MAT 117: MATHEMATICAL PROBLEM SOLVING IN LIBERAL ARTS



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Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

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1.1: Introduction

In many decision making situations, it is necessary to gather the group consensus. This happens when a group of friends decides which movie to watch, when a company decides which product design to manufacture, and when a democratic country elects its leaders.

While the basic idea of voting is fairly universal, the method by which those votes are used to determine a winner can vary. Amongst a group of friends, you may decide upon a movie by voting for all the movies you're willing to watch, with the winner being the one with the greatest approval. A company might eliminate unpopular designs then revote on the remaining. A country might look for the candidate with the most votes.

In deciding upon a winner, there is always one main goal: to reflect the preferences of the people in the most fair way possible.

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1.2: Preference Schedules

To begin, we're going to want more information than a traditional ballot normally provides. A traditional ballot usually asks you to pick your favorite from a list of choices. This ballot fails to provide any information on how a voter would rank the alternatives if their first choice was unsuccessful.

Preference ballot

A **preference ballot** is a ballot in which the voter ranks the choices in order of preference.

Example 1

A vacation club is trying to decide which destination to visit this year: Hawaii (H), Orlando (O), or Anaheim (A). Their votes are shown below:

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
1st choice	A	A	O	H	A	O	H	O	H	A
2nd choice	O	H	H	A	H	H	A	H	A	H
3rd choice	H	O	A	O	O	A	O	A	O	O

Solution

These individual ballots are typically combined into one **preference schedule**, which shows the number of voters in the top row that voted for each option:

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Notice that by totaling the vote counts across the top of the preference schedule we can recover the total number of votes cast: $1+3+3+3=10$ total votes.

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1.3: Plurality

The voting method we're most familiar with in the United States is the **plurality method**.

Plurality Method

In this method, the choice with the most first-preference votes is declared the winner. Ties are possible, and would have to be settled through some sort of run-off vote.

This method is sometimes mistakenly called the majority method, or "majority rules", but it is not necessary for a choice to have gained a majority of votes to win. A majority is over 50%; it is possible for a winner to have a **plurality** without having a majority.

Example 2

In our election from previous pages, we had the preference table:

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Solution

For the plurality method, we only care about the first choice options. Totaling them up:

Anaheim: 1+3 = 4 first-choice votes

Orlando: 3 first-choice votes

Hawaii: 3 first-choice votes

Anaheim is the winner using the plurality voting method.

Notice that Anaheim won with 4 out of 10 votes, 40% of the votes, which is a plurality of the votes, but not a majority.

Try it Now 1

Three candidates are running in an election for County Executive: Goings (G), McCarthy (M), and Bunney (B)[1]. The voting schedule is shown below. Which candidate wins under the plurality method?

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

Note: In the third column and last column, those voters only recorded a first-place vote, so we don't know who their second and third choices would have been.

Answer

Using plurality method:

G gets $44+14+20=78$ first-choice votes

M gets $70+22=92$ first-choice votes

B gets $80+39=119$ first-choice votes

Bunney (B) wins under plurality method.

[1] This data is loosely based on the 2008 County Executive election in Pierce County, Washington. See www.co.pierce.wa.us/xml/abtus...ec/summary.pdf

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1.4: What's Wrong with Plurality?

The election from Example 2 may seem totally clean, but there is a problem lurking that arises whenever there are three or more choices. Looking back at our preference table, how would our members vote if they only had two choices?

Anaheim vs Orlando: 7 out of the 10 would prefer Anaheim over Orlando

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Anaheim vs Hawaii: 6 out of 10 would prefer Hawaii over Anaheim

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

This doesn't seem right, does it? Anaheim just won the election, yet 6 out of 10 voters, 60% of them, would have preferred Hawaii! That hardly seems fair. Marquis de Condorcet, a French philosopher, mathematician, and political scientist wrote about how this could happen in 1785, and for him we name our first **fairness criterion**.

Fairness Criteria

The fairness criteria are statements that seem like they should be true in a fair election.

Condorcet Criterion

If there is a choice that is preferred in every one-to-one comparison with the other choices, that choice should be the winner. We call this winner the **Condorcet Winner**, or Condorcet Candidate.

Example 3

In the election from Example 2, what choice is the Condorcet Winner?

Solution

We see above that Hawaii is preferred over Anaheim. Comparing Hawaii to Orlando, we can see 6 out of 10 would prefer Hawaii to Orlando.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Since Hawaii is preferred in a one-to-one comparison to both other choices, Hawaii is the Condorcet Winner.

Example 4

Consider a city council election in a district that is historically 60% Democratic voters and 40% Republican voters. Even though city council is technically a nonpartisan office, people generally know the affiliations of the candidates. In this election there are three candidates: Don and Key, both Democrats, and Elle, a Republican. A preference schedule for the votes looks as follows:

	342	214	298
1st choice	Elle	Don	Key
2nd choice	Don	Key	Don
3rd choice	Key	Elle	Elle

Solution

We can see a total of $342 + 214 + 298 = 854$ voters participated in this election. Computing percentage of first place votes:

Don: $214/854 = 25.1\%$

Key: $298/854 = 34.9\%$

Elle: $342/854 = 40.0\%$

So in this election, the Democratic voters split their vote over the two Democratic candidates, allowing the Republican candidate Elle to win under the plurality method with 40% of the vote.

Analyzing this election closer, we see that it violates the Condorcet Criterion. Analyzing the one-to-one comparisons:

Elle vs Don: 342 prefer Elle; 512 prefer Don: Don is preferred

Elle vs Key: 342 prefer Elle; 512 prefer Key: Key is preferred

Don vs Key: 556 prefer Don; 298 prefer Key: Don is preferred

So even though Don had the smallest number of first-place votes in the election, he is the Condorcet winner, being preferred in every one-to-one comparison with the other candidates.

? Try it Now 2

Consider the election from Try it Now 1. Is there a Condorcet winner in this election?

Answer

Determining the Condorcet Winner:

G vs M: $44 + 14 + 20 = 78$ prefer G, $70 + 22 + 80 = 172$ prefer M: M preferred

G vs B: $44 + 14 + 20 + 70 = 148$ prefer G, $22 + 80 + 39 = 141$ prefer B: G preferred

M vs B: $44 + 70 + 22 = 136$ prefer M, $14 + 80 + 39 = 133$ prefer B: M preferred

M is the Condorcet winner, based on the information we have.

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1.5: Insincere Voting

Situations like the one in Example 4 above, when there are more than one candidate that share somewhat similar points of view, can lead to **insincere voting**. Insincere voting is when a person casts a ballot counter to their actual preference for strategic purposes. In the case above, the democratic leadership might realize that Don and Key will split the vote, and encourage voters to vote for Key by officially endorsing him. Not wanting to see their party lose the election, as happened in the scenario above, Don's supporters might insincerely vote for Key, effectively voting against Elle.

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1.6: Instant Runoff Voting

Instant Runoff Voting (IRV), also called Plurality with Elimination, is a modification of the plurality method that attempts to address the issue of insincere voting.

Instant Runoff Voting (IRV)

In IRV, voting is done with preference ballots, and a preference schedule is generated. The choice with the *least* first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voters' next choice. This continues until a choice has a majority (over 50%).

This is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed without requiring a second costly election.

This voting method is used in several political elections around the world, including election of members of the Australian House of Representatives, and was used for county positions in Pierce County, Washington until it was eliminated by voters in 2009. A version of IRV is used by the International Olympic Committee to select host nations.

Example 5

Consider the preference schedule below, in which a company's advertising team is voting on five different advertising slogans, called A, B, C, D, and E here for simplicity.

Initial votes

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	A	D	C	E	A
3rd choice	A	D	C	A	A	D
4th choice	D	B	A	E	C	B
5th choice	E	E	E	B	D	C

If this was a plurality election, note that B would be the winner with 9 first-choice votes, compared to 6 for D, 4 for C, and 1 for E.

There are total of 3+4+4+6+2+1 = 20 votes. A majority would be 11 votes. No one yet has a majority, so we proceed to elimination rounds.

Solution

Round 1: We make our first elimination. Choice A has the fewest first-place votes, so we remove that choice

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C		D	C	E	
3rd choice		D	C			D
4th choice	D	B		E	C	B
5th choice	E	E	E	B	D	C

We then shift everyone's choices up to fill the gaps. There is still no choice with a majority, so we eliminate again.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	D	D	C	E	D
3rd choice	D	B	C	E	C	B
4th choice	E	E	E	B	D	C

Round 2: We make our second elimination. Choice E has the fewest first-place votes, so we remove that choice, shifting everyone's options to fill the gaps.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	D
2nd choice	C	D	D	C	C	B
3rd choice	D	B	C	B	D	C

Notice that the first and fifth columns have the same preferences now, we can condense those down to one column.

	5	4	4	6	1
1st choice	B	C	B	D	D
2nd choice	C	D	D	C	B
3rd choice	D	B	C	B	C

Now B has 9 first-choice votes, C has 4 votes, and D has 7 votes. Still no majority, so we eliminate again.

Round 3: We make our third elimination. C has the fewest votes.

	5	4	4	6	1
1st choice	B	D	B	D	D
2nd choice	D	B	D	B	B

Condensing this down:

	9	11
1st choice	B	D
2nd choice	D	B

D has now gained a majority, and is declared the winner under IRV.

? Try it Now 3

Consider again the election from Try it Now 1. Find the winner using IRV.

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

Answer

G has the fewest first-choice votes, so is eliminated first. The 20 voters who did not list a second choice do not get transferred - they simply get eliminated

	136	133
1st choice	M	B
2nd choice	B	M

McCarthy (M) now has a majority, and is declared the winner.

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1.7: What's Wrong with IRV?

Example 6

Let's return to our City Council Election

	342	214	298
1st choice	Elle	Don	Key
2nd choice	Don	Key	Don
3rd choice	Key	Elle	Elle

Solution

In this election, Don has the smallest number of first place votes, so Don is eliminated in the first round. The 214 people who voted for Don have their votes transferred to their second choice, Key.

	342	512
1st choice	Elle	Key
2nd choice	Key	Elle

So Key is the winner under the IRV method.

We can immediately notice that in this election, IRV violates the Condorcet Criterion, since we determined earlier that Don was the Condorcet winner. On the other hand, the temptation has been removed for Don's supporters to vote for Key; they now know their vote will be transferred to Key, not simply discarded.

Example 7

Consider the voting system below.

	37	22	12	29
1st choice	Adams	Brown	Brown	Carter
2nd choice	Brown	Carter	Adams	Adams
3rd choice	Carter	Adams	Carter	Brown

Solution

In this election, Carter would be eliminated in the first round, and Adams would be the winner with 66 votes to 34 for Brown.

Now suppose that the results were announced, but election officials accidentally destroyed the ballots before they could be certified, and the votes had to be recast. Wanting to "jump on the bandwagon", 10 of the voters who had originally voted in the order Brown, Adams, Carter change their vote to favor the presumed winner, changing those votes to Adams, Brown, Carter.

	47	22	2	29
1st choice	Adams	Brown	Brown	Carter
2nd choice	Brown	Carter	Adams	Adams
3rd choice	Carter	Adams	Carter	Brown

In this re-vote, Brown will be eliminated in the first round, having the fewest first-place votes. After transferring votes, we find that Carter will win this election with 51 votes to Adams' 49 votes! Even though the only vote changes made *favored* Adams, the change ended up costing Adams the election. This doesn't seem right, and introduces our second fairness criterion:

Monotonicity Criterion

If voters change their votes to increase the preference for a candidate, it should not harm that candidate's chances of winning.

This criterion is violated by this election. Note that even though the criterion is violated in this particular election, it does not mean that IRV always violates the criterion; just that IRV has the potential to violate the criterion in certain elections.

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1.8: Borda Count

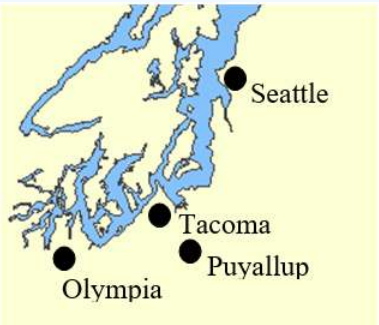
Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770.

Borda Count

In this method, points are assigned to candidates based on their ranking; 1 point for last choice, 2 points for second-to-last choice, and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

Example 8

A group of mathematicians are getting together for a conference. The members are coming from four cities: Seattle, Tacoma, Puyallup, and Olympia. Their approximate locations on a map are shown to the right.



The votes for where to hold the conference were:

	51	25	10	14
1st choice	Seattle	Tacoma	Puyallup	Olympia
2nd choice	Tacoma	Puyallup	Tacoma	Tacoma
3rd choice	Olympia	Olympia	Olympia	Puyallup
4th choice	Puyallup	Seattle	Seattle	Seattle

Solution

In each of the 51 ballots ranking Seattle first, Puyallup will be given 1 point, Olympia 2 points, Tacoma 3 points, and Seattle 4 points. Multiplying the points per vote times the number of votes allows us to calculate points awarded:

	51	25	10	14
1st choice 4 points	Seattle $4 \cdot 51 = 204$	Tacoma $4 \cdot 25 = 100$	Puyallup $4 \cdot 10 = 40$	Olympia $4 \cdot 14 = 56$
2nd choice 3 points	Tacoma $3 \cdot 51 = 153$	Puyallup $3 \cdot 25 = 75$	Tacoma $3 \cdot 10 = 30$	Tacoma $3 \cdot 14 = 42$
3rd choice 2 points	Olympia $2 \cdot 51 = 102$	Olympia $2 \cdot 25 = 50$	Olympia $2 \cdot 10 = 20$	Puyallup $2 \cdot 14 = 28$
4th choice 1 point	Puyallup $1 \cdot 51 = 51$	Seattle $1 \cdot 25 = 25$	Seattle $1 \cdot 10 = 10$	Seattle $1 \cdot 14 = 14$

Adding up the points:

- Seattle: $204 + 25 + 10 + 14 = 253$ points
- Tacoma: $153 + 100 + 30 + 42 = 325$ points
- Puyallup: $51 + 75 + 40 + 28 = 194$ points
- Olympia: $102 + 50 + 20 + 56 = 228$ points

Under the Borda Count method, Tacoma is the winner of this vote.

Try it Now 4

Consider again the election from Try it Now 1. Find the winner using Borda Count. Since we have some incomplete preference ballots, for simplicity, give every unranked candidate 1 point, the points they would normally get for last place.

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

Answer

Using Borda Count:

We give 1 point for 3rd place, 2 points for 2nd place, and 3 points for 1st place.

	44	14	20	70	22	80	39
1st choice	G 132pt	G 42pt	G 60pt	M 210pt	M 66pt	B 240pt	B 117pt
2nd choice	M 88pt	B 28pt		G 140pt	B 44pt	M 160pt	
3rd choice	B 44pt	M 14pt	M20pt B20pt	B 70pt	G 22pt	G 80pt	M39pt G39pt

G: $132 + 42 + 60 + 140 + 22 + 80 + 39 = 515$ pts

M: $88 + 14 + 20 + 210 + 66 + 160 + 39 = 597$ pts

B: $44 + 28 + 20 + 70 + 44 + 240 + 117 = 563$ pts

McCarthy (M) would be the winner using Borda Count.

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1.9: What's Wrong with Borda Count?

You might have already noticed one potential flaw of the Borda Count from the previous example. In that example, Seattle had a majority of first-choice votes, yet lost the election! This seems odd, and prompts our next fairness criterion:

Majority Criterion

If a choice has a majority of first-place votes, that choice should be the winner.

The election from Example 8 using the Borda Count violates the Majority Criterion. Notice also that this automatically means that the Condorcet Criterion will also be violated, as Seattle would have been preferred by 51% of voters in any head-to-head comparison.

Borda count is sometimes described as a consensus-based voting system, since it can sometimes choose a more broadly acceptable option over the one with majority support. In the example above, Tacoma is probably the best compromise location. This is a different approach than plurality and instant runoff voting that focus on first-choice votes; Borda Count considers every voter's entire ranking to determine the outcome.

Because of this consensus behavior, Borda Count, or some variation of it, is commonly used in awarding sports awards. Variations are used to determine the Most Valuable Player in baseball, to rank teams in NCAA sports, and to award the Heisman trophy.

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1.10: Copeland's Method (Pairwise Comparisons)

So far none of our voting methods have satisfied the Condorcet Criterion. The Copeland Method specifically attempts to satisfy the Condorcet Criterion by looking at pairwise (one-to-one) comparisons.

Copeland's Method

In this method, each pair of candidates is compared, using all preferences to determine which of the two is more preferred. The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded $\frac{1}{2}$ point. After all pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

Variations of Copeland's Method are used in many professional organizations, including election of the Board of Trustees for the Wikimedia Foundation that runs Wikipedia.

Example 9

Consider our vacation group example from the beginning of the chapter. Determine the winner using Copeland's Method.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Solution

We need to look at each pair of choices, and see which choice would win in a one-to-one comparison. You may recall we did this earlier when determining the Condorcet Winner. For example, comparing Hawaii vs Orlando, we see that 6 voters, those shaded below in the first table below, would prefer Hawaii to Orlando. Note that Hawaii doesn't have to be the voter's first choice – we're imagining that Anaheim wasn't an option. If it helps, you can imagine removing Anaheim, as in the second table below.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

	1	3	3	3
1st choice			O	H
2nd choice	O	H	H	
3rd choice	H	O		O

Based on this, in the comparison of Hawaii vs Orlando, Hawaii wins, and receives 1 point.

Comparing Anaheim to Orlando, the 1 voter in the first column clearly prefers Anaheim, as do the 3 voters in the second column. The 3 voters in the third column clearly prefer Orlando. The 3 voters in the last column prefer Hawaii as their first choice, but if they had to choose between Anaheim and Orlando, they'd choose Anaheim, their second choice overall. So, altogether 1+3+3=7 voters prefer Anaheim over Orlando, and 3 prefer Orlando over Anaheim. So, comparing Anaheim vs Orlando: 7 votes to 3 votes: Anaheim gets 1 point.

All together,

Hawaii vs Orlando: **6 votes to 4 votes: Hawaii gets 1 point**
Anaheim vs Orlando: **7 votes to 3 votes: Anaheim gets 1 point**
Hawaii vs Anaheim: **6 votes to 4 votes: Hawaii gets 1 point**

Hawaii is the winner under Copeland's Method, having earned the most points.

Notice this process is consistent with our determination of a Condorcet Winner.

Example 10

Consider the advertising group's vote we explored earlier. Determine the winner using Copeland's method.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	A	D	C	E	A
3rd choice	A	D	C	A	A	D
4th choice	D	B	A	E	C	B
5th choice	E	E	E	B	D	C

Solution

With 5 candidates, there are 10 comparisons to make:

A vs B: 11 votes to 9 votes	A gets 1 point
A vs C: 3 votes to 17 votes	C gets 1 point
A vs D: 10 votes to 10 votes	A gets $\frac{1}{2}$ point, D gets $\frac{1}{2}$ point
A vs E: 17 votes to 3 votes	A gets 1 point
B vs C: 10 votes to 10 votes	B gets $\frac{1}{2}$ point, C gets $\frac{1}{2}$ point
B vs D: 9 votes to 11 votes	D gets 1 point
B vs E: 13 votes to 7 votes	B gets 1 point
C vs D: 9 votes to 11 votes	D gets 1 point
C vs E: 17 votes to 3 votes	C gets 1 point
D vs E: 17 votes to 3 votes	D gets 1 point

Totaling these up:

A gets $2\frac{1}{2}$ points

B gets $1\frac{1}{2}$ points

C gets $2\frac{1}{2}$ points

D gets $3\frac{1}{2}$ points

E gets 0 points

Using Copeland's Method, we declare D as the winner.

Notice that in this case, D is not a Condorcet Winner. While Copeland's method will also select a Condorcet Candidate as the winner, the method still works in cases where there is no Condorcet Winner.

Try it Now 5

Consider again the election from Try it Now 1. Find the winner using Copeland's method. Since we have some incomplete preference ballots, we'll have to adjust. For example, when comparing M to B, we'll ignore the 20 votes in the third column which do not rank either candidate.

	44	14	20	70	22	80	39
1st choice	G	G	G	M	M	B	B
2nd choice	M	B		G	B	M	
3rd choice	B	M		B	G	G	

Answer

Using Copeland's Method:

Looking back at our work from Try it Now #2, we see

G vs M: $44+14+20=78$ prefer G, $70+22+80=172$ prefer M: M preferred – 1 point

G vs B: $44+14+20+70=148$ prefer G, $22+80+39=141$ prefer B: G preferred – 1 point

M vs B: $44 + 70 + 22 = 136$ prefer M, $14 + 80 + 39 = 133$ prefer B: M preferred – 1 point

M earns 2 points; G earns 1 point. M wins under Copeland's method.

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1.11: What's Wrong with Copeland's Method?

As already noted, Copeland's Method does satisfy the Condorcet Criterion. It also satisfies the Majority Criterion and the Monotonicity Criterion. So is this the perfect method? Well, in a word, no.

Example 11

A committee is trying to award a scholarship to one of four students, Anna (A), Brian (B), Carlos (C), and Dimitry (D). The votes are shown below:

	5	5	6	4
1st choice	D	A	C	B
2nd choice	A	C	B	D
3rd choice	C	B	D	A
4th choice	B	D	A	C

Solution

Making the comparisons:

A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
A vs C: 14 votes to 6 votes: A gets 1 point
A vs D: 5 votes to 15 votes: D gets 1 point
B vs C: 4 votes to 16 votes: C gets 1 point
B vs D: 15 votes to 5 votes: B gets 1 point
C vs D: 11 votes to 9 votes: C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $1\frac{1}{2}$ points
C has 2 points D has 1 point

So Carlos is awarded the scholarship. However, the committee then discovers that Dimitry was not eligible for the scholarship (he failed his last math class). Even though this seems like it shouldn't affect the outcome, the committee decides to recount the vote, removing Dimitry from consideration. This reduces the preference schedule to:

	5	5	6	4
1st choice	A	A	C	B
2nd choice	C	C	B	A
3rd choice	B	B	A	C

A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
A vs C: 14 votes to 6 votes A gets 1 point
B vs C: 4 votes to 16 votes C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $\frac{1}{2}$ point
C has 1 point

Suddenly Anna is the winner! This leads us to another fairness criterion.

The Independence of Irrelevant Alternatives (IIA) Criterion

If a non-winning choice is removed from the ballot, it should not change the winner of the election.

Equivalently, if choice A is preferred over choice B, introducing or removing a choice C should not cause B to be preferred over A.

In the election from Example 11, the IIA Criterion was violated.

This anecdote illustrating the IIA issue is attributed to Sidney Morgenbesser:

> After finishing dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says "In that case I'll have the blueberry pie."

Another disadvantage of Copeland's Method is that it is fairly easy for the election to end in a tie. For this reason, Copeland's method is usually the first part of a more advanced method that uses more sophisticated methods for breaking ties and determining the winner when there is not a Condorcet Candidate.

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1.12: So Where's the Fair Method?

At this point, you're probably asking why we keep looking at method after method just to point out that they are not fully fair. We must be holding out on the perfect method, right?

Unfortunately, no. A mathematical economist, Kenneth Arrow, was able to prove in 1949 that there is no voting method that will satisfy all the fairness criteria we have discussed.

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem states, roughly, that it is not possible for a voting method to satisfy every fairness criteria that we've discussed.

To see a very simple example of how difficult voting can be, consider the election below:

	5	5	5
1st choice	A	C	B
2nd choice	B	A	C
3rd choice	C	B	A

Notice that in this election:

- 10 people prefer A to B
- 10 people prefer B to C
- 10 people prefer C to A

No matter whom we choose as the winner, 2/3 of voters would prefer someone else! This scenario is dubbed **Condorcet's Voting Paradox**, and demonstrates how voting preferences are not transitive (just because A is preferred over B, and B over C, does not mean A is preferred over C). In this election, there is no fair resolution.

It is because of this impossibility of a totally fair method that Plurality, IRV, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

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1.13: Approval Voting

Up until now, we've been considering voting methods that require ranking of candidates on a preference ballot. There is another method of voting that can be more appropriate in some decision making scenarios. With Approval Voting, the ballot asks you to mark all choices that you find acceptable. The results are tallied, and the option with the most approval is the winner.

Example 12

A group of friends is trying to decide upon a movie to watch. Three choices are provided, and each person is asked to mark with an "X" which movies they are willing to watch. The results are:

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
Titanic		X	X			X		X		X
Scream	X		X	X		X	X		X	
The Matrix	X	X	X	X	X		X			X

Solution

Totaling the results, we find

Titanic received 5 approvals

Scream received 6 approvals

The Matrix received 7 approvals.

In this vote, The Matrix would be the winner.

Try it Now 6

Our mathematicians deciding on a conference location from earlier decide to use Approval voting. Their votes are tallied below. Find the winner using Approval voting.

	30	10	15	20	15	5	5
Seattle	X	X	X			X	
Tacoma	X		X	X	X	X	
Puyallup		X		X	X	X	
Olympia			X		X		X

Answer

Using Approval voting:

Seattle has $30+10+15+5=60$ approval votes

Tacoma has $30+15+20+15+5=85$ approval votes

Puyallup has $10+20+25+5=50$ approval votes

Olympia has $15+15+5=35$ approval votes

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1.14: What's Wrong with Approval Voting?

Approval voting can very easily violate the Majority Criterion.

Example 13

Consider the voting schedule:

	80	15	5
1st choice	A	B	C
2nd choice	B	C	B
3rd choice	C	A	A

Solution

Clearly A is the majority winner. Now suppose that this election was held using Approval Voting, and every voter marked approval of their top two candidates.

- A would receive approval from 80 voters
- B would receive approval from 100 voters
- C would receive approval from 20 voters

B would be the winner. Some argue that Approval Voting tends to vote the least disliked choice, rather than the most liked candidate.

Additionally, Approval Voting is susceptible to strategic insincere voting, in which a voter does not vote their true preference to try to increase the chances of their choice winning. For example, in the movie example above, suppose Bob and Alice would much rather watch Scream. They remove The Matrix from their approval list, resulting in a different result.

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
Titanic		X	X			X		X		X
Scream	X		X	X		X	X		X	
The Matrix		X	X		X		X			X

Totaling the results, we find Titanic received 5 approvals, Scream received 6 approvals, and The Matrix received 5 approvals. By voting insincerely, Bob and Alice were able to sway the result in favor of their preference.

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1.15: Voting in America

In American politics, there is a lot more to selecting our representatives than simply casting and counting ballots. The process of selecting the president is even more complicated, so we'll save that for the next chapter. Instead, let's look at the process by which state congressional representatives and local politicians get elected.

For most offices, a sequence of two public votes is held: a primary election and the general election. For non-partisan offices like sheriff and judge, in which political party affiliation is not declared, the primary election is usually used to narrow the field of candidates.

Typically, the two candidates receiving the most votes in the primary will then move forward to the general election. While somewhat similar to instant runoff voting, this is actually an example of **sequential voting** - a process in which voters cast totally new ballots after each round of eliminations. Sequential voting has become quite common in television, where it is used in reality competition shows like American Idol.

Congressional, county, and city representatives are partisan offices, in which candidates usually declare themselves a member of a political party, like the Democrats, Republicans, the Green Party, or one of the many other smaller parties. As with non-partisan offices, a primary election is usually held to narrow down the field prior to the general election. Prior to the primary election, the candidate would have met with the political party leaders and gotten their approval to run under that party's affiliation.

In some states a **closed primary** is used, in which only voters who are members of the Democrat party can vote on the Democratic candidates, and similar for Republican voters. In other states, an **open primary** is used, in which any voter can pick the party whose primary they want to vote in. In other states, **caucuses** are used, which are basically meetings of the political parties, only open to party members. Closed primaries are often disliked by independent voters, who like the flexibility to change which party they are voting in. Open primaries do have the disadvantage that they allow raiding, in which a voter will vote in their non-preferred party's primary with the intent of selecting a weaker opponent for their preferred party's candidate.

Washington State currently uses a different method, called a **top 2 primary**, in which voters select from the candidates from all political parties on the primary, and the top two candidates, regardless of party affiliation, move on to the general election. While this method is liked by independent voters, it gives the political parties incentive to select a top candidate internally before the primary, so that two candidates will not split the party's vote.

Regardless of the primary type, the general election is the main election, open to all voters. Except in the case of the top 2 primary, the top candidate from each major political party would be included in the general election. While rules vary state-to-state, for an independent or minor party candidate to get listed on the ballot, they typically have to gather a certain number of signatures to petition for inclusion.

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1.16: Exercises

1. To decide on a new website design, the designer asks people to rank three designs that have been created (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

   ABC, ABC, ACB, BAC, BCA, BCA, ACB, CAB, CAB, BCA, ACB, ABC

2. To decide on a movie to watch, a group of friends all vote for one of the choices (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

   CAB, CBA, BAC, BCA, CBA, ABC, ABC, CBA, BCA, CAB, CAB, BAC

3. The planning committee for a renewable energy trade show is trying to decide what city to hold their next show in. The votes are shown below.

Number of voters	9	19	11	8
1st choice	Buffalo	Atlanta	Chicago	Buffalo
2nd choice	Atlanta	Buffalo	Buffalo	Chicago
3rd choice	Chicago	Chicago	Atlanta	Atlanta

   a. How many voters voted in this election?
   b. How many votes are needed for a majority? A plurality?
   c. Find the winner under the plurality method.
   d. Find the winner under the Borda Count Method.
   e. Find the winner under the Instant Runoff Voting method.
   f. Find the winner under Copeland's method.

4. A non-profit agency is electing a new chair of the board. The votes are shown below.

Number of voters	11	5	10	3
1st choice	Atkins	Cortez	Burke	Atkins
2nd choice	Cortez	Burke	Cortez	Burke
3rd choice	Burke	Atkins	Atkins	Cortez

   a. How many voters voted in this election?
   b. How many votes are needed for a majority? A plurality?
   c. Find the winner under the plurality method.
   d. Find the winner under the Borda Count Method.
   e. Find the winner under the Instant Runoff Voting method.
   f. Find the winner under Copeland's method.

5. The student government is holding elections for president. There are four candidates (labeled A, B, C, and D for convenience). The preference schedule for the election is:

Number of voters	120	50	40	90	60	100
1st choice	C	B	D	A	A	D
2nd choice	D	C	A	C	D	B
3rd choice	B	A	B	B	C	A
4th choice	A	D	C	D	B	C

   a. How many voters voted in this election?
   b. How many votes are needed for a majority? A plurality?
   c. Find the winner under the plurality method.
   d. Find the winner under the Borda Count Method.
   e. Find the winner under the Instant Runoff Voting method.
   f. Find the winner under Copeland's method.

6. The homeowners association is deciding a new set of neighborhood standards for architecture, yard maintenance, etc. Four options have been proposed. The votes are:

Number of voters	8	9	11	7	7	5
1st choice	B	A	D	A	B	C
2nd choice	C	D	B	B	A	D
3rd choice	A	C	C	D	C	A
4th choice	D	B	A	C	D	B

a. How many voters voted in this election?
b. How many votes are needed for a majority? A plurality?
c. Find the winner under the plurality method.
d. Find the winner under the Borda Count Method.
e. Find the winner under the Instant Runoff Voting method.
f. Find the winner under Copeland's method.

7. Consider an election with 129 votes.
   a. If there are 4 candidates, what is the smallest number of votes that a plurality candidate could have?
   b. If there are 8 candidates, what is the smallest number of votes that a plurality candidate could have?

8. Consider an election with 953 votes.
   a. If there are 7 candidates, what is the smallest number of votes that a plurality candidate could have?
   b. If there are 8 candidates, what is the smallest number of votes that a plurality candidate could have?

9. Does this voting system having a Condorcet Candidate? If so, find it.

Number of voters	14	15	2
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

10. Does this voting system having a Condorcet Candidate? If so, find it.

Number of voters	8	7	6
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

11. The marketing committee at a company decides to vote on a new company logo. They decide to use approval voting. Their results are tallied below. Each column shows the number of voters with the particular approval vote. Which logo wins under approval voting?

Number of voters	8	7	6
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

12. The downtown business association is electing a new chairperson, and decides to use approval voting. The tally is below, where each column shows the number of voters with the particular approval vote. Which candidate wins under approval voting?

Number of voters	8	7	6	3	4	2	5
A	X	X			X		X
B	X		X	X			X
C		X	X	X		X	
D	X		X		X	X	

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1.17: Concepts

1. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If for some reason the election had to be held again and C decided to drop out of the election, which caused B to become the winner, which is the primary fairness criterion violated in this election?
2. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If for some reason the election had to be held again and many people who had voted for C switched their preferences to favor A, which caused B to become the winner, which is the primary fairness criterion violated in this election?
3. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If in a head-to-head comparison a majority of people prefer B to A or C, which is the primary fairness criterion violated in this election?
4. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If B had received a majority of first place votes, which is the primary fairness criterion violated in this election?

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1.18: Exploration

1. In the election shown below under the Plurality method, explain why voters in the third column might be inclined to vote insincerely. How could it affect the outcome of the election?

Number of voters	96	90	10
1st choice	A	B	C
2nd choice	B	A	B
3rd choice	C	C	A

2. In the election shown below under the Borda Count method, explain why voters in the second column might be inclined to vote insincerely. How could it affect the outcome of the election?

Number of voters	20	18
1st choice	A	B
2nd choice	B	A
3rd choice	C	C

3. Compare and contrast the motives of the insincere voters in the two questions above.

4. Consider a two party election with preferences shown below. Suppose a third candidate, C, entered the race, and a segment of voters sincerely voted for that third candidate, producing the preference schedule from #17 above. Explain how other voters might perceive candidate C.

Number of voters	96	100
1st choice	A	B
2nd choice	B	A

5. In question 18, we showed that the outcome of Borda Count can be manipulated if a group of individuals change their vote.
   a. Show that it is possible for a single voter to change the outcome under Borda Count if there are four candidates.
   b. Show that it is not possible for a single voter to change the outcome under Borda Count if there are three candidates.

6. Show that when there is a Condorcet winner in an election, it is impossible for a single voter to manipulate the vote to help a different candidate become a Condorcet winner.

7. The Pareto criterion is another fairness criterion that states: *If every voter prefers choice A to choice B, then B should not be the winner*. Explain why plurality, instant runoff, Borda count, and Copeland's method all satisfy the Pareto condition.

8. Sequential Pairwise voting is a method not commonly used for political elections, but sometimes used for shopping and games of pool. In this method, the choices are assigned an order of comparison, called an agenda. The first two choices are compared. The winner is then compared to the next choice on the agenda, and this continues until all choices have been compared against the winner of the previous comparison.
   a. Using the preference schedule below, apply Sequential Pairwise voting to determine the winner, using the agenda: A, B, C, D.

Number of voters	10	15	12
1st choice	C	A	B
2nd choice	A	B	D
3rd choice	B	D	C
4th choice	D	C	A

8. b. Show that Sequential Pairwise voting can violate the Pareto criterion.
   c. Show that Sequential Pairwise voting can violate the Majority criterion.

9. The Coombs method is a variation of instant runoff voting. In Coombs method, the choice with the most last place votes is eliminated. Apply Coombs method to the preference schedules from questions 5 and 6.

10. Copeland's Method is designed to identify a Condorcet Candidate if there is one, and is considered a Condorcet Method. There are many Condorcet Methods, which vary primarily in how they deal with ties, which are very common when a Condorcet winner does not exist. Copeland's method does not have a tie-breaking procedure built-in. Research the Schulze method,

another Condorcet method that is used by the Wikimedia foundation that runs Wikipedia, and give some examples of how it works.

11. The plurality method is used in most U.S. elections. Some people feel that Ross Perot in 1992 and Ralph Nader in 2000 changed what the outcome of the election would have been if they had not run. Research the outcomes of these elections and explain how each candidate could have affected the outcome of the elections (for the 2000 election, you may wish to focus on the count in Florida). Describe how an alternative voting method could have avoided this issue.
12. Instant Runoff Voting and Approval voting have supporters advocating that they be adopted in the United States and elsewhere to decide elections. Research comparisons between the two methods describing the advantages and disadvantages of each in practice. Summarize the comparisons, and form your own opinion about whether either method should be adopted.
13. In a primary system, a first vote is held with multiple candidates. In some states, each political party has its own primary. In Washington State, there is a "top two" primary, where all candidates are on the ballot and the top two candidates advance to the general election, regardless of party. Compare and contrast the top two primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.
14. In a primary system, a first vote is held with multiple candidates. In some many states, where voters must declare a party to vote in the primary election, and they are only able to choose between candidates for their declared party. The top candidate from each party then advances to the general election. Compare and contrast this primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.
15. Sometimes in a voting scenario it is desirable to rank the candidates, either to establish preference order between a set of choices, or because the election requires multiple winners. For example, a hiring committee may have 30 candidates apply, and need to select 6 to interview, so the voting by the committee would need to produce the top 6 candidates. Describe how Plurality, Instant Runoff Voting, Borda Count, and Copeland's Method could be extended to produce a ranked list of candidates.

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CHAPTER OVERVIEW

2: Weighted Voting

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2.1: Introduction

In a corporate shareholders meeting, each shareholders' vote counts proportional to the amount of shares they own. An individual with one share gets the equivalent of one vote, while someone with 100 shares gets the equivalent of 100 votes. This is called **weighted voting**, where each vote has some weight attached to it. Weighted voting is sometimes used to vote on candidates, but more commonly to decide "yes" or "no" on a proposal, sometimes called a motion. Weighted voting is applicable in corporate settings, as well as decision making in parliamentary governments and voting in the United Nations Security Council.

In weighted voting, we are most often interested in the power each voter has in influencing the outcome.

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2.2: Beginnings

We'll begin with some basic vocabulary for weighted voting systems.

Vocabulary for Weighted Voting

Each individual or entity casting a vote is called a **player** in the election. They're often notated as $\mathbf{P_1, P_2, P_3, ..., P_N}$ where N is the total number of voters.

Each player is given a **weight**, which usually represents how many votes they get.

The **quota** is the minimum weight needed for the votes or weight needed for the proposal to be approved.

A weighted voting system will often be represented in a shorthand form:

$[q : w_1, w_2, w_3, ..., w_N]$

In this form, q is the quota, w_1 is the weight for player 1, and so on.

Example 1

In a small company, there are 4 shareholders. Mr. Smith has a 30% ownership stake in the company, Mr. Garcia has a 25% stake, Mrs. Hughes has a 25% stake, and Mrs. Lee has a 20% stake. They are trying to decide whether to open a new location. The company by-laws state that more than 50% of the ownership has to approve any decision like this. This could be represented by the weighted voting system:

$[51 : 30, 25, 25, 20]$

Solution

Here we have treated the percentage ownership as votes, so Mr. Smith gets the equivalent of 30 votes, having a 30% ownership stake. Since more than 50% is required to approve the decision, the quota is 51, the smallest whole number over 50.

In order to have a meaningful weighted voting system, it is necessary to put some limits on the quota.

Limits on the Quota

The quota must be more than $\frac{1}{2}$ the total number of votes.

The quota can't be larger than the total number of votes.

Why? Consider the voting system $[q; 3, 2, 1]$

Here there are 6 total votes. If the quota was set at only 3, then player 1 could vote yes, players 2 and 3 could vote no, and both would reach quota, which doesn't lead to a decision being made. In order for only one decision to reach quota at a time, the quota must be at least half the total number of votes. If the quota was set to 7, then no group of voters could ever reach quota, and no decision can be made, so it doesn't make sense for the quota to be larger than the total number of voters.

Try it Now 1

In a committee there are four representatives from the management and three representatives from the workers' union. For a proposal to pass, four of the members must support it, including at least one member of the union. Find a voting system that can represent this situation.

Answer

If we represent the players as $M_1, M_2, M_3, M_4, U_I, U_2, U_3$ then we may be tempted to set up a system like $[4 : 1, 1, 1, 1, 1, 1, 1]$. While this system would meet the first requirement that four members must support a proposal for it to pass, this does not satisfy the requirement that at least one member of the union must support it.

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2.3: A Look at Power

Consider the voting system $[10 : 11, 3, 2]$. Notice that in this system, player 1 can reach quota without the support of any other player. When this happens, we say that player 1 is a **dictator**.

Dictator

A player will be a dictator if their weight is equal to or greater than the quota. The dictator can also block any proposal from passing; the other players cannot reach quota without the dictator.

In the voting system $[8 : 6, 3, 2]$, no player is a dictator. However, in this system, the quota can only be reached if player 1 is in support of the proposal; player 2 and 3 cannot reach quota without player 1's support. In this case, player 1 is said to have **veto power**. Notice that player 1 is not a dictator, since player 1 would still need player 2 or 3's support to reach quota.

Veto Power

A player has veto power if their support is necessary for the quota to be reached. It is possible for more than one player to have veto power, or for no player to have veto power.

With the system $[10 : 7, 6, 2]$, player 3 is said to be a **dummy**, meaning they have no influence in the outcome. The only way the quota can be met is with the support of both players 1 and 2 (both of which would have veto power here); the vote of player 3 cannot affect the outcome.

Dummy

A player is a dummy if their vote is never essential for a group to reach quota.

Example 2

In the voting system $[16 : 7, 6, 3, 3, 2]$, are any players dictators? Do any have veto power? Are any dummies?

Solution

No player can reach quota alone, so there are no dictators.

Without player 1, the rest of the players' weights add to 14, which doesn't reach quota, so player 1 has veto power. Likewise, without player 2, the rest of the players' weights add to 15, which doesn't reach quota, so player 2 also has veto power.

Since player 1 and 2 can reach quota with either player 3 or player 4's support, neither player 3 or player 4 have veto power. However they cannot reach quota with player 5's support alone, so player 5 has no influence on the outcome and is a dummy.

Try it Now 2

In the voting system $[q : 10, 5, 3]$, which players are dictators, have veto power, and are dummies if the quota is 10? 12? 16?

Answer

In the voting system $[q : 10, 5, 3]$, if the quota is 10, then player 1 is a dictator since they can reach quota without the support of the other players. This makes the other two players automatically dummies.

If the quota is 12, then player 1 is necessary to reach quota, so has veto power. Since at this point either player 2 or player 3 would allow player 1 to reach quota, neither player is a dummy, so they are regular players (not dictators, no veto power, and not a dummy).

If the quota is 16, then no two players alone can reach quota, so all three players have veto power.

To better define power, we need to introduce the idea of a **coalition**. A coalition is a group of players voting the same way. In the example above, $\{P_1, P_2, P_4\}$ would represent the coalition of players 1, 2 and 4. This coalition has a combined weight of $7 + 6 + 3 = 16$, which meets quota, so this would be a winning coalition.

A player is said to be **critical** in a coalition if them leaving the coalition would change it from a winning coalition to a losing coalition. In the coalition $\{P_1, P_2, P_4\}$, every player is critical. In the coalition $\{P_3, P_4, P_5\}$, no player is critical, since it wasn't a winning coalition to begin with. In the coalition $\{P_1, P_2, P_3, P_4, P_5\}$, only players 1 and 2 are critical; any other player could leave the coalition and it would still meet quota.

Coalitions and Critical Players

A coalition is any group of players voting the same way.

A coalition is a **winning coalition** if the coalition has enough weight to meet quota.

A player is **critical** in a coalition if them leaving the coalition would change it from a winning coalition to a losing coalition.

Example 3

In the Scottish Parliament in 2009 there were 5 political parties: 47 representatives for the Scottish National Party, 46 for the Labour Party, 17 for the Conservative Party, 16 for the Liberal Democrats, and 2 for the Scottish Green Party. Typically all representatives from a party vote as a block, so the parliament can be treated like the weighted voting system:

$[65 : 47, 46, 17, 16, 2]$

Solution

Consider the coalition $\{P_1, P_3, P_4\}$. No two players alone could meet the quota, so all three players are critical in this coalition.

In the coalition $\{P_1, P_3, P_4, P_5\}$, any player except P_1 could leave the coalition and it would still meet quota, so only P_1 is critical in this coalition.

Notice that a player with veto power will be critical in every winning coalition, since removing their support would prevent a proposal from passing.

Likewise, a dummy will never be critical, since their support will never change a losing coalition to a winning one.

Dictators, Veto, and Dummies and Critical Players

A player is a **dictator** if the single-player coalition containing them is a winning coalition.

A player has **veto power** if they are critical in every winning coalition.

A player is a **dummy** if they are not critical in any winning coalition.

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2.4: Calculating Power- Banzhaf Power Index

The **Banzhaf power index** was originally created in 1946 by Lionel Penrose, but was reintroduced by John Banzhaf in 1965. The power index is a numerical way of looking at power in a weighted voting situation.

Calculating Banzhaf Power Index

To calculate the Banzhaf power index:

1. List all winning coalitions
2. In each coalition, identify the players who are critical
3. Count up how many times each player is critical
4. Convert these counts to fractions or decimals by dividing by the total times any player is critical

Example 4

Find the Banzhaf power index for the voting system $[8:6,3,2]$.

Solution

We start by listing all winning coalitions. If you aren't sure how to do this, you can list all coalitions, then eliminate the non-winning coalitions. No player is a dictator, so we'll only consider two and three player coalitions.

$\{P_1, P_2\}$ Total weight: 9. Meets quota.

$\{P_1, P_3\}$ Total weight: 8. Meets quota.

$\{P_2, P_3\}$ Total weight: 5. Does not meet quota.

$\{P_1, P_2, P_3\}$ Total weight: 11. Meets quota.

Next we determine which players are critical in each winning coalition. In the winning two-player coalitions, both players are critical since no player can meet quota alone. Underlining the critical players to make it easier to count:

$\{\underline{P_1}, \underline{P_2}\}$

$\{\underline{P_1}, \underline{P_3}\}$

In the three-person coalition, either P_2 or P_3 could leave the coalition and the remaining players could still meet quota, so neither is critical. If P_1 were to leave, the remaining players could not reach quota, so P_1 is critical.

$\{\underline{P_1}, P_2, P_3\}$

Altogether,P_1 is critical 3 times, P_2 is critical 1 time, and $P_3$is critical 1 time.

Converting to percents:

$P_1 = 3/5 = 60\%$

$P_2 = 1/5 = 20\%$

$P_3 = 1/5 = 20\%$

Example 5

Consider the voting system $[16:7,6,3,3,2]$. Find the Banzhaf power index.

Solution

The winning coalitions are listed below, with the critical players underlined.

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$

$\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$

$\{\underline{P_1}, \underline{P_2}, P_3, P_4\}$

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_5\}$

$\{\underline{P_1}, \underline{P_2}, \underline{P_4}, P_5\}$

$\{\underline{P_1}, \underline{P_2}, P_3, P_4, P_5\}$

Counting up times that each player is critical:

$P_1 = 6$

$P_2 = 6$

$P_3 = 2$

$P_4 = 2$

$P_5 = 0$

Total of all: 16

Divide each player's count by 16 to convert to fractions or percents:

$P_1 = 6/16 = 3/8 = 37.5\%$

$P_2 = 6/16 = 3/8 = 37.5\%$

$P_3 = 2/16 = 1/8 = 12.5\%$

$P_4 = 2/16 = 1/8 = 12.5\%$

$P_5 = 0/16 = 0 = 0\%$

The Banzhaf power index measures a player's ability to influence the outcome of the vote. Notice that player 5 has a power index of 0, indicating that there is no coalition in which they would be critical power and could influence the outcome. This means player 5 is a dummy, as we noted earlier.

Example 6

Revisiting the Scottish Parliament, with voting system $[65 : 47, 46, 17, 16, 2]$, the winning coalitions are listed, with the critical players underlined.

Solution

$\{\underline{P_1}, \underline{P_2}\}$

$\{\underline{P_1}, \underline{P_2}, P_3\}$ $\{\underline{P_1}, \underline{P_2}, P_4\}$

$\{\underline{P_1}, \underline{P_2}, P_5\}$ $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$

$\{\underline{P_1}, \underline{P_3}, \underline{P_5}\}$ $\{\underline{P_1}, \underline{P_4}, \underline{P_5}\}$

$\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$ $\{\underline{P_2}, \underline{P_3}, \underline{P_5}\}$

$\{P_1, P_2, P_3, P_4\}$ $\{P_1, P_2, P_3, P_5\}$

$\{\underline{P_1}, P_2, P_4, P_5\}$ $\{\underline{P_1}, P_3, P_4, P_5\}$

$\{\underline{P_2}, \underline{P_3}, P_4, P_5\}$

$\{P_1, P_2, P_3, P_4, P_5\}$

Counting up times that each player is critical:

District	Times critical	Power index
P_1 (Scottish National Party)	9	$9/27 = 33.3\%$
P_2 (Labour Party)	7	$7/27 = 25.9\%$
P_3 (Conservative Party)	5	$5/27 = 18.5\%$
P_4 (Liberal Democrats Party)	3	$3/27 = 11.1\%$
P_5 (Scottish Green Party)	3	$3/27 = 11.1\%$

Interestingly, even though the Liberal Democrats party has only one less representative than the Conservative Party, and 14 more than the Scottish Green Party, their Banzhaf power index is the same as the Scottish Green Party's. In parliamentary governments, forming coalitions is an essential part of getting results, and a party's ability to help a coalition reach quota defines its influence.

? Try it Now 3

Find the Banzhaf power index for the weighted voting system **[36 : 20, 17, 16, 3]**.

Answer

The voting system tells us that the quota is 36, that Player 1 has 20 votes (or equivalently, has a weight of 20), Player 2 has 17 votes, Player 3 has 16 votes, and Player 4 has 3 votes.

A coalition is any group of one or more players. What we're looking for is winning coalitions - coalitions whose combined votes (weights) add to up to the quota or more. So the coalition **{P3, P4}** is not a winning coalition because the combined weight is $16 + 3 = 19$, which is below the quota.

So we look at each possible combination of players and identify the winning ones:

{P1, P2}(weight : 37)	{P1, P3} (weight: 36)
{P1, P2, P3} (weight: 53)	{P1, P2, P4} (weight: 40)
{P1, P3, P4} (weight: 39)	{P1, P2, P3, P4} (weight: 56)
{P2, P3, P4}(weight: 36)	

✔ Example 7

Banzhaf used this index to argue that the weighted voting system used in the Nassau County Board of Supervisors in New York was unfair. The county was divided up into 6 districts, each getting voting weight proportional to the population in the district, as shown below. Calculate the power index for each district.

Misplaced '#'

Solution

Translated into a weighted voting system, assuming a simple majority is needed for a proposal to pass:

[58 : 31, 31, 28, 21, 2, 2]

Listing the winning coalitions and marking critical players:

{H1, H2}	{H1, OB, NH}	{H2, OB, NH, LB}
{H1, OB}	{H1, OB, LB}	{H2, OB, NH, GC}
{H2, OB}	{H1, OB, GC}	{H2, OB, LB, GC}
{H1, H2, NH}	{H1, OB, NH, LB}	{H2, OB, NH, LB, GC}
{H1, H2, LB}	{H1, OB, NH, GC}	{H1, H2, OB}
{H1, H2, GC}	{H1, OB, LB, GC}	{H1, H2, OB, NH}
{H1, H2, NH, LB}	{H1, OB, NH, LB.GC}	{H1, H2, OB, LB}
{H1, H2, NH, GC}	{H2, OB, NH}	{H1, H2, OB, GC}
{H1, H2, LB, GC}	{H2, OB, LB}	{H1, H2, OB, NH, LB}
{H1, H2, NH, LB, GC}	{H2, OB, GC}	{H1, H2, OB, NH, GC}
		{H1, H2, OB, NH, LB, GC}

There are a lot of them! Counting up how many times each player is critical,

Misplaced '#'

It turns out that the three smaller districts are dummies. Any winning coalition requires two of the larger districts.

The weighted voting system that Americans are most familiar with is the Electoral College system used to elect the President. In the Electoral College, states are given a number of votes equal to the number of their congressional representatives (house + senate). Most states give all their electoral votes to the candidate that wins a majority in their state, turning the Electoral College into a weighted voting system, in which the states are the players. As I'm sure you can imagine, there are billions of possible winning coalitions, so the power index for the Electoral College has to be computed by a computer using approximation techniques.

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2.5: Calculating Power- Shapley-Shubik Power Index

The **Shapley-Shubik** power index was introduced in 1954 by economists Lloyd Shapley and Martin Shubik, and provides a different approach for calculating power.

In situations like political alliances, the order in which players join an alliance could be considered the most important consideration. In particular, if a proposal is introduced, the player that joins the coalition and allows it to reach quota might be considered the most essential. The Shapley-Shubik power index counts how likely a player is to be **pivotal**. What does it mean for a player to be pivotal?

First, we need to change our approach to coalitions. Previously, the coalition $\{P_1, P_2\}$ and $\{P_2, P_1\}$ would be considered equivalent, since they contain the same players. We now need to consider the *order* in which players join the coalition. For that, we will consider **sequential coalitions** – coalitions that contain all the players in which the order players are listed reflect the order they joined the coalition. For example, the sequential coalition

$< P_2, P_1, P_3 >$ would mean that P_2 joined the coalition first, then P_1, and finally P_3. The angle brackets < > are used instead of curly brackets to distinguish sequential coalitions.

Pivotal Player

A sequential coalition lists the players in the order in which they joined the coalition.

A **pivotal player** is the player in a sequential coalition that changes a coalition from a losing coalition to a winning one. Notice there can only be one pivotal player in any sequential coalition.

Example 8

In the weighted voting system $[8 : 6, 4, 3, 2]$, which player is pivotal in the sequential coalition $< P_3, P_2, P_4, P_1 >$?

Solution

The sequential coalition shows the order in which players joined the coalition. Consider the running totals as each player joins:

P_3	Total weight: 3	Not winning
P_3, P_2	Total weight: $3 + 4 = 7$	Not winning
P_3, P_2, P_4	Total weight: $3 + 4 + 2 = 9$	Winning
R_2, P_3, P_4, P_1	Total weight: $3 + 4 + 2 + 6 = 15$	Winning

Since the coalition *becomes* winning when P_4 joins, P_4 is the pivotal player in this coalition.

Calculating Shapley-Shubik Power Index

To calculate the Shapley-Shubik Power Index:

1. List all sequential coalitions
2. In each sequential coalition, determine the pivotal player
3. Count up how many times each player is pivotal
4. Convert these counts to fractions or decimals by dividing by the total number of sequential coalitions

How many sequential coalitions should we expect to have? If there are *N* players in the voting system, then there are N possibilities for the first player in the coalition, N–1 possibilities for the second player in the coalition, and so on. Combining these possibilities, the total number of coalitions would be:$N(N-1)(N-2)(N-3)\cdots(3)(2)(1)$. This calculation is called a **factorial**, and is notated $N!$ The number of sequential coalitions with N players is $N!$

Example 9

How many sequential coalitions will there be in a voting system with 7 players?

Solution

There will be $7!$ sequential coalitions. $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

As you can see, computing the Shapley-Shubik power index by hand would be very difficult for voting systems that are not very small.

Example 10

Consider the weighted voting system $[6 : 4, 3, 2]$. We will list all the sequential coalitions and identify the pivotal player. We will have 3! = 6 sequential coalitions. The coalitions are listed, and the pivotal player is underlined.

$< P_1, \underline{P_2}, P_3 >$ $< P_1, \underline{P_3}, P_2 >$ $< P_2, \underline{P_1}, P_3 >$
$< P_2, P_3, \underline{P_1} >$ $< P_3, P_2, \underline{P_1} >$ $< P_3, \underline{P_1}, P_2 >$

Solution

$\mathbf{P_1}$ is pivotal 4 times, $\mathbf{P_2}$ is pivotal 1 time, and $\mathbf{P_3}$ is pivotal 1 time.

Player	Times pivotal	Power index
P_1	4	$4/6 = 66.7\%$
P_2	1	$1/6 = 16.7\%$
P_3	1	$1/6 = 16.7\%$

For comparison, the Banzhaf power index for the same weighted voting system would be $\mathbf{P_1 : 60\%, P_2 : 20\%, P_3 : 20\%}$. While the Banzhaf power index and Shapley-Shubik power index are usually not terribly different, the two different approaches usually produce somewhat different results.

Try it Now 4

Find the Shapley-Shubik power index for the weighted voting system $\mathbf{[36 : 20, 17, 15]}$.

Answer

Listing all sequential coalitions and identifying the pivotal player:

$< P_1, \underline{P_2}, P_3 >$ $< P_1, P_3, \underline{P_2} >$ $< P_2, \underline{P_1}, P_3 >$
$< P_2, P_3, \underline{P_1} >$ $< P_3, P_2, \underline{P_1} >$ $< P_3, P_1, \underline{P_2} >$

$\mathbf{P_1}$ is pivotal 3 times, $\mathbf{P_2}$ is pivotal 3 times, and $\mathbf{P_3}$ is pivotal 0 times.

