Example 16.5.2

Find the derivative of $h(x) = (4x^3 - 11)(x + 3)$

This is the same function we found the derivative of in Example 1, but let's use the product rule and check to see if we get the same answer. For this first example, we will provide a lot more detail and steps than one usually actually shows when working a problem like this.

Notice we can think of $h(x)$ as the product of two functions $f(x) = 4x^3 - 11$ and $g(x) = x + 3$. Finding the derivative of each of these,

$$f'(x) = 12x^2 \text{ and } g'(x) = 1.$$

Using the product rule,

$$\begin{aligned} h'(x) &= (f')(g) + (f)(g') \\ &= \left(12x^2\right)(x + 3) + \left(4x^3 - 11\right)(1) \end{aligned}$$

To check if this is equivalent to the answer we found in Example 1 we could simplify:

$$\begin{aligned} h'(x) &= \left(12x^2\right)(x + 3) + \left(4x^3 - 11\right)(1) \\ &= 12x^3 + 36x^2 + 4x^3 - 11 \\ &= 16x^3 + 36x^2 - 11 \end{aligned}$$

From this, we can see the answers are equivalent.

Example 16.5.3

Find the derivative of $F(t) = e^t \ln(t)$

This is a product, so we need to use the product rule. I like to put down empty parentheses to remind myself of the pattern; that way I don't forget anything:

$$F'(t) = (\ )(\ ) + (\ )(\ )$$

Then I fill in the parentheses – the first set gets the derivative of e^t, the second gets $\ln t$ left alone, the third gets e^t left alone, and the fourth gets the derivative of $\ln t$.

$$F'(t) = \left(e^t\right)(\ln(t)) + \left(e^t\right)\left(\frac{1}{t}\right) = e^t \ln(t) + \frac{e^t}{t}.$$

Notice that this was one we could *not* have done by "multiplying out."

Example 16.5.4

Find the derivative of $y = \frac{x^4+4^x}{3+16x^3}$.

This is a quotient, so we need to use the quotient rule. Again, you find it helpful to put down the empty parentheses as a template:

$$y' = \frac{(\ )(\ ) - (\ )(\ )}{(\ )^2}$$

We can find the derivative of the numerator and denominator separately:

For the numerator: $f(x) = x^4 + 4^x$ so $f'(x) = 4x^3 + \ln(4) \cdot 4^x$

For the denominator: $g(x) = 3 + 16x^3$ so $g'(x) = 48x^2$

Then fill in all the pieces:

$$y' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$y' = \frac{\left(4x^3 + \ln(4) \cdot 4^x\right)\left(3 + 16x^3\right) - \left(x^4 + 4^x\right)\left(48x^2\right)}{\left(3 + 16x^3\right)^2}$$

Now for goodness' sake don't try to simplify that! Remember that "simple" depends on what you will do next; in this case, we were asked to find the derivative, and we've done that. I expect you to do any "basic" simplifications, such as multiplying constants together or doing obvious cancellations or combining of terms, but otherwise please STOP unless there is a reason to simplify further.

Example 16.5.5

Suppose a large tank contains 8 kg of a chemical dissolved in 50 liters of water. If a tap is opened and water is added to the tank at a rate of 5 liters per minute, at what rate is the concentration of chemical in the tank changing after 4 minutes?

First we need to set up a model for the concentration of chemical. The concentration would be measured as kg of chemical per liter of water, kg/L. The number of kg of chemical stays constant at 8 kg, but the quantity of water in the tank is increasing by 5 L/min. The total volume of water in the tank after t minutes is $50 + 5t$, so the concentration after t minutes is

$$c(t) = \frac{8}{50 + 5t}.$$

To find the rate at which the concentration is changing, we need the derivative:

$$\begin{aligned} c'(t) &= \frac{\frac{d}{dt}(8) \cdot (50 + 5t) - (8) \cdot \frac{d}{dt}(50 + 5t)}{(50 + 5t)^2} \\ &= \frac{(0) \cdot (50 + 5t) - (8)(5)}{(50 + 5t)^2} \\ &= -\frac{40}{(50 + 5t)^2} \end{aligned}$$

At $t = 4$,

$$c'(4) = -\frac{40}{(50 + 5(4))^2} \approx -0.00816.$$

Note that the units here are kg per liter, per minute, or $\frac{\text{kg/L}}{\text{min}}$. In other words, this tells us that after 4 minutes, the concentration of chemical is *decreasing* by 0.00816 kg/L each minute.

More Graphical Interpretations of the Basic Business Math Terms

Returning to our discussion of business and economics topics, in addition to total cost and marginal cost, we often also want to talk about average cost or average revenue.

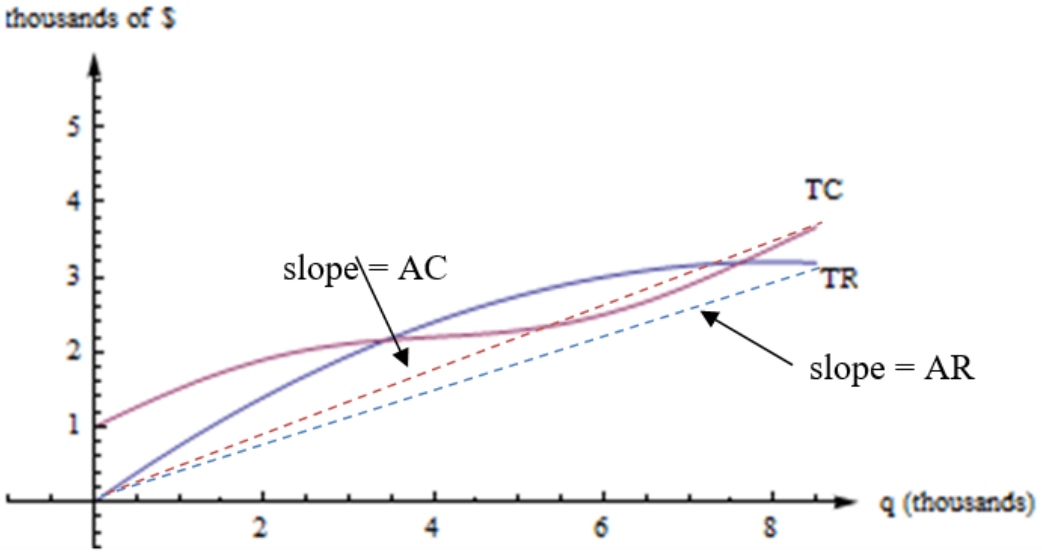
Recall from the previous section that the **Average Cost** (**AC**) for q items is the total cost divided by q, or $AC(q) = \frac{TC}{q}$. You can also talk about the average fixed cost, $\frac{FC}{q}$, or the average variable cost, $\frac{TVC}{q}$.

Also recall that the **Average Revenue** (**AR**) for q items is the total revenue divided by q, or $AR(q) = \frac{TR}{q}$.

We already know that we can find average rates of change by finding slopes of secant lines. AC, AR, MC, and MR are all rates of change, and we can find them with slopes, too.

$AC(q)$ is the slope of a diagonal line, from (0, 0) to $(q, TC(q))$.

$AR(q)$ is the slope of the line from (0, 0) to $(q, TR(q))$.



And just as we found marginal Total Cost, we can also find marginal Average Cost.

Example 16.5.6

The cost, in thousands of dollars, for producing x thousand cellphone cases is given by $C(x) = 22 + x - 0.004x^2$. Find

1. the Fixed Costs,
2. the Average Cost when 5 thousand, 10 thousand, or 20 thousand cases are produced,
3. the Marginal Average Cost when 5 thousand cases are produced.

1. The fixed costs are the costs when no items are produced: $C(0) = 22$ thousand dollars.
2. The average cost function is total cost divided by number of items, so

$$AC(x) = \frac{C(x)}{x} = \frac{22 + x - 0.004x^2}{x}.$$

Note the units are thousands of dollars per thousands of items, which simplifies to just dollars per item.

At a production of 5 thousand items: $AC(5) = \frac{22+5-0.004(5)^2}{5} = 5.38$ dollars per item.

At a production of 10 thousand items: $AC(10) = \frac{22+5-0.004(10)^2}{10} = 3.16$ dollars per item.

At a production of 20 thousand items: $AC(20) = \frac{22+5-0.004(20)^2}{20} = 2.02$ dollars per item.

Notice that while the total cost increases with production, the average cost per item decreases, because the initial fixed costs are being distributed across more items.

3. For the marginal average cost, we need to find the derivative of the average cost function. We can either calculate this using the quotient rule, or we could use algebra to simplify the equation first (this is the easier option – remember, simplifying before differentiating is almost always easier):

$$\begin{aligned} AC(x) &= \frac{22 + x - 0.004x^2}{x} \\ &= \frac{22}{x} + \frac{x}{x} - \frac{0.004x^2}{x} \\ &= \frac{22}{x} + 1 - 0.004x \\ &= 22x^{-1} + 1 - 0.004x \end{aligned}$$

(Note: we haven't differentiated yet, only simplified.)

Taking the derivative,

$$AC'(x) = -22x^{-2} - 0.004 = -\frac{22}{x^2} - 0.004.$$

When 5 thousand items are produced,

$$AC'(5) = -\frac{22}{5^2} - 0.004 = -0.884.$$

Since the units on AC are dollars per item, and the units on x are thousands of items, the units on AC' are dollars per item per thousands of items. This tells us that when 5 thousand items are produced, the average cost per item is *decreasing* by \$0.884 for each additional thousand items produced.

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16.6: Chain Rule

There is one more type of complicated function that we will want to know how to differentiate: composition. The Chain Rule will let us find the derivative of a composition. (This is the last derivative rule we will learn!)

Example 16.6.1

Find the derivative of $y = (4x^3 + 15x)^2$.

Solution

This is not a simple polynomial, so we can't use the basic building block rules yet. It is a product, so we could write it as $y = (4x^3 + 15x)^2 = (4x^3 + 15x)(4x^3 + 15x)$ and use the product rule. Or we could multiply it out and simply differentiate the resulting polynomial. I'll do it the second way:

$$\begin{aligned} y &= (4x^3 + 15x)^2 \\ &= 16x^6 + 120x^4 + 225x^2 \\ y' &= 96x^5 + 480x^3 + 450x \end{aligned}$$

Now suppose we want to find the derivative of $y = (4x^3 + 15x)^{20}$. We **could** write it as a product with 20 factors and use the product rule, or we **could** multiply it out. But I don't want to do that, do you?

We need an easier way, a rule that will handle a composition like this. The Chain Rule is a little complicated, but it saves us the much more complicated algebra of multiplying something like this out. It will also handle compositions where it wouldn't be possible to "multiply it out."

The Chain Rule is a common place for students to make mistakes. Part of the reason is that the notation takes a little getting used to. And part of the reason is that students often forget to use it when they should. When should you use the Chain Rule? Almost every time you take a derivative.

Derivative Rules: Chain Rule

In what follows, f and g are differentiable functions where $y = f(g(x))$. We could alternatively write $y = f(u)$ and $u = g(x)$.

Chain Rule (Leibniz notation)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Notice that the du's seem to cancel. This is one advantage of the Leibniz notation – it can remind you of how the chain rule chains together.

Chain Rule (using prime notation)

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Chain Rule (in words)

The derivative of a composition is the derivative of the outside (with the inside staying the same) TIMES the derivative of the inside.

I recite the version in words each time I take a derivative, especially if the function is complicated.

Example 16.6.2

Find the derivative of $y = (4x^3 + 15x)^2$.

Solution

This is the same one we did before by multiplying out. This time, let's use the Chain Rule: The inside function is what appears inside the parentheses: $4x^3 + 15x$. The outside function is the first thing we find as we come in from the outside – it's the square function, $(\text{inside})^2$.

The derivative of this outside function is $(2 \cdot \text{inside})$. Now using the chain rule, the derivative of our original function is $(2 \cdot \text{inside})$ TIMES the derivative of the inside (which is $12x^2 + 15$):

$$y' = 2\left(4x^3 + 15x\right)\left(12x^2 + 15\right)$$

If you multiply this out, you get the same answer we got before. Hurray! Algebra works!

Example 16.6.3

Find the derivative of $y = (4x^3 + 15x)^{20}$.

Solution

Now we have a way to handle this one. It's the derivative of the outside TIMES the derivative of the inside.

The outside function is $(\text{inside})^{20}$, which has derivative $20\,(\text{inside})^{19}$, so

$$y' = 20\left(4x^3 + 15x\right)^{19}\left(12x^2 + 15\right).$$

Example 16.6.4

Differentiate $y = e^{x^2+5}$.

Solution

This isn't a simple exponential function; it's a composition. Typical calculator or computer syntax can help you see what the "inside" function is here. On a TI calculator, for example, when you push the e^x key, it opens up parentheses: $\boxed{e\text{^}(}$. This tells you that the "inside" of the exponential function is the exponent. Here, the inside is the exponent $x^2 + 5$. Now we can use the Chain Rule: We want the derivative of the outside TIMES the derivative of the inside. The outside is the "e to the something" function, so its derivative is the same thing. The derivative of what's inside is $2x$. So

$$\frac{d}{dx}\left(e^{x^2+5}\right) = \left(e^{x^2+5}\right) \cdot (2x).$$

Example 16.6.5

The table gives values for f, f', g, and g' at a number of points. Use these values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -1$ and 0.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(g \circ f)(x)$
-1	2	3	1	0		
0	-1	1	3	2		
1	1	0	-1	3		

2	3	-1	0	1		
3	0	2	2	-1		

Solution

$$(f \circ g)(-1) = f\left(g(-1)\right) = f(3) = 0$$
$$(f \circ g)(0) = f\left(g(0)\right) = f(1) = 1$$
$$(f \circ g)'(-1) = f'\left(g(-1)\right) \cdot g'(-1) = f'(3) \cdot (0) = (2)(0) = 0 \text{ and}$$
$$(f \circ g)'(0) = f'\left(g(0)\right) \cdot g'(0) = f'(1) \cdot (2) = (-1)(2) = -2$$

Example 16.6.6

If 2400 people now have a disease, and the number of people with the disease appears to double every 3 years, then the number of people expected to have the disease in t years is $y = 2400 \cdot 2^{t/3}$.

1. How many people are expected to have the disease in 2 years?
2. When are 50,000 people expected to have the disease?
3. How fast is the number of people with the disease expected to grow now and 2 years from now?

1. In 2 years, $y = 2400 \cdot 2^{2/3} \approx 3,810$ people.
2. We know $y = 50,000$, and we need to solve $50,000 = 2400 \cdot 2^{t/3}$ for t. We could start by isolating the exponential by dividing both sides by 2400,
$$\frac{50000}{2400} = 2^{t/3}$$
$$\ln\left(\frac{50000}{2400}\right) = \ln\left(2^{t/3}\right) \qquad \text{(Taking the natural log of both sides.)}$$
$$\ln\left(\frac{50000}{2400}\right) = \frac{t}{3}\ln(2) \qquad \text{(Using the exponent property for logs.)}$$
$$t = \frac{3\ln\left(\frac{50000}{2400}\right)}{\ln(2)} \approx 13.14 \text{ years} \qquad \text{(Solving for } t\text{.)}$$
We expect 50,000 people to have the disease about 13.14 years from now.
3. This is asking for $\frac{dy}{dt}$ when $t = 0$ and 2 years. Using the chain rule,
$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}\left(2400 \cdot 2^{t/3}\right) \\ &= 2400 \cdot 2^{t/3} \cdot \ln(2) \cdot \frac{1}{3} \\ &\approx 554.5 \cdot 2^{t/3}\end{aligned}$$
So, at $t = 0$ the rate of growth of the disease is approximately $554.5 \cdot 2^0 \approx 554.5$ people/year. In 2 years the rate of growth will be approximately $554.5 \cdot 2^{2/3} \approx 880$ people/year.

Derivatives of Complicated Functions

You're now ready to take the derivative of some mighty complicated functions. But how do you tell what rule applies first? Work your way in from the outside – what do you encounter first? That's the first rule you need. Use the Product, Quotient, and Chain Rules to peel off the layers, one at a time, until you're all the way inside.

Example 16.6.7

Find $\frac{d}{dx}\left(e^{3x} \cdot \ln(5x+7)\right)$.

Solution

Coming in from the outside, we see that this is a product of two (complicated) functions. So we'll need the Product Rule first. we'll fill in the pieces we know, and then we can figure the rest as separate steps and substitute in at the end:

$$\frac{d}{dx}\left(e^{3x}\cdot\ln(5x+7)\right)=\left(\frac{d}{dx}\left(e^{3x}\right)\right)\cdot\ln(5x+7)+e^{3x}\cdot\left(\frac{d}{dx}\left(\ln(5x+7)\right)\right)$$

Now as separate steps, we'll find

$$\frac{d}{dx}\left(e^{3x}\right)=3e^{3x}\quad\text{(using the Chain Rule)}$$

and

$$\frac{d}{dx}\left(\ln(5x+7)\right)=\frac{1}{5x+7}\cdot 5\quad\text{(also using the Chain Rule).}$$

Finally, to substitute these in their places:

$$\frac{d}{dx}\left(e^{3x}\cdot\ln(5x+7)\right)=\left(3e^{3x}\right)\cdot\ln(5x+7)+e^{3x}\cdot\left(\frac{1}{5x+7}\cdot 5\right)$$

(We can stop here – we don't need to try to simplify any further.)

Example 16.6.8

Differentiate $z=\left(\dfrac{3t^3}{e^t(t-1)}\right)^4$.

Solution

Don't panic! As we come in from the outside, what's the first thing we encounter? It's that fourth power. That tells us that this is a composition, a (complicated) function raised to the fourth power.

Step One: Use the Chain Rule. The derivative of the outside TIMES the derivative of the inside:

$$\frac{dz}{dt}=\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)^4=4\left(\frac{3t^3}{e^t(t-1)}\right)^3\cdot\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)$$

Now we're one step inside, and we can concentrate on just the $\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)$ part. Now, as you come in from the outside, the first thing you encounter is a quotient – this is the quotient of two (complicated) functions.

Step Two: Use the Quotient Rule. The derivative of the numerator is straightforward, so we can just calculate it. The derivative of the denominator is a bit trickier, so we'll leave it for now:

$$\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)=\frac{\left(9t^2\right)\left(e^t(t-1)\right)-\left(3t^3\right)\left(\frac{d}{dt}\left(e^t(t-1)\right)\right)}{\left(e^t(t-1)\right)^2}$$

Now we've gone one more step inside, and we can concentrate on just the $\frac{d}{dt}\left(e^t(t-1)\right)$ part, which involves a product.

Step Three: Use the Product Rule:

$$\frac{d}{dt}\left(e^t(t-1)\right)=\left(e^t\right)(t-1)+\left(e^t\right)(1)$$

And now we're all the way in – no more derivatives to take!

Step Four: Now it's just a question of substituting back – be careful now!

$$\frac{d}{dt}\left(e^t(t-1)\right)=\left(e^t\right)(t-1)+\left(e^t\right)(1)$$

so

$$\frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right) = \frac{\left(9t^2\right)\left(e^t(t-1)\right) - \left(3t^3\right)\left(\left(e^t\right)(t-1) + \left(e^t\right)(1)\right)}{\left(e^t(t-1)\right)^2}$$

so

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{3t^3}{e^t(t-1)}\right)^4 = 4\left(\frac{3t^3}{e^t(t-1)}\right)^3 \cdot \left(\frac{\left(9t^2\right)\left(e^t(t-1)\right) - \left(3t^3\right)\left(\left(e^t\right)(t-1) + \left(e^t\right)(1)\right)}{\left(e^t(t-1)\right)^2}\right)$$

Phew!

What if the Derivative Doesn't Exist?

Differentiable

A function is called **differentiable** at a point if its derivative exists at that point.

We've been acting as if derivatives exist everywhere for every function. This is true for most of the functions that you will run into in this class. But there are some common places where the derivative doesn't exist.

Remember that the derivative is the slope of the tangent line to the curve. That's what to think about.

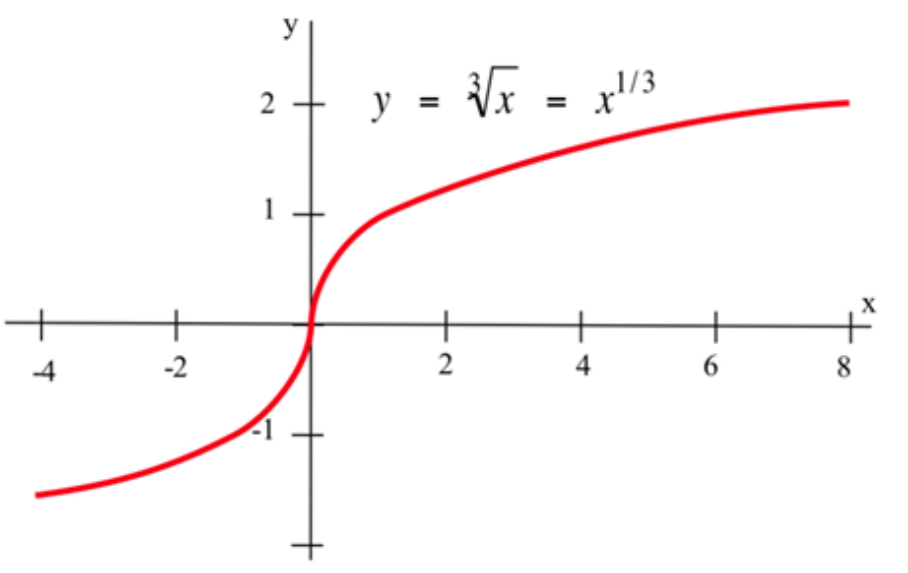
Where can a slope not exist? **If the tangent line is vertical, the derivative will not exist.**

Example 16.6.9

Show that $f(x) = \sqrt[3]{x} = x^{1/3}$ is not differentiable at $x = 0$.

Solution

Finding the derivative, $f(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$. At $x = 0$, this function is undefined. From the graph, we can see that the tangent line to this curve at $x = 0$ is vertical with undefined slope, which is why the derivative does not exist at $x = 0$.



Where can a tangent line not exist?

If there is a sharp corner (cusp) in the graph, the derivative will not exist at that point because there is no well-defined tangent line (a teetering tangent, if you will).

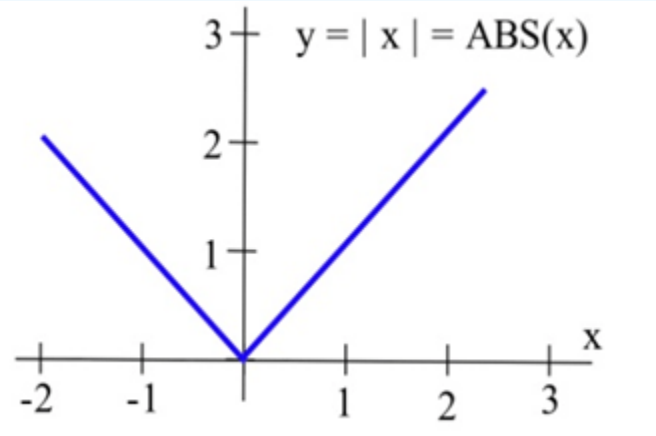
If there is a discontinuity in the graph (a jump, a break, a hole in the graph, or a vertical asymptote), the tangent line will be different on either side and **the derivative will not exist at that point**.

Example 16.6.10

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Solution

On the left side of the graph, the slope of the line is -1. On the right side of the graph, the slope is +1. There is no well-defined tangent line at the sharp corner at $x = 0$, so the function is not differentiable at that point.



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16.7: Second Derivative and Concavity

Second Derivative and Concavity

Graphically, a function is **concave up** if its graph is curved with the opening upward (Figure $16.7.1a$). Similarly, a function is **concave down** if its graph opens downward (Figure $16.7.1b$).

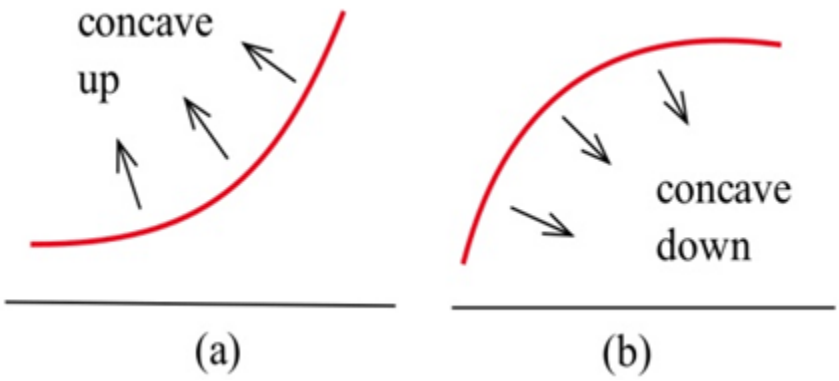
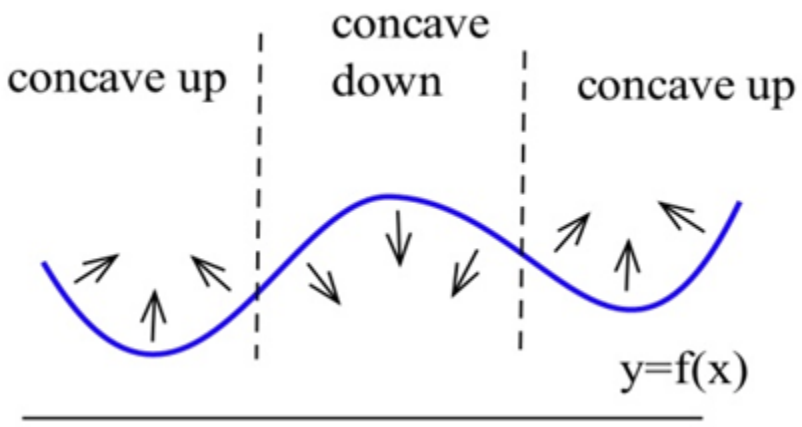


Figure $16.7.1$

This figure shows the concavity of a function at several points. Notice that a function can be concave up regardless of whether it is increasing or decreasing.



For example, **an epidemic**: Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In Figure $16.7.2$ below, $f(x)$ is the number of people who have the disease at time x, and two different situations are shown. In both Figure $16.7.2a$ and Figure $16.7.2b$, the number of people with the disease, $f(\text{now})$, and the rate at which new people are getting sick, $f'(\text{now})$, are the same. The difference in the two situations is the concavity of f, and that difference in concavity might have a big effect on your decision.

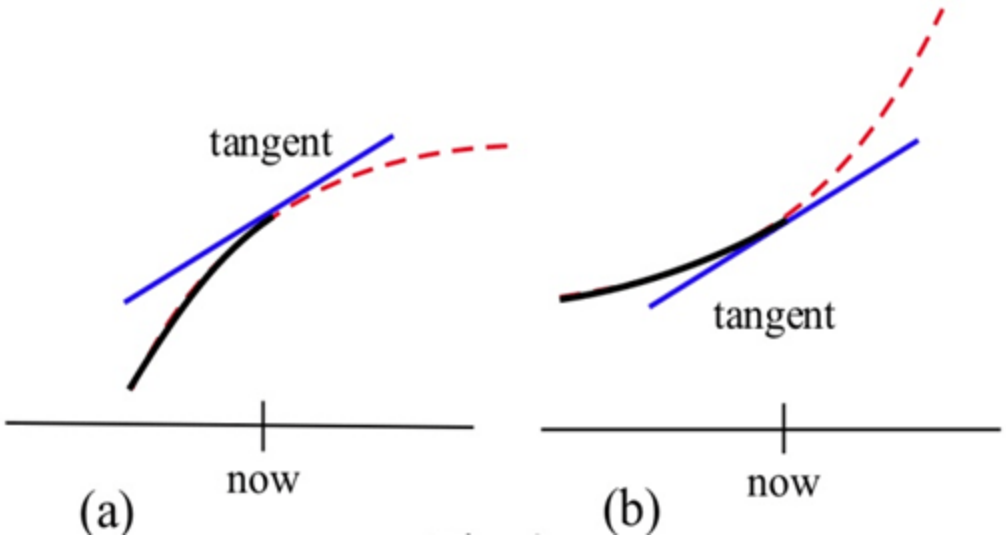


Figure 16.7.2

In Figure $16.7.2a$, f is concave down at "now", the slopes are decreasing, and it looks as if it is tailing off. We can say "f is increasing at a decreasing rate." It appears that the current methods are starting to bring the epidemic under control.

In Figure $16.7.2b$, f is concave up, the slopes are increasing, and it looks as if it will keep increasing faster and faster. It appears that the epidemic is still out of control.

The differences between the graphs come from whether the *derivative* is increasing or decreasing

The derivative of a function f is a function that gives information about the slope of f. **The derivative tells us if the original function is increasing or decreasing**.

Because f' is a function, we can take its derivative. This second derivative also gives us information about our original function f. The second derivative gives us a mathematical way to tell how the graph of a function is curved. **The second derivative tells us if the original function is concave up or down**.

Second Derivative

Let $y = f(x)$. The **second derivative of** f is the derivative of $y' = f'(x)$.

Using prime notation, this is $f''(x)$ or y''. You can read this aloud as "f double prime of x" or "y double prime."

