MATH 104 GREEN



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CHAPTER OVERVIEW

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1.1: Introduction

Think of a drawer in your kitchen used to store flatware. This drawer likely holds forks, spoons, and knives, and possibly other items such as a meat thermometer and a can opener. The drawer in this case represents a tool used to group a collection of objects. The members of the group are the individual items in the drawer, such as a fork or a spoon.



Figure 1.1 A flatware drawer is like a set in that it contains distinct objects. (credit: modification of work “silverware” by Jo Naylor/Flickr, CC BY 2.0)

The members of a set can be anything, such as people, numbers, or letters of the alphabet. In statistical studies, a set is a well-defined collection of objects used to identify an entire population of interest. For example, in a research study examining the effects of a new medication, there can be two sets of people: one set that is given the medication and a different set that is given a placebo (control group). In this chapter, we will discuss sets and Venn diagrams, which are graphical ways to show relationships between different groups.

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1.2: Basic Set Concepts



Figure 1.2.1: A spoon, fork, and knife are elements of the set of flatware. (credit: modification of work “Cupofjoy” Wikimedia CC0 1.0 Public Domain)

Learning Objectives

After completing this section, you should be able to:

- Represent sets in a variety of ways.
- Represent well-defined sets and the empty set with proper set notation.
- Compute the cardinal value of a set.
- Differentiate between finite and infinite sets.
- Differentiate between equal and equivalent sets.

Sets and Ways to Represent Them

Think back to your kitchen organization. If the drawer is the set, then the forks and knives are elements in the set. Sets can be described in a number of different ways: by roster, by set-builder notation, by interval notation, by graphing on a number line, and by Venn diagrams. Sets are typically designated with capital letters. The simplest way to represent a set with only a few members is the **roster** (or *listing*) **method**, in which the elements in a set are listed, enclosed by curly braces and separated by commas. For example, if F represents our set of flatware, we can represent F by using the following set notation with the roster method:

$$F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$$

Example 1.1: Writing a Set Using the Roster or Listing Method

Write a set consisting of your three favorite sports and label it with a capital S.

Answer

There are multiple possible answers depending on what your three favorite sports are, but any answer must list three different sports separated by commas, such as the following:

$$S = \{\text{hockey, basketball, soccer}\}$$

Your Turn 1.1

Write a set consisting of four small hand tools that might be in a toolbox and label it with a capital T.

All the sets we have considered so far have been well-defined sets. A well-defined set clearly communicates whether an element is a member of the set or not. The members of a well-defined set are fixed and do not change over time. Consider the following question. What are your top 10 songs of 2021? You could create a list of your top 10 favorite songs from 2021, but the list your friend creates will not necessarily contain the same 10 songs. So, the set of your top 10 songs of 2021 is not a well-defined set. On the other hand, the set of the letters in your name is a well-defined set because it does not vary (unless of course you change your name). The NFL wide receiver, Chad Johnson, famously changed his name to Chad Ochocinco to match his jersey number of 85.

Example 1.2: Identifying Well-Defined Sets

For each of the following collections, determine if it represents a well-defined set.

1. The group of all past vice presidents of the United States.
2. A group of old cats.

Answer

1. The group of all past vice presidents of the United States is a well-defined set, because you can clearly identify if any individual was or was not a member of that group. For example, Britney Spears is not a member of this set, but Joe Biden is a member of this set.
2. A group of old cats is not a well-defined set because the word old is ambiguous. Some people might consider a seven-year-old cat to be old, while others might think a cat is not old until it is 13 years old. Because people can disagree on what is and what is not a member of this group, the set is not well-defined.

Your Turn 1.2

For each of the following collections, determine if it represents a well-defined set.

A collection of medium-sized potatoes.

The original members of the Black Eyed Peas musical group.

On January 20, 2021, Kamala Harris was sworn in as the first woman vice president of the United States of America. If we were to consider the set of all women vice presidents of the United States of America prior to January 20, 2021, this set would be known as an empty set; the number of people in this set is 0, since there were no women vice presidents before Harris. The empty set, also called the null set, is written symbolically using a pair of braces, $\{\ \}$, or a zero with a slash through it, $\emptyset$.

Checkpoint

The set containing the number $0, \{0\}$*, is a set with one element in it. It is not the same as the empty set,* $\{\ \}$*, which does not have any elements in it. Symbolically:* $\{0\} \neq \{\ \}$.

Example 1.3: Representing the Empty Set Symbolically

Represent each of the following sets symbolically.

1. The set of prime numbers less than 2.
2. The set of birds that are also mammals.

Answer

1. A prime number is a natural number greater than 1 that is only divisible by one and itself. Since there are no prime numbers less than 2, this set is empty, and we can represent it symbolically as follows: $\{\ \}$ or $\emptyset$. These two different symbols for the empty set can be used interchangeably.

2. The set of birds and the set of mammals do not intersect, so the set of birds that are also mammals is empty, and we can represent it symbolically as $\emptyset$ or $\{\ \}$.

Your Turn 1.3

Represent the set of all numbers divisible by 0 symbolically.

Who Knew?: The Number Zero

We use the number zero to represent the concept of nothing every day. The machine language of computers is binary, consisting only of zeros and ones, and even way before that, the number zero was a powerful invention that allowed our understanding of mathematics and science to develop. The historical record shows the Babylonians first used zeros around 300 B.C., while the Mayans developed and began using zero separately around 350 A.D. What is considered the first formal use of zero in arithmetic operations was developed by the Indian mathematician Brahmagupta around 650 A.D.

Brahmagupta, Mathematician and Astronomer

Another interesting feature of the number zero is that although it is an even number, it is the only number that is neither negative nor positive.

For larger sets that have a natural ordering, sometimes an ellipsis is used to indicate that the pattern continues. It is common practice to list the first three elements of a set to establish a pattern, write the ellipsis, and then provide the last element. Consider the set of all lowercase letters of the English alphabet, A. This set can be written symbolically as $A = \{a, b, c, \ldots, z\}$.

The sets we have been discussing so far are finite sets. They all have a limited or fixed number of elements. We also use an ellipsis for infinite sets, which have an unlimited number of elements, to indicate that the pattern continues. For example, in set theory, the set of natural numbers, which is the set of all positive counting numbers, is represented as $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Notice that for this set, there is no element following the ellipsis. This is because there is no largest natural number; you can always add one more to get to the next natural number. Because the set of natural numbers grows without bound, it is an infinite set.

Example 1.4: Writing a Finite Set Using the Roster Method and an Ellipsis

Write the set of even natural numbers including and between 2 and 100, and label it with a capital E. Include an ellipsis.

Answer

Write the label, E, followed by an equal sign and then a bracket. Write the first three even numbers separated by commas, beginning with the number two to establish a pattern. Next, write the ellipsis followed by a comma and the last number in the list, 100. Finally, write the closing bracket to complete the set.

Write the label, E, followed by an equal sign and then a bracket.

$$E = \{$$

Write the first three even numbers separated by commas, beginning with the number 2 to establish a pattern.

$$E = \{2, 4, 6,$$

Next, write the ellipsis followed by a comma and the last number in the list, 100.

$$E = \{2, 4, 6, \ldots, 100$$

Finally, write the closing bracket to complete the set.

$$E = \{2, 4, 6, \ldots, 100\}$$

Your Turn 1.4

Use an ellipsis to write the set of single digit numbers greater than or equal to zero and label it with a capital D.

Our number system is made up of several different infinite sets of numbers. The set of integers, $\mathbb{Z}$, is another infinite set of numbers. It includes all the positive and negative counting numbers and the number zero. There is no largest or smallest integer.

Example 1.5: Writing an Infinite Set Using the Roster Method and Ellipses

Write the set of integers using the roster method, and label it with a $\mathbb{Z}$.

Answer

Step 1: As always, we write the label and then the opening bracket. Because the negative counting numbers are infinite, to represent that the pattern continues without bound to the left, we must use an ellipsis as the first element in our list.

Step 2: We place a comma and follow it with at least three consecutive integers separated by commas to establish a pattern.

Step 3: Add an ellipsis to the end of the list to show that the set of integers continues without bound to the right.

Complete the list with a closing bracket. The set of integers may be represented as follows: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Your Turn 1.5

Write the set of odd numbers greater than 0 and label it with a capital M.

A shorthand way to write sets is with the use of set builder notation, which is a verbal description or formula for the set. For example, the set of all lowercase letters of the English alphabet, A, written in set builder notation is:

$A = \{x | x \text{ is a lowercase letter of the English alphabet.}\}$

This is read as, "Set A is the set of all elements x such that x is a lowercase letter of the English alphabet."

Example 1.6: Writing a Set Using Set Builder Notation

Using set builder notation, write the set B of all types of balls. Explain what the notation means.

Answer

The verbal description of the set is, "Set B is the set of all elements b such that b is a ball." This set can be written in set builder notation as follows: $B = \{b | b \text{ is a ball.}\}$

Your Turn 1.6

Using set builder notation, write the set C of all types of cars.

Example 1.7: Writing Sets Using Various Methods

Consider the set of letters in the word "happy." Determine the best way to represent this set, and then write the set using either the roster method or set builder notation, whichever is more appropriate.

Answer

Because the letters in the word "happy" consist of a small finite set, the best way to represent this set is with the roster method. Choose a label to represent the set, such as H.

$H = \{\text{h}, \text{a}, \text{p}, \text{y}\}$.

Notice that the letter “p” is only represented one time. This occurs because when representing members of a set, each unique element is only listed once no matter how many times it occurs. Duplicate elements are never repeated when representing members of a set.

Your Turn 1.7

Use the roster method or set builder notation to represent the collection of all musical instruments.

Computing the Cardinal Value of a Set

Almost all the sets most people work with outside of pure mathematics are finite sets. For these sets, the cardinal value or cardinality of the set is the number of elements in the set. For finite set A, the cardinality is denoted symbolically as $n(A)$. For example, a set that contains four elements has a cardinality of 4.

How do we measure the cardinality of infinite sets? The ‘smallest’ infinite set is the set of natural numbers, or counting numbers, $\mathbb{N} = \{1, 2, 3, ...\}$. This set has a cardinality of $\aleph_0$ (pronounced "aleph-null"). All sets that have the same cardinality as the set of natural numbers are countably infinite. This concept, as well as notation using aleph, was introduced by mathematician Georg Cantor who once said, “A set is a Many that allows itself to be thought of as a One.”

Example 1.8: Computing the Cardinal Value of a Set

Write the cardinal value of each of the following sets in symbolic form.

1. $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set.

Answer

1. There are 5 distinct elements in set F: a fork, a spoon, a knife, a meat thermometer, and a can opener. Therefore, the cardinal value of set F is 5 and written symbolically as $n(F) = 5$.
2. Because the empty set does not have any elements in it, the cardinality of the empty set is zero. Symbolically we write this as: $n(\emptyset) = 0$.

Your Turn 1.8

Write the cardinal value of each of the following sets in symbolic form.

Set P is the set of prime numbers less than 2 .

2. Set A is the set of lowercase letters of the English alphabet, $A = \{a, b, c, ..., z\}$.

Now that we have learned to represent finite and infinite sets using both the roster method and set builder notation, we should also be able to determine if a set is finite or infinite based on its verbal or symbolic description. One way to determine if a set is finite or not is to determine the cardinality of the set. If the cardinality of a set is a natural number, then the set is finite.

Example 1.9: Differentiating Between Finite and Infinite Sets

Classify each of the following sets as infinite or finite.

1. $E = \{2, 4, 6, 8, 10\}$
2. A is the set of lowercase letters of the English Alphabet, $A = \{a, b, c, ..., z\}$.
3. $\mathbb{Q} = \left\{ \frac{p}{q} | p \text{ and } q \text{ are integers and } q \neq 0 \right\}$

Answer

1. $n(E) = 5$. Since 5 is a natural number, the set is finite.
2. $n(A)$ =26n(A)=26. Since 26 is a natural number, the set is finite.

3. Set $\mathbb{Q}$ is the set of rational numbers or fractions. Because the set of integers is a subset of the set of rational numbers, and the set of integers is infinite, the set of rational numbers is also infinite. There is no smallest or largest rational number.

Your Turn 1.9

Classify each of the following sets as infinite or finite.

$B = \{b, a, k, e\}$

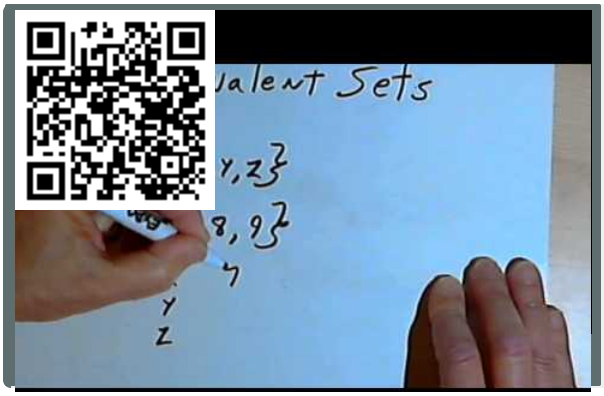
2. $\mathbb{R} = \{x \mid x \text{ is a real number }\}$

Equal versus Equivalent Sets

When speaking or writing we tend to use equal and equivalent interchangeably, but there is an important distinction between their meanings. Consider a new Ford Escape Hybrid and a new Toyota Rav4 Hybrid. Both cars are hybrid electric sport utility vehicles; in that sense, they are equivalent. They will both get you from place to place in a relatively fuel-efficient way. In this example we are comparing the single member set {Toyota Rav4 Hybrid} to the single member set {Ford Escape Hybrid}. Since these two sets have the same number of elements, they are also equivalent mathematically, meaning they have the same cardinality. But they are not equal, because the two cars have different looks and features, and probably even handle differently. Each manufacturer will emphasize the features unique to their vehicle to persuade you to buy it; if the SUVs were truly equal, there would be no reason to choose one over the other.

Now consider two Honda CR-Vs that are made with exactly the same parts, on the same assembly line within a few minutes of each other—these SUVs are equal. They are identical to each other, containing the same elements without regard to order, and the only differentiator when making a purchasing decision would be varied pricing at different dealerships. The set {Honda CR-V} is equal to the set {Honda CR-V}. Symbolically, we represent equal sets as $A = B$ and equivalent sets as $A \sim B$.

Now, let us consider a Toyota dealership that has 10 RAV4s on the lot, 8 Prii, 7 Highlanders, and 12 Camrys. There is a one-to-one relationship between the set of vehicles on the lot and the set consisting of the number of each type of vehicle on the lot. Therefore, these two sets are equivalent, but not equal. The set {RAV4, Prius, Highlander, Camry} is equivalent to the set {10, 8, 7, 12} because they have the same number of elements.



Checkpoint

If two sets are equal, they are also equivalent, because equal sets also have the same cardinality.

Example 1.10: Differentiating Between Equivalent and Equal Sets

Determine if the following pairs of sets are equal, equivalent, or neither.

1. $E = \{2, 4, 6, 8, 10\}$ and $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set and the set of prime numbers less than 2.

3. The set of vowels in the word happiness and the set of consonants in the word happiness.

Answer

1. Sets E and F both have a cardinal value of 5, but the elements in these sets are different. So, the two sets are equivalent, but they are not equal: $E \sim F$.
2. The set of prime numbers consists of the set of counting numbers greater than one that can only be divided evenly by one and itself. The set of prime numbers less than 2 is an empty set, since there are no prime numbers less than 2. Therefore, these two sets are equal (and equivalent).
3. The set of vowels in the word happiness is $\{a, i, e\}$ and the set of consonants in the word happiness is $\{h, p, n, s\}$. The cardinal value of these sets two sets is $n(\{a, i, e\}) = 3$ and $n(\{h, p, n, s\}) = 4$, respectively. Because the cardinality of the two sets differs, they are not equivalent. Further, their elements are not identical, so they are also not equal.

Your Turn 1.10

Determine if the following pairs of sets are equal, equivalent, or neither.

Set $B = \{b, a, k, e\}$ and set $A = \{a, b, e, k\}$

2. Set $B = \{b, a, k, e\}$ and set $F = \{f, l, a, k, e\}$

3. Set $B = \{b, a, k, e\}$ and set $C = \{c, a, k, e\}$

People in Mathematics: Georg Cantor

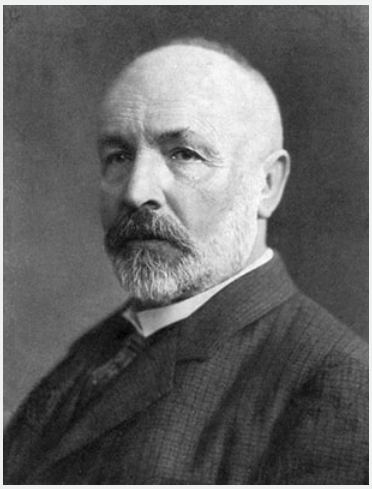


Figure 1.2.2: Georg Cantor (credit: Wikimedia, public domain)

Georg Cantor, the father of modern set theory, was born during the year 1845 in Saint Petersburg, Russa and later moved to Germany as a youth. Besides being an accomplished mathematician, he also played the violin. Cantor received his doctoral degree in Mathematics at the age of 22.

In 1870, at the age of 25 he established the uniqueness theorem for trigonometric series. His most significant work happened between 1874 and 1884, when he established the existence of transcendental numbers (also called irrational numbers) and proved that the set of real numbers are uncountably infinite—despite the objections of his former professor Leopold Kronecker.

Cantor published his final treatise on set theory in 1897 at the age of 52, and was awarded the Sylvester Medial from the Royal Society of London in 1904 for his contributions to the field. At the heart of Cantor's work was his goal to solve the continuum problem, which later influenced the works of David Hilbert and Ernst Zermelo.

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Check Your Understanding

1. A _____ is a well-defined collection of objects.
2. The _____ of a finite set A, denoted $n(A)$, is the number of elements in set A.
3. Determine if the following description describes a well-defined set: “The top 5 pizza restaurants in Chicago.”
4. The United States is the only country to have landed people on the moon as of March 21, 2021. What is the cardinality of the set of all people who have walked on the moon prior to this date?
5. Set A is a set of a dozen distinct donuts, and set B is a set of a dozen different types of apples. Is set A equal to set B, equivalent to set B, or neither?
6. Is the set of all butterflies in the world a finite set or an infinite set?
7. Represent the set of all upper-case letters of the English alphabet using both the roster method and set builder notation.

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1.2.1: Exercises

For the following exercises, represent each set using the roster method.

1. The set of primary colors: red, yellow, and blue.
2. A set of the following flowers: rose, tulip, marigold, iris, and lily.
3. The set of natural numbers between 50 and 100.
4. The set of natural numbers greater than 17.
5. The set of different pieces in a game of chess.
6. The set of natural numbers less than 21.

For the following exercises, represent each set using set builder notation.

7. The set of all types of lizards.
8. The set of all stars in the universe.
9. The set of all integer multiples of 3 that are greater than zero.
10. The set of all integer multiples of 4 that are greater than zero.
11. The set of all plants that are edible.
12. The set of all even numbers.

For the following exercises, represent each set using the method of your choice.

13. The set of all squares that are also circles.
14. The set of natural numbers divisible by zero.
15. The set of Mike and Carol's children on the TV show, *The Brady Bunch*.
16. The set of all real numbers.
17. The set of polar bears that live in Antarctica.
18. The set of songs written by Prince.
19. The set of children's books written and illustrated by Mo Willems.
20. The set of seven colors commonly listed in a rainbow.

For the following exercises, determine if the collection of objects represents a well-defined set or not.

21. The names of all the characters in the book, *The Fault in Our Stars* by John Green.
22. The five greatest soccer players of all time.
23. A group of old dogs that are able to learn new tricks.
24. A list of all the movies directed by Spike Lee as of 2021.
25. The group of all zebras that can fly an airplane.
26. The group of National Baseball League Hall of Fame members who have hit over 700 career home runs.

For the following exercises, compute the cardinal value of each set.

27. $P = \{\text{Snuzzle, Butterscotch, Blue Belle, Minty, Blossom, Cotton Candy}\}$
28. $T = \{\text{pepperoni, sausage, bacon, ham, mushrooms, olives, bell pepper, pineapple}\}$
29. $\emptyset$
30. $B = \{5, 6, 7, \ldots, 20\}$
31. $F = \left\{\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, \frac{9}{9}\right\}$
32. $\{\ \}$
33. $C = \{n^3 \mid n \text{ is a member of } \mathbb{N}\}$
34. $S = \{7n | n \text{ is an element of } \mathbb{N}\}$
35. $L = \{l, m, n, \ldots, y\}$
36. The set of numbers on a standard 6-sided die.

For the following exercises, determine whether set \(A and set \(B are equal, equivalent or neither.

37. $A = \{$ right, acute, obtuse $\}; B = \{$ equilateral, scalene, isoceles $\}$.
38. $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}; B = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$.

39. $A = \{$ red, orange, yellow $\}; B = \{$ green, blue, indigo, violet $\}$.
40. $A = \{5n \mid n \in \mathbb{N}\}; B = \mathbb{N}$.
41. $A = \{-2, -1, 0, ...\}; B = \{2, 3, 5, ...\}$.
42. $A = \{$ John, Paul, George, Ringo $\}; B = \{$ Bono, Larry, The Edge, Adam $\}$.
43. $A = \emptyset; B = \{\}$.
44. $A = \{$ lemon, lime, orange $\}; B = \{$ orange, lemon, lime, grape $\}$.

For the following exercises, determine if the set described is finite or infinite.

45. The set of natural numbers.
46. The empty set.
47. The set consisting of all jazz venues in New Orleans, Louisiana.
48. The set of all real numbers.
49. The set of all different types of cheeses.
50. The set of all words in *Merriam-Webster's Collegiate Dictionary*, Eleventh Edition, published in 2020.

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1.3: Subsets



Figure 1.4 The players on a soccer team who are actively participating in a game are a subset of the greater set of team members. (Credit: “PAFC-Mezokovesd-108” by Puskás Akadémia/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

- Represent subsets and proper subsets symbolically.
- Compute the number of subsets of a set.
- Apply concepts of subsets and equivalent sets to finite and infinite sets.

The rules of Major League Soccer (MLS) allow each team to have up to 30 players on their team. However, only 18 of these players can be listed on the game day roster, and of the 18 listed, 11 players must be selected to start the game. How the coaches and general managers form the team and choose the starters for each game will determine the success of the team in any given year.

The entire group of 30 players is each team’s set. The group of game day players is a subset of the team set, and the group of 11 starters is a subset of both the team set and the set of players on the game day roster.

Set A is a subset of set B if every member of set A is also a member of set B. Symbolically, this relationship is written as $A \subseteq B$.

Sets can be related to each other in several different ways: they may not share any members in common, they may share some members in common, or they may share all members in common. In this section, we will explore the way we can select a group of members from the whole set.

Checkpoint

Every set is also a subset of itself, $B \subseteq B$

Recall the set of flatware in our kitchen drawer from Section 1.1,

$F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$. Suppose you are preparing to eat dinner, so you pull a fork and a knife from the drawer to set the table. The set $D = \{\text{knife, fork}\}$ is a subset of set F, because every member or element of set D is also a

member of set F. More specifically, set D is a proper subset of set F, because there are other members of set F not in set D. This is written as $D \subset F$. The only subset of a set that is not a proper subset of the set would be the set itself.

Checkpoint

The empty set or null set, $\emptyset$, is a proper subset of every set, except itself.

Graphically, sets are often represented as circles. In the following graphic, set A is represented as a circle completely enclosed inside the circle representing set B, showing that set A is a proper subset of set B. The element x represents an element that is in both set A and set B.

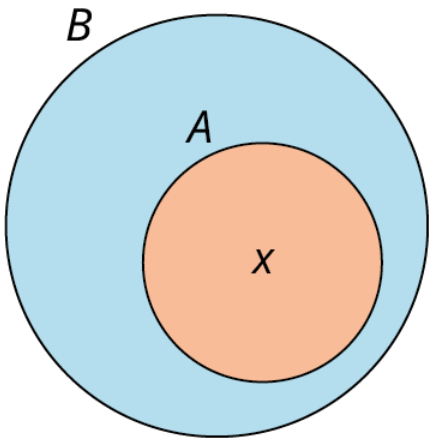


Figure 1.5

Checkpoint

While we can list all the subsets of a finite set, it is not possible to list all the possible subsets of an infinite set, as it would take an infinitely long time.

Example 1.11

Listing All the Proper Subsets of a Finite Set

Set L is a set of reading materials available in a shop at the airport, $L = \{\text{newspaper, magazine, book}\}$. List all the subsets of set L.

Answer

Step 1: It is best to begin with the set itself, as every set is a subset of itself. In our example, the cardinality of set L is $n(L) = 3$. There is only one subset of set L that has the same number of elements of set L: $\{\text{newspaper, magazine, book}\}$.

Step 2: Next, list all the proper subsets of the set containing $n(L) - 1$ elements. In this case, $3 - 1 = 2$. There are three subsets that each contain two elements: $\{\text{newspaper, magazine}\}$, $\{\text{newspaper, book}\}$, and $\{\text{magazine, book}\}$.

Step 3: Continue this process by listing all the proper subsets of the set containing $n(L) - 2$ elements. In this case, $3 - 2 = 1$. There are three subsets that contain one element: $\{\text{newspaper}\}$, $\{\text{magazine}\}$, and $\{\text{book}\}$.

Step 4: Finally, list the subset containing 0 elements, or the empty set: $\{\ \}$.

Your Turn 1.11

Consider the set of possible outcomes when you flip a coin, /**/S = \{ {\text{heads, tails}}\} /**/. List all the possible subsets of set /**/S./**/

Example 1.12

Determining Whether a Set Is a Proper Subset

Consider the set of common political parties in the United States, $P = \{\text{Democratic, Green, Libertarian, Republican}\}$. Determine if the following sets are proper subsets of P.

1. $M = \{\text{Democratic, Republican}\}$
2. $G = \{\text{Green}\}$
3. $V = \{\text{Republican, Libertarian, Green, Democratic}\}$

Answer

1. M is a proper subset of P, written symbolically as $M \subset P$ because every member of M is a member of set P, but P also contains at least one element that is not in M.
2. G is a single member proper subset of P, written symbolically as $G \subset P$, because Green is a member of set P, but P also contains other members (such as Democratic) that are not in G.
3. V is subset of P because every member of V is also a member of P, but it is not a proper subset of P because there are no members of V that are not also in set P. We can represent the relationship symbolically as $V \subseteq P$, or more precisely, set V is equal to set P, $V = P$.

Your Turn 1.12

Consider the set of generation I legendary Pokémon, /**/L = \{ {\text{Articuno, Zapdos, Moltres, Mewtwo}}\} /**/. Give an example of a proper subset containing:

1. one member.
2. three members.
3. no members.

Example 1.13

Expressing the Relationship between Sets Symbolically

Consider the subsets of a standard deck of cards: $S = \{\text{spades, hearts, diamonds, clubs}\}$; $R = \{\text{hearts, diamonds}\}$; $B = \{\text{spades, clubs}\}$; and $C = \{\text{clubs}\}$.

Express the relationship between the following sets symbolically.

1. Set S and set B.
2. Set C and set B.
3. Set R and R.

Answer

1. $B \subset S$. B is a proper subset of set S.
2. $C \subset B$. C is a proper subset of set B.
3. $R \subseteq$RorR=RR$\subseteq$RorR=R. R is subset of itself, but not a proper subset of itself because R is equal to itself.

Your Turn 1.13

Express the relationship between the set of natural numbers, $\mathbb{N} = \{1, 2, 3, ...\}$, and the set of even numbers, \(/E = \{ 2,4,6, \ldots \}\).

Exponential Notation

So far, we have figured out how many subsets exist in a finite set by listing them. Recall that in Example 1.11, when we listed all the subsets of the three-element set $L = \{$ newspaper, magazine, book\} we saw that there are eight subsets. In Your Turn 1.11, we discovered that there are four subsets of the two-element subset, $S = \{$ heads, tails $\}$. A one-element set has two subsets, the empty set and itself. The only subset of the empty set is the empty set itself. But how can we easily figure out the number of subsets in a very large finite set? It turns out that the number of subsets can be found by raising 2 to the number of elements in the set, using

exponential notation to represent repeated multiplication. For example, the number of subsets of the set $L = \{$ newspaper, magazine, book $\}$ is equal to $2^3 = 2 \cdot 2 \cdot 2 = 8$. Exponential notation is used to represent repeated multiplication, $b^n = b \cdot b \cdot b \cdot \ldots \cdot b$, where b appears as a factor n times.

FORMULA

The number of subsets of a finite set A is equal to 2 raised to the power of $n(A)$, where $n(A)$ is the number of elements in set A: **Number of Subsets of Set** $A = 2^{n(A)}$.

Checkpoint

Note that $2^0 = 1$*, so this formula works for the empty set, also.*

Example 1.14

Computing the Number of Subsets of a Set

Find the number of subsets of each of the following sets.

1. The set of top five scorers of all time in the NBA:
   $S = \{$ LeBron James, Kareem Abdul-Jabbar, Karl Malone, Kobe Bryant, Michael Jordan
2. The set of the top four bestselling albums of all time:
   $A = \{$ Thriller, Hotel California, The Beatles White Album, Led Zepplin IV $\}$.
3. $R = \{$ Snap, Crackle, Pop $\}$.

Answer

1. $n(S) = 5$. So, the total number of subsets of Sis25=2·2·2·2·2=32Sis25=2·2·2·2·2=32.
2. $n(A) = 4$. Therefore, the total number of subsets of Ais24=16Ais24=16.
3. $n(R) = 3$. So, the total number of subsets of Ris23=8Ris23=8.

Your Turn 1.14

Compute the total number of subsets in the set of the top nine tennis grand slam singles winners,
$T = \{$ Margaret Court, Serena Williams, Steffi Graff, Rafael Nadal, Novak Djokovic, Roger Federer, Helen Wills, Martina Navratilova, Chris Everett\}.

Equivalent Subsets

In the early 17th century, the famous astronomer Galileo Galilei found that the set of natural numbers and the subset of the natural numbers consisting of the set of square numbers, semanticsn2n2, are equivalent. Upon making this discovery, he conjectured that the concepts of less than, greater than, and equal to did not apply to infinite sets.

Sequences and series are defined as infinite subsets of the set of natural numbers by forming a relationship between the sequence or series in terms of a natural number, n. For example, the set of even numbers can be defined using set builder notation as $\{a | a = 2n \text{ where } n \text{ is a natural number}\}$. The formula in this case replaces every natural number with two times the number, resulting in the set of even numbers, $\{2, 4, 6, \ldots\}$. The set of even numbers is also equivalent to the set of natural numbers.

Who Knew?

Employment Opportunities

You can make a career out of working with sets. Applications of equivalent sets include relational database design and analysis.

Relational databases that store data are tables of related information. Each row of a table has the same number of columns as every other row in the table; in this way, relational databases are examples of set equivalences for finite sets. In a relational database, a primary key is set up to identify all related information. There is a one-to-one relationship between the primary key and any other information associated with it.

Database design and analysis is a high demand career with a median entry-level salary of about $85,000 per year, according to salary.com.

Example 1.15

Writing Equivalent Subsets of an Infinite Set

Using natural numbers, multiples of 3 are given by the sequence $\{3, 6, 9, ...\}$. Write this set using set builder notation by expressing each multiple of 3 using a formula in terms of a natural number, n.

Answer

$\{m|m = 3n \text{ where } n \text{ is a natural number}\}$ or $\{m|m = 3n \text{ where } n \in \mathbf{N}\}$. In this example, m is a multiple of 3 and n is a natural number. The symbol $\in$ is read as "is a member or element of." Because there is a one-to-one correspondence between the set of multiples of 3 and the natural numbers, the set of multiples of 3 is an equivalent subset of the natural numbers.

Your Turn 1.15

Using natural numbers, multiples of 5 are given by the sequence $\{5, 10, 15, ...\}$. Write this set using set builder notation by associating each multiple of 5 in terms of a natural number, n.

Example 1.16

Creating Equivalent Subsets of a Finite Set That Are Not Equal

A fast-food restaurant offers a deal where you can select two options from the following set of four menu items for $6: a chicken sandwich, a fish sandwich, a cheeseburger, or 10 chicken nuggets. Javier and his friend Michael are each purchasing lunch using this deal. Create two equivalent, but not equal, subsets that Javier and Michael could choose to have for lunch.

Answer

The possible two-element subsets are: {chicken sandwich, fish sandwich}, {chicken sandwich, cheeseburger}, {chicken sandwich, chicken nuggets}, {fish sandwich, cheeseburger}, {fish sandwich, chicken nuggets}, and {cheeseburger, chicken nuggets}. One possible solution is that Javier picked the set {chicken sandwich, chicken nuggets}, while Michael chose the {cheeseburger, chicken nuggets}. Because Javier and Michael both picked two items, but not exactly the same two items, these sets are equivalent, but not equal.

Your Turn 1.16

Serena and Venus Williams walk into the same restaurant as Javier and Michael, but they order the same pair of items, resulting in equal sets of choices. If Venus ordered a fish sandwich and chicken nuggets, what did Serena order?

Example 1.17

Creating Equivalent Subsets of a Finite Set

A high school volleyball team at a small school consists of the following players: {Angie, Brenda, Colleen, Estella, Maya, Maria, Penny, Shantelle}. Create two possible equivalent starting line-ups of six players that the coach could select for the next game.

Answer

There are actually 28 possible ways that the coach could choose his starting line-up. Two such equivalent subsets are {Angie, Brenda, Maya, Maria, Penny, Shantelle} and {Angie, Brenda, Colleen, Estella, Maria, Shantelle}. Each subset has six members, but they are not identical, so the two sets are equivalent but not equal.

Your Turn 1.17

> Consider the same group of volleyball players from above: {Angie, Brenda, Colleen, Estella, Maya, Maria, Penny, Shantelle}. The team needs to select a captain and an assistant captain from their members. List two possible equivalent subsets that they could select.

Check Your Understanding

8. Every member of a _____ of a set is also a member of the set.

9. Explain what distinguishes a proper subset of a set from a subset of a set.

10. The _____ set is a proper subset of every set except itself.

11. Is the following statement true or false? $A \subseteq A$.

12. If the cardinality of set A is $n(A) = 10$, then set A has a total of _____ subsets.

13. Set A is _____ to set B if $n(A) = n(B)$.

14. If every member of set A is a member of set B and every member of set B is also a member set A, then set A is _____ to set B.

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1.3.1: Exercises

For the following exercises, list all the proper subsets of each set.

1. {chocolate, vanilla, strawberry}
2. {true, false }
3. {mother, father, daughter, son }
4. $\{7\}$

For the following exercises, determine the relationship between the two sets and write the relationship symbolically.

$D = \{0, 1, 2, \ldots, 9\}, A = \{0, 2, 4, 6, 8\}, B = \{1, 3, 5, 7, 9\}, C = \{8, 6, 4, 2, 0\}, Z = \{0\}$, and $\emptyset$

5. D and A

6. B and D

7. C and D

8. Z and C

9. Z and $\emptyset$

10. A and B

11. A and C

12. $\emptyset$ and D

13. B and C

14. A and Z

For the following exercises, calculate the total number of subsets of each set.

15. {Adele, Beyonce, Cher, Madonna, Shakira }

16. { Art, Paul }

17. {Peter, Paul, Mary}

18. $\emptyset$

19. $\{3\}$

20. $\{l, o, v, e\}$

21. }

22. \{football, baseball, basketball, soccer, hockey, tennis, golf\}

23. Set A, if $n(A) = 12$.

24. Set B, if $n(B) = 9$.

For the following exercises, use the set of letters in the word largest as the set, $U = \{\mathrm{l}, \mathrm{a}, \mathrm{r}, \mathrm{g}, \mathrm{e}, \mathrm{s}, \mathrm{t}\}$.

25. Find a subset of U that is equivalent, but not equal, to the set: $\{l, \mathrm{a}, \mathrm{s}, \mathrm{t}\}$.

26. Find a subset of U that is equal to the set: $\{l, \mathrm{a}, \mathrm{s}, \mathrm{t}\}$.

27. Find a subset of U that is equal to the set: $\{\mathrm{a}, \mathrm{r}, \mathrm{t}\}$.

28. Find a subset of U that is equivalent, but not equal, to the set $\{\mathrm{a}, \mathrm{r}, \mathrm{t}, \mathrm{s}\}$.

29. Find a subset of U that is equivalent, but not equal, to the set: $\{\mathrm{r}, \mathrm{a}, \mathrm{t}, \mathrm{e}, \mathrm{s}\}$.

30. Find a subset of U that is equal to the set: $\{\mathrm{r}, \mathrm{a}, \mathrm{t}, \mathrm{e}, \mathrm{s}\}$.

31. Find two three-character subsets of set U that are equivalent, but not equal, to each other.

32. Find two three-character subsets of set U that are equal to each other.

33. Find two five-character subsets of set U that are equal to each other.

34. Find two five-character subsets of set U that are equivalent, but not equal, to each other.

Find two equal subsets of /**/U/**/ with a cardinality of 4.

35. Find two equivalent subset of U with a cardinality of 7 .
36. Find two equal subsets of U with a cardinality of 4 .
37. Find a subset of U that is equivalent, but not equal to, $\{0, 3, 6, 9, ...\}$.
38. Find a subset of U that is equivalent, but not equal to, $\{-1, -4, -9, -16, -25, ...\}$.
39. True or False. The set of natural numbers, $\mathbb{N} = \{1, 2, 3, ...\}$, is equivalent to set U.
40. True or False. Set U is an equivalent subset of the set of rational numbers, $\mathbb{Q} = \left\{ \frac{p}{q} \middle| \, p \text{ and } q \text{ are integers and } q \neq 0. \right\}$.

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1.4: Understanding Venn Diagrams

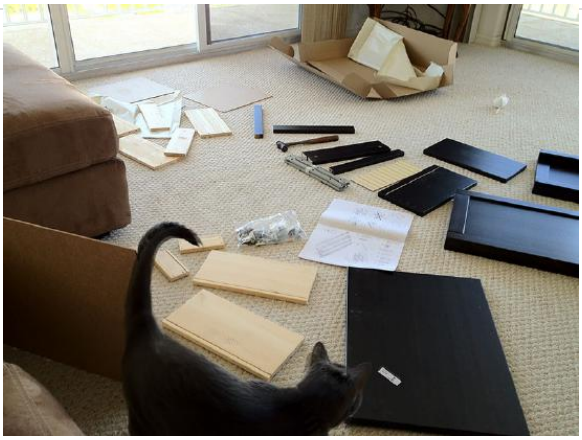


Figure 1.6: When assembling furniture, instructions with images are easier to follow, just like how set relationships are easier to understand when depicted graphically. (credit: "Time to assemble more Ikea furniture!" by Rod Herrea/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Utilize a universal set with two sets to interpret a Venn diagram.
- Utilize a universal set with two sets to create a Venn diagram.
- Determine the complement of a set.

Have you ever ordered a new dresser or bookcase that required assembly? When your package arrives you excitedly open it and spread out the pieces. Then you check the assembly guide and verify that you have all the parts required to assemble your new dresser. Now, the work begins. Luckily for you, the assembly guide includes step-by-step instructions with images that show you how to put together your product. If you are really lucky, the manufacturer may even provide a URL or QR code connecting you to an online video that demonstrates the complete assembly process. We can likely all agree that assembly instructions are much easier to follow when they include images or videos, rather than just written directions. The same goes for the relationships between sets.

Interpreting Venn Diagrams

Venn diagrams are the graphical tools or pictures that we use to visualize and understand relationships between sets. Venn diagrams are named after the mathematician John Venn, who first popularized their use in the 1880s. When we use a Venn diagram to visualize the relationships between sets, the entire set of data under consideration is drawn as a rectangle, and subsets of this set are drawn as circles completely contained within the rectangle. The entire set of data under consideration is known as the universal set.

Consider the statement: All trees are plants. This statement expresses the relationship between the set of all plants and the set of all trees. Because every tree is a plant, the set of trees is a subset of the set of plants. To represent this relationship using a Venn diagram, the set of plants will be our universal set and the set of trees will be the subset. Recall that this relationship is expressed symbolically as: $\textbf{Trees} \subset \textbf{Plants}$. To create a Venn diagram, first we draw a rectangle and label the universal set "$U = \textbf{Plants}$." Then we draw a circle within the universal set and label it with the word "Trees."

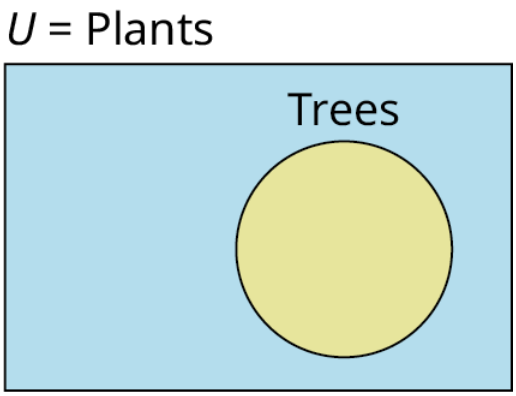


Figure 1.7:

This section will introduce how to interpret and construct Venn diagrams. In future sections, as we expand our knowledge of relationships between sets, we will also develop our knowledge and use of Venn diagrams to explore how multiple sets can be combined to form new sets.

Example 1.18: Interpreting the Relationship between Sets in a Venn Diagram

Write the relationship between the sets in the following Venn diagram, in words and symbolically.

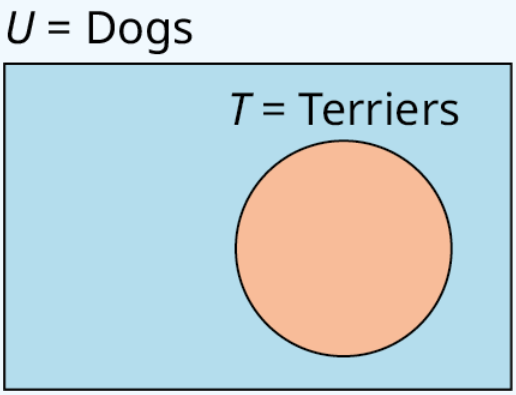


Figure 1.8:

Answer

The set of terriers is a subset of the universal set of dogs. In other words, the Venn diagram depicts the relationship that all terriers are dogs. This is expressed symbolically as $T \subset U$.

Your Turn 1.18

Write the relationship between the sets in the following Venn diagram, in words and symbolically.

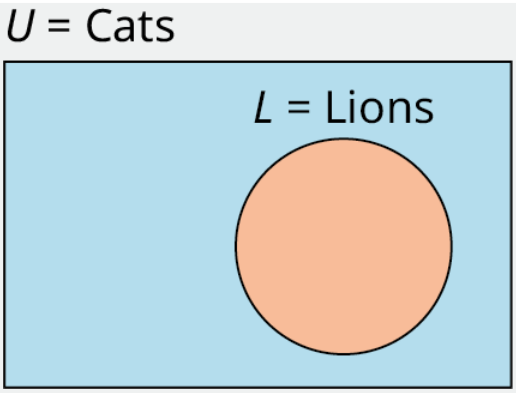


Figure 1.9

So far, the only relationship we have been considering between two sets is the subset relationship, but sets can be related in other ways. Lions and tigers are both different types of cats, but no lions are tigers, and no tigers are lions. Because the set of all lions and the set of all tigers do not have any members in common, we call these two sets disjoint sets, or non-overlapping sets.

Two sets A and B are disjoint sets if they do not share any elements in common. That is, if a is a member of set A, then a is not a member of set B. If b is a member of set B, then b is not a member of set A. To represent the relationship between the set of all cats and the sets of lions and tigers using a Venn diagram, we draw the universal set of cats as a rectangle and then draw a circle for the set of lions and a separate circle for the set of tigers within the rectangle, ensuring that the two circles representing the set of lions and the set of tigers do not touch or overlap in any way.

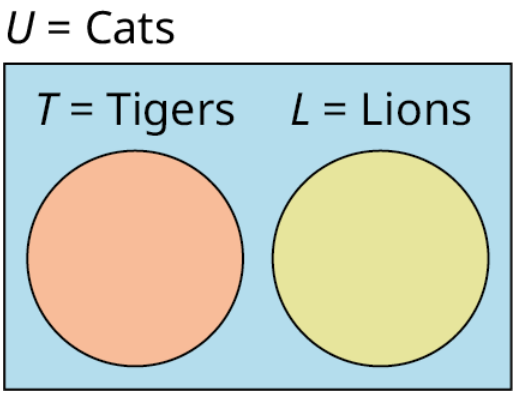


Figure 1.10:

Example 1.19: Describing the Relationship between Sets

Describe the relationship between the sets in the following Venn diagram.

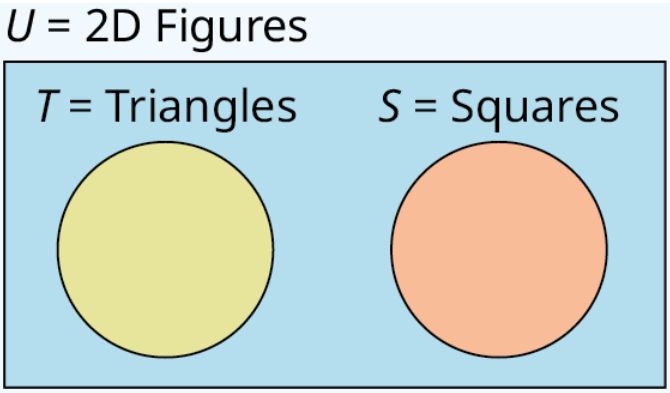


Figure 1.11:

Answer

The set of triangles and the set of squares are two disjoint subsets of the universal set of two-dimensional figures. The set of triangles does not share any elements in common with the set of squares. No triangles are squares and no squares are triangles, but both squares and triangles are 2D figures.

Your Turn 1.19

Describe the relationship between the sets in the following Venn diagram.

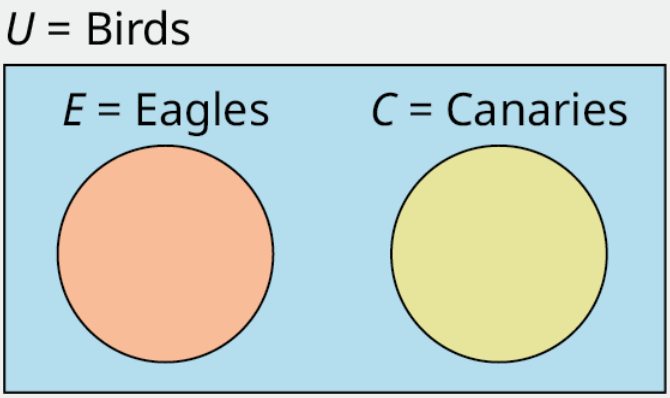


Figure 1.12

Creating Venn Diagrams

The main purpose of a Venn diagram is to help you visualize the relationship between sets. As such, it is necessary to be able to draw Venn diagrams from a written or symbolic description of the relationship between sets.

Procedure

To create a Venn diagram:

1. Draw a rectangle to represent the universal set, and label it $U = \text{set}$nameU=set name.
2. Draw a circle within the rectangle to represent a subset of the universal set and label it with the set name.

Checkpoint

If there are multiple disjoint subsets of the universal set, their separate circles should not touch or overlap.

Example 1.20: Drawing a Venn Diagram to Represent the Relationship Between Two Sets

Draw a Venn diagram to represent the relationship between each of the sets.

1. All rectangles are parallelograms.
2. All women are people.

Answer

1. The set of rectangles is a subset of the set of parallelograms.
   First, draw a rectangle to represent the universal set and label it with U =ParallelogramsU=Parallelograms, then draw a circle completely within the rectangle, and label it with the name of the set it represents, R =RectanglesR=Rectangles.

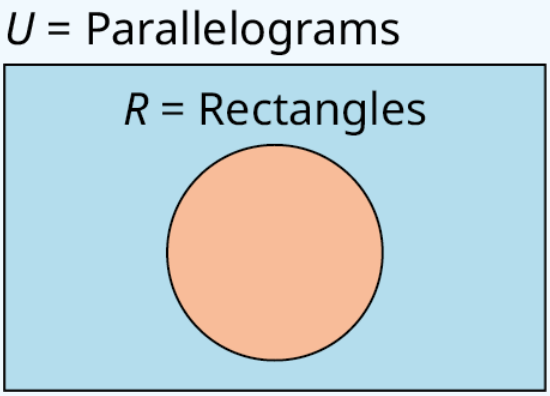


Figure 1.13:

In this example, both letters and names are used to represent the sets involved, but this is not necessary. You may use either letters or names alone, as long as the relationship is clearly depicted in the diagram, as shown below.

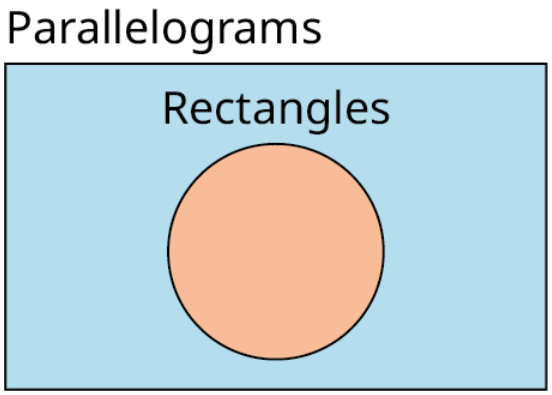


Figure 1.14:

or

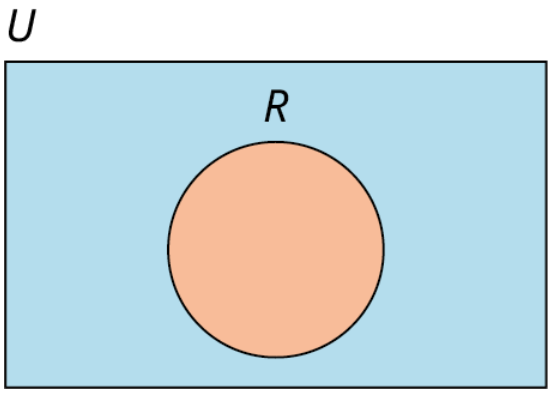


Figure 1.15:

2. The universal set is the set of people, and the set of all women is a subset of the set of people.

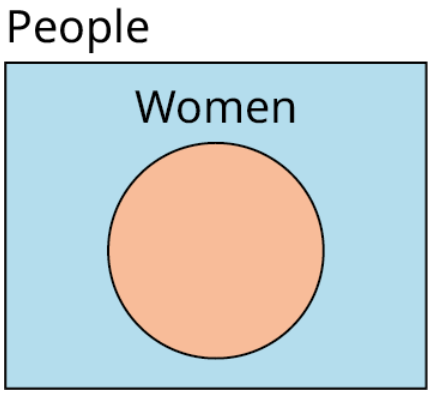


Figure 1.16:

Your Turn 1.20

Draw a Venn diagram to represent the relationship between each of the sets. All natural numbers are integers.

$A \subset U$. Draw a Venn diagram to represent this relationship.

Example 1.21: Drawing a Venn Diagram to Represent the Relationship Between Three Sets

All bicycles and all cars have wheels, but no bicycle is a car. Draw a Venn diagram to represent this relationship.

Answer

Step 1: The set of bicycles and the set of cars are both subsets of the set of things with wheels. The universal set is the set of things with wheels, so we first draw a rectangle and label it with $U = \textbf{Things with}$WheelsU=Things with Wheels.

Step 2: Because the set of bicycles and the set of cars do not share any elements in common, these two sets are disjoint and must be drawn as two circles that do not touch or overlap with the universal set.

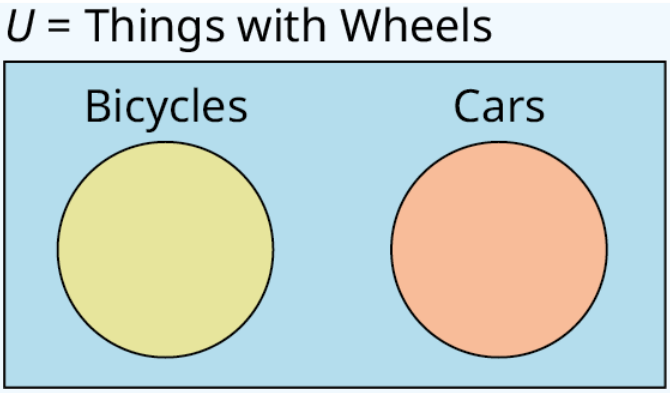


Figure 1.17:

Your Turn 1.21

Airplanes and birds can fly, but no birds are airplanes. Draw a Venn diagram to represent this relationship.

The Complement of a Set

Recall that if set A is a proper subset of set U, the universal set (written symbolically as $A \subset U$), then there is at least one element in set U that is not in set A. The set of all the elements in the universal set U that are not in the subset A is called the complement of set A, A'. In set builder notation this is written symbolically as: $A' = \{x \in U | x \notin A\}$. The symbol $\in$ is used to represent the phrase, "is a member of," and the symbol $\notin$ is used to represent the phrase, "is not a member of." In the Venn diagram below, the complement of set A is the region that lies outside the circle and inside the rectangle. The universal set U includes all of the elements in set A and all of the elements in the complement of set A, and nothing else.

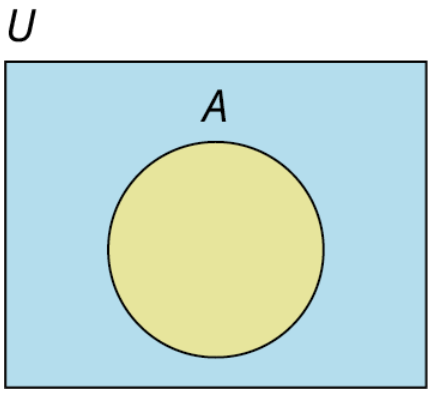


Figure 1.18:

Consider the set of digit numbers. Let this be our universal set, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Now, let set A be the subset of U consisting of all the prime numbers in set U, $A = \{2, 3, 5, 7\}$. The complement of set A is $A' = \{0, 1, 4, 6, 8, 9\}$. The following Venn diagram represents this relationship graphically.

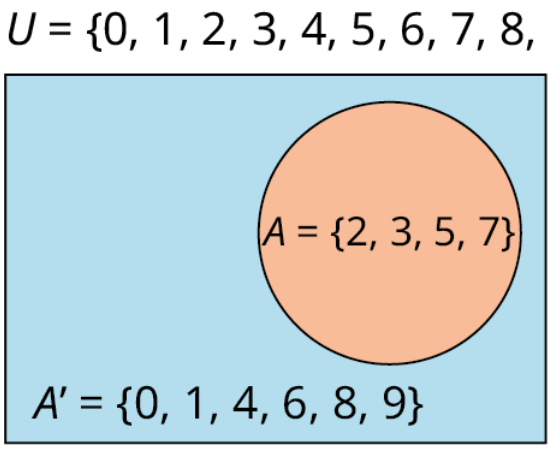


Figure 1.19:

Example 1.22: Finding the Complement of a Set

For both of the questions below, A is a proper subset of U.

1. Given the universal set $U =$ { Billie Eilish, Donald Glover, Bruno Mars, Adele, Ed Sheeran\} and set $A =$ { Donald Glover, Bruno Mars, Ed Sheeran }, find A'.

2. Given the universal set $U = \{\text{d} \mid \text{d}$ is a dog$\}$ and $B = \{\text{b} \in \text{U} \mid \text{b}$ is a beagle $\}$, find B'.

Answer

1. The complement of set A is the set of all elements in the universal set U that are not in set A. $A' =$ { Billie Eilish, Adele }.

2. The complement of set B is the set of all dogs that are not beagles. All members of set B' are in the universal set because they are dogs, but they are not in set B, because they are not beagles. This relationship can be expressed in set build notation as follows:
$B' =$ { All dogs that are not beagles. $\}, B' = \{d \in U \mid d$ is not a beagle. $\}$, or $B' = \{d \in U \mid d \notin B\}$.

Your Turn 1.22

For both of the questions below, A is a proper subset of U. 1. Given the universal set $U =$ { red, orange, yellow, green, blue, indigo, violet } and set $A =$ { yellow, red, blue }, find A'. 2. Given the universal set $U = \{c \mid c$ is a cat $\}$ and set $A = \{c \in U \mid c$ is not a lion $\}$, find A'.

Check Your Understanding

1. A Venn diagram is a graphical representation of the _____ between sets.
2. In a Venn diagram, the set of all data under consideration, the _____ set, is drawn as a rectangle.
3. Two sets that do not share any elements in common are _____ sets.
4. The _____ of a subset A or the universal set, U, is the set of all members of U that are not in A.
5. The sets A and A' are _____ subsets of the universal set.

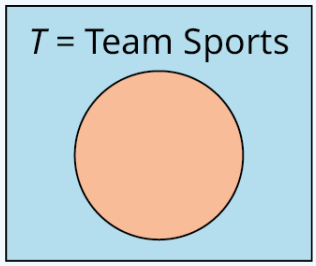
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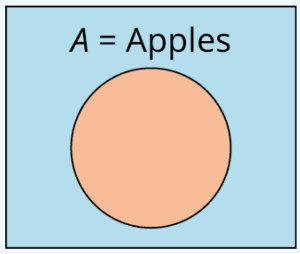
1.4.1: Exercises

For the following exercises, interpret each Venn diagram and describe the relationship between the sets, symbolically and in words.

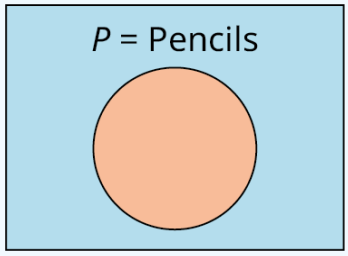
Exercise 1.4.1.1



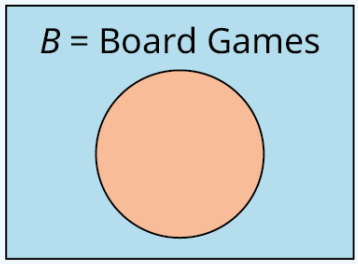
Exercise 1.4.1.2



Exercise 1.4.1.3

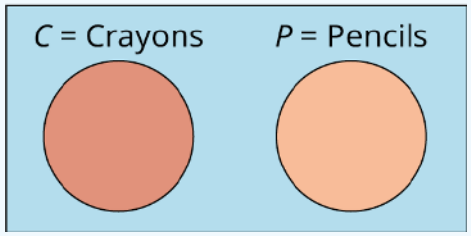


Exercise 1.4.1.4



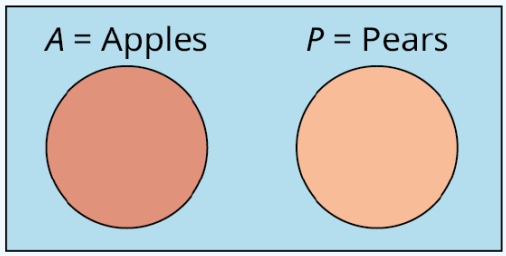
Exercise 1.4.1.5

U = Writing Utensils



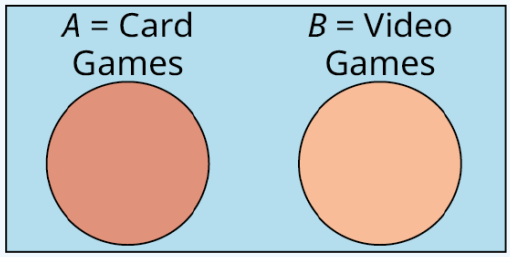
Exercise 1.4.1.6

U = Fruit



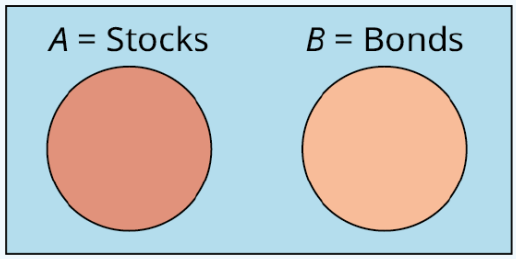
Exercise 1.4.1.7

U = Games



Exercise 1.4.1.8

U = Investments



For the following exercises, create a Venn diagram to represent the relationships between the sets.

Exercise 1.4.1.9

All birds have wings.

? Exercise 1.4.1.10

All cats are animals.

? Exercise 1.4.1.11

All almonds are nuts, and all pecans are nuts, but no almonds are pecans.

? Exercise 1.4.1.12

All rectangles are quadrilaterals, and all trapezoids are quadrilaterals, but no rectangles are trapezoids.

? Exercise 1.4.1.13

Lizards $\subset$ Reptiles.

? Exercise 1.4.1.14

Ladybugs $\subset$ Insects.

? Exercise 1.4.1.15

Ladybugs $\subset$ Insects and Ants $\subset$ Insects, but no Ants are Ladybugs.

? Exercise 1.4.1.16

Lizards $\subset$ Reptiles and Snakes $\subset$ Reptiles, but no Lizards are Snakes.

? Exercise 1.4.1.17

A and B are disjoint subsets of U.

? Exercise 1.4.1.18

C and D are disjoint subsets of U.

? Exercise 1.4.1.19

T is a subset of U.

? Exercise 1.4.1.20

S is a subset of U.

? Exercise 1.4.1.21

$J =$ Jazz, $M =$ Music, and $J \subset M$.

? Exercise 1.4.1.22

$R =$ Reggae, $M =$ Music, and $R \subset M$.

Exercise 1.4.1.23

$J =$ Jazz, $R =$ Reggae, and $M =$ Music are sets with the following relationships: $J \subset M, R \subset M$, and R is disjoint from J.

Exercise 1.4.1.24

$J =$ Jazz, $B =$ Bebop, and $M =$ Music are sets with the following relationships: $J \subset M$ and $B \subset J$.

For the following exercises, the universal set is the set of single digit numbers, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find the complement of each subset of U.

Exercise 1.4.1.25

$A = \{6, 7, 8\}$

Exercise 1.4.1.26

$A = \{0, 2, 4, 6, 8\}$

Exercise 1.4.1.27

$A = \{\}$

Exercise 1.4.1.28

$A = \{0, 1, 4, 6, 8, 9\}$

Exercise 1.4.1.29

$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Exercise 1.4.1.30

$A = \{0, 1, 3, 4, 5, 6, 7, 9\}$

Exercise 1.4.1.31

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Exercise 1.4.1.32

$A = \{0, 3, 6, 9\}$

For the following exercises, the universal set is $U =$ \{Bashful, Doc, Dopey, Grumpy, Happy, Sleepy, Sneezy\}. Find the complement of each subset of U.

Exercise 1.4.1.33

$A =$ { Happy, Bashful, Grumpy\}

Exercise 1.4.1.34

$A = \{$ Sleepy, Sneezy $\}$

Exercise 1.4.1.35

$A = \{\text{Doc}\}$

Exercise 1.4.1.36

$A = \{$ Doc, Dopey $\}$

Exercise 1.4.1.37

$A = \emptyset$

Exercise 1.4.1.38

$A = \{$ Doc, Grumpy, Happy, Sleepy, Bashful, Sneezy, Dopey\}

For the following exercises, the universal set is $U = \mathbb{N} = \{1, 2, 3, ...\}$. Find the complement of each subset of U.

Exercise 1.4.1.39

$A = \{1, 2, 3, 4, 5\}$

Exercise 1.4.1.40

$A = \{1, 3, 5, ...\}$

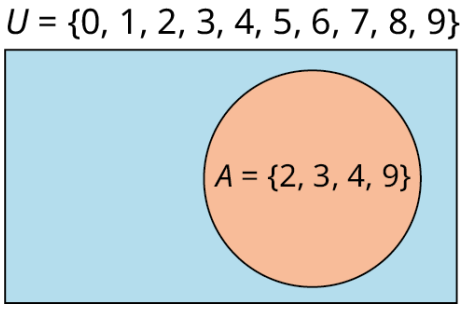
Exercise 1.4.1.41

$A = \{1\}$

Exercise 1.4.1.42

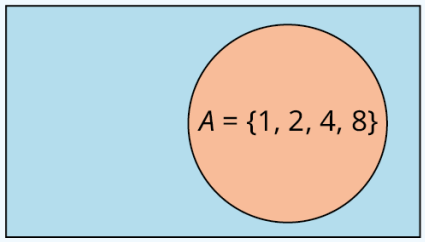
$A = \{4, 5, 6, ...\}$

For the following exercises, use the Venn diagram to determine the members of the complement of set A, A'.



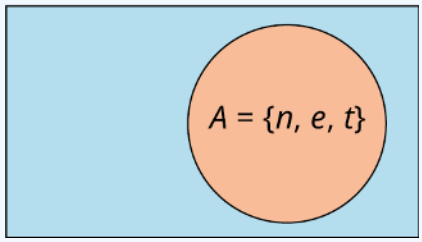
Exercise 1.4.1.44

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$



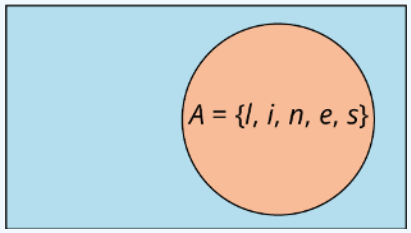
Exercise 1.4.1.45

$U = \{l, i, s, t, e, n\}$



Exercise 1.4.1.46

$U = \{l, i, s, t, e, n\}$



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1.5: Set Operations with Two Sets

Figure 1.20 A large, multigenerational family contains an intersection and a union of sets. (credit: “Family Photo Shoot Bani Syakur” by Mainur Risyada/Flickr, CC BY 2.0)



Figure **1.5.1**: A large, multigenerational family contains an intersection and a union of sets. (credit: “Family Photo Shoot Bani Syakur” by Mainur Risyada/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Determine the intersection of two sets.
- Determine the union of two sets.
- Determine the cardinality of the union of two sets.
- Apply the concepts of AND and OR to set operations.
- Draw conclusions from Venn diagrams with two sets.

The movie *Yours, Mine, and Ours* was originally released in 1968 and starred Lucille Ball and Henry Fonda. This movie, which is loosely based on a true story, is about the marriage of Helen, a widow with eight children, and Frank, a widower with ten children, who then have an additional child together. The movie is a comedy that plays on the interpersonal and organizational struggles of feeding, bathing, and clothing twenty people in one household.

If we consider the set of Helen's children and the set of Frank's children, then the child they had together is the intersection of these two sets, and the collection of all their children combined is the union of these two sets. In this section, we will explore the operations of union and intersection as it relates to two sets.

The Intersection of Two Sets

The members that the two sets share in common are included in the intersection of two sets. To be in the intersection of two sets, an element must be in both the first set and the second set. In this way, the intersection of two sets is a logical AND statement. Symbolically, A intersection B is written as: $A \cap B$. A intersection B is written in set builder notation as: $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

Let us look at Helen's and Frank's children from the movie Yours, Mine, and Ours. Helen's children consist of the set $H = \{$ Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph\} and Frank's children are included in the set $F = \{$ Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph \}. H intersection F is the set of children they had together. $H \cap F = \{$ Joseph $\}$, because Joseph is in both set H and set F.

Example 1.23

Finding the Intersection of Set A and Set B

Set $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$. Find A intersection B.

Answer

The intersection of sets A and B include the elements that set A and B have in common: 3, 5, and 7. $A \cap B = \{3, 5, 7\}$.

Your Turn 1.23

1. Set $A = \{h, a, p, y\}$ and $B = \{s, a, d\}$. Find A intersection B.

Notice that if sets A and B are disjoint sets, then they do not share any elements in common, and A intersection B is the empty set, as shown in the Venn diagram below.

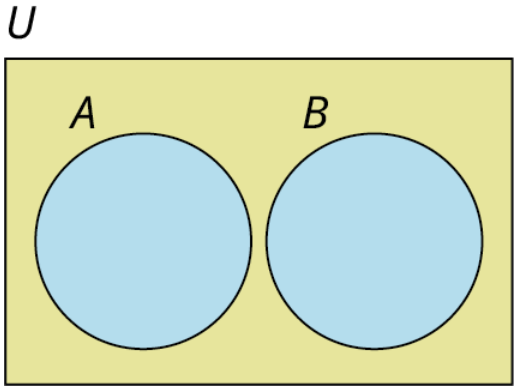


Figure 1.21

Example 1.24

Determining the Intersection of Disjoint Sets

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cap B$.

Answer

Because sets A and B are disjoint, they do not share any elements in common. So, the intersection of set A and set B is the empty set. $A \cap B = \emptyset$.

Your Turn 1.24

Set $A = \{$ red, yellow, blue $\}$ and set $B = \{$ orange, green, purple $\}$. Find $A \cap B$.

Notice that if set A is a subset of set B, then A intersection B is equal to set A, as shown in the Venn diagram below.

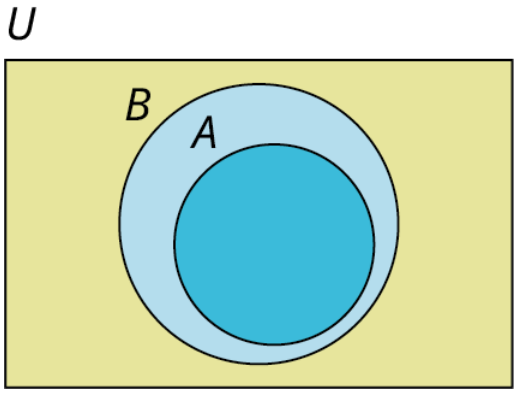


Figure 1.22

Example 1.25

Finding the Intersection of a Set and a Subset

Set $A = \{1, 3, 5, \ldots\}$ and set $B = \mathbb{N} = \{1, 2, 3, \ldots\}$ Find $A \cap B$.

Answer

Because set A is a subset of set B, A intersection B is equal to set A. $A \cap B = A = \{1, 3, 5, ...\}$, the set of odd natural numbers.

Your Turn 1.25

1. Set $A = \{a, b, c, \ldots, z\}$ and set $B = \{a, e, i, o, u\}$. Find $A \cap B$.

The Union of Two Sets

Like the union of two families in marriage, the union of two sets includes all the members of the first set and all the members of the second set. To be in the union of two sets, an element must be in the first set, the second set, or both. In this way, the union of two sets is a logical inclusive OR statement. Symbolically, A union B is written as: $A \cup B$. A union B is written in set builder notation as: $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

Let us consider the sets of Helen's and Frank's children from the movie Yours, Mine, and Ours again. Helen's children is set $H = \{$ Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph\} and Frank's children is set $F = \{$ Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph $\}$. The union of these two sets is the collection of all nineteen of their children, $H \cup F = \{$ Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph, Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan\}. Notice, Joseph is in both set H and set F, but he is only one child, so, he is only listed once in the union.

Example 1.26

Finding the Union of Sets A and B When A and B Overlap

Set $A = \{1, 3, 5, 7, 9\}$ and set $B = \{2, 3, 5, 7\}$. Find A union B.

Answer

A union B is the set formed by including all the unique elements in set A, set B, or both sets A and B: $A \cup B = \{1, 3, 5, 7, 9, 2\}$. The first five elements of the union are the five unique elements in set A. Even though 3, 5, and 7 are also members of set B, these elements are only listed one time. Lastly, set B includes the unique element 2, so 2 is also included as part of the union of sets A and B.

Your Turn 1.26

1. Set $A = \{h, a, p, y\}$ and set $B = \{s, a, d\}$. Find A union B.

When observing the union of sets A and B, notice that both set A and set B are subsets of A union B. Graphically, A union B can be represented in several different ways depending on the members that they have in common. If A and B are disjoint sets, then A union B would be represented with two disjoint circles within the universal set, as shown in the Venn diagram below.

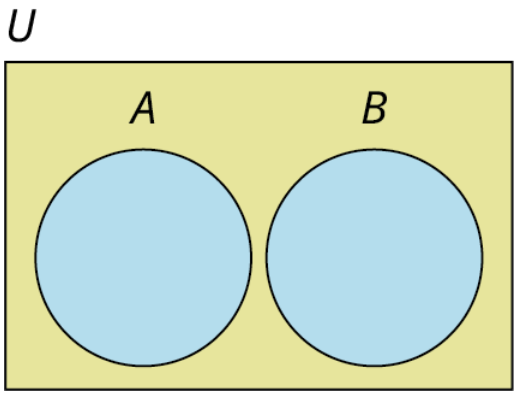


Figure 1.23 $A \cup B$

If sets A and B share some, but not all, members in common, then the Venn diagram is drawn as two separate circles that overlap.

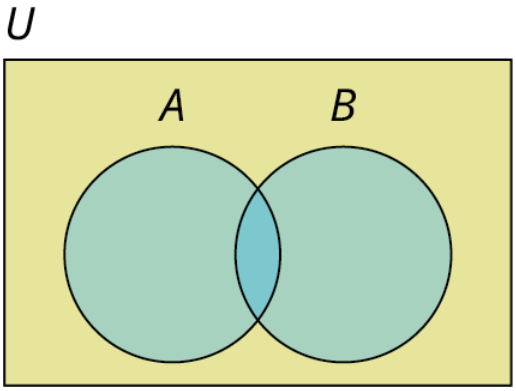


Figure 1.24

If every member of set A is also a member of set B, then A is a subset of set B, and A union B would be equal to set B. To draw the Venn diagram, the circle representing set A should be completely enclosed in the circle containing set B.

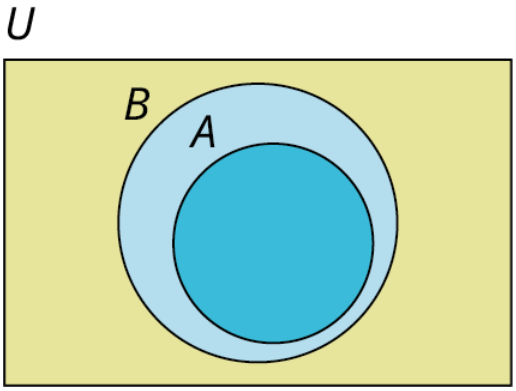


Figure 1.25

Example 1.27

Finding the Union of Sets A and B When A and B Are Disjoint

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cup B$.

Answer

Because sets A and B are disjoint, the union is simply the set containing all the elements in both set A and set B. $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Your Turn 1.27

Set $A =$ { red, yellow, blue } and set $B =$ \{orange, green, purple }. Find $A \cup B$.

Example 1.28

Finding the Union of Sets A and B When One Set is a Subset of the Other

Set $A = \{1, 3, 5, ...\}$ and set $B = \mathbb{N} = \{1, 2, 3, ...\}$. Find $A \cup B$.

Answer

Because set A is a subset of set B, A union B is equal to set B. $A \cup B = \mathbb{N} = \{1, 2, 3, ...\} = B$.

Your Turn 1.28

1. Set $A = \{a, b, c, \ldots, z\}$ and set $B = \{a, e, i, o, u\}$. Find $A \cup B$.

Video



The Basics of Intersection of Sets, Union of Sets and Venn Diagrams

Tech Check

Set Operation Practice

Sets Challenge is an application available on both Android and iPhone smartphones that allows you to practice and gain familiarity with the operations of set union, intersection, complement, and difference.

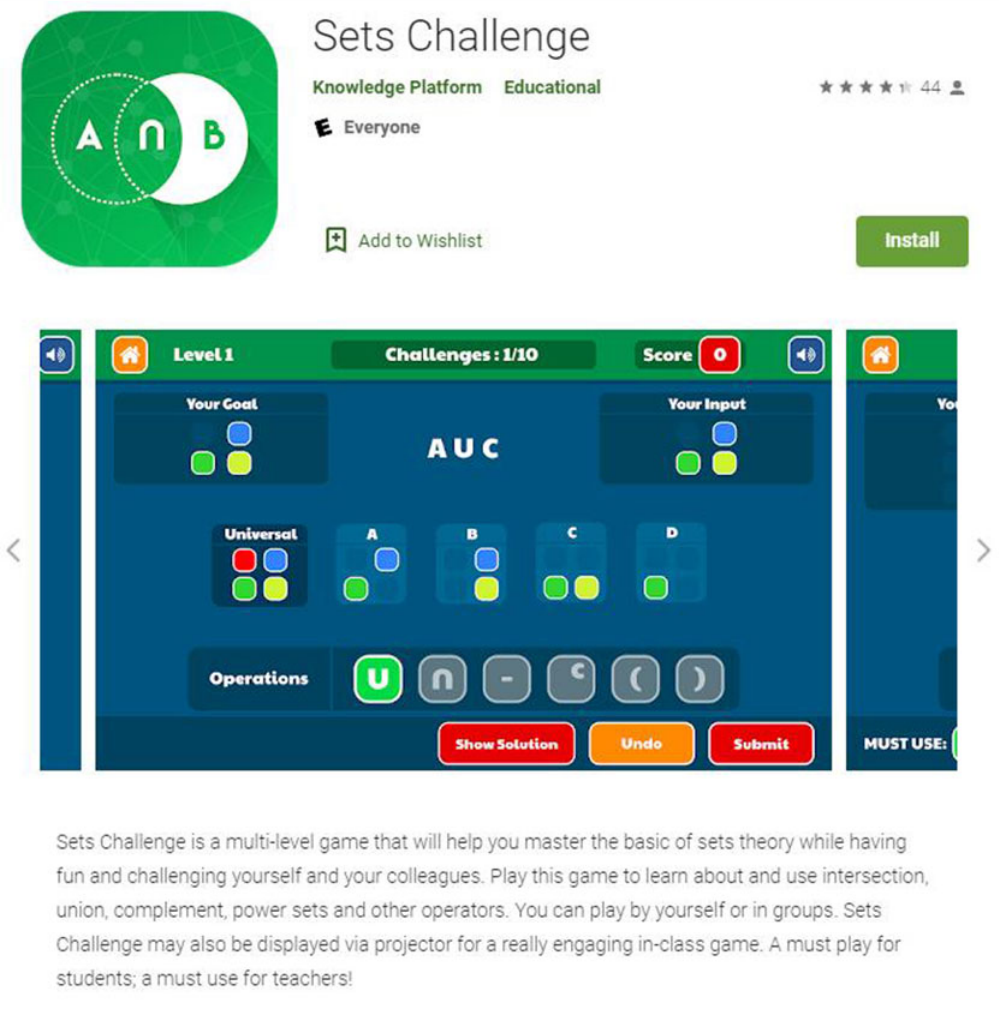


Figure 1.26 Google Play Store image of Sets Challenge game. (credit: screenshot from Google Play)

The Sets Challenge application/game uses some notation that differs from the notation covered in the text.

- The complement of set A in this text is written symbolically as A', but the Sets Challenge game uses A^C to represent the complement operation.
- In the text we do not cover set difference between two sets A and B, represented in the game as $A - B$. In the game this operation removes from set A all the elements in $A \cap B$. For example, if set $A = \{a, b, c, d\}$ and set $B = \{b, d, f, h\}$ are subsets of the universal set $U = \{a, b, c, \ldots, z\}$, then $A - B = \{a, b, c, d\} - \{b, d\} = \{a, c\}$, and $B - A = \{b, d, f, h\} - \{b, d\} = \{f, h\}$. There is a project at the end of the chapter to research the set difference operation.

Determining the Cardinality of Two Sets

The cardinality of the union of two sets is the total number of elements in the set. Symbolically the cardinality of A union B is written, $n(A \cup B)$. If two sets A and B are disjoint, the cardinality of A union B is the sum of the cardinality of set A and the cardinality of set B. If the two sets intersect, then A intersection B is a subset of both set A and set B. This means that if we add the cardinality of set A and set B, we will have added the number of elements in A intersection B twice, so we must then subtract it once as shown in the formula that follows.

FORMULA

The **cardinality of A union B** is found by adding the number of elements in set A to the number of elements in set B, then subtracting the number of elements in the intersection of set A and set B. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ or $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$.

Checkpoint

If sets A and B are disjoint, then $n(A \cap B) = n(A \text{ and } B) = 0$ and the formula is still valid, but simplifies to $n(A \cup B) = n(A) + n(B)$.

Example 1.29

Determining the Cardinality of the Union of Two Sets

The number of elements in set A is 10, the number of elements in set B is 20, and the number of elements in A intersection B is 4. Find the number of elements in A union B.

Answer

Using the formula for determining the cardinality of the union of two sets, we can say $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 10 + 20 - 4 = 26$.

Your Turn 1.29

1. If $n(A) = 23, n(B) = 17$, and $n(A \cap B) = 7$, then find $n(A \cup B)$.

Example 1.30

Determining the Cardinality of the Union of Two Disjoint Sets

If A and B are disjoint sets and the cardinality of set A is 37 and the cardinality of set B is 43, find the cardinality of A union B.

Answer

To find the cardinality of A union B, apply the formula, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Because sets A and B are disjoint, $A \cap B$ is the empty set, therefore $n(A \cap B) = n(\emptyset) = 0$ and $n(A \cup B) = 37 + 43 - 0 = 80$.

Your Turn 1.30

1. If $A \cap B = \emptyset, n(A) = 35$, and $n(B) = 78$, then find $n(A \cup B)$.

Applying Concepts of “AND” and “OR” to Set Operations

To become a licensed driver, you must pass some form of written test and a road test, along with several other requirements depending on your age. To keep this example simple, let us focus on the road test and the written test. If you pass the written test but fail the road test, you will not receive your license. If you fail the written test, you will not be allowed to take the road test and you will not receive a license to drive. To receive a driver's license, you must pass the written test AND the road test. For an “AND” statement to be true, both conditions that make up the statement must be true. Similarly, the intersection of two sets A and B is the set of elements that are in both set A and set B. To be a member of A intersection B, an element must be in set A and also must be in set B. The intersection of two sets corresponds to a logical "AND" statement.

The union of two sets is a logical inclusive "OR" statement. Say you are at a birthday party and the host offers Leah, Lenny, Maya, and you some cake or ice cream for dessert. Leah asks for cake, Lenny accepts both cake and ice cream, Maya turns down both, and you choose only ice cream. Leah, Lenny, and you are all having dessert. The “OR” statement is true if at least one of the components is true. Maya is the only one who did not have cake or ice cream; therefore, she did not have dessert and the “OR” statement is false. To be in the union of two sets A and B, an element must be in set A or set B or both set A and set B.

Example 1.31

Applying the "AND" or "OR" Operation

$A = \{0, 3, 6, 9, 12\}$, $B = \{0, 4, 8, 12, 16\}$, and $C = \{1, 2, 3, 5, 8, 13\}$.

Find the set consisting of elements in:

1. A and B.
2. A or B.
3. A or C.
4. (B and C) or A.

Answer

1. A and $B = A \cap B = \{0, 12\}$, because only the elements 0 and 12 are members of both set A and set B.
2. A or $B = A \cup B = \{0, 3, 4, 6, 8, 9, 12, 16\}$, because the set A or B is the collection of all elements in set A or set B, or both.
3. A or $C = A \cup C = \{0, 1, 2, 3, 5, 6, 8, 9, 12, 13\}$, because the set A or C is the collection of all elements in set A or set C, or both.
4. (B and C) or $A = (B \cap C) \cup A$. Parentheses are evaluated first: (B and C) $= B \cap C = \{8\}$, because the only member that both set B and set C share in common is 8. So, now we need to find $\{8\}$ or $\{0, 3, 6, 9, 12\}$, Because the word translates to the union operation, the problem becomes $\{8\} \cup \{0, 3, 6, 9, 12\}$, which is equal to $\{0, 3, 6, 8, 9, 12\}$.

Your Turn 1.31

$A = \{h, a, p, y\}$, and $B = \{a, w, e, s, o, m\}$, and $C = \{m, a, t, h\}$.
Find the set consisting of elements in:
1. A or B.
2. A and C.
3. B or C.
4. (A and C) and B.

Example 1.32

Determine and Apply the Appropriate Set Operations to Solve the Problem

Don Woods is serving cake and ice cream at his Juneteenth celebration. The party has a total of 54 guests in attendance. Suppose 30 guests requested cake, 20 guests asked for ice cream, and 12 guests did not have either cake or ice cream.

1. How many guests had cake or ice cream?
2. How many guests had cake and ice cream?

Answer

1. The total number of people at the party is 54, and 12 people did not have cake or ice cream. Recall that the total number of elements in the universal set is always equal to the number of elements in a subset plus the number of elements in the complement of the set, $n(U) = n(A) + n(A')$. That means $54 = n(\text{cake or ice cream}) + n(\text{not (cake or ice cream)})$, or equivalently, $n(\text{cake} \cup \text{ice cream}) = 54 - n((\text{cake} \cup \text{ice cream})') = 54 - 12 = 42$. A total of 42 people at the party had cake or ice cream.
2. To determine the number of people who had both cake and ice cream, we need to find the intersection of the set of people who had cake and the set of people who had ice cream. From Question 1, the number of people who had cake or ice cream is 42. This is the union of the two sets. The formula for the union of two sets is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Use the information given in the problem and substitute the known values into the formula to solve for the number of people in the intersection: $42 = 30 + 20 - n(A \cap B)$. Adding 30 and 20, the equation simplifies to $42 = 50 - n(\text{cake and ice cream})$. Which means $n(\text{cake and ice cream}) = 50 - 42 = 8$.

Your Turn 1.32

Ravi and Priya are serving soup and salad along with the main course at their wedding reception. The reception will have a total of 150 guests in attendance. A total of 92 soups and 85 salads were ordered, while 23 guests did not order any soup or salad.
1. How many guests had soup or salad or both?
2. How many guests had both soup and a salad?

Who Knew?

The Real Inventor of the Venn Diagram

John Venn, in his writings, references works by both John Boole and Augustus De Morgan, who referred to the circle diagrams commonly used to present logical relationships as Euler's circles. Leonhard Euler's works were published over 100 years prior to Venn's, and Euler may have been influenced by the works of Gottfried Leibniz.

So, why does John Venn get all the credit for these graphical depictions? Venn was the first to formalize the use of these diagrams in his book *Symbolic Logic*, published in 1881. Further, he made significant improvements in their design, including shading to highlight the region of interest. The mathematician C.L. Dodgson, also known as Lewis Carroll, built upon Venn's work by adding an enclosing universal set.

Invention is not necessarily coming up with an initial idea. It is about seeing the potential of an idea and applying it to a new situation.

References:

Margaret E. Baron. "A Note on the Historical Development of Logic Diagrams: Leibniz, Euler and Venn." *The Mathematical Gazette*, vol. 53, no. 384, 1969, pp. 113-125. *JSTOR*, www.jstor.org/stable/3614533. Accessed 15 July 2021.

Deborah Bennett. "Drawing Logical Conclusions." *Math Horizons*, vol. 22, no. 3, 2015, pp. 12-15. *JSTOR*, www.jstor.org/stable/10.4169/mathhorizons.22.3.12. Accessed 15 July 2021.

Drawing Conclusions from a Venn Diagram with Two Sets

All Venn diagrams will display the relationships between the sets, such as subset, intersecting, and/or disjoint. In addition to displaying the relationship between the two sets, there are two main additional details that Venn diagrams can include: the individual members of the sets or the cardinality of each disjoint subset of the universal set.

A Venn diagram with two subsets will partition the universal set into 3 or 4 sections depending on whether they are disjoint or intersecting sets. Recall that the complement of set A, written A', is the set of all elements in the universal set that are not in set A.

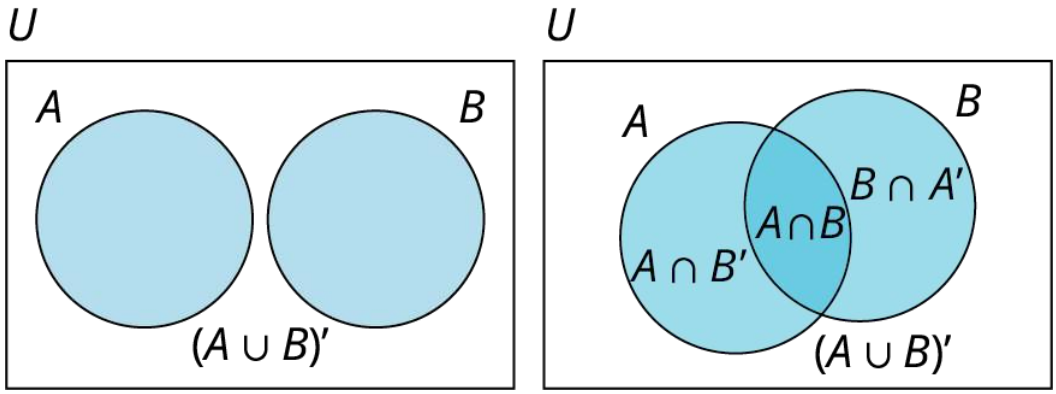


Figure 1.27 Side-by-side Venn diagrams with disjoint and intersecting sets, respectively.

Example 1.33

Using a Venn Diagram to Draw Conclusions about Set Membership

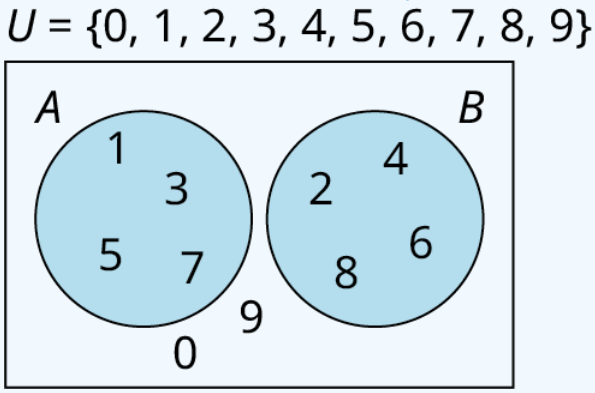


Figure 1.28

1. Find $A \cup B$.
2. Find $A \cap B$.
3. Find B'.
4. Find $n(B')$.

Answer

1. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, because A union B is the collection of all elements in set A or set B or both.
2. Because A and B are disjoint sets, there are no elements that are in both A and B. Therefore, A intersection B is the empty set, $A \cap B = \emptyset$.
3. The complement of set B is the set of all elements in the universal set that are not in set B: $B' = \{0, 1, 3, 5, 7, 9\}$.
4. The cardinality, or number of elements in set B', isn(B′)=6.

Your Turn 1.33

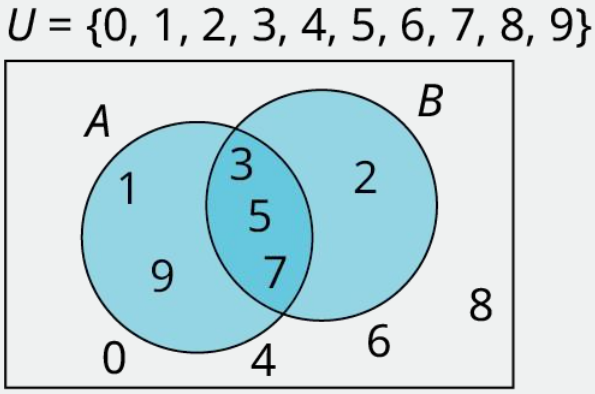


Figure 1.29 Venn diagram with two intersecting sets and members.
1. Find $A \cap B$.
2. Find $A \cup B$. 3. Find $A \cap B'$. 4. Find $n(A \cap B')$.

Example 1.34

Using a Venn Diagram to Draw Conclusions about Set Cardinality

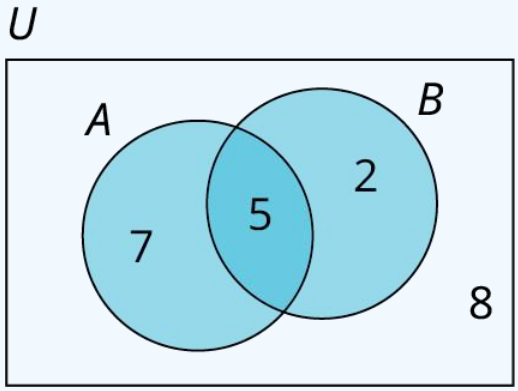


Figure 1.30 Venn diagram with two intersecting sets and number of elements in each section indicated.

1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$.
3. Find $n(A)$.

Answer

1. The number of elements in A or B is the number of elements in A union B: $n(A \cup B) = n(\{2, 5, 7\}) = 14$.
2. The number of elements in A and B is the number of elements in A intersection B: $n(A \cap B) = 5$.
3. The number of elements in set A is the sum of all the numbers enclosed in the circle representing set A: $n(A) = n(\{7, 5\}) = 12$.

Your Turn 1.34

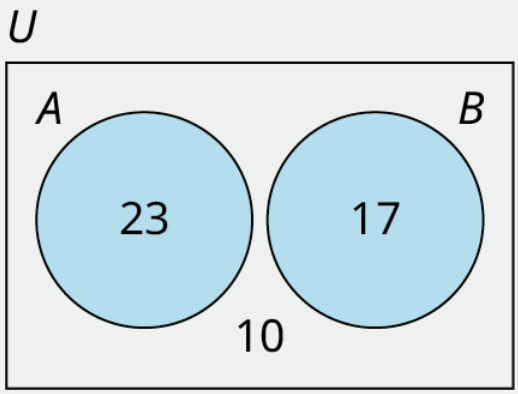


Figure 1.31 Venn diagram with two disjoint sets and number of elements in each section.
1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$. 3. Find $n(A')$.

Check Your Understanding

20. The _____ of two sets A and B is the set of all elements that they share in common.
21. The _____ of two sets A and B is the collection of all elements that are in set A or set B, or both set A and set B.
22. The union of two sets A and B is represented symbolically as _____ .
23. The intersection of two sets A and B is represented symbolically as _____ .
24. If set A is a subset of set B, then A intersection B is equal to set _____ .
25. If set A is a subset of set B, then A union B is equal to set _____ .
26. If set A and set B are disjoint sets, then A intersection B is the _____ set.
27. The cardinality of A union B, $n(A \cup B)$, is found using the formula: _____ .

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1.5.1: Exercises

For the following exercises, determine the union or intersection of the sets as indicated.

$$A = \{2, 4, 6, 8, 10, 12\}, B = \{4, 8, 12, 16, 20\}, C = \{8, 16, 24, 32, 40\}, \text{ and } D = \{10, 20, 30, 40, 50\}$$

Exercise 1.5.1.1

$B \cup C$

Exercise 1.5.1.2

$A \cap D$

Exercise 1.5.1.3

$D \cap C$

Exercise 1.5.1.4

$A \cup D$

Exercise 1.5.1.5

$A \cap (C \cup D)$

Exercise 1.5.1.6

$B \cup (A \cap D)$

Exercise 1.5.1.7

$D \cup (A \cap C)$

Exercise 1.5.1.8

$C \cap (A \cup D)$

Exercise 1.5.1.9

$B \cap (A \cap D)$

Exercise 1.5.1.10

$B \cap (A \cap C)$

Exercise 1.5.1.11

$B \cup (A \cup D)$

? Exercise 1.5.1.12

$B \cup (A \cup C)$

For the following exercises, use the sets provided to apply the "AND" or "OR" operation as indicated to find the resulting set.

$$U = \{a, b, c, \ldots, z\}, S = \{s, a, m, p, l, e\}, M = \{m, a, p\}, L = \{l, a, m, p\}, D = \{d, o, g\}, \text{ and } P = \{p, l, o, t\}$$

? Exercise 1.5.1.13

Find the set consisting of elements in S and P.

? Exercise 1.5.1.14

Find the set consisting of elements in M or D.

? Exercise 1.5.1.15

Find the set consisting of elements in P or M.

? Exercise 1.5.1.16

Find the set consisting of elements in M and D.

? Exercise 1.5.1.17

Find the set consisting of elements in L and M.

? Exercise 1.5.1.18

Find the set consisting of elements in L or M.

? Exercise 1.5.1.19

Find the set consisting of the elements in D or M or P.

? Exercise 1.5.1.20

Find the set consisting of the elements in S or M or P.

? Exercise 1.5.1.21

Find the set consisting of the elements in (S or D) and P.

? Exercise 1.5.1.22

Find the set consisting of the elements in S or (D and P).

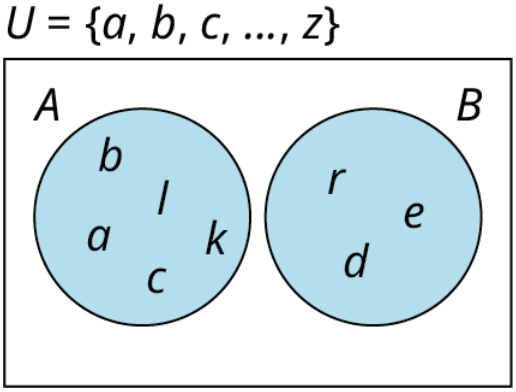
? Exercise 1.5.1.23

Find the set consisting of elements in U or (P and S).

Exercise 1.5.1.24

Find the set consisting of elements in (U or P) and S.

For the following exercises, use the Venn diagram provided to answer the following questions about the sets.



Exercise 1.5.1.25

Find $A \cup B$.

Exercise 1.5.1.26

Find $A \cap B$.

Exercise 1.5.1.27

Find $(A \cap B)'$.

Exercise 1.5.1.28

Find $(A \cup B)'$.

Exercise 1.5.1.29

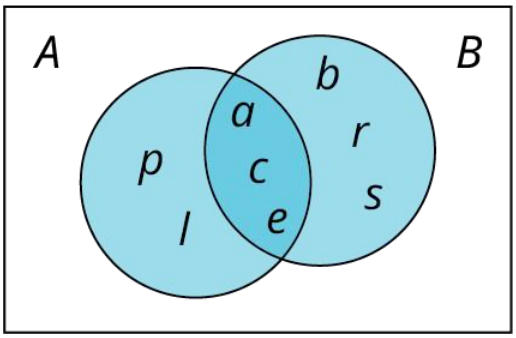
Find $A \cap B'$.

Exercise 1.5.1.30

Find $B \cap A'$.

For the following exercises, use the Venn diagram provided to answer the following questions about the sets.

$U = \{a, b, c, \ldots, z\}$



Exercise 1.5.1.31

Find $A \cap B$.

Exercise 1.5.1.32

Find $A \cup B$.

Exercise 1.5.1.33

Find $(A \cup B)'$.

Exercise 1.5.1.34

Find $(A \cap B)'$.

Exercise 1.5.1.35

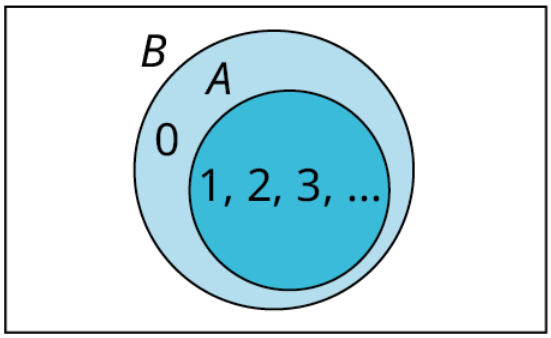
Find $B \cap A'$.

Exercise 1.5.1.36

Find $A \cap B'$.

For the following exercises, use the Venn diagram provided to answer the following questions about the sets.

$U = Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$



Exercise 1.5.1.37

Find $A \cup B$.

Exercise 1.5.1.38

Find $A \cap B$.

Exercise 1.5.1.39

Find $(A \cup B)'$.

Exercise 1.5.1.40

Find $(A \cap B)'$.

Exercise 1.5.1.41

Find $B \cap A'$.

Exercise 1.5.1.42

Find $A \cap B'$.

For the following exercises, determine the cardinality of the union of set A and set B.

Exercise 1.5.1.43

If set $A = \{$ red, white, blue $\}$ and set $B = \{$ green, white, red $\}$, find $n(A \cup B)$.

Exercise 1.5.1.44

If set $A = \{$ silver, gold, bronze $\}$ and set $B = \{$ silver, gold $\}$, find the number of elements in A or B.

Exercise 1.5.1.45

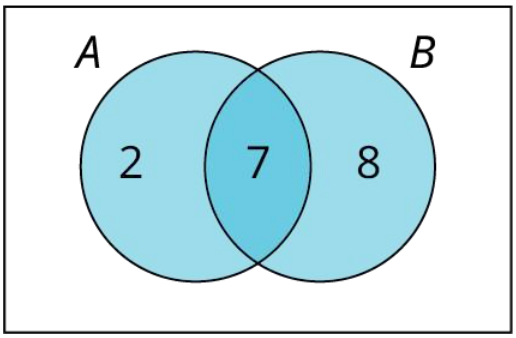
If set $A = \{$ glass, plate, fork, knife $\}$ and set $B = \{$ bowl, spoon, cup $\}$, find the number of elements in A or B.

Exercise 1.5.1.46

If set $A = \{$ Algebra, Geometry, Biology, Physics, Chemistry, English $\}$ and Set $B = \{$ Algebra, English, History, Spanish, French, Music $\}$, find $n(A \cup B)$.

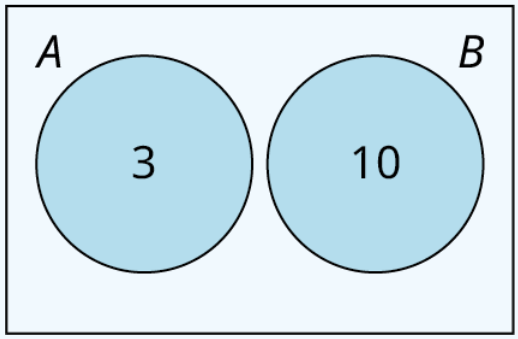
For the following exercises, use the Venn diagram to determine the cardinality of A union B.

$U = 21$



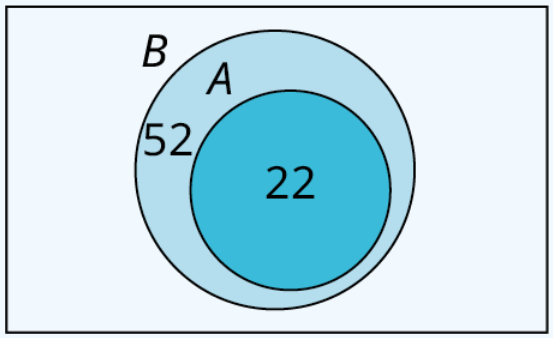
Exercise 1.5.1.48

$U = 15$

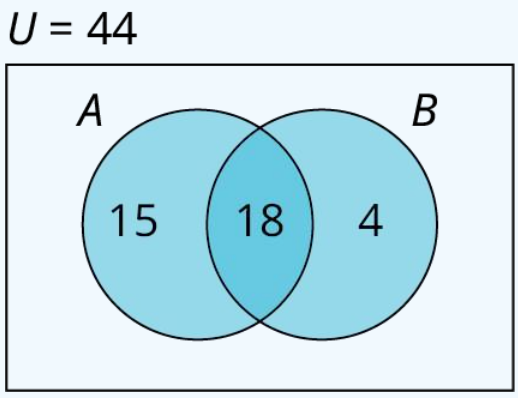


Exercise 1.5.1.49

$U = 88$



Exercise 1.5.1.50



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1.6: Set Operations with Three Sets

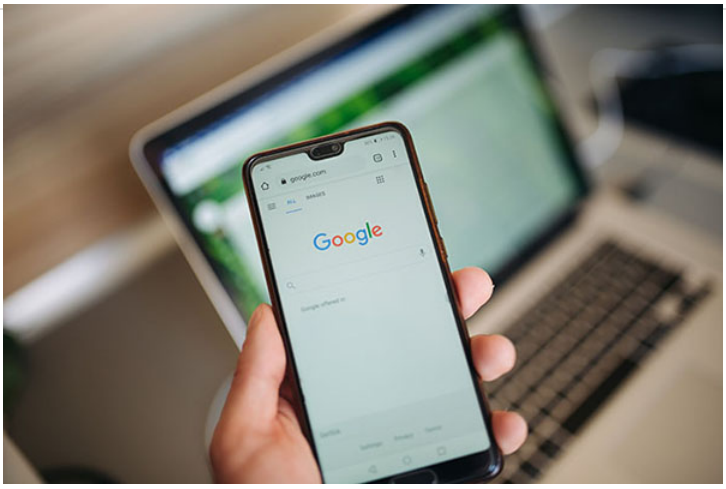


Figure 1.32: Companies like Google collect data on how you use their services, but the data requires analysis to really mean something. (credit: “Man holding smartphone and searches through google” by Nenad Stojkovic/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

- Interpret Venn diagrams with three sets.
- Create Venn diagrams with three sets.
- Apply set operations to three sets.
- Prove equality of sets using Venn diagrams.

Have you ever searched for something on the Internet and then soon after started seeing multiple advertisements for that item while browsing other web pages? Large corporations have built their business on data collection and analysis. As we start working with larger data sets, the analysis becomes more complex. In this section, we will extend our knowledge of set relationships by including a third set.

A Venn diagram with two intersecting sets breaks up the universal set into four regions; simply adding one additional set will increase the number of regions to eight, doubling the complexity of the problem.

Venn Diagrams with Three Sets

Below is a Venn diagram with two intersecting sets, which breaks the universal set up into four distinct regions.

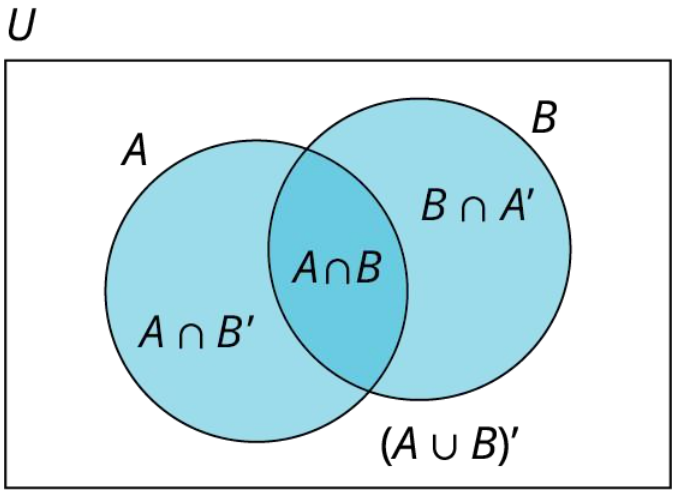


Figure 1.33:

Next, we see a Venn diagram with three intersecting sets, which breaks up the universal set into eight distinct regions.