Player	Times pivotal	Power index
P_1	3	$3/6 = 50\%$
P_2	3	$3/6 = 50\%$
P_3	0	$0/6 = 0\%$

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2.6: Exercises(Skills)

1. Consider the weighted voting system $[47 : 10, 9, 9, 5, 4, 4, 3, 2, 2]$
   a. How many players are there?
   b. What is the total number (weight) of votes?
   c. What is the quota in this system?
2. Consider the weighted voting system $[31 : 10, 10, 8, 7, 6, 4, 1, 1]$
   a. How many players are there?
   b. What is the total number (weight) of votes?
   c. What is the quota in this system?
3. Consider the weighted voting system $[q : 7, 5, 3, 1, 1]$
   a. What is the smallest value that the quota q can take?
   b. What is the largest value that the quota q can take?
   c. What is the value of the quota if at least two-thirds of the votes are required to pass a motion?
4. Consider the weighted voting system $[q : 10, 9, 8, 8, 8, 6]$
   a. What is the smallest value that the quota q can take?
   b. What is the largest value that the quota q can take?
   c. What is the value of the quota if at least two-thirds of the votes are required to pass a motion?
5. Consider the weighted voting system $[13 : 13, 6, 4, 2]$
   a. Identify the dictators, if any.
   b. Identify players with veto power, if any
   c. Identify dummies, if any.
6. Consider the weighted voting system $[11 : 9, 6, 3, 1]$
   a. Identify the dictators, if any.
   b. Identify players with veto power, if any
   c. Identify dummies, if any.
7. Consider the weighted voting system $[19 : 13, 6, 4, 2]$
   a. Identify the dictators, if any.
   b. Identify players with veto power, if any
   c. Identify dummies, if any.
8. Consider the weighted voting system $[17 : 9, 6, 3, 1]$
   a. Identify the dictators, if any.
   b. Identify players with veto power, if any
   c. Identify dummies, if any.
9. Consider the weighted voting system $[15 : 11, 7, 5, 2]$
   a. What is the weight of the coalition $\{P_1, P_2, P_4\}$
   b. In the coalition $\{P_1, P_2, P_4\}$ which players are critical?
10. Consider the weighted voting system $[17 : 13, 9, 5, 2]$
    a. What is the weight of the coalition $\{P_1, P_2, P_3\}$
    b. In the coalition $\{P_1, P_2, P_3\}$ which players are critical?
11. Find the Banzhaf power distribution of the weighted voting system

    $[27 : 16, 12, 11, 3]$
12. Find the Banzhaf power distribution of the weighted voting system

$[33 : 18, 16, 15, 2]$

13. Consider the weighted voting system $[q : 15, 8, 3, 1]$ Find the Banzhaf power distribution of this weighted voting system,
    a. When the quota is 15
    b. When the quota is 16
    c. When the quota is 18
14. Consider the weighted voting system $[q : 15, 8, 3, 1]$ Find the Banzhaf power distribution of this weighted voting system,
    a. When the quota is 19
    b. When the quota is 23
    c. When the quota is 26
15. Consider the weighted voting system $[17 : 13, 9, 5, 2]$. In the sequential coalition $< P_3, P_2, P_1, P_4 >$ which player is pivotal?
16. Consider the weighted voting system $[15 : 13, 9, 5, 2]$. In the sequential coalition $\langle P_1, P_4, P_2, P_3 \rangle$ which player is pivotal?
17. Find the Shapley-Shubik power distribution for the system $[24 : 17, 13, 11]$
18. Find the Shapley-Shubik power distribution for the system $[25 : 17, 13, 11]$

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2.7: Exercises(Concepts)

19. Consider the weighted voting system $[q : 7, 3, 1]$
    a. Which values of q result in a dictator (list all possible values)
    b. What is the smallest value for q that results in exactly one player with veto power but no dictators?
    c. What is the smallest value for q that results in exactly two players with veto power?
20. Consider the weighted voting system $[q : 9, 4, 2]$
    a. Which values of q result in a dictator (list all possible values)
    b. What is the smallest value for q that results in exactly one player with veto power?
    c. What is the smallest value for q that results in exactly two players with veto power?
21. Using the Shapley-Shubik method, is it possible for a dummy to be pivotal?
22. If a specific weighted voting system requires a unanimous vote for a motion to pass:
    a. Which player will be pivotal in any sequential coalition?
    b. How many winning coalitions will there be?
23. Consider a weighted voting system with three players. If Player 1 is the only player with veto power, there are no dictators, and there are no dummies:
    a. Find the Banzhaf power distribution.
    b. Find the Shapley-Shubik power distribution
24. Consider a weighted voting system with three players. If Players 1 and 2 have veto power but are not dictators, and Player 3 is a dummy:
    a. Find the Banzhaf power distribution.
    b. Find the Shapley-Shubik power distribution
25. An executive board consists of a president (P) and three vice-presidents (V_1, V_2, V_3). For a motion to pass it must have three yes votes, one of which must be the president's. Find a weighted voting system to represent this situation.
26. On a college's basketball team, the decision of whether a student is allowed to play is made by four people: the head coach and the three assistant coaches. To be allowed to play, the student needs approval from the head coach and at least one assistant coach. Find a weighted voting system to represent this situation.
27. In a corporation, the shareholders receive 1 vote for each share of stock they hold, which is usually based on the amount of money the invested in the company. Suppose a small corporation has two people who invested \$30,000 each, two people who invested \$20,000 each, and one person who invested \$10,000. If they receive one share of stock for each \$1000 invested, and any decisions require a majority vote, set up a weighted voting system to represent this corporation's shareholder votes.
28. A contract negotiations group consists of 4 workers and 3 managers. For a proposal to be accepted, a majority of workers and a majority of managers must approve of it. Calculate the Banzhaf power distribution for this situation. Who has more power: a worker or a manager?
29. The United Nations Security Council consists of 15 members, 10 of which are elected, and 5 of which are permanent members. For a resolution to pass, 9 members must support it, which must include all 5 of the permanent members. Set up a weighted voting system to represent the UN Security Council and calculate the Banzhaf power distribution.

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2.8: Exercises(Exploration)

1. In the U.S., the Electoral College is used in presidential elections. Each state is awarded a number of electors equal to the number of representatives (based on population) and senators (2 per state) they have in congress. Since most states award the winner of the popular vote in their state all their state's electoral votes, the Electoral College acts as a weighted voting system. To explore how the Electoral College works, we'll look at a mini-country with only 4 states. Here is the outcome of a hypothetical election:

State	Smalota	Medigan	Bigonia	Hugodo
Population	50,000	70,000	100,000	240,000
Votes for A	40,000	50,000	80,000	50,000
Votes for B	10,000	20,000	20,000	190,000

a. If this country did not use an Electoral College, which candidate would win the election?
b. Suppose that each state gets 1 electoral vote for every 10,000 people. Set up a weighted voting system for this scenario, calculate the Banzhaf power index for each state, then calculate the winner if each state awards all their electoral votes to the winner of the election in their state.
c. Suppose that each state gets 1 electoral vote for every 10,000 people, plus an additional 2 votes. Set up a weighted voting system for this scenario, calculate the Banzhaf power index for each state, then calculate the winner if each state awards all their electoral votes to the winner of the election in their state.
d. Suppose that each state gets 1 electoral vote for every 10,000 people, and awards them based on the number of people who voted for each candidate. Additionally, they get 2 votes that are awarded to the majority winner in the state. Calculate the winner under these conditions.
e. Does it seem like an individual state has more power in the Electoral College under the vote distribution from part c or from part d?
f. Research the history behind the Electoral College to explore why the system was introduced instead of using a popular vote. Based on your research and experiences, state and defend your opinion on whether the Electoral College system is or is not fair.

2. The value of the Electoral College (see previous problem for an overview) in modern elections is often debated. Find an article or paper providing an argument for or against the Electoral College. Evaluate the source and summarize the article, then give your opinion of why you agree or disagree with the writer's point of view. If done in class, form groups and hold a debate.

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CHAPTER OVERVIEW

3: Cryptography

3.1: Cryptography
3.2: Substitution Ciphers
3.3: Transposition Ciphers
3.4: Advanced shared symmetric-key methods
3.5: Public Key Cryptography
3.6: Other Secret Keeping Methods
3.7: The One-Way Function
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3.1: Cryptography

When people need to secretly store or communicate messages, they turn to cryptography. Cryptography involves using techniques to obscure a message so outsiders cannot read the message. It is typically split into two steps: encryption, in which the message is obscured, and decryption, in which the original message is recovered from the obscured form.

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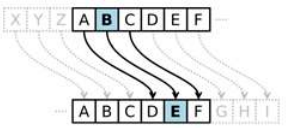
3.2: Substitution Ciphers

One simple encryption method is called a **substitution cipher**.

Substitution Cipher

A substitution cipher replaces each letter in the message with a different letter, following some established mapping.

A simple example of a substitution cipher is called the **Caesar cipher**, sometimes called a shift cipher. In this approach, each letter is replaced with a letter some fixed number of positions later in the alphabet. For example, if we use a shift of 3, then the letter A would be replaced with D, the letter 3 positions later in the alphabet. The entire mapping would look like: [1]



Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**

Maps to: **DEFGHIJKLMNOPQRSTUVWXYZABC**

Example 1

Use the Caesar cipher with shift of 3 to encrypt the message: “We ride at noon”

Solution

We use the mapping above to replace each letter. W gets replaced with Z, and so forth, giving the encrypted message: ZH ULGH DW QRRQ.

Notice that the length of the words could give an important clue to the cipher shift used. If we saw a single letter in the encrypted message, we would assume it must be an encrypted A or I, since those are the only single letters than form valid English words.

To obscure the message, the letters are often rearranged into equal sized blocks. The message ZH ULGH DW QRRQ could be written in blocks of three characters as

ZHU LGH DWQ RRQ.

Example 2

Decrypt the message GZD KNK YDX MFW JXA if it was encrypted using a shift cipher with shift of 5.

Solution

We start by writing out the character mapping by shifting the alphabet, with A mapping to F, five characters later in the alphabet.

Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**

Maps to: **FGHIJKLMNOPQRSTUVWXYZABCDE**

We now work backwards to decrypt the message. The first letter G is mapped to by B, so B is the first character of the original message. Continuing, our decrypted message is

BUY FIF TYS HAR ESA.

Removing spaces we get BUYFIFTYSHARESA. In this case, it appears an extra character was added to the end to make the groups of three come out even, and that the original message was “Buy fifty shares.”

Try it Now 1

Decrypt the message BNW MVX WNH if it was encrypted using a shift cipher with shift 9 (mapping A to J).

Answer

SEND MONEY



Notice that in both the ciphers above, the extra part of the alphabet wraps around to the beginning. Because of this, a handy version of the shift cipher is a cipher disc, such as the Alberti cipher disk shown here[2] from the 1400s. In a cipher disc, the inner wheel could be turned to change the cipher shift. This same approach is used for "secret decoder rings."

The security of a cryptographic method is very important to the person relying on their message being kept secret. The security depends on two factors:

1. The security of the method being used
2. The security of the encryption key used

In the case of a shift cipher, the method is "a shift cipher is used." The encryption key is the specific amount of shift used.

Suppose an army is using a shift cipher to send their messages, and one of their officers is captured by their enemy. It is likely the method and encryption key could become compromised. It is relatively hard to change encryption methods, but relatively easy to change encryption keys.

During World War II, the Germans' Enigma encryption machines were captured, but having details on the encryption method only slightly helped the Allies, since the encryption keys were still unknown and hard to discover. Ultimately, the security of a message cannot rely on the method being kept secret; it needs to rely on the key being kept secret.

Encryption Security

The security of any encryption method should depend only on the encryption key being difficult to discover. It is not safe to rely on the encryption method (algorithm) being kept secret.

With that in mind, let's analyze the security of the Caesar cipher.

Example 3

Suppose you intercept a message, and you know the sender is using a Caesar cipher, but do not know the shift being used. The message begins EQZP. How hard would it be to decrypt this message?

Solution

Since there are only 25 possible shifts, we would only have to try 25 different possibilities to see which one produces results that make sense. While that would be tedious, one person could easily do this by hand in a few minutes. A modern computer could try all possibilities in under a second.

Shift	Message	Shift	Message	Shift	Message	Shift	Message
1	DPYO	7	XJSI	13	RDMC	19	LXGW
2	COXN	8	WIRH	14	QCLB	20	KWFV
3	BNWM	9	VHQG	15	PBKA	21	JVEU
4	AMVL	10	UGPF	16	OAJZ	22	IUDT
5	ZLUK	11	TFOE	17	NZIY	23	HTCS
6	YKTJ	12	SEND	18	MYHX	24	GSBR
						25	FRAQ

In this case, a shift of 12 (A mapping to M) decrypts EQZP to SEND. Because of this ease of trying all possible encryption keys, the Caesar cipher is not a very secure encryption method.

Brute Force Attack

A brute force attack is a method for breaking encryption by trying all possible encryption keys.

To make a brute force attack harder, we could make a more complex substitution cipher by using something other than a shift of the alphabet. By choosing a random mapping, we could get a more secure cipher, with the tradeoff that the encryption key is harder to describe; the key would now be the entire mapping, rather than just the shift amount.

Example 4

Use the substitution mapping below to encrypt the message “March 12 0300”

Original: $mathrm{ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789}$

Maps to: **2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7P**

Solution

Using the mapping, the message would encrypt to 62SQT ZN Y1YY

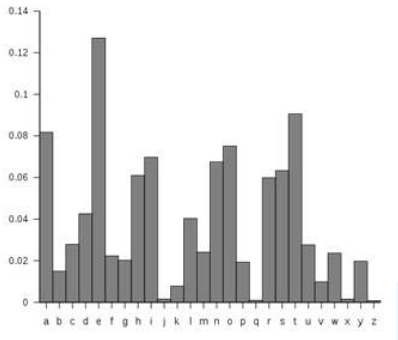
Try it Now 2

Use the substitution mapping from Example 4 to decrypt the message C2SVX2VP

Answer

PARTY AT 9

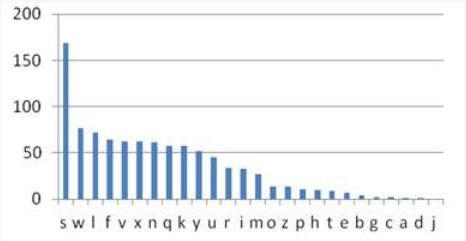
While there were only 25 possible shift cipher keys (35 if we had included numbers), there are about 10^{40} possible substitution ciphers[3]. That’s much more than a trillion trillions. It would be essentially impossible, even with supercomputers, to try every possible combination. Having a huge number of possible encryption keys is one important part of key security.



Unfortunately, this cipher is still not secure, because of a technique called frequency analysis, discovered by Arab mathematician Al-Kindi in the 9th century. English and other languages have certain letters than show up more often in writing than others.[4] For example, the letter E shows up the most frequently in English. The chart to the right shows the typical distribution of characters.

Example 5

The chart to the right shows the frequency of different characters in some encrypted text. What can you deduce about the mapping?



Solution

Because of the high frequency of the letter S in the encrypted text, it is very likely that the substitution maps E to S. Since W is the second most frequent character, it likely that T or A maps to W. Because C, A, D, and J show up rarely in the encrypted text, it is likely they are mapped to from J, Q, X, and Z.

In addition to looking at individual letters, certain pairs of letters show up more frequently, such as the pair “th.” By analyzing how often different letters and letter pairs show up an encrypted message, the substitution mapping used can be deduced[5].

[1] en.Wikipedia.org/w/index.php?title=File:Caesar3.svg&page=1. PD

[2] en.Wikipedia.org/wiki/File:Alberti_cipher_disk.JPG

[3] There are 35 choices for what *A* maps to, then 34 choices for what *B* maps to, and so on, so the total number of possibilities is 35*34*33*…*2*1 = 35! = about 10^{40}

[4] en.Wikipedia.org/w/index.php?title=File:English_letter_frequency_(alphabetic).svg&page=1 PD

[5] For an example of how this is done, see en.Wikipedia.org/wiki/Frequency_analysis

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3.3: Transposition Ciphers

Another approach to cryptography is **transposition cipher.**

Transposition Ciphers

A transposition cipher is one in which the order of characters is changed to obscure the message.

An early version of a transposition cipher was a Scytale[1], in which paper was wrapped around a stick and the message was written. Once unwrapped, the message would be unreadable until the message was wrapped around a same-sized stick again.



One modern transposition cipher is done by writing the message in rows, then forming the encrypted message from the text in the columns.

Example 6

Encrypt the message “Meet at First and Pine at midnight” using rows 8 characters long.

Solution

We write the message in rows of 8 characters each. Nonsense characters are added to the end to complete the last row.

MEETATFI

RSTANDPI

NEATMIDN

IGHTPXNR

We could then encode the message by recording down the columns. The first column, reading down, would be MRNI. All together, the encoded message would be MRNI ESEG ETAH TATT ANMP TDIX FPDN IINR. The spaces would be removed or repositioned to hide the size of table used, since that is the encryption key in this message.

Example 7

Decrypt the message CEE IAI MNL NOG LTR VMH NW using the method above with a table with rows of 5 characters.

Solution

Since there are total of 20 characters and each row should have 5 characters, then there will be $20/5 = 4$ rows.

We start writing, putting the first 4 letters, CEEI, down the first column.

CALLM

EINTH

EMORN

INGVW

We can now read the message: CALL ME IN THE MORNING VW. The VW is likely nonsense characters used to fill out the message.

More complex versions of this rows-and-column based transposition cipher can be created by specifying an order in which the columns should be recorded. For example, the method could specify that after writing the message out in rows that you should record the third column, then the fourth, then the first, then the fifth, then the second. This adds additional complexity that would make it harder to make a brute-force attack.

To make the encryption key easier to remember, a word could be used. For example, if the key word was “MONEY”, it would specify that rows should have 5 characters each. The order of the letters in the alphabet would dictate which order to read the columns in. Since E, the 4th letter in the word, is the earliest letter in the alphabet from the word MONEY, the 4th column would be used first, followed by the 1st column (M), the 3rd column (N), the 2nd column (O), and the 5th column (Y).

Example 8

Encrypt the message BUY SOME MILK AND EGGS using a transposition cipher with key word MONEY.

Solution

Writing out the message in rows of 5 characters:

BUYSO
MEMIL
KANDE
GGSPK

We now record the columns in order 4 1 3 2 5:

SIDP BMKG YMNS UEAG OLEK

As before, we’d then remove or reposition the spaces to conceal evidence of the encryption key.

Try it Now 3

Encrypt the message “Fortify the embassy” using a transposition cipher with key word HELP

Answer

HELP gives column order 2 1 3 4.

FORT

IFYT

HEEM

BASS

YPLR

Encrypted text: OFE APF IHB YRY ESL TTM SR

To decrypt a keyword-based transposition cipher, we’d reverse the process. In the example above, the keyword MONEY tells us to begin with the 4th column, so we’d start by writing SIDP down the 4th column, then continue to the 1st column, 3rd column, etc.

Example 9

Decrypt the message RHA VTN USR EDE AIE RIK ATS OQR using a row-and-column transposition cipher with keyword PRIZED.

Solution

The keyword PRIZED tells us to use rows with 6 characters. Since D comes first in the alphabet, we start with 6th column. Since E is next in the alphabet, we’d follow with the 5th column. Continuing, the word PRIZED tells us the message was recorded with the columns in order 4 5 3 6 2 1.

For the decryption, we set up a table with 6 characters in each row. Since the beginning of the encrypted message came from the last column, we start writing the encrypted message down the last column.

					R
					H
					A
					V

The 5th column was the second one the encrypted message was read from, so is the next one we write to.

				T	R
				N	H
				U	A
				S	V

Continuing, we can fill out the rest of the message.

A	I	R	S	T	R
I	K	E	O	N	H
E	A	D	Q	U	A
R	T	E	R	S	V

Reading across the rows gives our decrypted message: AIRSTRIKEONHEADQUARTERSV

Unfortunately, since the transposition cipher does not change the frequency of individual letters, it is still susceptible to frequency analysis, though the transposition does eliminate information from letter pairs.

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[1] en.Wikipedia.org/wiki/File:Skytala%26EmptyStrip-Shaded.png

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3.4: Advanced shared symmetric-key methods

Both the substitution and transposition methods discussed so far are shared **symmetric-key** methods, meaning that both sender and receiver would have to have agreed upon the same secret encryption key before any methods could be sent.

All of the methods so far have been susceptible to frequency analysis since each letter is always mapped to the same encrypted character. More advanced methods get around this weakness. For example, the Enigma machines used in World War II had wheels that rotated. Each wheel was a substitution cipher, but the rotation would cause the substitution used to shift after each character.

For a simplified example, in the initial setup, the wheel might provide the mapping

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789**
- Maps to: **2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7P**

After the first character is encrypted, the wheel rotates, shifting the mapping one space, resulting in a new shifted mapping:

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789**
- Maps to: **P2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7**

Using this approach, no letter gets encrypted as the same character over and over.

Example 10

Encrypt the message “See me”. Use a basic Caesar cipher with shift 3 as the initial substitution, but shift the substitution one place after each character.

Solution

The initial mapping is

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**
- Maps to: **DEFGHIJKLMNOPQRSTUVWXYZABC**

This would map the first letter, S to V. We would then shift the mapping by one.

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**
- Now maps to: **EFGHIJKLMNOPQRSTUVWXYZABCD**

Now the next letter, E, will map to I. Again we shift the cipher

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**
- Now maps to: **FGHIJKLMNOPQRSTUVWXYZABCDE**

The next letter, E, now maps to J. Continuing this process, the final message would be VIJSL.

Notice that frequency analysis is much less useful now, since the character E has been mapped to three different characters due to the shifting of the substitution mapping.

Try it Now 4

Decrypt the message KIQRV if it was encrypted using a basic Caesar cipher with shift 3 as the initial substitution, but shifting the substitution one place after each character.

Answer

The initial mapping was:

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**
- Now maps to: **DEFGHIJKLMNOPQRSTUVWXYZABC**

Using this, we can see the first character of the encrypted message, K, can be decrypted to the letter H. We now shift the mapping by one character.

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ
- Now maps to: EFGHIJKLMNOPQRSTUVWXYZABCD

The second character in the message, I, can be decrypted to the letter E. Continuing this process of shifting and decrypting, KIQRV decrypts to HELLO.



The actual Engima machines used in WWII were more complex. Each wheel consisted of a complex substitution cipher, and multiple wheels were used in a chain[1]. The specific wheels used, order of the wheels, and starting position of the wheels formed the encryption key. While captured Engima devices provided the Allied forces details on the encryption method, the keys still had to be broken to decrypt messages.

These code breaking efforts led to the development of some of the first electronic computers by Alan Turing at Bletchley Park in the United Kingdom. This is generally considered the beginnings of modern computing[2].

In the 1970s, the U.S. government had a competition and ultimately approved an algorithm deemed DES (Data Encryption Standard) to be used for encrypting government data. It became the standard encryption algorithm used. This method used a combination of multiple substitution and transposition steps, along with other steps in which the encryption key is mixed with the message. This method uses an encryption key with length 56 bits, meaning there are 2^{56} possible keys.

This number of keys make a brute force attack extremely difficult and costly, but not impossible. In 1998, a team was able to find the decryption key for a message in 2 days, using about $250,000 worth of hardware. However, the price and time will go down as computer power increases.

From 1997 to 2001 the government held another competition, ultimately adopting a new method, deemed AES (Advanced Encryption Standard). This method uses encryption keys with 128, 192, or 256 bits, providing up to 2^{256} possible keys, making brute force attacks essentially impossible.

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[1] http://en.wikipedia.org/wiki/File:En...abet_rings.jpg

[2] For a good overview, see http://www.youtube.com/watch?v=5nK_ft0Lf1s

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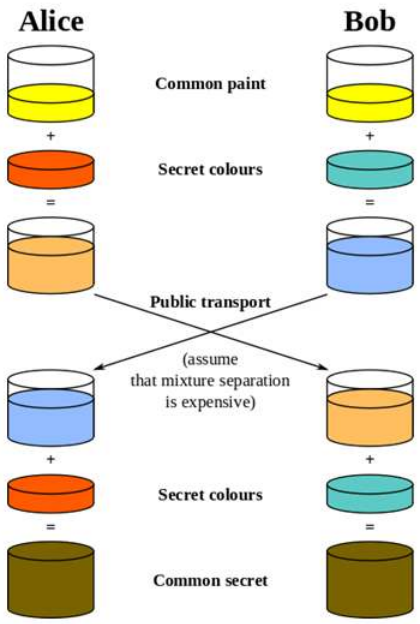
3.5: Public Key Cryptography

Suppose that you are connecting to your bank's website. It is possible that someone could intercept any communication between you and your bank, so you'll want to encrypt the communication. The problem is that all the encryption methods we've discussed require than both parties have already agreed on a shared secret encryption key. How can you and your bank agree on a key if you haven't already?

This becomes the goal of public key cryptography – to provide a way for two parties to agree on a key without a snooping third party being able to determine the key. The method relies on a one-way function; something that is easy to do one way, but hard to reverse. We will explore the Diffie-Hellman-Merkle key exchange method.

As an example, let's consider mixing paint. It's easy to mix paint to make a new color, but much harder to separate a mixed paint into the two original colors used.[1][2]

Using this analogy, Alice and Bob publically agree on a common starter color. Each then mixes in some of their own secret color. They then exchange their mixed colors.



Since separating colors is hard, even if a snooper were to obtain these mixed colors, it would be hard to obtain the original secret colors.

Once they have exchanged their mixed colors, Alice and Bob both add their secret color to the mix they obtained from the other person. In doing so, both Alice and Bob now have the same common secret color, since it contains a mix of the original common color, Alice's secret color, and Bob's secret color.

They now have a common secret color they can use as their encryption key, even though neither Alice nor Bob knows the other's secret color.

Likewise, there is no way for a snooper to obtain the common secret color without separating one of the mixed colors.

To get this process to work for computer communication, we need to have the process result in a share common number to act as the common secret encryption key. For this, we need a numerical one-way function.

Modular arithmetic

If you think back to doing division with whole numbers, you may remember finding the whole number result and the remainder after division.

Modulus [3]

The **modulus** is another name for the remainder after division.

For example, 17 mod 5 = 2, since if we divide 17 by 5, we get 3 with remainder 2.

Modular arithmetic is sometimes called clock arithmetic, since analog clocks wrap around times past 12, meaning they work on a modulus of 12. If the hour hand of a clock currently points to 8, then in 5 hours it will point to 1. While $8 + 5 = 13$, the clock wraps around after 12, so all times can be thought of as modulus 12. Mathematically, 13 mod 12 = 1.

Example 11

Compute: a) 10 mod 3 b) 15 mod 5 c) 2^7 mod 5

Solution

a) Since 10 divided by 3 is 3 with remainder 1,10 mod $3 = 1$

b) Since 15 divided by 5 is 3 with no remainder, 15 mod $5 = 0$

c) $2^7 = 128.128$ divide by 5 is 25 with remainder 3, so 2^7 mod $5 = 3$

Try it Now 5

Compute: a) 23 mod 7 b) 15 mod 7 c) 2034 mod 7

Answer

a) 2 b) 1 c) 4

Recall that when we divide 17 by 5, we could represent the result as 3 remainder 2, as the mixed number $3\frac{2}{5}$, or as the decimal 3.4. Notice that the modulus, 2, is the same as the numerator of the fractional part of the mixed number, and that the decimal part 0.4 is equivalent to the fraction . We can use these conversions to calculate the modulus of not-too-huge numbers on a standard calculator.

Modulus on a Standard Calculator

To calculate a mod n on a standard calculator

1. Divide a by n
2. Subtract the whole part of the resulting quantity
3. Multiply by n to obtain the modulus

Example 12

Calculate 31345 mod 419

Solution

$31345/419 = 74.8090692$	Now subtract 74 to get just the decimal remainder
$74.8090692 - 74 = 0.8090692$	Multiply this by 419 to get the modulus
$0.8090692 \times 419 = 339$	This tells us 0.8090692 was equivalent to $\frac{339}{419}$

In the text above, only a portion of the decimal value was written down. In practice, you should try to avoid writing down the intermediary steps, and instead allow your calculator to retain as many decimal values as it can.

The one-way function

When you use a prime number p as a modulus, you can find a special number called a generator, g, so that g^n mod p will result in all the values from 1 to $p-1$

n	3^n	3^nmod7
1	3	3
2	9	2
3	27	6
4	81	4
5	243	5
6	729	1

In the table to the top, notice that when we give values of n from 1 to 6, we get out all values from 1 to 6. This means 3 is a generator when 7 is the modulus.

This gives us our one-way function. While it is easy to compute the value of g^n mod p when we know n, it is difficult to find the exponent n to obtain a specific value.

For example, suppose we use $p = 23$ and $g = 5$. If I pick n to be 6, I can fairly easily calculate 5^6 mod $23 = 15625$ mod $23 = 8$

If someone else were to tell you $5^n \bmod 23 = 7$, it is much harder to find n. In this particular case, we'd have to try 22 different values for n until we found one that worked – there is no known easier way to find n other than brute-force guessing.

While trying 22 values would not take too long, when used in practice much larger values for p are used, typically with well over 500 digits. Trying all possibilities would be essentially impossible.

The key exchange

Before we can begin the key exchange process, we need a couple more important facts about modular arithmetic.

Modular Exponentialtion Rule

$(a^b \bmod n) = (a \bmod n)^b \bmod n$

Example 13

Compute 12^5 mod 7 using the exponentiation rule.

Solution

Evaluated directly: $12^5 = 248,832$, so $12^5 \bmod 7 = 248,823 \bmod 7 = 3$

Using the rule above, $12^5 \bmod 7 = (12 \bmod 7)^5 \bmod 7 = 5^5 \bmod 7 = 3125 \bmod 7 = 3$

You may remember a basic exponent rule from algebra:

$$(a^b)^c = a^{bc} = a^{cb} = (a^c)^b$$

For example:

$$64^2 = (4^3)^2 = 4^6 = (4^2)^3 = 16^3$$

We can combine the modular exponentiation rule with the algebra exponent rule to define the modular exponent power rule.

Modular Exponent Power Rule

$$(a^b \bmod n)^c \bmod n = (a^{bc} \bmod n) = (a^c \bmod n)^b \bmod n$$

Example 14

Verify the rule above if $a = 3, b = 4, c = 5$, and $n = 7$

Solution

$(3^4 \bmod 7)^5 \bmod 7 = (81 \bmod 7)^5 \bmod 7 = 4^5 \bmod 7 = 1024 \bmod 7 = 2$

$(3^5 \bmod 7)^4 \bmod 7 = (243 \bmod 7)^4 \bmod 7 = 5^4 \bmod 7 = 625 \bmod 7 = 2$, the same result.

Try it Now 6

Use the modular exponent rule to calculate $10000 \bmod 7$, by noting $10000 = 10^4$

Answer

$10000 \bmod 7 = 10^4 \bmod 7 = (10 \bmod 7)^4 \bmod 7 = 3^4 \bmod 7 = 81 \bmod 7 = 4$

This provides us the basis for our key exchange. While it will be easier to understand in the following example, here's the process:

1. Alice and Bob agree publically on values a prime p and generator g.
2. Alice picks some secret number a, while Bob picks some secret number b.
3. Alice computes $A = g^a \bmod p$ and sends it to Bob.
4. Bob computes $B = g^b \bmod p$ and sends it to Alice.

5. Alice computes $B^a \bmod p$, which is $(g^b \bmod p)^a \bmod p$.
6. Bob computes $A^b \bmod p$, which is $(g^a \bmod p)^b \bmod p$.

The modular exponent power rule tells us $(g^a \bmod p)^b \bmod p = (g^b \bmod p)^a \bmod p$, so Alice and Bob will arrive at the same shared value to use as a key, even though neither knows the other's secret number, and no eavesdropper can determine this value knowing only g, p, A, and B

Example 15

Alice and Bob publically share a generator and prime modulus. In this case, we'll use 3 as the generator and 17 as the prime.

Solution

	Alice		**Bob**
Alice and Bob publically share a generator and prime modulus.	$g = 3, p = 17$	Common info	$g = 3, p = 17$
Each then secretly picks a number n of their own.	$n = 8$	secret number	$n = 6$
Each calculates $g^n \bmod p$	$3^8 \bmod 17 = 16$		$3^6 \bmod 17 = 15$
They then exchange these resulting values.	$A = 16$ $B = 15$		$B = 15$ $A = 16$
Each then raises the value they received to the power of their secret $n \bmod p$	$B^n \bmod p =$ $15^8 \bmod 17 = 1$	mix in secret number	$A^n \bmod p =$ $16^6 \bmod 17 = 1$
The result is the shared secret key.	1	shared secret key	1

The shared secrets come out the same because of the modular exponent power rule

$(a^b \bmod n)^c \bmod n = (a^c \bmod n)^b \bmod n$. Alice computed $(3^6 \bmod 17)^8 \bmod 17$ while Bob computed $(3^8 \bmod 17)^6 \bmod 17$, which the rule says will give the same results.

Notice that even if a snooper were to obtain both exchanged values $A = 16$ and $B = 15$, there is no way they could obtain the shared secret key from these without having at least one of Alice or Bob's secret numbers. There is no easy way to obtain the secret numbers from the shared values, since the function was a one-way function.

Using this approach, Alice and Bob can now use the shared secret key obtained as the key for a standard encryption algorithm like DES or AES.

Try it Now 7

Suppose you are doing a key exchange with Kylie using generator 5 and prime 23. Your secret number is 2. What number do you send to Kylie? If Kylie sends you the value 8, determine the shared secret key.

Answer

To compute the number we'd send to Kylie, we raise the generator to the power of our secret number modulus the prime: $5^2 \bmod 23 = 25 \bmod 23 = 2$.

If Kylie sends us the value 8, we determine the shared secret by raising her number to the power of our secret number modulus the prime: $8^2 \bmod 23 = 64 \bmod 23 = 18.18$ would be the shared secret.

RSA

There are several other public-key methods used, including RSA, which is very commonly used. RSA involves distributing a public encryption key, which anyone can use to encrypt messages to you, but which can only be decrypted using a separate private key. You can think of this as sending an open padlock to someone – they can lock up information, but no one can unlock it without the key you kept secret.

RSA's security relies on the difficulty of factoring large numbers. For example, it's easy to calculate that 53 times 59 is 3127, but given the number 12,317 that is a product of two primes, it's much harder to find the numbers that multiply to give that number. It's exponentially harder when the primes each have 100 or more digits. Suppose we find two primes p and q and multiply them to get $n = pq$. This number will be very hard to factor. If we also know p and q, there are shortcuts to find two numbers e and d so that $m^{ed} \bmod n = m \bmod n$ for all numbers m. Without knowing the factorization of n, finding these values is very hard.

To use RSA, we generate two primes p and q and multiply them to get $n = pq$. since we know the factorization, we can easily find e and d so $m^{ed} = m \bmod n$. Now, we lock away p
q, and d. We then send the values e and n out publically. To encrypt a message m, the sender computes $S = m^e \bmod n$. As we saw earlier, the modulus is a one-way function which makes the original message very hard to recover from S. However, we have our private key d we can use to decrypt the message. When we receive the secret message S, we compute $S^d \bmod n = (m^e)^d \bmod n = m^{ed} \bmod n = m \bmod n$, recovering the original message[4].

Example 16

Suppose that Alice has computed $n = 3127, e = 3$, and $d = 2011$. Show how Bob would encrypt the message 50 and how Alice would then decrypt it.

Solution

Bob would only know his message, $m = 50$ and Alice's public key: $n = 3127$ and $e = 3$. He would encrypt the message by computing $m^e \bmod n : 50^3 \bmod 3127 = 3047$

Alice can then decrypt this message using her private key d by computing $S^d \bmod n$:

$3047^{2011} \bmod 3127 = 50$

This method differs from Diffie-Hellman-Merkle because no exchange process is needed; Bob could send Alice an encrypted message using Bob's public key without having to communicate with Alice beforehand to determine a shared secret key. This is especially handy for applications like encrypting email, where both parties might not be online at the same time to perform a Diffie-Hellman-Merkle style key exchange.

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[1] en.Wikipedia.org/w/index.php?...nge.svg&page=1

[2] For a video overview of this process, see http://www.youtube.com/watch?v=YEBfamv-_do

[3] Sometime, instead of seeing 17 mod 5 = 2, you'll see 17 ≡ 2 (mod 5). The ≡ symbol means "congruent to" and means that 17 and 2 are equivalent, after you consider the modulus 5.

[4] Many details have been left out, including how *e* and *d* are determined, and why this all works. For a bit more detail, see http://www.youtube.com/watch?v=wXB-V_Keiu8, or http://doctrina.org/How-RSA-Works-With-Examples.html

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3.6: Other Secret Keeping Methods

While this chapter has focused on cryptography methods for keeping a message secure, there is another area of secret keeping called **steganography** which focuses on hiding the existence of a message altogether. Historically, steganography included techniques like invisible ink, watermarks, or embedding a secret message inside a longer document that appeared unimportant. The digital age has provided many new ways to hide messages in digital photos or in the background noise of a music file.

Resources

There are several calculation tools available based on the material in this chapter available at: http://www.opentextbookstore.com/mathinsociety/apps/

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3.7: The One-Way Function

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3.8: The key exchange

Substitution ciphers

In the questions below, if it specifies an alphabetic cipher, then the original map used letters only: ABCDEFGHIJKLMNOPQRSTUVWXYZ. If it specifies an alphanumeric cipher, then the original map used letters and numbers: ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789

1. Encrypt the message “SEND SUPPLIES” using an alphabetic Caesar cipher with shift 7 (mapping A to H).

2. Encrypt the message “CANCEL CONTRACT” using an alphanumeric Caesar cipher with shift 16 (mapping A to Q).

3. Decrypt the message “2R1 ONO 5SN OXM O“ if it was encrypted using an alphanumeric Ceasar cipher with shift 10 (mapping A to K).

4. Decrypt the message “RJJY NSAJ SNHJ“ if it was encrypted using an alphabetic Ceasar cipher with shift 5 (mapping A to F).

For questions 5-8 use this substitution mapping:

Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789**
Maps to: **HLCO2BQF5WRTZN1G4D8IJ6SUVK3A0X9YME7P**

5. Use the substitution mapping to encrypt the message “DEAR DIARY”

6. Use the substitution mapping to encrypt the message “ATTACK AT SUNRISE”

7. Use the substitution mapping to decrypt the message “Z2DQ 2D1N”

8. Use the substitution mapping to decrypt the message “Z22 IHI3 YX3”

Transposition ciphers

9. Encrypt the message “Meet in the library at ten” using a tabular transposition cipher with rows of length 5 characters.

10. Encrypt the message “Fly surveillance over the northern county” using a tabular transposition cipher with rows of length 8 characters.

11. Decrypt the message “THE VHI NIE SAN SHT STI MQA DAN SDR S“ if it was encrypted using a tabular transposition cipher with rows of length 7 characters.

12. Decrypt the message “DOLR UTIR INON KVEY AZ“ if it was encrypted using a tabular transposition cipher with rows of length 6 characters.

13. Encrypt the message “Buy twenty million” using a tabular transposition cipher with the encryption keyword “RENT”.

14. Encrypt the message “Attack from the northeast” using a tabular transposition cipher with the encryption keyword “POWER”.

15. Decrypt the message “RYL OEN ONI TPM IEE YTE YDH WEA HRM S” if it was encrypted using a tabular transposition cipher with the encryption keyword “READING”.

16. Decrypt the message “UYH SRT ABV HLN SEE L” if it was encrypted using a tabular transposition cipher with the encryption keyword “MAIL”.

Shifting substitution ciphers

17. Encrypt the message “SEND SUPPLIES” using an alphabetic Caesar cipher that starts with shift 7 (mapping A to H), and shifts one additional space after each character is encoded.

18. Encrypt the message “CANCEL CONTRACT” using an alphabetic Caesar cipher that starts with shift 5 (mapping A to F), and shifts one additional space after each character is encoded.

Modular arithmetic

19. Compute

a. $15 \bmod 4$
b. $10 \bmod 5$
c. $257 \bmod 11$

20. Compute

a. $20 \bmod 4$
b. $14 \bmod 3$
c. $86 \bmod 13$

21. Determine if 4 is a generator modulus 11

22. Determine if 2 is a generator modulus 13

23. Use the modular exponent rule to calculate $157^{10} \bmod 5$

24. Use the modular exponent rule to calculate $133^{8} \bmod 6$

Diffie-Hellman-Merkle key exchange

25. Suppose you are doing a key exchange with Marc using generator 5 and prime 23. Your secret number is 7. Marc sends you the value 3. Determine the shared secret key.

26. Suppose you are doing a key exchange with Jen using generator 5 and prime 23. Your secret number is 4. Jen sends you the value 8. Determine the shared secret key.

RSA

27. Suppose that Alice has computed $n = 33, e = 7,$ and $d = 3$. Show how Bob would encrypt the message 5 and how Alice would then decrypt it.

28. Suppose that Alice has computed $n = 55, e = 7,$ and $d = 13$. Show how Bob would encrypt the message 8 and how Alice would then decrypt it.

Extensions

29. To further obscure a message, sometimes the usual alphabet characters are replaced with other symbols. Design a new set of symbols, and use it to encode a message. Exchange with a friend and see if they can decode your message.

30.To make an encryption harder to break, sometimes multiple substitution and transposition ciphers are used in sequence. For example, a method might specify that the first letter of the encryption keyword be used to determine the initial shift for a Caesar cipher (perhaps with a rotating cipher), and also be used for a transposition cipher. Design your own sequence of encryption steps and encrypt a message. Exchange with a friend and see if they can follow your process to decrypt the message.

31. When using large primes, computing values like $67^{24} \bmod 83$ can be difficult on a calculator without using additional tricks, since 67^{24} is a huge number. We will explore an approach used.

a. Notice that $67^2 \bmod 83$ is fairly easy to calculate: $67^2 \bmod 83 = 4489 \bmod 83 = 7$. Since $67^4 \bmod 83 = (67^2)^2 \bmod 83$ can be rewritten using the modular exponent rule as $(67^2 \bmod 83)^2 \bmod 83$, this is also easy to evaluate: $67^4 \bmod 83 = (67^2 \bmod 83)^2 \bmod 83 = 7^2 \bmod 83 = 49$ This process can be continued to find $67^8 \bmod 83$ as $(6^4)^2 \bmod 83$. Find this value, then find $67^{16} \bmod 83$ and $67^{32} \bmod 83$
b. There is a rule that $(ab) \bmod n = (a \bmod n)(b \bmod n) \bmod n$ Noting that $17000 = 170 \times 100$, calculate 17000 mod 83 using the rule above.
c. Note that $67^5 = 67^4 67$. Use this, along with the rule from above and the results from part a to compute $67^5 \bmod 83$
d. Note that $67^7 = 67^{4+2+1} = 67^4 67^2 67^1$. Compute $67^7 \bmod 83$.
e. Write 67^{24} as a product of powers of 67, and use this to compute $67^{24} \bmod 83$

32. Use the process from the previous question to evaluate $23^{34} \bmod 37$.

33. To encrypt text messages with RSA, the words are first converted into a string of numbers, and then encrypted. Several characters are usually combined together to produce a message number smaller than the modulus, but approximately the same size. Look up an ASCII table to convert the message “SCALE THE WALLS” to numbers, then encrypt it using the RSA public key $n = 10823, e = 5$. Since ASCII characters are two digits, pair up characters to form four-digit numbers before encoding. For example A is 65 and B is 66, so the character pair AB could be treated as the number 6566 and encrypted as 10148

34. Explore approaches to steganography that don’t require specialized software. Attempt to hide a message using one of these techniques, and see if a fellow student can detect the message.

35. When you visit a secure website, your web browser will report that the site’s identity has been verified by a third party, called a certificate authority. This is meant to assure you that you are visiting the actual company’s website. Research how these certificates work.



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CHAPTER OVERVIEW

4: Apportionment

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4.2: Hamilton’s Method
4.3: Jefferson’s Method
4.4: Webster’s Method
4.5: Huntington-Hill Method
4.6: Lowndes’ Method
4.7: Apportionment of Legislative Districts
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4.1: Introduction

Apportionment is the problem of dividing up a fixed number of things among groups of different sizes. In politics, this takes the form of allocating a limited number of representatives amongst voters. This problem, presumably, is older than the United States, but the best-known ways to solve it have their origins in the problem of assigning each state an appropriate number of representatives in the new Congress when the country was formed. States also face this apportionment problem in defining how to draw districts for state representatives. The apportionment problem comes up in a variety of non-political areas too, though. We face several restrictions in this process:

Apportionment Rules

1. The things being divided up can exist only in whole numbers.
2. We must use all of the things being divided up, and we cannot use any more.
3. Each group must get at least one of the things being divided up.
4. The number of things assigned to each group should be at least approximately proportional to the population of the group. (Exact proportionality isn't possible because of the whole number requirement, but we should try to be close, and in any case, if Group A is larger than Group B, then Group B shouldn't get more of the things than Group A does.)

In terms of the apportionment of the United States House of Representatives, these rules imply:

1. We can only have whole representatives (a state can't have 3.4 representatives)
2. We can only use the (currently) 435 representatives available. If one state gets another representative, another state has to lose one.
3. Every state gets at least one representative
4. The number of representatives each state gets should be approximately proportional to the state population. This way, the number of constituents each representative has should be approximately equal.

We will look at four ways of solving the apportionment problem. Three of them (Lowndes's method is the exception) have been used at various times to apportion the U.S. Congress, although the method currently in use (the Huntington-Hill method) is significantly more complicated.

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4.2: Hamilton's Method

Alexander Hamilton proposed the method that now bears his name. His method was approved by Congress in 1791, but was vetoed by President Washington. It was later adopted in 1852 and used through 1911. He begins by determining, to several decimal places, how many things each group should get. Since he was interested in the question of Congressional representation, we'll use the language of states and representatives, so he determines how many representatives each state should get. He follows these steps:

Hamilton's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**. Since we can only allocate whole representatives, Hamilton resolves the whole number problem, as follows:
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are called the **lower quotas**. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives (and the "or equal to" part happens only in very specific circumstances that are incredibly unlikely to turn up).
4. Assuming that the total from Step 3 was less than the total number of representatives, assign the remaining representatives, one each, to the states whose decimal parts of the quota were largest, until the desired total is reached.

Make sure that each state ends up with at least one representative!

Note on rounding: Today we have technological advantages that Hamilton (and the others) couldn't even have imagined. Take advantage of them, and keep several decimal places.

Example 1

The state of Delaware has three counties: Kent, New Castle, and Sussex. The Delaware state House of Representatives has 41 members. If Delaware wants to divide this representation along county lines (which is *not* required, but let's pretend they do), let's use Hamilton's method to apportion them. The populations of the counties are as follows (from the 2010 Census):

County	Population
Kent	162,310
New Castle	538,479
Sussex	197,145
Total	**897,934**

Solution

1. First, we determine the divisor: $897,934/41 = 21,900.82927$

2. Now we determine each county's quota by dividing the county's population by the divisor:

County	Population	Quota
Kent	162,310	7.4111
New Castle	538,479	24.5872
Sussex	197,145	9.0017
Total	**897,934**	

3. Removing the decimal parts of the quotas gives:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	**897,934**		**40**

4. We need 41 representatives and this only gives 40. The remaining one goes to the county with the largest decimal part, which is New Castle:

County	Population	Quota	Initial	Final
Kent	162,310	7.4111	7	7
New Castle	538,479	24.5872	24	25
Sussex	197,145	9.0017	9	9
Total	**897,934**		**40**	**41**

Example 2

Use Hamilton's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	**1,052,567**

Solution

1. The divisor is $1,052,567/75 = 14,034.22667$

2. Determine each county's quota by dividing its population by the divisor:

County	Population	Quota
Bristol	49,875	3.5538
Kent	166,158	11.8395
Newport	82,888	5.9061
Providence	626,667	44.6528
Washington	126,979	9.0478
Total	**1,052,567**	

3. Remove the decimal part of each quota:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	**1,052,567**		**72**

4. We need 75 representatives and we only have 72, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which are Newport, Kent, and Providence:

County	Population	Quota	Initial	Final
Bristol	49,875	3.5538	3	3
Kent	166,158	11.8395	11	12
Newport	82,888	5.9061	5	6
Providence	626,667	44.6528	44	45
Washington	126,979	9.0478	9	9
Total	**1,052,567**		**72**	**75**

Note that even though Bristol County's decimal part is greater than .5, it isn't big enough to get an additional representative, because three other counties have greater decimal parts.

Hamilton's method obeys something called the Quota Rule. The Quota Rule isn't a law of any sort, but just an idea that some people, including Hamilton, think is a good one.

Quota Rule

The Quota Rule says that the final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, that means that each state should either go up to the next whole number above its quota, or down to the next whole number below its quota.

Controversy

After seeing Hamilton's method, many people find that it makes sense, it's not that difficult to use (or, at least, the difficulty comes from the numbers that are involved and the amount of computation that's needed, not from the method), and they wonder why anyone would want another method. The problem is that Hamilton's method is subject to several paradoxes. Three of them happened, on separate occasions, when Hamilton's method was used to apportion the United States House of Representatives.

The **Alabama Paradox** is named for an incident that happened during the apportionment that took place after the 1880 census. (A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama.) The post-1880 apportionment had been completed, using Hamilton's method and the new population numbers from the census. Then it was decided that because of the country's growing population, the House of Representatives should be made larger. That meant that the apportionment would need to be done again, still using Hamilton's method and the same 1880 census numbers, but with more representatives. The assumption was that some states would gain another representative and others would stay with the same number they already had (since there weren't enough new representatives being added to give one more to every state). The paradox is that Alabama ended up *losing* a representative in the process, even though no populations were changed and the total number of representatives increased.

The **New States Paradox** happened when Oklahoma became a state in 1907. Oklahoma had enough population to quality for five representatives in Congress, so they raised the total number of representatives by five. When the apportionment was recalculated, unsurprisingly Oklahoma received five representatives. However, despite their populations not changing, Maine gained a representative and New York lost one.

The **Population Paradox** happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative *to Maine*.

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4.3: Jefferson's Method

Thomas Jefferson proposed a different method for apportionment. After Washington vetoed Hamilton's method, Jefferson's method was adopted, and used in Congress from 1791 through 1842. Jefferson, of course, had political reasons for wanting his method to be used rather than Hamilton's. Primarily, his method favors larger states, and his own home state of Virginia was the largest in the country at the time. He would also argue that it's the ratio of people to representatives that is the critical thing, and apportionment methods should be based on that. But the paradoxes we saw also provide mathematical reasons for concluding that Hamilton's method isn't so good, and while Jefferson's method might or might not be the best one to replace it, at least we should look for other possibilities.

The first steps of Jefferson's method are the same as Hamilton's method. He finds the same divisor and the same quota, and cuts off the decimal parts in the same way, giving a total number of representatives that is less than the required total. The difference is in how Jefferson resolves that difference. He says that since we ended up with an answer that is too small, our divisor must have been too big. He changes the divisor by making it smaller, finding new quotas with the new divisor, cutting off the decimal parts, and looking at the new total, until we find a divisor that produces the required total.

Jefferson's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are the **lower quotas**. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 3

We'll return to Delaware and apply Jefferson's method. We begin, as we did with Hamilton's method, by finding the quotas with the original divisor, $21,900.82927$:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
Newport	538,479	24.5872	24
Providence	197,145	9.0017	9
Total	**897,934**		**40**

Solution

We need 41 representatives, and this divisor gives only 40. We must reduce the divisor until we get 41 representatives. Let's try $21,500$ as the divisor:

County	Population	Quota	Initial
Kent	162,310	7.5493	7
Newport	538,479	25.0455	25
Providence	197,145	9.1695	9
Total	**897,934**		**41**

This gives us the required 41 representatives, so we're done. If we had fewer than 41, we'd need to reduce the divisor more. If we had more than 41, we'd need to choose a divisor less than the original but greater than the second choice.

Notice that with the new, lower divisor, the quota for New Castle County (the largest county in the state) increased by much more than those of Kent County or Sussex County.

Example 4

We'll apply Jefferson's method for Rhode Island. The original divisor of 14,034.22667 gave these results:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	**1,052,567**		**72**

Solution

We need 75 representatives and we only have 72, so we need to use a smaller divisor. Let's try **13,500**:

County	Population	Quota	Initial
Bristol	49,875	3.6944	3
Kent	166,158	12.3080	12
Newport	82,888	6.1399	6
Providence	626,667	46.4198	46
Washington	126,979	9.4059	9
Total	**1,052,567**		**76**

We've gone too far. We need a divisor that's greater than 13,500 but less than **14,034.22667**. Let's try **13,700**:

County	Population	Quota	Initial
Bristol	49,875	3.6405	3
Kent	166,158	12.1283	12
Newport	82,888	6.0502	6
Providence	626,667	45.7421	45
Washington	126,979	9.2685	9
Total	**1,052,567**		**75**

This works.

Notice, in comparison to Hamilton's method, that although the results were the same, they came about in a different way, and the outcome was almost different. Providence County (the largest) almost went up to 46 representatives before Kent (which is much smaller) got to 12. Although that didn't happen here, it can. Divisor-adjusting methods like Jefferson's are not guaranteed to follow the quota rule!

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4.4: Webster's Method

Daniel Webster (1782-1852) proposed a method similar to Jefferson's in 1832. It was adopted by Congress in 1842, but replaced by Hamilton's method in 1852. It was then adopted again in 1901. The difference is that Webster rounds the quotas to the nearest whole number rather than dropping the decimal parts. If that doesn't produce the desired results at the beginning, he says, like Jefferson, to adjust the divisor until it does. (In Jefferson's case, at least the first adjustment will always be to make the divisor smaller. That is not always the case with Webster's method.)

Webster's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Round all the quotas to the nearest whole number (but don't forget what the decimals were). Add up the remaining whole numbers.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 3 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 5

Again, Delaware, with an initial divisor of $21,900.82927$:

County	Population	Quota	Initial
Kent	166,310	7.4111	7
New Castle	538,479	24.5872	25
Sussex	197,145	9.0017	9
Total	**897,934**		**41**

Solution

This gives the required total, so we're done.

Example 6

Again, Rhode Island, with an initial divisor of $14,034.22667$:

County	Population	Quota	Initial
Bristol	49,875	3.5538	4
Kent	166,158	11.8395	12
Newport	82,888	5.9061	6
Providence	626,667	44.6528	45
Washington	126,979	9.0478	9
Total	**1,052,567**		**76**

Solution

This is too many, so we need to increase the divisor. Let's try $14,100$:

County	Population	Quota	Initial
Bristol	49,875	3.5372	4
Kent	166,158	11.7843	12
Newport	82,888	5.8786	6
Providence	626,667	5.8786	44
Washington	126,979	9.0056	9
Total	**1,052,567**		**75**

This works, so we're done.

Like Jefferson's method, Webster's method carries a bias in favor of states with large populations, but rounding the quotas to the nearest whole number greatly reduces this bias. (Notice that Providence County, the largest, is the one that gets a representative trimmed because of the increased quota.) Also like Jefferson's method, Webster's method does not always follow the quota rule, but it follows the quota rule much more often than Jefferson's method does. (In fact, if Webster's method had been applied to every apportionment of Congress in all of American history, it would have followed the quota rule every single time.)

In 1980, two mathematicians, Peyton Young and Mike Balinski, proved what we now call the Balinski-Young Impossibility Theorem.

Balinski-Young Impossibility Theorem

The Balinski-Young Impossibility Theorem shows that any apportionment method which always follows the quota rule will be subject to the possibility of paradoxes like the Alabama, New States, or Population paradoxes. In other words, we can choose a method that avoids those paradoxes, but only if we are willing to give up the guarantee of following the quota rule.

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4.5: Huntington-Hill Method

In 1920, no new apportionment was done, because Congress couldn't agree on the method to be used. They appointed a committee of mathematicians to investigate, and they recommended the Huntington-Hill Method. They continued to use Webster's method in 1931, but after a second report recommending Huntington-Hill, it was adopted in 1941 and is the current method of apportionment used in Congress.

The Huntington-Hill Method is similar to Webster's method, but attempts to minimize the percent differences of how many people each representative will represent.

Huntington-Hill Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off the decimal part of the quota to obtain the lower quota, which we'll call n. Compute $\sqrt{n(n+1)}$, which is the **geometric mean** of the lower quota and one value higher.
4. If the quota is larger than the geometric mean, round up the quota; if the quota is smaller than the geometric mean, round down the quota. Add up the resulting whole numbers to get the **initial allocation**.
5. If the total from Step 4 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 4 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 4 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 7

Again, Delaware, with an initial divisor of 21,900.82927:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Kent	162,310	7.4111	7	7.48	7
New Castle	538,479	24.5872	24	24.49	25
Sussex	197,145	9.0017	9	9.49	9
Total	**897,934**				**41**

Solution

This gives the required total, so we're done.

Example 8

Again, Rhode Island, with an initial divisor of 14,034.22667:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5538	3	3.46	4
Kent	166,158	11.8395	11	11.49	12
Newport	82,888	5.9061	5	5.48	6
Providence	626,667	44.6528	44	44.50	45
Washington	126,979	9.0478	9	9.49	9
Total	**1,052,567**				**76**

Solution

This is too many, so we need to increase the divisor. Let's try 14,100:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5372	3	3.46	4
Kent	166,158	11.7843	11	11.49	12
Newport	82,888	5.8786	5	5.48	6
Providence	626,667	44.4445	44	44.50	44
Washington	126,979	9.0056	9	9.49	9
Total	**1,052,567**				**75**

This works, so we're done.

In both these cases, the apportionment produced by the Huntington-Hill method was the same as those from Webster's method.

Example 9

Consider a small country with 5 states, two of which are much larger than the others. We need to apportion 70 representatives. We will apportion using both Webster's method and the Huntington-Hill method.

State	Population
A	300,500
B	200,000
C	50,000
D	38,000
E	21,500

Solution

1. The total population is 610,000. Dividing this by the 70 representatives gives the divisor: 8714.286.

2. Dividing each state's population by the divisor gives the quotas.

State	Population	Quota
A	300,500	34.48361
B	200,000	22.95082
C	50,000	5.737705
D	38,000	4.360656
E	21,500	2.467213

Webster's Method

3. Using Webster's method, we round each quota to the nearest whole number

State	Population	Quota	Initial
A	300,500	34.48361	34
B	200,000	22.95082	23
C	50,000	5.737705	6
D	38,000	4.360656	4
E	21,500	2.467213	2

4. Adding these up, they only total 69 representatives, so we adjust the divisor down. Adjusting the divisor down to 8700 gives an updated allocation totaling 70 representatives

State	Population	Quota	Initial
A	300,500	34.54023	35
B	200,000	22.98851	23
C	50,000	5.747126	6
D	38,000	4.367816	4
E	21,500	2.471264	2

Huntington-Hill Method

3. Using the Huntington-Hill method, we round down to find the lower quota, then calculate the geometric mean based on each lower quota. If the quota is less than the geometric mean, we round down; if the quota is more than the geometric mean, we round up.

State	Population	Quota	Lower Quota	Geometric Mean	Initial
A	300,500	34.48361	34	34.49638	34
B	200,000	22.95082	22	22.49444	23
C	50,000	5.737705	5	5.477226	6
D	38,000	4.360656	4	4.472136	4
E	21,500	2.467213	2	2.44949	3

These allocations add up to 70, so we're done.

Notice that this allocation is different than that produced by Webster's method. In this case, state E got the extra seat instead of state A.

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4.6: Lowndes' Method

William Lowndes (1782-1822) was a Congressman from South Carolina (a small state) who proposed a method of apportionment that was more favorable to smaller states. Unlike the methods of Hamilton, Jefferson, and Webster, Lowndes's method has never been used to apportion Congress.

Lowndes believed that an additional representative was much more valuable to a small state than to a large one. If a state already has 20 or 30 representatives, getting one more doesn't matter very much. But if it only has 2 or 3, one more is a big deal, and he felt that the additional representatives should go where they could make the most difference.

Like Hamilton's method, Lowndes's method follows the quota rule. In fact, it arrives at the same quotas as Hamilton and the rest, and like Hamilton and Jefferson, it drops the decimal parts. But in deciding where the remaining representatives should go, we divide the decimal part of each state's quota by the whole number part (so that the same decimal part with a smaller whole number is worth more, because it matters more to that state).

Lowndes's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). Add up the remaining whole numbers.
4. Assuming that the total from Step 3 was less than the total number of representatives, divide the decimal part of each state's quota by the whole number part. Assign the remaining representatives, one each, to the states whose **ratio** of decimal part to whole part were largest, until the desired total is reached.

Example 10

We'll do Delaware again. We begin in the same way as with Hamilton's method:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	**897,934**		**40**

Solution

We need one more representative. To find out which county should get it, Lowndes says to divide each county's decimal part by its whole number part, with the largest result getting the extra representative:

Kent: $0.4111/7 \approx 0.0587$
New Castle: $0.5872/24 \approx 0.0245$
Sussex: $0.0017/9 \approx 0.0002$

The largest of these is Kent's, so Kent gets the 41th representative:

County	Population	Quota	Initial	Ratio	Final
Kent	162,310	7.4111	7	0.0587	8
New Castle	538,479	24.5872	24	0.0245	24
Sussex	197,145	9.0017	9	0.0002	9
Total	**897,934**		**40**		**41**

Example 11

Rhode Island, again beginning in the same way as Hamilton:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	**1,052,567**		**72**

Solution

We divide each county's quota's decimal part by its whole number part to determine which three should get the remaining representatives:

Bristol:	$0.5538/3 \approx 0.1846$
Kent:	$0.8395/11 \approx 0.0763$
Newport:	$0.9061/5 \approx 0.1812$
Providence:	$0.6528/44 \approx 0.0148$
Washington:	$0.0478/9 \approx 0.0053$

The three largest of these are Bristol, Newport, and Kent, so they get the remaining three representatives:

County	Population	Quota	Initial	Ratio	Final
Bristol	49,875	3.5538	3	0.1846	4
Kent	166,158	11.8395	11	0.0763	12
Newport	82,888	5.9061	5	0.1812	6
Providence	626,667	44.6528	44	0.0148	44
Washington	126,979	9.0478	9	0.0053	9
Total	**1,052,567**		**72**		**75**

As you can see, there is no "right answer" when it comes to choosing a method for apportionment. Each method has its virtues, and favors different sized states.