Using Leibniz notation, the second derivative is written $\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$. This is read aloud as "the second derivative of y (or f)."

Derivatives and the Graph of a Function

The first derivative tells us if a function is increasing or decreasing

If $f'(x)$ is positive on an interval, the graph of $y = f(x)$ is **increasing** on that interval.

If $f'(x)$ is negative on an interval, the graph of $y = f(x)$ is **decreasing** on that interval.

The second derivative tells us if a function is concave up or concave down

If $f''(x)$ is positive on an interval, the graph of $y = f(x)$ is **concave up** on that interval. We can say that f is increasing (or decreasing) **at an increasing rate**.

If $f''(x)$ is negative on an interval, the graph of $y = f(x)$ is **concave down** on that interval. We can say that f is increasing (or decreasing) **at a decreasing rate**.

Example 16.7.1

Find $f''(x)$ for $f(x) = 3x^7$.

First, we need to find the first derivative:

$$f'(x) = 21x^6.$$

Then we take the derivative of that function:

$$f''(x) = \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}\left(21x^6\right) = 126x^5.$$

If $f(x)$ represents the position of a particle at time x, then $v(x) = f'(x)$ will represent the velocity (rate of change of the position) of the particle and $a(x) = v'(x) = f''(x)$ will represent the acceleration (the rate of change of the velocity) of the particle.

You are probably familiar with acceleration from driving or riding in a car. The speedometer tells you your velocity (speed). When you leave from a stop and press down on the accelerator, you are accelerating – increasing your speed.

Example 16.7.2

The height (feet) of a particle at time t seconds is $f(t) = t^3–4t^2 + 8t$. Find the height, velocity and acceleration of the particle when $t = 0$, 1, and 2 seconds.

$f(t) = t^3–4t^2 + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet, and $f(2) = 8$ feet.

The velocity is $v(t) = f'(t) = 3t^2–8t + 8$ so $v(0) = 8$ ft/s, $v(1) = 3$ ft/s, and $v(2) = 4$ ft/s. At each of these times the velocity is positive and the particle is moving upward, increasing in height.

The acceleration is $a(t) = f''(t) = 6t–8$ so $a(0) = –8$ ft/s^2, $a(1) = –2$ ft/s^2 and $a(2) = 4$ ft/s^2.

At time 0 and 1, the acceleration is negative, so the particle's velocity would be decreasing at those points - the particle was slowing down. At time 2, the velocity is positive, so the particle was increasing in speed.

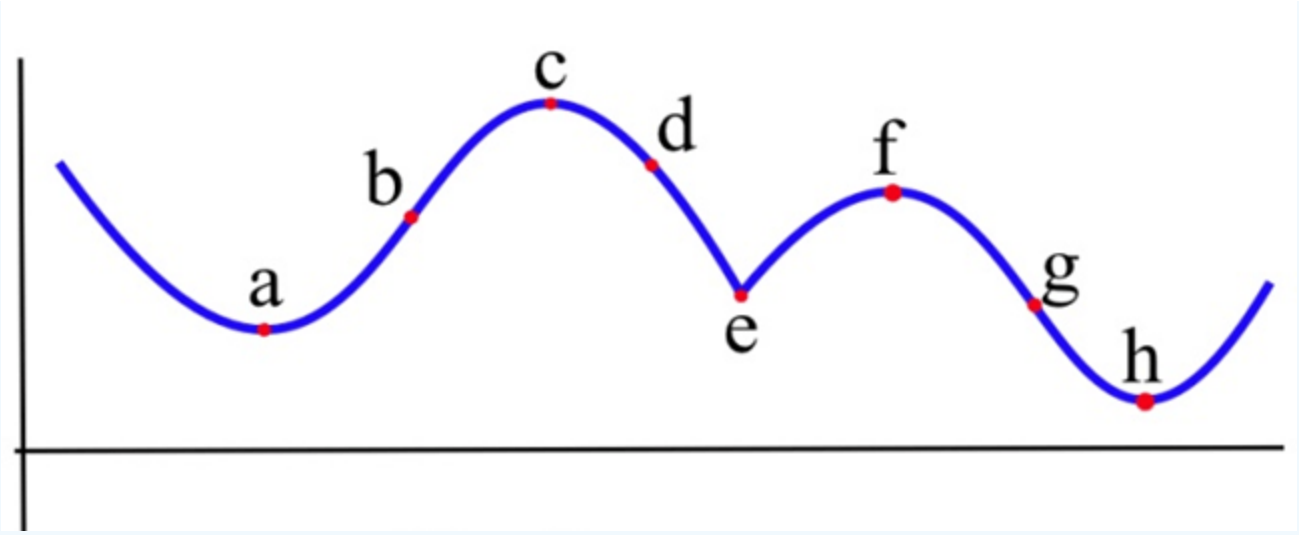
Inflection Points

Inflection Point

An **inflection point** is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

Example 16.7.3

Which of the labeled points in the graph below are inflection points?



The concavity changes at points b and g. At points a and h, the graph is concave up on both sides, so the concavity does not change. At points c and f, the graph is concave down on both sides. At point e, even though the graph looks strange there, the graph is concave down on both sides – the concavity does not change.

Inflection points happen when the concavity changes. Because we know the connection between the concavity of a function and the sign of its second derivative, we can use this to find inflection points.

Working Definition: Inflection Point

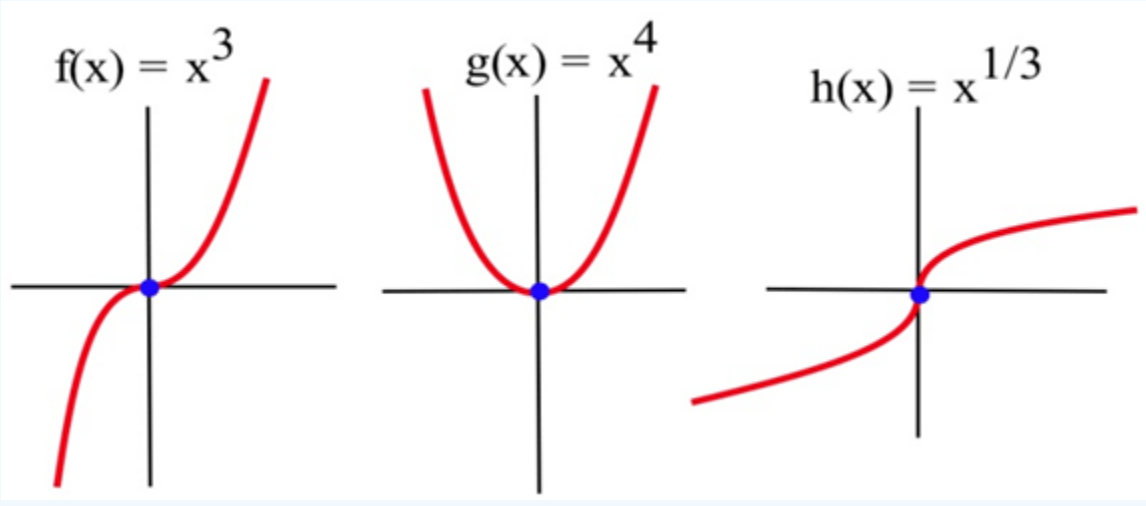
An **inflection point** is a point on the graph where the second derivative changes sign.

In order for the second derivative to change signs, it must either be zero or be undefined. So to find the inflection points of a function we only need to check the points where $f''(x)$ is 0 or undefined.

Note that it is not enough for the second derivative to be zero or undefined. We still need to check that the sign of f'' changes sign. The functions in the next example illustrate what can happen.

Example 16.7.4

Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = x^{1/3}$. For which of these functions is the point (0,0) an inflection point?



Graphically, it is clear that the concavity of $f(x) = x^3$ and $h(x) = x^{1/3}$ changes at (0,0), so (0,0) is an inflection point for f and h. The function $g(x) = x^4$ is concave up everywhere so (0,0) is not an inflection point of g.

We can also compute the second derivatives and check the sign change.

If $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$. The only point at which $f''(x) = 0$ or is undefined (f' is not differentiable) is at $x = 0$. If $x < 0$, then $f''(x) < 0$ so f is concave down. If $x > 0$, then $f''(x) > 0$ so f is concave up. At $x = 0$ the concavity changes so the point $(0, f(0)) = (0, 0)$ is an inflection point of $f(x) = x^3$.

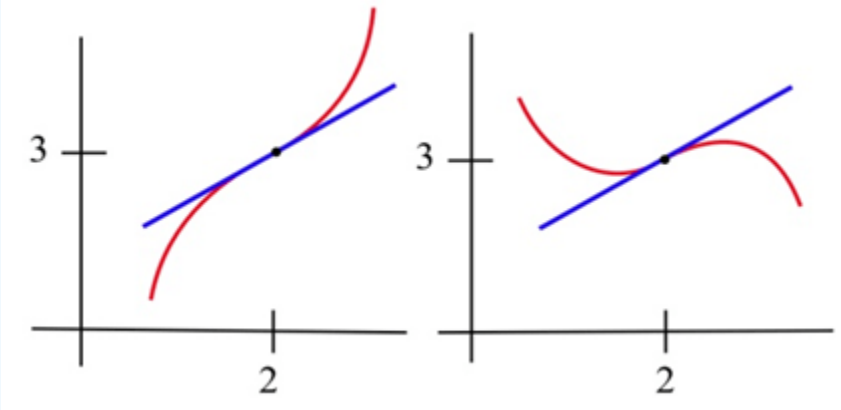
If $g(x) = x^4$, then $g'(x) = 4x^3$ and $g''(x) = 12x^2$. The only point at which $g''(x) = 0$ or is undefined is at $x = 0$. If $x < 0$, then $g''(x) > 0$ so g is concave up. If $x > 0$, then $g''(x) > 0$ so g is also concave up. At $x = 0$ the concavity **does not change** so the point $(0, g(0)) = (0, 0)$ is **not an inflection point** of $g(x) = x^4$. Keep this example in mind!

If $h(x) = x^{1/3}$, then $h'(x) = \frac{1}{3}x^{-2/3}$ and $h''(x) = -\frac{2}{9}x^{-5/3}$. h'' is not defined if $x = 0$, but $h''(\text{negative number}) > 0$ and $h''(\text{positive number}) < 0$ so h changes concavity at (0,0) and (0,0) is an inflection point of h.

Example 16.7.5

Sketch the graph of a function with $f(2) = 3$, $f'(2) = 1$, and an inflection point at (2,3).

Two possible solutions are shown here.



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16.8: Optimization

In theory and applications, we often want to maximize or minimize some quantity. An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance. A manufacturer may want to maximize profits and market share or minimize waste. A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

Without calculus, we only know how to find the optimum points in a few specific examples (for example, we know how to find the vertex of a parabola). But what if we need to optimize an unfamiliar function?

The best way we have without calculus is to examine the graph of the function, perhaps using technology. But our view depends on the viewing window we choose – we might miss something important. In addition, we'll probably only get an approximation this way. (In some cases, that will be good enough.)

Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums, while at the same time ensuring that we haven't missed anything important.

Local Maxima and Minima

Before we examine how calculus can help us find maximums and minimums, we need to define the concepts we will develop and use.

Definition: Local Maxima and Minima

$f(x)$ has a **local maximum** at $x = a$ if $f(a) \geq f(x)$ for all x near a.

$f(x)$ has a **local minimum** at $x = a$ if $f(a) \leq f(x)$ for all x near a.

$f(x)$ has a **local extreme** at $x = a$ if $f(a)$ is a **local maximum or minimum**.

The plurals of these are maxima and minima. We often simply say "max" or "min;" it saves a lot of syllables.

Some books say "relative" instead of "local."

The process of finding maxima or minima is called **optimization**.

A point is a local max (or min) if it is higher (lower) than all the **nearby points**. These points come from the shape of the graph.

Definition: Global Maxima and Minima

$f(x)$ has a **global maximum** at $x = a$ if $f(a) \geq f(x)$ for all x in the domain of $f(x)$.

$f(x)$ has a **global minimum** at $x = a$ if $f(a) \leq f(x)$ for all x in the domain of $f(x)$.

$f(x)$ has a **global extreme** at $x = a$ if $f(a)$ is a **global maximum or minimum**.

Some books say "absolute" instead of "global."

A point is a global max (or min) if it is higher (lower) than every point on the graph. These points come from the shape of the graph **and** the window through which we view the graph.

The local and global extremes of the function in Figure 1 are labeled. You should notice that every global extreme is also a local extreme, but there are local extremes that are not global extremes.

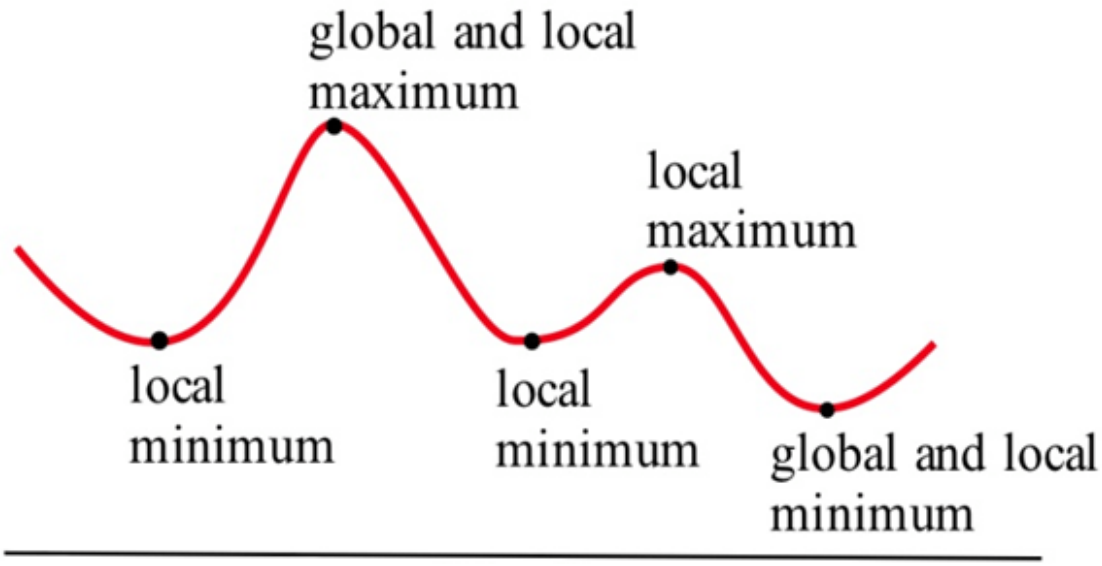


Figure 1

If $h(x)$ is the height of the earth above sea level at the location x, then the global maximum of h is h(summit of Mt. Everest) = 29,028 feet. The local maximum of h for the United States is h(summit of Mt. McKinley) = 20,320 feet. The local minimum of h for the United States is h(Death Valley) = -282 feet.

Example 16.8.1

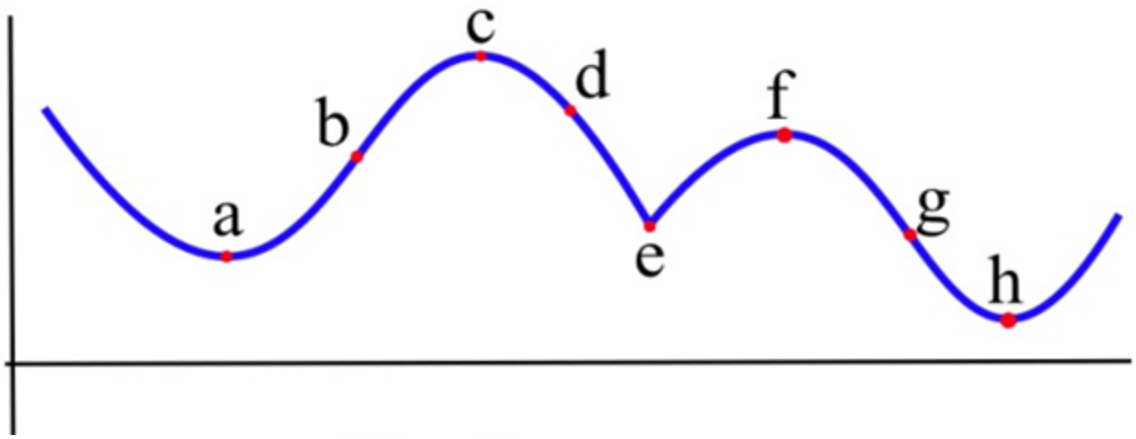
The table shows the annual calculus enrollments at a large university. Which years had local maximum or minimum calculus enrollments? What were the global maximum and minimum enrollments in calculus?

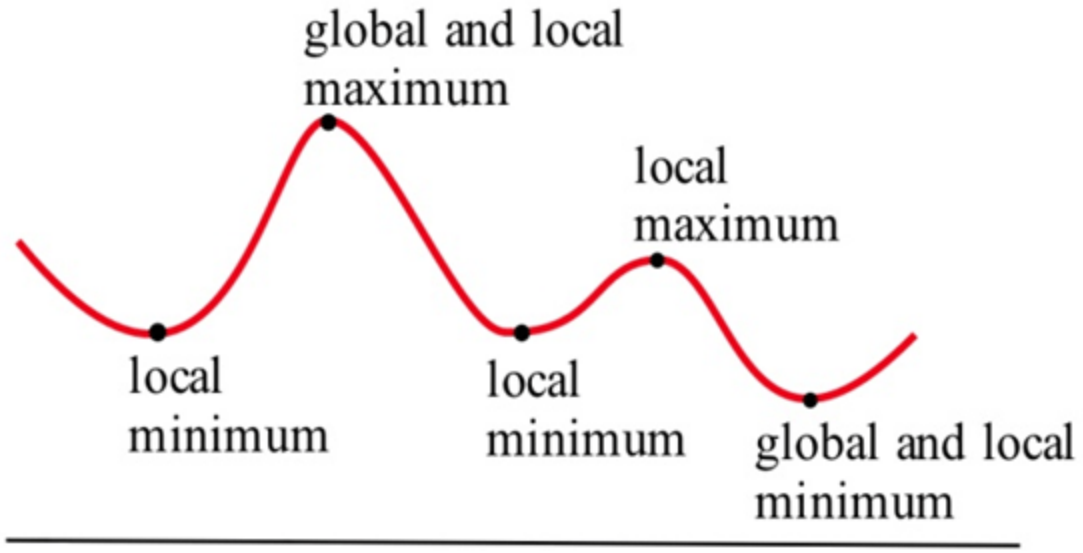
Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Enrollment	1257	1324	1378	1336	1389	1450	1523	1582	1567	1545	1571

There were local maxima in 2002 and 2007; the global maximum was 1582 students in 2007. There were local minima in 2003 and 2009; the global minimum was 1257 students in 2000. We choose not to think of 2000 as a local minimum or 2010 as a local maximum; however, some books would include the endpoints. We are allowed to have a global maximum or global minimum at an endpoint.

Finding Maxima and Minima of a Function

What must the tangent line look like at a local max or min? Look at these two graphs again – you'll see that at all the extreme points, the tangent line is horizontal (so $f' = 0$). There is one cusp in the blue graph – the tangent line is vertical there (so f' is undefined).





That gives us the clue how to find extreme values.

Critical number

A **critical number** for a function f is a value $x = a$ in the domain of f where either $f'(a) = 0$ **or** $f'(a)$ **is undefined.**

Critical point

A **critical point** for a function f is a point $(a, f(a))$ where a is a critical number of f.

A local max or min of f can only occur at a critical point.

Example 16.8.2

Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$.

A critical number of f can occur only where $f'(x) = 0$ or where f' does not exist.

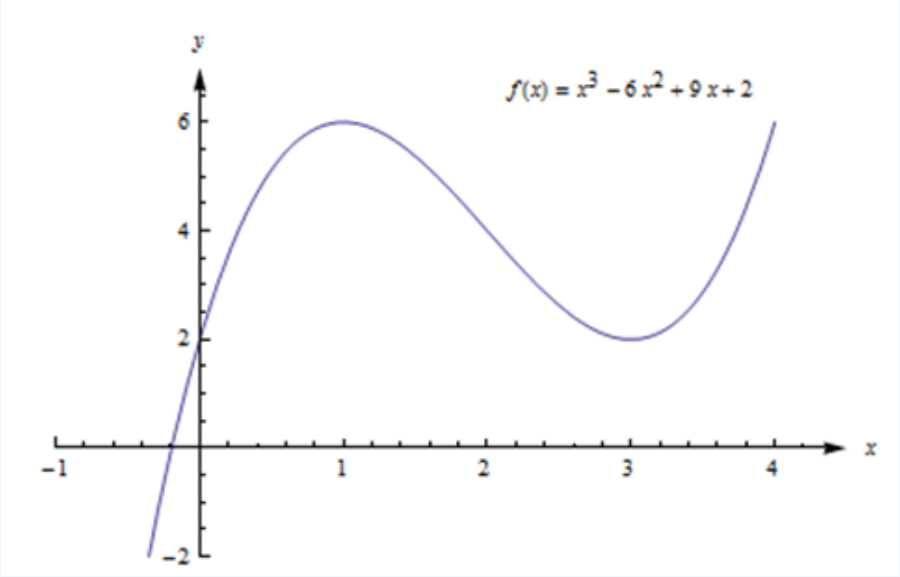
$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$$

So $f'(x) = 0$ at $x = 1$ and $x = 3$ (and no other values of x). There are no places where f' is undefined.

The critical numbers are $x = 1$ and $x = 3$. So the critical points are (1, 6) and (3, 2).

These are the only possible locations of local extremes of f. We haven't discussed yet how to tell whether either of these points is actually a local extreme of f, or which kind it might be. But we can be certain that no other point is a local extreme.

The graph of f below shows that $(1, f(1)) = (1, 6)$ is a local maximum and $(3, f(3)) = (3, 2)$ is a local minimum. This function does not have a global maximum or minimum.



Example 16.8.3

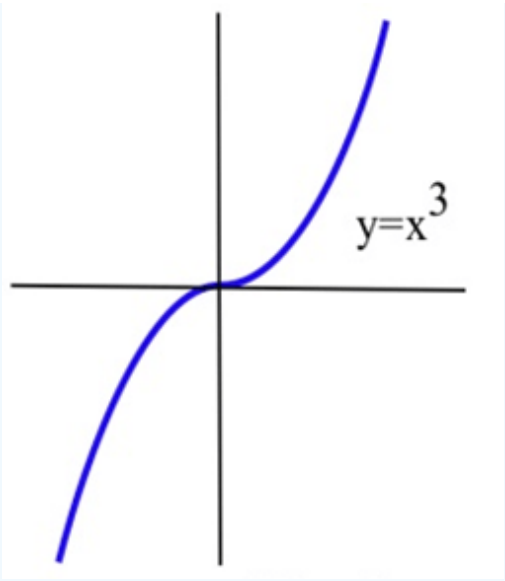
Find all local extremes of $f(x) = x^3$.

$f(x) = x^3$ is differentiable for all x, and $f'(x) = 3x^2$. The only place where $f'(x) = 0$ is at $x = 0$, so the only candidate is the critical point (0,0).

If $x > 0$ then $f(x) = x^3 > 0 = f(0)$, so $f(0)$ is not a local maximum.

Similarly, if $x < 0$ then $f(x) = x^3 < 0 = f(0)$ so $f(0)$ is not a local minimum.

The critical point (0,0) is the only candidate to be a local extreme of f, and, based on the graph, this candidate did not turn out to be a local extreme of f. The function $f(x) = x^3$ does not have any local extremes.



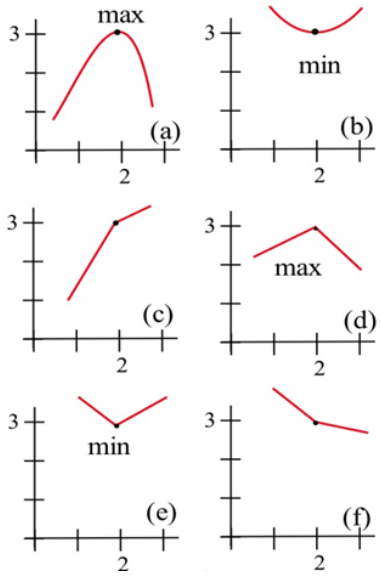
Remember this example! It is not enough to find the critical points – we can only say that f *might* have a local extreme at the critical points.

First and Second Derivative Tests

Is that critical point a Maximum or Minimum (or Neither)?

Once we have found the critical points of f, we still have the problem of determining whether these points are maxima, minima or neither.

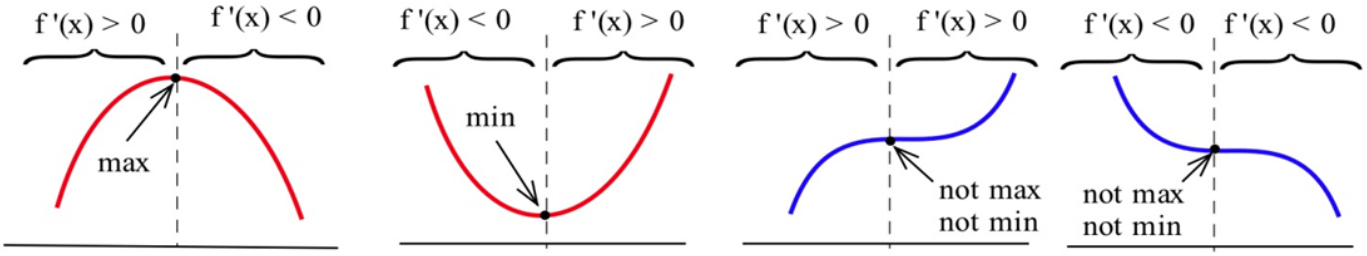
All of the graphs in the figure below have a critical point at (2, 3). It is clear from the graphs that the point (2,3) is a local maximum in (a) and (d), (2,3) is a local minimum in (b) and (e), and (2,3) is not a local extreme in (c) and (f).



The critical numbers only give the *possible* locations of extremes, and some critical numbers are not the locations of extremes. The critical numbers are the *candidates* for the locations of maxima and minima.

f' and Extreme Values of f

Four possible shapes of graphs are shown here – in each graph, the point marked by an arrow is a critical point, where $f'(x) = 0$. What happens to the derivative near the critical point?



At a local max, such as in the graph on the left, the function increases on the left of the local max, then decreases on the right. The derivative is first positive, then negative at a local max. At a local min, the function decreases to the left and increases to the right, so the derivative is first negative, then positive. When there isn't a local extreme, the function continues to increase (or decrease) right past the critical point – the derivative doesn't change sign.

The First Derivative Test for Extremes

Find the critical points of f.

For each critical number c, examine the sign of f' to the left and to the right of c. What happens to the sign as you move from left to right?

- If $f'(x)$ **changes from positive to negative** at $x = c$, then f has a **local maximum** at $(c, f(c))$.
- If $f'(x)$ **changes from negative to positive** at $x = c$, then f has a **local minimum** at $(c, f(c))$.
- If $f'(x)$ **does not change sign** at $x = c$, then $(c, f(c))$ is **neither** a local maximum nor a local minimum.

Example 16.8.4

Find the critical points of $f(x) = x^3 - 6x^2 + 9x + 2$ and classify them as local max, local min, or neither.

We already found the critical points; they are (1, 6) and (3, 2).

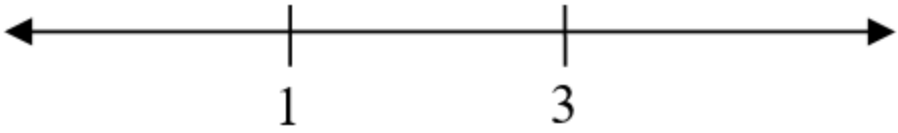
Now we can use the first derivative test to classify each. Recall that $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$. The factored form is easiest to work with here, so let's use that.

At (1, 6) we could choose a number slightly less than 1 to plug into the formula for f' – perhaps use $x = 0$, or $x = 0.9$. Then we could examine its sign. But we don't care about the numerical value, all we are interested in is its sign. And for that, we don't have to do any plugging in:

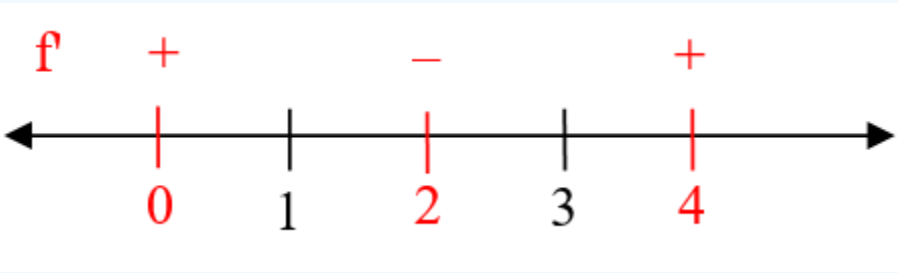
- If x is a little less than 1, then $x - 1$ is negative, and $x - 3$ is negative. So $f' = 3(x - 1)(x - 3)$ will be pos(neg)(neg) = positive.
- For x a little more than 1, we can evaluate f' at a number more than 1 (but less than 3, we don't want to go past the next critical point!) – perhaps $x = 2$. Or we can make a quick sign argument like what we did above: for x a little more than 1, $f' = 3(x - 1)(x - 3)$ will be pos(pos)(neg) = negative.