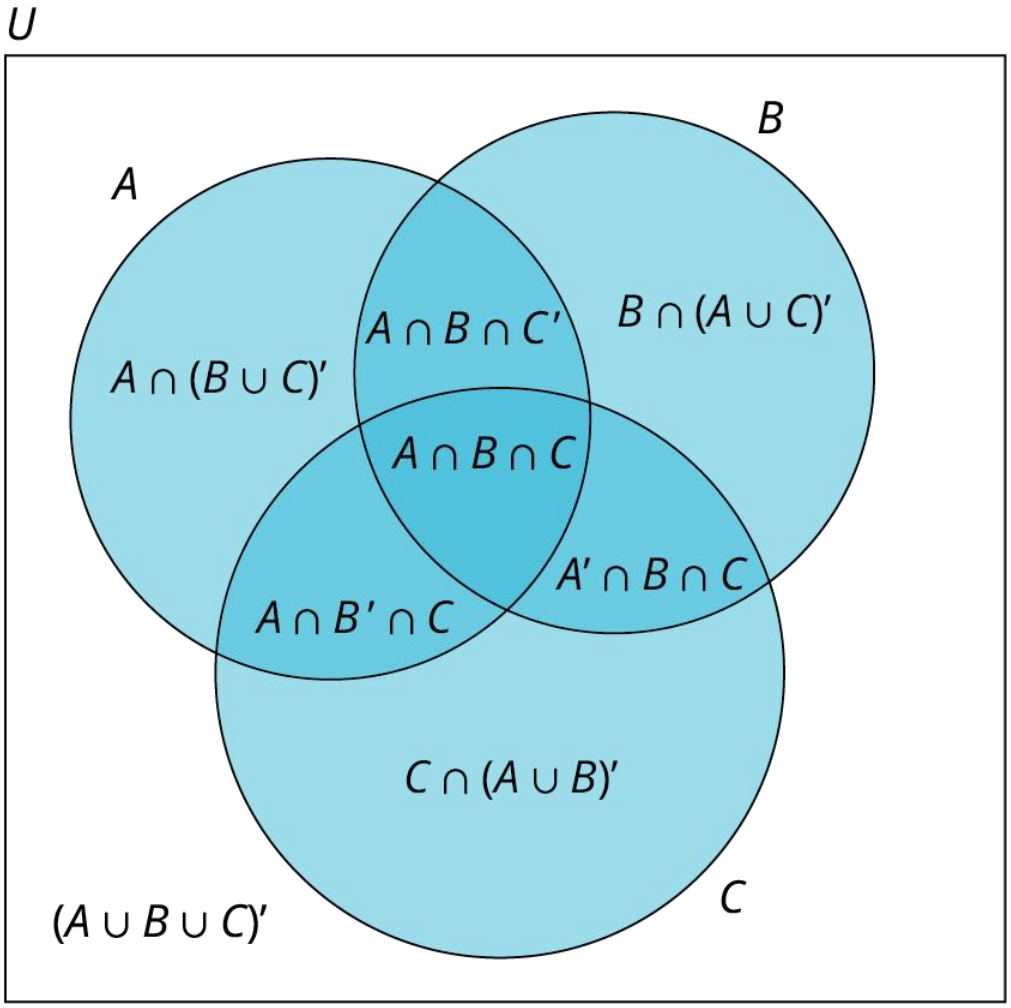


Figure 1.34:

Tech Check: Shading Venn Diagrams

Venn Diagram is an Android application that allows you to visualize how the sets are related in a Venn diagram by entering expressions and displaying the resulting Venn diagram of the set shaded in gray.

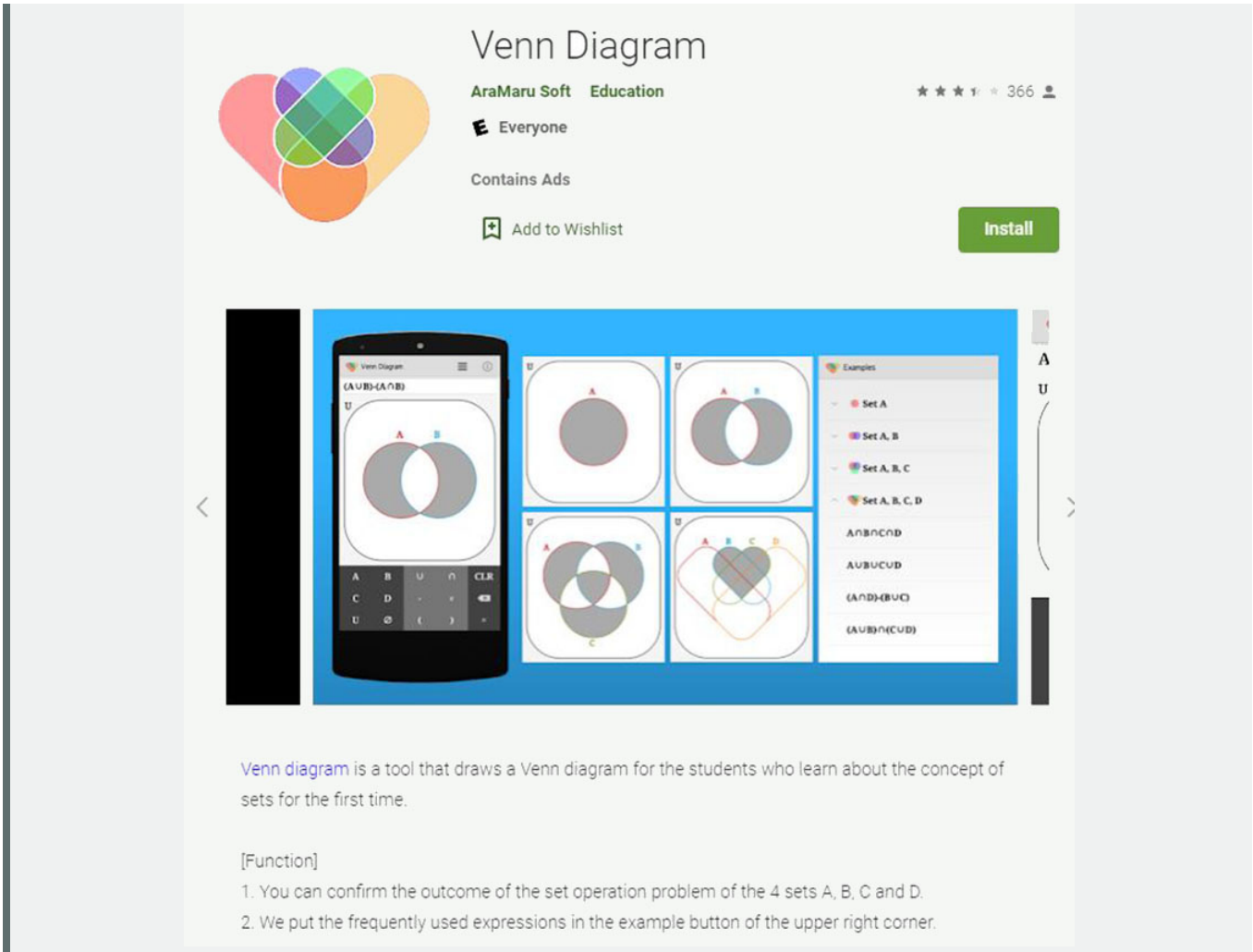


Figure 1.35: Google Play Store image of Venn Diagram app. (credit: screenshot from Google Play)

The Venn Diagram application uses some notation that differs from the notation covered in this text.

a. The complement of set A in this text is written symbolically as A', but the Venn Diagram app uses semanticsACAC to represent the complement operation.
b. The set difference operation, $-$, is available in the Venn Diagram app, although this operation is not covered in the text.

It is recommended that you explore this application to expand your knowledge of Venn diagrams prior to continuing with the next example.

In the next example, we will explore the three main blood factors, A, B and Rh. The following background information about blood types will help explain the relationships between the sets of blood factors. If an individual has blood factor A or B , those will be included in their blood type. The Rh factor is indicated with a + or $\mathfrak{a}-$. For example, if a person has all three blood factors, then their blood type would be AB^+. In the Venn diagram, they would be in the intersection of all three sets, $A \cap B \cap Rh^+$. If a person did not have any of these three blood factors, then their blood type would be O^-, and they would be in the set $(A \cup B \cup Rh^+)'$ which is the region outside all three circles.

Example 1.35: Interpreting a Venn Diagram with Three Sets

Use the Venn diagram below, which shows the blood types of 100 people who donated blood at a local clinic, to answer the following questions.

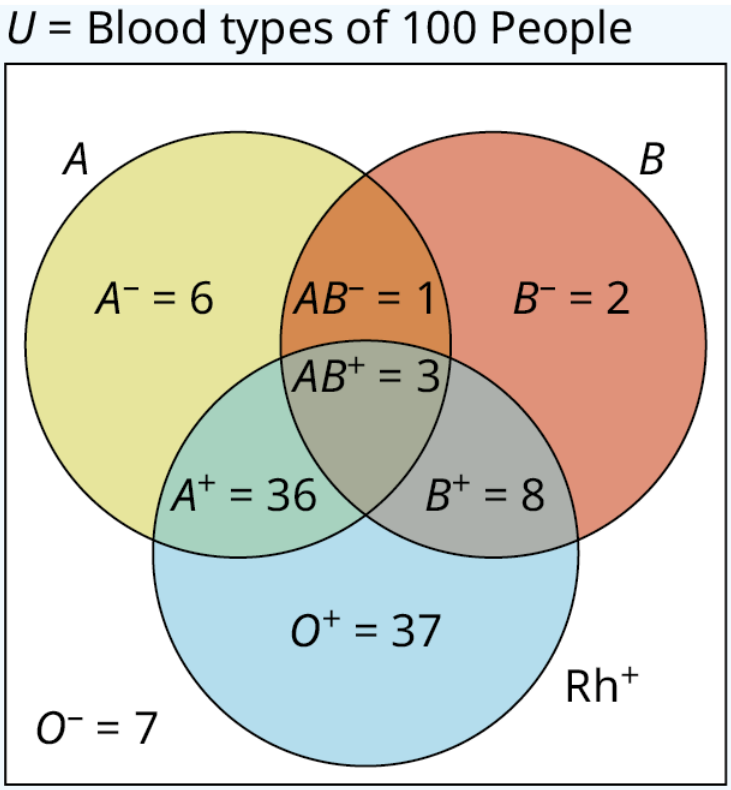


Figure 1.36

1. How many people with a type A blood factor donated blood?
2. Julio has blood type $\mathbf{B^+}$. If he needs to have surgery that requires a blood transfusion, he can accept blood from anyone who does not have a type A blood factor. How many people donated blood that Julio can accept?
3. How many people who donated blood do not have the $\mathbf{Rh^+}$ blood factor?
4. How many people had type A and type B blood?

Answer

1. The number of people who donated blood with a type A blood factor will include the sum of all the values included in the A circle. It will be the union of sets $A^-, A^+, AB^-$and AB^+. $n(A) = n(A^-) + n(A^+) + n(AB^-) + n(AB^+) = 6 + 36 + 1 + 3 = 46$.

2. In part 1, it was determined that the number of donors with a type A blood factor is 46 . To determine the number of people who did not have a type A blood factor, use the following property, A' union is equal to U, which means $n(A) + n(A') = n(U)$, and $n(A') = n(U) - n(A) = 100 - 46 = 54$. Thus, 54 people donated blood that Julio can accept.

3. This would be everyone outside the $\mathbf{Rh^+}$circle, or everyone with a negative Rh factor, $n(Rh^-) = n(O^-) + n(A^-) + n(AB^-) + n(B^-) = 7 + 6 + 1 + 2 = 16$.

4. To have both blood type A and blood type B , a person would need to be in the intersection of sets A and B. The two circles overlap in the regions labeled $AB^-$and AB^+. Add up the number of people in these two regions to get the total: $1 + 3 = 4$. This can be written symbolically as $n(A \text{ and } B) = n(A \cap B) = n(AB^-) + n(AB^+) = 1 + 3 = 4$.

Your Turn 1.35

Use the same Venn diagram in the example above to answer the following questions.

How many people donated blood with a type B blood factor?

2. How many people who donated blood did not have a type B blood factor?

How many people who donated blood had a type B blood factor or were Rh^+?

Who Knew?: Blood Types

Most people know their main blood type of A, B, AB, or O and whether they are $\mathbf{Rh}^{+}$ or $\mathbf{Rh}^{-}$, but did you know that the International Society of Blood Transfusion recognizes twenty-eight additional blood types that have important implications for organ transplants and successful pregnancy? For more information, check out this article:

Blood mystery solved: Two new blood types identified

Creating Venn Diagrams with Three Sets

In general, when creating Venn diagrams from data involving three subsets of a universal set, the strategy is to work from the inside out. Start with the intersection of the three sets, then address the regions that involve the intersection of two sets. Next, complete the regions that involve a single set, and finally address the region in the universal set that does not intersect with any of the three sets. This method can be extended to any number of sets. The key is to start with the region involving the most overlap, working your way from the center out.

Example 1.36: Creating a Venn Diagram with Three Sets

A teacher surveyed her class of 43 students to find out how they prepared for their last test. She found that 24 students made flash cards, 14 studied their notes, and 27 completed the review assignment. Of the entire class of 43 students, 12 completed the review and made flash cards, nine completed the review and studied their notes, and seven made flash cards and studied their notes, while only five students completed all three of these tasks. The remaining students did not do any of these tasks. Create a Venn diagram with subsets labeled: “Notes,” “Flash Cards,” and “Review” to represent how the students prepared for the test.

Answer

Step 1: First, draw a Venn diagram with three intersecting circles to represent the three intersecting sets: Notes, Flash Cards, and Review. Label the universal set with the cardinality of the class.

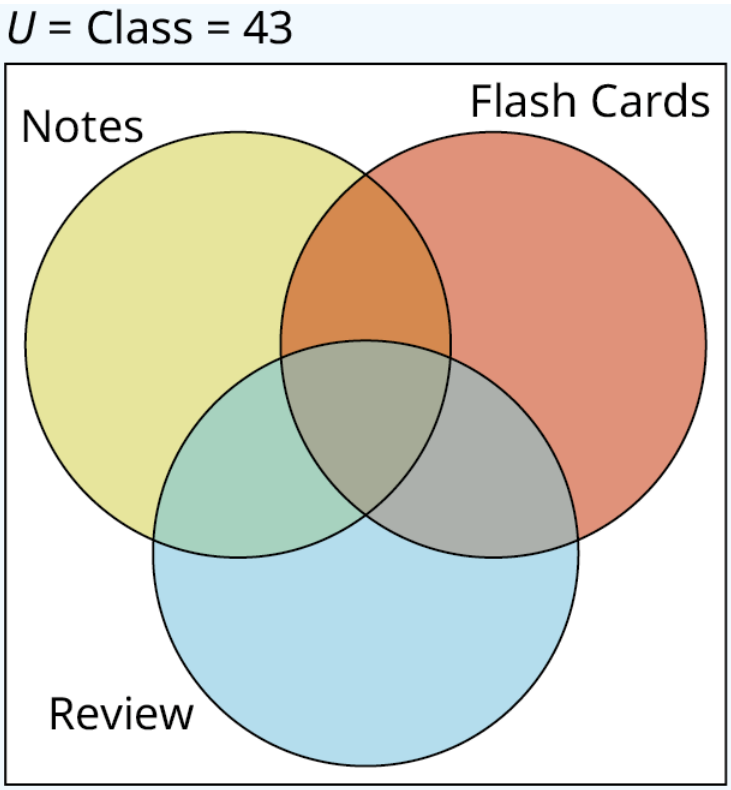


Figure 1.37:

Step 2: Next, in the region where all three sets intersect, enter the number of students who completed all three tasks.

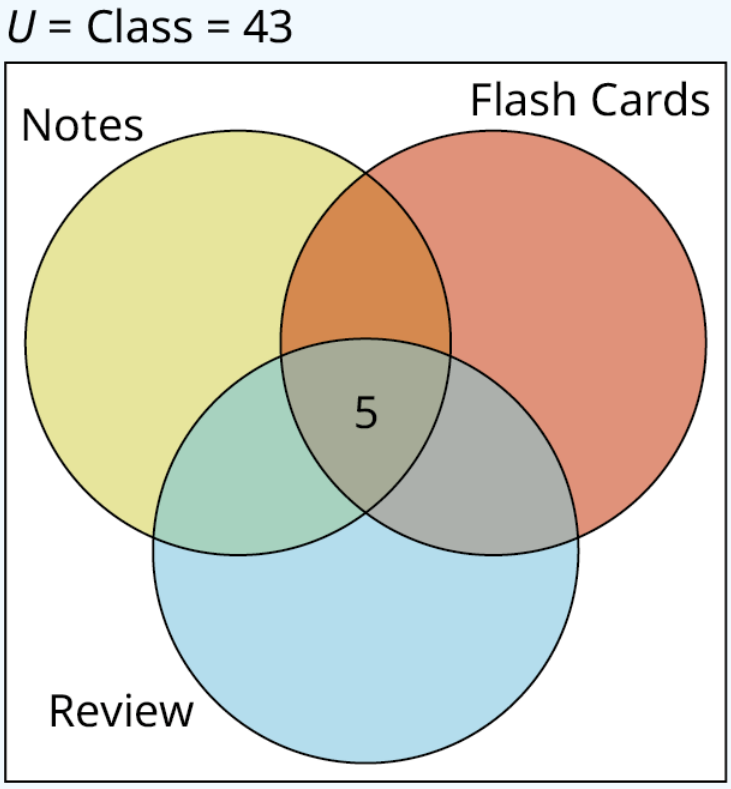


Figure 1.38:

Step 3: Next, calculate the value and label the three sections where just two sets overlap.

Review and flash card overlap

. A total of 12 students completed the review and made flash cards, but five of these twelve students did all three tasks, so we need to subtract: $12 - 5 = 7$ $12 - 5 = 7$. This is the value for the region where the flash card set intersects with the review set.

Review and notes overlap

. A total of 9 students completed the review and studied their notes, but again, five of these nine students completed all three tasks. So, we subtract: $9 - 5 = 4$ $9 - 5 = 4$. This is the value for the region where the review set intersects with the notes set.

Flash card and notes overlap

. A total of 7 students made flash cards and studied their notes; subtracting the five students that did all three tasks from this number leaves 2 students who only studied their notes and made flash cards. Add these values to the Venn diagram.

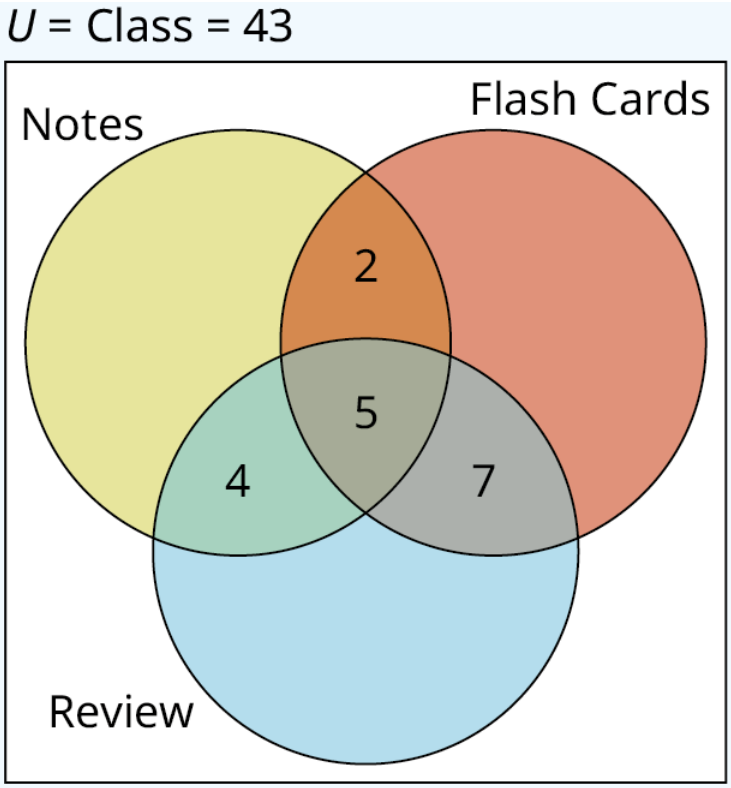


Figure 1.39:

Step 4: Now, repeat this process to find the number of students who only completed one of these three tasks.

A total of 24 students completed flash cards, but we have already accounted for 2 + 5 + 7 = 14 2 + 5 + 7 = 14 of these. Thus, 24 - 14 = 10 24 - 14 = 10 students who just made flash cards. A total of 14 students studied their notes, but we have already accounted for 4 + 5 + 2 = 11 4 + 5 + 2 = 11 of these. Thus, 14 - 11 = 3 14 - 11 = 3 students only studied their notes. A total of 27 students completed the review assignment, but we have already accounted for 4 + 5 + 7 = 16 4 + 5 + 7 = 16 of these, which means 27 - 16 = 11 27 - 16 = 11 students only completed the review assignment. Add these values to the Venn diagram.

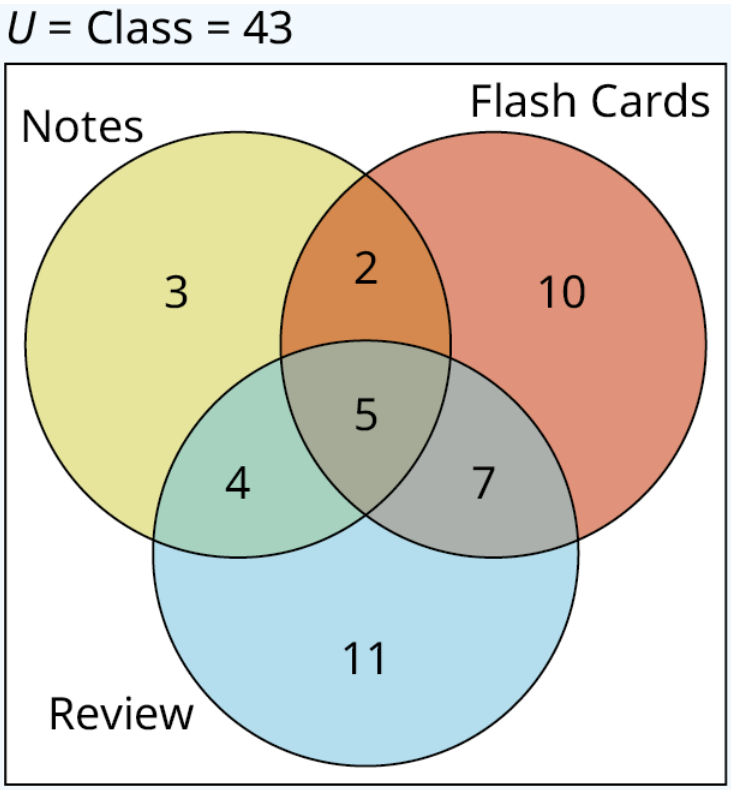


Figure 1.40:

Step 5: Finally, compute how many students did not do any of these three tasks. To do this, we add together each value that we have already calculated for the separate and intersecting sections of our three sets: $3+2+4+5+10+7+11=42$. Because there 43 students in the class, and $43-42=1$, this means only one student did not complete any of these tasks to prepare for the test. Record this value somewhere in the rectangle, but outside of all the circles, to complete the Venn diagram.

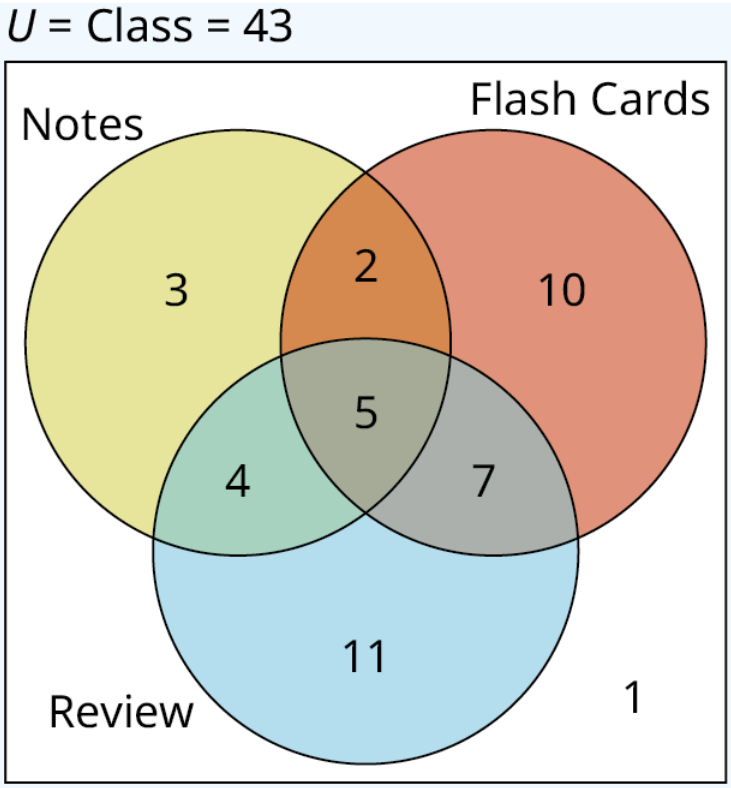


Figure 1.41

Your Turn 1.36

A group of 50 people attending a conference who preordered their lunch were able to select their choice of soup, salad, or sandwich. A total of 17 people selected soup, 29 people selected salad and 35 people selected a sandwich. Of these orders, 11 attendees selected soup and salad, 10 attendees selected soup and a sandwich, and 18 selected a salad and a sandwich, while eight people selected a soup, a salad, and a sandwich. Create a Venn diagram with subsets labeled "Soup," "Salad," and "Sandwich," and label the cardinality of each section of the Venn diagram as indicated by the data.

Applying Set Operations to Three Sets

Set operations are applied between two sets at a time. Parentheses indicate which operation should be performed first. As with numbers, the inner most parentheses are applied first. Next, find the complement of any sets, then perform any union or intersections that remain.

Example 1.37: Applying Set Operations to Three Sets

Perform the set operations as indicated on the following sets: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{0, 1, 2, 3, 4, 5, 6\}$, $B = \{0, 2, 4, 6, 8, 10, 12\}$, and $C = \{0, 3, 6, 9, 12\}$.

1. Find $(A \cap B) \cap C$.
2. Find $A \cap (B \cup C)$.
3. Find $(A \cap B) \cup C'$.

Answer

1. Parentheses first, A intersection B equals $A \cap B = \{0, 2, 4, 6\}$, the elements common to both A and B.
   $(A \cap B) \cap C = \{0, 2, 4, 6\} \cap \{0, 3, 6, 9, 12\} = \{0, 6\}$, because the only elements that are in both sets are 0 and 6.

2. Parentheses first, B union C equals $B \cup C = \{0, 2, 3, 4, 6, 8, 9, 10, 12\}$, the collection of all elements in set B or set C or both. $A \cap (B \cup C) = \{0, 1, 2, 3, 4, 5, 6\} \cap \{0, 2, 3, 4, 6, 8, 9, 10, 12\} = \{0, 2, 3, 4, 6\}$, because the intersection of these two sets is the set of elements that are common to both sets.
3. Parentheses first, A intersection B equals $A \cap B = \{0, 2, 4, 6\}$. Next, find C'. The complement of set C is the set of elements in the universal set U that are not in set C. $C' = \{1, 2, 4, 5, 7, 8, 10, 11\}$. Finally, find $(A \cap B) \cup C' = \{0, 2, 4, 6\} \cup \{1, 2, 4, 5, 7, 8, 10, 11\} = \{0, 1, 2, 4, 5, 6, 7, 8, 10, 11\}$.

Your Turn 1.37 Using the same sets from Example 1.37, perform the set operations indicated.

Find $A \cap (B \cap C)$.

Find $(A \cap B) \cup (A \cap C)$.

Find $(A \cup C') \cap (B \cup C')$.

Notice that the answers to the Your Turn are the same as those in the Example. This is not a coincidence. The following equivalences hold true for sets:

- $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$. These are the associative property for set intersection and set union.
- $A \cap B = B \cap A$ and $A \cup B = B \cup A$. These are the commutative property for set intersection and set union.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. These are the distributive property for sets over union and intersection, respectively.

Proving Equality of Sets Using Venn Diagrams

To prove set equality using Venn diagrams, the strategy is to draw a Venn diagram to represent each side of the equality, then look at the resulting diagrams to see if the regions under consideration are identical.

Augustus De Morgan was an English mathematician known for his contributions to set theory and logic. De Morgan's law for set complement over union states that $(A \cup B)' = A' \cap B'$. In the next example, we will use Venn diagrams to prove De Morgan's law for set complement over union is true. But before we begin, let us confirm De Morgan's law works for a specific example. While showing something is true for one specific example is not a proof, it will provide us with some reason to believe that it may be true for all cases.

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 3, 4\}$, and $B = \{3, 4, 5, 6\}$. We will use these sets in the equation $(A \cup B)' = A' \cap B'$. To begin, find the value of the set defined by each side of the equation.

Step 1: $A \cup B$ is the collection of all unique elements in set A or set B or both. $A \cup B = \{2, 3, 4, 5, 6\}$. The complement of A union B, $(A \cup B)'$, is the set of all elements in the universal set that are not in $A \cup B$. So, the left side the equation $(A \cup B)'$ is equal to the set $\{1, 7\}$.

Step 2: The right side of the equation is $A' \cap B'$. A' is the set of all members of the universal set U that are not in set A. $A' = \{1, 5, 6, 7\}$. Similarly, $B' = \{1, 2, 7\}$.

Step 3: Finally, $A' \cap B'$ is the set of all elements that are in both A' and B'. The numbers 1 and 7 are common to both sets, therefore, $A' \cap B' = \{1, 7\}$. Because, $\{1, 7\} = \{1, 7\}$ we have demonstrated that De Morgan's law for set complement over union works for this particular example. The Venn diagram below depicts this relationship.

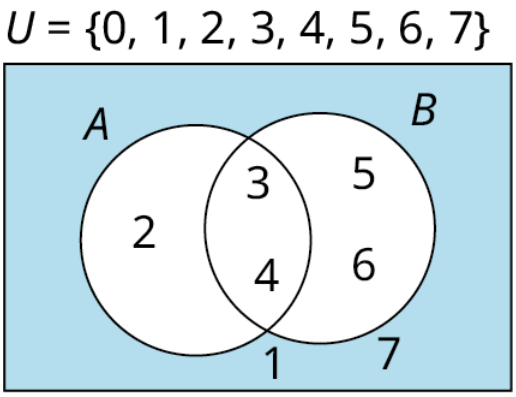


Figure 1.42:

Example 1.38: Proving De Morgan's Law for Set Complement over Union Using a Venn Diagram

De Morgan's Law for the complement of the union of two sets A and B states that: $(A \cup B)' = A' \cap B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Answer

Step 1: First, draw a Venn diagram representing the left side of the equality. The regions of interest are shaded to highlight the sets of interest. $A \cup B$ is shaded on the left, and $(A \cup B)'$ is shaded on the right.

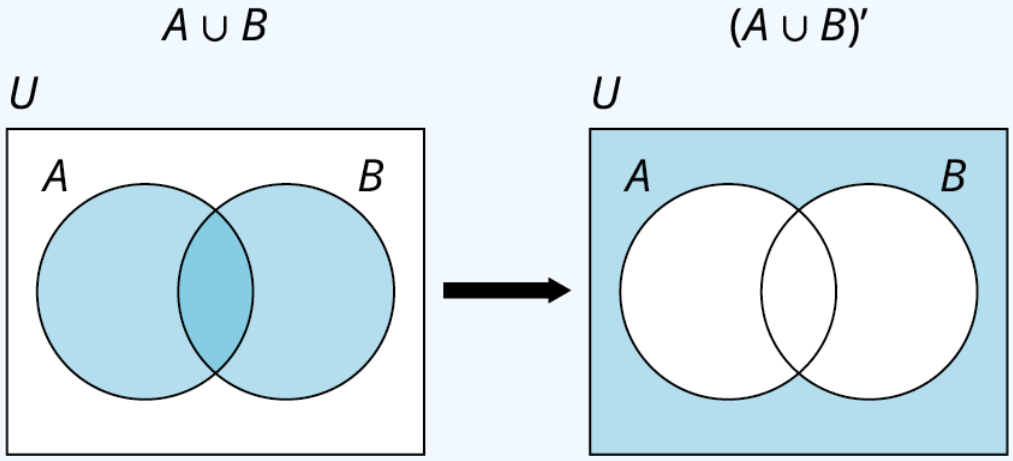
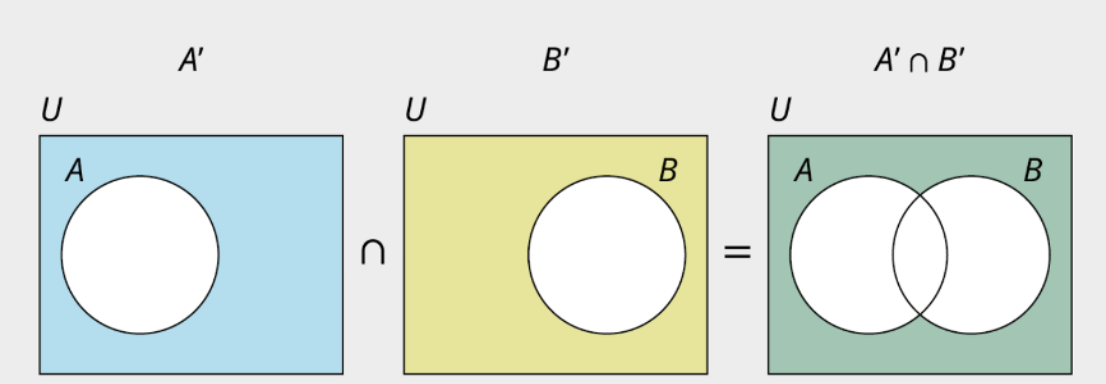


Figure 1.43:

Step 2: Next, draw a Venn diagram to represent the right side of the equation. A' is shaded and B' is shaded. Because A' and B' mix to form $A' \cap B'$ is also shaded.



Step 3: Verify the conclusion. Because the shaded region in the Venn diagram for $(A \cup B)'$ matches the shaded region in the Venn diagram for $A' \cap B'$, the two sides of the equation are equal, and the statement is true. This completes the proof that De Morgan's law is valid.

Your Turn 1.38

De Morgan's Law for the complement of the intersection of two sets A and B states that $(A \cap B)' = A' \cup B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Check Your Understanding

When creating a Venn diagram with two or more subsets, you should begin with the region involving the most _____ , then work your way from the center outward.

To construct a Venn diagram with three subsets, draw and label three circles that overlap in a common _____ region inside the rectangle of the universal set to represent each of the three subsets.

In a Venn diagram with three sets, A, B, and C overlap is equal to the set _____ .

When performing set operations with three or more sets, the order of operations is inner most _____ first, then find the _____ of any sets, and finally perform any union or intersection operations that remain.

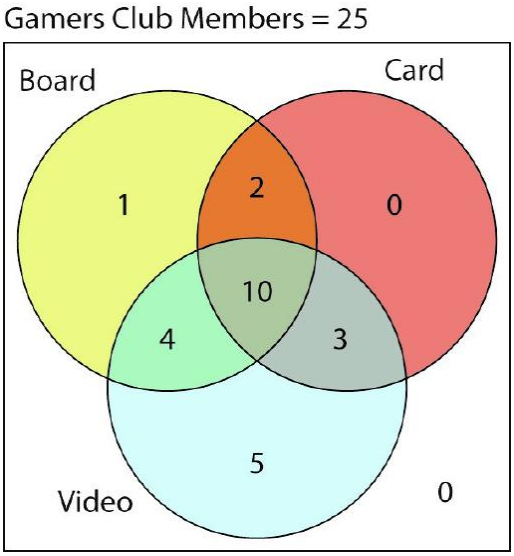
To prove set equality using Venn diagrams, draw a Venn diagram to represent each side of the _____ and then compare the diagrams to determine if they match or not. If they match, the statement is _____, otherwise it is not.

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1.6.1: Exercises

A gamers club at Baily Middle School consisting of 25 members was surveyed to find out who played board games, card games, or video games. Use the results depicted in the Venn diagram below to answer the following exercises.



Exercise 1.6.1.1

How many gamers club members play all three types of games: board games, card games, and video games?

Exercise 1.6.1.2

How many gamers are in the set Board ∩ Video?

Exercise 1.6.1.3

If Javier is in the region with a total of three members, what type of games does he play?

Exercise 1.6.1.4

How many gamers play video games?

Exercise 1.6.1.5

How many gamers are in the set Board ∪ Card?

Exercise 1.6.1.6

How many members of the gamers club do not play video games?

Exercise 1.6.1.7

How many members of this club only play board games?

Exercise 1.6.1.8

How many members of this club only play video games?

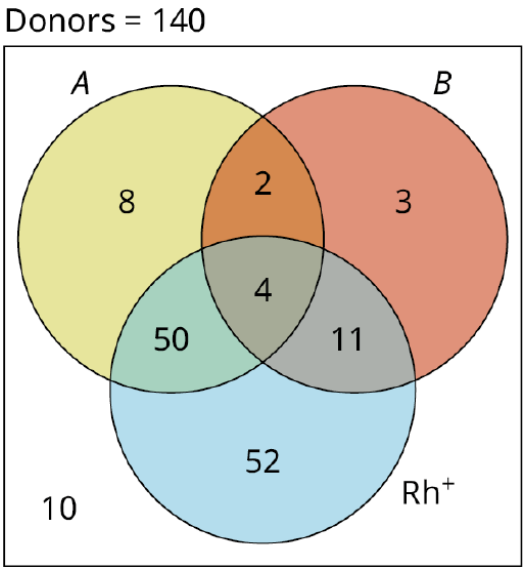
Exercise 1.6.1.9

How many members of the gamers club play video and card games?

Exercise 1.6.1.10

How many members of the gamers club are in the set Card'?

A blood drive at City Honors High School recently collected blood from 140 students, staff, and faculty. Use the results depicted in the Venn diagram below to answer the following exercises.



Exercise 1.6.1.11

Blood type $\mathbf{AB^+}$is the universal acceptor. Of the 140 people who donated at City Honors, how many had blood type $\mathbf{AB^+}$?

Exercise 1.6.1.12

Blood type $\mathbf{O^-}$is the universal donor. Anyone needing a blood transfusion can receive this blood type. How many people who donated blood during this drive had $\mathbf{O^-}$blood?

Exercise 1.6.1.13

How many people donated with a type $\boldsymbol{A}$ blood factor?

Exercise 1.6.1.14

How many people donated with a type $\boldsymbol{A}$ and type $\boldsymbol{B}$ blood factor (that is, they had type $\boldsymbol{AB}$ blood).

Exercise 1.6.1.15

How many donors were $\mathbf{O^+}$?

Exercise 1.6.1.16

How many donors were not $\mathbf{Rh^+}$?

Exercise 1.6.1.17

Opal has blood type $\mathbf{A^+}$. If she needs to have surgery that requires a blood transfusion, she can accept blood from anyone who does not have a type B blood factor. How many people donated blood during this drive at City Honors that Opal can accept?

Exercise 1.6.1.18

Find $n\left(A \cap \mathbf{Rh^+}\right)$.

Exercise 1.6.1.19

Find $n\left(A \cup \mathbf{Rh^+}\right)$.

Exercise 1.6.1.20

Find $n\left(A \cap B \cap \mathbf{Rh^-}\right)$.

For the following exercises, create a three circle Venn diagram to represent the relationship between the described sets.

Exercise 1.6.1.21

The number of elements in the universal set, U, is $n(U) = 48$. Sets A, B, and C are subsets of U : $n(A) = 23, n(B) = 25$, and $n(C) = 17$. Also, $n(A \cap B) = 15, n(B \cap C) = 12, n(C \cap A) = 11$, and $n(A \cap B \cap C) = 8$.

Exercise 1.6.1.22

The number of elements in the universal set, U, is $n(U) = 88$. Sets A, B, and C are subsets of U : $n(A) = 31, n(B) = 46$: $n(C) = 33$. Also, $n(A \cap B) = 24, n(B \cap C) = 24, n(C \cap A) = 26$, and $n(A \cap B \cap C) = 22$.

Exercise 1.6.1.23

The number of elements in the universal set, U, is $n(U) = 52$. Sets A, B, and C are subsets of U : $n(A) = 23, n(B) = 27$, and $n(C) = 29$. Also, $n(A \cap B) = 22, n(B \cap C) = 21, n(C \cap A) = 19$, and $n(A \cap B \cap C) = 18$.

Exercise 1.6.1.24

The number of elements in the universal set, U, is $n(U) = 144$. Sets A, B, and C are subsets of U : $n(A) = 36, n(B) = 64$, and $n(C) = 81$. Also, $n(A \cap B) = 26, n(B \cap C) = 61, n(C \cap A) = 29$, and $n(A \cap B \cap C) = 25$.

Exercise 1.6.1.25

The universal set, U, has a cardinality of 36 . $n(A) = 12, n(B) = 12, n(C) = 15, n(A$ and $B) = 3, n(B$ and $C) = 4, n(C$ and $A) = 5$, $n(C$ and $A) = 5$, and $n(A$ and B and $C) = 1$.

Exercise 1.6.1.26

The universal set, U, has a cardinality of $63. n(A) = 29$, $n(B) = 31, n(C) = 41, n(A \text{ and } B) = 12, n(B \text{ and } C) = 16, n(C \text{ and } A) = 18$, and $n(A \text{ and } B \text{ and } C) = 5$.

Exercise 1.6.1.27

The universal set, U, has a cardinality of 72 . $n(A) = 32, n(B) = 32, n(C) = 44, n(A \text{ and } B) = 18, n(B \text{ and } C) = 22, n(C \text{ and } A) = 26$, and $n(A \text{ and } B \text{ and } C) = 14$.

Exercise 1.6.1.28

The universal set, U, has a cardinality of 81 . $n(A) = 54, n(B) = 41, n(C) = 52, n(A \text{ and } B) = 32, n(B \text{ and } C) = 28, n(C \text{ and } A) = 30$, and $n(A \text{ and } B \text{ and } C) = 21$.

Exercise 1.6.1.29

The anime drawing club at Pratt Institute conducted a survey of its 42 members and found that 23 of them sketched with pastels, 28 used charcoal, and 17 used colored pencils. Of these, 10 club members used all three mediums, 18 used charcoal and pastels, 11 used colored pencils and charcoal, and 12 used colored pencils and pastels. The remaining club members did not use any of these three mediums.

Exercise 1.6.1.30

A new SUV is selling with three optional packages: a sport package, a tow package, and an entertainment package. A dealership gathered the following data for all 31 of these vehicles sold during the month of July. A total of 18 SUVs included the entertainment package, 11 included the tow package, and 16 included the sport package. Of these, five SUVs included all three packages, seven were sold with both the tow package and sport package, 11 were sold with the entertainment and sport package, and eight were sold with the tow package and entertainment package. The remaining SUVs sold did not include any of these optional packages.

For the following exercises, perform the set operations as indicated on the following sets: $U = \{$ red, orange, yellow, green, blue, indigo, violet $\}, A = \{$ red, yellow, blue $\}, B = \{$ orange, green, violet $\}$, and $C = \{$ red, green, indigo $\}$.

Exercise 1.6.1.31

Find $(A \cup B) \cap C$.

Exercise 1.6.1.32

Find $(A \cap C) \cup B$.

Exercise 1.6.1.33

Find $U \cap (B \cup C)$.

Exercise 1.6.1.34

Find $(B \cap A) \cap U$.

? Exercise 1.6.1.35

Find $A \cap (B \cap C)'$.

? Exercise 1.6.1.36

Find $A' \cap (B \cup C)$.

For the following exercises, perform the set operations as indicated on the following sets: $U = \{20, 21, 22, \ldots, 29\}$, $A = \{21, 24, 27\}$, $B = \{20, 22, 24, 28\}$, and $C = \{21, 23, 25, 27\}$.

? Exercise 1.6.1.37

Find A and B and C'.

? Exercise 1.6.1.38

Find A' or B or C.

? Exercise 1.6.1.39

Find (A or B) and C'.

? Exercise 1.6.1.40

Find (A or B) or C'.

? Exercise 1.6.1.41

Find (A and C) and B'.

? Exercise 1.6.1.42

Find (A **or** B)$'$ and C.

For the following exercises, use Venn diagrams to prove the following properties of sets:

? Exercise 1.6.1.43

Commutative property for the union of two sets: $A \cup B = B \cup A$.

? Exercise 1.6.1.44

Commutative property for the intersection of two sets: $A \cap B = B \cap A$.

? Exercise 1.6.1.45

Associative property for the intersection of three sets: $(A \cap B) \cap C = A \cap (B \cap C)$.

? Exercise 1.6.1.46

Associative property for the union of three sets: $A \cup (B \cup C) = (A \cup B) \cup C$.

Exercise 1.6.1.47

Distributive property for set intersection over set union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.6.1.48

Distributive property for set union over set intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

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CHAPTER OVERVIEW

2: Logic

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2.1: Introduction

What is logic? Logic is the study of reasoning, and it has applications in many fields, including philosophy, law, psychology, digital electronics, and computer science.

In law, constructing a well-reasoned, logical argument is extremely important. The main goal of arguments made by lawyers is to convince a judge and jury that their arguments are valid and well-supported by the facts of the case, so the case should be ruled in their favor. Think about Thurgood Marshall arguing for desegregation in front of the U.S. Supreme Court during the *Brown v. Board of Education of Topeka* lawsuit in 1954, or Ruth Bader Ginsburg arguing for equality in social security benefits for both men and women under the law during the mid-1970s. Both these great minds were known for the preparation and thoroughness of their logical legal arguments, which resulted in victories that advanced the causes they fought for. Thurgood Marshall and Ruth Bader Ginsburg would later become well respected justices on the U.S. Supreme Court themselves.



Figure **2.1.1**: Logic is key to a well-reasoned argument, in both math and law. (credit: modification of work "Lady Justicia holding sword and scale bronze figurine with judge hammer on wooden table" by Jernej Furman/Flickr, CC BY 2.0)

In this chapter, we will explore how to construct well-reasoned logical arguments using varying structures. Your ability to form and comprehend logical arguments is a valuable tool in many areas of life, whether you're planning a dinner date, negotiating the purchase of a new car, or persuading your boss that you deserve a raise.

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2.2: Statements and Quantifiers



Figure **2.2.1**: Construction of a logical argument, like that of a house, requires you to begin with the right parts. (credit: modification of work “Barn Raising” by Robert Stinnett/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Identify logical statements.
2. Represent statements in symbolic form.
3. Negate statements in words.
4. Negate statements symbolically.
5. Translate negations between words and symbols.
6. Express statements with quantifiers of all, some, and none.
7. Negate statements containing quantifiers of all, some, and none.

Have you ever built a club house, tree house, or fort with your friends? If so, you and your friends likely started by gathering some tools and supplies to work with, such as hammers, saws, screwdrivers, wood, nails, and screws. Hopefully, at least one member of your group had some knowledge of how to use the tools correctly and helped to direct the construction project. After all, if your house isn't built on a strong foundation, it will be weak and could possibly fall apart during the next big storm. This same foundation is important in logic.

In this section, we will begin with the parts that make up all logical arguments. The building block of any logical argument is a logical statement, or simply a statement. A logical statement has the form of a complete sentence, and it must make a claim that can be identified as being true or false.

When making arguments, sometimes people make false claims. When evaluating the strength or validity of a logical argument, you must also consider the truth values, or the identification of true or false, of all the statements used to support the argument. While a false statement is still considered a logical statement, a strong logical argument starts with true statements.



Figure **2.2.2**: Not all roses are red. (credit: “assorted pink yellow white red roses macro” by ProFlowers/Flickr, CC BY 2.0)

An example of logical statement with a false truth value is, “All roses are red.” It is a logical statement because it has the form of a complete sentence and makes a claim that can be determined to be either true or false. It is a false statement because not all roses are red: some roses are red, but there are also roses that are pink, yellow, and white. Requests, questions, or directives may be complete sentences, but they are not logical statements because they cannot be determined to be true or false. For example, suppose someone said to you, “Please, sit down over there.” This request does not make a claim and it cannot be identified as true or false; therefore, it is not a logical statement.

? Exercise 2.2.1: Identifying Logical Statements

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. Tiger Woods won the Master’s golf championship at least five times.
2. Please, sit down over there.
3. All cats dislike dogs.

Answer

1. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. As of 2021, this statement is true: Tiger Woods won the Master’s in 1997, 2001, 2002, 2005 and 2019.
2. This is not a logical statement because, although it is a complete sentence, this request does not make a claim that can be identified as being either true or false.
3. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. This statement is false because some cats do like some dogs.

✔ Your Turn 2.2.1

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. The Buffalo Bills defeated the New York Giants in Super Bowl XXV
2. Michael Jackson’s album *Thriller* was released in 1982.

3. Would you like some coffee or tea?

Representing Statements in Symbolic Form

When analyzing logical arguments that are made of multiple logical statements, symbolic form is used to reduce the amount of writing involved. Symbolic form also helps visualize the relationship between the statements in a more concise way in order to determine the strength or validity of an argument. Each logical statement is represented symbolically as a single lowercase letter, usually starting with the letter $\boldsymbol{p}$.

To begin, you will practice how to write a single logical statement in symbolic form. This skill will become more useful as you work with compound statements in later sections.

? Exercise 2.2.2: Representing Statements Using Symbolic Form

Write each of the following logical statements in symbolic form.

1. Barry Bonds holds the Major League Baseball record for total career home runs.
2. Some mammals live in the ocean.
3. Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020.

Answer

1. p : Barry Bonds holds the Major League Baseball record for total career home runs. The statement is labeled with a $\boldsymbol{p}$. Once the statement is labeled, use $\boldsymbol{p}$ as a replacement for the full written statement to write and analyze the argument symbolically.
2. q : Some mammals live in the ocean. The letter $\boldsymbol{q}$ is used to distinguish this statement from the statement in question 1, but any lower-case letter may be used.
3. r : Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020. When multiple statements are present in later sections, you will want to be sure to use a different letter for each separate logical statement.

✓ Your Turn 2.2.2

Write each of the following logical statements in symbolic form.

1. The movie *Gandhi* won the Academy Award for Best Picture in 1982.
2. Soccer is the most popular sport in the world.
3. All oranges are citrus fruits.

Who Knew?

Mathematics is not the only language to use symbols to represent thoughts or ideas. The Chinese and Japanese languages use symbols known as Hanzi and Kanji, respectively, to represent words and phrases. At one point, the American musician Prince famously changed his name to a symbol representing love.

BBC Prince Symbol Article

Negating Statements

Consider the false statement introduced earlier, “All roses are red.” If someone said to you, “All roses are red,” you might respond with, “Some roses are not red.” You could then strengthen your argument by providing additional statements, such as, “There are also white roses, yellow roses, and pink roses, to name a few.”

The negation of a logical statement has the opposite truth value of the original statement. If the original statement is false, its negation is true, and if the original statement is true, its negation is false. Most logical statements can be negated by simply adding or removing the word *not*. For example, consider the statement, “Emma Stone has green eyes.” The negation of this statement would be, “Emma Stone does not have green eyes.” The table below gives some other examples.

Table 2.2.1

Logical Statement	Negation
Gordon Ramsey is a chef.	Gordon Ramsey is not a chef.
Tony the Tiger does not have spots.	Tony the Tiger has spots.

The way you phrase your argument can impact its success. If someone presents you with a false statement, the ability to rebut that statement with its negation will provide you with the tools necessary to emphasize the correctness of your position.

? Exercise 2.2.3: Negating Logical Statements

Write the negation of each logical statement in words.

1. Michael Phelps was an Olympic swimmer.
2. Tom is a cat.
3. Jerry is not a mouse.

Answer

1. Add the word *not* to negate the affirmative statement: “Michael Phelps was not an Olympic swimmer.”
2. Add the word *not* to negate the affirmative statement: “Tom is not a cat.”
3. Remove the word *not* to negate the negative statement: “Jerry is a mouse.”

✓ Your Turn 2.2.3

Write the negation of each logical statement in words.

1. Ted Cruz was not born in Texas
2. Adele has a beautiful singing voice.
3. Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

The symbol for negation, or not, in logic is the tilde, ~. So, not p is represented as $\sim p$. To negate a statement symbolically, remove or add a tilde. The negation of not (not p) is p. Symbolically, this equation is $\sim(\sim p) = p$.

? Exercise 2.2.4: Negating Logical Statements Symbolically

Write the negation of each logical statement symbolically.

1. p: Michael Phelps was an Olympic swimmer.
2. r: Tom is not a cat.
3. $\sim q$: Jerry is not a mouse.

Answer

1. To negate an affirmative logical statement symbolically, add a tilde: $\sim p$.
2. Because the symbol for this statement is r, its negation is $\sim r$.
3. The symbol for this statement is $\sim q$, so to negate it we simply remove the ~, because $\sim(\sim q) = q$. The answer is q.

✓ Your Turn 2.2.4

Write the negation of each logical statement symbolically.

1. p: Ted Cruz was not born in Texas
2. q: Adele has a beautiful voice.
3. r: Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

Translating Negations Between Words and Symbols

In order to analyze logical arguments, it is important to be able to translate between the symbolic and written forms of logical statements.

? Exercise 2.2.5

Given the statements:

r: Elmo is a red Muppet.

p: Ketchup is not a vegetable.

1. Write the symbolic form of the statement, “Elmo is not a red Muppet.”
2. Translate the statement $\sim p$ into words.

Answer

1. “Elmo is not a red muppet” is the negation of “Elmo is a red muppet,” which is represented as r. Thus, we would write “Elmo is not a red muppet” symbolically as $\sim r$.
2. Because $p$is the symbol representing the statement, “Ketchup is not a vegetable,” $\sim p$ is equivalent to the statement, “Ketchup is a vegetable.”

✓ Your Turn 2.2.5

Given the statements:
r: Woody and Buzz Lightyear are best friends.
p: Wonder Woman is not stronger than Captain Marvel.

1. Write the symbolic form of the statement, "Wonder Woman is stronger than Captain Marvel."
2. Translate the statement r into words

Expressing Statements with Quantifiers of All, Some, or None

A quantifier is a term that expresses a numerical relationship between two sets or categories. For example, all squares are also rectangles, but only some rectangles are squares, and no squares are circles. In this example, *all*, *some*, and *none* are quantifiers. In a logical argument, the logical statements made to support the argument are called premises, and the judgment made based on the premises is called the conclusion. Logical arguments that begin with specific premises and attempt to draw more general conclusions are called inductive arguments.

Consider, for example, a parent walking with their three-year-old child. The child sees a cardinal fly by and points it out. As they continue on their walk, the child notices a robin land on top of a tree and a duck flying across to land on a pond. The child recognizes that cardinals, robins, and ducks are all birds, then excitedly declares, "All birds fly!" The child has just made an inductive argument. They noticed that three different specific types of birds all fly, then synthesized this information to draw the more general conclusion that all birds can fly. In this case, the child's conclusion is false.

The specific premises of the child's argument can be paraphrased by the following statements:

- Premise: Cardinals are birds that fly.
- Premise: Robins are birds that fly.
- Premise: Ducks are birds that fly.

The general conclusion is: “All birds fly!”

All inductive arguments should include at least three specific premises to establish a pattern that supports the general conclusion. To counter the conclusion of an inductive argument, it is necessary to provide a counter example. The parent can tell the child about penguins or emus to explain why that conclusion is false.

On the other hand, it is usually impossible to prove that an inductive argument is true. So, inductive arguments are considered either strong or weak. Deciding whether an inductive argument is strong or weak is highly subjective and often determined by the background knowledge of the person making the judgment. Most hypotheses put forth by scientists using what is called the "scientific method" to conduct experiments are based on inductive reasoning.

In the following example, we will use quantifiers to express the conclusion of a few inductive arguments.

? Exercise 2.2.6: Drawing General Conclusions to Inductive Arguments Using Quantifiers

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. Squares and rectangles have four sides. A square is a parallelogram, and a rectangle is a parallelogram. What conclusion can be drawn from these premises?

2. $1+2=3, 6+7=13$, and $23+24=47$. Of these, 1 and 2,6 and 7 , and 23 and 24 are consecutive integers; 3,13 , and 47 are odd numbers. What conclusion can be drawn from these premises?

3. Sea urchins live in the ocean, and they do not breathe air. Sharks live in the ocean, and they do not breathe air. Eels live in the ocean, and they do not breath air. What conclusion can be drawn from these premises?

Answer

1. The conclusion you would likely come to here is "Some four-sided figures are parallelograms." However, it would be incorrect to say that all four-sided figures are parallelograms because there are some four-sided figures, such as trapezoids, that are not parallelograms. This is a false conclusion.
2. From these premises, you may draw the conclusion "All sums of two consecutive counting numbers result in an odd number." Most inductive arguments cannot be proven true, but several mathematical properties can be. If we let n n represent our first counting number, then $n + 1$ would be the next counting number and $n + (n + 1) = 2 n + 1$. Because $2n$ is divisible by two, it is an even number, and if you add one to any even number the result is always an odd number. Thus, the conclusion is true!
3. Based on the premises provided, with no additional knowledge about whales or dolphins, you might conclude "No creatures that live in the ocean breathe air." Even though this conclusion is false, it still follows from the known premises and thus is a logical conclusion based on the evidence presented. Alternatively, you could conclude "Some creatures that live in the ocean do not breathe air." The quantifier you choose to write your conclusion with may vary from another person's based on how persuasive the argument is. There may be multiple acceptable answers.

✓ Your Turn 2.2.6

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. $1+2=3$, $5+6=11$, and $14+15=29$. Of these, 1 and 2 are consecutive integers, 5 and 6 are consecutive integers, and 14 and 15 are consecutive integers. Also, their sums, 3, 11, and 29 are all prime numbers. Prime numbers are positive integers greater than one that are only divisible by one and the number itself. What conclusion can you draw from these premises?
2. A robin is a bird that lays blue eggs. A chicken is a bird that typically lays white and brown eggs. An ostrich is a bird that lays exceptionally large eggs. If a bird lays eggs, then they do not give live birth to their young. What conclusion can you draw from these premises?
3. All parallelograms have four sides. All rectangles are parallelograms. All squares are rectangles. What additional conclusion can you make about squares from these premises?

Checkpoint

It is not possible to prove definitively that an inductive argument is true or false in most cases.

Negating Statements Containing Quantifiers

Recall that the negation of a statement will have the opposite truth value of the original statement. There are four basic forms that logical statements with quantifiers take on.

- All A are B.
- Some A are B.
- No A are B.
- Some A are not B.

The negation of logical statements that use the quantifiers *all*, *some*, or *none* is a little more complicated than just adding or removing the word *not*.

For example, consider the logical statement, “All oranges are citrus fruits.” This statement expresses as a subset relationship. The set of oranges is a subset of the set of citrus fruit. This means that there are no oranges that are outside the set of citrus fruit. The negation of this statement would have to break the subset relationship. To do this, you could say, “At least one orange is not a citrus fruit.” Or, more concisely, “Some oranges are not citrus fruit.” It is tempting to say "No oranges are citrus fruit," but that would be incorrect. Such a statement would go beyond breaking the subset relationship, to stating that the two sets have nothing in common. The negation of "semanticsA is a subset of B" would be to state that "A is not a subset of B," as depicted by the Venn diagram in Figure **2.2.3**.

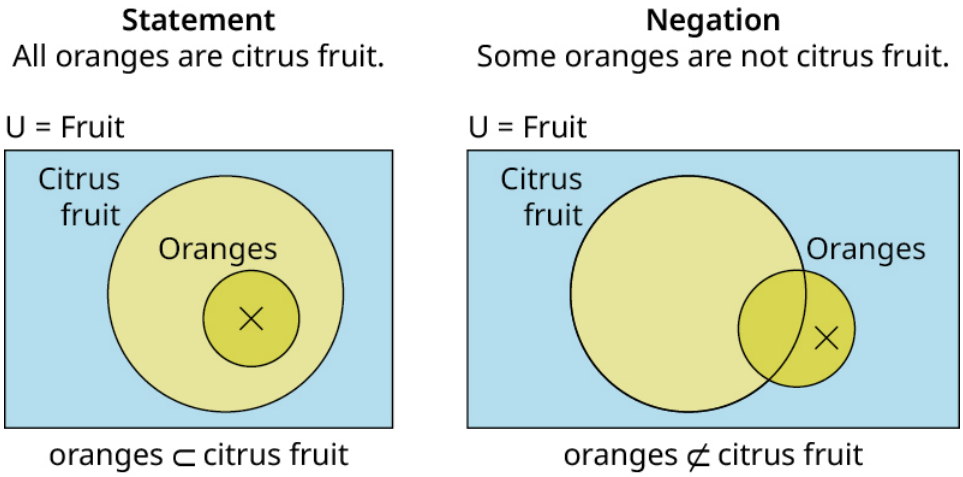


Figure **2.2.3**: The statement, “All oranges are citrus fruit,” is true, so its negation, “Some oranges are not citrus fruit,” is false.

Now, consider the statement, “No apples are oranges.” This statement indicates that the set of apples is disjointed from the set of oranges. The negation must state that the two are not disjoint sets, or that the two sets have a least one member in common. Their intersection is not empty. TThe negation of the statement, “A intersection B is the empty set,” is the statement that " A intersection B is not empty," as depicted in the Venn diagram in Figure **2.2.4**.

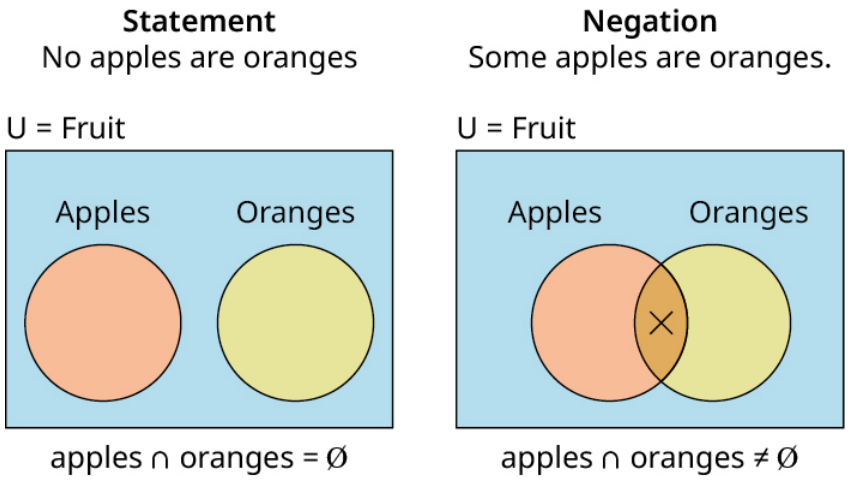


Figure 2.2.4:The negation of the true statement “No apples are oranges,” is the false statement, “Some apples are oranges.”

Table 2.2.2 summarizes the four different forms of logical statements involving quantifiers and the forms of their associated negations, as well as the meanings of the relationships between the two categories or sets A and B.

Table 2.2.2

Logical Statements with Quantifiers	Negation of Logical Statements w/Quantifiers
Form: All A are B. Means: A is a subset of B, $A \subset B$. All zebras have stripes. (True)	Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some zebras do not have stripes. (False)
Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \emptyset$. Some fish are sharks. (True)	Form: No A are B. Means: A intersection B is empty, $A \cap B = \emptyset$. No fish are sharks. (False)
Form: No A are B. Means: A intersection B is empty, $A \cap B = \emptyset$. No trees are evergreens. (False)	Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \emptyset$. Some trees are evergreens. (True)
Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some horses are not mustangs. (True)	Form: All A are B. Means: A is a subset of B, $A \subset B$. All horses are mustangs. (False)

We covered sets in great detail in Chapter 1. To review, " A is a subset of B" means that every member of set A is also a member of set B. The intersection of two sets A and B is the set of all elements that they share in common. If A intersection B is the empty set, then sets A and B do not have any elements in common. The two sets do not overlap. They are disjoint.



Exercise 2.2.7: Negating Statements Containing Quantifiers *All*, *Some*, or *None*

Given the statements:

p: All leopards have spots.
r: Some apples are red.
s: No lemons are sweet.

Write each of the following symbolic statements in words.

1. $\sim p$
2. $\sim r$
3. $\sim s$

Answer

1. The statement "All leopards have spots" is p and has the form "All A are B." According to Table 2.2.2, the negation will have the form "Some A are not B." The negation of p is the statement, "Some leopards do not have spots."
2. The statement "Some apples are red" has the form "Some A are B." This indicates that the categories A and B overlap or intersect. According to Table 2.2.2, the negation will have the form, "No A are B," indicating that A and B do not intersect. This results in the opposite truth value of the original statement, so the negation of "Some apples are red" is the statement: "No apples are red."
3. Because s is the statement: "No lemons are sweet," s is asserting that the set of lemons does not intersect with the set of sweet things. The negation of $s, \sim s$, must make the opposite claim. It must indicate that the set of lemons intersects with the set of sweet things. This means at least one lemon must be sweet. The statement, "Some lemons are sweet" is $\sim s$. The negation of the statement, "No A are B," is the statement, "Some A are B," as indicated in Table 2.2.2.

Your Turn 2.2.7

Given the statements:
$\sim p$: Some apples are not sweet.
r : No triangles are squares.
s : Some vegetables are green. Write each of the following symbolic statements in words.

1. p
2. $\sim r$
3. $\sim s$

Check Your Understanding

1. A _____ _____ is a complete sentence that makes a claim that may be either true or false.
2. The _____ of a logical statement has the opposite truth value of the original statement.
3. If p represents the logical statement, “Marigolds are yellow flowers,” then _____ represents the statement, “Marigolds are not yellow flowers.”
4. The statement (p) has the same truth value as the statement _____.
5. The logical statements used to support the conclusion of an argument are called _____.
6. _____ arguments attempt to draw a general conclusion from specific premises.
7. All, some, and none are examples of _____, words that assign a numerical relationship between two or more groups.
8. The negation of the statement, “All giraffes are tall,” is _____.

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2.2.1: Exercises

For the following exercises, determine whether the sentence represents a logical statement. If it is a logical statement, determine whether it is true or false.

Exercise 2.2.1.1

A loan used to finance a house is called a mortgage.

Exercise 2.2.1.2

All odd numbers are divisible by 2.

Exercise 2.2.1.3

Please, bring me that notebook.

Exercise 2.2.1.4

Robot, what's your function?

Exercise 2.2.1.5

In English, a conjunction is a word that connects two phrases or parts of a sentence together.

Exercise 2.2.1.6

$8 - 3 = 5$.

Exercise 2.2.1.7

$7 + 3 = 11$.

Exercise 2.2.1.8

What is 7 plus 3?

For the following exercises, write each statement in symbolic form.

Exercise 2.2.1.9

Grammy award winning singer, Lady Gaga, was not born in Houston, Texas.

Exercise 2.2.1.10

Bruno Mars performed during the Super Bowl halftime show twice.

Exercise 2.2.1.11

Coco Chanel said, "The most courageous act is still to think for yourself. Aloud."

? Exercise 2.2.1.12

Bruce Wayne is not Superman.

For the following exercises, write the negation of each statement in words.

? Exercise 2.2.1.13

Bozo is not a clown.

? Exercise 2.2.1.14

Ash is Pikachu's trainer and friend.

? Exercise 2.2.1.15

Vanilla is the most popular flavor of ice cream.

? Exercise 2.2.1.16

Smaug is a fire breathing dragon.

? Exercise 2.2.1.17

Elephant and Piggy are not best friends.

? Exercise 2.2.1.18

Some dogs like cats.

? Exercise 2.2.1.19

Some donuts are not round.

? Exercise 2.2.1.20

All cars have wheels.

? Exercise 2.2.1.21

No circles are squares.

? Exercise 2.2.1.22

Nature's first green is not gold.

? Exercise 2.2.1.23

The ancient Greek philosopher Plato said, "The greatest wealth is to live content with little."

Exercise 2.2.1.24

All trees produce nuts.

For the following exercises, write the negation of each statement symbolically and in words.

Exercise 2.2.1.25

p: Their hair is red.

Exercise 2.2.1.26

q: My favorite superhero does not wear a cape.

Exercise 2.2.1.27

s: All wolves howl at the moon.

Exercise 2.2.1.28

t: Nobody messes with Texas.

Exercise 2.2.1.29

u: I do not love New York.

Exercise 2.2.1.30

v: Some cats are not tigers.

Exercise 2.2.1.31

q: No squares are not parallelograms.

Exercise 2.2.1.32

p: The President does not like broccoli.

For the following exercises, write each of the following symbolic statements in words.

Exercise 2.2.1.33

Given: p: Kermit is a green frog; translate p into words.

Exercise 2.2.1.34

Given: r: Mick Jagger is not the lead singer for The Rolling Stones; translate r into words.

Exercise 2.2.1.35

Given: q: All dogs go to heaven; translate q into words.

Exercise 2.2.1.36

Given: s: Some pizza does not come with pepperoni on it; translate s into words.

Exercise 2.2.1.37

Given: p: No pizza comes with pineapple on it; translate (p) into words.

Exercise 2.2.1.38

Given: r: Not all roses are red; translate (r) into words.

Exercise 2.2.1.39

Given: t: Thelonious Monk is not a famous jazz pianist; translate (t) into words.

Exercise 2.2.1.40

Given: v: Not all violets are blue; translate (v) into words.

For the following exercises, draw a logical conclusion from the premises that includes one of the following quantifiers: all, some, or none.

Exercise 2.2.1.41

The Ford Motor Company builds cars in Michigan. General Motors builds cars in Michigan. Chrysler builds cars in Michigan. What conclusion can be drawn from these premises?

Exercise 2.2.1.42

Michelangelo Buonarroti was a great Renaissance artist from Italy. Raphael Sanzio was a great Renaissance artist from Italy. Sandro Botticelli was a great Renaissance artist from Italy. What conclusion can you draw from these premises?

Exercise 2.2.1.43

Four is an even number and it is divisible by 2. Six is an even number and it is divisible by 2. Eight is an even number and it is divisible by 2. What conclusion can you draw from these premises?

Exercise 2.2.1.44

Three is an odd number and it is not divisible by 2. Seven is an odd number and it is not divisible by 2. Twenty-one is an odd number and it is not divisible by 2. What conclusion can you draw from these premises?

Exercise 2.2.1.45

The odd number 5 is not divisible by 3. The odd number 7 is not divisible by 3. The odd number 29 is not divisible by 3. What conclusion can you draw from these premises?

Exercise 2.2.1.46

Penguins are flightless birds. Emus are flightless birds. Ostriches are flightless birds. What conclusion can you draw from these premises?

Exercise 2.2.1.47

Plants need water to survive. Animals need water to survive. Bacteria need water to survive. What conclusion can you draw from these premises?

Exercise 2.2.1.48

A chocolate chip cookie is not sour. An oatmeal cookie is not sour. An Oreo cookie is not sour. What conclusion can you draw from these premises?

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2.3: Compound Statements



2.3.1:A person seeking their driver's license must pass two tests. A compound statement can be used to explain performance on both tests at once. (credit: modification of work “Drivers License -Teen driver” by State Farm/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Translate compound statements into symbolic form.
2. Translate compound statements in symbolic form with parentheses into words.
3. Apply the dominance of connectives.

Suppose your friend is trying to get a license to drive. In most places, they will need to pass some form of written test proving their knowledge of the laws and rules for driving safely. After passing the written test, your friend must also pass a road test to demonstrate that they can perform the physical task of driving safely within the rules of the law.

Consider the statement: "My friend passed the written test, but they did not pass the road test." This is an example of a compound statement, a statement formed by using a connective to join two independent clauses or logical statements. The statement, “My friend passed the written test,” is an independent clause because it is a complete thought or idea that can stand on its own. The second independent clause in this compound statement is, “My friend did not pass the road test.” The word "but" is the connective used to join these two statements together, forming a compound statement. So, did your friend acquire their driving license?.

This section introduces common logical connectives and their symbols, and allows you to practice translating compound statements between words and symbols. It also explores the order of operations, or dominance of connectives, when a single compound statement uses multiple connectives.

Common Logical Connectives

Understanding the following logical connectives, along with their properties, symbols, and names, will be key to applying the topics presented in this chapter. The chapter will discuss each connective introduced here in more detail.

The joining of two logical statements with the word "and" or "but" forms a compound statement called a conjunction. In logic, for a conjunction to be true, all the independent logical statements that make it up must be true. The symbol for a conjunction is $\wedge$. Consider the compound statement, “Derek Jeter played professional baseball for the New York Yankees, and he was a shortstop.” If p represents the statement, “Derrick Jeter played professional baseball for the New York Yankees,” and if q represents the statement, “Derrick Jeter was a short stop,” then the conjunction will be written symbolically as $p \wedge q$.

The joining of two logical statements with the word “or” forms a compound statement called a disjunction. Unless otherwise specified, a disjunction is an inclusive *or* statement, which means the compound statement formed by joining two independent clauses with the word *or* will be true if a least one of the clauses is true. Consider the compound statement, "The office manager ordered cake for for an employee’s birthday or they ordered ice cream.” This is a disjunction because it combines the independent clause, “The office manager ordered cake for an employee’s birthday,” with the independent clause, “The office manager ordered

ice cream," using the connective, *or*. This disjunction is true if the office manager ordered only cake, only ice cream, or they ordered both cake and ice cream. Inclusive *or* means you can have one, or the other, or both!

Joining two logical statements with the word *implies*, or using the phrase "if *first statement*, then *second statement,"* is called a conditional or *implication*. The clause associated with the "if" statement is also called the hypothesis or *antecedent*, while the clause following the "then" statement or the word *implies* is called the **conclusion** or *consequent*. The conditional statement is like a one-way contract or promise. The only time the conditional statement is false, is if the hypothesis is true and the conclusion is false. Consider the following conditional statement, "If Pedro does his homework, then he can play video games." The hypothesis/antecedent is the statement following the word *if*, which is "Pedro does/did his homework." The conclusion/consequent is the statement following the word *then*, which is "Pedro can play his video games."

Joining two logical statements with the connective phrase "if and only if" is called a biconditional. The connective phrase "if and only if" is represented by the symbol, $\leftrightarrow$. In the biconditional statement, $p \leftrightarrow q$, p is called the hypothesis or antecedent and q is called the conclusion or consequent. For a biconditional statement to be true, the truth values of p and q must match. They must both be true, or both be false.

The table below lists the most common connectives used in logic, along with their symbolic forms, and the common names used to describe each connective.

Table **2.3.1**

Connective	Symbol	Name
and but	$\wedge$	conjunction
or	$\vee$	disjunction, inclusive or
not	~	negation
if ..., then implies	$\rightarrow$	conditional, implication
if and only if	$\leftrightarrow$	biconditional

Checkpoint

These connectives, along with their names, symbols, and properties, will be used throughout this chapter and should be memorized.

Exercise 2.3.1

Associate Connectives with Symbols and Names

For each of the following connectives, write its name and associated symbol.

1. or
2. implies
3. but

Answer

A compound statement formed with the connective word *or* is called a disjunction, and it is represented by the $\vee$ symbol.

A compound statement formed with the connective word *implies* or phrase "*if …, then*" is called a conditional statement or implication and is represented by the $\rightarrow$ symbol.

A compound statement formed with the connective words *but* or *and* is called a conjunction, and it is represented by the $\wedge$ symbol.

Your Turn 2.3.1

For each connective write its name and associated symbol.

1. not
2. and
3. if and only if

Translating Compound Statements to Symbolic Form

To translate a compound statement into symbolic form, we take the following steps.

1. Identify and label all independent affirmative logical statements with a lower case letter, such as p, q, or r.
2. Identify and label any negative logical statements with a lowercase letter preceded by the negation symbol, such as $\sim p$, $\sim q$, or $\sim r$.
3. Replace the connective words with the symbols that represent them, such as $\wedge, \vee, \rightarrow,$ or $\leftrightarrow$.

Consider the previous example of your friend trying to get their driver's license. Your friend passed the written test, but they did not pass the road test. Let p represent the statement, "My friend passed the written test." And, let $\sim q$ represent the statement, "My friend did not pass the road test." Because the connective *but* is logically equivalent to the word *and*, the symbol for *but* is the same as the symbol for *and*; replace *but* with the symbol $\wedge$. The compound statement is symbolically written as: $p \wedge \sim q$. My friend passed the written test, but they did not pass the road test.

Checkpoint

When translating English statements into symbolic form, do not assume that every connective "and" means you are dealing with a compound statement. The statement, "The stripes on the U.S. flag are red and white," is a simple statement. The word "white" doesn't express a complete thought, so it is not an independent clause and does not get its own symbol. This statement should be represented by a single letter, such as s: The stripes on the U.S. flag are red and white.

Exercise 2.3.2: Translating Compound Statements into Symbolic Form

Let p represent the statement, "It is a warm sunny day," and let q represent the statement, "the family will go to the beach." Write the symbolic form of each of the following compound statements.

1. If it is a warm sunny day, then the family will go to the beach.
2. The family will go to the beach, and it is a warm sunny day.
3. The family will not go to the beach if and only if it is not a warm sunny day.
4. The family not go to the beach, or it is a warm sunny day.

Answer

1. Replace "it is a warm sunny day" with p. Replace "the family will go to the beach." with q. Next. Next, because the connective is if ..., then place the conditional symbol, $\rightarrow$, between p and q. The compound statement is written symbolically as: $p \rightarrow q$.
2. Replace "The family will go to the beach" with q. Replace "it is a warm sunny day." with p. Next, because the connective is and, place the $\wedge$ symbol between q and p. The compound statement is written symbolically as: $q \wedge p$.
3. Replace "The family will not go to the beach. with $\sim q$. Replace "it is not a warm sunny day" with $\sim p$. Next, because the connective is or, if and only if, place the biconditional symbol, $\leftrightarrow$ between $\sim q$ and $\sim p$. The compound statement is written symbolically as: $\sim q \leftrightarrow \sim p$.
4. Replace "The family will not go to the beach" with $\sim q$. Replace "it is a warm sunny day" with p. Next, because the connective is or, place the $\vee$ symbol between $\sim q$ and p. The compound statement is written symbolically as: $\sim q \vee p$.

Your Turn 2.3.2

Let p represent the statement, "Last night it snowed," and let q represent the statement, "Today we will go skiing." Write the symbolic form of each of the following compound statements:

1. Today we will go skiing, but last night it did not snow.
2. Today we will go skiing if and only if it snowed last night.
3. Last night is snowed or today we will not go skiing.
4. If it snowed last night, then today we will go skiing.

Translating Compound Statements in Symbolic Form with Parentheses into Words

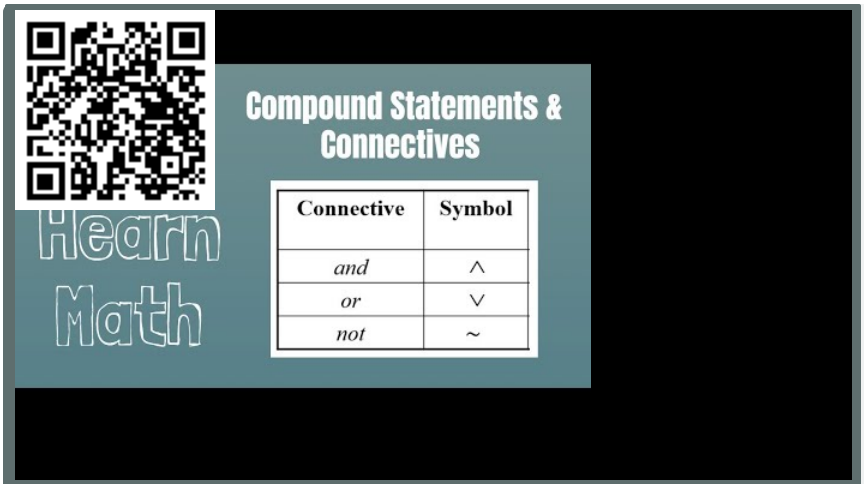
When parentheses are written in a logical argument, they group a compound statement together just like when calculating numerical or algebraic expressions. Any statement in parentheses should be treated as a single component of the expression. If multiple parentheses are present, work with the inner most parentheses first.

Consider your friend's struggles to get their license to drive. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." The statement $(p \wedge q) \rightarrow r$ can be translated into words as follows: the statement $p \wedge q$ is grouped together to form the hypothesis of the conditional statement and r is the conclusion. The conditional statement has the form "if $p \wedge q$, then r." Therefore, the written form of this statement is: "If my friend passed the written test and they passed the road test, then my friend received a driver's license."

Sometimes a compound statement within parentheses may need to be negated as a group. To accomplish this, add the phrase, "it is not the case that" before the translation of the phrase in parentheses. For example, using p, q, and r of your friend obtaining a license, let's translate the statement $\sim (p \wedge q) \rightarrow \sim r$ into words.
In this case, the hypothesis of the conditional statement is $\sim (p \wedge q)$ and the conclusion is $\sim r$. To negate the hypothesis, add the phrase "it is not the case" before translating what is in parentheses. The translation of the hypothesis is the sentence, "It is not the case that my friend passed the written test and they passed the road test," and the translation of the conclusion is, "My friend did not receive a driver's license." So, a translation of the complete conditional statement, $\sim (p \wedge q) \rightarrow \sim r$ is: "If it is not the case that my friend passed the written test and the road test, then my friend did not receive a driver's license."

Checkpoint

It is acceptable to interchange proper names with pronouns and remove repeated phrases to make the written statement more readable, as long the meaning of the logical statement is not changed.



Logic Part 1B: Compound Statements, Connectives and Symbols

? Exercise 2.3.3

Let p represent the statement, "My child finished their homework," let q represent the statement, "My child cleaned her room," let r represent the statement, "My child played video games," and let s represent the statement, "My child streamed a movie." Translate each of the following symbolic statements into words.

1. $\sim (p \wedge q)$
2. $(p \wedge q) \rightarrow (r \vee s)$
3. $\sim (r \vee s) \leftrightarrow \sim (p \wedge q)$

Answer

1. Replace $\sim$ with "It is not the case," and $\wedge$ with "and." One possible translation is: "It is not the case that my child finished their homework and cleaned their room."
2. The hypothesis of the conditional statement is, "My child finished their homework and cleaned their room." The conclusion of the conditional statement is, "My child played video games or streamed a movie." One possible translation of the entire statement is: "If my child finished their homework and cleaned their room, then they played video games or streamed a movie."
3. The hypothesis of the biconditional statement is $\sim (r \vee s)$ and is written in words as: "It is not the case that my child played video games or streamed a movie." The conclusion of the biconditional statement is $\sim (p \wedge q)$, which translates to: "It is not the case that my child finished their homework and cleaned their room." Because the biconditional, $\leftrightarrow$ translates to if and only if, one possible translation of the statement is: "It is not the case that my child played video games or streamed a movie if and only if it is not the case that my child finished their homework and cleaned their room."

✓ Your Turn 2.3.3

Let p represent the statement, "My roommates ordered pizza," let q represent the statement, "I ordered wings," and let r be the statement, "Our friends came over to watch the game." Translate the following statements into words.

1. $\sim r \rightarrow (p \vee q)$
2. $(p \wedge q) \rightarrow r$
3. $\sim (p \vee r)$

The Dominance of Connectives

The order of operations for working with algebraic and arithmetic expressions provides a set of rules that allow consistent results. For example, if you were presented with the problem $1 + 3 \times 2$, and you were not familiar with the order of operation, you might assume that you calculate the problem from left to right. If you did so, you would add 1 and 3 to get 4, and then multiply this answer by 2 to get 8, resulting in an incorrect answer. Try inputting this expression into a scientific calculator. If you do, the calculator should return a value of 7, not 8.

Use the Scientific Calculator found under the Resources tab to the far left

The order of operations for algebraic and arithmetic operations states that all multiplication must be applied prior to any addition. Parentheses are used to indicate which operation—addition or multiplication—should be done first. Adding parentheses can change and/or clarify the order. The parentheses in the expression $1 + (3 \times 2)$ indicate that 3 should be multiplied by 2 to get 6, and then 1 should be added to 6 to get 7: $1 + (3 \times 2) = 7$.

As with algebraic expressions, there is a set of rules that must be applied to compound logical statements in order to evaluate them with consistent results. This set of rules is called the dominance of connectives. When evaluating compound logical statements, connectives are evaluated from least dominant to most dominant as follows:

1. Parentheses are the least dominant connective. So, any expression inside parentheses must be evaluated first. Add as many parentheses as needed to any statement to specify the order to evaluate each connective.
2. Next, we evaluate negations.

3. Then, we evaluate conjunctions and disjunctions from left to right, because they have equal dominance.
4. After evaluating all conjunctions and disjunctions, we evaluate conditionals.
5. Lastly, we evaluate the most dominant connective, the biconditional. If a statement includes multiple connectives of equal dominance, then we will evaluate them from left to right.

See Figure **2.3.2** for a visual breakdown of the dominance of connectives.

Dominance	Connective	Symbol	Evaluate
Least Dominant	Parentheses	( )	First
↓	Negation	~	↓
↓	Disjunction/Conjunction	∨, ∧	Left to right or add parentheses to specify order because or/and have equal dominance.
↓	Conditional	→	↓
Most Dominant	Biconditional	↔	Last

Figure **2.3.2**

Let's revisit your friend's struggles to get their driver's license. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." Let's use the dominance of connectives to determine how the compound statement $p \wedge \sim q \rightarrow r$ should be evaluated.

Step 1: There are no parentheses, which is least dominant of all connectives, so we can skip over that.

Step 2: Because negation is the next least dominant, we should evaluate $\sim q$ first. We could add parentheses to the statement to make it clearer which connecting needs to be evaluated first: $p \wedge \sim q \rightarrow r$ is equivalent to $p \wedge (\sim q) \rightarrow r$.

Step 3: Next, we evaluate the conjunction, $\wedge$. $p \wedge (\sim q) \rightarrow r$ is equivalent to $(p \wedge (\sim q)) \rightarrow r$.

Step 4: Finally, we evaluate the conditional, $\rightarrow$, as this is the most dominant connective in this compound statement.

Who Knew?

When using spreadsheet applications, like Microsoft Excel or Google Sheets, have you noticed that core functions, such as sum, average, or rate, have the same name and syntax for use? This is not a coincidence. Various standards organizations have developed requirements that software developers must implement to meet the conditions set by governments and large customers worldwide. The OpenDocument Format OASIS Standard enables transferring data between different office productivity applications and was approved by the International Standards Organization (ISO) and IEC on May 6, 2006.

Prior to adopting these standards, a government entity, business, or individual could lose access to their own data simply because it was saved in a format no longer supported by a proprietary software product—even data they were required by law to preserve, or data they simply wished to maintain access to.

Just as rules for applying the dominance of connectives help maintain understanding and consistency when writing and interpreting compound logical statements and arguments, the standards adopted for database software maintain global integrity and accessibility across platforms and user bases.

Exercise 2.3.4

For each of the following compound logical statements, add parentheses to indicate the order to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \wedge \sim q \vee r$
2. $q \rightarrow \sim p \wedge r$
3. $\sim (p \vee q) \leftrightarrow \sim p \wedge \sim q$

Answer

1. Because negation is the least dominant connective, we evaluate it first: $p \wedge (\sim q) \vee r$. Because conjunction and disjunction have the same dominance, we evaluate them left to right. So, we evaluate the conjunction next, as indicated by the additional set of parentheses: $(p \wedge (\sim q)) \vee r$. The only remaining connective is the disjunction, so it is evaluated last, as indicated by the third set of parentheses. The complete solution is: $((p \wedge (\sim q)) \vee r)$.
2. Negation has the lowest dominance, so it is evaluated first: $q \rightarrow (\sim p) \wedge r$. The remaining connectives are the conditional and the conjunction. Because conjunction has a lower precedence than the conditional, it is evaluated next, as indicated by the second set of parentheses: $q \rightarrow ((\sim p) \wedge r)$. The last step is to evaluate the conditional, as indicated by the third set of parentheses: $(q \rightarrow ((\sim p) \wedge r))$.
3. This statement is known as De Morgan's Law for the negation of a disjunction. It is always true. Section 2.6 of this chapter will explore De Morgan's Laws in more detail.
   - First, we evaluate the negations on the right side of the biconditional prior to the conjunction.
   - Then, we evaluate the disjunction on the left side of the biconditional, followed by the negation of the disjunction on the left side.
   - Lastly, after completely evaluating each side of the biconditional, we evaluate the biconditional. It does not matter which side you begin with.
   The final solution is: $(\sim (p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$.

Example 2.3.4

For each of the following compound logical statements, add parentheses to indicate the order in which to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \vee q \wedge \sim r$
2. $\sim p \rightarrow q \vee r$
3. $\sim p \vee \sim q \leftrightarrow \sim (p \wedge q)$

WORK IT OUT: Logic Terms and Properties – Matching Game

Materials: For every group of four students, include 30 cards (game, trading, or playing cards), 30 individual clear plastic gaming card sleeves, and 30 card-size pieces of paper. Prepare a list of 60 questions and answers ahead of time related to definitions and problems in Statements and Quantifiers and Compound Statements. Provide each group of four students with 20 questions and their associated answers. Instruct each group to select 15 of the 20 questions, and then, for each problem selected, create one card with the question and one card with the answer. Instruct the groups to then place each problem and answer in a separate card sleeve. Once they create 15 problem cards and 15 answer cards, have students shuffle the set of cards.

To play the game, groups should exchange their card decks to ensure no team begins playing with the deck that they created. Split each four-person group into teams of two students. After shuffling the cards, one team lays cards face down on their desk in a five-by-six grid. The other team will go first.

Each player will have a turn to try matching the question with the correct answer by flipping two cards to the face up position. If a team makes a match, they get to flip another two cards; if they do not get a match, they flip the cards face down and it is the other team's turn to flip over two cards. The game continues in this manner until teams match all question cards with their corresponding answer cards. The team with the most set of matching cards wins.

In the first module of this chapter, we evaluated the truth value of negations. In the following modules, we will discuss how to evaluate conjunctions, disjunctions, conditionals, and biconditionals, and then evaluate compound logical statements using truth tables and our knowledge of the dominance of connectives.

Check Your Understanding

1. A _____ _____ is a logical statement formed by combining two or more statements with connecting words, such as and, or, but, not, and if …, then.
2. A _____ is a word or symbol used to join two or more logical statements together to form a compound statement.
3. The most dominant connective is the _____.
4. _____ are used to specify which logical connective should be evaluated first when evaluating a compound statement.
5. Both _____ and _____ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

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2.3.1: New Page

For the following exercises, translate each compound statement into symbolic form. Given p: "Layla has two weeks for vacation," q: "Marcus is Layla's friend," r: "Layla will travel to Paris, France," and s: "Layla and Marcus will travel together to Niagara Falls, Ontario."

Exercise 2.3.1.1

If Layla has two weeks for vacation, then she will travel to Paris, France.

Exercise 2.3.1.2

Layla and Marcus will travel together to Niagara Falls, Ontario or Layla will travel to Paris, France.

Exercise 2.3.1.3

If Marcus is not Layla's friend, then they will not travel to Niagara Falls, Ontario together."

Exercise 2.3.1.4

Layla and Marcus will travel to Niagara Falls, Ontario together if and only if Layla and Marcus are friends.

Exercise 2.3.1.5

If Layla does not have two weeks for vacation and Marcus is Layla's friend, then Marcus and Layla will travel together to Niagara Falls, Ontario.

Exercise 2.3.1.6

If Layla has two weeks for vacation and Marcus is not her friend, then she will travel to Paris, France.

For the following exercises, translate each compound statement into symbolic form. Given p: "Tom is a cat," q: "Jerry is a mouse," r: "Spike is a dog," s: "Tom chases Jerry," and t: "Spike catches Tom."

Exercise 2.3.1.7

Jerry is a mouse and Tom is a cat.

Exercise 2.3.1.8

If Tom chases Jerry, then Spike will catch Tom.

Exercise 2.3.1.9

If Spike does not catch Tom, then Tom did not chase Jerry.

Exercise 2.3.1.10

Tom is a cat and Spike is a dog, or Jerry is not a Mouse.

Exercise 2.3.1.11

It is not the case that Tom is not a cat and Jerry is not a mouse.

Exercise 2.3.1.12

Spike is not a dog and Jerry is a mouse if and only if Tom chases Jerry, but Spike does not catch Tom.

For the following exercises, translate the symbolic form of each compound statement into words. Given p: "Tracy Chapman plays guitar," q: "Joan Jett plays guitar," r: "All rock bands include guitarists," and s: "Elton John plays the piano."

Exercise 2.3.1.13

$p \vee r$

Exercise 2.3.1.14

$\sim s \rightarrow \sim q$

Exercise 2.3.1.15

$(p \wedge q) \leftrightarrow r$

Exercise 2.3.1.16

$\sim r \wedge (q \vee s)$

Exercise 2.3.1.17

$\sim (p \wedge \sim q)$

Exercise 2.3.1.18

$(q \rightarrow \sim r) \leftrightarrow (\sim p \vee \sim r)$

For the following exercises, translate the symbolic form of each compound statement into words. Given p: "The median is the middle number," q: "The mode is the most frequent number," r: "The mean is the average number," s: "The median, mean, and mode are equal," and t: "The data set is symmetric."

Exercise 2.3.1.19

$t \rightarrow s$

Exercise 2.3.1.20

$p \wedge (q \wedge r)$

Exercise 2.3.1.21

$\sim t \rightarrow \sim s$

Exercise 2.3.1.22

$(r \wedge p) \leftrightarrow q$

Exercise 2.3.1.23

$(t \rightarrow \sim q) \vee (r \rightarrow s)$

Exercise 2.3.1.24

$\sim (q \vee r) \rightarrow t$

For the following exercises, apply the proper dominance of connectives by adding parentheses to indicate the order in which the statement must be evaluated.

Exercise 2.3.1.25

$p \rightarrow q \vee r$

Exercise 2.3.1.26

$p \wedge q \leftrightarrow \sim r$

Exercise 2.3.1.27

$p \vee r \vee \sim q$

Exercise 2.3.1.28

$p \wedge \sim q \wedge r$

Exercise 2.3.1.29

$p \wedge r \vee s \wedge t$

Exercise 2.3.1.30

$q \rightarrow \sim r \leftrightarrow \sim p \vee \sim r$

Exercise 2.3.1.31

$p \rightarrow r \vee s \leftrightarrow \sim t$

Exercise 2.3.1.32

$\sim (t \wedge s) \vee (p \rightarrow q) \wedge \sim r$

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2.4: Constructing Truth Tables



Figure 2.4.1: Just like solving a puzzle, a computer programmer must consider all potential solutions in order to account for each possible outcome. (credit: modification of work “Deadline Xmas 2010” by Allan Henderson/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Interpret and apply negations, conjunctions, and disjunctions.
2. Construct a truth table using negations, conjunctions, and disjunctions.
3. Construct a truth table for a compound statement and interpret its validity.

Are you familiar with the Choose Your Own Adventure book series written by Edward Packard? These gamebooks allow the reader to become one of the characters and make decisions that affect what happens next, resulting in different sequences of events in the story and endings based on the choices made. Writing a computer program is a little like what it must be like to write one of these books. The programmer must consider all the possible inputs that a user can put into the program and decide what will happen in each case, then write their program to account for each of these possible outcomes.

A truth table is a graphical tool used to analyze all the possible truth values of the component logical statements to determine the validity of a statement or argument along with all its possible outcomes. The rows of the table correspond to each possible outcome for the given logical statement identified at the top of each column. A single logical statement p has two possible truth values, true or false. In truth tables, a capital T will represent true values, and a capital F will represent false values.

In this section, you will use the knowledge built in Statements and Quantifiers and Compound Statements to analyze arguments and determine their truth value and validity. A logical argument is valid if its conclusion follows from its premises, regardless of whether those premises are true or false. You will then explore the truth tables for negation, conjunction, and disjunction, and use these truth tables to analyze compound logical statements containing these connectives.

Interpret and Apply Negations, Conjunctions, and Disjunctions

The negation of a statement will have the opposite truth value of the original statement. When p is true, $\sim p$ is false, and when p is false, $\sim p$ is true.

? Exercise 2.4.1

For each logical statement, determine the truth value of its negation.

1. p: $3+5=8$.
2. q: All horses are mustangs.
3. $\sim r$: New Delhi is not the capital of India.

Answer

p is true because $3 + 5$ does equal 8; therefore, the negation of p, $\sim p$, $3 + 5 \neq 8$, $\sim p$, $3 + 5 \neq 8$, is false.
q is false because there are other types of horses besides mustangs, such as Clydesdales or Arabians; therefore, the negation of q, $\sim q$, is true.

$\sim r$ is false because New Delhi is the capital of India; therefore, the negation of $\sim r$, r , is true.

Your Turn 2.4.1

For each logical statement, determine the truth value of its negation.

1. p: $3 \times 5 = 14$.
2. q: Some houses are built with bricks
3. r: Abuja is the capital of Nigeria.

A conjunction is a logical *and* statement. For a conjunction to be true, both statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

1. $p \wedge q$
2. $\sim q \wedge r$
3. $\sim p \wedge q$

Answer

1. p is true, and q is false. Because one statement is true, and the other statement is false, this makes the complete conjunction false.
2. q is false, so $\sim q$ is true, and r is true. Therefore, both statements are true, making the complete conjunction true.
3. $\sim p$ is false, and q is false. Because both statements are false, the complete conjunction is false.

Your Turn 2.4.2

Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each conjunction.

1. $p \wedge q$
2. $q \wedge r$
3. $\sim r \wedge p$.

Checkpoint

The only time a conjunction is true is if both statements that make up the conjunction are true.

A disjunction is a logical inclusive *or* statement, which means that a disjunction is true if one or both statements that form it are true. The only way a logical inclusive *or* statement is false is if both statements that form the disjunction are false.

1. $p \vee q$
2. $\sim q \vee r$
3. $\sim p \vee q$

Answer

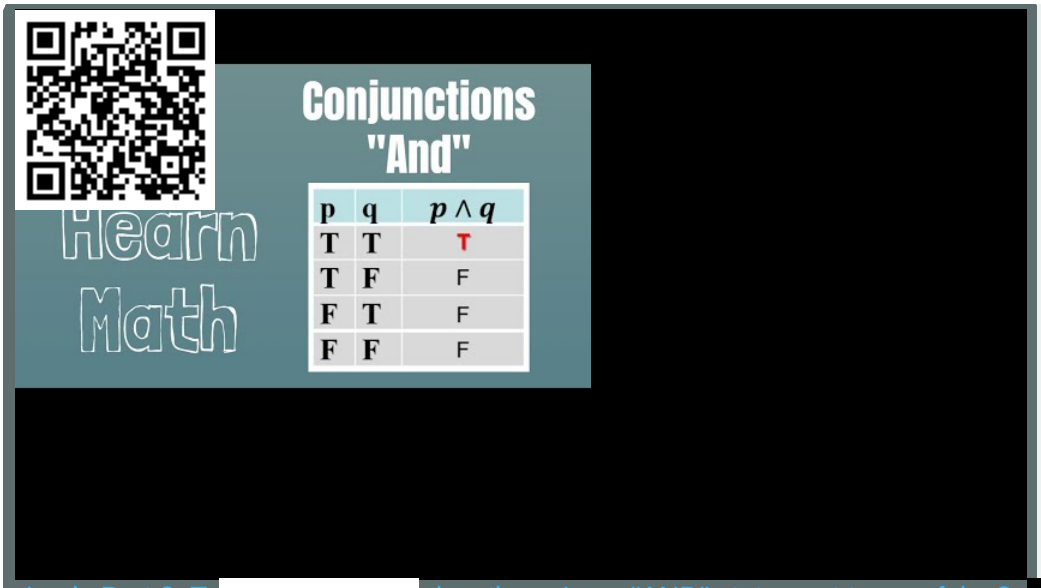
1. **p is true, and q is false. One statement is true, and one statement is false, which makes the complete disjunction true.**
2. **q is false, so $\sim q$ is true, and r is true. Therefore, one statement is true, and the other statement is true, which makes the complete disjunction true.**
3. **$\sim p$ is false, and q is false. When all of the component statements are false, the disjunction is false.**

Your Turn 2.4.3

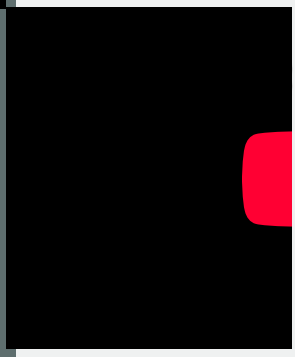
Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each disjunction.

1. $p \vee q$
2. $\sim p \vee r$
3. $q \vee r$

In the next example, you will apply the dominance of connectives to find the truth values of compound statements containing negations, conjunctions, and disjunctions and use a table to record the results. When constructing a truth table to analyze an argument where you can determine the truth value of each component statement, the strategy is to create a table with two rows. The first row contains the symbols representing the components that make up the compound statement. The second row contains the truth values of each component below its symbol. Include as many columns as necessary to represent the value of each compound statement. The last column includes the complete compound statement with its truth value below it.



Logic Part 2: Tr njunctions: Is an "AND" statement true or false?



Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false?

1. $\sim p \wedge q \vee r$
2. $\sim p \vee q \wedge r$
3. $\sim (p \wedge r) \vee q$

Answer

1. **Step 1:** The statement " $\sim p \wedge q \vee r$ " contains three basic logical statements, p, q, and r, and three connectives, $\sim, \wedge, \vee$. When we place parentheses in the statement to indicate the dominance of connectives, the statement becomes $(((\sim p) \wedge q) \vee r)$.
Step 2: After we have applied the dominance of connectives, we create a two row table that includes a column for each basic statement that makes up the compound statement, and an additional column for the contents of each parentheses. Because we have three sets of parentheses, we include a column for $\sim p$, the innermost parentheses, a column for $(\sim p) \wedge q$, the next set of parentheses, and $((\sim p) \wedge q) \vee r$ in the last column for the third parentheses.
Step 3: Once the table is created, we determine the truth value of each statement starting from left to right. The truth values of p, q, and r are true, false, and true, respectively, so we place T, F, and T in the second row of the table. Because p is true, $\sim p$ is false.
Step 4: Next, evaluate $\sim p \wedge q$ from the table: $\sim p$ is false, and q is also false, so $\sim p \wedge q$ is false, because a conjunction is only true if both of the statements that make it are true. Place an F in the table below its heading.
Step 5: Finally, using the table, we understand that $(\sim p \wedge q)$ is false and r is true, so the complete statement $(\sim p \wedge q) \vee r$ is false or true, which is true (because a disjunction is true whenever at least one of the statements that make it is true). Place a T in the last column of the table. The complete statement $\sim p \wedge q \vee r$ is true.

2. **Step 1**: Applying the dominance of connectives to the original compound statement $\sim p \vee q \wedge r$, we get $(((\sim p) \vee q) \wedge r)$.
Step 2: The table needs columns for $p, q, r, \sim p, (\sim p) \vee q$, and $((\sim p) \vee q) \wedge r$.
Step 3:The truth values of p, q, r, and $\sim p$ are the same as in Question 1.
Step 4: Next, $\sim p \vee q$ is false or false, which is false, so we place an F below this statement in the table. This is the only time that a disjunction is false.
Step 5: Finally, $\sim p \vee q$ and r are the conjunction of the statements, $\sim p \vee q$ and r, and so the expressions evaluate to false and true, which is false. Recall that the only time an "and" statement is true is when both statements that form it are also true. The complete statement $\sim p \vee q \wedge r$ is false.

3. **Step 1**: Applying the dominance of connectives to the original statement, we have: $((\sim (p \wedge r)) \vee q)$.
Step 2: So, the table needs the following columns: $p, q, r, p \wedge r, \sim (p \wedge r)$, and $\sim (p \wedge r) \vee q$.
Step 3: The truth values of p, q, and r are the same as in Questions 1 and 2.
Step 4: From the table it can be seen that $p \wedge r$ is true and true, which is true. So the negation of p and r is false, because the negation of a statement always has the opposite truth value of the original statement.
Step 5: Finally, $\sim (p \wedge r) \vee q$ is the disjunction of $\sim (p \wedge r)$ with q, and so we have false or false, which makes the complete statement false.

T	F	T	T	F	F

✔ Your Turn ✔ 2.4.4

Given p : Yellow is a primary color, q : Blue is a primary color, and r : Green is a primary color, determine the truth value of each compound statement, by constructing a truth table.

1. $\sim q \wedge p \vee r$
2. $p \vee q \wedge \sim r$
3. $\sim (p \wedge r) \wedge q$

Construct Truth Tables to Analyze All Possible Outcomes

Recall from Statements and Questions that the negation of a statement will always have the opposite truth value of the original statement; if a statement p is false, then its negation $\sim p$ is true, and if a statement p is true, then its negation $\sim p$ is false. To create a truth table for the negation of statement p, add a column with a heading of $\sim p$, and for each row, input the truth value with the opposite value of the value listed in the column for p, as depicted in the table below.

Negation	
p	~ p
T	F
F	T

Conjunctions and disjunctions are compound statements formed when two logical statements combine with the connectives “and” and “or” respectively. How does that change the number of possible outcomes and thus determine the number of rows in our truth table? The multiplication principle, also known as the *fundamental counting principle*, states that the number of ways you can select an item from a group of n items and another item from a group with m items is equal to the product of m and n. Because each proposition p and q has two possible outcomes, true or false, the multiplication principle states that there will be $2 \times 2 = 4$ possible outcomes: {TT, TF, FT, FF}.

The tree diagram and table in Figure 2.4.2 demonstrate the four possible outcomes for two propositions p and q.