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4.7: Apportionment of Legislative Districts

In most states, there are a fixed number of representatives to the state legislature. Rather than apportioning each county a number of representatives, legislative districts are drawn so that each legislator represents a district. The apportionment process, then, comes in the drawing of the legislative districts, with the goal of having each district include approximately the same number of constituents. Because of this goal, a geographically small city may have several representatives, while a large rural region may be represented by one legislator.

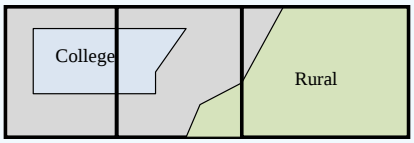
When populations change, it becomes necessary to redistrict the regions each legislator represents (Incidentally, this also occurs for the regions that federal legislators represent). The process of redistricting is typically done by the legislature itself, so not surprisingly it is common to see **gerrymandering**.

Gerrymandering

Gerrymandering is when districts are drawn based on the political affiliation of the constituents to the advantage of those drawing the boundary.

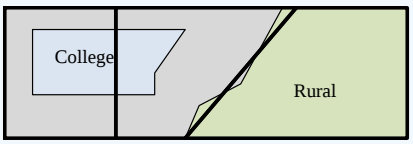
Example 12

Consider three districts, simplified to the three boxes below. On the left there is a college area that typically votes Democratic. On the right is a rural area that typically votes Republican. The rest of the people are more evenly split. The middle district has been voting 50% Democratic and 50% Republican.



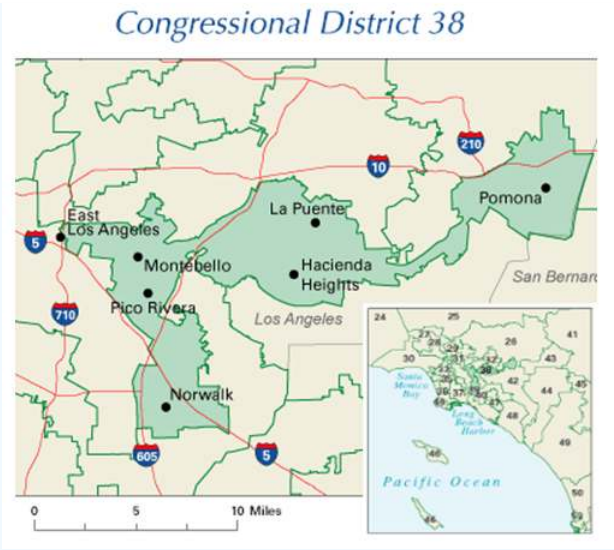
Solution

As part of a redistricting, a Democratic led committee could redraw the boundaries so that the middle district includes less of the typically Republican voters, thereby making it more likely that their party will win in that district, while increasing the Republican majority in the third district.



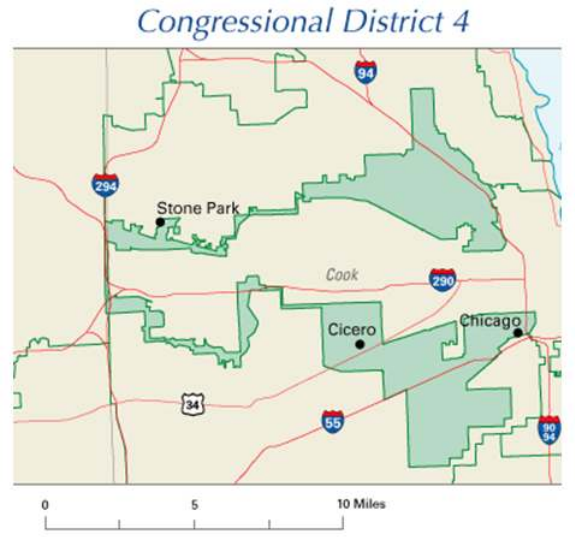
Example 13

The map to the right shows the 38th congressional district in California in 2004[1]. This district was created through a bi-partisan committee of incumbent legislators. This gerrymandering leads to districts that are not competitive; the prevailing party almost always wins with a large margin.



Solution

The map to the right shows the 4th congressional district in Illinois in 2004.[2] This district was drawn to contain the two predominantly Hispanic areas of Chicago. The largely Puerto Rican area to the north and the southern Mexican areas are only connected in this districting by a piece of the highway to the west.



[1] http://en.Wikipedia.org/wiki/File:Ca...ct_38_2004.png

[2] en.Wikipedia.org/wiki/File:Il...ict_4_2004.png

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4.8: Exercises

In exercises 1-8, determine the apportionment using

a. Hamilton's Method
b. Jefferson's Method
c. Webster's Method
d. Huntington-Hill Method
e. Lowndes' method

1. A college offers tutoring in Math, English, Chemistry, and Biology. The number of students enrolled in each subject is listed below. If the college can only afford to hire 15 tutors, determine how many tutors should be assigned to each subject.

Math: 330 English: 265 Chemistry: 130 Biology: 70

2. Reapportion the previous problem if the college can hire 20 tutors.

3. The number of salespeople assigned to work during a shift is apportioned based on the average number of customers during that shift. Apportion 20 salespeople given the information below.

Shift	Morning	Midday	Afternoon	Evening
Average number of customers	95	305	435	515

4. Reapportion the previous problem if the store has 25 salespeople.

5. Three people invest in a treasure dive, each investing the amount listed below. The dive results in 36 gold coins. Apportion those coins to the investors.

Alice: $7,600 Ben: $5,900 Carlos: $1,400

6. Reapportion the previous problem if 37 gold coins are recovered.

7. A small country consists of five states, whose populations are listed below. If the legislature has 119 seats, apportion the seats.

A: 810,000 B: 473,000 C: 292,000 D: 594,000 E: 211,000

8. A small country consists of six states, whose populations are listed below. If the legislature has 200 seats, apportion the seats.

A: 3,411 B: 2,421 C: 11,586 D: 4,494 E: 3,126 F: 4,962

9. A small country consists of three states, whose populations are listed below.

A: 6,000 B: 6,000 C: 2,000

a. If the legislature has 10 seats, use Hamilton's method to apportion the seats.

b. If the legislature grows to 11 seats, use Hamilton's method to apportion the seats.

c. Which apportionment paradox does this illustrate?

10. A state with five counties has 50 seats in their legislature. Using Hamilton's method, apportion the seats based on the 2000 census, then again using the 2010 census. Which apportionment paradox does this illustrate?

County	2000 Population	2010 Population
Jefferson	60,000	60,000
Clay	31,200	31,200
Madison	69,200	72,400
Jackson	81,600	81,600
Franklin	118,000	118,400

11. A school district has two high schools: Lowell, serving 1715 students, and Fairview, serving 7364. The district could only afford to hire 13 guidance counselors.

a. Determine how many counselors should be assigned to each school using Hamilton's method.

b. The following year, the district expands to include a third school, serving 2989 students. Based on the divisor from above, how many additional counselors should be hired for the new school?

c. After hiring that many new counselors, the district recalculates the reapportion using Hamilton's method. Determine the outcome.

d. Does this situation illustrate any apportionment issues?

12. A small country consists of four states, whose populations are listed below. If the legislature has 116 seats, apportion the seats using Hamilton's method. Does this illustrate any apportionment issues?

A: 33,700 B: 559,500 C: 141,300 D: 89,100

Exploration

13. Explore and describe the similarities, differences, and interplay between weighted voting, fair division (if you've studied it yet), and apportionment.

14. In the methods discussed in the text, it was assumed that the number of seats being apportioned was fixed. Suppose instead that the number of seats could be adjusted slightly, perhaps 10% up or down. Create a method for apportioning that incorporates this additional freedom, and describe why you feel it is the best approach. Apply your method to the apportionment in Exercise 7.

15. Lowndes felt that small states deserved additional seats more than larger states. Suppose you were a legislator from a larger state, and write an argument refuting Lowndes.

16. Research how apportionment of legislative seats is done in other countries around the world. What are the similarities and differences compared to how the United States apportions congress?

17. Adams's method is similar to Jefferson's method, but rounds quotas *up* rather than *down*. This means we usually need a modified divisor that is smaller than the standard divisor. Rework problems 1-8 using Adam's method. Which other method are the results most similar to?

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CHAPTER OVERVIEW

5: Sequences

In this chapter, we will explore the mathematics behind situations involving probabilities and counting. We will take an in-depth look at annuities. We will also look at the branch of mathematics that would allow us to calculate the number of ways to choose lottery numbers and the probability of winning.

5.1: Prelude to Sequences, Probability and Counting Theory

5.2: Sequences and Their Notations

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5.1: Prelude to Sequences, Probability and Counting Theory

A lottery winner has some big decisions to make regarding what to do with the winnings. Buy a villa in Saint Barthélemy? A luxury convertible? A cruise around the world? The likelihood of winning the lottery is slim, but we all love to fantasize about what we could buy with the winnings. One of the first things a lottery winner has to decide is whether to take the winnings in the form of a lump sum or as a series of regular payments, called an annuity, over the next 30 years or so.



Figure 5.1.1: (credit: Robert S. Donovan, Flickr.)

This decision is often based on many factors, such as tax implications, interest rates, and investment strategies. There are also personal reasons to consider when making the choice, and one can make many arguments for either decision. However, most lottery winners opt for the lump sum. In this chapter, we will explore the mathematics behind situations such as these. We will take an in-depth look at annuities. We will also look at the branch of mathematics that would allow us to calculate the number of ways to choose lottery numbers and the probability of winning.

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5.2: Sequences and Their Notations

Learning Objectives

- Write the terms of a sequence defined by an explicit formula.
- Write the terms of a sequence defined by a recursive formula.
- Use factorial notation.

A video game company launches an exciting new advertising campaign. They predict the number of online visits to their website, or hits, will double each day. The model they are using shows **2** hits the first day, **4** hits the second day, **8** hits the third day, and so on (Table **5.2.1**).

Table **5.2.1**

Day	1	2	3	4	5	…
Hits	2	4	8	16	32	…

If their model continues, how many hits will there be at the end of the month? To answer this question, we'll first need to know how to determine a list of numbers written in a specific order. In this section, we will explore these kinds of ordered lists.

Writing the Terms of a Sequence Defined by an Explicit Formula

One way to describe an ordered list of numbers is as a *sequence*. A sequence is a function whose domain is a subset of the counting numbers. The sequence established by the number of hits on the website is

$$\{2, 4, 8, 16, 32, \ldots\}$$

The *ellipsis* (…) indicates that the sequence continues indefinitely. Each number in the sequence is called a term. The first five terms of this sequence are **2**, **4**, **8**, **16**, and **32**.

Listing all of the terms for a sequence can be cumbersome. For example, finding the number of hits on the website at the end of the month would require listing out as many as **31** terms. A more efficient way to determine a specific term is by writing a formula to define the sequence.

One type of formula is an *explicit formula*, which defines the terms of a sequence using their position in the sequence. Explicit formulas are helpful if we want to find a specific term of a sequence without finding all of the previous terms. We can use the formula to find the nth *term of the sequence*, where nn is any positive number. In our example, each number in the sequence is double the previous number, so we can use powers of 2 to write a formula for the nth term.

$$\{2,\ 4,\ 8,\ 16,\ 32,\ \ldots,\ ?,\ \ldots\}$$
$$\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \qquad \downarrow$$
$$\{2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5,\ \ldots,\ 2^n,\ \ldots\}$$

The first term of the sequence is $2^1 = 2$, the second term is $2^2 = 4$, the third term is $2^3 = 8$, and so on. The nth term of the sequence can be found by raising 2 to the nth power. An explicit formula for a sequence is named by a lower case letter a, b, c... with the subscript n. The explicit formula for this sequence is

$$a_n = 2^n$$

Now that we have a formula for the nth term of the sequence, we can answer the question posed at the beginning of this section. We were asked to find the number of hits at the end of the month, which we will take to be 31 days. To find the number of hits on the last day of the month, we need to find the 31st term of the sequence. We will substitute 31 for n in the formula.

$$a_{31} = 2^{31}$$
$$[5pt] = 2,147,483,648$$

If the doubling trend continues, the company will get $2,147,483,648$ hits on the last day of the month. That is over **2.1** billion hits! The huge number is probably a little unrealistic because it does not take consumer interest and competition into account. It does, however, give the company a starting point from which to consider business decisions.

Another way to represent the sequence is by using a table. The first five terms of the sequence and the nth term of the sequence are shown in Table **5.2.2**.

Table 5.2.2

n	1	2	3	4	5	n
n^{th} term of the sequence, a_n	2	4	8	16	32	2^n

Graphing provides a visual representation of the sequence as a set of distinct points. We can see from the graph in Figure **5.2.1** that the number of hits is rising at an exponential rate. This particular sequence forms an exponential function.

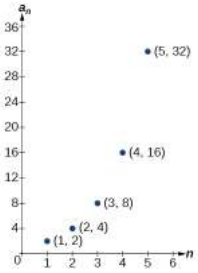


Figure **5.2.1**

Lastly, we can write this particular sequence as

$$\{2, 4, 8, 16, 32, \ldots, 2^n, \ldots\}$$

A sequence that continues indefinitely is called an **infinite sequence**. The domain of an infinite sequence is the set of counting numbers. If we consider only the first **10** terms of the sequence, we could write

$$\{2, 4, 8, 16, 32, \ldots, 2^n, \ldots, 1024\}$$

This sequence is called a **finite sequence** because it does not continue indefinitely.

SEQUENCE

A *sequence* is a function whose domain is the set of positive integers. A *finite sequence* is a sequence whose domain consists of only the first n positive integers. The numbers in a sequence are called *terms*. The variable a with a number subscript is used to represent the terms in a sequence and to indicate the position of the term in the sequence.

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{5.2.1}$$

We call a_1 the first term of the sequence, a_2 the second term of the sequence, a_3 the third term of the sequence, and so on. The term a_n is called the nth term of the sequence, or the general term of the sequence. An explicit formula defines the nth term of a sequence using the position of the term. A sequence that continues indefinitely is an infinite sequence.

Q&A: Does a sequence always have to begin with a_1?

No. In certain problems, it may be useful to define the initial term as a_0 instead of a_1. In these problems, the domain of the function includes 0.

How to: Given an explicit formula, write the first n terms of a sequence

1. Substitute each value of n into the formula. Begin with $n = 1$ to find the first term, a_1.
2. To find the second term, a_2, use $n = 2$.
3. Continue in the same manner until you have identified all nn terms.

Example 5.2.1: Writing the Terms of a Sequence Defined by an Explicit Formula

Write the first five terms of the sequence defined by the explicit formula $a_n = -3n + 8$.

Solution

Substitute $n = 1$ into the formula. Repeat with values 2 through 5 for n.

$$\begin{aligned} n = 1 \quad & a_1 = -3(1) + 8 = 5 \\ n = 2 \quad & a_2 = -3(2) + 8 = 2 \\ n = 3 \quad & a_3 = -3(3) + 8 = -1 \\ n = 4 \quad & a_4 = -3(4) + 8 = -4 \\ n = 5 \quad & a_5 = -3(5) + 8 = -7 \end{aligned}$$

The first five terms are $\{5, 2, -1, -4, -7\}$.

Analysis

The sequence values can be listed in a table. A table, such as Table 5.2.3, is a convenient way to input the function into a graphing utility.

Table 5.2.3

n	1	2	3	4	5
a_n	5	2	–1	–4	–7

A graph can be made from this table of values. From the graph in Figure 5.2.2, we can see that this sequence represents a linear function, but notice the graph is not continuous because the domain is over the positive integers only.

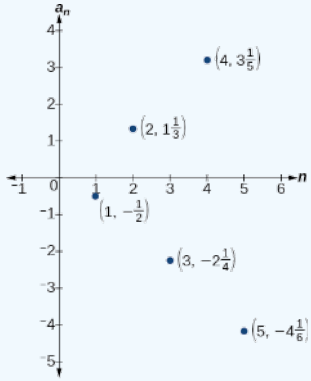


Figure 5.2.2

Exercise 5.2.1

Write the first five terms of the sequence defined by the *explicit formula* $t_n = 5n - 4$.

Answer

The first five terms are $\{1, 6, 11, 16, 21\}$.

Investigating Alternating Sequences

Sometimes sequences have terms that are alternate. In fact, the terms may actually alternate in sign. The steps to finding terms of the sequence are the same as if the signs did not alternate. However, the resulting terms will not show increase or decrease as n increases. Let's take a look at the following sequence.

$$\{2, -4, 6, -8\}$$

Notice the first term is greater than the second term, the second term is less than the third term, and the third term is greater than the fourth term. This trend continues forever. Do not rearrange the terms in numerical order to interpret the sequence.

How to: Given an explicit formula with alternating terms, write the first nn terms of a sequence

1. Substitute each value of nn into the formula. Begin with $n = 1$ to find the first term, a_1. The sign of the term is given by the $(-1)^n$ in the explicit formula.
2. To find the second term, a_2, use $n = 2$.
3. Continue in the same manner until you have identified all n terms.

Example 5.2.2: Writing the Terms of an Alternating Sequence Defined by an Explicit Formula

Write the first five terms of the sequence.

$$a_n = \frac{(-1)^n n^2}{n+1}$$

Solution

Substitute $n = 1$, $n = 2$, and so on in the formula.

$$n = 1 \quad a_1 = \frac{(-1)^1 2^2}{1+1} = -\frac{1}{2}$$
$$n = 2 \quad a_2 = \frac{(-1)^2 2^2}{2+1} = \frac{4}{3}$$
$$n = 3 \quad a_3 = \frac{(-1)^3 3^2}{3+1} = -\frac{9}{4}$$
$$n = 4 \quad a_4 = \frac{(-1)^4 4^2}{4+1} = \frac{16}{5}$$
$$n = 5 \quad a_5 = \frac{(-1)^5 5^2}{5+1} = -\frac{25}{6}$$

The first five terms are $\{-12, 43, -94, 165, -256\}$.

Analysis

The graph of this function, shown in Figure 5.2.3, looks different from the ones we have seen previously in this section because the terms of the sequence alternate between positive and negative values.

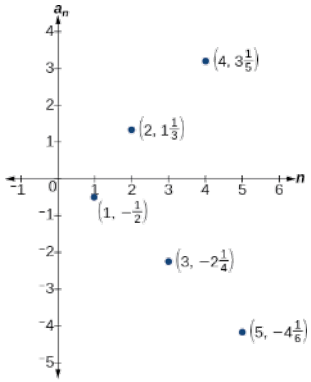


Figure 5.2.3

Q&A: In the example above, does the (-1) to the power of n account for the oscillations of signs?

Yes, the power might be n, $n+1$, $n-1$, and so on, but any odd powers will result in a negative term, and any even power will result in a positive term.

Exercise 5.2.2

Write the first five terms of the sequence:

$$a_n = \frac{4n}{(-2)^n}$$

Answer

The first five terms are $\{-2, 2, -32, 1, -58\}$.

Investigating Piecewise Explicit Formulas

We've learned that sequences are functions whose domain is over the positive integers. This is true for other types of functions, including some piecewise functions. Recall that a *piecewise function* is a function defined by multiple subsections. A different formula might represent each individual subsection.

How to: Given an explicit formula for a piecewise function, write the first n terms of a sequence

1. Identify the formula to which $n = 1$ applies.
2. To find the first term, a_1, use $n = 1$ in the appropriate formula.
3. Identify the formula to which $n = 2$ applies.
4. To find the second term, a_2, use $n = 2$ in the appropriate formula.
5. Continue in the same manner until you have identified all n terms.

Example 5.2.3: Writing the Terms of a Sequence Defined by a Piecewise Explicit Formula

Write the first six terms of the sequence.

$$a_n = \begin{cases} n^2 \text{ if n is not divisible by 3} \\ \frac{n}{3} \text{ if n is divisible by 3} \end{cases}$$

Solution

Substitute $n=1$, $n=2$, and so on in the appropriate formula. Use n^2 when n is not a multiple of 3. Use n^3 when n is a multiple of 3.

$$\begin{aligned}
a_1 &= 1^2 = 1 && \text{1 is not a multiple of 3. Use } n^2.\\
a_2 &= 2^2 = 4 && \text{2 is not a multiple of 3 . Use } n^2.\\
a_3 &= \frac{3}{3} = 1 && \text{3 is a multiple of 3. Use } \frac{n}{3}.\\
a_4 &= 4^2 = 16 && \text{4 is not a multiple of 3. Use } n^2.\\
a_5 &= 5^2 = 25 && \text{5 is not a multiple of 3.Use } n^2.\\
a_6 &= \frac{6}{3} = 2 && \text{6 is a multiple of 3.Use } \frac{n}{3}
\end{aligned}$$

The first six terms are $\{1, 4, 1, 16, 25, 2\}$.

Analysis

Every third point on the graph shown in Figure 5.2.4 stands out from the two nearby points. This occurs because the sequence was defined by a piecewise function.

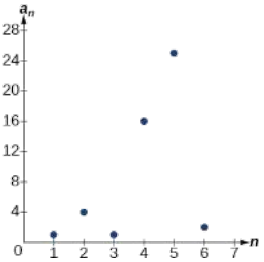


Figure 5.2.4

? Exercise 5.2.3

Write the first six terms of the sequence.

$$a_n = \begin{cases} 2n^3 & \text{if } n \text{ is odd} \\ \frac{5n}{2} & \text{if } n \text{ is even} \end{cases}$$

Answer

The first six terms are $\{2, 5, 54, 10, 250, 15\}$.

Finding an Explicit Formula

Thus far, we have been given the explicit formula and asked to find a number of terms of the sequence. Sometimes, the explicit formula for the nth term of a sequence is not given. Instead, we are given several terms from the sequence. When this happens, we can work in reverse to find an explicit formula from the first few terms of a sequence. The key to finding an explicit formula is to look for a pattern in the terms. Keep in mind that the pattern may involve alternating terms, formulas for numerators, formulas for denominators, exponents, or bases.

How to: Given the first few terms of a sequence, find an explicit formula for the sequence

1. Look for a pattern among the terms.
2. If the terms are fractions, look for a separate pattern among the numerators and denominators.
3. Look for a pattern among the signs of the terms.
4. Write a formula for a_n in terms of n. Test your formula for $n=1$, $n=2$, and $n=3$.

Example 5.2.4: Writing an Explicit Formula for the *n*th Term of a Sequence

Write an explicit formula for the nth term of each sequence.

a. $\{-\frac{2}{11}, \frac{3}{13}, -\frac{4}{15}, \frac{5}{17}, -\frac{6}{19}, \ldots\}$

b. $\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3,125}, -\frac{2}{15,625}, \ldots\}$

c. $\{e^4, e^5, e^6, e^7, e^8, \ldots\}$

Solution

Look for the pattern in each sequence.

a. The terms alternate between positive and negative. We can use $(-1)^n$ to make the terms alternate. The numerator can be represented by $n+1$. The denominator can be represented by $2n+9$.

$$a_n = \frac{(-1)^n(n+1)}{2n+9}$$

b. The terms are all negative. $\left\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3,125}, -\frac{2}{15,625}, \ldots\right\}$ The numerator is 2.

$\left\{-\frac{2}{5^2}, -\frac{2}{5^3}, -\frac{2}{5^4}, -\frac{2}{5^5}, -\frac{2}{5^6}, \ldots, -\frac{2}{5^n}\right\}$ The denominators are increasing powers of 5.

So we know that the fraction is negative, the numerator is 2, and the denominator can be represented by $5n+1$.

$$a_n = -\frac{2}{5^{n+1}}$$

c. The terms are powers of e. For $n=1$, the first term is e^4 so the exponent must be $n+3$.

$$a_n = e^{n+3}$$

? Exercise 5.2.4A

Write an explicit formula for the nth term of the sequence.

$$\{9, -81, 729, -6,561, 59,049, \ldots\}$$

Answer

$a_n = (-1)^{n+1}9^n$

? Exercise 5.2.4B

Write an explicit formula for the nth term of the sequence.

$$\{-34, -98, -2712, -8116, -24320, \ldots\}$$

Answer

$a_n = -\frac{3^n}{4n}$

? Exercise 5.2.4C

Write an explicit formula for the nth term of the sequence.

$$\left\{\frac{1}{e^2}, \frac{1}{e}, 1, e, e^2, \ldots\right\}$$

Answer

$a_n = e^{n-3}$

Writing the Terms of a Sequence Defined by a Recursive Formula

Sequences occur naturally in the growth patterns of nautilus shells, pinecones, tree branches, and many other natural structures. We may see the sequence in the leaf or branch arrangement, the number of petals of a flower, or the pattern of the chambers in a nautilus shell. Their growth follows the Fibonacci sequence, a famous sequence in which each term can be found by adding the preceding two terms. The numbers in the sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Other examples from the natural world that exhibit the Fibonacci sequence are the Calla Lily, which has just one petal, the Black-Eyed Susan with 13 petals, and different varieties of daisies that may have 21 or 34 petals.

Each term of the Fibonacci sequence depends on the terms that come before it. The Fibonacci sequence cannot easily be written using an explicit formula. Instead, we describe the sequence using a **recursive formula**, a formula that defines the terms of a sequence using previous terms.

A recursive formula always has two parts: the value of an initial term (or terms), and an equation defining a_n in terms of preceding terms. For example, suppose we know the following:

$$a_1 = 3$$
$$a_n = 2a_{n-1} - 1, \text{ for } n \geq 2$$

We can find the subsequent terms of the sequence using the first term.

$$a_1 = 3$$
$$a_2 = 2a_1 - 1 = 2(3) - 1 = 5$$
$$a_3 = 2a_2 - 1 = 2(5) - 1 = 9$$
$$a_4 = 2a_3 - 1 = 2(9) - 1 = 17$$

So the first four terms of the sequence are $\{3, 5, 9, 17\}$.

The recursive formula for the Fibonacci sequence states the first two terms and defines each successive term as the sum of the preceding two terms.

$$a_1 = 1$$
$$a_2 = 1$$
$$a_n = a_{n-1} + a_{n-2}, \text{ for } n \geq 3$$

To find the tenth term of the sequence, for example, we would need to add the eighth and ninth terms. We were told previously that the eighth and ninth terms are 21 and 34, so

$$a_{10} = a_9 + a_8 = 34 + 21 = 55$$

RECURSIVE FORMULA

A recursive formula is a formula that defines each term of a sequence using preceding term(s). Recursive formulas must always state the initial term, or terms, of the sequence.

Q&A: Must the first two terms always be given in a recursive formula?

No. The Fibonacci sequence defines each term using the two preceding terms, but many recursive formulas define each term using only one preceding term. These sequences need only the first term to be defined.

How to: Given a recursive formula with only the first term provided, write the first n terms of a sequence

1. Identify the initial term, a_1, which is given as part of the formula. This is the first term.
2. To find the second term, a_2, substitute the initial term into the formula for a_{n-1}. Solve.
3. To find the third term, a_3, substitute the second term into the formula. Solve.
4. Repeat until you have solved for the nth term.

✓ Example 5.2.5: Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first five terms of the sequence defined by the recursive formula.

$$a_1 = 9$$
$$a_n = 3a_{n-1} - 20, \text{ for } n \geq 2$$

Solution

The first term is given in the formula. For each subsequent term, we replace $a_n - 1$ with the value of the preceding term.

$$n=1 \quad a_1=9$$
$$n=2 \quad a_2=3a_1-20=3(9)-20=27-20=7$$
$$n=3 \quad a_3=3a_2-20=3(7)-20=21-20=1$$
$$n=4 \quad a_4=3a_3-20=3(1)-20=3-20=-17$$
$$n=5 \quad a_5=3a_4-20=3(-17)-20=-51-20=-71$$

The first five terms are $\{9,7,1,-17,-71\}$. See Figure 5.2.5.

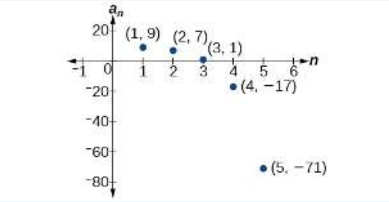


Figure 5.2.5

Exercise 5.2.5

Write the first five terms of the sequence defined by the recursive formula.

$$a_1=2$$
$$a_n=2a_{n-1}+1, \text{ for } n\geq 2$$

Answer

$\{2,5,11,23,47\}$

How to: Given a recursive formula with two initial terms, write the first n terms of a sequence

1. Identify the initial term, a_1, which is given as part of the formula.
2. Identify the second term, a_2, which is given as part of the formula.
3. To find the third term, substitute the initial term and the second term into the formula. Evaluate.
4. Repeat until you have evaluated the nth term.

Example 5.2.6: Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first six terms of the sequence defined by the recursive formula.

$$a_1=1$$
$$a_2=2$$
$$a_n=3a_{n-1}+4a_{n-2}, \text{ for } n\geq 3$$

Solution

The first two terms are given. For each subsequent term, we replace a_{n-1} and a_{n-2} with the values of the two preceding terms.

$$n=3 \quad a_3=3a_2+4a_1=3(2)+4(1)=10$$
$$n=4 \quad a_4=3a_3+4a_2=3(10)+4(2)=38$$
$$n=5 \quad a_5=3a_4+4a_3=3(38)+4(10)=154$$
$$n=6 \quad a_6=3a_5+4a_4=3(154)+4(38)=614$$

The first six terms are $\{1,2,10,38,154,614\}$. See Figure 5.2.6.

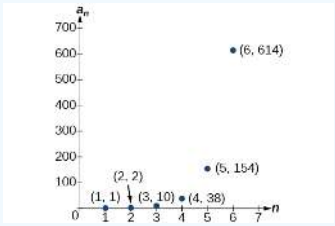


Figure 5.2.6

Exercise 5.2.6

Write the first 8 terms of the sequence defined by the recursive formula.

$$a_1=0$$
$$a_2=1$$
$$a_3=1$$
$$a_n=\frac{a_{n-1}}{a_{n-2}}+a_{n-3}, \text{ for } n\geq 4$$

Answer

$\{0,1,1,1,2,3,52,176\}$.

Using Factorial Notation

The formulas for some sequences include products of consecutive positive integers. n **factorial**, written as $n!$, is the product of the positive integers from 1 to n. For example,

$$4!=4\cdot 3\cdot 2\cdot 1=24$$
$$5!=5\cdot 4\cdot 3\cdot 2\cdot 1=120$$

An example of formula containing a **factorial** is $a_n=(n+1)!$. The sixth term of the sequence can be found by substituting 6 for n.

$$a_6 = (6+1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

The factorial of any whole number n is $n(n-1)!$ We can therefore also think of $5!$ as $5 \cdot 4!$.

FACTORIAL

n factorial is a mathematical operation that can be defined using a recursive formula. The factorial of n, denoted $n!$, is defined for a positive integer n as:

$$0! = 1 \quad (5.2.2)$$
$$1! = 1 \quad (5.2.3)$$
$$n! = n(n-1)(n-2)\cdots(2)(1), \text{ for } n \geq 2 \quad (5.2.4)$$

The special case $0!$ is defined as $0! = 1$.

Q&A: Can factorials always be found using a calculator?

No. Factorials get large very quickly—faster than even exponential functions! When the output gets too large for the calculator, it will not be able to calculate the factorial.

Example 5.2.7: Writing the Terms of a Sequence Using Factorials

Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{5n}{(n+2)!}$.

Solution

Substitute $n = 1$, $n = 2$, and so on in the formula.

$$n=1 \quad a_1 = \frac{5(1)}{(1+2)!} = \frac{5}{3!} = \frac{5}{3 \cdot 2 \cdot 1} = \frac{5}{6}$$
$$n=2 \quad a_2 = \frac{5(2)}{(2+2)!} = \frac{10}{4!} = \frac{10}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{12}$$
$$n=3 \quad a_3 = \frac{5(3)}{(3+2)!} = \frac{15}{5!} = \frac{15}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{8}$$
$$n=4 \quad a_4 = \frac{5(4)}{(4+2)!} = \frac{20}{6!} = \frac{20}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{36}$$
$$n=5 \quad a_5 = \frac{5(5)}{(5+2)!} = \frac{25}{7!} = \frac{25}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{1,008}$$

The first five terms are $\left\{\frac{5}{6}, \frac{5}{12}, \frac{1}{8}, \frac{1}{36}, \frac{5}{1,008}\right\}$.

Analysis

Figure 5.2.7 shows the graph of the sequence. Notice that, since factorials grow very quickly, the presence of the factorial term in the denominator results in the denominator becoming much larger than the numerator as nn increases. This means the quotient gets smaller and, as the plot of the terms shows, the terms are decreasing and nearing zero.

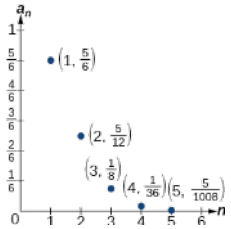


Figure 5.2.7

Exercise 5.2.7

Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{(n+1)!}{2n}$.

Answer

The first five terms are $\left\{1, \frac{3}{2}, 4, 15, 72\right\}$.

Media

Access this online resource for additional instruction and practice with sequences.

- Finding Terms in a Sequence

Key Equations

Formula for a factorial	$0! = 1$ $1! = 1$ $n! = n(n-1)(n-2)\cdots(2)(1)$, for $n \geq 2$

Key Concepts

- A sequence is a list of numbers, called terms, written in a specific order.
- Explicit formulas define each term of a sequence using the position of the term. See Example 5.2.1, Example 5.2.2, and Example 5.2.3.
- An explicit formula for thenth nth term of a sequence can be written by analyzing the pattern of several terms. See Example 5.2.4.
- Recursive formulas define each term of a sequence using previous terms.
- Recursive formulas must state the initial term, or terms, of a sequence.
- A set of terms can be written by using a recursive formula. See Example 5.2.5 and Example 5.2.6.
- A factorial is a mathematical operation that can be defined recursively.
- The factorial of n is the product of all integers from 1 to n. See Example 5.2.7.

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CHAPTER OVERVIEW

6: Growth Models

6.1: Introduction
6.2: Linear (Algebraic) Growth
6.3: When good models go bad
6.4: Exponential (Geometric) Growth
6.5: Solving Exponentials for Time- Logarithms
6.6: Logistic Growth
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6.1: Introduction

Populations of people, animals, and items are growing all around us. By understanding how things grow, we can better understand what to expect in the future. In this chapter, we focus on time-dependant change.

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6.2: Linear (Algebraic) Growth

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that P_n represents the number, or population, of bottles Marco has after n years. So P_0 would represent the number of bottles now, P_1 would represent the number of bottles after 1 year, P_2 would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$P_0 = 437$$

$$P_n = P_{n-1} + 32$$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, P_{n-1}. Using this relationship, we could calculate:

$$P_1 = P_0 + 32 = 437 + 32 = 469$$

$$P_2 = P_1 + 32 = 469 + 32 = 501$$

$$P_3 = P_2 + 32 = 501 + 32 = 533$$

$$P_4 = P_3 + 32 = 533 + 32 = 565$$

$$P_5 = P_4 + 32 = 565 + 32 = 597$$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An **explicit equation** allows us to calculate P_n directly, without needing to know P_{n-1}. While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$\begin{aligned} P_1 &= 437 + 32 &&= 437 + 1(32) \\ P_2 &= P_1 + 32 = 437 + 32 + 32 &&= 437 + 2(32) \\ P_3 &= P_2 + 32 = (437 + 2(32)) + 32 &&= 437 + 3(32) \\ P_4 &= P_3 + 32 = (437 + 3(32)) + 32 &&= 437 + 4(32) \end{aligned}$$

You can probably see the pattern now, and generalize that

$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for P_n and solving for n.

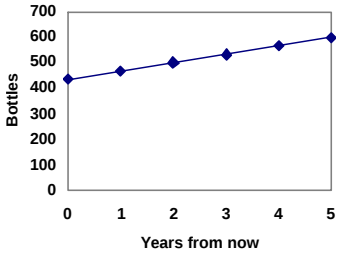
$$1000 = 437 + 32n$$

$$563 = 32n$$

$$n = 563/32 = 17.59$$

So Marco will reach 1000 bottles in 18 years.

In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.



Linear Growth

If a quantity starts at size P_0 and grows by d every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form:

$P_n = P_{n-1} + d$

Explicit form:

$P_n = P_0 + dn$

In this equation, represents the **common difference** – the amount that the population changes each time n increases by 1

Connection to prior learning – slope and intercept

You may recognize the common difference, d, in our linear equation as slope. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as $y = mx + b$.

In the standard algebraic equation $y = mx + b$, b was the y-intercept, or the y value when x was zero. In the form of the equation we're using, we are using P0 to represent that initial amount.

In the $y = mx + b$ equation, recall that m was the slope. You might remember this as "rise over run", or the change in y divided by the change in x. Either way, it represents the same thing as the common difference, d, we are using – the amount the output Pn changes when the input n increases by 1.

The equations $y = mx + b$ and $P_n = P_0 + dn$ mean the same thing and can be used the same ways, we're just writing it somewhat differently.

Example 1

The population of elk in a national forest was measured to be 12,000 in 2003, and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

Solution

To begin, we need to define how we're going to measure n. Remember that P_0 is the population when $n = 0$, so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define $n = 0$ to be the year 2003. Then

$P_0 = 12,000$.

Next we need to find d. Remember d is the growth per time period, in this case growth per year. Between the two measurements, the population grew by $15,000 - 12,000 = 3,000$, but it took $2007 - 2003 = 4$ years to grow that much. To find the growth per year, we can divide: 3000 elk / 4 years = 750 elk in 1 year.

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

d =slope= $\frac{\text{change in output}}{\text{change in input}} = \frac{15,000-12,000}{2007-2003} = \frac{3000}{4} = 750$

We can now write our equation in whichever form is preferred.

Recursive form:

$P_0 = 12,000$

$P_n = P_{n-1} + 750$

Explicit form:

$P_n = 12,000 + 750n$

To answer the question, we need to first note that the year 2014 will be $n = 11$, since 2014 is 11 years after 2003. The explicit form will be easier to use for this calculation:

$P_{11} = 12,000 + 750(11) = 20,250$ elk

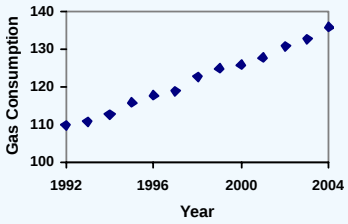
Example 2

Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below[1]. Find a model for this data, and use it to predict consumption in 2016. If the trend continues, when will consumption reach 200 billion gallons?

Year	'92	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billion of gallons)	110	111	113	116	118	119	123	125	126	128	131	133	136

Solution

Plotting this data, it appears to have an approximately linear relationship:



While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.

Letting $n = 0$ correspond with 1993 would give $P_0 = 111$ billion gallons.

To find d, we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of $133–111 = 22$ billion gallons, over 10 years. This gives us an average change of 22 billion gallons / 10 year = 2.2 billion gallons per year.

Equivalently,

$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{133–111}{10–0} = \frac{22}{10} = 2.2$ billion gallons per year

We can now write our equation in whichever form is preferred.

Recursive form:

$P_0 = 111$

$P_n = P_{n–1} + 2.2$

Explicit form:

$P_n = 111 + 2.2n$

Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2016:

$n = 23 (2016–1993 = 23$ years later)

$P_{23} = 111 + 2.2(23) = 161.6$

Our model predicts that the US will consume 161.6 billion gallons of gasoline in 2016 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set $P_n = 200$, and solve for n:

$P_n = 200$	Replace P_n with our model
$111 + 2.2n = 200$	Subtract 111 from both sides
$2.2n = 89$	Divide both sides by 2.2
$n = 40.4545$	

This tells us that consumption will reach 200 billion about 40 years after 1993, which would be in the year 2033.

Example 3

The cost, in dollars, of a gym membership for n months can be described by the explicit equation $P_n = 70 + 30n$. What does this equation tell us?

Solution

The value for P_0 in this equation is 70, so the initial starting cost is \$70. This tells us that there must be an initiation or start-up fee of \$70 to join the gym.

The value for d in the equation is 30, so the cost increases by \$30 each month. This tells us that the monthly membership fee for the gym is \$30 a month.

Try it Now 1

The number of stay-at-home fathers in Canada has been growing steadily[2]. While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers, then use it to predict the number if 2020.

Year	1976	1984	1991	2000	2010
Number of stay-at-home fathers	20,610	28,725	43,530	47,665	53,555

Answer

Letting $n = 0$ correspond with 1976, then $P_0 = 20,610$.

From 1976 to 2010 the number of stay-at-home fathers increased by

$53,555 - 20,610 = 32,945$

This happened over 34 years, giving a common different d of $32,945/34 = 969$.

$P_n = 20,610 + 969n$

Predicting for 2020, we use $n = 44$

$P_{44} = 20,610 + 969(44) = 63,246$ stay-at-home fathers in 2020.

[1] www.bts.gov/publications/nati...ble_04_10.html

[2] www.fira.ca/article.php?id=140

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6.3: When good models go bad

When using mathematical models to predict future behavior, it is important to keep in mind that very few trends will continue indefinitely.

Example 4

Suppose a four year old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year.

Solution

We can set up a growth model, with $n = 0$ corresponding to 4 years old.

Recursive form:

$P_0 = 39$

$P_n = P_{n-1} + 2.5$

Explicit form:

$P_n = 39 + 2.5n$

So at 6 years old, we would expect him to be

$P_2 = 39 + 2.5(2) = 44$ inches tall

Any mathematical model will break down eventually. Certainly, we shouldn't expect this boy to continue to grow at the same rate all his life. If he did, at age 50 he would be

$P_{46} = 39 + 2.5(46) = 154$ inches tall $= 12.8$ feet tall!

When using any mathematical model, we have to consider which inputs are reasonable to use. Whenever we extrapolate, or make predictions into the future, we are assuming the model will continue to be valid.

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6.4: Exponential (Geometric) Growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. If there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent any inhibiting factors, populations of people and animals tend to grow by a percent of the existing population each year.

Suppose our lake began with 1000 fish, and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P_0 = 1000$. How do we calculate P_1? The new population will be the old population, plus an additional 10%. Symbolically:

$$P_1 = P_0 + 0.10P_0$$

Notice this could be condensed to a shorter form by factoring:

$$P_1 = P_0 + 0.10P_0 = 1P_0 + 0.10P_0 = (1 + 0.10)P_0 = 1.10P_0$$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as "the original 100% plus an additional 10%"

For our fish population,

$$P_1 = 1.10(1000) = 1100$$

We could then calculate the population in later years:

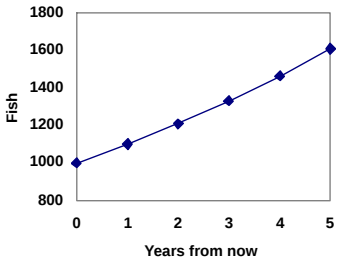
$$P_2 = 1.10P_1 = 1.10(1100) = 1210$$

$$P_3 = 1.10P_2 = 1.10(1210) = 1331$$

Notice that in the first year, the population grew by 100 fish, in the second year, the population grew by 110 fish, and in the third year the population grew by 121 fish.

While there is a constant percentage growth, the actual increase in number of fish is increasing each year.

Graphing these values we see that this growth doesn't quite appear linear.



To get a better picture of how this percentage-based growth affects things, we need an explicit form, so we can quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$$P_1 = 1.10P_0 = 1.10(1000)$$

$$P_2 = 1.10P_1 = 1.10(1.10(1000)) = 1.10^2(1000)$$

$$P_3 = 1.10P_2 = 1.10\left(1.10^2(1000)\right) = 1.10^3(1000)$$

$$P_4 = 1.10P_3 = 1.10\left(1.10^3(1000)\right) = 1.10^4(1000)$$

Observing a pattern, we can generalize the explicit form to be:

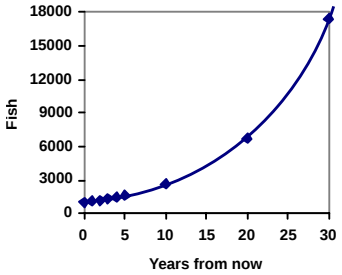
$P_n = 1.10^n(1000)$, or equivalently, $P_n = 1000\left(1.10^n\right)$

From this, we can quickly calculate the number of fish in 10, 20, or 30 years:

$$P_{10} = 1.10^{10}(1000) = 2594$$

$$P_{20} = 1.10^{20}(1000) = 6727$$

$$P_{30} = 1.10^{30}(1000) = 17449$$



Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years compared to almost 18000 with this percent-based growth, called exponential growth.

In exponential growth, the population grows proportional to the size of the population, so as the population gets larger, the same percent growth will yield a larger numeric growth.

Exponential Growth

If a quantity starts at size P_0 and grows by $R\%$ (written as a decimal, r) every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form:

$P_n = (1 + r)P_{n-1}$

Explicit form:

$P_n = (1 + r)^n P_0$ or equivalently, $P_n = P_0(1 + r)^n$

We call r the **growth rate**.

The term $(1 + r)$ is called the **growth multiplier**, or common ratio.

Example 5

Between 2007 and 2008, Olympia, WA grew almost 3% to a population of 245 thousand people. If this growth rate was to continue, what would the population of Olympia be in 2014?

Solution

As we did before, we first need to define what year will correspond to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P_0 = 245,000$. The year 2014 would then be $n = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the explicit form:

$P_6 = (1 + 0.03)^6(245,000) = 1.19405(245,000) = 292,542.25$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

Evaluating exponents on the calculator

To evaluate expressions like $(1.03)^6$, it will be easier to use a calculator than multiply 1.03 by itself six times. Most scientific calculators have a button for exponents. It is typically either labeled like:

[^], $[y^x]$, or $[x^y]$.

To evaluate 1.03^6 we'd type 1.03 [^] 6, or 1.03 $[y^x]$ 6. Try it out - you should get an answer around 1.1940523.

Try it Now 2

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, what will India's population grow to by 2020?

Answer

Using $n = 0$ corresponding with 2008,

$P_{12} = (1 + 0.0134)^{12}(1.14) =$ about 1.337 billion people in 2020

Example 6

A friend is using the equation $P_n = 4600(1.072)^n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

Solution

In the equation, $P_0 = 4600$, which is the starting value of the tuition when $n = 0$. This tells us that the tuition in 2010 was $4,600.

The growth multiplier is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% each year.

Putting this together, we could say that the tuition in 2010 was $4,600, and is expected to grow by 7.2% each year.

Example 7

In 1990, the residential energy use in the US was responsible for 962 million metric tons of carbon dioxide emissions. By the year 2000, that number had risen to 1182 million metric tons[1]. If the emissions grow exponentially and continue at the same rate, what will the emissions grow to by 2050?

Solution

Similar to before, we will correspond $n = 0$ with 1990, as that is the year for the first piece of data we have. That will make $P_0 = 962$ (million metric tons of CO_2). In this problem, we are not given the growth rate, but instead are given that $P_{10} = 1182$.

When $n = 10$, the explicit equation looks like:

$$P_{10} = (1 + r)^{10} P_0$$

We know the value for P_0, so we can put that into the equation:

$$P_{10} = (1 + r)^{10} 962$$

We also know that $P_{10} = 1182$, so substituting that in, we get

$$1182 = (1 + r)^{10} 962$$

We can now solve this equation for the growth rate, r. Start by dividing by 962.

$$\frac{1182}{962} = (1 + r)^{10} \qquad \text{Take the 10th root of both sides}$$

$$\sqrt[10]{\frac{1182}{962}} = 1 + r \text{Subtract 1 from both sides}$$

$$r = \sqrt[1]{\frac{1182}{962}} - 1 = 0.0208 = 2.08\%$$

So if the emissions are growing exponentially, they are growing by about 2.08% per year. We can now predict the emissions in 2050 by finding P_{60}

$P_{60} = (1 + 0.0208)^{60} 962 = 3308.4$ million metric tons of CO_2 in 2050

Rounding

As a note on rounding, notice that if we had rounded the growth rate to 2.1%, our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

If you need to round, **keep at least three significant digits** - numbers after any leading zeros. So 0.4162 could be reasonably rounded to 0.416. A growth rate of 0.001027 could be reasonably rounded to 0.00103.

Evaluating roots on the calculator

In the previous example, we had to calculate the 10th root of a number. This is different than taking the basic square root, $\sqrt{}$. Many scientific calculators have a button for general roots. It is typically labeled like:

[$\sqrt[y]{}$], [$\sqrt[x]{}$], or [$\sqrt[y]{x}$]

To evaluate the 3rd root of 8, for example, we'd either type 3 [$\sqrt[x]{}$] 8, or 8 [$\sqrt[x]{}$] 3, depending on the calculator. Try it on yours to see which to use – you should get an answer of 2.

If your calculator does not have a general root button, all is not lost. You can instead use the property of exponents which states that $\sqrt[n]{a} = a^{1/n}$. So, to compute the 3rd root of 8, you could use your calculator's exponent key to evaluate $(2)^2 - 1 =$. To do this, type:

8 [y^x] (1 [÷] 3)

The parentheses tell the calculator to divide 1/3 before doing the exponent.

Try it Now 3

The number of users on a social networking site was 45 thousand in February when they officially went public, and grew to 60 thousand by October. If the site is growing exponentially, and growth continues at the same rate, how many users should they expect two years after they went public?

Answer

Here we will measure n in months rather than years, with $n = 0$ corresponding to the February when they went public. This gives $P_0 = 45$ thousand. October is 8 months later, so P8 = 60.

$P_8 = (1 + r)^8 P_0$

$60 = (1 + r)^8 45$

$\frac{60}{45} = (1 + r)^8$

$\sqrt[8]{\frac{60}{45}} = 1 + r$

$r = \sqrt[8]{\frac{60}{45}} - 1 = 0.0366$, or 3.66%

The general explicit equation is $P_n = (1.0366)^n 45$

Predicting 24 months (2 years) after they went public:

$P_{24} = (1.0366)^{24} 45 = 106.63$ thousand users.

Example 8

Looking back at the last example, for the sake of comparison, what would the carbon emissions be in 2050 if emissions grow linearly at the same rate?

Solution

Again we will get $n = 0$ correspond with 1990, giving $P_0 = 962$. To find d, we could take the same approach as earlier, noting that the emissions increased by 220 million metric tons in 10 years, giving a common difference of 22 million metric tons each year.

Alternatively, we could use an approach similar to that which we used to find the exponential equation. When $n = 10$, the explicit linear equation looks like:

$P_{10} = P_0 + 10d$

We know the value for P_0, so we can put that into the equation:

$P_{10} = 962 + 10d$

Since we know that $P_{10} = 1182$, substituting that in we get

$1182 = 962 + 10d$

We can now solve this equation for the common difference, d.

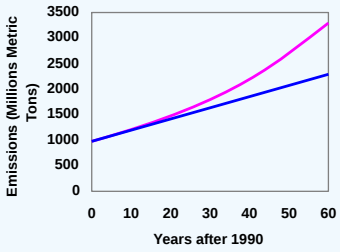
$1182 - 962 = 10d$

$220 = 10d$

$d = 22$

This tells us that if the emissions are changing linearly, they are growing by 22 million metric tons each year. Predicting the emissions in 2050,

$P_{60} = 962 + 22(60) = 2282$ million metric tons.



You will notice that this number is substantially smaller than the prediction from the exponential growth model. Calculating and plotting more values helps illustrate the differences.

So how do we know which growth model to use when working with data? There are two approaches which should be used together whenever possible:

1) Find more than two pieces of data. Plot the values, and look for a trend. Does the data appear to be changing like a line, or do the values appear to be curving upwards?

2) Consider the factors contributing to the data. Are they things you would expect to change linearly or exponentially? For example, in the case of carbon emissions, we could expect that, absent other factors, they would be tied closely to population values, which tend to change exponentially.

[1] www.eia.doe.gov/oiaf/1605/ggrpt/carbon.html

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6.5: Solving Exponentials for Time- Logarithms

Earlier, we found that since Olympia, WA had a population of 245 thousand in 2008 and had been growing at 3% per year, the population could be modeled by the equation

$$P_n = (1 + 0.03)^n(245,000)$$

or equivalently,

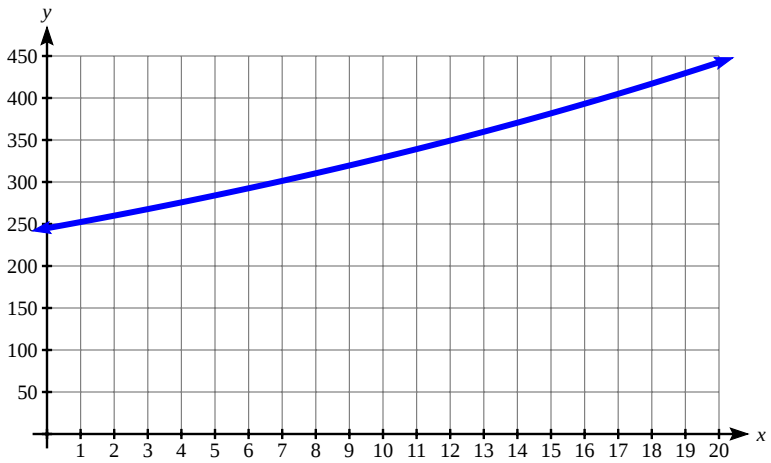
$$P_n = 245,000(1.03)^n.$$

Using this equation, we were able to predict the population in the future.

Suppose we wanted to know when the population of Olympia would reach 400 thousand. Since we are looking for the year n when the population will be 400 thousand, we would need to solve the equation

$$\begin{aligned} &400,000 = 245,000(1.03)^n \quad \text{dividing both sides by } 245,000 \text{ gives} \\ &1.6327 = 1.03^n \end{aligned} \tag{6.5.1}$$

One approach to this problem would be to create a table of values, or to use technology to draw a graph to estimate the solution.



From the graph, we can estimate that the solution will be around 16 to 17 years after 2008 (2024 to 2025). This is pretty good, but we'd really like to have an algebraic tool to answer this question. To do that, we need to introduce a new function that will undo exponentials, similar to how a square root undoes a square. For exponentials, the function we need is called a logarithm. It is the inverse of the exponential, meaning it undoes the exponential. While there is a whole family of logarithms with different bases, we will focus on the common log, which is based on the exponential 10^x.

Common Logarithm

The **common logarithm**, written $\log(x)$, undoes the exponential 10^x

This means that

$$\log(10^x) = x$$

and likewise

$$10^{\log(x)} = x$$

This also means the statement $10^a = b$ is equivalent to the statement $\log(b) = a$

$\log(x)$ is read as "log of x", and means "the logarithm of the value x". It is important to note that this is *not* multiplication – the log doesn't mean anything by itself, just like $\sqrt{}$ doesn't mean anything by itself; it has to be applied to a number.

Example 9

Evaluate each of the following

a. $\log(100)$
b. $\log(1000)$
c. $\log(10000)$
d. $\log(1/100)$
e. $\log(1)$

Solution

a. $\log(100)$ can be written as $\log(10^2)$. Since the log undoes the exponential, $\log(10^2) = 2$
b. $\log(1000) = \log(10^3) = 3$
c. $\log(10000) = \log(10^4) = 4$
d. Recall that $x^{-n} = \frac{1}{x^n}$. $\log\left(\frac{1}{100}\right) = \log\left(10^{-2}\right) = -2$
e. Recall that $x^0 = 1$. $\log(1) = \log(10^0) = 0$

It is helpful to note that from the first three parts of the previous example that the number we're taking the log of has to get 10 *times bigger* for the log to increase in value by 1.

Of course, most numbers cannot be written as a nice simple power of 10. For those numbers, we can evaluate the log using a scientific calculator with a log button.

Example 10

Evaluate $\log(300)$

Solution

Using a calculator, $\log(300)$ is approximately 2.477121

With an equation, just like we can add a number to both sides, multiply both sides by a number, or square both sides, we can also take the logarithm of both sides of the equation and end up with an equivalent equation. This will allow us to solve some simple equations.

Example 11

a. Solve $10^x = 1000$
b. Solve $10^x = 3$
c. Solve $2(10^x) = 8$

Solution

a) Taking the log of both sides gives $\log(10^x) = \log(1000)$

Since the log undoes the exponential, $\log(10^x) = x$. Similarly $\log(1000) = \log(10^3) = 3$.

The equation simplifies then to $x = 3$.

b) Taking the log of both sides gives $\log(10^x) = \log(3)$.

On the left side, $\log(10^x) = x$, so $x = \log(3)$. We can approximate this value with a calculator. $x \approx 0.477$

c) Here we would first want to isolate the exponential by dividing both sides of the equation by 2, giving

$10^x = 4$.

Now we can take the log of both sides, giving $\log(10^x) = \log(4)$, which simplifies to

$x = \log(4) \approx 0.602$

This approach allows us to solve exponential equations with powers of 10, but what about problems like $2 = 1.03^n$ from earlier, which have a base of 1.03? For that, we need the exponent property for logs.

Properties of Logs: Exponent Property

$\log(A^r) = r\log(A)$

To show why this is true, we offer a proof.

Since the logarithm and exponential undo each other, $10^{\log A} = A$. So

$$A^r = \left(10^{\log \cdot A}\right)^r$$

Utilizing the exponential rule that states $(x^a)^b = x^{ab}$,

$$A^r = \left(10^{\log A}\right)^r = 10^{r\log A}$$

So then

$$\log(A^r) = \log\left(10^{r\log A}\right)$$

Again utilizing the property that the log undoes the exponential on the right side yields the result

$$\log(A^r) = r\log A$$

Example 12

Rewrite log(25) using the exponent property for logs

Solution

$$\log(25) = \log\left(5^2\right) = 2\log(5)$$

This property will finally allow us to answer our original question.

Solving exponential equations with logarithms

1. Isolate the exponential. In other words, get it by itself on one side of the equation. This usually involves dividing by a number multiplying it.
2. Take the log of both sides of the equation.
3. Use the exponent property of logs to rewrite the exponential with the variable exponent multiplying the logarithm.
4. Divide as needed to solve for the variable.

Example 13

If Olympia is growing according to the equation, $P_n = 245(1.03)^n$, where n is years after 2008, and the population is measured in thousands. Find when the population will be 400 thousand.

Solution

We need to solve the equation

$400 = 245(1.03)^n$	Begin by dividing both sides by 245 to isolate the exponential
$1.633 = 1.03^n$	Now take the log of both sides
$\log(1.633) = \log\left(1.03^n\right)$	Use the exponent property of logs on the right side
$\log(1.633) = n\log(1.03)$	Now we can divide by $\log(1.03)$
$\frac{\log(1.633)}{\log(1.03)} = n$	We can approximate this value on a calculator
$n \approx 16.591^{\circ}$	

Alternatively, after applying the exponent property of logs on the right side, we could have evaluated the logarithms to decimal approximations and completed our calculations using those approximations, as you'll see in the next example. While the final answer may come out slightly differently, as long as we keep enough significant values during calculation, our answer will be close enough for most purposes.