So f' changes from positive to negative, which means there is a local max at (1, 6).

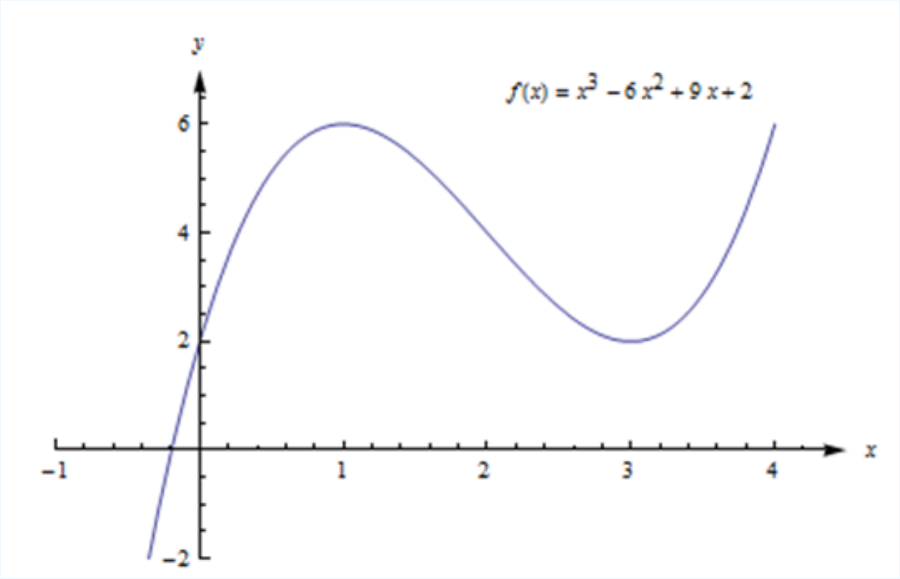
As another approach, we could draw a number line, and mark the critical numbers:



We already know the derivative is zero or undefined at the critical numbers. On each interval between these values, the derivative will stay the same sign. To determine the sign, we could pick a test value in each interval, and evaluate the derivative at those points (or use the sign approach used above).

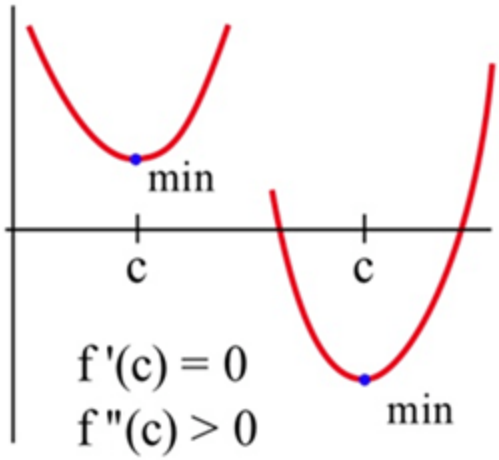


At (3, 2) f' changes from negative to positive, so there is a local min at (3, 2). This confirms what we saw before in the graph.



f'' and Extreme Values of f

The concavity of a function can also help us determine whether a critical point is a maximum or minimum or neither. For example, if a point is at the bottom of a concave up function, then the point is a minimum.



The Second Derivative Test for Extremes

Find all critical points of f. For those critical points where $f'(c) = 0$, find $f''(c)$.

- If $f''(c) < 0$ (negative) then f is concave down and has a local maximum at $x = c$.
- If $f''(c) > 0$ (positive) then f is concave up and has a local minimum at $x = c$.
- If $f''(c) = 0$ then f may have a local maximum, a minimum or neither at $x = c$.

The cartoon faces can help you remember the Second Derivative Test.





Example 16.8.5

$f(x) = 2x^3 - 15x^2 + 24x - 7$ has critical numbers $x = 1$ and 4. Use the Second Derivative Test for Extremes to determine whether $f(1)$ and $f(4)$ are maximums or minimums or neither.

We need to find the second derivative:

$$\begin{aligned} f(x) &= 2x^3 - 15x^2 + 24x - 7 \\ f'(x) &= 6x^2 - 30x + 24 \\ f''(x) &= 12x - 30 \end{aligned}$$

Then we just need to evaluate f'' at each critical number:

$x = 1$: $f''(1) = 12(1) - 30 < 0$, so there is a local maximum at $x = 1$.

$x = 4$: $f''(4) = 12(4) - 30 > 0$, so there is a local minimum at $x = 4$.

Many students like the Second Derivative Test. The Second Derivative Test is often easier to use than the First Derivative Test. You only have to find the sign of one number for each critical number rather than two. And if your function is a polynomial, its second derivative will probably be a simpler function than the derivative.

However, if you needed a product rule, quotient rule, or chain rule to find the first derivative, finding the second derivative can be a lot of work. Also, even if the second derivative is easy, the Second Derivative Test doesn't always give an answer. The First Derivative Test will always give you an answer.

Use whichever test you want to. But remember – you have to do some test to be sure that your critical point actually is a local max or min.

Global Maxima and Minima

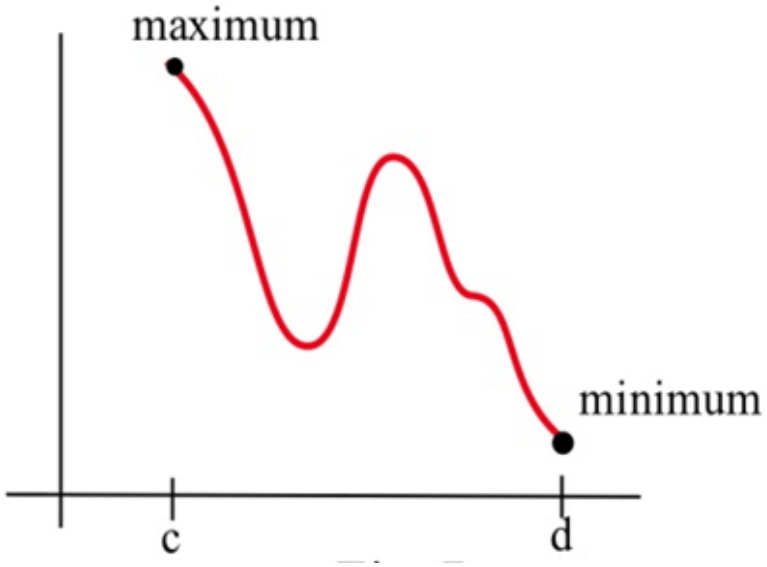
In applications, we often want to find the global extreme; knowing that a critical point is a local extreme is not enough.

For example, if we want to make the greatest profit, we want to make the absolutely greatest profit of all. How do we find global max and min?

There are just a few additional things to think about.

Endpoint Extremes

The local extremes of a function occur at critical points – these are points in the function that we can find by thinking about the shape (and using the derivative to help us). But if we're looking at a function on a closed interval, the endpoints could be extremes. These endpoint extremes are not related to the shape of the function; they have to do with the interval, the window through which we're viewing the function.



In the graph above, it appears that there are three critical points – one local min, one local max, and one that is neither one. But the global max, the highest point of all, is at the left endpoint. The global min, the lowest point of all, is at the right endpoint.

How do we decide if endpoints are global max or min? It's easier than you expected – simply plug in the endpoints, along with all the critical numbers, and compare y-values.

To find Global Extremes

This technique can be used when you are trying to find global extrema over a limited interval.

The only places where a function can have a global extreme on a limited interval are at critical points or endpoints.

- If the function has only one critical point, and it's a local extreme, then it is also the global extreme.
- If there are endpoints, find the global extremes by comparing y-values at all the critical points and at the endpoints.
- When in doubt, graph the function to be sure. (However, unless the problem explicitly tells you otherwise, it is **not** enough to just use the graph to get your answer.)

Example 16.8.6

Find the global max and min of $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \leq x \leq 6$.

$f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$. We need to find critical points, and we need to check the endpoints.

$f'(x) = 3(x+1)(x-3) = 0$ when $x = -1$ and $x = 3$. The endpoints of the interval are $x = -2$ and $x = 6$.

Now we simply compare the values of f at these four values of x:

x	$f(x)$
-2	3
-1	10
3	-22
6	59

The global minimum of f on $[-2, 6]$ is -22, when $x = 3$, and the global maximum of f on $[-2, 6]$ is 59, when $x = 6$.

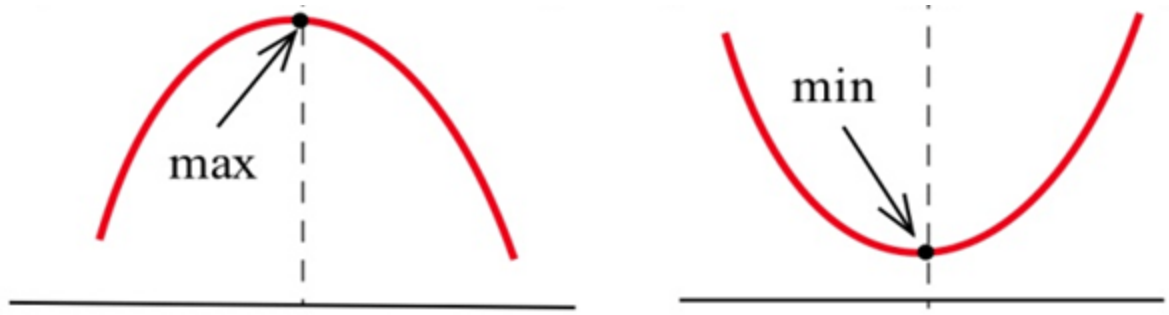
Caution

Be careful! You can **only** use this technique (evaluating the function at endpoints and critical values) to determine **global** extrema over a limited interval.

If a question asks for **local** extrema, you **must** use the first or second derivative test described earlier in the section.

If there's only one critical point

If the function has only one critical point and it's a local max (or min), then it must be the global max (or min). To see this, think about the geometry. Look at the graph on the left – there is a local max, and the graph goes down on either side of the critical point. Suppose there was some other point that was higher – then the graph would have to turn around. But that turning point would have shown up as another critical point. If there's only one critical point, then the graph can never turn back around.



When in doubt, graph it and look.

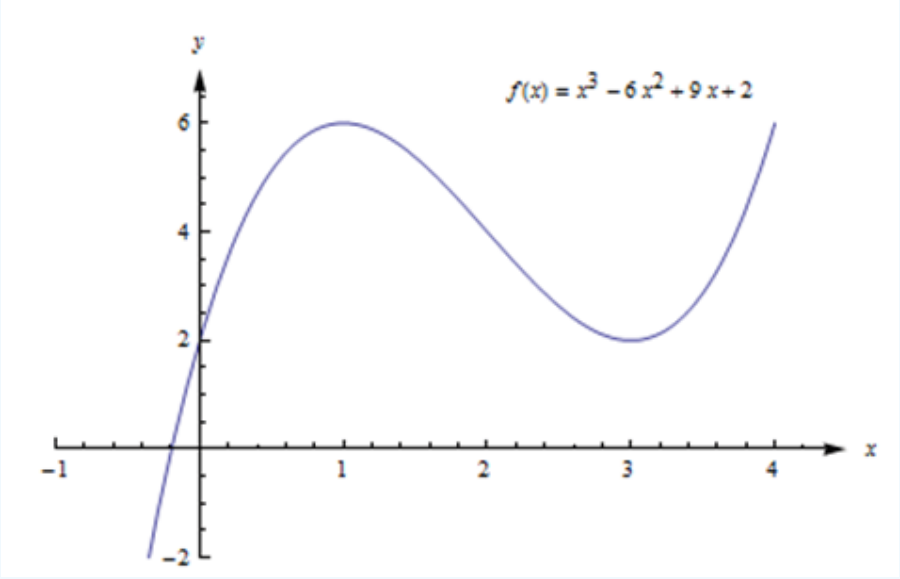
If you are trying to find a global max or min on an open interval (or the whole real line), and there is more than one critical point, then you need to look at the graph to decide whether there is a global max or min. Be sure that all your critical points show in your graph, and that you graph beyond them – that will tell you what you want to know.

Example 16.8.7

Find the global max and min of $f(x) = x^3 - 6x^2 + 9x + 2$.

We have previously found that (1, 6) is a local max and (3, 2) is a local min. This is not a closed interval, and there are two critical points, so we must turn to the graph of the function to find global max and min.

The graph of f shows that points to the left of $x = 4$ have y-values greater than 6, so (1, 6) is not a global max. Likewise, if x is negative, y is less than 2, so (3, 2) is not a global min. There are no endpoints, so we've exhausted all the possibilities. This function does not have a global maximum or minimum.



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16.9: Curve Sketching

This section examines some of the interplay between the shape of the graph of f and the behavior of f'. If we have a graph of f, we will see what we can conclude about the values of f'. If we know values of f', we will see what we can conclude about the graph of f. We will also utilize the information from f'' that we learning in the last section.

First Derivative Information

Definitions

The function $f(x)$ is **increasing** on (a,b) if $a < x_1 < x_2 < b$ implies $f(x_1) < f(x_2)$.

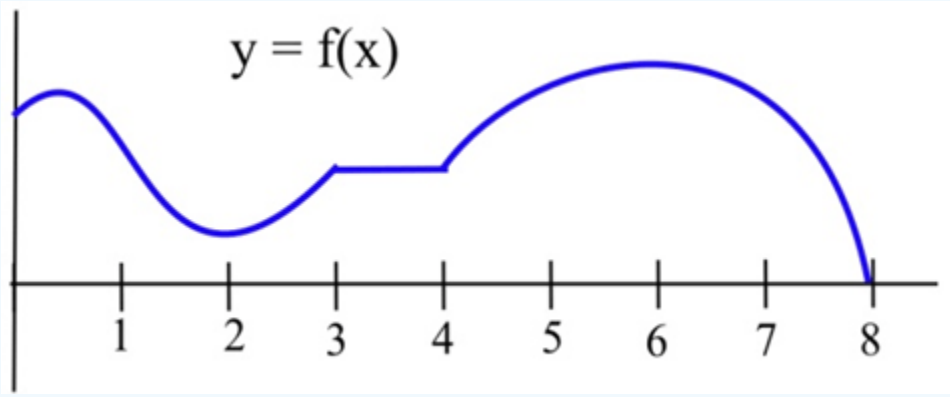
The function $f(x)$ is **decreasing** on (a,b) if $a < x_1 < x_2 < b$ implies $f(x_1) > f(x_2)$.

Graphically, f is **increasing** (decreasing) if, as we move from left to right along the graph of f, the height of the graph **increases** (decreases).

These same ideas make sense if we consider $h(t)$ to be the height (in feet) of a rocket at time t seconds. We naturally say that the rocket is rising or that its height is increasing if the height $h(t)$ increases over a period of time, as t increases.

Example 16.9.1

List the intervals on which the function shown is increasing or decreasing.



f is increasing on the intervals [0,0.5], [2,3] and [4,6].

f is decreasing on [0.5,2] and [6,8].

On the interval [3,4] the function is not increasing or decreasing – it is constant.

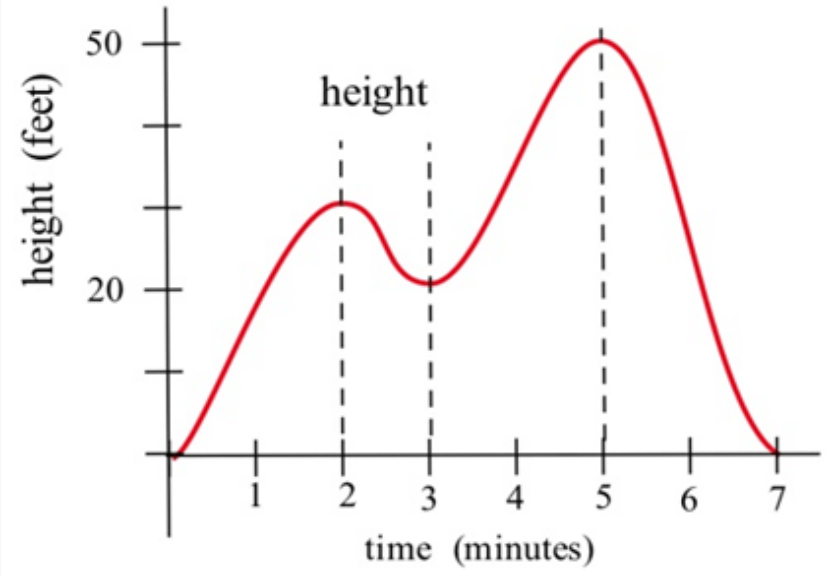
First Derivative Information about Shape

For a function f which is differentiable on an interval (a,b);

- if f is increasing on (a,b), then $f'(x) \geq 0$ for all x in (a,b).
- if f is decreasing on (a,b), then $f'(x) \leq 0$ for all x in (a,b).
- if f is constant on (a,b), then $f'(x) = 0$ for all x in (a,b).

Example 16.9.2

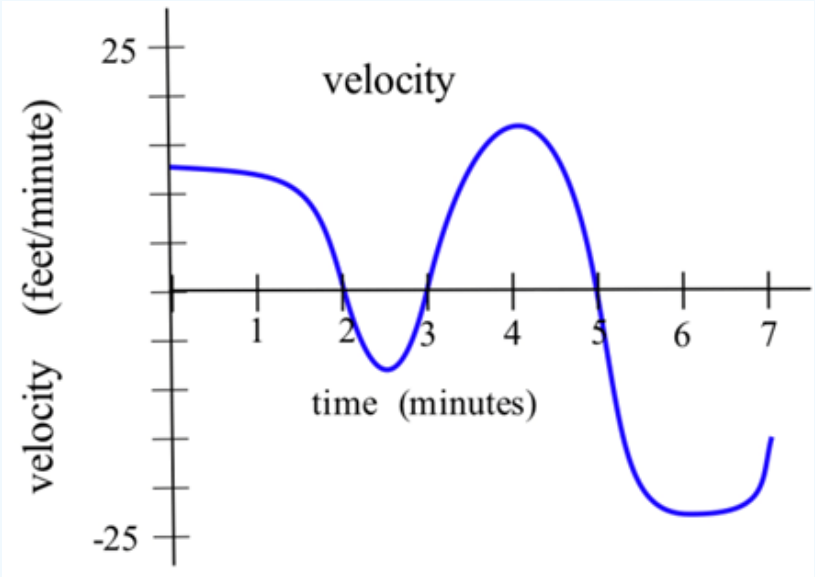
The graph shows the height of a helicopter during a period of time. Sketch the graph of the upward velocity of the helicopter, $\frac{dh}{dt}$.



Notice that the $h(t)$ has a local maximum when $t = 2$ and $t = 5$, and so $v(2) = 0$ and $v(5) = 0$. Similarly, $h(t)$ has a local minimum when $t = 3$, so $v(3) = 0$.

When h is increasing, v is positive. When h is decreasing, v is negative.

Using this information, we can sketch a graph of $v(t) = \frac{dh}{dt}$.



The next theorem is almost the converse of the First Shape Theorem and explains the relationship between the values of the derivative and the graph of a function from a different perspective. It says that if we know something about the values of f', then we can draw some conclusions about the shape of the graph of f.

First Derivative Information about Shape (part 2)

For a function f which is differentiable on an interval I;

- if $f'(x) > 0$ for all x in the interval I, then f is increasing on I.
- if $f'(x) < 0$ for all x in the interval I, then f is decreasing on I.
- if $f'(x) = 0$ for all x in the interval I, then f is constant on I.

Example 16.9.3

Use information about the values of f' to help graph $f(x) = x^3 - 6x^2 + 9x + 1$.

Solution

$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ so $f'(x) = 0$ only when $x = 1$ or $x = 3$. f' is a polynomial so it is always defined.

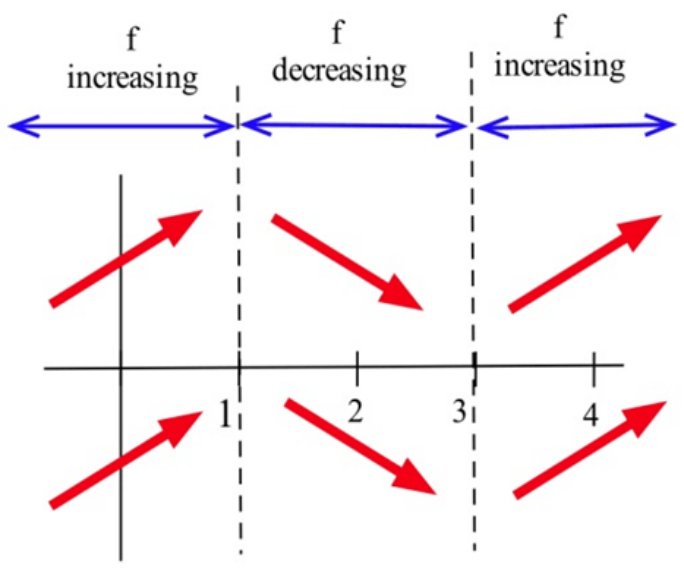
The only critical numbers for f are $x = 1$ and $x = 3$, and they divide the real number line into three intervals: $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$. On each of these intervals, the function is either always increasing or always decreasing.

If $x < 1$, then $f'(x) =$ 3(negative number)(negative number) > 0 so f is increasing.

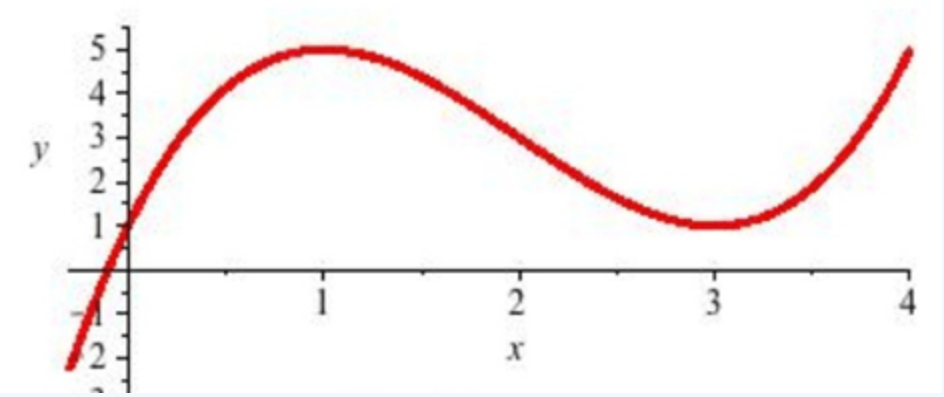
If $1 < x < 3$, then $f'(x) =$ 3(positive number)(negative number) < 0 so f is decreasing.

If $x > 3$, then $f'(x) =$ 3(positive number)(positive number) > 0 so f is increasing.

Even though we don't know the value of f anywhere yet, we do know a lot about the shape of the graph of f: as we move from left to right along the x-axis, the graph of f increases until $x = 1$, then the graph decreases until $x = 3$, and then the graph increases again. The graph of f makes "turns" when $x = 1$ and $x = 3$; it has a local maximum at $x = 1$, and a local minimum at $x = 3$.



To plot the graph of f, we still need to evaluate f at a few values of x, but only at a very few values. $f(1) = 5$, and (1,5) is a local maximum of f. $f(3) = 1$, and (3,1) is a local minimum of f. The resulting graph of f is shown here.



Second Derivative Information

Until now, we've only used first derivative information, but we could also use information from the second derivative to provide more information about the shape of the function.

Second Derivative Information about Shape

- If f is concave up on (a, b), then $f''(x) \geq 0$ for all x in (a, b).
- If f is concave down on (a, b), then $f''(x) \leq 0$ for all x in (a, b).

The converse of both of these are also true:

- If $f''(x) \geq 0$ for all x in (a, b), then f is concave up on (a, b).
- If $f''(x) \leq 0$ for all x in (a, b), then f is concave down on (a, b).

Example 16.9.4

Use information about the values of f'' to help determine the intervals on which the function $f(x) = x^3 - 6x^2 + 9x + 1$ is concave up and concave down.

Solution

For concavity, we need the second derivative: $f'(x) = 3x^2 - 12x + 9$, so $f''(x) = 6x - 12$.

To find possible inflection points, set the second derivative equal to zero. $6x - 12 = 0$, so $x = 2$. This divides the real number line into two intervals: $(-\infty, 2)$ and $(2, \infty)$.

For $x < 2$, the second derivative is negative (for example, $f''(0) = 6(0) - 12 = -12$), so f is concave down. For $x > 2$, the second derivative is positive, so f is concave up.

We could have incorporated this concavity information when sketching the graph for the previous example, and indeed we can see the concavity reflected in the graph shown.

Example 16.9.5

Use information about the values of f' and f'' to help graph $f(x) = x^{2/3}$.

Solution

$f'(x) = \frac{2}{3}x^{-1/3}$. This is undefined at $x = 0$.

$f''(x) = -\frac{2}{9}x^{-4/3}$. This is also undefined at $x = 0$.

This creates two intervals: $x < 0$, and $x > 0$.

On the interval $x < 0$, we could test out a value like $x = -1$:

$$f'(-1) = \frac{2}{3}(-1)^{-1/3} = -\frac{2}{3}$$

and

$$f''(-1) = -\frac{2}{9}(-1)^{-4/3} = -\frac{2}{9}.$$

$f'(x)$ is negative and $f''(x)$ is negative, so we can conclude that the function is decreasing and concave down on this interval.

On the interval $x > 0$, we could test out a value like $x = 1$:

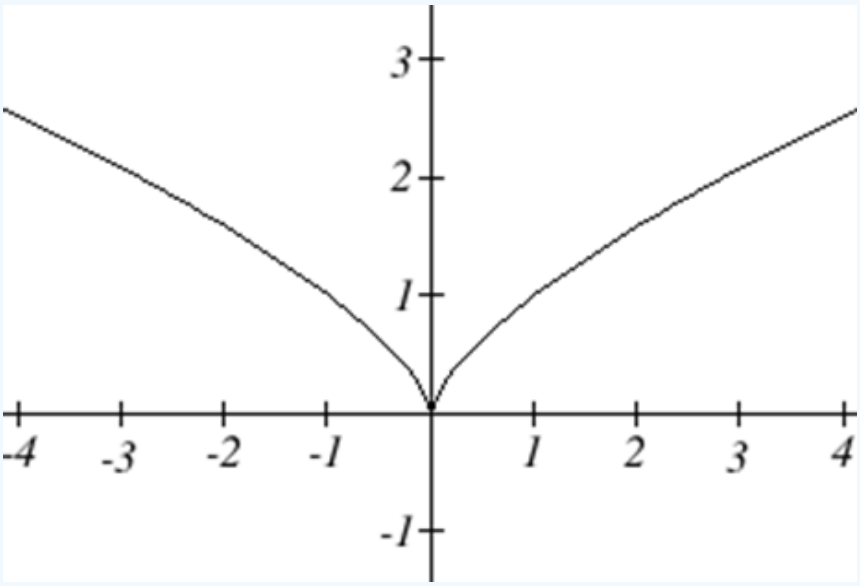
$$f'(1) = \frac{2}{3}(1)^{-1/3} = \frac{2}{3}$$

and

$$f''(1) = -\frac{2}{9}(1)^{-4/3} = -\frac{2}{9}.$$

$f'(x)$ is positive and $f''(x)$ is negative, so we can conclude that the function is increasing and concave down on this interval.

We can also calculate that $f(0) = 0$, giving us a base point for the graph. Using this information, we can conclude the graph must look like this:



Sketching without an Equation

Of course, graphing calculators and computers are great at graphing functions. Calculus provides a way to illuminate what may be hidden or out of view when we graph using technology. More importantly, calculus gives us a way to look at the derivatives of functions for which there is no equation given. We already saw the idea of this back in Section 2.3 where we sketched the derivative of two graphs by estimating slopes on the curves.

We can summarize all the derivative information about shape in a table.

Summary of Derivative Information about the Graph

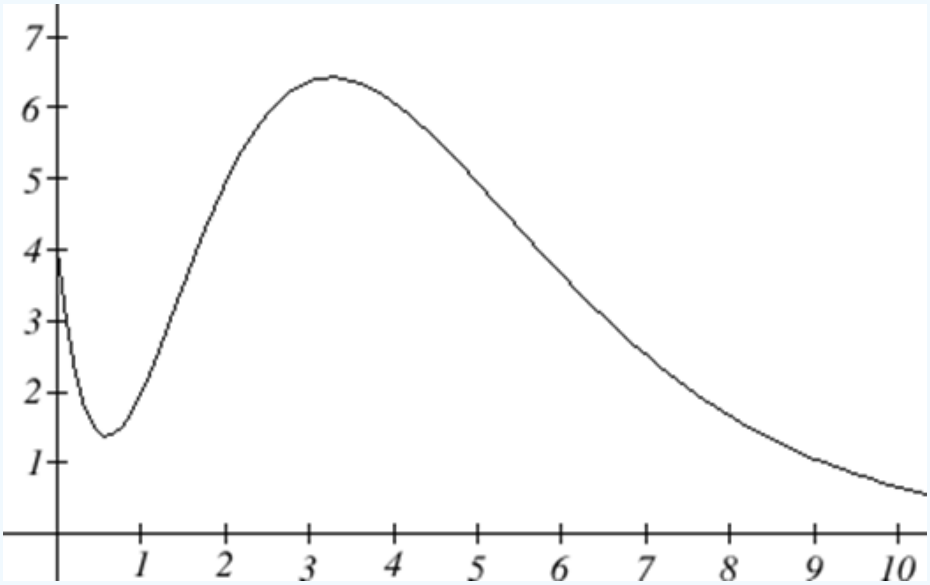
$f(x)$	Increasing	Decreasing	Concave Up	Concave Down
$f'(x)$	$+$	$-$	Increasing	Decreasing
$f''(x)$			$+$	$-$

When $f'(x) = 0$, the graph of f may have a local max or min.