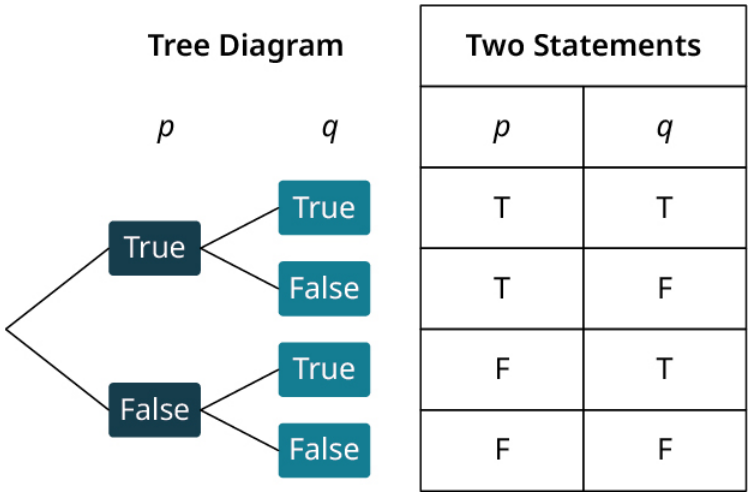
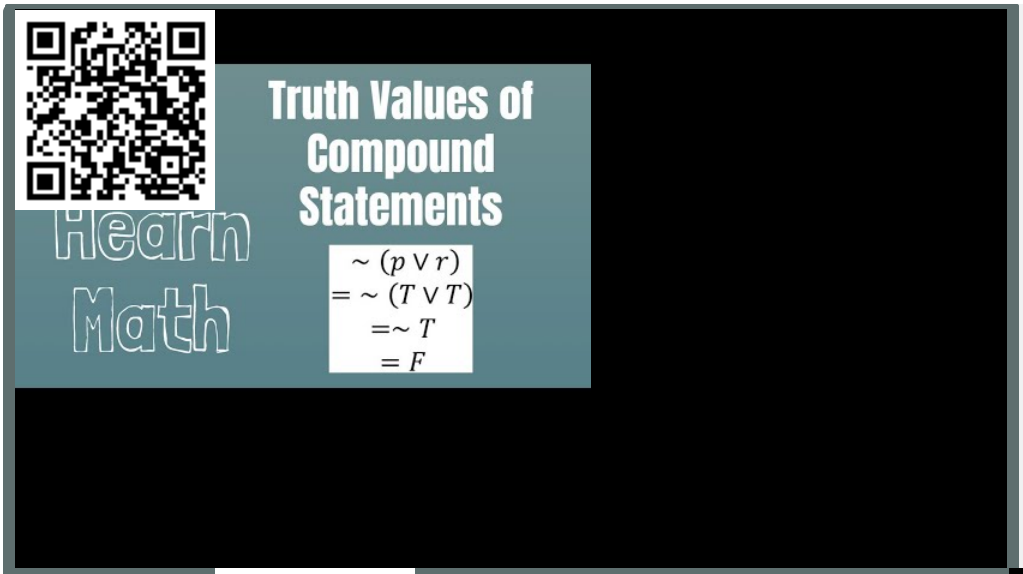


Figure 2.4.2:

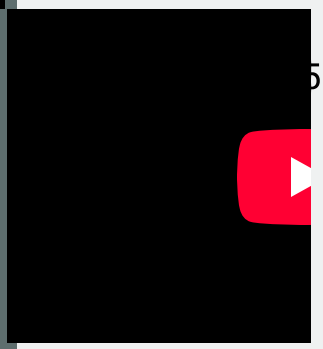
A conjunction is a logical *and* statement. For a conjunction to be true, both the statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

A disjunction is a logical inclusive *or* statement. Which means that a disjunction is true if one or both statements that make it are true. The only way a logical inclusive *or* statement is false is if both statements that make up the disjunction are false.

Conjunction (AND)				Disjunction (OR)		
p	q	$p \wedge q$		p	q	$p \vee q$
T	T	T		T	T	T
T	F	F		T	F	T
F	T	F		F	T	T
F	F	F		F	F	F



Logic Part 4: Tr ompound Statements with "and", "or", and "not"



Logic Part 5: What are truth tables? How do you set them up?

? Exercise 2.4.5: Constructing Truth Tables to Analyze Compound Statements

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \vee \sim q$
2. $\sim (p \wedge q)$
3. $(p \vee \sim q) \wedge r$

Answer

1. Step 1: Because there are two basic statements, p and q, and each of these has two possible outcomes, we will have $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF. The columns will include $p, q, \sim q$, and $p \vee \sim q$.
Step 2: Every value in column $\sim q$ will have the opposite truth value of the corresponding value in column q : F, T, F, and T.

Step 3: To complete the last column, evaluate each element in column p with the corresponding element in column $\sim q$ using the connective or.

p	q	$\sim q$	$p \vee \sim q$.
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

2. Step 1: The columns will include $p, q, p \wedge q$, and $\sim (p \wedge q)$. Because there are two basic statements, p and q, the table will have four rows to account for all possible outcomes.
Step 2: The $p \wedge q$ column will be completed by evaluating the corresponding elements in columns p and q respectively with the and connective.
Step 3: The final column, $\sim (p \wedge q)$, will be the negation of the $p \wedge q$ column.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

3. Step 1: This statement has three basic statements, p, q, and r. Because each basic statement has two possible truth values, true or false, the multiplication principal indicates there are $2(2)(2) = 8$ possible outcomes. So eight rows of outcomes are needed in the truth table to account for each possibility. Half of the eight possibilities must be true for the first statement, and half must be false.
Step 2: So, the first column for statement p, will have four T's followed by four F's. In the second column for statement q, when p is true, half the outcomes for q must be true and the other half must be false, and the same pattern will repeat for when p is false. So, column q will have TT, FF, FF, FF.

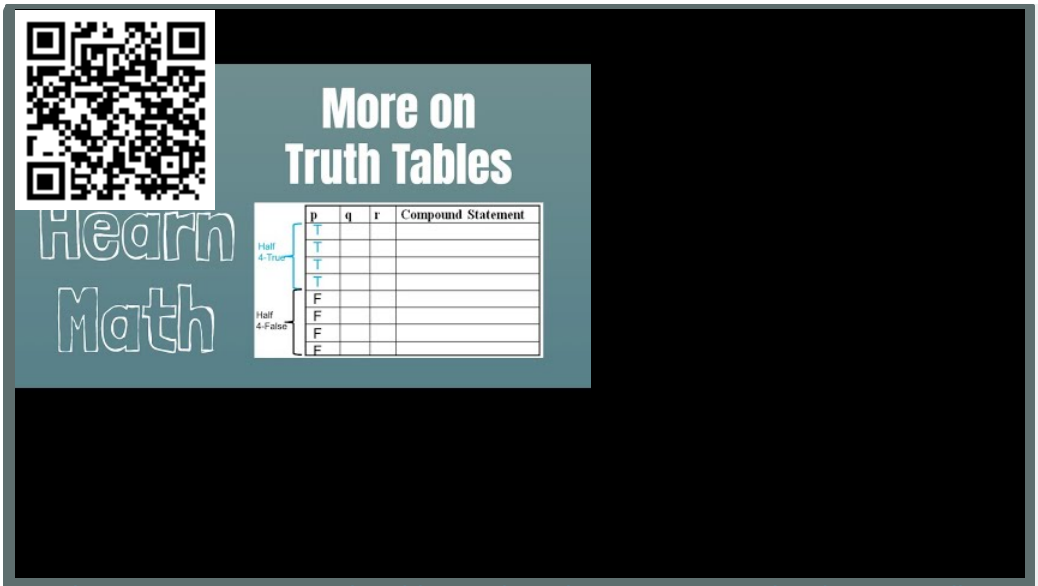
Step 3: The column for the third statement, r, must alternate between T and F. Once, the three basic propositions are listed, you will need a column for $\sim, p \vee \sim q$, and $(p \vee \sim q) \wedge r$.
Step 4: The column for the negation of $q, \sim q$, will have the opposite truth value of each value in column q.
Step 5: Next, fill in the truth values for the column containing the statement $p \vee \sim q$. The or statement is true if at least one of p or $\sim q$ is true, otherwise it is false.
Step 6: Finally, fill in the column containing the conjunction $(p \vee \sim q) \wedge r$. To evaluate this statement, combine column $p \vee \sim q$ and column r with the and connective. Recall, that only time "and" is true is when both values are true, otherwise the statement is false. The complete truth table is:

p	q	r	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	T	T	T
F	F	F	T	T	F

Your Turn 2.4.5

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \wedge \sim q$
2. $\sim (p \vee q)$
3. $(p \wedge \sim q) \vee r$



Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

Determine the Validity of a Truth Table for a Compound Statement

A logical statement is valid if it is always true regardless of the truth values of its component parts. To test the validity of a compound statement, construct a truth table to analyze all possible outcomes. If the last column, representing the complete statement, contains only true values, the statement is valid.

Exercise 2.4.6: Determining the Validity of Compound Statements

Construct a truth table to determine the validity of each of the following statements.

1. $\sim p \wedge q$
2. $\sim (p \wedge \sim p)$

Answer

1. Step 1: Because there are two statements, p and q, and each of these has two possible outcomes, there will be $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, **FT**, and FF.
Step 2: The columns, will include $p, q, \sim p$ and $\sim p \wedge q$. Every value in column $\sim p$ will have the opposite truth value of the corresponding value in column p.
Step 3: To complete the last column, evaluate each element in column $\sim p$ with the corresponding element in column q using the connective and. The last column contains at least one false, therefore the statement $\sim p \wedge q$ is not valid.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

2. Step 1: Because the statement $\sim (p \wedge \sim p)$ only contains one basic proposition, the truth table will only contain two rows. Statement p may be either true or false.
Step 2: The columns will include $p, \sim p, p \wedge \sim p$, and $\sim (p \wedge \sim p)$. Evaluate column $p \wedge \sim p$ with the and connective, because the symbol $\wedge$ represents a conjunction or logical and statement. True and false is false, and false and true is also false.
Step 3: The final column is the negation of each entry in the third column, both of which are false, so the negation of false is true. Because all the truth values in the final column are true, the statement $\sim (p \wedge \sim p)$ is valid.

p	$\sim p$	$p \wedge \sim p$	$\sim (p \wedge \sim p)$
T	F	F	T
F	T	F	T

Your Turn 2.4.6

Construct a truth table to determine the validity of each of the following statements.
1. $p \vee \sim p$
2. $\sim p \vee \sim q$

Check Your Understanding

1. A logical argument is _____ if its conclusion follows from its premises.
2. A logical statement is valid if it is always _____.
3. A _____ _____ is a tool used to analyze all the possible outcomes for a logical statement.
4. The truth table for the conjunction, $p \wedge q$, has _____ rows of truth values.
5. The truth table for the negation of a logical statement, p, has _____ rows of truth values.

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2.4.1: Exercises

For the following exercises, find the truth value of each statement.

Exercise 2.4.1.1

$p: 7 \times 3 = 21$. What is the truth value of $\sim p$?

Exercise 2.4.1.2

q : The sun revolves around the Earth. What is the truth value of $\sim q$?

Exercise 2.4.1.3

$\sim r$: The acceleration of gravity is $9.81\ \mathrm{m/sec^2}$. What is the truth value of r ?

Exercise 2.4.1.4

s : Dan Brown is not the author of the book, The Davinci Code. What is the truth value of $\sim (\sim s)$?

Exercise 2.4.1.5

t : Broccoli is a vegetable. What is the truth value of $\sim (\sim t)$?

For the following exercises, given $p: 1 + 2 = 3, q$: Five is an even number, and r : Seven is a prime number, find the truth value of each of the following statements.

Exercise 2.4.1.6

$\sim q$

Exercise 2.4.1.7

$p \wedge q$

Exercise 2.4.1.8

$p \vee q$

Exercise 2.4.1.9

$\sim p \vee \sim q$

Exercise 2.4.1.10

$p \wedge \sim q$

Exercise 2.4.1.11

$p \wedge r$

Exercise 2.4.1.12

$q \wedge r$

Exercise 2.4.1.13

$q \wedge \sim r$

Exercise 2.4.1.14

$q \vee \sim r$

Exercise 2.4.1.15

$\sim (p \wedge r)$

Exercise 2.4.1.16

$p \vee q \wedge r$

Exercise 2.4.1.17

$\sim p \vee (q \wedge r)$

Exercise 2.4.1.18

$\sim (q \wedge r) \vee \sim p$

Exercise 2.4.1.19

$q \vee r \vee p$

Exercise 2.4.1.20

$\sim q \wedge r \wedge p$

For the following exercises, complete the truth table to determine the truth value of the proposition in the last column.

Exercise 2.4.1.21

p	q	r	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \wedge r$
T	T	T			

Exercise 2.4.1.22

p	q	r	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \wedge r$
F	T	F			

Exercise 2.4.1.23

p	q	r	$\sim p$	$\sim r$	$\sim p \wedge q (\sim p \wedge q) \vee \sim r$
F	F	F			

Exercise 2.4.1.24

p	q	r	$\sim p$	$\sim r$	$\sim p \vee q$	$(\sim p \vee q) \vee \sim r$
F	F	F				

For the following exercises, given p : All triangles have three sides, q : Some rectangles are not square, and r : A pentagon has eight sides, determine the truth value of each compound statement by constructing a truth table.

Exercise 2.4.1.25

$\sim r \wedge q \wedge p$

Exercise 2.4.1.26

$\sim (q \wedge p) \vee r$

Exercise 2.4.1.27

$\sim p \vee q \wedge r$

Exercise 2.4.1.28

$\sim p \vee \sim q \vee r$ For the following exercises, construct a truth table to analyze all the possible outcomes for the following arguments.

Exercise 2.4.1.29

$\sim q \wedge q$

Exercise 2.4.1.30

$\sim p \vee \sim q$

Exercise 2.4.1.31

$\sim p \wedge \sim q$

Exercise 2.4.1.32

$p \wedge q \vee r$ For the following exercises, construct a truth table to determine the validity of each statement.

Exercise 2.4.1.33

$\sim q \vee q$

Exercise 2.4.1.34

$p \wedge \sim q$

Exercise 2.4.1.35

$p \wedge q \vee \sim p$

Exercise 2.4.1.36

$(p \wedge q) \vee (\sim p \wedge \sim q)$

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2.5: Truth Tables for the Conditional and Biconditional

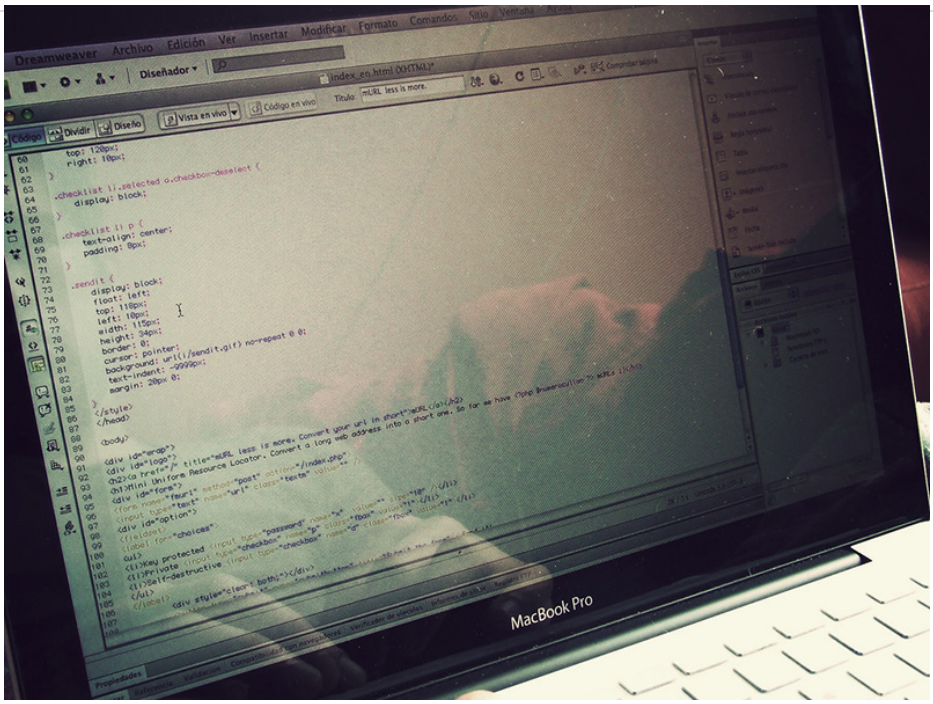


Figure **2.5.1**: If-then statements use logic to execute directions. (credit: “Coding” by Carlos Varela/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use and apply the conditional to construct a truth table.
2. Use and apply the biconditional to construct a truth table.
3. Use truth tables to determine the validity of conditional and biconditional statements.

Computer languages use if-then or if-then-else statements as decision statements:

- *If* the hypothesis is true, *then* do something.
- Or, *if* the hypothesis is true, *then* do something; else do something else.

For example, the following representation of computer code creates an if-then-else decision statement:

Check value of variable i.

If $i < 1$, then print "Hello, World!" else print "Goodbye".

In this imaginary program, the if-then statement evaluates and acts on the value of the variable i. For instance, if $i = 0$, the program would consider the statement $i < 1$ as true and “Hello, World!” would appear on the computer screen. If instead, $i = 3$, the program would consider the statement $i < 1$ as false (because 3 is greater than 1), and print “Goodbye” on the screen.

In this section, we will apply similar reasoning without the use of computer programs.

People in Mathematics

The Countess of Lovelace, Ada Lovelace, is credited with writing the first computer program. She wrote an algorithm to work with Charles Babbage’s Analytical Engine that could compute the Bernoulli numbers in 1843. In doing so, she became the first person to write a program for a machine that would produce more than just a simple calculation. The computer programming language ADA is named after her.

Reference: Posamentier, Alfred and Spreitzer Christian, “Chapter 34 Ada Lovelace: English (1815-1852)” pp. 272-278, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Use and Apply the Conditional to Construct a Truth Table

A conditional is a logical statement of the form if p, then q. The conditional statement in logic is a promise or contract. The only time the conditional, $p \rightarrow q$, is false is when the contract or promise is broken.

For example, consider the following scenario. A child's parent says, "If you do your homework, then you can play your video games." The child really wants to play their video games, so they get started right away, finish within an hour, and then show their parent the completed homework. The parent thanks the child for doing a great job on their homework and allows them to play video games. Both the parent and child are happy. The contract was satisfied; true implies true is true.

Now, suppose the child does not start their homework right away, and then struggles to complete it. They eventually finish and show it to their parent. The parent again thanks the child for completing their homework, but then informs the child that it is too late in the evening to play video games, and that they must begin to get ready for bed. Now, the child is really upset. They held up their part of the contract, but they did not receive the promised reward. The contract was broken; true implies false is false.

So, what happens if the child does not do their homework? In this case, the hypothesis is false. No contract has been entered, therefore, no contract can be broken. If the conclusion is false, the child does not get to play video games and might not be happy, but this outcome is expected because the child did not complete their end of the bargain. They did not complete their homework. False implies false is true. The last option is not as intuitive. If the parent lets the child play video games, even if they did not do their homework, neither parent nor child are going to be upset. False implies true is true.

The truth table for the conditional statement below summarizes these results.

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Checkpoint

Notice that the only time the conditional statement, $p \rightarrow q$, is false is when the hypothesis, p, is true and the conclusion, q, is false.

Video



Logic Part 8: The Conditional and Tautologies

Exercise 2.5.1: Constructing Truth Tables for Conditional Statements

Assume both of the following statements are true: p: My sibling washed the dishes, and q: My parents paid them $5.00. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$
2. $p \rightarrow \sim q$
3. $\sim p \rightarrow q$

Answer

Because p is true and q is true, the statement p → q is, “If my sibling washed the dishes, then my parents paid them $5.00.” My sibling did wash the dishes, since p p is true, and the parents did pay the sibling $5.00, so the contract was entered and completed. The conditional statement is true, as indicated by the truth table representing this case:
T → T = T.

p	q	$p \rightarrow q$
T	T	**T**

p → ~ q p

2. p → ~ q p → ~ q translates to the statement, “If my sibling washed the dishes, then my parents did not pay them $5.00.” p p is true, but ~ q ~ q is false. The sibling completed their end of the contract, but they did not get paid. The contract was broken by the parents. The conditional statement is false, as indicated by the truth table representing this case:

T → F = F.

p	q	$\sim q$	$p \rightarrow \sim q$
T	T	F	**F**

3. ~ p → q translates to the statement, “If my sibling did not wash the dishes, then my parents paid them $5.00.” ~ p is false, but q is true. The sibling did not do the dishes. No contract was entered, so it could not be broken. The parents decided to pay them $5.00 anyway. The conditional statement is true, as indicated by the truth table representing this case: F → T = T.

p	q	$\sim p$	$\sim p \rightarrow q$
T	T	F	**T**

Your Turn 2.5.1

Assume p is true and q is false. p: Kevin vacuumed the living room, and q: Kevin's parents did not let him borrow the car. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$
2. $p \rightarrow \sim q$
3. $\sim p \rightarrow q$

Exercise 2.5.2: Determining Validity of Conditional Statements

Construct a truth table to analyze all possible outcomes for each of the following statements then determine whether they are valid.

1. $p \wedge q \rightarrow \sim q$
2. $p \rightarrow \sim p \vee q$

Answer

1. Applying the dominance of connectives, the statement $p \wedge q \rightarrow \sim q$ is equivalent to $(p \wedge q) \rightarrow (\sim q)$. So, the columns of the truth table will include p, q, $p \wedge q$, $\sim q$, and $p \wedge q \rightarrow \sim q$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4\ 2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge q \rightarrow \sim q$
T	T	T	F	**F**
T	F	F	T	**T**
F	T	F	F	**T**
F	F	F	T	**T**

Applying the dominance of connectives, the statement $p \rightarrow \sim p \vee q$ is equivalent to $(p) \rightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include p, q, ~ p, ~ p ∨ q, and $p \rightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4\ 2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow (\sim p \vee q)$
T	T	F	T	**T**
T	F	F	F	**F**
F	T	T	T	**T**
F	F	T	T	**T**

Your Turn 2.5.2

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $q \rightarrow \sim p \vee q$
2. $\sim p \rightarrow q \wedge p$

Use and Apply the Biconditional to Construct a Truth Table

The biconditional, $p \leftrightarrow q$, is a two way contract; it is equivalent to the statement $(p \rightarrow q) \wedge (q \rightarrow p)$. A biconditional statement, $p \leftrightarrow q$, is true whenever the truth value of the hypothesis matches the truth value of the conclusion, otherwise it is false.

The truth table for the biconditional is summarized below.

p	q	$p \leftrightarrow q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

Video

https://math.libretexts.org/@go/page/214377



Logic Part 11B: Biconditional and Summary of Truth Value Rules in Logic

Exercise 2.5.3: Constructing Truth Tables for Biconditional Statements

Assume both of the following statements are true: p : The plumber fixed the leak, and q : The homeowner paid the plumber $150.00. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \sim q$
3. $\sim p \leftrightarrow \sim q$

Answer

1. Because p p is true and q q is true, the statement p ↔ q p ↔ q is "The plumber fixed the leak if and only if the homeowner paid them $150.00." Because both p p and q q are true, the leak was fixed and the plumber was paid, meaning both parties satisfied their end of the bargain. The biconditional statement is true, as indicated by the truth table representing this case: T ↔ T = T.

p	q	$p \leftrightarrow q$
T	T	**T**

2. p → ~ q p → ~ q translates to the statement, "The plumber fixed the leak if and only if the homeowner did not pay them $150." If the plumber fixed the leak and the homeowner did not pay them, the homeowner will have broken their end of the contract. The biconditional statement is false, as indicated by the truth table representing this case:
T ↔ F = F.

p	q	$\sim q$	$p \leftrightarrow \sim q$
T	T	F	**F**

3. ~ p ↔ ~ q ~ p ↔ ~ q translates to the statement, "The plumber did not fix the leak if and only if the homeowner did not pay them $150." In this case, neither party—the plumber nor the homeowner—entered into the contract. The leak was not repaired, and the plumber was not paid. No agreement was broken. The biconditional statement is true, as indicated by the truth table representing this case: F ↔ F = T.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$
T	T	F	F	**T**

Your Turn 2.5.3

Assume p is true and q is false: p : The contractor fixed the broken window, and q : The homeowner paid the contractor $200. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \sim q$
3. $\sim p \leftrightarrow q$

Checkpoint

The biconditional, $p \leftrightarrow q$, is true whenever the truth values of p and q match, otherwise it is false.

Video



Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

Exercise 2.5.1: Determining Validity of Biconditional Statements

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $p \wedge q \leftrightarrow p \wedge \sim q$
2. $p \vee q \leftrightarrow \sim p \vee q$
3. $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$
4. $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$

Answer

1. Applying the dominance of connectives, the statement $p \wedge q \leftrightarrow p \wedge \sim q$ is equivalent to $(p \wedge q) \leftrightarrow (p \wedge (\sim q))$. So, the columns of the truth table will include $p, q, p \wedge q, \sim q, p \wedge \sim q$ and $(p \wedge q) \leftrightarrow (p \wedge \sim q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge \sim q$	$(p \wedge q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	**F**
T	F	F	T	T	**F**
F	T	F	F	F	**T**
F	F	F	T	F	**T**

2. Applying the dominance of connectives, the statement $p \vee q \leftrightarrow \sim p \vee q$ is equivalent to $(p \vee q) \leftrightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include $p, q, p \vee q, \sim p, \sim p \vee q$, and $(p \vee q) \leftrightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \vee q$	$\sim p$	$\sim p \vee q$	$(p \vee q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	T	F	F	**F**
F	T	T	T	T	**T**
F	F	F	T	T	**F**

3. Applying the dominance of connectives, the statement $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$ is equivalent to $(p \rightarrow q) \leftrightarrow ((\sim q) \rightarrow (\sim p))$. So, the columns of the truth table will include $p, q, p \rightarrow q, \sim q, \ \sim p, \sim q \rightarrow \sim p$, and $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$. Because there are only two basic propositions, p and q the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is valid because the last column is all true.

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	F	F	T	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	T	**T**
F	F	T	T	T	T	**T**

4. Applying the dominance of connectives, the statement $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$ is equivalent to $((p \wedge q) \rightarrow (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. So, the columns of the truth table will include $p, q, r, \ \sim r, p \wedge q, (p \wedge q) \wedge \sim r, (p \wedge q) \wedge r$, and $((p \wedge q) \rightarrow (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. Because there are three basic propositions, p, q, and r, the table will have $2(2)(2) = 8$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \rightarrow \sim r$	$(p \wedge q) \wedge r$	$(p \wedge q \rightarrow \sim r) \leftrightarrow (p \wedge$
T	T	T	F	T	F	T	**T**
T	T	F	T	T	T	F	**T**
T	F	T	F	F	T	F	**F**
T	F	F	T	F	T	F	**F**
F	T	T	F	F	T	F	**F**
F	T	F	T	F	T	F	**F**
F	F	T	F	F	T	F	**F**
F	F	F	T	F	T	F	**F**

Your Turn 2.5.1

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
2. $\sim p \leftrightarrow q \wedge p$
3. $p \rightarrow q \leftrightarrow \sim p \vee q$
4. $p \wedge q \rightarrow r \leftrightarrow \sim p \vee \sim q \vee r$

Check Your Understanding

1. 1. In logic, a conditional statement can be thought of as a _____.
2. If the hypothesis, p, of a conditional statement is true, the _____, q, must also be true for the conditional statement \($p \to q$ to be true.
3. If the _____ of a conditional statement is false, the conditional statement is true.
4. The symbolic form of the _____ statement is $p \leftrightarrow q$
5. The _____ statement is equivalent to the statement $(p \to q) \wedge (q \to p)$.
6. p if and only if q is _____ whenever the truth value of p matches the truth value of q, otherwise it is false.

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2.5.1: Exercises

For the following exercises, complete the truth table to determine the truth value of the proposition in the last column.

Exercise 2.5.1.1

p	q	p	$p \to q$
T	T		

Exercise 2.5.1.2

p	q	q	$p \to q$
T	T		

Exercise 2.5.1.3

p	q	p	$p \leftrightarrow q$
F	T		

Exercise 2.5.1.4

p	q	q	$p \leftrightarrow q$
F	T		

Exercise 2.5.1.5

p	q	r	p	$p \wedge q$	$(p \wedge q) \to r$
F	T	F			

Exercise 2.5.1.6

p	q	r	p	r	$p \wedge q$	$(p \wedge q) \to r$
F	F	F				

Exercise 2.5.1.7

p	q	r	p	r	$p \vee q$	$(p \vee q) \leftrightarrow r$
F	F	F				

Exercise 2.5.1.8

p	q	r	p	r	$p \wedge q$	$(p \wedge q) \leftrightarrow r$
T	F	F				

Exercise 2.5.1.9

p	q	r	p	r	$p \vee q$	$p \rightarrow r$	$(p \vee r) \leftrightarrow (p \rightarrow r)$
F	F	F					

Exercise 2.5.1.10

p	q	r	p	r	$p \wedge q$	$p \rightarrow r$	$(p \wedge q) \leftrightarrow (p \rightarrow r)$
T	T	T					

For the following exercises, assume these statements are true: p: Faheem is a software engineer, q: Ann is a project manager, r: Giacomo works with Faheem, and s: The software application was completed on time. Translate each of the following statements to symbols, then construct a truth table to determine its truth value.

Exercise 2.5.1.11

If Giacomo works with Faheem, then Faheem is not a software engineer.

Exercise 2.5.1.12

If the software application was not completed on time, then Ann is not a project manager.

Exercise 2.5.1.13

The software application was completed on time if and only if Giacomo worked with Faheem.

Exercise 2.5.1.14

Ann is not a project manager if and only if Faheem is a software engineer.

Exercise 2.5.1.15

If the software application was completed on time, then Ann is a project manager, but Faheem is not a software engineer.

Exercise 2.5.1.16

If Giacomo works with Faheem and Ann is a project manager, then the software application was completed on time.

Exercise 2.5.1.17

The software application was not completed on time if and only if Faheem is a software engineer or Giacomo did not work with Faheem.

Exercise 2.5.1.18

Faheem is a software engineer or Ann is not a project manager if and only if Giacomo did not work with Faheem and the software application was completed on time.

Exercise 2.5.1.19

Ann is a project manager implies Faheem is a software engineer if and only if the software application was completed on time implies Giacomo worked with Faheem.

Exercise 2.5.1.20

If Giacomo did not work with Faheem implies that the software application was not completed on time, then Ann was not the project manager.

For the following exercises, construct a truth table to analyze all the possible outcomes and determine the validity of each argument.

Exercise 2.5.1.21

$p \vee \sim q \rightarrow q$

Exercise 2.5.1.22

$\sim q \rightarrow p \wedge \sim q$

Exercise 2.5.1.23

$(p \rightarrow q) \leftrightarrow q$

Exercise 2.5.1.24

$(p \rightarrow q) \leftrightarrow p$

Exercise 2.5.1.25

$\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$

Exercise 2.5.1.26

$(p \rightarrow q) \wedge p \rightarrow q$

Exercise 2.5.1.27

$p \rightarrow q \rightarrow r$

Exercise 2.5.1.28

$(p \rightarrow q) \wedge (q \rightarrow r) \leftrightarrow (p \rightarrow r)$

Exercise 2.5.1.29

$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

? Exercise 2.5.1.30

$p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$

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2.6: Equivalent Statements



Figure **2.6.1**: How your logical argument is stated affects the response, just like how you speak when holding a conversation can affect how your words are received. (credit: modification of work by Goelshivi/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

1. Determine whether two statements are logically equivalent using a truth table.
2. Compose the converse, inverse, and contrapositive of a conditional statement

Have you ever had a conversation with or sent a note to someone, only to have them misunderstand what you intended to convey? The way you choose to express your ideas can be as, or even more, important than what you are saying. If your goal is to convince someone that what you are saying is correct, you will not want to alienate them by choosing your words poorly.

Logical arguments can be stated in many different ways that still ultimately result in the same valid conclusion. Part of the art of constructing a persuasive argument is knowing how to arrange the facts and conclusion to elicit the desired response from the intended audience.

In this section, you will learn how to determine whether two statements are **logically equivalent** using truth tables, and then you will apply this knowledge to compose logically equivalent forms of the conditional statement. Developing this skill will provide the additional skills and knowledge needed to construct well-reasoned, persuasive arguments that can be customized to address specific audiences.

Checkpoint

An alternate way to think about logical equivalence is that the truth values have to match. That is, whenever p is true, q is also true, and whenever p is false, q is also false.

Determine Logical Equivalence

Two statements, p and q, are logically equivalent when $p \leftrightarrow q$ is a valid argument, or when the last column of the truth table consists of only true values. When a logical statement is always true, it is known as a tautology. To determine whether two statements p and

q are logically equivalent, construct a truth table for $p \leftrightarrow q$ and determine whether it valid. If the last column is all true, the argument is a tautology, it is valid, and p is logically equivalent to q; otherwise, p is not logically equivalent to q.

? Exercise 2.6.1: Determining Logical Equivalence with a Truth Table

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \rightarrow q; \sim p \rightarrow \sim q$
2. $p \rightarrow q; \sim p \vee q$

Answer

1. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, (p → q) ↔ (~ p → ~ q).

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$(p \rightarrow q) \leftrightarrow (\sim p \rightarrow \sim q)$
T	T	T	F	F	T	**T**
T	F	F	F	T	T	**F**
F	T	T	T	F	F	**F**
F	F	T	T	T	T	**T**

Because the last column it not all true, the biconditional is not valid and the statement $p \rightarrow q$ is not logically equivalent to the statement $\sim p \rightarrow \sim q$.

2. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion,
(p → q) ↔ (~ p ∨ q) .

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	F	F	F	**T**
F	T	T	T	T	**T**
F	F	T	T	T	**T**

Because the last column is true for every entry, the biconditional is valid and the statement $p \rightarrow q$ is logically equivalent to the statement $\sim p \vee q$. Symbolically, $p \rightarrow q \equiv \sim p \vee q$.

✓ Your Turn 2.6.1

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \rightarrow q; q \rightarrow \sim p$
2. $p \rightarrow q; p \vee \sim q$

Compose the Converse, Inverse, and Contrapositive of a Conditional Statement

The converse, inverse, and contrapositive are variations of the conditional statement, $p \rightarrow q$.

- The converse is if q then p, and it is formed by interchanging the hypothesis and the conclusion. The converse is logically equivalent to the inverse.

- The inverse is if $\sim p$ then $\sim q$, and it is formed by negating both the hypothesis and the conclusion. The inverse is logically equivalent to the converse.
- The contrapositive is if $\sim q$ then $\sim p$, and it is formed by interchanging and negating both the hypothesis and the conclusion. The contrapositive is logically equivalent to the conditional.

The table below shows how these variations are presented symbolically.

				Conditional	Contrapositive	Converse	Inverse
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$q \rightarrow p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Exercise 2.6.2: Writing the Converse, Inverse, and Contrapositive of a Conditional Statement

Use the statements, p: Harry is a wizard and q: Hermione is a witch, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

Answer

1. The conditional statement takes the form, “if p p , then q q ,” so the conditional statement is: “If Harry is a wizard, then Hermione is a witch.” Remember the *if … then …* words are the connectives that form the conditional statement.
2. The converse swaps or interchanges the hypothesis, p p , with the conclusion, q q . It has the form, “if q q , then p p .” So, the converse is: "If Hermione is a witch, then Harry is a wizard."
3. To construct the inverse of a statement, negate both the hypothesis and the conclusion. The inverse has the form, “if ~ p ~ p , then ~ q ~ q ,” so the inverse is: "If Harry is not a wizard, then Hermione is not a witch."
4. The contrapositive is formed by negating and interchanging both the hypothesis and conclusion. It has the form, “if ~ q ~ q , then ~ p ~ p ," so the contrapositive statement is: "If Hermione is not a witch, then Harry is not a wizard."

Your Turn 2.6.2

Use the statements, p : Elvis Presley wore capes and q : Some superheroes wear capes, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

Exercise 2.6.3

Use the conditional statement, “If all dogs bark, then Lassie likes to bark,” to identify the following.

1. Write the hypothesis of the conditional statement and label it with a p.
2. Write the conclusion of the conditional statement and label it with a q.
3. Identify the following statement as the converse, inverse, or contrapositive: “If Lassie likes to bark, then all dogs bark.”
4. Identify the following statement as the converse, inverse, or contrapositive: “If Lassie does not like to bark, then some dogs do not bark.”

5. Which statement is logically equivalent to the conditional statement?

Answer

1. The hypothesis is the phrase following the *if*. The answer is *p*: All dogs bark. Notice, the word *if* is not included as part of the hypothesis.
2. The conclusion of a conditional statement is the phrase following the *then*. The word *then* is not included when stating the conclusion. The answer is: q: Lassie likes to bark.
3. "Lassie likes to bark" is *q* and "All dogs bark" is *p*. So, "If Lassie likes to bark, then all dogs bark," has the form "if *q*, then *p*," which is the form of the converse. "Lassie does not like to bark" is $\sim q$ and "Some dogs do not bark" is $\sim p$. The statement, "If Lassie does not like to bark, then some dogs do not bark," has the form "if $\sim q$, then $\sim p$," which is the form of the contrapositive.
4. The contrapositive $\sim q \rightarrow \sim p$ is logically equivalent to the conditional statement $p \rightarrow q$.

Your Turn 2.6.3

Use the conditional statement, "If Dora is an explorer, then Boots is a monkey," to identify the following:

1. Write the hypothesis of the conditional statement and label it with a p.

2. Write the conclusion of the conditional statement and label it with a q.

3. Identify the following statement as the converse, inverse, or contrapositive: "If Dora is not an explorer, then Boots is not a monkey."

4. Identify the following statement as the converse, inverse, or contrapositive: "If Boots is a monkey, then Dora is an explorer."

5. Which statement is logically equivalent to the inverse?

Exercise 2.6.4

Assume the conditional statement, p→q: "If Chadwick Boseman was an actor, then Chadwick Boseman did not star in the movie *Black Panther*" is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.
3. Write the contrapositive of the statement in words and determine its truth value.

Answer

The only way the conditional statement can be false is if the hypothesis, *p*: Chadwick Boseman was an actor, is true and the conclusion, q: Chadwick Boseman did not star in the movie *Black Panther*, is false. The converse is $q \rightarrow p$, and it is written in words as: "If Chadwick Boseman did not star in the movie *Black Panther*, then Chadwick Boseman was an actor." This statement is true, because false → true is true.

The inverse has the form "$\sim p \rightarrow \sim q$." The written form is: "If Chadwick Boseman was not an actor, then Chadwick Boseman starred in the movie *Black Panther*." Because *p* is true, and *q* is false, $\sim p$ is false, and $\sim q$ is true. This means the inverse is false → true, which is true. Alternatively, from Question 1, the converse is true, and because the inverse is logically equivalent to the converse it must also be true.

The contrapositive is logically equivalent to the conditional. Because the conditional is false, the contrapositive is also false. This can be confirmed by looking at the truth values of the contrapositive statement. The contrapositive has the form "$\sim q \rightarrow \sim p$." Because q is false and p is true, $\sim q$ is true and $\sim p$ is false. Therefore, $\sim q \rightarrow \sim p$ is true → false, which is false. The written form of the contrapositive is "If Chadwick Boseman starred in the movie *Black Panther*, then Chadwick Boseman was not an actor."

Your Turn 2.6.4

Assume the conditional statement $p \to q$: “If my friend lives in San Francisco, then my friend does not live in California” is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.
3. Write the contrapositive of the statement in words and determine its truth value.

Check Your Understanding

1. Two statements p and q are logically equivalent to each other if the biconditional statement, $p \leftrightarrow q$ is _____.
2. The _____ statement has the form, “p then q.”
3. The contrapositive is _____ _____ to the conditional statement, and has the form, "if q, then p."
4. The _____ of the conditional statement has the form, "if p, then q."
5. The _____ of the conditional statement is logically equivalent to the _____ and has the form, "if q then p."

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2.6.1: Exercises

For the following exercises, determine whether the pair of compound statements are logically equivalent by constructing a truth table.

Exercise 2.6.1.1

Converse: $q \to p$ and inverse: $\sim p \to \sim q$

Exercise 2.6.1.2

Conditional: $p \to q$ and contrapositive: $\sim q \to \sim p$

Exercise 2.6.1.3

Inverse: $\sim p \to \sim q$ and contrapositive: $\sim q \to \sim p$

Exercise 2.6.1.4

Conditional: $p \to q$ and converse: $q \to p$

Exercise 2.6.1.5

$\sim p \to q$ and $p \vee \sim q$

Exercise 2.6.1.6

$\sim p \to q$ and $p \vee q$

Exercise 2.6.1.7

$\sim (p \wedge q)$ and $\sim p \wedge \sim q$

Exercise 2.6.1.8

$\sim (p \wedge q)$ and $\sim p \vee \sim q$

Exercise 2.6.1.9

$p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$

Exercise 2.6.1.10

$p \wedge (q \vee r)$ and $(p \wedge q) \vee r$

For the following exercises, answer the following:

a. Write the conditional statement $p \to q$ in words.
b. Write the converse statement $q \to p$ in words.
c. Write the inverse statement $\sim p \to \sim q$ in words.
d. Write the contrapositive statement $\sim q \to \sim p$ in words.

Exercise 2.6.1.11

p : Six is afraid of Seven and q : Seven ate Nine.

Exercise 2.6.1.12

p : Hope is eternal and q : Despair is temporary.

Exercise 2.6.1.13

p : Tom Brady is a quarterback and q : Tom Brady does not play soccer.

Exercise 2.6.1.14

p : Shakira does not sing opera and q : Shakira sings popular music.

Exercise 2.6.1.15

p :The shape does not have three sides and q : The shape is not a triangle.

Exercise 2.6.1.16

p : All birds can fly and q : Emus can fly.

Exercise 2.6.1.17

p : Penguins cannot fly and q : Some birds can fly.

Exercise 2.6.1.18

p : Some superheroes do not wear capes and q : Spiderman is a superhero.

Exercise 2.6.1.19

p : No Pokémon are little ponies and q : Bulbasaur is a Pokémon.

Exercise 2.6.1.20

p : Roses are red, and violets are blue and q : Sugar is sweet, and you are sweet too.

For the following exercises, use the conditional statement: "If Clark Kent is Superman, then Lois Lane is not a reporter," to answer the following questions.

Exercise 2.6.1.21

Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

Exercise 2.6.1.22

Write the conclusion of the conditional statement, label it with a q, and determine its truth value.

Exercise 2.6.1.23

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Clark Kent is not Superman, then Lois Lane is a reporter."

Exercise 2.6.1.24

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Lois Lane is a reporter, then Clark Kent is not Superman."

Exercise 2.6.1.25

Which form of the conditional is logically equivalent to the converse?

For the following exercises, use the conditional statement: "If The Masked Singer is not a music competition, then Donnie Wahlberg was a member of New Kids on the Block," to answer the following questions.

Exercise 2.6.1.26

Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

Exercise 2.6.1.27

Write the conclusion of the conditional statement, label it with a q, and determine its truth value.

Exercise 2.6.1.28

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Donnie Wahlberg was a member of New Kids on the Block, then The Masked Singer is not a music competition."

Exercise 2.6.1.29

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If The Masked Singer is a music competition, then Donnie Wahlberg was not a member of New Kids on the Block."

Exercise 2.6.1.30

Which form of the conditional is logically equivalent to the contrapositive, $\sim q \rightarrow \sim p$?

For the following exercises, use the conditional statement: "If all whales are mammals, then no fish are whales," to answer the following questions.

Exercise 2.6.1.31

Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

Exercise 2.6.1.32

Write the conclusion of the conditional statement, label it with a q, and determine its truth value.

Exercise 2.6.1.33

Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If some fish are whales, then some whales are not mammals."

Exercise 2.6.1.34

Write the inverse in words and determine its truth value.

Exercise 2.6.1.35

Write the converse in words and determine its truth value.

For the following exercises, use the conditional statement: "If some parallelograms are rectangles, then some circles are not symmetrical," to answer the following questions.

Exercise 2.6.1.36

Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

Exercise 2.6.1.37

Write the conclusion of the conditional statement, label it with a q, and determine its truth value.

Exercise 2.6.1.38

Write the converse in words and determine its truth value.

Exercise 2.6.1.39

Write the contrapositive in words and determine its truth value.

Exercise 2.6.1.40

Write the inverse in words and determine its truth value.

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2.7: De Morgan's Laws

Figure **2.7.1**: De Morgan's Laws were key to the rise of logical mathematical expression and helped serve as a bridge for the invention of the computer. (credit: modification of work "Golden Gate Bridge (San Francisco Bay, California, USA)" by James St. John/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use De Morgan's Laws to negate conjunctions and disjunctions.
2. Construct the negation of a conditional statement.
3. Use truth tables to evaluate De Morgan's Laws.

The contributions to logic made by Augustus De Morgan and George Boole during the 19th century acted as a bridge to the development of computers, which may be the greatest invention of the 20th century. **Boolean** logic is the basis for computer science and digital electronics, and without it the technological revolution of the late 20th and early 21st centuries—including the creation of computer chips, microprocessors, and the Internet—would not have been possible. Every modern computer language uses Boolean logic statements, which are translated into commands understood by the underlying electronic circuits enabling computers to operate. But how did this logic get its name?

People in Mathematics: George Boole

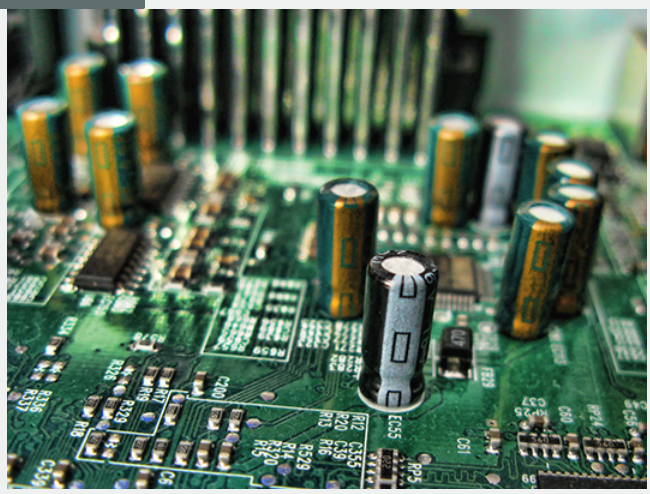


Figure **2.7.2**: Boole's algebra of logic was foundational in the design of digital computer circuits. (credit: "Circuit Board" by Squeezyboy/Flickr, CC BY 2.0)

George Boole was born in Lincolnshire, England in 1815. He was the son of a cobbler who provided him some initial education, but Boole was mostly self-taught. He began teaching at 16 years of age, and opened his own school at the age of 20. In 1849, at the age of 34, he was appointed Professor of Mathematics at Queens College in Cork, Ireland. In 1853, he published the paper, *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*, which is the treatise that the field of Boolean algebra and digital circuitry was built on.

Reference: Posamentier, Alfred and Spreitzer Christian, "Chapter 35 George Boole: English (1815-1864)" pp. 279-283, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Negation of Conjunctions and Disjunctions

In Chapter 1, Example 1.37 used a Venn diagram to prove De Morgan's Law for set complement over union. Because the complement of a set is analogous to negation and union is analogous to an *or* statement, there are equivalent versions of De Morgan's Laws for logic.

FORMULA

De Morgan's Law for negation of a conjunction: $\sim(p \wedge q) \equiv \sim p \vee \sim q$

De Morgan's Law for the negation of a disjunction: $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Negation of a conditional: $\sim(p \rightarrow q) \equiv p \wedge \sim q$

Writing conditional as a disjunction: $p \rightarrow q \equiv \sim p \vee q$

Checkpoint

Recall that the symbol for logical equivalence is: $\equiv$.

De Morgan's Laws allow us to write the negation of conjunctions and disjunctions without using the phrase, "It is not the case that …" to indicate the parentheses. Avoiding this phrase often results in a written or verbal statement that is clearer or easier to understand.

Exercise 2.7.1: Applying De Morgan's Law for Negation of Conjunctions and Disjunctions

Write the negation of each statement in words without using the phrase, "It is not the case that."

1. Kristin is a biomedical engineer and Thomas is a chemical engineer.
2. A person had cake or they had ice cream.

Answer

1. Kristin is a biomedical engineer and Thomas is a chemical engineer has the form "$p \wedge q$," where p is the statement, "Kristin is a biomedical engineer," and q is the statement, "Thomas is a chemical engineer." By De Morgan's Law, the negation of a conjunction, $\sim(p \wedge q)$, is logically equivalent to $\sim p \vee \sim q$. $\sim p$ is "Kristen is not a biomedical engineer," and $\sim q$ is "Thomas is not a chemical engineer." By De Morgan's Law, the solution has the form "$\sim p \vee \sim q$," so the answer is: "Kristin is not a biomedical engineer or Tom is not a chemical engineer."
2. A person had cake or they had ice cream has the form "$p \vee q$," where p is the statement, "A person had cake," and q is the statement, "A person had ice cream." By De Morgan's Law for the negations of a disjunction, $\sim(p \vee q) \equiv \sim p \wedge \sim q$. The solution is the statement: "A person did not have cake and they did not have ice cream."

Your Turn 2.7.1

Write the negation of each statement in words without using the phrase, *it is not the case that*.

1. Jackie played softball or she ran track.
2. Brandon studied for his certification exam, and he passed his exam.

is not a chemical engineer."

Negation of a Conditional Statement

The negation of any statement has the opposite truth values of the original statement. The **negation of a conditional**, $\sim(p \rightarrow q)$, is the conjunction of p and not q, $p \wedge \sim q$. Consider the truth table below for the negation of the conditional.

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

The only time the negation of the conditional statement is true is when p is true, and q is false. This means that $\sim(p \rightarrow q)$ is logically equivalent to $p \wedge \sim q$, as the following truth table demonstrates.

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \wedge \sim q$	$\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	F	**T**
T	F	F	T	T	T	**T**
F	T	T	F	F	F	**T**
F	F	T	F	T	F	**T**

Exercise 2.7.2: Constructing the Negation of a Conditional Statement

Write the negation of each conditional statement.

1. If Adele won a Grammy, then she is a singer.
2. If Henrik Lundqvist played professional hockey, then he did not win the Stanley Cup.

Answer

1. The negation of the conditional statement, $p \rightarrow q$, is the statement, $p \wedge \sim q$. The hypothesis of the conditional statement is p : "Adele won a Grammy," and conclusion of the conditional statement is q : "Adele is a singer." The negation of the conclusion, $\sim q$, is the statement: "She is not a singer." Therefore, the answer is $p \wedge \sim q$: "Adele won a Grammy, and she is not a singer."

2. The hypothesis is p : "Henrik Lundqvist played professional hockey," and the conclusion of the conditional statement is q : "He did not win the Stanley Cup." The negation of q is the statement: "He won the Stanley Cup." The negation of the conditional statement is equal to $p \wedge \sim q$: "Henrick Lundqvist played professional hockey, and he won the Stanley Cup."

Your Turn 2.7.2

Write the negation of each conditional statement.

1. If Edna Mode makes a new superhero costume, then it will not include a cape.
2. If I had pancakes for breakfast, then I used maple syrup.

Exercise 2.7.3: Constructing the Negation of a Conditional Statement with Quantifiers

Write the negation of each conditional statement.

1. If all cats purr, then my partner's cat purrs.
2. If a penguin is a bird, then some birds do not fly.

Answer

1. The negation of the conditional statement $p \rightarrow q$ is the statement $p \wedge \sim q$. The hypothesis of the conditional statement is p : "All cats purr," and the conclusion of the conditional statement is q : "My partner's cat purrs." The negation of the conclusion, $\sim q$, is the statement: "My partner's cat does not purr." Therefore, the answer is $p \wedge \sim q$: "All cats purr, but my partner's cat does not purr."

2. The hypothesis is p : "A penguin is a bird," and the conclusion of the conditional statement is q : "Some birds do not fly." The negation of q is the statement: "All birds fly." Therefore, the negation of the conditional statement is equal to $p \wedge \sim q$: "A penguin is a bird, and all birds fly."

Your Turn 2.7.3

Write the negation of each conditional statement.

1. If some people like ice cream, then ice cream makers will make a profit.

2. If Raquel cannot play video games, then nobody will play video games.

Many of the properties that hold true for number systems and sets also hold true for logical statements. The following table summarizes some of the most useful properties for analyzing and constructing logical arguments. These properties can be verified using a truth table.

Property	Conjunction (AND)	Disjunction (OR)
Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \wedge (q \wedge r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
Conditional	$\sim(p \rightarrow q) \equiv p \wedge \sim q$	$p \rightarrow q \equiv \sim p \vee q$

Exercise 2.7.4

Negating a Conditional Statement with a Conjunction or Disjunction

Write the negation of each conditional statement applying De Morgan's Law.

1. If mom needs to buy chips, then Mike had friends over and Bob was hungry.
2. If Juan had pizza or Chris had wings, then dad watched the game.

Answer

1. The conditional has the form "If p then q or r," where p is "Mom needs to buy chips," q is "Mike had friends over," and r is "Bob was hungry." The negation of $p \rightarrow (q \wedge r)$ is $p \wedge \sim (q \wedge r)$. Applying De Morgan's Law to the statement $\sim (q \wedge r)$ the result is $\sim q \vee \sim r$, so our conditional statement becomes $p \wedge (\sim q \vee \sim r)$. By the distributive property for conjunction over disjunction, this statement is equivalent to $(p \wedge \sim q) \vee (p \wedge \sim r)$. Translating the statement $(p \wedge \sim q) \vee (p \wedge \sim r)$ into words, the solution is: "Mom needs to buy chips and Mike did not have friends over, or Mom needs to buy chips and Bob was not hungry."

2. The conditional has the form "If p or q, then r," where p is "Juan had pizza," q is "Chris had wings," and r is "Dad watched the game." The negation of $(p \vee q) \rightarrow r$ is $(p \vee q) \wedge \sim r$. By the distributive property for disjunction over conjuction, the statement is equivalent to $(p \vee \sim r) \wedge (q \vee \sim r)$. Translating the statement $(p \vee \sim r) \wedge (q \vee \sim r)$ into words, the solution is: "Juan had pizza or dad did not watch the game, and Chris had wings or dad did not watch the game."

Your Turn 2.7.4

Write the negation of each conditional statement applying De Morgan's Law.

1. If Eric needs to replace the light bulb, then Marcus left the light on all night or Dan broke the bulb.
2. If Trenton went to school and Regina went to work, then Merika cleaned the house.

Evaluating De Morgan's Laws with Truth Tables

In Chapter 1, you learned that you could prove the validity of De Morgan's Laws using Venn diagrams. Truth tables can also be used to prove that two statements are logically equivalent. If two statements are logically equivalent, you can use the form of the statement that is clearer or more persuasive when constructing a logical argument.

The next example will prove the validity of one of De Morgan's Laws using a truth table. The same procedure can be applied to any two logical statement that you believe are equivalent. If the last column of the truth table is a tautology, then the two statements are logically equivalent.

Exercise 2.7.5: Verifying De Morgan's Law for Negation of a Conjunction

Construct a truth table to verify De Morgan's Law for the negation of a conjunction, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, is valid.

Answer

Step 1: To verify any logical equivalence, you must first replace the logical equivalence symbol, $\equiv$, with the biconditional symbol, $\leftrightarrow$. The statement $\sim(p \wedge q) \equiv \sim p \vee \sim q$ becomes $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$

Step 2: Next, you create a truth table for the statement. Because we have two basic statements, p, and q, the truth table will have four rows to account for all the possible outcomes. The columns will be $p, q, \sim p, \sim q, p \wedge q, \sim(p \wedge q), \sim p \vee \sim q$, and the biconditional statement is $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	T	F	F	F	F	**T**
T	F	F	T	F	T	T	**T**
F	T	F	T	T	F	T	**T**
F	F	F	T	T	T	T	**T**

Step 3: Finally, verify that the statement is valid by confirming it is a tautology. In this instance, the last column is all true. Therefore, the statement is valid and De Morgan's Law for the negation of a conjunction is verified.

Your Turn 2.7.5

1. Construct a truth table to verify De Morgan's Law for the negation of a disjunction, $\sim(p \vee q) \equiv \sim p \wedge \sim q$, is valid.

Check Your Understanding

1. De Morgan's Law for the negation of a conjunction states that \($(p \wedge q)$ is logically equivalent to _____.

2. De Morgan's Law for the negation of a disjunction states that $(p \vee q)$ is logically equivalent to _____.

3. The negation of the conditional statement, $(p \rightarrow q)$, is logically equivalent to _____.

$(\ (p \rightarrow q)) \equiv p \rightarrow q$, which means the conditional statement is logically equivalent to $(p \wedge\ q)$. Apply _____ to the statement $(p \wedge\ q)$ to show that the conditional statement $p \rightarrow q \equiv\ p \vee q$.

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2.7.1: Exercises

For the following exercises, use De Morgan's Laws to write each statement without parentheses.

Exercise 2.7.1.1

$\sim(\sim p \vee q)$

Exercise 2.7.1.2

$\sim(\sim p \wedge \sim q)$

Exercise 2.7.1.3

$\sim(p \wedge \sim q)$

Exercise 2.7.1.4

$\sim(\sim p \vee \sim q)$

For the following exercises, use De Morgan's Laws to write the negation of each statement in words without using the phrase, "It is not the case that, ..."

Exercise 2.7.1.5

Sergei plays right wing and Patrick plays goalie.

Exercise 2.7.1.6

Mario is a carpenter, or he is a plumber.

Exercise 2.7.1.7

Luigi is a plumber, or he is not a video game character.

Exercise 2.7.1.8

Ralph Macchio was the original Karate Kid, and karate is not for defense only.

Exercise 2.7.1.9

Some people like broccoli, but my siblings did not like broccoli.

Exercise 2.7.1.10

Some people do not like chocolate or all people like pizza.

For the following exercises, write each statement as a conjunction or disjunction in symbolic form by applying the property for the negation of a conditional.

Exercise 2.7.1.11

$\sim (p \rightarrow q)$

Exercise 2.7.1.12

$\sim (p \rightarrow \sim q)$

Exercise 2.7.1.13

$\sim (\sim p \rightarrow q)$

Exercise 2.7.1.14

$\sim (\sim p \rightarrow \sim q)$

Exercise 2.7.1.15

$\sim (p \wedge q \rightarrow \sim r)$

Exercise 2.7.1.16

$\sim (p \rightarrow q \vee r)$

Exercise 2.7.1.17

$\sim (p \vee q \rightarrow \sim r)$

Exercise 2.7.1.18

$(p \rightarrow q \wedge r)$

For the following exercises, write the negation of each conditional in words by applying the property for the negation of a conditional.

Exercise 2.7.1.19

If a student scores an 85 on the final exam, then they will receive an A in the class.

Exercise 2.7.1.20

If a person does not pass their road test, then they will not receive their driver's license.

Exercise 2.7.1.21

If a student does not do their homework, then they will not play video games.

Exercise 2.7.1.22

If a commuter misses the bus, then they will not go to work today.

? Exercise 2.7.1.23

If a racecar driver gets pulled over for speeding, then they will not make it to the track on time for the race.

? Exercise 2.7.1.24

If Rene Descartes was a philosopher, then he was not a mathematician.

? Exercise 2.7.1.25

If George Boole invented Boolean algebra and Thomas Edison invented the light bulb, then Pacman is not the best video game ever.

? Exercise 2.7.1.26

If Jonas Salk created the polio vaccine, then his child received the vaccine or his child had polio.

? Exercise 2.7.1.27

If Billie Holiday sang the blues or Cindy Lauper sang about true colors, then John Lennon was not a Beatle.

? Exercise 2.7.1.28

If Percy Jackson is the lightning thief and Artemis Fowl is a detective, then Artemis Fowl will catch Percy Jackson.

? Exercise 2.7.1.29

If all rock stars are men, then Pat Benatar is not a rock star.

? Exercise 2.7.1.30

If Lady Gaga is a rock star, then some rock stars are women.

? Exercise 2.7.1.31

If yellow combined with blue makes green, then all colors are beautiful.

? Exercise 2.7.1.32

If leopards have spots and zebras have stripes, then some animals are not monotone in color. For the following exercises, construct a truth table to verify that the logical property is valid.

? Exercise 2.7.1.33

$p \rightarrow q \equiv \sim p \vee q$

? Exercise 2.7.1.34

$p \rightarrow \sim q \equiv \sim p \vee \sim q$

Exercise 2.7.1.35

$\sim p \rightarrow q \equiv p \vee q$

Exercise 2.7.1.36

$\sim (p \rightarrow \sim q \vee \sim r) \equiv p \wedge q \wedge r$

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2.8: Logical Arguments

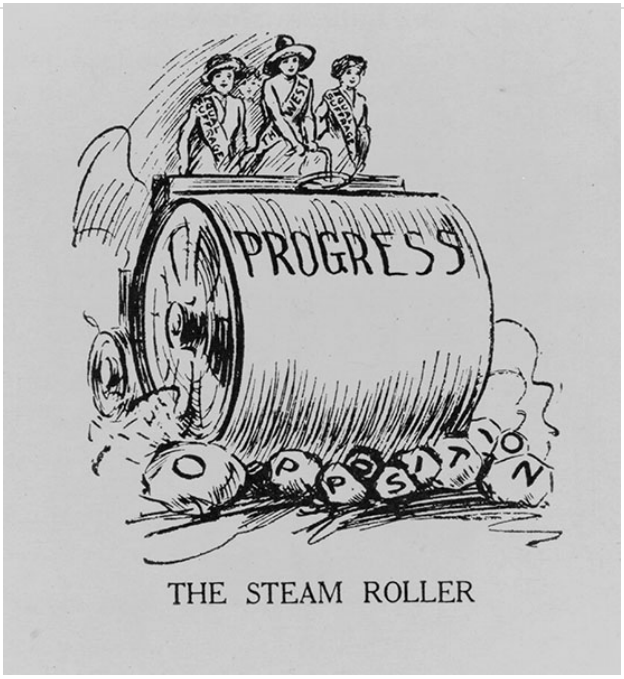


Figure **2.8.1**: Not all logical arguments are valid, and the ongoing fight for equal rights proves that much progress has yet to be made. (credit: "The Steam Roller" by Library of Congress Prints and Photographs Division, public domain) ;

Learning Objectives

After completing this section, you should be able to:

1. Apply the law of detachment to determine the conclusion of a pair of statements.
2. Apply the law of denying the consequent to determine the conclusion for pairs of statements.
3. Apply the chain rule to determine valid conclusions for pairs of true statements.

The previous sections of this chapter provide the foundational skills for constructing and analyzing logical arguments. All logical arguments include a set of premises that support a claim or conclusion; but not all logical arguments are valid and sound. A logical argument is valid if its conclusion follows from the premises, and it is sound if it is valid and all of its premises are true. A false or deceptive argument is called a fallacy. Many types of fallacies are so common that they have been named.

Who Knew?

In 1936, Dale Carnegie published his first book, titled *How to Win Friends and Influence People*. It was marketed as training materials for the improvement of public speaking and negotiation skills, and the methods it presented are still used today. Carnegie famously said, “When dealing with people, remember you are not dealing with creatures of logic, but creatures of emotion.”

People who put forth fallacious logical arguments often take advantage of our susceptibility to emotional appeals, to try to convince us that what they are saying is true. The study of logic helps us combat this weakness through recognition and learning to focus on the facts and structure of the argument.

This section focuses on the two main forms that logical arguments can take. While inductive arguments attempt to draw a more general conclusion from a pattern of specific premises, deductive arguments attempt to draw specific conclusions from at least one or more general premises. Deductive arguments can be proven to be valid using Venn diagrams or truth tables.

Inductive arguments generally cannot be proven to be true. They are judged as being strong or weak, but, like any opinion, whether you believe an argument is strong or weak often depends on your knowledge of the topic being discussed along with the evidence being provided in the premises. *Hasty generalization* is the name given to any fallacy that presents a weak inductive argument.

Checkpoint

Be careful! Premises may be true or false. If a premise is false, the claims made by the argument should be questioned.

Law of Detachment

The law of detachment is a valid form of a conditional argument that asserts that if both the conditional, $p \to q$, and the hypothesis, p, are true, then the conclusion q must also be true. The law of detachment is also called affirming the hypothesis (or antecedent) and modus ponens. Symbolically, it has the form $((p \to q) \wedge p) \to q$.

Law of Detachment	
Premise:	$p \to q$
Premise:	p
Conclusion:	$\therefore q$

Checkpoint

The $\therefore$ is read as the word, "therefore."

Looking at the truth table for the conditional statement, the only time the conditional is true is when the hypothesis p is also true. The only place this happens is in the first row, where q is also true, confirming that the law of detachment is a valid argument.

p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Another way to verify that the law of detachment is a valid argument is to construct a truth table for the argument $((p \to q) \wedge p) \to q$ and verify that it is a tautology.

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Venn diagrams may also be used to verify deductive arguments, which include conditional premises. Consider the statement $p \to q$:Then verify if the conclusion is also represented by the Venn diagram of the premises. If it is, the argument is valid. If it is not, the argument is not valid. The set of guitarists is drawn as a subset of the set of musicians to represent the premise.

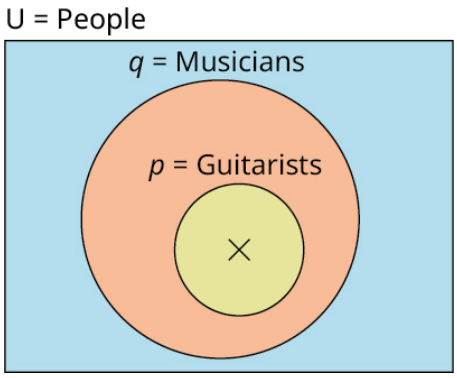


Figure 2.8.2

Now, examine the Venn diagram to verify if the conclusion is included in the picture. The conclusion is q. Because the $\times$ is in the set p, and p is a subset of q, $\times$ is also in q; therefore, the law of detachment is a valid argument.

Checkpoint

Remember that an argument can be valid without being true. For the argument to be proven true, it must be both valid and sound. An argument is sound if all its premises are true.

Video



Logic Part 14: Common Argument Forms like Modus Ponens and Tollens

Exercise 2.8.1: Applying the Law of Detachment to Determine a Valid Conclusion

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci was an artist.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan played for the Chicago Bulls.
3. If all fish have gills, then clown fish have gills. All fish have gills.

Answer

1. The premises are $p \rightarrow q$: If Leonardo da Vinci was an artist, then he painted the Mona Lisa, and p : Leonardo da Vinci was an artist. This argument has the form of the law of detachment, so, the conclusion is q : Leonardo da Vinci painted the Mona Lisa.

2. The premises follow the form of the law of detachment, so a valid conclusion would be q. The premises are $p \rightarrow q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and p : Michael Jordan played for the Chicago Bulls. The conclusion that follows from the premises is q : Michael Jordan was not a soccer player.

3. The premises are $p \to q$: If all fish have gills, then clown fish have gills, and p : All fish have gills. This argument has the form of the law of detachment, so the conclusion is q : clown fish have gills.

Your Turn 2.8.1

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If my classmate likes history, then some people like history. My classmate likes history.
2. If you do not like to read, then some people do not like reading. You do not like to read.
3. If the polygon has five sides, then it is not an octagon. The polygon has five sides.

Another form of a valid conditional argument is called the law of denying the consequent, or modus tollens. Recall, that the conditional statement, $p \to q$, is logically equivalent to the contrapositive, $\sim q \to \sim p$. So, if the conditional statement is true, then the contrapositive statement is also true. By the law of detachment, if $\sim q$ is also true, then it follows that $\sim p$ must also be true. Symbolically, it has the form $((p \to q) \wedge \sim q) \to \sim p$.

Law of Denying the Consequent	
Premise:	$p \to q$
Premise:	$\sim q$
Conclusion:	$\therefore \sim p$

Checkpoint

The conditional statement can also be described as, "If antecedent, then consequent."This is where the law of denying the consequent gets its name.

To verify if the law of denying the consequent is a valid argument, construct a truth table for the argument, $((p \to q) \wedge \sim q) \to \sim p$, and verify that it is a tautology.

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim q$	$((p \to q) \wedge \sim q) \to \sim p$
T	T	F	F	T	F	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	F	**T**
F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise: $p \to q$

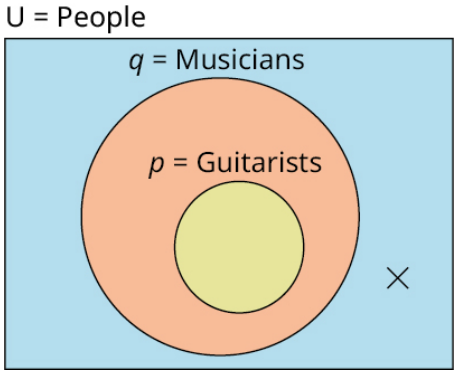


Figure 2.8.3

Because the × is also outside the set of guitarists, the statement $\sim p$ follows from the premises and the argument is valid.

Exercise 2.8.2: Applying the Law of Denying the Consequent to Determine a Valid Conclusion

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci did not paint the *Mona Lisa*.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan was a soccer player.
3. If all fish have gills, then clown fish have gills. Clown fish do not have gills.

Answer

1. The premises are $p \rightarrow q$: If Leonardo da Vinci was an artist, then he painted the Mona Lisa, and $\sim q$: Leonardo da Vinci did not paint the Mona Lisa. This argument has the form of the law of denying the consequent, so the conclusion is $\sim p$: Leonardo da Vinci was not an artist.
2. The premises follow the form of the law of denying the consequent, so a valid conclusion would be $\sim p$. The premises are: $p \rightarrow q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and $\sim q$: Michael Jordan was a soccer player. The conclusion that follows from the premises is $\sim p$: Michael Jordan did not play for the Chicago Bulls.
3. The premises are $p \rightarrow q$: If all fish have gills, then clown fish have gills, and $\sim q$: Clown fish do not have gills. This argument has the form of the law denying the consequent, so the conclusion is $\sim p$: Some fish do not have gills.

Your Turn 2.8.2

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If my classmate likes history, then some people like history. Nobody likes history.
2. If Homer does not like to read, then some people do not like reading. All people like reading.
3. If the polygon has five sides, then it is not an octagon. The polygon is an octagon.

Chain Rule for Conditional Arguments

The chain rule for conditional arguments is another form of a valid conditional argument. It is also called hypothetical syllogism or the transitivity of implication. Recall that the conditional statement $p \rightarrow q$ can also be read as p implies q. This is where the name *transitivity of implication* comes from. The transitive property for numbers states that, if $3 < 4$ and $4 < 5$, then it follows that $3 < 5$. The chain rule extends this property to conditional statements. If the premises of the argument consist of two conditional statements, with the form "$p \rightarrow q$" and "$q \rightarrow r$," then it follows that $p \rightarrow r$. Symbolically, it has the form $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$.

Chain Rule for Conditional Arguments	
Premise:	$p \rightarrow q$
Premise:	$q \rightarrow r$
Conclusion:	$\therefore p \rightarrow r$

To verify the chain rule for conditional arguments, construct a truth table for the argument, $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$, and verify that it is a tautology.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (p \to r)$	$p \to r$	$((p \to q) \wedge (q \to r)) -$
T	T	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	T	F	T	F	T	**T**
T	F	F	F	T	F	F	**T**
F	T	T	T	T	T	T	**T**
F	T	F	T	F	F	T	**T**
F	F	T	T	T	T	T	**T**
F	F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise $p \to q$:

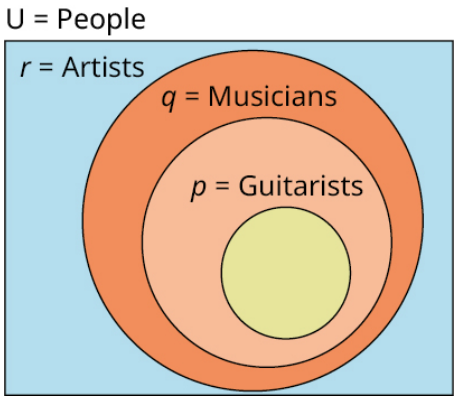


Figure 2.8.4

? Exercise 2.8.3: Applying the Chain Rule for Conditional Arguments to Determine a Valid and Sound Conclusion

Each pair of statements represents true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate goes to work, then my roommate will get paid. If my roommate gets paid, then my roommate will pay their bills.
2. If robins can fly, then some birds can fly. If some birds can fly, then we will watch birds fly.
3. If Irma is a teacher, then Irma has a college degree. If Irma has a college degree, then Irma graduated from college.

Answer

1. The premises are $p \to q$: "If my roommate goes to work, then they will get paid," and $q \to r$: "If my roommate gets paid, then my roommate will pay their bills." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If my roommate goes to work, then my roommate will pay their bills."

2. The premises are $p \to q$: "If robins can fly, then some birds can fly," and $q \to r$: "If some birds can fly, then we will watch them fly." This argument has the form of the chain rule for conditional arguments, so, the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If robins can fly, then we will watch birds fly."

3. The premises are $p \to q$: (see line 1 of solution 1 and 2 above) "If Irma is a teacher, then Irma has a college degree," and $q \to r$: "If Irma has a college degree, then Irma graduated from college." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form " $p \to r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If Irma is a teacher, then Irma graduated from college."

Your Turn 2.8.3

Each pair of statements represent true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate does not go to work, then my roommate will not get paid. If my roommate does not get paid, then they will not be able to pay their bills.

2. If penguins cannot fly, then some birds cannot fly. If some birds cannot fly, then we will watch the news.

3. If Marcy goes to the movies, then Marcy will buy popcorn. If Marcy buys popcorn, then she will buy water.

Check Your Understanding

1. A _____ is a logical statement used as a fact to support the conclusion of an argument.
2. A logical argument is _____ if its conclusion follows from the premises.
3. A logical argument that attempts to draw a more general conclusion from a pattern of specific premises is called an _____ argument.
4. A _____ argument draws specific conclusions from more general premises.
5. Not all arguments are true. A false or deceptive argument is called a _____.
6. If an argument is valid and all of its premises are true, then it is considered _____.

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2.8.1: Exercises

For the following exercises, analyze the argument and identify the form of the argument as the law of detachment, the law of denying the consequent, the chain rule for conditional arguments, or none of these.

Exercise 2.8.1.1

If Apple Inc. releases a new iPhone, then customers will buy it. Customers did not buy a new iPhone. Therefore, Apple Inc. did not release a new iPhone.

Exercise 2.8.1.2

In the animated movie Toy Story, if Paul Newman turned down the role of voicing Woody, then Tom Hanks was chosen for the role. Tom Hanks was chosen as the voice for Woody, therefore, Paul Newman turned down the role of voicing Woody in Toy Story.

Exercise 2.8.1.3

$p \rightarrow q$ and $q \rightarrow r$. $\therefore p \rightarrow r$.

Exercise 2.8.1.4

$p \rightarrow q$ and p. $\therefore q$.

Exercise 2.8.1.5

$p \rightarrow q$ and $\sim q$. $\therefore \sim p$.

Exercise 2.8.1.6

If all people are created equal, then all people are the same with respect to the law. If all people are the same with respect to the law, then justice is blind. Therefore, if all people are created equal, then justice is blind.

Exercise 2.8.1.7

If I mow the lawn, then my caregiver will pay me twenty dollars. I mowed the lawn. Therefore, my caregiver paid me twenty dollars.

Exercise 2.8.1.8

If Robin Williams was a comedian, then some comedians are funny. No comedians are funny. Therefore, Robin Williams was not a comedian.

For the following exercises, each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine and write a valid conclusion.

Exercise 2.8.1.9

$p \rightarrow \sim q$ and p.

Exercise 2.8.1.10

$\sim p \rightarrow q$ and $\sim p$.

Exercise 2.8.1.11

If Richard Harris played Dumbledore, then Daniel Radcliffe played Harry Potter. Richard Harris played Dumbledore.

Exercise 2.8.1.12

If Emma Watson is an actor, then Emma Watson starred as Belle in the movie Beauty and the Beast. Emma Watson is an actor.

Exercise 2.8.1.13

If some Granny Smiths are available, then we will make an apple pie. Some Granny Smiths are available.

Exercise 2.8.1.14

If Peter Rabbit lost his coat, then all rabbits must avoid Mr. McGregor's garden. Peter Rabbit lost his coat. For the following exercises, each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine and write a valid conclusion.

Exercise 2.8.1.15

If Greg and Ralph are friends, then Greg will not play a prank on Ralph. Greg played a prank on Ralph.

Exercise 2.8.1.16

If Drogon is not a dragon, then Daenerys ruled Westeros. Daenerys did not rule Westeros.

Exercise 2.8.1.17

$p \rightarrow \sim q$ and q.

Exercise 2.8.1.18

$\sim p \rightarrow q$ and $\sim q$.

Exercise 2.8.1.19

If all dragons breathe fire, then rainwings are not dragons. Rainwings are dragons.

Exercise 2.8.1.20

If some pirates have parrots as pets, then some parrots do not like crackers. All parrots like crackers.

For the following exercises, each pair of statements represent true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

Exercise 2.8.1.21

$\sim p \rightarrow q$ and $q \rightarrow \sim r$.

Exercise 2.8.1.22

$r \rightarrow \sim q$ and $\sim q \rightarrow \sim p$.

Exercise 2.8.1.23

$q \rightarrow r$ and $p \rightarrow q$.

Exercise 2.8.1.24

$\sim r \rightarrow p$ and $q \rightarrow \sim r$.

Exercise 2.8.1.25

If Mr. Spock is a science officer, then Montgomery Scott is an engineer. If Montgomery Scott is an engineer, then James T. Kirk is the captain.

Exercise 2.8.1.26

If Prince Charles is a character from Star Wars, then Luke Skywalker is not a Jedi. If Luke Skywalker is not a Jedi, then Darth Vader is not his father.

For the following exercises, each pair of statements represent true premises in a logical argument. Based on these premises, state a valid conclusion based on the form of the argument.

Exercise 2.8.1.27

If my siblings drink milk out of the carton, then they will leave the carton on the counter. My siblings did not leave the carton on the counter.

Exercise 2.8.1.28

If my friend likes to bowl, then my partner does not like to play softball. My friend likes to bowl.

Exercise 2.8.1.29

If mathematics is fun, then students will study algebra. If students study algebra, then they will score a 100 on their final exam.

Exercise 2.8.1.30

If all fleas bite and our dog has fleas, then our dog will scratch a lot. Our dog will not scratch a scratch a lot.

Exercise 2.8.1.31

If the toddler is not tall, then they will use a stepladder to reach the cookie jar. If the toddler will use a stepladder to reach the cookie jar, then they will drop the jar. If they drop the cookie jar, then they will not eat any cookies.

Exercise 2.8.1.32

If you do not like to dance, then you will not go to the club. You went to the club.

For the following exercises, use a truth table or construct a Venn diagram to prove whether the following arguments are valid.

Exercise 2.8.1.33

Denying the hypothesis: $p \rightarrow q$ and $\sim p$. Therefore, $\sim q$.

Exercise 2.8.1.34

Affirming the consequent: $p \rightarrow q$ and q. Therefore, p.

Exercise 2.8.1.35

$\sim p \vee q$ and p. Therefore, q.

Exercise 2.8.1.36

$p \wedge q \rightarrow r$ and $\sim r$. Therefore, $\sim p \vee \sim q$.

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CHAPTER OVERVIEW

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3.1: Introduction

In this chapter, you will learn the fundamental skills needed to work with graphs used in an area of mathematics known as graph theory. You can think of these graphs as a kind of map. Maps have served many purposes over the course of history. You probably use GPS maps to navigate to various destinations. A scientist from ancient Greece named Ptolemy wanted an accurate map of the world to make more accurate astrological predications. In recent years, neurobiologists have mapped the cerebral cortex to better understand the human brain. Social network analysts map online interactions to assist advertisers in reaching target audiences. Like other maps, the graphs you will study in this chapter can serve many purposes, but they do not have a lot of the details you might expect in a map such as size, shape, and distance between objects. All of that is stripped away so that we can focus on one element of maps, the connections between objects.



Figure **3.1.1** Networks connect cities around the globe. (credit: "Globalization" by faith.e.murphy/Flickr, Public Domain)

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3.2: Graph Basics



Figure **3.2.1**: Cell phone networks connect individuals. (credit: "Business people using their phones" by Rawpixel Ltd./Flickr, CC BY 2.0)

Learning Objectives

1. Identify parts of a graph.
2. Model applications of graph basics.

When you hear the word, *graph*, what comes to mind? You might think of the semanticsxyxy-coordinate system you learned about earlier in this course, or you might think of the line graphs and bar charts that are used to display data in news reports. The graphs we discuss in this chapter are probably very different from what you think of as a graph. They look like a bunch of dots connected by short line segments. The dots represent a group of objects and the line segments represent the connections, or relationships, between them. The objects might be bus stops, computers, Snapchat accounts, family members, or any other objects that have direct connections to each other. The connections can be physical or virtual, formal or casual, scientific or social. Regardless of the kind of connections, the purpose of the graph is to help us visualize the structure of the entire network to better understand the interactions of the objects within it.

Parts of a Graph

In a graph, the objects are represented with dots and their connections are represented with lines like those in Figure 12.3. Figure 12.3 displays a simple graph labeled G and a multigraph labeled H. The dots are called vertices; an individual dot is a **vertex**, which is one object of a set of objects, some of which may be connected. We often label vertices with letters. For example, Graph G has vertices a, b, c, and d, and Multigraph H has vertices, e, f, g, and h. Each line segment or connection joining two vertices is referred to as an edge. H is considered a multigraph because it has a double edge between f and h, and a double edge between h and g. Another reason H is called a multigraph is that it has a loop connecting vertex e to itself; a loop is an edge that joins a vertex to itself. Loops and double edges are not allowed in a simple graph.

To sum up, a simple graph is a collection of vertices and any edges that may connect them, such that every edge connects two vertices with no loops and no two vertices are joined by more than one edge. A multigraph is a graph in which there may be loops

or pairs of vertices that are joined by more than one edge. In this chapter, most of our work will be with simple graphs, which we will call graphs for convenience.

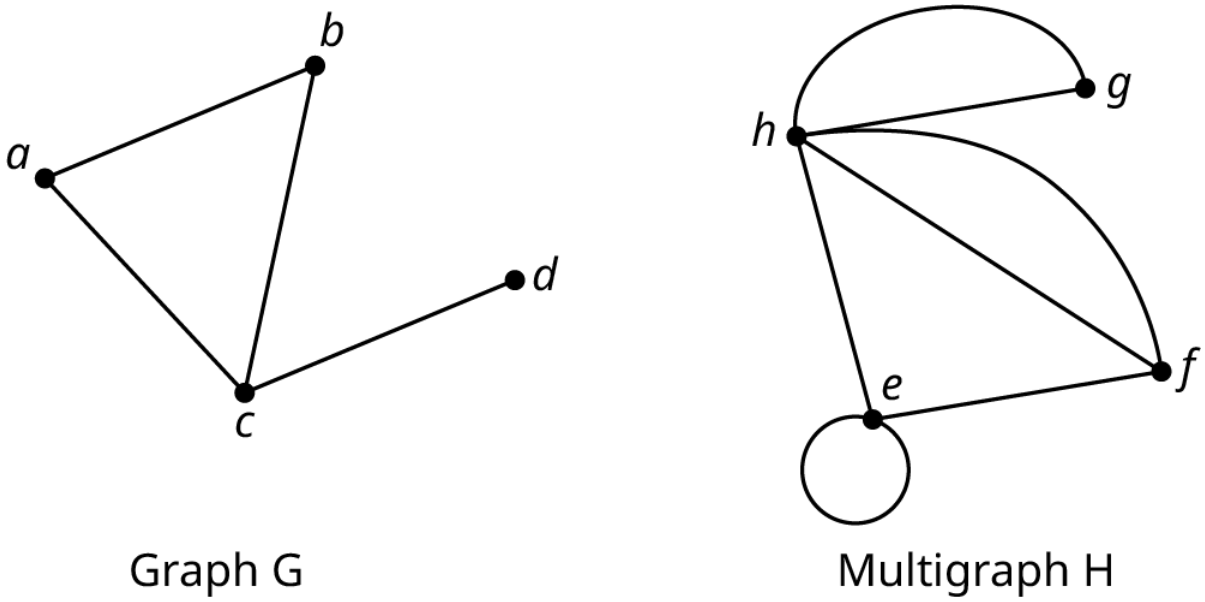


Figure **3.2.2**: A Graph and a Multigraph

It is not necessary for the edges in a graph to be straight. In fact, you can draw an edge any way you want. In graph theory, the focus is on which vertices are connected, not how the connections are drawn (see Figure 12.4). In a graph, each edge can be named by the two letters of the associated vertices. The four edges in Graph *X* in Figure 12.4 are *ab*, *ac*, *ad*, and *ae*. The order of the letters is not important when you name the edge of a graph. For example, *ab* refers to the same edge as *ba*.

Checkpoint

A graph may have vertices that are not joined to other vertices by edges, such as vertex f in Graph X in Figure 12.4, but any edge must have a vertex at each end.

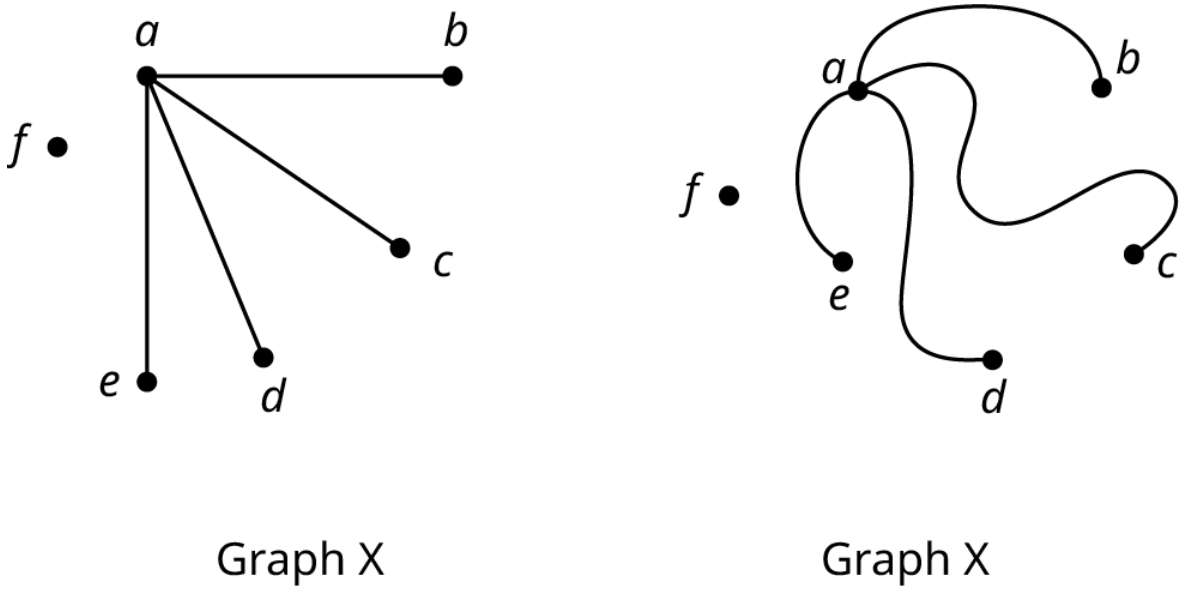


Figure **3.2.3**: Different Representations of the Same Graph

Example 3.2.1: Identifying Edges and Vertices

Name all the vertices and edges of graph *F* in Figure **3.2.4**.

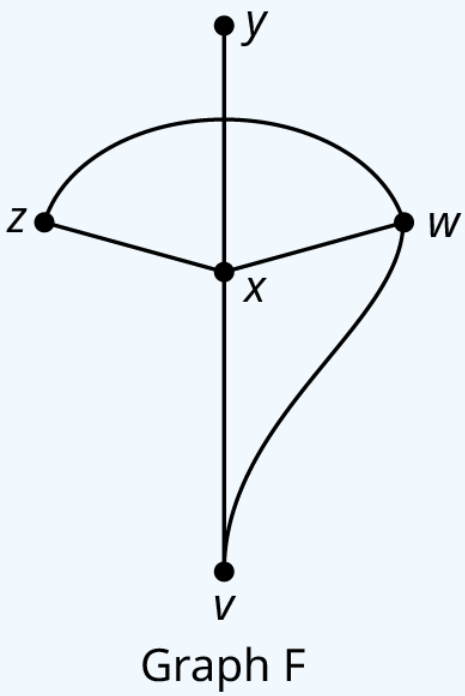


Figure **3.2.4** Graph *F*

Answer

The vertices are *v*, *w*, *x*, *y*, and *z*. The edges are *vw*, *vx*, *wx*, *wz*, *xy*, and *xz*.

Checkpoint

When listing the vertices and edges in a graph, work in alphabetical order to avoid accidentally listing the same item twice. When you are finished, count the number of vertices or edges you listed and compare that to the number of vertices or edges on the graph to ensure you didn't miss any.

Your Turn 3.2.1

Name all the vertices and edges of Graph *A*.

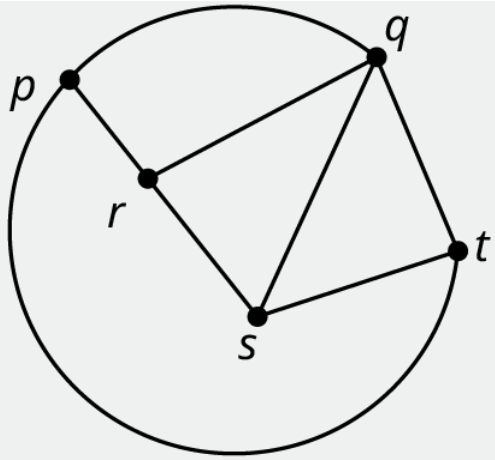


Figure **3.2.5**: Graph *A*

Since the purpose of a graph is to represent the connections between objects, it is very important to know if two vertices share a common edge. The two vertices at either end of a given edge are referred to as neighboring, or adjacent. For example, in Figure 12.5, vertices *x* and *w* are adjacent, but vertices *y* and *w* are not.

Example 3.2.2: Identifying Vertices That Are Not Adjacent

Name all the pairs of vertices of graph *F* in Figure 12.5 that are *not* adjacent.

Answer

The pairs of vertices that are not adjacent in graph *F* are *v* and *y*, *v* and *z*, *w* and *y*, and *y* and *z*.

Your Turn 3.2.2

Name all the pairs of vertices of graph *A* in Figure 12.6 that are *not* adjacent.

People in Mathematics: Sergey Brin and Laurence Page

The “Google boys,” Sergey Brin and Laurence Page, transformed the World Wide Web in 1998 when they used the mathematics of graph theory to create an algorithm called Page Rank, which is known as the Google Search Engine today. The two computer scientists identified webpages as vertices and hyperlinks on those pages as edges because hyperlinks connect one website to the next. The number of edges influences the ranking of a website on the Google Search Engine because the websites with more links to “credible sources” are ranked higher. ("Page Rank: The Graph Theory-based Backbone of Google," September 20, 2011, Cornell University, Networks Blog.

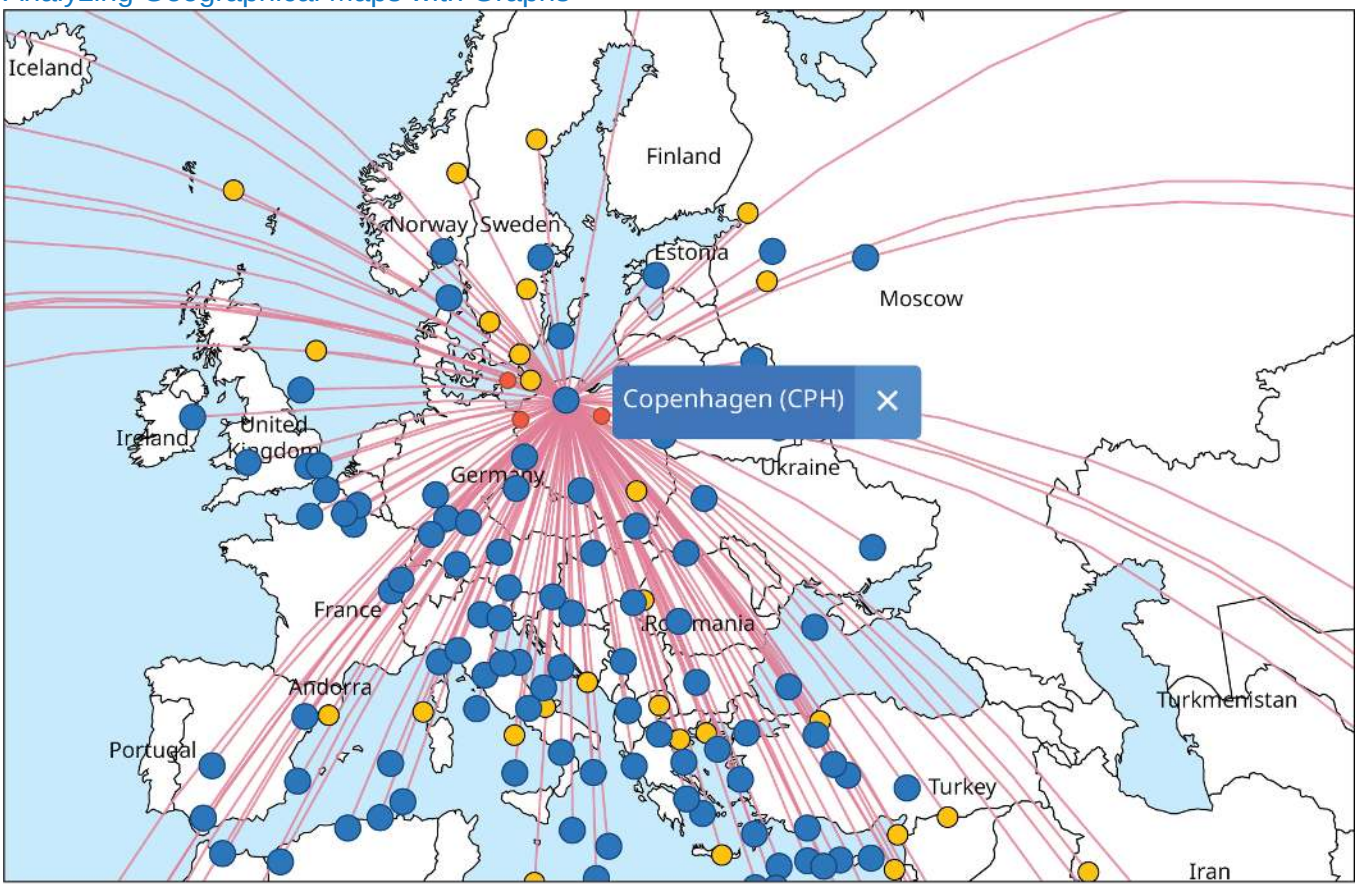


Figure 3.2.6: Commercial airlines' route systems create a global network.

When graphs are used to model and analyze real-world applications, the number of edges that meet at a particular vertex is important. For example, a graph may represent the direct flight connections for a particular airport as in Figure 3.2.7. Representing the connections with a graph rather than a map shifts the focus away from the relative positions and toward which airports are connected. In Figure 3.2.7, the vertices are the airports, and the edges are the direct flight paths. The number of flight connections between a particular airport and other South Florida airports is the number of edges meeting at a particular vertex. For example, Key West has direct flights to three of the five airports on the graph. In graph theory terms, we would say that vertex FYW has degree 3. The degree of a vertex is the number of edges that connect to that vertex.

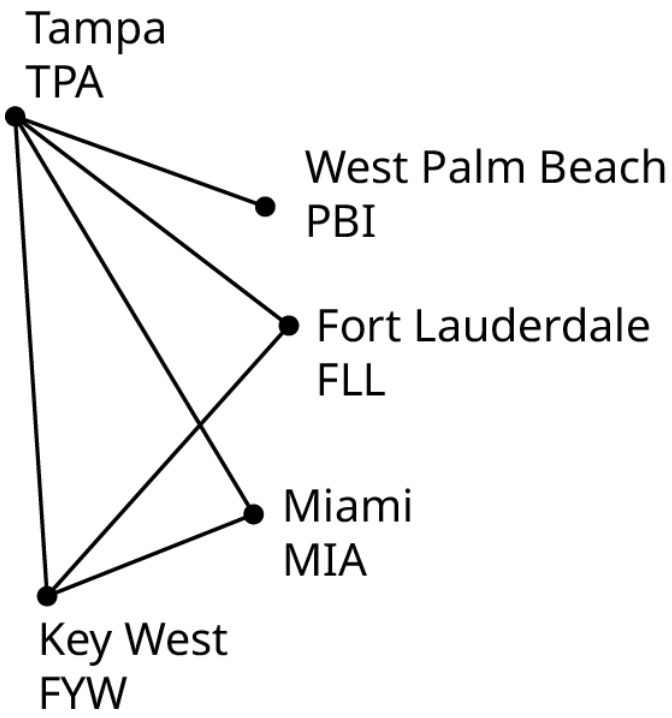


Figure **3.2.7**: Direct Flights between South Florida Airports

Example 3.2.3: Determining the Degree of a Vertex

Determine the degree of each vertex of Graph *J* in Figure **3.2.8**. If graph *J* represents direct flights between a set of airports, do any of the airports have direct flights to two or more of the other cities on the graph?

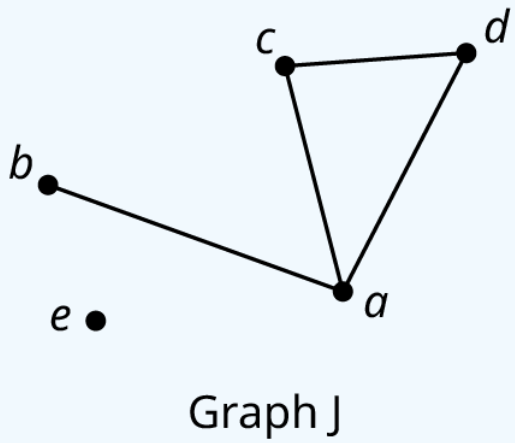


Figure 1**3.2.8** Graph *J*

Answer

For each vertex, count the number of edges that meet at that vertex. This value is the degree of the vertex. In Figure 12.10, the dashed edges indicate the edges that meet at the marked vertex.

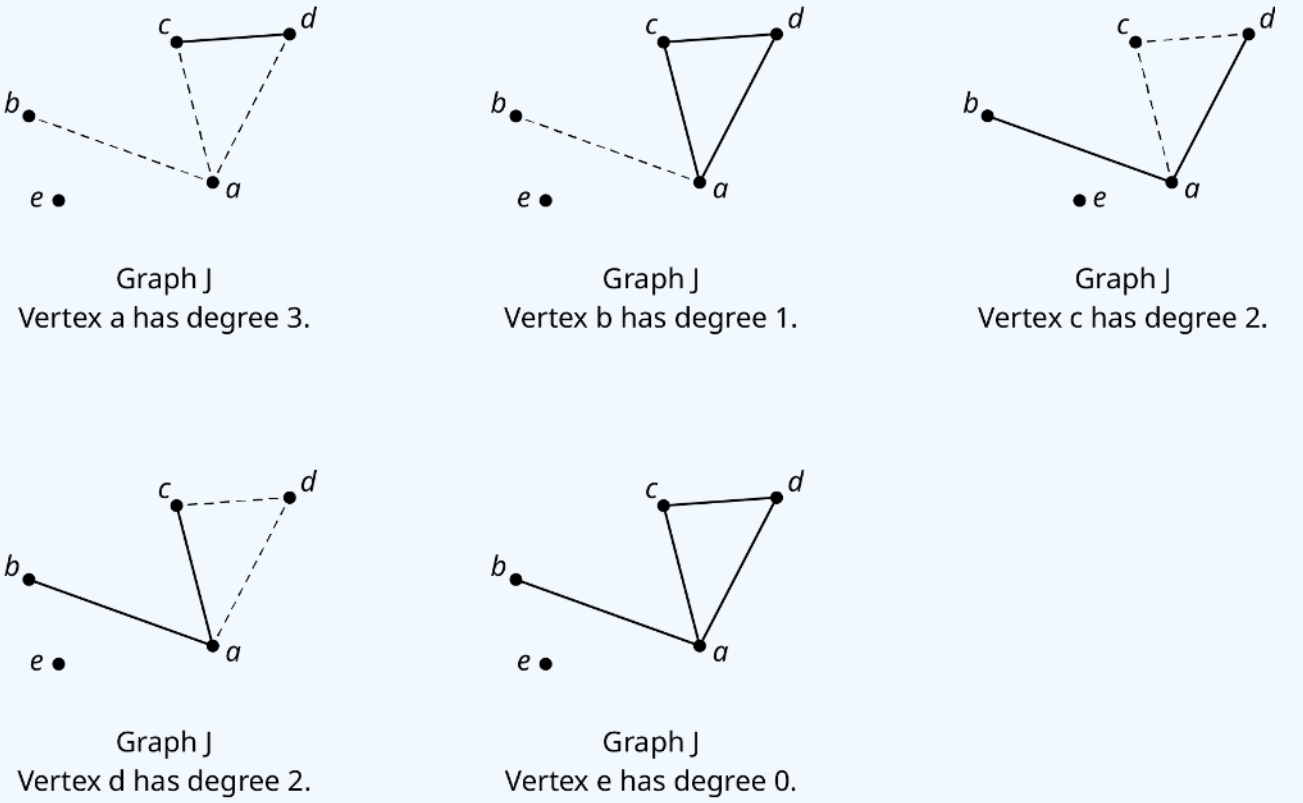


Figure **3.2.9** Degrees of Vertices of Graph *J*

Vertex *a* has degree 3, vertex *b* has degree 1, vertices *c* and *d* each have degree 2, and vertex *e* has degree 0. Airports *a*, *c*, and *d* have direct flights to two or more of the other airports.

Your Turn 3.2.3

Name a vertx of Graph *A* in Figure **3.2.6** with degree 4.

Graphs are also used to analyze regional boundaries. The states of Utah, Colorado, Arizona, and New Mexico all meet at a single point known as the “Four Corners,” which is shown in the map in Figure 12.11.

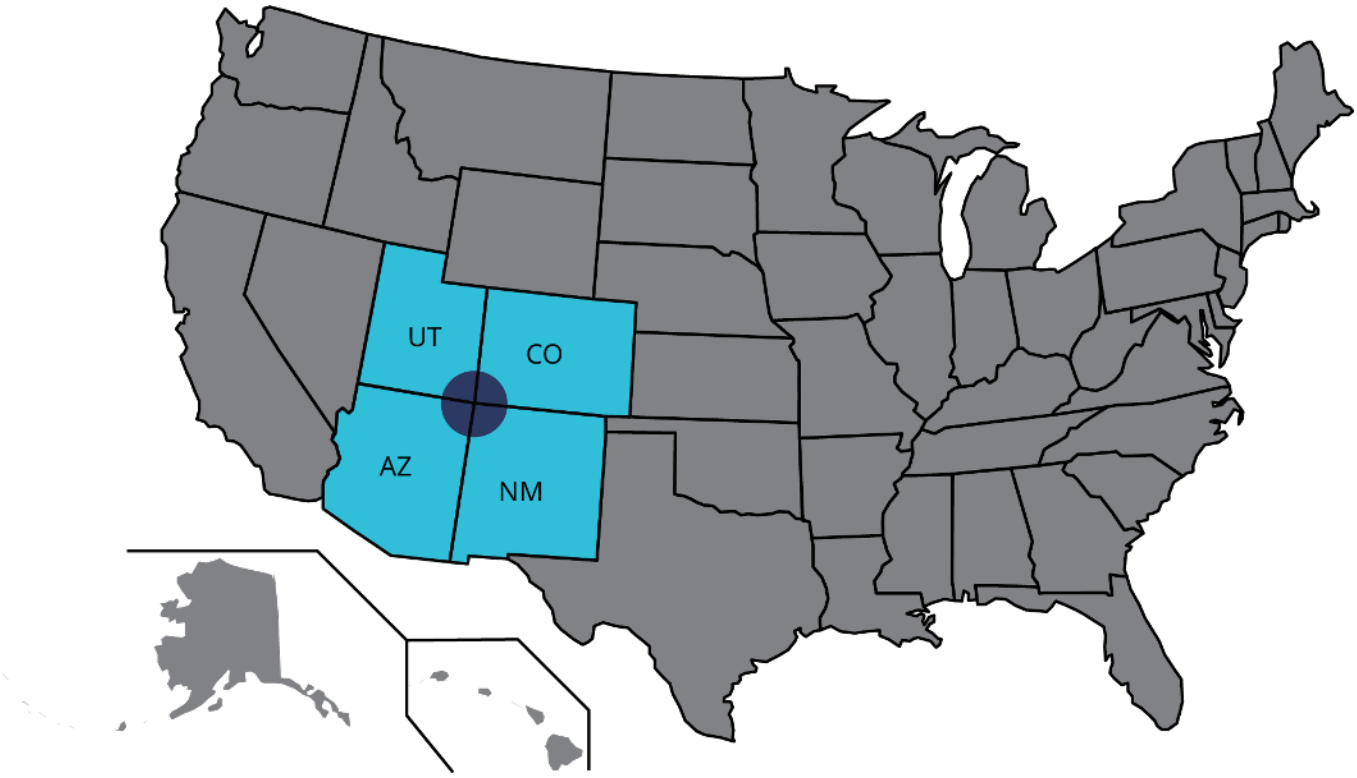


Figure **3.2.10**: Map of the Four Corners

In Figure **3.2.11**, each vertex represents one of these states, and each edge represents a shared border. States like Utah and New Mexico that meet at only a single point are *not* considered to have a shared border. By representing this map as a graph, where the connections are shared borders, we shift our perspective from physical attributes such as shape, size and distance, toward the existence of the relationship of having a shared boundary.