Example 14

Polluted water is passed through a series of filters. Each filter removes 90% of the remaining impurities from the water. If you have 10 million particles of pollutant per gallon originally, how many filters would the water need to be passed through to reduce the pollutant to 500 particles per gallon?

Solution

In this problem, our "population" is the number of particles of pollutant per gallon. The initial pollutant is 10 million particles per gallon, so $P_0 = 10,000,000$. Instead of changing with time, the pollutant changes with the number of filters, so n will represent the number of filters the water passes through.

Also, since the amount of pollutant is decreasing with each filter instead of increasing, our "growth" rate will be negative, indicating that the population is decreasing instead of increasing, so $r = -0.90$.

We can then write the explicit equation for the pollutant:

$$\begin{aligned} P_n &= 10,000,000(1-0.90)^n \\ &= 10,000,000(0.10)^n \end{aligned}$$

To solve the question of how many filters are needed to lower the pollutant to 500 particles per gallon, we can set P_n equal to 500, and solve for n.

$500 = 10,000,000(0.10)^n$	Divide both sides by$10,000,000$
$0.00005 = 0.10^n$	Take the log of both sides
$\log(0.00005) = \log(0.10^n)$	Use the exponent property of logs on the right side
$\log(0.00005) = n\log(0.10)$	Evaluate the logarithms to a decimal approximation
$-4.301 = n(-1)$	Divide by -1, the value multiplying n
$4.301 = n$	

It would take about 4.301 filters. Of course, since we probably can't install 0.3 filters, we would need to use 5 filters to bring the pollutant below the desired level.

Try it Now 4

India had a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, when will India's population reach 1.2 billion?

Answer

$1.14(1.0134)^n = 1.2$. $n = 3.853$, which is during 2011

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6.6: Logistic Growth

In our basic exponential growth scenario, we had a recursive equation of the form

$$P_n = P_{n-1} + rP_{n-1}$$

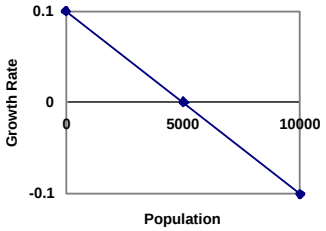
In a confined environment, however, the growth rate may not remain constant. In a lake, for example, there is some *maximum sustainable population* of fish, also called a **carrying capacity**.

Carrying Capacity

The **carrying capacity**, or **maximum sustainable population**, is the largest population that an environment can support.

For our fish, the carrying capacity is the largest population that the resources in the lake can sustain. If the population in the lake is far below the carrying capacity, then we would expect the population to grow essentially exponentially. However, as the population approaches the carrying capacity, there will be a scarcity of food and space available, and the growth rate will decrease. If the population exceeds the carrying capacity, there won't be enough resources to sustain all the fish and there will be a negative growth rate, causing the population to decrease back to the carrying capacity.

If the carrying capacity was 5000, the growth rate might vary something like that in the graph shown. Note that this is a linear equation with intercept at 0.1 and slope $-\frac{0.1}{5000}$, so we could write an equation for this adjusted growth rate as:



$$r_{\text{adjusted}} = 0.1 - \frac{0.1}{5000}P = 0.1\left(1 - \frac{P}{5000}\right)$$

Substituting this in to our original exponential growth model for r gives

$$P_n = P_{n-1} + 0.1\left(1 - \frac{P_{n-1}}{5000}\right)P_{n-1}$$

Logistic Growth

If a population is growing in a constrained environment with carrying capacity K, and absent constraint would grow exponentially with growth rate r, then the population behavior can be described by the logistic growth model:

$$P_n = P_{n-1} + r\left(1 - \frac{P_{n-1}}{K}\right)P_{n-1}$$

Unlike linear and exponential growth, logistic growth behaves differently if the populations grow steadily throughout the year or if they have one breeding time per year. The recursive formula provided above models generational growth, where there is one breeding time per year (or, at least a finite number); there is no explicit formula for this type of logistic growth.

Example 15

A forest is currently home to a population of 200 rabbits. The forest is estimated to be able to sustain a population of 2000 rabbits. Absent any restrictions, the rabbits would grow by 50% per year. Predict the future population using the logistic growth model.

Solution

Modeling this with a logistic growth model, $r = 0.50$, $K = 2000$, and $P0 = 200$. Calculating the next year:

$$P_1 = P_0 + 0.50\left(1 - \frac{P_0}{2000}\right)P_0 = 200 + 0.50\left(1 - \frac{200}{2000}\right)200 = 290$$

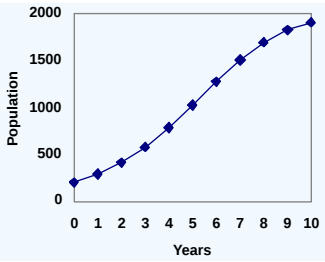
We can use this to calculate the following year:

$$P_2 = P_1 + 0.50\left(1 - \frac{P_1}{2000}\right)P_1 = 290 + 0.50\left(1 - \frac{290}{2000}\right)290 \approx 414$$

A calculator was used to compute several more values:

n	0	1	2	3	4	5	6	7	8	9	10
P_n	200	290	414	578	784	1022	1272	1503	1690	1821	1902

Plotting these values, we can see that the population starts to increase faster and the graph curves upwards during the first few years, like exponential growth, but then the growth slows down as the population approaches the carrying capacity.



Example 16

On an island that can support a population of 1000 lizards, there is currently a population of 600. These lizards have a lot of offspring and not a lot of natural predators, so have very high growth rate, around 150%. Calculating out the next couple generations:

Solution

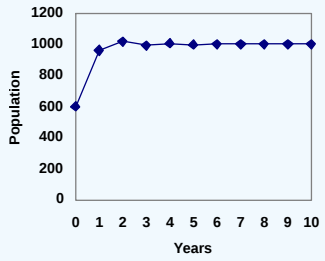
$P_1 = P_0 + 1.50\left(1 - \frac{P_0}{1000}\right)P_0 = 600 + 1.50\left(1 - \frac{600}{1000}\right)600 = 960$

$P_2 = P_1 + 1.50\left(1 - \frac{P_1}{1000}\right)P_1 = 960 + 1.50\left(1 - \frac{960}{1000}\right)960 = 1018$

Interestingly, even though the factor that limits the growth rate slowed the growth a lot, the population still overshot the carrying capacity. We would expect the population to decline the next year.

$P_3 = P_2 + 1.50\left(1 - \frac{P_3}{1000}\right)P_3 = 1018 + 1.50\left(1 - \frac{1018}{1000}\right)101$

Calculating out a few more years and plotting the results, we see the population wavers above and below the carrying capacity, but eventually settles down, leaving a steady population near the carrying capacity.



Try it Now 5

A field currently contains 20 mint plants. Absent constraints, the number of plants would increase by 70% each year, but the field can only support a maximum population of 300 plants. Use the logistic model to predict the population in the next three years.

Answer

$P_1 = P_0 + 0.70\left(1 - \frac{P_0}{300}\right)P_0 = 20 + 0.70\left(1 - \frac{20}{300}\right)20 = 33$

$P_2 = 54$

$P_3 = 85$

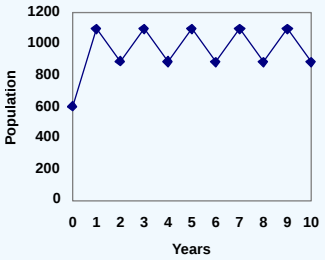
Example 17

On a neighboring island to the one from the previous example, there is another population of lizards, but the growth rate is even higher – about 205%.

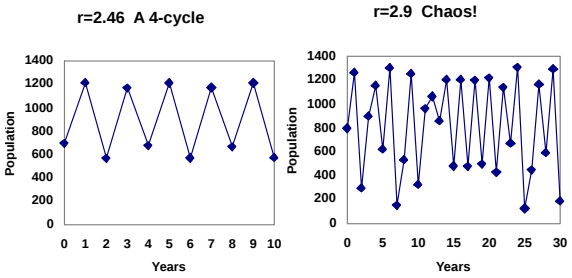
Solution

Calculating out several generations and plotting the results, we get a surprise: the population seems to be oscillating between two values, a pattern called a 2-cycle.

While it would be tempting to treat this only as a strange side effect of mathematics, this has actually been observed in nature. Researchers from the University of California observed a stable 2-cycle in a lizard population in California[1].



Taking this even further, we get more and more extreme behaviors as the growth rate increases higher. It is possible to get stable 4-cycles, 8-cycles, and higher. Quickly, though, the behavior approaches chaos (remember the movie Jurassic Park?).



[1] users.rcn.com/jkimball.ma.ult...ulations2.html

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6.7: Exercises

Skills

1. Marko currently has 20 tulips in his yard. Each year he plants 5 more.
   a. Write a recursive formula for the number of tulips Marko has
   b. Write an explicit formula for the number of tulips Marko has
2. Pam is a Disc Jockey. Every week she buys 3 new albums to keep her collection current. She currently owns 450 albums.
   a. Write a recursive formula for the number of albums Pam has
   b. Write an explicit formula for the number of albums Pam has
3. A store's sales (in thousands of dollars) grow according to the recursive rule $P_n = P_{n-1} + 15$, with initial population $P_0 = 40$.
   a. Calculate P_1 and P_2
   b. Find an explicit formula for P_n
   c. Use your formula to predict the store's sales in 10 years
   d. When will the store's sales exceed $100,000?
4. The number of houses in a town has been growing according to the recursive rule $P_n = P_{n-1} + 30$, with initial population $P_0 = 200$.
   a. Calculate P_1 and P_2
   b. Find an explicit formula for P_n
   c. Use your formula to predict the number of houses in 10 years
   d. When will the number of houses reach 400 houses?
5. A population of beetles is growing according to a linear growth model. The initial population (week 0) was $P_0 = 3$, and the population after 8 weeks is $P_8 = 67$.
   a. Find an explicit formula for the beetle population in week n
   b. After how many weeks will the beetle population reach 187?
6. The number of streetlights in a town is growing linearly. Four months ago $(n = 0)$ there were 130 lights. Now $(n = 4)$ there are 146 lights. If this trend continues,
   a. Find an explicit formula for the number of lights in month n
   b. How many months will it take to reach 200 lights?
7. Tacoma's population in 2000 was about 200 thousand, and had been growing by about 9% each year.
   a. Write a recursive formula for the population of Tacoma
   b. Write an explicit formula for the population of Tacoma
   c. If this trend continues, what will Tacoma's population be in 2016?
   d. When does this model predict Tacoma's population to exceed 400 thousand?
8. Portland's population in 2007 was about 568 thousand, and had been growing by about 1.1% each year.
   a. Write a recursive formula for the population of Portland
   b. Write an explicit formula for the population of Portland
   c. If this trend continues, what will Portland's population be in 2016?
   d. If this trend continues, when will Portland's population reach 700 thousand?
9. Diseases tend to spread according to the exponential growth model. In the early days of AIDS, the growth rate was around 190%. In 1983, about 1700 people in the U.S. died of AIDS. If the trend had continued unchecked, how many people would have died from AIDS in 2005?
10. The population of the world in 1987 was 5 billion and the annual growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2015.

11. A bacteria culture is started with 300 bacteria. After 4 hours, the population has grown to 500 bacteria. If the population grows exponentially,
    a. Write a recursive formula for the number of bacteria
    b. Write an explicit formula for the number of bacteria
    c. If this trend continues, how many bacteria will there be in 1 day?
    d. How long does it take for the culture to triple in size?
12. A native wolf species has been reintroduced into a national forest. Originally 200 wolves were transplanted. After 3 years, the population had grown to 270 wolves. If the population grows exponentially,
    a. Write a recursive formula for the number of wolves
    b. Write an explicit formula for the number of wolves
    c. If this trend continues, how many wolves will there be in 10 years?
    d. If this trend continues, how long will it take the population to grow to 1000 wolves?
13. One hundred trout are seeded into a lake. Absent constraint, their population will grow by 70% a year. The lake can sustain a maximum of 2000 trout. Using the logistic growth model,
    a. Write a recursive formula for the number of trout
    b. Calculate the number of trout after 1 year and after 2 years.
14. Ten blackberry plants started growing in my yard. Absent constraint, blackberries will spread by 200% a month. My yard can only sustain about 50 plants. Using the logistic growth model,
    a. Write a recursive formula for the number of blackberry plants in my yard
    b. Calculate the number of plants after 1, 2, and 3 months
15. In 1968, the U.S. minimum wage was $1.60 per hour. In 1976, the minimum wage was $2.30 per hour. Assume the minimum wage grows according to an exponential model where n represents the time in years after 1960.
    a. Find an explicit formula for the minimum wage.
    b. What does the model predict for the minimum wage in 1960?
    c. If the minimum wage was $5.15 in 1996, is this above, below or equal to what the model predicts?

Concepts

16. The population of a small town can be described by the equation $P.n = 4000 + 70n$, where n is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.
17. The population of a small town can be described by the equation $P.n = 4000(1.04)n$, where n is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.

Exploration

Most of the examples in the text examined growing quantities, but linear and exponential equations can also describe decreasing quantities, as the next few problems will explore.

18. A new truck costs $32,000. The car's value will depreciate over time, which means it will lose value. For tax purposes, depreciation is usually calculated linearly. If the truck is worth $24,500 after three years, write an explicit formula for the value of the car after n years.
19. Inflation causes things to cost more, and for our money to buy less (hence your grandparents saying, "In my day, you could buy a cup of coffee for a nickel"). Suppose inflation decreases the value of money by 5% each year. In other words, if you have $1 this year, next year it will only buy you $0.95 worth of stuff. How much will $100 buy you in 20 years?
20. Suppose that you have a bowl of 500 M&M candies, and each day you eat $\frac{1}{4}$ of the candies you have. Is the number of candies left changing linearly or exponentially? Write an equation to model the number of candies left after n days.
21. A warm object in a cooler room will decrease in temperature exponentially, approaching the room temperature according to the formula where Tn is the temperature after n minutes, r is the rate at which temperature is changing, a is a constant, and Tr is the temperature of the room. Forensic investigators can use this to predict the time of death of a homicide victim. Suppose that

when a body was discovered ($n = 0$) it was 85 degrees. After 20 minutes, the temperature was measured again to be 80 degrees. The body was in a 70 degree room.

a. Use the given information with the formula provided to find a formula for the temperature of the body.
b. When did the victim die, if the body started at 98.6 degrees?

22. Recursive equations can be very handy for modeling complicated situations for which explicit equations would be hard to interpret. As an example, consider a lake in which 2000 fish currently reside. The fish population grows by 10% each year, but every year 100 fish are harvested from the lake by people fishing.
    a. Write a recursive equation for the number of fish in the lake after n years.
    b. Calculate the population after 1 and 2 years. Does the population appear to be increasing or decreasing?
    c. What is the maximum number of fish that could be harvested each year without causing the fish population to decrease in the long run?
23. The number of Starbucks stores grew after first opened. The number of stores from 1990-2007, as reported on their corporate website[1], is shown below.
    a. Carefully plot the data. Does is appear to be changing linearly or exponentially?
    b. Try finding an equation to model the data by picking two points to work from. How well does the equation model the data?
    c. Try using an equation of the form , where k is a constant, to model the data. This type of model is called a Power model. Compare your results to the results from part **b**. *Note: to use this model, you will need to have 1990 correspond with* $n = 1$ *rather than* $n = 0$.

Year	Number of Starbucks stores
1990	84
1991	116
1992	165
1993	272
1994	425
1995	677
1996	1015
1997	1412
1998	1886

Year	Number of Starbucks stores
1999	2498
2000	3501
2001	4709
2002	5886
2003	7225
2004	8569
2005	10241
2006	12440
2007	15756

24. Thomas Malthus was an economist who put forth the principle that population grows based on an exponential growth model, while food and resources grow based on a linear growth model. Based on this, Malthus predicted that eventually demand for food and resources would out outgrow supply, with doom-and-gloom consequences. Do some research about Malthus to answer these questions.
    a. What societal changes did Malthus propose to avoid the doom-and-gloom outcome he was predicting?
    b. Why do you think his predictions did not occur?
    c. What are the similarities and differences between Malthus's theory and the logistic growth model?

[1] www.starbucks.com/aboutus/Company_Timeline.pdf retrieved May 2009

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CHAPTER OVERVIEW

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7.1: Introduction

Where do we find math around us? Math can be found in areas that are expected and sometimes in areas that are surprising. There are many ways that mathematical concepts, such as those in this text, are infused in the world around us. In this chapter, we will explore a sampling of five distinct areas from everyday life where math's impact plays a meaningful role.



Figure **7.1.1** Math can be found in many different areas and subjects. (credit: modification of work "Chemistry/Physics Library" by University Libraries/Flickr, CC BY 2.0)

Math's impact on art can be found in numerical relationships that are known to create or enhance beauty. The Fibonacci numbers are one mathematical example that can be found in nature such as in petal count of a rose. On a different note, a mathematical exploration can aid in making a convincing argument on how we can positively impact our environment. Whether looking at the choices of a single individual or the larger impact offered from a collaborative effort, there are measurable responses to positively address climate change. Turning to medicine, which has been a topic of global importance in recent years, we will explore how math is used to determine drug dosage rates and test the validity of a vaccine. Switching back to items of aesthetic nature, we will examine some foundational components of music which, like art, brings beauty and joy to our lives. Finally, we will explore some ways that math is used in sports to predict future performance and analyze tournaments styles.

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7.2: Math and Art

Learning Objectives

1. Identify and describe the golden ratio.
2. Identify and describe the Fibonacci sequence and its application to nature.
3. Apply the golden ratio and the Fibonacci sequence relationship.
4. Identify and compute golden rectangles.

> *Art is the expression or application of human creative skill and imagination, typically in a visual form such as painting or sculpture, producing works to be appreciated primarily for their beauty or emotional power. -Oxford Dictionary*

Art, like other disciplines, is an area that combines talent and experience with education. While not everyone considers themself skilled at creating art, there are mathematical relationships commonly found in artistic masterpieces that drive what is considered attractive to the eye. Nature is full of examples of these mathematical relationships.



Figure **7.2.1**: Sunflower seeds appear in a pattern that involves the Fibonacci sequence. (credit: "Sunflower Surprise" by frankieleon/Flickr, CC BY 2.0)

Enroll in a cake decorating class and, when you learn how to create flowers out of icing, you will likely be directed as to the number of petals to use. Depending on the desired size of a rose flower, the recommendation for the number of petals to use is commonly 5, 8, or 13 petals. If learning to draw portraits, you may be surprised to learn that eyes are approximately halfway between the top of a person's head and their chin. Studying architecture, we find examples of buildings that contain golden rectangles and ratios that add to the beautifying of the design. The Parthenon (Figure **7.2.2**), which was built around 400 BC, as well as modern-day structures such the Washington Monument are two examples containing these relationships. These seemingly unrelated examples and many more highlight mathematical relationships that we associate with beauty in artistic form.



Figure **7.2.2**: The Parthenon in Greece demonstrates the golden ratio. (credit: "Parthénon" by Julien Maury/Flickr, Public Domain Mark 1.0)

Golden Ratio

The golden ratio, also known as the golden proportion, is a ratio aspect that can be found in beauty from nature to human anatomy as well as in golden rectangles that are commonly found in building structures. The golden ratio is expressed in nature from plants to creatures such as the starfish, honeybees, seashells, and more. It is commonly noted by the Greek letter ϕ (pronounced “fee”). $\phi = \text{mfrac}52\phi=1+52$, which has a decimal value approximately equal to 1.618.

Consider Figure **7.2.2**: Note how the building is balanced in dimension and has a natural shape. The overall structure does not appear as if it is too wide or too tall in comparison to the other dimensions.

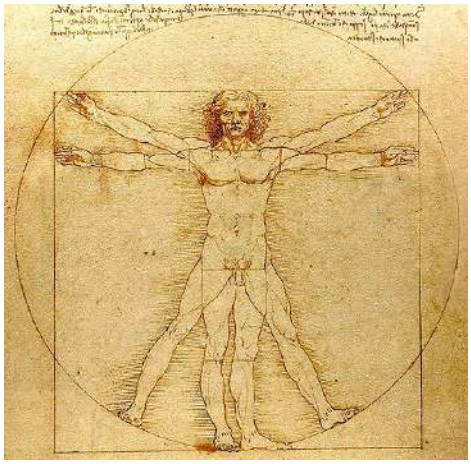


Figure **7.2.3**: *Vitruvian Man* by Leonardo da Vinci (credit: "Vitruvian Man" by Leonardo da Vinci/Wikimedia Commons, Public Domain)

The golden ratio has been used by artists through the years and can be found in art dating back to 3000 BC. Leonardo da Vinci is considered one of the artists who mastered the mathematics of the golden ratio, which is prevalent in his artwork such as *Virtuvian Man* (Figure **7.2.4**). This famous masterpiece highlights the golden ratio in the proportions of an ideal body shape.

The golden ratio is approximated in several physical measurements of the human body and parts exhibiting the golden ratio are simply called golden. The ratio of a person’s height to the length from their belly button to the floor is ϕ or approximately 1.618. The bones in our fingers (excluding the thumb), are golden as they form a ratio that approximates ϕ. The human face also includes several ratios and those faces that are considered attractive commonly exhibit golden ratios.

Example 7.2.1: Using Golden Ratio and a Person’s Height

If a person’s height is 5 ft 6 in, what is the approximate length from their belly button to the floor rounded to the nearest inch, assuming the ratio is golden?

Answer

Step 1: Convert the height to inches

$$\textbf{5 ft 6 in} = \textbf{66 in}$$

Step 2: Calculate the length from the belly button to the floor, L.

$$66/L = 1.618$$
$$L = 40.8 \text{ in}$$

The length from the person’s belly button to the floor would be approximately 41 in.

Your Turn 7.2.1

If a person’s height is 6 ft 2 in, what is the approximate length from their belly button to the floor rounded to the nearest inch if the ratio is golden?

Fibonacci Sequence and Application to Nature



Figure **7.2.4**: Rose petals appear in a Fibonacci spiral. (credit: “rilke4” by monchoohcnom/Flickr, Public Domain Mark 1.0)

The Fibonacci sequence can be found occurring naturally in a wide array of elements in our environment from the number of petals on a rose flower to the spirals on a pine cone to the spines on a head of lettuce and more. The Fibonacci sequence can be found in artistic renderings of nature to develop aesthetically pleasing and realistic artistic creations such as in sculptures, paintings, landscape, building design, and more. It is the sequence of numbers beginning with 1, 1, and each subsequent term is the sum of the previous two terms in the sequence (1, 1, 2, 3, 5, 8, 13, …).

The petal counts on some flowers are represented in the Fibonacci sequence. A daisy is sometimes associated with plucking petals to answer the question “They love me, they love me not.” Interestingly, a daisy found growing wild typically contains 13, 21, or 34 petals and it is noted that these numbers are part of the Fibonacci sequence. The number of petals aligns with the spirals in the flower family.

Example **7.2.2**: Applying the Fibonacci Sequence to Rose Petals

Suppose you were creating a rose out of icing, assuming a Fibonacci sequence in the petals, how many petals would be in the row following a row containing 13 petals?

Answer

The number of petals on a rose is often modeled with the numbers in the Fibonacci sequence, which is 1, 1, 2, 3, 5, 8, 13,…, where the next number in the sequence is the sum of $8 + 13 = 21$. There would be 21 petals on the next row of the icing rose.

Your Turn **7.2.2**

If a circular row on a pinecone contains 21 scales and models the Fibonacci sequence, approximately how many scales would be found on the next circular row?

Golden Ratio and the Fibonacci Sequence Relationship

Mathematicians for years have explored patterns and applications to the world around us and continue to do so today. One such pattern can be found in ratios of two adjacent terms of the Fibonacci sequence.

Recall that the Fibonacci sequence = 1, 1, 3, 5, 8, 13,… with 5 and 8 being one example of adjacent terms. When computing the ratio of the larger number to the preceding number such as 8/5 or 13/8, it is fascinating to find the golden ratio emerge. As larger numbers from the Fibonacci sequence are utilized in the ratio, the value more closely approaches ϕ, the golden ratio.

Example 7.2.3: Finding Golden Ratio in Adjacent Fibonacci Terms

The 24th Fibonacci number is 46,368 and the 25th is 75,025. Show that the ratio of the 25th and 24th Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.

Answer

$$\frac{75,025}{46,368} = 1.618$$

The ratio of the 25th and 24th term is approximately equal to the value of ϕ rounded to the nearest thousandth, 1.618.

Your Turn 7.2.3

The 23rd Fibonacci number is 28,657 and the 24th is 46,368. Show that the ratio of the 24th and 23rd Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.



Figure 7.2.5: The pyramids of Giza in Egypt (credit: "Giza Pyramids" by Vincent Brown/Flickr, CC BY 2.0)

Golden Rectangles

Turning our attention to man-made elements, the golden ratio can be found in architecture and artwork dating back to the ancient pyramids in Egypt (Figure 13.6) to modern-day buildings such as the UN headquarters. The ancient Greeks used golden rectangles —any rectangles where the ratio of the length to the width is the golden ratio—to create aesthetically pleasing as well as solid structures, with examples of the golden rectangle often being used multiple times in the same building such as the Parthenon, which is shown in Figure 7.2.1. Golden rectangles can be found in twentieth-century buildings as well, such as the Washington Monument.

Looking at another man-made element, artists paintings often contain golden rectangles. Well-known paintings such as Leonardo da Vinci's *The Last Supper* and the *Vitruvian Man* contain multiple golden rectangles as do many of da Vinci's masterpieces.

Whether framing a painting or designing a building, the golden rectangle has been widely utilized by artists and are considered to be the most visually pleasing rectangles.

Example 7.2.4: Finding Golden Rectangle in Frames

A frame has dimensions of 8 in by 6 in. Calculate the ratio of the sides rounded to the nearest thousandth and determine if the size approximates a golden rectangle.

Answer

8/6 = 1.333; A golden rectangle's ratio is approximately 1.618. The frame dimensions are close to a golden rectangle.

Your Turn 7.2.4

A frame has dimensions of 10 in by 8 in. Calculate the ratio of the sides rounded to the nearest thousandth and determine if the size approximates a golden rectangle.

People in Mathematics: M.C. Escher

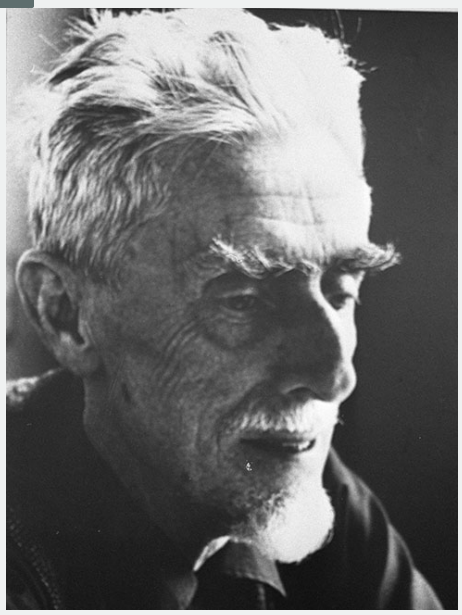


Figure 7.2.6: M.C. Escher (credit: "M.C. Escher" by Hans Peters (ANEFO)/Dutch National Archives, CC0 1.0 Public Domain)

Mauritis Cornelis Escher was a Dutch-born world-famous graphic artist and his work can be found in murals, stamps, wallpaper designs, illustrations in books, and even carpets. Over his lifetime, M.C. Escher created hundreds of lithographs and wood engravings as well as more than 2,000 sketches.

Escher's work is characterized with the infusion of geometric designs that obey most of the mathematical rules. If you study his work closely, you can see where he breaks a mathematical relationship to create famous illusions such as soldiers marching around the top of a square turret where the soldiers appear to be always going uphill but are contained on a single set of stairs in a square. Look closely and the golden ratio as well as golden rectangles abound in Escher's work.

Like many famous people, M.C. Escher did not find success in his early school years. Before finding success, Escher failed his final school exam and quit a short stint in architecture. Finding a graphic arts teacher who recognized Escher's talent, Escher completed art school and enjoyed traveling through Italy, where he found much of his inspiration for his work.

Check Your Understanding

1. What is the value of the golden ratio to the nearest thousandth?
2. What are the first 10 terms of the Fibonacci sequence?
3. What is a golden rectangle?

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7.2.1: Exercises

Exercise 7.2.1.1

A person's height is 5 ft 2 in. What is the approximate length from their belly button to the floor rounded to the nearest inch?

Exercise 7.2.1.2

A person's height is 6 ft 3 in. What is the approximate length from their belly button to the floor rounded to the nearest inch?

Exercise 7.2.1.3

A person's length from their belly button to the floor is 3 ft 11 in. What is the person's approximate height rounded to the nearest inch?

Exercise 7.2.1.4

A person's length from their belly button to the floor is 58 in. What is the person's approximate height rounded to the nearest inch?

Exercise 7.2.1.5

The spikes on a pineapple mirror the Fibonacci sequence. If a row on a pineapple contains five spikes, approximately how many spikes would be found on the next larger row of spikes?

Exercise 7.2.1.6

The leaves on a plant mirror the Fibonacci sequence. If a set of leaves on the plant contains 5 leaves, how many leaves would be found on the previous smaller set of leaves?

Exercise 7.2.1.7

The spines on a head of lettuce mirror the Fibonacci sequence. If a head of lettuce contains 13 spines, approximately how many spines would be found on the next inside layer?

Exercise 7.2.1.8

The seeds on a sunflower mirror the Fibonacci sequence. If a circular layer on the sunflower contains 55 seeds, approximately how many seeds would be found on the next larger circular layer?

Exercise 7.2.1.9

The segments on a palm frond mirror the Fibonacci sequence. If a palm frond contains 89 segments, approximately how many segments would be found on the next larger palm frond?

Exercise 7.2.1.10

The 19th term of the Fibonacci sequence is 4,181 and the 20th term is 6,765. What is the 21st term of the sequence?

Exercise 7.2.1.11

The 23rd term of the Fibonacci sequence is 28,657 and the 24th term is 46,368. What is the 22nd term of the sequence?

Exercise 7.2.1.12

The 18th term of the Fibonacci sequence is 2,584 and the 20th term is 6,765. What is the 19th term of the sequence?

Exercise 7.2.1.13

The 25th term of the Fibonacci sequence is 75,025 and the 20th term is 6,765. What is the 24th term of the sequence?

Exercise 7.2.1.14

The 10th Fibonacci number is 55 and the 11th is 89. Show that the ratio of the 11th and 10th Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.

Exercise 7.2.1.15

The 23rd Fibonacci number is 28,657 and the 24th is 46,368. Show that the ratio of the 24th and 23rd Fibonacci numbers is approximately ϕ. Round your answer to the nearest ten-thousandth.

Exercise 7.2.1.16

The 22nd Fibonacci number is 17,711 and the 21st is 10,946. Show that the ratio of the 22nd and 21st Fibonacci numbers is approximately ϕ. Round your answer to the nearest ten-thousandth.

Exercise 7.2.1.17

The 16th term of the Fibonacci sequence is 987. Use the approximate value of ϕ of 1.618 to estimate the 15th term. Round your answer to the nearest whole number.

Exercise 7.2.1.18

The 26th term of the Fibonacci sequence is 121,393. Use the approximate value of ϕ of 1.618 to estimate the 25th term. Round your answer to the nearest whole number.

Exercise 7.2.1.19

A frame has dimensions of 20 in by 24 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

Exercise 7.2.1.20

A fence has dimensions of 75 in by 45 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

Exercise 7.2.1.21

A frame has a length of 50 in. Calculate the width rounded to the nearest inch if the frame is to be a golden rectangle.

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7.3.1: Key Terms

13.1 Math and Art

- golden ratio
- ϕ
- Fibonacci sequence
- golden rectangle

13.2 Math and the Environment

- greenhouse gas
- CO_2 emissions
- watt
- kilowatt (kW)

13.3 Math and Medicine

- concentrations/dosages of drugs
- mathematical modeling

13.4 Math and Music

- frequency
- pitch
- Hertz
- decibel
- half-step
- whole step
- flat
- sharp
- octave

13.5 Math and Sports

- data analytics
- seeding

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7.3.2: Key Concepts

13.1 Math and Art

- The golden ratio, ϕ, can be found in nature, and the relationship is often associated with beauty and balance.
- The Fibonacci sequence reflects a pattern of numbers that can be found in various places in nature. The sequence can be used to predict other values that follow the Fibonacci pattern.
- State some naturally occurring applications of the Fibonacci sequence.
- State some naturally occurring applications of the golden ratio.
- Determine if a rectangle is golden.
- State some artistic applications of the golden rectangle.

13.2 Math and the Environment

- Recognize how water conservation by one person, family, community, or nation can positively impact the world's freshwater supply.
- Recall components from the history of solar use by mankind.
- Calculate electrical demand given watts.
- Recognize advantages of residential solar power.

13.3 Math and Medicine

- Compute volumes of prescription drugs in liquid and pill form.
- Validate the effectiveness of a new drug.
- Mathematical modeling can be used to describe and track the spread of a virus.

13.4 Math and Music

- As frequency of a sound increases, the pitch of the sound increases.
- Hertz is a unit of measurement for frequency.
- Decibel is a unit of measurement for the intensity of sound.
- Half-steps and whole-steps describe one type of movement between two notes on a keyboard.
- Octaves are a collection of any 12 consecutive notes, which is a foundation in music theory.

13.5 Math and Sports

- Describe how a round-robin tournament is organized.
- Compute the number of games played in a round-robin tournament.
- Describe how a single-elimination tournament is organized.
- Compute the number of games played in a single-elimination tournament.

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7.3.3: Formula Review

13.2 Math and the Environment

$\mathbf{1{,}000/watts = time\ needed\ to}$

$\mathbf{watts/1{,}000 = kilowatt}$

13.3 Math and Medicine

$\mathbf{Volume\ needed = (medicine\ dosage\ required)\ (weight\ of\ drug\ by\ volume)}$.

13.5 Math and Sports

Number of games in a single round-robin tournament with n teams is $n(n-1)/2$.

Number of games in a single-elimination tournament with n teams is $(n-1)$.

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7.3.4: Projects

Lucas Sequence and Fibonacci Sequence

The Lucas numbers bear some similarity to the Fibonacci numbers and exhibit a stronger link to the golden ratio. Edouard Lucas is credited with naming the Fibonacci numbers and the Lucas numbers were so named in his honor. The Lucas numbers play a role in finding prime numbers that are utilized in encrypting data for actions such as using your debit card to obtain money at a cash machine or when making a credit card purchase for point of sale as well as when shopping online.

Complete the following questions to explore numbers in the Lucas sequence as well as their relationships to the numbers in the Fibonacci sequence.

1. Conduct an Internet search to find out what a Lucas number is and how the Lucas numbers are related to the Fibonacci numbers.
2. What are the first two numbers in the Lucas sequence?
3. Describe how the next number in the Lucas sequence is determined and compare this to how the next number is determined in the Fibonacci sequence.
4. Complete the following table listing the first 10 terms in the Fibonacci and Lucas sequence:

Term	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Fibonacci Numbers	**0**	**1**	**1**	**2**		**5**		**13**		**34**
Lucas Numbers			**3**		**7**		**18**		**47**	

- Interestingly, many patterns can be found in looking at the relationships in the Fibonacci and Lucas numbers. Look closely at the chart in question 3 to discover one such pattern. Observe the Fibonacci numbers in the third and fifth terms and compare with the Lucas number in the fourth term of the sequence. Describe the pattern found. Does this pattern continue in the table?
- Research the Fibonacci or the Lucas numbers to find an application in our world distinct from what has been shared in this project and section. Write a paragraph sharing the findings of your research.

Solar Array for a Residence

One of the first steps in adding solar to a residence is determining the size of a system to achieve the desired output. In this project, we will explore the solar needs of a residence and estimate needs of a solar array to supply electrical output to meet various percentages of electrical need.

Step 1: Obtain an electric bill from your apartment/home. Find the average monthly or yearly usage if listed or call the electric company to inquire. If an electric bill is not available, use the Internet to find an average monthly or yearly electric usage for your area.

Step 2: Determine a daily and hourly usage. Divide the average monthly usage by 30 or the yearly average by 365. Divide again by 24 to calculate an average hourly electric usage, which will yield the average kilowatt-hours for how much electrical power your is being utilized in an hour.

Step 3: Multiply your average hourly use (kilowatts) by 1,000 to convert to watts.

Step 4: Use the Internet to determine the average daily peak hours of sunlight where you live.

Step 5: Divide your average hourly watts (Step 3) by the average daily peak hours (Step 4) to calculate the average energy needed for a solar array to produce every hour.

Step 6: Determine the average energy needed in a solar array per hour to meet each of the following:

a. 100% coverage of average energy (Step 5)
b. 80% coverage of average energy
c. 50% coverage of average energy

Step 7: Using the values computed in Step 6, compute the residential savings based on an average cost of 12 cents per watt.

Vaccine Validation

Validation of vaccines is a topic that exploded in the news when the Covid-19 pandemic spread across the world. As governments and organizations looked for a vaccine to curb the spread and minimize the severity of infection, concern was expressed by some for what appeared to be a quick discovery for a Covid-19 vaccine.

Conduct an Internet search to explore the following questions. Pay special attention to the sources you select to ensure that they are credible sources.

1. Research the term *efficacy rates*. Express what efficacy means in your own words.
2. Using a minimum of two sources, compose a well-developed paragraph describing what validation of a vaccine means.
3. Using a minimum of two sources, compose a well-developed paragraph sharing the steps in validating a vaccine.
4. Using a minimum of two sources, compose a well-developed paragraph describing how a vaccine is determined to be validated.
5. Using your research for Questions 1–3, write a summary paragraph sharing your reflections on the validation of the Covid-19 vaccine or on validating a vaccine in general. What key components did you learn? What would you like to learn more about related to validating a vaccine?

Frequency and Ultrasonic Sounds

Ultrasonic sounds have been utilized for a variety of reasons, from purportedly repelling rodents and other animals as well as a variety of other applications. Using an Internet search, complete the following questions to explore some of these applications and examine the validity of various claims.

Repelling Insects, Rodents, and Small Animals

Some radio stations purport to play a high pitch sound dually with their music to aid in deterring insects and other annoying bugs to aid in providing a bug-reduced listening environment. To deter small rodents, some products claim to emit ultrasonic sounds that drive away mice and other similar pests.

1. Research the science behind ultrasonic pest controls, paying attention to the source of the information that you find. Compare the information found on advertisements, reviews, and scientific articles.
   a. What frequency ranges do ultrasonic pest deterrent devices utilize and how do these frequencies compare to the audible range that humans can hear?
   b. Write a short paragraph comparing the claims in the advertisements with independent reviews and scientific articles.
   c. What do you conclude about ultrasonic pest controls and why?

Disbursing Teenagers

Some business owners and communities have turned to products such as the “mosquito” sonic deterrent device to discourage groups of teenagers from loitering around storefronts and community landmarks, citing a public nuisance issue and public safety concerns.

2. Research the science behind ultrasonic deterrent devices as they apply to dispensing teenagers. Compare the information found on advertisements, reviews, and scientific articles.
   a. What frequency ranges do ultrasonic teenager deterrent devices utilize and how do these frequencies compare to the audible range that adults can hear?
   b. Write a short paragraph comparing the claims in advertisements with independent reviews and scientific articles.
   c. What are some of the ethical debates surrounding the use of ultrasonic teenager deterrent devices?
   d. What do you conclude about business owners or communities using ultrasonic teenager deterrent devices and why?

Jewelry Cleaner

Use of pastes and liquid chemicals to clean jewelry can be harsh on stones as well as metals. So how can we safely obtain the sparkling clean look at home that jewelry stores provide? Some would say the answer is to use an ultrasonic jewelry cleaner, but do these really work?

3. Research the science behind ultrasonic jewelry cleaners, including the phrase “cavitation process” in your search. Compare the information found on advertisements, reviews, and scientific articles.
    a. Write a short paragraph detailing how ultrasonic jewelry cleaners work and what role the cavitation process plays in the claims for ultrasonic cleaning.
    b. Do ultrasonic jewelry cleaners utilize low or high frequencies? How do these frequencies compare to the audible range that humans can hear?
    c. Write a short paragraph comparing the claims in the advertisements with independent reviews and scientific articles.
    d. What do you conclude about ultrasonic jewelry cleaners and why?

Specialized Ringtones

As the use of cell phones has become commonplace and families grow towards each member having their own cell phone, specialized ringtones have become popular and can aid in identifying who is calling just by the ringtone.

Ever hear of ringtones that can be heard by teens but often not their teachers? The banning of cell phone use by K–12 students during class time as been implemented across a wide array of schools and some students have purportedly found ways to get around teachers hearing a cell phone ring through the use of ultrasonic ringtones.

4. Research the science behind ultrasonic ringtones. Compare the information found on advertisements, reviews, and scientific articles.
    a. Do ultrasonic ringtones utilize low or high frequencies?
    b. Write a short paragraph detailing how ultrasonic ringtones work. How do these frequencies compare to the audible range that adults can hear? Include ages that are purported to hear and not hear the ringtones as well as the frequency ranges utilized.
    c. Write a short paragraph comparing the claims in the advertisements with independent reviews and scientific articles.
    d. What do you conclude about ultrasonic ringtones and why?

Streaming Services and Math

With the ability to stream music virtually anywhere you are, it is not surprising that Google Play Music, Apple Music, and a slew of other companies such as Spotify, Amazon Music, YouTube Music, Sound Cloud, Pandora, Deezer Music, Tidal, Napster, and Bandcamp have invested heavily to bring streaming service to users worldwide. Streaming services have expanded to offer virtually every genre of music with vast libraries to meet diverse user requests.

Considering all of the choices available for streaming music, there is a wide array of options for subscribing. Conduct an Internet research to review your current streaming choices, if any, and evaluate competitors’ products.

1. In this project, you will explore options for music streaming service subscriptions. Select a minimum of five streaming services listed above that you are not currently utilizing and determine the below components. Format your findings in an easy-to-read format such as a table similar to the one shown below.
    a. Monthly subscription cost
    b. Available features

Available Features	Service Choice 1:	Service Choice 2:	Service Choice 3:	Service Choice 4:	Service Choice 5:
Able to stream unlimited music					
Able to purchase individual songs					
Able to purchase individual whole albums					
Ability to create personalized play lists					
Ability to select specific songs to play					
Ability to listen					
Other =					

2. Premium cost per month, if available = _____
    a. Feature(s) offered with premium monthly subscription = _____
3. What is (are) your current music streaming services, if any?
    a. What is your current monthly service charge(s)?
    b. What features does your current streaming service offer?
4. Write a paragraph summarizing your findings and include if your current streaming service(s) meets your needs or if research for this project has you considering changing streaming services. Support your rationale.

Math and Baseball

Baseball is known to have one of the largest pools of statistics related to the game and its players. Managers, coaches, and pitchers study the statistics of the players on opposing teams to give their team an edge by knowing what pitches to throw for the best probability to be missed by a batter. In similar fashion, batters study pitchers' statistics to learn a pitcher's strength and how to predict what a pitcher will throw and how to best hit against a pitcher.

The three primary baseball statistics are batting average, home runs, and runs batted in (RBIs), which are the components of the title of Triple Crown winner that is awarded to players who dominate in these three areas. However, there is a wealth of other statistics to evaluate when studying the performance of a player.

Conduct an Internet search to research statistics and how they are calculated in the following categories:

Batting Statistics

1. There are about 30 batting statistics. Select a minimum of 10 batting statistics. Compose an organized list including the name of the statistic, abbreviation, explanation of what it represents, as well as how it is calculated.
   As an example: AB/HR represents *at bats per home run* and is calculated by the number of times a player is at bat divided by home runs.

Pitching Statistics

2. There are about 40 pitching statistics. Select a minimum of 10 pitching statistics. Compose an organized list including the name of the statistic, abbreviation, explanation of what it represents, as well as how it is calculated.
   As an example: K/9 represents *strikeouts per nine innings* and is calculated by the number of strikeouts times nine divided by the number of innings pitched.

Fielding Statistics

3. There are around 10 fielding statistics. Select a minimum of five fielding statistics. Compose an organized list including the name of the statistic, abbreviation, explanation of what it represents, as well as how it is calculated.
   As an example: FP represents *fielding percentage* is calculated by the number of total plays divided by the number of total chances.

Overall

4. Select a player from recent years to evaluate. The player can be one that you have followed, one from a favorite team, or any current player. Find the statistics shared in Questions 1–3 to use in evaluating the potential strengths and weaknesses of the player. Write a short paragraph analyzing your selected player, supported by the statistics from the answers to Questions 1–3.

Math and Fantasy Football

Fantasy football offers spectators an added dimension to football season with a competitive math-based game where the active components are real-life players in the current season. For clarity in this exercise, the actual fantasy football players will be denoted as FFP and actual professional team members will be denoted as players.

While some fantasy football leagues have slightly different setups or scoring systems, most share some common elements.

Often using a lottery system to determine who picks first, second, and so on, FFPs select 15 current players to comprise their personal fantasy football team. The players selected can be from any professional teams and a FFP can utilize any team recognized by their league. FFPs can elect to keep the same players on their team for the whole league play or trade for any player not selected by another FFP in their league.

At the start of each week during football season, each fantasy football player selects their roster of actual players to comprise their roster of starting players. Typically, a starting roster consists of the following players:

Number to Select	Position Abbreviation	Position Title
1	QB	Quarterback
1	K	Kicker
1	TE	Tight End
2	RB	Running Back
1	D/ST	Defense
2	WR	Wide receiver
1	RB or WR	Flex

As actual professional games are played, points are tallied based on your league's scoring system. The points the team members on your starting roster make during the week are computed and whichever FFP has the highest score for the week wins that week.

The FFPs with the best records of wins versus losses enters fantasy football playoffs to determine the ultimate league champion and collects the pot.

The above overview of fantasy football describes the basic game play. The fun comes in understanding and analyzing the math behind the scoring.

1. Talk to people you know who have played fantasy football in the past and interview them. Inquire about how they select players and how league(s) they have played in have computed the weekly scoring.
   a. Was a league manager recruited to manage the record keeping of scores?
   b. Was an online scoring system utilized or did the fantasy football use a streamlined or specialized scoring system?
   c. What strategies were used to select the team of 15 players and who would be on the weekly starting roster?
   d. Were any online resources utilized to aid in choosing which players to select?

e. How was math utilized to select team members or who to place as a starter each week? Be as specific as possible, citing examples when available.

2. Conduct an Internet research to find two different resources that a FFP could utilize when selecting players for their team or for their starting roster. Describe the math involved in the resources.
3. Conduct an Internet research to learn more about the scoring utilized by fantasy football leagues. Write a short paragraph sharing how the use of math and knowledge of football statistics is used and how they are calculated to aid a FFP in selecting their team and starting roster.

Math and Hockey

Hockey is full of math from obvious components such as scoring and statistics to the shape of the rink and the angles involved in puck movement.

Collegiate and professional hockey games are 60 minutes long and are divided into three periods of 20 (60/3) minutes each. At any one time, there are five players and one goalie on the ice for each team. If a player is called on a penalty and is placed in the penalty box, that team now has four players, which is 20% less players on the ice competing against five opponents. In some instances, a team may have two players in the penalty box at one time, resulting in three players, or 40% less players on the ice compared to a full team. Being one player down for a 2-minute penalty or potentially 5 minutes for a major penalty leads to an imbalance on the ice and calls for a quick change of offense and defense strategy.

Rink Composition—North American

An ice rink is comprised of various geometrical shapes, each with precise dimensions.

1. Research the dimensions of a hockey rink and draw an accurate scale model on graph paper. Be sure to include the scale for your model and indicate all units.
2. List the shapes and the numbers of each kind of hockey rink. You should find four different shapes, some with multiple dimensions.
3. Describe the shape and dimensions of a hockey puck using geometric vocabulary.

Statistics

Using an Internet search, select two top hockey players from the same league to answer the following questions:

4. Write a paragraph sharing a minimum of five statistics for each player you have selected. Describe how each of the statistics are calculated and what each statistic means.
5. Write a paragraph comparing the two players and determine who you believe is the better player. Support your choice.

Scoring

The basics of scoring in ice hockey is simple, the team with the most goals is the winner. But, how to score the most goals involves much math!

6. Research one of the following components involved in hockey puck movement on the ice and write a paragraph summarizing your findings. Be specific and detailed in your summary.
    a. Angles
    b. Velocity vectors
    c. Angle of incident and angle of return
    d. Speed and acceleration (player as well as puck movement)

Math and Soccer

As the world's most popular sport, you'll be excited to confirm that soccer is full of mathematics ranging from scoring and statistics to footwork, angles, and field shape.

Soccer requires understanding of mathematical concepts and equations as well as skill, fitness, and game knowledge. One such example is angles, which you all will remember from your geometry class. While players are not carrying protractors and measuring angles during play, mental calculation of angles is a constant in any successful player's thinking. A goalie is not

physically able to cover the entire open net region and a player must calculate an angle to kick the ball consistent with the net opening while predicting the ability of the goalie to stop the ball from entering the net.

1. Research angles as they apply to soccer play. Provide two examples of different plays indicating angle of a player's body, angle of foot striking the ball, and angle to the net. Include relevant dimensions. Adding a diagram that may aid in clarity is an option.
2. Draw a scale model of a soccer field including dimensions with labels.

Using an Internet search, select two top soccer players from the same league to answer the following questions:

3. Write a paragraph sharing a minimum of five statistics for each player you have selected. Describe how each of the statistics are calculated and what each statistic means.
4. Write a paragraph comparing the two players and determine who you believe is the better player. Support your choice.

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7.3.5: Chapter Review

Section 13.1 Math and Art

Exercise 7.3.5.1

A person's length from their belly button to the floor is 25 in. What is the person's approximate height rounded to the nearest inch?

Exercise 7.3.5.2

The spiral seeds on a fruit mirror the Fibonacci sequence. If a row of the spiral contains 34 leaves, how many leaves would be found on the next larger spiral of seeds?

Exercise 7.3.5.3

The seeds on a sunflower mirror the Fibonacci sequence. If a circular layer on the sunflower contains 89 seeds, approximately how many seeds would be found on the next larger circular layer?

Exercise 7.3.5.4

A term of the Fibonacci sequence is 17,711 and the previous term is 10,946. What is the following term of the sequence?

Exercise 7.3.5.5

The 35th term of the Fibonacci sequence is 9,227,465. Use the approximate value of ϕ of 1.618 to estimate the 34th term. Round your answer to the nearest whole number.

Exercise 7.3.5.6

A frame has dimensions of 16 in by 24 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

Section 13.2 Math and the Environment

Exercise 7.3.5.7

An average toilet uses 5 gal of water per flush. A high-efficiency toilet uses about 1.25 gal per flush. How much water would a household save in a month (30 days) if the toilet was flushed 6 times a day?

Exercise 7.3.5.8

When washing dishes, leaving the faucet running utilizes about 15 gal every 5 minutes, whereas filling the sink and turning the faucet off except to rinse the dishes uses about 6 gal for washing a dishpan-sized load of dishes. How much water is saved by not leaving the faucet running if it takes 10 minutes to wash a dishpan-sized load of dishes each day for a year containing 30 days in each month?

Exercise 7.3.5.9

Leaving the water running when washing your hands consumes about 4 gal of water, whereas turning the water off when lathering reduces the water used to 1 gal. How much water is saved in an apartment of 4 people in 6 months (180 days) if each person averages washing their hands twice a day?

Exercise 7.3.5.10

How long would it take for an energy-saving light bulb to consume 1 kW of power if the bulb is rated at 2.5 W? Round your answer to the nearest hour.

Exercise 7.3.5.11

How long would it take for a 60 W light bulb to consume 1 kW of power? Round your answer to the nearest hour.

Exercise 7.3.5.12

A portable fan uses about 40 W. How many kilowatt-hours are needed to run the fan during a 10-day trip if the fan is run for an average of 3.5 hours a day?

Section 13.3 Math and Medicine

Exercise 7.3.5.13

In a 2% solution of lidocaine, how many milligrams are there per milliliter?

Exercise 7.3.5.14

You are prescribed a nebulizer solution to aid with asthma. If the solution is 0.7% w/v, what volume is needed to deliver a 7 mg dose?

Exercise 7.3.5.15

A prescription calls for a patient to receive 17 mg of a drug to be taken in oral form. If the pills are available in 3.4 mg, how many pills will the patient need daily?

Exercise 7.3.5.16

A prescription calls for a patient to receive 9 mg daily of a drug to be taken daily for 10 days. If the pills are available in 2.25 mg, how many pills are needed to fill the prescription?

Exercise 7.3.5.17

A patient is prescribed 2 mg/kg of a drug to be delivered daily intramuscularly, divided into three doses. If the patient weighs 30 kg, how many milligrams of the drug is needed for each dose?

Exercise 7.3.5.18

A patient is prescribed 0.75 mg/kg of a drug to be delivered intramuscularly once per day. If the drug is available in 2.5 mg/mL and the patient weighs 63 kg, how many milliliters of the drug should be given per day?

Section 13.4 Math and Music

Exercise 7.3.5.19

Select the most representative sound registering 50 dB: airport gate noise, light rain on a car, dirt bike racing.

Exercise 7.3.5.20

Select the most representative sound registering 100 dB: city traffic when in a car with windows rolled up, children playing on a school playground, professional sporting event such as football.

Exercise 7.3.5.21

What frequency range can teenagers typically hear but adults over 20 are often not able to hear?

Exercise 7.3.5.22

What is three whole steps up from $G^{\#}$?

Exercise 7.3.5.23

What is one half-step down from F?

Exercise 7.3.5.24

What is an octave above D_3?

Exercise 7.3.5.25

Given that the frequency of $\mathbf{A_4 = 440\ Hz}$, what is the frequency of $\mathbf{A_6}$?

Section 13.5 Math and Sports

For the following exercises, compute how many games would be played in the style of tournament and number of teams given in each question. Assume all tournaments are single round-round or single-elimination.

Exercise 7.3.5.26

How many games would be played with 10 teams using a round-robin tournament?

Exercise 7.3.5.27

How many games would be played with 25 teams using a round-robin tournament?

Exercise 7.3.5.28

How many games would be played with 12 teams using a round-robin tournament?

Exercise 7.3.5.29

How many games would be played with 4 teams using a single-elimination tournament?

Exercise 7.3.5.30

How many games would be played with 9 teams using a single-elimination tournament?

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7.3.6: Chapter Test

Exercise 7.3.6.1

A person's length from their belly button to the floor is 3 ft 8 in, assuming the ratio is golden. What is the person's approximate height rounded to the nearest inch?

Exercise 7.3.6.2

The leaves on an invasive weed mirror the Fibonacci sequence. If a set of leaves on the plant contains 13 leaves, how many leaves would be found on the previous smaller set of leaves?

Exercise 7.3.6.3

The 14th term of the Fibonacci sequence is 377 and the 15th term is 610. What is the 13th term of the sequence?

Exercise 7.3.6.4

A fence has dimensions of 40 in by 25 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

Exercise 7.3.6.5

An average toilet uses 5 gal of water per flush. A high-efficiency toilet uses about 1.25 gal per flush. How much water would a dorm save in an academic year (32 weeks) if the toilet was flushed 22 times a day?

Exercise 7.3.6.6

When washing dishes, leaving the faucet running utilizes about 15 gal every 5 minutes, whereas filling the sink and turning the faucet off except to rinse the dishes uses about 5 gal for washing a dishpan-sized load of dishes. How much water is saved by not leaving the faucet running if it takes 10 minutes to wash a dishpan-sized load of dishes each day for a week?

Exercise 7.3.6.7

How long would it take for a 75 W light bulb to consume 1 kW of power? Round your answer to the nearest hour.

Exercise 7.3.6.8

A phone charger uses about 20 W. How many kilowatt-hours are needed to use the charger during a month if the charger is used for an average of 1.5 hours a day?

Exercise 7.3.6.9

A pharmacy has a 1 L bottle of 0.9% solution of sodium chloride. How many grams of sodium chloride are contained in the bottle?

Exercise 7.3.6.10

A prescription calls for a patient to receive 5.1 mg of a drug to be taken in oral form. If the pills are available in 1.7 mg, how many pills will the patient need daily?

Exercise 7.3.6.11

A prescription calls for a patient to receive 7 mg daily of a drug to be taken daily for 7 days. If the pills are available in 1.75 mg, how many pills are needed to fill the prescription?

Exercise 7.3.6.12

A patient is prescribed 3.8 mg/kg of a drug to be delivered daily intramuscularly, divided into two doses. If the patient weighs 20 kg, how many milligrams of the drug are needed for each dose?

Exercise 7.3.6.13

Select the most representative decibel value for a college classroom with students working in groups: 10 dB, 30 dB, 50 dB.

Exercise 7.3.6.14

What is typically the top frequency that adults over 20 can hear: 17 Hz, 170 Hz, 1,700 Hz, 17,000 Hz?

Exercise 7.3.6.15

What note is three whole steps down from $F^{\#}$?

Exercise 7.3.6.16

Given that the approximate frequency of $C_4{}^{\#} = 277\ \text{Hz}$, what is the approximate frequency of $C_2{}^{\#}$ rounded to the nearest whole number?

For the following exercises, compute how many games would be played in the style of tournament and number of teams given in each question. Assume all tournaments are single round-robin or single-elimination.

Exercise 7.3.6.17

How many games would be played with 22 teams using a round-robin tournament?

Exercise 7.3.6.18

How many games would be played with 7 teams using a round-robin tournament?

Exercise 7.3.6.19

How many games would be played with 8 teams using a single-elimination tournament?

Exercise 7.3.6.20

How many games would be played with 17 teams using a single-elimination tournament?

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CHAPTER OVERVIEW

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8.1: Introduction

The topic of money management is a broad and sometimes complex one. Ultimately, personal money management involves managing both our debt and also our savings and investments.

In 2021, the average American had consumer debt balance of $96,371. Nearly $100,000 per person. And less than 25% of Americans are debt free. Consumer debt can include mortgages, credit cards, as well as student loans. A key question all consumers should consider is how to manage debt and not become overburdened by it. The first step is to create a budget, which puts earnings into perspective, indicating what we can, and cannot, afford. A budget also entails setting aside certain funds for savings and investment, which help us achieve our short- and long-term goals.



Figure **8.1.1** Financial health helps you realize your goals. (credit: modification of work "Budget and Bills" by Alabama Extension/Flickr, Public Domain)

Creating a budget requires an understanding of how money—debt and savings—works. Initially, percentages and interest need to be understood. They drive most of what happens with debt and savings. With that understanding, discussions of buying a house, a car, or incurring credit card debt can be addressed from a financial perspective. All the while, retirement is waiting. Preparing for retirement involves saving and saving earlier rather than later. The power of compound interest is on full display when saving early.

This chapter covers some of the basics of money management: percentages, interest, budgeting, debt (student loans, mortgage, car, credit cards), savings, investments, and taxes.