When $f''(x) = 0$, the graph of f may have an inflection point.

Example 16.9.6

A company's bank balance, B, in millions of dollars, t weeks after releasing a new product is shown in the graph below. Sketch a graph of the marginal balance – the rate at which the bank balance was changing over time.



Solution

Notice that since the tangent line will be horizontal at about $t = 0.6$ and $t = 3.2$, the derivative will be 0 at those points.

We can then identifying intervals on which the original function is increasing or decreasing. This will tell us when the derivative function is positive or negative.

Interval	$B(t)$	$B'(t)$
$0 < t < 0.6$	Decreasing	Negative
$0.6 < t < 3.2$	Increasing	Positive
$t > 3.2$	Decreasing	Negative

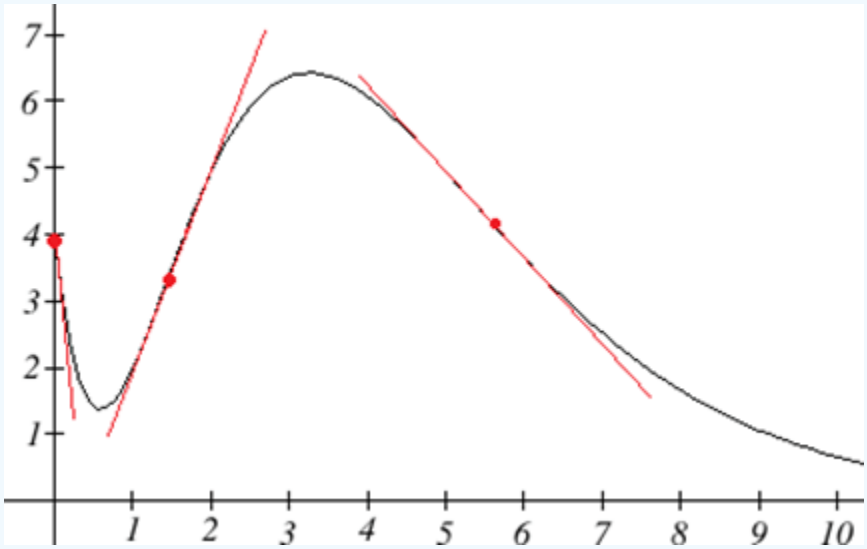
There appear to be inflection points at about $t = 1.5$ and $t = 5.5$. At these points, the derivative will be changing from increasing to decreasing or vice versa, so the derivative will have a local max or min at those points.

Looking at the intervals of concavity:

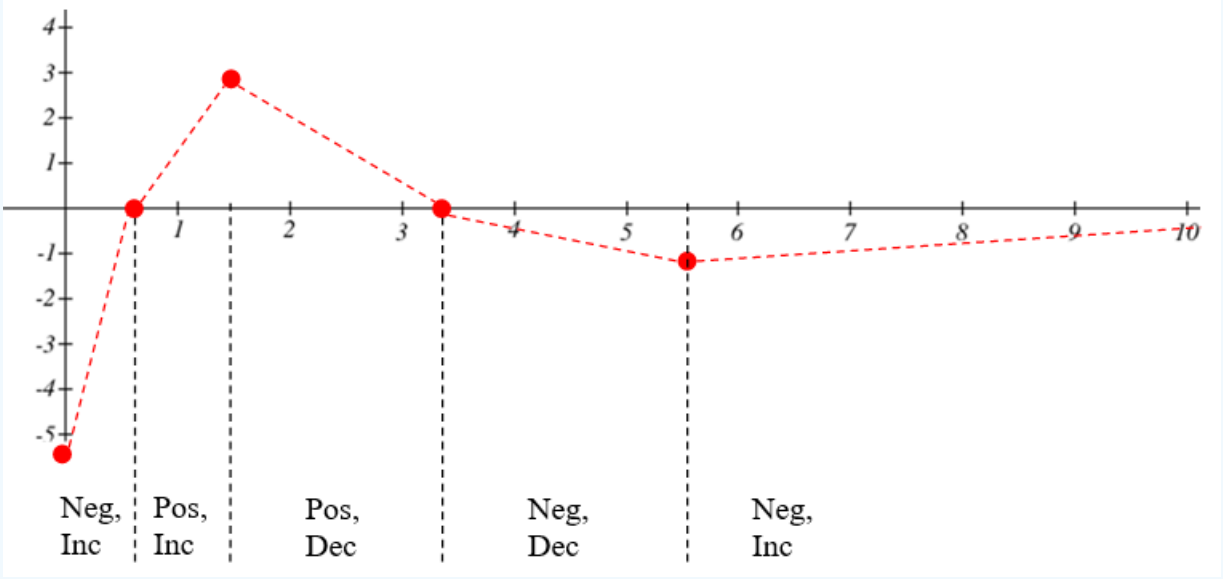
Interval	$B(t)$	$B'(t)$
$0 < t < 1.5$	Concave Up	Increasing
$1.5 < t < 5.5$	Concave Down	Decreasing
$t > 5.5$	Concave Up	Increasing

If we wanted a more accurate sketch of the derivative function, we could also estimate the derivative at a few points:

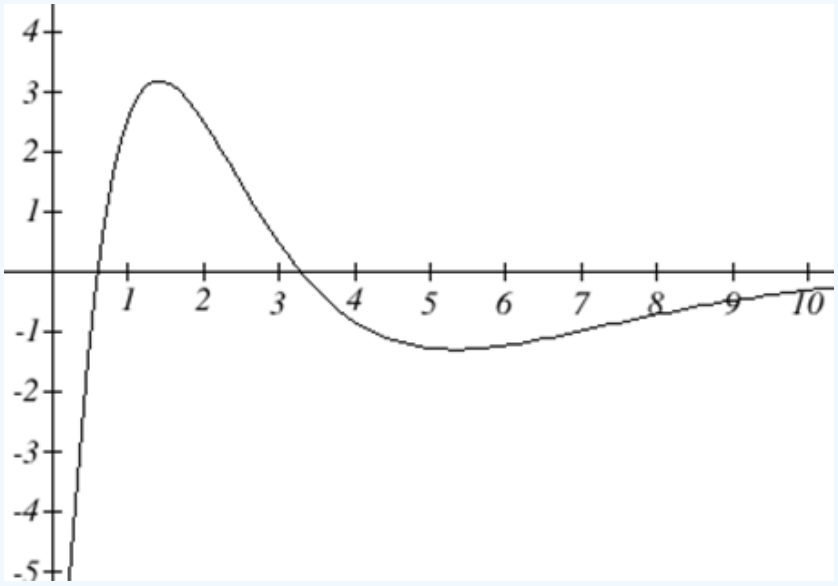
t	$B'(t)$
0	-10
1.5	3
6	-1



Now we can sketch the derivative. We know a few values for the derivative function, and on each interval we know the shape we need. We can use this to create a rough idea of what the graph should look like.



Smoothing this out gives us a good estimate for the graph of the derivative.



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16.10: Applied Optimization

We have used derivatives to help find the maximums and minimums of some functions given by equations, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values. More typically, someone will describe a problem and ask your help in maximizing or minimizing something: “What is the largest volume package which the post office will take?”; “What is the quickest way to get from here to there?”; or “What is the least expensive way to accomplish some task?” In this section, we’ll discuss how to find these extreme values using calculus.

Max/Min Applications

Example

The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store’s parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of $7 per running foot. The fourth side will be built of cement blocks, at a cost of $14 per running foot. Find the dimensions of the least costly such enclosure.

The process of finding maxima or minima is called optimization. The function we're optimizing is called the **objective function** (or **objective equation**). The objective function can be recognized by its proximity to “est” words (greatest, least, highest, farthest, most, ...). Look at the garden store example; the cost function is the objective function.

In many cases, there are two (or more) variables in the problem. In the garden store example again, the length and width of the enclosure are both unknown. If there is an equation that relates the variables we can solve for one of them in terms of the others, and write the objective function as a function of just one variable. Equations that relate the variables in this way are called **constraint equations**. The constraint equations are always equations, so they will have equals signs. For the garden store, the fixed area relates the length and width of the enclosure. This will give us our constraint equation.

Max-Min Story Problem Technique

1. Translate the English statement of the problem line by line into a picture (if that applies) and into math. This is often the hardest step!
2. Identify the objective function. Look for words indicating a largest or smallest value.
    1. If you seem to have two or more variables, find the constraint equation. Think about the English meaning of the word “constraint”, and remember that the constraint equation will have an equals sign.
    2. Solve the constraint equation for one variable and substitute into the objective function. Now you have an equation of one variable.
3. Use calculus to find the optimum values. (Take derivative, find critical points, test. Don't forget to check the endpoints!)
4. Look back at the question to make sure you answered what was asked. Translate your number answer back into English.

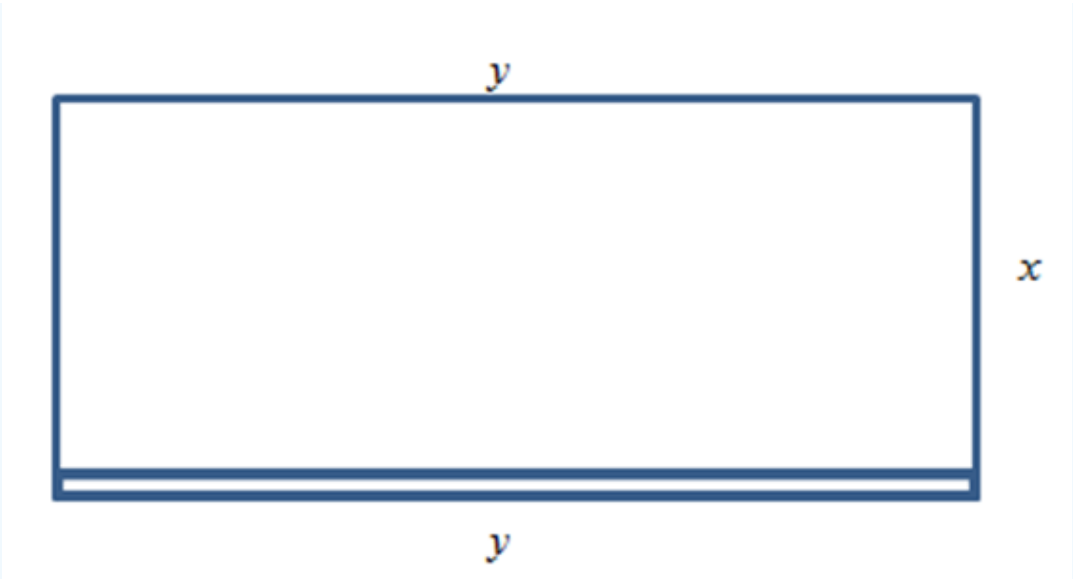
Example 16.10.1

The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of $7 per running foot. The fourth side will be built of cement blocks, at a cost of $14 per running foot. Find the dimensions of the least costly such enclosure.

First, translate line by line into math and a picture:

Text	Translation

The manager of a garden store wants to build a 600 *square foot rectangular* enclosure on the store's parking lot in order to display some equipment. *Three sides of the enclosure* will be built of redwood fencing, at a *cost of $7 per running foot*. The *fourth side* will be built of cement blocks, at a *cost of $14 per running foot*. Find the dimensions of the least costly such enclosure.	Let x and y be the dimensions of the enclosure, with y being the length of the side made of blocks. The $$\text{Area} = A = xy = 600.$$ $2x + y$ costs $7 per foot, y costs $14 per foot, so $$\text{Cost} = C = 7(2x + y) + 14y = 14x + 21y.$$ Find x and y so that C is minimized.



The objective function is the cost function, and we want to minimize it. As it stands, though, it has two variables, so we need to use the constraint equation. The constraint equation is the fixed area $A = xy = 600$. Solve A for x to get $x = \frac{600}{y}$, and then substitute into C:

$$C = 14\left(\frac{600}{y}\right) + 21y = \frac{8400}{y} + 21y.$$

Now we have a function of just one variable, so we can find the minimum using calculus.

$$C' = -\frac{8400}{y^2} + 21$$

C' is undefined for $y = 0$, and $C' = 0$ when $y = 20$ or $y = -20$.

Of these three critical numbers, only $y = 20$ makes sense (is in the domain of the actual function) – remember that y is a length, so it can't be negative, and $y = 0$ would mean there was no enclosure at all, so it couldn't have an area of 600 square feet.

Test $y = 20$ (here we chose the second derivative test):

$$C'' = \frac{16800}{y^3} > 0,$$

so this is a local minimum.

Since this is the only critical point in the domain, this must be the global minimum. Going back to our constraint function, we can find that when $y = 20$, $x = 30$. The dimensions of the enclosure that minimize the cost are 20 feet by 30 feet.

When trying to maximize their revenue, businesses also face the constraint of consumer demand. While a business would love to see lots of products at a very high price, typically demand decreases as the price of goods increases. In simple cases, we can construct that demand curve to allow us to maximize revenue.

Example 16.10.2

A concert promoter has found that if she sells tickets for \$50 each, she can sell 1200 tickets, but for each \$5 she raises the price, 50 less people attend. What price should she sell the tickets at to maximize her revenue?

We are trying to maximize revenue, and we know that $R = pq$, where p is the price per ticket, and q is the quantity of tickets sold.

The problem provides information about the demand relationship between price and quantity – as price increases, demand decreases. We need to find a formula for this relationship. To investigate, let's calculate what will happen to attendance if we raise the price:

Price, p	50	55	60	65
Quantity, q	1200	1150	1100	1050

You might recognize this as a linear relationship. We can find the slope for the relationship by using two points:

$$m = \frac{1150 - 1200}{55 - 50} = \frac{-50}{5} = -10.$$

You may notice that the second step in that calculation corresponds directly to the statement of the problem: the attendance drops 50 people for every $5 the price increases.

Using the point-slope form of the line, we can write the equation relating price and quantity:

$$q - 1200 = -10(p - 50).$$

Simplifying to slope-intercept form gives the demand equation

$$q = 1700 - 10p.$$

Substituting this into our revenue equation, we get an equation for revenue involving only one variable:

$$R = pq = p(1700 - 10p) = 1700p - 10p^2.$$

Now, we can find the maximum of this function by finding critical numbers. $R' = 1700 - 20p$, so $R' = 0$ when $p = 85$.

Using the second derivative test, $R'' = -20 < 0$ (for any value of p), so the critical number is a local maximum. Since it is the only critical number, we can also conclude that it is the global maximum.

The promoter will be able to maximize revenue by charging $85 per ticket. At this price, she will sell $q = 1700 - 10(85) = 850$ tickets, generating $72,250 in revenue.

"Marginal Revenue = Marginal Cost"

You may have heard before that "profit is maximized when marginal cost and marginal revenue are equal." Now you can see why people say that! (Even though it's not completely true.)

Example 16.10.3

Suppose we want to maximize profit.

Now we know what to do – find the profit function, find its critical points, test them, etc.

But remember that Profit = Revenue - Cost. So Profit' = Revenue' - Cost'. That is, the derivative of the profit function is $MR - MC$.

Now let's find the critical points – those will be where Profit' = 0 or is undefined. Profit' = 0 when $MR - MC = 0$, or where $MR = MC$.

Profit has critical points when Marginal Revenue and Marginal Cost are equal.

In all the cases we'll see in this class, Profit will be very well behaved, and we won't have to worry about looking for critical points where Profit' is undefined. But remember that not all critical points are local max! The places where $MR = MC$ could represent local max, local min, or neither one.

Example 16.10.4

A company sells q ribbon winders per year at $\$p$ per ribbon winder. The demand function for ribbon winders is given by: $p = 300 - 0.02q$. The ribbon winders cost $30 apiece to manufacture, plus there are fixed costs of $9000 per year. Find the quantity where profit is maximized.

We want to maximize profit, but there isn't a formula for profit given. So let's make one. We can find a function for Revenue = pq using the demand function for p.

$$R(q) = (300 - 0.02q)q = 300q - 0.02q^2.$$

We can also find a function for Cost, using the variable cost of $30 per ribbon winder, plus the fixed cost:

$$C(q) = 9000 + 30q.$$

Putting them together, we get a function for Profit:

$$P(q) = R(q) - C(q) = \left(300q - 0.02q^2\right) - (9000 + 30q) = -0.02q^2 + 270q - 9000$$

Now we have two choices. We can find the critical points of Profit by taking the derivative of $P(q)$ directly, or we can find MR and MC and set them equal. (Naturally, we'll get the same answer either way.)

Let's use $MR = MC$ this time.

$$\begin{aligned} MR &= 300 - 0.04q \\ MC &= 30 \\ 300 - 0.04q &= 30 \\ 270 &= 0.04q \\ q &= 6750 \end{aligned}$$

The only critical point is at $q = 6750$. Now we need to be sure this is a local max and not a local min. In this case, we'll look to the graph of $P(q)$ – it's a downward opening parabola, so this must be a local max. And since it's the only critical point, it must also be the global max.

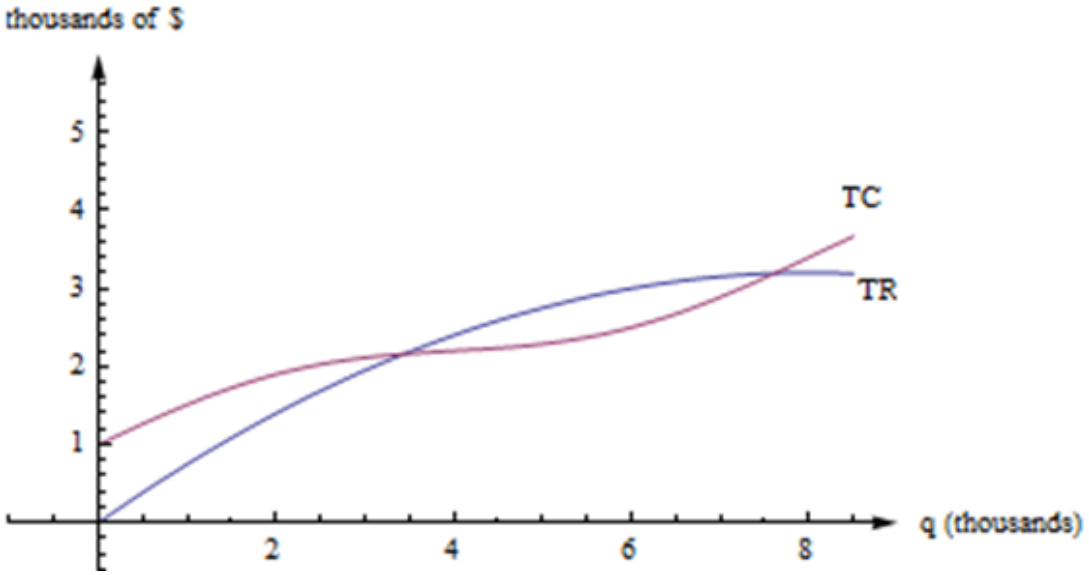
Profit is maximized when they sell 6750 ribbon winders.

"Average Cost = Marginal Cost"

"Average cost is minimized when average cost = marginal cost" is another saying that isn't quite true; in this case, the correct statement is:

Average Cost has critical points when Average Cost and Marginal Cost are equal.

Let's look at a geometric argument here. Remember that the average cost is the slope of the diagonal line, the line from the origin to the point on the total cost curve. If you move your clear plastic ruler around, you'll see (and feel) that the slope of the diagonal line is smallest when the diagonal line just touches the cost curve – when the diagonal line is actually a tangent line – when the average cost is equal to the marginal cost.



Example 16.10.5

The cost in dollars to produce q gift baskets is given by $C(q) = 160 + 2q + .1q^2$. Find the quantity where the average cost is minimum.

Solution

$A(q) = \frac{C(q)}{q} = \frac{160}{q} + 2 + .1q$. We could find the critical points by finding A', or by setting average cost to marginal cost; we'll do the latter this time.

$$MC(q) = 2 + .2q.$$

So we want to solve:

$$\begin{aligned} \frac{160}{q} + 2 + .1q &= 2 + .2q \\ \frac{160}{q} &= .1q \\ 1600 &= q^2 \\ q &= 40 \end{aligned}$$

The critical point of average cost is when $q = 40$.

Notice that we still have to confirm that the critical point is a minimum. For this, we can use the first or second derivative test on $A(q)$.

$$\begin{aligned} A'(q) &= \frac{-160}{q^2} + .1 \\ A''(q) &= \frac{320}{q^3} > 0 \end{aligned}$$

The second derivative is positive for all positive q, so that means this is a local min. Average cost is minimized when they produce 40 gift baskets; at that quantity, the average cost is $10 per basket.

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16.11: Other Applications

Tangent Line Approximation

Back when we first thought about the derivative, we used the slope of secant lines over tiny intervals to approximate the derivative:

$$f'(a) \cong \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

Now that we have other ways to find derivatives, we can exploit this approximation to go the other way. Solve the expression above for $f(x)$, and you'll get the tangent line approximation:

The Tangent Line approximation (TLA)

To approximate the value of $f(x)$ using TLA, find some a where

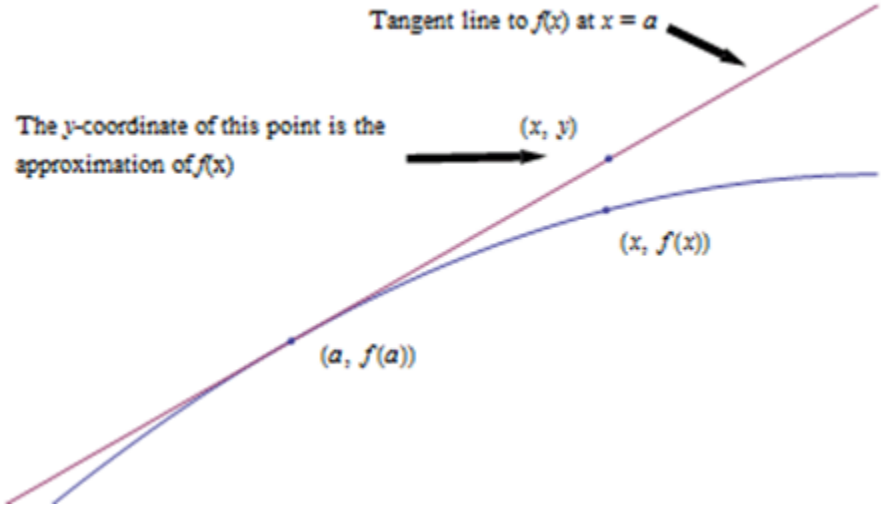
1. a and x are "close," and
2. You know the exact values of both $f(a)$ and $f`(a)$. Then

$$f(x) \cong f(a) + f'(a)(x - a).$$

Another way to look at the same formula:

$$\Delta y \cong f'(a)\Delta x$$

How close is close? It depends on the shape of the graph of f. In general, the closer the better.



Example 16.11.1

Suppose we know that $g(20) = 5$ and $g'(20) = 1.4$. Use this information to approximate $g(23)$ and $g(18)$.

Solution

Using the tangent line approximation:

$g(23) \approx 5 + (1.4)(23–20) = 9.2$

$g(18) \approx 5 + (1.4)(18–20) = 2.2$

Note that we don't know if these approximations are close – but they're the best we can do with the limited information we have to start with. Note also that 18 and 23 are sort of close to 20, so we can hope these approximations are pretty good. We'd feel more confident using this information to approximate $g(20.003)$. We'd feel very unsure using this information to approximate $g(55)$.

Elasticity

We know that demand functions are decreasing, so when the price increases, the quantity demanded goes down. But what about revenue = price $\times$ quantity? When the price increases will revenue go down because the demand dropped so much? Or will revenue increase because demand didn't drop very much?

Example 16.11.2

Suppose a company's demand function is $D(p) = 100 - p^2$, and the company's current price is \$5. What will happen to revenue if they raise the price \$0.05?

Solution

We are more interested in how the price change compares to the demand change, so we are going to convert everything to relative (percent) changes.

The price has increased by

$$\frac{\Delta p}{p} = \frac{\$0.05}{\$5} = 0.01 = 1\%$$

The demand decreased from $D(5) = 100 - 5^2 = 75$ to $D(5.05) = 100 - 5.05^2 = 74.4975$, a total change of $75 - 74.4975 = -0.5025$. As a relative change, that's

$$\frac{\Delta q}{q} = \frac{-0.05025}{75} = -0.0067 = -0.67\%$$

By raising the price 1%, the demand only fell by 0.67%. Since the price is increasing more than the demand falls, we would expect total revenue to increase. In this case we say that the demand is inelastic.

Elasticity of demand is a measure of how demand reacts to price changes. It's normalized – that means the particular prices and quantities don't matter, and everything is treated as a percent change.

Following the logic of the example above, we want to compare the relative change in demand to the relative change in price, or in other words, we want to look at

$$\frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}}.$$

By approximating the changes in demand and price by the derivatives, this formula simplifies to

$$\frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q} \cdot \frac{dq}{dp}$$

Elasticity of Demand

Given a demand function that gives q in terms of p, so $q = D(p)$, the **elasticity of demand** is

$$E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right| = \left| \frac{p}{q} \cdot D'(p) \right|$$

(Note that since demand is a decreasing function of p, the derivative is negative. That's why we have the absolute values – so E will always be positive.)

You may also see this formula written as

$$E = -\frac{p \cdot D'(p)}{D(p)}$$

The two forms of the equation are equivalent, and you can use either.

- If $E < 1$, we say demand is **inelastic.** In this case, raising prices increases revenue.
- If $E > 1$, we say demand is **elastic.** In this case, raising prices decreases revenue.
- If $E = 1$, we say demand is **unitary.** $E = 1$ at critical points of the revenue function.

Interpretation of elasticity

If the price increases by 1%, the demand will decrease by **E**%.

Example 16.11.3

A company sells q ribbon winders per year at \$$p$ per ribbon winder. The demand function for ribbon winders is given by $p = 300 - 0.02q$. Find the elasticity of demand when the price is \$70 apiece. Will an increase in price lead to an increase in revenue?

Solution

First, we need to solve the demand equation so it gives q in terms of p, so that we can find $\frac{dq}{dp}$: $p = 300 - 0.02q$, so $q = 15000 - 50p$. Then $\frac{dq}{dp} = -50$.

We need to find q when $p = 70$:

$$q = 11500.$$

Now compute

$$E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right| = \left| \frac{70}{11500} \cdot (-50) \right| \approx 0.3$$

$E < 1$, so demand is inelastic. Increasing the price by 1% would only cause a 0.3% drop in demand. Increasing the price would lead to an increase in revenue, so it seems that the company should increase its price.

The demand for products that people have to buy, such as onions, tends to be inelastic. Even if the price goes up, people still have to buy about the same amount of onions, and revenue will not go down. The demand for products that people can do without, or put off buying, such as cars, tends to be elastic. If the price goes up, people will just not buy cars right now, and revenue will drop.

Example 16.11.4

A company finds the demand q, in thousands, for their kites to be $q = 400 - p^2$ at a price of p dollars. Find the elasticity of demand when the price is \$5 and when the price is \$15. Then find the price that will maximize revenue.

Solution

Calculating the derivative, $\frac{dq}{dp} = -2p$. The elasticity equation as a function of p will be:

$$E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right| = \left| \frac{p}{400 - p^2} \cdot (-2p) \right| = \left| \frac{-2p^2}{400 - p^2} \right|$$

Evaluating this to find the elasticity at \$5 and at \$15:

$$E(5) = \left| \frac{-2(5)^2}{400 - (5)^2} \right| \approx 0.133$$

So the demand is inelastic when the price is \$5.

At a price of \$5, a 1% increase in price would decrease demand by only 0.133%. Revenue could be raised by increasing prices.

$$E(15) = \left| \frac{-2(15)^2}{400 - (15)^2} \right| \approx 2.571$$

So the demand is elastic when the price is \$15.

At a price of \$15, a 1% increase in price would decrease demand by 2.571%. Revenue could be raised by decreasing prices.

To maximize the revenue, we could solve for when $E = 1$:

$$\begin{aligned} \left| \frac{-2p^2}{400 - p^2} \right| &= 1 \\ 2p^2 &= 400 - p^2 \\ 3p^2 &= 400 \\ p &= \sqrt{\frac{400}{3}} \approx 11.55. \end{aligned}$$

A price of \$11.55 will maximize the revenue.

As you saw the in last example, elasticity provides another way to determine the price that will maximize revenue given a demand function. It is not quicker or easier than the methods learned earlier in the course, but you are welcome to use either technique.