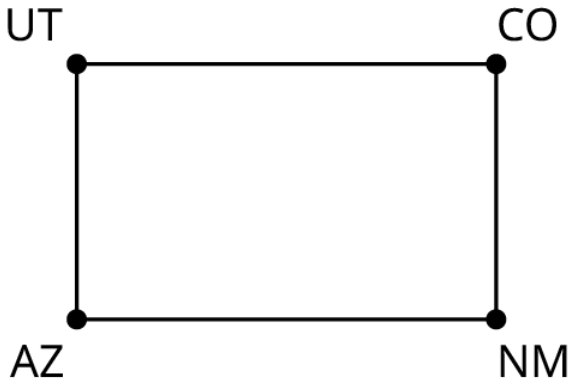


Figure **3.2.11**: Graph of the Shared Boundaries of Four Corners States

Example 3.2.4: Graphing the Midwestern States

A map of the Midwest is given in Figure **3.2.12**. Create a graph of the region in which each vertex represents a state and each edge represents a shared border.

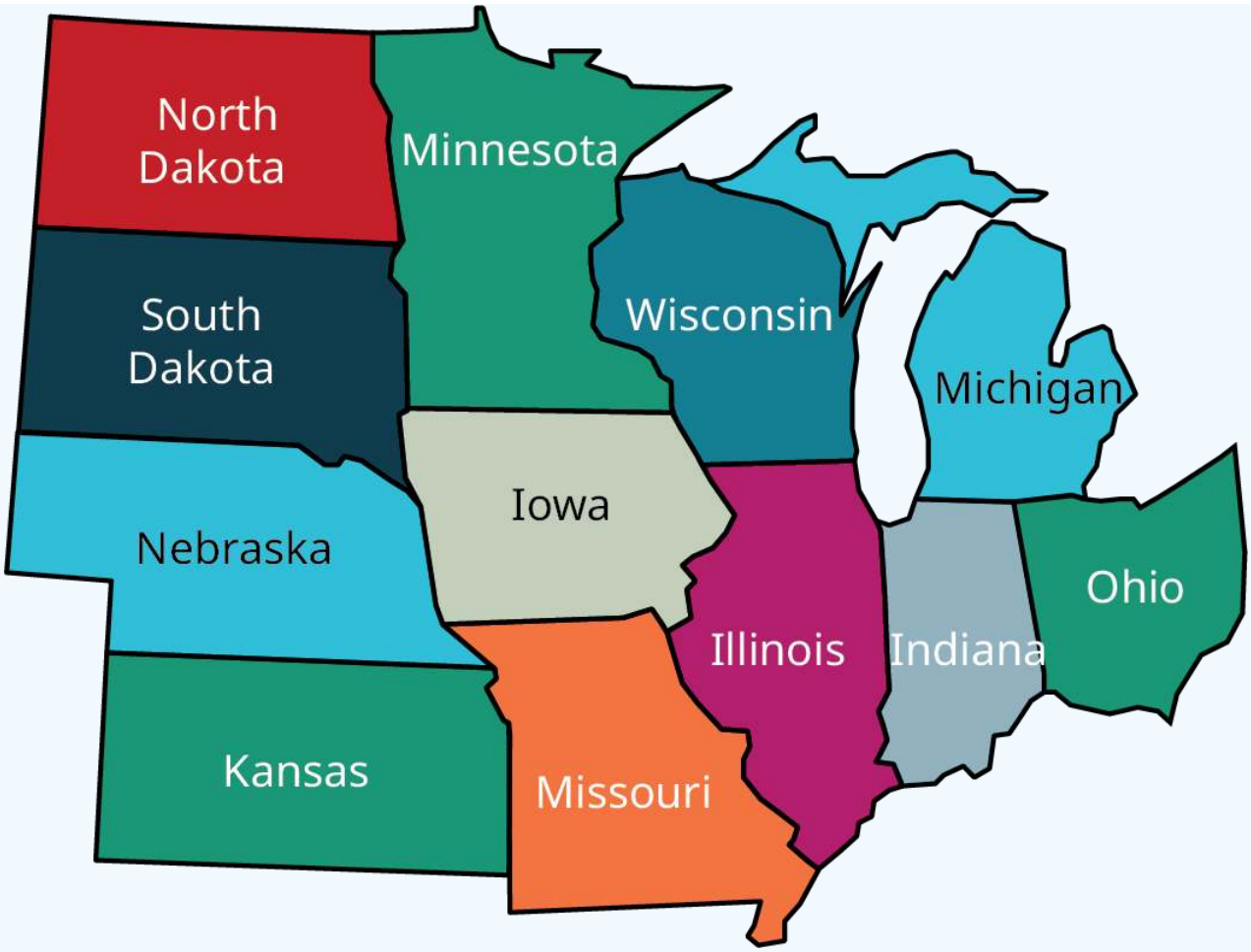


Figure 3.2.12 Map of Midwestern States

Answer

Step 1: For each state, draw and label a vertex as in Figure 3.2.13

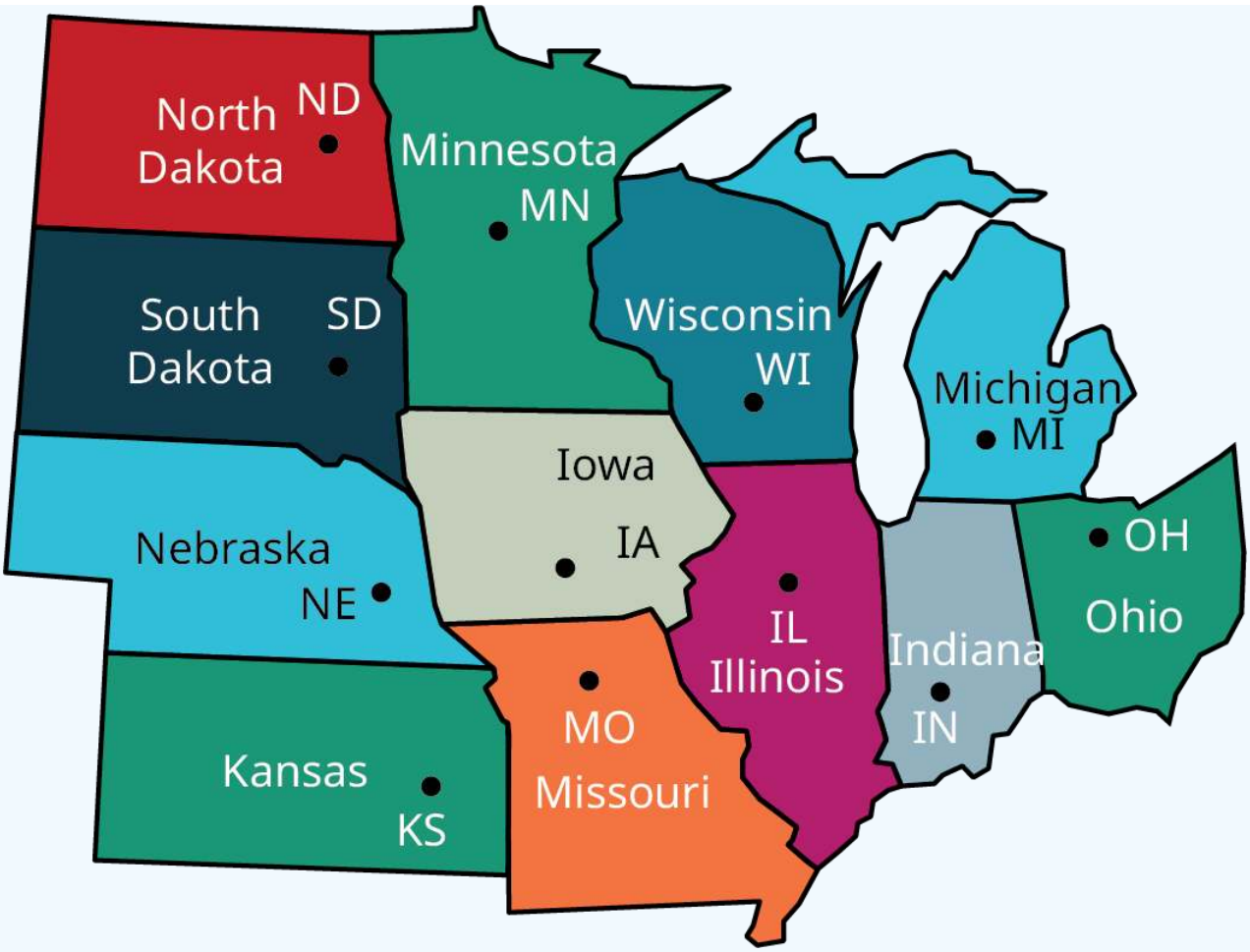


Figure 3.2.13 Vertex Assigned to Each Midwestern State

https://math.libretexts.org/@go/page/214452

Step 2:

Draw edges between any two states that share a common land border as in Figure 3.2.14

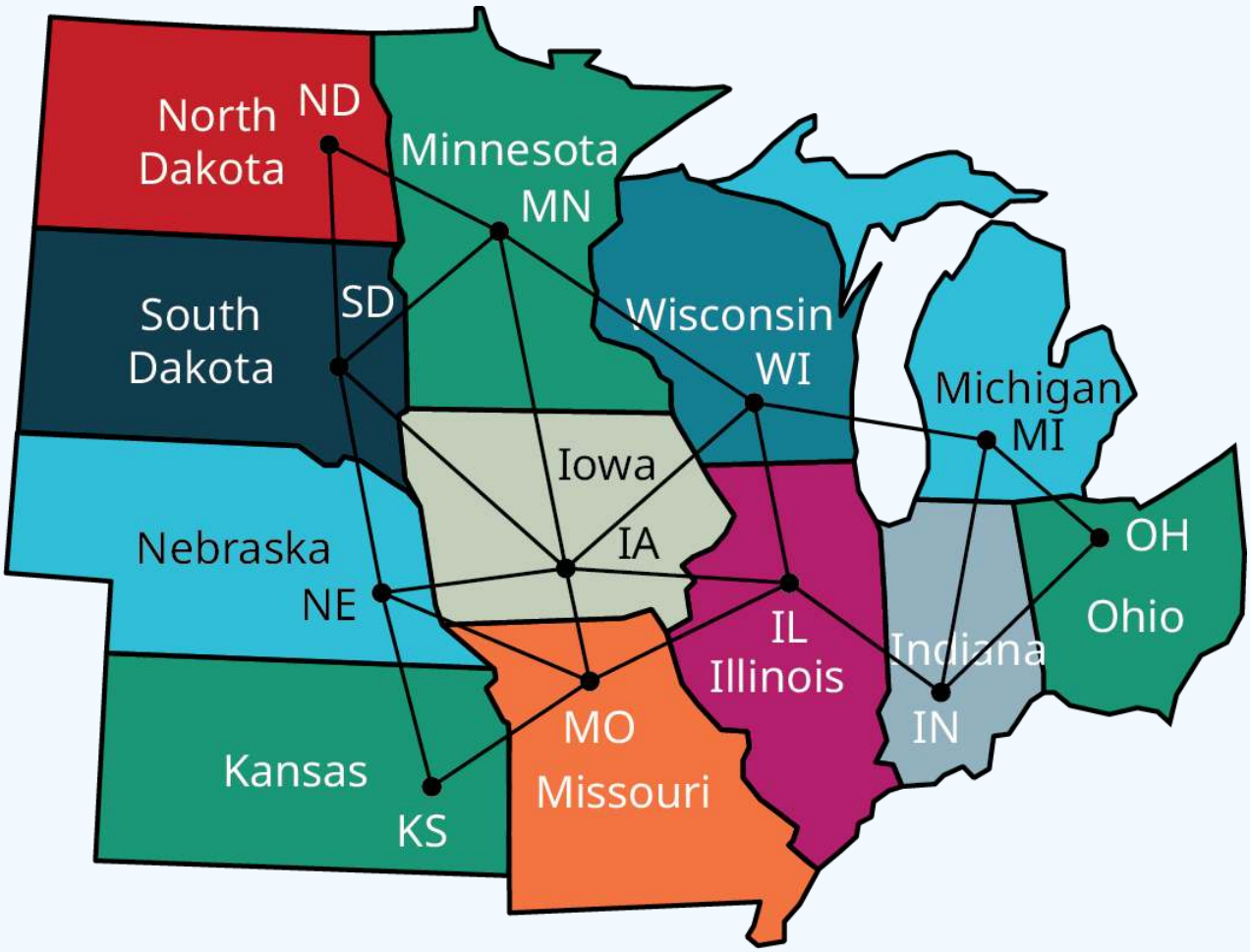


Figure 3.2.14: Edge Assigned to Each Pair of Midwestern States with Common Border

The graph is given in Figure 3.2.15.

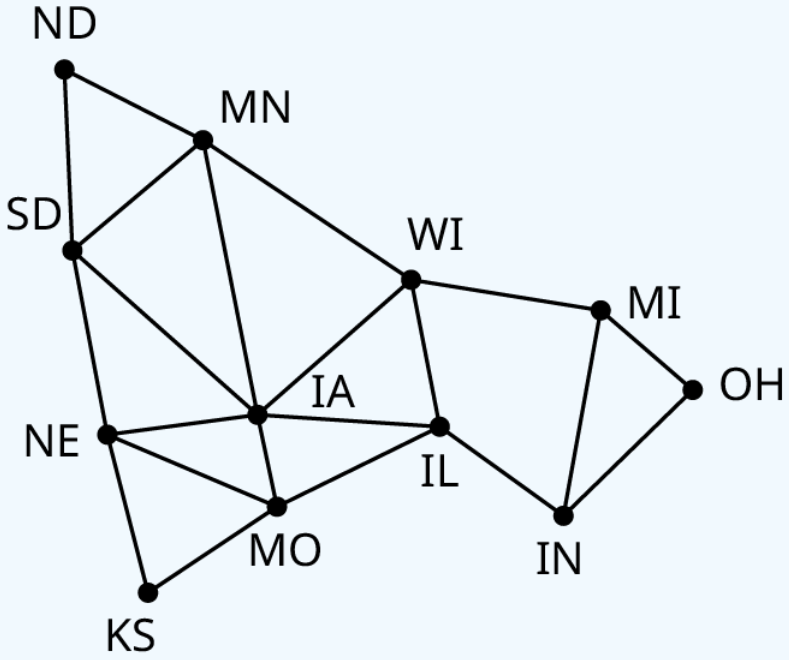


Figure **3.2.15**: Final Graph Representing Common Boundaries between Midwestern States

Your Turn 3.2.4

The figure shows a map of the Island of Oahu in the State of Hawaii divided into regions. Draw a graph in which each vertex represents one of the regions and each edge represents a shared land border.



Figure **3.2.16**: Map of Oahu

Video



Graph Theory: Create a Graph to Represent Common Boundaries on a Map

Graphs of Social Interactions

Geographical maps are just one of many real-world scenarios which graphs can depict. Any scenario in which objects are connected to each other can be represented with a graph, and the connections don't have to be physical. Just think about all the connections you have to people around the world through social media! Who is in your network of Twitter followers? Whose Snapchat network are you connected to?

Example 3.2.5: Graphing Chloe's *Roblox* Friends

Roblox is an online gaming platform. Chloe is interested to know how many people in her network of *Roblox* friends are also friends with each other so she polls them. Explain how a graph or multigraph might be drawn to model this scenario by identifying the objects that could be represented by vertices and the connections that could be represented by edges. Indicate whether a graph or a multigraph would be a better model.

Answer

The objects that are represented with vertices are *Roblox* friends. A *Roblox* friendship between two friends will be represented as an edge between a pair of vertices. There will be no double edges because it is not possible for two friends to be linked twice in *Roblox*; they are either friends or they are not. Also, a player cannot be a friend to themself, so there is no need for a loop. Since there are no double edges or loops, this is best represented as a graph.

Your Turn 3.2.5

In a particular poker tournament, five groups of five players will play at a table until one player wins, then the five winning players will play each other at a table in a final round. Explain how a graph or multigraph might be drawn to model this scenario by identifying the objects that could be represented by vertices and the connections that could be represented by edges. Indicate whether a graph or a multigraph would be a better model.

Who Knew?: Using Graph Theory to Reduce Internet Fraud

Could graphs be used to reduce Internet fraud? At least one researcher thinks so. Graph theory is used every day to analyze our behavior, particularly on social network sites. Alex Buetel, a computer scientist from Carnegie Mellon University in Pittsburgh, Pennsylvania, published a research paper in 2016 that discussed the possibilities of distinguishing the normal interactions from those that might be fraudulent using graph theory. Buetel wrote, "To more effectively model and detect abnormal behavior, we model *how* fraudsters work, catching previously undetected fraud on Facebook, Twitter, and Tencent Weibo and improving classification accuracy by up to 68%." In the same paper, the researcher discusses how similar techniques can be used to model many other applications and even, "predict *why* you like a particular movie." (Alex Beutel, "User Behavior Modeling with Large-Scale Graph Analysis," http://reports-archive.adm.cs.cmu.ed...-CS-16-105.pdf, May 2016, CMU-CS-16-105, Computer Science Department, School of Computer Science, Carnegie Mellon University, Pittsburgh, PA)

Check Your Understanding

1. A simple graph has no loops.
   a. True
   b. False

2. It is not possible to have a vertex of degree 0.
   a. True
   b. False

3. A graph with three vertices has at most three edges.
   a. True
   b. False

4. A multigraph with three vertices has at most three edges.

a. True
b. False

5. If vertex *a* is adjacent to vertex *b*, and vertex *b* is adjacent to vertex *c*, then vertex *a* must be adjacent to vertex *c*.

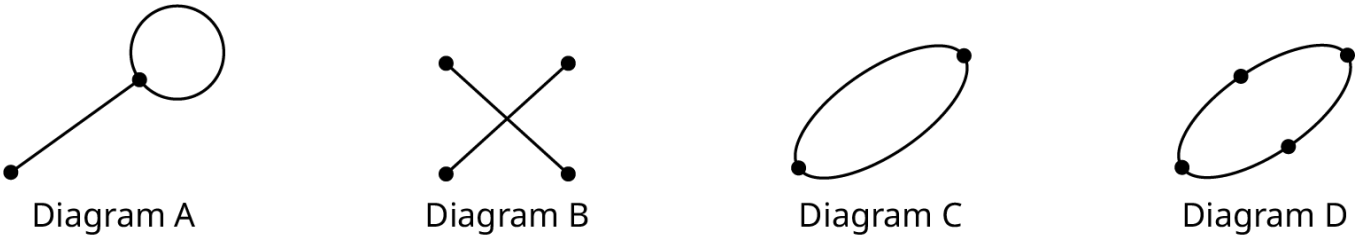
a. True
b. False

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3.2.1: Exercises

For the following exercises, use the figure showing Diagrams *A*, *B*, *C*, and *D*.



Exercise 3.2.1.1

Identify any multigraphs.

Exercise 3.2.1.2

Identify any graphs.

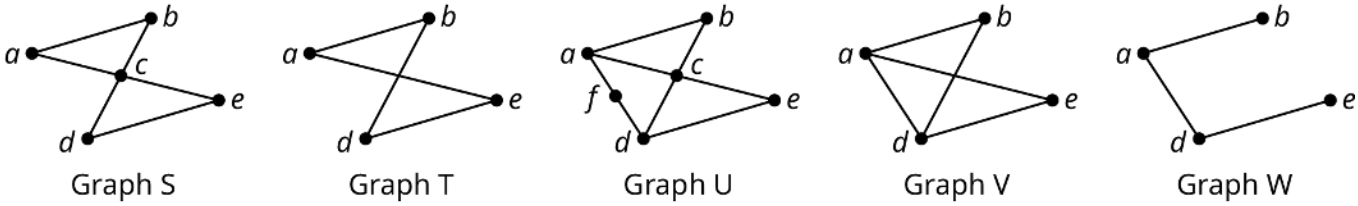
Exercise 3.2.1.3

Identify any multigraph that has a loop.

Exercise 3.2.1.4

Identify any multigraph that has a double edge.

For the following exercises, use the figure showing Graphs *S*, *T*, *U*, *V*, and *W*.



Exercise 3.2.1.5

Determine the number of vertices in Graph *T*.

Exercise 3.2.1.6

Determine the number of vertices in Graph *U*.

Exercise 3.2.1.7

Identify the graph with the most vertices.

Exercise 3.2.1.8

Identify the graphs with four vertices.

Exercise 3.2.1.9

Identify the graph with the most edges.

Exercise 3.2.1.10

Identify the graph with the fewest edges.

Exercise 3.2.1.11

Name the vertices in Graph *S*.

Exercise 3.2.1.12

Name the vertices in Graph *V*.

Exercise 3.2.1.13

Determine the number of edges in Graph *U*.

Exercise 3.2.1.14

Determine the number of edges in Graph *T*.

Exercise 3.2.1.15

Name the edges in Graph *V*.

Exercise 3.2.1.16

Name the edges in Graph *W*.

Exercise 3.2.1.17

Identify any pairs of vertices in Graph *T* that are not adjacent.

Exercise 3.2.1.18

Identify any vertices in Graph *V* that are not adjacent.

Exercise 3.2.1.19

Find the degree of vertex *a* in graph *U*.

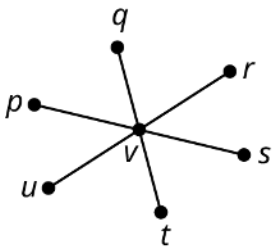
Exercise 3.2.1.20

Find the degree of vertex *a* in graph *S*.

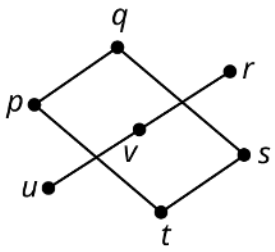
Exercise 3.2.1.21

Which two graphs have a vertex of degree 4?

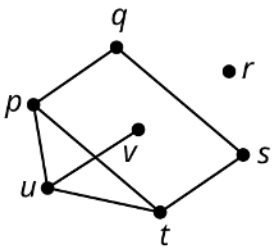
For the following exercises, use the figure showing Graphs *A*, *B*, *C*, and *D*.



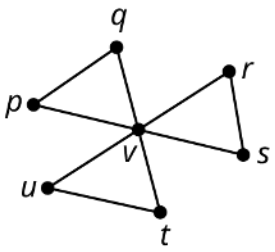
Graph A



Graph B



Graph C



Graph D

? Exercise 3.2.1.22

Identify the graph with 1 vertex of degree 6 and 6 vertices of degree 1.

? Exercise 3.2.1.23

Identify the graph with 1 vertex of degree 6 and 6 vertices of degree 2.

? Exercise 3.2.1.24

Identify the graph with exactly 1 vertex of degree 0.

? Exercise 3.2.1.25

Identify the graph with exactly 2 vertices of degree 1.

? Exercise 3.2.1.26

Name all the pairs of adjacent vertices in Graph *B*.

? Exercise 3.2.1.27

Name all the pairs of adjacent vertices in Graph *A*.

? Exercise 3.2.1.28

Identify the graph in which the sum of the degrees of the vertices is 14.

For each of the following exercises, a scenario is given. Explain how a graph or multigraph might be drawn to model the scenario by identifying the objects that could be represented by vertices and the connections that could be represented by edges. Indicate whether a graph or a multigraph would be a better model.

? Exercise 3.2.1.29

The Centers for Disease Control and Prevention (CDC) is tracking the spread of a virus. The CDC is attempting to determine where the virus began by identifying where each known carrier contracted the virus.

? Exercise 3.2.1.30

The faculty members in a college mathematics department have a “telephone tree” that assigns each faculty member two other faculty members to call with information in the event of an emergency. The chairperson of the department contacts two faculty

members. Each of these faculty members contacts two faculty members, and so on, until all faculty members have been notified.

Exercise 3.2.1.31

There is a theory that any two people on earth are no more than six social connections apart, which is known as “six degrees of separation” or the “six handshake rule.” It means that there is a chain of a “friend of a friend” that connects any two people through at most six steps.

Exercise 3.2.1.32

The U.S. Postal Service has a network of post offices that move mail around the United States. Mail trucks from one post office follow routes deliver mail to other locations. These mail trucks also pick up mail to bring back to their home post office.

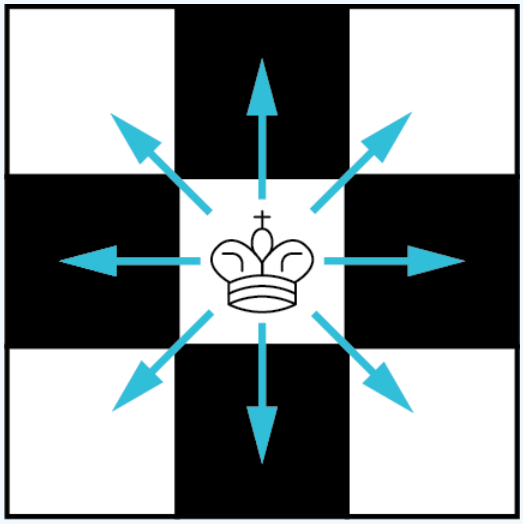
In the following exercises , draw a graph to represent the given scenario.

Exercise 3.2.1.33

Draw a graph to model the scenario described in Your Turn 12.5.

Exercise 3.2.1.34

In the game of chess, a king can move one space in any direction, vertically, horizontally, or diagonally, as indicated in the figure. Draw a graph to represent the possible movements of a king on a three-by-three section of a chess board such that each edge represents one move, and each vertex represents a position where the king might be at the beginning or end of a turn. Assume that the king can have unlimited turns.



Exercise 3.2.1.35

Chloe is interested to know how many in her network of *Roblox* friends are also friends with each other, so she polls them. The following is a list of each of Chloe’s friends and their common friends. Create a graph of the *Roblox* friends in Chloe’s network, including Chloe.

- Aria is friends with no one else in Chloe’s network.
- Benicio is friends with Dakshayani, Eun-ah, and Fukashi.
- Dakshayani is friends with Benicio.
- Eun-ah is friends with Benicio and Fukashi.

- Fukashi is friends with Benicio and Eun-ah.

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3.3: Graph Structures

Learning Objectives

1. Describe and interpret relationships in graphs.
2. Model relationships with graphs.

Graph theory is used in neuroscience to study how different parts of the brain connect. Neurobiologists use functional magnetic resonance imaging (fMRI) to measure levels of blood in different parts of the brain, called nodes. When nodes are active at the same time, it suggests there is a functional connection between them so they form a network. This network can be represented as a graph where the vertices are the nodes and the functional connections are the edges between them. (Mikey Taylor, "Graph Theory & Machine Learning in Neuroscience," Medium.com, June 24, 2020.

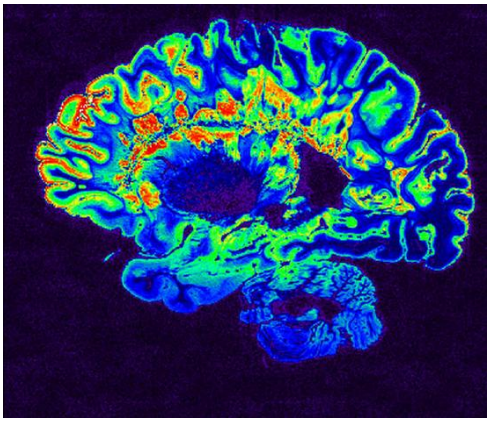


Figure **3.3.1**: Neuroimaging shows brain activity. (credit: "MRI Scan" by NIH Image Gallery/Flickr, Public Domain)

Importance of the Degrees of Vertices

One reason scientists study these networks is to determine how successful the communication within a network continues to be when it experiences failures in nodes and connections. Graphs can be used to study the resilience of these networks. (Mikey Taylor, "Graph Theory & Machine Learning in Neuroscience," Medium.com, June 24, 2020)

✔ Example ✔ 3.3.1: Using Graphs to Understand Relationships

The graphs in Figure ✔ **3.3.2** represent neural networks, where the vertices are the nodes, and the edges represent functional connections between them. Which graph do you think would represent a network with the highest resistance to failure? Which graph would probably be the most vulnerable to failure? How might this relate to the degrees of the vertices?

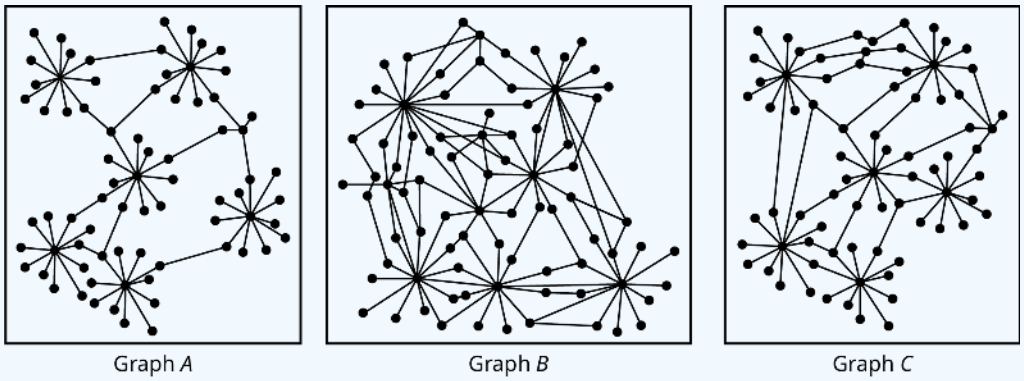


Figure ✔ **3.3.2** Models of Neural Networks

Answer

Graph *B* represents a network that would have the highest resistance to failure because there are more edges connecting the nodes indicating there are more connections between nodes than on either of the other graphs. This would make the network more resistant to failure because there are more options to work around a faulty edge or node. Graph *A* would probably be the most susceptible to failure because the network depends on a few particular edges and vertices for connections. There are more vertices of higher degree in Graph *B* than in Graph *A* because there are more edges connecting the nodes.

Your Turn 3.3.1

Sociologists use graphs to study connections between people and to identify characteristics that make communities more resilient, more likely to stay connected over time. In the graphs shown, the edges represent social connections and the vertices are individuals.

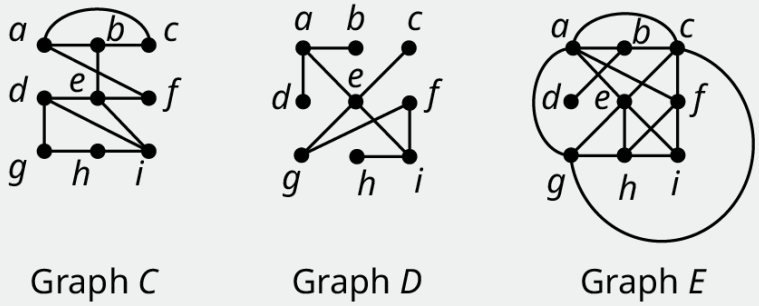


Figure **3.3.3**: Models of Communities

Which graph do you think represents the most resilient community? What is the sum of the degrees of all the vertices in this graph?

Which graph do you think represents the community that is least likely to stay connected over time? What is the sum of the degrees of all the vertices in this graph?

Is the sum of the degrees higher or lower in a more resilient community?

Relating the Number of Edges to the Degrees of Vertices

In the applications of graph theory to neuroscience and sociology in Example **3.3.1** and YOUR TURN **3.3.1** , there was a correlation between the degrees of vertices and the resilience of a network. Researchers in many fields have also observed a direct relationship between the number of edges in a graph and the degrees of the vertices. To begin to understand this relationship, consider a graph with five vertices and zero edges as in Figure **3.3.4** .

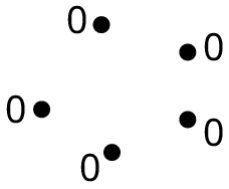


Figure **3.3.4**: Graph with Five Vertices and zero Edges

Instead of being marked with a name, each vertex in Figure 12.21 is marked with its degree. In this case, all of the degrees are 0 so the sum of the degrees is also zero. Suppose that we add an edge between any two existing vertices and indicate the degrees of the vertices. This gives us a graph with five vertices and one edge like the graph in Figure **3.3.5** .

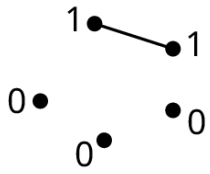


Figure **3.3.5**: Graph with Five Vertices and One Edge

Note that the degrees of two vertices increased, each by 1. So, the sum of the degrees is now 2. Suppose that we continue in this way, adding one edge at a time and making note of the number of edges and the sum of the degrees of the vertices as in Figure 12.23.

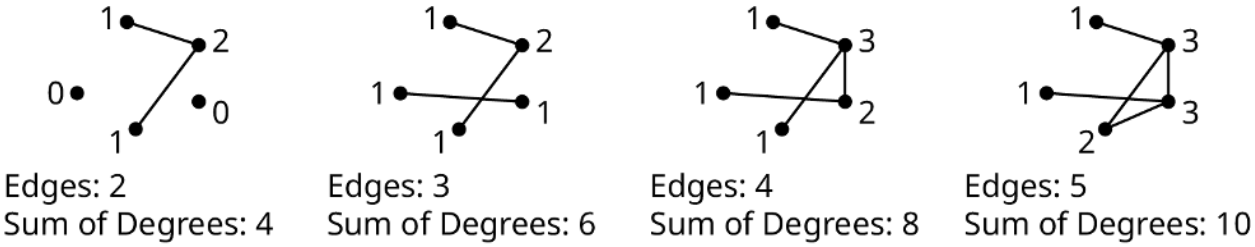


Figure **3.3.6**: Comparing Number of Edges to Sum of Degrees

Figure **3.3.7** demonstrates a characteristic that is true of all graphs of any shape or size. When the number of edges is increased by one, the sum of the degrees increases by two. This happens because each edge has two ends and each end increases the degree of one vertex by one unit. As a result, the sum of the degrees of the vertices on any graph is always twice the number of edges. This relationship is known as the **Sum of Degrees Theorem**.

FORMULA

For the Sum of Degrees Theorem,

$$sumofthedegrees\(= 2 \times \)numberofedges$$

or

$$\text{number of edges} = \frac{\text{sum of degrees}}{2}$$

mfrac

Example 3.3.2: Finding the Sum of Degrees

Suppose that a graph has five edges.

1. Find the sum of the degrees of the vertices.
2. Draw two different graphs that demonstrate this conclusion.

Answer

1. The sum of the degrees is twice the number of edges: 2 × 5=10. The sum of the degrees is 10.
2. See Figure ✔ **3.3.8**.

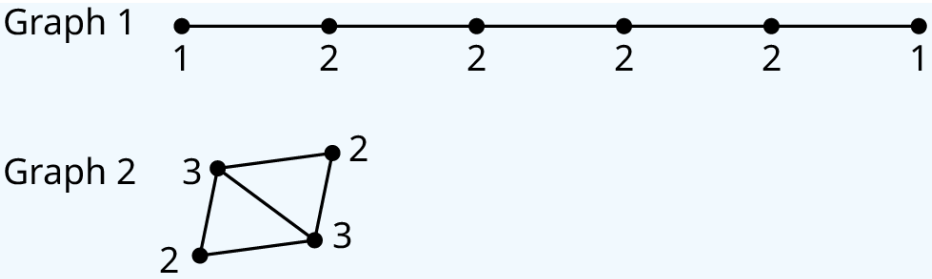


Figure ✔ 3.3.8

Your Turn 3.3.2

Suppose that the sum of the degrees of a graph is six.

Find the number of edges.

Draw two graphs that demonstrate your conclusion.

Completeness

Suppose that there were five strangers in a room, *A*, *B*, *C*, *D*, and *E*, and each one would be introduced to each of the others. How many introductions are necessary? One way to begin to answer this question is to draw a graph with each vertex representing an individual in the room and each edge representing an introduction as in Figure 3.3.9.

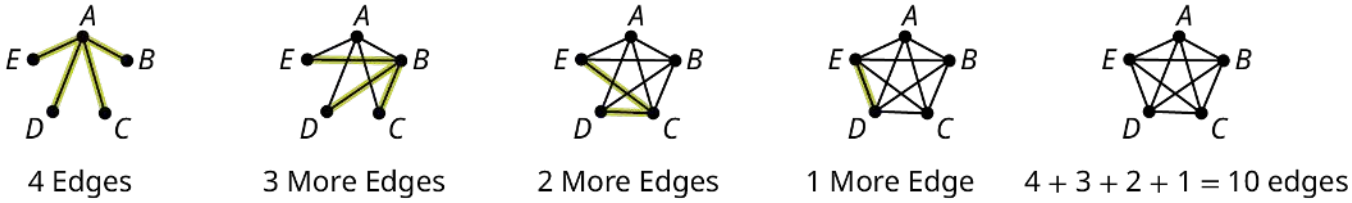


Figure 3.3.9: Model of Introductions between Five Strangers

Let's approach the problem by thinking about how many new people Person A would meet, then Person B, and so on, making sure not to repeat any introductions. The first graph in Figure 3.3.9 shows Person A meeting Persons B, C, D, and E, for a total of 4 introductions. The next graph shows that Person B still has to meet Persons C, D, and E, for a total of 3 more introductions. The next graph shows that Person C still has to meet Persons D and E, which is 2 more introductions. The next graph shows that Person D only remains to meet Person E, which is 1 more introduction. The final graph has $4+3+2+1=10$ edges representing 10 introductions. The last graph is an example of a complete graph because each pair of vertices is joined by an edge. Another way of saying this is that the graph is complete because each vertex is adjacent to every other vertex. Figure 3.3.10 shows complete graphs with three, four, five, and six vertices. Math input error

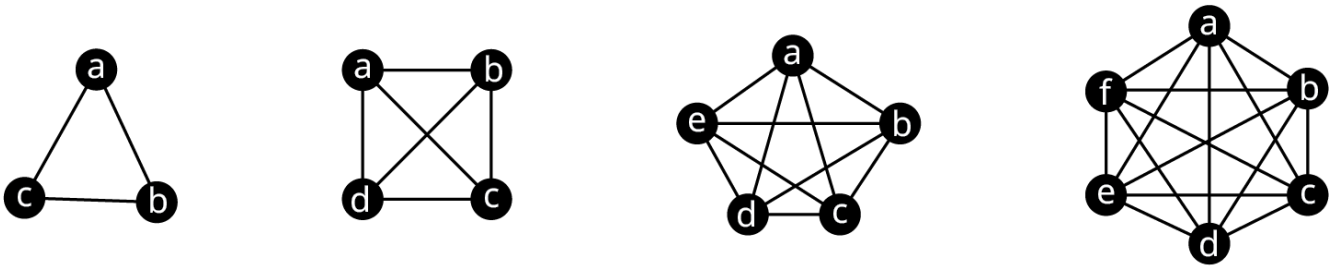


Figure 3.3.106: Complete Graphs with Up to Six Vertices

Suppose we want to know the number of introductions necessary in a room with six people. This would be represented by a complete graph with six vertices, and the total number of introductions would be $5+4+3+2+1=15$, the number of edges in the graph. In fact, you can always find the number of introductions in a room with n people by adding all the whole numbers from 1 to $n-1$.

FORMULA

The number of edges in a complete graph with n vertices is the sum of the whole numbers from 1 to $n-1$, $1+2+3+\cdots+(n-1)$.

Suppose that we want to determine how many introductions are necessary in a room with 500 strangers. In other words, suppose that we want to determine the number of edges in a complete graph with 500 vertices. Adding up all the numbers from 1 to 499 could take a long time! In the next example, we use the Sum of Degrees Theorem to make the problem more manageable.

Example 3.3.3: Using the Sum of Degrees Theorem

Use the Sum of Degrees Theorem to determine the number of introductions required in a room with

1. 6 strangers.
2. 10 strangers.
3. n strangers.

Answer

1. Since there are 6 strangers, there are 6 vertices. Since each individual must meet 5 other individuals, there are 5 edges meeting at each vertex which means each vertex has degree 5. Since there are 6 vertices of degree 5 , the sum of degrees is $6 \cdot 5 = 30$. By the Sum of Degrees Theorem, the number of edges is half the sum of the degrees, which is $\frac{30}{2} = 15$. So, the total number of introductions is 15.
2. Since there are 10 strangers, there are 10 vertices. Since each individual must meet 9 other individuals, there are 9 edges meeting at each vertex which means each vertex has degree 9. Since there are 10 vertices of degree 9 , the sum of degrees is $10 \cdot 9 = 90$. By the Sum of Degrees Theorem, the number of edges is half the sum of the degrees, which is $\frac{90}{2} = 45$. So, the total number of introductions is 45.
3. Since there are n strangers, there are n vertices. Since each individual must meet $n-1$ other individuals, there are $n-1$ edges meeting at each vertex which means each vertex has degree $n-1$. Since there are n vertices of degree $n-1$, the sum of degrees is $n(n-1)$. By the Sum of Degrees Theorem, the number of edges is half the sum of the degrees, which is $\frac{n(n-1)}{2}$. So, the total number of introductions is $\frac{n(n-1)}{2}$.

Your Turn 3.3.3

Determine the number of introductions necessary in a room with 500 strangers using the Sum of Degrees Theorem.

Now we have a shorter way to calculate the number of introductions in a room with n strangers, and the number of edges on a complete graph with n vertices. Let's update our formula.

FORMULA

The number of edges in a complete graph with n vertices is $1+2+3+\cdots+(n-1)=\frac{n(n-1)}{2}$.

Subgraphs

Sometimes a graph is a part of a larger graph. For example, the graph of South Florida Airports from Figure 12.8 is part of a larger graph that includes Orlando International Airport in Central Florida, which is shown in Figure 3.3.11

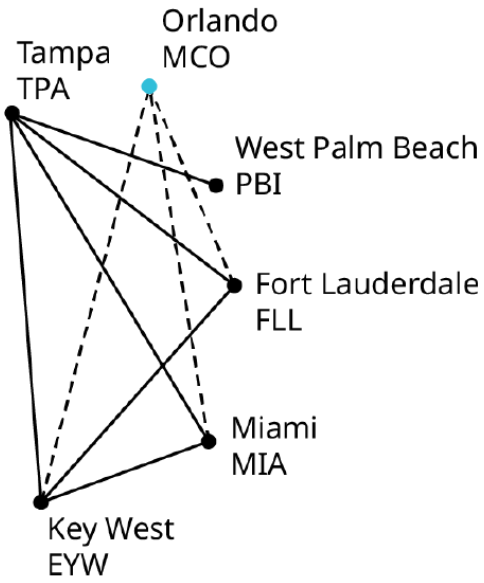


Figure **3.3.11**: Orlando and South Florida Airports

The graph in Figure **3.3.11** includes an additional vertex, MCO, and additional edges shown with dashed lines. The graph of direct flights between South Florida airports from Figure 12.1.7 is called a **subgraph** of the graph that also includes direct flights between Orlando and the same South Florida airports in Figure **3.3.11**. In general terms, if Graph *B* consists entirely of a set of edges and vertices from a larger Graph *A*, then *B* is called a subgraph of *A*.

✔ Example **3.3.4**: Identifying a Subgraph

In Figure ✔ **3.3.12**, Graph *G* is given, along with four diagrams. Determine whether each diagram is or is not a subgraph of Graph *G* and explain why.

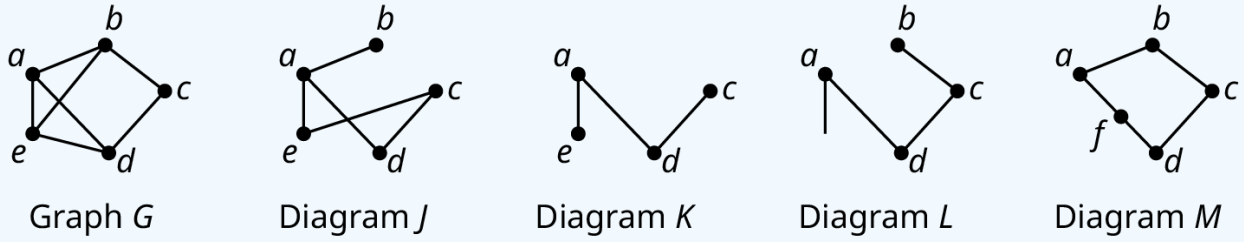


Figure ✔ **3.3.12** Graph *G* and Potential Subgraphs

Answer

- Diagram *J* is not a subgraph of Graph *G* because edge *ec* is not in Graph *G*.
- Diagram *K* is a subgraph of Graph *G* because all of its vertices and edges were part of Graph *G*.
- Diagram *L* is not a graph at all, because there is a line extending from vertex *a* that does not connect *a* to another vertex. So, Diagram *L* cannot be a subgraph.
- Diagram *M* is not a subgraph of Graph *G* because it contains a vertex *f*, which is not part of *G*.

Your Turn **3.3.4**

Draw a subgraph of Graph *F* from Figure 12.1.3 that has exactly four vertices and five edges.

Identifying and Naming Cycles

When you think of a cycle in everyday life, you probably think of something that begins and ends the same way. For example, the water cycle (Figure 12.29) begins with water in a lake or ocean, which evaporates into water vapor, condenses into clouds, and then

returns to earth as rain or some other form of precipitation that settles into lakes or oceans. A cycle in graph theory is similar in that it begins and ends in the same way: It is a series of connected edges that begin and end at the same vertex but otherwise never repeat any vertices.

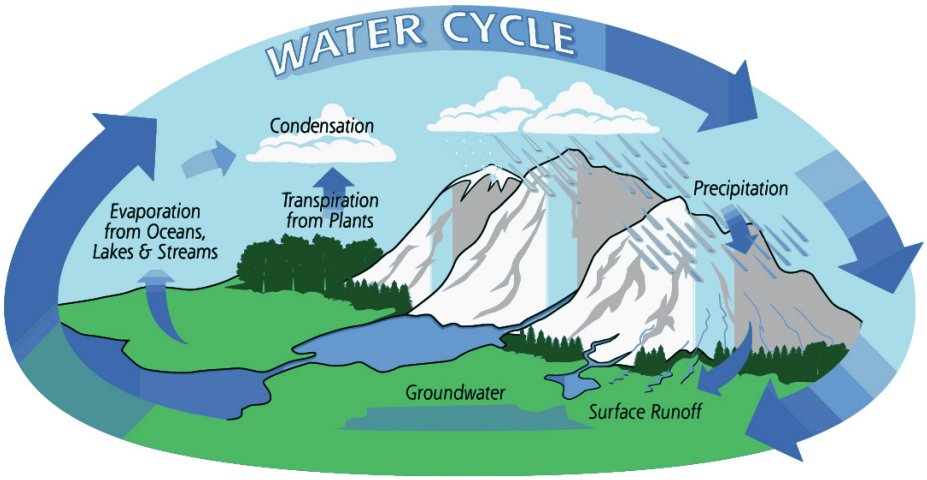
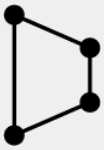
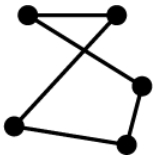
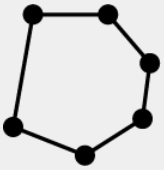
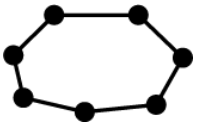
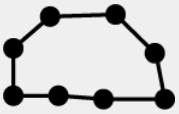
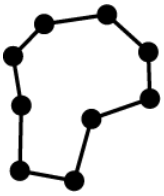
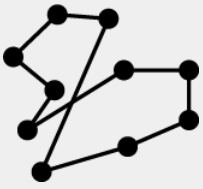


Figure **3.3.13**: The water cycle begins and ends with water. (credit: "Diagram of the water cycle" by NASA, Public Domain)

In a cycle, there are always the same number of vertices as edges, and all vertices must be of degree 2. Cycles are often referred to by the number of vertices. For example, a cycle with 5 vertices can be called a 5-cycle. Cycles can also be named after polygons based on the number of edges. For example a 5-cycle is also called a pentagon. Table **3.3.1** lists these names for cycles of size 3 through size 10.

Table **3.3.1** Categories of Cycles

Cycle Category	Number of Edges	Example
triangle	3	
quadrilateral	4	
pentagon	5	
pentagon	6	
heptagon	7	
octagon	8	
octagon	9	
octagon	10	

There are many more polygon names, including a megagon that has a million edges and a googolgon that has 10^{100} edges, but usually we just say n-gon when the number n is past 10 . For example, a cycle with 11 edges could be called an 11-gon.

Notice that the 10-cycle, or decagon, appears to cross over itself in Table 3.3.1. Remember, graphs can be drawn differently but represent the same connections. In Figure 3.3.14, the same decagon is transformed into a graph that does not appear to overlap itself. We have done this without changing any of the connections so both diagrams represent the same relationships, and both diagrams are considered decagons.

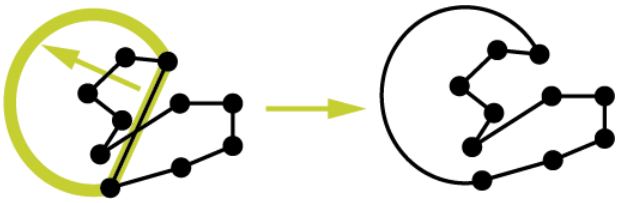


Figure **3.3.14**: Transformation of a Decagon

Who Knew?: Googol to Google

Did the name "googolgon" ring a bell for you? If so, there is a good reason for it. The creators of Google.com renamed their search engine Google, a misspelled reference to the number "googol" alluding to the enormous number of calculations their algorithm can tackle. A googol is a 1 followed by 100 zeroes, or 10^{100}. William L. Hosch, "Google, American Company," https://www.britannica.com/topic/Google-Inc, Encyclopedia Britannica online)Math input error

Cyclic Subgraphs and Cliques

When cycles appear as subgraphs within a larger graph, they are called **cyclic subgraphs**. Cyclic subgraphs are named by listing their vertices sequentially. The vertex where you begin is not important. Graph *K* in Figure **3.3.15** has two triangle cycles (*g*, *h*, *j*) and (*h*, *i*, *j*), and one quadrilateral cycle (*g*, *h*, *i*, *j*).

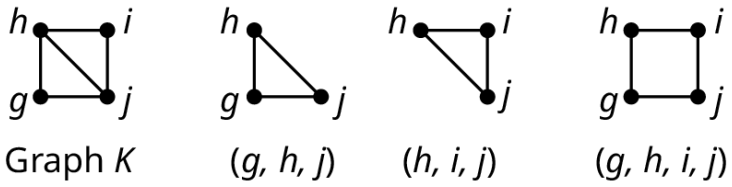


Figure **3.3.15**: Cycles in Graph *K*

Checkpoint

The same cycle can be named in many ways, but we must keep in mind that listing two vertices consecutively indicates an edge exists between them. For example, (g, h, i, j) can also be named (h, i, j, g) or (j, i, h, g). However, you cannot name it (g, i, h, j,) which doesn't reflect the correct order of the vertices. We can see that the order (g, i, h, j) is incorrect because the cycle has no edges gi or hj.

Example 3.3.5: Identifying Cyclic Subgraphs

Identify the types of cyclic subgraphs in Graph *H* in Figure ✔ **3.3.16** and name them.

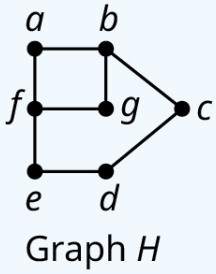


Figure ✔ **3.3.16** Graph *H*

Answer

In order to avoid missing a cycle, begin by analyzing the vertex *a*, then proceed in alphabetical order. Vertex *a* is part of two cycles, the quadrilateral cycle (*a*, *b*, *f*, *g*,) and the hexagonal cycle (*a*, *b*, *c*, *d*, *e*, *f*). Next look at vertex *b*. It is part of the hexagonal cycle (*b*, *c*, *d*, *e*, *f*, *g*). After checking each of the remaining vertices, we see there are no other cycles.

Checkpoint

In order to avoid naming the same cycle twice, consider naming cycles beginning with the letter closest to the beginning of the alphabet and then progressing clockwise.

Your Turn 3.3.5

Which types of cycles are in Graph G in Figure 3.3.16? Use the vertices of the graph to give a name for one of each type that you find.

We have seen that sociologists use graphs to study the structures of social networks. In sociology, there is a principle known as Triadic Closure. It says that if two individuals in a social network have a friend in common, then it is more likely those two individuals will become friends too. Sociologists refer to this as an open triad becoming a closed triad. This concept can be visualized as graphs in Figure a and person b each has a friendship with person c. In the closed triad, person a and person a have also developed a friendship. Notice that the graph representing the closed triad is a three-cycle, or a triangle, in graph theory terms.

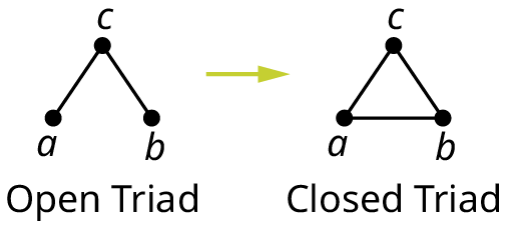


Figure 3.3.17: Graphs of Open Triad and Closed Triad

Another topic of interest to sociologists, as well as computer scientists and scientists in many fields, is the concept of a **clique**. In a social group, a clique is a subgroup who are all friends. A triad is an example of a clique with three people, but there can be cliques of any size. In graph theory, a clique is a complete subgraph.

Example 3.3.6: Identifying Triads and Cliques

The graphs in YOUR TURN 3.3.1 represent social communities. The vertices are individuals and the edges are social connections. Use the graphs to answer the questions.

1. Which graph has the most triads? Name the triangles.
2. Which graph has the fewest triads? Name the triangles.
3. How might an increase in the number of triads in a graph affect the resilience of a community? Explain your reasoning.
4. Identify a clique with more than three vertices in Graph E by naming its vertices.

Answer

1. Graph E has the most triads. The triangles are (a, b, c), (a, e, g), (c, f, i), (e, g, h), (e, h, i), and (f, h, i).
2. Graph D has the fewest triads. There are no triangles in the graph.
3. More triads means vertices of greater degree, which indicates greater resilience. Specifically, if an edge is removed from a closed triad, there the two individuals who are no longer adjacent are still connected by way of the third member of the triad.
4. The vertices a, e, c, and g form a clique with four vertices.

Your Turn 3.3.6

From Exercise 35 in Section 12.1.0 Exercises, Chloe is interested to know how many in her network of Roblox friends are also friends with each other, so she polls them. The following is a list of each of Chloe's friends and their common friends. Find a pentagon cycle in the graph of the Roblox friends in Chloe's network that includes Chloe.

- Aria is friends with no one else in Chloe's network.

- Benicio is friends with Dakshayani, Eun-ah, and Fukashi.
- Dakshayani is friends with Benicio.
- Eun-ah is friends with Benicio and Fukashi.
- Fukashi is friends with Benicio and Eun-ah.

Three-Cycles in Complete Graphs

Just as complete graphs have a predictable number of edges, complete graphs have a predictable number of cyclic subgraphs. Let's look at the three-cycles within complete graphs with up to six vertices, which are shown in Figure **3.3.18**.

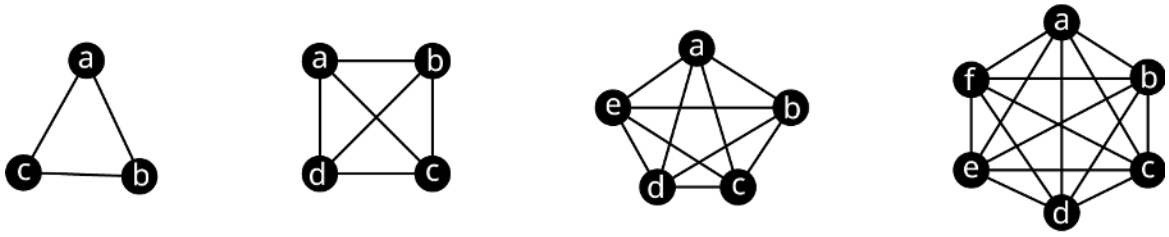


Figure **3.3.18**: Complete Graphs with Up to Six Vertices

Let's list the names of all the triangles in each graph. ***Since every pair of vertices is adjacent, any three vertices on a complete graph form a triangle***. There is only one triangle in the complete graph with three vertices, (*a*, *b*, *c*). For the rest of the graphs, it is important that we take an organized approach. Start with the vertex that is first alphabetically, listing any triangles that include that vertex also in alphabetical order. Then, proceed to the next vertex in the alphabet, and list any triangles that include that vertex, except those that are already listed. Keep going in this way as shown in Table 12.2.

Table **3.3.2** Listing Triangles in Complete Graphs

Complete Graph With:	3 Vertices	4 Vertices	5 Vertices	6 Vertices
All *a* triangles	(*a*, *b*, *c*)	(*a*, *b*, *c*), (*a*, *b*, *d*), (*a*, *c*, *d*)	(*a*, *b*, *c*), (*a*, *b*, *d*), (*a*, *b*, *e*) (*a*, *c*, *d*), (*a*, *c*, *e*) (*a*, *d*, *e*)	(*a*, *b*, *c*), (*a*, *b*, *d*), (*a*, *b*, *e*), (*a*, *b*, *f*) (*a*, *c*, *d*), (*a*, *c*, *e*), (*a*, *c*, *f*) (*a*, *d*, *e*), (*a*, *d*, *f*) (*a*, *e*, *f*)
Other *b* triangles	None	(*b*, *c*, *d*)	(*b*, *c*, *d*), (*b*, *c*, *e*), (*b*, *d*, *e*)	(*b*, *c*, *d*), (*b*, *c*, *e*), (*b*, *c*, *f*) (*b*, *d*, *e*), (*b*, *d*, *f*) (*b*, *e*, *f*)
Other *c* triangles	None	None	(*c*, *d*, *e*)	(*c*, *d*, *e*), (*c*, *d*, *f*) (*c*, *e*, *f*)
Other *d* triangles	None	None	None	(*d*, *e*, *f*)
Total	1	$(2+1)+1$ $=3+1$ $=4$	$(3+2+1)+(2+1)+1$ $=6+3+1$ $=10$	(4+3+2+1)+(3+2+1)+(2+1)+1 =10+6+3+1 =20

Look at the last row in Table **3.3.2**. Do you see a pattern emerge for counting triangles in a complete graph? Without drawing a complete graph with 7 vertices, we can predict that it will have $(5+4+3+2+1)+(4+3+2+1)+(3+2+1)+(2+1)+1=35$ triangles inside it. This pattern also appears in a famous diagram known to Western mathematicians as "Pascal's Triangle." Figure 12.35 displays the first 11 rows of Pascal's Triangle. Row 0 of Pascal's Triangle only has the number 1 in it. The first and last entries in each of the other rows are also 1s. Otherwise, all the entries are formed by adding two entries from the previous row. For example, in row 6, entry 1 is 6, which was found by adding 1 and the 5 from the previous row, and entry 2 is 15, which was found by adding the 5 and the 10 from the previous row, as shown in Table **3.3.2**.

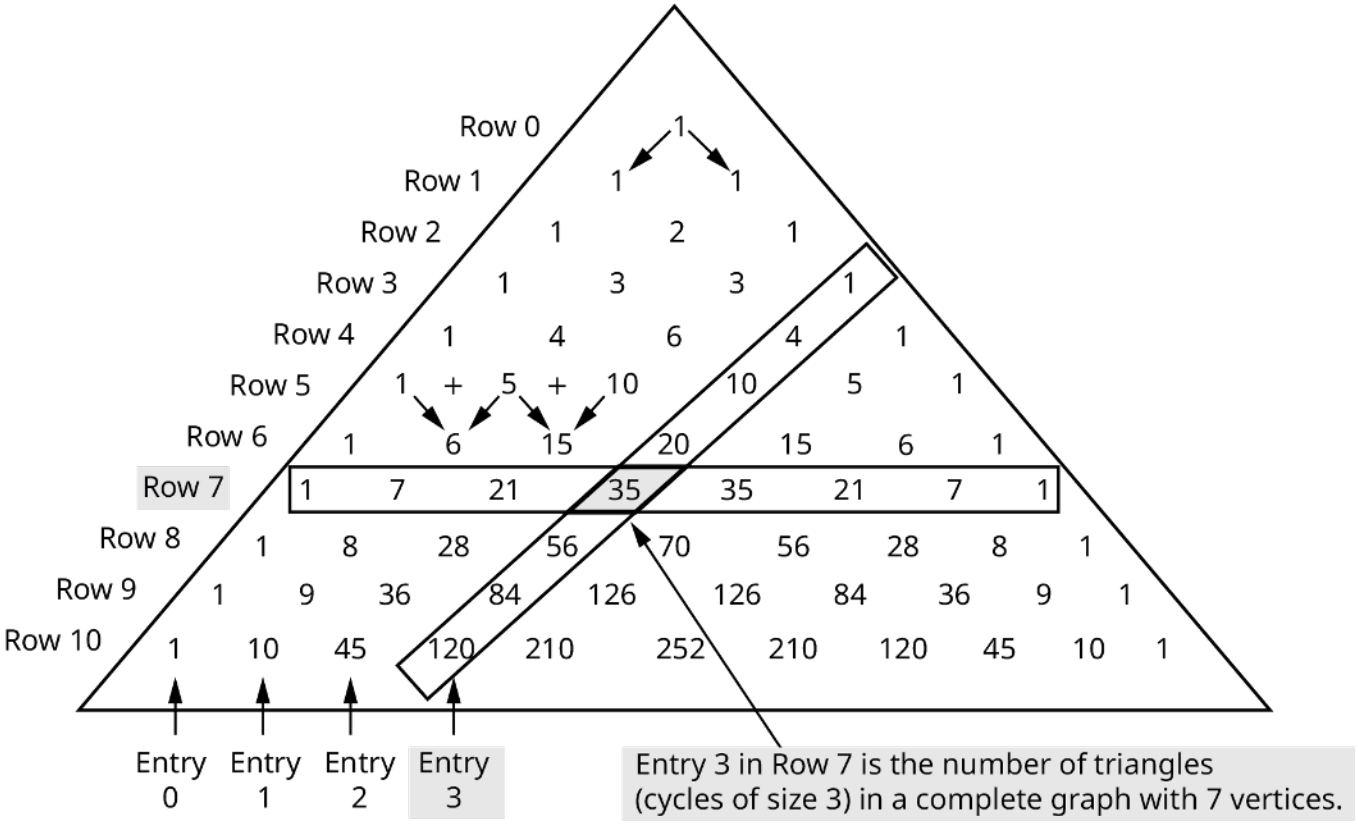


Figure **3.3.19**: Pascal's Triangle

Checkpoint

IMPORTANT! When you count the rows and entries of Pascal's Triangle begin with row 0 and entry 0, not row 1 and entry 1.

Video



The Mathematical Secrets of Pascal's Triangle by Wajdi Mohamed Ratemi

If we begin to list the third entries in each row of Pascal's triangle from the top down, we see 1, 4, 10, 20, 35, and so on. Notice that these values are exactly the number of triangles in a complete graph of sizes 3, 4, 5, 6, and 7, respectively. Specifically, entry 3 in row 7 is 35, the number of triangles in a complete graph of size 7. Let's practice using Pascal's triangle to find the number of triangles in a complete graph of a particular size.

STEPS TO FIND THE NUMBER OF TRIANGLES IN A COMPLETE GRAPH OF SIZE *n*

- **Step 1** Calculate the entries in row *n* of Pascal's Triangle using sums of pairs of entries in previous rows as needed.
- **Step 2** The number of triangles is entry 3 in row *n*. Remember to count the entries 0, 1, 2, and 3, from left to right.

Example 3.3.7: Using Pascal's Triangle to Find the Number of Triangles in a Complete Graph

Use Pascal's Triangle in Figure 12.35 to find the number of triangles in a complete graph with 11 vertices.

Answer

Step 1: Identify the row number and calculate its entries. Since the complete graph has 11 vertices, we will need to look in row 11 of Pascal's Triangle. Use row 10 in Figure 12.35 to find the entries in row 11 as shown in Figure 3.3.20.

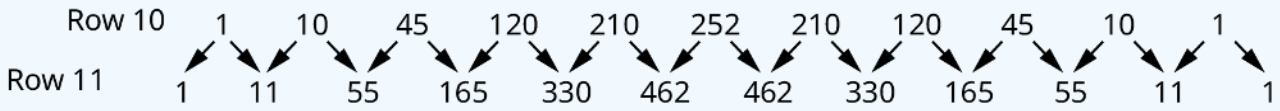


Figure 3.3.20 Rows 10 and 11 of Pascal's Triangle

Step 2: Find entry number 3. The number of triangles is in entry 3. Remember to count the entries beginning with entry 0 as shown in Figure 3.3.20.

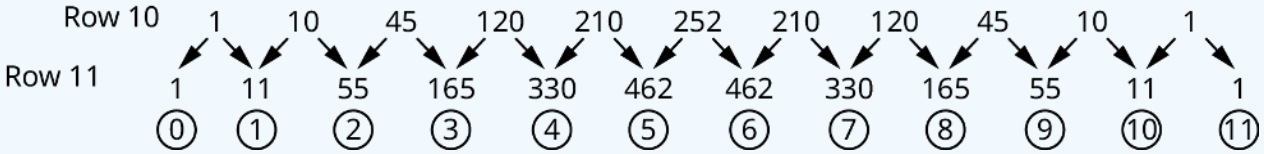


Figure 3.3.21: Identifying an Entry in Pascal's Triangle

Entry 3 in row 11 is 165. So, there are 165 triangles in a complete graph with 11 vertices.

Your Turn 3.3.7

A sociologist is studying a community of 13 individuals in which every person has a social connection to every other person. How many closed triads are there in the community?

Tech Check

Since the entries in Pascal's Triangle are useful in many applications, many resources such as online and traditional calculator functions have been developed to assist in calculating the values. The website dCode.xyz has a tool that automatically calculates entries in Pascal's Triangle using these steps:

Step 1: Make sure to select Index 0.

Step 2: Select your preference, display the triangle until a particular row (line), display only one row (line), or calculate the value at coordinates (which means give the value of a single entry).

Step 3: Enter row number (line number) you are interested, or enter the row number and entry number in parentheses: (row number, entry number).

Step 4: Select Calculate.

Step 5: Retrieve the answer from the results box.

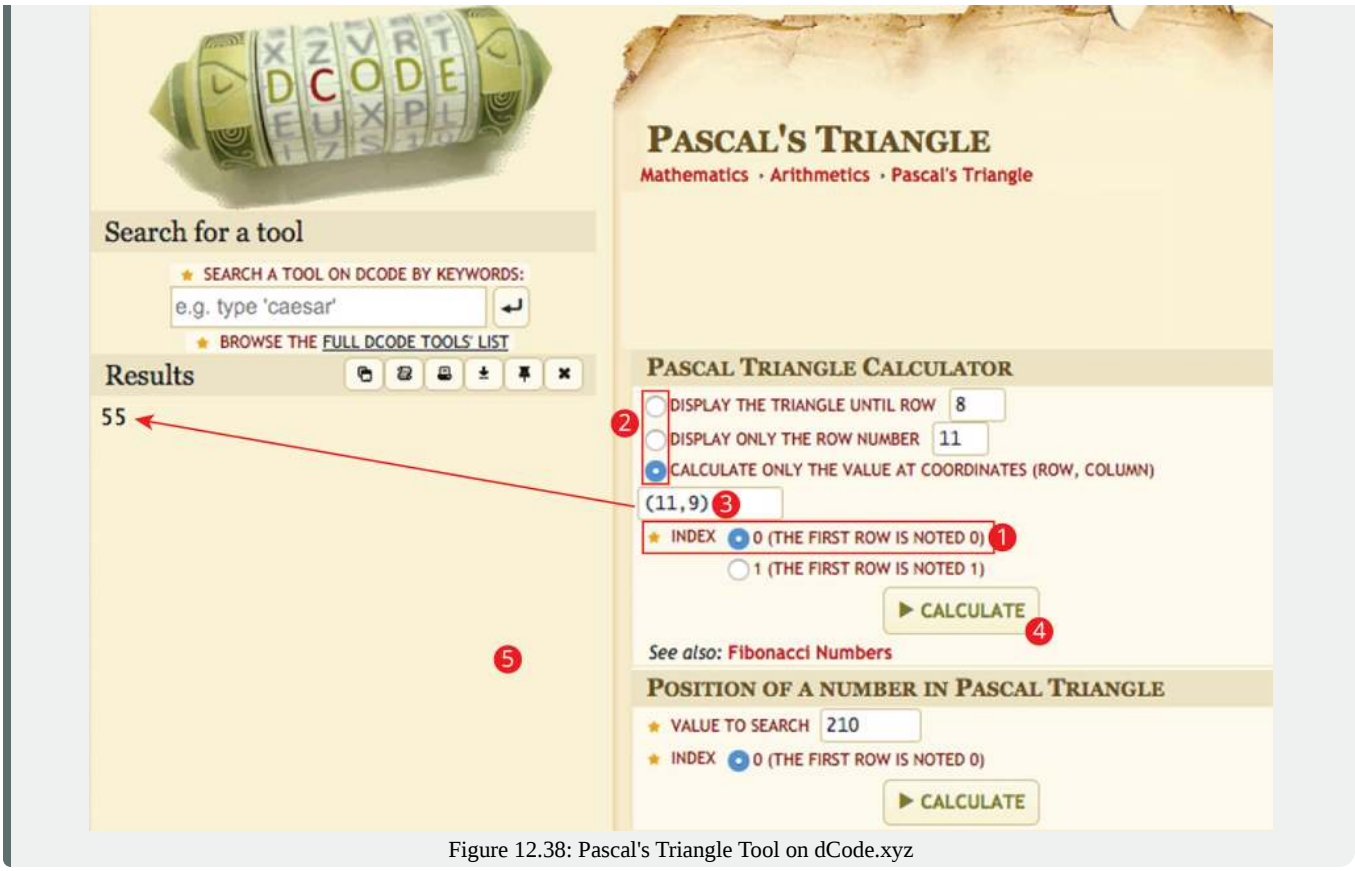


Figure 12.38: Pascal's Triangle Tool on dCode.xyz

Check Your Understanding

1. A complete graph with four vertices must have exactly four edges.

a. True
b. False

2. In a cycle, every vertex has degree two.

a. True
b. False

3. The number of vertices in a graph is twice the number of edges.

a. True
b. False

4 . The cycle (a, b, c, d) can also be called (c, b, a, d).

a. True
b. False

5. The cycle (a, b, c, d) can also be called (a, b, d, c).

a. True
b. False

6. The only cliques that are cycles are cliques of three vertices.

a. True
b. False

7. A student found that the sum of the degrees of the vertices in a graph was 13. Why is that impossible?

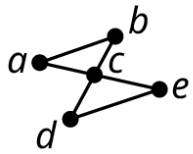
8. A student finds the number of edges in a complete graph by taking half the number of vertices, then subtracting one from the number of vertices, then multiplying these two values together. Will the student get the correct result?

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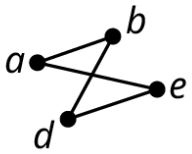
- **12.2: Graph Structures** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/contemporary-mathematics.

3.3.1: Exercises

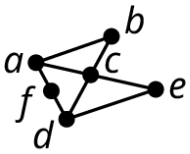
Use the figure to answer the following exercises.



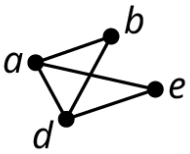
Graph S



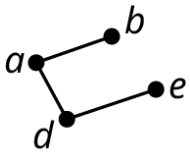
Graph T



Graph U



Graph V



Graph W

Exercise 3.3.1.1

List any graphs that are subgraphs of Graph U.

Exercise 3.3.1.2

List any graphs that are subgraphs of Graph V.

Exercise 3.3.1.3

Explain why Graph T is not a subgraph of Graph S.

Exercise 3.3.1.4

List any graphs that have a quadrilateral cyclic subgraph and name the quadrilaterals.

Use the Sum of Degrees Theorem as needed to answer the following exercises.

Exercise 3.3.1.5

How many edges are in a graph if the sum of the degrees of its vertices is 18.

Exercise 3.3.1.6

Draw two possible graphs to demonstrate that the sum of the degrees of the vertices in a graph with 7 edges is 14. Label the degrees of the vertices.

Exercise 3.3.1.7

What is the sum of the degrees of the vertices of a graph that has 122 edges?

Exercise 3.3.1.8

A graph has 3 vertices of degree 2, 4 vertices of degree 1, and 2 vertices of degree 3. How many edges are in the graph?

Exercise 3.3.1.9

There are 6 vertices and 11 edges in a graph, 2 of degree 5, 1 of degree 4, and 2 of degree 3. What is the degree of the remaining vertex?

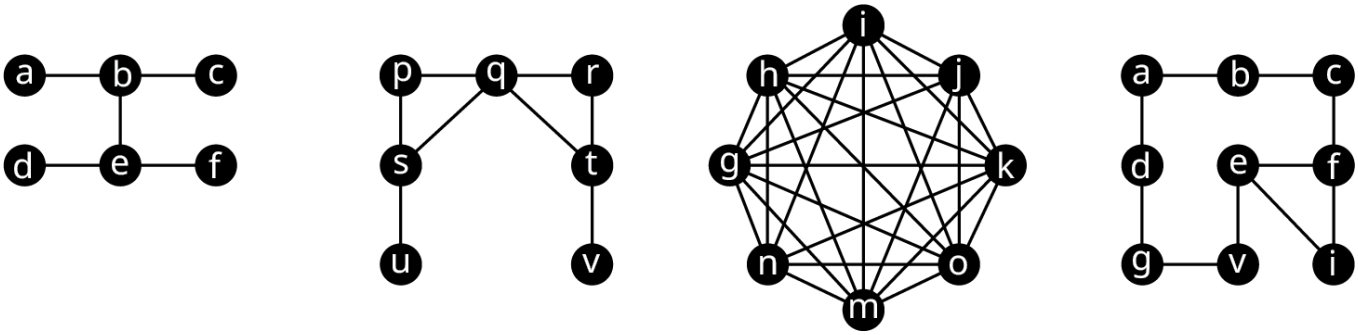
Exercise 3.3.1.10

A complete graph has 5 vertices. What is the sum of the degrees of the vertices?

Exercise 3.3.1.11

A complete graph has 120 edges. How many vertices does it have?

Use the figure to answer the following exercises. Identify the graph or graphs with the given characteristic.



Graph 1 Graph 2 Graph 3 Graph 4

Exercise 3.3.1.12

The sum of the degrees of the vertices is 16.

Exercise 3.3.1.13

The graph is complete.

Exercise 3.3.1.14

The graph has no cyclic subgraphs

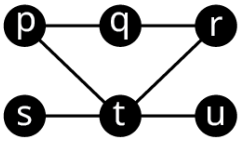
Exercise 3.3.1.15

The graph contains at least one octagon.

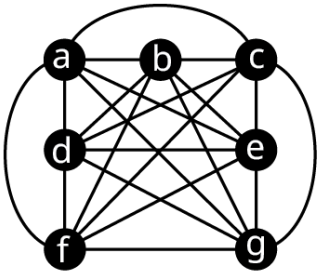
Exercise 3.3.1.16

The graph contains exactly two 3-cycles.

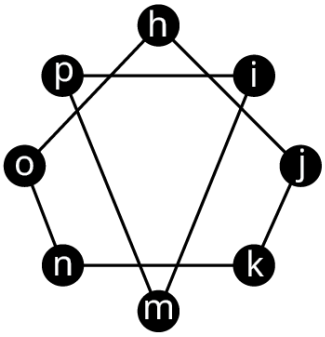
Use the figure to answer the following exercises. Identify the graph or graphs with the given characteristic.



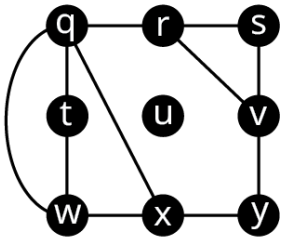
Graph 5



Graph 6



Graph 7



Graph 8

? Exercise 3.3.1.17

The graph is a subgraph of Graph 6.

? Exercise 3.3.1.18

The sum of the degrees of the vertices is 16.

? Exercise 3.3.1.19

The graph is complete.

? Exercise 3.3.1.20

The graph has no pentagons.

? Exercise 3.3.1.21

The graph contains at least one octagon.

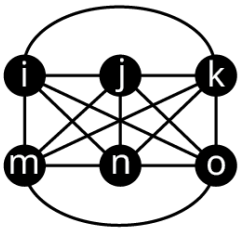
? Exercise 3.3.1.22

The graph contains exactly two cyclic subgraphs.

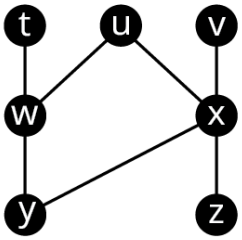
? Exercise 3.3.1.23

Use Table 12.1 to name a heptagon in Graph 8.

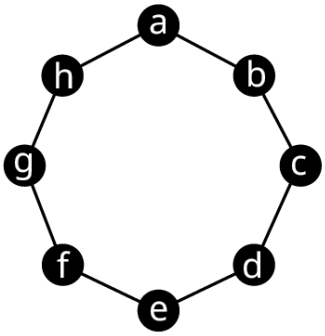
Use the figure to answer the following exercises. Identify the graph or graphs with the given characteristic.



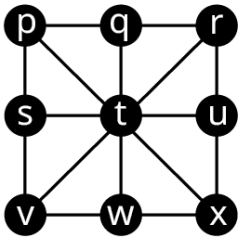
Graph 9



Graph 10



Graph 11



Graph 12

? Exercise 3.3.1.24

The sum of the degrees of the vertices is 16.

? Exercise 3.3.1.25

The graph is complete.

? Exercise 3.3.1.26

The graph contains exactly one quadrilateral.

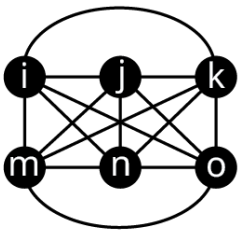
? Exercise 3.3.1.27

The graph contains at least one octagon.

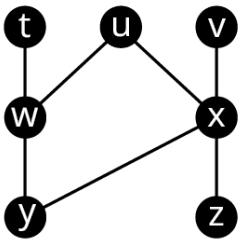
? Exercise 3.3.1.28

The graph contains exactly one cyclic subgraph.

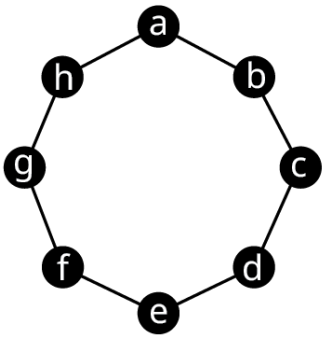
Use Table 12.1 to answer the following exercises about the below figure.



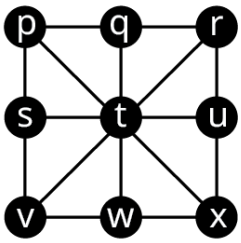
Graph 9



Graph 10



Graph 11



Graph 12

? Exercise 3.3.1.29

List every quadrilateral in Graph 12.

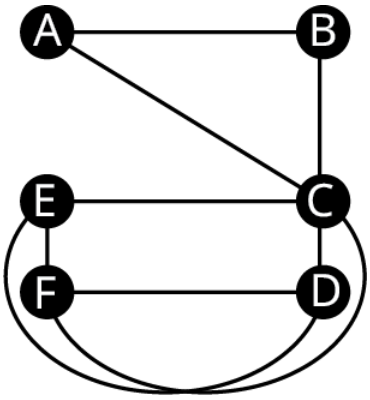
Exercise 3.3.1.30

List every quadrilateral in Graph 10.

Exercise 3.3.1.31

Determine the number of quadrilaterals in Graph 9.

Use the figure to answer the following exercises.



Exercise 3.3.1.32

Name five triangles in the graph.

Exercise 3.3.1.33

Identify a clique with more than three vertices in the graph by listing its vertices.

For the following exercises, draw a graph with the given characteristics.

Exercise 3.3.1.34

4 vertices, 6 edges, a subgraph that is a 4-cycle.

Exercise 3.3.1.35

11 vertices, the only cyclic subgraphs are triangles.

Exercise 3.3.1.36

Complete graph, no quadrilateral subgraph

Exercise 3.3.1.37

4 vertices, 4 edges, no cyclic subgraphs

Exercise 3.3.1.38

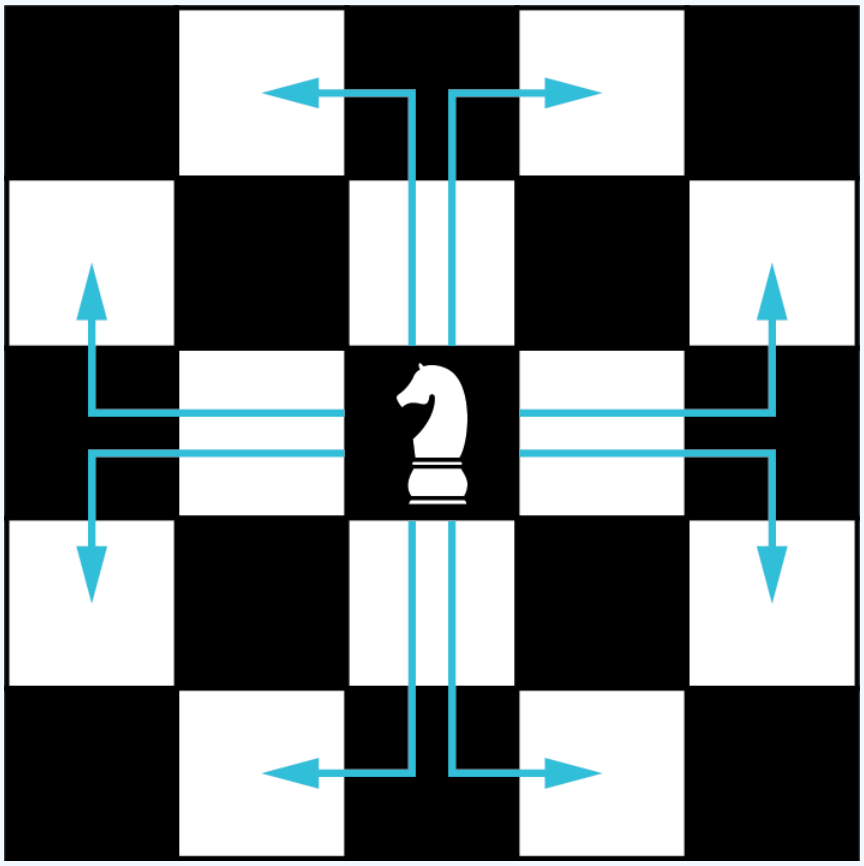
Complete graph, sum of the degrees of the vertices is 20.

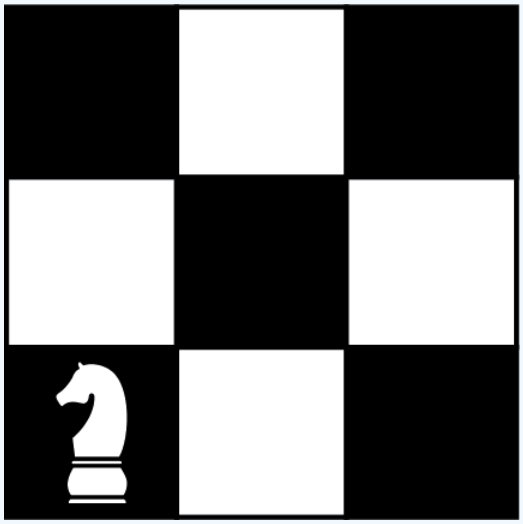
Exercise 3.3.1.39

7 vertices, largest clique has 5 vertices.

Exercise 3.3.1.40

In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The eight possible moves a knight can make from a space in the center of a five-by-five grid are shown in the first figure. Draw a graph that represents all the legal moves of a knight on a three-by-three grid starting from the lower left corner as shown in the second figure where the vertices will represent the spaces occupied by the knight and the edges will indicate a move from one space to the next.





Exercise 3.3.1.41

What kind of cycle is the resulting graph you drew for Exercise 40?

Exercise 3.3.1.42

Use Pascal's triangle to find number of triangles in a complete graph with 14 vertices.

Exercise 3.3.1.43

Do you think that a graph representing network of friends on Facebook is likely to be complete or not? Explain your reasoning.

Exercise 3.3.1.44

Would the graph of a family tree, in which edges represent parent-child lineage, contain any cycles? Why or why not?

Exercise 3.3.1.45

We have seen that the number of triangles in a complete graph with 7 vertices can be calculated using the pattern $(5+4+3+2+1)+(4+3+2+1)+(3+2+1)+(2+1)+1=35$. This pattern gets very long for complete graphs with more vertices, but we have seen sums from 1 to a number before, and we had a shortcut! Recall from Example 12.8 that $1+2+3+\cdots+(n-1)=\frac{n(n-1)}{2}$. This makes finding the sum from 1 up to $n-1$ shorter. We can write each of the sums we need in the form $\frac{n(n-1)}{2}$. To find the sum from 1 to 5 , since $n-1=5$, use $n=6: 1+2+3+4+5=\frac{6(5)}{2}$ To find the sum from 1 to 4 , since $n-1=4$, use $n=5: 1+2+3+4=\frac{5(4)}{2}$

To find the sum from 1 to 3 , since $n-1=3$, use $n=4: 1+2+3=\frac{4(3)}{2}$

To find the sum from 1 to 2 , since $n-1=2$, use $n=3: 1+2=\frac{3(2)}{2}$

To find the sum from 1 to 1 , (Sounds silly, but it helps to see the pattern!) since $n-1=1$, use $n=2$:

$$1=\frac{2(1)}{2}$$

Use these to rewrite the calculation for the number of triangles in a graph with 7 vertices.

$$\begin{aligned}&(5+4+3+2+1)+(4+3+2+1)+(3+2+1)+(2+1)+1\\&=\underbrace{1}_{\frac{2(1)}{2}}+\underbrace{(1+2)}_{\frac{3(2)}{2}}+\underbrace{(1+2+3)}_{\frac{4(3)}{2}}+\underbrace{(1+2+3+4)}_{\frac{5(4)}{2}}+\underbrace{(1+2+3+4+5)}_{\frac{6(5)}{2}}\\&=\frac{2(1)}{2}+\frac{3(2)}{2}+\frac{4(3)}{2}+\frac{5(4)}{2}+\frac{6(5)}{2}\\&=\underbrace{\frac{1(2)+2(3)+3(4)+4(5)+5(6)}{2}}=35\end{aligned}$$

The number of triangles in complete graph with 7 vertices.

Similarly, for the number of triangles in a complete graph with 8 vertices, instead of

$$(6+5+4+3+2+1)+(5+4+3+2+1)+(4+3+2+1)+(3+2+1)+(2+1)+1=56$$

use the shorter formula $\frac{1(2)+2(3)+3(4)+4(5)+5(6)+6(7)}{2}=56$.

Use this pattern to find the number of triangles in a complete graph with 10 vertices. In which row and entry of Pascal's triangle can you also find this result?

Exercise 3.3.1.46

Use the fact that the sum from 1 to $n-1$ is $\frac{n(n-1)}{2}$ to write a formula for the number of triangles in a complete graph with n vertices.

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3.4: Comparing Graphs

Learning Objectives

1. Identify the characteristics used to compare graphs.
2. Determine when two graphs represent the same relationships
3. Explore real-world examples of graph isomorphisms
4. Find the complement of a graph

Maps of the same region may not always look the same. For example, a map of Earth on a flat surface looks distorted at the poles. When the same regions are mapped on a spherical globe, the countries that are closer to the polls appear smaller without the distortion. Despite these differences, the two maps still communicate the same relationships between nations such as shared boundaries, and relative position on the earth. In essence, they are the same map. This means that for every point on one map, there is a corresponding point on the other map in the same relative location. In this section, we will determine exactly what characteristics need to be the shared by two graphs in order for us to consider them "the same."

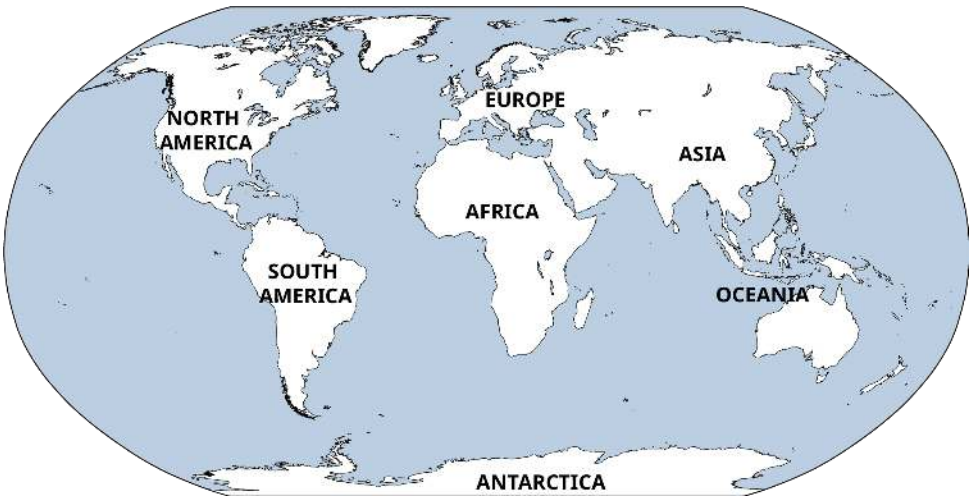


Figure **3.4.1** A flat map represents the surface of Earth in two dimensions. (credit: modification of work "World Map" by Scratchinghead/Wikimedia Commons, Public Domain)



Figure **3.4.2** A globe represents the surface of Earth in three dimensions. (credit: "Globe" by Wendy Cope/Flickr, CC BY 2.0)

When Are Two Graphs Really the Same Graph?

In arithmetic, when two numbers have the same value, we say they are equal, like ½ = 0.5. Although ½ and 0.5 look different, they have the same value, because they are assigned the same position on the real number line. When do we say that two graphs are

equal?

Figure 3.4.3 shows Graphs *A* and *F*, which are identical except for the labels. Graphs are visual representations of connections. As long as two graphs indicate the same pattern of connections, like Graph *A* and Graph *F*, they are considered to be equal, or in graph theory terms, isomorphic.

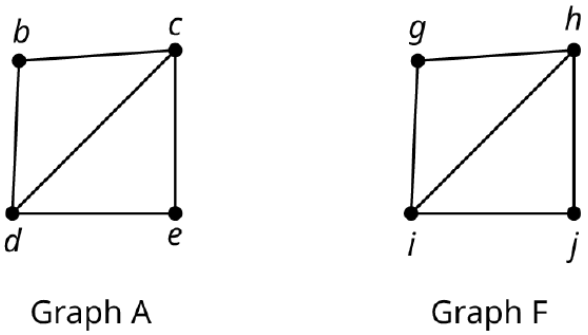


Figure 3.4.3: Identical Graphs with Different Labels

Two graphs are isomorphic if either one of these conditions holds:

1. One graph can be transformed into the other without breaking existing connections or adding new ones.
2. There is a correspondence between their vertices in such a way that any adjacent pair in one graph corresponds to an adjacent pair in the other graph.

It is important to note that if either one of the isomorphic conditions holds, then both of them do. When we need to decide if two graphs are isomorphic, we will need to make sure that one of them holds. For example, Figure 3.4.4 shows how Graph *T* can be bent and flipped to look like Graph *Z*, which means that Graphs *T* and *Z* satisfy condition 1 and are isomorphic.

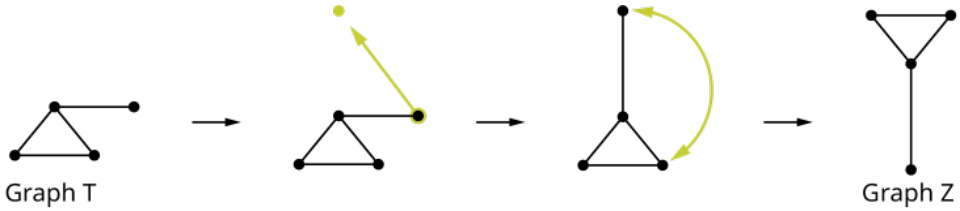


Figure 3.4.4 Graph *T* Transformed Into Graph *Z*

Also, notice that the vertices that were adjacent in the first graph are still adjacent in the transformed graph as shown in Figure 3.4.5. For example, vertex 3 is still adjacent to vertex 4, which means they are still neighboring vertices joined by a single edge.

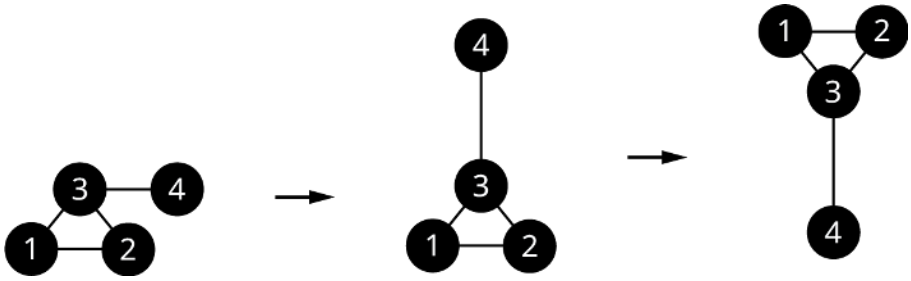
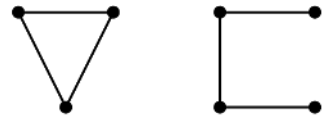
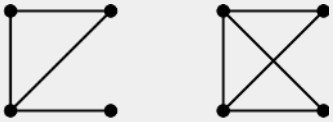
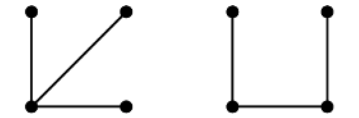
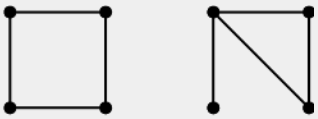


Figure 3.4.5: Adjacent Vertices Are Still Adjacent

When Are Two Graphs Really Different?

Verifying that two graphs are isomorphic can be a challenging process, especially for larger graphs. It makes sense to check for any obvious ways in which the graphs might differ so that we don't spend time trying to verify that graphs are isomorphic when they are not. If two graphs have any of the differences shown in Table 12.3, then they cannot be isomorphic.

Table **3.4.1** Characteristics of Graphs That Are Not Isomorphic

Unequal number of vertices	
Unequal number of edges	
Unequal number of vertices of a particular degree	
Different cyclic subgraphs	

Recognizing Isomorphic Graphs

Isomorphic graphs that represent the same pattern of connections can look very different despite having the same underlying structure. The edges can be stretched and twisted. The graph can be rotated or flipped. For example, in Figure **3.4.6**, each of the diagrams represents the same pattern of connections.

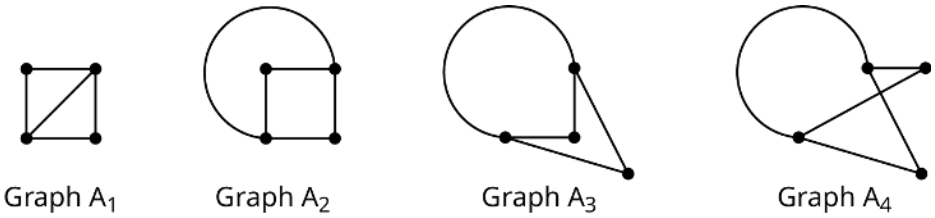


Figure **3.4.6**4: Four Representations of the Same Graph

Looking at Figure 12.44, how can we know that these graphs are isomorphic? We will start by checking for any obvious differences. Each of the graphs in Figure **3.4.6** has four vertices and five edges; so, there are no differences there. Next, we will focus on the degrees of the vertices, which have been labeled in Figure **3.4.7**

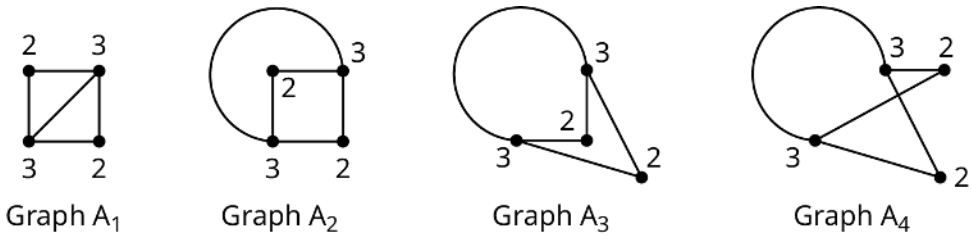


Figure **3.4.7**: Graphs with Vertices of the Same Degrees

As shown in Figure **3.4.7**, each graph has two vertices of degree 2 and two vertices of degree 3; so, there are no differences there. Now, let's check for cyclic subgraphs. These are highlighted in Figure **3.4.8**6.

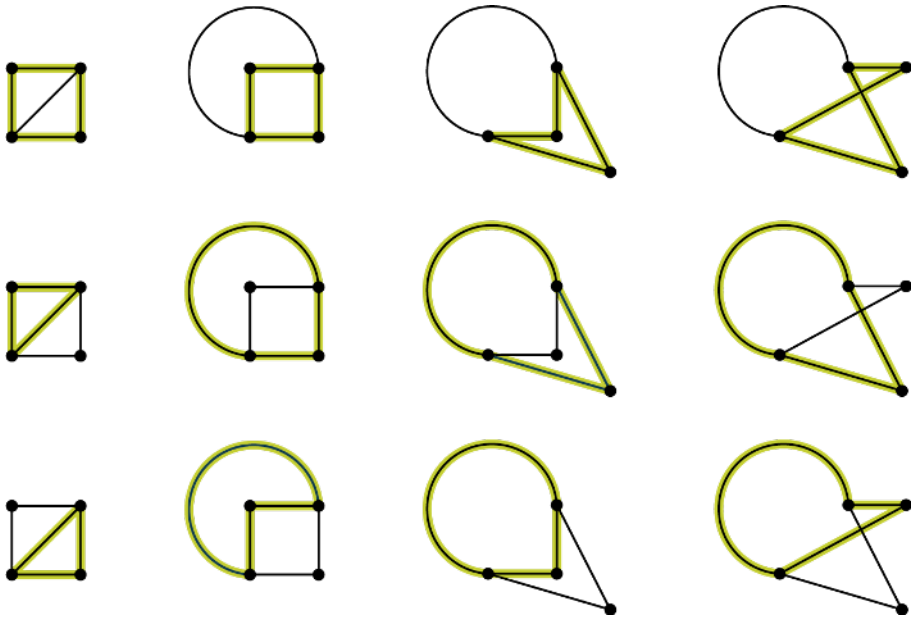


Figure **3.4.8**: Graphs with the Same Cyclic Subgraphs

As shown in Figure **3.4.8**, each graph has two triangles and one quadrilateral; so, no differences there either. It is beginning to look likely that these graphs are isomorphic, but we will have to look further to be sure.

To know with certainty that these graphs are isomorphic, we need to confirm one of the two conditions from the definition of isomorphic graphs. With smaller graphs, you may be able to visualize how to stretch and twist one graph to get the other to see if condition 1 holds. Imagine the edges are stretchy and picture how to pull and twist one graph to form the other. If you can do this without breaking or adding any connections, then the graphs are really the same. Figure **3.4.9** demonstrates how to change graph A_4 to get A_3, graph A_3 to get A_2, and graph A_2 to get A_1.

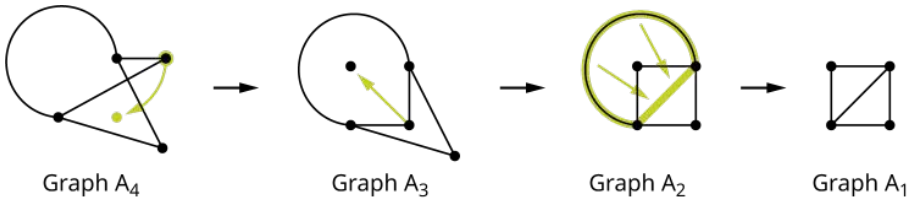


Figure **3.4.9**: Transforming Graphs

Now that we have used visual analysis to see that condition 1 holds for graphs A_1, A_2, A_3, and A_4 in Figure 12.46, we know that they are isomorphic. In Figure 12.47, one of the edges of graph A_4 crossed another edge of the graph. By transforming it into graph A_3, we have "untangled" it. Graphs that can be untangled are called planar graphs. The complete graph with five vertices is an example of a nonplanar graph-that means that, no matter how hard you try, you can't untangle it. But, when you try to figure out if two graphs are the same, it can be helpful to untangle them as much as possible to make the similarities and differences more obvious.

✔ Example **3.4.1**: Identifying Isomorphic Graphs

Which of the three graphs in Figure ✔ **3.4.10** are isomorphic, if any? Justify your answer.

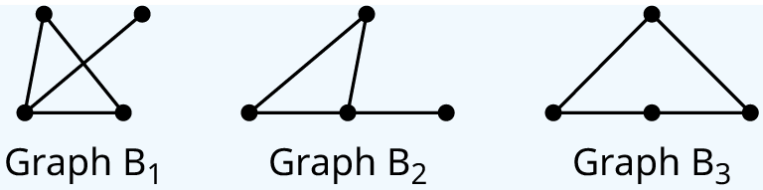


Figure ✔ 3.4.10: Three Similar Graphs

Answer

Step 1: Check for differences in number of vertices, number of edges, degrees of vertices, and types of cycles to see if an isomorphism is possible.

- Vertices: They all have the same number of vertices, 4.
- Edges: They all have the same number of edges, 4.
- Degrees: Graph B_1 and Graph B_2 each have a vertex of degree 3, while Graph B_3 does not. So, Graph B_3 is not isomorphic to either of the other two graphs, but Graph B_1 and Graph B_2 could possibly be isomorphic.
- Cycles: Focus on any cycles in Graph B_1 and Graph B_2. Each graph has a triangle as shown in Figure ✔ 3.4.11.

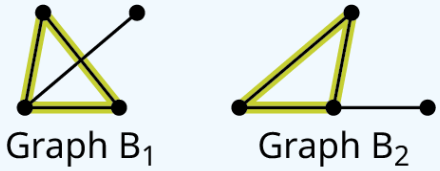


Figure ✔ 3.4.11 Cycles of Graph B_1 and Graph B_2

Step 2: If no differences were found and an isomorphism is possible, verify one of the conditions in the definition of isomorphic.

Since we were able to determine that Graph B_1 and Graph B_2 have no obvious differences, and they are relatively small graphs, we will attempt to transform one graph into the other, which would verify condition 1. Graph B_1 can be transformed into Graph B_2 without breaking or adding connections as shown in Figure ✔ 3.4.12. Begin by untangling graph B_1. Then rotate or flip as needed to see that the graphs match.

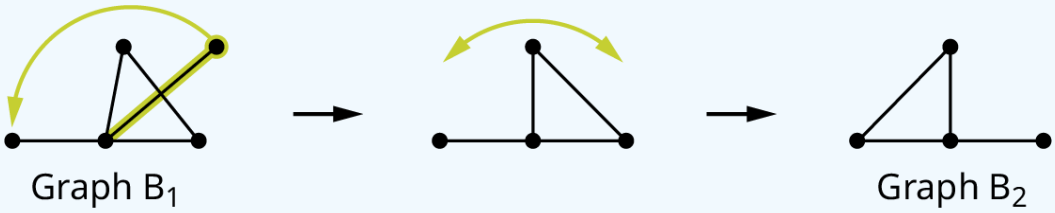


Figure ✔ 3.4.12: Transformation of Graph B_1

So, Graph B_1 and Graph B_2 have the same structure and are isomorphic.

Your Turn 3.4.1

Name four differences between Graph C_1 and Graph C_2 that show they cannot be isomorphic.

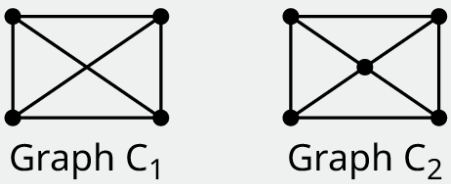


Figure 3.4.13: Two Distinct Graphs

Have you ever noticed that many popular board games may look different but are really the same game? A good example is the many variations of the board game *Monopoly*®, which was submitted to the U.S. Patent Office in 1935. Although the rules have been revised a bit, a very similar game board is still in use today. There have been many versions of Monopoly over the years. Many have been stylized to reflect a popular theme, such as a show or sports team, while retaining the same game board structure. If we were to represent these different versions of the game using a graph, we would find that the graphs are isomorphic. Let's analyze some game boards using graph theory to determine if they have the same structure despite having different appearances.

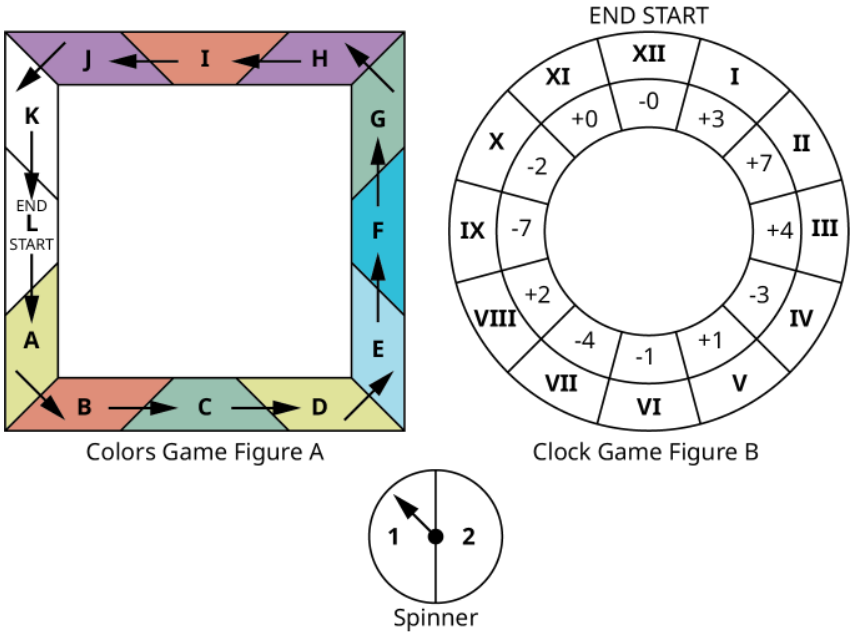


Figure **3.4.14**: Two Game Boards

✔ Example **3.4.2**: Deciding If Graphs are Isomorphic

A teacher uses games to teach her students about colors and numbers as shown in Figure ✔ **3.4.14**.