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8.2: Understanding Percent

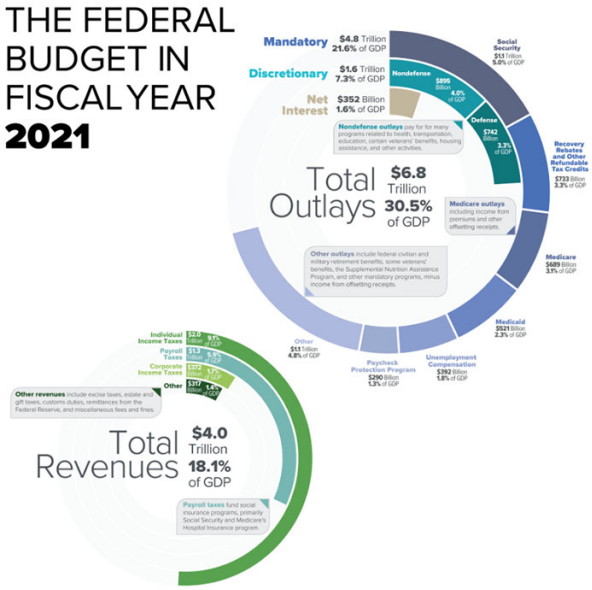


Figure **8.2.1**: The federal budget describes how money is spent and how money is earned. (credit: "Breakdown of revenues and outlays in 2021 US Federal budget" Wikimedia Commons, Public Domain)

Learning Objectives

1. Define and calculate percent.
2. Convert between percent, decimal, and fractional values.
3. Calculate the total, percent, or part.
4. Solve application problems involving percents.

In 2020, the U.S. federal government budgeted $3.5 billion for the National Park Service, which appears to be a very large number (and is!) and a large portion of the total federal budget. However, the total outlays from the U.S. federal government in 2020 was $6.6 trillion. So, the amount budgeted for the National Park Service was less than one-tenth of 1 percent, or 1/10%, of the total outlays. This **percent** describes a specific number. Understanding that ratio puts the $3.5 billion budgeted to the National Park Service in perspective.

This chapter focuses on percent as a primary tool for understanding money management. The interest paid on debt, the interest earned through investments, and even taxes are entirely determined using percent. This section introduces the basics of working with this invaluable tool.

Define and Calculate Percent

The word percent comes from the Latin phrase per centum, which means "by the hundred." So any percent is a number divided by 100. Changing a percent to a fraction is to write the percent in its fractional form. To write $n\%$ in its fractional form is to write the percent as the fraction $\frac{n}{100}$.

Checkpoint

A percent need not be an integer and does not have to be less than 100.

Example 8.2.1: Rewriting a Percent as a Fraction

Rewrite the following as fractions:

1. 18%

2. 84%
3. 38.7%
4. 213%

Answer

1. Using the definition and n = 18, 18% in fractional form is $\frac{18}{100}$.
2. Using the definition and n = 84, 84% in fractional form is $\frac{84}{100}$.
3. Using the definition and n = 38.7, 38.7% in fractional form is $\frac{38.7}{100}$.
4. Using the definition and n = 213, 213% in fractional form is $\frac{213}{100}$.

Your Turn 8.2.1

Rewrite the following as fractions:

1. 3%
2. 94%
3. 67.2%
4. 670%

Convert Between Percent, Decimal, and Fractional Values

When any calculation with a percent is to be performed, the form of the percent must be changed, either to its fractional form or its **decimal form**. We can change a percent into decimal form by dividing the percent by 100 and representing the result as a decimal.

FORMULA

The decimal form of n% is found by calculating the decimal value of semanticsn÷100.

Example 8.2.2: Converting a Percent to Decimal Form

Convert the following percents to decimal form:

1. 17%
2. 7%
3. 18.45%

Answer

1. To convert 17% to its decimal form, divide 17 by 100. This moves the decimal two places to the left, resulting in 0.17. The decimal form of 17% is 0.17.
2. To convert 7% to its decimal form, divide 7 by 100. This moves the decimal two places to the left, resulting in 0.07. The decimal form of 7% is 0.07.
3. To convert 18.45% to its decimal form, divide 18.45 by 100. This moves the decimal two places to the left, resulting in 0.1845. The decimal form of 18.45% is 0.1845.

Your Turn 8.2.2

Convert the following percents to decimal form:

1. 9%
2. 24%
3. 2.18%

You should notice that, to convert from percent to decimal form, you can simply move the decimal two places to the left without performing the division.

FORMULA

To convert the number x from decimal form to percent, multiply x by 100 and place a percent sign, %, after the number, $(x \times 100)\%$.

Example 8.2.3: Converting the Decimal Form of a Percent to Percent

Convert each of the following to percent:

1. 0.34
2. 4.15
3. 0.0391

Answer

1. Using the formula and x = 0.34, we calculate $(0.34 \times 100)\%$, which gives us 34%.
2. Using the formula and x = 4.15, we calculate $(4.15 \times 100)\%$, which gives us 415%.
3. Using the formula and x = 0.0391, we calculate $(0.0391 \times 100)\%$, which gives us 3.91%.

Your Turn 8.2.3

Convert the following to percent:

1. 0.41
2. 0.02
3. 9.2481

You should notice that, to convert from decimal form to percent form, you can simply move the decimal two places to the right without performing the multiplication.

Calculate the Total, Percent, or Part

The word “of” is used to indicate multiplication using fractions, as in “one-fourth of 56.” To find “one-fourth of 56” we would multiply 56 by one-fourth. We can think of percents as fractions with a specific denominator—100. So, to calculate “25% of 52,” we multiply 52 by 25%. But, first we need to convert the percent to either fractional form (25/100) or decimal form. Using the decimal form of 25% we have 0.25 × 52, which equals 13.

In this problem, 52 is the **total** or **base**, 25 is the **percentage**, and 13 is **the percentage of** 52, or the **part** of 52. This is sometimes referred to as the **amount**.

FORMULA

The mathematical formula relating the total (base), the percent in decimal form, and the part (amount) is $part =$ percent×totalpart=percent×total, or, $amount =$percent×baseamount=percent×base.

Checkpoint

In all calculations, the percent is expressed in decimal form.

Knowing any two of the values in our formula allows us to calculate the third value. In the following example, we know the total and the percent, and are asked to find the percentage of the total.

Example 8.2.4: Finding the Percent of a Total

1. Determine 70% of 3,500
2. Determine 156% of 720

Answer

1. The total is $x = 3,500$, and the percent is $r = 70$. The decimal form of 70% is 0.70 . To find the part, or percent of the total, substitute those values into the formula and calculate.

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ &= 0.70 \times 3500 \\ &= 2450 \end{aligned}$$

From this, we say that 70% of 3,500 is 2,450 .

2. The total is $x = 720$, and the percent is $r = 156$. The decimal form of 156% is 1.56 . To find the part, or percent of the total, substitute those values into the formula and calculate.

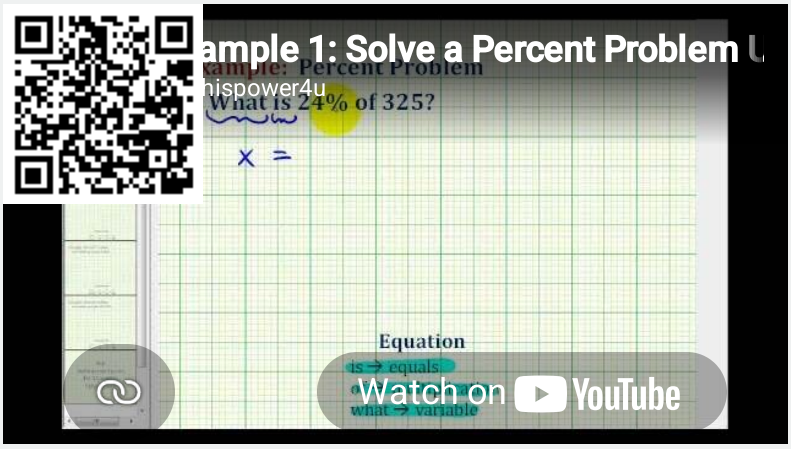
$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ &= 1.56 \times 720 \\ &= 1,123.2 \end{aligned}$$

From this, we say that 156% of 720 is $1,123.2$.

Your Turn 8.2.4

1. Determine 26% of 1,300.
2. Determine 225% of 915.

Video



Finding Percent of a Total

In the previous example, we knew the total and the percent and found the part using our formula. We may instead know the percent and the part, but not the total. We can use our formula again to solve for the total.

Example 8.2.5: Finding the Total from the Percent and the Part

1. What is the total if 35% of the total is 70?
2. What is the total if 10% of the total is 4,000?

Answer

1. **Step 1**: The percent is 35 , which in decimal form is 0.35 . We were given that 35% of the total is 70 , so the part is 70 . We are to find the total. Substituting into the formula, we have

<

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 70 &= 0.35 \times \text{total} \end{aligned}$$

Step 2: To find the total, we solve the equation for the total.

$$\begin{aligned} 70 &= 0.35 \times \text{total} \\ \frac{70}{0.35} &= \frac{0.35 \times \text{total}}{0.35} \\ 200 &= \frac{0.35 \times \text{total}}{0.35} \\ 200 &= \text{total} \end{aligned}$$

From this we see that 200 is the total, or, that **35%** of 200 is 70.

2. **Step 1**: The percent is 10 , which in decimal form is 0.1 . We were given that **10%** of the total is 4,000 , so the part is 4,000 . Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 4,000 &= 0.1 \times \text{total} \end{aligned}$$

Step 2: To find the total, we solve the equation for the total.

Misplaced &

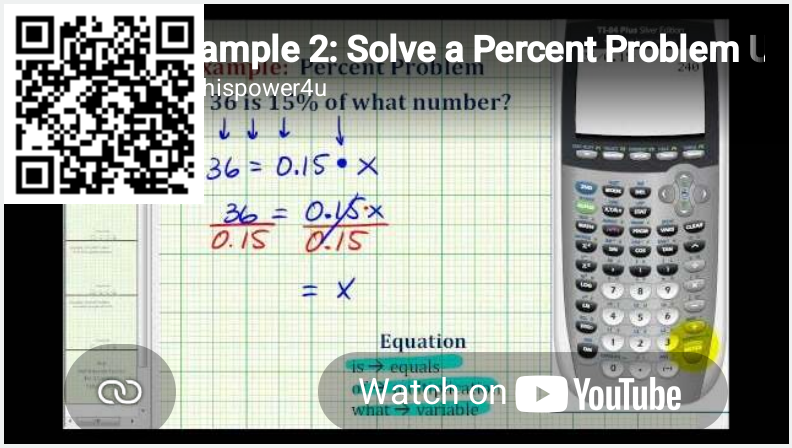
From this we see that 40,000 is the total, or that **10%** of 40,000 is 4,000 .

Your Turn 8.2.5

What is the total if 18% of the total is 45?

What is the total if 15% of the total is 900?

Video



Finding the Total from the Percent and the Part

Similarly, the percent can be found if the total and the percent of the total (the part) are known. This will result in the decimal form of the percent, so it must be converted to percent form.

Example 8.2.6: Finding the Percent from the Total and the Part

1. What percent of 500 is 175?
2. What percent of 228 is 155?

Answer

1. Step 1: The total is 500 , the percent of the total is 175 . Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 175 &= \text{percent} \times 500 \end{aligned}$$

Step 2: To find the percent, we solve the equation for the percent.

$$\begin{aligned} 175 &= \text{percent} \times 500 \\ \frac{175}{500} &= \frac{\text{percent} \times 500}{500} \\ 0.35 &= \frac{\text{percent} \times 500}{500} \\ 0.35 &= \text{percent} \end{aligned}$$

We see the percent in decimal form is 0.35 . Converting from the decimal form yields 35%. We say that 175 is 35% of 500 .

2. Step 1: The total is 228 , the percent of the total is 155 . Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 155 &= \text{percent} \times 228 \end{aligned}$$

Step 2: To find the percent, we solve the equation for the percent.

$$\begin{aligned} 155 &= \text{percent} \times 228 \\ \frac{155}{228} &= \frac{\text{percent} \times 228}{228} \\ 0.6798 &= \frac{\text{percent} \times 228}{228} \\ 0.6798 &= \text{percent} \end{aligned}$$

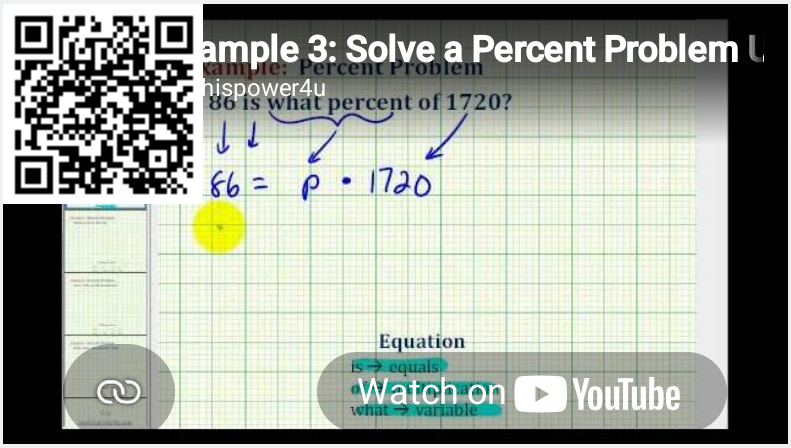
We see the percent is 0.6798 (rounded to four decimal places). Converting from the decimal form yields 67.98%. We say that 155 is 67.98% of 228.

Your Turn 8.2.6

Find the percent in the following:

1. Total is 40, percent of the total is 25
2. Total is 730, percent of the total is 292

Video



Finding the Percent When the Total and the Part Are Known

Solve Application Problems Involving Percents

Percents are frequently used in finance, research, science experiments, and even casual conversation. Understanding these types of values helps when consuming media or discussing finances, for instance. Effectively working with and interpreting numbers and percents will help you become an informed consumer of this information.

In most cases, working through what is presented requires you to identify that you are indeed working with a question of percents, which two of the three values that are related through percents are known, and which of the three values you need to find.

Example 8.2.7: Retention Rate at College

Justine applies to a medium size university outside her hometown and finds out that the retention rate (percent of students who return for their sophomore year) for the 2021 academic year at the university was 84%. During a visit to the registrar's office, she finds out that 1,350 people had enrolled in academic year 2021. How many students from the academic year 2021 are returning for the 2022 academic year?

Answer

The percent of students who will return for the 2022 academic year (the retention rate) is 84%. The total number of students who enrolled in the 2021 academic year was 1,350. This means the percent is known and the total is known. From this, we can determine the number of students who will return (percent of the total) for the 2022 academic year using the formula

$$\text{part} =$$

percent×totalpart=percent×total. Substituting into the formula and calculating, we find that the number of students that are returning is

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ &= 0.84 \times 1{,}350 \\ &= 1{,}134 \end{aligned}$$

So 1,134 students will return for the 2022 academic year.

Your Turn 8.2.7

Harris works the bookstore in their hometown. During one particular day, the store had total sales of $1,765, of which Harris sold 30%. What were Harris's total sales that day?

Example 8.2.8: Percent of Chemistry Majors

In this situation, the percent is to be determined. We know the total number of students, 45 , and the part of the students that are chemistry majors, 18 . Using that information and the formula

$$\text{part} = \text{percent} \times \text{total}$$

, the percent can be found. Substituting and solving, we have

$$\begin{aligned} 18 &= \text{percent} \times 45 \\ \frac{18}{45} &= \frac{\text{percent} \times 45}{45} \\ 0.4 &= \frac{\text{percent} \times 45}{45} \\ 0.4 &= \text{percent} \end{aligned}$$

Converting the 0.4 from decimal form, we find that 40% of the students in the calculus class are chemistry majors.

Your Turn 8.2.8

At the Fremont County fair, there were 2,532 adult visitors. Of these, 1,679 purchased the Adult Mega Pass. What percent of the adult visitors purchased the Adult Mega Pass?

Example 8.2.9: Total Sales and Commission

Mariel makes a 20% commission on every sale she makes. One week, her commission check is for $153.00. What were her total sales that week?

Answer

In this problem, Mariel's total sales is to be determined. We know the percent she earns is **20%**. We also know that her sales commission was **$153.00**, which is the percent of the total. Using this information and the formula

$$\text{part} = \text{percent} \times \text{total}$$

we can find Mariel's total sales. The decimal form of **20%** is 0.2 . The part, or percent of the total, is 153. Substituting and solving, we obtain

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 153 &= 0.2 \times \text{total} \\ \frac{153}{0.2} &= \frac{0.2 \times \text{total}}{0.2} \\ 765 &= \frac{0.2 \times \text{total}}{0.2} \end{aligned}$$

Mariel's total sales were $765.00.

Your Turn 8.2.9

Mina's family has replaced 65% of their home's older light bulbs with LED bulbs. If they now have 52 LED bulbs, how many total lightbulbs are in Mina's house?

Who Knew?: LED Lightbulbs

According to the energy website from the U.S. government, LED lightbulbs use at least 75% less energy than incandescent bulbs. They also last up to 25 times as long as an incandescent bulb. If lighting is a significant percent of your electrical use, replacing all incandescent bulbs with LED bulbs will significantly reduce your electric bill.

Check Your Understanding

1. What is the denominator for any percent?

2. Convert 38.7% to decimal form.

3. What is 68% of 280?

4. Find the total if 41% of the total is 342. If necessary, round to two decimal places.

5. TikTok has an estimated 80,000,000 (80 million) registered users in the United States. The population of the United States is 332,403,650. What percent of the U.S. population are registered TikTok users? If necessary, round to two decimal places.

6. An Amazon fulfillment center needs to hire 20% more drivers. If there are currently 80 drivers, how many more drivers will be hired?

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8.2.1: Exercises

For any answer, round to two decimal places, if necessary.

In the following exercises, rewrite the percent as a fraction

Exercise 8.2.1.1

45%

Exercise 8.2.1.2

9.1%

Exercise 8.2.1.3

8%

Exercise 8.2.1.4

673%

In the following exercises, rewrite the percent in decimal form.

Exercise 8.2.1.5

18%

Exercise 8.2.1.6

9%

Exercise 8.2.1.7

71.2%

Exercise 8.2.1.8

934%

Exercise 8.2.1.9

Find 35% of 250

Exercise 8.2.1.10

Calculate 83.1% of 390

Exercise 8.2.1.11

Calculate 3.1% of 500

Exercise 8.2.1.12

Calculate 750% of 620

Exercise 8.2.1.13

If 40% of the total is 32, how much is the total?

Exercise 8.2.1.14

If 3% of the total is 6.32, how much is the total?

Exercise 8.2.1.15

If 150% of the total is 61.9, how much is the total?

Exercise 8.2.1.16

If 18.1% of the total is 18.5, how much is the total?

Exercise 8.2.1.17

13 is what percent of 40?

Exercise 8.2.1.18

89 is what percent of 500?

Exercise 8.2.1.19

31 is what percent of 73?

Exercise 8.2.1.20

593.2 is what percent of 184.5?

Exercise 8.2.1.21

36 people in a village of 150 want to install a new splashpad at the local playground. What percent of the village wants to install the new splashpad?

Exercise 8.2.1.22

Mitena is enrolled in a movie appreciation course. There are 84 students (including Mitena) in the course. After having the students fill out a survey, the professor informs the students that 45.2% chose horror as their favorite movie genre. How many students in Mitena's class chose horror as their favorite movie genre? Round off to the nearest integer.

Exercise 8.2.1.23

Jadyn's dorm has a "Rick and Morty night" every Wednesday during the semester. One Wednesday, 27 students from the dorm come to watch the TV show Rick and Morty. Jadyn knows this is 30% of the dorm's residents. How many students reside in the dorm?

24. **Percent Error.** When performing a scientific experiment that results in quantities of some sort, such as mass in chemistry or momentum in physics, the percent error is often computed. Percent error, $\%E$, is the percent by which the value obtained in an experiment, the observed value O, is different than the value that was expected, the expected value E, in the experiment. The formula is below.

\(\% E = \frac

{E}\)
Jim and Kelly are working on a chemistry experiment and expect the result to be 50 grams. However, their result was 48.7 grams. Find Jim and Kelly's percent error.

25. Percent Error. See Exercise 24 for the definition of percent error.

Hailey and Elsbeth are using an experiment to determine Earth's gravity. The expected value is $9.807\ \mathrm{m/s^2}$. Their experiment gives them a value of $9.457\ \mathrm{m/s^2}$. Find the percent error for Hailey and Elsbeth's experiment.

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8.3: Discounts, Markups, and Sales Tax



Figure **8.3.1**: Sale prices are often described as percent discounts. (credit: "Close-up of a discount sign" by Ivan Radic/Flickr, CC BY 2.0)

Learning Objectives

1. Calculate discounts.
2. Solve application problems involving discounts.
3. Calculate markups.
4. Solve application problems involving markups.
5. Compute sales tax.
6. Solve application problems involving sales tax.

Many people first encounter percentages during a retail transaction such as a percent discount (SALE! 25% off!!), or through sales tax ("Wait, I thought this was $1.99?"), a report that something has increased by some percentage of the previous value (NOW! 20% more!!). These are examples of percent decreases and percent increases. In this section, we discuss decrease, increase, and then the case of sales tax.

Calculating Discounts

Retailers frequently hold sales to help move merchandise. The sale price is almost always expressed as some amount off the original price. These are discounts, a reduction in the price of something. The price after the discount is sometimes referred to as the reduced price or the sale price.

When a reduction is a percent discount, it is an application of percent, which was introduced in Understanding Percent. The formula used was $\text{part} = $. In a discount application, the discount plays the role of the part, the percent discount is the percentage, and the original price plays the role of the total.

FORMULA

The formula for a discount based on a percentage is $\textbf{discount} = \textbf{percent discount} \times \textbf{original}$, with the percent discount expressed as a decimal. The price of the item after the discount is $\textbf{sale price} = \textbf{original price} -$.

These are often combined into the following formula

$$\begin{aligned}\textbf{sale price} &= \textbf{original price} - \textbf{percent discount} \times \textbf{original price} \\ &= \textbf{original price} \times (1 - \textbf{percent discount})\end{aligned}$$

When the original price and the percent discount are known, the discount and the sale price can be directly computed.

Example 8.3.1: Calculating Discount for a Percent Discount

Calculate the discount for the given price and discount percentage. Then calculate the sale price.

1. Original price = $75.80; percent discount is 25%
2. Original price = $168.90; percent discount is 30%

Answer

1. Substituting the values into the formula $\textbf{discount} = \textbf{percent discount} \times \textbf{original}$, we find that the discount is $\textbf{discount} = 0.25 \times 75.80 = 18.95$. The discount is $18.95.

   The sale price of the item is then $\textbf{sale price} = \textbf{original price} - \textbf{discount} = 75.80 - 18.95 = 56.85$, or $56.85.

2. Substituting the values into the formula $\textbf{discount} = \textbf{percent discount} \times \textbf{original}$, we find that the discount is $\textbf{discount} = 0.30 \times 168.90 = 50.67$. The discount is $50.67.

   The sale price of the item is then $\textbf{sale price} = \textbf{original price} - \textbf{discount} = 168.90 - 50.67 = 118.23$, or $118.23.

Your Turn 8.3.1

Calculate the discount for the given original price and discount percentage. Then calculate the sale price.

Original price = $1,550.00; percent discount is 32%

Original price = $27.50; percent discount is 10%

Sometimes the original price and the sale price of an item is known. From this, the percent discount can be computed using the formula $\textbf{discount} = \textbf{percent discount} \times \textbf{original}$, by solving for the percent discount.

Example 8.3.2: Calculating the Percent Discount from the Original and Sale Prices

Determine the percent discount based on the given original and sale prices.

1. Original price = $1,200.00; sale price = $900.00
2. Original price = $36.70; sale price = $29.52

Answer

1. **Step 1.** Find the discount. Using the original price and the sale price, we can find the discount with the formula $\textbf{sale price} = \textbf{original price} -$. Substituting and calculating, we find the discount to be $\textbf{900.00} = \textbf{1,200.00} - \textbf{discount}$. Solving for the discount gives $300.00.

   Step 2. Find the percent discount. Substituting the discount of $300.00 and the original price of $1,200.00, into the formula $\textbf{discount} = \textbf{percent discount} \times \textbf{original}$, we can find the percent discount.

$$300.00 = \text{percent discount} \times 1{,}200.00$$
$$\frac{300.00}{1{,}200.00} = \text{percent discount}$$
$$0.25 = \text{percent discount}$$

Converting to percent form, the percent discount is 25%.

2. **Step 1.** Find the discount. Using the original price and the sale price, we can find the discount with the formula $\text{sale price} = \text{original price} -$. Substituting and calculating, we find the discount to be $29.52 = 36.70 - \text{discount}$. Solving for the discount gives $7.38.

   Step 2. Find the percent discount. Substituting the discount of $7.38 and the original price of $36.70, into the formula $\text{discount} = \text{percent discount} \times \text{original}$, we can find the percent discount.

$$29.52 = \text{percent discount} \times 36.70$$
$$\frac{29.52}{36.70} = \text{percent discount}$$
$$0.2 = \text{percent discount}$$

   Converting to percent form, the percent discount is 20%.

Your Turn 8.3.2

Determine the percent discount based on the given original and sale prices.

Original price = $250.00; sale price = $162.50

Original price = $19.50; sale price = $17.16

Sometimes the sale price and the percent discount of an item are known. From this, the original price can be found. To avoid multiple steps, though, the formula that we will use is $\text{sale price} = \text{original price} \times (1 - \text{percent discount})$. The original price can be found by solving this equation for the original price.

Example 8.3.3: Calculating the Original Price from the Percent Discount and Sale Price

Determine the original price based on the percent discount and sale price.

1. Percent discount 10%; sale price = $450.00
2. Percent discount 75%, sale price = $90.00

Answer

1. Using the percent discount and the sale price, we can find the original price with the formula $\text{sale price} = \text{original price} \times (1 - \text{percent discount})$. Substituting and solving for the original price, we find

$$\begin{aligned}\text{sale price} &= \text{original price} \times (1 - \text{percent discount})\\ 450.00 &= \text{original price} \times (1 - 0.10)\\ 450.00 &= \text{original price} \times (0.90)\\ 500.00 &= \text{original}\end{aligned}$$

   The original price of the item was $500.00.

2. Using the percent discount and the sale price, we can find the original price with the formula$\text{sale price} = \text{original price} \times (1 - \text{percent discount})$. Substituting and solving for the original price, we find

$$\begin{aligned}\text{sale price} &= \text{original price} \times (1 - \text{percent discount})\\ 90.00 &= \text{original price} \times (1 - 0.75)\\ 90.00 &= \text{original price} \times (0.25)\\ 360.00 &= \text{original}\end{aligned}$$

The original price of the item was $360.00.

Your Turn 8.3.3

Determine the original price based on the percent discount and sale price.

Percent discount 15%; sale price = $11.05

Percent discount 9%; sale price = $200.20

Solve Application Problems Involving Discounts

In application problems, identify what is given and what is to be found, using the terms that have been learned, such as discount, original price, percent discount, and sale price. Once you have identified those, use the appropriate formula (or formulas) to find the solution(s).

Example 8.3.4: Determine Discount and New Price a Sale Rack Item

The sale rack at a clothing store is marked "All Items 30% off." Ian finds a shirt that had an original price of $80.00. What is the discount on the shirt? What is the sale price of the shirt?

Answer

We are asked to find the discount, and the sale price. We know the percent discount is 30%, or 0.30 in decimal form. The original price was $80.

Substituting into the percent discount formula, we find that the discount is $\text{discount} = \text{percent discount} \times \text{original price} = 0.30 \times 80 =$.

The discount is $24 on that shirt. The sale price is the original price minus the discount, so the sale price is $80 – $24 = $56.

Your Turn 8.3.4

A bed originally priced at $550, but is on sale, with a 60% discount. What is the discount on the bed? What is the sale price of the bed?

Example 8.3.5: Determine the Percent Discount of a Bus Pass

An annual pass on the city bus is priced at $240. The student price, though, is $168. What is the percent discount for students for the bus pass?

Answer

We know the original price of the item, $240. We also know the sale price of the item, $168. From this we know the discount is $\$240 - \$168 = \$72$. Substituting these values into the formula $\text{discount} = \text{percent discount} \times \text{original}$, we can find the percent discount.

$$\begin{aligned} \text{discount} &= \text{percent discount} \times \text{original price} \\ 72 &= \text{percent discount} \times 240 \\ \frac{72}{240} &= \frac{\text{percent discount} \times 240}{240} \\ 0.3 &= \text{percent} \end{aligned}$$

The student percent discount on the bus pass is 30%.

Your Turn 8.3.51

A pharmacy offers students at a nearby college a discount. Jerry purchases ibuprofen, which had an original price of $15.80. The cost to Jerry after the student discount was $13.43. What is the percent discount for students at the pharmacy?

Example 8.3.6: Finding the Original Price of a New Pair of Tires

Kendra's car developed a flat, and the tire store told her that two tires had to be replaced. She got a 10% discount on the pair of tires, and the sale price came to $189.00. What was the original price of the tires?

Answer

Using the percent discount and the sale price, we can find the original price with the formula $\textbf{sale price} = \textbf{original price} \times (1 - \textbf{percent discount})$. Substituting and solving for the original price, we find

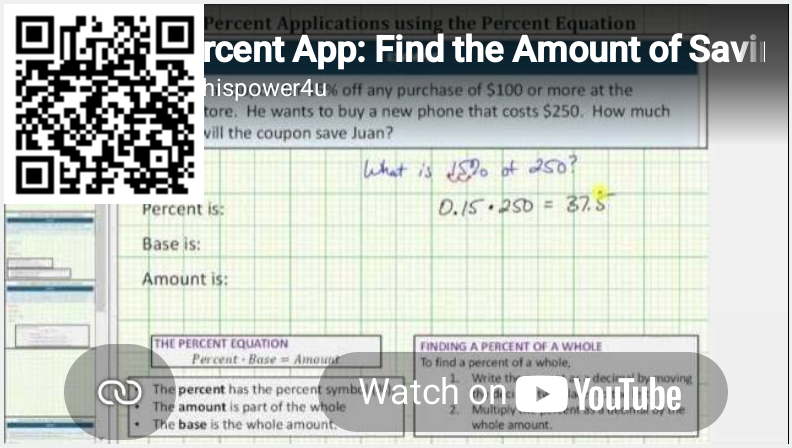
$$\begin{aligned} \textbf{sales price} &= \textbf{original price} \times (1 - \textbf{percent discount}) \\ 189.00 &= \textbf{original price} \times (1 - 0.10) \\ 189.00 &= \textbf{original price} \times (0.90) \\ 210.00 &= \textbf{original price} \end{aligned}$$

The original price of the two tires Kendra bought was $210.00.

Your Turn 8.3.6

Marisol needed to buy a new microwave. She got a 26% discount. The sale price Marisol paid was $43.66. What was the original price of the microwave?

Video



Computing Price Based on a Percent Off Coupon

WORK IT OUT

There are cases where retailers allow multiple discounts to be applied. However, it is rare that the discount percentages are added together. For example, if you have a 15% coupon and qualify for a 20% price reduction, the retailer typically does not add those two percentages together to determine the new price. The retailer instead applies one discount, then applies the second discount to the price obtained after the first discount was deducted.

Research the original prices of two different laptops offered by one retail outlet. Assume you will receive a student discount of 12% and your outlet of choice is having a 15% off sale on all laptops.

For each laptop:

1. List the original price and calculate the price after applying the student discount (12%) only.
2. Then find the price after applying the sale discount (15% off) to the price found in Step 1.
3. Determine the total saved on the laptop and what percent discount the total savings represents.
4. Now, apply the discounts in reverse order (first the sale discount, then the student discount).
5. Note anything interesting about your findings.

Calculate Markups

When retailers purchase goods to sell, they pay a certain price, called the **cost**. The retailer then charges more than that amount for the goods. This increase is called the **markup**. This selling price, or **retail price**, is what the retailer charges the consumer in order to pay their own costs and make a profit. Markup, then is very similar to discount, except we add the markup, while we subtract the discount.

FORMULA

The formula for a markup based on a percentage is $\textbf{markup} = \textbf{percent}$, with the percent markup expressed as a decimal. The price of the item after the markup is $\textbf{retail price} = \textbf{cost}+$.

These are often combined into the following formula

$$\textbf{retail price} = \textbf{cost} + \textbf{percent markup} \times \textbf{cost} = \textbf{cost} \times (1 + \textbf{percent markup})$$

It should be noted that the formulas used for a markup are very similar to those for a discount, with addition replacing the subtraction.

Example 8.3.7: Determining the Retail Price Based on the Cost and the Percent Markup

Calculate the markup for the given cost and markup percentage. Then calculate the retail price.

1. Cost = $62.00; percent markup is 15%
2. Cost = $750.00; percent markup is 45%

Answer

1. Substituting the values into the formula $\textbf{markup} = \textbf{percent}$, we find that the markup is $\textbf{markup} = 0.15 \times 62.00 = 9.30$. The markup is $9.30.
   The retail price of the item is then $\textbf{retail price} = \textbf{cost}+$, or $62.00 + $9.30 = $71.30.
2. Substituting the values into the formula $\textbf{markup} = \textbf{percent}$, we find that the markup is $\textbf{markup} = 0.45 \times 750.00 = 337.50$. The markup is $337.50.
   The retail price of the item is then $\textbf{retail price} = \textbf{cost}+$, or $750.00 + $337.50 = $1,087.50.

Your Turn 8.3.7

Calculate the markup for the given cost and markup percentage. Then calculate the retail price.

Cost = $1,800.00; percent markup is 22%

Cost = $10.50; percent markup is 10%

Sometimes the cost and the retail price of an item are known. From this, the percent markup can be computed using the formula $\textbf{markup} = \textbf{percent}$, by solving for the percent markup.

Example 8.3.8: Calculating the Percent Markup from the Cost and Retail Price

Determine the percent markup based on the given cost and retail price. Round percentages to two decimal places.

1. Cost = $90.00; retail price = $103.50
2. Cost = $5.20; retail price = $9.90

Answer

1. **Step 1:** Using the cost and the retail price, we can find the markup with the formula $\textbf{retail price} = \textbf{cost} +$. Substituting and calculating, we find the markup to be $\textbf{103.50} = \textbf{90.00} + \textbf{markup}$. Solving for the markup gives $13.50.
   Step 2: After substituting the markup, $13.50, and the original price, $90.00, into the formula $\textbf{markup} = \textbf{percent}$, we can find the percent markup.

$$\begin{aligned} \textbf{13.50} &= \textbf{percent markup} \times \textbf{90.00} \\ \frac{\textbf{13.50}}{\textbf{90.00}} &= \textbf{percent markup} \\ \textbf{0.15} &= \textbf{percent markup} \end{aligned}$$

   Converting to percent form, the percent markup is 15%.

2. **Step 1:** Using the cost and the retail price, we can find the markup with the formula $\textbf{retail price} = \textbf{cost} +$. Substituting and calculating, we find the markup to be $\textbf{9.90} = \textbf{5.20} + \textbf{markup}$. Solving for the markup gives $4.70.
   Step 2: After substituting the markup, $4.70, and the original price, $5.20, into the formula $\textbf{markup} = \textbf{percent}$, we can find the percent markup.

$$\begin{aligned} \textbf{4.70} &= \textbf{percent markup} \times \textbf{5.20} \\ \frac{\textbf{4.70}}{\textbf{5.20}} &= \textbf{percent markup} \\ \textbf{0.9038} &= \textbf{percent markup} \end{aligned}$$

   Converting to percent form, the percent markup is 90.38%.

Your Turn 8.3.8

Determine the percent markup based on the given cost and retail price. Round percentages to two decimal places.

Cost = $120.00; retail price = $190.00

Cost = $0.38; retail price = $1.14

Sometimes the retail price and the percent markup of an item are known. From this, the cost can be found. To avoid multiple steps, though, the formula that we will use is $\textbf{retail price} = \textbf{cost} \times (\textbf{1} + \textbf{percent markup})$. The cost can be found by solving this equation for the cost.

Example 8.3.9: Calculating the Cost from the Percent Markup and Retail Price

Determine the cost based on the percent markup and retail price.

1. Percent markup 20%; retail price = $10.62
2. Percent markup 125%; retail price = $26.55

Answer

1. Using the percent markup and the retail price, we can find the cost with the formula $\textbf{retail price} = \textbf{cost} \times (\textbf{1} + \textbf{percent markup})$. Substituting and solving for the cost, we find

$$\begin{aligned} \text{retail price} &= \text{cost} \times (1 + \text{percent markup}) \\ 10.62 &= \text{cost} \times (1 + 0.2) \\ 10.62 &= \text{cost} \times (1.2) \\ 8.85 &= \end{aligned}$$

The cost of the item was $8.85.

2. Using the percent markup and the retail price, we can find the cost with the formula $\text{retail price} = \text{cost} \times (1 + \text{percent markup})$. Substituting and solving for the original price, we find

$$\begin{aligned} \text{retail price} &= \text{cost} \times (1 + \text{percent markup}) \\ 26.55 &= \text{cost} \times (1+2.25) \\ 26.55 &= \text{cost} \times (3.25) \\ 11.80 &= \end{aligned}$$

The cost of the item was $11.80.

Your Turn 8.3.9

Determine the cost based on the percent markup and retail price.

Percent markup 15%; retail price = $40.25

Percent markup 300%; retail price = $35.96

Solve Application Problems Involving Markups

As before when working with application problems, be sure to look for what is given and identify what you are to find. Once you have evaluated the problem, use the appropriate formula to find the solution(s). These application problems address markups.

Example 8.3.10: Determine Retail Price of a Power Bar

Janice works at a convenience store near campus. It sells protein bars at a 60% markup. If a bar costs the store $1.30, how much is the retail price at the convenience store?

Answer

We are asked to find the retail price. We know the percent markup is 60%. The cost of the bar was $1.30. Substituting into the percent markup formula, we find that the markup is $\text{markup} = \text{percent markup} \times \text{cost} = 0.60 \times 1.30 = 0.78$. The markup is $0.78 on that protein bar. The retail price is the cost plus the markup, so the retail price is $\text{retail price} = \text{cost} + \text{markup} = 1.30 + 0.78 = 2.08$. The retail price is $2.08.

Your Turn 8.3.10

A furniture outlet spends $360.00 to buy a bed. The store marks up the bed by 250%. What is the retail price of the bed?

Example 8.3.11: Determine the Percent Markup of a Phone

Javi began working at a phone outlet. In a recent shipment, he noticed that the cost of the phone to the store was $480.00. The phone sells for $840.00 in the store. What is the percent markup on the phone?

Answer

We know the cost of the phone, $480. We also know the retail price of the phone, $840.00. From this we know the markup is $\$840.00 - \$480.00 = \$360.00$. Substituting these values into the formula $\text{markup} = \text{percent}$, we can find the percent markup.

$$\begin{aligned} \text{markup} &= \text{percent markup} \times \text{cost} \\ 360 &= \text{percent markup} \times 480 \\ \frac{360}{480} &= \frac{\text{percent markup} \times 480}{480} \\ 0.75 &= \text{percent markup} \end{aligned}$$

The markup on the phone is 75%.

Your Turn 8.3.11

Maggie does some research into textbook costs. The *Sociology of the Family* text she finds sells for $234.36 but costs the store only $189.00. What is the percent markup on the sociology book?

Example 8.3.12: Finding the Cost of a T-Shirt

Bob decided to order a t-shirt for his gaming friend online for $29.50. He knows the markup on such t-shirts is 18%. What was the t-shirt's cost before the markup?

Answer

Using the percent markup and the retail price, $29.50, we can find the cost with the formula $\text{retail price} = \text{original price} \times (1 + \text{percent markup})$. Substituting and solving for cost we find

$$\begin{aligned} \text{retail price} &= \text{cost} \times (1 + \text{percent markup}) \\ 29.50 &= \text{cost} \times (1+0.18) \\ 29.50 &= \text{cost} \times (1.18) \\ 25.00 &= \text{cost} \end{aligned}$$

The cost of the t-shirt was $25.00.

Your Turn 8.3.12

Tina has opened a retail shop and purchased a unique hat for resale. Tina uses a 50% markup and sells the hat for $57.00. How much did the hat cost Tina?

Compute Sales Tax

Sales tax is applied to the sale or lease of some goods and services in the United States but is not determined by the federal government. It is most often set, collected, and spent by individual states, counties, parishes, and municipalities. None of these sales tax revenues go to the federal government.

For example, North Carolina has a state sales tax of 4.75% while New Mexico has a state sales tax of 5%. Additionally, many counties in North Carolina charge an additional 2% sales tax, bringing the total sales tax for most (72 of the 100) counties in North Carolinians to 6.75%. However, in Durham, the county sales tax is 2.25% plus an additional 0.5% tax used to fund public transportation, bringing Durham County's sales tax to 7%. To find the sales tax in a particular place, then, add other locality sales taxes to the base state sales tax rate.

How much we pay in sales tax depends on where we are, and what we are buying.

To determine the amount of sales tax on taxable purchase, we need to find the product of the purchase price, or marked price, and the sales tax rate for that locality.

FORMULA

To calculate the amount of sales tax paid on the purchase price in a locality with sales tax given in decimal form, calculate $\text{sales tax} = \text{purchase price} \times \text{tax}$ The total price is then $\text{Total price} = \text{purchase price} + \text{purchase price} \times \text{tax rate} = \text{purchase price} \times (1 + \text{tax rate})$

Checkpoint

When the sales tax calculation results in a fraction of a penny, then normal rounding rules apply, round up for half a penny or more, but round down for less than half a penny.

You should notice that this the same as markup, except using a different term. Sales tax plays the role of markup, the purchase price plays the role of cost, and the tax rate plays the role of percent markup. This means all the strategies developed for markups apply to this situation, with the changes indicated.

Example 8.3.13: Sales Tax in Kankakee Illinois

The sales tax in Kankakee, Illinois, is 8.25%. Find the sales tax and total price of items based on the purchase price listed.

1. Purchase price = $428.99
2. Purchase price = $34.88

Answer

1. The sales tax is found using $\textbf{sales tax} = \textbf{purchase price} \times \textbf{tax}$. The purchase price is $428.99 and the tax rate is 8.25%. Substituting and calculating, the sales tax is $\textbf{sales tax} = \$428.99 \times 0.0825 = \$35.391675$. The sales tax needs to be rounded off. Since the third decimal place (fraction of a penny) is 1, we round down and the sales tax is $35.39. The total price is the sales tax plus the purchase price, so is $\$428.99 + \$35.88 = \$464.87$.
2. The sales tax on the item is found using $\textbf{sales tax} = \textbf{purchase price} \times \textbf{tax}$. The purchase price is $34.88 and the tax rate is 8.25%. Substituting and calculating, the sales tax is $\textbf{sales tax} = \$34.88 \times 0.0825 = \$2.8776$. The sales tax needs to be rounded off. Since the third decimal place (fraction of a penny) is 7, we round up and the sales tax is $2.88. The total price of the item is the sales tax plus the purchase price, so is $\$34.88 + \$2.88 = \$37.76$.

Your Turn 8.3.13

The sales tax in Union County, Oregon, is 7%. Find the sales tax and total price of items based on the purchase price listed.

Purchase price = $1,499.00

Purchase price = $26.89

As before, the information available might be different than only the purchase price and the sales tax rate. In these cases, use either $\textbf{sales tax} = \textbf{purchase price} \times \textbf{tax}$ or $\textbf{Total price} = \textbf{purchase price} \times (1 + \textbf{tax rate})$ and solve for the indicated tax, price, or rate. These problems mirror those for percent markup.

Be aware, almost all sales tax rates are structured as full percentages, or half percent, or one-quarter percent, or three-quarter percent. This means the decimal value of the sales tax rate, written as a percent, will be either 0, as in 5.0%, 5 as in 7.5%, 25 as in 3.25%, or 75 as in 4.75%. When rounding for the sales tax percentage, be sure to use this guideline.

Example 8.3.14: Calculating the Sales Tax from the Purchase Price and the Total Price

Find the sales tax rate for the indicated purchase price and total price. Round using the guideline for sales tax percentages.

1. Purchase price = $329.50; total price = $354.21
2. Purchase Price = $13.77; total price = $14.39

Answer

1. **Step 1.** Find the sales tax paid. First, the amount of sales tax must be found. Subtracting the purchase price from the total price, the amount of sales tax is $24.71.
   Step 2. Find the sales tax rate. Using the purchase price, the sales tax, and the formula $\textbf{sales tax} = \textbf{purchase price} \times \textbf{tax}$, the sales tax rate can be found. Substituting and solving yields

$$\begin{aligned}\text{Sales Tax} &= \text{purchase price} \times \text{tax rate}\\ \$24.71 &= \$329.50 \times \text{tax rate}\\ \frac{\$24.71}{\$329.50} &= \text{tax rate}\\ 0.07499 &= \text{tax}\end{aligned}$$

Keeping in mind the guideline for rounding sales tax rate, the sales tax rate is 7.5%.

2. **Step 1.** Find the sales tax paid. First, the amount of sales tax must be found. Subtracting the purchase price from the total price, the amount of sales tax is $0.62.
   Step 2. Find the sales tax rate. Using the purchase price, the sales tax, and the formula $\text{sales tax} = \text{purchase price} \times \text{tax}$, the sales tax rate can be found. Substituting and solving yields

$$\begin{aligned}\text{sales tax} &= \text{purchase price} \times \text{tax rate}\\ \$0.62 &= \$13.77 \times \text{tax rate}\\ \frac{\$0.62}{\$13.77} &= \text{tax rate}\\ 0.04503 &= \text{tax}\end{aligned}$$

Keeping in mind the guideline for rounding sales tax rate, the sales tax rate is 4.5%.

Your Turn 8.3.14

Find the sales tax rate for the indicated purchase price and total price. Round using the guideline for sales tax percentages.

Purchase price = $83.90; total price = $88.30

Purchase price = $477.00; total price = $509.20

Example 8.3.15: Calculating the Purchase Price from the Sales Tax and Total Price

Find the purchase price for the indicated sales tax rate and total price.

1. Sales tax rate = 5.75%; total price = $36.56
2. Sales tax rate = 4.25%; total price = $97.17

Answer

1. When the sales tax rate and the total price are known, the formula $\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$ can be used to find the purchase price. Substituting the tax rate and total price into the formula and solving, we find

$$\begin{aligned}\text{Total price} &= \text{purchase price} \times (1 + \text{tax rate})\\ \$36.56 &= \text{purchase price} \times (1 + 0.0575)\\ \frac{\$36.56}{1.0575} &= \text{purchase price}\\ \$34.57 &= \text{purchase}\end{aligned}$$

The purchase price, the price before tax, was $34.57.

2. When the sales tax rate and the total price are known, the formula $\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$ can be used to find the purchase price. Substituting the tax rate and total price into the formula and solving, we find

$$\begin{aligned}\text{Total price} &= \text{purchase price} \times (1 + \text{tax rate})\\ \$97.17 &= \text{purchase price} \times (1 + 0.0425)\\ \frac{\$97.17}{1.0425} &= \text{purchase price}\\ \$93.21 &= \text{purchase}\end{aligned}$$

The purchase price, the price before tax, was $93.21.

Your Turn 8.3.15

Find the purchase price for the indicated sales tax rate and total price.

Sales tax rate = 8.25%; total price = $157.81

Sales tax rate = 6.75%; total price = $522.01

Solve Application Problems Involving Sales Tax

Solving problems involving sales tax follows the same ideas and steps as solving problems for markups. But here we will use the following formula:

$$\text{total price} = \text{purchase price} + \text{sales tax}$$

We can also use the formula:

$$\text{total price} = \text{purchase price} \times (1 + \text{sales tax rate})$$

.

This can be seen in the following examples.

Example 8.3.16: Compute Sales Tax for Denver, Colorado

The sales tax rate in Denver Colorado is 8.81%. Keven buys a TV in Denver, and the purchase price (before taxes) is $499.00. How much will Keven pay in sales tax and what will be the total amount he spends when he buys the TV?

Answer

The sales tax rate in Denver is 8.81%. To find the sales tax Keven will pay, find 8.81% of the purchase price. In decimal form, that sales tax rate is 0.0881. Using the formula and substituting 499.00 for purchase price, we find that Keven will pay $\text{purchase price} \times \text{tax rate} = \$499 \times 0.0881 = \$43.96$ in sales tax for the TV.

The total price that Keven will pay is the purchase price plus the sales tax, or $\$499.00 + \$43.96 = \$542.96$.

Your Turn 8.3.16

Daryl decides to buy a new scooter in St. Louis, Missouri, where the sales tax is 9.68%. The scooter he chooses has a purchase price of $1,149. How much will Daryl pay in sales tax and what is the total price he spends on the scooter?

Example 8.3.17: Compute Sales Tax for Austin, Texas

Jillian visits Austin, Texas, and purchases a new set of weights for her home. She spends, including sales tax, $467.64. The sales tax rate in Austin Texas is 8.25%. How much of the total price is sales tax?

Answer

The sales tax paid for this purchase is the difference in the total price and the purchase price. We know the total price is $467.64. We also know the sales tax rate, which is 8.25%. In decimal form, this is 0.0825. Using these values and the formula $\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$ to find the purchase price.

$$\begin{aligned} \text{total price} &= \text{purchase price} \times (1 + \text{tax rate}) \\ \$467.64 &= \text{purchase price} \times (1 + 0.0825) \\ \$467.64 &= \text{purchase price} \times (1.0825) \\ \$432 &= \text{purchase price} \end{aligned}$$

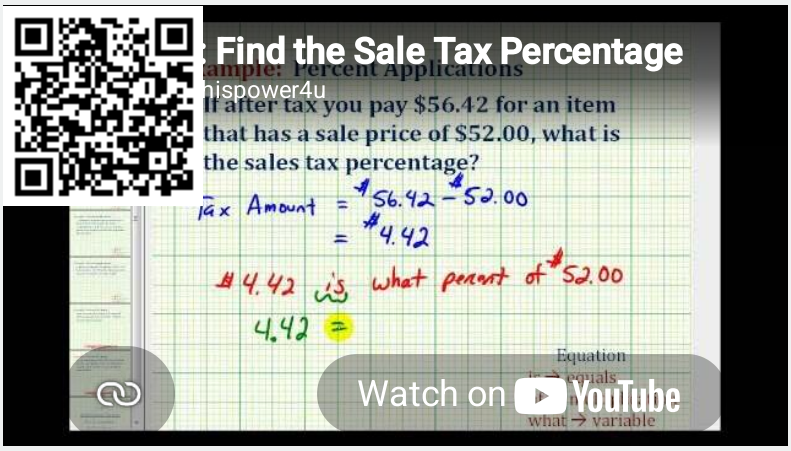
Knowing both the total price and the now the purchase price, we can find the difference, which is the sales tax.

The total price was $467.64. The purchase price was $432. The difference of the total price and the purchase price, or the sales tax, is then $467.64 − $432.00, which is $35.64. Jillian pays $35.64 in sales tax.

Your Turn 8.3.17

Elizabeth decides to buy new running shoes in her hometown of Springfield, Illinois, where the sales tax rate is 6.25%. If her total bill comes to $153, how much of the total price is sales tax?

Video



Finding Sales Tax Percentage

Who Knew?

West Virginia was the first state to impose a sales tax. This happened on May 3, 1921.

Look up your locality on this website that lists standard state-level sales tax rates and compare the sales tax structure in your state to two nearby states (for the lower 48) and for any two states (Alaska and Hawaii).

Check Your Understanding

1. What is a discount?
2. What is a markup?
3. An item has a retail price of $45.00. What is the sale price after a 32% discount?
4. A retailer buys an item for $311.00. What is the retail price if their markup is 60%?
5. Does sales tax have the same formula as markup?
6. If the sales tax is 6.8%, what is the total price for an item that has a purchase price of $39.95?

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8.3.1: Exercises

For the following exercises, use the given values to find the indicated value. Round percent results to 2 decimal places. Round money results to the penny (2 decimal places).

Exercise 8.3.1.1

Retail price = $399.00, percent discount = 30%, find the sale price.

Exercise 8.3.1.2

Retail Price = $75.00, percent discount = 65%, find the sale price.

Exercise 8.3.1.3

Retail price = $125.00, sale price = $90.00, find the percent discount.

Exercise 8.3.1.4

Retail price = $47.00, sale price = $41.50, find the percent discount.

Exercise 8.3.1.5

Sale price = $145.70, percent discount = 20%, find the retail price.

Exercise 8.3.1.6

Sale price = $1,208.43, percent discount = 13%, find the retail price.

Exercise 8.3.1.7

Retail price = $26,790.00, percent discount = 8%, find the sale price.

Exercise 8.3.1.8

Sale price = $314.06, percent discount = 33%, find the retail price.

Exercise 8.3.1.9

Retail price = $145.50, sale price = $117.90, find the percent discount.

Exercise 8.3.1.10

Retail price = $28.90, percent discount = 18%, find the sale price.

Exercise 8.3.1.11

Sale price = $17.59, percent discount = 12%, find the retail price.

Exercise 8.3.1.12

Retail price = $57.50, sale price = $46.00, find the percent discount.

Exercise 8.3.1.13

Cost = $130.00, percent markup = 34%, find the retail price.

Exercise 8.3.1.14

Cost = $2.27, percent markup = 42%, find the retail price.

Exercise 8.3.1.15

Cost = $68.45, retail price = $109.90, find the percent markup.

Exercise 8.3.1.16

Cost = $466.16, retail price = $699.00, find the percent markup.

Exercise 8.3.1.17

Retail price = $98.99, percent markup = 25%, find the cost.

Exercise 8.3.1.18

Retail price = $799.00, percent markup = 55%, find the cost.

In the following exercises, find the sales tax and total paid.

Exercise 8.3.1.19

Retail price = $17.99; sales tax = 7.5%

Exercise 8.3.1.20

Retail price = $799.00; sales tax = 8.5%

Exercise 8.3.1.21

Retail price = $176.83; sales tax = 6.25%

Exercise 8.3.1.22

Retail price = $223.93; sales tax = 4.5%

In the following exercises, find the sales tax rate.

Exercise 8.3.1.23

Purchase price = $257.45; total price = $273.54

Exercise 8.3.1.24

Purchase price = $14.99; total price = $15.74

Exercise 8.3.1.25

Purchase price = $26.83; total price = $28.84

Exercise 8.3.1.26

Purchase price = $2,399.90; total price = $2,609.89

In the following exercises, find the purchase price.

Exercise 8.3.1.27

Sales tax rate = 4.75%; total price = $50.15

Exercise 8.3.1.28

Sales tax rate = 8%; total price = $1,069.20

Exercise 8.3.1.29

Sales tax rate = 9.5%; total price = $51.45

Exercise 8.3.1.30

Sales tax rate = 5.75%; total price = $3,065.69

Exercise 8.3.1.31

Harris has a coupon for 20% off for any purchase. She finds a new tennis racket for $278.00. How much is the price after the coupon is applied?

Exercise 8.3.1.32

After the employee discount, Mariam will pay $46.55. What is her employee discount rate if the retail price was $53.50? Round to nearest full percent.

Exercise 8.3.1.33

Resa purchased a new game for her cousin. After sales tax, she paid $41.13. Find the sales tax rate she paid if the purchase price of the game was $38.99.

Exercise 8.3.1.34

Larissa opens a new secondhand bookstore. She buys a book for $2.75. What is her percent markup if the sells the book for $8.50. Round to the nearest percent.

Exercise 8.3.1.35

Doug opens a used auto parts store. He pays $30 for a car door. How much will he charge if his percent markup is 60%?

Exercise 8.3.1.36

Gaia and Seth live in Osceola County in Florida, where the sales tax rate is 7.5%. They purchased some new camping gear. The price before taxes came to $784.62. How much do they pay after the sales tax is applied?

Exercise 8.3.1.37

Theresa decides to purchase a new phone, which has a retail price of $799.00. Her discount is 20% through a friends and family plan. The sales tax in her county is 6.75%. How much will she pay after the discount? How much will she pay after the tax is applied?

Exercise 8.3.1.38

Sakari manages a retail outlet. They receive a shipment of shirts. She sees on the shipping list that each shirt cost 24.50 to the store. The store marks up the shirts by 45%. The county in which she lives charges sales tax of 6.5%. What is the retail price of one of the shirts? After sales tax, how much will a customer pay for the shirt?

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8.4: Credit Cards



Figure **8.4.1**: Credit cards are both a convenience and a danger. (credit: "Credit Cards" by Sean MacEntee/Flickr, CC BY 2.0)

Learning Objectives

1. Apply for a credit card armed with basic knowledge.
2. Distinguish between three basic types of credit cards.
3. Compare and contrast the benefits and drawbacks of credit cards.
4. Read and understand the basic parts of a credit card statement.
5. Compute interest, balance due, and minimum payment due for a credit card.

It can be difficult to get along these days without at least one credit card. Most hotels and rental car agencies require that a credit card is used. There are even a number of retailers and restaurants that no longer accept cash. They make online purchasing easier. And nothing contributes more to a good credit rating than a solid history of making credit card payments on time.

Being granted a credits card is a privilege. Used unwisely that privilege can become a curse and the privilege may be withdrawn. In this section, we will talk about the different types of credit cards and their advantages and disadvantages. The more knowledge a cardholder has about the credit card industry, the better able credit accounts can be managed, and that knowledge may cause major adjustments to a cardholder's lifestyle.

All credit cards are not equal, but they all represent consumers borrowing money, usually from a bank, to pay for needs and "wants." As such, they are a type of loan, and your repayment may include interest. (You might want to review Section 6.8, which discusses loans and repayment plans.)

There are many institutions and credit cards to choose from. Use caution as you shop around for a credit card that suits you. Your top concern is likely the interest rates on purchases and cash advances. But be careful to also read the small print regarding charges for late payments, and other fees such as an **annual fee,** where the credit card charges you (the cardholder) a fee each year for the privilege of using the cards. Many cards charge no such fee, but there are many that charge modest to heavy fees. Make sure to understand rules for **reward programs**, where the credit card issuer grants benefits based on one's spending. Finally, once one applies for and is granted a credit card, pay attention to the credit limit the bank offers. Once a company is owed that much money, use of the card for purchases should be curtailed until some of the debt is paid off.

Checkpoint

The interest rate will not matter if the balance is paid every month. When the balance is paid every month, there is NO INTEREST charged.

Types of Credit Cards

There are basically three types of credit cards: bank-issued credit cards, store-issued credit cards, and travel/entertainment credit cards. We will look at all three and explain the good and the bad qualities of each.

Bank-Issued Credit Cards

Perhaps the most widely used credit card type is the **bank-issued credit card**, like Visa or MasterCard (and even American Express and Discover cards). These types of cards are an example of revolving credit, meaning that additional credit is extended before the previous balance is paid—but only up to the assigned credit limit. Bank-issued cards are considered the most convenient, as they can be used to purchase anything, including apparel, furniture, groceries, fuel for automobiles, meals, hotel bills, and so on, just as if paying with cash. The interest rates on bank-issued credit cards are usually lower than those for other credit cards we'll discuss, and the credit limits are generally higher. Currently, bank-issued cards have an average 20.09% APR.

Store-Issued Credit Cards

Store-issued credit cards are issued by retailers. One can hardly walk into a store these days without being offered a discount on purchases if one applies for the store credit card. These cards can only be used in that store or family of stores that issues the card. However, if a store credit card is associated with Visa, MasterCard, or American Express, then the card might be used the same way that the bank-issued cards are used. This is called cobranding. The logo of the bank-issued card will be present on the store card. Many stores offer both types. Like other credit cards, they may come with an annual fee.