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16.12: Implicit Differentiation and Related Rates

Implicit Differentiation

In our work up until now, the functions we needed to differentiate were either given **explicitly**, such as $y = x^2 + e^x$, or it was possible to get an explicit formula for them, such as solving $y^3 - 3x^2 = 5$ to get $y = \sqrt[3]{5 + 3x^2}$. Sometimes, however, we will have an equation relating x and y which is either difficult or impossible to solve explicitly for y, such as $y + e^y = x^2$. In any case, we can still find $y' = f'(x)$ by using implicit differentiation.

The key idea behind implicit differentiation is to **assume that y is a function of x** even if we cannot explicitly solve for y. This assumption does not require any work, but we need to be very careful to treat y as a function when we differentiate and to use the Chain Rule.

Example 16.12.1

Assume that y is a function of x. Calculate

1. $\frac{d}{dx}\left(y^3\right)$
2. $\frac{d}{dx}\left(x^3y^2\right)$
3. $\frac{d}{dx}\left(\ln(y)\right)$

Solution

1. We need the chain rule since y is a function of x:
$$\frac{d}{dx}\left(y^3\right) = 3y^2\frac{dy}{dx} \overset{\text{or}}{=} 3y^2y'$$
2. We need to use the product rule and the Chain Rule:
$$\begin{aligned}\frac{d}{dx}\left(x^3y^2\right) &= x^3\frac{d}{dx}\left(y^2\right) + y^2\frac{d}{dx}\left(x^3\right)\\ &= x^3 2y\frac{dy}{dx} + y^2 3x^2\\ &\overset{\text{or}}{=} 2x^3yy' + 3y^2x^2\end{aligned}$$
3. We know $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$, so we use that and the Chain Rule:
$$\frac{d}{dx}\left(\ln(y)\right) = \frac{1}{y}\cdot y'$$

Implicit Differentiation

To determine y', differentiate each side of the defining equation, **treating y as a function of x**, and then algebraically solve for y'.

Example 16.12.2

Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point (3,4) using implicit differentiation.

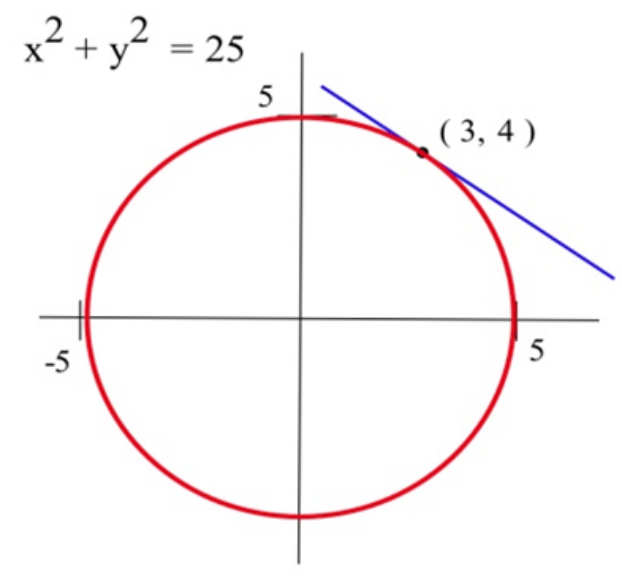
Solution

We differentiate each side of the equation $x^2 + y^2 = 25$ and then solve for y':

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(25)$$
$$2x+2yy'=0$$

Solving for y', we have $y'=-\frac{2x}{2y}=-\frac{x}{y}$, and, at the point (3,4),

$$y'=-\frac{3}{4}.$$



In the previous example, it would have been easy to explicitly solve for y, and then we could differentiate y to get y'. Because we could explicitly solve for y, we had a choice of methods for calculating y'. Sometimes, however, we cannot explicitly solve for y, and the only way of determining y' is to use implicit differentiation.

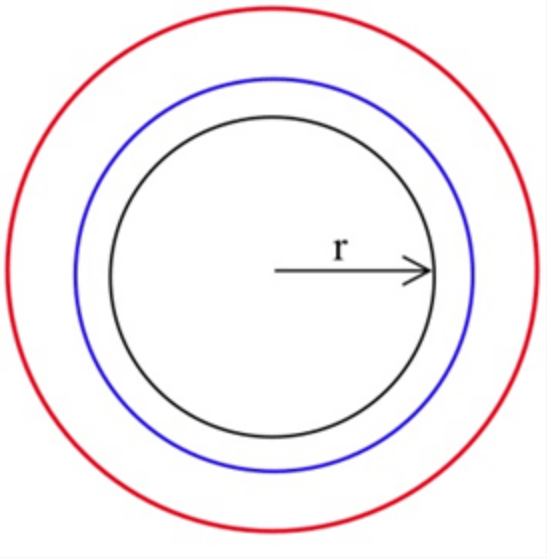
Related Rates

If several variables or quantities are related to each other and some of the variables are changing at a known rate, then we can use derivatives to determine how rapidly the other variables must be changing.

Example 16.12.3

Suppose the border of a town is roughly circular, and the radius of that circle has been increasing at a rate of 0.1 miles each year. Find how fast the area of the town has been increasing when the radius is 5 miles.

Solution



We could get an *approximate* answer by calculating the area of the circle when the radius is 5 miles ($A = \pi r^2 = \pi(5 \text{ miles})^2 \approx 78.6 \text{ miles}^2$) and 1 year later when the radius is 0.1 feet larger than before ($A = \pi r^2 = \pi(5.1 \text{ miles})^2 \approx 81.7 \text{ miles}^2$) and then finding

$$\frac{\Delta \text{Area}}{\Delta \text{time}} = \frac{81.7 \text{ mi}^2 - 78.6 \text{ mi}^2}{1 \text{ year}} = 3.1 \text{ mi}^2/\text{yr}.$$

This approximate answer represents the average change in area during the 1 year period when the radius increased from 5 miles to 5.1 miles, and would correspond to the secant slope on the area graph.

To find the exact answer, though, we need derivatives. In this case both radius and area are functions of time:

$$r(t) = \text{ radius at time } t \qquad A(t) = \text{ area at time } t$$

We know how fast the radius is changing, which is a statement about the derivative: $\frac{dr}{dt} = 0.1 \frac{\text{mile}}{\text{year}}$. We also know that $r = 5$ at our moment of interest.

We are looking for how fast the area is increasing, which is $\frac{dA}{dt}$.

Now we need an equation relating our variables, which is the area equation:

$$A = \pi r^2.$$

Taking the derivative of both sides of that equation with respect to t, we can use implicit differentiation:

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$
$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

Plugging in the values we know for r and $\frac{dr}{dt}$,

$$\frac{dA}{dt} = \pi 2(5 \text{ miles}) \left(0.1 \frac{\text{miles}}{\text{year}}\right) = \pi \frac{\text{miles}^2}{\text{year}}$$

So the area of the town is increasing by approximately 3.14 square miles per year when the radius is 5 miles.

Related Rates

When working with a related rates problem,

1. Draw a picture (if possible).
2. Identify the quantities that are changing, and assign them variables.
3. Find an equation that relates those quantities.
4. Differentiate both sides of that equation with respect to time.
5. Plug in any known values for the variables or rates of change.
6. Solve for the desired rate.

Example 16.12.4

A company has determined the demand curve for their product is $q = \sqrt{5000 - p^2}$, where p is the price in dollars, and q is the quantity in millions. If weather conditions are driving the price up $2 a week, find the rate at which demand is changing when the price is $40.

Solution

The quantities changing are p and q, and we assume they are both functions of time, t, in weeks. We already have an equation relating the quantities, so we can implicitly differentiate it.

$$\begin{aligned}\frac{d}{dt}(q) &= \frac{d}{dt}\left(5000 - p^2\right)^{1/2} \\ \frac{dq}{dt} &= \frac{1}{2}\left(5000 - p^2\right)^{-1/2}\frac{d}{dt}\left(5000 - p^2\right) \\ \frac{dq}{dt} &= \frac{1}{2}\left(5000 - p^2\right)^{-1/2}\left(-2p\frac{dp}{dt}\right)\end{aligned}$$

Using the given information, we know the price is increasing by $2 per week when the price is $40, giving $\frac{dp}{dt} = 2$ when $p = 40$. Plugging in these values,

$$\frac{dq}{dt} = \frac{1}{2}\left(5000 - 40^2\right)^{-1/2}(-2(40)(2)) \approx -1.37$$

Demand is falling by 1.37 million items per week.

Example 16.12.5

The total daily cost for producing x items in a day is $TC(x) = 300,000 + 4x + \frac{200,000}{x}$. If production has been ramping up by 20 items a day, find the rate at which total daily cost is increasing, if they are currently producing 2,000 items.

Solution

The quantities changing are x and TC, and we assume they are both functions of time, t, in days. We already have an equation relating the quantities, so we can implicitly differentiate it.

$$\begin{aligned}\frac{d}{dt}(TC) &= \frac{d}{dt}\left(300,000 + 4x + 200,000x^{-1}\right) \\ \frac{dTC}{dt} &= 4\frac{dx}{dt} - 200,000x^{-2}\frac{dx}{dt} \\ \frac{dTC}{dt} &= \left(4 - \frac{200,000}{x^2}\right)\frac{dx}{dt}\end{aligned}$$

We know the quantity produced is increasing by 20 items per week when the production is 2,000 items, giving $\frac{dx}{dt} = 20$ when $x = 2000$. Plugging in these values,

$$\frac{dTC}{dt} = \left(4 - \frac{200,000}{2000^2}\right)(20) = 79$$

Total daily cost is increasing by $79 each day.

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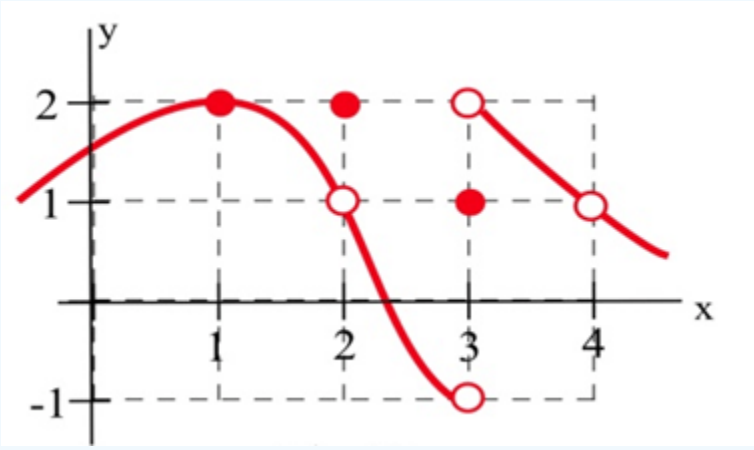
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16.E: The Derivative (Exercises)

2.2 Exercises

Exercise 16.E.1

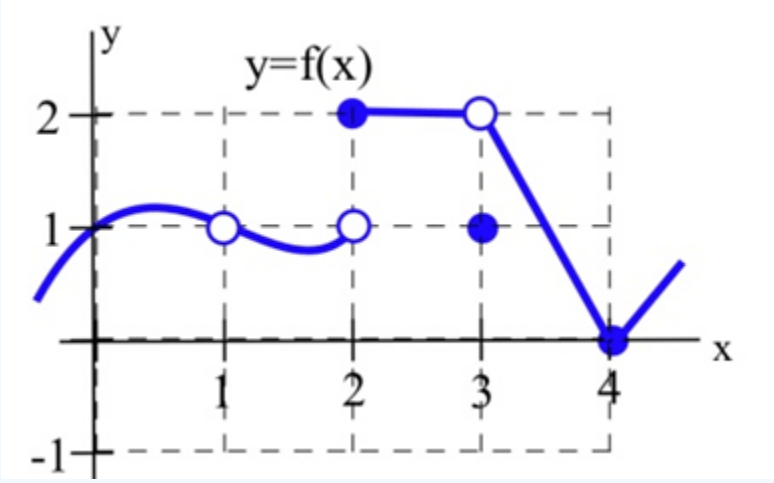
Use the graph to determine the following limits.



a. $\lim_{x\to 1} f(x)$	b. $\lim_{x\to 2} f(x)$
c. $\lim_{x\to 3} f(x)$	d. $\lim_{x\to 4} f(x)$

Exercise 16.E.2

Use the graph to determine the following limits.



a. $\lim_{x\to 1} f(x)$	b. $\lim_{x\to 2} f(x)$
c. $\lim_{x\to 3} f(x)$	d. $\lim_{x\to 4} f(x)$

Exercise 16.E.3

Evaluate

a. $\lim_{x\to 1} \frac{x^2+3x+3}{x-2}$	b. $\lim_{x\to 2} \frac{x^2+3x+3}{x-2}$

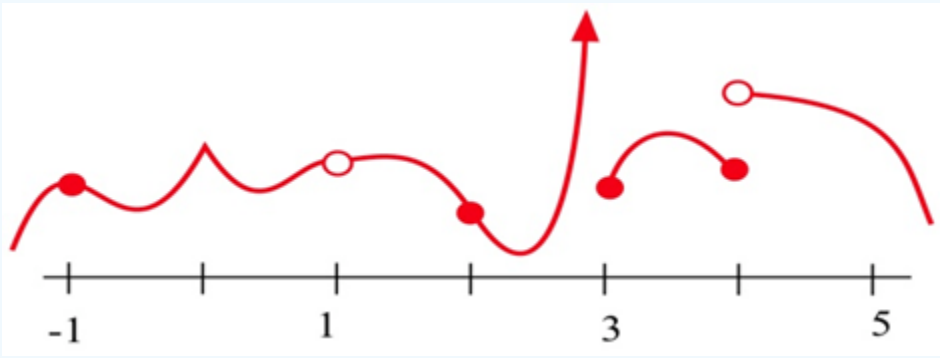
Exercise 16.E.4

Evaluate

a. $\lim_{x\to 0} \frac{x+7}{x^2+9x+14}$	b. $\lim_{x\to 3} \frac{x+7}{x^2+9x+14}$
c. $\lim_{x\to 4} \frac{x+7}{x^2+9x+14}$	d. $\lim_{x\to 7} \frac{x+7}{x^2+9x+14}$

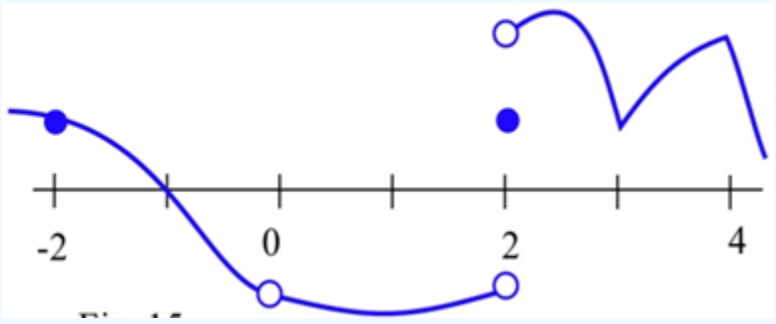
Exercise 16.E.5

At which points is the function shown discontinuous?



Exercise 16.E.6

At which points is the function shown discontinuous?



Exercise 16.E.7

Find at least one point at which each function is not continuous and state which of the 3 conditions in the definition of continuity is violated at that point

a. $\frac{x+5}{x-3}$	b. $\frac{x^2+x-6}{x-2}$	c. $\frac{x}{x}$

d. $\frac{\pi}{x^2-6x+9}$	e. $\ln(x^2)$	

2.3 Exercises

Exercise 16.E.1

What is the slope of the line through (3,9) and (x, y) for $y = x^2$ and $x = 2.97$? $x = 3.001$? $x = 3 + h$? What happens to this last slope when h is very small (close to 0)? Sketch the graph of $y = x^2$ for x near 3.

Exercise 16.E.2

What is the slope of the line through (–2,4) and (x, y) for $y = x^2$ and $x = -1.98$? $x = -2.03$? $x = -2 + h$? What happens to this last slope when h is very small (close to 0)? Sketch the graph of $y = x^2$ for x near –2.

Exercise 16.E.3

What is the slope of the line through (2,4) and (x, y) for $y = x^2 + x–2$ and $x = 1.99$?

$x = 2.004$? $x = 2 + h$? What happens to this last slope when h is very small? Sketch the graph of $y = x^2 + x–2$ for x near 2.

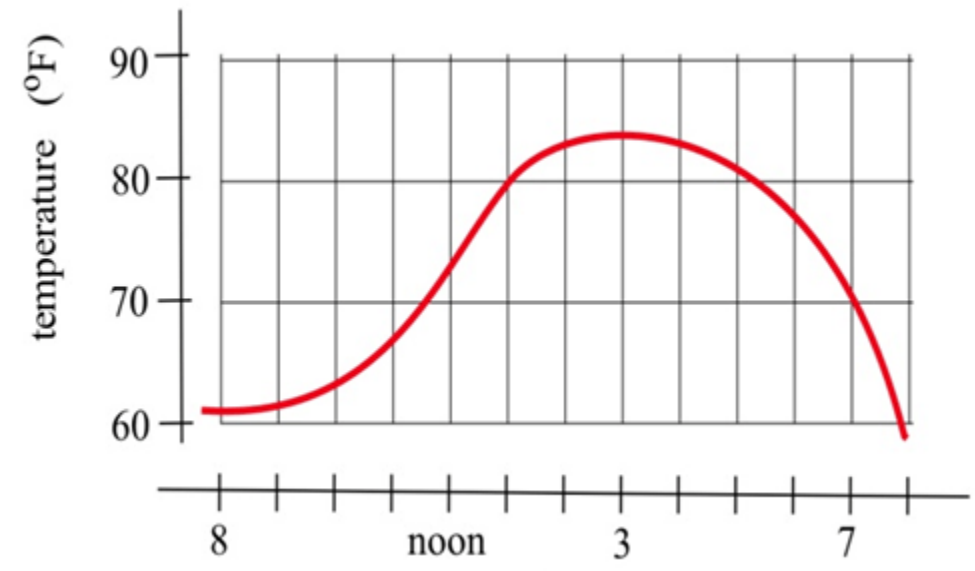
Exercise 16.E.4

What is the slope of the line through (–1,–2) and (x, y) for $y = x^2 + x–2$ and $x = –.98$?

$x = -1.03$? $x = -1 + h$? What happens to this last slope when h is very small? Sketch the graph of $y = x^2 + x–2$ for x near –1.

Exercise 16.E.5

The graph to the right shows the temperature during a day in Ames.

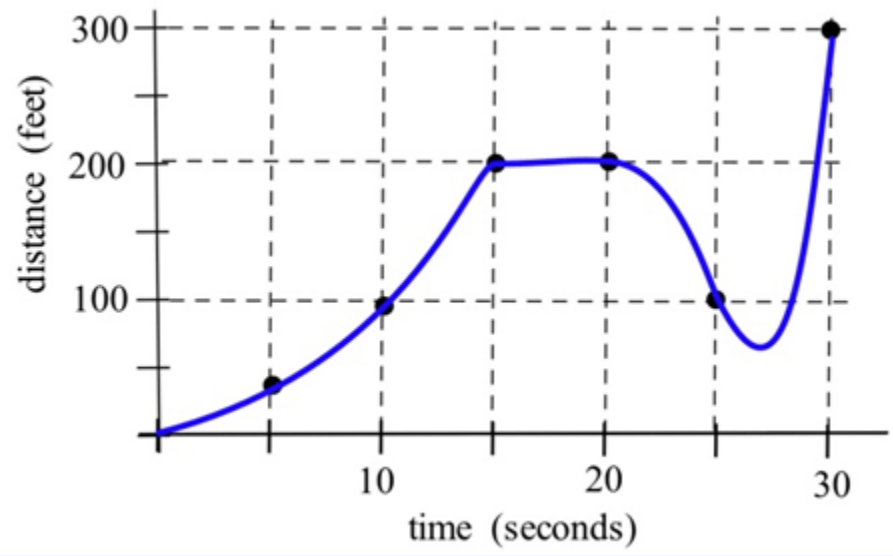


(a) What was the average change in temperature from 9 am to 1 pm?

(b) Estimate how fast the temperature was rising **at** 10 am and **at** 7 pm?

Exercise $16.E.6$

The graph shows the distance of a car from a measuring position located on the edge of a straight road.



(a) What was the average velocity of the car from $t = 0$ to $t = 30$ seconds?

(b) What was the average velocity of the car from $t = 10$ to $t = 30$ seconds?

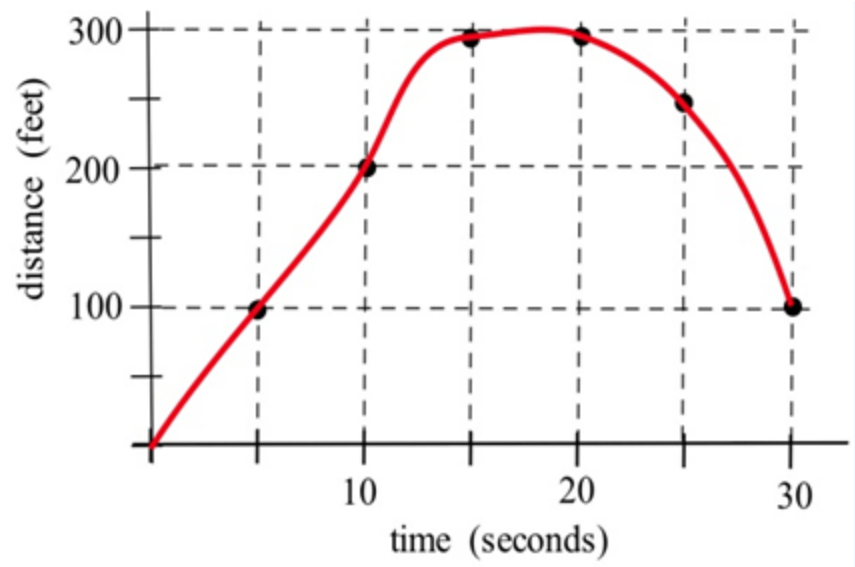
(c) About how fast was the car traveling **at** $t = 10$ seconds? **at** $t = 20$ s? **at** $t = 30$ s?

(d) What does the horizontal part of the graph between $t = 15$ and $t = 20$ seconds mean?

(e) What does the negative velocity at $t = 25$ represent?

Exercise $16.E.7$

The graph shows the distance of a car from a measuring position located on the edge of a straight road.



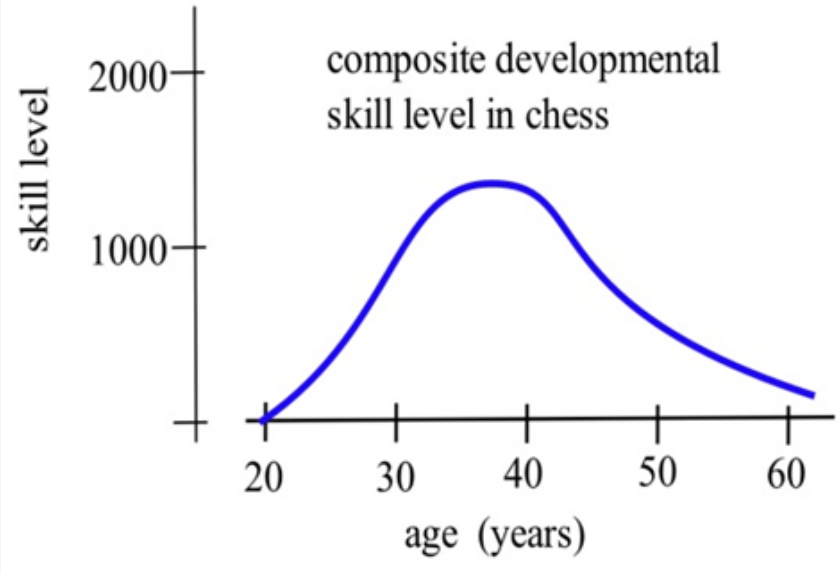
(a) What was the average velocity of the car from $t = 0$ to $t = 20$ seconds?

(b) What was the average velocity from $t = 10$ to $t = 30$ seconds?

(c) About how fast was the car traveling **at** $t = 10$ seconds? **at** $t = 20$ s? **at** $t = 30$ s?

Exercise $16.E.8$

The graph shows the composite developmental skill level of chessmasters at different ages as determined by their performance against other chessmasters. (From "Rating Systems for Human Abilities", by W.H. Batchelder and R.S. Simpson, 1988. UMAP Module 698.)



(a) At what age is the "typical" chessmaster playing the best chess?

(b) At approximately what age is the chessmaster's skill level increasing most rapidly?

(c) Describe the development of the "typical" chessmaster's skill in words.

(d) Sketch graphs which you **think** would reasonably describe the performance levels versus age for an athlete, a classical pianist, a rock singer, a mathematician, and a professional in your major field.

Exercise 16.E.9

Use the function in the graph to fill in the table and then graph $m(x)$.

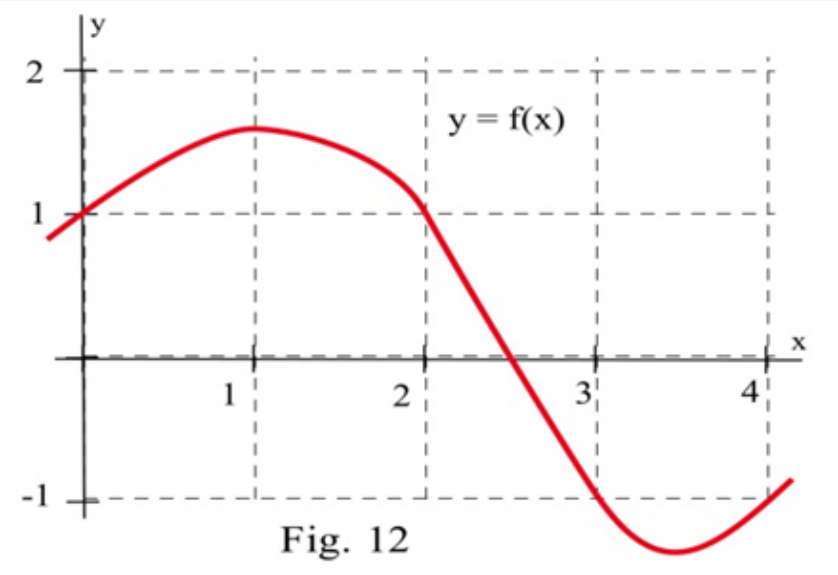
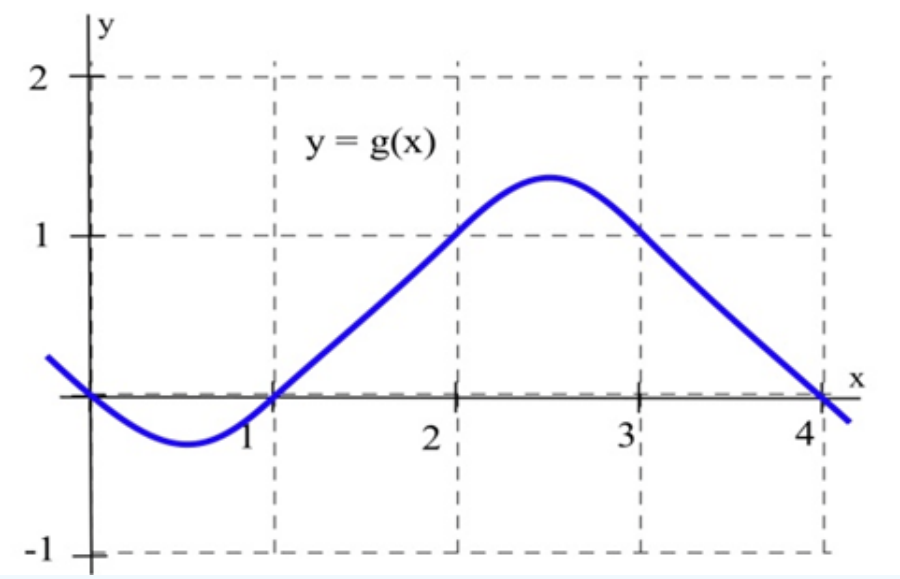


Fig. 12

x	$y = f(x)$	$m(x) =$ the estimated slope of the tangent line to $y = f(x)$ at the point (x, y)
0		
0.5		
1.0		
1.5		
2.0		
2.5		
3.0		
3.5		
4.0		

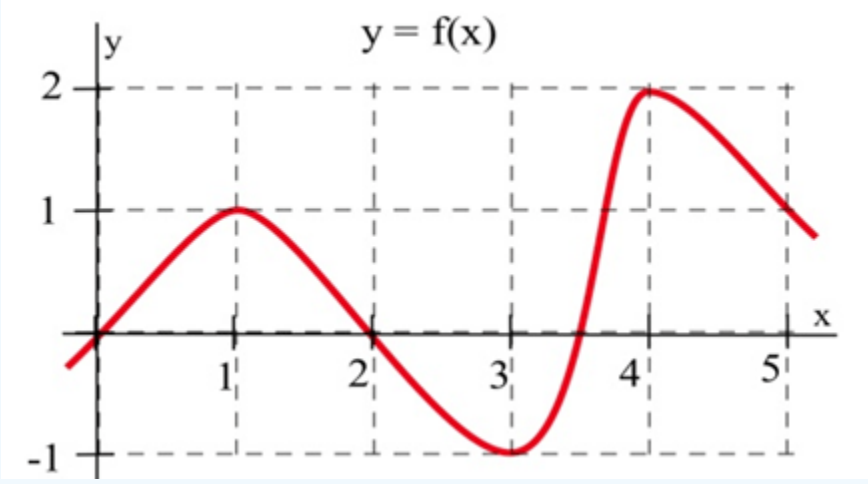
Exercise 16.E.10

Use the function in the graph to fill in the table and then graph $m(x)$.



x	$y = g(x)$	$m(x) =$ the estimated slope of the tangent line to $y = g(x)$ at the point (x, y)
0		
0.5		
1.0		
1.5		
2.0		
2.5		
3.0		
3.5		
4.0		

Exercise 16.E.11

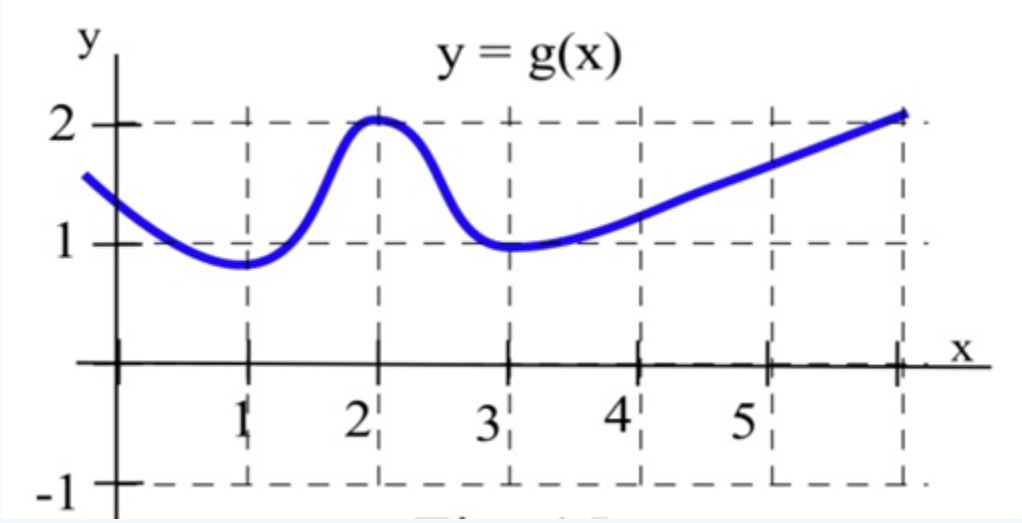


(a) At what values of x does the graph of f in the graph have a horizontal tangent line?