In the Colors Game, shown in Figure *A*, each player begins in the space marked START and proceeds in a counterclockwise direction. On each turn, the player spins a spinner marked 1 and 2 and moves forward the number of spaces shown on the spinner. If the player lands on a space marked with any color other than white, the player must move forward or back to the other space of the same color. The first player to land in or pass the space marked END wins.

In the Clock Game, shown in Figure *B*, each player begins in the space marked START and proceeds in a clockwise direction. On each turn, the player spins a spinner marked 1 and 2 and moves forward the number of spaces shown on the spinner. In the same turn, the player must read the number in the space and move forward the number of spaces indicated by a positive value or backward the number of spaces indicated by a negative value. Then the turn ends. The first player to land in or pass the space marked END wins.

Draw a graph or multigraph to represent each game in which the vertices are the spaces and the edges represent the ability of a player to move between the spaces either by a spin or as dictated by a marked color or number. (We will ignore the direction of motion for simplicity.) Transform one of the graphs to show that it is isomorphic to the other and explain what this tells you about the games.

This tells us that the games are essentially the same game with the same moves even though the games appear to be different.

Answer

The graph representing the Clock Game can be transformed as shown in Figure ✔ **3.4.15** so that we can see that the graphs are isomorphic. There is a correspondence between their vertices in such a way that any adjacent pair in one graph corresponds to an adjacent pair in the other graph.

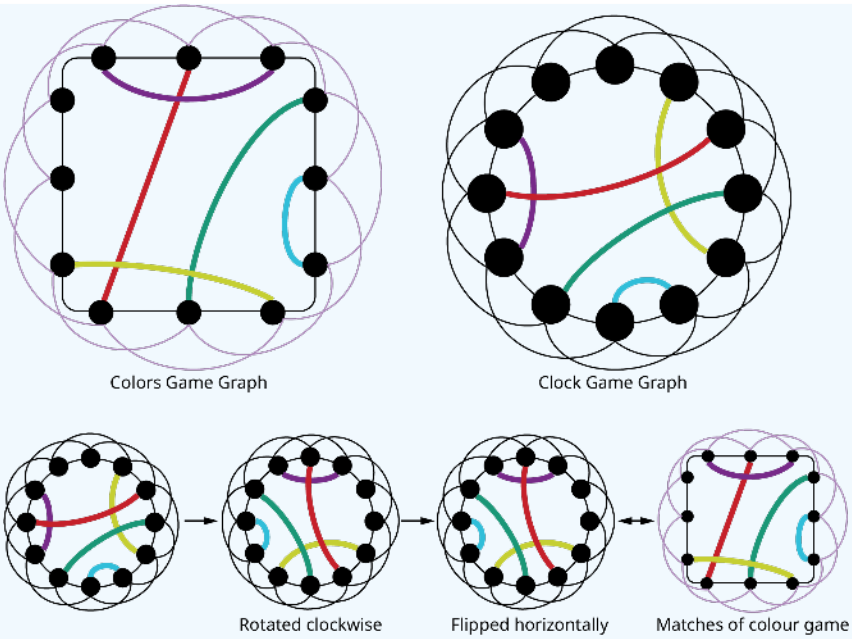


Figure ✔ **3.4.15** Graphs for Clock Game and Colors Game

Your Turn 3.4.2

Let's compare the games in Figures *A* and *B*. In the game Diamonds, shown in Figure *A*, each player begins in the space marked START and proceeds in a counterclockwise direction. On each turn, the player rolls a six-sided die and moves forward the number of spaces shown on the die. If the player lands on a space marked with a diamond, the player jumps to the next space marked with a diamond and stops there. The first player to land in or pass the space marked END wins. In the Dots game, shown in Figure *B*, each player begins in the space marked 1 and proceeds in numerical order through the numbered spaces. On each turn, the player rolls a six-sided die and moves forward the number of spaces shown on the die. When a player lands on a space marked with a green dot, they immediately move to the space indicated by the arrow. Construct a graph to represent each gameboard such that each vertex represented a space on the game board and each edge represented the ability of a player to move directly from one space to the next. For example, in the Dots game, the vertex representing space 3 would be adjacent to 2, 4, and 6. Identify at least three differences between a graph based on Diamonds and a graph based on Dots to show the graphs are not isomorphic.

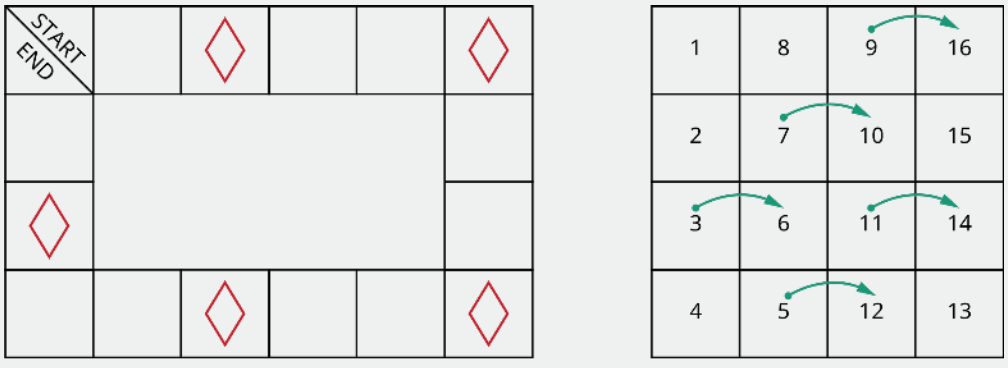


Figure *A*
Diamond game

Figure *B*
Dots game

Figure 📌 **3.4.16**: Two Game Boards

People in Mathematics: Elizabeth Magie

Game designer, engineer, comedian, and political activist, Elizabeth Magie designed a game called “Landlord’s Game” to educate fellow citizens about the dangers of monopolies and the benefits of wealth redistribution through a land tax. Magie, whose father had campaigned with Abraham Lincoln, was a proponent of a land tax, an idea popularized by Henry George’s 1879 book, *Progress and Poverty*. She designed the game to be played by two sets of rules for comparison. In one version, the goal was to dominate opponents by creating monopolies, leaving one wealthy player standing in the end. In the other version, all the players were rewarded when a monopoly was created through a simulated land tax. She patented this game for the first time in 1904. Does the game board in Figure 3.4.16 look familiar?

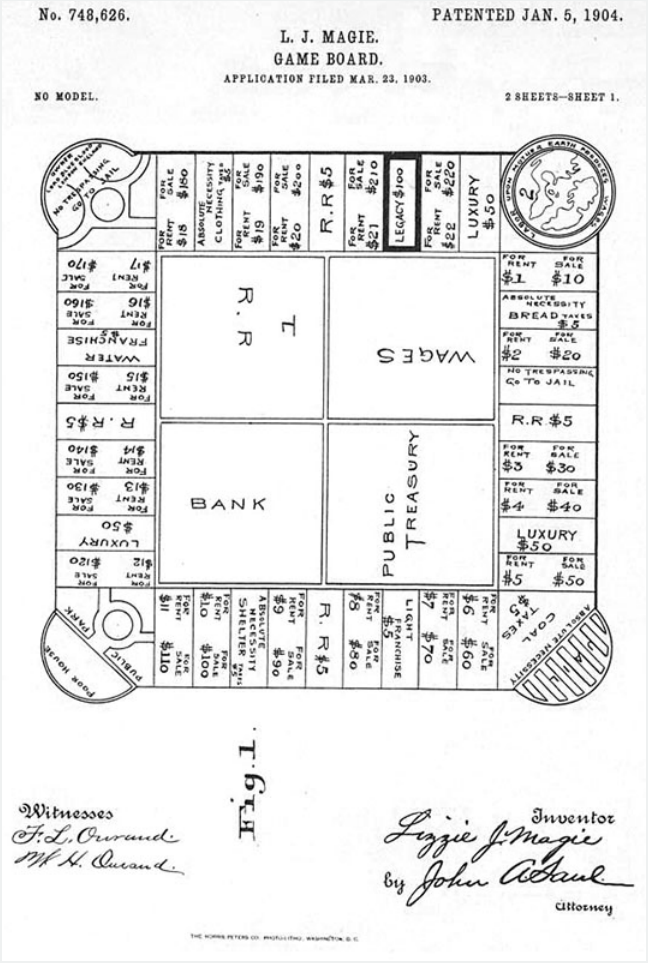


Figure 3.4.16 *Landlord’s Game* Patent (credit: "Landlord's Game Patent" Wikimedia Commons, Public Domain)

That’s right! A modified version of the game was obtained from friends by a man named Charles Darrow who renamed it Monopoly and sold to Parker Brothers. Parker Brothers later paid Magie $500 for the right to her patent on it. The property tax version was left behind, and the modern game of Monopoly was born. (Mary Pilon, *Monopoly’s Lost Female Inventor*, September 1, 2018, National Women’s History Museum, *“Monopoly’s* Lost Female Inventor,”

Identifying and Naming Isomorphisms

When two graphs are isomorphic, meaning they have the same structure, there is a correspondence between their vertices, which can be named by listing corresponding pairs of vertices. This list of corresponding pairs of vertices in such a way that any adjacent pair in one graph corresponds to an adjacent pair in the other graph is called an **isomorphism**. Consider the isomorphic graphs in Figure 3.4.8. In Figure 3.4.17, we could replace the labels Graph *F* with the labels from Graph *A* and have an identical graph, as in Figure 3.4.18.

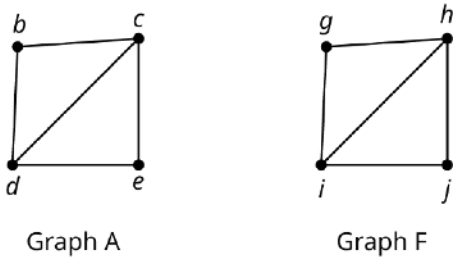


Figure **3.4.17**: Identical Graphs with Different Labels

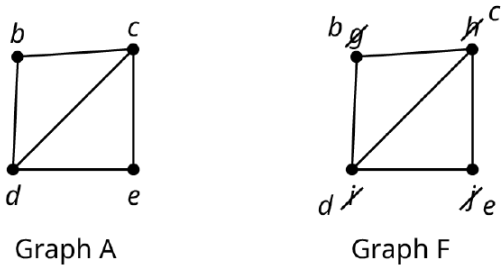


Figure **3.4.18**: Corresponding Vertices

So, we can identify an isomorphism between Graph *A* and Graph *F* by listing the corresponding pairs of vertices: *b*-*g*, *c*-*h*, *d*-*i*, and *e*-*j*. Notice that *b* is adjacent to *c* and *g* is adjacent to *h*. This must be the case since *b* corresponds to *g* and *c* corresponds to *h*. The same is true for other pairs of adjacent vertices.

An isomorphism between graphs is not necessarily unique. There can be more than one isomorphism between two graphs. We can see how to form a different isomorphism between Graph *A* and Graph *F* from Figure 12.36 by rotating Graph *F* clockwise and comparing the rotated version of *F* to Graph *A* as in YOUR TURN **3.4.13**. Now, we can see that a second isomorphism exists, which has the correspondence: *b*-*j*, *d*-*h*, *c*-*i*, and *e*-*g* as shown in Figure **3.4.19**.

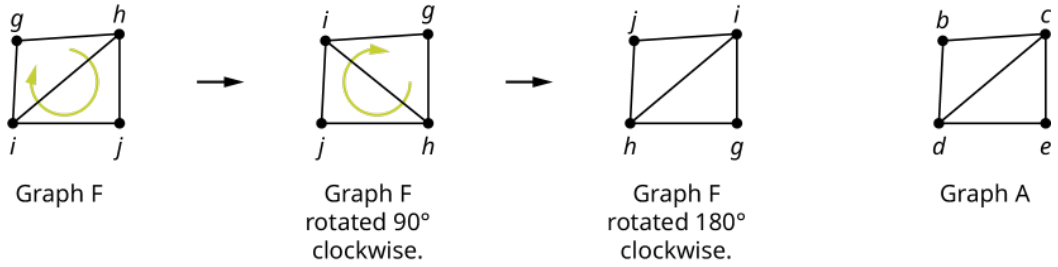


Figure **3.4.19** Transforming Graph *F* into Graph *A*

Checkpoint

When you name isomorphisms, one way to check that your answer is reasonable is to make sure that the degrees of corresponding vertices are equal.

Video



Determine If Two Graphs Are Isomorphic and Identify the Isomorphism

Example 3.4.31: Identifying Isomorphisms

We showed that the Graphs B_1 and B_2 in Figure ✔ 3.4.8 are isomorphic. In Figure ✔ 3.4.20, labels have been assigned to the vertices of Graphs B_1 and B_2. Identify an isomorphism between them by listing corresponding pairs of vertices.

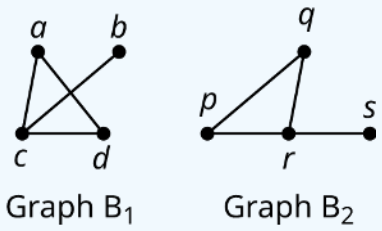


Figure ✔ 3.4.20: Two Isomorphic Graphs

Answer

Figure ✔ 3.4.8 showed how to transform Graph B_1 to get Graph B_2. In Figure ✔ 3.4.21, we will do the same, but this time we will include the labels.

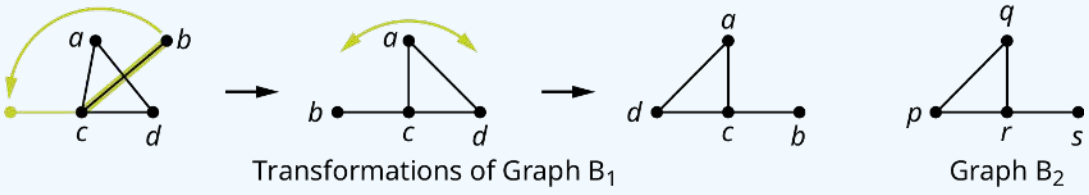


Figure ✔ 3.4.21 Transform Graph B_1 into Graph B_2

From YOUR TURN ✔ 3.4.8, we can see the corresponding vertices: *a*-*q*, *d*-*p*, *c*-*r*, and *b*-*s*, which is an isomorphism of the two graphs.

Your Turn 3.4.3

Graphs *E* and *T* are isomorphic. Name two isomorphisms.

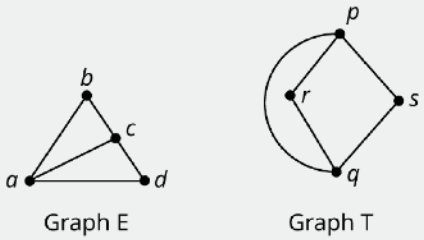


Figure 3.4.22: Graph *E* and Graph *T*

Example 3.4.4: Recognizing an Isomorphism

Determine whether Graphs *G* and *S* in Figure 3.4.23 are isomorphic. If not, explain how they are different. If so, name the isomorphism.

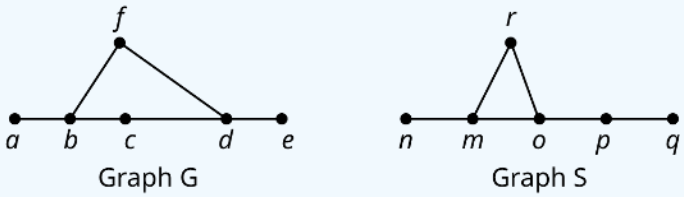


Figure 3.4.23 Graph *G* and Graph *S*

Step 2: This step is not necessary because we now know Graphs *G* and *S* are not isomorphic.

Answer

Step 1: Check for any differences.

- Vertices: Graphs *G* and *S* each have six vertices.
- Edges: Graphs *G* and *S* each have six edges.
- Degrees: Each graph also has two vertices of degree 1, two vertices of degree 2, and two vertices of degree 3.
- Cycles: From Figure 3.4.24, we can see that Graph *G* contains a quadrilateral cycle (*b*, *f*, *d*, *c*) but Graph *S* has no quadrilaterals. Also, Graph *S* contains a triangle cycle (*m*, *r*, *o*) but Graph *G* has no triangles. This means that the graphs are not isomorphic.

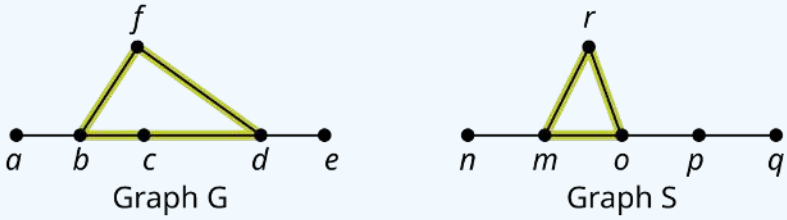


Figure 3.4.24 Comparing Subgraphs

Your Turn 3.4.4

Three students have been asked to determine if Graphs *E* and *T* are isomorphic and justify their answers.

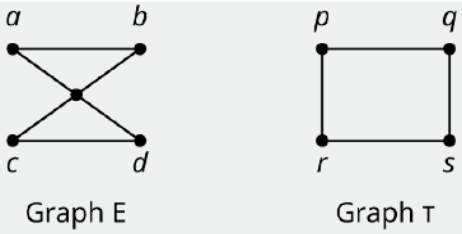


Figure 3.4.25: Graph *E* and Graph *T*

Javier believes that Graphs *E* and *T* are not isomorphic because Graph *E* contains a triangle cycle and Graph *T* does not. He supports his conclusion with the graphs shown.

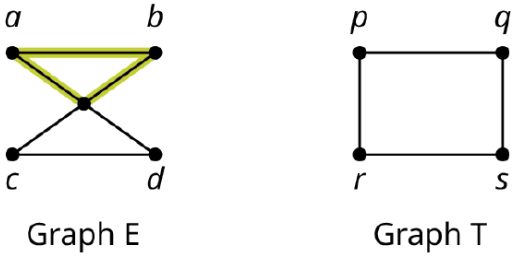


Figure 3.4.26: Javier's Highlighted Cycle

Maubi also believes that Graphs *E* and *T* are not isomorphic, but says that it is because Graph *T* has a quadrilateral cycle (*p*, *q*, *r*, *s*) while Graph *E* does not. Caden believes that Graphs *E* and *T* are isomorphic. She said one isomorphism is *a*-*p*, *b*-*q*, *c*-*s*, and *d*-*r*. Determine who is correct, who is incorrect, and explain how you know.

Complementary Graphs

Suppose that you are a camp counselor at Camp Woebegone and you are holding a camp Olympics with four events. The campers have signed up for the events. You drew a graph in Figure 3.4.27 to help you visualize which events have campers in common.

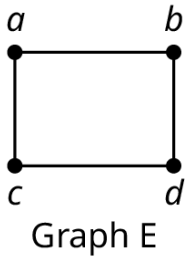


Figure 3.4.27: Graph of Events with Campers in Common

Graph *E* in Figure 3.4.27 shows that some of the same campers will be in events *a* and *b*, as well as *b* and *d*, *c* and *d*, and *a* and *c*. What do you think the graph would look like that represented the events that do not have campers in common? It would have the same vertices, but any pair of adjacent edges in Graph *E*, would not be adjacent in the new graph, and vice versa. This is called a **complementary** graph, as shown in Figure 3.4.28. Two graphs are complementary if they have the same set of vertices, but any vertices that are adjacent in one, are not adjacent in the other. In this case, we can say that one graph is the **complement** of the other.

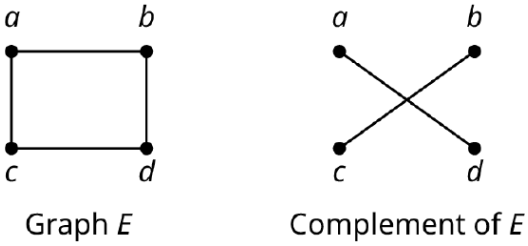


Figure 3.4.28: Graphs of Camp Olympics

One way to find the complement of a graph is to draw the complete graph with the same number of vertices and remove all the edges that were in the original graph. Let's say you wanted to find the complement of Graph *E* from Figure **3.4.28**, and you didn't already know it was Graph *F*. You could start with the complete graph with four vertices and remove the edges that are in Graph *E* as shown in Figure **3.4.29**.

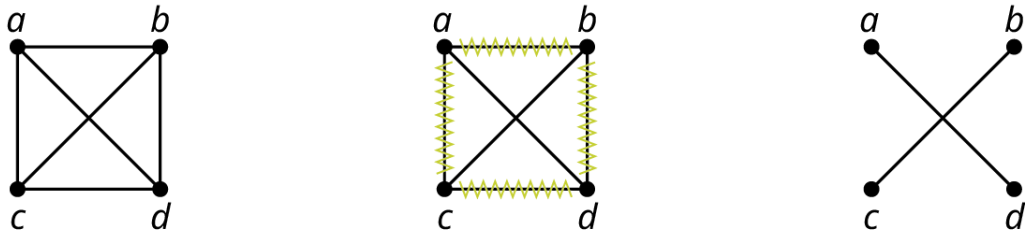


Figure **3.4.29**: Use Complete Graph to Find Complement

✔ Example 3.4.5: Finding a Complement

A particular high school has end-of-course exams in (E3) English 3, (E4) English 4, (M) Advanced Math, (C) Calculus, (W) World History, (U) U.S. History, (B) Biology, and (P) Physics. No English 3 students are taking English 4, World History, or Biology; no English 4 students are also in Calculus, Advanced Math, U.S. History, or Physics; no Physics students are also taking Advanced Math; No World History students are also taking U.S. History; and no Advanced Math students are also taking Calculus.

1. Create a graph in which the vertices represent the exams, and an edge between a pair of vertices indicates that there are no students taking both exams.
2. Find the complement of the graph in part 1.
3. Explain what the graph in part 2 represents.

Answer

In Figure ✔ **3.4.30**, we have drawn a vertex for each exam and edges between any vertices that have no students in common.

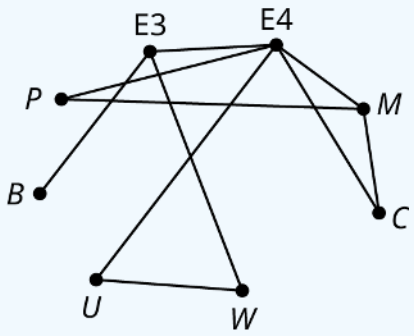


Figure ✔ **3.4.30** Graph of Exams with No Students in Common

One way to get the complement of the graph in Figure ✔ **3.4.30** is to draw a complete graph with the same number of vertices and remove the edges they have in common as shown in Figure ✔ **3.4.31**.

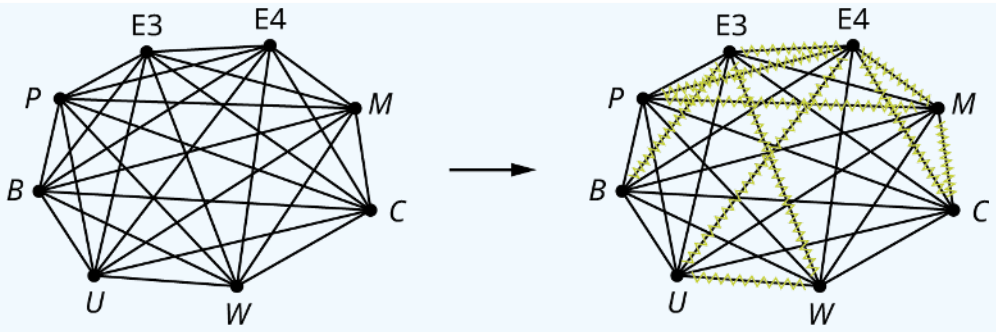


Figure ✔ 3.4.31: Remove Unwanted Edges from Complete Graph

The final graph of the complement is in Figure ✔ 3.4.32.

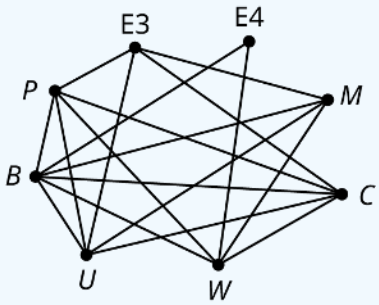
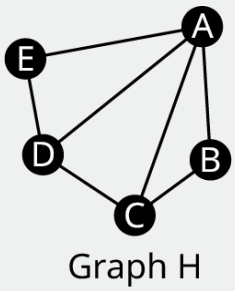


Figure ✔ 3.4.32: Graph of Exams with Students in Common

In the graph in Figure 12.71, the vertices are still the exams, and a pair of adjacent vertices represents a pair of exams that have students in common.

Your Turn 3.4.5

Find the complement of the Graph *H*.



Graph H

Figure 3.4.33

When two graphs are really the same graph, they have the same missing edges. So, when two graphs have a lot of edges, it may actually be easier to determine if they are isomorphic by looking at which edges are missing rather than which edges are included. In other words, we can determine if two graphs are isomorphic by checking if their complements are isomorphic.

✔ Example 3.4.6: Using a Complement to Find an Isomorphism

Use Figure ✔ 3.4.34 to answer each question.

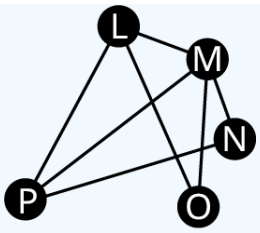


Figure ✔ **3.4.34** Graph *K*

1. Find the complement of Graph *K*.
2. Identify an isomorphism between the complement of Graph *K* from part 1, and the complement of Graph *H* in YOUR TURN 12.17.
3. Confirm that the correspondence between the vertices you found in part 2 also gives an isomorphism between Graph *H* from YOUR TURN 12.17, and Graph *K* from Figure ✔ **3.4.34**.

Answer

1. The complement of Graph *K* can be found by removing the edges of Graph *K* from a complete graph with the same vertices as shown in Figure ✔ **3.4.35**.

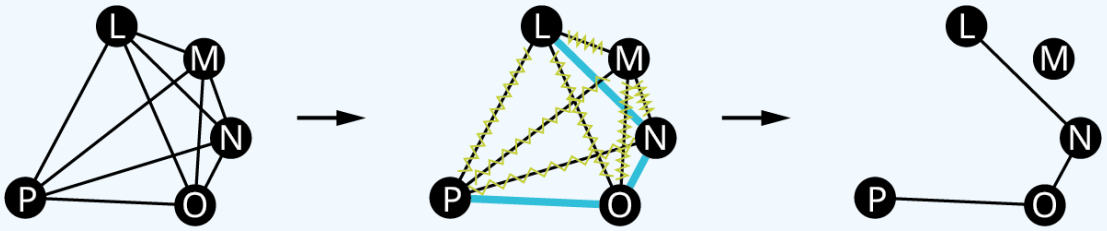


Figure ✔ **3.4.35** Find Complement of Graph *K*

2. An isomorphism between the complement of Graph *K* and the complement of Graph *H* is *A-M*, *C-L*, *E-N*, *B-O*, and *D-P*, which is confirmed by transforming the complement of Graph *K* in Figure ✔ **3.4.36**.

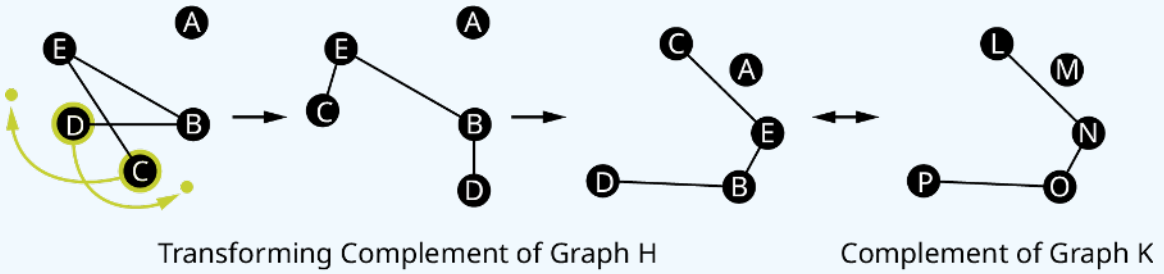


Figure ✔ **3.4.36**: Isomorphism between Complements of *K* and *H*

3. Figure ✔ **3.4.37** shows how Graph *K* can be transformed into Graph *H* to confirm that the correspondence is *A-M*, *C-L*, *E-N*, *B-O*, and *D-P* also gives an isomorphism between Graph *K* and Graph *H*.

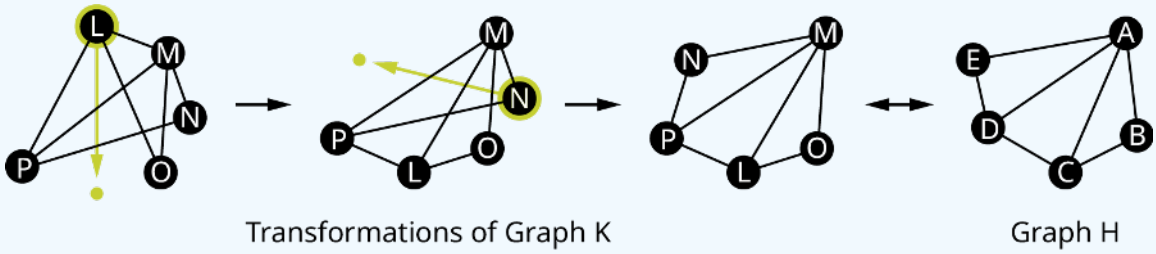


Figure ✔ **3.4.37** Isomorphism between *K* and *H*

This means we now have three conditions that guarantee two graphs are isomorphic.

First Way: One graph can be transformed into the other without breaking existing connections or adding new ones.

Second Way: There is a correspondence between their vertices in such a way that any adjacent pair in one graph corresponds to an adjacent pair in the other graph.

Third Way: Their complements are isomorphic.

If any one of these statements is true, then they are all true. If any one of these statements is false, then they are all false.

Your Turn 3.4.6

Suppose that a Graph *M* is a complete graph and Graph *N* is the complement of *M*. What are the degrees of the vertices of Graph *N*? How do you know?

WORK IT OUT

Here is an activity you can do with a few of your classmates that will build your graph comparison skills.

Step 1: Draw a planar graph with the following characteristics: exactly five vertices, one vertex of degree four, at least two vertices of degree three, and exactly eight edges. Give names to the vertices. Make sure you do not use the same letters or numbers to label your vertices as your classmates do.

Step 2: Analyze your graph. What is the degree of each vertex? Does your graph have any cyclic subgraphs? If so, list them and indicate their sizes.

Step 3: Draw and analyze the complement of your graph. How many edges and vertices does it have? What is the degree of each vertex? Does the complementary graph have any cyclic subgraphs? If so, list them and indicate their sizes.

Step 4: Compare your graphs to each of your classmates' graphs. Does your graph have the same number of edges and vertices as the graph of your classmate? Does your graph have the same size cyclic subgraphs as the graph of your classmate? How does the complement of your graph compare to the complement of the graph of your classmate? Determine if your graph is isomorphic to your classmates' graph. If so, give a correspondence that demonstrates the isomorphism. If not, explain how you know.

Check Your Understanding

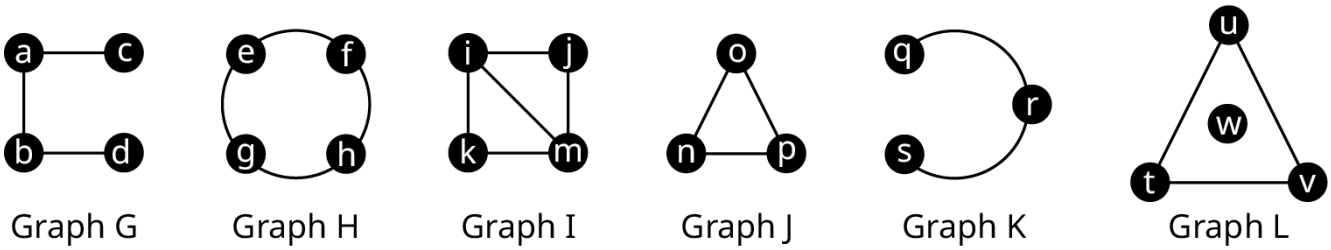
1. For the following exercises, determine whether each statement is always true, sometimes true, or never true.
2. Two graphs are isomorphic, and the graphs have the same structure.
3. A graph with four vertices is isomorphic to a graph with five vertices.
4. The sums of the degrees of the vertices of two graphs are equal, but the two graphs are not isomorphic.
5. Two graphs are isomorphic, but the graphs have a different number of edges.
6. One graph can be transformed to look like a second graph without removing or adding any connections, and the two graphs are isomorphic.
7. Two graphs are isomorphic, and there is more than one isomorphism between the two graphs.
8. Two graphs have the same number of vertices, but there is no isomorphism between them.
9. There is a correspondence between the vertices of Graph *A* and the vertices of Graph *B* such that the adjacent vertices in Graph *A* always correspond to vertices of Graph *B*, but the two graphs are not isomorphic
10. Two graphs have the same number of edges, the same number of vertices, vertices of the same degree, and have all the same subgraphs, but they are not isomorphic
11. Two graphs are isomorphic, and the sum of the degrees of the vertices of one equals the sum of the degrees of the other graph.
12. If two graphs are isomorphic, then their complements are isomorphic.

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3.4.1: Exercises

Use the figure to answer the following exercises. A pair of graphs is given. Identify three differences between them that demonstrate the graphs are not isomorphic.



Graph G Graph H Graph I Graph J Graph K Graph L

? Exercise 3.4.1.1

G and *H*

? Exercise 3.4.1.2

G and *I*

? Exercise 3.4.1.3

G and *J*

? Exercise 3.4.1.4

G and *K*

? Exercise 3.4.1.5

G and *L*

? Exercise 3.4.1.6

H and *I*

? Exercise 3.4.1.7

H and *J*

? Exercise 3.4.1.8

H and *K*

? Exercise 3.4.1.9

H and *L*

Exercise 3.4.1.10

I and *J*

Exercise 3.4.1.11

I and *K*

Exercise 3.4.1.12

I and *L*

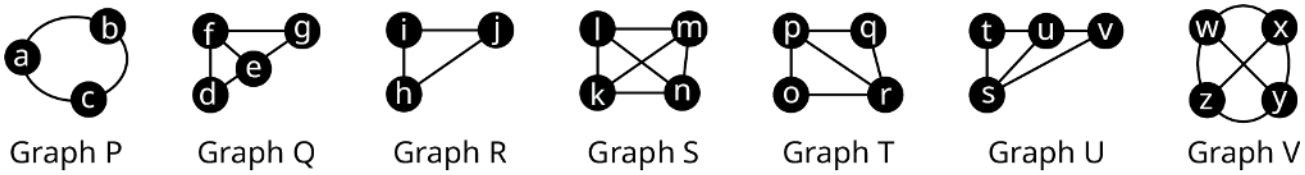
Exercise 3.4.1.13

J and *K*

Exercise 3.4.1.14

K and *L*

Use the figure to answer the following exercises. In each exercise, a pair of graphs is given. Determine if one graph is a subgraph of the other graph, or the two graphs are isomorphic.



Exercise 3.4.1.15

P and *Q*

Exercise 3.4.1.16

P and *R*

Exercise 3.4.1.17

P and *U*

Exercise 3.4.1.18

Q and *R*

Exercise 3.4.1.19

Q and *S*

Exercise 3.4.1.20

Q and *T*

Exercise 3.4.1.21

Q and *V*

Exercise 3.4.1.22

S and *V*

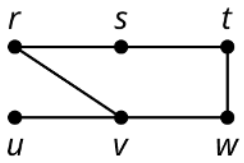
Exercise 3.4.1.23

T and *U*

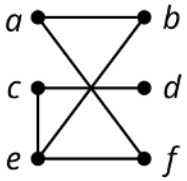
Exercise 3.4.1.24

U and *V*

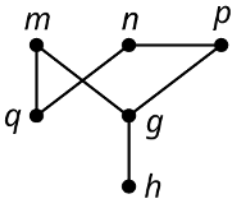
Use the figure to answer the following exercises. In each exercise, a pair of graphs is given. Either give a reason that the graphs are not isomorphic, or show how one of the graphs can be transformed to look like the other.



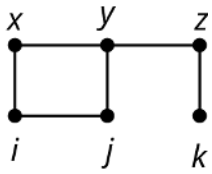
Graph A



Graph B



Graph C



Graph D

Exercise 3.4.1.25

A and *B*

Exercise 3.4.1.26

A and *C*

Exercise 3.4.1.27

A and *D*

Exercise 3.4.1.28

B and *C*

Exercise 3.4.1.29

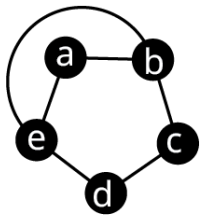
B and *D*

Exercise 3.4.1.30

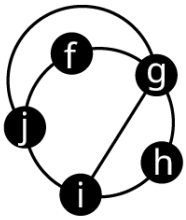
C and *D*

Use the figure to answer the following exercises. Determine if one graph is a subgraph of the other graph, or the two graphs are isomorphic. If they are isomorphic, name an isomorphism. If one is a subgraph of the other, indicate a correspondence between the

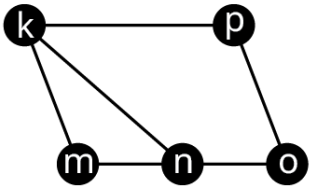
vertices that demonstrates the relationship.



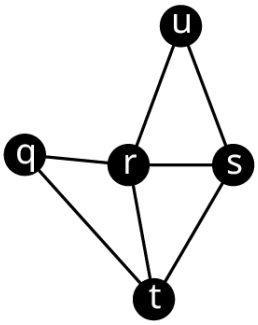
Graph W



Graph X



Graph Y



Graph Z

? Exercise 3.4.1.31

W and *Z*

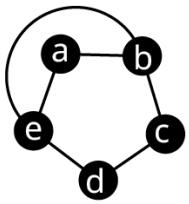
? Exercise 3.4.1.32

X and *Y*

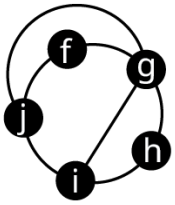
? Exercise 3.4.1.33

X and *Z*

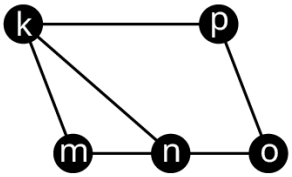
Use the figure to answer the following exercises.



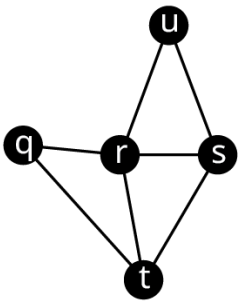
Graph W



Graph X



Graph Y



Graph Z

? Exercise 3.4.1.34

Find the complement of Graph *W*.

? Exercise 3.4.1.35

Find the complement of Graph *Y*.

? Exercise 3.4.1.36

Find the complement of Graph *X*.

? Exercise 3.4.1.37

Find an isomorphism between the complement of W and the complement of Y if one exists. If not, explain how you know.

? Exercise 3.4.1.38

Are W and Y isomorphic? Explain how you know.

? Exercise 3.4.1.39

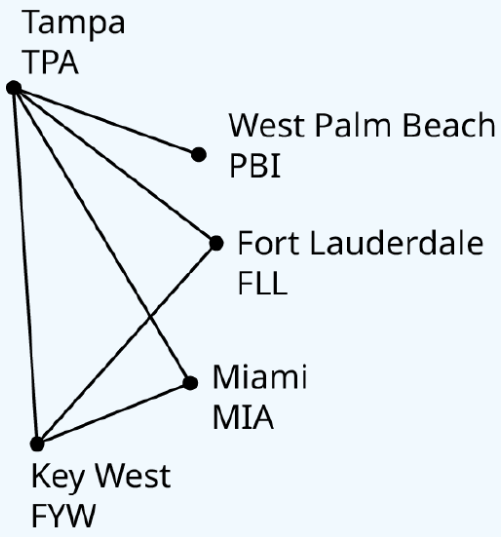
Find an isomorphism between the complement of W and the Complement of X if one Exists. If not, explain how you know.

? Exercise 3.4.1.40

Are W and X isomorphic? Explain how you know.

? Exercise 3.4.1.41

Find the complement of the graph in the given figure representing direct flights between south Florida airports. Explain what the graph represents.



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3.5: Navigating Graphs

Learning Objectives

1. Describe and identify walks, trails, paths, and circuits.
2. Solve application problems using walks, trails, paths, and circuits.
3. Identify the chromatic number of a graph.
4. Describe the Four-Color Problem.
5. Solve applications using graph colorings.

Now that we know the basic parts of graphs and we can distinguish one graph from another, it is time to really put our graphs to work for us. Many applications of graph theory involve navigating through a graph very much like you would navigate through a maze. Imagine that you are at the entrance to a maze. Your goal is to get from one point to another as efficiently as possible. Maybe there are treasures hidden along the way that make straying from the shortest path worthwhile, or maybe you just need to get to the end fast. Either way, you definitely want to avoid any wrong turns that would cause unnecessary backtracking. Luckily, graph theory is here to help!



Figure 3.5.1: Visitors navigate a garden maze. (credit: “Longleat Maze” by Niki Odolphie/Wikimedia, CC BY 2.0)

Walks

Suppose Figure 3.5.2 is a maze you want to solve. You want to get from the start to the end.

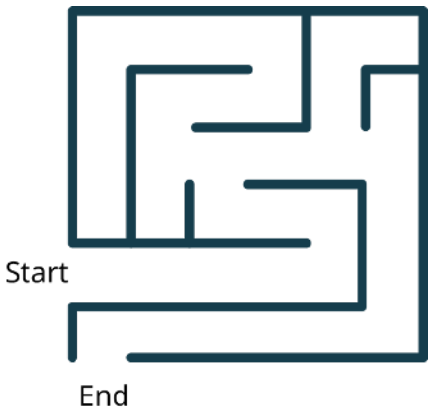


Figure 3.5.28: Your Maze

You can approach this task any way you want. The only rule is that you can't climb over the wall. To put this in the context of graph theory, let's imagine that at every intersection and every turn, there is a vertex. The edges that join the vertices must stay within the walls. The graph within the maze would look like Figure **3.5.3**.

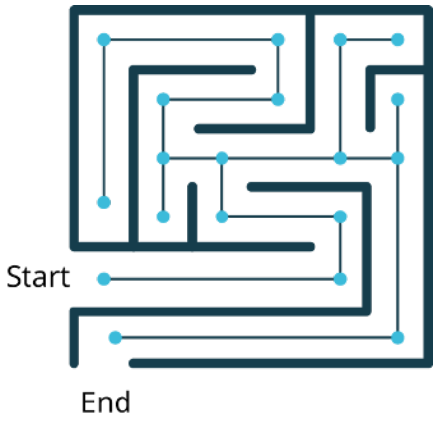


Figure **3.5.3**: The Graph in Your Maze

One approach to solving a maze is to just start walking. It is not the most efficient approach. You might cross through the same intersection twice. You might backtrack a bit. It's okay. We are just out for a walk. It might look something like the black sequence of vertices and edges in Figure **3.5.4**.

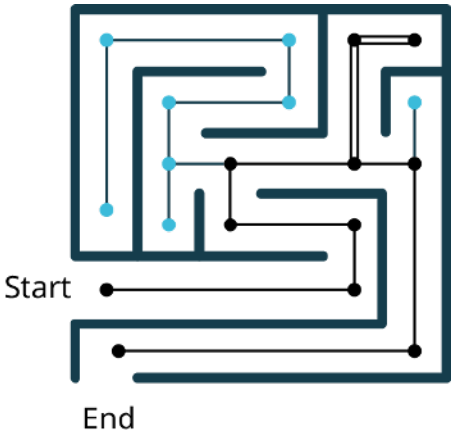


Figure **3.5.4**: The Walk Through the Graph in Your Maze

This type of sequence of adjacent vertices and edges is actually called a walk (or directed walk) in graph theory too!

A walk can be identified by naming the sequence of its vertices (or by naming the sequence of its edges if those are labeled). Let's take the graph out of the context of the maze and give each vertex a name and each edge of the walk a direction as in Figure **3.5.5**.

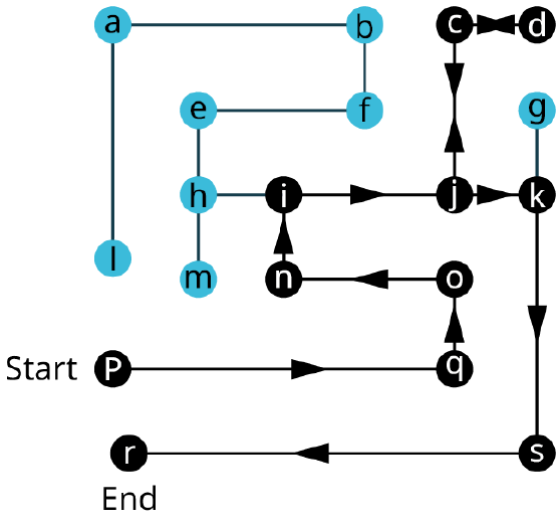


Figure 3.5.5: The Graph without Your Maze

The name of this walk from p to r is $p \to q \to o \to n \to i \to j \to c \to d \to c \to j \to k \to s \to r$. When a particular edge on our graph was traveled in both directions, it had arrows in both directions and the letters of vertices that were visited more than once had to be repeated in the name of the walk.

Checkpoint

Not every list of vertices or list of edges makes a path. The order must take into account the way in which the edges and vertices are connected. The list must be a sequence, which means they are in order. No edges or vertices can be skipped and you cannot go off the graph.

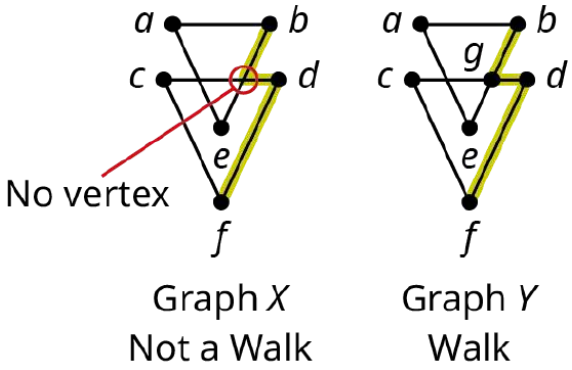


Figure 3.5.6: A Walk or Not a Walk?

The highlighted edges Graph Y in Figure 12.82 represent a walk between f and b. The highlighted edges in Graph X do **not** represent a walk between f and b, because there is a turn at a point that is not a vertex. This is like climbing over a wall when you are walking through a maze. Another way of saying this is that $b \to d \to f$ is not a walk, because there is no edge between b and d.

Checkpoint

A point on a graph where two edges cross is not a vertex.

Example 3.5.1: Naming a Walk Through A House

Figure 3.5.7 shows the floor plan of a house. Use the floor plan to answer each question.

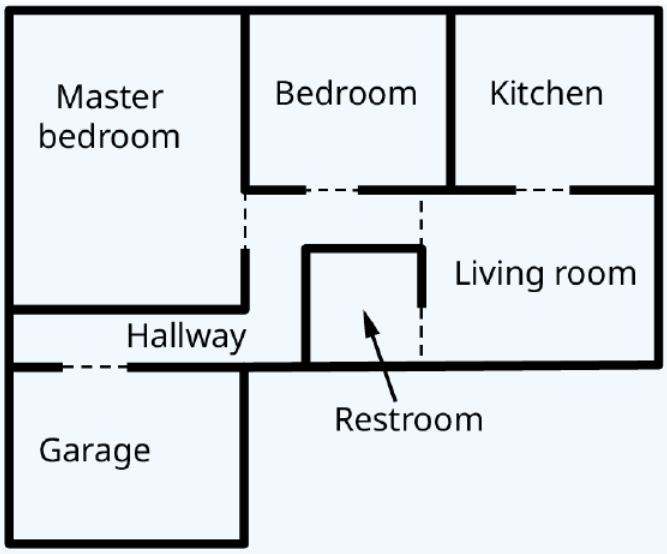


Figure ✔ 3.5.7 Floor Plan of a House

1. Draw a graph to represent the floor plan in which each vertex represents a different room (or hallway) and edges represent doorways between rooms.
2. Name a walk through the house that begins in the living room, ends in the garage and visits each room (or hallway) at least once.

Answer

1. **Step 1:** We will need a vertex for each room and it is convenient to label them according to the names of the rooms. Visualize the scenario in your head as shown in Figure ✔ 3.5.8. You don't have to write this step on your paper.

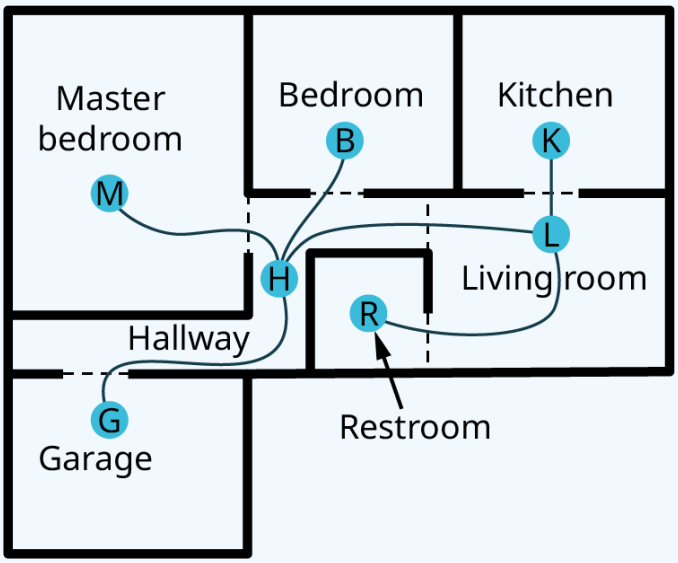


Figure ✔ 3.5.8 Assigning Vertices to Rooms

Step 2: Draw a graph to represent the scenario. Start with the vertices. Then connect those vertices that share a doorway in the floorplan as shown in Figure ✔ 3.5.9.

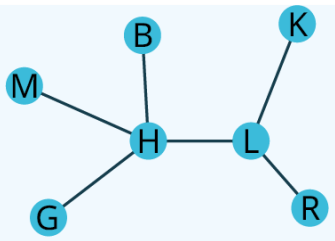


Figure ✔ 3.5.9: Graph of the Floor plan

2. **Step 1:** Draw a path that begins at vertex L, representing the living room, and ending at vertex G, representing the garage making sure to visit every room at least once. There are many ways this can be done. You may want to number the edges to keep track of their order. One example is shown in Figure ✔ 3.5.10.

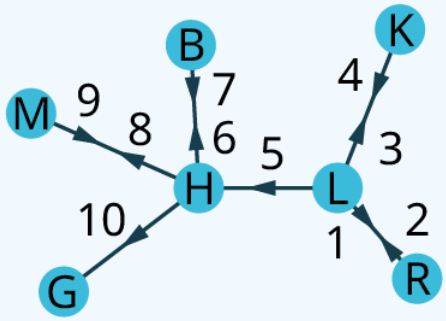


Figure ✔ 3.5.10: Draw the Path from L to G

Step 2: Name the path that you followed by listing the vertices in the order you visited them.

$L \to R \to L \to K \to L \to H \to B \to H \to M \to H \to G$

Your Turn 3.5.1

Recall from the Graph Basics section that a graph in which there are multiple edges between the same pair of vertices is called a multigraph. The figure shows a map of the four bridges that link Staten Island to Brooklyn and New Jersey and a multigraph representing the map. In the multigraph, the edges are labeled instead of the vertices. The edges represent the bridges, G (Goethals Bridge), B (Bayonne Bridge), C (Outerbridge Crossing), and V (Verrazzano-Narrows Bridge). The vertices represent New Jersey, Staten Island, and Brooklyn.

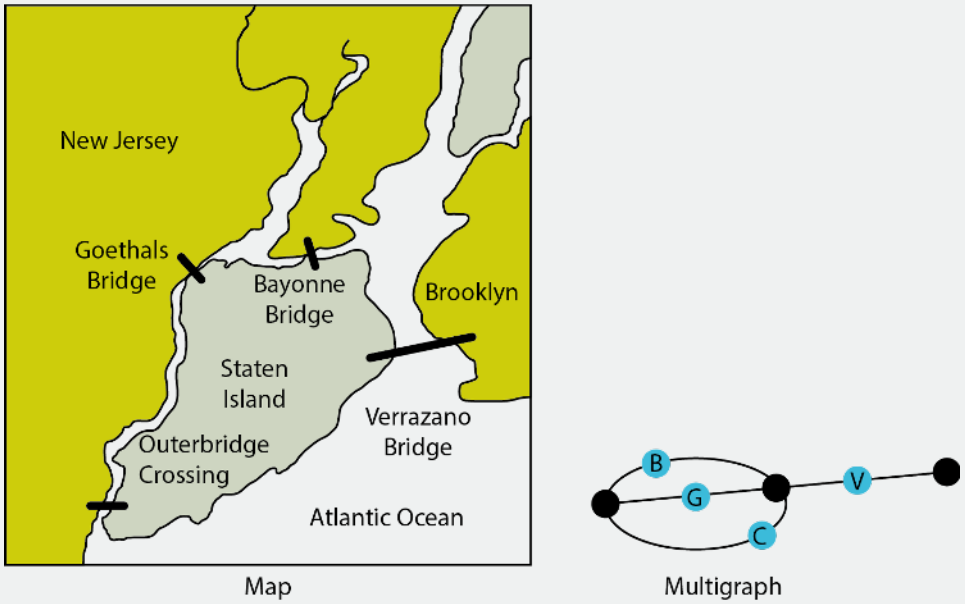


Figure 📌 3.5.11: Map and Multigraph of Staten Island Bridges

A path can be named by a sequences of edges instead of a sequence of vertices. Determine which of the following sequence of edges is a walk.

a. $B \to V \to C \to G \to C$
b. $V \to C \to B \to G \to C$
c. $C \to V \to G \to B \to B$
d. $G \to V \to B \to V \to C$

Paths and Trails

A walk is the most basic way of navigating a graph because it has no restrictions except staying on the graph. When there are restrictions on which vertices or edges we can visit, we will call the walk by a different name. For example, if we want to find a walk that avoids travelling the same edge twice, we will say we want to find a trail (or directed trail). If we want to find a walk that avoids visiting the same vertex twice, we will say, we want to find a path (or directed path).

Walks, trails, and paths are all related.

1. All paths are trails, but trails that visit the same vertex twice are not paths.
2. All trails are walks, but walks in which an edge is visited twice would not be trails.

We can visualize the relationship as in Figure **3.5.12**.

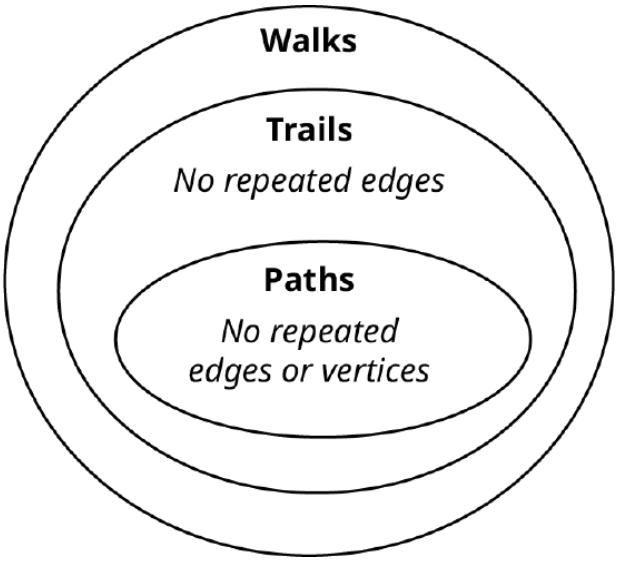


Figure **3.5.12**: Walks, Trails, and Paths

Let's practice identifying walks, trails, and paths using the graphs in Figure **3.5.13**.

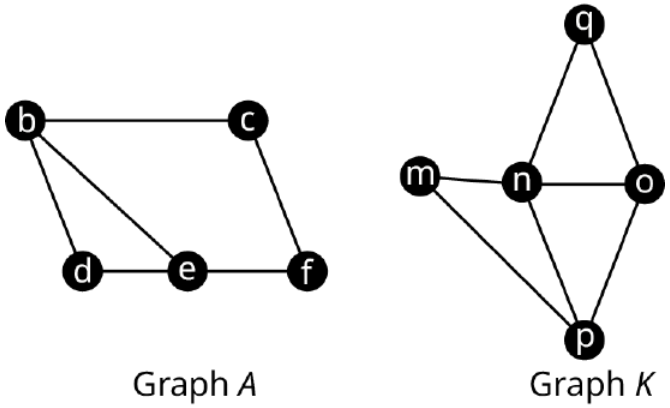


Figure **3.5.13**: Graphs *A* and *K*

✔ Example 3.5.2: Identifying Walks, Paths, and Trails

Consider each sequence of vertices from Graph *A* in Figure 12.89. Determine if it is only a walk, both a walk and a path, both a walk and a trail, all three, or none of these.

1. $b \to c \to d \to e \to f$
2. $c \to b \to d \to b \to e$
3. $c \to f \to e \to d \to b \to c$
4. $b \to e \to f \to c \to b \to d$

Answer

1. First, check to see if the sequence of vertices is a walk by making sure that the vertices are consecutive. As you can see in Figure ✔ 3.5.14, there is no edge between vertex *c* and vertex *d*.

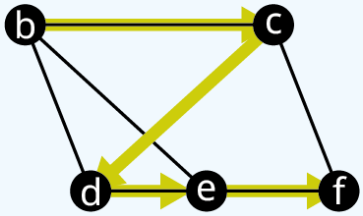


Figure ✔ 3.5.14

This means that the sequence is not a walk. If it is not a walk, then it can't be a path and it cannot be a trail, so, it is none of these.

- First, check to see if the sequence is a walk. As you can see in Figure ✔ 3.5.15, the vertices are consecutive.

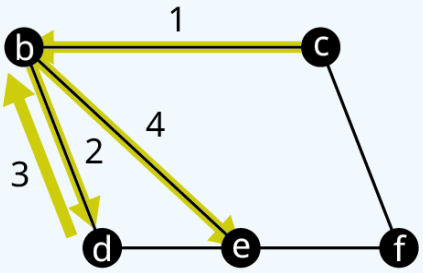


Figure ✔ 3.5.5:

This means that the sequence is a walk. Since the vertex *b* is visited twice, this walk is not a path. Since edge semantics bdbd is traveled twice, this walk is not a trail. So, the sequence is only a walk.

- First, check to see if the sequence is a walk. We can see in Figure ✔ 3.5.16 that the vertices are consecutive, which means it is a walk.

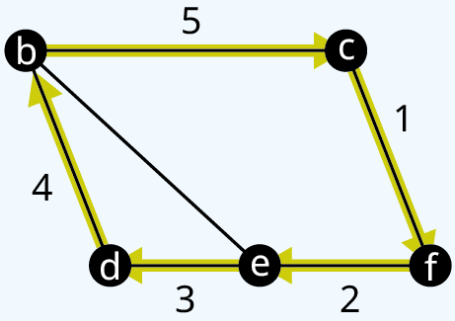


Figure ✔ 3.5.16:

Next, check to see if any vertex is visited twice. Remember, we do **not** consider beginning and ending at the same vertex to be visiting a vertex twice. So, no vertex was visited twice. This means we have a walk that is also a path. Next

check to see if any edge was visited twice; none were. So, the sequence is a walk, a path, and a trail.

- First, check to see if the sequence is a walk. We can see in Figure ✔ 3.5.17 that the vertices are consecutive, which means it is a walk.

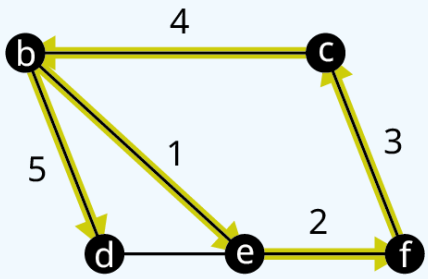


Figure ✔ 3.5.17:

Next check to see if any vertex is visited twice. Since vertex *b* is visited twice, this is not a path. Finally, check to see if any edges are traveled twice. Since no edges are traveled twice, this is a trail. So, the sequence of vertices is a walk and a trail.

Your Turn 3.5.2

Consider each sequence of vertices from Graph *K* in Figure 12.106. Determine if it is only a walk, both a walk and a path, both a walk and a trail, all three, or none of these.

$n \to q \to o \to p$

$p \to n \to q \to o \to n \to m$

$m \to n \to o \to p \to q$

Video

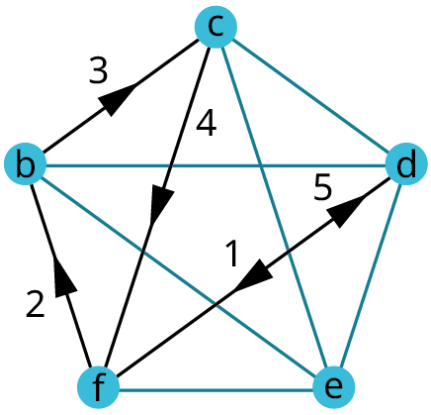
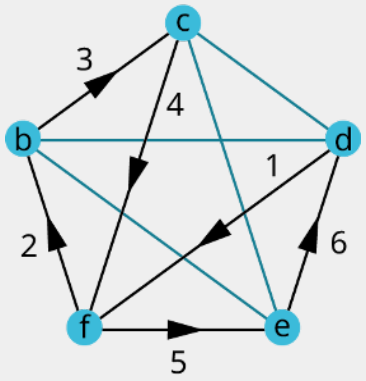
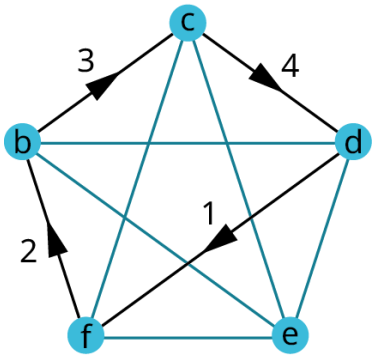


Walks, Trails, and Paths in Graph Theory

Circuits

In many applications of graph theory, such as creating efficient delivery routes, beginning and ending at the same location is a requirement. When a walk, path, or trail end at the same location or vertex they began, we call it closed. Otherwise, we call it open (does not begin and end at the same location or vertex). Some examples of closed walks, closed trails, and closed paths are given in in the following table.

Table 3.5.1: Closed Walks, Trails, and Paths

DESCRIPTION	EXAMPLE	CHARACTERISTICS
A closed walk is a walk that begins and ends at the same vertex.	 $d \to f \to b \to c \to f \to d$	Alternating sequence of vertices and edges Begins and ends at the same vertex
A closed trail is a trail that begins and ends at the same vertex. It is commonly called a circuit.	 $d \to f \to b \to c \to f \to e \to d$	No repeated edges Begins and ends at the same vertex
A closed path is a path that begins and ends at the same vertex. It is also referred to as a directed cycle because it travels through a cyclic subgraph.	 $d \to f \to b \to c \to d$	No repeated edges or vertices Begins and ends at the same vertex

Since walks, trails, and paths are all related, closed walks, circuits, and directed cycles are related too.

1. All circuits are closed walks, but closed walks that visit the same edge twice are not circuits.
2. All directed cycles are circuits, but circuits in which a vertex is visited twice are not directed cycles.

We can visualize the relationship as in Figure **3.5.18**.

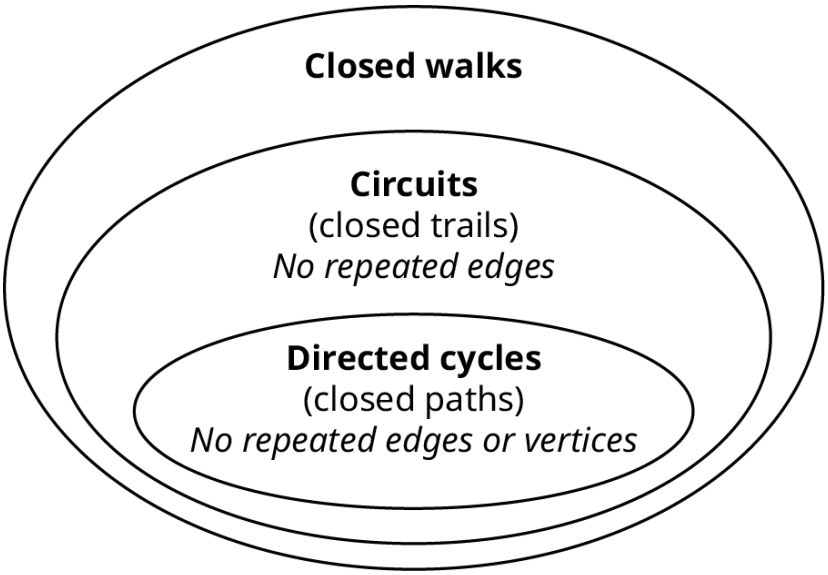


Figure 3.5.18: Closed Walks, Circuits, and Directed Cycles

The same circuit can be named using any of its vertices as a starting point. For example, the circuit $d \to f \to b \to c \to d$ can also be referred to in the following ways.

$a \to b \to c \to d \to a$ is the same as

$$\begin{cases} b \to c \to d \to a \to b \\ c \to d \to a \to b \to c \\ d \to a \to b \to c \to d \end{cases}$$

Let's practice working with closed walks, circuits (closed trails), and directed cycles (closed paths). In the graph in Figure 3.5.19, the vertices are major central and south Florida airports. The edges are direct flights between them.

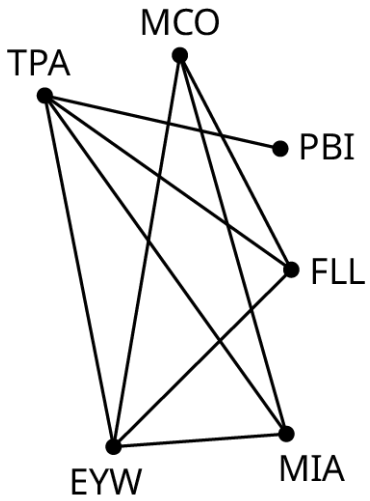


Figure 3.5.19: Major Central and South Florida Airports

Example 3.5.3: Determining a Closed Walk, Circuit, or Directed Cycle

Suppose that you need to travel by air from Miami (MIA) to Orlando (MCO) and you were restricted to flights represented on the graph. For the trip to Orlando, you decide to purchase tickets with a layover in Key West (EYW) as shown in Figure 3.5.20, but you still have to decide on the return trip. Determine if your roundtrip itinerary is a closed walk, a circuit, and/or a directed cycle, based on the return trip described in each part.

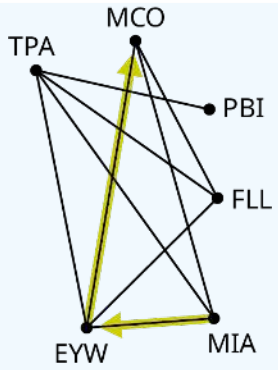


Figure ✔ **3.5.20** MIA to EYW to MCO

1. You returned to Miami (MIA) by reversing your route.
2. Your direct flight back left Orlando (MCO) but was diverted to Fort Lauderdale (FLL)! From there you flew to Tampa (TPA) before returning to Miami (MIA).

Answer

1. The whole trip was *MIA* → *EYW* → *MCO* → *EYW* → *MIA*. This is a closed walk, because it is a walk that begins and ends at the same vertex. It is not a circuit, because it repeats edges. If it is not a circuit, then it cannot be a directed cycle.
2. The whole trip was *MIA* → *EYW* → *MCO* → *FLL* → *TPA* → *MIA*. This is a closed walk, because it is a walk that begins and ends at the same vertex. It is a circuit because no edges were repeated. It is also a directed cycle because no vertices were repeated either. So, it is all three!

Your Turn 3.5.3

Suppose that you want to travel from Palm Beach (PBI) to any other city listed in Figure 12.113 and then return to Palm Beach.

Why is it impossible to find an itinerary that is a circuit?

How is this related to the degree of the vertex PBI?

Video



Closed Walks, Closed Trails (Circuits), and Closed Paths (Directed Cycles) in Graph Theory

Graph Colorings

In this section so far, we have looked at how to navigate graphs by proceeding from one vertex to another in a sequence that does not skip any vertices, but in some applications we may want to skip vertices. Remember the camp Olympics at Camp Woebegone

in Comparing Graphs? You were planning a camp Olympics with four events. The campers signed up for the events. You drew a graph to help you visualize which events have campers in common. The vertices of Graph *E* in Figure **3.5.21** represent the events and adjacent vertices indicate that there are campers who are participating in both.

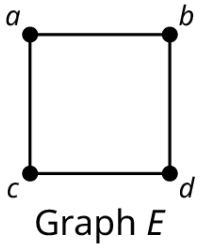


Figure **3.5.21**: Graphs of Camp Olympics

In this case, we do NOT want events represented by two adjacent vertices to occur in the same timeslot, because that would prevent the campers who wanted to participate in both from doing so. We can use the graph in Figure **3.5.21** to count the timeslots we need so there are no conflicts. Let's assign each timeslot a different color. We could categorize events that happen at 1 pm as Red; 2 pm, Purple; 3 pm, Blue; and 4 pm, Green. Then assign different colors to any pair of adjacent vertices to ensure that the events they represent do not end up in the same timeslot. Figure **3.5.22** shows several of the ways to do this while obeying the rule that no pair of adjacent vertices can be the same color.

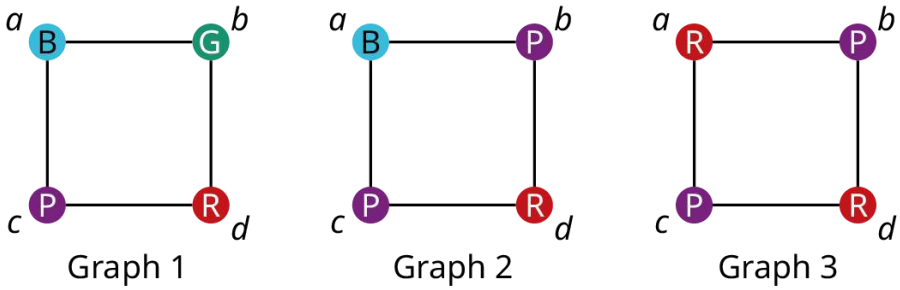


Figure **3.5.22**: Graph Colorings

In Figure **3.5.22**, the graphs with vertices colored so that no adjacent vertices are the same color are called graph colorings. Notice that Graph 3 has the fewest colors, which means it shows us how to have the fewest number of timeslots. The events marked in red, *a* and *d*, can be held at the same time because they are not adjacent and do not have conflicts. Also, the events marked in purple, *b* and *c*, can be held at the same time. We would not need green or blue timeslots at all!

A graph that uses n colors is called an n-coloring. The smallest number of colors needed to color a particular graph is called its chromatic number.

Graph colorings can be used in many applications like the scheduling scenario at camp Woebegone. Let's look at how they work in more detail. Figure **3.5.23** shows two different colorings of a particular graph. Coloring A is called a four-coloring, because it uses four colors, red (R), green (G), blue (B), and purple (P). Coloring B is called a three-coloring because it uses three. The colors allow us to visually subdivide the graphs into groups. The only rule is that adjacent vertices are different colors so that they are in different groups.

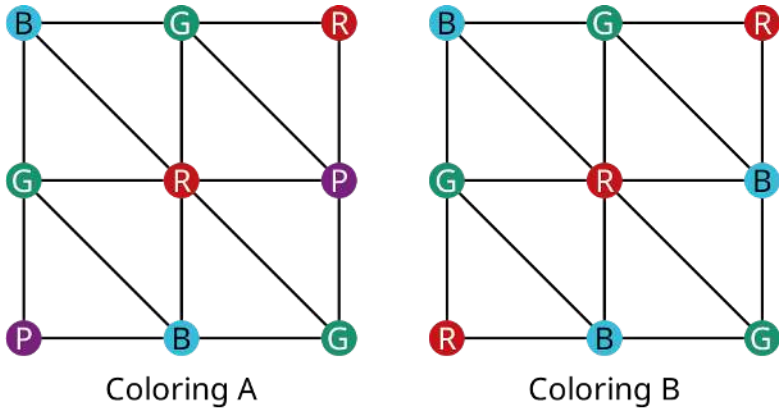
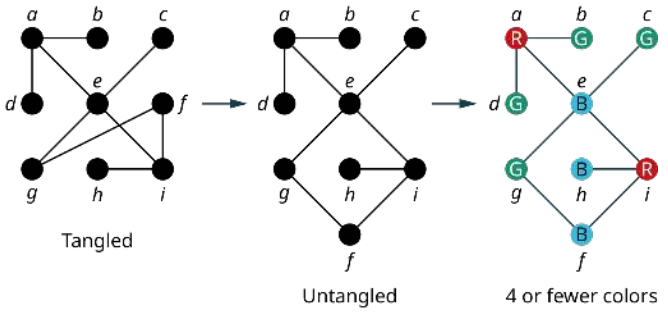
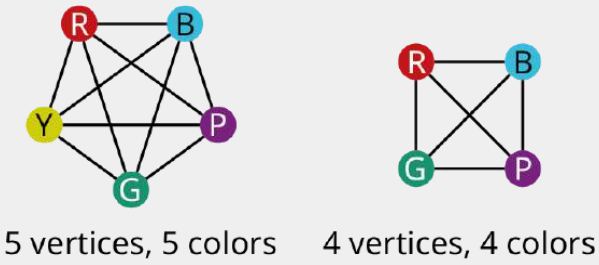
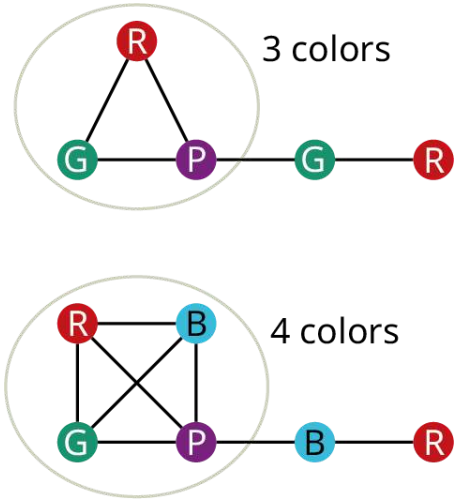


Figure **3.5.23**: Two Colorings of the Same Graph

It turns out that a three-coloring is the best we can do with the graph in Figure **3.5.23**. No matter how many different patterns you try with only two colors, you will never find one in which the adjacent vertices are always different colors. In other words, the graph has a chromatic number of three. For large graphs, computer assistance is usually required to find the chromatic numbers. There is no formula for finding the chromatic number of a graph, but there are some facts that are helpful in Table **3.5.2**.

Table **3.5.2**: Facts about Graph Colorings

Fact	Example
Recall that planar graphs are untangled; that is, they can be drawn on a flat surface so that no two edges are crossing. **If a graph is planar, it can be colored with four colors or possibly fewer.**	
Each vertex in a complete graph is adjacent to every other vertex forcing us to color them all different colors.**The chromatic number of a complete graph is the same as the number of vertices.**	
If a graph has a clique, a complete subgraph, then each vertex in the clique must have a different color and the vertices outside the clique may or may not have even more colors. **The chromatic number of a graph is *at least* the number of vertices in its biggest clique.**	

Creating Colorings to Solve Problems

Let's see how these facts can help us color the graph in Figure **3.5.24**.

- Since the graph is planar, the chromatic number is *no more than* four.
- The graph is not complete, but it has complete subgraphs of three vertices. In other words, it has triangles like the one shown with blue vertices in Figure 12.100. This means that the chromatic number is *at least* three.

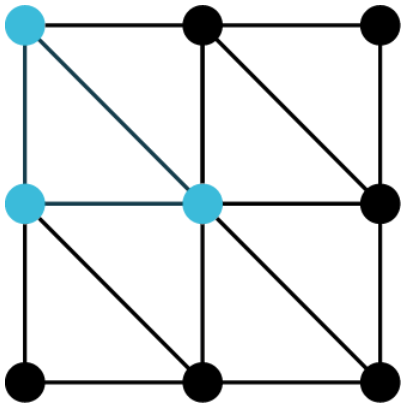


Figure 3.5.4: A Graph with Triangles

We know we can color this graph in three or four colors. It is usually best to start by coloring the vertex of highest degree as shown in Figure 3.5.25. In this case, we used red (R). The color is not important.

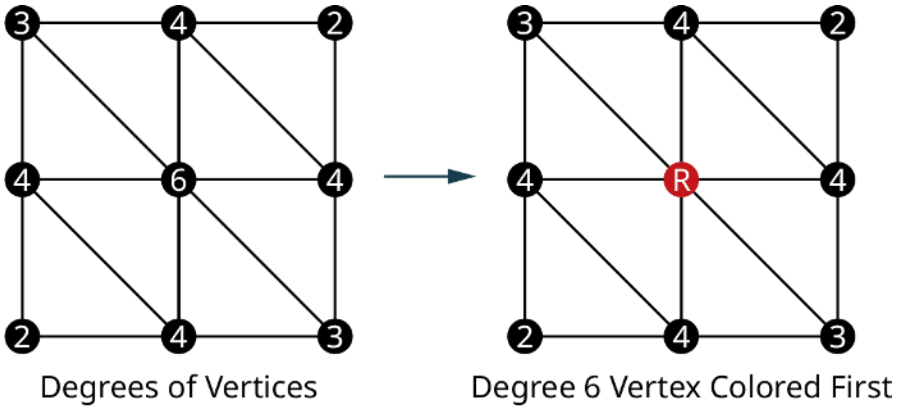


Figure 3.5.25: Color Highest Degree Vertex First

We want to color as many of the vertices the same color as possible; so, we look at all the vertices that are not adjacent to the red vertex and begin to color them, red starting with the one among them of highest degree. Since the only vertices that are not adjacent are both degree 2, choose either one and color it red as shown in Figure 3.5.26.

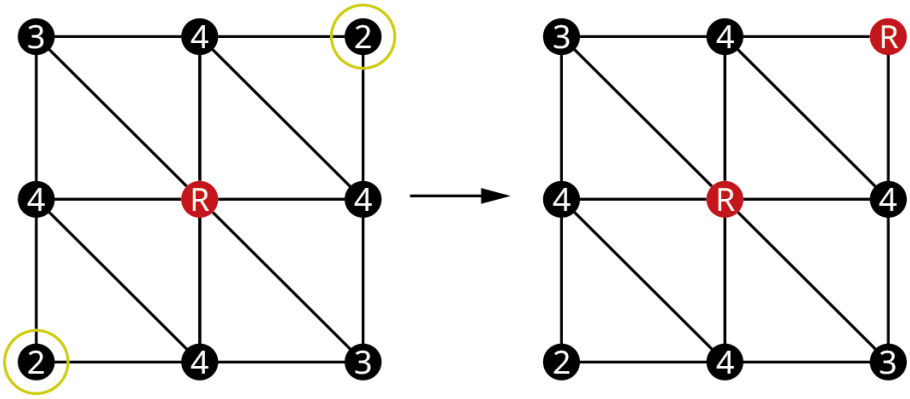


Figure 3.5.26: Color More Red from Highest to Lowest Degree

Now, there is only one vertex left that is not adjacent to a red; so, color it red. Of the remaining vertices, the highest degree is four; so, color one of the vertices of degree four in a different color. These two steps are shown in Figure 3.5.27.

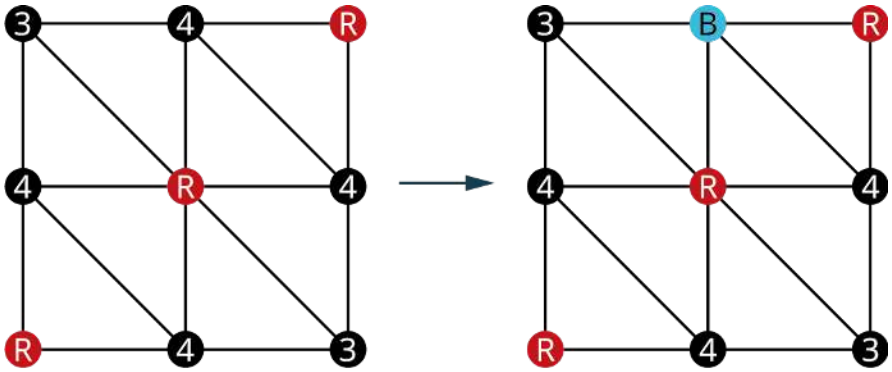


Figure **3.5.27**: Complete the Reds and Start Another Color

Repeat the same procedure. There are three remaining vertices that are not adjacent to a blue. Color as many blue as possible, with priority going to vertices of higher degree as shown in Figure **3.5.28**.

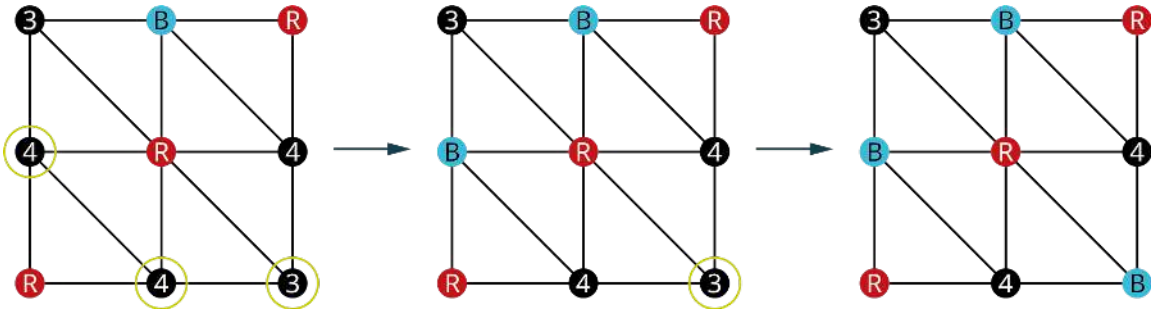


Figure **3.5.28**: Repeat Procedure for Color Blue

All the remaining vertices are adjacent to blue. So, it is time to repeat the procedure with another color as shown in Figure **3.5.29**.

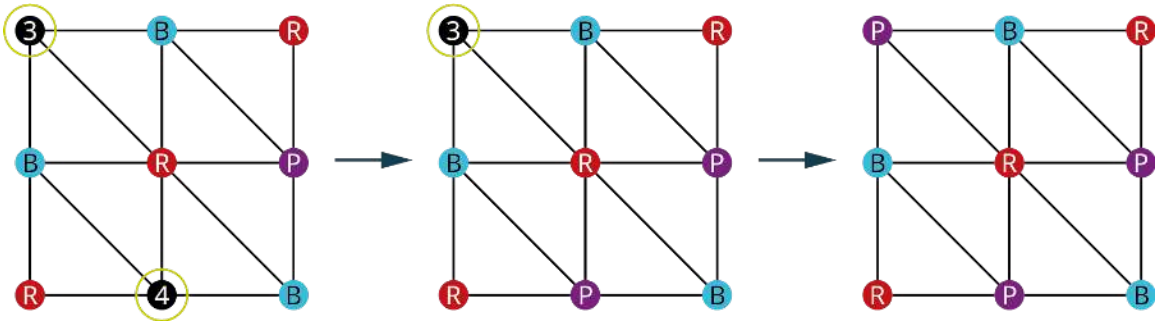


Figure **3.5.29**: Repeat Procedure for Color Purple

All the vertices are now colored with a three-coloring so we know the chromatic number is *at most* three, but we knew the chromatic number was *at least* three because the graph has a triangle. So, we are now certain it is *exactly* three.

https://math.libretexts.org/@go/page/214458

Video



Coloring Graphs Part 1: Coloring and Identifying Chromatic Number

Suppose the vertices of the graph in Figure 3.5.30 represented the nine events in the Camp Woebegone Olympics, and edges join any events that have campers in common, but nonadjacent vertices do not.

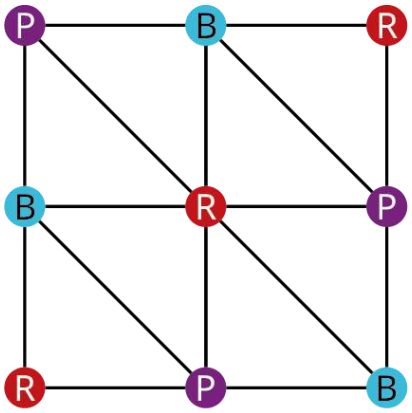


Figure 3.5.30: Coloring for Nine Event Camp Woebegone Olympics

Any vertices of the same color are nonadjacent; so, they have no conflicts. Since this graph is a three-coloring, all nine events could be scheduled in just three timeslots!

Example 3.5.4: Understanding Chromatic Numbers

We previously discussed a high school, which holds end-of-course exams in (E3) English 3, (E4) English 4, (M) Advanced Math, (C) Calculus, (W) World History, (U) U.S. History, (B) Biology, and (P) Physics. We were given a list of courses that had no students in common. We used that information to find the graph in Figure ✔ 3.5.31, which shows edges between exams with students in common. Use the graph we found to answer each question.

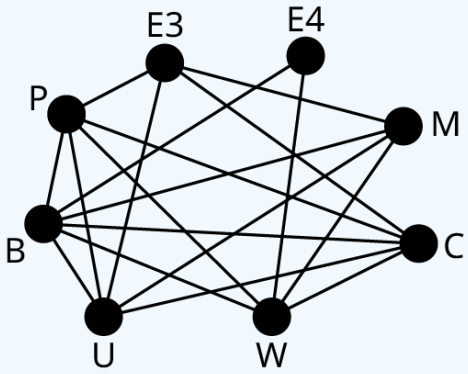


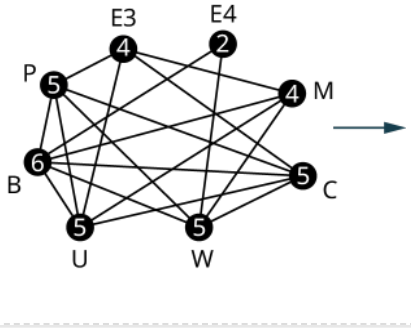
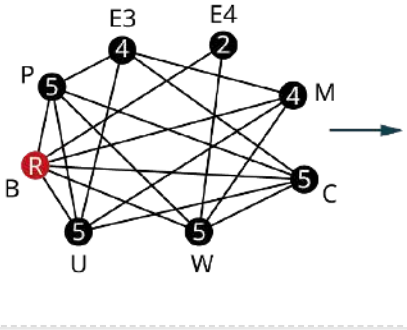
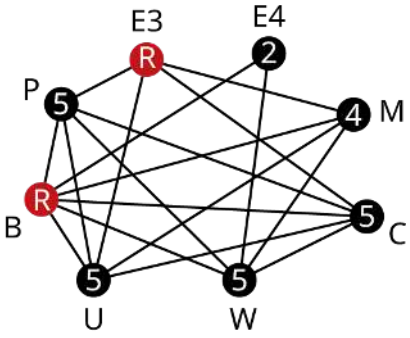
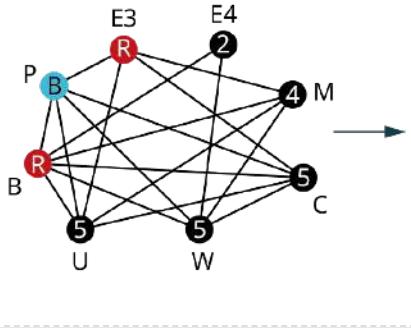
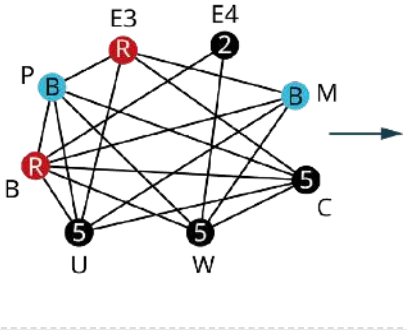
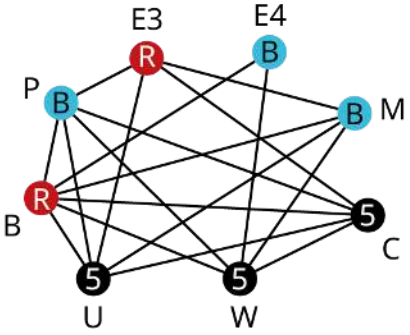
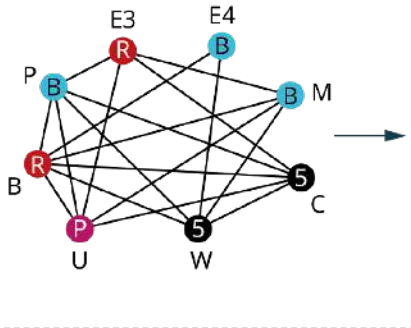
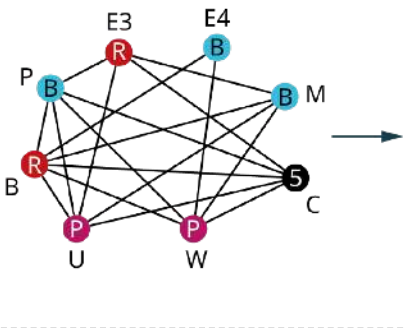
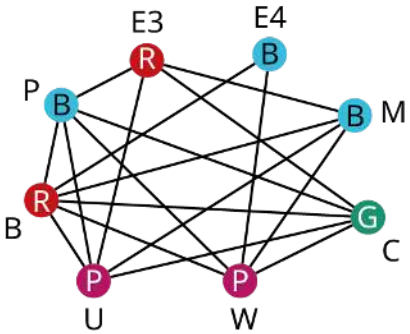
Figure ✔ **3.5.31** Graph of Exams with Students in Common

1. The graph contains a clique of size 4 formed by the vertices P, E3, C, and U. What does this tell you about the chromatic number?
2. The graph is not planar, meaning that you cannot untangle it. What does this tell you about the chromatic number?
3. Create a coloring by coloring vertex of highest degree first, coloring as many other vertices as possible each color from highest to lowest degree, then repeating this process for the remaining vertices.
4. Do you know what the minimum number of timeslots is? If so, what is it and how do you know? If not, what are the possibilities?

Answer

1. We would need four different colors just for the clique with four vertices; so, the chromatic number is *at least* four.
2. It is possible for the chromatic number to be greater than four.
3. The process is shown in Table ✔ **3.5.3**.

Table ✔ **3.5.3** Coloring the Graph

		
This is the original graph. Vertex B has highest degree.	Color vertex B. Vertex E3 is the only remaining vertex that is NOT adjacent to B.	Color vertex E3 the same color. Vertices P, U, W, and C are the remaining vertices with highest degree. Pick one to color.
		
Color vertex P a new color. Vertices E4 and M are NOT adjacent to P, and M has higher degree.	Color vertex M the same color. Vertex E4 is the only remaining vertex NOT adjacent to P or M.	Color vertex E4 the same color. Vertices U, W, and C are the remaining vertices with highest degree. Pick one to color.
		
Color U a new color. Vertex W is the only remaining vertex NOT adjacent to U.	Color vertex W the same color. Vertex C is the only remaining vertex that has NOT been colored.	Color vertex C a new color. The coloring is final. We used four colors.

The last graph in Table ✔ **3.5.3** is the final coloring.

4. Yes, the minimum number of times slots is the chromatic number. We knew the chromatic number had to be at least four because there was a clique with four vertices. Now we have found a four-coloring of the graph which tells us that the chromatic number is at most four. So, we know four must be the chromatic number.

Checkpoint

The procedure used in Example 22 to color the graph is not guaranteed to result in a graph that has the minimum number of colors possible, but it is usually results in a coloring that is close to the chromatic number.

Your Turn 3.5.4

The given figure shows the same graph colored in several ways. The largest clique of the graph has three vertices. Use this information to answer each question.

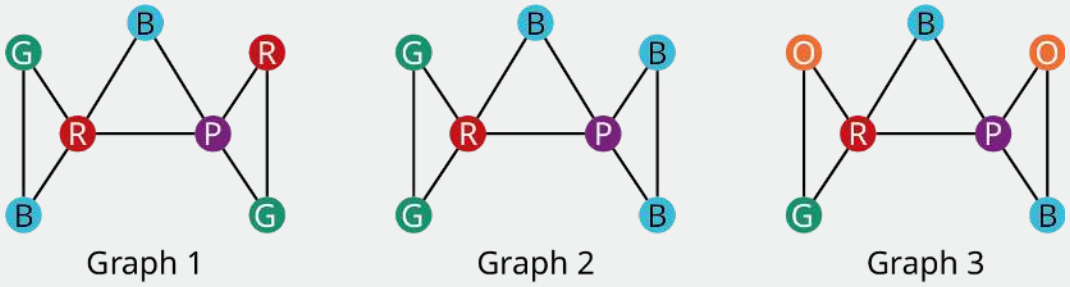


Figure 3.5.32: Possible Colorings of the Same Graph

Is the graph planar? What does your answer tell you about the chromatic number?

What is the lowest possible chromatic number of this graph based on the size of any cliques?

What are the possible chromatic numbers?

Which graph or graphs meet the definition of a valid graph coloring?

Draw an n-coloring for the graph where n is the answer to Exercise 1.

The Four Color Problem

The idea of coloring graphs to solve problems was inspired by one of the most famous problems in mathematics, the "four color problem." The idea was that, no matter how complicated a map might be, only four colors were needed to color the map so that no two regions that shared a boundary would be the same color. For many years, everyone suspected this to be true, because no one could create a map that needed more than four colors, but they couldn't prove it was true in general. Finally, graphs were used to solve the problem!

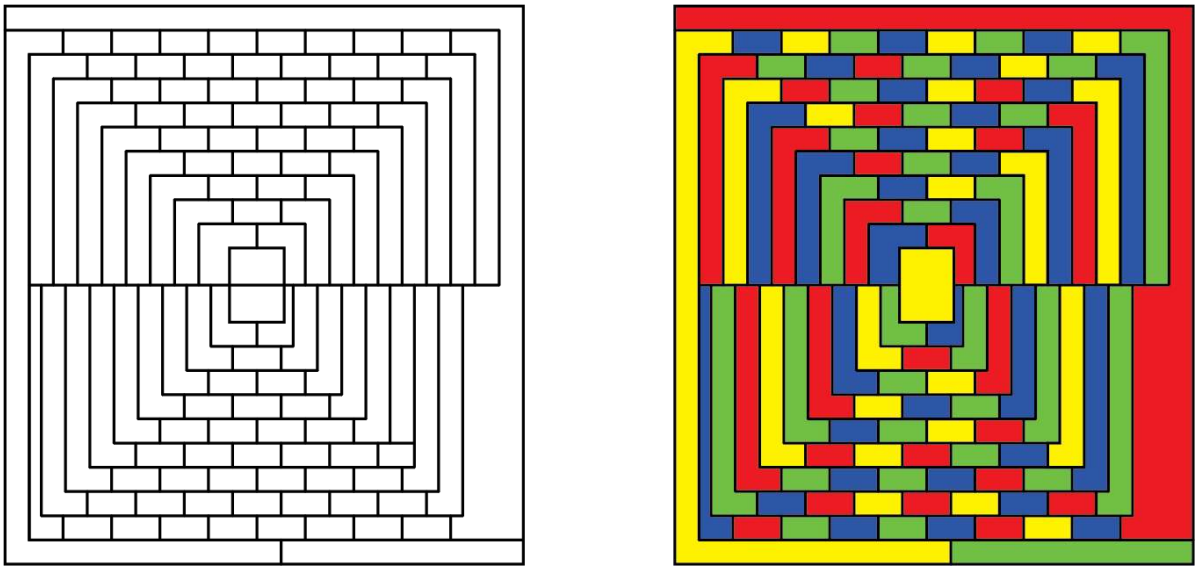


Figure 3.5.33: Using only four colors, no two adjacent regions have the same color.

Video



The Four Color Map Theorem – Numberphile

We saw how maps can be represented as graphs in Graph Basics. Figure 3.5.34 shows a map of the midwestern region of the United States.

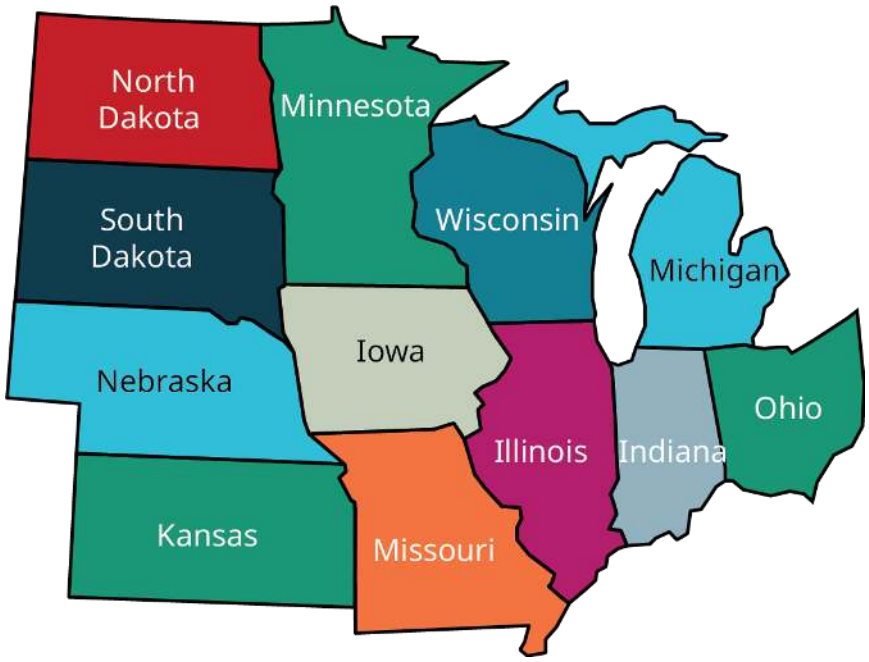


Figure 3.5.34: Map of Midwestern States

Figure 3.5.34 shows how this map can be associated with a graph in which each vertex represents a state and each edge indicates the states that share a common boundary. Figure3.5.35 shows the final graph.

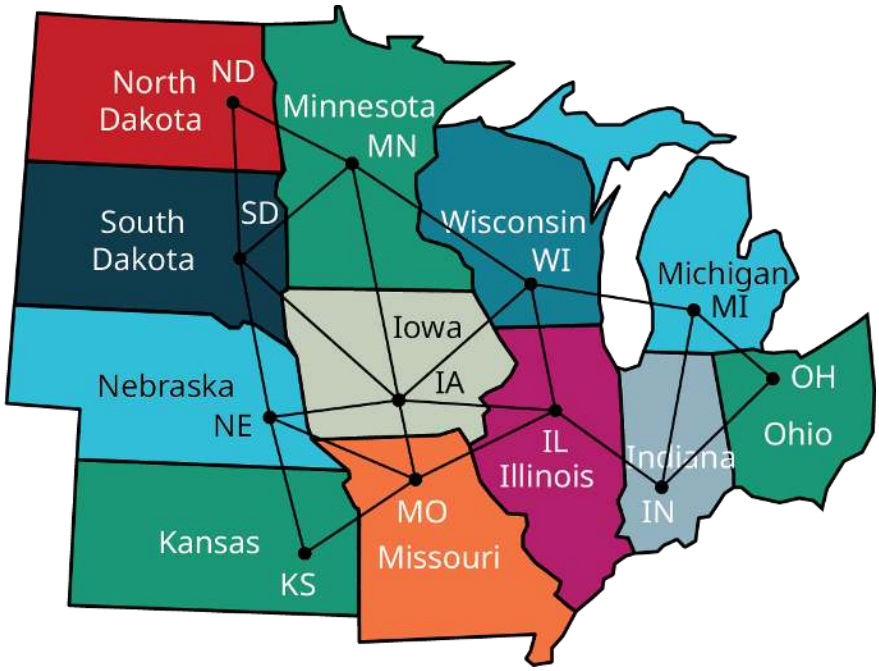


Figure **3.5.35**: Edge Assigned to Each Pair of Midwestern States with Common Border

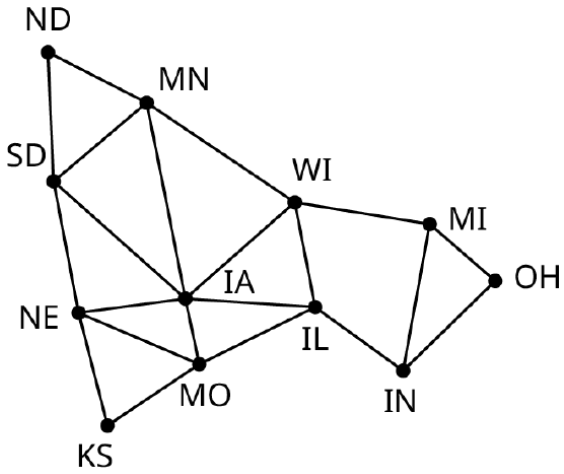


Figure **3.5.36**: Final Graph Representing Common Boundaries between Midwestern States

Notice that the graph representing the common boundaries between midwestern states is planar, meaning that it can be drawn on a flat surface without edges crossing. As we have seen, any planar graph has a chromatic number of four or less. This very well-known fact is called the Four-Color Theorem, or Four-Color Map Theorem.

Video



Coloring Graphs Part 2: Coloring Maps and the Four Color Problem

People in Mathematics: Four-Color Theorem

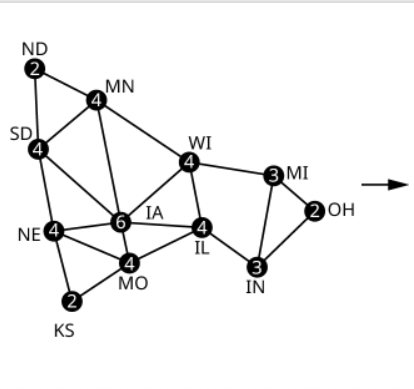
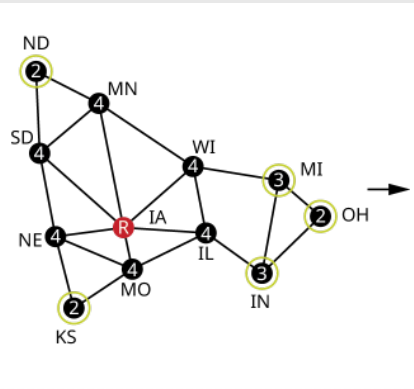
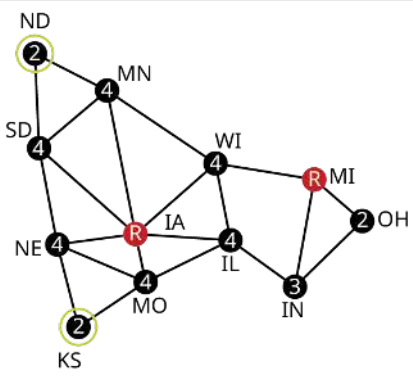
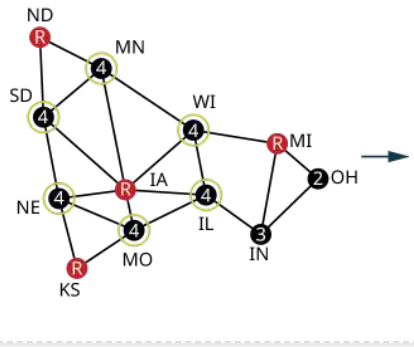
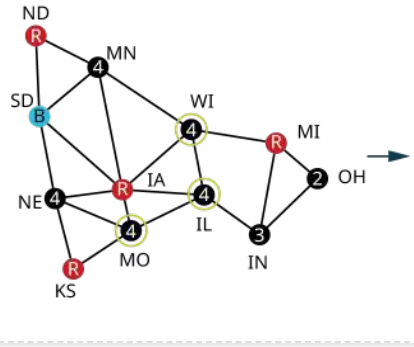
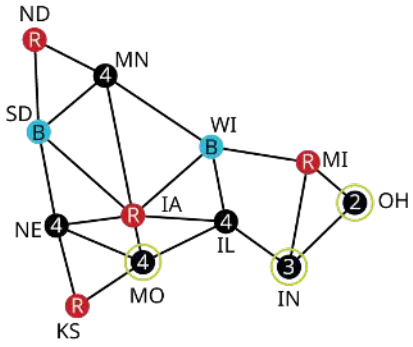
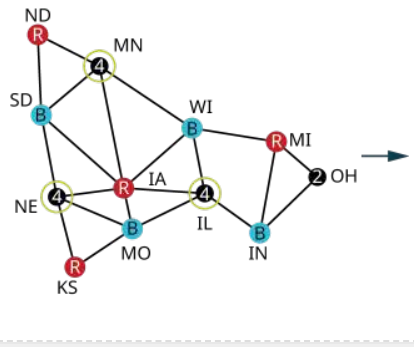
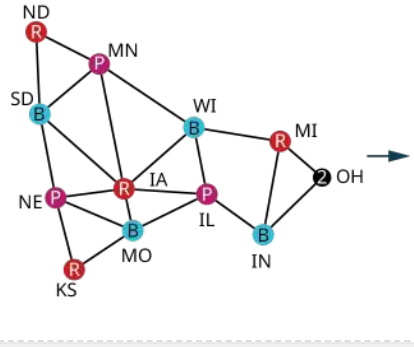
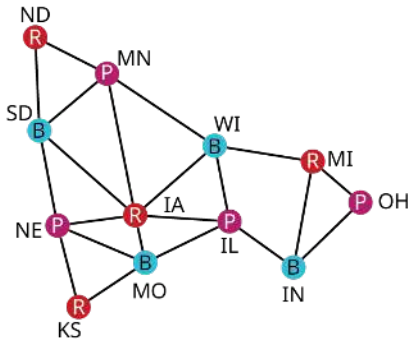
In the 19th century, Francis Guthrie was coloring a map of British counties and noticed that he could color so that no two adjacent counties were the same color using only four colors. This seemed to be the case for other maps as well, but he could not figure out why. He shared this with his brother, Frederick, who subsequently took the problem to his professor, August De Morgan, one of the most famous mathematicians of all time. De Morgan never was able to solve the problem, but he shared it with his colleagues. The problem continued to intrigue and perplex mathematicians for over a hundred years, inspiring discoveries in several areas of mathematics. It was finally proven by Kenneth Appel and Wolfgang Haken from the University of Illinois in 1976 using a combination of computer-aided calculations and graph theory. This was the first time in history that a mathematical theorem was proven using a computer! (Jesus Najera, "The Four-Color Theorem," *Cantor's Paradise*, a Medium publication, October 22, 2019)

? Example 3.5.5: Using Four Colors or Fewer to Color a Graph

Find a coloring of the graph in Figure ? 3.5.34, which uses four colors or fewer. Use the resulting coloring as a guide to recolor the map in Figure ? 3.5.34. How many colors did you use? Does this support the conclusion of the Four-Color Theorem? If so, how?