Store credit cards usually charge higher interest rates than bank-issued cards. Currently, store credit cards have an APR (annual percentage rate) of 24.15%. Any rewards offered by store credit cards are usually limited to purchases made in their own store, and it typically takes longer to accumulate enough rewards or points to redeem them, whereas cobranded credit cards offer opportunities to earn rewards on all purchases, regardless of whether purchases are made in the issuing store or not.

Store credit cards usually offer lower credit limits, at least in the beginning. After being proved to be a responsible credit card owner, credit limits can be raised. Nevertheless, store credit cards are a good choice for those new to the credit card industry. If on-time payments are consistently made, it is an excellent way to get started building a credit history.

Travel/Entertainment Cards, or Charge Cards

This is the third type of credit card. The **travel and entertainment cards,** also known as **charge cards,** first and foremost offer very high limits or unlimited credit, but they must be paid in full every month. They generally charge high annual fees and impose expensive penalties should a payment be late. On the other hand, they typically have longer grace periods and offer many and various kinds of rewards.

Check out this nerdwallet article about the differences between a charge card and a credit card.

Checkpoint

An interest rate will greatly depend on credit score. Responsible use of credit cards will increase a credit score. See the WHO KNEW? from The Basics of Loans.

Video



Choosing a Credit Card

Who Knew?: Top Travel and Entertainment Cards

At one time, Diner's Club was the premier entertainment card. To be accepted into the Diner's Club and be rewarded with a charge card meant one was special. Today, an American Express card is held with the same reverence as Diner's Club was in the past. It was a privilege to own one of these cards. There is a lot of competition going on to supplant Diner's Club, as shown by the Chase Sapphire Preferred Card's travel rewards and benefits.

Another thing worth mentioning here and something that appears to be an unusual offering is part of the American Express Gold and higher-level cards. Cardholders can actually open a savings account, buy a CD, or apply for a personal or business loan from American Express. The company boasts a higher interest rate than what is paid on traditional savings accounts and CDs.

Example 8.4.1: Comparing Credit Cards

1. Which type of credit card is paid off every month so has no interest to be paid, but comes with high fees?
2. Which type of credit cards are the most widely accepted?
3. Which type of credit cards are the most limited?

Answer

1. Charge cards are to be paid off completely each month.
2. Bank-issued credit cards are the most flexible to use, because they are not limited to which retailers or service providers accept them.
3. Store issues credit cards are the most limited, since they only work in that family of stores.

Your Turn 8.4.1

Which type of credit card typically have the highest interest rates?

What should be considered other than interest rate when selecting a card?

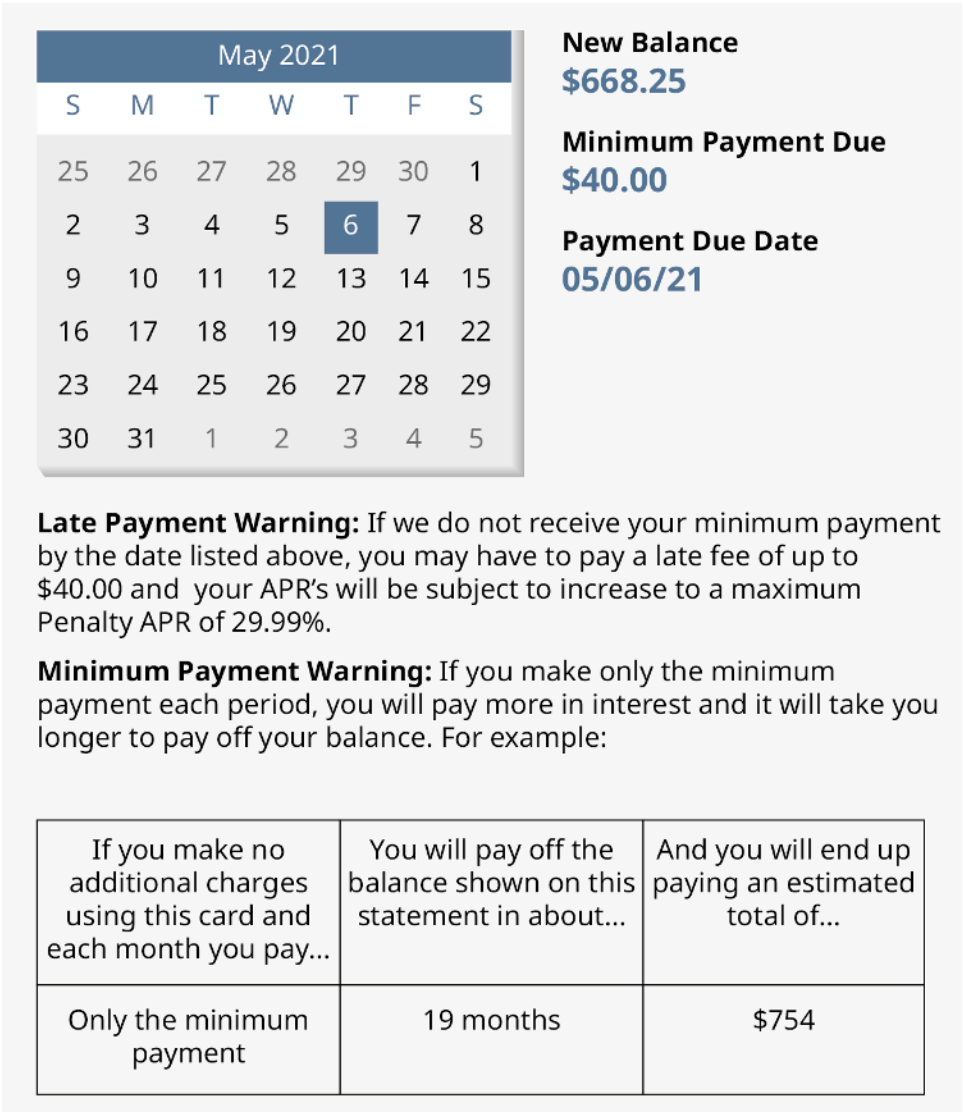
Which type of cards typically have high annual fees?

Credit Card Statements

Cardholders usually receive monthly statements and have 21 days to pay the minimum amount due. The statements itemize and summarize activity on the credit card for that statement's billing period. The billing period for a credit card is generally a month long, but typically does not start and end on the first and last days of the month. The statement will include the current balance,

interest rate, the minimum payment due, and the due date. Be aware, different companies produce statements that are laid out differently. The information will be clearly labeled though.

The due date is a top concern. Missing a due date is one of the worst things a cardholder can do financially, and this is by far the biggest downfall of owning a credit card. Not only is the cardholder subject to late fees, but when a payment is late more than once there is a high probability that the cardholder will be negatively reported to the credit bureaus, which can quickly erode a credit score. Figure 6.25 shows an excerpt from an actual statement from a Chase Bank Visa card, based on the current $668.25 balance.



May 2021

S	M	T	W	T	F	S
25	26	27	28	29	30	1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31	1	2	3	4	5

New Balance
$668.25

Minimum Payment Due
$40.00

Payment Due Date
05/06/21

Late Payment Warning: If we do not receive your minimum payment by the date listed above, you may have to pay a late fee of up to $40.00 and your APR's will be subject to increase to a maximum Penalty APR of 29.99%.

Minimum Payment Warning: If you make only the minimum payment each period, you will pay more in interest and it will take you longer to pay off your balance. For example:

If you make no additional charges using this card and each month you pay...	You will pay off the balance shown on this statement in about...	And you will end up paying an estimated total of...
Only the minimum payment	19 months	$754

Figure 8.4.2: Credit card statement

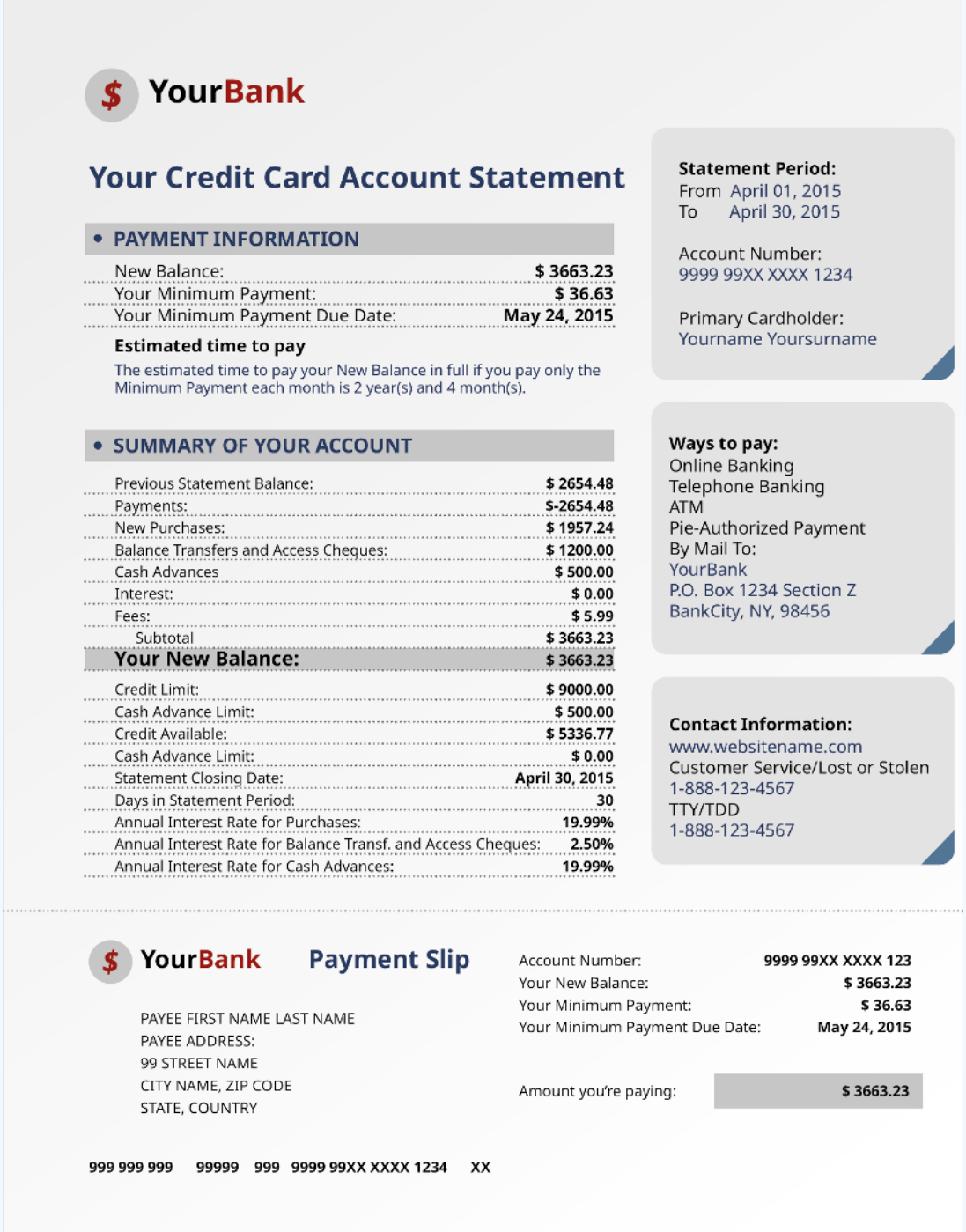
Specifically pay attention to the late payment penalty and minimum payment warning statements. stating that if no other purchases are made and you continue making only the minimum payment, it will take 19 months to pay off the balance and you will pay $754.00. You can't say you were not warned.

It is critical that you examine your statement every month because it is always a possibility that your account may have been compromised. If you should notice fraudulent charges on your statement, notifying the credit card company is often enough to have those charges researched by the company and removed. The card with the fraudulent charges will be canceled and a new card with a new account number will be sent to you.

Example 8.4.2: Reading a Credit Card Statement

On the credit card statement Figure 6.26, identify

1. The balance due
2. The minimum required payment
3. The length of time it takes to pay off the balance by paying the minimum payments and without charging more to the card
4. The interest rate for purchases



$ YourBank

Your Credit Card Account Statement

PAYMENT INFORMATION

New Balance:	$ 3663.23
Your Minimum Payment:	$ 36.63
Your Minimum Payment Due Date:	May 24, 2015

Estimated time to pay

The estimated time to pay your New Balance in full if you pay only the Minimum Payment each month is 2 year(s) and 4 month(s).

SUMMARY OF YOUR ACCOUNT

Previous Statement Balance:	$ 2654.48
Payments:	$-2654.48
New Purchases:	$ 1957.24
Balance Transfers and Access Cheques:	$ 1200.00
Cash Advances	$ 500.00
Interest:	$ 0.00
Fees:	$ 5.99
Subtotal	$ 3663.23
Your New Balance:	$ 3663.23
Credit Limit:	$ 9000.00
Cash Advance Limit:	$ 500.00
Credit Available:	$ 5336.77
Cash Advance Limit:	$ 0.00
Statement Closing Date:	April 30, 2015
Days in Statement Period:	30
Annual Interest Rate for Purchases:	19.99%
Annual Interest Rate for Balance Transf. and Access Cheques:	2.50%
Annual Interest Rate for Cash Advances:	19.99%

Statement Period:
From April 01, 2015
To April 30, 2015

Account Number:
9999 99XX XXXX 1234

Primary Cardholder:
Yourname Yoursurname

Ways to pay:
Online Banking
Telephone Banking
ATM
Pie-Authorized Payment
By Mail To:
YourBank
P.O. Box 1234 Section Z
BankCity, NY, 98456

Contact Information:
www.websitename.com
Customer Service/Lost or Stolen
1-888-123-4567
TTY/TDD
1-888-123-4567

$ YourBank Payment Slip

PAYEE FIRST NAME LAST NAME
PAYEE ADDRESS:
99 STREET NAME
CITY NAME, ZIP CODE
STATE, COUNTRY

Account Number:	9999 99XX XXXX 123
Your New Balance:	$ 3663.23
Your Minimum Payment:	$ 36.63
Your Minimum Payment Due Date:	May 24, 2015
Amount you're paying:	$ 3663.23

999 999 999 99999 999 9999 99XX XXXX 1234 XX

Figure 8.4.3: Credit card account statement

Answer

1. The balance due is under the payment information heading and is $3,663.23.
2. The minimum payment due is also under the payment information heading, and is $36.63.
3. The time to pay off the balance using only minimum payments is below the payment information, and says it takes 2 years and 4 months to pay off the balance.
4. The interest rate for purchases is toward the bottom of the statement. It is 19.99%.

Your Turn 8.4.2

Referring to the statement above, answer the following:

What is the statement period?

What is the credit limit?

How much in fees were charged?

Video



Reading Credit Card Statements

Compute Interest, Balance Due, and Minimum Payment Due for a Credit Card

Computing all of these values depends on understanding and computing the average daily balance on a credit card. Once that is known, the interest, balance due, and minimum payment can be found.

Above all else, if you pay off the entire balance each month, interest is not charged.

Average Daily Balance

Most credit card companies compute interest using the **average daily balance** method.

To find the average daily balance on your credit card, determine the balance on the card each day of the billing period (often that month), and take the average. One process to find that average daily balance follows these steps:

1. Start with a list of transactions with their dates and amounts.
2. For each day that had transactions, find the total of the transactions for the day. Expenditures are treated as positive values, payments are treated as negative values.
3. Create a table containing each day with a different balance. The balance is the previous balance plus the day's total transactions.
4. Add a column for the number of days those balances until the balance changed.
5. Add a column that contains the balances multiplied by the number of days until the balance changed.
6. Find the sum of that last column.
7. Divide the sum by the number of days in the billing period (often the number of days in the month). This is the average daily balance.

Example 8.4.3: Computing Average Daily Balance

The billing cycle goes from May 1 to May 31. The balance at the start of the billing cycle is $450.21. The list of transactions on the card is below.

Date	Activity	Amount
1-May	Billing Date Balance	$450.21
10-May	Payment	$120.00
15-May	Groceries	$83.43
26-May	Auto Parts	$45.12
26-May	Restaurant	$85.34
30-May	Shoes	$98.23

Find the average daily balance for the credit card during the month of May.

Answer

To find the average daily balance, we use the following steps.

1. Start with a list of transactions with their dates and amounts.
   This list is provided.
2. For each day that had transactions, find the total of the transactions for the day.
   The only day with more than one transaction was May 26. The sum of those transactions is $130.46. Treating the payment as a negative value, the daily transaction amounts are

Date	Amount
1-May	$450.21
10-May	-$120.00
15-May	$83.43
26-May	$130.46
30-May	$98.23

3. Create a table containing each day with a different balance.
   The new table with dates that had different balances is below.

Date	Balance
1-May	$450.21
10-May	$330.21
15-May	$413.64
26-May	$544.10
30-May	$642.33

4. Now, add a column for the number of days those balances until the balance changed. The days until the balance changes is found by finding the difference in the dates. For instance, from May 15 to May 26 was 11. Adding that column to the table we have

Date	Balance	Days Until Balance Changes
1-May	$450.21	9
10-May	$330.21	5
15-May	$413.64	11
26-May	$544.10	4
30-May	$642.33	2

The last entry was 2 since there are 31 days in May.

5. Add a column that contains the balances multiplied by the number of days until the balance changed. We create the column and multiply the values.

Date	Balance	Days Until Balance Changes	Balance Times Days
1-May	$450.21	9	$4,051.89
10-May	$330.21	5	$1,651.05
15-May	$413.64	11	$4,550.04
26-May	$544.10	4	$2,176.40
30-May	$642.33	2	$1,284.66

6. Find the sum of that last column. Adding that last column we have a sum of $13,714.04.
7. There are 31 days in May, so divide the sum by 31, which gives an average of $442.39, which is the average daily balance.

Your Turn 8.4.3

The billing cycle goes from June 1 to June 30. The previous month's balance is $563.80. The transactions are in the table below.

Date	**Activity**	**Amount**
1-Jun	Balance	$563.80
2-Jun	Gasoline	$47.50
2-Jun	Groceries	$63.42
15-Jun	Movie	$38.75
15-Jun	Payment	$250.00
27-Jun	Pharmacy	$31.21
28-Jun	Auto fuel	$48.00

Find the average daily balance for this credit card.

Calculating the Interest for a Credit Card

The interest charged for a credit card is based on the daily interest rate of the card, the number of days in the billing cycle, and the average daily balance on the card.

FORMULA

The interest charge, I, for a credit card during a billing cycle is $I =$, where ADB is the average daily balance, r is the annual percentage rate, and d is the number of days in the billing cycle. As before, interest is rounded up to the next penny.

Example 8.4.4: Calculating Interest for a Credit Card Billing Cycle

Compute the interest charged for the credit card based on the given average daily balance (ABD), annual interest rate, and number of days in the billing cycle.

1. ADB = $2,765.00, annual interest rate 13.99%, billing cycle of 30 days
2. ADB = $789.30, annual interest rate 17.99%, billing cycle of 31 days
3. ADB = $1,037.85, annual interest rate 11.99%, billing cycle of 28 days

Answer

1. Substituting $2,765.00 for ADB, 0.1399 for r and 30 for d and calculating, we find the interest charge to be $I = \frac{\text{ADB} \times r \times d}{365} = \frac{\$2{,}765.00 \times 0.1399 \times 30}{365} = \$31.80$.
2. Substituting $789.30 for ADB, 0.1799 for r and 31 for d and calculating, we find the interest charge to be $I = \frac{\text{ADB} \times r \times d}{365} = \frac{\$789.30 \times 0.1799 \times 31}{365} = \$12.06$.
3. Substituting $1,037.85 for ADB, 0.1199 for r and 28 for d and calculating, we find the interest charge to be $I = \frac{\text{ADB} \times r \times d}{365} = \frac{\$1{,}037.85 \times 0.1199 \times 28}{365} = \$9.55$.

Your Turn 8.4.4

Compute the interest charged for the credit card based on the given average daily balance (ABD), annual interest rate, and number of days in the billing cycle.

ADB = $2,135.00, annual interest rate 12.9%, billing cycle of 30 days

ADB = $1,589.63, annual interest rate 9.99%, billing cycle of 31 days

ADB = $6,803.41, annual interest rate 14.9%, billing cycle of 28 days

Who Knew?: Credit Cards Charge Stores Fees

The interest you pay is not the only way a credit card company generates revenue. It also charges fees to the retailers, online stores, and service providers that allow the consumer, you, to use your credit card to pay them. These are called processing fees. Currently they typically range from 2.87% to 4.35% of each transaction. That means if you use your credit card at a store and spend $100.00, the store will have to pay the credit card company somewhere between $2.87 and $4.35.

One type of processing fee is the interchange fee. Mastercard charges the vendor 1.35% of the sale, plus an additional percentage up to 3.25%, and a fixed $001 fee for each transaction.

Added to that is an assessment fee. This fee is 0.14% for Visa cards.

Calculating the Balance of a Credit Card

The **balance,** or sometimes balance due, on a credit card is the previous balance, plus all expenses, minus all payments and credits, plus the interest on the card. As stated before, if the card was paid off, there is no interest to be paid.

Example 8.4.5: Calculating the Balance of a Credit Card

Find the balance on the credit card with the given interest charge and balance before interest was charged. The cards were not paid off previously.

1. Balance before interest is $708.50, interest charge is $8.15

2. Balance before interest is $1,395.10, interest charge is $21.32

Answer

1. Adding the balance before interest to the interest charge, we find the balance to be $716.65.
2. Adding the balance before interest to the interest charge, we find the balance to be $1,416.42.

Your Turn 8.4.5

Find the balance on the credit card with the given interest charge and balance before interest was charged. The cards were not paid off previously.

Balance before interest is $560.00, interest charge is $6.44

Balance before interest is $3,218.00, interest charge is $49.17

The next example puts all those steps together.

Example 8.4.6: Find Balance Due from Transactions and Interest Rate

Kaylen's credit card charges 16.9% annual interest. His current billing period is from November 1 to November 30. The balance on November 1 was $1,845.23. Use Kaylen's following transactions to determine his balance due at the end of the billing cycle.

Date	Activity	Amount
1-Nov	Billing Date Balance	$1,845.23
3-Nov	Groceries	$78.50
4-Nov	Tablet	$159.00
4-Nov	Online Game Purchase	$39.99
4-Nov	Restaurant	$47.10
10-Nov	Payment	$300.00
13-Nov	Gasoline	$58.75
13-Nov	Clothing	$135.00
18-Nov	Gift	$30.00
18-Nov	Restaurant	$21.75
28-Nov	Gasoline	$43.79

Answer

The first step is to find Kaylen's average daily balance. To find the average daily balance, we use the following steps.

1. Start with a list of transactions with their dates and amounts.
   This list is provided.
2. For each day that had transactions, find the total of the transactions for the day. The days with more than one transaction were Nov. 4, Nov. 13, and Nov. 18. Treating the payment on November 10 as a negative value, the daily transaction amounts are

Date	Amount
1-Nov	$1,845.23
3-Nov	$78.50
4-Nov	$246.09
10-Nov	-$300.00
13-Nov	$193.75
18-Nov	$51.75
28-Nov	$43.79

3. Create a table containing each day with a different balance. The new table with dates that had different balances is below.

Date	Balance
1-Nov	$1,845.23
3-Nov	$1,923.73
4-Nov	$2,169.82
10-Nov	$1,869.82
13-Nov	$2,063.57
18-Nov	$2,115.32
28-Nov	$2,159.11

4. Now, add a column for the number of days those balances until the balance changed. The days until the balance changes is found by finding the difference in the dates. For instance, from May 15 to May 26 was 11. Adding that column to the table we have
   The last entry was 3 since there are 30 days in November (the 28th, 29th, and 30th).

Date	Balance	Days Until Balance Changes
1-Nov	$1,845.23	2
3-Nov	$1,923.73	1
4-Nov	$2,169.82	6
10-Nov	$1,869.82	3
13-Nov	$2,063.57	5
18-Nov	$2,115.32	10
28-Nov	$2,159.11	3

5. Add a column that contains the balances multiplied by the number of days until the balance changed. We create the column and multiply the values.

Date	Balance	Days Until Balance Changes	Balance Times Days
1-Nov	$1,845.23	2	$3,690.46
3-Nov	$1,923.73	1	$1,923.73
4-Nov	$2,169.82	6	$13,018.92
10-Nov	$1,869.82	3	$5,609.46
13-Nov	$2,063.57	5	$10,317.85
18-Nov	$2,115.32	10	$21,153.20
28-Nov	$2,159.11	3	$6,477.33

6. Find the sum of that last column. Adding that last column we have a sum of $62,190.95.
7. There are 30 days in November, so divide the sum by 30, which gives an average of $2,073.03, which is the average daily balance.

With the average daily balance, we can determine the interest that is charged for November. Substituting ADB = $2,073.03, r = 0.169, and d = 30 into the formula $I =$ and calculating, we find the interest to be $I = \frac{\text{ADB} \times r \times d}{365} = \frac{\$2{,}073.03 \times 0.169 \times 30}{365} = \$28.80$.
This interest is added to the final balance from the table in step 3, $2,159.11, which yields a balance due of $2,101.83.

Your Turn 8.4.6

Angel's credit card charges 16.9% annual interest. His current billing period is from August 1 to August 31. The balance on August 1 was $982.45. Use Angel's following transactions to determine his balance due at the end of the billing cycle.

Date	**Activity**	**Amount**
1-Aug	Billing Date Balance	$982.45
5-Aug	Food	$125.31
13-Aug	Payment	$500.00
14-Aug	Gasoline	$51.65
14-Aug	Pizza	$36.99
14-Aug	Shoes	$89.45
19-Aug	Electric bill	$178.34
21-Aug	Internet	$36.99
21-Aug	Food	$93.45
30-Aug	Gasoline	$43.18

Minimum Payment Due

The **minimum payment** due is the smallest required amount to be paid on a credit card to avoid late fees and penalties, such as an increased interest rate. The calculations for this may differ from card to card. They also depend in the balance of the credit card. General guidelines for minimum payment due are:

- For larger balances (usually over $1,000), the minimum payment will be some percentage of the balance due.
- For moderate balances (between $25 and $1,000), the minimum would be a specified dollar amount. $25 seems to be a common value.

- If the balance is small (under $25 for instance), then the minimum payment is the balance.

Those are just guidelines. Individual cards may vary in these values.

Minimum payments should only be paid if money is short in a given month. The length of time to pay off a credit card using minimum payments is quite long, and results in paying a lot of interest. It is strongly discouraged.

Example 8.4.7: Calculate the Minimum Payment Due

The FYA credit card company has the following minimum payment policy. For balances over $1,000, the minimum payment is 2.5% of the balance due plus fees, but not interest. For balances between $500.00 and $999.99, the minimum payment is $50.00. For balances $499.99 and under, the minimum payment is $25.00 or the balance due, whichever is smaller.

In the following, calculate the minimum payment due given the credit card minimum payment policy, the balance due and fees charged.

1. Balance due is $1,309.00, no fees
2. Balance due is $265.50, $35 in fees
3. Balance due is $784.90, no fees

Answer

1. The balance is over $1,000, so the minimum payment is 2.5% of the balance due plus fees. 2.5% of the balance due is $0.025 \times \$1{,}309.00 = \$32.73$. Since there are no fees, the minimum payment due is $32.73.
2. The balance is under $499.99, so the minimum payment due is $25.00.
3. The balance is between $500.00 and $999.00, so the minimum payment due is $50.00.

Your Turn 8.4.7

The CLH credit card company has the following minimum payment policy. For balances over $1,000, the minimum payment is 1% of the balance due plus interest and fees. For balances between $25.00 and $999.99, the minimum payment is $25.00 plus any fees. For balances under $25.00, the minimum payment is the balance due plus any fees.
In the following, calculate the minimum payment due given the credit card minimum payment policy, the balance due and fees charged.

Balance due is $2,308.00, billing cycle interest is $24.39, no fees

Balance due is $265.50, $59.00 in fees

Balance due is $19.90, no fees

Check out this nerdwallet article about minimum payments for more!

Check Your Understanding

1. What are the three main types of credit cards?
2. Which type of credit card typically has the highest interest rate?
3. Name three important pieces of information that are included in a credit card statement.
4. A credit card has an average daily balance of $1,428.50 during a 31-day billing cycle. Find the interest if the interest rate is 15.9%.
5. The billing cycle goes from June 15 to July 14, or 30 days. The balance at the start of the billing cycle is $3,825.50. The list of transactions on the card is below. Find the average daily balance.

Date	Activity	Amount
15-Jun	Billing Date Balance	$3,825.50
20-Jun	Food	$125.31
20-Jun	Clothing	$345.00
29-Jun	Payment	$750.00
1-Jul	Restaurant	$94.80
6-Jul	Gasoline	$49.75
6-Jul	Home Repair Supplies	$683.94
6-Jul	Internet	$49.99
10-Jul	Cell Phone	$85.00
10-Jul	Food	$175.24

6. Find the balance due on a credit card statement if ADB = $487.65, the annual interest rate is 16.8%, the billing cycle was 30 days, and the balance at the end of the billing cycle, before interest is added, was $689.47.

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8.4.1: Exercises

Exercise 8.4.1.1

Name three main criteria to choose a credit card by.

Exercise 8.4.1.2

Which type of credit card is most convenient to use?

Exercise 8.4.1.3

Which type of credit card comes with no preset spending limit?

Exercise 8.4.1.4

How much interest does a credit card holder pay if they pay off the balance every month?

Exercise 8.4.1.5

Which type of credit card comes with the highest annual fees?

Use the following credit card statement for the following exercises.

September 2022						
S	M	T	W	T	F	S
28	29	30	31	1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	1
2	3	4	5	6	7	8

New Balance
$2,144.89

Minimum Payment Due
$50.00

Payment Due Date
09/26/22

Late Payment Warning: If we do not receive your minimum payment by the date listed above, you may have to pay a late fee of up to $39.00.

Minimum Payment Warning: If you make only the minimum payment each period, you will pay more in interest and it will take you longer to pay off your balance. For example:

If you make no additional charges using this card and each month you pay...	You will pay off the balance shown on this statement in about...	And you will end up paying an estimated total of...
Only the minimum payment	8 years	$3,837
$76	3 years	$2,738 (Savings = $1,099)

Previous Balance	$1,937.30
Payment, Credits	–$300.00
Purchases	+$478.31
Cash Advances	$0.00
Balance Transfers	$0.00
Fees Charged	$0.00
Interest Charged	+$29.28
New Balance	**$2,144.89**
Opening/Closing Date	08/02/22 - 09/01/22
Credit Access Line	$7,000
Available Credit	$4,855
Cash Access Line	$350
Available for Cash	$350
Past Due Amount	**$0.00**
Balance over the Credit Access Line	**$0.00**

? Exercise 8.4.1.6

What is the minimum payment due?

? Exercise 8.4.1.7

How long will it take to pay the balance if the minimum only is paid and no new purchases are made?

? Exercise 8.4.1.8

How much interest was charged in this billing cycle?

? Exercise 8.4.1.9

What is the balance on the credit card?

Exercise 8.4.1.10

What is the credit limit on this card?

Exercise 8.4.1.11

What is the late payment fee for this card?

Exercise 8.4.1.12

When is the payment due?

Exercise 8.4.1.13

The billing cycle for a credit card goes from April 1 to April 30. The balance at the start of the billing cycle is $1,598.00. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
1-Apr	Billing Date Balance	$1,598.00
9-Apr	Gasoline	$51.24
9-Apr	Food	$105.56
9-Apr	Payment	$675.00
13-Apr	Camping Trip	$229.75
21-Apr	Gasoline	$38.45
22-Apr	Gifts	$148.88
22-Apr	Food	$49.75
30-Apr	Gym Payment	$74.99

Exercise 8.4.1.14

The billing cycle for a credit card goes from September 1 to September 30. The balance at the start of the billing cycle is $384.25. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
1-Sep	Billing Date Balance	$384.25
2-Sep	Food	$94.54
5-Sep	Gasoline	$25.65
5-Sep	Internet	$39.99
6-Sep	Payment	$380.00
9-Sep	Insurance	$174.52
16-Sep	Food	$83.54
16-Sep	Day Care	$350.00
20-Sep	Tires	$2,337.56
21-Sep	Child Clothing	$27.65
21-Sep	Gasoline	$31.00
28-Sep	Television	$299.95

? Exercise 8.4.1.15

The billing cycle for a credit card goes from October 10 to November 9. The balance at the start of the billing cycle is $930.50. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
10-Oct	Billing Date Balance	$930.50
11-Oct	Clothing	$350.00
14-Oct	Computer	$865.84
17-Oct	Food	$106.51
21-Oct	Payment	$700.00
21-Oct	Restaurant	$134.52
21-Oct	Hotel	$387.56
30-Oct	Hockey Game	$76.47
5-Nov	Memorabilia	$150.00
5-Nov	Restaurant	$94.45
6-Nov	Gasoline	$49.19

? Exercise 8.4.1.16

The billing cycle for a credit card goes from February 15 to March 16 during a non-leap year. The balance at the start of the billing cycle is $292.82. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
15-Feb	Billing Date Balance	$292.82
21-Feb	Food	$64.57
22-Feb	Gasoline	$31.50
22-Feb	Food	$71.94
28-Feb	Insurance	$133.25
28-Feb	Payment	$100.00
3-Mar	Gasoline	$26.61
12-Mar	School Trip	$300.00

In the following exercises, compute the interest charged for the credit card based on the given average daily balance (ABD), annual interest rate, and number of days in the billing cycle.

Exercise 8.4.1.17

ADB = $350.00, annual interest rate 14.9%, billing cycle of 30 days.

Exercise 8.4.1.18

ADB = $4,312.00, annual interest rate 9.99%, billing cycle of 31 days.

Exercise 8.4.1.19

ADB = $563.38, annual interest rate 17.9%, billing cycle of 30 days.

Exercise 8.4.1.20

ADB = $1,043.53, annual interest rate 11.9%, billing cycle of 31 days.

In the following exercises, find the balance on the credit card with the given interest charge and balance before interest was charged. The cards were not paid off previously.

Exercise 8.4.1.21

Balance before interest is $1630.00, interest charge is $16.48.

Exercise 8.4.1.22

Balance before interest is $621.00, interest charge is $7.81.

Exercise 8.4.1.23

Balance before interest is $1,380.00, interest charge is $15.35

Exercise 8.4.1.24

Balance before interest is $2,774.00, interest charge is $44.05.

Exercise 8.4.1.25

Alsaggr's credit card charges 11.9% annual interest. His current billing period is from April 1 to April 30. The balance on April 1 was $1,598.00. Use Alsaggr's following transactions to determine his balance due at the end of the billing cycle.

Date	Activity	Amount
1-Apr	Billing Date Balance	$1,598.00
9-Apr	Gasoline	$51.24
9-Apr	Food	$105.56
9-Apr	Payment	$675.00
13-Apr	Camping Trip	$229.75
21-Apr	Gasoline	$38.45
22-Apr	Gifts	$148.88
22-Apr	Food	$49.75
30-Apr	Gym Payment	$74.99

Exercise 8.4.1.26

Marisa's credit card charges 8.9% annual interest. Her current billing period is from September 1 to September 30. The balance on September 1 was $384.25. Use Marisa's following transactions to determine her balance due at the end of the billing cycle.

Date	Activity	Amount
1-Sep	Billing Date Balance	$384.25
2-Sep	Food	$94.54
5-Sep	Gasoline	$25.65
5-Sep	Internet	$39.99
9-Sep	Insurance	$174.52
16-Sep	Food	$83.54
16-Sep	Day Care	$350.00
20-Sep	Tires	$2,337.56
21-Sep	Child Clothing	$27.65
21-Sep	Gasoline	$31.00
28-Sep	Television	$299.95

Exercise 8.4.1.27

Haley's credit card charges 18.9% annual interest. Her current billing period is from October 10 to November 9. The balance on October 10 was $930.50. Use Haley's following transactions to determine her balance due at the end of the billing cycle.

Date	Activity	Amount
10-Oct	Billing Date Balance	$930.50
11-Oct	Clothing	$350.00
14-Oct	Computer	$865.84
17-Oct	Food	$106.51
21-Oct	Payment	$700.00
21-Oct	Restaurant	$134.52
21-Oct	Hotel	$387.56
30-Oct	Hockey Game	$76.47
5-Nov	Memorabilia	$150.00
5-Nov	Restaurant	$94.45
6-Nov	Gasoline	$49.19

? Exercise 8.4.1.28

Pavly's credit card charges 10.9% annual interest. His current billing period is from February 15 to March 16 in a non-leap year. The balance on February 15 was $292.82. Use Pavly's following transactions to determine his balance due at the end of the billing cycle.

Date	Activity	Amount
15-Feb	Billing Date Balance	$292.82
21-Feb	Food	$64.57
22-Feb	Gasoline	$31.50
22-Feb	Food	$71.94
28-Feb	Insurance	$133.25
28-Feb	Payment	$100.00
3-Mar	Gasoline	$26.61
12-Mar	School Trip	$300.00

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SECTION OVERVIEW

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8.5.1: Key Terms

6.1 Understanding Percent

- Percent
- Fractional form
- Decimal form
- Total
- Base
- Percent of the total
- Part
- Amount

6.2 Discounts, Markups, and Sales Tax

- Discount
- Cost
- Markup
- Retail price

6.3 Simple Interest

- Interest
- Principal
- Annual percentage rate
- Simple interest
- Term
- Due
- Origination date
- Payoff amount
- Future value
- Partial payment
- Present value

6.4 Compound Interest

- Compound interest
- Effective annual yield

6.5 Making a Personal Budget

- Budget
- Necessary expenses
- Fixed expenses
- Variable expenses
- 50-30-20 budget philosophy

6.6 Methods of Savings

- Savings account
- 1099 form
- Certificate of deposit
- Money market account
- Return on investment
- Ordinary annuity

6.7 Investments

- Bonds
- Maturity date
- Stocks
- Dividend
- Mutual fund
- Prospectus
- Issue price
- Shares
- Stock table
- Individual retirement account
- Roth IRA
- 401(k)

6.8 The Basics of Loans

- Fixed interest rate
- Variable interest rate
- Installment loan
- Loan amortization
- Revolving credit
- Amortization table
- Cost of finance

6.9 Understanding Student Loans

- FAFSA
- College funding gap
- Subsidized loan
- Unsubsidized loan
- Parent loan for undergraduate students
- Private student loan
- School-channel loan
- Direct-to-consumer loan
- Standard repayment plan
- Federal consolidation
- Refinancing
- Private consolidation
- Graduated repayment plan
- Extended repayment plan
- Discretionary income
- Pay as you earn (PAYE) repayment plan
- Revised pay as you earn (REPAYE) repayment plan
- Income-based (IBR) repayment plan
- Income-contingent (ICR) repayment plan
- Delinquent
- Default
- Rehabilitation

6.10 Credit Cards

- Reward program
- Annual fee

- Credit limit
- Bank-issued credit card
- Store-issued credit card
- Travel and entertainment cards
- Charge cards
- Billing period
- Balance
- Minimum payment
- Average daily balance

6.11 Buying or Leasing a Car

- Title and registration fees
- Destination fee
- Documentation fee
- Dealer preparation fee
- Extended warranty
- Down payment
- Acquisition fee
- Security deposit
- Disposition fees
- Liability insurance
- Collision insurance
- Comprehensive insurance
- Uninsured or underinsured motorist insurance
- Medical payment insurance
- Personal injury insurance
- Gap insurance
- Rental reimbursement insurance

6.12 Renting and Homeownership

- Lease
- Landlord
- Mortgage
- Escrow account
- Assessed value

6.13 Income Tax

- Gross income
- Adjusted gross income
- Exemption
- Taxable income
- Deduction
- Tax credit
- Earned income credit
- American opportunity credit
- Lifetime learning credit
- Child tax credit
- Child and dependent care tax credit
- Premium tax credit

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8.5.2: Key Concepts

6.1 Understanding Percent

- Know what a percent is as a fraction, a decimal, and as a part of the whole.
- Use the percent equation to find any of the three values that are related by the equation.
- Apply the percent equation in applications.

6.2 Discounts, Markups, and Sales Tax

- Discounts are markdowns from an original price.
- Mark-ups are increases to the price paid by a retailer to cover their costs.
- be able to calculate the markup based on a percentage of the cost
- Sales taxes vary from state to state and often county to county.
- Retail prices, sales prices and percent discounts can be calculated if the other two values are known.
- Original costs, retail prices, and percent markup can be calculated if the other two values are known.
- In calculations, sales tax acts like a markup.

6.3 Simple Interest

- Interest is money that is paid by a borrower for the privilege of borrowing the money.
- Simple interest is computed by substituting the principal, interest rate, and number of years into the formula $I =$
- The payoff for a loan is the amount of principal remaining on a loan plus the interest that accumulated on the loan since the last payment.
- The future value of an investment yielding simple interest is the original principal plus the interest earned on the investment.
- When making a partial payment, some of the payment pays off all the accumulated interest, while the remainder of the payment is applied to the principal of the loan.
- Finding the present value of an investment is used to determine how much should be invested now in order to achieve a specific goal.

6.4 Compound Interest

- Compound interest means that the interest earned during one period will earn interest in later periods. Essentially, the amount of the principal grows from period to period.
- The important values in computing compound interest are the interest rate, the principal, the length of time the investment, and the number of times the investment is compounded.
- Compound interest has minimal impact early, but later has a very large impact.
- You can determine how much to invest today in order to reach a goal for some time later.
- Compound interest can be translated into an effective annual yield, which allows for comparison between investment options.

6.5 Making a Personal Budget

- A budget is a set of guidelines for how to allocate your income.
- Budgeting helps to plan for many of life's expenses
- Budgets are used to compare income to expenses. When expenses exceed income, changes have to be made.
- Budgets can help evaluate the affordability of life changes.
- One guideline for setting a budget is the 50-30-20 budget philosophy. The guidelines suggest that 50% of income is allocated to necessary expenses, 30% to expenses that wants, and 20% to savings and other debt reduction.

6.6 Methods of Savings

- There are three main types of savings accounts, saving accounts, certificates of deposit (CD), and money market accounts.
- Savings account are very risk free, and so yield low interest rates.
- The differences in the three types of savings accounts relate to their convenience.
- Savings account typically have a lower interest rate that money market accounts, which typically have lower interest rates than CDs.

- Ordinary annuities more accurately reflect how we save, in that money is deposited repeatedly over time.
- Spreadsheet software, such as Google Sheets, have built in functions that can be used to quickly calculate both the future value of an ordinary annuity account, but also the payment necessary to reach a goal using an ordinary annuity.

6.7 Investments

- There are many different investments with different returns and risks.
- Bonds are loans form the purchaser to the entity selling the bond.
- Bonds have some tax benefits, low to no risk, and a low return.
- Stocks represent part ownership in a company. As such, stock holders share in the profits, and losses, of the company.
- Information, including price, P/E, yearly highs and lows, and dividend amount can be found in online stock tables available on many websites.
- Mutual funds represent collections of professionally administered investment vehicles. Have shares in a mutual fund has lower risk than ownership of stocks.
- Retirement accounts employ some of the same strategies as mutual funds, in that they spread the risk and are professionally managed.
- IRAs and Roth IRAs differ on when taxes are paid on the money, and who can use them. Roth IRAs have income limits while traditional IRAs do not.

6.8 The Basics of Loans

- There are many reasons for a loan, but primarily it is taken out for a large expense when cash is not available.
- Each payment for an installment loan consists of an interest portion and a principal portion.
- There is a formula to calculate the payment necessary to pay off a loan in installments.
- Amortization schedules, or tables, show how each payment is applied to principal and interest. It also includes other details such as remaining balance and total interest paid.
- Loans often have other fees associated with them such as origination fees or application fees. The total of the interest paid and the fees is the cost of finance.

6.9 Understanding Student Loans

- The FAFSA must be filled out each year that a student wishes to borrow for.
- A student's funding gap determines how much they need in loans to pay for college.
- Federal subsidized student loans defer payments until after graduation and interest does not accrue on these loans.
- Unsubsidized student loans defer payment until after graduation but interest begins accruing as soon as the loan finds are disbursed.
- There are both yearly and aggregate limits for student loans to prevent over-borrowing, among other reasons.
- Federal direct loans have a low interest rate set by the government, but other student loans have varying rates of interest set by the banks.
- The standard repayment plan lasts 10 years and is made up of monthly payments.
- Consolidating or refinancing student loans merges many student loans into one loan.
- If only federal loans are consolidated, the interest rate is the same as the individual loans, currently set at 4.99%.
- If other loans are refinanced together, the interest rate may be lower with the new loan.
- Other repayment plans are available. Such a plan may have payment that start small and grow as the loan is paid off, or it may have a longer term, or may be based on the discretionary income of the student.
- Being delinquent on a student loan is a precursor to being in default. Making payments in a timely fashion allows the student to avoid this situation.

6.10 Credit Cards

- Credit cards can be a flexible way to pay for almost anything, but can become a financial hazard if used unwisely.
- When deciding which credit card to apply for, evaluate the interest rate, fees (annual and late), reward programs and credit limit. Be sure they meet your criteria.
- Paying off the balance of your credit card every month will control your spending and will never result in paying interest.

- Credit card statements hold all important information about your credit card, including payment, balances, charges and billing cycle dates.
- Although the minimum payment is attractive precisely because it is so small, paying only the minimum results is a long payoff term and higher interest costs.

6.11 Buying or Leasing a Car

- There are many factors to consider when choosing to buy or lease a car.
- The cost of the car is increased by a number of fees and sales tax.
- There are advantages to buying a car and advantages to leasing a car. The decision between the two depends on the preference of the buyer.
- Insurance covers costs associate with accidents. It is made up of various components.
- The costs of owning a car, including insurance and maintenance, should be a part of the budgeting process.
- Budgeting for unexpected repairs can ease the stress of encountering large repair bill.

6.12 Renting and Homeownership

- There are many points of comparison between renting and buying a house.
- Before deciding to buy a house, you should carefully consider all the responsibilities that come with home ownership.
- Renting comes with more restrictions on the renter, but with fewer costs and is easier to move from.
- Owning a house has more costs but has more freedom, plus the owner creates equity.
- Mortgages are loans, and payments are calculated in the same way as any other loan.
- Amortization tables help a homeowner understand the mortgage and how the payments are applied to the principal and interest.
- In addition to paying the amount financed for a mortgage, the monthly payment will include an escrow payment, which covers insurance and taxes.

6.13 Income Tax

- Federal income tax is based on income after certain adjustments.
- Gross income is income from all sources, including gifts and winnings.
- Before taxes are calculated, the taxable income is found by subtracting deductions and exemptions from gross income.
- Income tax is progressive, increasing in rate as income increases.
- Being in the 32% tax bracket means some of your income is taxed at 10%, some at 12%, some at 22%, some at 24%, and the rest at 32%.
- Income in each tax bracket is taxed at that bracket's rate, which means in 2022 the first $10,275 earned is taxed at 10% only.
- Tax credits are subtracted from the taxes that are owed.
- Some tax credits are refundable, which means they can make the amount you owe negative, which results in a refund.

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8.5.3: Videos

6.1 Understanding Percent

- Finding Percent of a Total
- Finding the Total from the Percent and the Part
- Finding the Percent When the Total and the Part Are Known

6.2 Discounts, Markups, and Sales Tax

- Computing Price Based on a Percent Off Coupon
- Finding Sales Tax Percentage

6.4 Compound Interest

- Compound Interest
- Compare Simple Interest to Interest Compounded Annually
- Compare Simple Interest and Compound Interest for Different Number of Periods Per Year

6.5 Making a Personal Budget

- Creating a Budget
- 50-30–20 Budget Philosophy

6.6 Methods of Savings

- Return on Investment, ROI
- Future Value Using Google Sheets

6.7 Investments

- Bonds
- Reading Stock Summary Online
- Mutual Funds
- 401(k) Accounts

6.8 The Basics of Loans

- Credit Scores Explained
- Reading an Amortization Table

6.9 Understanding Student Loans

- Types of Student Loans
- Repayment Plans

6.10 Credit Cards

- Choosing a Credit Card
- Reading Credit Card Statements

6.12 Renting and Homeownership

- Rent or Buy
- Closing Costs

6.13 Income Tax

- Deductions Versus Credits
- Tax Brackets

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8.5.4: Formula Review

Formula Review

6.1 Understanding Percent

part = percent $\cdot$ total

6.2 Discounts, Markups, and Sales Tax

discount = percent discount $\times$ original price

sale price = original price - discount

sale price = original price - percent discount $\times$ original price = original price $\times$(1 $-$ percent discount)

markup = percent markup $\times$ cost

retail price = cost + markup

retail price = cost + percent markup $\times$ cost = cost $\times$(1 + percent markup)

sales tax = purchase price $\times$ tax rate

Total price = purchase price + purchase price $\times$ tax rate = purchase price $\times$(1 + tax rate)

6.3 Simple Interest

$$I = P \times r \times t$$
$$T = P + I$$
$$T = P + P \times r \times t$$
$$I = P \times \frac{r}{365} \times t$$
$$FV = P + I = P + P \times r \times t$$
$$A = P \times \frac{r \times (1+r)^t}{(1+r)^t - 1}$$
$$PV = \frac{FV}{(1+rt)}$$

6.4 Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$PV = \frac{A}{\left(1 + \frac{r}{n}\right)^{n \times t}}$$
$$Y = \left(1 + \frac{r}{n}\right)^{n} - 1$$

6.6 Methods of Savings

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$\text{ROI} = \frac{FV - P}{P}$$
$$FV = pmt \times \frac{(1 + r/n)^{n \times t} - 1}{r/n}$$
$$pmt = \frac{FV \times (r/n)}{(1 + r/n)^{n \times t} - 1}$$

6.7 Investments

$$\text{annual return } = \left(\frac{FV}{P}\right)^{\left(\frac{1}{t}\right)} - 1$$

$$\text{P/E} = \frac{\text{Share Price}}{\text{Dividend}}$$

$$\text{Yld\%} = \frac{\text{Annual Dividend}}{\text{Share Price}} \times 100\%$$

6.8 The Basics of Loans

$$I = P \times \frac{r}{n}$$

$$pmt = \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} - 1}$$

6.9 Understanding Student Loans

funding gap = total cost - all aid

$$A = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

$$pmt = \tfrac{P \times (r/n) \times (1+r/n)^{n \times t}}{(1+r/n)^{n \times t} - 1}$$

discretionary income $=$ gross income $-1.5\times$ poverty guideline

6.10 Credit Cards

$$I = \frac{\text{ADB} \times r \times d}{365}$$

6.11 Buying or Leasing a Car

$$pmt = \frac{P \times (r/n) \times (1 + r/n)^{n \times t}}{(1 + r/n)^{n \times t} - 1}$$

$$\text{MD} = \frac{P - R}{n}$$

$$\text{APR} = 2400 \times \text{MF}$$

$$\text{MF} = \text{APR}/2,400$$

$$PMT = \frac{(P - R)}{n} + (P + R) \times \text{MF}$$

6.12 Renting and Home Ownership

$$pmt = \frac{P \times (r/12) \times (1 + r/12)^{12 \times t}}{(1 + r/12)^{12 \times t} - 1}$$

$$T = pmt \times 12 \times t$$

$$\text{CoF} = T - P$$

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8.5.5: Projects

Creating Your Future Budget

In this project, you will create a budget based on a job you are likely to have after you graduate.

1. Go online and research the average starting salary for the profession you are studying for. Use at least two sources. Be sure to record the web address from your search.
2. Approximate your monthly take-home pay. You may use the SmartAsset website to estimate this.
3. Use the 50-30-20 budget philosophy to determine how much you should budget for needs, wants and savings, or extra debt reduction.
4. Create a list of likely expenses. This list must include rent/mortgage, utilities, food, and school loan repayment. You may also want to include car payments, gasoline, and other items.
5. Categorize each expense as need, want, or savings.
6. Using the amounts found in step 3, decide how much to allocate to each of your expenses. It may help to quickly research how much rent is where you want to live.
7. Discuss the choices you had to make, and why you prioritized some expenses over others.

Interest Rate and Time: What Is the Relationship?

The interplay between interest rate and time for an annuity is not easily seen. How the amount that must be deposited per compounding period, pmt, changes based on the time and interest rate would be useful to understand. In this project, you will explore this relationship. We will use a fixed future value of FV = \$1,000,000 and a fixed number of periods per year, 12 (monthly compounding). With those, we'll find various annuity payments that must be made to reach the goal.

The annual interest rate for the investment is in the top row. The number of years for the investment is in the left column. In each cell (or box), find the monthly payment necessary to reach the goal of \$1,000,000.

		Annual Interest Rate					
		1.5%	2.0%	3.0%	5.0%	7.5%	10.0%
Number of Years	10						
	15						
	20						
	30						
	40						
	45						

Describe how the interest rates and number of years impact the payment necessary to reach the goal of \$1,000,000.

Finding a Home

In this project you will identify a home you like, and then estimate the costs associated with that home.

1. Find a home in your region that you would like to buy using an online search of listings in your area. Zillow is a good place to begin.
2. Find the asking price for this home. Assume you would pay that price.
3. Find an estimation for closing costs in your area. Assume you finance those costs also.
4. Estimate the taxes to be paid on the home per year. It is likely that the online listing of the home has an estimate for the taxes for the house.
5. Use Google to determine the average homeowner's insurance cost in your region.

6. Use the internet to determine the average interest rate for a 30-year mortgage.
7. Find how much you would pay per month, based on the answers to the previous questions, including the escrow payments for taxes and insurance.
8. Assume you will pay $50 per $100,000 borrowed in PMI. Add this to the monthly payment.

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8.5.6: Chapter Review

Understanding Percent

Exercise 8.5.6.1

Convert 0.45 to percent form.

Exercise 8.5.6.2

What is 70% of 200?

Exercise 8.5.6.3

Rewrite the fraction $\frac{2}{4}0100$ as a percentage.

Exercise 8.5.6.4

20 is what percent of 500?

Discounts, Markups, and Sales Tax

Exercise 8.5.6.5

An item with a retail price of $250.00 is on sale for 30% off. What is the sale price of the item? Round to the nearest penny.

Exercise 8.5.6.6

The sales price of an item is $38.50 after a 40% discount. What was the retail price of the item? Round to the nearest penny.

Exercise 8.5.6.7

A retailer buys an item for $1,000.00 and has a 65% markup. What is the retail price of the item? Round to the nearest penny.

Exercise 8.5.6.8

The sale price of an item is $46.00 and the retail price is $65.00. What is the percent discount? Round to two decimal places.

Exercise 8.5.6.9

The retail price of an item is $345.38 and sales tax in the region is 8.25%. How much is the sales tax?

Simple Interest

Exercise 8.5.6.10

$3,000 is invested in a 5-year CD earning 2.25% interest. How much is the CD worth in 5 years?

Exercise 8.5.6.11

A $4,000.00 simple interest loan is taken out for 3 years at 12.5%. How much is owed when the loan comes due?

Exercise 8.5.6.12

A $10,000 loan with annual simple interest of 14.9% is taken out for 90 days. How much is due in 90 days?

Exercise 8.5.6.13

A simple interest loan for $20,000 is taken out at 13.9% annual interest rate. A partial payment of $13,500.00 is made 30 days into the loan period. After this payment, what will the remaining balance be?

Exercise 8.5.6.14

Find the present value of an investment with future value of $15,000 if the investment earns 3.55% simple interest for 10 years?

Compound Interest

Exercise 8.5.6.15

Find the future value after 20 years of $15,000 deposited in an account bearing 4.26% interest compounded quarterly.

Exercise 8.5.6.16

What is the effective annual yield for an account bearing 3.21% interest compounded quarterly? Round to two decimal places if necessary.

Exercise 8.5.6.17

Find the present value of $100,000 in an account bearing 5.25% interest compounded daily after 30 years.

Making a Personal Budget

Exercise 8.5.6.18

In budgeting, what is the difference between an expense that is a need and an expense that is a want?

Exercise 8.5.6.19

Apply the 50-30-20 budget philosophy to a monthly income of $6,000.00.

Exercise 8.5.6.20

Create the budget for a person with the following income and bills, and determine how much the income exceeds or falls short of the expenses:
Job = $5,250, Side job = $550, rent = $1,150.00, utilities = $150.00, internet = $39.99, student loans = $375.00, food = $550.00, bus and subway = $112.00, credit cards = $200.00, entertainment = $400.00, clothing = $150.00.

Exercise 8.5.6.21

In question 20, the budget for a person was created. Can the person afford to buy a car if the monthly payments will be $365.50, monthly car insurance will be $114.75, and gasoline will be $200.00?

Methods of Savings

Exercise 8.5.6.22

Why is a CD less flexible than a money market account?

Exercise 8.5.6.23

Which is likely to have the highest interest rate, a savings account, CD, or money market account?

Exercise 8.5.6.24

$3,000 is deposited in a money market account bearing 3.88% interest compounded monthly. How much is the account worth in 15 years?

Exercise 8.5.6.25

Refer to question 24. What is the return on investment for the money market account? Round to two decimal places, if necessary.

Exercise 8.5.6.26

$300 is deposited monthly in an ordinary annuity that bears 6.5% interest compounded monthly. How much is in the annuity after 30 years?

Exercise 8.5.6.27

How much must be deposited quarterly in an account bearing 4.89% interest compounded quarterly for 30 years so the account has a future value of $500,000?

Investments

Exercise 8.5.6.28

Why is investing in a stock riskier than in a mutual fund?

Exercise 8.5.6.29

Which investment offers high rates of return with reduced risk?

Exercise 8.5.6.30

Which investment offer a fixed rate of return over time?

Exercise 8.5.6.31

Which is the riskiest investment?

Exercise 8.5.6.32

How much is earned on a 10-year bond with issue price of $1,000 that pays 3.9% annually?

Exercise 8.5.6.33

Henri's employer matches up to 6% of salary for IRA contributions. Henri's annual income is $52,500. If Henri wants to deposit $3,000 annually in his IRA, how much in matching funds does the employer deposit in the IRA?

Exercise 8.5.6.34

How much must be deposited quarterly in a mutual fund that is expected to earn 11.2% interest compounded quarterly if the account is to be worth $1,000,000 after 38 years?

Exercise 8.5.6.35

Lorraine purchases stocks for $5,000. The stocks paid dividends by reinvesting the dividends in stock. She sold the stock for $13,750 after 6 years. What was Lorraine's annual return on that investment? Round to two decimal places if necessary.

Exercise 8.5.6.36

What are two ways stocks earn money?

The Basics of Loans

Exercise 8.5.6.37

What is an installment loan?

Exercise 8.5.6.38

An installment loan with monthly payments has an outstanding balance of $3,560. If the annual interest rate is 9%, how much interest will be paid that month?

Exercise 8.5.6.39

A loan for $42,000 is taken out for 6 years at 7.5% interest. What are the payments for that loan?

Exercise 8.5.6.40

The total interest paid on a loan was $4,500. The loan had an origination fee of $400.00, a $125.00 processing fee, and a filing fee of $250.00. How much was the cost of financing for that loan?

Exercise 8.5.6.41

Using the amortization table below, what is the remaining balance after payment 25?