(b) At what value(s) of x is the value of f the largest? smallest?

(c) Sketch the graph of $m(x)$ = the slope of the line tangent to the graph of f at the point (x, y)

Exercise 16.E.12



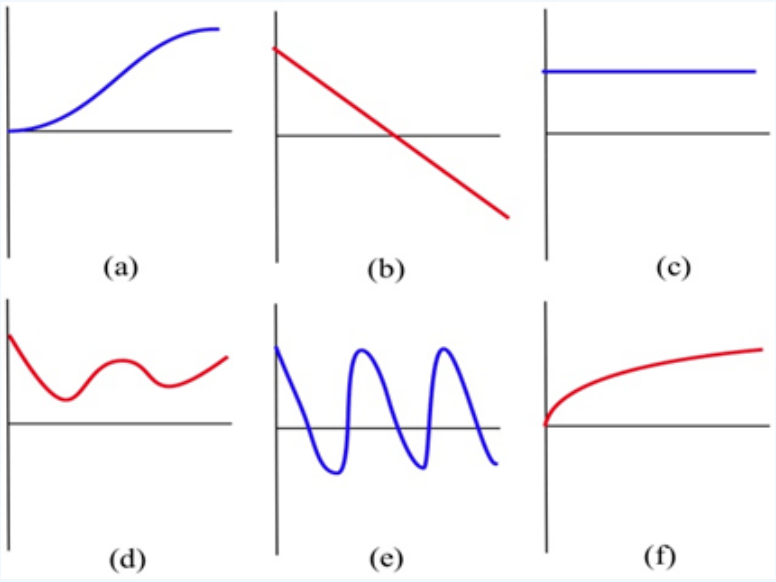
(a) At what values of x does the graph of g have a horizontal tangent line?

(b) At what value(s) of x is the value of g the largest? smallest?

(c) Sketch the graph of $m(x) =$ the slope of the line tangent to the graph of g at the point (x, y).

Exercise 16.E.13

Match the situation descriptions with the corresponding **time-velocity** graph.



(a) A car quickly leaving from a stop sign.

(b) A car sedately leaving from a stop sign.

(c) A student bouncing on a trampoline.

(d) A ball thrown straight up.

(e) A student confidently striding across campus to take a calculus test.

(f) An unprepared student walking across campus to take a calculus test.

Exercise $16.E.14 - 16.E.19$

For each function $f(x)$ in problems 14 – 19, perform steps (a) – (d):

(a) calculate $m_{\text{sec}} = \frac{f(x+h)-f(x)}{h}$ and simplify

(b) determine $m_{\tan} = \lim_{h\to 0} m_{\text{sec}}$

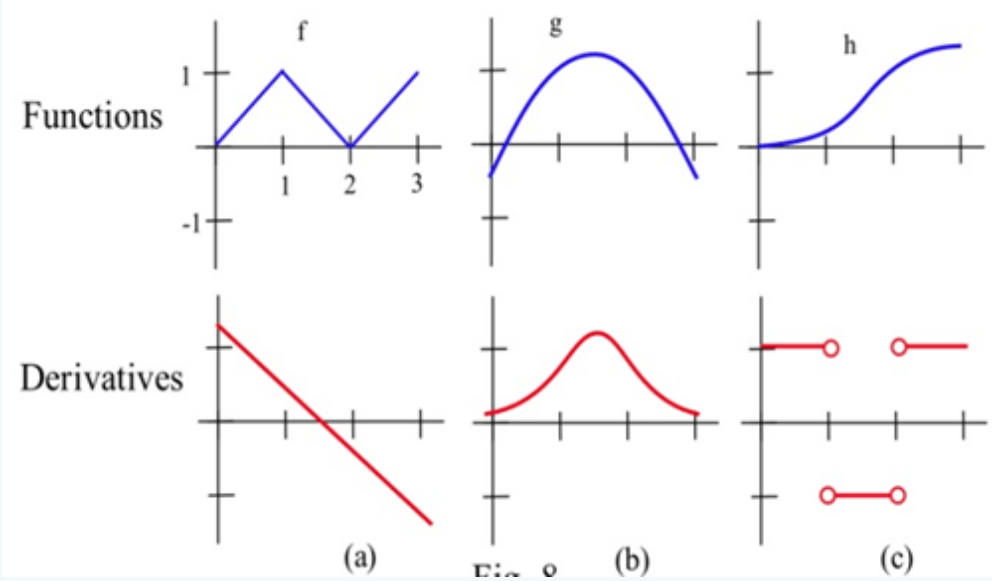
(c) evaluate $m_{\tan}$ at $x = 2$,

(d) find the equation of the line tangent to the graph of f at $(2, f(2))$

14. $f(x) = 3x–7$	15. $f(x) = 2–7x$	16. $f(x) = ax + b$ where a and b are constants
17. $f(x) = x^2 + 3x$	18. $f(x) = 8–3x^2$	19. $f(x) = ax^2 + bx + c$ where a, b and c are constants

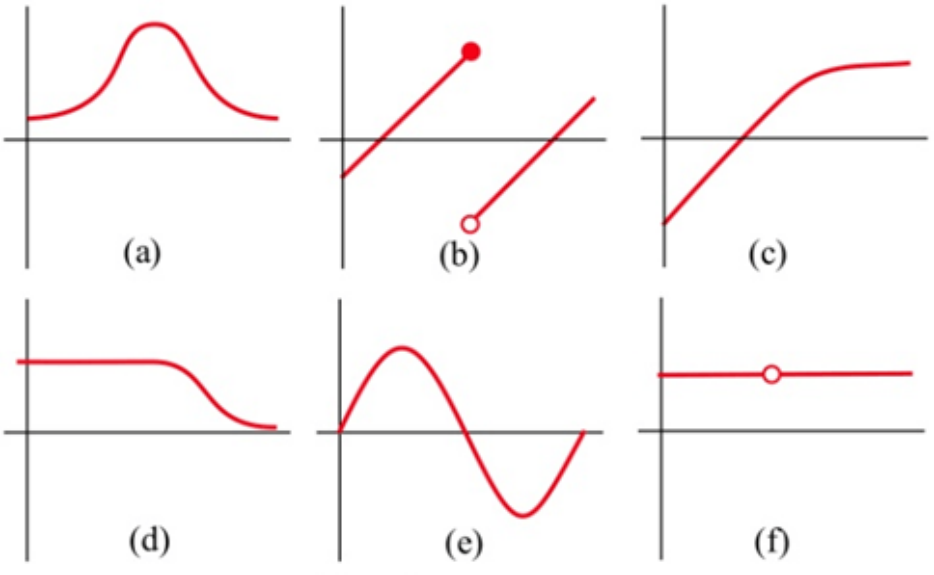
Exercise $16.E.20$

Match the graphs of the three functions below with the graphs of their derivatives.



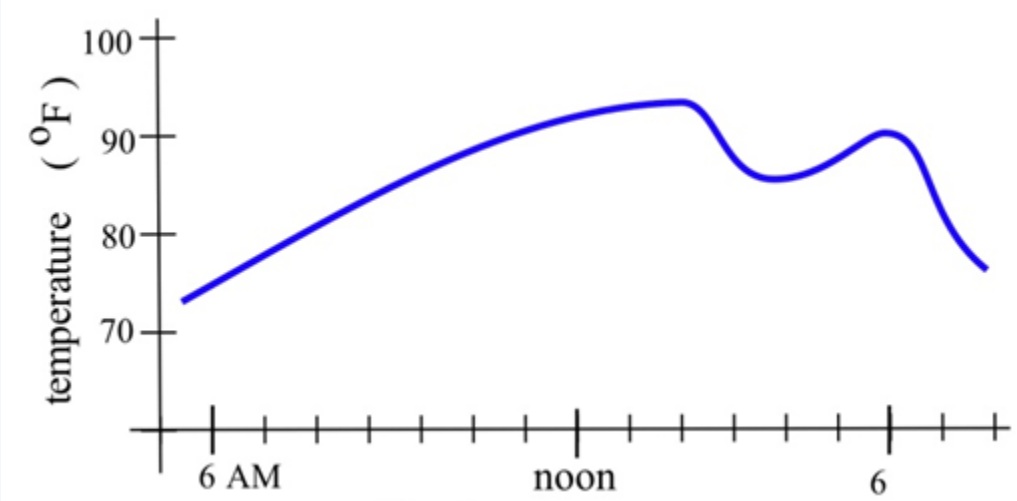
Exercise $16.E.21$

Below are six graphs, three of which are derivatives of the other three. Match the functions with their derivatives.



Exercise 16.E.22

The graph below shows the temperature during a summer day in Chicago. Sketch the graph of the **rate** at which the temperature is changing. (This is just the graph of the **slopes** of the lines which are tangent to the temperature graph.)



Exercise 16.E.23

Fill in the table with the appropriate units for $f'(x)$.

units for x	units for $f(x)$	units for $f'(x)$
hours	miles	
people	automobiles	

dollars	pancakes	
days	trout	
seconds	miles per second	
seconds	gallons	
study hours	test points	

Exercise 16.E.24

If $C(x)$ is the total cost, in millions, of producing x thousand items, interpret $C'(4) = 2$.

Exercise 16.E.25

Suppose $P(t)$ is the number of individuals infected by a disease t days after it was first detected. Interpret $P'(50) = -200$.

2.4 Exercises

Exercise 16.E.1

Fill in the values in the table for $\frac{d}{dx}(3(f(x))$, $\frac{d}{dx}(2f(x) + g(x))$, and $\frac{d}{dx}(3(g(x) - f(x))$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\frac{d}{dx}(3(f(x))$	$\frac{d}{dx}(2f(x) + g(x))$	$\frac{d}{dx}(3(g(x) - f(x))$
0	3	-2	-4	3			
1	2	-1	1	0			
2	4	2	3	1			

Exercise 16.E.2

Find

(a) $D(x^{12})$

(b) $\frac{d}{dx}(\sqrt[7]{x})$

(c) $D(\frac{1}{x^3})$

(d) $\frac{dx^e}{dx}$

Exercise $16.E.3$

Find

(a) $D(x^9)$

(b) $\frac{dx^{2/3}}{dx}$

(c) $D(\frac{1}{x^4})$

(d) $D(x^\pi)$

Exercise $16.E.4 - 16.E.8$

In problems 4 – 8, (a) calculate $f'(1)$ and (b) determine when $f'(x) = 0$.

4. $f(x) = x^2–5x + 13$
5. $f(x) = 5x^2–40x + 73$
6. $f(x) = x^3 + 9x^2 + 6$
7. $f(x) = x^3 + 3x^2 + 3x–1$
8. $f(x) = x^3 + 2x^2 + 2x–1$

Exercise $16.E.9$

Where do $f(x) = x^2–10x + 3$ and $g(x) = x^3–12x$ have horizontal tangent lines?

Exercise $16.E.10$

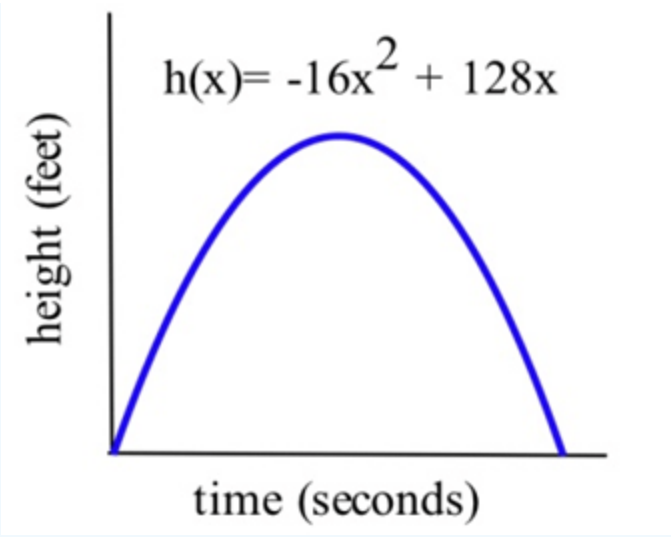
It takes $T(x) = x^2$ hours to weave x small rugs. What is the marginal production time to weave a rug? (Be sure to include the units with your answer.)

Exercise $16.E.11$

It costs $C(x) = \sqrt{x}$ dollars to produce x golf balls. What is the marginal production cost to make a golf ball? What is the marginal production cost when $x = 25$? when $x = 100$? (Include units.)

Exercise $16.E.12$

An arrow shot straight up from ground level with an initial velocity of 128 feet per second will be at height $h(x) = –16x^2 + 128x$ feet at x seconds.



(a) Determine the velocity of the arrow when $x = 0$, 1 and 2 seconds.

(b) What is the velocity of the arrow, $v(x)$, at any time x?

(c) At what time x will the velocity of the arrow be 0?

(d) What is the greatest height the arrow reaches?

(e) How long will the arrow be aloft?

(f) Use the answer for the velocity in part (b) to determine the acceleration, $a(x) = v'(x)$, at any time x.

Exercise $16.E.13$

If an arrow is shot straight up from ground level on the moon with an initial velocity of 128 feet per second, its height will be $h(x) = -2.65x^2 + 128x$ feet at x seconds. Do parts (a) – (e) of problem 40 using this new equation for h.

Exercise $16.E.14$

$f(x) = x^3 + Ax^2 + Bx + C$ with constants A, B and C. Can you find conditions on the constants A, B and C which will guarantee that the graph of $y = f(x)$ has two distinct "vertices"? (Here a "vertex" means a place where the curve changes from increasing to decreasing or from decreasing to increasing.)

2.5 Exercises

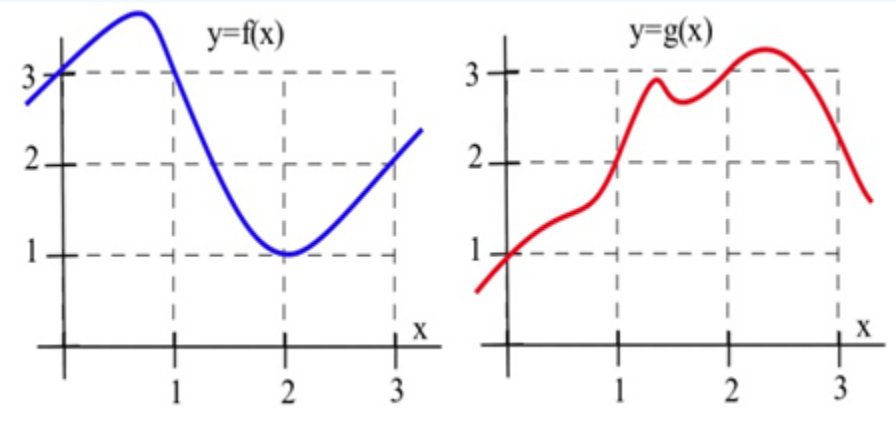
Exercise $16.E.1$

Use the values in the table to fill in the rest of the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\frac{d}{dx}(f(x) \cdot g(x))$	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$	$\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right)$
0	3	-2	-4	3			
1	2	-1	1	0			

2	4	2	3	1			

Exercise $16.E.2 - 16.E.3$



2. Use the information in the graph to plot the values of the functions $f+g$, $f \cdot g$ and f/g and their derivatives at $x = 1$, 2 and 3.

3. Use the information in the graph to plot the values of the functions $2f$, $f–g$ and g/f and their derivatives at $x = 1$, 2 and 3.

Exercise $16.E.4$

Calculate $\frac{d}{dx}((x-5)(3x+7))$ by (a) using the product rule and (b) expanding the product and then differentiating. Verify that both methods give the same result.

Exercise $16.E.5$

If the product of f and g is a constant $(f(x) \cdot g(x) = k$ for all $x)$, then how are $\frac{\frac{d}{dx}(f(x))}{f(x)}$ and $\frac{\frac{d}{dx}(g(x))}{g(x)}$ related?

Exercise $16.E.6$

If the quotient of f and g is a constant ($\frac{f(x)}{g(x)} = k$ for all $x)$, then how are $g \cdot f'$ and $f \cdot g'$ related?

Exercise $16.E.7 - 16.E.8$

In problems 7 – 8, (a) calculate $f'(1)$ and (b) determine when $f'(x) = 0$

7. $f(x) = \frac{7x}{x^2+4}$	8. $f(x) = \frac{3x^2}{2x-3}$

Exercise $16.E.9$

Determine $\frac{d}{dx}(x^2+1)(7x-3)$ and $\frac{d}{dt}(\frac{3t-2}{5t+1})$.

Exercise 16.E.10

Find (a) $\frac{d}{dx}(x^3e^x)$ and (b) $\frac{d}{dx}(e^x)^3$.

Exercise 16.E.11

Find (a) $\frac{d}{dt}(te^t)$, (b) $d(e^x)^5$

Exercise 16.E.12

A manufacturer has determined that an employee with d days of production experience will be able to produce approximately $P(d) = 3 + 15(1–e^{-0.2d})$ items per day. Graph $P(d)$.

(a) Approximately how many items will a beginning employee be able to produce each day?

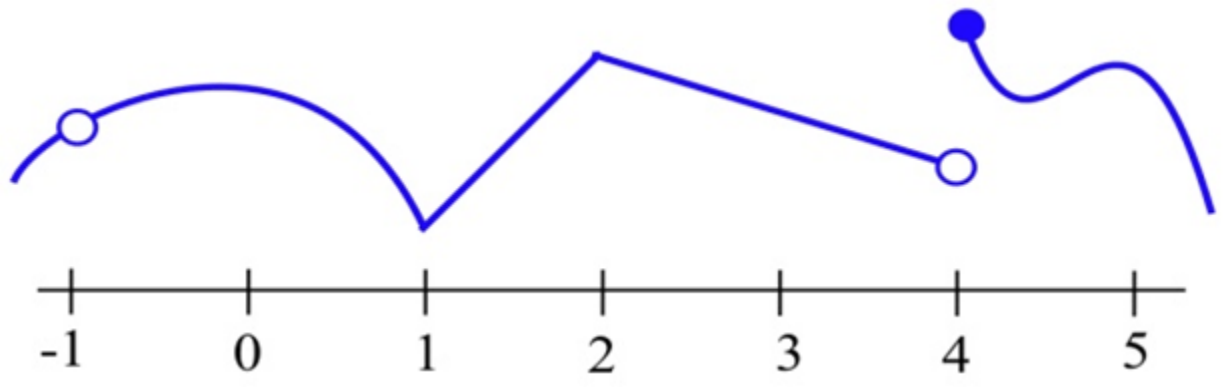
(b) How many items will an experienced employee be able to produce each day?

(c) What is the marginal production rate of an employee with 5 days of experience? (What are the units of your answer, and what does this answer mean?

2.6 Exercises

Exercise 16.E.1

The graph of $y = f(x)$ is shown.

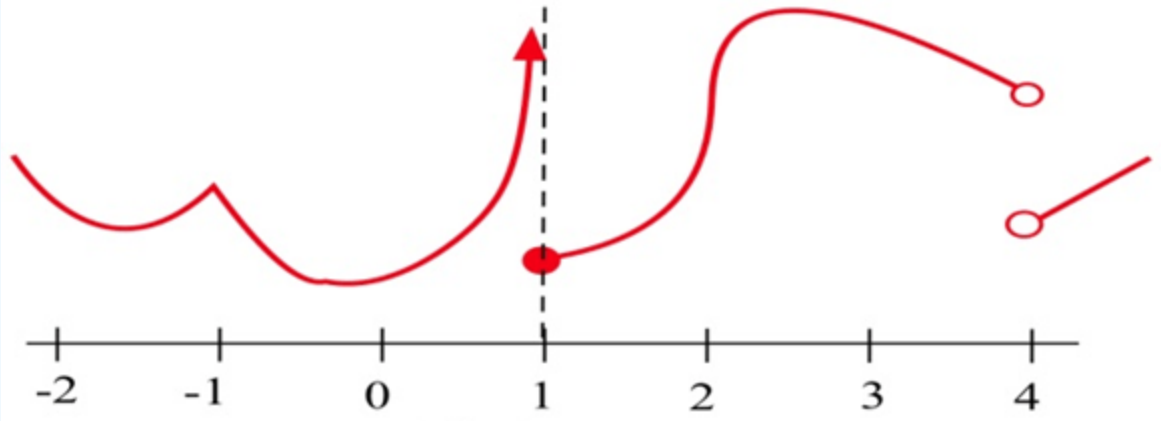


(a) At which integers is f continuous?

(b) At which integers is f differentiable?

Exercise 16.E.2

The graph of $y = g(x)$ is shown.



(a) At which integers is g continuous?

(b) At which integers is g differentiable?

Exercise $16.E.3 - 16.E.4$

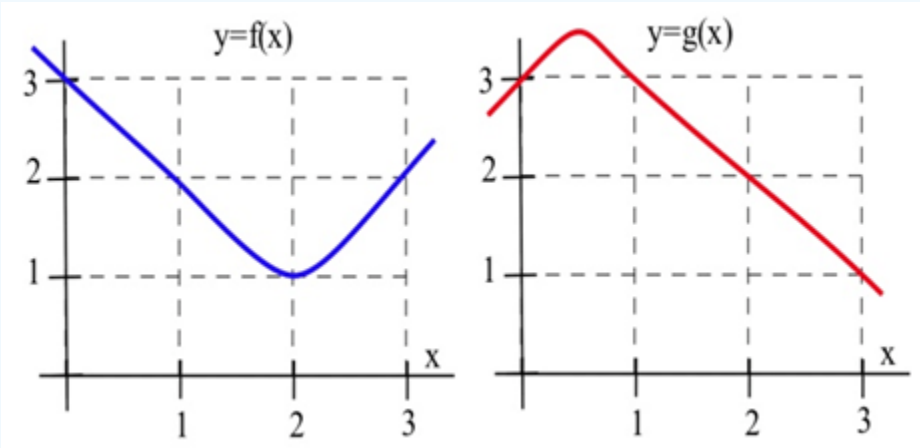
Problems 3 and 4 refer to the values given in this table:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
-2	2	-1	1	1		
-1	1	2	0	2		
0	-2	1	2	-1		
1	0	-2	-1	2		
2	1	0	1	-1		

3. Use the table of values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1$ and 2.

4. Use the table of values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x =$ –2, –1 and 0.

Exercise $16.E.5 - 16.E.6$



5. Use the graphs to estimate the values of $g(x)$, $g'(x)$, $(f \circ g)(x)$, $f'(g(x))$, and $(f \circ g)'(x)$ at $x = 1$.

6. Use the graphs to estimate the values of $g(x)$, $\mathbf{g'(x)}$, $(f \circ g)(x)$, $\mathbf{f'(g(x))}$, and $\mathbf{(f \circ g)'(x)}$ for $x = 2$.

Exercise $16.E.7 - 16.E.12$

In problems 7 – 12, find the derivative of each function.

7. $f(x) = (2x–8)^5$	8. $f(x) = (6x–x^2)^{10}$	9. $f(x) = x \cdot (3x + 7)^5$
10. $f(x) = (2x + 3)^6 \cdot (x–2)^4$	11. $f(x) = \sqrt{x^2 + 6x - 1}$	12. $f(x) = \frac{x–5}{(x+3)^4}$

Exercise $16.E.13$

If f is a differentiable function,

(a) how are the graphs of $y = f(x)$ and $y = f(x) + k$ related?

(b) how are the derivatives of $f(x)$ and $f(x) + k$ related?

2.7 Exercises

Exercise $16.E.1 - 16.E.2$

In problems 1 and 2, each quotation is a statement about a quantity of something changing over time. Let $f(t)$ represent the quantity at time t. For each quotation, tell what f represents and whether the first and second derivatives of f are positive or negative.

1. (a) "Unemployment rose again, but the rate of increase is smaller than last month."

(b) "Our profits declined again, but at a slower rate than last month."

(c) "The population is still rising and at a faster rate than last year."

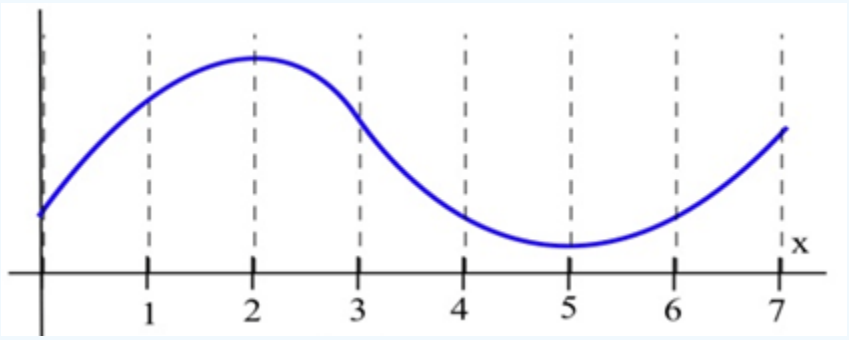
2. (a) "The child's temperature is still rising, but slower than it was a few hours ago."

(b) "The number of whales is decreasing, but at a slower rate than last year."

(c) "The number of people with the flu is rising and at a faster rate than last month."

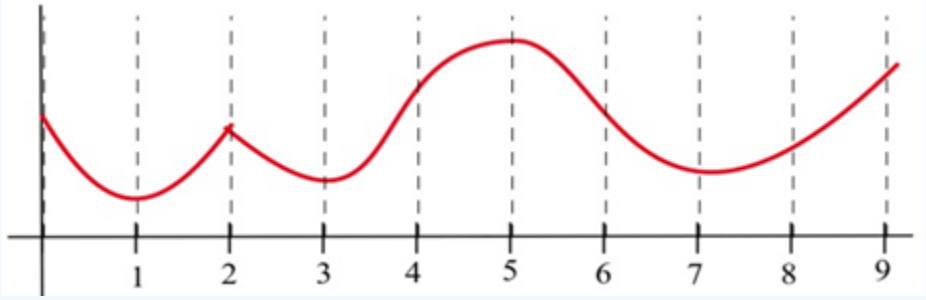
Exercise $16.E.3$

On which intervals is the function in the graph (a) concave up? (b) concave down?



Exercise 16.E.4

On which intervals is the function in graph (a) concave up? (b) concave down?



Exercise 16.E.5

Sketch the graphs of functions which are defined and concave up everywhere and which have

(a) no roots.

(b) exactly 1 root.

(c) exactly 2 roots.

(d) exactly 3 roots.

Exercise 16.E.6 – 16.E.9

In problems 6 – 9, a function and values of x so that $f'(x) = 0$ are given. Use the Second Derivative Test to determine whether each point $(x, f(x))$ is a local maximum, a local minimum or neither

6. $f(x) = 2x^3–15x^2 + 6, x = 0, 5.$
7. $g(x) = x^3–3x^2–9x + 7, x = –1, 3.$
8. $h(x) = x^4–8x^2–2, x = –2, 0, 2.$
9. $f(x) = x \cdot \ln(x), x = 1/e.$

Exercise 16.E.10

Which of the labeled points in the graph are inflection points?