Answer

Table ? **3.5.4** Steps to Color Graph of Midwestern States

		
Step 1: Graph with degrees of vertices labeled. Vertex IA has highest degree.	**Step 2:** Color vertex IA any color. Vertices ND, KS, MI, OH, and IN are NOT adjacent to IA. MI and IN have highest degree, 3.	**Step 3:** Color either MI or IN the same color as IA. Vertex ND and KS are the only remaining vertices not adjacent to a red vertex. They both have degree 2.
		
Step 4: Since KS and ND are not adjacent to each other, we can save a step and color both red. The highest degree of remaining vertices is four.	**Step 5:** Choose one of the vertices of degree 4, SD, to color with a new color, blue. The vertices WI, IL, MO, IN,and OH are NOT adjacent to blue. WI, IL, and MO have the highest degree.	**Step 6:** Choose one of WI, IL, or MO to color. We color WI blue. MO, IN and OH are NOT adjacent to blue. MO has the highest degree of these.
		
Step 7: Color MO blue. All remaining vertices are adjacent to blue.	**Step 8:** Since MN, IL, and NE are not adjacent, save steps and color all three the new color, purple.	**Step 9:** Since vertex OH is not adjacent to purple, color it purple. This is the final graph. We used three colors.

Choose a new color. Four vertices remain, MN, NE, IL, and OH. MN, NE, and IL have the highest degree.	Vertex OH is the only remaining vertex that has NOT been colored.	

The final graph in ? 3.5.4 shows how we would color the map. In Figure ? 3.5.37 we have colored the map to correspond to the colors on the graph.

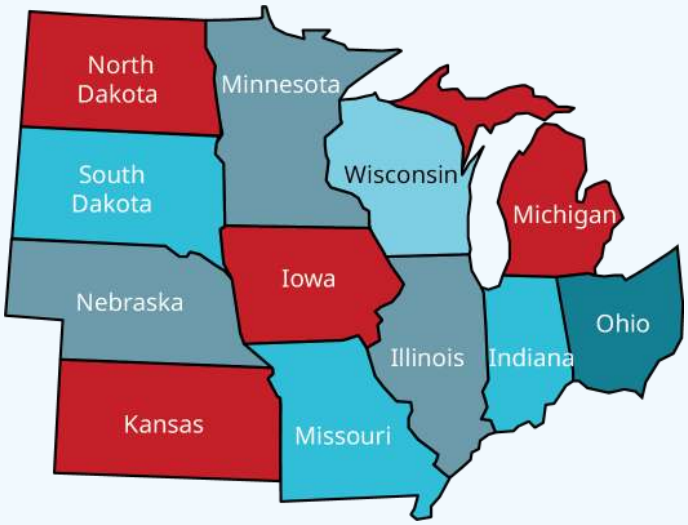


Figure ? 3.5.37: Midwestern States in Three Colors

We used three colors to color the graph. This supports the Four-Color Theorem, because the graph is planar and its chromatic number is less than four.

Your Turn 3.5.5

The first figure shows a map of the Island of Oahu in the State of Hawaii divided into regions, and the second figure shows a graph of the map. What is the chromatic number? Find a graph coloring that reflects the chromatic number of the graph.



Figure 3.5.38: Map of Oahu

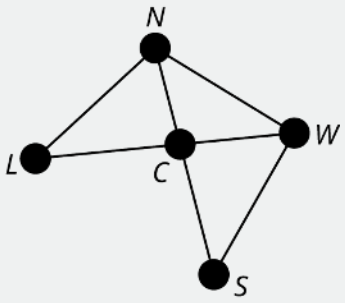


Figure 3.5.39: Graph Representing Common Boundaries between Regions of Oahu

Who Knew?: Coloring A Möbius Strip

Have you ever heard of a möbius strip? It is a flat object with only one side. This might sound theoretical, but you can make one for yourself. Take a narrow strip of paper, make a half twist in it, and tape the ends together. To prove to yourself that it has only one side, pick a point anywhere on the paper and start drawing a line. You can draw the line continuously without lifting your pen from the paper and eventually, you will get back to the point at which you started. Since you never had to flip over the paper, you have just proved it has one side!

Now, you might be wondering what this has to do with Graph Theory. It turns out that a map drawn on a möbius strip cannot necessarily be drawn using four colors. This doesn't contradict the Four-Color Theorem, which only applies to graphs that are planar. A möbius strip does not live on a "plane," or flat surface, which means the maps are not always planar like those drawn on a flat piece of paper. In the case of a möbius strip, it is the Six-Color Map Theorem! Figure 3.5.40 is an example of a möbius strip with a map that requires five colors.

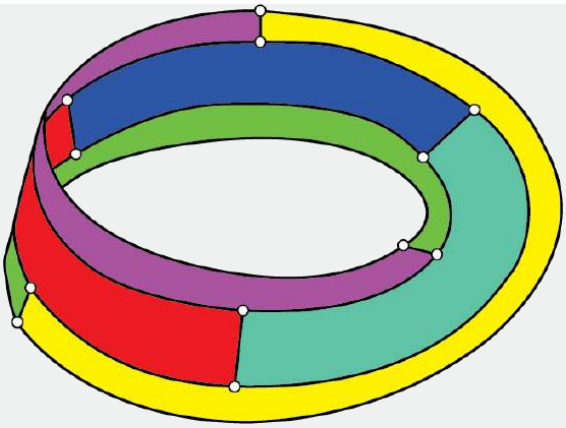


Figure 3.5.40: A Five-Coloring on a Möbius Strip

Video



Neil deGrasse Tyson Explains the Möbius Strip

Check Your Understanding

For the following exercises, determine whether each statement is always true or sometimes true.

1. A trail is a path.
2. A trail is a walk.
3. A walk is a path.
4. A circuit is a trail.
5. A directed cycle is a path.
6. A circuit is a directed cycle.
7. A directed cycle is a circuit.
8. If a graph has an n-coloring, then its chromatic number is n.
9. If the chromatic number of a graph is n, then it has an n-coloring.
10. If a graph is planar, then it has a chromatic number of at most four.

For the following exercise, fill in the blanks to make the statement true.

11. A walk that _____ is a trail.
12. A trail that _____ is a circuit.
13. A circuit that _____ is a directed cycle.
14. A closed walk that _____ is a circuit.
15. A complete graph with n vertices has a chromatic number of_____.

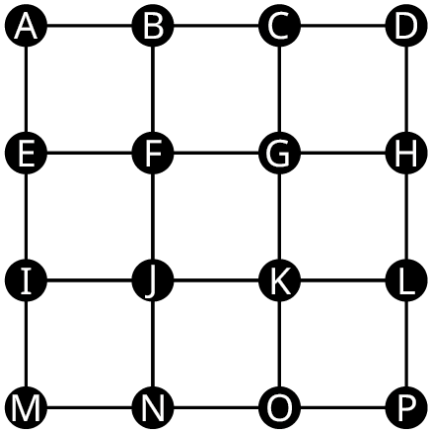
16. A graph with a clique with n-vertices has a chromatic number of _____ n.

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3.5.1: Exercises

For the following exercises, identify each sequence of vertices from the figure as a walk, trail, and/or path. Select all that apply.



Exercise 3.5.1.1

$A \to B \to F \to G \to K \to J \to F \to B$

Exercise 3.5.1.2

$G \to K \to O \to N \to J \to K \to L$

Exercise 3.5.1.3

$F \to J \to K \to G \to B \to A$

Exercise 3.5.1.4

$I \to J \to K \to L \to K \to J \to N$

Exercise 3.5.1.5

$M \to N \to O \to K \to L \to H$

Exercise 3.5.1.6

$A \to F \to K \to P$

Exercise 3.5.1.7

$N \to J \to F \to B \to C \to G \to F \to E$

Exercise 3.5.1.8

$E \to F \to J \to I \to E$

For the following exercises, identify each sequence of vertices from Figure 12.134 as a closed walk, circuit (closed trail), and/or directed cycle (closed path). Select all that apply.

Exercise 3.5.1.9

$A \to B \to F \to G \to K \to J \to F \to B \to A$

Exercise 3.5.1.10

$G \to K \to O \to N \to J \to K \to L$

Exercise 3.5.1.11

$F \to J \to N \to O \to K \to J \to I \to E \to F$

Exercise 3.5.1.12

$I \to J \to K \to G \to F \to E \to I$

Exercise 3.5.1.13

$M \to N \to O \to K \to J \to I \to M$

Exercise 3.5.1.14

$N \to J \to F \to B \to C \to G \to F \to J \to N$

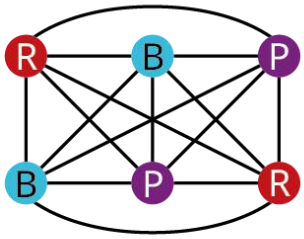
Exercise 3.5.1.15

$A \to B \to G \to F \to E \to A$

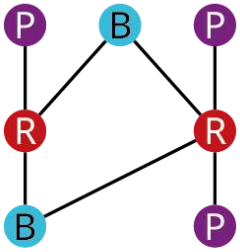
Exercise 3.5.1.16

$E \to F \to G \to K \to J \to F \to B \to A \to E$

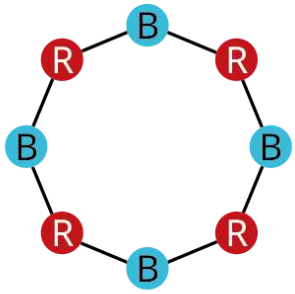
For the following exercises, use the graphs shown. Identify the graph or graphs with the given characteristics.



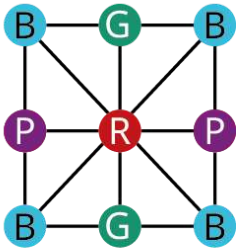
Graph 1



Graph 2



Graph 3



Graph 4

Exercise 3.5.1.17

The colors do not follow the definition of a graph coloring.

Exercise 3.5.1.18

The graph is a two-coloring.

Exercise 3.5.1.19

The graph is a three-coloring.

Exercise 3.5.1.20

The graph is a four-coloring.

Exercise 3.5.1.21

The graph is planar.

Exercise 3.5.1.22

The graph has a chromatic number of two.

Exercise 3.5.1.23

Fewer colors could be used to color the graph.

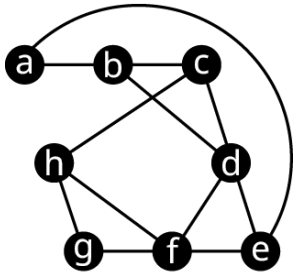
Exercise 3.5.1.24

The chromatic number is more than four.

Exercise 3.5.1.25

The Four-Color Theorem applies to the graph.

For the following exercises, complete the sequence of vertices from the graph to create the indicated kind of closed walk.



Exercise 3.5.1.26

Closed path: $a \to b \to \square \to \square \to \square \to a$

Exercise 3.5.1.27

Closed path that visits edge ed twice: $e \to d \to \square \to \square \to \square \to e$

Exercise 3.5.1.28

Directed cycle: $e \to a \to \square \to \square \to \square \to e$

Exercise 3.5.1.29

Directed cycle: $b \to d \to \square \to \square \to \square \to b$

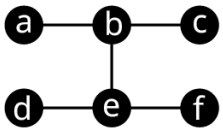
Exercise 3.5.1.30

Circuit that visits vertex d twice: $b \to c \to d \to \square \to \square \to \square \to b$

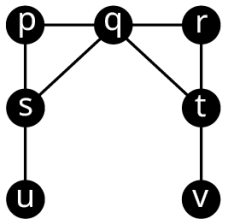
Exercise 3.5.1.31

Circuit that visits vertex f twice: $g \to h \to f \to \square \to \square \to \square \to g$

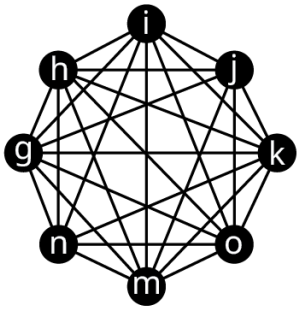
For the following exercises, use the graphs shown.



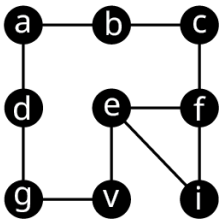
Graph 1



Graph 2



Graph 3



Graph 4

Exercise 3.5.1.32

Find the chromatic number n for graph 1 and give an n-coloring.

Exercise 3.5.1.33

Find the chromatic number n for graph 2 and give an n-coloring.

Exercise 3.5.1.34

Find the chromatic number n for graph 3 and give an n-coloring.

Exercise 3.5.1.35

Find the chromatic number n for graph 4 and give an n-coloring.

For the following exercises, indicate the smallest and largest possible chromatic number of the graph described.

Exercise 3.5.1.36

The graph has 15 vertices and contains a clique with 9 vertices.

Exercise 3.5.1.37

A planar graph with 100 vertices and a clique with 3 vertices.

Exercise 3.5.1.38

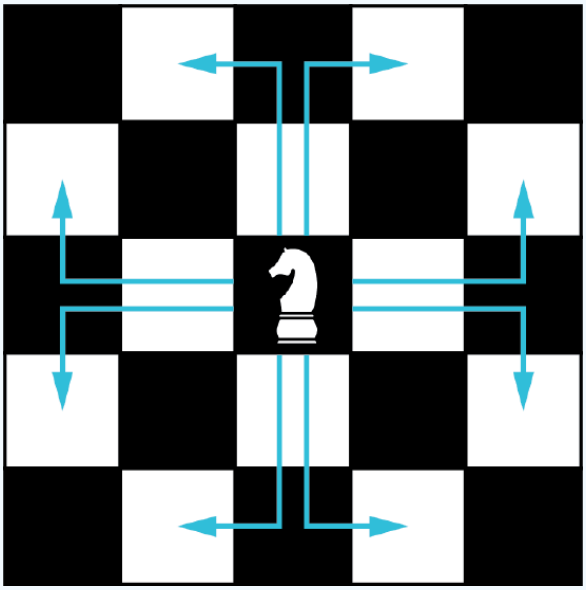
A planar graph with more than 2 vertices and no cliques.

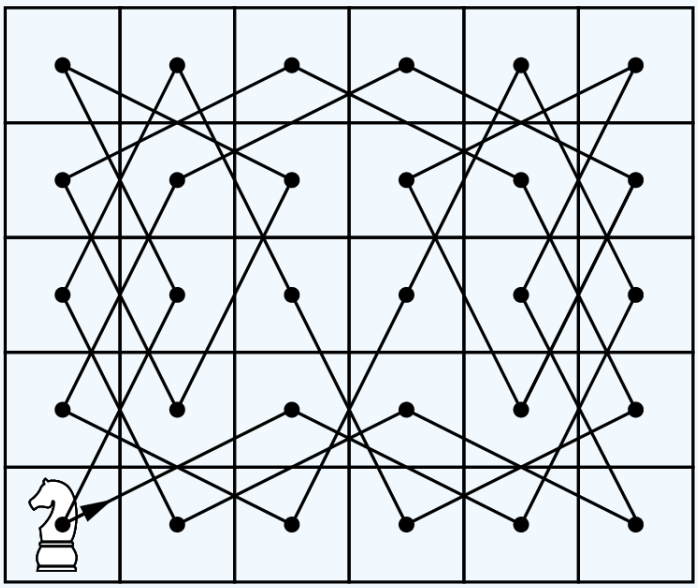
Exercise 3.5.1.39

A complete graph with 2,123 vertices.

Exercise 3.5.1.40

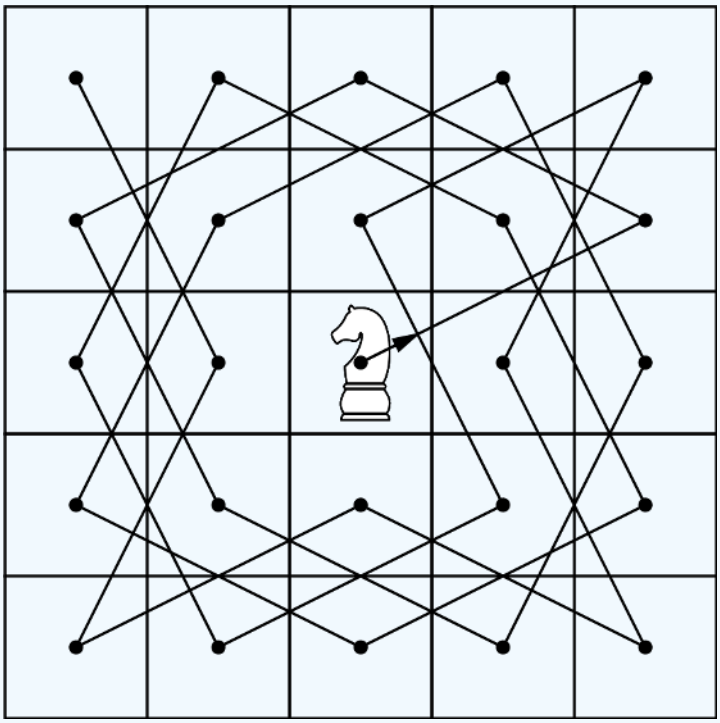
In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The 8 possible moves a knight can make from a space in the center of a five-by-five grid are shown in the first figure. A knight's tour is a sequence of moves by a knight on a chessboard (of any size) such that the knight visits every square exactly once. If the knight's tour brings the knight back to its starting position on the board, it is called a closed knight's tour. Otherwise, it is called an open knight's tour. Determine if the closed knight's tour in the second figure is most accurately described as a closed walk, a circuit, or a directed cycle. Explain your reasoning.





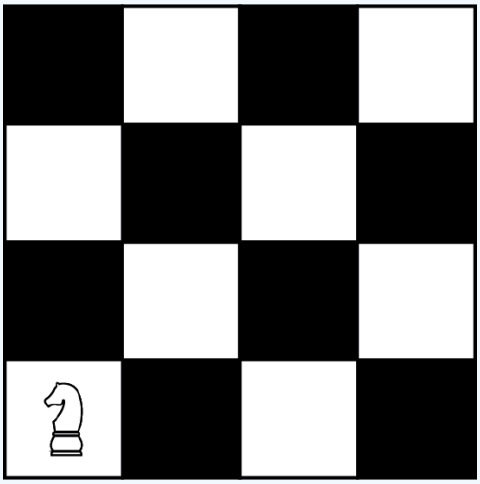
? Exercise 3.5.1.41

An open knight's tour on a five-by-five board is shown in the figure. Is it most accurately described as a walk, a trail, or a path?



? Exercise 3.5.1.42

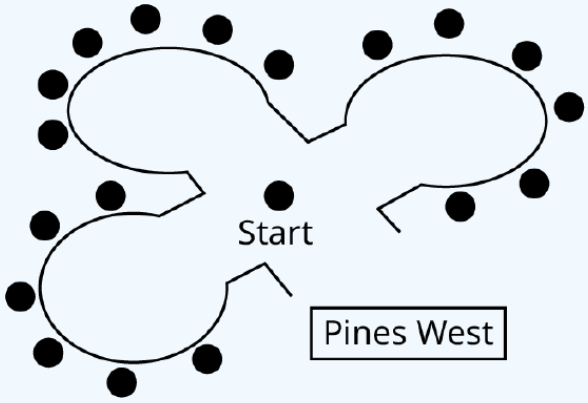
A knight's tour is not possible on a four-by-four board like the one shown in the figure. Find an open tour of the board in which the knight is permitted to travel through a given space more than once to achieve the goal of visiting every space. Is your tour most accurately described as a walk, trail, or path? Explain why.



The neighborhood of Pines West has three cul-de-sacs that meet at an intersection as shown in the figure. A postal delivery person starts at the intersection and visits each house in a cul-de-sac once, returns to the intersection, visits each house in the next cul-de-sac, and so on, returning to the intersection when finished.

Exercise 3.5.1.43

Describe how the route can be represented as a graph. If there is no backtracking, in other words, the person never reverses direction, is the route followed by the postal delivery person best described as a walk, trail, path, closed walk, circuit, or directed cycle? Explain your reasoning.

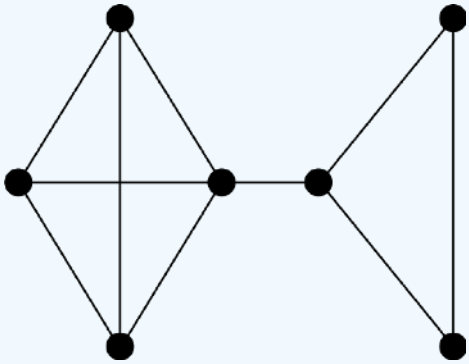


Exercise 3.5.1.44

Draw the graph for the route described in the neighborhood. Find its chromatic number. Find a map coloring that shows the chromatic number.

Exercise 3.5.1.45

When radio towers are within range of each other, they must use different frequencies. In the graph, the vertices represent the towers, and the edges indicate the towers are in range of each other. Use graph colorings to determine the number of frequencies needed to avoid two towers in the same range being assigned the same frequency. Give an example of a coloring to support your conclusion.



Exercise 3.5.1.46

A Sudoku puzzle consists of a nine-by-nine grid which is subdivided into nine three-by-three smaller grids. Given an incomplete grid, the goal is to complete it in such a way that each row contains the numbers 1 through 9, each column contains the numbers 1 through 9, and each three by three grid contains the numbers 1 through 9 as shown in the figure. If a graph were drawn with a vertex to represent each entry in the grid, what should the edges represent so that a nine-coloring of the graph would be a solution to the puzzle?

Sudoku Puzzle

8			9	3				2
		9					4	
7		2	1			9	6	
2							9	
	6						7	
	7				6			5
	2	7			8	4		6
	3					5		
5				6	2			8

Solution to Puzzle

8	4	6	9	3	7	1	5	2
3	1	9	6	2	5	8	4	7
7	5	2	1	8	4	9	6	3
2	8	5	7	1	3	6	9	4
4	6	3	8	5	9	2	7	1
9	7	1	2	4	6	3	8	5
1	2	7	5	9	8	4	3	6
6	3	8	4	7	1	5	2	9
5	9	4	3	6	2	7	1	8

At a wedding reception, the bride and groom would like their family and friends to get to know each other better. They have decided that if two people know each other, they will not seat them at the same table. The following is a list of guests and who they know. Guests *A*, *B*, *C*, and *D* all know each other; guests *E*, *F*, *G*, and *H* all know each other; guests *I*, *J*, and *K* all know each other; guests *P*, *Q* and *R* all know each other, guests *L*, *M*, and *O* all know each other. Additionally, *I* knows *D* and *I* knows *G*, *J* knows *M*, *K* knows *P*, *N* knows *L*, and *N* knows *O*.

? Exercise 3.5.1.47

Draw a graph to show the relationships.

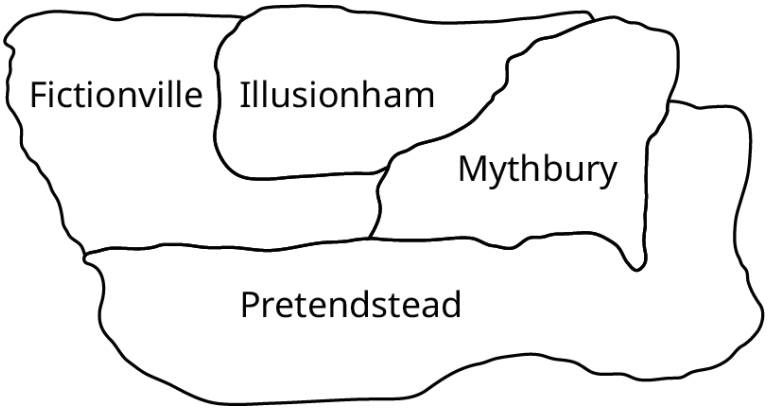
? Exercise 3.5.1.48

Determine the minimum number of tables needed to seat the guests so that they will sit with people they don't know.

Exercise 3.5.1.49

Give a coloring to support your conclusion.

The map of the states of Imaginaria is given. Use the figure to answer the following exercises.



Exercise 3.5.1.50

Draw a graph to represent the map.

Exercise 3.5.1.51

Determine the chromatic number of the graph you found.

Exercise 3.5.1.52

Give a graph coloring that supports your answer in the previous question.

Exercise 3.5.1.53

Color the provided map to correspond with the coloring you found in the previous question.

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3.6: Euler Circuits



Figure **3.6.1**: Delivery trucks move goods from place to place. (credit: “Mack Midliner” by Jason Lawrence/Flickr, CC BY 2.0)

Learning Objectives

1. Determine if a graph is connected.
2. State the Chinese postman problem.
3. Describe and identify Euler Circuits.
4. Apply the Euler Circuits Theorem.
5. Evaluate Euler Circuits in real-world applications.

The delivery of goods is a huge part of our daily lives. From the factory to the distribution center, to the local vendor, or to your front door, nearly every product that you buy has been shipped multiple times to get to you. If the cost and time of that delivery is too great, you will not be able to afford the product. Delivery personnel have to leave from one location, deliver the goods to various places, and then return to their original location and do all of this in an organized way without losing money. How do delivery services find the most efficient delivery route? The answer lies in graph theory.

Connectedness

Before we can talk about finding the best delivery route, we have to make sure there is a route at all. For example, suppose that you were tasked with visiting every airport on the graph in Figure **3.6.2** by plane. Could you accomplish that task, only taking direct flight paths between airports listed on this graph? In other words, are all the airports connected by paths? Or are some of the airports disconnected from the others?

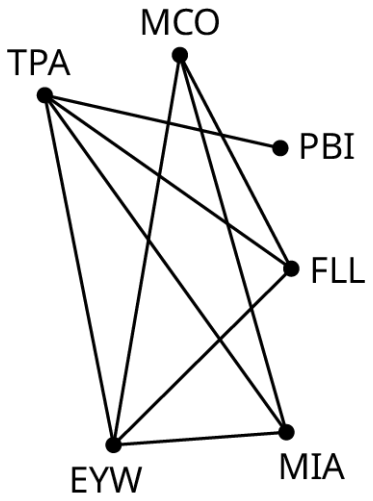


Figure **3.6.2**: Major Central and South Florida Airports

In Figure 12.118, we can see TPA is adjacent to PBI, FLL, MIA, and EYW. Also, there is a path between TPA and MCO through FLL. This indicates there is a path between each pair of vertices. So, it is possible to travel to each of these airports only taking direct flight paths and visiting no other airports. In other words, the graph is **connected** because there is a path joining every pair of vertices on the graph. Notice that if one vertex is connected to each of the others in a graph, then all the vertices are connected to each other. So, one way to determine if a graph is connected is to focus on a single vertex and determine if there is a path between that vertex and each of the others. If so, the graph is connected. If not, the graph is **disconnected,** which means at least one pair of vertices in a graph is not joined by a path. Let's take a closer look at graph *X* in Figure **3.6.3** . Focus on vertex *a*. There is a path between vertices *a* and *b*, but there is no path between vertex *a* and vertex *c*. So, Graph *X* is disconnected.

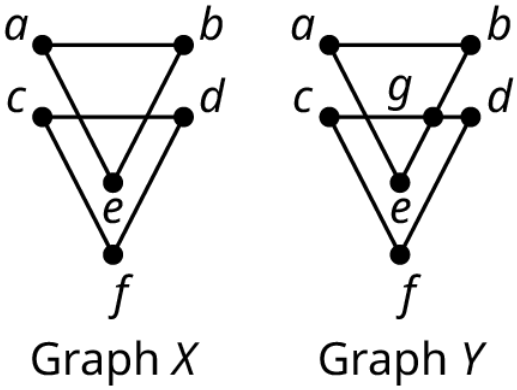


Figure **3.6.3**: Connected vs. Disconnected

When you are working with a planar graph, you can also determine if a graph is connected by untangling it. If you draw it so that so that none of the edges are overlapping, like we did with Graph *X* in Figure **3.6.4** , it is easier to see that the graph is disconnected.

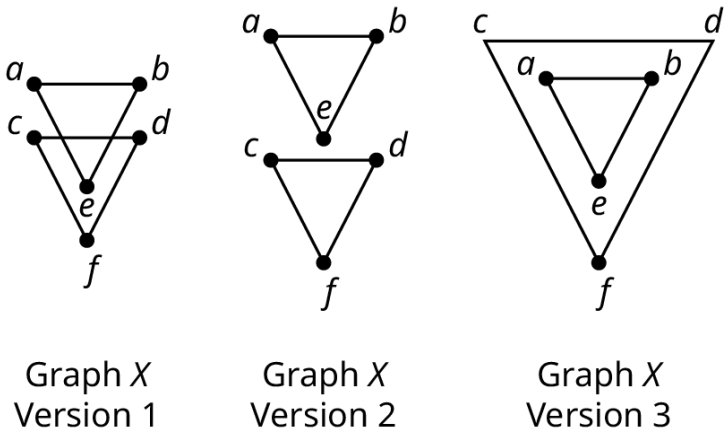


Figure **3.6.4**: Untangling Graph *X*

Versions 2 and 3 of Graph *X* in Figure **3.6.4** each have the same number of vertices, number of edges, degrees of the vertices, and pairs of adjacent vertices as version 1. In other words, each version retains the same structure as the original graph. Since versions 2 and 3 of Graph *X*, do not have overlapping edges, it is easier to identify pairs of vertices that do not have paths between them, and it is more obvious that Graph *X* is disconnected. In fact, there are two completely separate, disconnected subgraphs, one with the vertices in $\{a, b, e\}$, and the other with the vertices $\{c, d, f\}$

These sets of vertices together with all of their edges are called **components** of Graph *X*. A component of a graph is a subgraph in which there is a path between each pair of vertices in the subgraph, but no edges between any of the vertices in the subgraph and a vertex that is not in the subgraph.

Now let's focus on Graph *Y* from Figure **3.6.3** . As in Graph *X*, there is a path between vertices *a* and *b*, as well as between vertices *a* and *e*, but Graph *Y* is different from Graph *X* because of vertex *g*. Not only is there a path between vertices *a* and *g*, but vertex *g* bridges the gap between *a* and *c* with the path $a \to b \to g \to c$. Similarly, there is a path between vertices *a* and *d* and vertices *a* and *f*: $a \to b \to g \to d$, $a \to b \to g \to d \to f$. Since there is a path between vertex *a* and every other vertex, Graph *Y* is connected. You can also see this a bit more clearly by untangling Graph *Y* as in Figure 12.121. Even when *Y* is drawn so that the edges do not overlap, the graph cannot be drawn as two separate, unconnected pieces. In other words, Graph *Y* has only one component with the vertices $\{a, b, c, d, e, f\}$.

We can give an alternate definition of **connected** and **disconnected** using the idea of components. A graph is connected if it has only one component. A graph is disconnected if it has more than one component. These alternate definitions are equivalent to the previous definitions. This means that you can confirm a graph is connected or disconnected either by checking to see if there is a path between each vertex and each other vertex, or by identifying the number of components. A graph is connected if it has only one component.

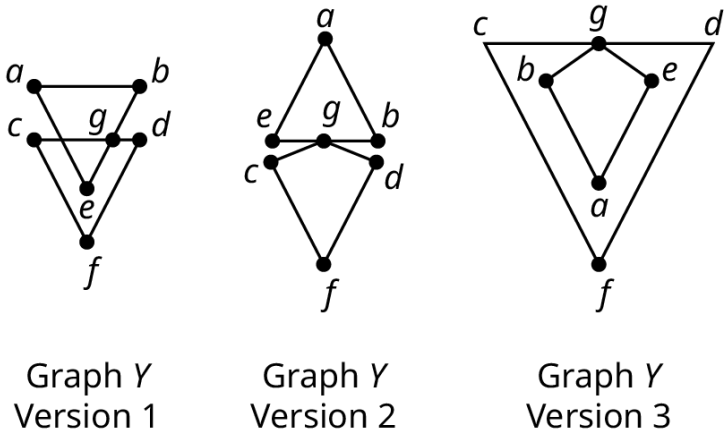


Figure **3.6.5**: Untangling Graph *Y*

Example 3.6.1: Determining If a Graph Is Connected or Disconnected

Use Figure 3.6.6 to answer each question.

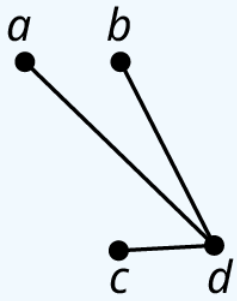


Figure 3.6.6 Graph E

1. Find a path between vertex a and every other vertex on the graph, if possible.
2. Identify all the components of Graph E.
3. Determine whether the graph is connected or disconnected and explain how you know.

Answer

1. The paths are $a \to d \to b$, $a \to d \to c$, and $a \to d$.
2. There is only one component in Graph E. It has the vertices $\{a, b, c, d\}$.
3. The graph is connected, because there is a path between vertex a and every other vertex. We can also so that Graph E is connected because it has only one component.

Your Turn 3.6.1

Determine whether each graph is connected or disconnected and identify the set of vertices that make up each of its components.

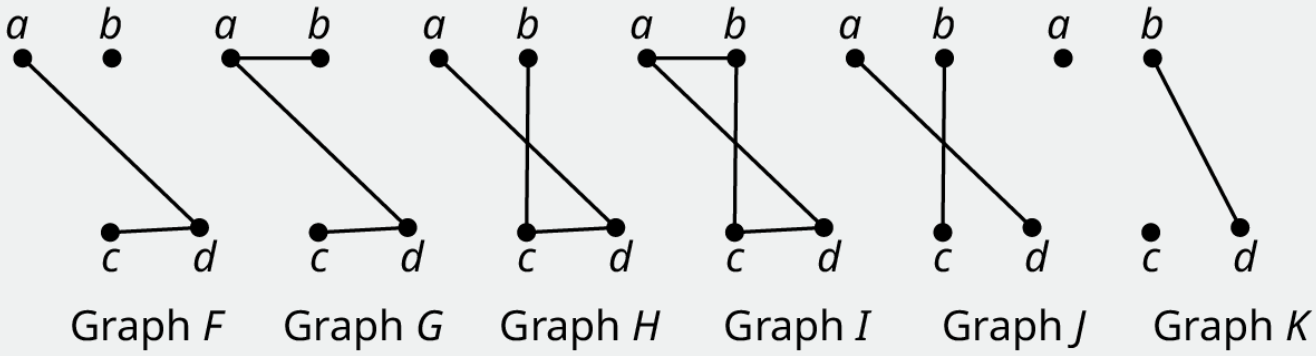


Figure 3.6.7

Video



Connected and Disconnected Graphs in Graph Theory

Example 3.6.2: Applying Connectedness

The U.S. Interstate Highway System extends throughout the 48 contiguous states. It also has routes in the states of Hawaii and Alaska, and the commonwealth of Puerto Rico. Consider a graph representing the U.S. Interstate Highway System, in which there is a vertex for each of the 50 states and Puerto Rico, and an edge is drawn between any two vertices representing states that are connected by a highway in that system. Would this graph be connected or disconnected? Explain your reasoning.

Answer

Hawaii, Alaska, and Puerto Rico are geographically separate from the 48 contiguous states, and each from each other. This means that there are vertices on the graph with no path joining them to the other vertices. So the graph is disconnected.

Your Turn 3.6.2

"Six degrees of separation" is the idea that any two people on Earth are connected by at most six social connections. Assume that this is true. Consider a graph in which each vertex is a person on Earth, and each edge is a social connection. Would this graph be connected or disconnected? Explain your reasoning.

Origin of Euler Circuits

The city of Konigsberg, modern day Kaliningrad, Russia, has waterways that divide up the city. In the 1700s, the city had seven bridges over the various waterways. The map of those bridges is shown in Figure **3.6.8** . The question as to whether it was possible to find a route that crossed each bridge exactly once and return to the starting point was known as the Konigsberg Bridge Problem. In 1735, one of the most influential mathematicians of all time, Leonard Euler, solved the problem using an area of mathematics that he created himself, graph theory!

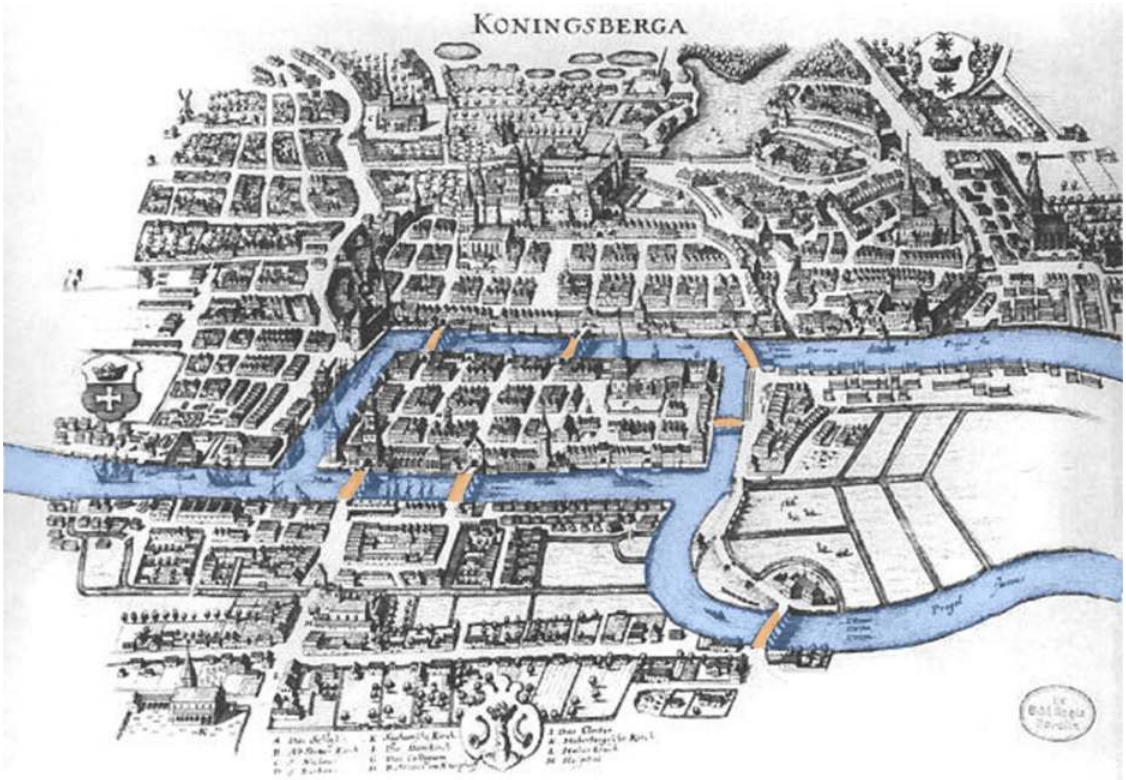


Figure 3.6.8: Map of the Bridges of Konigsberg in 1700s (credit: "Konigsberg Bridge" by Merian Erben/Wikimedia Commons, Public Domain)

Euler drew a multigraph in which each vertex represented a land mass, and each edge represented a bridge connecting them, as shown in Figure 3.6.9 . Remember from Navigating Graphs that a circuit is a trail, so it never repeats an edge, and it is closed, so it begins and ends at the same vertex. Euler pointed out that the Konigsberg Bridge Problem was the same as asking this graph theory question: Is it possible to find a circuit that crosses every edge? Since then, circuits (or closed trails) that visit every edge in a graph exactly once have come to be known as Euler circuits in honor of Leonard Euler.

Video



Recognizing Euler Trails and Euler Circuits

Euler was able to prove that, in order to have an Euler circuit, the degrees of all the vertices of a graph have to be even. He also proved that any graph with that characteristic must have an Euler circuit. So, saying that a connected graph is Eulerian is the same as saying it has vertices with all even degrees, known as the **Eulerian circuit theorem**.

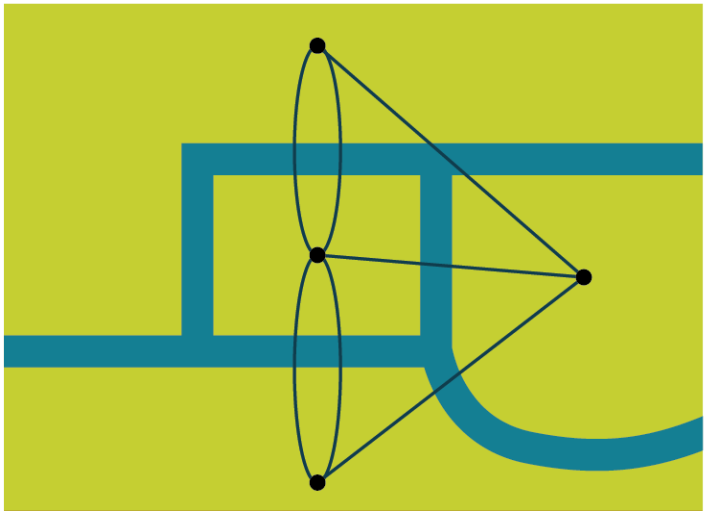


Figure **3.6.9**: Graph of Konigsberg Bridges

To understand why the Euler circuit theorem is true, think about a vertex of degree 3 on any graph, as shown in Figure **3.6.10** .

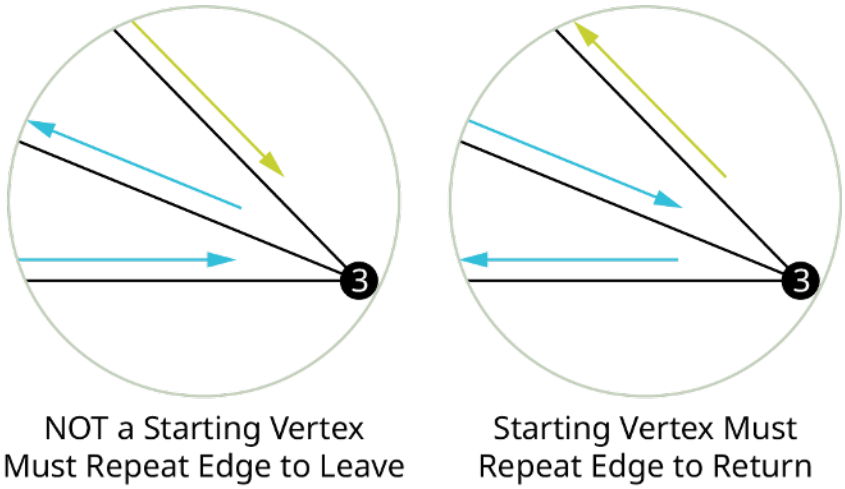


Figure **3.6.10**: A Vertex of Degree 3

First imagine the vertex of degree 3 shown in Figure **3.6.10** is not the starting vertex. At some point, each edge must be traveled. The first time one of the three edges is traveled, the direction will be toward the vertex, and the second time it will be away from the vertex. Then, at some point, the third edge must be traveled coming in toward the vertex again. This is a problem, because the only way to get back to the starting vertex is to then visit one of the three edges a second time. So, this vertex cannot be part of an Euler circuit.

Next imagine the vertex of degree 3 shown in Figure **3.6.10** is the starting vertex. The first time one of the edges is traveled, the direction is away from the vertex. At some point, the second edge will be traveled coming in toward the vertex, and the third edge will be the way back out, but the starting vertex is also the ending vertex in a circuit. The only way to return to the vertex is now to travel one of the edges a second time. So, again, this vertex cannot be part of an Euler circuit.

For the same reason that a vertex of degree 3 can never be part of an Euler circuit, a vertex of any odd degree cannot either. We can use this fact and the graph in Figure **3.6.11** to solve the Konigsberg Bridge Problem. Since the degrees of the vertices of the graph in Figure **3.6.10** are not even, the graph is not Eulerian and it cannot have an Euler circuit. This means it is not possible to travel through the city of Konigsberg, crossing every bridge exactly once, and returning to your starting position.

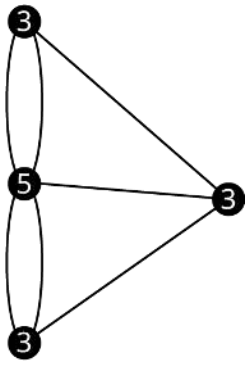


Figure 3.6.11: Degrees of Vertices in Konigsberg Bridge Multigraph

Video

Existence of Euler Circuits in Graph Theory

Chinese Postman Problem

At Camp Woebegone, campers travel the waterways in canoes. As part of the Camp Woebegone Olympics, you will hold a canoeing race. You have placed a checkpoint on each of the 11 different streams. The competition requires each team to travel each stream, pass through the checkpoints in any order, and return to the starting line, as shown in the Figure 3.6.12 .

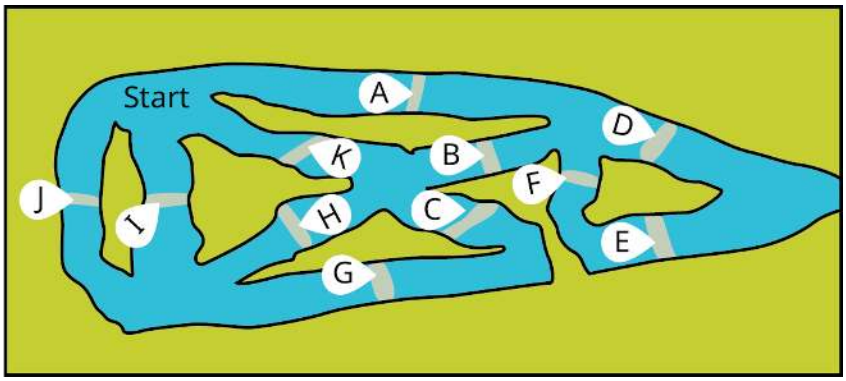


Figure 3.6.12: Map of Canoe Event Checkpoints

Since the teams want to go as fast as possible, they would like to find the shortest route through the course that visits each checkpoint and returns to the starting line. If possible, they would also like to avoid backtracking. Let's visualize the course as a multigraph in which the vertices represent turns and the edges represent checkpoints as in Figure 3.6.13 .

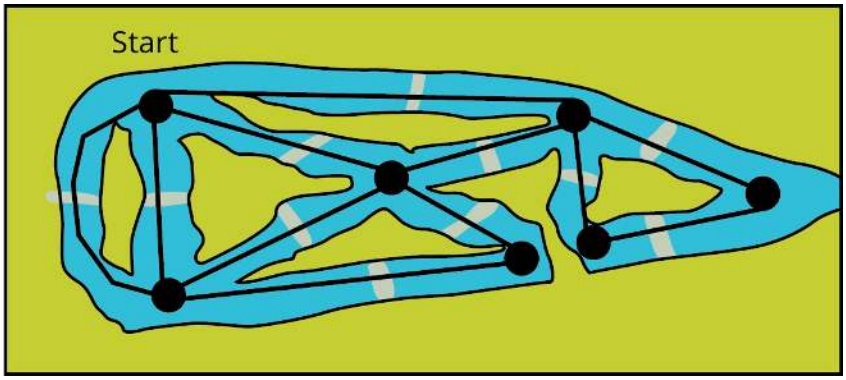


Figure 3.6.13: Multigraph of Canoe Event

The teams would like to find a closed walk that repeats as few edges as possible while still visiting every edge. If they never repeat an edge, then they have found a closed trail, which is a circuit. That circuit must cover all edges; so, it would be an Euler circuit. The task of finding a shortest circuit that visits every edge of a connected graph is often referred to as the **Chinese postman**

problem. The name Chinese postman problem was coined in honor of the Chinese mathematician named Kwan Mei-Ko in 1960 who first studied the problem.

If a graph has an Euler circuit, that will always be the best solution to a Chinese postman problem. Let's determine if the multigraph of the course has an Euler circuit by looking at the degrees of the vertices in Figure **3.6.14** . Since the degrees of the vertices are all even, and the graph is connected, the graph is Eulerian. It is possible for a team to complete the canoe course in such a way that they pass through each checkpoint exactly once and return to the starting line.

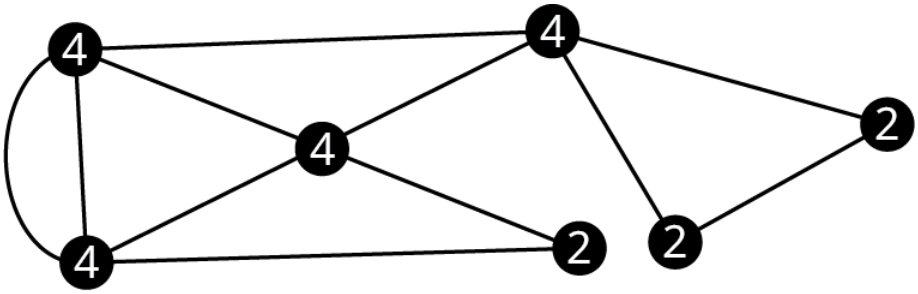


Figure **3.6.14**: Degrees of Vertices in Graph of Canoe Event

✔ Example **3.6.3**: Understanding Eulerian Graphs

A postal delivery person must deliver mail to every block on every street in a local subdivision. Figure ✔ **3.6.15** is a map of the subdivision. Use the map to answer each question.

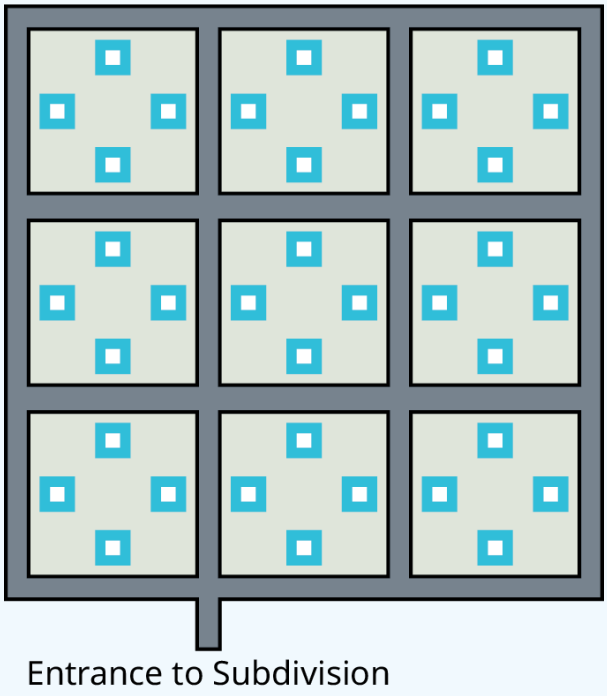


Figure ✔ **3.6.15** Map of Subdivision

1. Draw a graph or multigraph to represent the subdivision in which the vertices represent the intersections, and the edges represent streets.
2. Is your graph connected? Explain how you know.
3. Determine the degrees of the vertices in the graph.
4. Is your graph an Eulerian graph?
5. Is it possible for the postal delivery person to visit each block on each street exactly once? Justify your answer.

Answer

1. The graph is in Figure ✔ 3.6.16 .

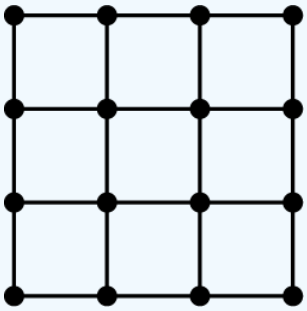


Figure ✔ 3.6.16 Graph of Subdivision

2. The graph is connected. It has only one component and there is a path between each pair of vertices.
3. There are four corner vertices of degree 2, there are eight exterior vertices of degree 3, and there are four interior vertices of degree 4.
4. The graph is not Eulerian because it has vertices of odd degree.
5. No. Since the graph is not Eulerian, there is no Euler circuit, which means that there is no route that would pass through every edge exactly once.

Your Turn 3.6.3

A pest control service has at least one customer on every block of every street or cul-de-sac in a neighborhood. Use the map of the neighborhood to answer each question.

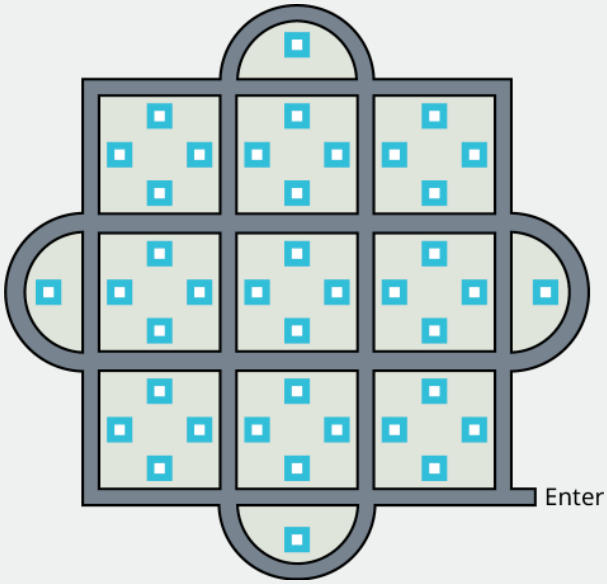


Figure 📌 3.6.17: Map of Neighborhood

Draw a graph or multigraph of the neighborhood in which the vertices represent intersections, and the edges represent the streets between them.

1. Is your graph connected? Explain how you know.
2. Determine the degrees of the vertices in the graph.
3. Is your graph an Eulerian graph?
4. Is it possible for the postal delivery person to visit each block on each street exactly once and start and end at the same position? Justify your answer.

Identifying Euler Circuits

Solving the Chinese postman problem requires finding a shortest circuit through any graph or multigraph that visits every edge. In the case of Eulerian graphs, this means finding an Euler circuit. The method we will use is to find any circuit in the graph, then find a second circuit starting at a vertex from the first circuit that uses only edges that were not in the first circuit, then find a third circuit starting at a vertex from either of the first two circuits that uses only edges that were not in the first two circuits, and then continue this process until all edges have been used. In the end, you will be able to link all the circuits together into one large Euler circuit.

Let's find an Euler circuit in the map of the Camp Woebegone canoe race. In Figure **3.6.18**, we have labeled the edges of the multigraph so that the circuits can be named. In a multigraph it is necessary to name circuits using edges and vertices because there can be more than one edge between adjacent vertices.

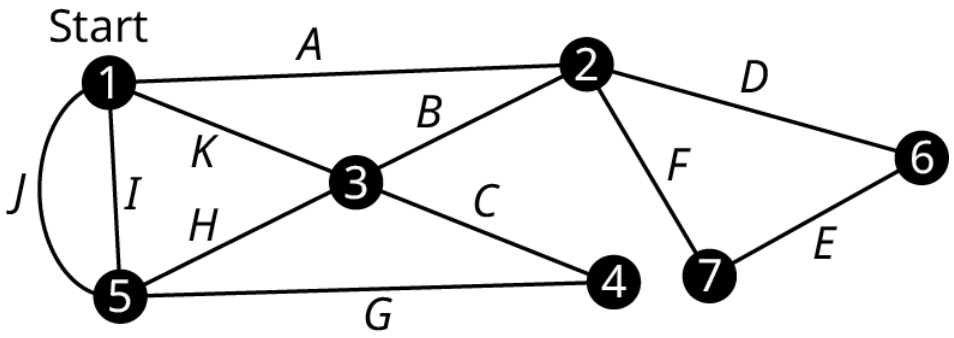


Figure **3.6.18**: Multigraph of Canoe Race with Vertices and Edges Labeled

We will begin with vertex 1 because it represents the starting line in this application. In general, you can start at any vertex you want. Find any circuit beginning and ending with vertex 1. Remember, a circuit is a trail, so it doesn't pass through any edge more than once. Figure **3.6.19** shows one possible circuit that we could use as the first circuit, $1 \rightarrow A \rightarrow 2 \rightarrow B \rightarrow 3 \rightarrow C \rightarrow 4 \rightarrow G \rightarrow 5 \rightarrow J \rightarrow 1$.

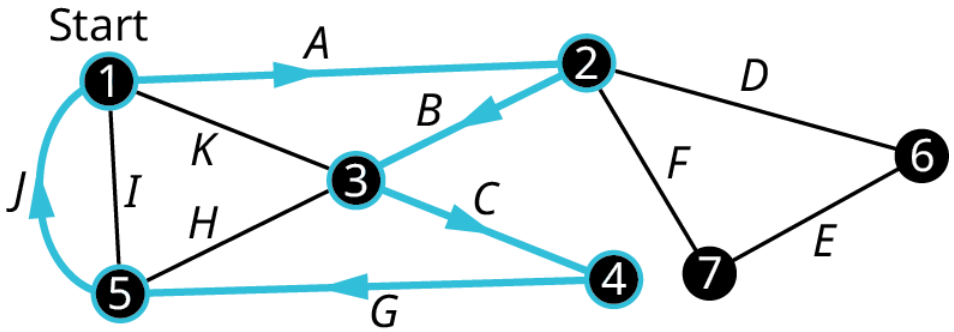


Figure **3.6.19**: First Circuit

From the edges that remain, we can form two more circuits that each start at one of the vertices along the first circuit. Starting at vertex 3 we can use $3 \rightarrow H \rightarrow 5 \rightarrow I \rightarrow 1 \rightarrow K \rightarrow 3$ and starting at vertex 2 we can use $2 \rightarrow D \rightarrow 6 \rightarrow E \rightarrow 7 \rightarrow F \rightarrow 2$, as shown in Figure **3.6.20**.

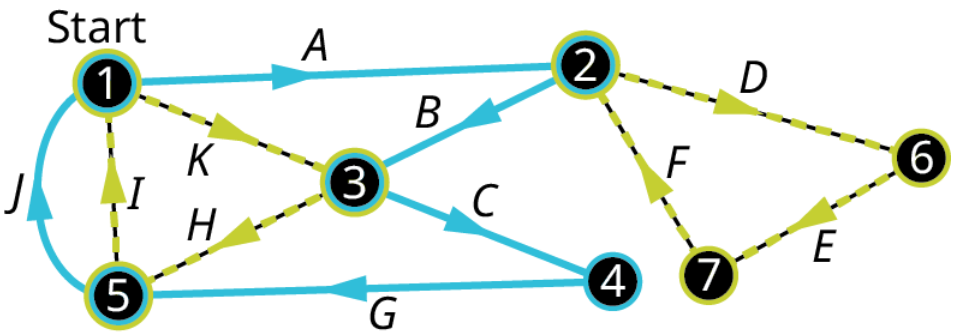


Figure **3.6.20**: Second and Third Circuits

Now that each of the edges is included in one and only once circuit, we can create one large circuit by inserting the second and third circuits into the first. We will insert them at their starting vertices 2 and 3

$$1 \to A \to \boxed{2} \to B \to \boxed{3} \to C \to 4 \to G \to 5 \to J \to 1$$

becomes

$$1 \to A \to \boxed{2 \to D \to 6 \to E \to 7 \to F \to 2} \to B \to$$
$$\boxed{3 \to H \to 5 \to I \to 1 \to K \to 3} \to C \to 4 \to G \to 5 \to J \to 1$$

Finally, we can name the circuit using vertices, 1 → 2 → 6 → 7 → 2 → 3 → 5 → 1 → 3 → 4 → 5 → 1, or edges, $A \to D \to E \to F \to B \to H \to I \to K \to C \to$ G → J.

Let's review the steps we used to find this Eulerian Circuit.

Steps to Find an Euler Circuit in an Eulerian Graph

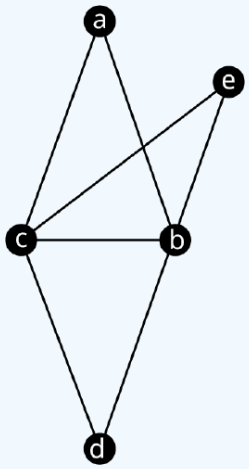
Step 1 - Find a circuit beginning and ending at any point on the graph. If the circuit crosses every edges of the graph, the circuit you found is an Euler circuit. If not, move on to step 2.

Step 2 - Beginning at a vertex on a circuit you already found, find a circuit that only includes edges that have not previously been crossed. If every edge has been crossed by one of the circuits you have found, move on to Step 3. Otherwise, repeat Step 2.

Step 3 - Now that you have found circuits that cover all of the edges in the graph, combine them into an Euler circuit. You can do this by inserting any of the circuits into another circuit with a common vertex repeatedly until there is one long circuit.

✔ Example **3.6.4**: Finding an Euler Circuit

Use Figure ✔ **3.6.21** to answer each question.



Graph *F*

Figure ✔ **3.6.21** Graph *F*

1. Verify the Graph F is Eulerian.
2. Find an Euler circuit that begins and ends at vertex c.

Answer

1. The graph is connected because it has only one component. Also, each of the vertices in graph F has even degree as shown in Figure ✔ 3.6.22. So, the graph is Eulerian.

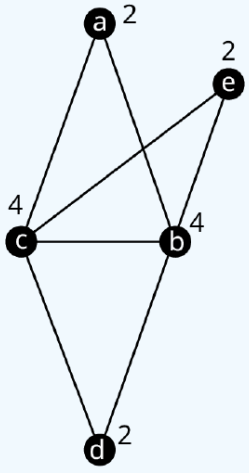


Figure ✔ 3.6.22 Degrees of Vertices in Graph F

2. **Step 1:** Beginning at vertex c, identify circuit $c \rightarrow e \rightarrow b \rightarrow c$ as shown in Figure ✔ 3.6.23. Since this circuit does not cover every edge in the graph, move on to Step 2.

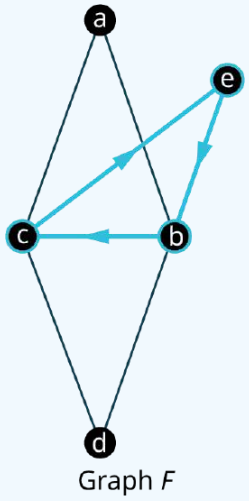
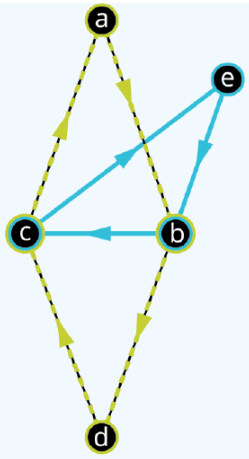


Figure ✔ 3.6.23: First Circuit in Graph F

Step 2: Find another circuit beginning at one of the vertices in the first circuit, using only edges that have not been used in the first circuit. It is possible to do this using either vertex c or vertex b. In Figure ✔ 3.6.24, we have used vertex b as the starting and ending point for a second circuit, $b \rightarrow d \rightarrow c \rightarrow a \rightarrow b$. Since all edges have been crossed by one of the two circuits, move on to Step 3.



Graph F

Figure ✔ **3.6.24**: Second Circuit in Graph F

Step 3: Combine the two circuits into one. Replace vertex b in the first circuit with the whole second circuit that begins at vertex b.

$$c \to e \to \boxed{b} \to$$

cbecomesc → e → b → d → c → a → b → cc → e → b → cbecomesc → e → b → d → c → a → b → c.
An Euler circuit that begins at vertex c is $c \to e \to b \to d \to c \to a \to b \to c$.

Your Turn 3.6.4

Use Graphs X and Y to answer each question.

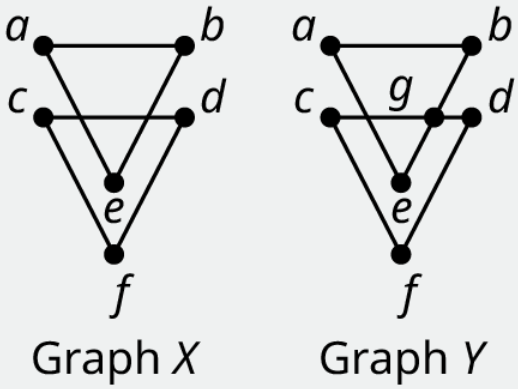


Figure 📌 **3.6.25**

Which graph does not have an Euler circuit? Explain how you know.

Find an Euler circuit in the other graph.

Eulerization

The Chinese postman problem doesn't only apply to Eulerian graphs. Recall the postal delivery person who needed to deliver mail to every block of every street in a subdivision. We used the map in Figure **3.6.26** to create the graph in Figure **3.6.27**.

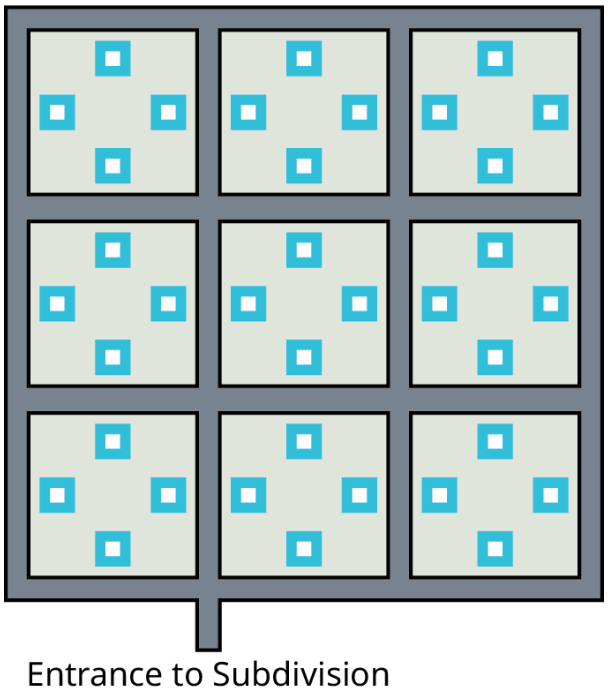


Figure **3.6.26**: Map of Subdivision

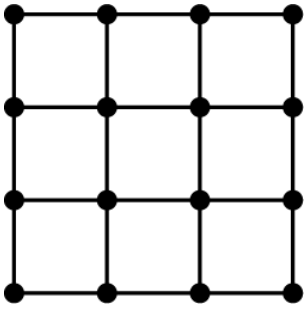


Figure **3.6.27**: Graph of Subdivision

Since the graph of the subdivision has vertices of odd degree, there is no Euler circuit. This means that there is no route through the subdivision that visits every block of every street without repeating a block. What should our delivery person do? They need to repeat as few blocks as possible. The technique we will use to find a closed walk that repeats as few edges as possible is called **eulerization**. This method adds duplicate edges to a graph to create vertices of even degree so that the graph will have an Euler circuit.

In Figure **3.6.28**, the eight vertices of odd degree in the graph of the subdivision are circled in green. We have added duplicate edges between the pairs of vertices, which changes the degrees of the vertices to even degrees so the resulting multigraph has an Euler circuit. In other words, we have eulerized the graph.

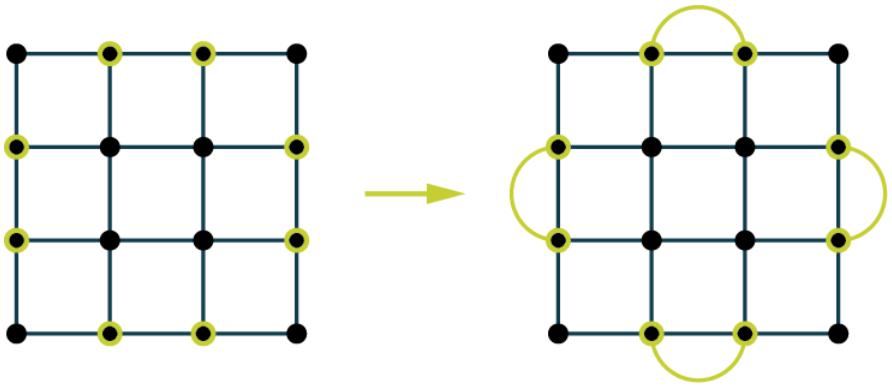


Figure 3.6.28: An Eulerized Graph

The duplicate edges in the eulerized graph correspond to blocks that our delivery person would have to travel twice. By keeping these duplicate edges to a minimum, we ensure the shortest possible route. It can be challenging to determine the fewest duplicate edges needed to eulerize a graph, but you can never do better than half the number of odd vertices. In the graph in Figure 3.6.28, we have found a way to fix the eight vertices of odd degree with only four duplicate edges. Since four is half of eight, we will never do better.

Checkpoint

IMPORTANT! The duplicate edges that we add indicate places where the route will pass twice. An entirely new edge between two vertices that were not previously adjacent would indicate that our postal delivery person created a new road through someone's property! So, ***we can duplicate existing edges, but we cannot create new ones****.*

Example 3.6.5: Eulerizing Graphs

Use Graph *A* and multigraphs *B*, *C*, *D*, and *E* given in Figure ✔ 3.6.29 to answer the questions.

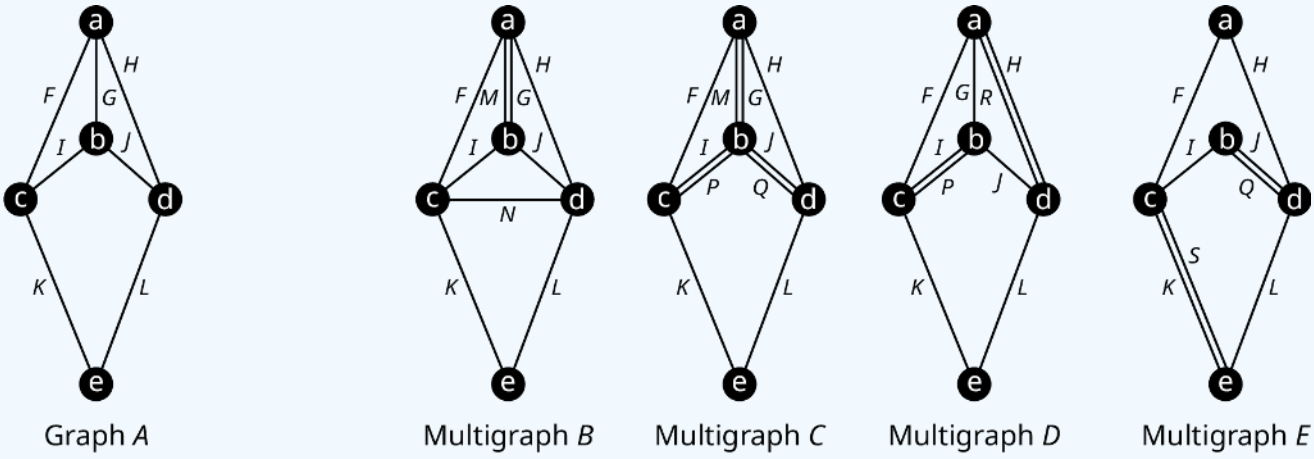


Figure ✔ 3.6.29 Graph *A* and Multigraphs *B* through *E*

1. Which of the multigraphs are not eulerizations of Graph *A*? Explain your answer.
2. Which eulerization of Graph *A* uses the fewest duplicate edges? How many does it use?
3. Is it possible to eulerize Graph *A* using fewer duplicate edges than your answer to part 2? If so, give an example. If not, explain why not.

Answer

1. Multigraph *B* is not an eulerization of *A* because the edge *N* between vertices *c* and *d* is not a duplicate of an existing edge. Multigraph *E* is not an eulerization of *A* because vertices *b* and *e* have odd degree.
2. Multigraph *C* uses 3 duplicate edges while multigraph *D* only uses 2. So, *D* uses the fewest.

3. Since there were four vertices in Graph *A*, the fewest number of edges that could possibly eulerize the graph is half of four, which is two. So, it is not possible to eulerize Graph *A* using fewer edges.

Your Turn 3.6.5

Create an eulerization using the fewest duplicate edges possible for Graph *Z*.

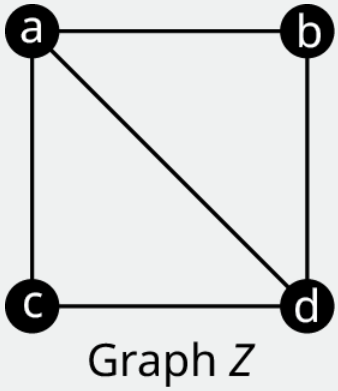


Figure 3.6.30: Graph *Z*

Checkpoint

IMPORTANT! Since only duplicate edges can be added to eulerize a graph, it is not possible to eulerize a disconnected graph.

WORK IT OUT

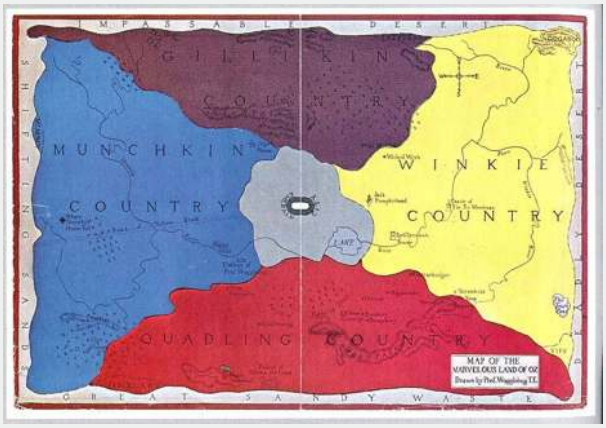


Figure 3.6.31: Map of the Magical Land of Oz (credit: "Map of Oz within the surrounding deserts" by L. Frank Baum/Wikimedia Commons, Public Domain)

In *The Wonderful Wizard of Oz*, written by L. Frank Baum and illustrated by W. W. Denslow, the region of the Emerald City lies at the center of the magical land of Oz, with Gillikin Country to the north, Winkie Country to the east, Munchkin Country to the west, and Quadling Country to the south. Munchkin Country and Winkie Country each shares a border with Gillikin Country and Quadling Country. Let's apply graph theory to Dorothy's famous journey through Oz. Draw a graph in which each vertex is one of the regions of Oz. Then answer each question:

1. Is there an Euler trail circuit that Dorothy could follow, instead of the yellow brick road, to lead her from the land of the Munchkins, through all the regions of Oz and back, passing over each border exactly once? If not, how could the graph be Eulerized most efficiently?
2. What is the chromatic number of the graph? Find a graph coloring that demonstrates your answer and use it to draw and color a graph of Oz.

Check Your Understanding

For the following exercises, determine whether each statement is always true, sometimes true, or never true.

Exercise 3.6.1

A disconnected graph has only one component.

Exercise 3.6.2

A graph that has all vertices of even degree is connected.

Exercise 3.6.3

There is a route through the city of Konigsberg that a person may pass over each bridge exactly once and return to the starting point.

Exercise 3.6.4

A graph with vertices of all even degree is Eulerian.

Exercise 3.6.5

An Eulerian graph has all vertices of even degree.

Exercise 3.6.6

An Euler circuit is a closed trail.

Exercise 3.6.7

An Euler circuit is a closed path

Exercise 3.6.8

To eulerize a graph, add new edges between previously nonadjacent vertices until no vertices have odd degree.

Exercise 3.6.9

To eulerize a graph, add duplicate edges between adjacent vertices until no vertices have odd degree.

Exercise 3.6.10

The number of duplicate edges required to eulerize a graph is half the number of vertices of odd degree.

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3.6.1: Exercises

Use Graphs *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, and *I* to answer the following exercises. Identify any graph or graphs with the given characteristics or indicate that none do.

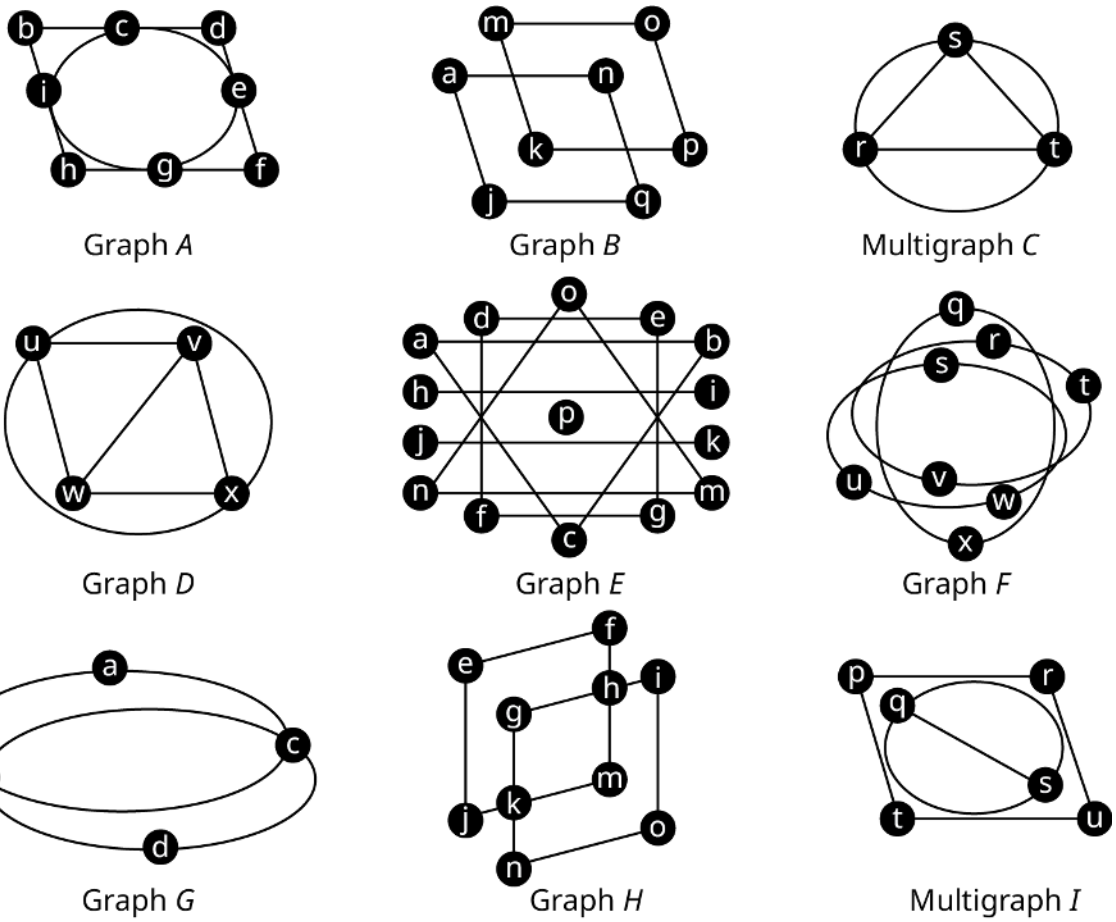


Figure 12.148

Exercise 3.6.1.1

Connected

Exercise 3.6.1.2

Disconnected

Exercise 3.6.1.3

Exactly two components

Exercise 3.6.1.4

Exactly three components

Exercise 3.6.1.5

Exactly four components

Exercise 3.6.1.6

Exactly five components

Exercise 3.6.1.7

At least one vertex of odd degree

Exercise 3.6.1.8

All vertices of even degree

Exercise 3.6.1.9

An Euler circuit

Exercise 3.6.1.10

A path between vertex j and each other vertex on the same graph

Use the graphs in the previous exercise to answer the following exercises. For each exercise, list the set of vertices for each component in the given graph.

Exercise 3.6.1.11

Graph B

Exercise 3.6.1.12

Graph E

Exercise 3.6.1.13

Graph F

Exercise 3.6.1.14

Multigraph I

Use the graphs in the initial exercise to answer the following exercises. For each exercise, a graph and a vertex on the graph are given. Find a path between the given vertex and each other vertex on the graph. If this is not possible, indicate that it is not.

Exercise 3.6.1.15

Graph A, vertex c.

Exercise 3.6.1.16

Graph *B*, vertex *m*.

Exercise 3.6.1.17

Graph *D*, vertex *x*.

Exercise 3.6.1.18

Graph *F*, vertex *w*.

Exercise 3.6.1.19

Graph *G*, vertex *a*.

Exercise 3.6.1.20

Graph *H*, vertex *e*.

Use the graphs in the initial exercise to answer the following exercises. For each exercise, a graph and a vertex on the graph are given. Find an Euler circuit beginning and ending at the given vertex if one exists. If no Euler circuits exist, explain how you know that they do not.

Exercise 3.6.1.21

Graph *A*, vertex *c*.

Exercise 3.6.1.22

Graph *B*, vertex *k*.

Exercise 3.6.1.23

Graph *D*, vertex *w*.

Exercise 3.6.1.24

Graph *G*, vertex *b*.

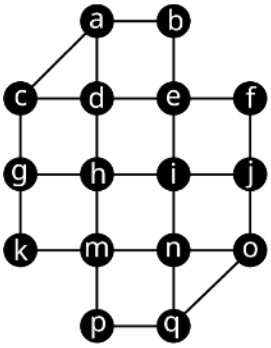
Exercise 3.6.1.25

Graph *H*, vertex *o*.

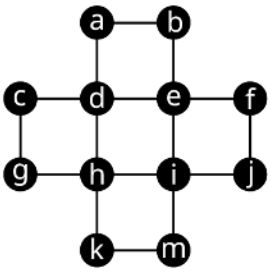
Exercise 3.6.1.26

Multigraph *I*, vertex *p*.

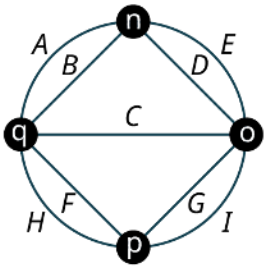
For the following exercises, use the connected graphs. In each exercise, a graph is indicated. Determine if the graph is Eulerian or not and explain how you know. If it is Eulerian, give an example of an Euler circuit. If it is not, state which edge or edges you would duplicate to eulerize the graph.



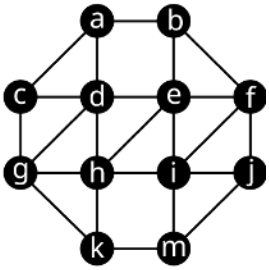
Graph J



Graph K



Multigraph L



Graph M

? Exercise 3.6.1.27

Graph J

? Exercise 3.6.1.28

Graph K

? Exercise 3.6.1.29

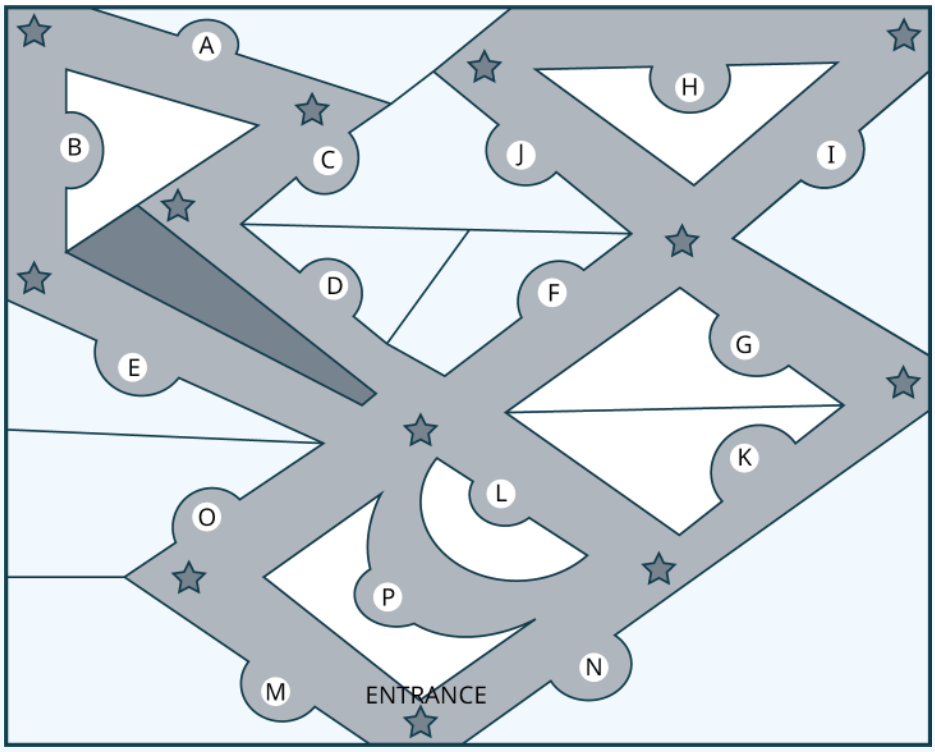
Multigraph L

? Exercise 3.6.1.30

Graph M

? Exercise 3.6.1.31

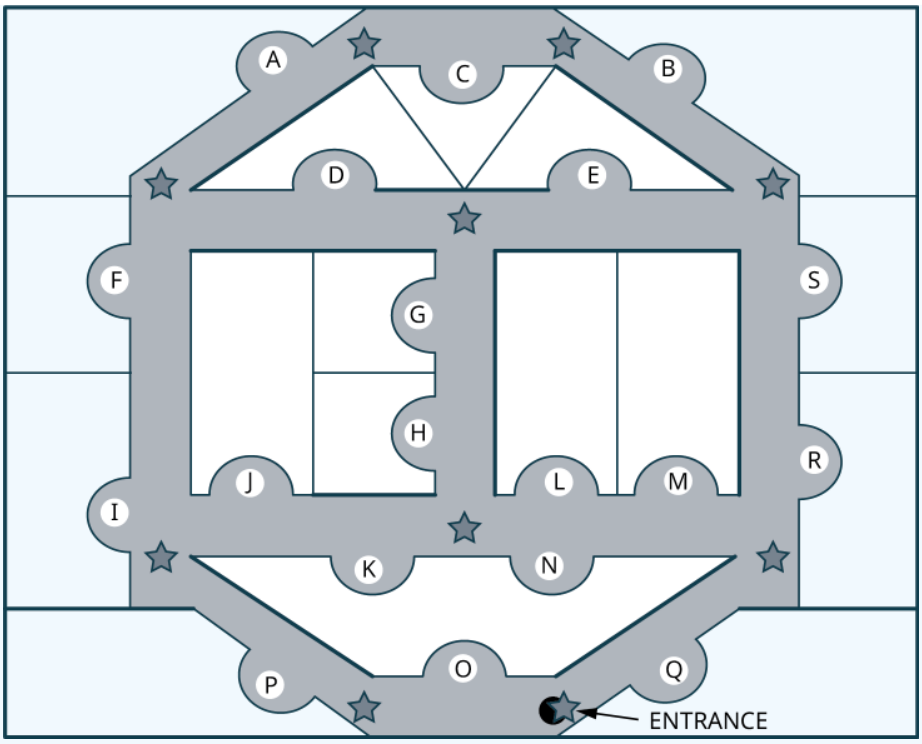
The figure shows a map of zoo exhibits A through P with walkways shown in gray. Draw a graph, or multigraph, to represent the routes through the zoo in which the edges represent walkways and the vertices represent turns and intersections, which are each marked with a star. Notice that there is exactly one exhibit between each pair of adjacent vertices. Label the edges with the corresponding exhibit. Use it to determine if it is possible for a visitor to begin at the entrance, view each exhibit exactly once, and end back at the entrance. If it is possible, give an example of a circuit on the graph that would represent a route the visitor could take. If it is not possible, explain why.



The figure shows the map of the exhibits at an indoor aquarium with hallways shown in gray. Turns and intersections of hallways are marked with stars.

? Exercise 3.6.1.32

Use the Euler circuit theorem and a graph in which the edges represent hallways and the vertices represent turns and intersections to explain why a visitor to the aquarium cannot start at the entrance, visit every exhibit exactly once, and return to the entrance.



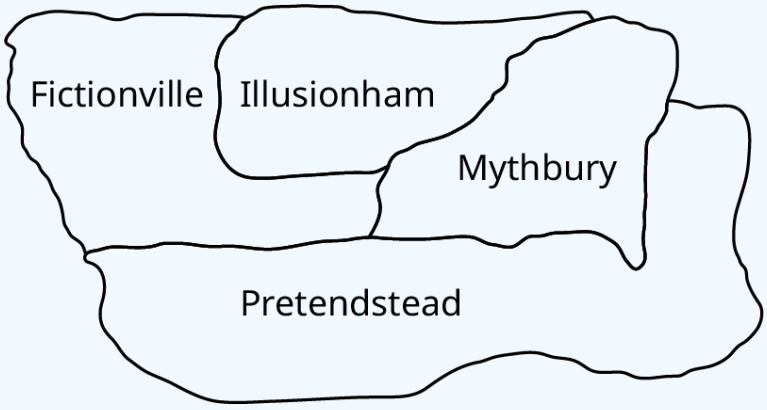
Exercise 3.6.1.33

Use an eulerization of the graph you found to determine the least amount of backtracking necessary to allow a visitor to begin at the entrance, see every exhibit at least once, and return to the entrance. How many hallways must be covered twice? Explain your reasoning.

The map of the states of Imaginaria is given.

Exercise 3.6.1.34

Use a graph to determine if it is possible to travel through Imaginaria crossing each the border between each pair of states exactly once, and returning to the state in which you started.



Exercise 3.6.1.35

Use an eulerization of the graph you found to determine the fewest borders that must be covered twice in order to cross each border at least once and return to the state in which you started. Explain your reasoning.

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3.7: Euler Trails



Figure **3.7.1** The Pony Express mail route spanned from California to Missouri. (credit: "Map of Pony Express" by Nathan Hughes Hamiltonh/Flickr, CC BY 2.0)

Learning Objectives

1. Describe and identify Euler trails.
2. Solve applications using Euler trails theorem.
3. Identify bridges in a graph.
4. Apply Fleury's algorithm.
5. Evaluate Euler trails in real-world applications.

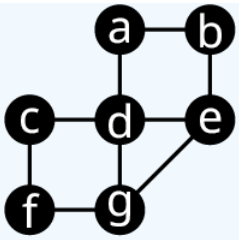
We used Euler circuits to help us solve problems in which we needed a route that started and ended at the same place. In many applications, it is not necessary for the route to end where it began. In other words, we may not be looking at circuits, but trails, like the old Pony Express trail that led from Sacramento, California in the west to St. Joseph, Missouri in the east, never backtracking.

Euler Trails

If we need a trail that visits every edge in a graph, this would be called an Euler trail. Since trails are walks that do not repeat edges, an Euler trail visits every edge exactly once.

Example 3.7.1: Recognizing Euler Trails

Use Figure **3.7.2** to determine if each series of vertices represents a trail, an Euler trail, both, or neither. Explain your reasoning.



Graph H

Figure ✔ 3.7.2
Graph H

1. $a \to b \to e \to g \to f \to c \to d \to e$
2. $a \to b \to e \to g \to f \to c \to d \to e \to b \to a \to d \to g$
3. $g \to d \to a \to b \to e \to d \to c \to f \to g \to e$

Answer

1. It is a trail only. It is a trail because it is a walk that doesn't cover any edges twice, but it is not an Euler trail because it didn't cover edges *ad* or *dg*.
2. It is neither. It is not a trail because it visits *ab* and *be* twice. Since it is not a trail, it cannot be an Euler trail.
3. It is both. It is a trail because it is a walk that doesn't cover any edges twice, and it is an Euler trail because it visits all the edges.

Your Turn 3.7.1

Use the figure to determine if each sequence of vertices represents an Euler trail or not. If not, explain why.

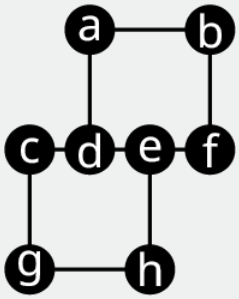


Figure 1 3.7.3: Graph I

1. $e \to f \to b \to a \to d \to c \to g \to h \to e \to d$
2. $d \to a \to b \to f \to e \to h \to g \to c$
3. $d \to a \to b \to e \to f \to b \to e \to d \to c \to g \to h \to e$

The Five Rooms Puzzle

Just as Euler determined that only graphs with vertices of even degree have Euler circuits, he also realized that the only vertices of odd degree in a graph with an Euler trail are the starting and ending vertices. For example, in Figure **3.7.3**, Graph H has exactly two vertices of odd degree, vertex g and vertex e. Notice the Euler trail we saw in Exercise 3 of Example **3.7.1** began at vertex g and ended at vertex e.

This is consistent with what we learned about vertices off odd degree when we were studying Euler circuits. We saw that a vertex of odd degree couldn't exist in an Euler circuit as depicted in Figure **3.7.3**. If it was a starting vertex, at some point we would leave the vertex and not be able to return without repeating an edge. If it was not a starting vertex, at some point we would return and not be able to leave without repeating an edge. Since the starting and ending vertices in an Euler trail are not the same, the start is a

vertex we want to leave without returning, and the end is a vertex we want to return to and never leave. Those two vertices must have odd degree, but the others cannot.

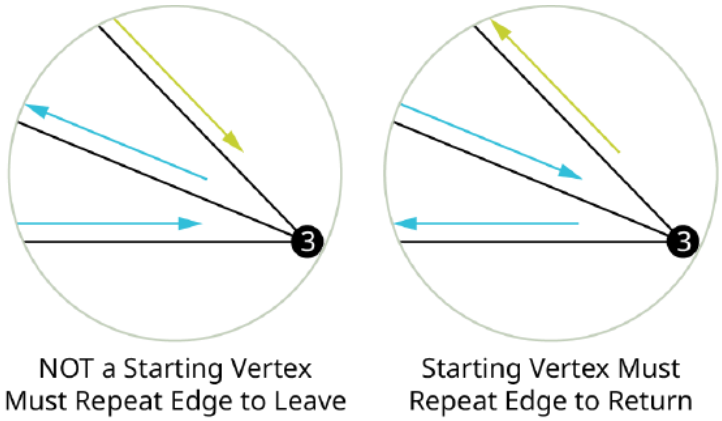


Figure **3.7.4**: A Vertex of Degree 3

Let's use the Euler trail theorem to solve a puzzle so you can amaze your friends! This puzzle is called the "Five Rooms Puzzle." Suppose that you were in a house with five rooms and the exterior. There is a doorway in every shared wall between any two rooms and between any room and the exterior as shown in Figure **3.7.5**. Could you find a route through the house that passes through each doorway exactly once?

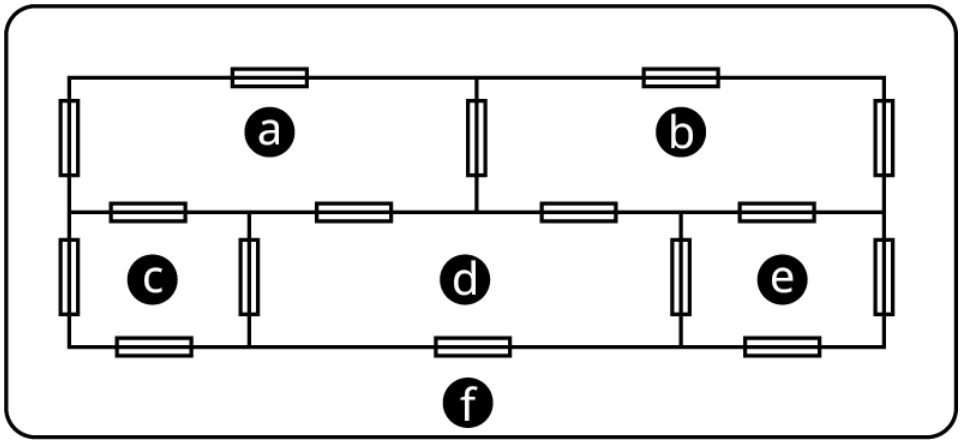


Figure **3.7.5**: Five Rooms Puzzle

Let's represent the puzzle with a graph in which vertices are rooms (or the exterior) and an edge indicates a door between two rooms as shown in Figure **3.7.6**.

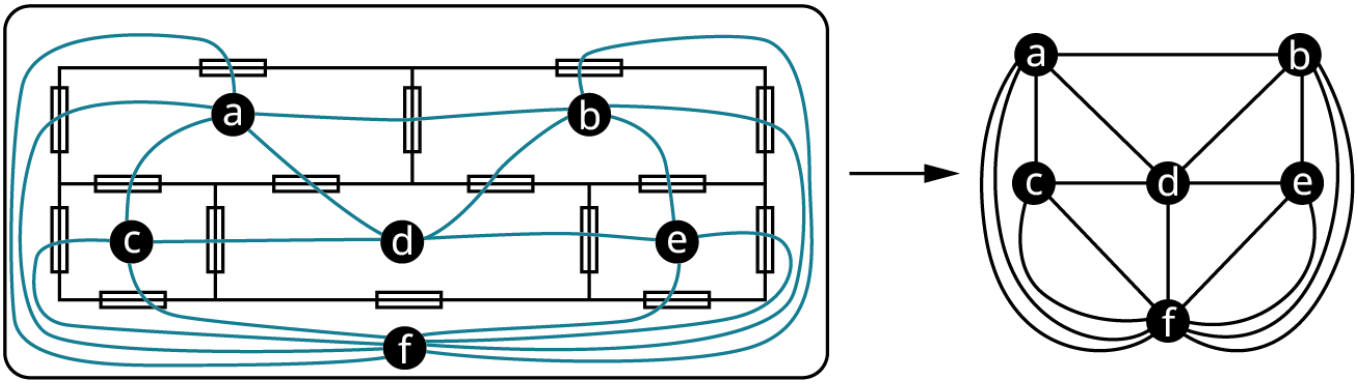


Figure **3.7.6**: Graph of Five Rooms Puzzle

To pass through each doorway exactly once means that we cross every edge in the graph exactly once. Since we have not been asked to start and end at the same position, but to visit each edge exactly once, we are looking for an Euler trail. Let's check the

degrees of the vertices.

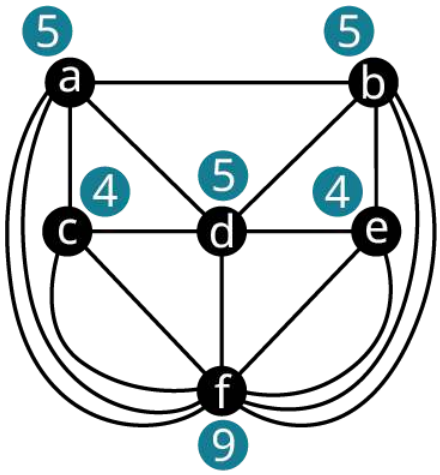


Figure **3.7.7**: Degrees of Vertices in Five Rooms Puzzle

Since there are more than two vertices of odd degree as shown in Figure **3.7.7**, the graph of the five rooms puzzle contains no Euler path. Now you can amaze and astonish your friends!

Bridges and Local Bridges

Now that we know which graphs have Euler trails, let's work on a method to find them. The method we will use involves identifying bridges in our graphs. A bridge is an edge which, if removed, increases the number of components in a graph. Bridges are often referred to as cut-edges. In Figure **3.7.8**, there are several examples of bridges. Notice that an edge that is not part of a cycle is always a bridge, and an edge that is part of a cycle is never a bridge.

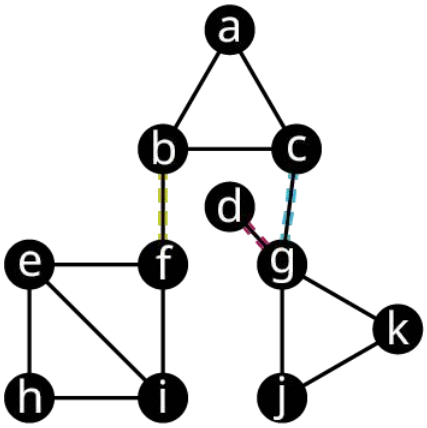


Figure **3.7.8**: Graph with Bridges

Edges *bf*, *cg*, and *dg* are "bridges"

The graph in Figure **3.7.8** is connected, which means it has exactly one component. Each time we remove one of the bridges from the graph the number of components increases by one as shown in Figure **3.7.9**. If we remove all three, the resulting graph in Figure 12.157 has four components.

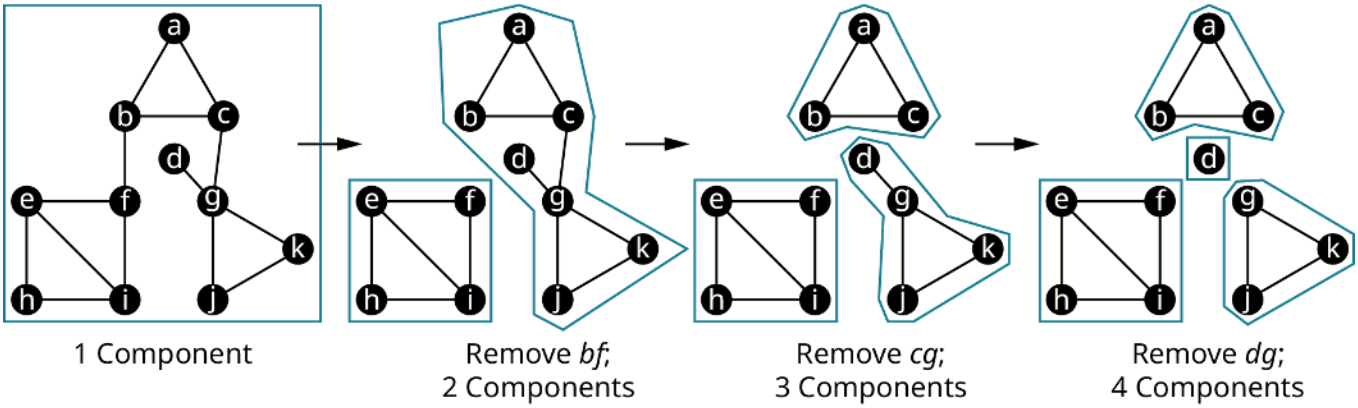


Figure **3.7.9** Removing a Bridge Increases Number of Components

In sociology, bridges are a key part of social network analysis. Sociologists study two kinds of bridges: local bridges and regular bridges. Regular bridges are defined the same in sociology as in graph theory, but they are unusual when studying a large social network because it is very unlikely a group of individuals in a large social network has only one link to the rest of the network. On the other hand, a local bridge occurs much more frequently. A local bridge is a friendship between two individuals who have no other friends in common. If they lose touch, there is no single individual who can pass information between them. In graph theory, a local bridge is an edge between two vertices, which, when removed, increases the length of the shortest path between its vertices to more than two edges. In Figure 12.158, a local bridge between vertices b and e has been removed. As a result, the shortest path between b and e is $b \to i \to j \to k \to e$, which is four edges. On the other hand, if edge ab were removed, then there are still paths between a and b that cover only two edges, like $a \to i \to b$.

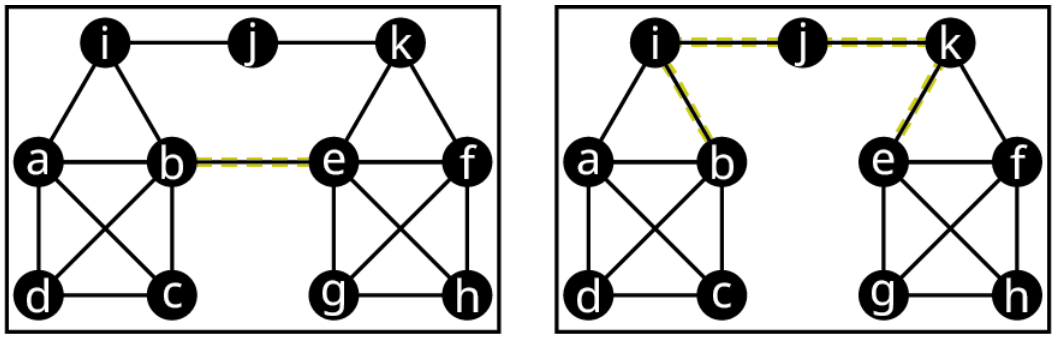


Figure **3.7.10**: Removing a Local Bridge

The significance of a local bridge in sociology is that it is the shortest communication route between two groups of people. If the local bridge is removed, the flow of information from one group to another becomes more difficult. Let's say that vertex b is Brielle and vertex e is Ella. Now, Brielle is less likely to hear about things like job opportunities, that Ella many know about. This is likely to impact Brielle as well as the friends of Brielle.

✔ Example 3.7.2: Identifying Bridges and Local Bridges

Use the graph of a social network in Figure ✔ **3.7.11** to answer each question.