Amortization Schedule Calculator					
Loan Amount	$17,500.00				
Interest Rate (annual)	8.25%				
Term (Years)	5				
Monthly Payment	$356.93				
Month	Payment	Principal	Interest	Total Interest	Balance
					$17,500.00
1	$356.93	$236.62	$120.31	$120.31	$17,263.38
2	$356.93	$238.25	$118.69	$239.00	$17,025.13
3	$356.93	$239.89	$117.05	$356.05	$16,785.24
4	$356.93	$241.54	$115.40	$471.44	$16,543.71
5	$356.93	$243.20	$113.74	$585.18	$16,300.51
6	$356.93	$244.87	$112.07	$697.25	$16,055.64
7	$356.93	$246.55	$110.38	$807.63	$15,809.09
8	$356.93	$248.25	$108.69	$916.32	$15,560.84
9	$356.93	$249.95	$106.98	$1,023.30	$15,310.89
10	$356.93	$251.67	$105.26	$1,128.56	$15,059.22
11	$356.93	$253.40	$103.53	$1,232.09	$14,805.82
12	$356.93	$255.14	$101.79	$1,333.88	$14,550.67
13	$356.93	$256.90	$100.04	$1,433.92	$14,293.77
14	$356.93	$258.66	$98.27	$1,532.19	$14,035.11
15	$356.93	$260.44	$96.49	$1,628.68	$13,774.66
16	$356.93	$262.23	$94.70	$1,723.38	$13,512.43
17	$356.93	$264.04	$92.90	$1,816.28	$13,248.39
18	$356.93	$265.85	$91.08	$1,907.36	$12,982.54
19	$356.93	$267.68	$89.25	$1,996.62	$12,714.86
20	$356.93	$269.52	$87.41	$2,084.03	$12,445.34
21	$356.93	$271.37	$85.56	$2,169.59	$12,173.97
22	$356.93	$273.24	$83.70	$2,253.29	$11,900.73
23	$356.93	$275.12	$81.82	$2,335.11	$11,625.62
24	$356.93	$277.01	$79.93	$2,415.03	$11,348.61
25	$356.93	$278.91	$78.02	$2,493.06	$11,069.69
26	$356.93	$280.83	$76.10	$2,569.16	$10,788.86

Understanding Student Loans

Exercise 8.5.6.42

If a college program takes 5 years, how long is the student's eligibility period for student loans?

Exercise 8.5.6.43

What is the funding gap for a student with cost of college equal to $35,750 and non-loan financial aid of $24,150?

Exercise 8.5.6.44

What is the maximum amount of federal subsidized and unsubsidized loans for a dependent student in their third year of an undergraduate data science program?

Exercise 8.5.6.45

A single person has gross income of $28,500. Their poverty guideline is $12,000. What is their discretionary income?

Credit Cards

Exercise 8.5.6.46

The billing cycle for a credit card goes from March 15 to April 14. The balance at the start of the billing cycle is $450.00. The list of transactions on the card is below. Find the average daily balance for the billing cycle.

Date	Activity	Amount
15-Mar	Billing Date Balance	$450.00
21-Mar	Movie	$50.00
28-Mar	Gasoline	$65.00
31-Mar	Snacks	$15.50
1-Apr	Dinner	$63.60
1-Apr	Payment	$300.00
1-Apr	Gasoline	$48.90
9-Apr	Plane Tickets	$288.50

Exercise 8.5.6.47

The average daily balance for Greg's last credit card statement was $1,403.50. The card charges 15.9% interest. If the billing cycle for that statement was 30 days, how much interest is Greg charged?

Exercise 8.5.6.48

The billing cycle for a credit card goes from March 15 to April 14. The balance at the start of the billing cycle is $450.00. The list of transactions on the card is below. The interest rate is 15.9%. What is the balance due at the end of the billing cycle?

Date	Activity	Amount
15-Mar	Billing Date Balance	$450.00
21-Mar	Movie	$50.00
28-Mar	Gasoline	$65.00
31-Mar	Snacks	$15.50
1-Apr	Dinner	$63.60
1-Apr	Payment	$300.00
1-Apr	Gasoline	$48.90
9-Apr	Plane Tickets	$288.50

Buying or Leasing a Car

Exercise 8.5.6.49

Name three common fees when leasing a car.

Exercise 8.5.6.50

Name three common fees when buying a car.

Exercise 8.5.6.51

If the cost of a car is $35,500, and the residual value of the car after 3 years is $27,800, what is the monthly depreciation for the car?

Exercise 8.5.6.52

What is the annual interest rate if the money factor is 0.000045?

Exercise 8.5.6.53

What is the payment for a lease on a car if the price of the car is $29,900, the residual price of the car is $16,500, the lease is for 24 months, and the APR is 8.9%?

Renting and Homeownership

Exercise 8.5.6.54

Name two advantages of renting a residence.

Exercise 8.5.6.55

Name two advantages of buying a home.

Exercise 8.5.6.56

What is PMI?

Exercise 8.5.6.57

What are the monthly payments for a 25-year mortgage for $165,000 at 4.8% interest, not including escrow?

Exercise 8.5.6.58

What are the monthly payments, including escrow, for a 25-year mortgage for $165,000 at 4.8% interest, if the property taxes are $5,200 annually, the homeowner's insurance is $825 every 6 months, and the PMI is $45 per month?

Income Tax

Exercise 8.5.6.59

What is the maximum gift that a person can receive that is an exemption on federal income taxes?

Exercise 8.5.6.60

Find the gross income if a person earns $51,500 at their full-time job, they earn $5,300 for contracted services, and $5,000 in gambling winnings.

Exercise 8.5.6.61

Find adjusted gross income for a person with a gross income of $67,850, if they had a $2,000 gift, they paid $1,000 in mortgage interest, $4,250 in property taxes, and they contributed $750 to a qualifying charity.

Exercise 8.5.6.62

How much SSI is paid by a person with an annual income of $183,500?

Exercise 8.5.6.63

How much in FICA taxes are paid from a person's paycheck with gross income of $3,450?

Exercise 8.5.6.64

How much in federal income tax does a person filing single pay if their taxable income is $205,240? Use the tax table below.

Bracket	Lower Income Limit	Upper Income Limit	Tax Rate
1	0	$10,275	10%
2	$10,276	$41,775	12%
3	$41,776	$89,075	22%
4	$89,076	$170,050	24%
5	$170,051	$215,950	32%
6	$215,951	$539,900	35%
7	$539,901		37%

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8.5.7: Chapter Test

Exercise 8.5.7.1

45 is what percent of 180?

Exercise 8.5.7.2

Find 47% of 31. Round to two decimal places.

Exercise 8.5.7.3

What is the sale price of a $90 shirt if there is a 25% discount for the shirt?

Exercise 8.5.7.4

A small boutique buys purses for $27.50 per purse. What is the boutique's retail price for the purses if they add a 90% markup?

Exercise 8.5.7.5

What was the discount on a computer is the retail price was $1,200 and the sale price is $950? Round to two decimal places if necessary.

Exercise 8.5.7.6

A 15% simple interest loan of $2,000 is taken out for 75 days. How much is owed when the loan is due?

Exercise 8.5.7.7

What is the future value of a $15,000 investment made at 4.65% interest compounded yearly for 10 years?

Exercise 8.5.7.8

Apply the 50-30-20 budget philosophy to a monthly income of $3,920.

Exercise 8.5.7.9

$2,500 is deposited in an account that bears 5.25% interest compounded quarterly for 15 years. What is the return on investment for that investment? Round to two decimal places if necessary.

Exercise 8.5.7.10

How much per month must be paid into an ordinary annuity so its future value is $150,000 after 10 years if the annuity earns 3.85% interest compounded monthly?

Exercise 8.5.7.11

Bethanie's employer matches up to 7% of annual salary for deposits into the company sponsored IRA. Bethanie earns $62,400 annually. She deposits $8,000 annually inter her IRA. How much does the company deposit in Bethanie's IRA?

Exercise 8.5.7.12

If $360 is deposited monthly in a mutual fund that is expected to earn 9.35% compounded monthly, how much will the mutual fund be worth after 23 years?

Exercise 8.5.7.13

A $14,250, 4-year loan is taken out at 9.15% interest. What are the monthly payments for the loan?

Exercise 8.5.7.14

Using the amortization table below, how much total interest has been paid after payment 20?

Amortization Schedule Calculator					
Loan Amount	$17,500.00				
Interest Rate (annual)	8.25%				
Term (Years)	5				
Monthly Payment	$356.93				
Month	Payment	Principal	Interest	Total Interest	Balance
					$17,500.00
1	$356.93	$236.62	$120.31	$120.31	$17,263.38
2	$356.93	$238.25	$118.69	$239.00	$17,025.13
3	$356.93	$239.89	$117.05	$356.05	$16,785.24
4	$356.93	$241.54	$115.40	$471.44	$16,543.71
5	$356.93	$243.20	$113.74	$585.18	$16,300.51
6	$356.93	$244.87	$112.07	$697.25	$16,055.64
7	$356.93	$246.55	$110.38	$807.63	$15,809.09
8	$356.93	$248.25	$108.69	$916.32	$15,560.84
9	$356.93	$249.95	$106.98	$1,023.30	$15,310.89
10	$356.93	$251.67	$105.26	$1,128.56	$15,059.22
11	$356.93	$253.40	$103.53	$1,232.09	$14,805.82
12	$356.93	$255.14	$101.79	$1,333.88	$14,550.67
13	$356.93	$256.90	$100.04	$1,433.92	$14,293.77
14	$356.93	$258.66	$98.27	$1,532.19	$14,035.11
15	$356.93	$260.44	$96.49	$1,628.68	$13,774.66
16	$356.93	$262.23	$94.70	$1,723.38	$13,512.43
17	$356.93	$264.04	$92.90	$1,816.28	$13,248.39
18	$356.93	$265.85	$91.08	$1,907.36	$12,982.54
19	$356.93	$267.68	$89.25	$1,996.62	$12,714.86
20	$356.93	$269.52	$87.41	$2,084.03	$12,445.34
21	$356.93	$271.37	$85.56	$2,169.59	$12,173.97
22	$356.93	$273.24	$83.70	$2,253.29	$11,900.73
23	$356.93	$275.12	$81.82	$2,335.11	$11,625.62
24	$356.93	$277.01	$79.93	$2,415.03	$11,348.61
25	$356.93	$278.91	$78.02	$2,493.06	$11,069.69
26	$356.93	$280.83	$76.10	$2,569.16	$10,788.86

Exercise 8.5.7.15

What is the maximum amount of federal subsidized and unsubsidized loans per year for an independent student in a graduate program?

Exercise 8.5.7.16

Braden takes out a private student loan for $6,250 in his second year of his undergraduate IT program. He begins paying the loan after 39 months. If Braden's interest rate was 8%, how much is his balance when he begins to pay off the loan?

Exercise 8.5.7.17

The billing cycle for Therese's credit card is July 7 to August 6 (31 days). The interest rate on the card is 17.9%. The balance on her card at the start of the billing cycle is $925.00. The table below has her transactions for the billing cycle. What is her balance due at the end of the cycle?

Date	Activity	Amount
7-Jul	Billing Date Balance	$925.00
10-Jul	Child Care	$850.00
10-Jul	Gasoline	$51.00
12-Jul	Food	$135.50
18-Jul	Payment	$750.00
18-Jul	Pizza	$35.00
18-Jul	Gift	$25.00
21-Jul	Food	$121.75
30-Jul	Gasoline	$47.50
1-Aug	Department store	$173.00
4-Aug	Food	$98.00

Exercise 8.5.7.18

The cost of a car is $45,750. The residual price after 2 years is $28,822.50. What is the monthly depreciation for the car?

Exercise 8.5.7.19

Find the payment for a 3-year lease on a car if the price of the car is $30,000, the residual price of the car is $20,000, and the APR is 8.5%.

Exercise 8.5.7.20

What are the monthly payments, excluding escrow, for a 20-year mortgage for $131,500 at 5.15% interest?

Exercise 8.5.7.21

What are the monthly payments, including escrow, for a 20-year mortgage for $131,500 at 5.15% interest, with property taxes of $3,170 per year and homeowner's insurance of $715 every 6 months?

Exercise 8.5.7.22

If a person earns $2700 gross on a paycheck, how much is taken out in FICA taxes?

Exercise 8.5.7.23

Find the taxable income for a person who has $81,200 in earnings, $1,250 reported on a 1099-INT for a savings account, $5,400 reported on a 1099-MISC for a contract job they did, paid $5,250 in mortgage interest, paid $4,700 in property taxes, and has $4,200 in bond interest.

Exercise 8.5.7.24

Find the federal income tax owed by a person filing single who has a taxable income of $114,750. Use the table below to determine the taxes.

Bracket	Lower Income Limit	Upper Income Limit	Tax Rate
1	0	$10,275	10%
2	$10,276	$41,775	12%
3	$41,776	$89,075	22%
4	$89,076	$170,050	24%
5	$170,051	$215,950	32%
6	$215,951	$539,900	35%
7	$539,901		37%

Exercise 8.5.7.25

If a person owes $20,000 in federal income tax, but qualifies for $4,000 in child tax credit and $8,000 for child and dependent care credit, how much federal tax does the person actually pay?

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CHAPTER OVERVIEW

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9.1: Introduction

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

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9.2: Simple Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $\$100(0.05) = \$5$. The total amount you would repay would be $105, the original principal plus the interest.

Simple One-time Interest

$$I = P_0 r$$

$$A = P_0 + I = P_0 + P_0 r = P_0(1 + r)$$

where

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate (in decimal form. Example: $5\% = 0.05$)

Example 1

A friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Solution

$P_0 = \$300$	the principal
$r = 0.03$	3% rate
$I = \$300(0.03) = \$9.$	You will earn $9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Solution

Each year, you would earn 5% interest: $\$1000(0.05) = \$50$ in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$$I = P_0 r t$$

$$A = P_0 + I = P_0 + P_0 r t = P_0(1 + rt)$$

where

- I is the interest
- A is the end amount: principal plus interest

- P_0 is the principal (starting amount)
- r is the interest rate in decimal form
- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.
A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3: Treasury Notes

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Solution

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P_0 = \$1000$	the principal
$r = 0.02$	2% rate per half-year
$t = 8$	4 years = 8 half-years
$I = \$1000(0.02)(8) = \$160.$	You will earn $160 interest total over the four years.

Try it Now 1

A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

Answer

$I = \$30$ of interest

$P_0 = \$500$ principal

$r = \frac{30}{500} = 0.06$ per month

$(0.06)(12) = 0.72$ per year

They are charging an annual interest rate of 72%.

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9.3: Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,

$P_0 = \$1000$

$r = 0.0025(0.25\%)$

$I = \$1000(0.0025) = \$2.50$

$A = \$1000 + \$2.50 = \$1002.50$

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

$P_0 = \$1002.50$

$I = \$1002.50(0.0025) = \$2.51$ (rounded)

$A = \$1000 + \$2.50 = \$1002.50$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

Month	Starting balance	Interest earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$P_0 = \$1000$

$P_m = (1 + 0.0025)P_{m-1}$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$P_0 = \$1000$

$P_1 = 1.0025P_0 = 1.0025(1000)$

$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$

$P_3 = 1.0025P_2 = 1.0025\left(1.0025^2(1000)\right) = 1.0025^3(1000)$

$P_4 = 1.0025P_3 = 1.0025\left(1.0025^3(1000)\right) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_m = (1.0025)^m(\$1000)$

Notice that the \$1000 in the equation was P_0, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$P_m = P_0\left(1 + \frac{r}{k}\right)^m$

In this formula:

m is the number of compounding periods (months in our example)

r is the annual interest rate

k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If N is the number of years, then $m = Nk$. Making this change gives us the standard formula for compound interest.

Compound Interest

$P_N = P_0\left(1 + \frac{r}{k}\right)^{Nk}$

P_N is the balance in the account after N years.

P_0 is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate in decimal form

k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Solution

In this example,

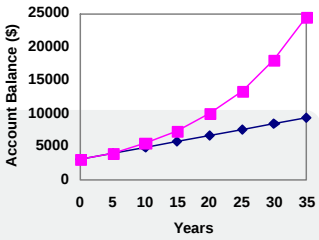
$P_0 = \$3000$	the initial deposit
$r = 0.06$	6% annual rate
$k = 12$	12 months in 1 year
$N = 20$	since we're looking for how much we'll have after 20 years

So $P_{20} = 3000\left(1 + \frac{0.06}{12}\right)^{20\times 12} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest ($15 per month)	6% compounded monthly $= 0.5\%$ each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65

As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.



Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$[\wedge]$, $[y^x]$, or $[x^y]$

To evaluate 1.005^{240} we'd type 1.005 $[\wedge]$ 240, or 1.005 $[y^x]$ 240. Try it out - you should get something around 3.3102044758.

Example 5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

We're looking for P_0.

$r = 0.04$	4%
$k = 4$	4 quarters in 1 year
$N = 18$	Since we know the balance in 18 years
$P_{18} = \$40,000$	The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for P_0.

$40000 = P_0\left(1 + \frac{0.04}{4}\right)^{18\times 4}$

$40000 = P_0(2.0471)$

$P_0 = \frac{40000}{2.0471} = \$19539.84$

So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

Example 6

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

Solution

$P_0 = \$1000$	the initial deposit
$r = 0.05$	5%
$k = 12$	12 months in 1 year
$N = 30$	since we're looking for the amount after 30 years

If we first compute $\frac{r}{k}$, we find $\frac{0.05}{12} = 0.00416666666667$

Here is the effect of rounding this to different values:

r/k rounded to:	Gives P_{30} to be:	Error
0.004	$4208.59	$259.15
0.0042	$4521.45	$53.71
0.00417	$4473.09	$5.35
0.004167	$4468.28	$0.54
0.0041667	$4467.80	$0.06
no rounding	$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - $5 off of $4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$P_{30} = 1000\left(1 + \frac{0.05}{12}\right)^{12\times 30}$

We can quickly calculate $12 \times 30 = 360$, giving $P_{30} = 1000\left(1 + \frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type this	Calculator shows
0.05[÷]12[=]	0.00416666666667
[+]1[=]	1.00416666666667
[y^x]360[=]	4.46774431400613
[×]1000[=]	4467.74431400613

Using your calculator continued

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 [×] (1 [+] 0.05 [÷] 12) [y^x] 360 [=]

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9.4: Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$

For a savings annuity, we simply need to add a deposit, d, to the account with each compounding period:

$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit $100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$r = 0.06$ (6%)

$k = 12$ (12 compounds/deposits per year)

$d = \$100$ (our deposit per month)

Writing out the recursive equation gives

$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005) P_{m-1} + 100$

Assuming we start with an empty account, we can begin using this relationship:

$P_0 = 0$

$P_1 = (1.005) P_0 + 100 = 100$

$P_2 = (1.005) P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$

$P_3 = (1.005) P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)^2 + 100(1.005) + 100$

Continuing this pattern, after m deposits, we'd have saved:

$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \cdots + 100(1.005) + 100$

In other words, after m months, the first deposit will have earned compound interest for $m - 1$ months. The second deposit will have earned interest for $m - 2$ months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$1.005 P_m = 1.005 \left(100(1.005)^{m-1} + 100(1.005)^{m-2} + \cdots + 100(1.005) + 100\right)$

Distributing on the right side of the equation gives

$1.005 P_m = 100(1.005)^m + 100(1.005)^{m-1} + \cdots + 100(1.005)^2 + 100(1.005)$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{array}{rcccc} 1.005 P_m & = 100(1.005)^m + & 100(1.005)^{m-1} + \cdots + & 100(1.005) \\ P_m & = & 100(1.005)^{m-1} + \cdots + & 100(1.005) + 100 \end{array}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$1.005 P_m - P_m = 100(1.005)^m - 100$

Solving for P_m

$0.005P_m = 100\left((1.005)^m - 1\right)$

$P_m = \frac{100((1.005)^m - 1)}{0.005}$

Replacing m months with $12N$, where N is measured in years, gives

$P_N = \frac{100((1.005)^{12N} - 1)}{0.005}$

Recall 0.005 was $\frac{r}{k}$ and 100 was the deposit d. 12 was k, the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$P_N = \frac{d\left(\left(1+\frac{r}{k}\right)^{Nk} - 1\right)}{\left(\frac{r}{k}\right)}$

P_N is the balance in the account after N years.

d is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $k = 12$.

If you make your deposits every year, use yearly compounding, $k = 1$.

If you make your deposits every quarter, use quarterly compounding, $k = 4$.

Etc.

When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Example 7

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

$d = \$100$	the monthly deposit
$r = 0.06$	6% annual rate
$k = 12$	since we're doing monthly deposits, we'll compound monthly
$N = 20$	we want the amount after 20 years

Putting this into the equation:

$$P_{20} = \frac{100\left(\left(1+\frac{0.06}{12}\right)^{20(12)}-1\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_{20} = \frac{100((1.005)^{240}-1)}{(0.005)}$$

$$P_{20} = \frac{100(3.310-1)}{(0.005)}$$

$$P_{20} = \frac{100(2.310)}{(0.005)} = \$46200$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $\$46,200 - \$24,000 = \$22,200$.

Example 8

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

In this example,

We're looking for d.

$r = 0.08$	8% annual rate
$k = 12$	since we're doing monthly deposits, we'll compound monthly
$N = 30$	30 years
$P_{30} = \$200,000$	The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for d.

$$200,000 = \frac{d\left(\left(1+\frac{0.08}{12}\right)^{30(12)}-1\right)}{\left(\frac{0.08}{12}\right)}$$

$$200,000 = \frac{d\left((1.00667)^{360}-1\right)}{(0.00667)}$$

$$200,000 = d(1491.57)$$

$$d = \frac{200,000}{1491.57} = \$134.09$$

So you would need to deposit $\$134.09$ each month to have $\$200,000$ in 30 years if your account earns 8% interest

Try it Now 2

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

Answer

$d = \$5$	the daily deposit
$r = 0.03$	3% annual rate
$k = 365$	since we're doing daily deposits, we'll compound daily
$N = 10$	we want the amount after 10 years

$$P_{10} = \frac{5\left(\left(1+\frac{0.03}{365}\right)^{365\times10}-1\right)}{\frac{0.03}{365}} = \$21,282.07$$

We would have deposited a total of $\$5 \cdot 365 \cdot 10 = \$18,250$, so $3,032.07 is from interest

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9.5: Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P_0 = \frac{d\left(1-\left(1+\frac{r}{k}\right)^{-Nk}\right)}{\left(\frac{r}{k}\right)}$$

P_0 is the balance in the account at the beginning (starting amount, or principal).

d is the regular withdrawal (the amount you take out each year, each month, etc.)

r is the annual interest rate (in decimal form. Example: $5\% = 0.05$)

k is the number of compounding periods in one year.

N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Example 9

After retiring, you want to be able to take $1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

In this example,

$d = \$1000$ the monthly withdrawal
$r = 0.06$ 6% annual rate
$k = 12$ since we're doing monthly withdrawals, we'll compound monthly
$N = 20$ since were taking withdrawals for 20 years

We're looking for P_0; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P_0 = \frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-20(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_0 = \frac{1000\times(1-(1.005)^{-240})}{(0.005)}$$

$P_0 = \frac{1000\times(1-0.302)}{(0.005)} = \$139,600$

You will need to have $139,600 in your account when you retire.

Notice that you withdrew a total of $240,000 ($1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $\$240,000 - \$139,600 = \$100,400$ in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this [(-)], some with [+/-] . The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240}you'd type something like: $1.005[\wedge][(-)]240$

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: $1.005[y^x]240[(-)] =$

Give it a try - you should get $1.005^{-240} = 0.302096$

Example 10

You know you will have $500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Solution

In this example,

We're looking for d.

$r = 0.08$	8% annual rate
$k = 12$	since we're doing monthly withdrawals
$N = 30$	since were taking withdrawals for 30 years
$P_0 = \$500,000$	we are beginning with $500,000

In this case, we're going to have to set up the equation, and solve for d.

$$500,000 = \frac{d\left(1-\left(1+\frac{0.08}{12}\right)^{-30(12)}\right)}{\left(\frac{0.08}{12}\right)}$$

$$500,000 = \frac{d(1-(1.00667)^{-360})}{(0.00667)}$$

$$500,000 = d(136.232)$$

$$d = \frac{500,000}{136.232} = \$3670.21$$

You would be able to withdraw $3,670.21 each month for 30 years.

Try it Now 3

A donor gives $100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Answer

$d =$ unknown	
$r = 0.04$	4% annual rate
$k = 1$	since we're doing annual scholarships
$N = 20$	since were taking withdrawals for 20 years
$P_0 = \$100,000$	we are starting with $100,000

$$100,000 = \frac{d\left(1-\left(1+\frac{0.04}{1}\right)^{-20\times 1}\right)}{\frac{0.04}{1}}$$

Solving for d gives $7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $\$100,000(0.04) = \$4,000$ a year would have been available.

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9.6: Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had $10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests $10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P_0 = \frac{d\left(1-\left(1+\frac{r}{k}\right)^{-Nk}\right)}{\left(\frac{r}{k}\right)}$$

P_0 is the balance in the account at the beginning (the principal, or amount of the loan).

d is your loan payment (your monthly payment, annual payment, etc)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

N is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

Example 11

You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

Solution

In this example,

$d = \$200$ **the monthly loan payment**
$r = 0.03$ **3% annual rate**
$k = 12$ **since we're doing monthly payments, we'll compound monthly**
$N = 5$ **since we're making monthly payments for 5 years**

We're looking for P_0, the starting amount of the loan.

$$P_0 = \frac{200\left(1-\left(1+\frac{0.03}{12}\right)^{-5(12)}\right)}{\left(\frac{0.03}{12}\right)}$$

$$P_0 = \frac{200(1-(1.0025)^{-60})}{(0.0025)}$$

$P_0 = \frac{200(1-0.861)}{(0.0025)} = \$11,120$

You can afford a $\$11,120$ loan.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $\$12,000 - \$11,120 = \$880$ interest total.

Example 12

You want to take out a $140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example,

We're looking for d.

$r = 0.06$	6% annual rate
$k = 12$	since we're doing monthly payments, we'll compound monthly
$N = 30$	since we're making monthly payments for 30 years
$P_0 = \$140,000$	the starting loan amount

In this case, we're going to have to set up the equation, and solve for d.

$$140,000 = \frac{d\left(1-\left(1+\frac{0.06}{12}\right)^{-30(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$140,000 = \frac{d(1-(1.005)^{-360})}{(0.005)}$$

$$140,000 = d(166.792)$$

$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of $839.37 per month for 30 years.

You're paying a total of $\$302,173.20$ to the loan company: $\$839.37$ per month for 360 months. You are paying a total of $\$302,173.20 - \$140,000 = \$162,173.20$ in interest over the life of the loan.

Try it Now 4

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Answer

$d =$	unknown
$r = 0.16$	16% annual rate
$k = 12$	since we're doing monthly payments, we'll compound monthly
$N = 2$	2 year to repay
$P_0 = 3,000$	the starting loan amount $3,000 loan

$$3,000 = \frac{d\left(1-\left(1+\frac{0.16}{12}\right)^{-2\times 12}\right)}{\frac{0.16}{12}}$$

Solving for d gives $\$146.89$ as monthly payments.

In total, she will pay $\$3,525.36$ to the store, meaning she will pay $\$525.36$ in interest over the two years.

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9.7: Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were $1,000 a month, after a year you will *not* have paid off $12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the remaining time on the loan?"

Example 13

If a mortgage at a 6% interest rate has payments of $1,000 a month, how much will the loan balance be 10 years from the end the loan?

Solution

To determine this, we are looking for the amount of the loan that can be paid off by $1,000 a month payments in 10 years. In other words, we're looking for P_0 when

$d = \$1,000$	the monthly loan payment
$r = 0.06$	6% annual rate
$k = 12$	since we're doing monthly payments, we'll compound monthly
$N = 10$	since we're making monthly payments for 10 more years

$$\begin{aligned} P_0 &= \frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-10(12)}\right)}{\left(\frac{0.06}{12}\right)} \\ &= \frac{1000\left(1-(1.005)^{-120}\right)}{(0.005)} \\ &= \frac{1000\left(1-(1.005)^{-120}\right)}{(0.005)} \\ &= \frac{1000(1-0.5496)}{(0.005)} \\ &= \$90,073.45 \end{aligned}$$

The loan balance with 10 years remaining on the loan will be $\$90,073.45$

Often times answering remaining balance questions requires two steps:

1. Calculating the monthly payments on the loan
2. Calculating the remaining loan balance based on the *remaining time* on the loan

Example 14

A couple purchases a home with a $\$180,000$ mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

Solution

First we will calculate their monthly payments.

We're looking for d.

$r = 0.04$	4% annual rate
$k = 12$	since they're paying monthly
$N = 30$	30 years
$P_0 = \$180,000$	the starting loan amount

We set up the equation and solve for d.

$$\begin{aligned} 180,000 &= \frac{d\left(1-\left(1+\frac{0.04}{12}\right)^{-30(12)}\right)}{\left(\frac{0.04}{12}\right)} \\ 180,000 &= \frac{d\left(1-(1.00333)^{-360}\right)}{(0.00333)} \\ 180,000 &= d(209.562) \\ d &= \frac{180,000}{209.562} \\ &= \$858.93 \end{aligned}$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$d = \$858.93$	the monthly loan payment we calculated above
$r = 0.04$	4% annual rate
$k = 12$	since they're paying monthly
$N = 25$	since they'd be making monthly payments for 25 more years

$$\begin{aligned} P_0 &= \frac{858.93\left(1-\left(1+\frac{0.04}{12}\right)^{-25(12)}\right)}{\left(\frac{0.04}{12}\right)} \\ &= \frac{858.93\left(1-(1.00333)^{-300}\right)}{(0.00333)} \\ &= \frac{858.93(1-0.369)}{(0.00333)} \\ &\approx \$162,758 \end{aligned}$$

The loan balance after 5 years, with 25 years remaining on the loan, will be $\$162,758$

Over that 5 years, the couple has paid off $\$180,000 - \$162,758 = \$17,242$ of the loan balance. They have paid a total of $\$858.93$ a month for 5 years (60 months), for a total of $\$51,535.80$, so $\$51,535.80 - \$17,242 = \$34,292.80$ of what they have paid so far has been interest.

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9.8: Which equation to use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

? Try it Now 5

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

a. Marcy received an inheritance of $20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each quarter. How much can she take out each quarter if she has 3 years of school left?
b. Paul wants to buy a new car. Rather than take out a loan, he decides to save $200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
c. Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have $30,000 in the account in 6 years. How much should Keisha deposit into the account?
d. Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of $100, how much new equipment can she buy?
e. How much would you need to save every month in an account earning 4% interest to have $5,000 saved up in two years?

Answer

a. This is a payout annuity problem. She can pull out $1833.60 a quarter.
b. This is a savings annuity problem. He will have saved up $7,524.11.
c. This is compound interest problem. She would need to deposit $22,386.46.
d. This is a loans problem. She can buy $4,609.33 of new equipment.
e. This is a savings annuity problem. You would need to save $200.46 each month

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9.9: Solving for time

Often we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

Note: This section assumes you've covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

Example 15

If you invest $2000 at 6% compounded monthly, how long will it take the account to double in value?

Solution

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$P_0 = \$2000$	the initial deposit
$r = 0.06$	6% annual rate
$k = 12$	12 months in 1 year

So our general equation is $P_N = 2000\left(1+\frac{0.06}{12}\right)^{N\times 12}$. We also know that we want our ending amount to be double of $2000, which is $4000, so we're looking for N so that $P_N = 4000$. To solve this, we set our equation for P_N equal to 4000.

$4000 = 2000\left(1+\frac{0.06}{12}\right)^{N\times 12}$	Divide both sides by 2000
$2 = (1.005)^{12N}$	To solve for the exponent, take the log of both sides
$\log(2) = \log\left((1.005)^{12N}\right)$	Use the exponent property of logs on the right side
$\log(2) = 12N\log(1.005)$	Now we can divide both sides by $12\log 1.005$
$\frac{\log(2)}{12\log(1.005)} = N$	Approximating this to a decimal
$N = 11.581$	

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded log(2) to 0.301 and log(1.005) to 0.00217, then your final answer would have been about 11.577 years.

Example 16

If you invest $100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to $10,000?

Solution

This is a savings annuity problem since we are making regular deposits into the account.

$d = \$1000$	the monthly deposit
$r = 0.03$	3% annual rate
$k = 12$	since we're doing monthly deposits, we'll compound monthly

We don't know N, but we want P_N to be $\$10,000$

Putting this into the equation:

$10,000 = \frac{100\left(\left(1+\frac{0.03}{12}\right)^{N(12)}-1\right)}{\left(\frac{0.03}{12}\right)}$	Simplifying the fractions a bit
$10,000 = \frac{100\left((1.0025)^{12N}-1\right)}{0.0025}$	

We want to isolate the exponential term, 1.0025^{12N}, so multiply both sides by 0.0025

$25 = 100\left((1.0025)^{12N} - 1\right)$	Divide both sides by 100
$0.25 = (1.0025)^{12N} - 1$	Add 1 to both sides
$1.25 = (1.0025)^{12N}$	Now take the log of both sides
$\log(1.25) = \log\left((1.0025)^{12N}\right)$	Use the exponent property of logs
$\log(1.25) = 12N\log(1.0025)$	Divide by $12\log(1.0025)$
$\frac{\log(1.25)}{12\log(1.0025)} = N$	Approximating to a decimal
$N = 7.447$ years	

It will take about 7.447 years to grow the account to $\$10,000$.

? Try it Now 6

Joel is considering putting a $1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of $30 a month?

Answer

$d = \$30$	The monthly payments
$r = 0.12$	12% annual rate
$k = 12$	since we're doing monthly deposits
$P_0 = \$1000$	we're starting with a $1,000 loan

We are solving for N, the time to pay off the loan

$$1,000 = \frac{30\left(1-\left(1+\frac{0.12}{12}\right)^{-N(12)}\right)}{\frac{0.12}{12}}$$

Solving for N gives 3.396. It will take about 3.4 years to pay off the purchase.

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9.10: Exercises

Skills

1. A friend lends you $200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?
2. Suppose you obtain a $3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?
3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of $10,000 for $9,800. This means that in 13 weeks, the government will give you the face value, earning you $200. What annual interest rate have you earned?
4. Suppose you are looking to buy a $5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?
5. You deposit $300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
6. How much will $1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
7. You deposit $2000 in an account earning 3% interest compounded monthly.
   a. How much will you have in the account in 20 years?
   b. How much interest will you earn?
8. You deposit $10,000 in an account earning 4% interest compounded monthly.
   a. How much will you have in the account in 25 years?
   b. How much interest will you earn?
9. How much would you need to deposit in an account now in order to have $6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
10. How much would you need to deposit in an account now in order to have $20,000 in the account in 4 years? Assume the account earns 5% interest.
11. You deposit $200 each month into an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 30 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
12. You deposit $1000 each year into an account earning 8% compounded annually.
    a. How much will you have in the account in 10 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
13. Jose has determined he needs to have $800,000 for retirement in 30 years. His account earns 6% interest.
    a. How much would he need to deposit in the account each month?
    b. How much total money will he put into the account?
    c. How much total interest will he earn?
14. You wish to have $3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
15. You want to be able to withdraw $30,000 each year for 25 years. Your account earns 8% interest.
    a. How much do you need in your account at the beginning
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?
16. How much money will I need to have at retirement so I can withdraw $60,000 a year for 20 years from an account earning 8% compounded annually?

a. How much do you need in your account at the beginning
b. How much total money will you pull out of the account?
c. How much of that money is interest?

17. You have $500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?
18. Loren already knows that he will have $500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?
19. You can afford a $700 per month mortgage payment. You've found a 30 year loan at 5% interest.
    a. How big of a loan can you afford?
    b. How much total money will you pay the loan company?
    c. How much of that money is interest?
20. Marie can afford a $250 per month car payment. She's found a 5 year loan at 7% interest.
    a. How expensive of a car can she afford?
    b. How much total money will she pay the loan company?
    c. How much of that money is interest?
21. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
22. You decide finance a $12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?
23. You want to buy a $200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.
    a. How much is the loan amount going to be?
    b. What will your monthly payments be if the interest rate is 5%?
    c. What will your monthly payments be if the interest rate is 6%?
24. Lynn bought a $300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
    a. Find her monthly payments.
    b. How much interest will she pay over the life of the loan?
25. Emile bought a car for $24,000 three years ago. The loan had a 5 year term at 3% interest rate, making monthly payments. How much does he still owe on the car?
26. A friend bought a house 15 years ago, taking out a $120,000 mortgage at 6% for 30 years, making monthly payments. How much does she still owe on the mortgage?
27. Pat deposits $6,000 into an account earning 4% compounded monthly. How long will it take the account to grow to $10,000?
28. Kay is saving $200 a month into an account earning 5% interest. How long will it take her to save $20,000?
29. James has $3,000 in credit card debt, which charges 14% interest. How long will it take to pay off the card if he makes the minimum payment of $60 a month?
30. Chris has saved $200,000 for retirement, and it is in an account earning 6% interest. If she withdraws $3,000 a month, how long will the money last?

Concepts

31. Suppose you invest $50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 25 years. How much will you have in the end?
32. Suppose you put off making investments for the first 5 years, and instead made deposits of $50 a month for 25 years into an account earning 8% compounded monthly. How much will you have in the end?

33. Mike plans to make contributions to his retirement account for 15 years. After the last contribution, he will start withdrawing $10,000 a quarter for 10 years. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the first 15 years, in order to accomplish his goal?
34. Kendra wants to be able to make withdrawals of $60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?
35. You have $2,000 to invest, and want it to grow to $3,000 in two years. What interest rate would you need to find to make this possible?
36. You have $5,000 to invest, and want it to grow to $20,000 in ten years. What interest rate would you need to find to make this possible?
37. You plan to save $600 a month for the next 30 years for retirement. What interest rate would you need to have $1,000,000 at retirement?
38. You really want to buy a used car for $11,000, but can only afford $200 a month. What interest rate would you need to find to be able to afford the car, assuming the loan is for 60 months?

Exploration

39. Pay day loans are short term loans that you take out against future paychecks: The company advances you money against a future paycheck. Either visit a pay day loan company, or look one up online. Be forewarned that many companies do not make their fees obvious, so you might need to do some digging or look at several companies.
    a. Explain the general method by which the loan works.
    b. We will assume that we need to borrow $500 and that we will pay back the loan in 14 days. Determine the total amount that you would need to pay back and the effective loan rate. The effective loan rate is the percentage of the original loan amount that you pay back. It is not the same as the APR (annual rate) that is probably published.
    c. If you cannot pay back the loan after 14 days, you will need to get an extension for another 14 days. Determine the fees for an extension, determine the total amount you will be paying for the now 28 day loan, and compute the effective loan rate.
40. Suppose that 10 years ago you bought a home for $110,000, paying 10% as a down payment, and financing the rest at 9% interest for 30 years.
    a. Let's consider your existing mortgage:
        i. How much money did you pay as your down payment?
        ii. How much money was your mortgage (loan) for?
        iii. What is your current monthly payment?
        iv. How much total interest will you pay over the life of the loan?

40. b. This year, you check your loan balance. Only part of your payments have been going to pay down the loan; the rest has been going towards interest. You see that you still have $88,536 left to pay on your loan. Your house is now valued at $150,000.

How much of the loan have you paid off? (i.e., how much have you reduced the loan balance by? Keep in mind that interest is charged each month - it's not part of the loan balance.)

        i. How much money have you paid to the loan company so far?
        ii. How much interest have you paid so far?
        iii. How much equity do you have in your home (equity is value minus remaining debt)

40. c. Since interest rates have dropped, you consider refinancing your mortgage at a lower 6% rate.
        i. If you took out a new 30 year mortgage at 6% for your remaining loan balance, what would your new monthly payments be?
        ii. How much interest will you pay over the life of the new loan?

40. d. Notice that if you refinance, you are going to be making payments on your home for another 30 years. In addition to the 10 years you've already been paying, that's 40 years total.
        i. How much will you save each month because of the lower monthly payment?

ii. How much total interest will you be paying (you need to consider the amount from 2c and 3b)

iii. Does it make sense to refinance? (there isn't a correct answer to this question. Just give your opinion and your reason)

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CHAPTER OVERVIEW

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10.1: Introduction

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle (we'll look at it in a bit). The field has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

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10.2: Graphs

Drawing Graphs

Example 1

Here is a portion of a housing development from Missoula, Montana[1]. As part of her job, the development's lawn inspector has to walk down every street in the development making sure homeowners' landscaping conforms to the community requirements.

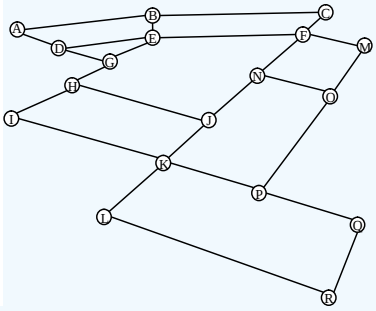


Solution

Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without having to do any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity.

To do that, we first need to simplify the picture into a form that is easier to work with. We can do that by drawing a simple line for each street. Where streets intersect, we will place a dot.



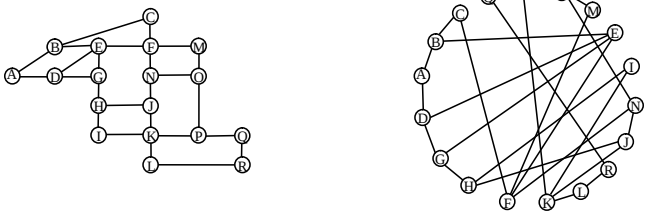


This type of simplified picture is called a **graph**.

Graphs, Vertices, and Edges

A **graph** consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices.

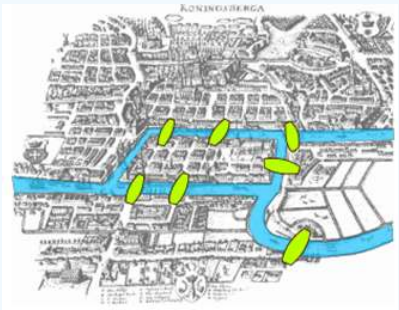
While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph. Both of the graphs below are equivalent to the one drawn above since they show the same edge connections between the same vertices as the original graph.



You probably already noticed that we are using the term *graph* differently than you may have used the term in the past to describe the graph of a mathematical function.

Example 2

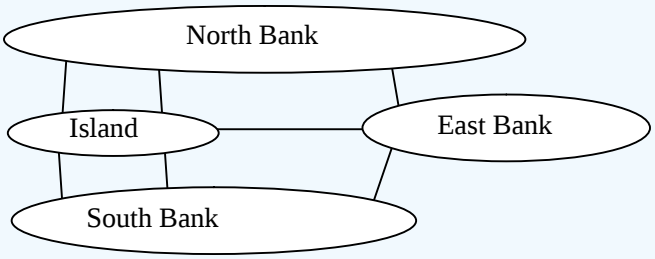
Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right[2].



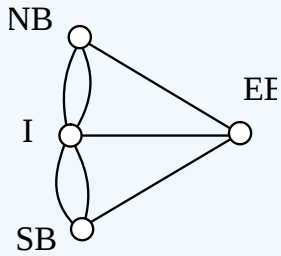
As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

Solution

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

While we haven't answered the actual question yet of whether or not there is a route which crosses every bridge once and returns to the starting location, the graph provides the foundation for exploring this question.

Definitions

While we loosely defined some terminology earlier, we now will try to be more specific.

Vertex

A vertex is a dot in the graph that could represent an intersection of streets, a land mass, or a general location, like “work” or “school”. Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass – the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

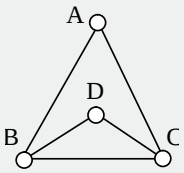
Edges

Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.

Adjacency Table

An alternate way to represent a graph is with an **adjacency table**. This lists the vertices on the row and column headers of a table, and has a value in each cell if the vertices are connected with an edge. Later the value might represent something like the length or cost of the edge, but for now we will just put a 1 where there is an edge.

For example, this graph



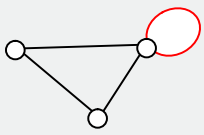
could be represented with the adjacency table

Adjacency table

	A	B	C	D
A		1	1	
B	1		1	1
C	1	1		1
D		1	1	

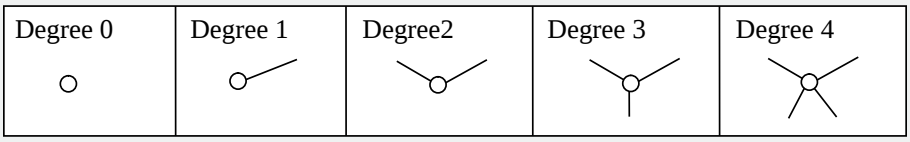
Loop

A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.



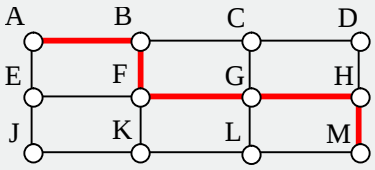
Degree of a vertex

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.



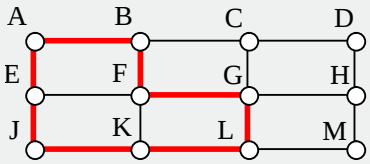
Path

A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph.



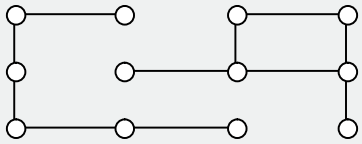
Circuit

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



Connected

A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices on the left to the vertices on the right.



Weights

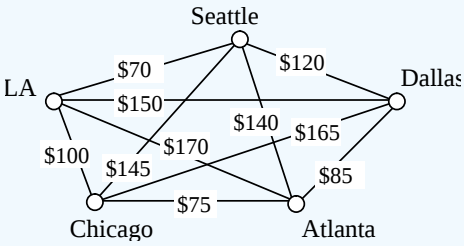
Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Try it Now 1

The graph below shows 5 cities. The weights on the edges represent the airfare for a one-way flight between the cities.

a. How many vertices and edges does the graph have?
b. Is the graph connected?
c. What is the degree of the vertex representing LA?
d. If you fly from Seattle to Dallas to Atlanta, is that a path or a circuit?

e. If you fly from LA to Chicago to Dallas to LA, is that a path or a circuit?



▼ Adjacency Table

Adjacency table with cities as vertices and edges representing flight cost

	Seattle	Dallas	Atlanta	Chicago	LA
Seattle	*	$120	$140	$145	$70
Dallas	$120	*	$85	$165	$150
Atlanta	$140	$85	*	$75	$170
Chicago	$145	$165	$75	*	$100
LA	$70	$150	$170	$100	*

Answer

a. 5 vertices, 10 edges

b. Yes, it is connected

c. The vertex is degree 4

d. A path

e. A circuit

[1] Sam Beebe. http://www.flickr.com/photos/sbeebe/2850476641/, CC-BY

[2] Bogdan Giuşcă. http://en.Wikipedia.org/wiki/File:Ko...rg_bridges.png

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10.3: Shortest Path

When you visit a website like Google Maps or use your Smartphone to ask for directions from home to your Aunt's house in Pasadena, you are usually looking for a shortest path between the two locations. These computer applications use representations of the street maps as graphs, with estimated driving times as edge weights.

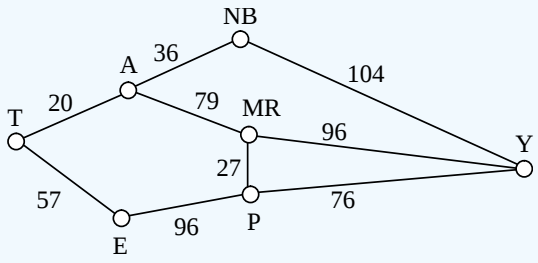
While often it is possible to find a shortest path on a small graph by guess-and-check, our goal in this chapter is to develop methods to solve complex problems in a systematic way by following **algorithms**. An algorithm is a step-by-step procedure for solving a problem. Dijkstra's (pronounced dike-stra) algorithm will find the shortest path between two vertices.

Dijkstra's Algorithm

1. Mark the ending vertex with a distance of zero. Designate this vertex as current.
2. Find all vertices leading to the current vertex. Calculate their distances to the end. Since we already know the distance the current vertex is from the end, this will just require adding the most recent edge. Don't record this distance if it is longer than a previously recorded distance.
3. Mark the current vertex as visited. We will never look at this vertex again.
4. Mark the vertex with the smallest distance as current, and repeat from step 2.

Example 3

Suppose you need to travel from Tacoma, WA (vertex T) to Yakima, WA (vertex Y). Looking at a map, it looks like driving through Auburn (A) then Mount Rainier (MR) might be shortest, but it's not totally clear since that road is probably slower than taking the major highway through North Bend (NB). A graph with travel times in minutes is shown below. An alternate route through Eatonville (E) and Packwood (P) is also shown.



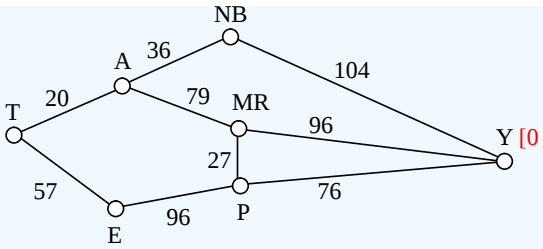
▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T	A	E	NB	MR	P	Y
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Solution

Step 1: Mark the ending vertex with a distance of zero. The distances will be recorded in [brackets] after the vertex name.



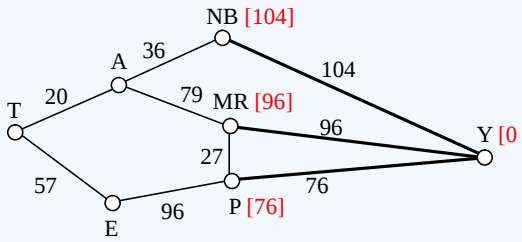
▼ Adjacency Table

In the adjacency table, we will record the distances in the column headers

Adjacency table with travel time in minutes between cities

	T	A	E	NB	MR	P	Y [0]
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 2: For each vertex leading to Y, we calculate the distance to the end. For example, NB is a distance of 104 from the end, and MR is 96 from the end. Remember that distances in this case refer to the travel time in minutes.



▼ Adjacency Table

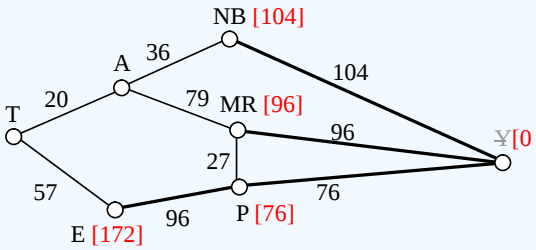
Adjacency table with travel time in minutes between cities

	T	A	E	NB [104]	MR [96]	P [76]	Y [0]
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4: We mark Y as visited, and mark the vertex with the smallest recorded distance as current. At this point, P will be designated current. Back to step 2.

Step 2 (#2): For each vertex leading to P (and not leading to a visited vertex) we find the distance from the end. Since E is 96 minutes from P, and we've already calculated P is 76 minutes from Y, we can compute that E is 96+76 = 172 minutes from Y.

If we make the same computation for MR, we'd calculate 76+27 = 103. Since this is larger than the previously recorded distance from Y to MR, we will *not* replace it.



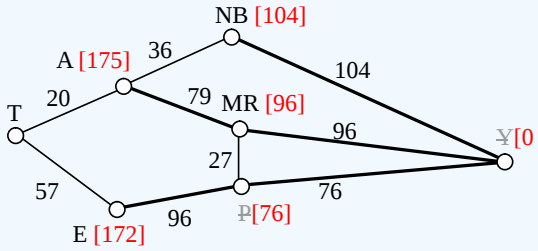
▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T	A	E [172]	NB [104]	MR [96]	P [76] current	Y [0] visited
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4 (#2): We mark P as visited, and designate the vertex with the smallest recorded distance as current: MR. Back to step 2.

Step 2 (#3): For each vertex leading to MR (and not leading to a visited vertex) we find the distance to the end. The only vertex to be considered is A, since we've already visited Y and P. Adding MR's distance 96 to the length from A to MR gives the distance 96+79 = 175 minutes from A to Y.



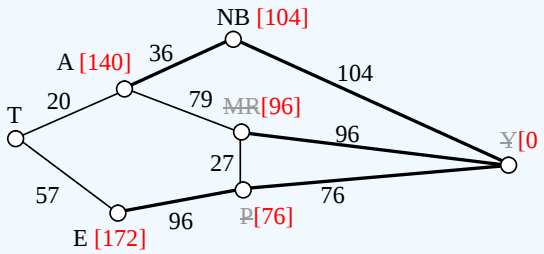
▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T	A [175]	E [172]	NB [104]	MR [96] current	P [76] visited	Y [0] visited
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4 (#3): We mark MR as visited, and designate the vertex with smallest recorded distance as current: NB. Back to step 2.

Step 2 (#4): For each vertex leading to NB, we find the distance to the end. We know the shortest distance from NB to Y is 104 and the distance from A to NB is 36, so the distance from A to Y through NB is 104+36 = 140. Since this distance *is* shorter than the previously calculated distance from Y to A through MR, we replace it.



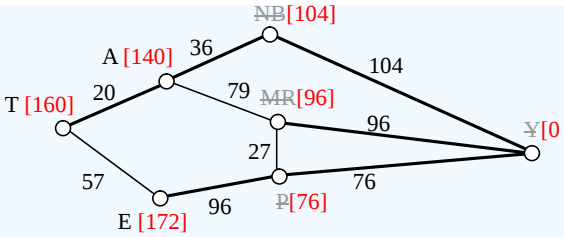
▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T	A [140]	E [172]	NB [104] current	MR [96] visited	P [76] visited	Y [0] visited
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4 (#4): We mark NB as visited, and designate A as current, since it now has the shortest distance.

Step 2 (#5): T is the only non-visited vertex leading to A, so we calculate the distance from T to Y through A: 20+140 = 160 minutes.



▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T [160]	A [140] current	E [172]	NB [104] visited	MR [96] visited	P [76] visited	Y [0] visited
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4 (#5): We mark A as visited, and designate E as current.

Step 2 (#6): The only non-visited vertex leading to E is T. Calculating the distance from T to Y through E, we compute $172 + 57 = 229$ minutes. Since this is longer than the existing marked time, we do not replace it.

Step 3 (#6): We mark E as visited. Since all vertices have been visited, we are done.

From this, we know that the shortest path from Tacoma to Yakima will take 160 minutes. Tracking which sequence of edges yielded 160 minutes, we see the shortest path is T-A-NB-Y.

Dijkstra's algorithm is an **optimal algorithm**, meaning that it always produces the actual shortest path, not just a path that is pretty short, provided one exists. This algorithm is also **efficient**, meaning that it can be implemented in a reasonable amount of time. Dijkstra's algorithm takes around V^2 calculations, where V is the number of vertices in a graph[1]. A graph with 100 vertices would take around 10,000 calculations. While that would be a lot to do by hand, it is not a lot for computer to handle. It is because of this efficiency that your car's GPS unit can compute driving directions in only a few seconds.

In contrast, an **inefficient** algorithm might try to list all possible paths then compute the length of each path. Trying to list all possible paths could easily take 10^{25} calculations to compute the shortest path with only 25 vertices; that's a 1 with 25 zeros after it! To put that in perspective, the fastest computer in the world would still spend over 1000 years analyzing all those paths.

Example 4

A shipping company needs to route a package from Washington, D.C. to San Diego, CA. To minimize costs, the package will first be sent to their processing center in Baltimore, MD then sent as part of mass shipments between their various processing centers, ending up in their processing center in Bakersfield, CA. From there it will be delivered in a small truck to San Diego.

The travel times, in hours, between their processing centers are shown in the table below. Three hours has been added to each travel time for processing.

Travel times adjacency table

	Baltimore	Denver	Dallas	Chicago	Atlanta	Bakersfield
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Find the shortest path from Baltimore to Bakersfield.

Solution

While we could draw a graph, we can also work directly from the table.

Step 1: The ending vertex, Bakersfield, is marked as current.

Step 2: All cities connected to Bakersfield, in this case Denver and Dallas, have their distances calculated; we'll mark those distances in the column headers.

Step 3 & 4: Mark Bakersfield as visited. Here, we are labeling it as such in the column header and putting a line through the corresponding row and column of the table as a quick visual indicator. We mark Denver as current, labeled as such in the column header, since it is the vertex with the shortest distance.

Travel times adjacency table

	Baltimore	**Denver** [19] current	Dallas [25]	Chicago	Atlanta	~~Bakersfield [0] visited~~
Baltimore	*			15	14	
Denver		*		18	24	~~19~~
Dallas			*	18	15	~~25~~
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
~~Bakersfield~~		~~19~~	~~25~~			~~*~~

Step 2 (#2): For cities connected to Denver, calculate distance to the end. For example, Chicago is 18 hours from Denver, and Denver is 19 hours from the end, the distance for Chicago to the end is 18+19 = 37 (Chicago to Denver to Bakersfield). Atlanta is 24 hours from Denver, so the distance to the end is 24+19 = 43 (Atlanta to Denver to Bakersfield).

Step 3 & 4 (#2): We mark Denver as visited and mark Dallas as current.

Travel times adjacency table

	Baltimore	~~Denver~~ ~~[19] visited~~	**Dallas** [25] current	Chicago [37]	Atlanta [43]	~~Bakersfield~~ ~~[0] visited~~
Baltimore	*			15	14	
~~Denver~~		~~*~~		~~18~~	~~24~~	~~19~~
Dallas			*	18	15	~~25~~
Chicago	15	~~18~~	18	*	14	
Atlanta	14	~~24~~	15	14	*	
~~Bakersfield~~		~~19~~	~~25~~			~~*~~

Step 2 (#3): For cities connected to Dallas, calculate the distance to the end. For Chicago, the distance from Chicago to Dallas is 18 and from Dallas to the end is 25, so the distance from Chicago to the end through Dallas would be 18+25 = 43. Since this is longer than the currently marked distance for Chicago, we do not replace it. For Atlanta, we calculate 15+25 = 40. Since this is shorter than the currently marked distance for Atlanta, we replace the existing distance.

Step 3 & 4 (#3): We mark Dallas as visited, and mark Chicago as current.

Travel times adjacency table

	Baltimore	~~Denver~~ ~~[19] visited~~	~~Dallas~~ ~~[25] visited~~	**Chicago** [37] current	Atlanta [40]	~~Bakersfield~~ ~~[0] visited~~
Baltimore	*			15	14	
~~Denver~~		~~*~~		~~18~~	~~24~~	~~19~~
~~Dallas~~			~~*~~	~~18~~	~~15~~	~~25~~
Chicago	15	~~18~~	~~18~~	*	14	
Atlanta	14	~~24~~	~~15~~	14	*	
~~Bakersfield~~		~~19~~	~~25~~			~~*~~

Step 2 (#4): Baltimore and Atlanta are the only non-visited cities connected to Chicago. For Baltimore, we calculate 15+37 = 52 and mark that distance. For Atlanta, we calculate 14+37 = 51. Since this is longer than the existing distance of 40 for Atlanta, we do not replace that distance.

Step 3 & 4 (#4): Mark Chicago as visited and Atlanta as current.