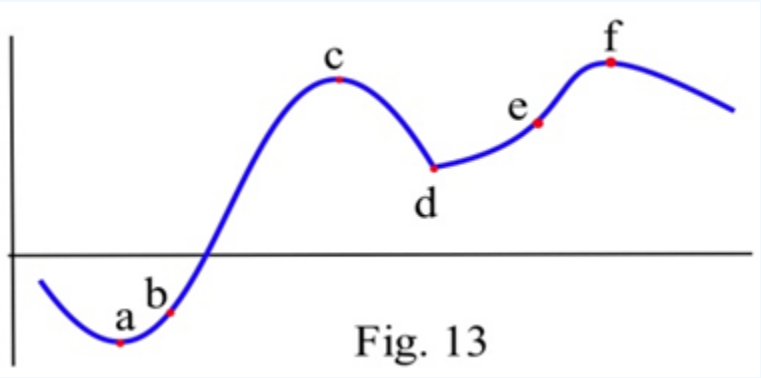
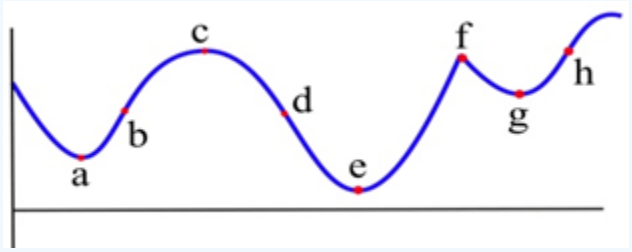


Fig. 13

Exercise 16.E.11

Which of the labeled points in the graph are inflection points?



Exercise 16.E.12

How many inflection points can a

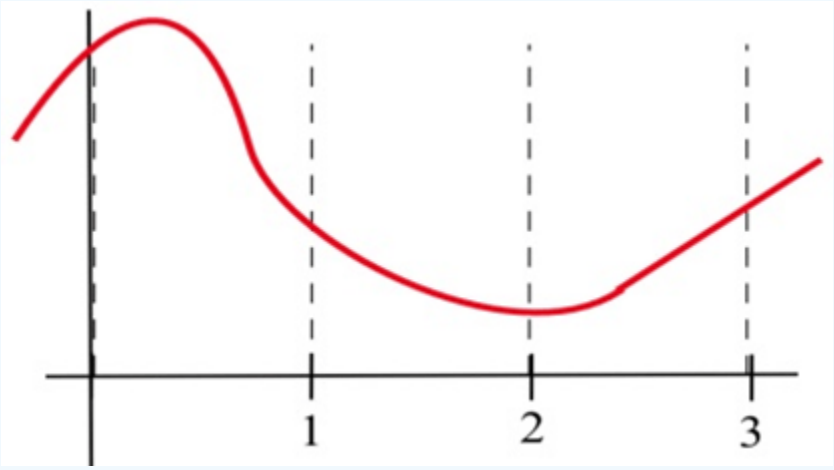
(a) quadratic polynomial have?

(b) cubic polynomial have?

(c) polynomial of degree n have?

Exercise 16.E.13

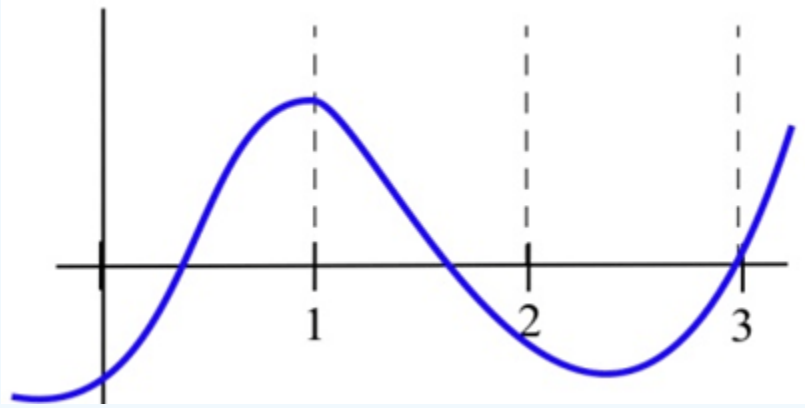
Fill in the table with "+", "−", or "0" for the function shown.



x	$f(x)$	$f'(x)$	$f''(x)$
0			
1			
2			
3			

Exercise 16.E.14

Fill in the table with "+", "−", or "0" for the function shown.



x	$g(x)$	$g'(x)$	$g''(x)$
0			
1			
2			
3			

Exercise 16.E.15 – 16.E.21

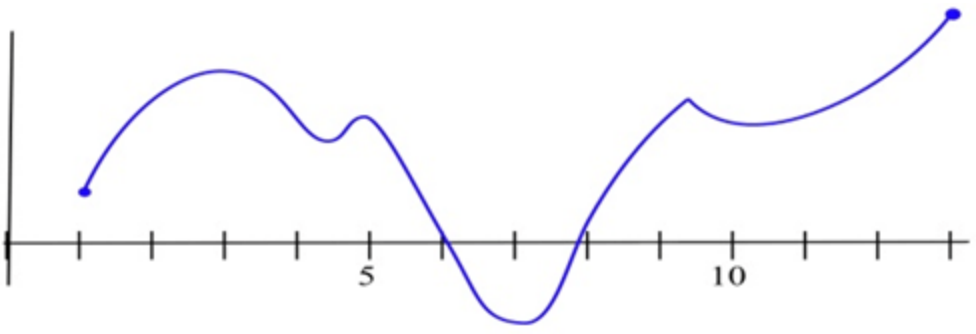
In problems 15 – 21, find the derivative and second derivative of each function.

15. $f(x) = 7x^2 + 5x - 3$
16. $f(x) = (2x - 8)^5$
17. $f(x) = (6x - x^2)^{10}$
18. $f(x) = x \cdot (3x + 7)^5$
19. $f(x) = (2x^3 + 3)^6$
20. $f(x) = \sqrt{x^2 + 6x - 1}$
21. $f(x) = \ln(x^2 + 4)$

2.8 Exercises

Exercise 16.E.1

Find all of the critical points of the function shown and identify them as local max, local min, or neither. Find the global max and min on the interval.



Exercise $16.E.2$

Find all of the critical points of the function shown and identify them as local max, local min, or neither. Find the global max and min on the interval.

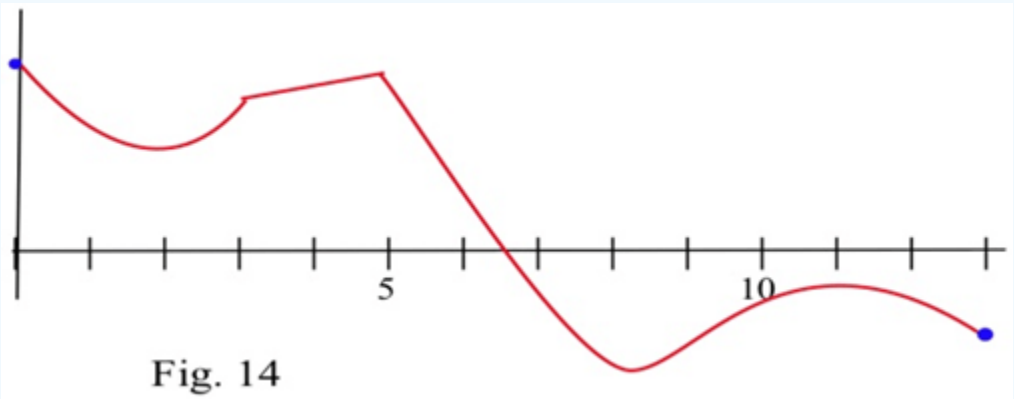


Fig. 14

Exercise $16.E.3 - 16.E.8$

In problems 3 – 8, find all of the critical points and local maximums and minimums of each function.

3. $f(x) = x^2 + 8x + 7$	4. $f(x) = 2x^2–12x + 7$
5. $f(x) = x^3–6x^2 + 5$	6. $f(x) = (x–1)^2(x–3)$
7. $f(x) = \ln(x^2–6x + 11)$	8. $f(x) = 2x^3–96x + 42$

Exercise $16.E.9 - 16.E.16$

In problems 9 – 16, find all critical points and global extremes of each function on the given intervals.

9. $f(x) = x^2–6x + 5$ on the entire real number line.
10. $f(x) = 2–x^3$ on the entire real number line.
11. $f(x) = x^3–3x + 5$ on the entire real number line.
12. $f(x) = x - e^x$ on the entire real number line.
13. $f(x) = x^2–6x + 5$ on [–2, 5].
14. $f(x) = 2–x^3$ on [–2, 1].
15. $f(x) = x^3–3x + 5$ on [–2, 1].

16. $f(x) = x - e^x$ on [1, 2].

Exercise 16.E.17

Suppose $f(1) = 5$ and $f'(1) = 0$. What can we conclude about the point (1,5) if

(a) $f'(x) < 0$ for $x < 1$, and $f'(x) > 0$ for $x > 1$?

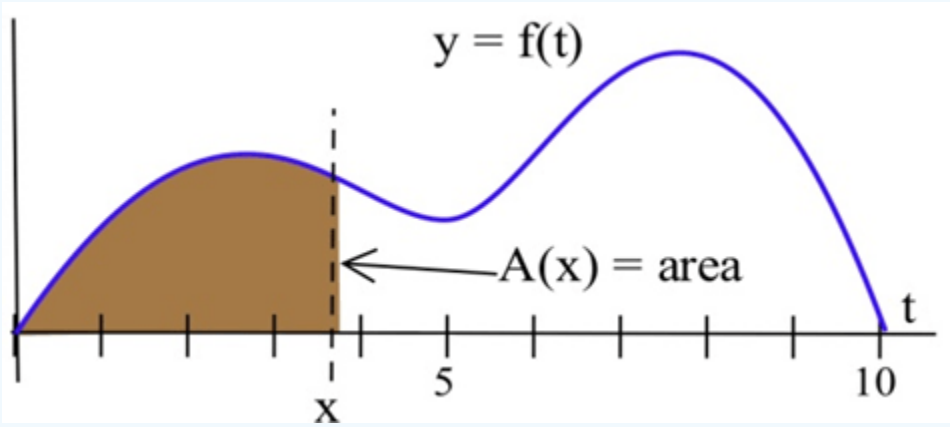
(b) $f'(x) < 0$ for $x < 1$, and $f'(x) < 0$ for $x > 1$?

(c) $f'(x) > 0$ for $x < 1$, and $f'(x) < 0$ for $x > 1$?

(d) $f'(x) > 0$ for $x < 1$, and $f'(x) > 0$ for $x > 1$?

Exercise 16.E.18

Define $A(x)$ to be the **area** bounded between the x–axis, the graph of f, and a vertical line at x.

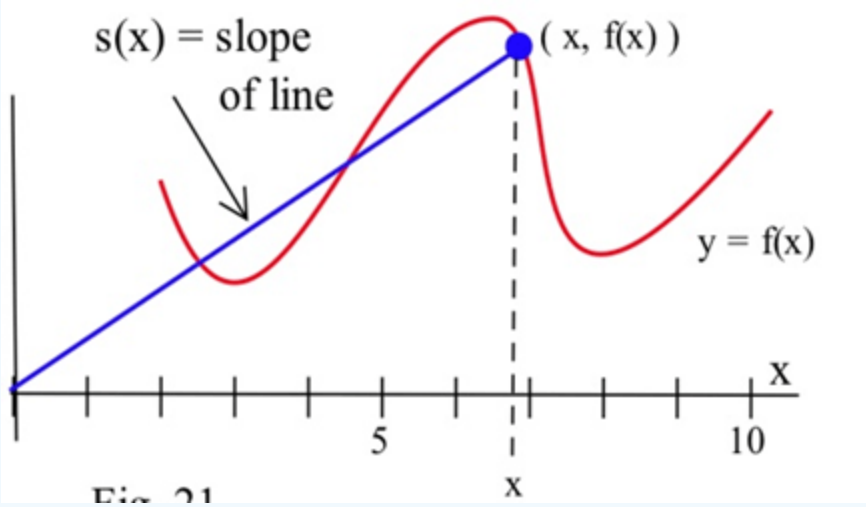


(a) At what value of x is $A(x)$ minimum?

(b) At what value of x is $A(x)$ maximum?

Exercise 16.E.19

Define $S(x)$ to be the **slope** of the line through the points $(0, 0)$ and $(x, f(x))$ based on the graph of f shown.

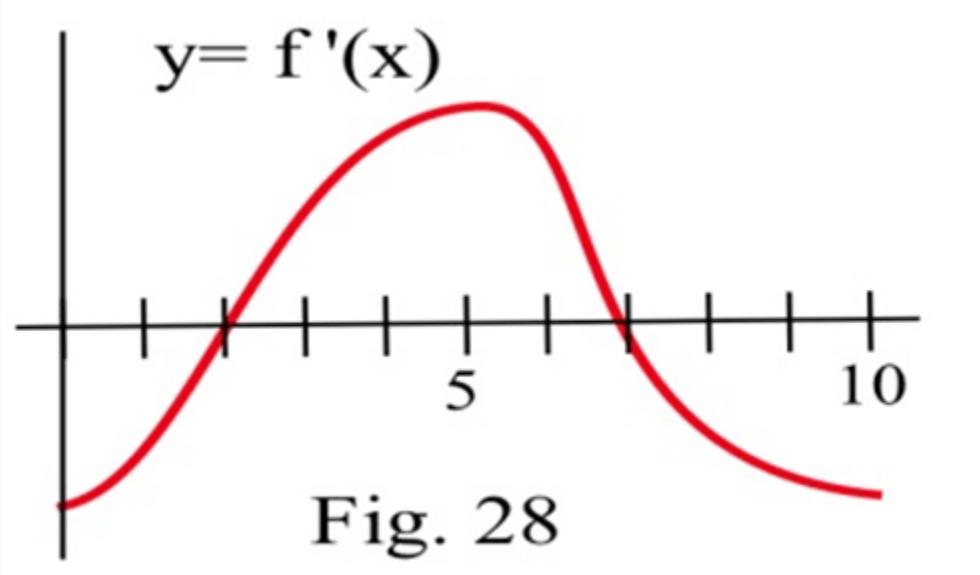


(a) At what value of x is $S(x)$ minimum?

(b) At what value of x is $S(x)$ maximum?

Exercise $16.E.20$

The graph of the **derivative** of a continuous function f.



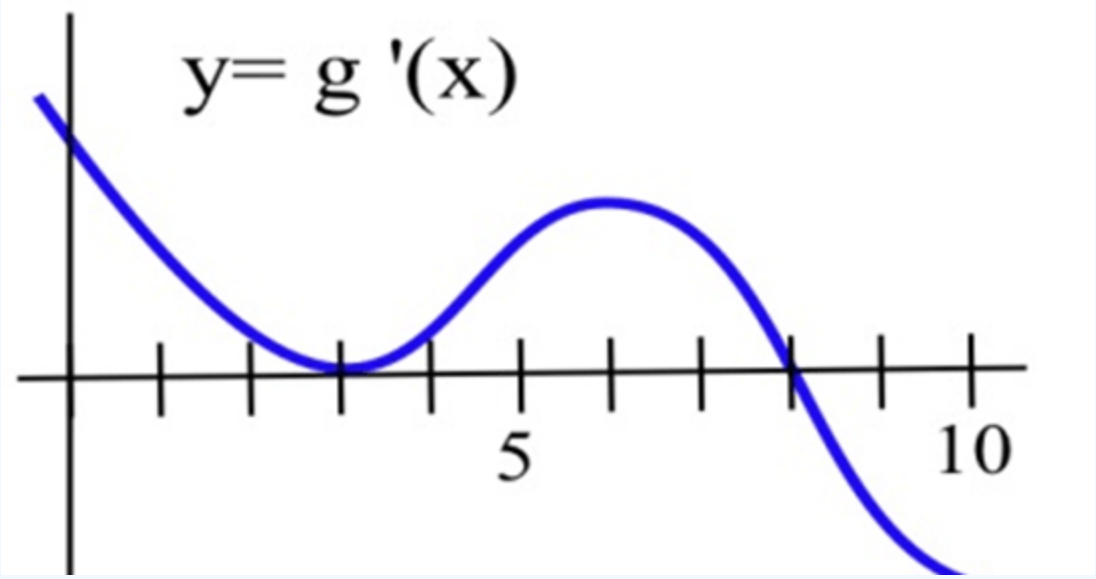
(a) List the critical numbers of f.

(b) For what values of x does f have a local maximum?

(c) For what values of x does f have a local minimum?

Exercise $16.E.21$

The graph of the **derivative** of a continuous function g.



(a) List the critical numbers of g.

(b) For what values of x does g have a local maximum?

(c) For what values of x does g have a local minimum?

Exercise $16.E.22 - 16.E.24$

In problems 22 – 24, a function and values of x so that $f'(x) = 0$ are given. Use the Second Derivative Test to determine whether each point $(x, f(x))$ is a local maximum, a local minimum or neither

22. $f(x) = 2x^3 – 15x^2 + 6, x = 0, 5.$
23. $g(x) = x^3 – 3x^2 – 9x + 7, x = –1, 3.$
24. $h(x) = x^4 – 8x^2 – 2, x = –2, 0, 2.$

2.9 Exercises

Exercise $16.E.1$

Sketch the graph of a continuous function f so that

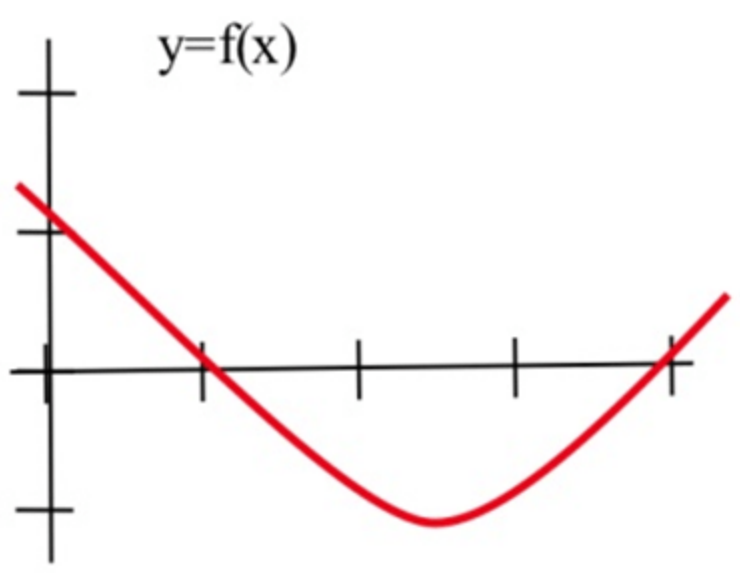
(a) $f(1) = 3$, $f'(1) = 0$, and the point (1,3) is a local maximum of f.

(b) $f(2) = 1$, $f'(2) = 0$, and the point (2,1) is a local minimum of f.

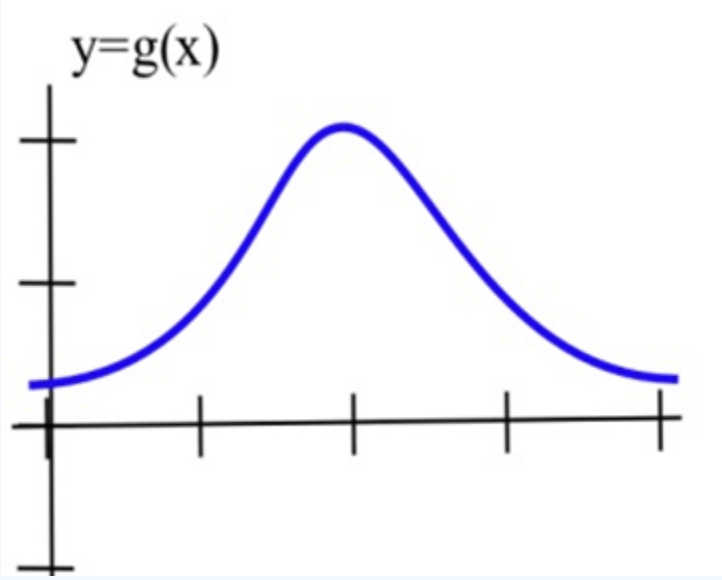
(c) $f(5) = 4$, $f'(5) = 0$, and the point (5,4) is not a local minimum or maximum of f.

Exercise $16.E.2 - 16.E.4$

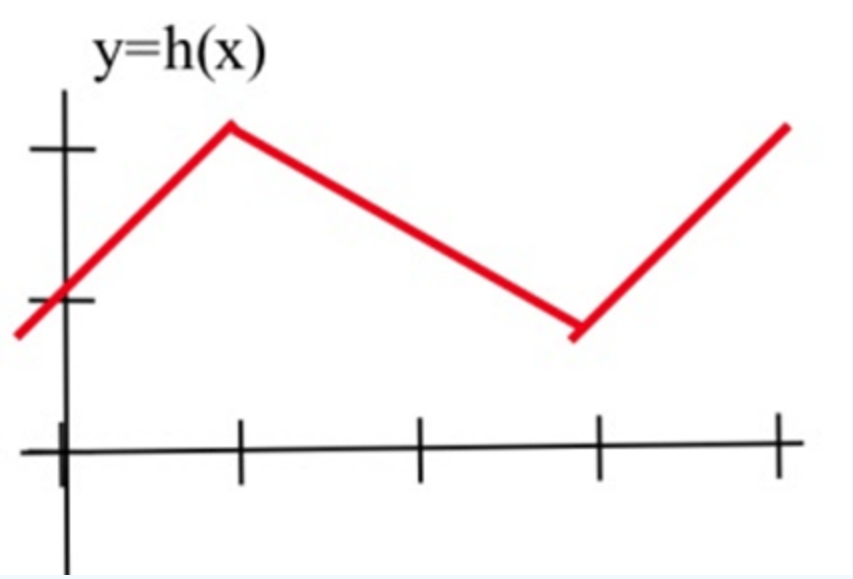
In problems 2–4, sketch the graph of the derivative of each function.



2.



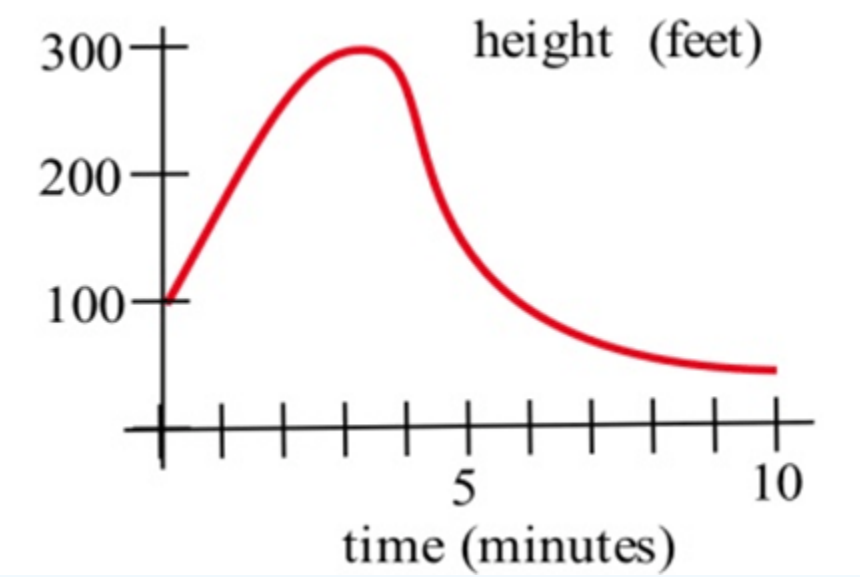
3.



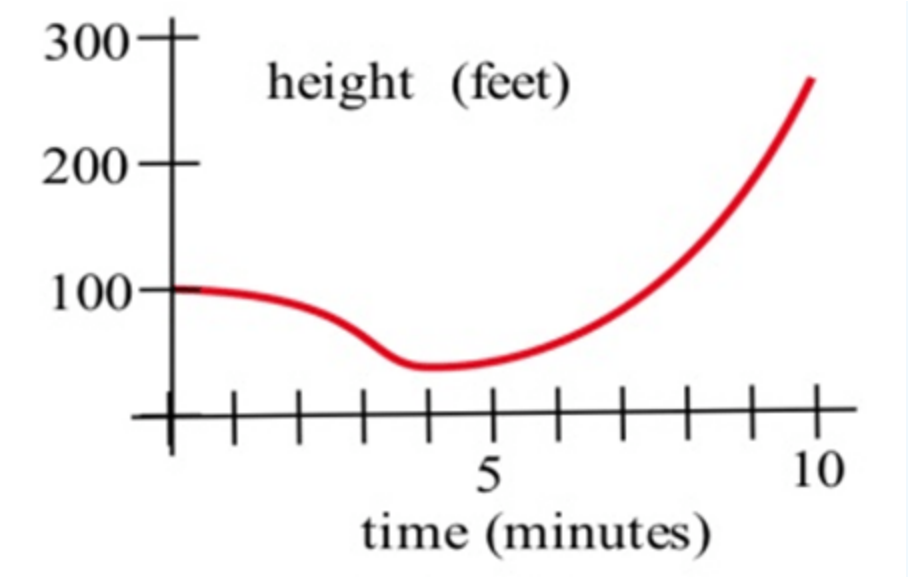
4.

Exercise $16.E.5 - 16.E.7$

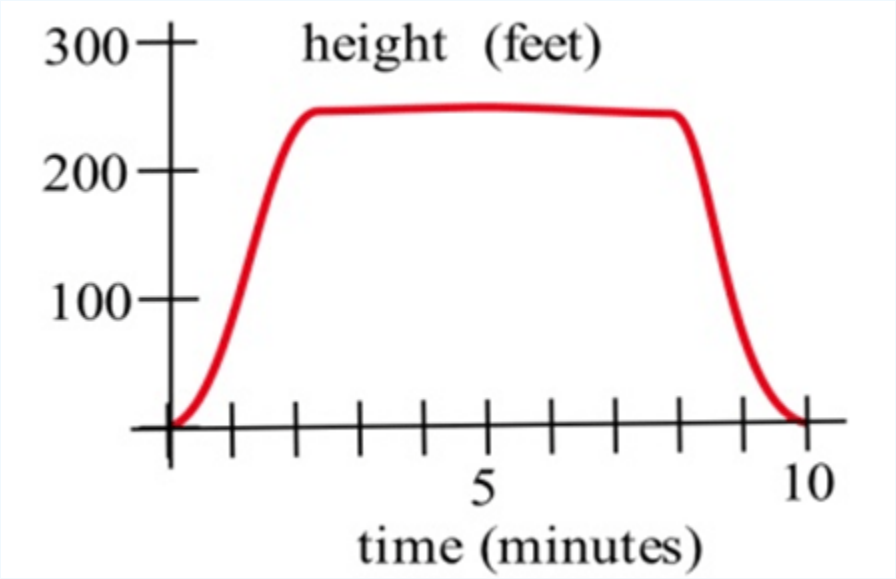
In problems 5–7, the graph of the height of a helicopter is shown. Sketch the graph of the upward velocity of the helicopter.



5.



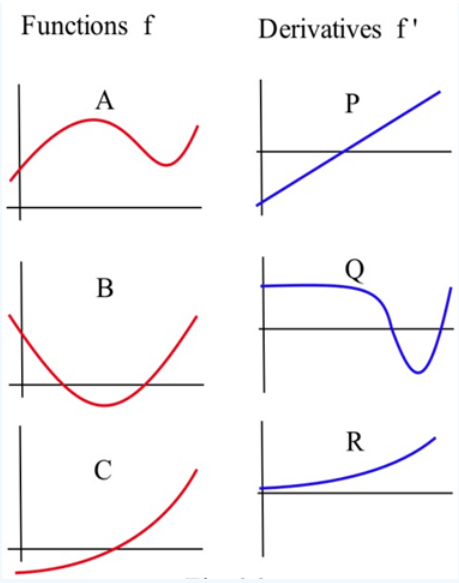
6.



7.

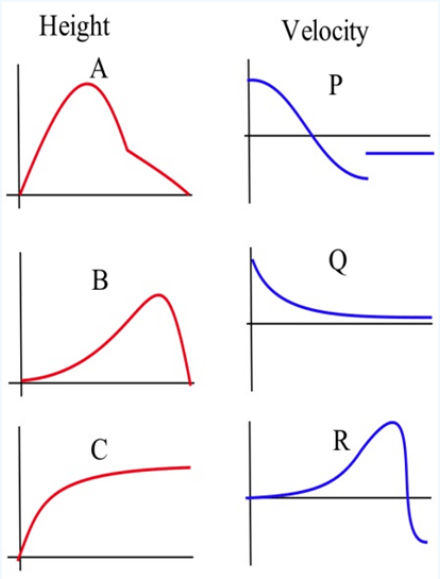
Exercise 16.E.8

In the graphs to the right, match the graphs of the functions with those of their derivatives



Exercise $16.E.9$

In the graphs below, match the graphs showing the heights of rockets with those showing their velocities.



Exercise $16.E.10$ – $16.E.14$

In problems 10 – 14, use information from the derivatives of each function to help you graph the function. Find all local maximums and minimums of each function.