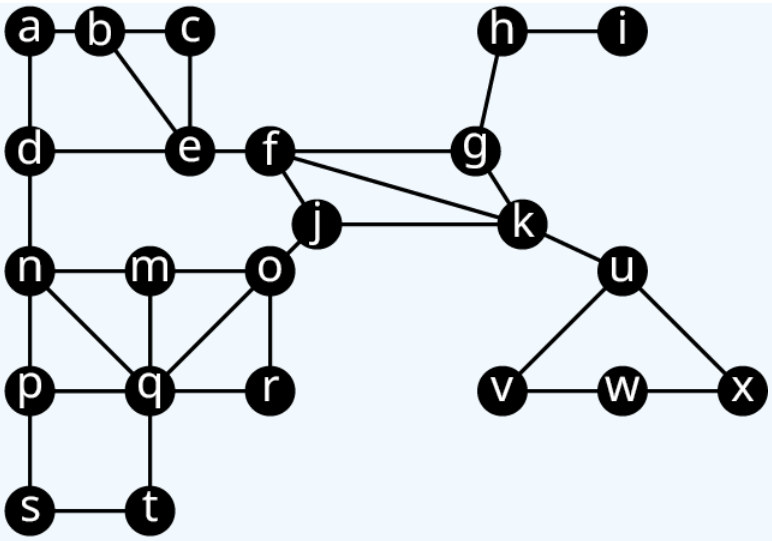


Figure ✔ **3.7.11** Graph of a Social Network

1. Identify any bridges.
2. If all bridges were removed, how many components would there be in the resulting graph?
3. Identify one local bridge.
4. For the local bridge you identified in part 3, identify the shortest path between the vertices of the local bridge if the local bridge were removed.

Answer

1. The edges *ku*, *gh*, and *hi* are bridges.
2. If the bridges were all removed, there would be four components in the resulting graph, $\{i\}$, $\{h\}$, $\{u, v, w, x\}$, and $\{a, b, c, d, e, f, g, j, k, m, n, o, p, q, r, s, t\}$ as shown in Figure 12.160.

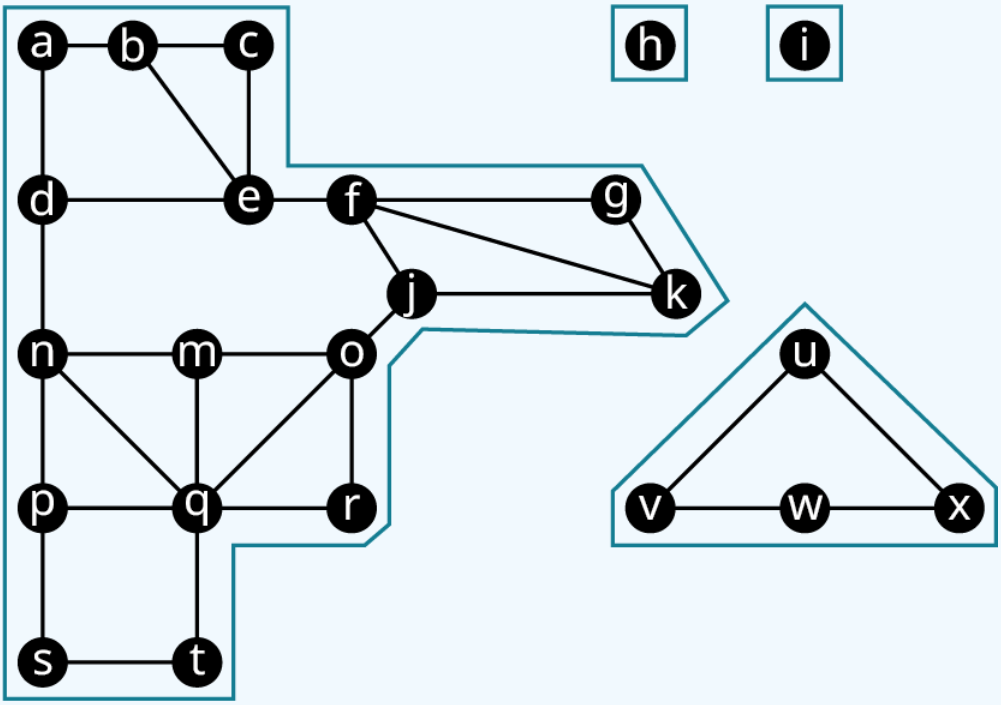


Figure ✔ **3.7.12** Graph of Social Network without Bridges

3. Three local bridges are *dn*, *ef*, and *qt*, among others.
4. If *dn* were removed, the shortest path between *d* and *n* would be $d \to e \to f \to j \to o \to m \to n$.

Your Turn 3.7.2

How many bridges and local bridges are in a complete graph with three or more vertices? Explain your reasoning.

Video



Bridges and Local Bridges in Graph Theory

Finding an Euler Trail with Fleury's Algorithm

Now that we are familiar with bridges, we can use a technique called Fleury's algorithm, which is a series of steps, or algorithm, used to find an Euler trail in any graph that has exactly two vertices of odd degree. Here are the steps involved in applying Fleury's algorithm.

Here are the steps involved in applying Fleury's algorithm.

Step 1: Begin at either of the two vertices of odd degree.

Step 2: Remove an edge between the vertex and any adjacent vertex that is NOT a bridge, unless there is no other choice, making a note of the edge you removed. Repeat this step until all edges are removed.

Step 3: Write out the Euler trail using the sequence of vertices and edges that you found. For example, if you removed *ab*, *bc*, *cd*, *de*, and *ef*, in that order, then the Euler trail is $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f$.

Figure 3.7.13 shows the steps to find an Euler trail in a graph using Fleury's algorithm.

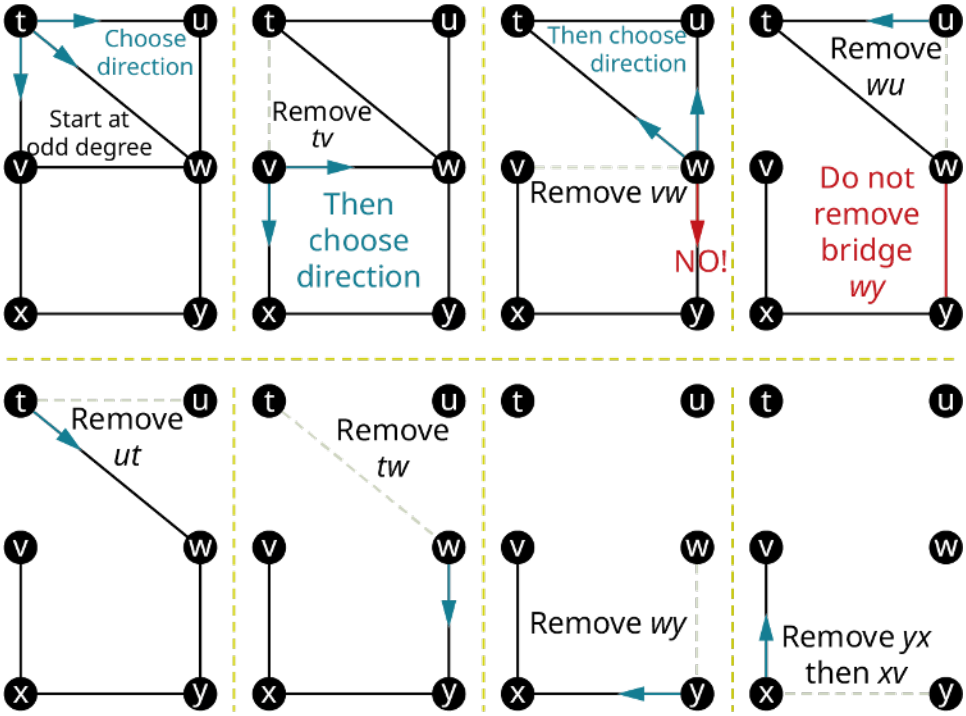
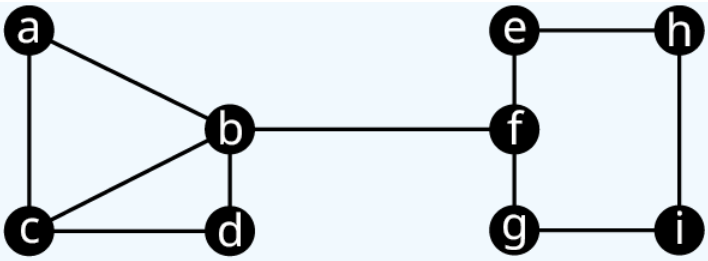


Figure **3.7.13** Using Fleury's Algorithm To Find Euler Trail

The Euler trail that was found in Figure **3.7.13** is $t \to v \to w \to u \to t \to w \to y \to x \to v$.

✔ Example **3.7.3**: Finding an Euler Trail with Fleury's Algorithm

Use Fleury's Algorithm to find an Euler trail for Graph J in Figure ✔ **3.7.14**



Graph *J*

Figure ✔ 3.7.14: Graph *J*

Answer

Step 1: Choose one of the two vertices of odd degree, *c* or *f*, as your starting vertex. We will choose *c*.

Step 2: Remove edge *ca*, *cb*, or *cd*. None of these are cut edges so we can select any of the three. We will choose *cb* as shown in Figure ✔ 3.7.15 to be the first edge removed.

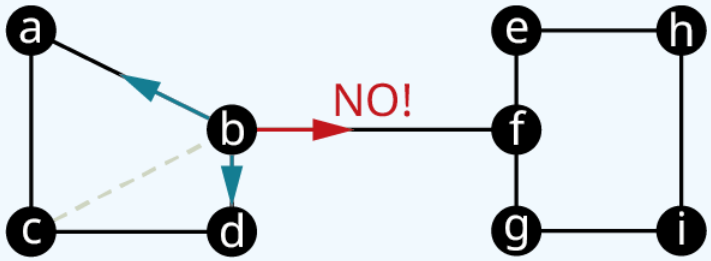


Figure ✔ 3.7.15 Graph *J* with *cb* Removed

Repeat Step 2 The next choice is to remove edge *ba*, *bd*, or *bf* as shown in Figure ✔ 3.7.16, but *bf* is not an option since it is a bridge. We will choose *ba* as shown in Figure 12.164 to be the second edge removed.

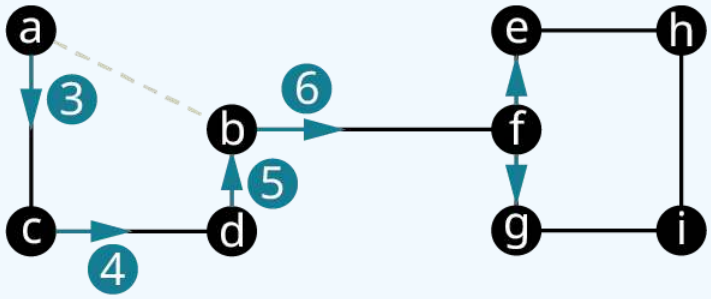


Figure ✔ 3.7.16 Graph *J* with *cb* and *ba* Removed

Repeat Step 2 for the third, fourth, fifth, sixth, and seventh edges. As shown in Figure ✔ 3.7.16, until we get to the seventh edge there is only one option each time, *ac*, *cd*, *db*, and *bf* in that order. For the seventh edge, we must choose between *fe* and *fg*. Neither of these are bridges. We choose *fe*. Figure ✔ 3.7.17 shows that *ac*, *cd*, *db*, *bf*, and *fe* have been removed.

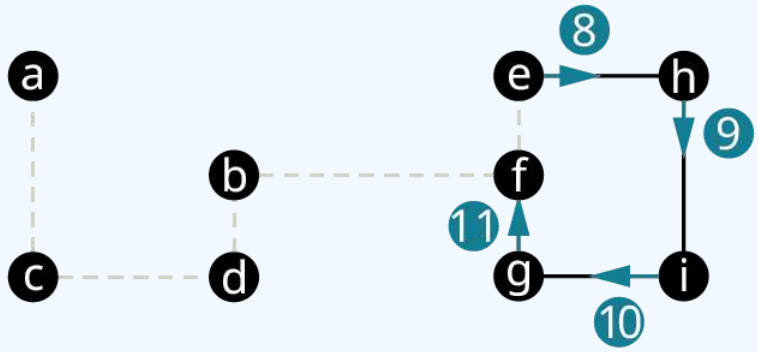


Figure ✔ 3.7.17 Graph *J* with Seven Edges Removed

Step 3: Write out the Euler trail using the vertices in the sequence that the edges were removed. We removed *cb*, *ba*, *ac*, *cd*, *db*, *bf*, *fe*, *eh*, *hi*, *ig*, and *gf*, in that order. The Euler trail is $c \to b \to a \to c \to d \to b \to f \to e \to h \to i \to g \to f$.Checkpoint

TIP! To avoid errors, count the number of edges in your graph and make sure thatyour Euler trail represents that number of edges.

Your Turn 3.7.3

Use Graph L to fill in the blanks to complete the steps in Fleury's algorithm.

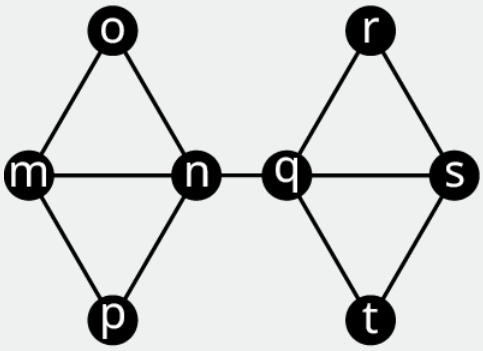


Figure 3.7.18: Graph L

1. The two vertices that can be used as the starting vertex are ____ and s.
2. If sq is the first edge removed, the three options for the second edge to be removed are qr, ___, and ___; however, ___ cannot be chosen because it is a _____.
3. If qr is the second edge removed, the next four edges to be removed must be ___, ___, ___, and ___, in that order.
4. After qn is removed, the three options for the next edge to be removed are no, ___, and ___, however, ___ cannot be chosen because it is a _____
5. If no is the next edge removed, the last four edges removed will be ___, ___, ___, and ___, in that order.
6. The final Euler trail using the answers to parts 1 through 5 is _____.

In the previous section, we found Euler circuits using an algorithm that involved joining circuits together into one large circuit. You can also use Fleury's algorithm to find Euler circuits in any graph with vertices of all even degree. In that case, you can start at any vertex that you would like to use.

Step 1: Begin at any vertex.

Step 2: Remove an edge between the vertex and any adjacent vertex that is NOT a bridge, unless there is no other choice, making a note of the edge you removed. Repeat this step until all edges are removed.

Step 3: Write out the Euler circuit using the sequence of vertices and edges that you found. For example, if you removed ab, bc, cd, de, and ea, in that order, then the Euler circuit is $a \to b \to c \to d \to e \to a$.

Video



Fluery's Algorithm to Find an Euler Circuit

Checkpoint

IMPORTANT! Since a circuit is a closed trail, every Euler circuit is also an Euler trail, but when we say Euler trail in this chapter, we are referring to an open Euler trail that begins and ends at different vertices.

Example 3.7.4: Finding an Euler Circuit or Euler Trail Using Fleury's Algorithm

Use Fleury's algorithm to find either an Euler circuit or Euler trail in Graph *G* in Figure ✔ 3.7.19.

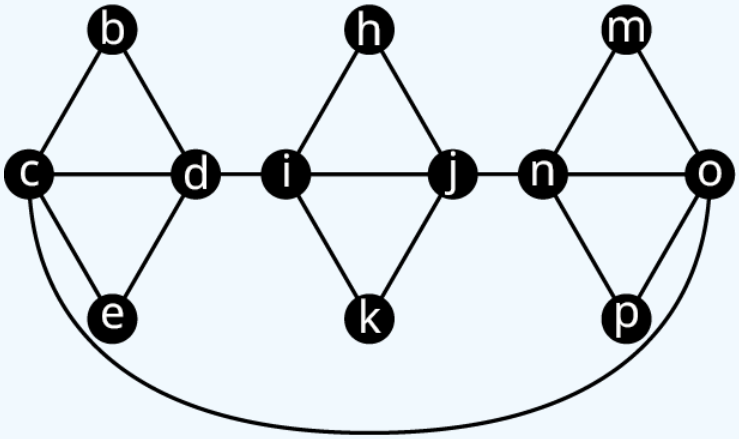


Figure ✔ 3.7.19: Graph *G*

Answer

Graph *G* has all vertices of even degree so it has an Euler circuit.

Step 1: Choose any vertex. We will choose vertex *j*.

Step 2: Remove one of the four edges that meet at vertex *j*. Since *jn* is a bridge, we must remove either *jh*, *ji*, or *jk*. We remove *ji* as shown in Figure 3.7.20.

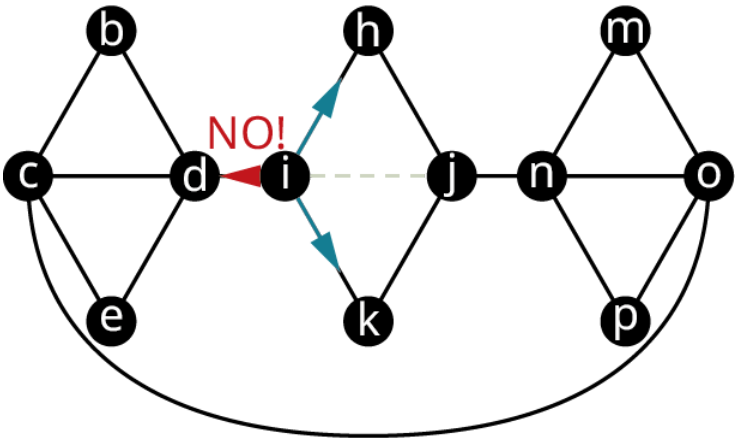


Figure 3.7.20 Graph *G* with 1 Edge Removed

Repeat Step 2: Since *id* is a bridge, we can remove either *ih* or *ik* next. We remove *ih*, and then the only option is to remove *hj* as shown in Figure 3.7.21.

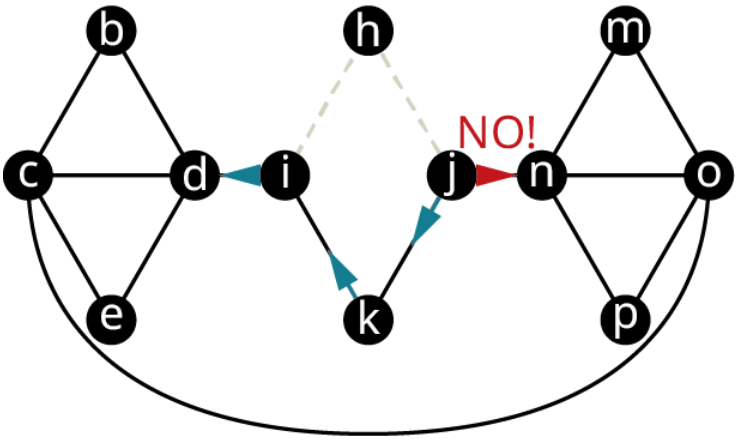


Figure 3.7.21 Graph *G* with 3 Edges Removed

Repeat Step 2: Since *jn* is a bridge, the next edge removed must be *jk*, and then the only option is to remove *ki* followed by *id* as shown in Figure 3.7.21. Even though *id* is a bridge, it can be removed because it is the only option at this point. Figure 3.7.22 shows Graph *G* with these additional edges removed.

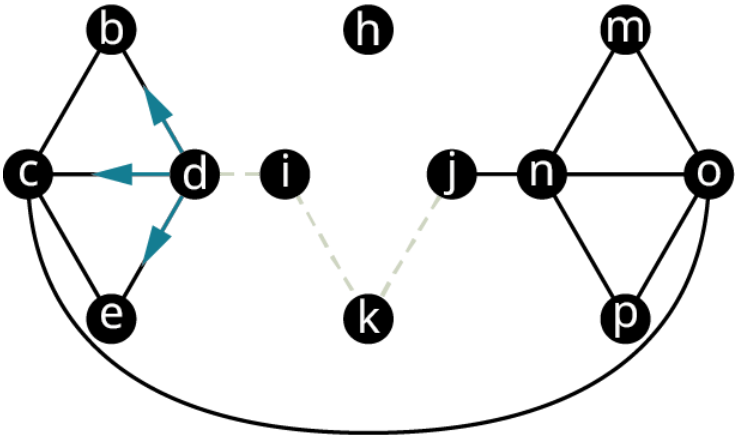


Figure 3.7.22 Graph *G* with 6 Edges Removed

Repeat Step 2: Choose any one of the edges *db*, *dc*, or *de*. We remove *dc* as shown in Figure 3.7.23.

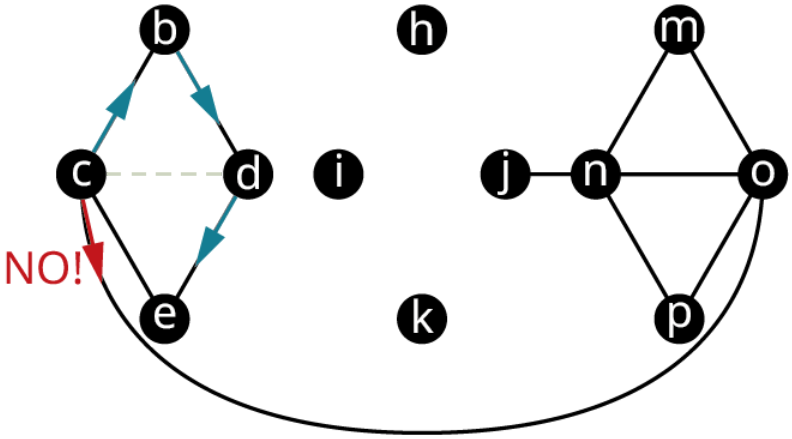


Figure 3.7.23 Graph *G* with 7 Edges Removed

Repeat Step 2: Since *co* is a bridge, choose *cb* next. We remove *cb*, then *bd*, and then *de* as shown in Figure 3.7.24.

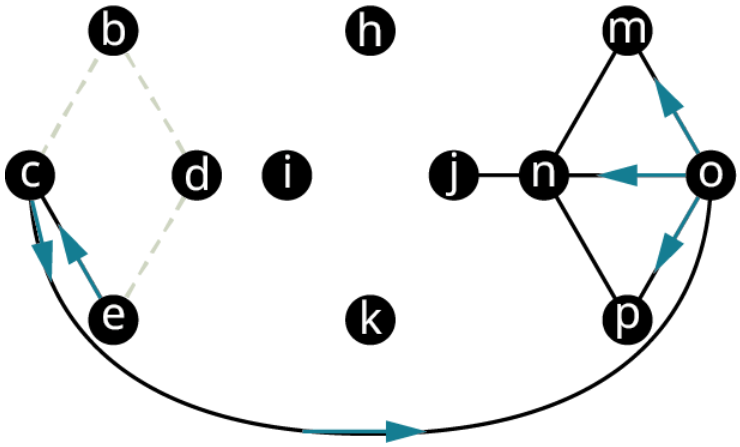


Figure **3.7.24** Graph *G* with 10 Edges Removed

Repeat Step 2: Next, remove *ec* and *co*. Then choose any of *op*, *pn*, or *om*. We remove on as shown in Figure **3.7.25**.

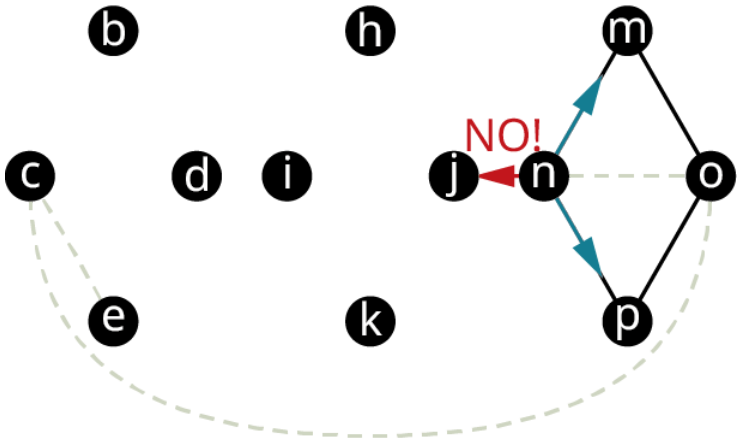


Figure **3.7.26** Graph *G* with 13 Edges Removed

Repeat Step 2: Next, remove either *nm*, *np*, or *nj*, but *nj* is a So, we remove *nm* as shown in Figure **3.7.27**.

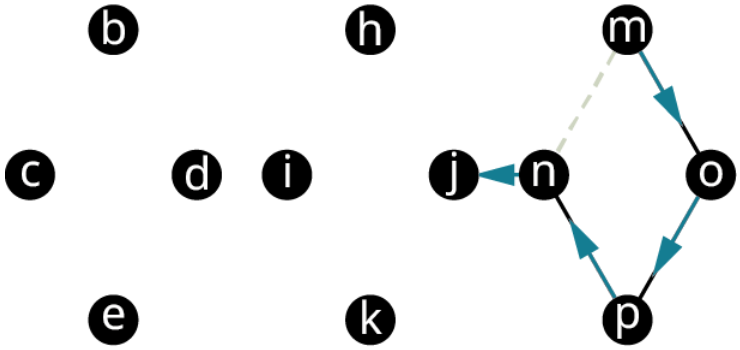


Figure **3.7.27** Graph *G* with 14 Edges Removed

Repeat Step 2: Next, remove *mo*, *op*, *pn*, and *nj*. And we are done!

Step 3: Notice that the algorithm brought us back to the vertex where we started, forming an Euler circuit. Write out the Euler circuit:

$j \to i \to h \to j \to k \to i \to d \to c \to b \to d \to e \to c \to o \to n \to m \to o \to p \to n \to j$

Your Turn 3.7.4

Find an Euler circuit or trail through the graph using Fleury's algorithm.

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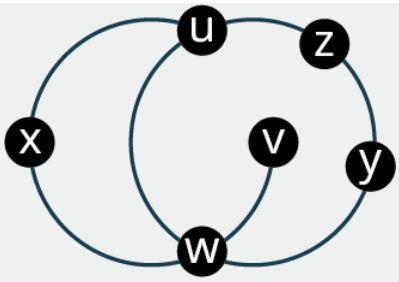


Figure 3.7.28: Graph T

WORK IT OUT

We have discussed a lot of subtle concepts in this section. Let's make sure we are all on the same page. Work with a partner to explain why each of the following facts about bridges are true. Support your explanations with definitions and graphs.

1. When a bridge is removed from a graph, the number of components increases.
2. A bridge is never part of a circuit.
3. An edge that is part of a triangle is never a local bridge.

Check Your Understanding

Fill in the blank to make the statement true.

1. An Euler trail is a trail that visits each _____ exactly once.

2. _____ algorithm is a procedure for finding an Euler trail or circuit.

3. An Euler _____ always begins and ends at the same vertex, but an Euler _____ does not.

4. When a bridge is removed from a graph, the number of _____ is increased by one.

5. When a _____ is removed from a graph, the shortest path between its vertices will be greater than two.

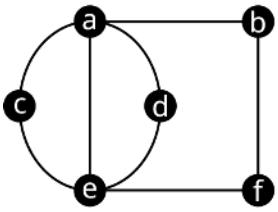
6. When using Fleury's algorithm to find an Euler trail, never remove a _____ unless it is the only option.

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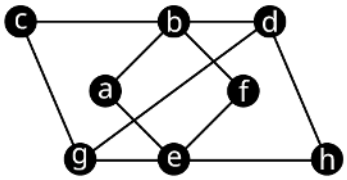
- **12.6: Euler Trails** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/contemporary-mathematics.

3.7.1: Exercises

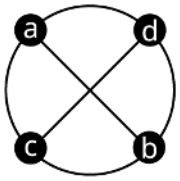
Use the figure to answer the following exercises. Identify the graph(s) with the given characteristics, if any.



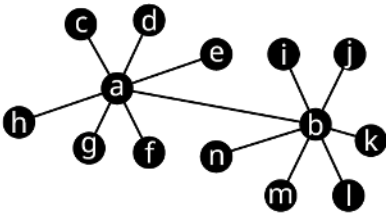
Graph *T*



Graph *U*



Graph *V*



Graph *W*

? Exercise 3.7.1.1

Connected

? Exercise 3.7.1.2

All vertices of even degree

? Exercise 3.7.1.3

Exactly two vertices of odd degree

? Exercise 3.7.1.4

Has an Euler trail

? Exercise 3.7.1.5

Has an Euler circuit

? Exercise 3.7.1.6

Has neither an Euler trail nor an Euler circuit

? Exercise 3.7.1.7

ab is a bridge

? Exercise 3.7.1.8

ef is a bridge

? Exercise 3.7.1.9

ab is a local bridge

Exercise 3.7.1.10

ef is a local bridge

Use the figure to answer the following exercises. In each exercise, a graph and a sequence of vertices are given. Determine whether each sequence of vertices is an Euler trail, an Euler circuit, or neither for the graph. If it is neither, explain why.

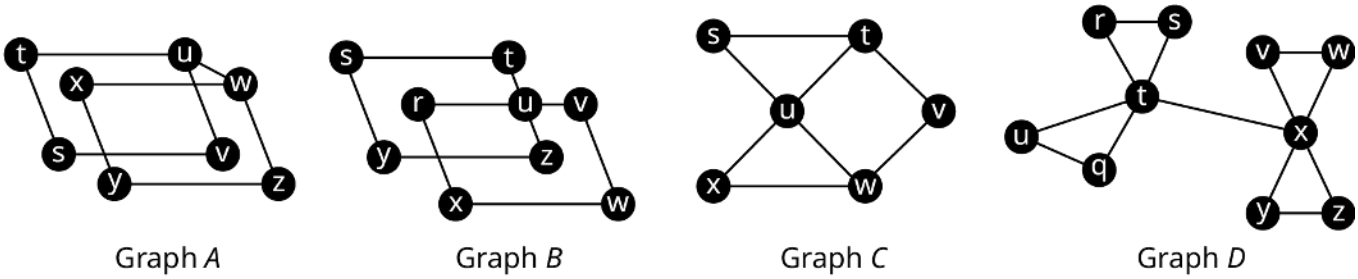


Figure 12.176

Exercise 3.7.1.11

Graph *A*, $w \to x \to y \to z \to w \to u \to t \to s \to v \to u$

Exercise 3.7.1.12

Graph *A*, $u \to v \to s \to t \to u \to w \to z \to y \to x \to w$

Exercise 3.7.1.13

Graph *A*, $s \to t \to u \to v \to u \to w \to z \to y \to x \to w$

Exercise 3.7.1.14

Graph *A*, $w \to x \to y \to z \to w \to v \to u \to t \to s \to v$

Exercise 3.7.1.15

Graph *B*, $u \to v \to w \to x \to r \to u \to t \to s \to y \to z \to u$

Exercise 3.7.1.16

Graph *B*, $v \to w \to x \to r \to u \to z \to y \to s \to t \to u$

Exercise 3.7.1.17

Graph *C*, $s \to t \to u \to v \to w \to x \to s$

Exercise 3.7.1.18

Graph *C*, $t \to u \to x \to w \to u \to s \to t \to v \to w$

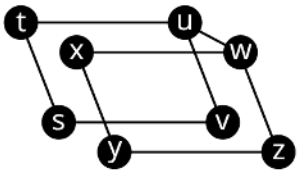
Exercise 3.7.1.19

Graph *D*, $t \to r \to s \to t \to u \to v \to t \to x \to v \to w \to x \to y \to z \to x$

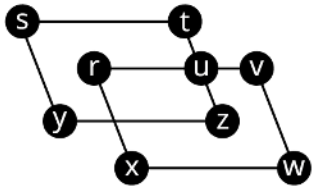
Exercise 3.7.1.20

Graph D, $x \to v \to w \to x \to y \to z \to x \to t \to r \to s \to t \to u \to v \to t$

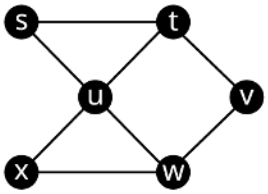
Use the figure to answer the following exercises. For each graph, identify a bridge if one exists. If it does not, state so. If it does, identify any components that are created when the bridge is removed.



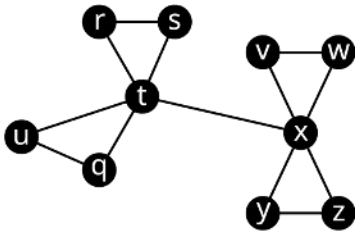
Graph A



Graph B



Graph C



Graph D

Exercise 3.7.1.21

Graph A

Exercise 3.7.1.22

Graph B

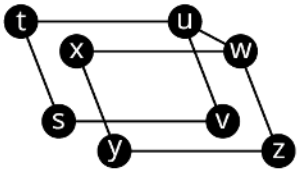
Exercise 3.7.1.23

Graph C

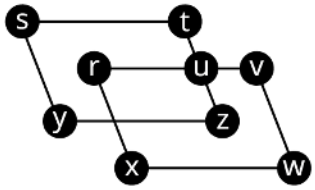
Exercise 3.7.1.24

Graph D

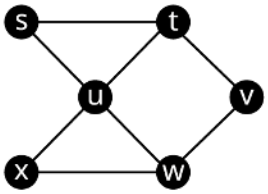
Use the figuree to answer the following exercises. For each graph, identify a local bridge if one exists. If it does not, state so. If it does, find a shortest path between the vertices of the local bridge if the local bridge is removed.



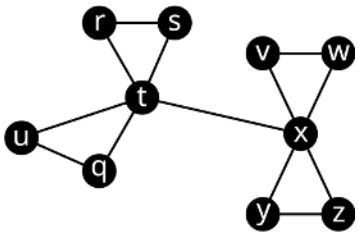
Graph A



Graph B



Graph C



Graph D

Exercise 3.7.1.25

Graph A

Exercise 3.7.1.26

Graph B

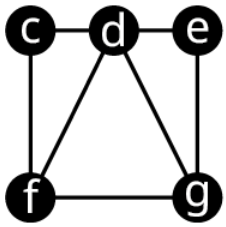
Exercise 3.7.1.27

Graph *C*

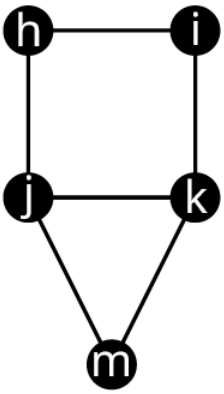
Exercise 3.7.1.28

Graph *D*

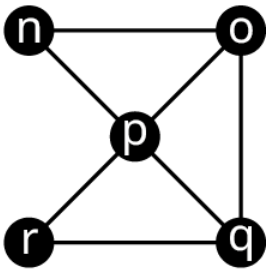
Use the graphs to answer the following exercises. In each exercise, a graph is given. Find two Euler trails in each graph using Fleury's algorithm.



Graph *Q*



Graph *R*



Graph *S*

Exercise 3.7.1.29

Graph *Q*

Exercise 3.7.1.30

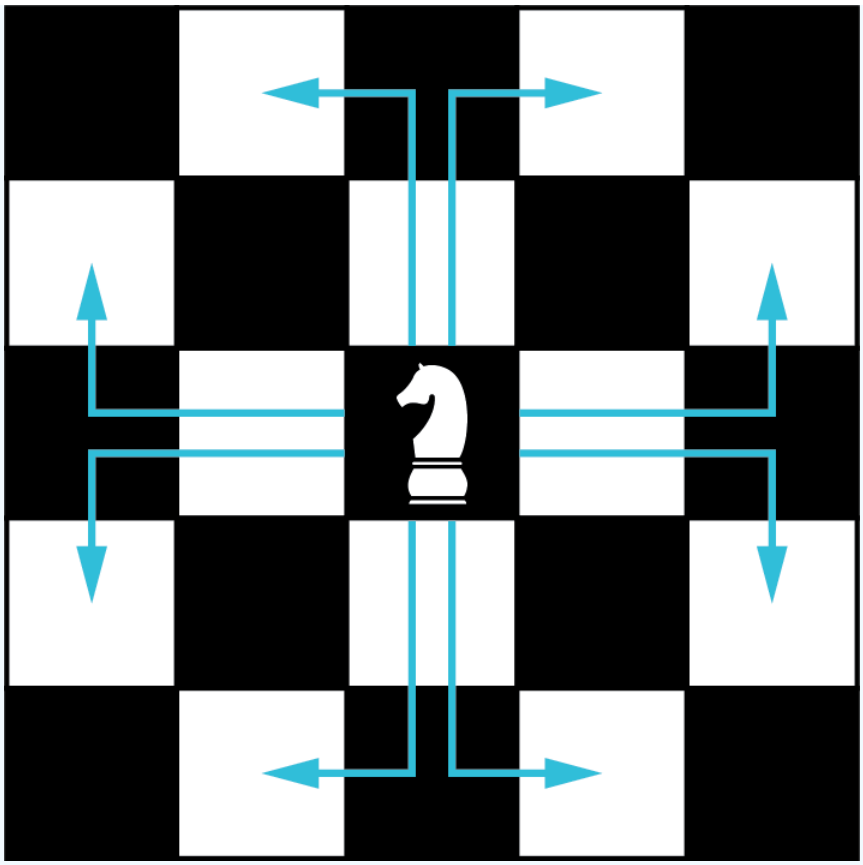
Graph *R*

Exercise 3.7.1.31

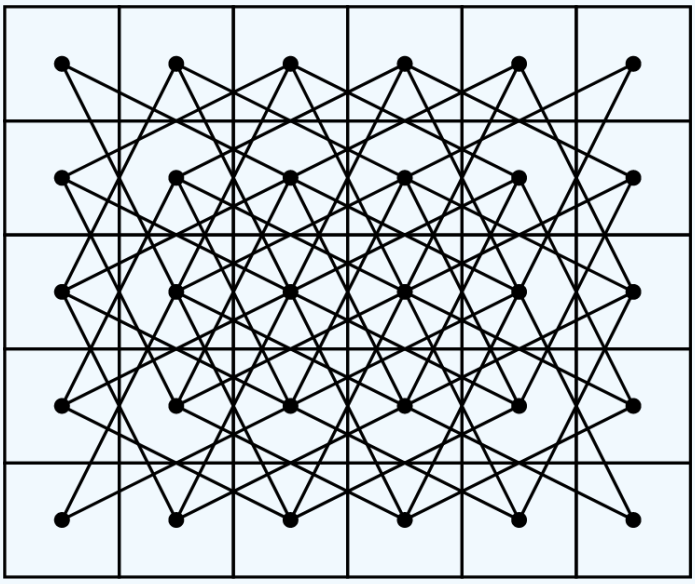
Graph *S*

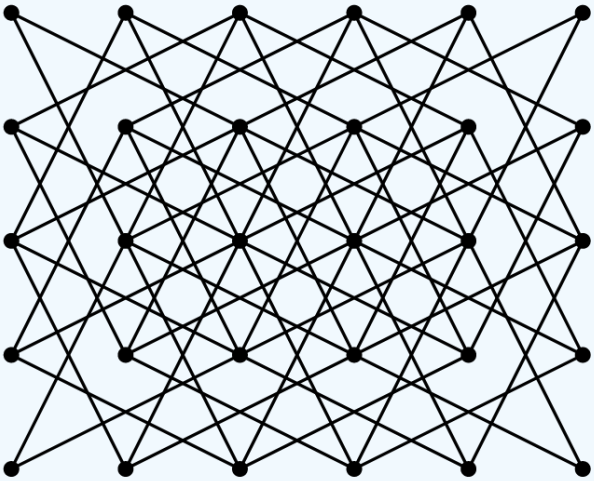
Exercise 3.7.1.32

In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The eight possible moves a knight can make from a given space are shown in the figure.

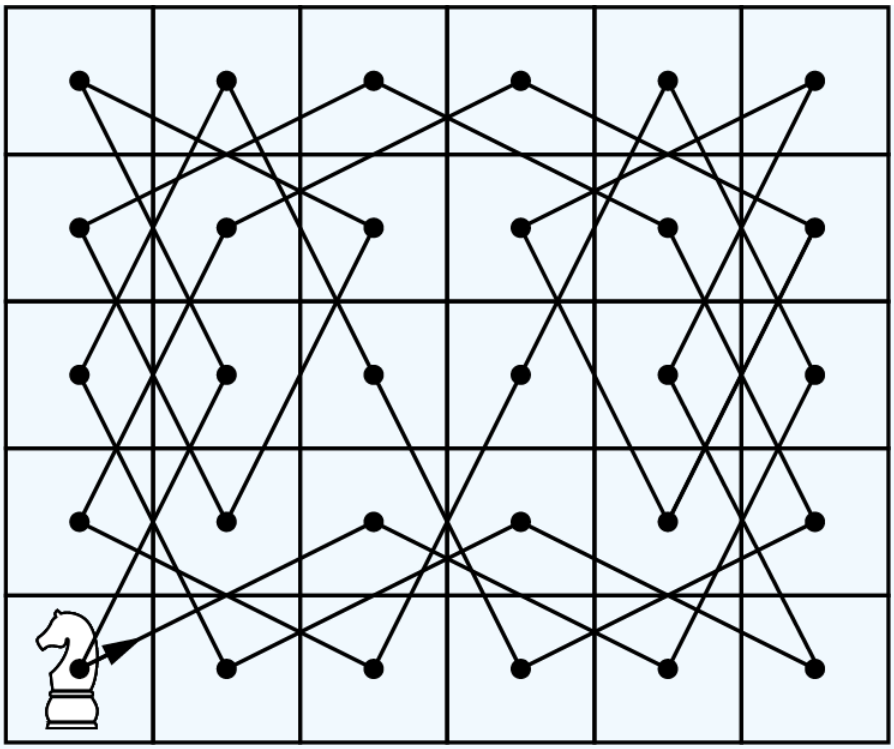


A graph in which each vertex represents a space on a five-by-six game board and each edge represents a move a knight could make is shown in the figure.



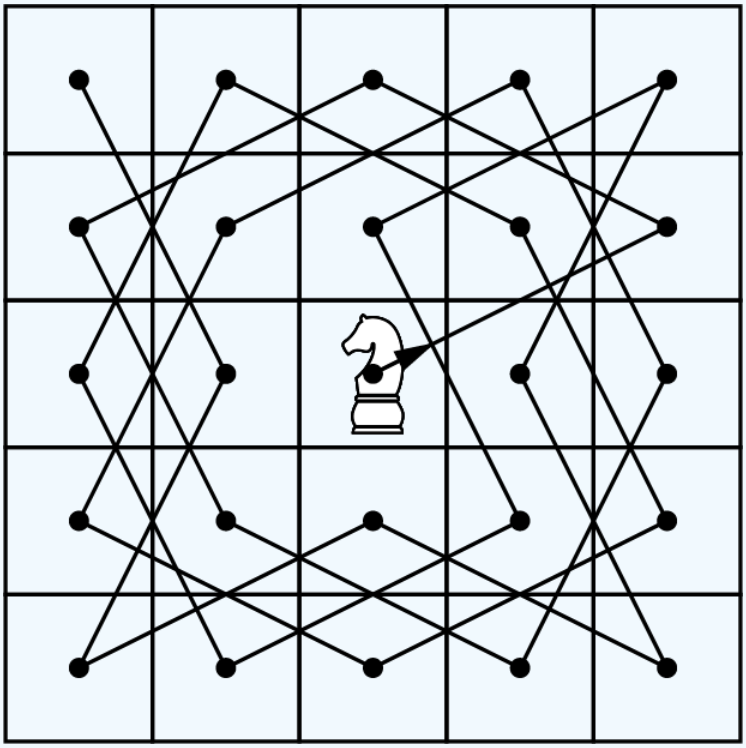


A knight's tour is a sequence of moves by a knight on a chessboard (of any size) such that the knight visits every square exactly once. If the knight's tour brings the knight back to its starting position on the board, it is called a closed knight's tour. Otherwise, it is called an open knight's tour. Determine if the closed knight's tour in the figure is most accurately described as a trail, a circuit, an Euler trail, or an Euler circuit of the graph of all possible knight moves. Explain your reasoning.



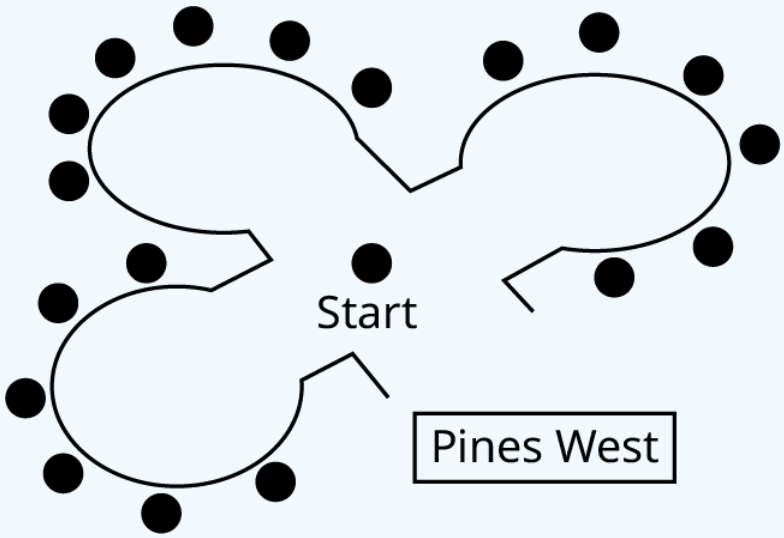
Exercise 3.7.1.33

Determine if the open knight's tour in the figure is most accurately described as a trail, a circuit, an Euler trail, or an Euler circuit of the graph of all possible knight moves on a five-by-five game board. Explain your reasoning.



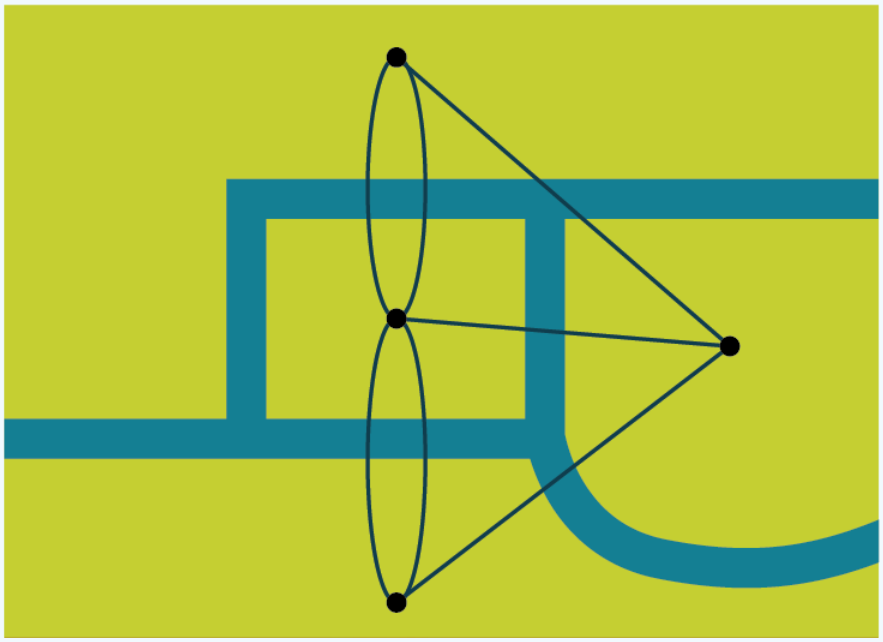
Exercise 3.7.1.34

The neighborhood of Pines West has three cul-de-sacs that meet at an intersection as shown in the figure. A postal delivery person starts at the intersection and visits each house in a cul-de-sac once, returns to the intersection, visits each house in the next cul-de-sac, and so on, returning to the intersection when finished. Describe how the route can be represented as a graph. If there is no backtracking, in other words, the person never reverses direction, is the route followed by the postal delivery person best described as a trail, a circuit, an Euler trail, or an Euler circuit? Explain your reasoning.



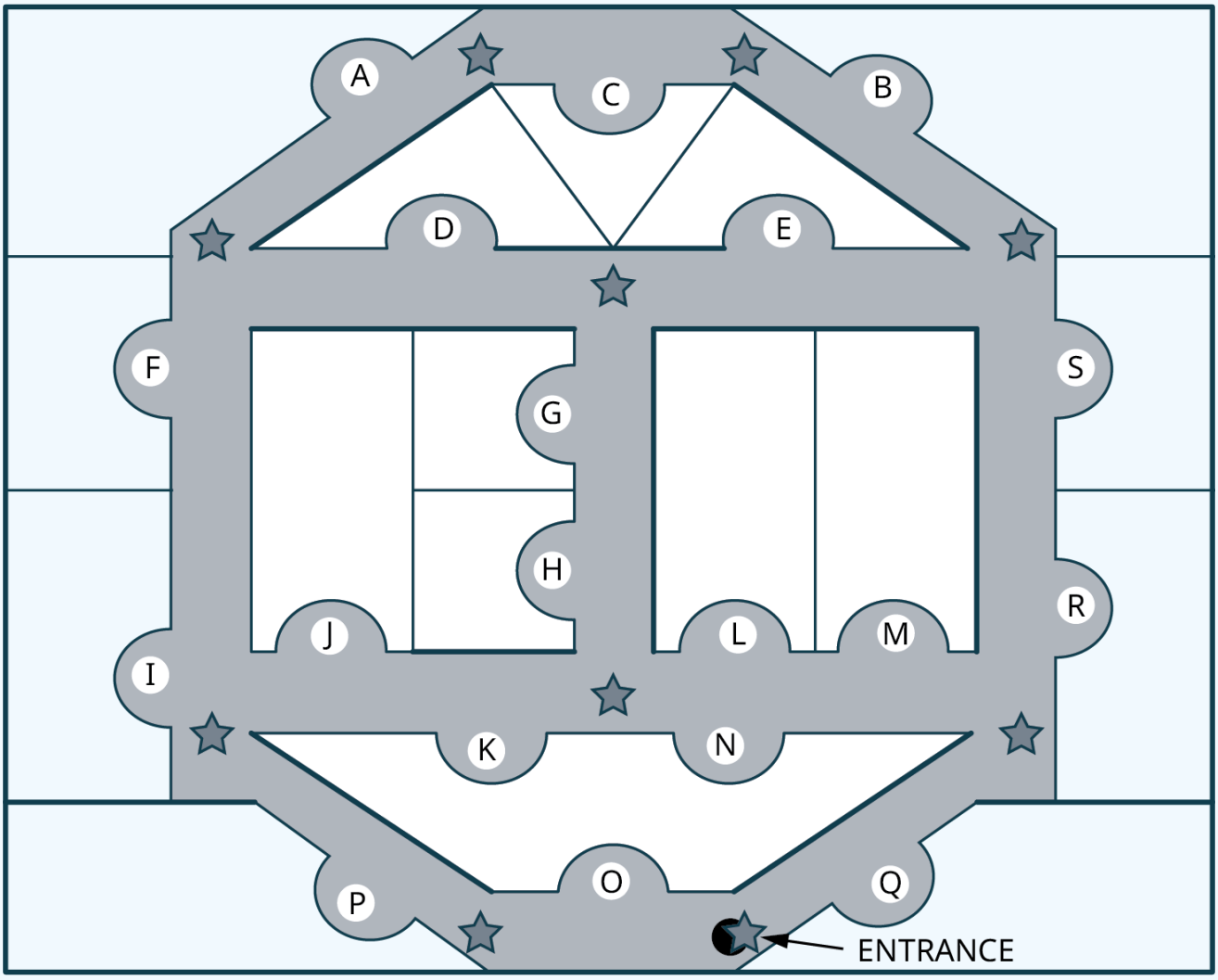
Exercise 3.7.1.35

Recall that the bridges of Konigsberg can be represented as a multigraph as shown in the figure. We have seen that no route through Konigsberg passes over each bridge exactly once and returns to the starting point. Is there a route that passes over each bridge exactly once but does not begin and end at the same point? Explain your reasoning.



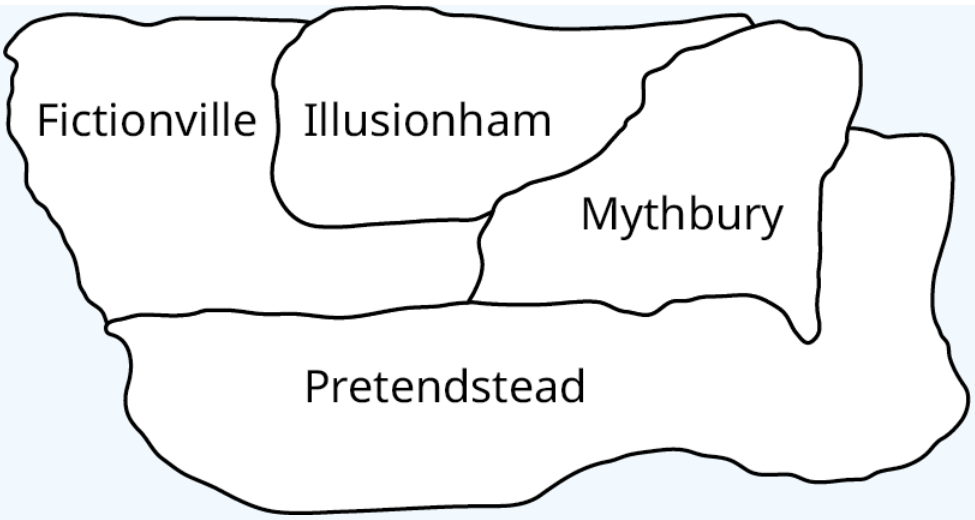
Exercise 3.7.1.36

The figure shows the map of the exhibits at an indoor aquarium. Use a graph in which the edges represent hallways and the vertices represent turns and intersections to explain why a visitor to the aquarium cannot start at one of the turns or intersections, passes by every exhibit exactly once, and end at one of the turns or intersections.



Exercise 3.7.1.37

The map of the states of Imaginaria is given. Use a graph to determine if it is possible to begin in one state, travel through Imaginaria crossing the border between each pair of states exactly once, and end in a different state. If it is possible, find such a route. If it is not, explain why.



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3.8: Hamilton Cycles

Learning Objectives

1. Describe and identify Hamilton cycles.
2. Compute the number of Hamilton cycles in a complete graph.
3. Apply and evaluate weighted graphs.

In Euler Circuits and Euler Trails, we looked for circuits and paths that visited each *edge* of a graph exactly once. In this section, we will look for circuits that visit each *vertex* exactly once. Like many concepts in graph theory, the idea of a circuit that visits each vertex once was inspired by games and puzzles. As early as the 9th century, Indian and Islamic intellectuals wondered whether it was possible for a knight to visit every space on a chess board of a given size, which is equivalent to visiting every vertex of a graph that represents the chess board.

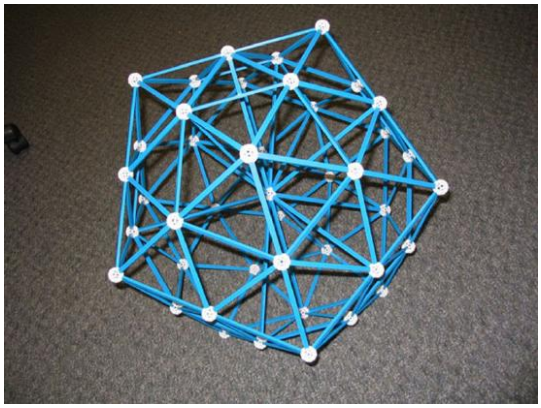


Figure **3.8.1**: The symmetries of an icosahedron, with 30 edges, 20 faces, and 12 vertices, can be analyzed using graph theory. (credit: "Really big icosahedron" by Clayton Shonkwiler/Wikimedia, CC BY 2.0)

In 1857, a mathematician named William Rowan Hamilton invented a puzzle in which players were asked to find a route along the edges of a dodecahedron (see Figure 12.178), which visited every vertex exactly once. Let's explore how graph theory provides insight into these games as well as practical applications such as the Traveling Salesperson Problem.

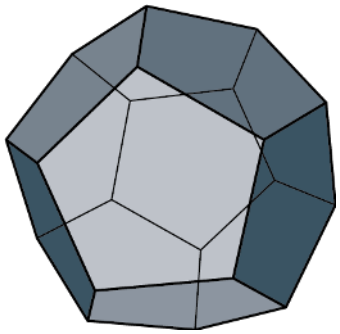


Figure **3.8.2**: A Dodecahedron

Hamilton's Puzzle

Before we look at the solution to Hamilton's puzzle, let's review some vocabulary we used in Figure **3.8.3** . It will be helpful to remember that directed cycle is a type of circuit that doesn't repeat any edges or vertices.

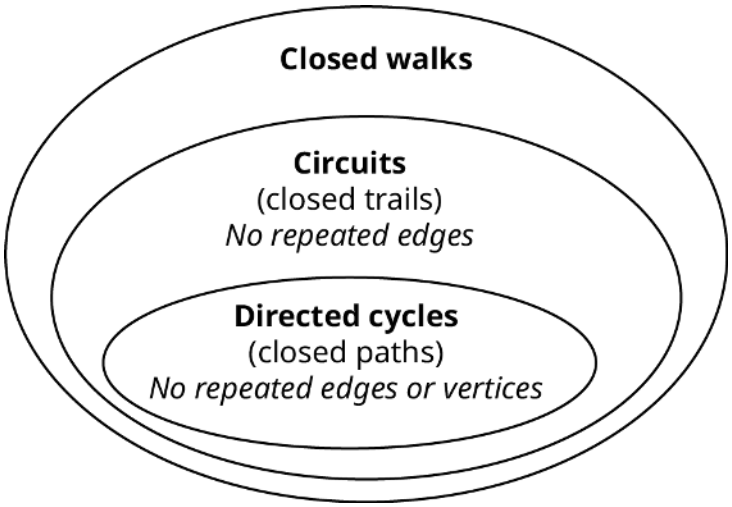


Figure **3.8.3**: Closed Walks, Circuits, and Directed Cycles

The goal of Hamilton's puzzle was to find a route along the edges of the dodecahedron, which visits each vertex exactly once. A dodecahedron is a three-dimensional space figure with faces that are all pentagons as we saw in Figure 12.178.

Since it is easier to visualize two dimensions rather than three, we will flatten out the dodecahedron and look at the edges and vertices on a flat surface. Graph *A* in Figure 12.180 shows a two-dimensional graph of the edges and vertices, and Graph *B* shows an untangled version of Graph *A* in which no edges are crossing. Graph *B* in Figure 12.180 is very similar to the design of the game board that Hamilton invented for his puzzle.

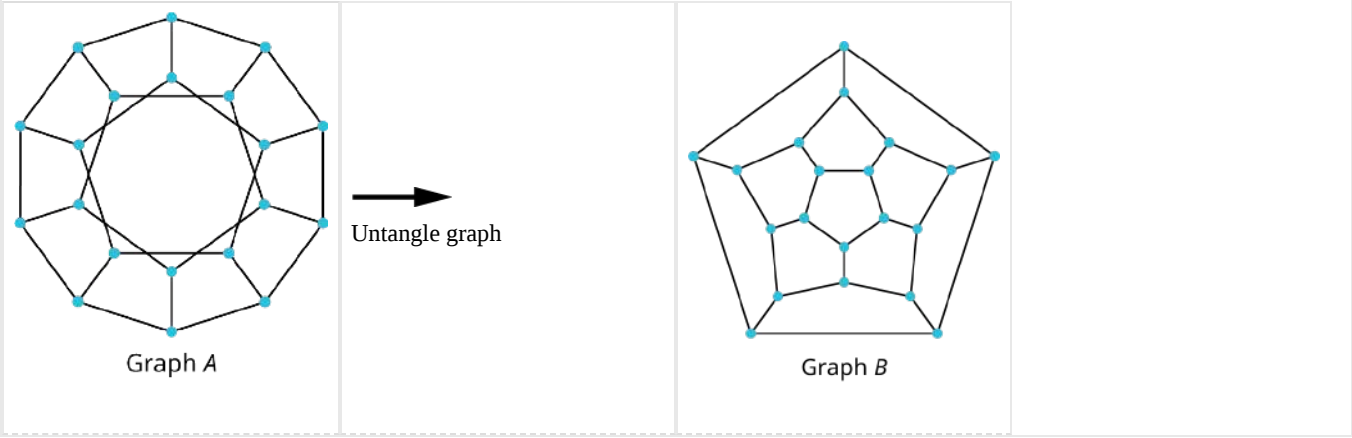


Figure **3.8.4**: Graph of Edges and Vertices of Dodecahedron

We can see that this is a planar graph because it can be “untangled.” In order to solve Hamilton’s puzzle, we need to find a circuit that visits every vertex once. A solution is shown in Figure 12.181.

.



Figure 3.8.5: A Solution to Hamilton's Puzzle

A circuit that doesn't repeat any vertices, like the one in Figure 12.181, is called a directed cycle. So, we can most accurately say that Hamilton's puzzle asks us to find a directed cycle that visits every vertex in a graph exactly once. Because Hamilton created and solved this puzzle, these special circuits were named Hamilton cycles, or Hamilton circuits.

Who Knew?: Icosian Game

When Hamilton invented his puzzle in the 19th century, it was originally called the Icosian game. This was a reference to an icosahedron, not a dodecahedron. Why? Hamilton was actually interested in the symmetries of icosahedrons. It turns out that these two space figures are related to each other. The dodecahedron has 30 edges, 20 faces, and 12 vertices, while the icosahedron has 30 edges, 12 faces, and 20 vertices. By relating the vertices of the dodecahedron to the faces of the icosahedron, Hamilton was able to make the mathematical connections necessary to use graph theory and dodecahedrons to make discoveries about the symmetries of icosahedrons.

Hamilton Cycles vs. Euler Circuits

Let's practice naming and identifying Hamilton cycles, as well as distinguishing them from Euler circuits. It is important to remember that Euler circuits visit all edges without repetition, while Hamilton cycles visit all vertices without repetition. Hamilton cycles are named by their vertices just like all circuits. An example is given in Figure 3.8.6.

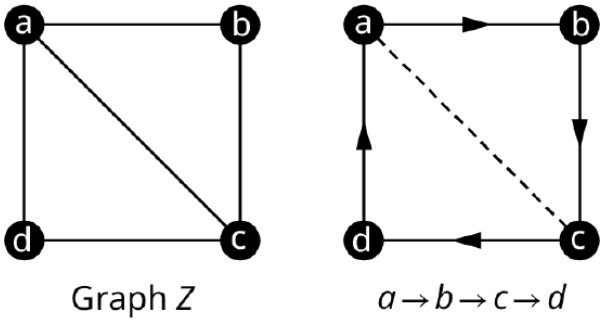


Figure 3.8.6: Hamilton Cycle in Graph Z

Notice that the Hamilton cycle $a \rightarrow b \rightarrow c \rightarrow d$ for Graph Z in Figure 12.182 is NOT an Euler circuit, because it does not visit edge semanticsacac. Some Hamilton cycles are also Euler circuits while some are not, and some Euler circuits are Hamilton cycles while some are not.

Checkpoint

TIP! Sometimes students confuse Euler circuits with Hamilton cycles. To help you remember, think of the E in Euler as standing for Edge.

Example 3.8.1: Differentiating between Hamilton Cycles or Euler Circuits

Use Figure ✔ 3.8.7 to determine whether the given circuit is a Hamilton cycle, an Euler circuit, both, or neither.

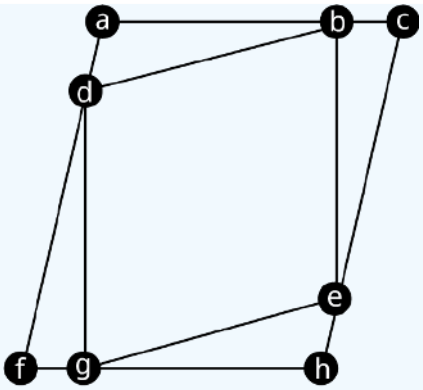


Figure 12.183 Graph Q

1. $a \to b \to c \to e \to h \to g \to f \to d \to a$
2. $g \to e \to h \to g \to f \to d \to a \to b \to d \to g$
3. $a \to b \to c \to e \to h \to g \to f \to d \to b \to e \to g \to d \to a$

Answer

1. This circuit is a Hamilton cycle only. It visits each vertex exactly once, so, it is a Hamilton cycle. It is not an Euler circuit because it doesn't visit all of the edges.
2. This circuit is neither Hamilton cycle nor an Euler circuit. It doesn't visit vertex c, so, it is not a Hamilton cycle. It also doesn't visit edges semanticsbebe and semanticscece, so, it is not an Euler circuit.
3. This circuit is an Euler circuit only. It visits several vertices more than once; so, it is not a Hamilton cycle. It visits every edge exactly once, so, it is an Euler circuit.

Checkpoint

TIP! Euler circuits never skip a vertex, so, they only fail to be Hamilton cycles when they visit a vertex more than once. Hamilton cycles never visit any edges twice, so, they only fail to be Euler circuits when they skip edges.

Your Turn 3.8.1

Consider the figure shown. Is the circuit $a \to b \to f \to e \to h \to i \to g \to d \to c \to a$ an Euler circuit, a Hamilton cycle, or both?

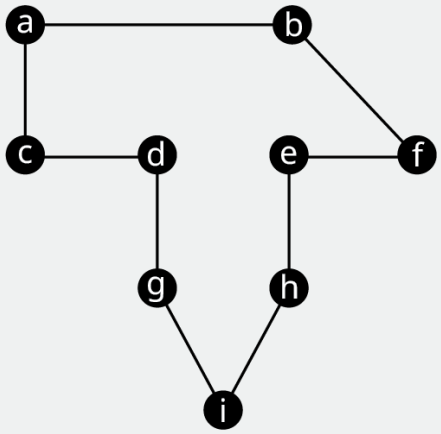


Figure 3.8.8: Graph T

Notice that the graph is a cycle. A cycle will always be Eulerian because all vertices are degree 2. Moreover, any circuit in the graph will always be both an Euler circuit and a Hamilton cycle. It is not always as easy to determine if a graph has a Hamilton

cycle as it is to see that it has an Euler circuit, but there is a large group of graphs that we know will always have Hamilton cycles, the complete graphs. Since all vertices in a complete graph are adjacent, we can always find a directed cycle that visits all the vertices. For example, look at the directed six-cycle, $n \to o \to p \to q \to r \to s$, in the complete graph with six vertices in Figure 3.8.9.

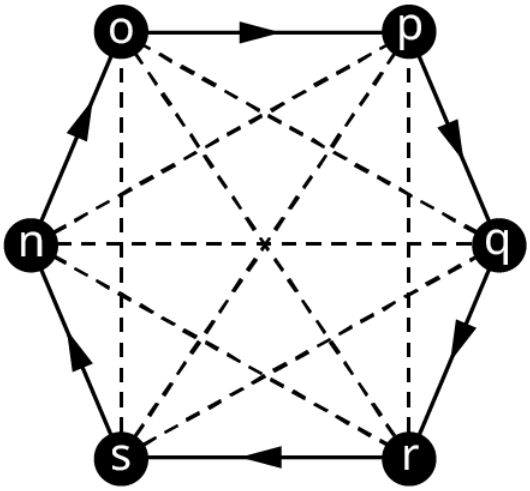


Figure 3.8.9: Directed Cycle in Complete Graph

That is not the only directed six-cycle in the graph though. We could find another just be reversing the direction, and we could find even more by using different edges. So, how many Hamilton cycles are in a complete graph with n vertices? Before we tackle this problem, let's look at a shorthand notation that we use in mathematics which will be helpful to us.

Factorials

In many areas of mathematics, we must calculate products like $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or $11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, products that involve multiplying all the counting numbers from a particular number down to 1. Imagine that the product happened to be all the numbers from 100 down to 1. That's a lot of writing! Instead of writing all of that out, mathematicians came up with a shorthand notation. For example, instead of $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, we write $7!$, which is read "7 factorial." In other words, the product of all the counting numbers from n down to 1 is called n factorial and it is written $n!$

Example 3.8.2: Evaluating Factorials

Evaluate $n!$ and $(n-1)!$ for $n = 4$.

Answer

$n! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 =$24n!=4!=4·3·2·1=24 and $(n-1)! = (4-1)! = 3! = 3 \cdot 2 \cdot 1 = 6$

Your Turn 3.8.12

Evaluate $n!$ and $(n-1)!$ for $n = 6$.

A common use for factorials is counting the number of ways to arrange objects. Suppose that there were three students, Aryana, Byron, and Carlos, who wanted to line up in a row. How many arrangements are possible? There are six possibilities: ABC, ACB, BAC, BCA, CAB, or CBA. Notice that there were three students being arranged, and the number of possible arrangements is three.

FORMULA

The number of ways to arrange n distinct objects is $n!$.

Example 3.8.3: Counting Arrangements of Letters

Find the number of ways to arrange the letters a, b, c, and d.

Answer

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

Your Turn 3.8.3

Find the number of ways to arrange the letters v, w, x, y, and z.

Counting Hamilton Cycles in Complete Graphs

Now, let's get back to answering the question of how many Hamilton cycles are in a complete graph. In Table 12.8, we have drawn all the four cycles in a complete graph with four vertices. Remember, cycles can be named starting with any vertex in the cycle, but we will name them starting with vertex a.

Table 3.8.1 Four-Cycles in Complete Graph with Four Vertices

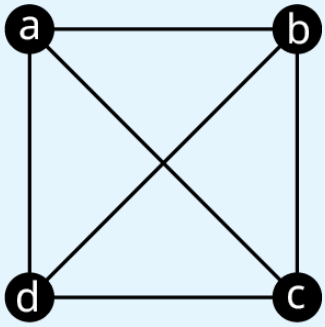
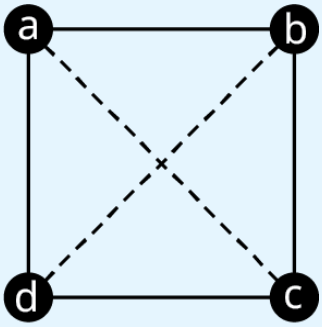
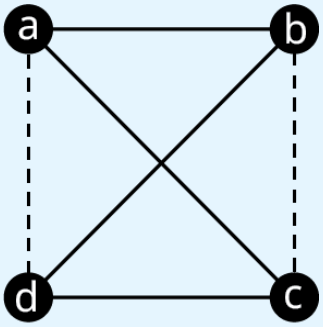
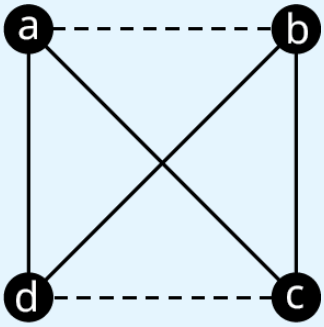
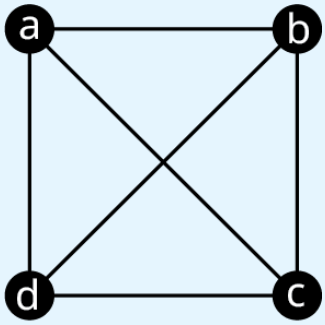
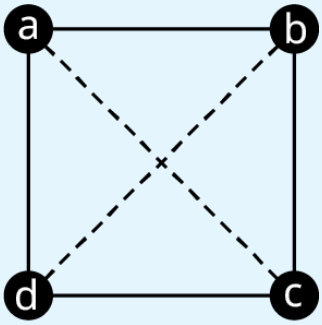
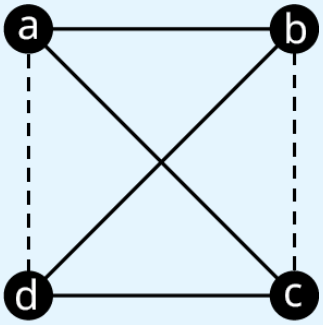
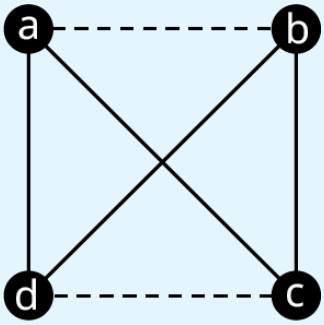
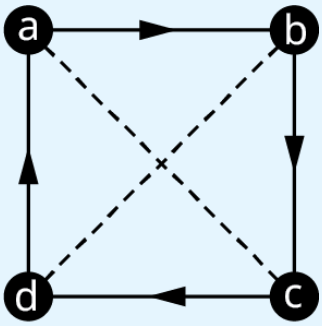
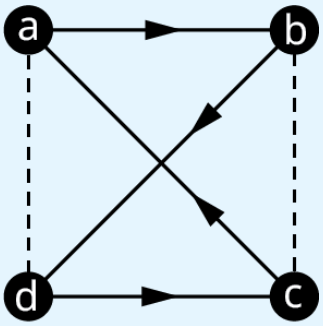
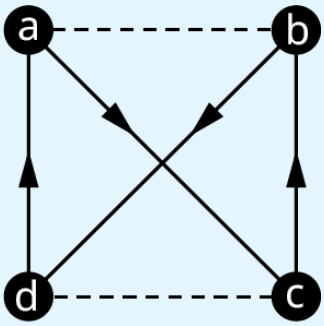
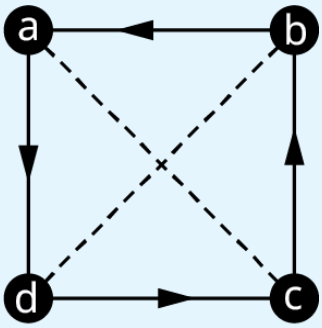
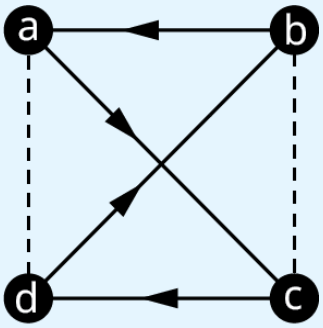
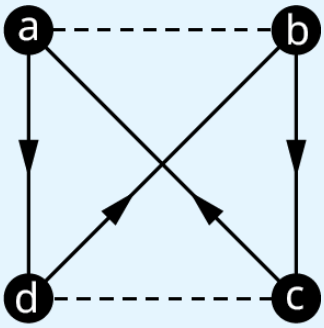
Complete Graph	Cycle	Cycle	Cycle
			
Cycle Name Clockwise	*a, b, c, d*	*a, b, d, c*	*a, c, b, d*
Cycle Name Counterclockwise	*a, d, c, b*	*a, c, d, b*	*a, d, b, c*

Table 3.8.1 shows that there are three unique four-cycles in a complete graph with four vertices. Notice that there were two ways to name each cycle, one reading the vertices in a clockwise direction and one reading the vertices in a counterclockwise direction. This is important to us because we are interested in Hamilton cycles, which are *directed* cycles. Although the cycles *(a, b, c, d)* and *(a, d, c, b)* are the same cycle, the directed cycles, $a \to b \to c \to d \to a$ and $a \to d \to c \to b \to a$, which travel the same route in reverse order are considered different directed cycles, as shown in Table 3.8.2.

Table 3.8.2 Hamilton Cycles in a Complete Graph with Four Vertices

Complete Graph	Cycle	Cycle	Cycle
			
Clockwise Hamilton Cycle	 $a \to b \to c \to d \to a$	 $a \to b \to d \to c \to a$	 $a \to c \to b \to d \to a$
Counter-clockwise Hamilton Cycle	 $a \to d \to c \to b \to a$	 $a \to c \to d \to b \to a$	 $a \to d \to b \to c \to a$

The six directed four-cycles in Table **3.8.2** are the only distinct Hamilton cycles in a complete graph with four vertices. Six is also the number of ways to arrange the three letters b, c, and d. (Do you see why?) The number of ways to arrange three letters is $3! = 3 \cdot 2 \cdot 1 = 6$. Similarly, the number of Hamilton cycles in a graph with five vertices is the number of ways to arrange four letters, which is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. In general, to find the number of Hamilton cycles in a graph, we take one less than the number of vertices and find its factorial.

FORMULA

The number of distinct Hamilton cycles in a complete graph with n vertices is $(n-1)!$

Example 3.8.4: Counting Hamilton Cycles in a Complete Graph

How many Hamilton cycles are in the complete graph in Figure ✔ 3.8.10?

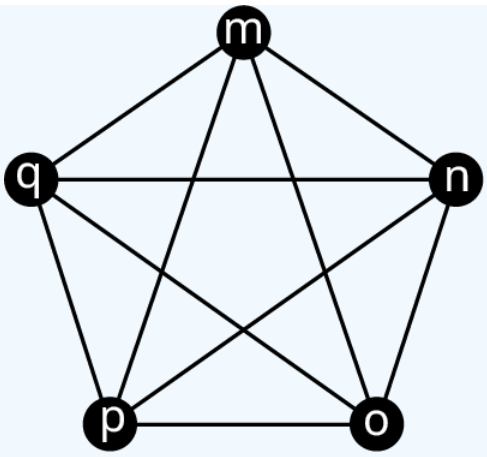


Figure ✔ **3.8.10** Complete Graph *L*

Answer

There are five vertices in the graph. Using $n = 5$, we have $(n-1)! = (5-1)! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ Hamilton cycles.

Your Turn 3.8.4

How many Hamilton cycles are in the graph?

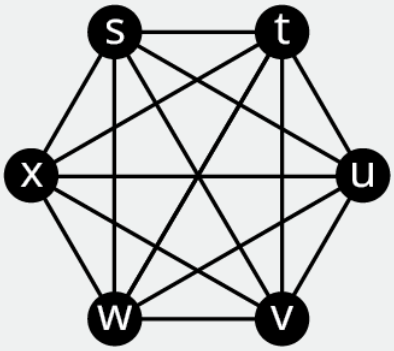


Figure **3.8.11**: Complete Graph *R*

Weighted Graphs

Suppose that an officer in the U.S. Air Force who is stationed at Vandenberg Air Force base must drive to visit three other California Air Force bases before returning to Vandenberg. The officer needs to visit each base once. The vertices in the graph in Figure **3.8.12** represent the four U.S. Air Force bases, Vandenberg, Edwards, Los Angeles, and Beale. The edges are labeled to with the driving distance between each pair of cities.

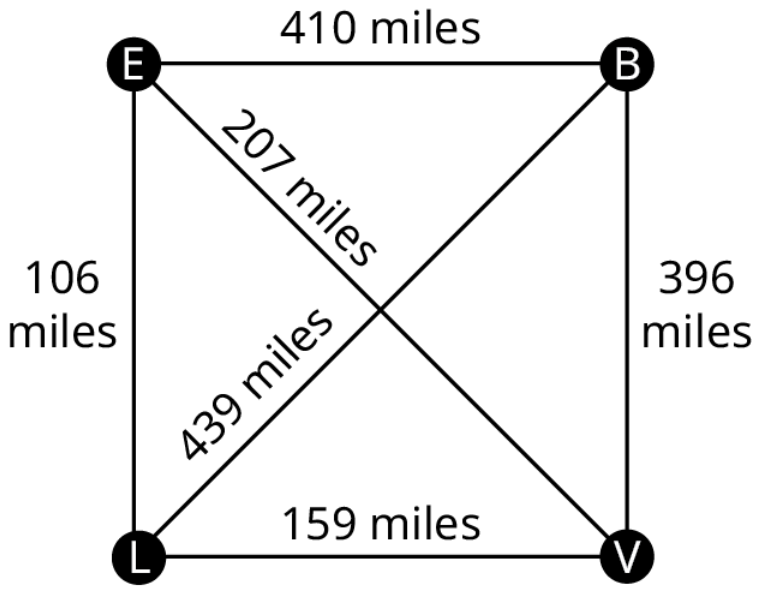


Figure 3.8.12: Graph of Four California Air Force Bases

The graph in Figure 3.8.12 is called a weighted graph, because each edge has been assigned a value or weight. The weights can represent quantities such as time, distance, money, or any quantity associated with the adjacent vertices joined by the edges. The total weight of any walk, trail, or path is the sum of the weights of the edges it visits.

Notice that the officer's trip can be represented as a Hamilton cycle, because each of the four vertices in the graph is visited exactly once.

✔ Example 3.8.5: Finding Hamilton Cycles of Lowest Weight

Use Figure ✔ 3.8.12 and the given Hamilton cycles to answer the following questions.

$V \to L \to E \to B \to V$

$V \to L \to B \to E \to V$

$V \to E \to L \to B \to V$

$V \to B \to E \to L \to V$

1. Which of the Hamilton cycles (directed cycles) lie on the same cycle (undirected cycle) in the graph?
2. Find the total weight of each cycle.
3. Of the four, which of the Hamilton cycles describes the shortest trip for the officer? Describe the route.

Answer

1. $V \to L \to E \to B \to V$ and $V \to B \to E \to L \to V$ follow the same edges in reverse order.
2. Any Hamilton cycles that lie on the same cycle will have the same edges and the same total weight.

   $V \to L \to E \to B \to V$ and $V \to B \to E \to L \to V$ each have total weight $159 + 106 + 410 + 396 = 1071$.

   $V \to L \to B \to E \to V$ has a total weight $159 + 439 + 410 + 207 = 1215$.

   $V \to E \to L \to B \to V$ has a total weight $396 + 439 + 106 + 207 = 1148$.
3. Hamilton cycles $V \to L \to E \to B \to V$ and $V \to B \to E \to L \to V$ each have the lowest total weight. The officer would take the route from Vandenberg, to Los Angeles, to Edwards, to Beale, and back to Vandenberg, or reverse that route.

Your Turn 3.8.5

Find the weight of the Hamilton cycle $m \to o \to p \to n \to q \to m$ in the given figure.

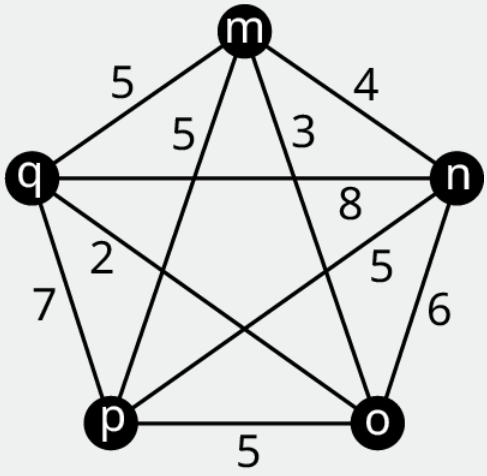


Figure 3.8.13: Weighted Complete Graph with Five Vertices

Check Your Understanding

For the following exercises, fill in the blank to make the statement true.

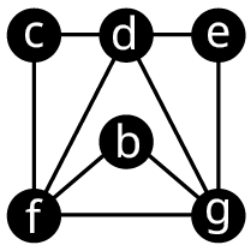
1. A Hamilton cycle is a circuit that visits each _____ exactly once.
2. A _____ graph with $n \geq 3$ vertices has $(n-1)!$ Hamilton cycles.
3. For the following exercises, fill in the blank with **is** or **is not** to make the statement true.
4. A Hamilton cycle _____ a circuit.
5. A Hamilton cycle that visits every edge _____ an Euler circuit.
6. A Hamilton cycle _____ different from a Hamilton circuit.
7. An Euler circuit that visits every vertex _____ a Hamilton cycle.
8. The total weight of a trail _____ the sum of the weights of the edges visited by the trail.
9. A weighted graph _____ always a complete graph.
10. The number of ways to arrange n objects _____ $(n-1)!$
11. Every cycle ____ a circuit.

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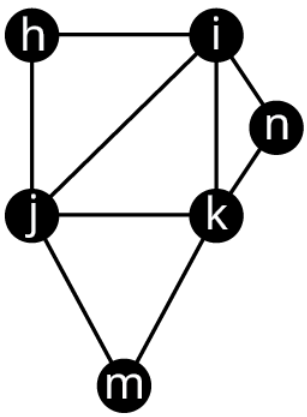
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3.8.1: Exercises

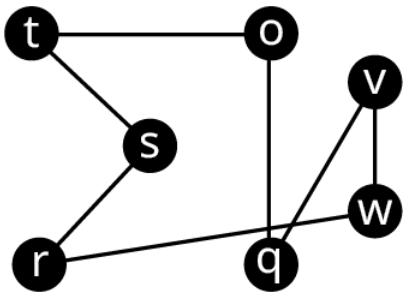
For the following exercises, use the figure to determine whether the sequence of vertices in the given graph is a Hamilton cycle, an Euler circuit, both, or neither.



Graph *A*



Graph *L*



Graph *U*

Exercise 3.8.1.1

Graph *A*: $f \to b \to g \to e \to d \to c \to f$

Exercise 3.8.1.2

Graph *A*: $g \to b \to f \to c \to d \to e \to g$

Exercise 3.8.1.3

Graph *A*: $d \to e \to g \to d \to f \to b \to g \to f \to c \to d$

Exercise 3.8.1.4

Graph *L*: $h \to i \to k \to n \to j \to h$

Exercise 3.8.1.5

Graph *L*: $n \to i \to h \to j \to m \to k \to n$

Exercise 3.8.1.6

Graph *L*: $j \to i \to n \to k \to i \to j \to k \to m \to j$

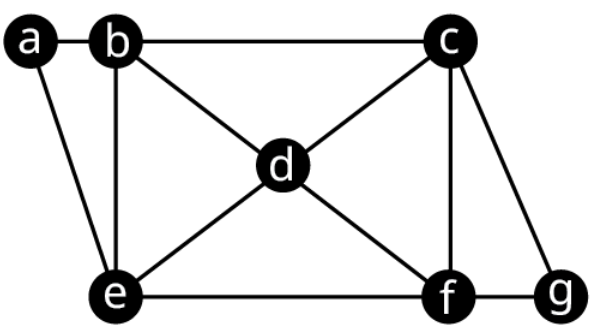
Exercise 3.8.1.7

Graph *U*: $v \to w \to r \to s \to t \to o \to q \to v$

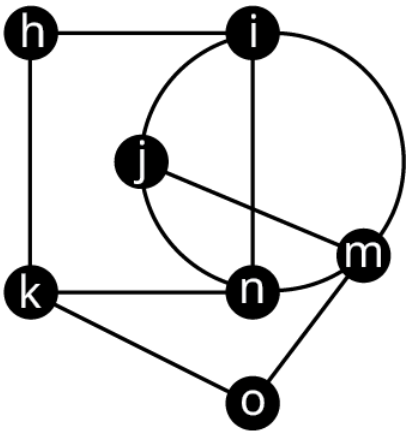
Exercise 3.8.1.8

Graph *U*: $w \to q \to r \to s \to t \to o \to v \to w$

For the following exercises, use the figure to find a circuit that fits the description.



Graph P



Graph Q

Exercise 3.8.1.9

A Hamilton cycle in Graph *P* that begins at vertex *c*.

Exercise 3.8.1.10

An Euler circuit in Graph *P* that begins at vertex *c*.

Exercise 3.8.1.11

A directed cycle in Graph *P* that is NOT a Hamilton cycle, and explain why it is not a Hamilton cycle.

Exercise 3.8.1.12

A directed circuit in Graph *P* that is NOT an Euler circuit, and explain why it is not an Euler circuit.

Exercise 3.8.1.13

A Hamilton cycle in Graph *Q* that begins at vertex *n*.

Exercise 3.8.1.14

An Euler circuit in Graph *Q* that begins at vertex *n*.

Exercise 3.8.1.15

A directed cycle in Graph *Q* that is NOT a Hamilton cycle, and explain why it is not a Hamilton cycle.

Exercise 3.8.1.16

A directed circuit in Graph *Q* that is NOT an Euler circuit, and explain why it is not an Euler circuit.

Exercise 3.8.1.17

Use the results of Exercises 9–16 to make an observation about which tends to involve a longer sequence of vertices, Hamilton cycles or Euler circuits. Explain why you think this is.

For the following exercises, evaluate the factorial expression for the given value of n.

Exercise 3.8.1.18

$n!,\ n = 10$

Exercise 3.8.1.19

$n!,\ n = 11$

Exercise 3.8.1.20

$(n-1)!,\ n = 10$

Exercise 3.8.1.21

$(n-1)!,\ n = 11$

For the following exercises, find the number of arrangements letters in the given word.

Exercise 3.8.1.22

have

Exercise 3.8.1.23

teamwork

Exercise 3.8.1.24

anime

Exercise 3.8.1.25

making

For the following exercises, find the number of Hamilton cycles in a complete graph with the given number of vertices.

Exercise 3.8.1.26

12 vertices

Exercise 3.8.1.27

13 vertices

? Exercise 3.8.1.28

9 vertices

? Exercise 3.8.1.29

8 vertices

? Exercise 3.8.1.30

x vertices

For the following exercises, given the number of Hamilton cycles in a complete graph, determine the number of vertices.

? Exercise 3.8.1.31

$5! = 120$ Hamilton cycles

? Exercise 3.8.1.32

$6! = 720$ Hamilton cycles

? Exercise 3.8.1.33

$7! = 5040$ Hamilton cycles

? Exercise 3.8.1.34

$x!$ Hamilton cycles

For the following exercises, all the distinct Hamilton cycles for a complete graph are given. Indicate which pairs of Hamilton cycles (directed cycles) lie on the same cycle (undirected cycle) in the graph.

? Exercise 3.8.1.35

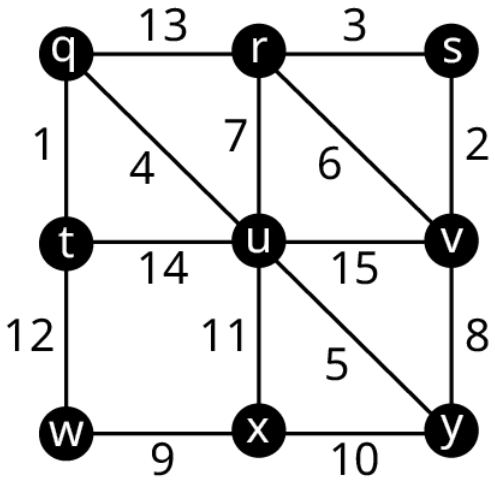
1. $b \to a \to c \to d \to b$
2. $b \to a \to d \to c \to b$
3. $b \to c \to a \to d \to b$
4. $b \to c \to d \to a \to b$
5. $b \to d \to a \to c \to b$
6. $b \to d \to c \to a \to b$

? Exercise 3.8.1.36

1. $i \to f \to g \to h \to e \to i$
2. $i \to f \to g \to e \to h \to i$
3. $i \to f \to h \to g \to e \to i$
4. $i \to f \to h \to e \to g \to i$
5. $i \to f \to e \to g \to h \to i$
6. $i \to f \to e \to h \to g \to i$
7. $i \to g \to f \to h \to e \to i$
8. $i \to g \to f \to e \to h \to i$
9. $i \to g \to h \to f \to e \to i$

10. $i \to g \to h \to e \to f \to i$
11. $i \to g \to e \to f \to h \to i$
12. $i \to g \to e \to h \to f \to i$
13. $i \to h \to g \to f \to e \to i$
14. $i \to h \to g \to e \to f \to i$
15. $i \to h \to f \to g \to e \to i$
16. $i \to h \to f \to e \to g \to i$
17. $i \to h \to e \to g \to f \to i$
18. $i \to h \to e \to f \to g \to i$
19. $i \to e \to g \to h \to f \to i$
20. $i \to e \to g \to f \to h \to i$
21. $i \to e \to h \to g \to f \to i$
22. $i \to e \to h \to f \to g \to i$
23. $i \to e \to f \to g \to h \to i$
24. $i \to e \to f \to h \to g \to i$

For the following exercises, use the figure to find the weight of the given Hamilton cycle.



? Exercise 3.8.1.37

$q \to r \to s \to v \to y \to x \to w \to t \to u \to q$

? Exercise 3.8.1.38

$u \to y \to x \to w \to t \to q \to r \to s \to v \to u$

? Exercise 3.8.1.39

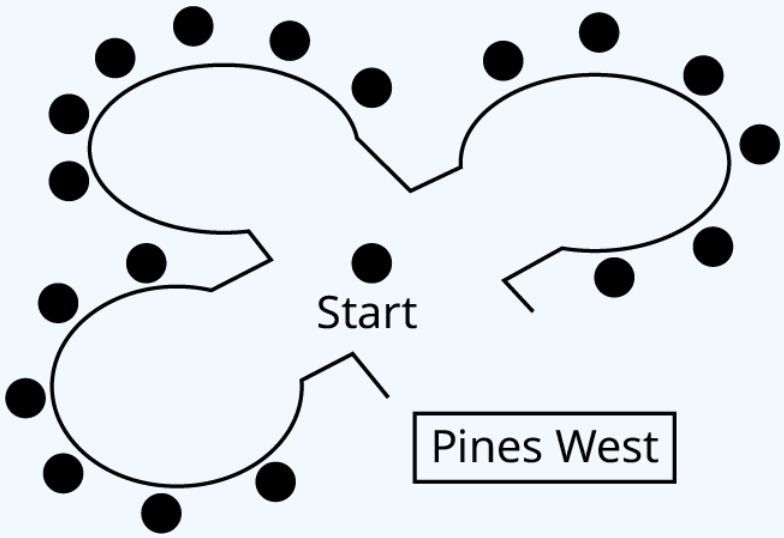
$y \to v \to s \to r \to u \to q \to t \to w \to x \to y$

? Exercise 3.8.1.40

$u \to v \to s \to r \to q \to t \to w \to x \to y \to u$

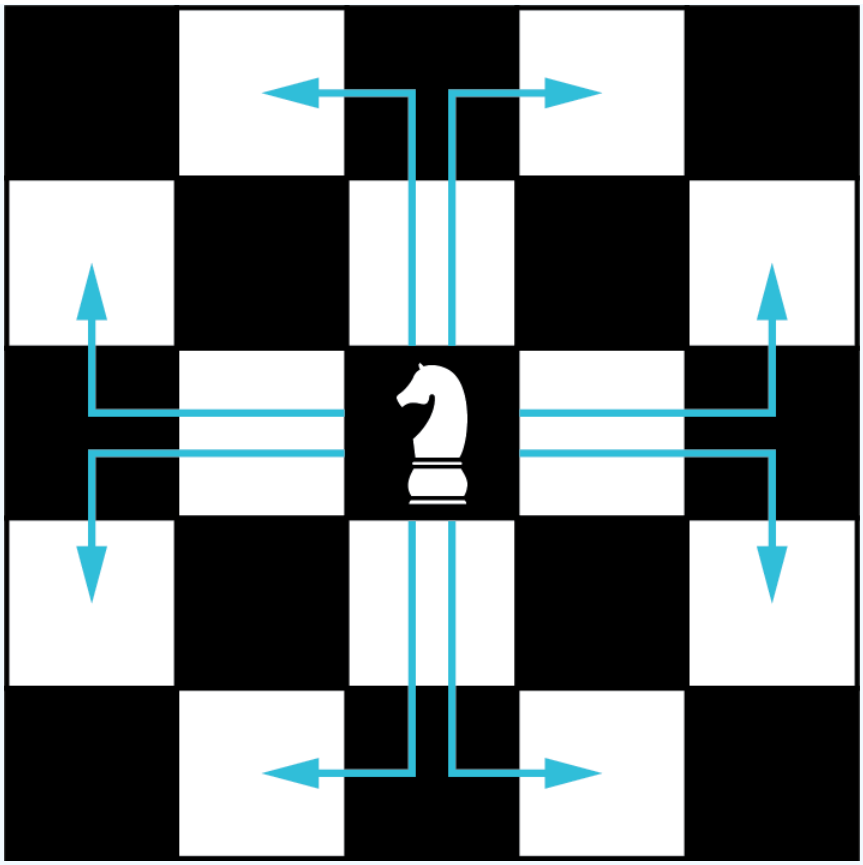
Exercise 3.8.1.41

The neighborhood of Pines West has three cul-de-sacs that meet at an intersection as shown. A postal delivery person starts at the intersection and visits each house in a cul-de-sac once, returns to the intersection, visits each house in the next cul-de-sac, and so on, returning to the intersection when finished. Describe how the route can be represented as a graph. If there is no backtracking, in other words, the person never reverses direction, is the route followed by the postal delivery person best described as a trail, an Euler circuit, a Hamilton cycle, or neither? Explain your reasoning.

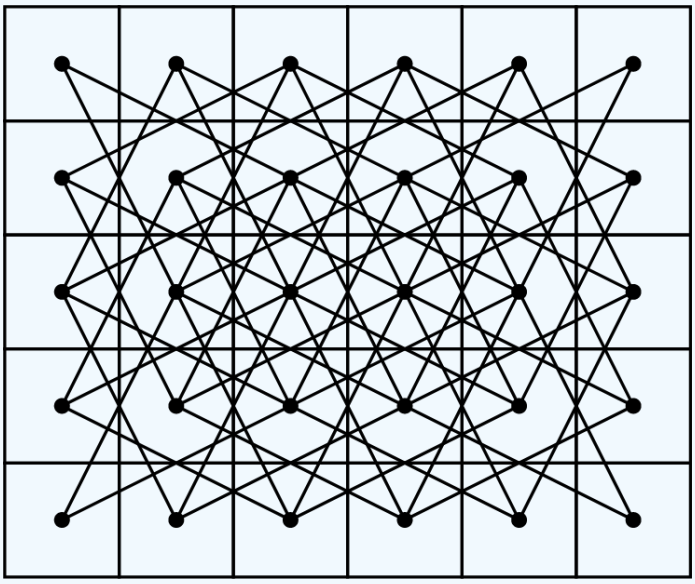


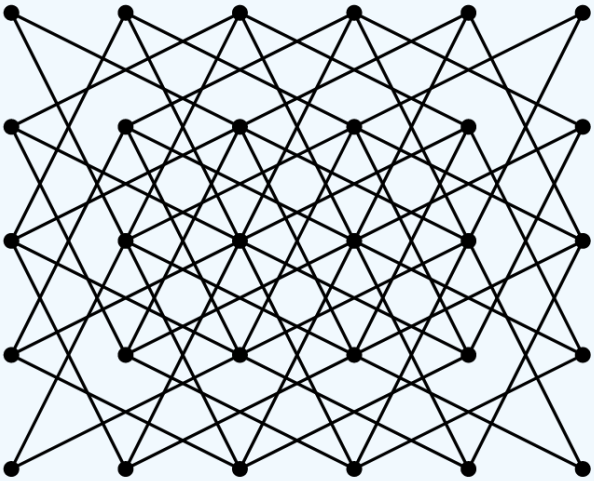
Exercise 3.8.1.42

In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The 8 possible moves a knight can make from a given space are shown in the figure.

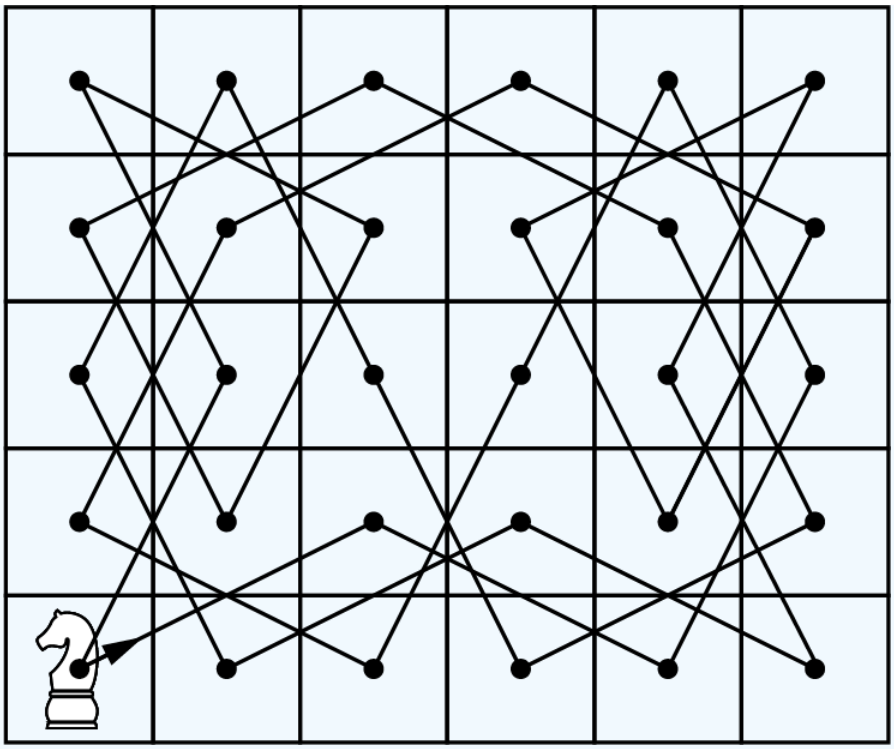


A graph in which each vertex represents a space on a five-by-six game board and each edge represents a move a knight could make is shown in the figure.





A knight's tour is a sequence of moves by a knight on a chessboard (of any size) such that the knight visits every square exactly once. If the knight's tour brings the knight back to its starting position on the board, it is called a closed knight's tour. Otherwise, it is called an open knight's tour.



Determine if the closed knight's tour in Figure 12.239 is most accurately described as an Euler circuit or a Hamilton cycle, or both, of the graph of all possible knight moves. Explain your reasoning.

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3.9: Hamilton Paths



Figure **3.9.1** A school bus picks up children along a planned route. (credit: "Kids at School Bus Stop" by Ty Hatch/Flickr, CC BY 2.0)

Learning Objectives

1. Describe and identify Hamilton paths.
2. Evaluate Hamilton paths in real-world applications.
3. Distinguish between Hamilton paths and Euler trails.

In the United States, school buses carry 25 million children between school and home every day. The total distance they travel is around 6 billion kilometers per year. In the city of Boston, Massachusetts, the 2016 budget for running those buses was $120 million dollars. In 2017, the city held a competition to find ways to cut costs and the Quantum Team from the MIT Operations Research Center came to the rescue, using a computer algorithm to identify the most efficient and least costly routes, which saved the city of Boston $5 million each year and even reduced daily CO_2 emissions by 9,000 kilograms! (*This* U.S. *city put an algorithm in charge of its school bus routes and saved $5 million*, Sean Fleming, World Economic Forum)

The problem the Quantum Team tackled involves graph theory. Imagine a graph in which vertices are the bus depot, the school, and the bus stops along a particular route. The bus must start at the depot, visit every stop exactly once, and end at the school. The route is a special kind of path that visits every vertex exactly once. Can you guess what those paths are called?

Hamilton Paths

Just as circuits that visit each vertex in a graph exactly once are called Hamilton cycles (or Hamilton circuits), paths that visit each vertex on a graph exactly once are called Hamilton paths. As we explore Hamilton paths, you might find it helpful to refresh your memory about the relationships between walks, trails, and paths by looking at Figure **3.9.1**. We know that paths are walks that don't repeat any vertices or edges. So, a Hamilton path visits every vertex without repeating any vertices or edges. Figure **3.9.2** shows a path from vertex *A* to vertex *E* and a Hamilton path from vertex *A* to vertex *E*.

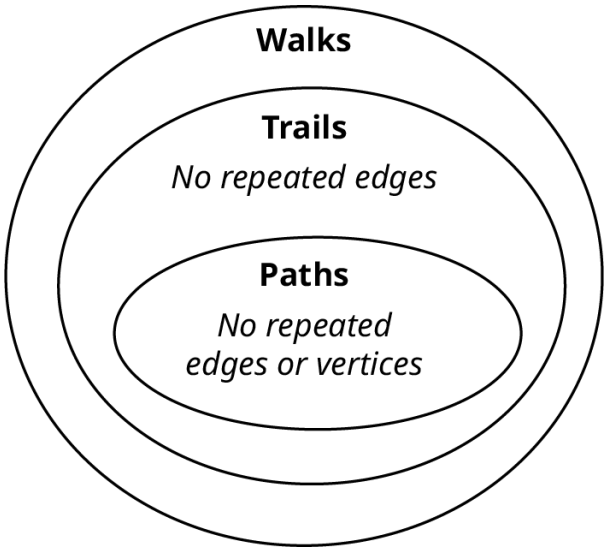


Figure **3.9.1**: Walks, Trails, and Paths

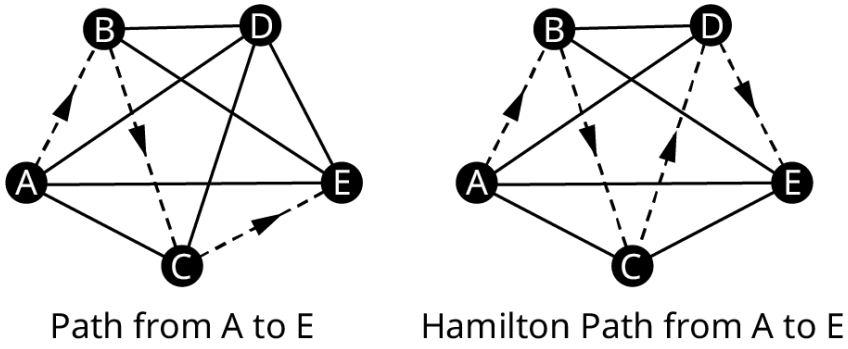


Figure **3.9.2**: Path or Hamilton Path?

✔ Example **3.9.1**: Identifying Hamilton Paths

Which of the following sequences of vertices is a Hamilton path for Graph Q in Figure ✔ **3.9.3**?

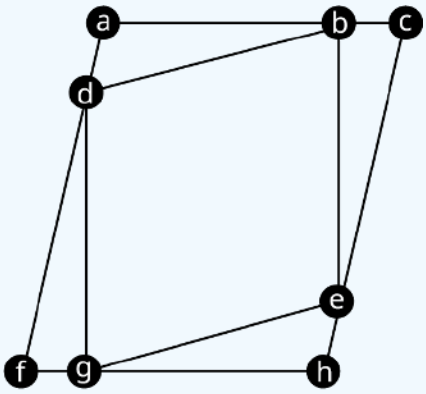


Figure ✔ **3.9.3** Graph Q

1. $a \to d \to b \to c \to e \to g \to f$
2. $c \to b \to e \to h \to g \to f \to d \to a$
3. $h \to e \to g \to d \to b \to e \to g \to f \to d \to a \to b \to c$

Answer

Sequence 1 is a path, because it is a walk that doesn't repeat any vertices or edges, but not a Hamilton path because it skips vertex h. Sequence 2 is a path that visits each vertex exactly once; so, it is a Hamilton path. Sequence 3 is a walk, but it is

not a path because it visits vertices g, e, and b each more than once; so, it cannot be a Hamilton path. So, we can see that only sequence 2 is a Hamilton path.

Your Turn 3.9.1

Use the graphs to determine if the given sequence of vertices represents a Hamilton path or not.