Travel times adjacency table

	Baltimore [52]	~~Denver~~ ~~[19] visited~~	~~Dallas~~ ~~[25] visited~~	~~Chicago~~ ~~[37] visited~~	**Atlanta** [40] current	~~Bakersfield~~ ~~[0] visited~~
Baltimore	*			~~15~~	14	
~~Denver~~		~~*~~		~~18~~	~~24~~	~~19~~
~~Dallas~~			~~*~~	~~18~~	~~15~~	~~25~~
~~Chicago~~	~~15~~	~~18~~	~~18~~	~~*~~	~~14~~	
Atlanta	14	~~24~~	~~15~~	~~14~~	*	
~~Bakersfield~~		~~19~~	~~25~~			~~*~~

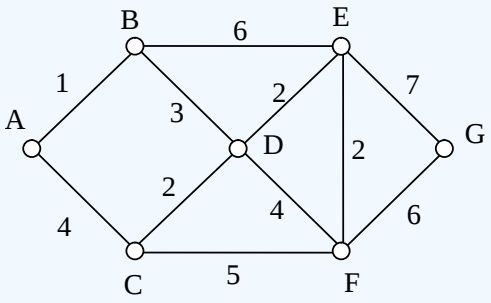
Step 2 (#5): The distance from Atlanta to Baltimore is 14. Adding that to the distance already calculated for Atlanta gives a total distance of 14+40 = 54 hours from Baltimore to Bakersfield through Atlanta. Since this is larger than the currently calculated distance, we do not replace the distance for Baltimore.

Step 3 & 4 (#5): We mark Atlanta as visited. All cities have been visited and we are done.

The shortest route from Baltimore to Bakersfield will take 52 hours, and will route through Chicago and Denver.

Try it Now 2

Find the shortest path between vertices A and G in the graph below.



▼ Adjacency Table

Adjacency table with edge weights

	A	B	C	D	E	F	G
A	*	1	4				
B	1	*		4	6		
C	4		*	2		5	
D		4	2	*	2	4	
E		6		2	*	2	7
F			5	4	2	*	6
G					7	6	*

Answer

The shortest path is ABDEG, with length 13

[1] It can be made to run faster through various optimizations to the implementation.

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10.4: Euler Circuits and the Chinese Postman Problem

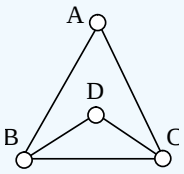
In the first section, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

Euler Path

An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

Example 5

In the graph shown below, there are several Euler paths.



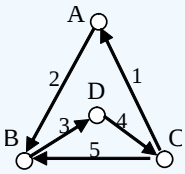
▼ Adjacency Table

Adjacency table

	A	B	C	D
A	*	1	1	
B	1	*	1	1
C	1	1	*	1
D		1	1	*

Solution

One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

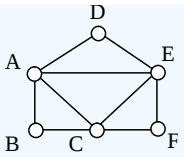


Euler Circuit

An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

Example 6

The graph below has several possible Euler circuits.



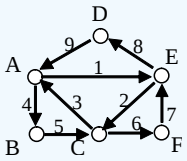
▼ Adjacency Table

Adjacency table

	A	B	C	D	E	F
A	*	1	1	1	1	
B	1	*	1			
C	1	1	*		1	1
D	1			*	1	
E	1		1	1	*	1
F			1		1	*

Solution

Here's a couple, starting and ending at vertex A: ADEACEFCBA and AECABCFEDA. The second is shown in arrows.



Look back at the example used for Euler paths – does that graph have an Euler circuit? A few tries will tell you no; that graph does not have an Euler circuit. When we were working with shortest paths, we were interested in the optimal path. With Euler paths and circuits, we're primarily interested in whether an Euler path or circuit *exists*.

Why do we care if an Euler circuit exists? Think back to our housing development lawn inspector from the beginning of the chapter. The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit! Luckily, Euler solved the question of whether or not an Euler path or circuit will exist.

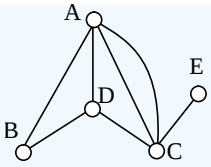
Euler's Path and Circuit Theorems

A graph will contain an Euler path if it contains at most two vertices of odd degree.

A graph will contain an Euler circuit if all vertices have even degree

Example 7

In the graph below, vertices A and C have degree 4, since there are 4 edges leading into each vertex. B is degree 2, D is degree 3, and E is degree 1. This graph contains two vertices with odd degree (D and E) and three vertices with even degree (A, B, and C), so Euler's theorems tell us this graph has an Euler path, but not an Euler circuit.



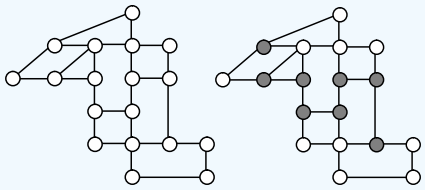
▼ Adjacency Table

Adjacency table, with values showing the number of edges connecting each pair

	A	B	C	D	E
A	*	1	2	1	
B	1	*		1	
C	2		*	1	1
D	1	1	1	*	
E			1		*

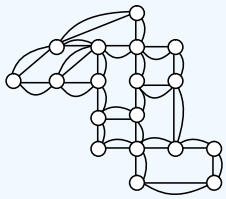
Example 8

Is there an Euler circuit on the housing development lawn inspector graph we created earlier in the chapter? All the highlighted vertices have odd degree. Since there are more than two vertices with odd degree, there are no Euler paths or Euler circuits on this graph. Unfortunately our lawn inspector will need to do some backtracking.



Example 9

When it snows in the same housing development, the snowplow has to plow both sides of every street. For simplicity, we'll assume the plow is out early enough that it can ignore traffic laws and drive down either side of the street in either direction. This can be visualized in the graph by drawing two edges for each street, representing the two sides of the street.



Notice that every vertex in this graph has even degree, so this graph does have an Euler circuit.

Now we know how to determine if a graph has an Euler circuit, but if it does, how do we find one? While it usually is possible to find an Euler circuit just by pulling out your pencil and trying to find one, the more formal method is **Fleury's algorithm.**

Fleury's Algorithm

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit, and delete it from the graph.
4. Continue until you're done.

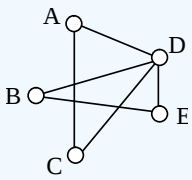
Example 10

Let's find an Euler Circuit on this graph using Fleury's algorithm, starting at vertex A.

Solution

Step 1:

Original Graph. Choosing edge AD.



▼ Adjacency Table

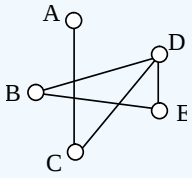
Adjacency table

	A	B	C	D	E
A	*		1	1	
B		*		1	1
C	1		*	1	
D	1	1	1	*	1
E		1		1	*

Circuit so far: AD

Step 2:

AD deleted. D is current. Can't choose DC since that would disconnect graph. Choosing DE



▼ Adjacency Table

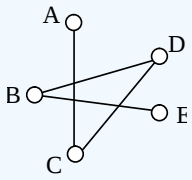
Adjacency table

	A	B	C	D	E
A	*		1	deleted	
B		*		1	1
C	1		*	1	
D	deleted	1	1	*	1
E		1		1	*

Circuit so far: ADE

Step 3:

E is current. From here, there is only one option, so the rest of the circuit is determined.



▼ Adjacency Table

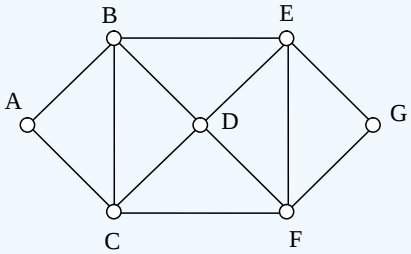
Adjacency table

	A	B	C	D	E
A	*		1		
B		*		1	1
C	1		*	1	
D		1	1	*	deleted
E		1		deleted	*

Circuit: ADEBDCA

Try it Now 3

Does the graph below have an Euler Circuit? If so, find one.



▼ Adjacency Table

Adjacency table

	A	B	C	D	E
A	*		1	1	
B		*		1	1
C	1		*	1	
D	1	1	1	*	1
E		1		1	*

Answer

Yes, all vertices have even degree so this graph has an Euler Circuit. There are several possibilities. One is: ABEGFCDFEDBCA

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10.5: Eulerization and the Chinese Postman Problem

Not every graph has an Euler path or circuit, yet our lawn inspector still needs to do her inspections. Her goal is to minimize the amount of walking she has to do. In order to do that, she will have to duplicate some edges in the graph until an Euler circuit exists.

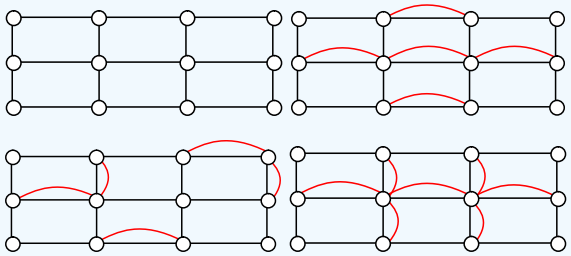
Eulerization

Eulerization is the process of adding edges to a graph to create an Euler circuit on a graph. To eulerize a graph, edges are duplicated to connect pairs of vertices with odd degree. Connecting two odd degree vertices increases the degree of each, giving them both even degree. When two odd degree vertices are not directly connected, we can duplicate all edges in a path connecting the two.

Note that we can only duplicate edges, not create edges where there wasn't one before. Duplicating edges would mean walking or driving down a road twice, while creating an edge where there wasn't one before is akin to installing a new road!

Example 11

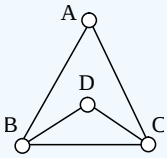
For the rectangular graph shown, three possible eulerizations are shown. Notice in each of these cases the vertices that started with odd degrees have even degrees after eulerization, allowing for an Euler circuit.



In the example above, you'll notice that the last eulerization required duplicating seven edges, while the first two only required duplicating five edges. If we were eulerizing the graph to find a walking path, we would want the eulerization with minimal duplications. If the edges had weights representing distances or costs, then we would want to select the eulerization with the minimal total added weight.

Try it Now 4

Eulerize the graph shown, then find an Euler circuit on the eulerized graph.



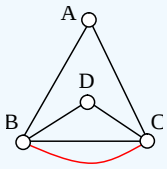
▼ Adjacency Table

Adjacency Table

	A	B	C	D
A	*	1	1	
B	1	*	1	1
C	1	1	*	1
D		1	1	*

Answer

This graph can be eulerized by duplicating the edge BC, as shown. One possible Euler circuit on the eulerized graph is ACDBCBA



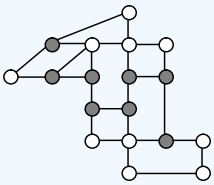
▼ Adjacency Table

Adjacency Table

	A	B	C	D
A	*	1	1	
B	1	*	2	1
C	1	2	*	1
D		1	1	*

Example 12

Looking again at the graph for our lawn inspector from Examples 1 and 8, the vertices with odd degree are shown highlighted. With eight vertices, we will always have to duplicate at least four edges.



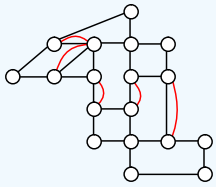
▼ Adjacency Table

Adjacency Table, odd vertices indicated in row headers

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
A		1		1														
B odd	1		1		1													
C		1				1												
D odd	1				1		1											
E		1		1		1	1											
F			1		1								1	1				
G odd				1	1			1										
H odd							1		1	1								
I								1			1							
J odd								1			1			1				
K									1	1		1				1		
L											1							1
M						1									1			
N odd						1				1					1			
O odd													1	1		1		
P odd											1				1		1	
Q																1		1
R												1					1	

Solution

In this case, we need to duplicate five edges since two odd degree vertices are not directly connected. Without weights we can't be certain this is the eulerization that minimizes walking distance, but it looks pretty good.



▼ Adjacency Table

Adjacency Table after Eulerization

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
A		1		1														
B	1		1		2													
C		1				1												
D	1				2		1											
E		2		2		1	1											
F			1		1								1	1				
G				1	1			2										
H							2		1	1								
I								1			1							
J								1			1			2				
K									1	1		1				1		
L											1							1
M						1									1			
N						1				2					1			
O													1	1		2		
P											1				2		1	
Q																1		1
R												1					1	

The problem of finding the optimal eulerization is called the Chinese Postman Problem, a name given by an American in honor of the Chinese mathematician Mei-Ko Kwan who first studied the problem in 1962 while trying to find optimal delivery routes for postal carriers. This problem is important in determining efficient routes for garbage trucks, school buses, parking meter checkers, street sweepers, and more.

Unfortunately, algorithms to solve this problem are fairly complex. Some simpler cases are considered in the exercises.

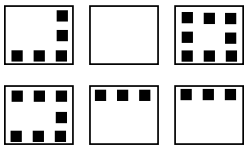
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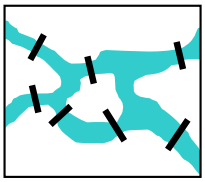
10.6: Exercise

Skills

1. To deliver mail in a particular neighborhood, the postal carrier needs to walk along each of the streets with houses (the dots). Create a graph with edges showing where the carrier must walk to deliver the mail.



2. Suppose that a town has 7 bridges as pictured below. Create a graph that could be used to determine if there is a path that crosses all bridges once.



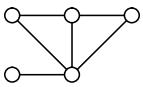
3. The table below shows approximate driving times (in minutes, without traffic) between five cities in the Dallas area. Create a weighted graph representing this data.

	Plano	Mesquite	Arlington	Denton
Fort Worth	54	52	19	42
Plano		38	53	41
Mesquite			43	56
Arlington				50

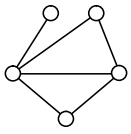
4. Shown in the table below are the one-way airfares between 5 cities[1]. Create a graph showing this data.

	Honolulu	London	Moscow	Cairo
Seattle	$159	$370	$654	$684
Honolulu		$830	$854	$801
London			$245	$323
Moscow				$329

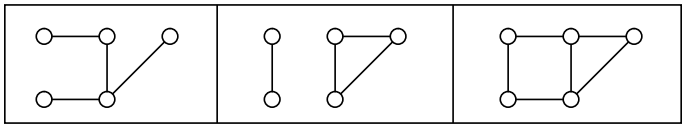
5. Find the degree of each vertex in the graph below.



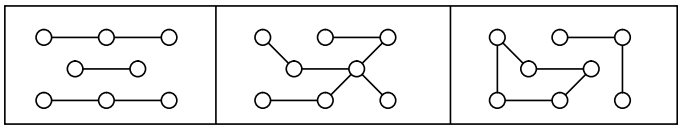
6. Find the degree of each vertex in the graph below.



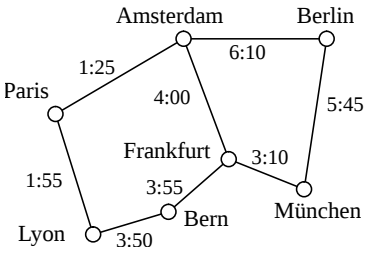
7. Which of these graphs are connected?



8. Which of these graphs are connected?

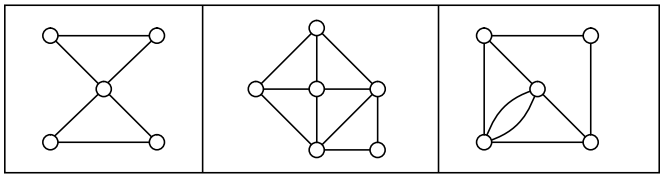


9. Travel times by rail for a segment of the Eurail system is shown below with travel times in hours and minutes[2]. Find path with shortest travel time from Bern to Berlin by applying Dijkstra's algorithm.

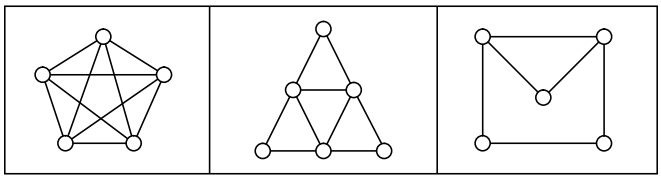


10. Using the graph from the previous problem, find the path with shortest travel time from Paris to München.

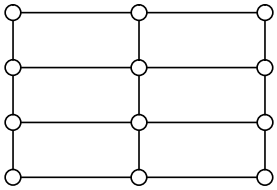
11. Does each of these graphs have an Euler circuit? If so, find it.



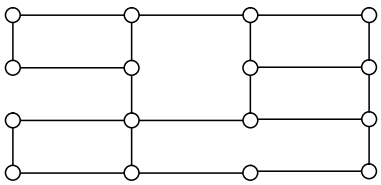
12. Does each of these graphs have an Euler circuit? If so, find it.



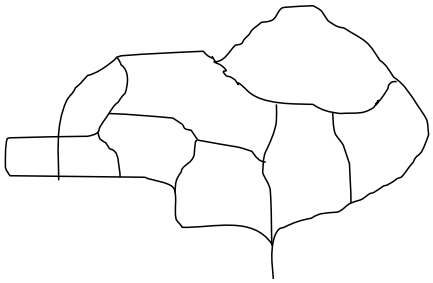
13. Eulerize this graph using as few edge duplications as possible. Then, find an Euler circuit.



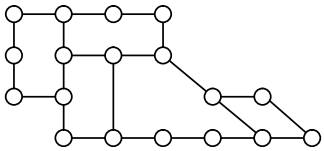
14. Eulerize this graph using as few edge duplications as possible. Then, find an Euler circuit.



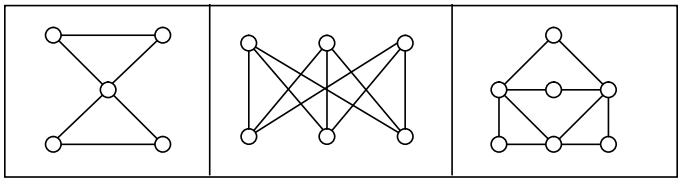
15. The maintenance staff at an amusement park need to patrol the major walkways, shown in the graph below, collecting litter. Find an efficient patrol route by finding an Euler circuit. If necessary, eulerize the graph in an efficient way.



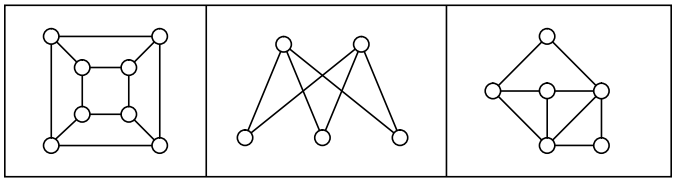
16. After a storm, the city crew inspects for trees or brush blocking the road. Find an efficient route for the neighborhood below by finding an Euler circuit. If necessary, eulerize the graph in an efficient way.



17. Does each of these graphs have at least one Hamiltonian circuit? If so, find one.



18. Does each of these graphs have at least one Hamiltonian circuit? If so, find one.



19. A company needs to deliver product to each of their 5 stores around the Dallas, TX area. Driving distances between the stores are shown below. Find a route for the driver to follow, returning to the distribution center in Fort Worth:

a. Using Nearest Neighbor starting in Fort Worth
b. Using Repeated Nearest Neighbor
c. Using Sorted Edges

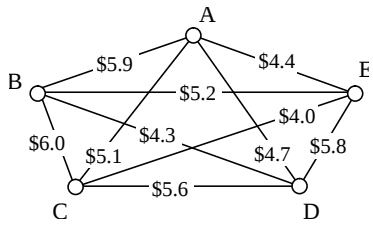
	Plano	Mesquite	Arlington	Denton
Fort Worth	54	52	19	42
Plano		38	53	41
Mesquite			43	56
Arlington				50

20. A salesperson needs to travel from Seattle to Honolulu, London, Moscow, and Cairo. Use the table of flight costs from problem #4 to find a route for this person to follow:

a. Using Nearest Neighbor starting in Seattle
b. Using Repeated Nearest Neighbor
c. Using Sorted Edges

21. When installing fiber optics, some companies will install a sonet ring; a full loop of cable connecting multiple locations. This is used so that if any part of the cable is damaged it does not interrupt service, since there is a second connection to the hub. A company has 5 buildings. Costs (in thousands of dollars) to lay cables between pairs of buildings are shown below. Find the circuit that will minimize cost:

a. Using Nearest Neighbor starting at building A
b. Using Repeated Nearest Neighbor
c. Using Sorted Edges



22. A tourist wants to visit 7 cities in Israel. Driving distances, in kilometers, between the cities are shown below[3]. Find a route for the person to follow, returning to the starting city:

a. Using Nearest Neighbor starting in Jerusalem
b. Using Repeated Nearest Neighbor
c. Using Sorted Edges

	Jerusalem	Tel Aviv	Haifa	Tiberias	Beer Sheba	Eilat
Jerusalem	_					
Tel Aviv	58	_				
Haifa	151	95	_			
Tiberias	152	134	69	_		
Beer Sheba	81	105	197	233	_	
Eilat	309	346	438	405	241	_
Nazareth	131	102	35	29	207	488

23. Find a minimum cost spanning tree for the graph you created in problem #3

24. Find a minimum cost spanning tree for the graph you created in problem #22

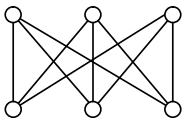
25. Find a minimum cost spanning tree for the graph from problem #21

Concepts

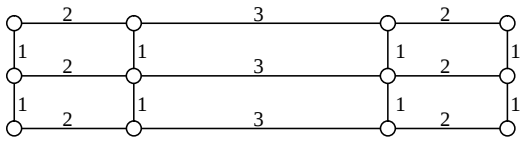
26. Can a graph have one vertex with odd degree? If not, are there other values that are not possible? Why?

27. A complete graph is one in which there is an edge connecting every vertex to every other vertex. For what values of n does complete graph with n vertices have an Euler circuit? A Hamiltonian circuit?

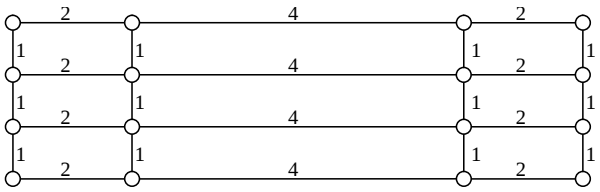
28. Create a graph by drawing n vertices in a row, then another n vertices below those. Draw an edge from each vertex in the top row to every vertex in the bottom row. An example when n=3 is shown below. For what values of n will a graph created this way have an Euler circuit? A Hamiltonian circuit?



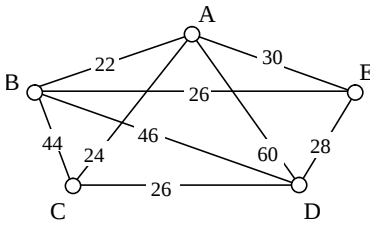
29. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



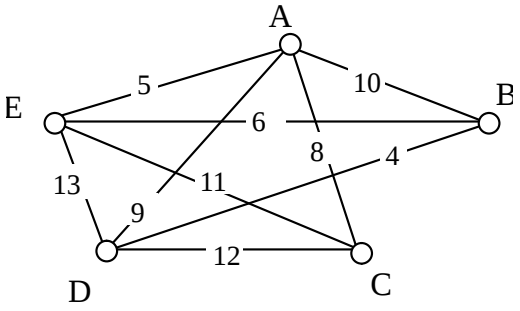
30. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



31. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



32. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



Explorations

33. Social networks such as Facebook and LinkedIn can be represented using graphs in which vertices represent people and edges are drawn between two vertices when those people are “friends.” The table below shows a friendship table, where an X shows that two people are friends.

34.

	A	B	C	D	E	F	G	H	I
A		X	X			X	X		
B			X		X				
C					X				
D					X				X
E							X		X
F								X	X
G								X	
H									X

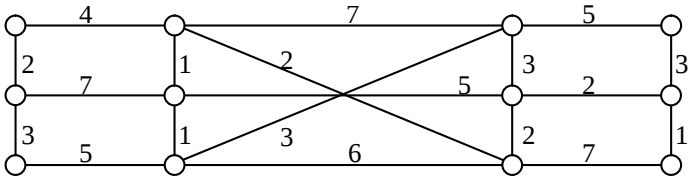
a. Create a graph of this friendship table
b. Find the shortest path from A to D. The length of this path is often called the “degrees of separation” of the two people.
c. Extension: Split into groups. Each group will pick 10 or more movies, and look up their major actors (www.imdb.com is a good source). Create a graph with each actor as a vertex, and edges connecting two actors in the same movie (note the movie name on the edge). Find interesting paths between actors, and quiz the other groups to see if they can guess the connections.

35. A spell checker in a word processing program makes suggestions when it finds a word not in the dictionary. To determine what words to suggest, it tries to find similar words. One measure of word similarity is the Levenshtein distance, which measures the number of substitutions, additions, or deletions that are required to change one word into another. For example, the words spit and spot are a distance of 1 apart; changing spit to spot requires one substitution (i for o). Likewise, spit is distance 1 from pit since the change requires one deletion (the s). The word spite is also distance 1 from spit since it requires one addition (the e). The word soot is distance 2 from spit since two substitutions would be required.

a. Create a graph using words as vertices, and edges connecting words with a Levenshtein distance of 1. Use the misspelled word “moke” as the center, and try to find at least 10 connected dictionary words. How might a spell checker use this graph?
b. Improve the method from above by assigning a weight to each edge based on the likelihood of making the substitution, addition, or deletion. You can base the weights on any reasonable approach: proximity of keys on a keyboard, common language errors, etc. Use Dijkstra’s algorithm to find the length of the shortest path from each word to “moke”. How might a spell checker use these values?

36. The graph below contains two vertices of odd degree. To eulerize this graph, it is necessary to duplicate edges connecting those two vertices.

1. Use Dijkstra’s algorithm to find the shortest path between the two vertices with odd degree. Does this produce the most efficient eulerization and solve the Chinese Postman Problem for this graph?



2. Suppose a graph has n odd vertices. Using the approach from part a, how many shortest paths would need to be considered? Is this approach going to be efficient?

---

[1] Cheapest fares found when retrieved Sept 1, 2009 for travel Sept 22, 2009

[2] From www.eurail.com/eurail-railway-map

[3] From http://www.ddtravel-acc.com/Israel-c...s-distance.htm

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CHAPTER OVERVIEW

11: Solutions to Selected Exercises

11.1: Voting Theory
11.2: Weighted Voting
11.3: Cryptography
11.4: Apportionment
11.5: Growth Models
11.6: Math and Art
11.7: Money Management
11.8: Finance
11.9: Graph Theory

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11.1: Voting Theory

1.

Number of voters	3	3	1	3	2
1st choice	A	A	B	B	C
2nd choice	B	C	A	C	A
3rd choice	C	B	C	A	B

3.

a. $9+19+11+8=47$
b. 24 for majority; 16 for plurality (though a choice would need a minimum of 17 votes to actually win under the Plurality method)
c. Atlanta, with 19 first-choice votes
d. Atlanta 94, Buffalo 111, Chicago 77. Winner: Buffalo
e. Chicago eliminated, 11 votes go to Buffalo. Winner: Buffalo
f. A vs B: B. A vs C: A. B vs C: B. B gets 2 pts, A 1 pt. Buffalo wins.

5.

a. $120+50+40+90+60+100=460$
b. 231 for majority; 116 for plurality
c. A with 150 first choice votes
d. A 1140, B 1060, C 1160, D 1240. Winner: D
e. B eliminated, votes to C. D eliminated, votes to A. Winner: A
f. A vs B: B. A vs C: A. A vs D: D. B vs C: C. B vs D: D. C vs D: C. A 1pt, B 1pt, C 2pt, D 2pt. Tie between C and D.

Winner would probably be C since C was preferred over D

7.

a. 33
b. 17

9. Yes, B

11. B, with 17 approvals

13. Independence of Irrelevant Alternatives Criterion

15. Condorcet Criterion

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11.2: Weighted Voting

1.

a. 9 players
b. $10+9+9+5+4+4+3+2+2=48$
c. 47

3.

a. 9, a majority of votes
b. 17, the total number of votes
c. 12, which is 2/3 of 17, rounded up

5.

a. P1 is a dictator (can reach quota by themselves)
b. P1, since dictators also have veto power
c. P2, P3, P4

7.

a. none
b. P1
c. none

9.

a. 11+7+2 = 20
b. P1 and P2 are critical

11. Winning coalitions, with critical players underlined:

$\{\underline{P1},\underline{P2}\}\ \{\underline{P1},P2,P3\}\{\underline{P1},\underline{P2},P4\}\{\underline{P1},\underline{P2},P3,P4\}\{\underline{P1},\underline{P3}\}\{\underline{P1},\underline{P3},P4\}$

P1: 6 times, P2: 2 times, P3: 2 times, P4: 0 times. Total: 10 times

Power: $P1: 6/10=60\%, P2: 2/10=20\%, P3: 2/10=20\%, P4: 0/10=0\%$

13.

a. $\{\underline{P1}\}\{P1,P2\}\{\underline{P1},P3\}\{\underline{P1},P4\}\{\underline{P1},P2,P3\}\{\underline{P1},P2,P4\}\{\underline{P1},P3,P4\}\{\underline{P1},P2,P3,P4\}$ P1: 100%, P2: 0%, P3: 0%, P4: 0%
b. $\{\underline{P1},\underline{P2}\}\{\underline{P1},P3\}\{\underline{P1},P4\}\{\underline{P1},P2,P3\}\{\underline{P1},P2,P4\}\{\underline{P1},P3,P4\}\{\underline{P1},P2,P3,P4\}$ P1: 7/10 = 70%, P2: 1/10 = 10%, P3: 1/10 = 10%, P4: 1/10 = 10%
c. $\{\underline{P1},\underline{P2}\}\{\underline{P1},\underline{P3}\}\{\underline{P1},P2,P3\}\{\underline{P1},\underline{P2},P4\}\{\underline{P1},\underline{P3},P4\}\{\underline{P1},P2,P3,P4\}$ P1: 6/10 = 60%, P2: 2/10 = 20%, P3: 2/10 = 20%, P4: 0/10 = 0%

15. $P3=5.P3+P2=14.P3+P2+P1=27$, reaching quota. $P1$ is critical.

17. Sequential coalitions with pivotal player underlined

$<P1,\underline{P2},P3><P1,\underline{P3},P2><P2,\underline{P1},P3><P2,\underline{P3},P1><P3,\underline{P1},P2><P3,\underline{P2},P1>$

$P1: 2/6=33.3\%, P2: 2/6=33.3\%, P3: 2/6=33.3\%$

19.

a. 6, 7
b. 8, given P1 veto power
c. 9, given P1 and P2 veto power

21. If adding a player to a coalition could cause it to reach quota, that player would also be critical in that coalition, which means they are not a dummy. So a dummy cannot be pivotal.

23. We know P2+P3 can't reach quota, or else P1 wouldn't have veto power.

P1 can't reach quota alone.

P1+P2 and P1+P3 must reach quota or else P2/P3 would be dummy.

a. $\{\underline{P1}, \underline{P2}\}\ \{P1, \underline{P3}\}\ \{\underline{P1}, P2, P3\}$. P1: 3/5, P2: 1/5, P3: 1/5
b. $< P1, \underline{P2}, P3 >< P1, \underline{P3}, P2 >< P2, \underline{P1}, P3 >< P2, P3, \underline{P1} >< P3, \underline{P1}, P2 >< P3, P2, \underline{P1} >$

$\mathbf{P1 : 4/6, \quad P2 : 1/6, \quad P3 : 1/6}$

25. $[4 : 2, 1, 1, 1]$ is one of many possibilities

27. $[56 : 30, 30, 20, 20, 10]$

29. $[54 : 10, 10, 10, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1]$ is one of many possibilities

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11.3: Cryptography

1. ZLU KZB WWS PLZ 3. SHRED EVIDENCE

5. O2H DO5 HDV 7. MERGER ON

9. MNI YNE TBA AEH RTA TEA TAI LRE A

11. THE STASH IS HIDDEN AT MARVINS QNS

13. UEM IYN IOB WYL TTL N

15. HIRE THIRTY NEW EMPLOYEES MONDAY

17. ZMW NDG CDA YVK

19. a) 3 b) 0 c) 4

21. We test out all n from 1 to 10

n	4^n	$4^n \bmod 11$
1	4	4
2	16	5
3	64	9
4	256	3
5	1024	1
6	4096	4
7	16384	5
8	65536	9
9	262144	3
10	1048576	1

Since we have repeats, and not all values from 1 to 10 are produced (for example, there is no n is $4^n \bmod 11 = 7$), 4 is *not* a generator $\bmod 11$.

23. $157^{10} \bmod 5 = (157 \bmod 5)^{10} \bmod 5 = 2^{10} \bmod 5 = 1024 \bmod 5 = 4$

25. $3^7 \bmod 23 = 2$

27. Bob would send $5^7 \bmod 33 = 14$. Alice would decrypt it as $14^3 \bmod 33 = 5$

31.

a. $67^8 \bmod 83 = \left(67^4 \bmod 83\right)^2 \bmod 83 = 49^2 \bmod 83 = 2401 \bmod 83 = 77$

$67^{16} \bmod 83 = \left(67^8 \bmod 83\right)^2 \bmod 83 = 77^2 \bmod 83 = 5929 \bmod 83 = 36$

b. $17000 \bmod 83 = (100 \bmod 83)*(170 \bmod 83) \bmod 83 = (17)(4) \bmod 83 = 68$

c. $67^5 \bmod 83 = \left(67^4 \bmod 83\right)(67 \bmod 83) \bmod 83 = (49)(67) \bmod 83 = 3283 \bmod 83 = 46$

d. $67^7 \bmod 83 = \left(67^4 \bmod 83\right)\left(67^2 \bmod 83\right)(67 \bmod 83) \bmod 83 = (49)(7)(67) \bmod 83 = 22981 \bmod 83 = 73$

e. $67^{24} = 67^{16}67^8$ so $67^{24} \bmod 83 = \left(67^{16} \bmod 83\right)\left(67^8 \bmod 83\right) \bmod 83 = (77)(36) \bmod 83 = 2272 \bmod 83 = 33$

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11.4: Apportionment

1.

a. Math: 6, English: 5, Chemistry: 3, Biology: 1
b. Math: 7, English: 5, Chemistry: 2, Biology: 1
c. Math: 6, English: 5, Chemistry: 3, Biology: 1
d. Math: 6, English: 5, Chemistry: 3, Biology: 1
e. Math: 6, English: 5, Chemistry: 2, Biology: 2

3.

a. Morning: 1, Midday: 5, Afternoon: 6, Evening: 8
b. Morning: 1, Midday: 4, Afternoon: 7, Evening: 8
c. Morning: 1, Midday: 5, Afternoon: 6, Evening: 8
d. Morning: 1, Midday: 5, Afternoon: 6, Evening: 8
e. Morning: 2, Midday: 5, Afternoon: 6, Evening: 7

5.

a. Alice: 18, Ben: 14, Carlos: 4
b. Alice: 19, Ben: 14, Carlos: 3
c. Alice: 19, Ben: 14, Carlos: 3
d. Alice: 19, Ben: 14, Carlos: 3
e. Alice: 18, Ben: 14, Carlos: 4

7.

a. A: 40, B: 24, C: 15, D: 30, E: 10
b. A: 41, B: 24, C: 14, D: 30, E: 10
c. A: 40, B: 24, C: 15, D: 30, E: 10
d. A: 40, B: 24, C: 15, D: 30, E: 10
e. A: 40, B: 24, C: 15, D: 29, E: 11

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11.5: Growth Models

1.

a. $P_0 = 20. P_n = P_{n-1} + 5$
b. $P_n = 20 + 5n$

3.

a. $P_1 = P_0 + 15 = 40 + 15 = 55. P_2 = 55 + 15 = 70$
b. $P_n = 40 + 15n$
c. $P_{10} = 40 + 15(10) = 190$ thousand dollars
d. $40 + 15n = 100$ when $n = 4$ years.

5. Grew 64 in 8 weeks: 8 per week

a. $P_n = 3 + 8n$
b. $187 = 3 + 8n. n = 23$ weeks

7.

a. $P_0 = 200$ (thousand), $P_n = (1 + .09)P_{n-1}$ where n is years after 2000
b. $P_n = 200(1.09)^n$
c. $P_{16} = 200(1.09)^{16} = 794.061(\text{ thousand }) = 794,061$
d. $200(1.09)^n = 400. \quad n = \log(2)/\log(1.09) = 8.043.$ In 2008

9. Let $n = 0$ be $1983. \quad P_n = 1700(2.9)^n. \quad 2005$ is $n = 22. \quad P_{22} = 1700(2.9)^{22} = 25,304,914,552,324$ people. Clearly not realistic, but mathematically accurate.

11. If n is in hours, better to start with the explicit form. $P_0 = 300. P_4 = 500 = 300(1 + r)^4$

$500/300 = (1 + r)^4. \quad 1 + r = 1.136. \quad r = 0.136$

a. $P_0 = 300. \quad P_n = (1.136)P_{n-1}$
b. $P_n = 300(1.136)^n$
c. $P_{24} = 300(1.136)^{24} = 6400$ bacteria
d. $300(1.136)^n = 900. n = \log(3)/\log(1.136) =$ about 8.62 hours

13.

a. $P_0 = 100 \quad P_n = P_{n-1} + 0.70\,(1 - P_{n-1}/2000)\,P_{n-1}$
b. $P_1 = 100 + 0.70(1 - 100/2000)(100) = 166.5$
c. $P_2 = 166.5 + 0.70(1 - 166.5/2000)(166.5) = 273.3$

15. To find the growth rate, suppose $n = 0$ was 1968. Then P_0 would be 1.60 and $P_8 = 2.30 = 1.60(1 + r)^8, r = 0.0464.$ since we want $n = 0$ to correspond to 1960 , then we don't know P_0, but P_8 would $1.60 = P_0(1.0464)^8$. $P_0 = 1.113$

a. $P_n = 1.113(1.0464)^n$
b. $P_0 = \$1.113,$ or about $\$1.11$
c. 1996 would be $n = 36. \quad P_{36} = 1.113(1.0464)^{36} = \$5.697.$ Actual is slightly lower.

17. The population in the town was 4000 in 2005, and is growing by 4% per year.

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11.6: Math and Art

Your Turn

13.1
1. 46 in
13.2
1. 34 scales
13.3
1. 1.618
13.4
1. 1.25, which is close to a golden rectangle
13.5
1. 210 gal
13.6
1. 1,050,000 gal
13.7
1. 9,828 million gal of water
13.8
1. 15.8 kW
13.9
1. $14.18
13.10
1. $0.06
13.11
1. 288 kW
13.12
1. 1.8 mg in 200 mL
13.13
1. 6 pills
13.14
1. 42 mg
13.15
1. 13 mL
13.16
1. Yara, Jeffrey
2. 5
13.17
1. birds chirping
2. window air conditioner
3. garbage disposal
4. live outdoor concert
13.18
1. up G, down F
2. up C, down $B^{\flat}$
3. up G, down F
13.19
1. up $E^{\flat}$, down B
2. up $D^{\#}$, down B
3. up $F^{\#}$, down D
13.20
1. G, $G^{\#}$, A, $A^{\#}$, B, C, $C^{\#}$, D, $D^{\#}$, E, F, $F^{\#}$
13.21
1. 83 Hz
13.22
1. 10 games
2. 66 games
3. 300 games
13.23

1. 4 games
2. 11 games
3. 24 games

Check Your Understanding

1. 1.618
2. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55
3. A rectangle whose length divided by the width is approximately equal to ϕ or 1.618.
4. $\mathbf{1\,kW = 1{,}000\,watts}$
5. 100 gal
6. 33 1/3 hours
7. 17 hours
8. medicine dosage required, weight of drug by volume
9. b Research on coronavirus vaccines began approximately 10 years before the COVID-19 pandemic hit the world.
10. Mathematical modeling is a graphical representation of a behavior, such as exposure to a virus. It can be helpful in mapping contract tracing with virus exposure during a pandemic.
11. Frequency is a static measured value, whereas pitch is subjective.
12. higher
13. 20–20,000 Hz
14. increase
15. 0 dB
16. 20–17,000Hz
17. It doubles.
18. Data analytics is used in sports to analyze data on team's players, potential recruits, opposing teams, and opposing players. This helps to find patterns and make predictions for future results.
19. Soccer (football)
20. $\boldsymbol{n\,(n-1)\,/2}$
21. $\boldsymbol{n-1}$

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11.7: Money Management

Your Turn

6.1
1. $\frac{3}{1}00$
2. $\frac{9}{4}100$
3. $\frac{6}{7}.2100$
4. $\frac{6}{7}0100$

6.2
1. 0.09
2. 0.24
3. 0.0218

6.3
1. 41%
2. 2%
3. 924.81%

6.4
1. 338
2. 2058.75

6.5
1. 250
2. 6,000

6.6
1. 62.5%
2. 40%

6.7
1. $529.50

6.8
1. 66.31%

6.9
1. 80 bulbs.

6.10
1. Discount is $496; sale price is $1,054.00
2. Discount is $2.75; sale price is $24.75

6.11
1. 35%
2. 12%

6.12
1. $13.00
2. $220.00

6.13
1. Discount is $330; sale price of the bed is $220.

6.14
1. 15%

6.15
1. $59.00

6.16
1. Markup is $396; retail price is $2,196.00.
2. Markup is $1.05; retail price is $11.55.

6.17
1. 58.33%
2. 200%

6.18
1. $35.00
2. $8.99

6.19
1. The retail price of the bed is $1,260.00.

6.20

1. 24%
6.21
1. $38.00
6.22
1. Sales tax is $104.93; total price is $1,603.93.
2. Sales tax is $1.88; total price is $28.77.
6.23
1. Sales tax rate = 5.25%
2. Sales tax rate = 6.75%
6.24
1. $145.78
2. $489.00
6.25
1. Sales tax is $111.22. The total price Daryl pays is $1,260.22.
6.26
1. The sales tax is $9.
6.27
1. Interest = $2,409.99, the loan payoff is $9,109.99
2. Interest = $8,901, the loan payoff is $34,701.00
6.28
1. She will pay $878.75 in interest, total repayment is $10,378.75.
6.29
1. He will pay $727.86 in interest, payoff is $9,127.86.
6.30
1. The cost to borrow is $115.51 and their payoff is $3,815.51.
6.31
1. Interest = $253.8, FV = $4,753.80
2. Interest = $406.00, FV = $2,406.00
3. Interest= $1019.18, FV = $121,019.18
4. Interest = $253.89, FV = $4,933.89
6.32
1. Mia will receive $6,254.85.
6.33
1. $846.58
2. $172.91
6.34
1. $709.59
2. $13,182.74
6.35
1. $2,313.74
6.36
1. $443.90
6.37
1. $14,285.72. $14,285.72 needs to be invested so that, after 10 years at 7.5% interest, the investment will be worth $25,000.
2. $97,709.93. $97,709.93 needs to be invested so that, after 35 years at 6.5% interest, the investment will be worth $320,000.
3. $48,813.56. $48,813.56 needs to be invested so that, after 270 months at 3.75% interest, the investment will be worth $90,000.
6.38
1. $10,549.46
6.39
1. The CD is worth $5,627.54 after 4 years.
6.40
1. CD would have been worth $5,600. Oksana earned $27.54 more with compound interest.
6.41
1. $11,443.81
2. $26,551.23
6.42
1. $5,813.64
6.43

1. $3,555.88
6.44
1. $100,811.07
2. $82,135.75
6.45
1. $96,271.23
6.46
1. A rate of 7% compounded quarterly is equivalent to a simple interest rate of 7.19%.
6.47
1. The effective annual yield is 2.53%.
6.48
1. The effective annual yields for the banks are 3.1158% for Smith Bank, 3.11% for Park Bank, 3.1381% for Town Bank, and 3.144% for Community Bank. Community Bank has the best yield.
6.49
1. Mateo's budget is below.

Income Source	Amount	Expense	Amount
Electrician	$3,375.00	Mortgage	$987
Side jobs	$300.00	Truck payment	$589
		Truck insurance	$312
		Utilities	$167
		Clothing	$150
		Entertainment	$400
		Credit Card	$325
		Food	$470
		Gasoline	$375

His total monthly income is $3,675.00, and his monthly expenses are $3,775. Mateo falls $100.00 short each month.
6.50
1. Maddy's budget is below.

Income Source	Amount	Expense	Amount
Engineer	$6,093.75	Mortgage	$1,452.89
		Car payment	$627.38
		Car insurance	$179.00
		Health Insurance	$265.00
		Utilities	$320.00
		Clothing	$150.00
		Entertainment	$400.00
		Credit Card	$450.00
		Food	$370.00
		Gasoline	$175.00

Her total monthly income is $6,093.75, and her monthly expenses are $4,389.27. Maddy has $1,704.48 in extra income per month.

This is her cushion in the budget.
6.51
1. Heather's budget is now

Income Source	Amount	Expense	Amount
Nursing	$3,765.40	Mortgage	$1,240
Part-time	$672.00	Car Payment	$489
		Student Loan	$728
		Car Insurance	$139
		Utilities	$295
		Clothing	$150
		Entertainment	$300
		Credit Card	$200
		Food	$400
		Gasoline	$250

The changes have added $355.00 to her budget. As her extra income is $601.40, she can afford the changes.
6.52
1. Heather's total income is $4,437.40.
For the necessities, Heather should budget $2,218.70.
For her wants, she should budget $1,331.22.
For savings and extra debt service, she should budget $887.48.
Her necessities total $3,536.00, which exceeds the suggested budget amount of $2,218.70.
Her wants total $300.00, which is below the suggested budget amount of $1,331.22.
Her excess income is $601.40, which is below the suggested budget amount of $887.48.
Heather should make some changes.
6.53
1. Elijah should budget $1,631.72 for needs, $979.03 for wants, and $652.69 for savings and debt service. When choosing where to live, what to eat, and what to drive, he should make choices that keep those costs, combined with debt service costs, gasoline, and utilities, below $1,631.72. This means he will have to make decisions about what his priorities are. Elijah should then figure out his wants, and stay within the limits here, that is, keep those costs below $979.03. Finally, he can begin to build his savings with the remaining $652.69.
6.54
1. Her monthly income is $3,641.66.
Needs (50%): $1,820.83, Wants (30%): $1,092.50, Savings (20%): $728.33
2. $1,061.83
3. $1,092.50, $728.33
4. She only has budgeted $1,061.83 for other necessities. It is difficult to imagine Fran being able to afford to change jobs and move, unless she reallocates money that she would want to save or use for entertainment or takes on another job. If Fran uses all the money that the 50-30-20 budget sets aside for savings, she then would have $1,790.16 to spend on those other necessities. If she moves away from the 50-30-20 philosophy, she may be able to afford the move. However, that means changing her priorities.
6.55
1. $5,952.97
6.56
1. $3,741.46
6.57
1. $8,978.57
6.58
1. 17.31%
2. 13.2%
6.59

1. $42,499.63
6.60
1. $93,037.59
6.61
1. $1,227.30
6.62
1. $237.50 per year, $5,237.50 on maturity date.
6.63
1. $108,501.30
6.64
1. 4.28%
2. 3.29%
6.65
1. Looking at the table, the current price of a share is $30.68.
2. The high was $56.28, and the low was $30.05.
3. August 4, 2022
4. 4.77%
5. $4.66
6.66
1. $336.00
6.67
1. $52
2. 2.6%
6.68
1. $108,352.43
6.69
1. $1,815.83
6.70
1. 23.75%
6.71
1. 4.35%
6.72
1. 371.74%
2. 13.8%
6.73
1. $3,336.13
6.74
1. Traditional
6.75
1. $5,460
2. $10,920
3. 100% return on the $5,460 deposit
6.76
1. $813,128.60
2. $457,384.83
6.77
1. $355.91
2. $62.48
6.78
1. $213.07
2. $209.96
6.79
1. $34,700, 3.79%, 6 years
2. $539.57
3. $25,837.42
4. $104.14
5. $2,246.62
6. The amount paid to principal increases each month

6.80
1. $6,437.18
6.81
1. $9,745
6.82
1. Tiana can take out $12,500, the maximum in year 6.
6.83
1. Gap = $10,850, Federal loans = $7,500, Remaining $3,350
6.84
1. $2,987.63
6.85
1. $82.70
2. $137.42
6.86
1. $259.69
6.87
1. There is no impact on Ryann's interest rate. The total Ryann pays back increases since the interest is extended to more periods.
6.88
1. $188.03
6.89
1. $5,310
6.90
1. $11,735
2. $97.80
6.91
1. Store-issued credit cards
2. Fees (both annual and penalty fees), reward programs, credit limits
3. Travel and entertainment cards
6.92
1. April 01 2015 to April 30 2015
2. $9,000.00
3. $5.99
6.93
1. $567.32
6.94
1. $22.64
2. $13.49
3. $77.77
6.95
1. $566.44
2. $3,267.17
6.96
1. $1,152.35
6.97
1. $47.47
2. $84.00
3. $19.90
6.98
1. $33,838.48
6.99
1. $32,777.75
6.100
1. $436.70
2. $649.47
6.101
1. $1,516.05
6.102
1. $10,567.00

6.103
1. $401.05
6.104
1. 6.25%
2. $0.000032916\overline{6}$
6.105
1. $74.49
2. $1,015.25
6.106
1. Purchase
2. Purchase
6.107
1. Personal injury protection
2. Liability insurance
6.108
1. $593.36
6.109
1. $42.02
6.110
1. $949.72
6.111
1. $319,620.60
6.112
1. $93,741.60
6.113
1. 6.1%
2. $1,383.61
3. $730.38
4. $723.66
5. $133,513.05
6. Payment 225
6.114
1. $1,521.20
6.115
1. $47,280
6.116
1. $80,300
6.117
1. $39,700
6.118
1. $25
6.119
1. Refund of $1,920
6.120
1. FICA is $244.80, SSI is $198.40
6.121
1. $9,114
6.122
1. $11,787
6.123
1. His credits exceed his taxes owed, so he receives a refund, which is for $1,673.94.

Check Your Understanding

1. 100
2. 0.387
3. 190.4
4. 834.15 (rounded to two decimal places)
5. 24.07%

6. 16 new drivers will be hired.
7. a reduction in price
8. the amount a retailer adds to the cost of goods to cover overhead and profit
9. $30.60
10. $497.60
11. Yes
12. $42.67
13. Interest is money paid by a borrower to a lender for the privilege of borrowing money.
14. The amount borrowed is the principal.
15. $1,169.10
16. $11,374.87
17. $266.82
18. $21,818.19
19. interest that is earned on both the principal and the interest from previous periods
20. compound
21. $15,698.69
22. $405.62
23. $30,266.48
24. Compare their effective annual yield.
25. 4.98%
26. A budget is an estimation of income and expenses over some period of time.
27. Expenses that represent basic living requirement and debt service.
28.

Income	Amount	Expenses	Amount
Job	$3,450.00	Rent	$925.00
		Car payment	$178.54
		Car insurance	$129.49
		Credit card	$117.00
		Gas	$195.00
		Food	$290.00
		Amazon Prime	$21.99
		Internet	$49.99
		Going out	$400.00
		Total	$2,307.01

29. $1,142.99
30. David should allocate $1,735 to necessary expenses, $1,035 to expenses that are wants, and $690 to savings and extra debt service.
31. David spends $1,835.03 on necessary expenses, which is fairly close to the 50-30-20 budget philosophy guidelines. He spends $471.98 on expenses that are wants, which is well below the guideline of $1,035. He has $1,149.99 in extra income, which is well above the savings and debt service guidelines.
32. A deposit account, held at a bank or other financial institution, which bears some interest on the deposited money.
33. Once money is deposited in a CD, the money cannot be withdrawn until the end of the CD period.
34. A money market account is more flexible. Transactions can be made after creating the money market account.
35. $8,549.93
36. 25.6%
37. $21,441.03
38. $2,030.65
39. Stocks
40. Income must be below a threshold for a person to use a Roth IRA.

41. They invest in different investment vehicles so that no single investment can have a large impact on the value of the mutual fund.
42. Each year, the holder of the bond receives $200. For 10 years, that's $2,000 total earned with the bond.
43. Yes
44. $3,500, which is 4% of her income
45. $114.00
46. 9.93%
47. $5,207.59
48. Varies
49. Loan with a fixed period, and the borrower pays a fixed amount per period until the loan is paid off
50. More is applied to the principal.
51. $59.11
52. $2,233.12
53. $42,962.51
54. $39,544.50
55. Spring (March) before the start of the fall term
56. 9 years
57. The difference between the cost of college (including books, fees, room and board, etc.) and all non-loan financial aid they receive
58. In subsidized loans, the interest does not accrue while the student is still in school. In unsubsidized student loans, the interest begins accruing as soon as the money is disbursed.
59. $31,000
60. $13,028.61
61. 10 years
62. 10%
63. Bank issued, store issued, charge (Travel/entertainment) cards
64. Store issued
65. Due date, minimum payment due, and balance due
66. $19.30
67. $4,104.27
68. $696.21
69. Cost of delivering the vehicle to the dealer
70. Registers your car with the state, gets the license plate, and assigns the title of the car to the lender
71. $628.89
72. Interest rate higher loan term shorter
73. Title and registration, documentation fee, sales tax
74. $333.34
75. 3.75%
76. Purchasing
77. Cover the damage caused to your car if involved in an accident with another vehicle
78. Buying
79. Renting
80. Buying
81. Mortgage
82. $663.32
83. $129,802.20
84. $554.06
85. Escrow
86. Any and all income, including wages, salary, interest earned, gifts, and winnings
87. After deductions are subtracted
88. $37,000
89. After taxes are calculated, credits are subtracted from taxes owed.
90. No, only for up to $147,000
91. $4,813
92. $1,600

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11.8: Finance

1. $A = 200 + .05(200) = \$210$

3. $\mathrm{I} = 200.\mathrm{t} = 13/52(13 \text{ weeks out of } 52 \text{ in a year }).\mathrm{P}_0 = 9800$

$200 = 9800(\mathrm{r})(13/52)\mathrm{r} = 0.0816 = 8.16\%$ annual rate

5. $P_{10} = 300(1 + .05/1)^{10(1)} = \$488.67$

7.

a. $P_{20} = 2000(1 + .03/12)^{20(12)} = \$3641.51$ in 20 years
b. $3641.51 - 2000 = \$1641.51$ in interest

9. $P_8 = P_0(1 + .06/12)^{8(12)} = 6000 \cdot P_0 = \$3717.14$ would be needed

11.

a. $P_{30} = \frac{200((1+0.03/12)^{30(12)}-1)}{0.03/12} = \$116,547.38$
b. $200(12)(30) = \$72,000$
c. $\$116,547.40 - \$72,000 = \$44,547.38$ of interest

13.

a. $P_{30} = 800,000 = \frac{d((1+0.06/12)^{30(12)}-1)}{0.06/12}\mathrm{d} = \$796.40$ each month
b. $\$796.40(12)(30) = \$286,704$
c. $\$800,000 - \$286,704 = \$513,296$ in interest

15.

a. $P_0 = \frac{30000(1-(1+0.08/1)^{-25(1)})}{0.08/1} = \$320,243.29$
b. $30000(25) = \$750,000$
c. $\$750,000 - \$320,243.29 = \$429,756.71$

17. $P_0 = 500,000 = \frac{d(1-(1+0.06/12)^{-20(12)})}{0.06/12}\mathrm{d} = \$3582.16$ each month

19.

$P_0 = \frac{700(1-(1+0.05/12)^{-30(12)})}{0.05/12} =$ a $\$130,397.13$ loan

$700(12)(30) = \$252,000$

$\$252,200 - \$130,397.13 = \$121,602.87$ in interest

21. $P_0 = 25,000 = \frac{d(1-(1+0.02/12)^{-48})}{0.02/12} = \$542.38$ a month

23.

Down payment of 10% is $\$20,000$, leaving $\$180,000$ as the loan amount

$P_0 = 180,000 = \frac{d(1-(1+0.05/12)^{-30(12)})}{0.05/12}\mathrm{d} = \$966.28\mathrm{amonth}$

$P_0 = 180,000 = \frac{d(1-(1+0.06/12)^{-30(12)})}{0.06/12}\mathrm{d} = \$1079.19$ a month

25. First we find the monthly payments:

$P_0 = 24,000 = \frac{d(1-(1+0.03/12)^{-5(12)})}{0.03/12} \cdot d = \$431.25$

Remaining balance: $P_0 = \frac{431.25(1-(1+0.03/12)^{-2(12)})}{0.03/12} = \$10,033.45$

27. $6000(1 + 0.04/12)^{12N} = 10000$

$(1.00333)^{12N} = 1.667$

$\log\left((1.00333)^{12N}\right) = \log(1.667)$

$12N\log(1.00333) = \log(1.667)$

$N = \frac{\log(1.667)}{12\log(1.00333)} =$ about 12.8 years

29. $3000 = \frac{60(1-(1+0.14/12)^{-12N})}{0.14/12}$

$3000(0.14/12) = 60\left(1-(1.0117)^{-12N}\right)$

$\frac{3000(0.14/12)}{60} = 0.5833 = 1-(1.0117)^{-12N}$

$0.5833 - 1 = -(1.0117)^{-12N}$

$-(0.5833-1) = (1.0117)^{-12N}$

$\log(0.4167) = \log\left((1.0117)^{-12N}\right)$

$\log(0.4167) = -12N\log(1.0117)$

$N = \frac{\log(0.4167)}{-12\log(1.0117)} =$ about 6.3 years

31. First 5 years: $P_5 = \frac{50((1+0.08/12)^{5(12)}-1)}{0.08/12} = \$3673.84$

Next 25 years: $3673.84(1+.08/12)^{25(12)} = \$26,966.65$

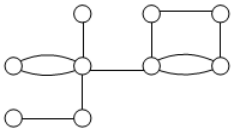
33. Working backwards, $P_0 = \frac{10000(1-(1+0.08/4)^{-10(4)})}{0.08/4} = \$273,554.79$ needed at retirement. To end up with that amount of money, $273,554.70 = \frac{d((1+0.08/4)^{15(4)}-1)}{0.08/4}$. He'll need to contribute $d = \$2398.52$ a quarter.

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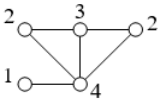
11.9: Graph Theory

1.



3.

5.

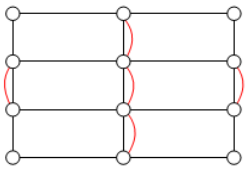


7. The first and the third graphs are connected

9. Bern to Frankfurt to Munchen to Berlin: 12hrs 50 min. (Though trip through Lyon, Paris and Amsterdam only adds 30 minutes)

11. The first graph has an Euler circuit. The last two graphs each have two vertices with odd degree.

13. One of several possible eulerizations requiring 5 duplications:



17. Only the middle graph has a Hamiltonian circuit.

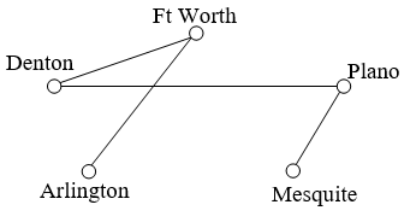
19.

a. Ft Worth, Arlington, Mesquite, Plano, Denton, Ft Worth: 183 miles
b. Same as part a
c. Same as part a

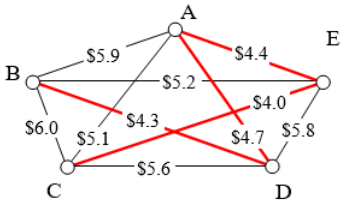
21.

a. ABDCEA
b. ACEBDA
c. ADBCEA

23.



25.



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