10. $f(x) = x^3 - 3x^2 - 9x - 5$	11. $g(x) = 2x^3 - 15x^2 + 6$	12. $h(x) = x^4 - 8x^2 + 3$
13. $r(t) = \frac{2}{t^2+1}$	14. $f(x) = \frac{x^2+3}{x}$	

2.10 Exercises

Exercise 16.E.1

(a) You have 200 feet of fencing available to construct a rectangular pen with a fence divider down the middle (see below). What dimensions of the pen enclose the largest total area?

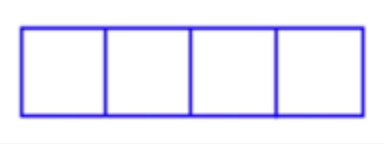
(b) If you need 2 dividers, what dimensions of the pen enclose the largest area?

(c) What are the dimensions in parts (a) and (b) if one edge of the pen borders on a river and does not require any fencing?



Exercise 16.E.2

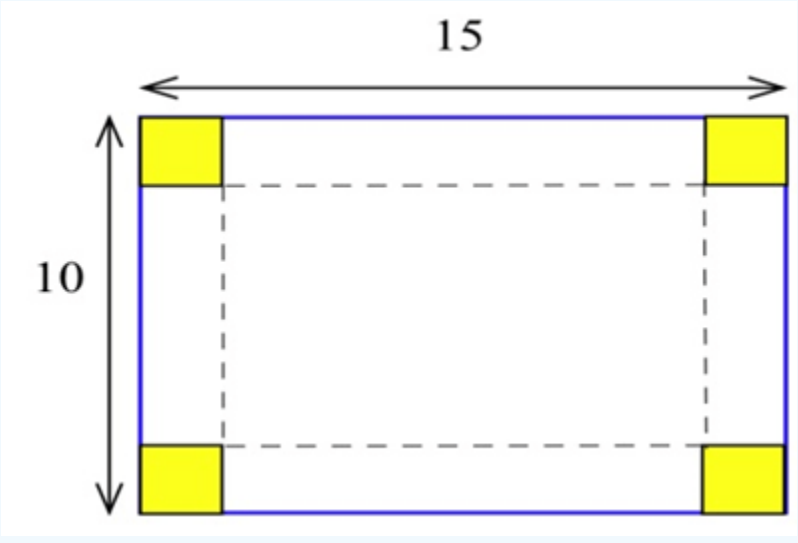
You have 120 feet of fencing to construct a pen with 4 equal sized stalls. If the pen is rectangular and shaped like the one below, what are the dimensions of the pen of largest area and what is that area?



Exercise 16.E.3

Suppose you decide to fence the rectangular garden in the corner of your yard. Then two sides of the garden are bounded by the yard fence which is already there, so you only need to use the 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions of the rectangular garden of largest area in the corner of the yard if you have F feet of new fencing available?

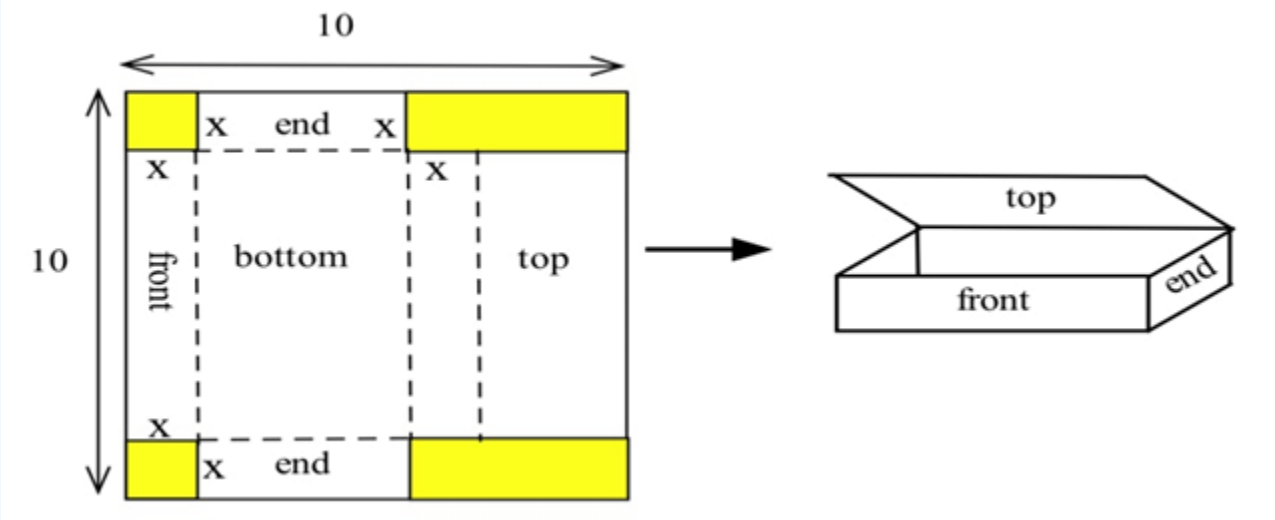
Exercise 16.E.4



(a) You have a 10 inch by 15 inch piece of tin which you plan to form into a box (without a top) by cutting a square from each corner and folding up the sides. How much should you cut from each corner so the resulting box has the greatest volume?

(b) If the piece of tin is A inches by B inches, how much should you cut from each corner so the resulting box has the greatest volume?

Exercise $16.E.5$



You have a 10 inch by 10 inch piece of cardboard which you plan to cut and fold as shown to form a box with a top. Find the dimensions of the box which has the largest volume.

Exercise $16.E.6$

(a) You have been asked to bid on the construction of a square-bottomed box with no top which will hold 100 cubic inches of water. If the bottom and sides are made from the same material, what are the dimensions of the box which uses the least material? (Assume that no material is wasted.)

(b) Suppose the box in part (a) uses different materials for the bottom and the sides. If the bottom material costs 5¢ per square inch and the side material costs 3¢ per square inch, what are the dimensions of the least expensive box which will hold 100 cubic inches of water?

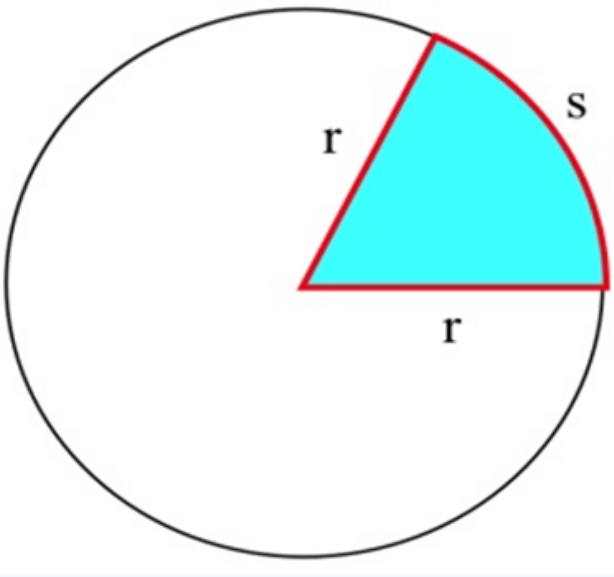
Exercise $16.E.7$

(a) Determine the dimensions of the least expensive cylindrical can which will hold 100 cubic inches if the materials cost 2¢, 5¢ and 3¢ respectively for the top, bottom and sides.

(b) How do the dimensions of the least expensive can change if the bottom material costs more than 5¢ per square inch?

Exercise $16.E.8$

You have 100 feet of fencing to build a pen in the shape of a circular sector, the "pie slice" shown. The area of such a sector is $(rs)/2$. What value of r maximizes the enclosed area?



Exercise 16.E.9

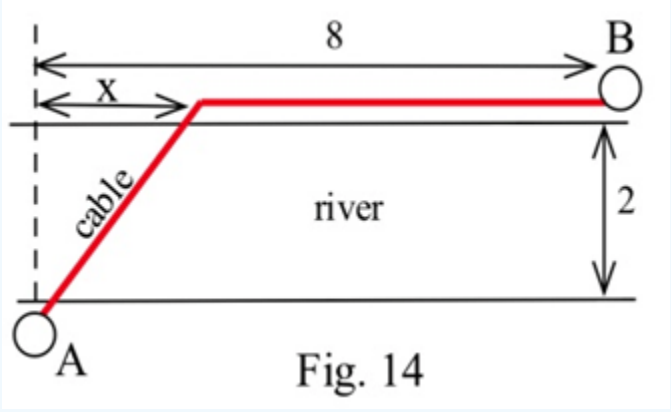


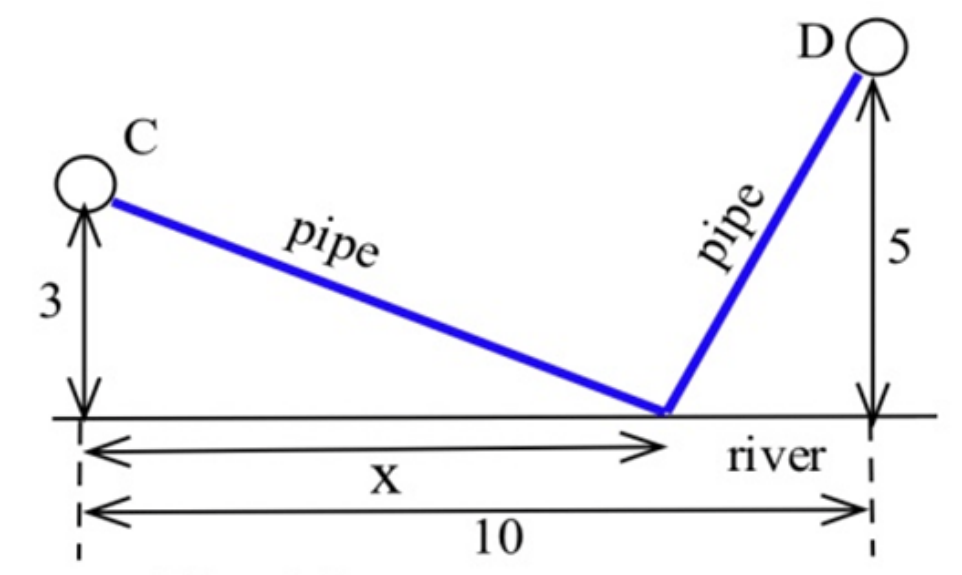
Fig. 14

(a) You have been asked to determine the least expensive route for a telephone cable which connects Andersonville with Beantown. If it costs $5000 per mile to lay the cable on land and $8000 per mile to lay the cable across the river and the cost of the cable is negligible, find the least expensive route.

(b) What is the least expensive route if the cable costs $7000 per mile plus the cost to lay it.

Exercise 16.E.10

You have been asked to determine where a water works should be built along a river between Chesterville and Denton to minimize the total cost of the pipe to the towns.

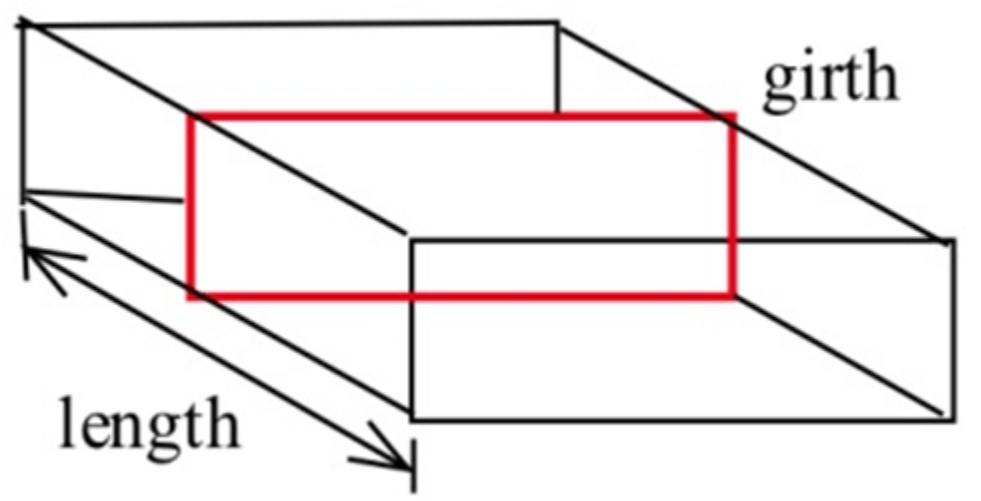


(a) Assume that the same size (and cost) pipe is used to each town. (This part can be done quickly without using calculus.)

(b) Assume that the pipe to Chesterville costs $3000 per mile and to Denton it costs $7000 per mile.

Exercise 16.E.11

U.S. postal regulations state that the sum of the length and girth (distance around) of a package must be no more than 108 inches.



(a) Find the dimensions of the acceptable box with a square end which has the largest volume.

(b) Find the dimensions of the acceptable box which has the largest volume if its end is a rectangle twice as long as it is width.

(c) Find the dimensions of the acceptable box with a circular end which has the largest volume.

Exercise $16.E.12$

D. Simonton claims that the "productivity levels" of people in different fields can be described as a function of their "career age" t by $p(t) = e^{-at} - e^{-bt}$ where a and b are constants which depend on the field of work, and career age is approximately 20 less than the actual age of the individual.

(a) Based on this model, at what ages do mathematicians $(a = .03, b = .05)$, geologists $(a = .02, b = .04)$, and historians $(a = .02, b = .03)$ reach their maximum productivity?

(b) Simonton says "With a little calculus we can show that the curve $(p(t))$ maximizes at $t = \frac{1}{b-a}\ln(\frac{b}{a})$." Use calculus to show that Simonton is correct.

Note: Models of this type have uses for describing the behavior of groups, but it is dangerous and usually invalid to apply group descriptions or comparisons to **individuals** in the group.

(**Scientific Genius**, by Dean Simonton, Cambridge University Press, 1988, pp. 69 – 73)

Exercise $16.E.13$

You own a small airplane which holds a maximum of 20 passengers. It costs you \$100 per flight from St. Thomas to St. Croix for gas and wages plus an additional \$6 per passenger for the extra gas required by the extra weight. The charge per passenger is \$30 each if 10 people charter your plane (10 is the minimum number you will fly), and this charge is reduced by \$1 per passenger for each passenger over 10 who goes (that is, if 11 go they each pay \$29, if 12 go they each pay \$28, etc.). What number of passengers on a flight will maximize your profits?

Exercise $16.E.14$

In the planning of a coffee shop, we estimate that if there is seating for between 40 and 80 people, the daily profit will be \$50 per seat. However, if the seating capacity is more than 80 places, the daily profit per seat will be decreased by \$1 for each additional seat over 80. What should the seating capacity be in order to maximize the coffee shop's total profit?

Exercise $16.E.15$

In the planning of a taco restaurant, we estimate that if there is seating for between 10 and 40 people, the daily profit will be \$10 per seat. However, if the seating capacity is more than 40 places, the daily profit per seat will be decreased by \$0.20 per seat. What should the seating capacity be in order to maximize the taco restaurant's total profit?

Exercise $16.E.16$

The total cost in dollars for Alicia to make q oven mitts is given by $C(q) = 64 + 1.5q + .01q^2$.

(a) What is the fixed cost?

(b) Find a function that gives the marginal cost.

(c) Find a function that gives the average cost.

(d) Find the quantity that minimizes the average cost.

(e) Confirm that the average cost and marginal cost are equal at your answer to part (d).

Exercise 16.E.17

Shaki makes and sells backpack danglies. The total cost in dollars for Shaki to make q danglies is given by $C(q) = 75 + 2q + .015q^2$. Find the quantity that minimizes Shaki's average cost for making danglies.

2.11 Exercises

Exercise 16.E.1

If $g(20) = 35$ and $g'(20) = -2$, estimate the value of $g(22)$.

Exercise 16.E.2

If $g(1) = -17$ and $g'(1) = 5$, estimate the value of $g(1.2)$.

Exercise 16.E.3

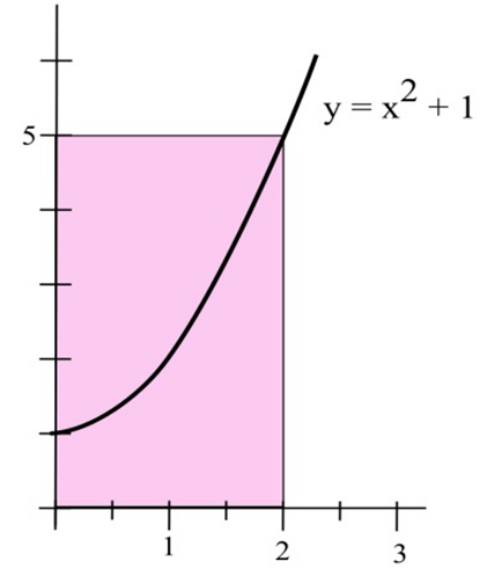
Use the Tangent Line Approximation to estimate the cube root of 9.

Exercise 16.E.4

Use the Tangent Line Approximation to estimate the fifth root of 30.

Exercise 16.E.5

A rectangle has one side on the x–axis, one side on the y–axis, and a corner on the graph of $y = x^2 + 1$.



(a) Use Linear Approximation of the area formula to estimate the increase in the area of the rectangle if the base grows from 2 to 2.3 inches.

(b) Calculate exactly the increase in the area of the rectangle as the base grows from 2 to 2.3 inches.

Exercise $16.E.6$

You can measure the diameter of a circle to within 0.3 cm.

(a) How large is the "error" in the calculated area of a circle with a measured diameter of 7.4 cm?

(b) How large is the "error" in the calculated area of a circle with a measured diameter of 13.6 cm?

(c) How large is the percentage error in the calculated area of a circle with a measured diameter of d?

Exercise $16.E.7$

The demand function for Alicia's oven mitts is given by $q = -8p + 80$ (q is the number of oven mitts, p is the price in dollars). Find the elasticity of demand when $p =$ \$7.50. Will revenue increase if Alicia raises her price from \$7.50?

Exercise $16.E.8$

The demand function for Shaki's danglies is given by $q = -35p + 205$ (q is the number of danglies, p is the price in dollars per dangly). Find the elasticity of demand when $p =$ \$5. Should Shaki raise or lower his price to increase revenue?

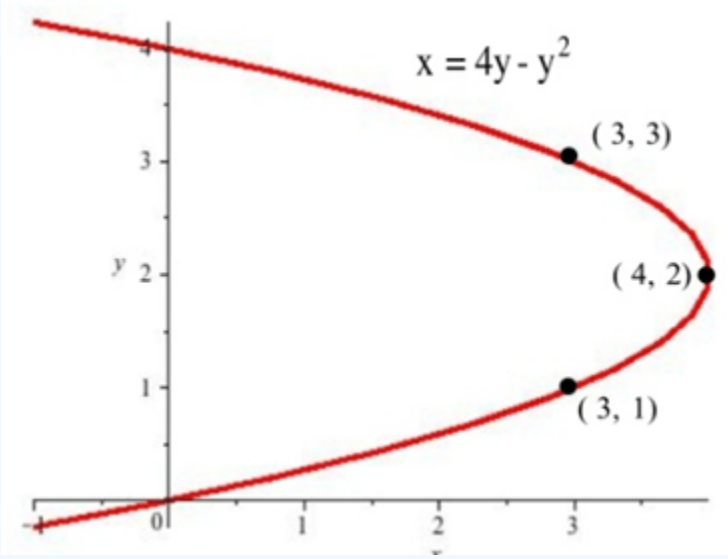
2.12 Exercises

Exercise $16.E.1 - 16.E.10$

In problems 1 – 10 find dy/dx by differentiating implicitly then find the value of dy/dx at the given point.

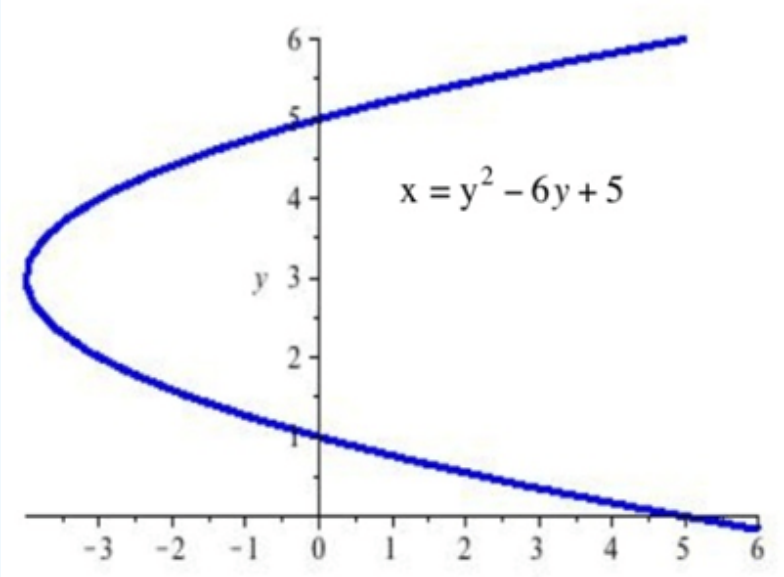
1. $x^2 + y^2 = 100$, point (6, 8)	2. $x^2 + 5y^2 = 45$, point (5, 2)
3. $x^2 - 3xy + 7y = 5$, point (2,1)	4. $\sqrt{x} + \sqrt{y} = 5$, point (4,9)
5. $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point (0,4)	6. $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point (3,0)
7. $\ln(y) + 3x - 7 = 0$, point $(2, e)$	8. $x^2 - y^2 = 16$, point (5,3)
9. $x^2 - y^2 = 16$, point (5, -3)	10. $y^2 + 7x^3 - 3x = 8$, point (1,2)

Exercise $16.E.11 - 16.E.12$



11. Find the slopes of the lines tangent to the graph in shown at the points (3,1), (3,3), and (4,2).
12. Find the slopes of the lines tangent to the graph in shown where the graph crosses the y–axis.

Exercise $16.E.13 - 16.E.14$



13. Find the slopes of the lines tangent to the graph in graph shown at the points ((5,0), (5,6), and (−4,3).
14. Find the slopes of the lines tangent to the graph in the graph shown where the graph crosses the y–axis.

Exercise $16.E.15 - 16.E.16$

In problems 15 – 16, find dy/dx using implicit differentiation and then find the slope of the line tangent to the graph of the equation at the given point.

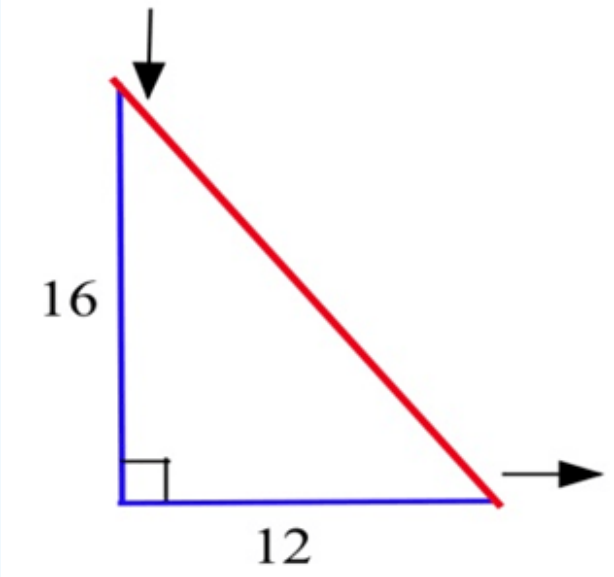
15. $y^3 – 5y = 5x^2 + 7$, point (1,3)	16. $y^2 – 5xy + x^2 + 21 = 0$, point (2,5)

Exercise $16.E.17$

An expandable sphere is being filled with liquid at a constant rate from a tap (imagine a water balloon connected to a faucet). When the radius of the sphere is 3 inches, the radius is increasing at 2 inches per minute. How fast is the liquid coming out of the tap? $(V = \frac{4}{3}\pi r^3)$

Exercise $16.E.18$

The 12 inch base of a right triangle is growing at 3 inches per hour, and the 16 inch height is shrinking at 3 inches per hour.



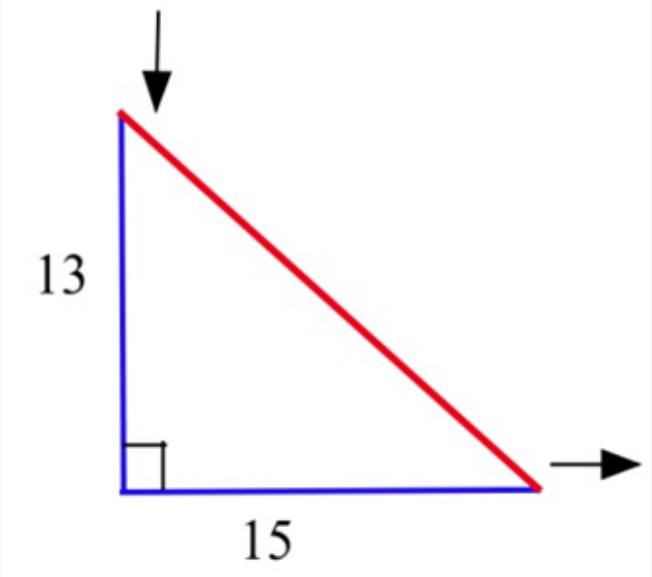
(a) Is the area increasing or decreasing?

(b) Is the perimeter increasing or decreasing?

(c) Is the hypotenuse increasing or decreasing?

Exercise $16.E.19$

One hour later the right triangle in Problem 2 is 15 inches long and 13 inches high, and the base and height are changing at the same rate as in Problem 18.



(a) Is the area increasing or decreasing now?

(b) Is the hypotenuse increasing or decreasing now?

(c) Is the perimeter increasing or decreasing now?

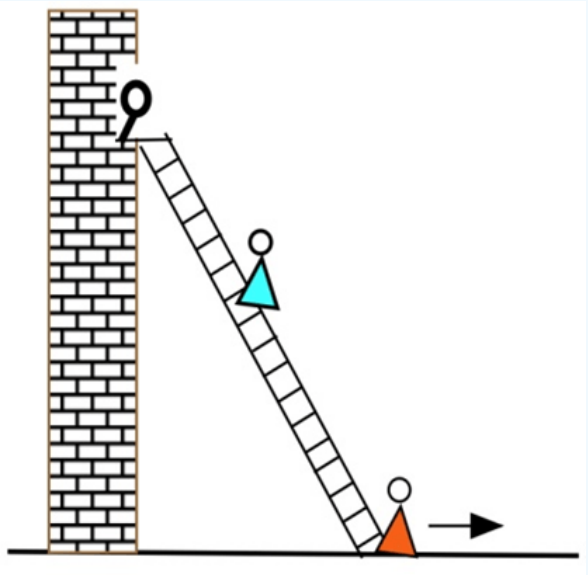
Exercise $16.E.20$

A young woman and her boyfriend plan to elope, but she must rescue him from his mother who has locked him in his room. The young woman has placed a 20 foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second. How fast is the top of the ladder (and the young couple) falling when the bottom of the ladder is

(a) 12 feet from the bottom of the wall?

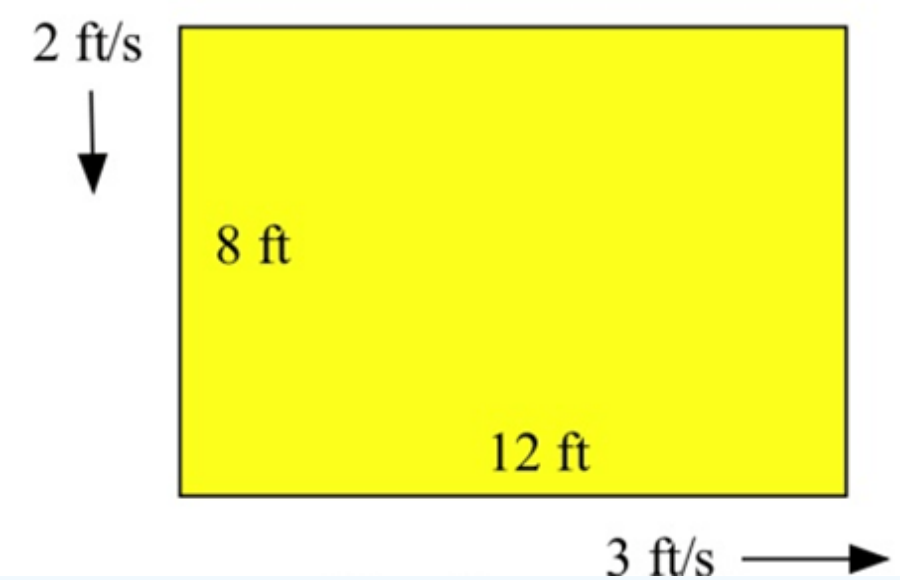
(b) 16 feet from the bottom of the wall?

(c) 19 feet from the bottom of the wall?



Exercise 16.E.21

The length of a 12 foot by 8 foot rectangle is increasing at a rate of 3 feet per second and the width is decreasing at 2 feet per second.



(a) How fast is the perimeter changing?

(b) How fast is the area changing?

Exercise 16.E.22

An oil tanker in Puget Sound has sprung a leak, and a circular oil slick is forming. The oil slick is 4 inches thick everywhere, is 100 feet in diameter, and the diameter is increasing at 12 feet per hour. Your job, as the Coast Guard commander or the tanker's captain, is to determine how fast the oil is leaking from the tanker.

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