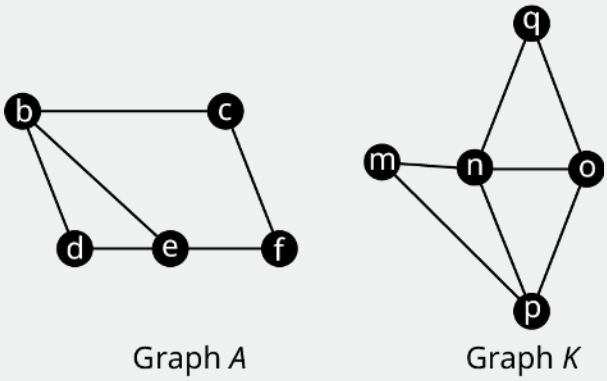


Figure 3.9.4: Graphs A and K

Graph A, $b \to c \to d \to e \to f$

Graph A, $b \to e \to f \to c \to b \to d$

Graph K, $n \to q \to o \to p \to m$

Checkpoint

TIP! Since a Hamilton path visits each vertex exactly once, it must have the same number of vertices listed as appear in the graph.

Finding Hamilton Paths

Suppose you were visiting an aquarium with some friends. The map of the aquarium is given in Figure 12.195. The letters represent the exhibits.

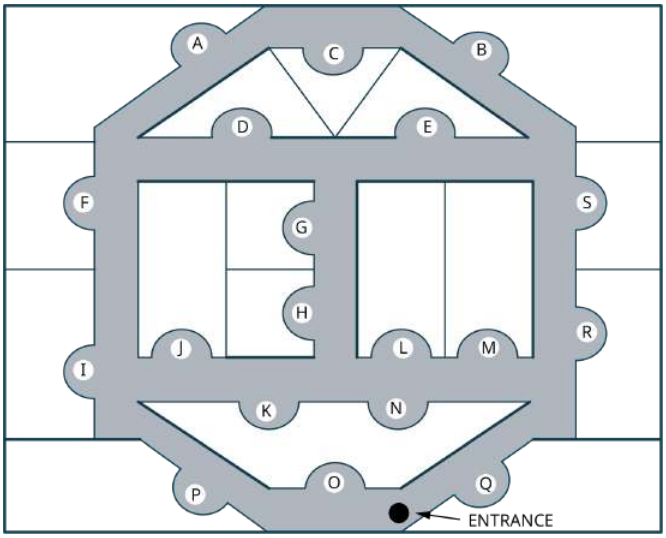


Figure 3.9.5: Map of Aquarium Exhibits

Figure 3.9.6 shows a graph of the aquarium in which each vertex represents an exhibit and each edge is a route between the pair of exhibits that doesn't bypass another exhibit.

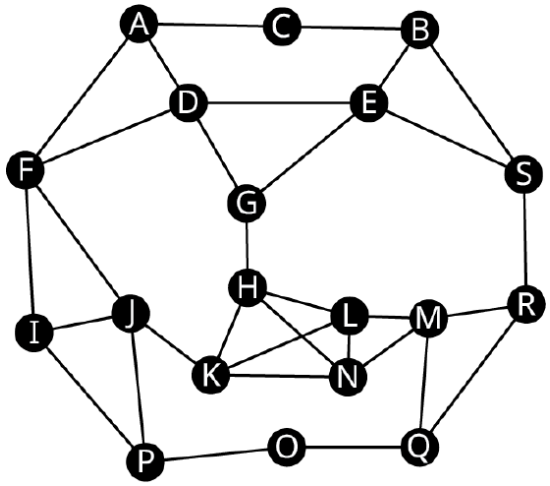


Figure **3.9.6**: Graph of Aquarium

Let's see if we can plan a tour of the exhibits that visits each exhibit exactly once, beginning at exhibit *O* and ending at exhibit *C*. Suppose that, after exhibit *O*, we plan to visit exhibit *Q* and then exhibit *M*. After *M*, should we plan to visit *N*, *L*, or *R*? Take a look at Figure **3.9.7**. If *R* is not chosen next, that will cause a problem later on. Do you see what it is?

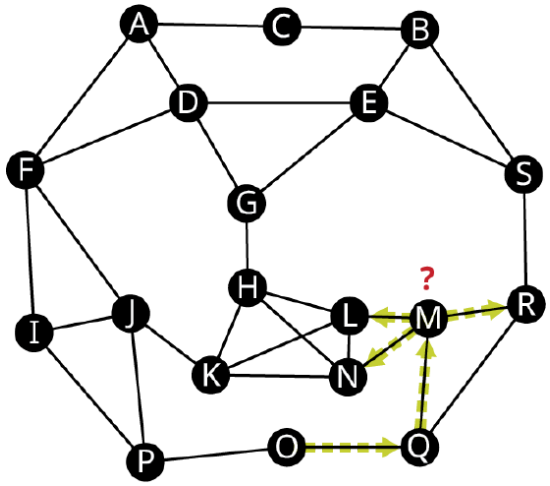


Figure **3.9.7** Choosing Vertex *L*, *N*, or *R*

If *L* or *N* is chosen next, the only way to get to *R* later will be to go from *S* to *R*, and then we will not be able to continue without repeating a vertex. So, we will pick *R* next, and then the only option is *S*. After *S* we have another choice to make. As shown in Figure **3.9.8**, the next choice is between *B* and *E*. Keeping in mind that the goal is to end at *C*, which would be the better choice?

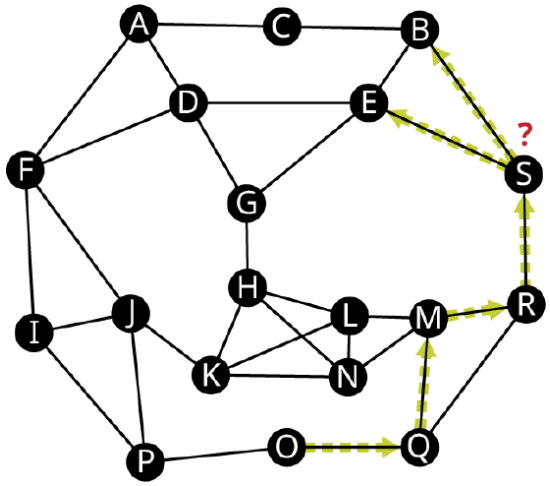


Figure **3.9.8** Choosing Vertex *B* or *E*

If you said vertex *B*, you are right! Otherwise, we will not be able to visit *B* later. After *B*, the only option is *E*. Then we can choose either *D* or *G*. Either will work fine. Let's choose *G* as shown in Figure **3.9.9**. After *G*, you must visit *H*, but should you visit *K* or *L* after that?

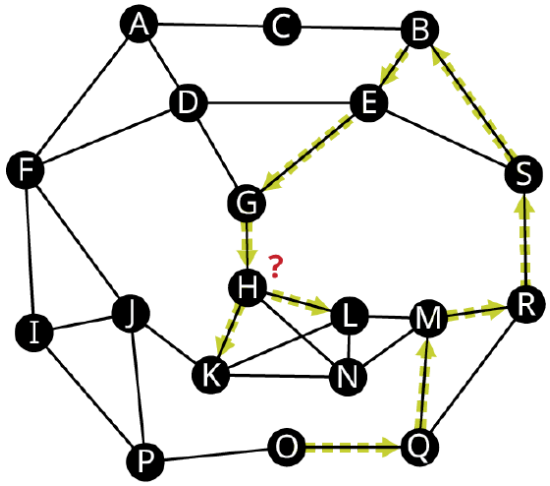


Figure **3.9.9** Choose Vertex *L* or *K*

If you said to go to vertex *L* next, you are right! Otherwise, it will be impossible to visit *N* without repeating a vertex. So, next is *L*, then *N*, then *K*, and then at *J* you have another decision to make we can see in Figure **3.9.10**. Should you choose *F*, *I*, or *P* next?

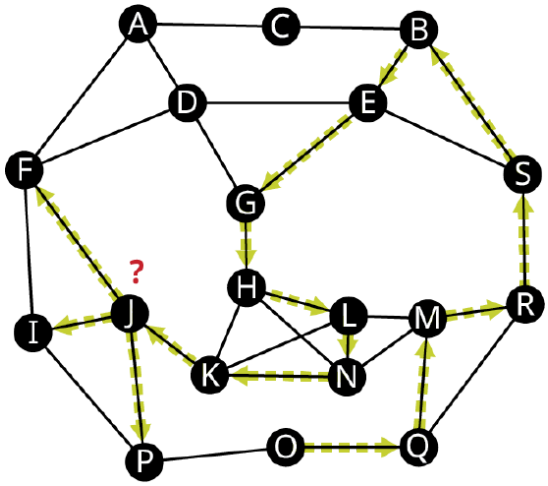


Figure **3.9.10** Choose Vertex *F*, *I*, or *P*

If you said *P*, you are right! If you choose either of the other two vertices, you will not be able to visit *P* later without passing through another vertex twice. Once *P* is chosen, vertex *I* must be next followed by *F*. Then you have to choose between *A* and *D* as shown in Figure **3.9.11**.

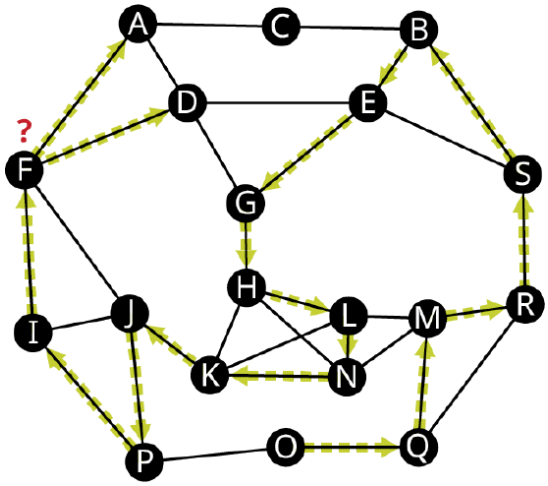


Figure **3.9.11** Choose Vertex *A* or *D*

In this case, we must go to *D* then to *A* so that we can visit *C* without backtracking. The complete Hamilton path is shown in Figure **3.9.12**.

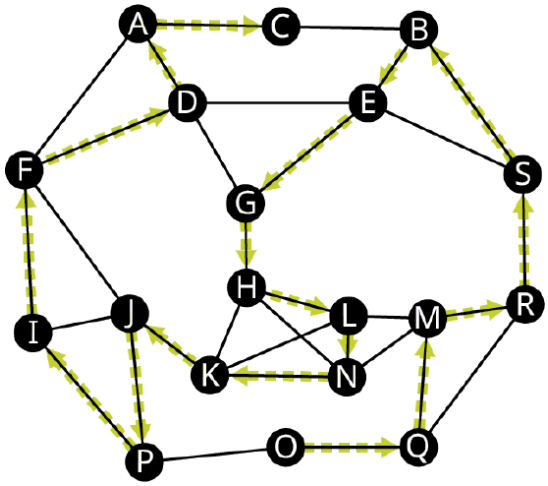


Figure **3.9.12**: Complete Hamilton Path from O to C

So, one Hamilton path that begins at O and ends at C is $O \to Q \to M \to R \to S \to B \to E \to G \to H \to L \to N \to K \to J \to P \to I \to F \to D \to A \to C$.

There is no set sequence of steps that can be used to find a Hamilton path if it exists, but it does help to keep in mind where we are headed and avoid choices that will make returning to a particular vertex impossible without repeating vertices. Let's practice finding Hamilton paths.

✔ Example 3.9.2: Finding a Hamilton Path

Use Figure ✔ **3.9.13** to find a Hamilton path between vertices C and D.

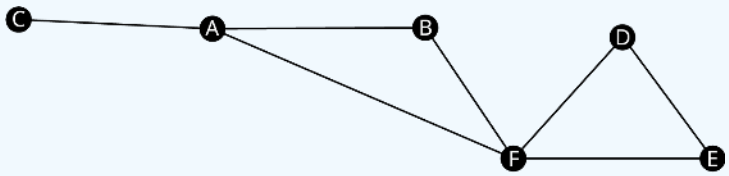


Figure ✔ **3.9.13**: Graph G

Answer

If we start at vertex C, A must be next. Then we must choose between B and F. If we choose F, we will have to backtrack to get to include B; so, we must choose B. Once we choose B, we must choose F next. After F, we choose E, because we want to end at D. So, a Hamilton path between C and D is $C \to A \to B \to F \to E \to D$

Your Turn 3.9.2

Use Figure **3.9.13** to find a Hamilton path between vertices C and E.

Existence of a Hamilton Path

It turns out that there is no Hamilton path between vertices A and E in Graph G in Figure **3.9.13**. To understand why, let's imagine there is a red apple tree on one side of a bridge and a green apple tree on the other side of the bridge. Now suppose someone asked you to pick up all the fallen apples under each tree without crossing the bridge more than once, and making sure that the first apple you pick up and the last apple you pick up are both red. You would say, that is impossible! To have the first and last apple be red would either require leaving the green apples on the ground or crossing the bridge twice.

Let's see how this relates to finding a Hamilton path between A and E in Graph G. The edge AC is a bridge because, if it were removed, the graph would become disconnected with two components, the component $\{C\}$ and the component $\{A, B, D, E, F\}$. So,

we can think of the vertices A, B, D, E, and F as the red apples, vertex C as the green apple, and the edge AC is the bridge between them as in Figure **3.9.14**.

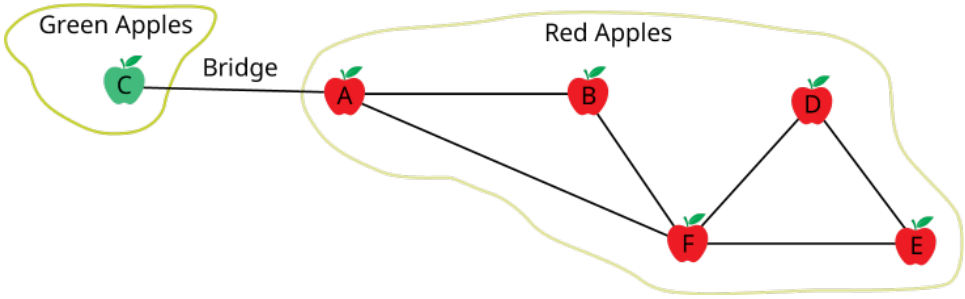


Figure **3.9.14**: Bridge between Red and Green Apples

The creation of a Hamilton path requires a visit to each vertex, just as picking up all the apples requires a visit to each apple. A and E are both red apples; so, a path from A to E would both start and end at a red apple, just as you were asked to do. And you wouldn't be able to cross the bridge twice because that would mean visiting A twice, which is not allowed in a Hamilton path. So, it is impossible to find a Hamilton path from A to E just as it was impossible to pick up all the apples without crossing the bridge more than once. By the same reasoning, if a graph has a bridge, there will never be a Hamilton path that begins and ends on the same side of that bridge, meaning beginning and ending at vertices that would be in the same component if the bridge were removed from the graph.

✔ Example **3.9.3**: Finding a Hamilton Path If One Exists

Find a Hamilton path from vertex s to vertex v for each graph in Figure ✔ **3.9.15** or indicate that there is none.

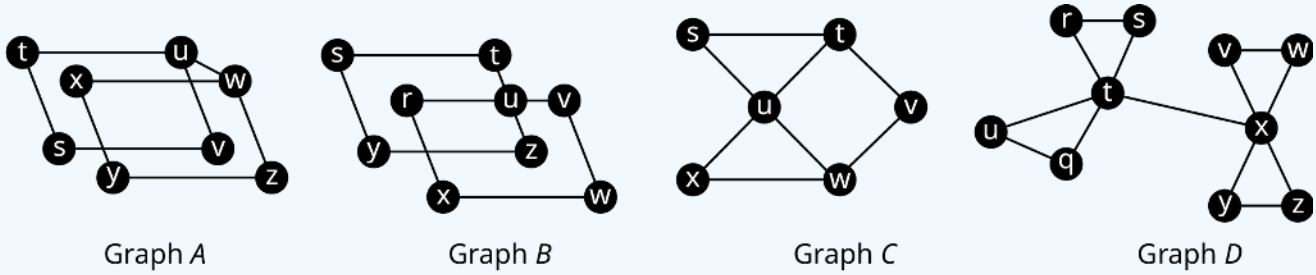


Figure ✔ **3.9.15** Graphs A, B, C, and D

Answer

Graph A: Edge uw is a bridge connecting component $\{s, t, u, v\}$ to the component $\{w, x, y, z\}$. There is no Hamilton path from vertex s to vertex v because they would be part of the same component if the bridge uw were removed.

Graph B: There are no bridges in Graph B. The only method we have to determine if a Hamilton path from vertex s to vertex v exists is to try every possibility. From vertex s, we can visit either vertex y or vertex t. We will try vertex y first and then come back to see what happens with vertex t. After visiting y, we must visit z and then u, but then we have to decide between vertices r, t, and v next as shown in Figure ✔ **3.9.16**.

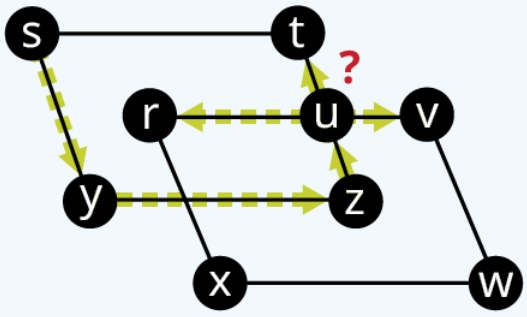


Figure ✔ **3.9.16** Choose Vertex r, t, or v

Vertex v is not an option since we want to end at v. Vertex t is not an option since that would force us to go to visit s a second time. So, we must go to vertex r next. After vertex r, we must visit x, then w, then v, but we missed vertex t as shown in Figure ✔ 3.9.17.

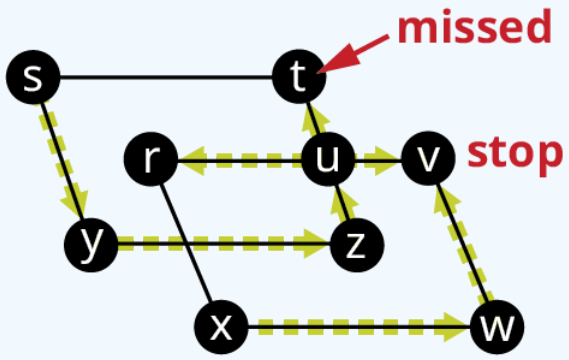


Figure ✔ 3.9.17: Missed Vertex t

Let's go back to the beginning and choose t instead of y. After t, we must go to u and then we have a choice to make between r, v, and z as shown in Figure ✔ 3.9.18.

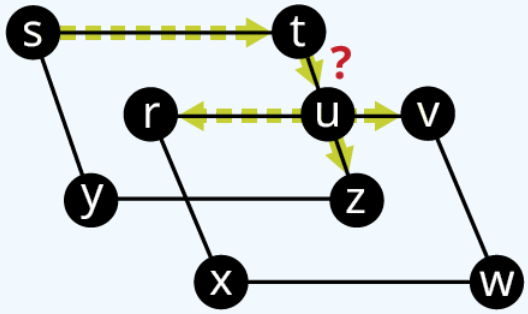


Figure ✔ 3.9.18 Choose Vertex v, r, or z

Vertex v is not an option since we want to end at v. Vertex z is not an option since that would force us to go y and then to visit s a second time. So, we must go to vertex r next. After r, we must go to x then w then v, where we have to stop even though we have missed vertices y and z, as shown in Figure ✔ 3.9.19.

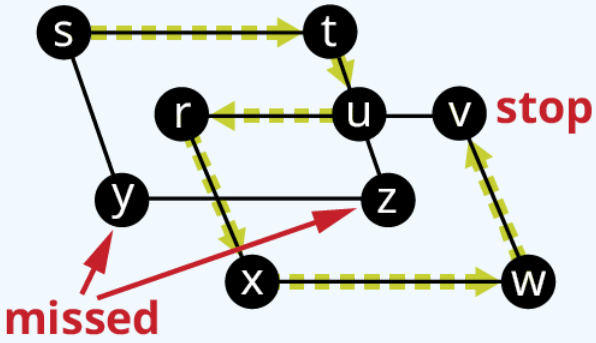


Figure ✔ 3.9.19 Missed Vertices y and z

So, we have tried every possible route and there are no Hamilton paths between s and v in Graph B.

Graph C: In Graph C, there is a Hamilton path, $s \to t \to u \to x \to w \to v$.

Graph D: In Graph D, there is a bridge, tx, which would form components $\{r, s, t, u, q\}$ and $\{v, w, x, y, z\}$ if it were removed. Since s and v would be in different components, it is possible there is a Hamilton path between them. The only way to know is to try all possibilities. If we begin at s, we can go to r then t, or we can go directly to t, either way, we have a problem as you can see in Figure ✔ 3.9.20.

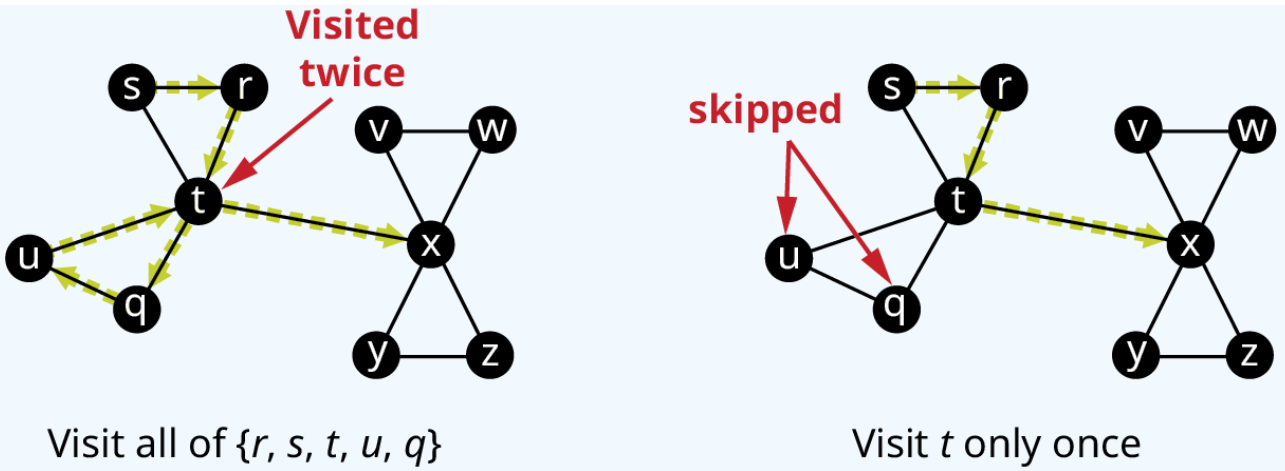


Figure ✔ **3.9.20**: Vertices Visited Twice or Skipped

If we visit all the vertices in the component $\{r, s, t, u, q\}$, we will have to visit t a second time in order to cross the bridge. If we visit t only once, we have to skip some of the vertices. So, there is no Hamilton path between s and v.

Your Turn 3.9.3

Use Graph L to find a Hamilton path between each pair of vertices or indicate that there isn't one.

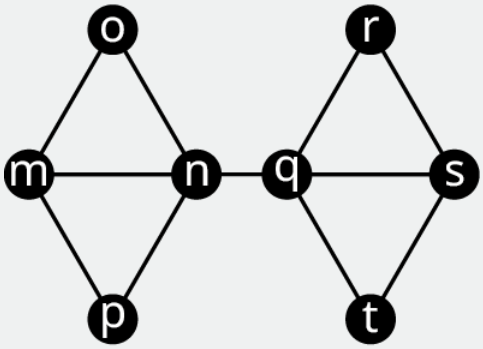
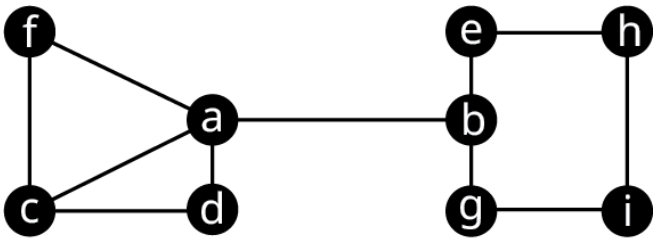
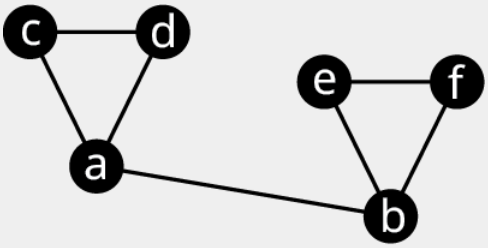
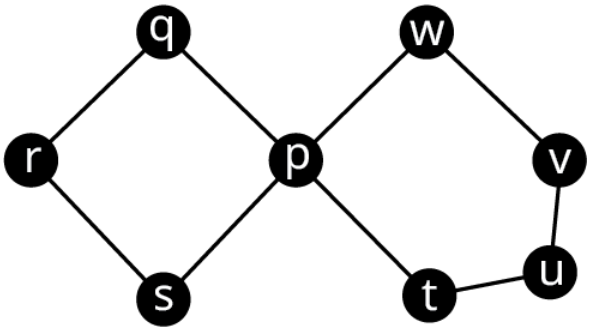
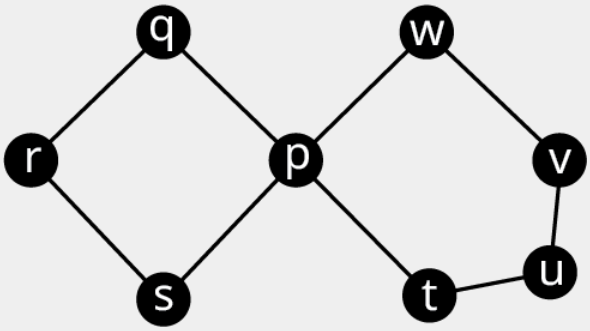


Figure 📌 **3.9.11**: Graph L

1. p to r
2. m to p
3. o to q

There is not a short way to determine if there is a Hamilton path between two vertices on a graph that works in every situation. However, there are a few common situations that can help us to quickly determine that there is no Hamilton path. Some of these are listed in Table **3.9.1**.

Table **3.9.1** Some Impossible Hamilton Paths

Scenario	Diagram
Scenario 1 If an edge *ab* is a bridge, then there is no Hamilton path between a pair of vertices that are on the same side of edge *ab*. We saw this in Graph *A* of Example 12.40.	 No Hamilton path between any two vertices in the component $\{a, c, d, f\}$. No Hamilton path between any two vertices in $\{b, e, h, g, i\}$.
Scenario 2 If an edge *ab* is a bridge with at least three components on each side, then there is no Hamilton path beginning or ending at *a* or *b*. We saw this in Graph *D* of Example 12.40.	 No Hamilton path beginning or ending at *a* or *b*.
	
Scenario 3 If a graph is composed of two cycles joined only at a single vertex *p*, and *v* is any vertex that is NOT adjacent to *p*, then there are no Hamilton paths beginning or ending at *p*. We saw this in Graph *B* of Example 12.40.	 No Hamilton path can be formed starting or ending at vertices, *r*, *v*, or *u* because they are not adjacent to *p*.

Checkpoint

There are also many other situations in which a Hamilton path is not possible. These are just a few that you will encounter.

Hamilton Path or Euler Trail?

We learned in Euler Trails that an Euler trail visits each edge exactly once, whereas a Hamilton path visits each vertex exactly once. Let's practice distinguishing between the two.

Example 3.9.4: Distinguishing between Hamilton Path and Euler Trail

Use Figure 3.9.12 to determine if the given sequence of vertices is a Hamilton path, an Euler trail, both, or neither.

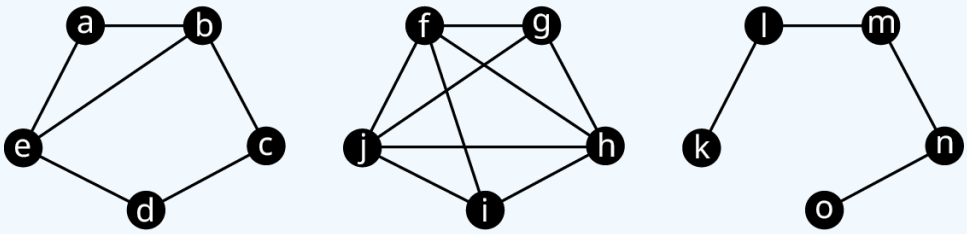


Figure 3.9.12 Graphs A, F, and K

1. Graph A, $e \to b \to a \to e \to d \to c \to b$
2. Graph F, $f \to g \to j \to h \to i$
3. Graph K, $k \to l \to m \to n \to o\backslash$

Answer

1. Since the sequence covers every edge once but visits vertices more than once, it is only an Euler trail.
2. Since the sequence visits every vertex exactly once but skips some edges, it is only a Hamilton path.
3. Since the sequence visits each edge and each vertex exactly once, it is both an Euler trail and a Hamilton path.

Your Turn 3.9.4

Use Figure 12.262 to determine if the given sequence of vertices is a Hamilton path, an Euler trail, both, or neither.

Graph A, $a \to b \to e \to d \to c$

Graph F, $g \to h \to i \to j \to h \to f \to g \to j \to f \to i$

Graph K, $o \to n \to m \to l$

WORK IT OUT

As we saw the Emerald City lies at the center of the Magical Land of Oz, with Gillikin Country to the north, Winkie Country to the east, Munchkin Country to the west, and Quadling Country to the south. Munchkin Country and Winkie Country each shares a border with Gillikin Country and Quadling Country. Let's apply graph theory to Dorothy's famous journey through Oz one more time!

Draw a graph in which each vertex is one of the regions of Oz. Is there a Hamilton path that Dorothy could follow, instead of the yellow brick road, to lead her from the land of the Munchkins, through all the regions of Oz exactly once, and end in the Emerald City? If so, what might it be? Compare your results with those of a classmate.

Check Your Understanding

1. For the following exercises, fill in the blank with *the same as* or *different from* to make the statement true.
2. Unlike in a Hamilton cycle, the vertex where the Hamilton path begins is _____ the vertex where the Hamilton path ends.
3. If a sequence of vertices represents a Hamilton path, the number of vertices listed should be _____ the number of vertices in the whole graph.
4. To determine if a graph has a Hamilton path, use a method that is _____ the method used to determine if a graph has an Euler trail.
5. If a graph with a bridge has a Hamilton path, the starting vertex should be on the side of the bridge that is _____ the side of the bridge with the ending vertex.

6. A path between two vertices of a graph that visits each vertex of the graph exactly once is called an Euler path

    a. True
    b. False

7. Any graph that has exactly two vertices of odd degree has a Hamilton path.

a. True
b. False

8. If a graph is composed of two cycles joined only at a single vertex *p*, then no Hamilton path can be formed starting or ending at any vertex that is adjacent to *p*.

a. True
b. False

9. If an edge *ab* is a bridge with at least three components on each side, then there is no Hamilton path between vertex *a* and any vertex on the other side of edge *ab*.

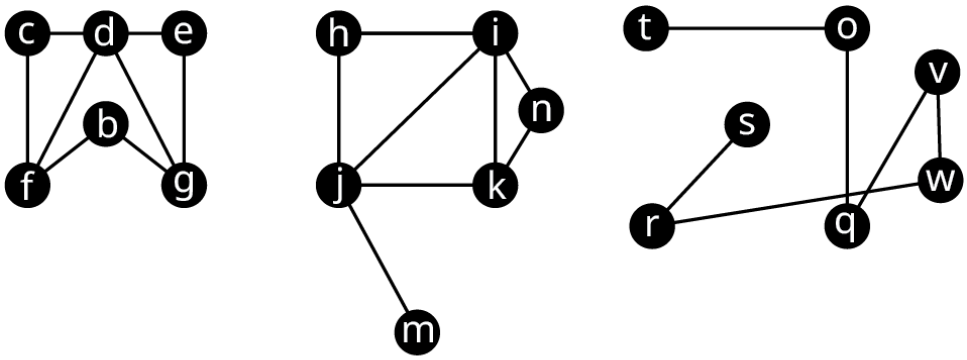
a. True
b. False

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3.9.1: Exercises

For the following exercises, use the figure to determine whether the sequence of vertices in the given graph is a Hamilton path, an Euler trail, both, or neither.



Graph *G* Graph *N* Graph *W*

Exercise 3.9.1.1

Graph *G*: $f \to b \to g \to e \to d \to c$

Exercise 3.9.1.2

Graph *G*: $g \to b \to f \to c \to d \to e$

Exercise 3.9.1.3

Graph *G*: $f \to b \to g \to d \to f \to c \to d \to e \to g$

Exercise 3.9.1.4

Graph *W*: $v \to w \to r \to s \to t \to o \to q$

Exercise 3.9.1.5

Graph *W*: $s \to r \to w \to v \to q \to o \to t$

Exercise 3.9.1.6

Graph *N*: $h \to i \to k \to n \to j \to h$

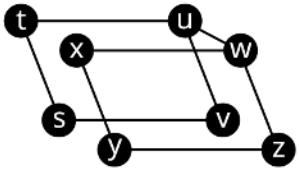
Exercise 3.9.1.7

Graph *N*: $n \to i \to h \to j \to m$

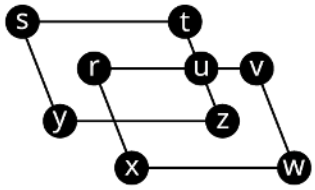
Exercise 3.9.1.8

Graph *N*: $m \to j \to h \to i \to k \to n \to i \to j \to k$

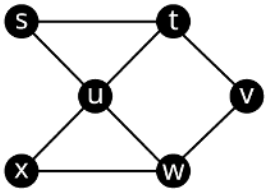
For the following exercises, use the figure to explain why the given sequence of vertices does not represent a Hamilton path.



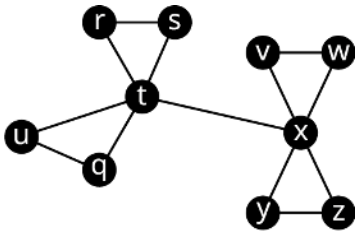
Graph *A*



Graph *B*



Graph *C*



Graph *D*

? Exercise 3.9.1.9

Graph *A*: $t \to s \to v \to u \to x \to w \to y \to z$

? Exercise 3.9.1.10

Graph *B*: $w \to x \to r \to u \to z \to y \to s \to t \to u \to v$

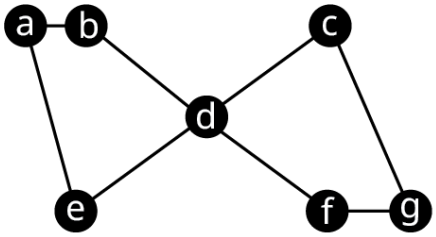
? Exercise 3.9.1.11

Graph *C*: $s \to u \to w \to v \to t$

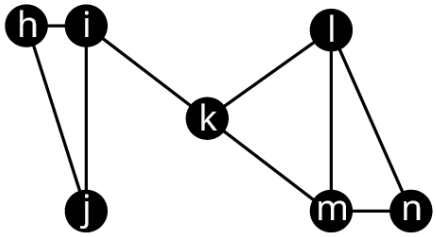
? Exercise 3.9.1.12

Graph *D*: $r \to \ \to t \to q \to u \to t \to x \to v \to w \to x \to z \to y$

For the following exercises, use the figure to find a path that fits the description or indicate which scenario from the figure makes it impossible.



Graph *H*



Graph *Q*

13.

A Hamilton path in Graph H that begins at vertex c and ends at vertex e.

? Exercise 3.9.1.14

A Hamilton path in Graph *Q* that begins at vertex *n* and ends at vertex *h*.

? Exercise 3.9.1.15

A Hamilton path in Graph *H* that begins at vertex *c* and ends at vertex *g*.

? Exercise 3.9.1.16

A Hamilton path in Graph *Q* that begins at vertex *m* and ends at vertex *j*.

Exercise 3.9.1.17

A Hamilton path in Graph H that begins at vertex g.

Exercise 3.9.1.18

A Hamilton path in Graph Q that begins at vertex i.

Exercise 3.9.1.19

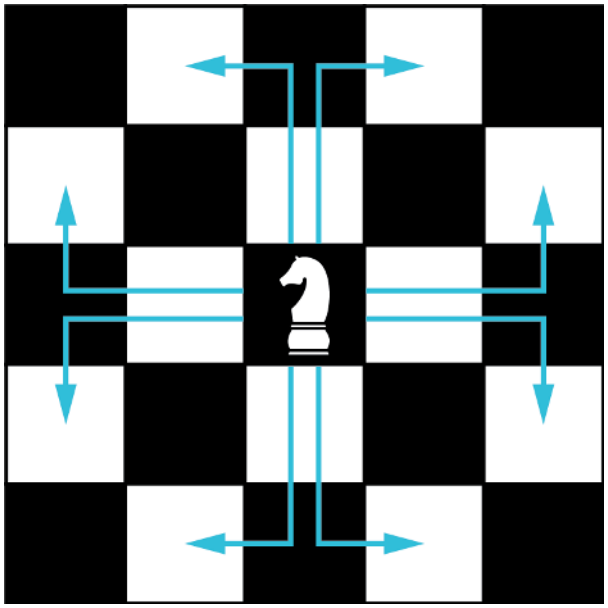
A path between n and j in Graph Q that is NOT a Hamilton path, and explain why it is not a Hamilton path.

Exercise 3.9.1.20

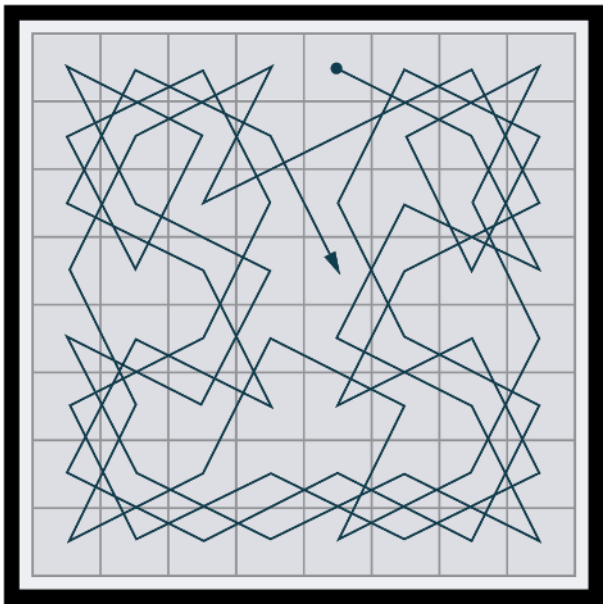
A path between a and c in Graph H that is NOT a Hamilton path, and explain why it is not a Hamilton path.

21.

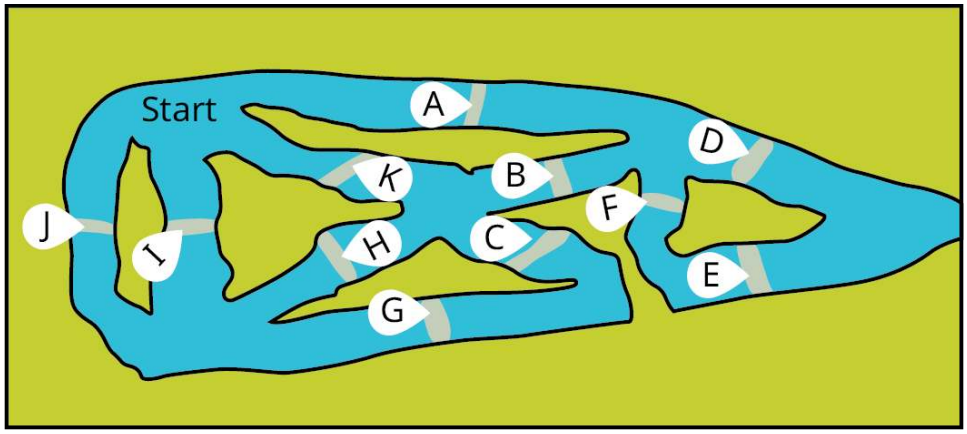
In chess, a knight can move in any direction, but it must move two spaces then turn and move one more space. The eight possible moves a knight can make from a given space are shown in the figure.



A knight's tour is a sequence of moves by a knight on a chessboard (of any size) such that the knight visits every square exactly once. If the knight's tour brings the knight back to its starting position on the board, it is called a closed knight's tour. Otherwise, it is called an open knight's tour. Determine if the Knight's tour shown in the figure is a Hamilton path, an Euler trail, or both, for the graph of all possible knight moves on an eight-by-eight chess board in which the vertices are spaces on the board and the edges indicate that the knight can move directly from one space to the other. Explain your reasoning.



Recall from the section Euler Circuits, as part of the Camp Woebegone Olympics, there is a canoeing race with a checkpoint on each of the 11 different streams as shown in the figure. The contestants must visit each checkpoint.



Draw a graph in which the vertices represent checkpoints, and an edge indicates that it is possible to travel from one checkpoint to the next without passing through another checkpoint.

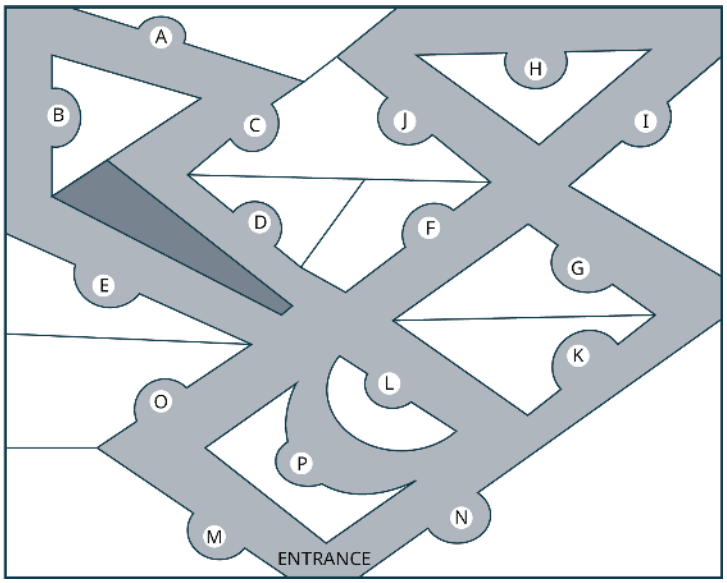
? Exercise 3.9.1.23

Find a Hamilton path beginning at vertex *A* and ending at vertex *E*.

? Exercise 3.9.1.24

What does this Hamilton path represent in the context of the race?

The figure shows a map of zoo exhibits A through P. Use it to answer each question.



25.

Draw a graph to represent the routes through the zoo in which the edges represent walkways and the vertices represent exhibits. Two vertices are connected if a person can walk between the exhibits they represent without passing another exhibit.

? Exercise 3.9.1.26

Use the graph you created to find a route that begins at exhibit *M*, ends at exhibit *J*, and visits each exhibit exactly once.

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3.10: Traveling Salesperson Problem



Figure **3.10.1**: Each door on the route of a traveling salesperson can be represented as a vertex on a graph. (credit: "Three in a row, Heriot Row" by Jason Mason/Flickr, CC BY 2.1)

Learning Objectives

1. Distinguish between brute force algorithms and greedy algorithms.
2. List all distinct Hamilton cycles of a complete graph.
3. Apply brute force method to solve traveling salesperson applications.
4. Apply nearest neighbor method to solve traveling salesperson applications.

We looked at Hamilton cycles and paths in the previous sections Hamilton Cycles and Hamilton Paths. In this section, we will analyze Hamilton cycles in complete weighted graphs to find the shortest route to visit a number of locations and return to the starting point. Besides the many routing applications in which we need the shortest distance, there are also applications in which we search for the route that is least expensive or takes the least time. Here are a few less common applications that you can read about on a website set up by the mathematics department at the University of Waterloo in Ontario, Canada:

- Design of fiber optic networks
- Minimizing fuel expenses for repositioning satellites
- Development of semi-conductors for microchips
- A technique for mapping mammalian chromosomes in genome sequencing

Before we look at approaches to solving applications like these, let's discuss the two types of algorithms we will use.

Brute Force and Greedy Algorithms

An algorithm is a sequence of steps that can be used to solve a particular problem. We have solved many problems in this chapter, and the procedures that we used were different types of algorithms. In this section, we will use two common types of algorithms, a brute force algorithm and a greedy algorithm. A brute force algorithm begins by listing every possible solution and applying each one until the best solution is found. A greedy algorithm approaches a problem in stages, making the apparent best choice at each stage, then linking the choices together into an overall solution which may or may not be the best solution.

To understand the difference between these two algorithms, consider the tree diagram in Figure 12.214. Suppose we want to find the path from left to right with the largest total sum. For example, branch A in the tree diagram has a sum of $10+2+11+13=36$.

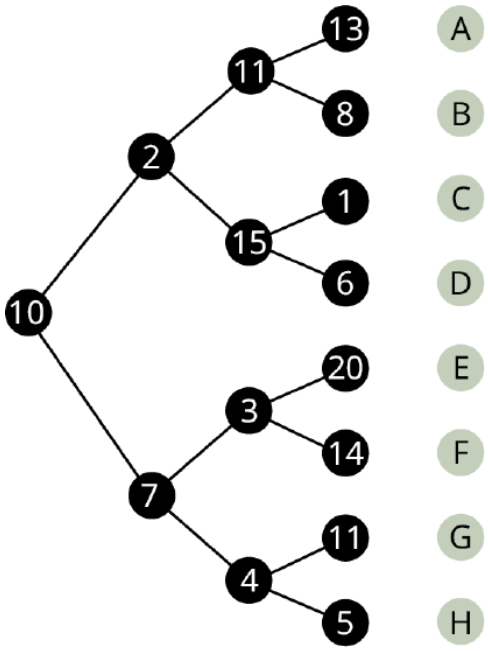


Figure 3.10.2: Points Along Different Paths

To be certain that you pick the branch with greatest sum, you could list each sum from each of the different branches:

A: $10+2+11+13=36$

B: $10+2+11+8=31$

C: $10+2+15+1=28$

D: $10+2+15+6=33$

E: $10+7+3+20=40$

F: $10+7+3+14=34$

G: $10+7+4+11=32$

H: $10+7+4+5=26$

Then we know with certainty that branch *E* has the greatest sum.

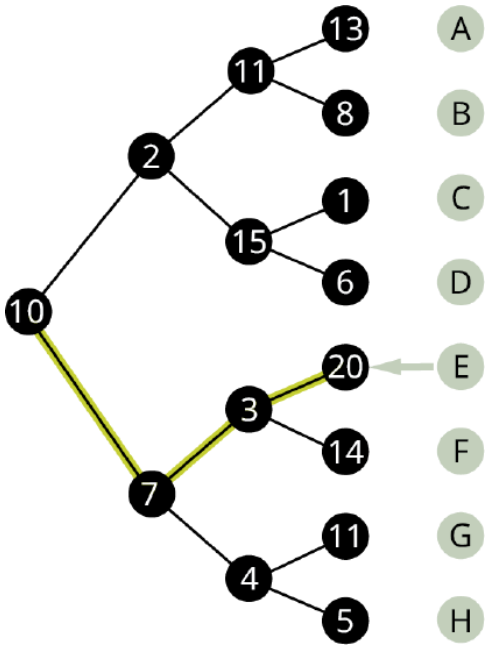


Figure **3.10.3**: Branch *E*

Now suppose that you wanted to find the branch with the highest value, but you only were shown the tree diagram in phases, one step at a time.

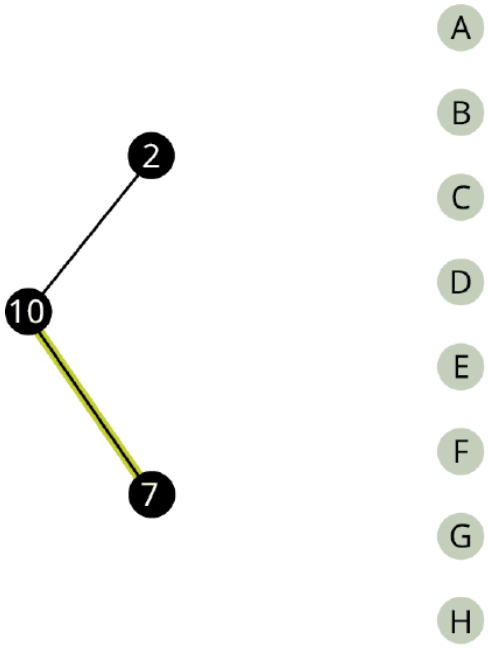


Figure **3.10.4**: Tree Diagram Phase 1

After phase 1, you would have chosen the branch with 10 and 7. So far, you are following the same branch. Let's look at the next phase.

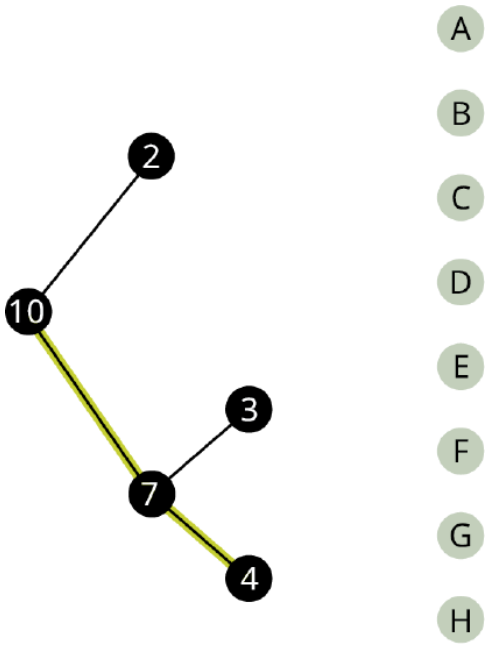


Figure **3.10.5**: Tree Diagram Phase 2

After phase 2, based on the information you have, you will choose the branch with 10, 7 and 4. Now, you are following a different branch than before, but it is the best choice based on the information you have. Let's look at the last phase.

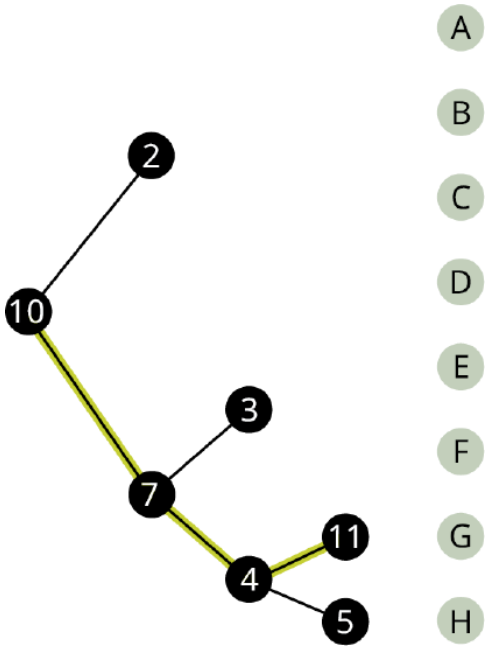


Figure **3.10.6**: Tree Diagram Phase 3

After phase 3, you will choose branch *G* which has a sum of 32.

The process of adding the values on each branch and selecting the highest sum is an example of a brute force algorithm because all options were explored in detail. The process of choosing the branch in phases, based on the best choice at each phase is a greedy algorithm. Although a brute force algorithm gives us the ideal solution, it can take a very long time to implement. Imagine a tree diagram with thousands or even millions of branches. It might not be possible to check all the sums. A greedy algorithm, on the other hand, can be completed in a relatively short time, and generally leads to good solutions, but not necessarily the ideal solution.

Example 3.10.1: Distinguishing between Brute Force and Greedy Algorithms

A cashier rings up a sale for $4.63 cents in U.S. currency. The customer pays with a $5 bill. The cashier would like to give the customer $0.37 in change using the fewest coins possible. The coins that can be used are quarters ($0.25), dimes ($0.10), nickels ($0.05), and pennies ($0.01). The cashier starts by selecting the coin of highest value less than or equal to $0.37, which is a quarter. This leaves $\$0.37 - \$0.25 = \$0.12$. The cashier selects the coin of highest value less than or equal to $0.12, which is a dime. This leaves $\$0.12 - \$0.10 = \$0.02$. The cashier selects the coin of highest value less than or equal to $0.02, which is a penny. This leaves $\$0.02 - \$0.01 = \$0.01$. The cashier selects the coin of highest value less than or equal to $0.01, which is a penny. This leaves no remainder. The cashier used one quarter, one dime, and two pennies, which is four coins. Use this information to answer the following questions.

1. Is the cashier's approach an example of a greedy algorithm or a brute force algorithm? Explain how you know.
2. The cashier's solution is the best solution. In other words, four is the fewest number of coins possible. Is this consistent with the results of an algorithm of this kind? Explain your reasoning.

Answer

1. The approach the cashier used is an example of a greedy algorithm, because the problem was approached in phases and the best choice was made at each phase. Also, it is not a brute force algorithm, because the cashier did not attempt to list out all possible combinations of coins to reach this conclusion.
2. Yes, it is consistent. A greedy algorithm does not always yield the best result, but sometimes it does.

Your Turn 3.10.1

Suppose that you lost the combination to a combination lock that consisted of three digits, and each was selected from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. You desperately need to open the lock without breaking it. You decide to check all possible combinations methodically, 000, then 001, then 002, and so on until you find the right combination. Is this an example of a brute force algorithm or a greedy algorithm?

The Traveling Salesperson Problem

Now let's focus our attention on the graph theory application known as the traveling salesperson problem (TSP) in which we must find the shortest route to visit a number of locations and return to the starting point.

Recall from Hamilton Cycles, the officer in the U.S. Air Force who is stationed at Vandenberg Air Force base and must drive to visit three other California Air Force bases before returning to Vandenberg. The officer needed to visit each base once. We looked at the weighted graph in Figure **3.10.7** representing the four U.S. Air Force bases: Vandenberg, Edwards, Los Angeles, and Beal and the distances between them.

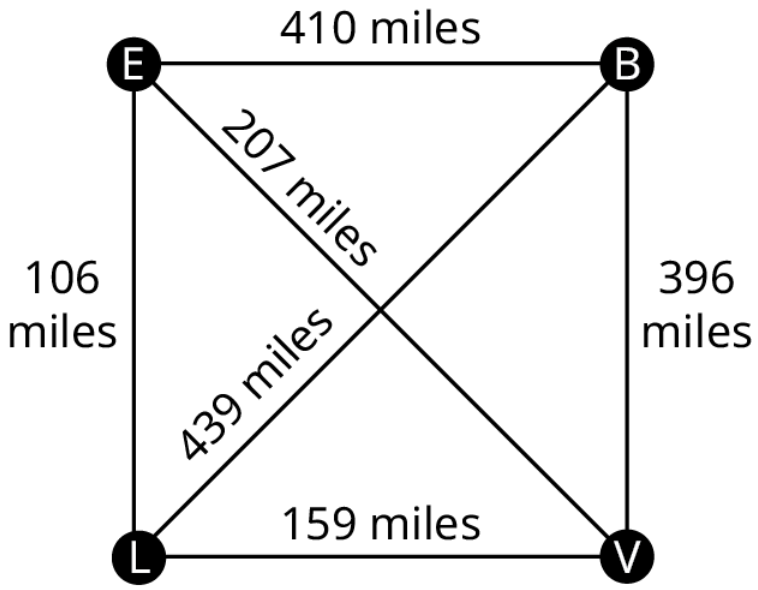
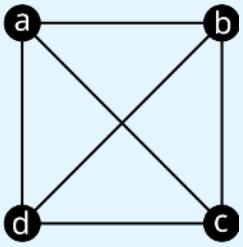
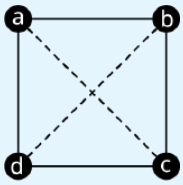
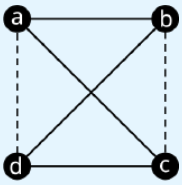
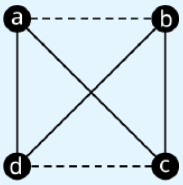
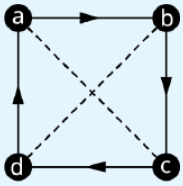
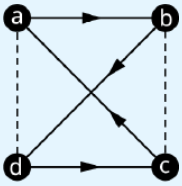
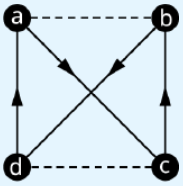
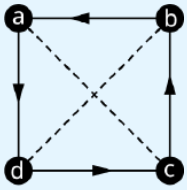
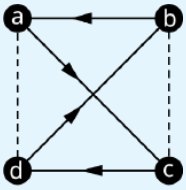
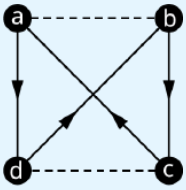


Figure 3.10.7: Graph of Four California Air Force Bases

Any route that visits each base and returns to the start would be a Hamilton cycle on the graph. If the officer wants to travel the shortest distance, this will correspond to a Hamilton cycle of lowest weight. We see in Table 3.10.1 that there are six distinct Hamilton cycles (directed cycles) in a complete graph with four vertices, but some lie on the same cycle (undirected cycle) in the graph.

Table 3.10.1 Hamilton Cycles in a Complete Graph with Four Vertices

Complete Graph	Cycle	Cycle	Cycle
			
Clockwise Hamilton Cycle	 $a \to b \to c \to d \to a$	 $a \to b \to d \to c \to a$	 $a \to c \to b \to d \to a$
Counterclockwise Hamilton Cycle	 $a \to d \to c \to b \to a$	 $a \to c \to d \to b \to a$	 $a \to d \to b \to c \to a$

Since the distance between bases is the same in either direction, it does not matter if the officer travels clockwise or counterclockwise. So, there are really only three possible distances as shown in Figure 3.10.8

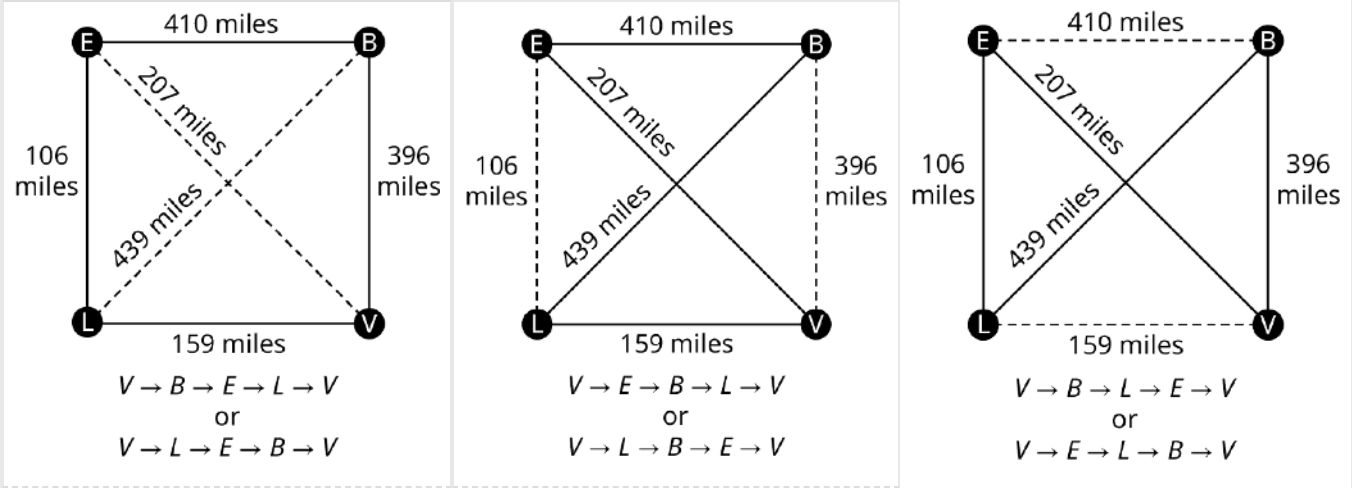


Figure 3.10.8: Three Possible Distances

The possible distances are:

$$396 + 410 + 106 + 159 = 1071$$
$$207 + 410 + 439 + 159 = 1215$$
$$396 + 439 + 106 + 207 = 1148$$

So, a Hamilton cycle of least weight is $V \to B \to E \to L \to V$ (or the reverse direction). The officer should travel from Vandenberg to Beal to Edwards, to Los Angeles, and back to Vandenberg.

Finding Weights of All Hamilton Cycles in Complete Graphs

Notice that we listed all of the Hamilton cycles and found their weights when we solved the TSP about the officer from Vandenberg. This is a skill you will need to practice. To make sure you don't miss any, you can calculate the number of possible Hamilton cycles in a complete graph. It is also helpful to know that half of the directed cycles in a complete graph are the same cycle in reverse direction, so, you only have to calculate half the number of possible weights, and the rest are duplicates.

FORMULA

In a complete graph with n vertices,

- The number of distinct Hamilton cycles is $(n-1)!$.
- There are at most $\frac{(n-1)!}{2}$ different weights of Hamilton cycles.

Checkpoint

TIP! When listing all the distinct Hamilton cycles in a complete graph, you can start them all at any vertex you choose. Remember, the cycle $a \to b \to c \to a$ is the same cycle as $b \to c \to a \to b$ so there is no need to list both.

Example 3.10.2: Calculating Possible Weights of Hamilton Cycles

Suppose you have a complete weighted graph with vertices *N*, *M*, *O*, and *P*.

1. Use the formula $(n-1)!$ to calculate the number of distinct Hamilton cycles in the graph.
2. Use the formula $\frac{(n-1)!}{2}$ to calculate the greatest number of different weights possible for the Hamilton cycles.

3. Are all of the distinct Hamilton cycles listed here? How do you know?
   Cycle 1: $N \to M \to O \to P \to N$
   Cycle 2: $N \to M \to P \to O \to N$
   Cycle 3: $N \to O \to M \to P \to N$
   Cycle 4: $N \to O \to P \to M \to N$
   Cycle 5: $N \to P \to M \to O \to N$
   Cycle 6: $N \to P \to O \to M \to N$
4. Which pairs of cycles must have the same weights? How do you know?

Answer

1. There are 4 vertices; so, $n = 4$. This means there are $(n-1)! = (4-1)! = 3 \cdot 2 \cdot 1 = 6$ distinct Hamilton cycles beginning at any given vertex.
2. Since $n = 4$, there are $\frac{(n-1)!}{2} = \frac{(4-1)!}{2} = \frac{6}{2} = 3$ possible weights.
3. Yes, they are all distinct cycles and there are 6 of them.
4. Cycles 1 and 6 have the same weight, Cycles 2 and 4 have the same weight, and Cycles 3 and 5 have the same weight, because these pairs follow the same route through the graph but in reverse.

Checkpoint

TIP! When listing the possible cycles, ignore the vertex where the cycle begins and ends and focus on the ways to arrange the letters that represent the vertices in the middle. Using a systematic approach is best; for example, if you must arrange the letters M, O, and P, first list all those arrangements beginning with M, then beginning with O, and then beginning with P, as we did in Example 12.42.

Your Turn 3.10.2

A complete weighted graph has vertices *V*, *W*, *X*, *Y*, and *Z*. List all the distinct Hamilton cycles. Use *V* as the starting vertex. Identify the pairs of cycles that have the same weight because they are reverses of each other.

The Brute Force Method

The method we have been using to find a Hamilton cycle of least weight in a complete graph is a brute force algorithm, so it is called the brute force method. The steps in the brute force method are:

Step 1: Calculate the number of distinct Hamilton cycles and the number of possible weights.

Step 2: List all possible Hamilton cycles.

Step 3: Find the weight of each cycle.

Step 4: Identify the Hamilton cycle of lowest weight.

Example 3.10.3: Applying the Brute Force Method

On the next assignment, the air force officer must leave from Travis Air Force base, visit Beal, Edwards, and Vandenberg Air Force bases each exactly once and return to Travis Air Force base. There is no need to visit Los Angeles Air Force base. Use Figure ✔ 3.10.9 to find the shortest route.

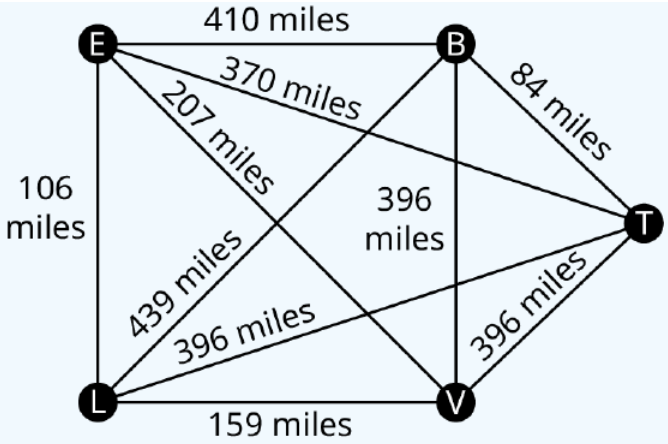


Figure ✔ **3.10.9** Distances between Five California Air Force Bases

Answer

Step 1: Since there are 4 vertices, there will be $(4-1)!=3!=6$ cycles, but half of them will be the reverse of the others; so, there will be $\frac{(4-1)!}{2}=\frac{6}{2}=3$ possible distances.

Step 2: List all the Hamilton cycles in the subgraph of the graph in Figure 12.222.

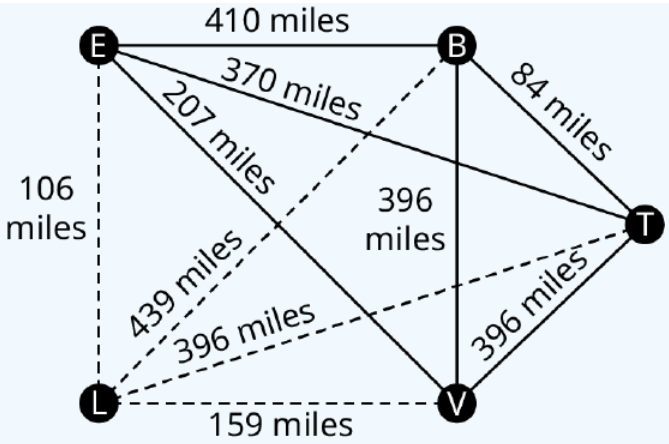


Figure ✔ **3.10.10** Subgraph with Cities *B*, *E*, *T*, and *V*

To find the 6 cycles, focus on the three vertices in the middle,
B, E,
and
V
. The arrangements of these vertices are
BEV, BVE, EBV, EVB, VBE
, and
VEB
. These would correspond to the 6 cycles:

1: $T \to B \to E \to V \to T$

2: $T \to B \to V \to E \to T$

3: $T \to E \to B \to V \to T$

4: $T \to E \to V \to B \to T$

5: $T \to V \to B \to E \to T$

6: $T \to V \to E \to B \to T$

Step 3: Find the weight of each path. You can reduce your work by observing the cycles that are reverses of each other.

1: $84 + 410 + 207 + 396 = 1097$

2: $84 + 396 + 207 + 370 = 1071$

3: $370 + 410 + 396 + 396 = 1572$

4: Reverse of cycle 2, 1071

5: Reverse of cycle 3, 1572

6: Reverse of cycle 1, 1097

Step 4: Identify a Hamilton cycle of least weight.

The second path, $T \to B \to V \to E \to T$, and its reverse, $T \to E \to V \to B \to T$, have the least weight. The solution is that the officer should travel from Travis Air Force base to Beal Air Force Base, to Vandenberg Air Force base, to Edwards Air Force base, and return to Travis Air Force base, or the same route in reverse.

Your Turn 3.10.3

Suppose that the Air Force officer needed to leave from Travis Air Force base, visit each of Beal, Edwards, and Los Angeles Air Force bases exactly once and return to Travis. Use Figure 12.278 to find the shortest route.

Now suppose that the officer needed a cycle that visited all 5 of the Air Force bases in Figure 12.221. There would be $(5-1)! = 4! = 4 \times 3 \times 2 \times 1 = 24$ different arrangements of vertices and $\frac{(5-1)!}{2} = \frac{4!}{2} = \frac{24}{2} = 12$ distances to compare using the brute force method. If you consider 10 Air Force bases, there would be $(10-1)! = 9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362,880$ different arrangements and $\frac{(10-1)!}{2} = \frac{9!}{2} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2} = 181,440$ distances to consider. There must be another way!

The Nearest Neighbor Method

When the brute force method is impractical for solving a traveling salesperson problem, an alternative is a greedy algorithm known as the nearest neighbor method, which always visit the closest or least costly place first. This method finds a Hamilton cycle of relatively low weight in a complete graph in which, at each phase, the next vertex is chosen by comparing the edges between the current vertex and the remaining vertices to find the lowest weight. Since the nearest neighbor method is a greedy algorithm, it usually doesn't give the best solution, but it usually gives a solution that is "good enough." Most importantly, the number of steps will be the number of vertices. That's right! A problem with 10 vertices requires 10 steps, not 362,880. Let's look at an example to see how it works.

Suppose that a candidate for governor wants to hold rallies around the state. They plan to leave their home in city *A*, visit cities *B*, *C*, *D*, *E*, and *F* each once, and return home. The airfare between cities is indicated in the graph in Figure **3.10.11**.

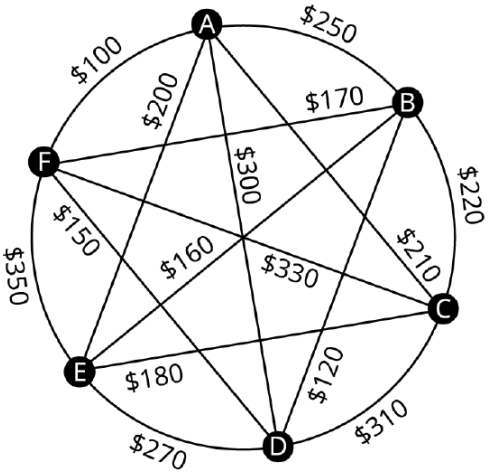


Figure **3.10.11**: Airfares between Cities *A*, *B*, *C*, *D*, *E*, and *F*

Let's help the candidate keep costs of travel down by applying the nearest neighbor method to find a Hamilton cycle that has a reasonably low weight. Begin by marking starting vertex as V_1 for "visited 1st." Then to compare the weights of the edges between A and vertices adjacent to A : $\$250, \$210, \$300, \$200$, and $\$100$ as shown in Figure **3.10.12**. The lowest of these is $\$100$, which is the edge between A and F.

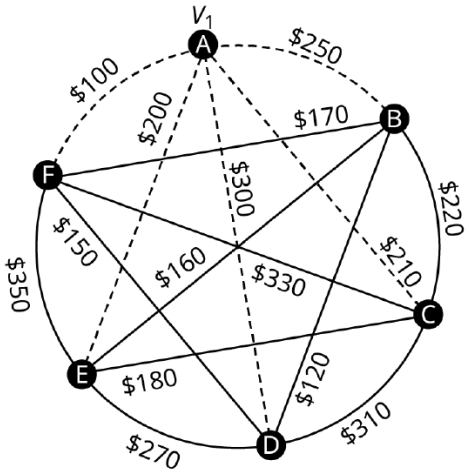


Figure **3.10.12**: Finding the Second Vertex

Mark F as V_2 for "visited 2nd" then compare the weights of the edges between F and the remaining vertices adjacent to F : $\$170, \$330, \$150$ and $\$350$ as shown in Figure **3.10.13**. The lowest of these is $\$150$, which is the edge between F and D.

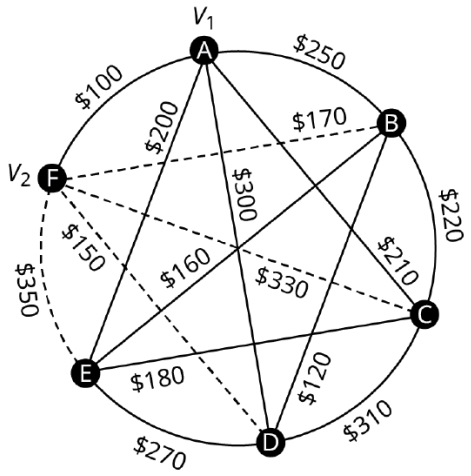


Figure **3.10.13**: Finding the Third Vertex

Mark D as V_3 for "visited 3rd." Next, compare the weights of the edges between D and the remaining vertices adjacent to D : **$120, $310**, and **$270** as shown in Figure **3.10.14**. The lowest of these is **$120**, which is the edge between D and B.

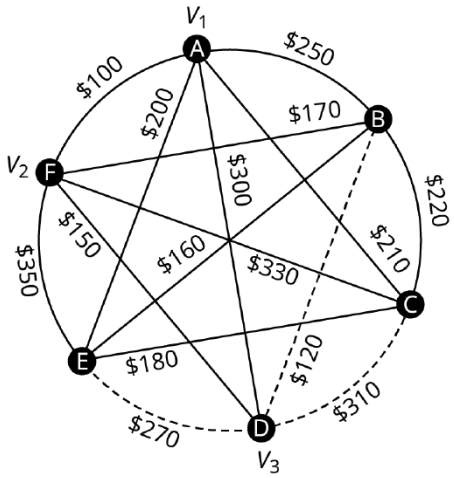


Figure **3.10.14**: Finding the Fourth Vertex

So, mark B as V_4 for "visited 4th." Finally, compare the weights of the edges between B and the remaining vertices adjacent to B : **$160** and **$220** as shown in Figure **3.10.15**. The lower amount is **$160**, which is the edge between B and E

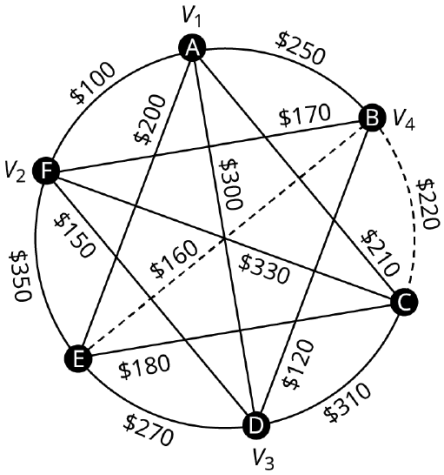


Figure **3.10.15**: Finding the Fifth Vertex

Now you can mark E as V_5 and mark the only remaining vertex, which is C, as V_6. This is shown in Figure 3.10.16. Make a note of the weight of the edge from E to C, which is \$180, and from C back to A, which is \$210.

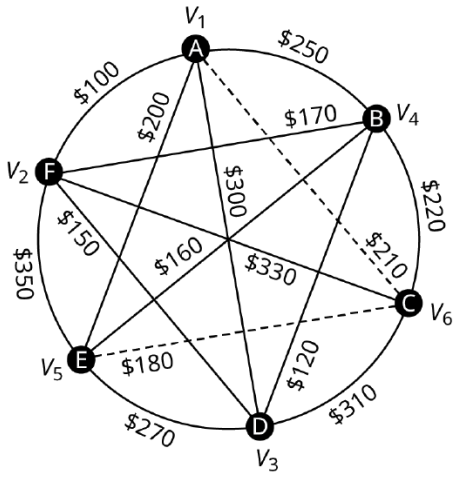


Figure 3.10.16: Finding the Sixth Vertex

The Hamilton cycle we found is $A \rightarrow F \rightarrow D \rightarrow B \rightarrow E \rightarrow C \rightarrow A$. The weight of the circuit is $\$100 + \$150 + \$120 + \$160 + \$180 + \$210 = \$920$. This may or may not be the route with the lowest cost, but there is a good chance it is very close since the weights are most of the lowest weights on the graph and we found it in six steps instead of finding 120 different Hamilton cycles and calculating 60 weights. Let's summarize the procedure that we used.

Step 1: Select the starting vertex and label semanticsV1V1 for "visited 1st." Identify the edge of lowest weight between semanticsV1V1 and the remaining vertices.

Step 2: Label the vertex at the end of the edge of lowest weight that you found in previous step as semanticsVnVn where the subscript n indicates the order the vertex is visited. Identify the edge of lowest weight between semanticsVnVn and the vertices that remain to be visited.

Step 3: If vertices remain that have not been visited, repeat Step 2. Otherwise, a Hamilton cycle of low weight is $V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_n \rightarrow$V1V1→V2→⋯→Vn→V1.

✔ Example 3.10.4: Using the Nearest Neighbor Method

Suppose that the candidate for governor wants to hold rallies around the state but time before the election is very limited. They would like to leave their home in city *A*, visit cities *B*, *C*, *D*, *E*, and *F* each once, and return home. The airfare between cities is not as important as the time of travel, which is indicated in Figure ✔ 3.10.17. Use the nearest neighbor method to find a route with relatively low travel time. What is the total travel time of the route that you found?

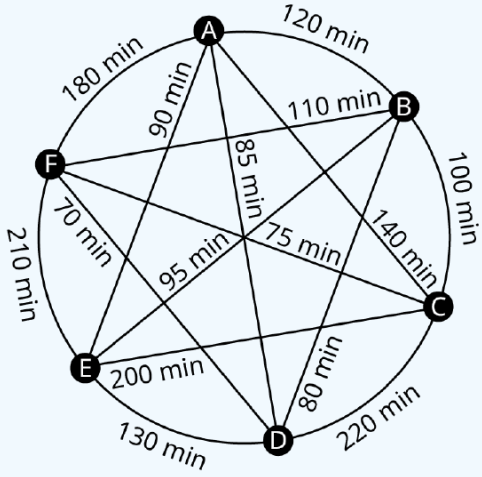


Figure ✔ **3.10.17** Travel Times between Cities *A*, *B*, *C*, *D*, *E* and *F*

Answer

Step 1: Label vertex *A* as semanticsV1V1. The edge of lowest weight between *A* and the remaining vertices is 85 min between *A* and *D*.

Step 2: Label vertex *D* as semanticsV2V2. The edge of lowest weight between *D* and the vertices that remain to be visited, *B*, *C*, *E*, and *F*, is 70 min between *D* and *F*.

Repeat Step 2: Label vertex *F* as semanticsV3V3. The edge of lowest weight between *F* and the vertices that remain to be visited, *B*, *C*, and *E*, is 75 min between *F* and *C*.

Repeat Step 2: Label vertex *C* as semanticsV4V4. The edge of lowest weight between *C* and the vertices that remain to be visited, *B* and *E*, is 100 min between *C* and *B*.

Repeat Step 2: Label vertex *B* as semanticsV5V5. The only vertex that remains to be visited is *E*. The weight of the edge between *B* and *E* is 95 min.

Step 3: A Hamilton cycle of low weight is $A \to D \to F \to C \to B \to E \to A$. So, a route of relatively low travel time is *A* to *D* to *F* to *C* to *B* to *E* and back to *A*. The total travel time of this route is: 85min+70min+75min+100min+95min+90min=515minor8hrs35min85min+70min+75min+100min+95min+90min=515minor8hrs35min

Your Turn 3.10.4

Use the nearest neighbor method to find a Hamilton cycle of relatively low weight beginning and ending at vertex *D* in Figure 12.240 and find its total weight. Give your answer in hours and minutes.

Check Your Understanding

1. The advantage of a greedy algorithm is that it is more efficient.

a. True
b. False

2. The disadvantage of a brute force algorithm is that it does not always give the ideal solution.

a. True
b. False

3. The nearest neighbor method is an example of a brute force algorithm.

a. True
b. False

4. The brute force method is an example of a greedy algorithm.

a. True
b. False

5. The brute force method is used to find a Hamilton cycle of least weight in a complete graph.

a. True
b. False

6. The nearest neighbor method is used to find the ideal solution to the traveling salesperson problem.

a. True
b. False

7. The traveling salesman problem involves finding the shortest route to travel between two points.

a. True
b. False

8. The traveling salesman problem can be represented as finding a Hamilton cycle of least weight on a weighted graph.

a. True
b. False

9. There is always more than one Hamilton cycle of least weight, a given Hamilton cycle and the reverse of that Hamilton cycle.

a. True
b. False

10. The greatest possible number of distinct weights for the Hamilton cycles of a complete graph with n vertices is $(n\text{-}1)!$

a. True
b. False

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3.10.1: Exercises

For the following exercises, determine whether the algorithm described is a greedy algorithm or a brute force algorithm.

Exercise 3.10.1.1

The algorithm for creating graph colorings in the section Navigating Graphs involved coloring the vertex of highest degree first, coloring as many other vertices as possible each color from highest to lowest degree, then repeating this process for the remaining vertices.

Exercise 3.10.1.2

Pallets of goods are to be transported on 10 flatbed trucks which have weight limits. To determine which goods will be shipped together, all the possible ways to divide the goods into 10 groups is listed and the total weight of each group is calculated.

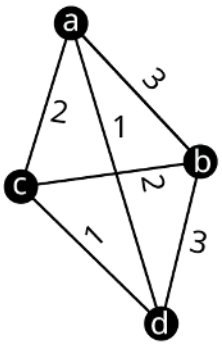
Exercise 3.10.1.3

A wedding planner is creating a seating arrangement for the reception dinner. The couple has provided a list of which guests must be seated together. The wedding planner prefers to use the fewest tables possible so that there is more space to mingle at the reception. The planner creates a list of all possible seating arrangements and selects one that meets these criteria.

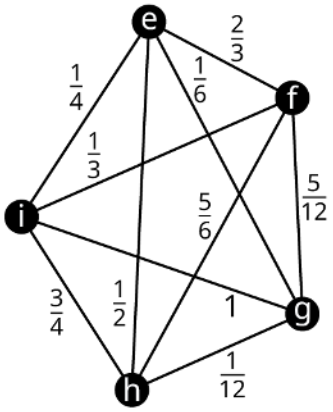
Exercise 3.10.1.4

Packages must be loaded into freight cars to be transported by train. It is preferred to use the fewest freight cars possible to keep the costs down. As each freight car is packed, the package with the largest girth that will fit in the freight car is loaded and this is repeated until the freight car can hold no more packages. When a freight car can hold no more packages, the next freight car is loaded.

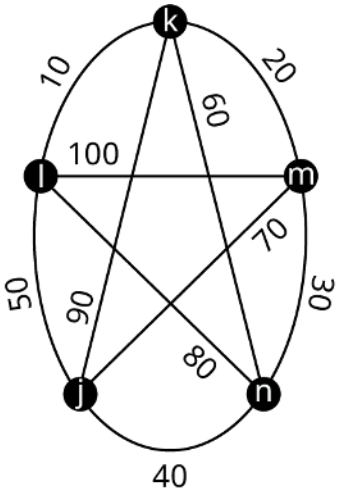
For the following exercises, use the figure to calculate the number of distinct Hamilton cycles beginning at the given vertex in the given graph. How many of those could possibly result in a different weight?



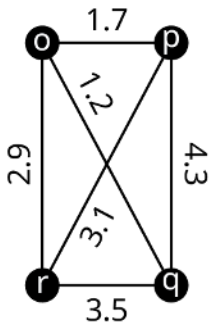
Graph A



Graph B



Graph C



Graph D

Exercise 3.10.1.5

Graph *A*, vertex *a*

Exercise 3.10.1.6

Graph *B*, vertex *e*

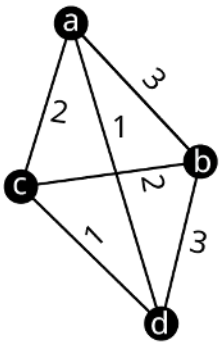
Exercise 3.10.1.7

Graph *C*, vertex *k*

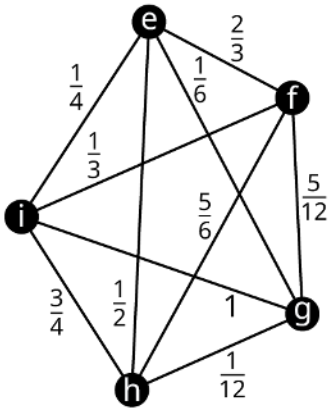
Exercise 3.10.1.8

Graph *D*, vertex *o*

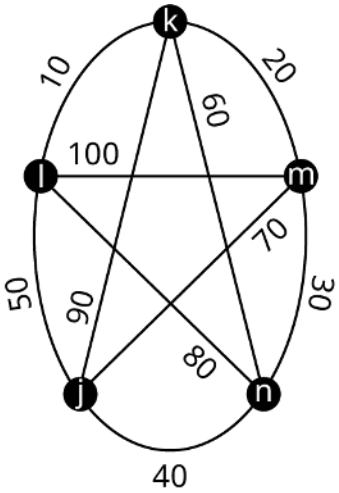
For the following exercises, use the figure to list all the distinct Hamilton cycles beginning at the given vertex in the given graph. Indicate which pairs of Hamilton cycles are reverses of each other.



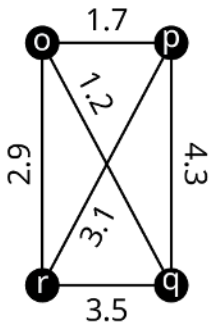
Graph A



Graph B



Graph C



Graph D

Exercise 3.10.1.9

Graph *A*, vertex *a*

Exercise 3.10.1.10

Graph *B*, vertex *e*

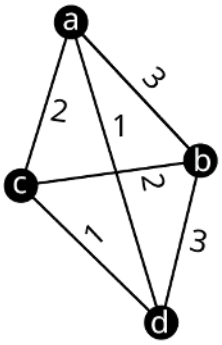
Exercise 3.10.1.11

Graph *C*, vertex *k*

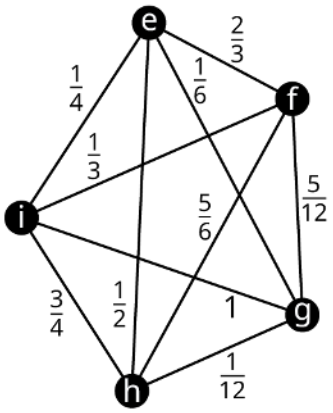
Exercise 3.10.1.12

Graph *D*, vertex *o*

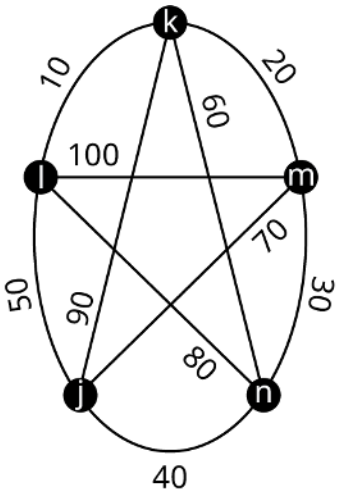
For the following exercises, use the figure to find a Hamilton cycle of least weight for the given graph, beginning at the given vertex, and using the brute force method. What is the weight of the cycle?



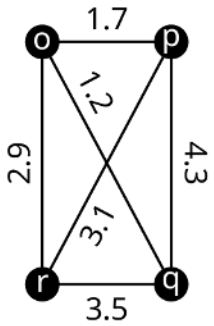
Graph A



Graph B



Graph C



Graph D

? Exercise 3.10.1.13

Graph *A*, vertex *a*

? Exercise 3.10.1.14

Graph *B*, vertex *e*

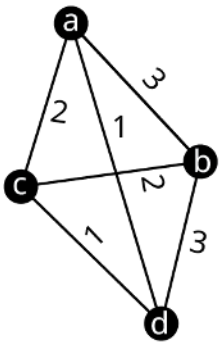
? Exercise 3.10.1.15

Graph *C*, vertex *k*

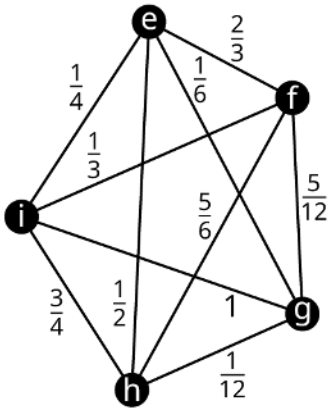
? Exercise 3.10.1.16

Graph *D*, vertex *o*

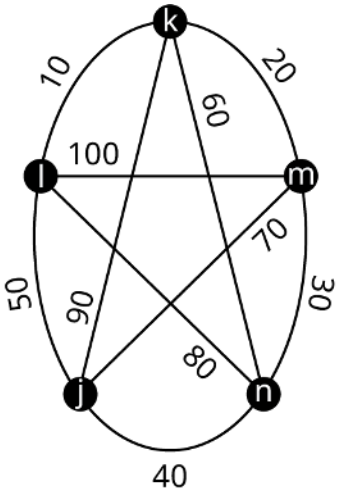
For the following exercises, use the figure to find a Hamilton cycle of low weight for the given graph, beginning at the given vertex, and using the nearest neighbor method. What is the weight of the cycle?



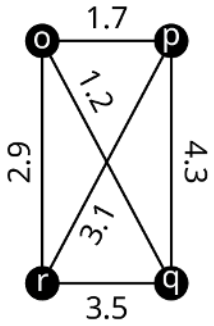
Graph A



Graph B



Graph C



Graph D

Exercise 3.10.1.17

Graph *A*, vertex *a*

Exercise 3.10.1.18

Graph *B*, vertex *e*

Exercise 3.10.1.19

Graph *C*, vertex *k*

Exercise 3.10.1.20

Graph *D*, vertex *o*

For the following exercises, use your solutions to Exercises 13–20 to compare the results of the brute force method to the results of the nearest neighbor method for each graph. Indicate whether the Hamilton cycle was the same or different and whether the weights were the same or different. If the weights are different, indicate which method gave the lower weight. Are your observations consistent with the characteristics of brute force algorithms and greedy algorithms? Explain your reasoning.

Exercise 3.10.1.21

Graph *A*, vertex *a*

Exercise 3.10.1.22

Graph *B*, vertex *e*

Exercise 3.10.1.23

Graph *C*, vertex *k*

Exercise 3.10.1.24

Graph *D*, vertex *o*

For the following exercises, use the table to create a complete weighted graph in which the vertices are the given cities, and the weights are the distances between them.

Cities	*U*	*V*	*W*	*X*	*Y*	*Z*
U	0	89	37	49	54	28
V	89	0	76	68	92	112
W	37	76	0	45	52	49
X	49	68	45	0	66	47
Y	54	92	52	66	0	29
Z	28	112	49	47	29	0

Distances between Cities *U*, *V*, *W*, *X*, *Y*, and *Z* in kilometers

Exercise 3.10.1.25

U, V, W, X

Exercise 3.10.1.26

U, W, Y, Z

Exercise 3.10.1.27

U, X, Y, Z

Exercise 3.10.1.28

U, V, W, X, Y

Exercise 3.10.1.29

U, W, X, Y, Z

Exercise 3.10.1.30

U, V, W, X, Y, Z

For the following exercises, use your solutions to Exercises 25–30 and the nearest neighbor method to find a Hamilton cycle to solve the traveling salesperson problem of finding a reasonably short route to leave from city *U*, visit each of the other cities listed and return to city *U*. Indicate the distance required to travel the route you found.

Exercise 3.10.1.31

U, V, W, X

Exercise 3.10.1.32

U, W, Y, Z

Exercise 3.10.1.33

U, X, Y, Z

Exercise 3.10.1.34

U, V, W, X, Y

Exercise 3.10.1.35

U, W, X, Y, Z

Exercise 3.10.1.36

U, V, W, X, Y, Z

For the following exercises, use your solutions to Exercises 25–30 and the brute force method to find a Hamilton cycle of lowest weight to solve the traveling salesperson problem of finding a shortest route to leave from city *U*, visit each of the other cities listed and return to city *U*. Indicate the distance required to travel the route you found.

Exercise 3.10.1.37

U, *V*, *W*, *X*

Exercise 3.10.1.38

U, *W*, *Y*, *Z*

Exercise 3.10.1.39

U, *X*, *Y*, *Z*

Exercise 3.10.1.40

U, *V*, *W*, *X*, *Y*

Exercise 3.10.1.41

U, *W*, *X*, *Y*, *Z*

For the following exercises, use your solutions to the indicated exercises to compare the results of the brute force method to the results of the nearest neighbor method for each traveling salesman problem of finding a reasonably short route to leave from city *U*, visit each of the other cities listed and return to city *U*. Indicate whether the greedy algorithm resulted in a Hamilton cycle of the same weight, lower weight, or higher weight. Is this consistent with the characteristics of brute force algorithms and greedy algorithms? Explain your reasoning

Exercise 3.10.1.42

Exercises 32 and 38: *U*, *W*, *Y*, *Z*

Exercise 3.10.1.43

Exercises 31 and 37: *U*, *V*, *W*, *X*

Exercise 3.10.1.44

Exercises 34 and 40: *U*, *V*, *W*, *X*, *Y*

Exercise 3.10.1.45

Exercises 35 and 41: *U*, *W*, *X*, *Y*, *Z*

Exercise 3.10.1.46

Exercises 33 and 39: *U*, *X*, *Y*, *Z*

The products at a particular factory are manufactured in phases. The same equipment is utilized for each phase, but it must be formatted differently to accomplish different tasks. The transition time to convert between a format for one task and another task varies. The times are given in Table 12.13. In the following exercises, use the table and the nearest neighbor method to find an

order in which to complete the tasks, which keeps the transition times down and ends with the same set up as it began so that the factory is ready to start the next batch. Assume that there are no restrictions on which tasks can be completed in which order. Hint: The nearest neighbor algorithm may give a different result depending on which vertex is the starting vertex, so, you must check all possibilities.

Task	*A*	*B*	*C*	*D*	*E*	*F*	*G*
A	0	75	130	45	120	70	100
B	75	0	140	65	115	25	60
C	130	140	0	35	55	20	125
D	45	65	35	0	50	30	40
E	120	115	55	50	0	95	145
F	70	25	20	30	95	0	15
G	100	60	125	40	145	15	0

Manufacturing Equipment Transition Times in Minutes

? Exercise 3.10.1.47

A, C, D, F, G

? Exercise 3.10.1.48

B, D, E, F, G

? Exercise 3.10.1.49

A, B, C, D, E, F

? Exercise 3.10.1.50

B, C, D, E, F, G

? Exercise 3.10.1.51

A, B, C, D, E, F, G

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3.11: Trees



Figure **3.11.1**: In graph theory, graphs known as trees have structures in common with live trees. (credit: “Row of trees in Roslev” by AKA CJ/Flickr, Public Domain)

Learning Objectives

1. Describe and identify trees.
2. Determine a spanning tree for a connected graph.
3. Find the minimum spanning tree for a weighted graph.
4. Solve application problems involving trees.

We saved the best for last! In this last section, we will discuss arguably the most fun kinds of graphs, trees. Have you every researched your family tree? Family trees are a perfect example of the kind of trees we study in graph theory. One of the characteristics of a family tree graph is that it never loops back around, because no one is their own grandparent!

What Is A Tree?

Whether we are talking about a family tree or a tree in a forest, none of the branches ever loops back around and rejoins the trunk. This means that a tree has no cyclic subgraphs, or is acyclic. A tree also has only one component. So, a tree is a connected acyclic graph. Here are some graphs that have the same characteristic. Each of the graphs in Figure **3.11.2** is a tree.

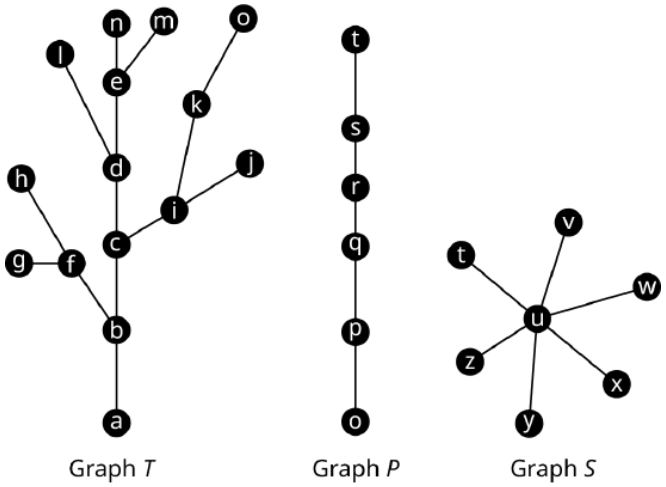


Figure **3.11.2**: Graphs *T*, *P*, and *S*

Let’s practice determining whether a graph is a tree. To do this, check if a graph is connected and has no cycles.

Example 3.11.1: Identifying Trees

Identify any trees in Figure 3.11.3. If a graph is not a tree, explain how you know.

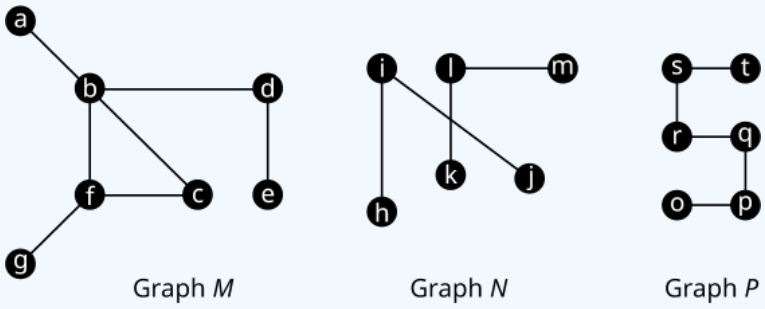


Figure 3.11.3: Graphs *M*, *N*, and *P*

Answer

- Graph *M* is not a tree because it contains the cycle (*b*, *c*, *f*).
- Graph *N* is not a tree because it is not connected. It has two components, one with vertices *h*, *i*, *j*, and another with vertices *k*, *l*, *m*.
- Graph *P* is a tree. It has no cycles and it is connected.

Your Turn 3.11.1

There are some configurations that are commonly used when setting up computer networks. Several of them are shown in the given figure. Which of the configurations in the figure appear to have the characteristics of a tree graph? If a configuration does not appear to have the characteristics of a tree graph, explain how you know.

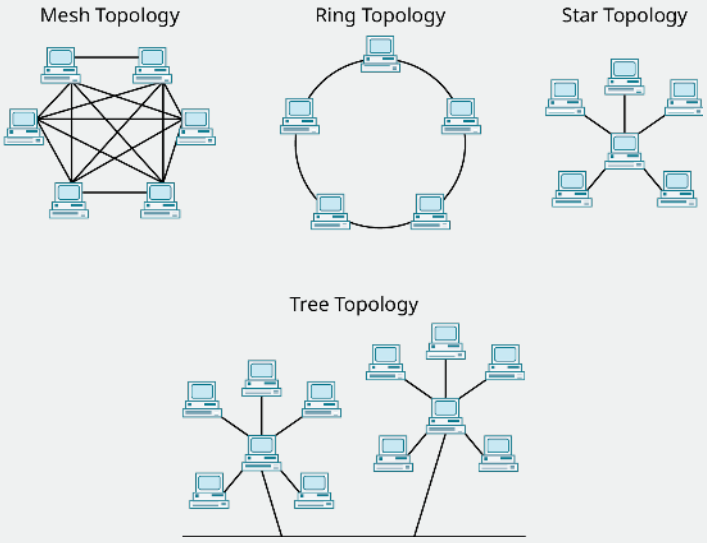


Figure 3.11.4: Common Network Configurations

Types of Trees

Mathematicians have had a lot of fun naming graphs that are trees or that contain trees. For example, the graph in Figure 3.11.5 is not a tree, but it contains two components, one containing vertices *a* through *d*, and the other containing vertices *e* through *g*, each of which would be a tree on its own. This type of structure is called a forest. There are also interesting names for trees with certain characteristics.

- A **path graph or linear graph** is a tree graph that has exactly two vertices of degree 1 such that the only other vertices form a single path between them, which means that it can be drawn as a straight line.

- A **star tree** is a tree that has exactly one vertex of degree greater than 1 called a **root,** and all other vertices are adjacent to it.
- A **starlike tree** is a tree that has a single **root** and several paths attached to it.
- A **caterpillar tree** is a tree that has a central path that can have vertices of any degree, with each vertex not on the central path being adjacent to a vertex on the central path and having a degree of one.
- A **lobster tree** is a tree that has a central path that can have vertices of any degree, with paths consisting of either one or two edges attached to the central path.

Examples of each of these types of structures are given in Figure **3.11.6**.

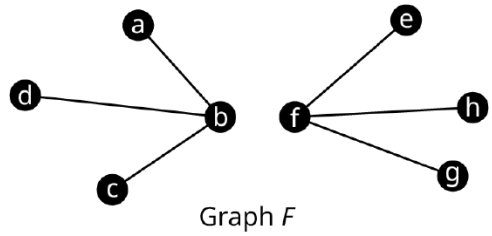


Figure **3.11.5**: Forest Graph F

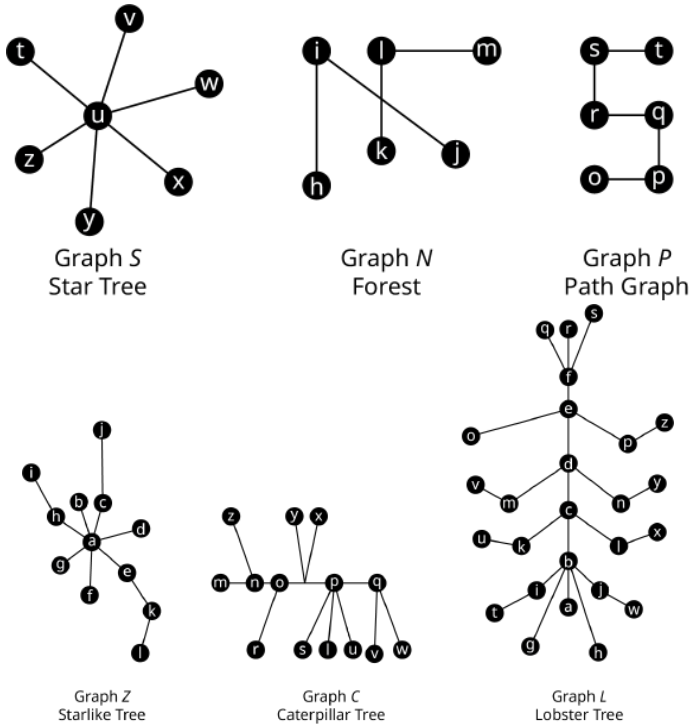


Figure **3.11.6**: Six Types of Trees

✔ Example 3.11.2: Identifying Types of Trees

Each graph in Figure ✔ **3.11.7** is one of the special types of trees we have been discussing. Identify the type of tree.

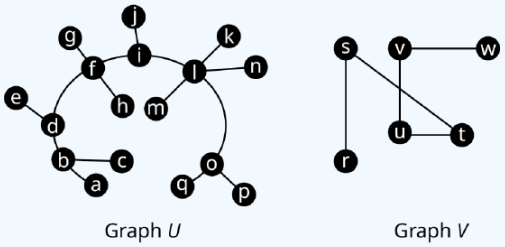


Figure ✔ **3.11.7** Graphs *U* and *V*

Answer

Graph U has a central path $a \to b \to d \to f \to i \to l \to o \to q$. Each vertex that is not on the path has degree 1 and is adjacent to a vertex that is on the path. So, U is a caterpillar tree.

Graph V is a path graph because it is a single path connecting exactly two vertices of degree one, $r \to s \to u \to v \to w$.

Your Turn 3.11.2

Of the network configurations from Figure 3.11.7, which, if any, has the characteristics of a

1. Star tree?
2. Caterpillar tree?
3. Path graph?

Characteristics of Trees

As we study trees, it is helpful to be familiar with some of their characteristics. For example, if you add an edge to a tree graph between any two existing vertices, you will create a cycle, and the resulting graph is no longer a tree. Some examples are shown in Figure 3.11.8. Adding edge bj to Graph T creates cycle (b, c, i, j). Adding edge rt to Graph P creates cycle (r, s, t). Adding edge tv to Graph S creates cycle (t, u, v).

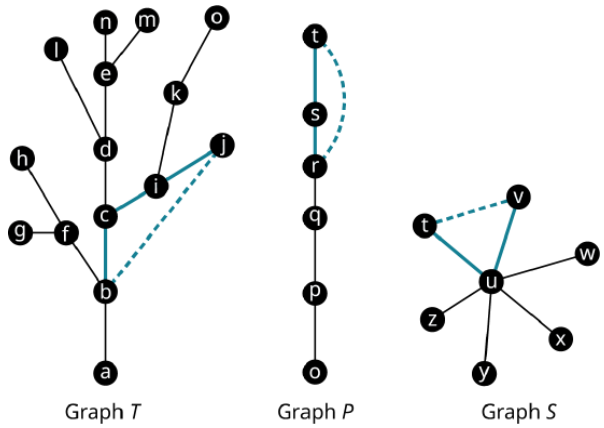


Figure 3.11.8: Adding Edges to Trees

It is also true that removing an edge from a tree graph will increase the number of components and the graph will no longer be connected. In fact, you can see in Figure \(\PageIndex{8}\ that removing one or more edges can create a forest. Removing edge qr from Graph P creates a graph with two components, one with vertices o, p and q, and the other with vertices r, s, and t. Removing edge uw from Graph S creates two components, one with just vertex w and the other with the rest of the vertices. When two edges were removed from Graph T, edge bf and edge cd, creates a graph with three components as shown in Figure 3.11.9.

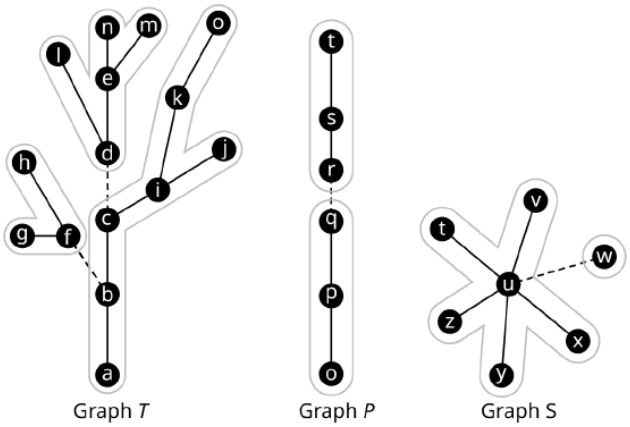


Figure 3.11.9: Removing Edges from Trees

A very useful characteristic of tree graphs is that the number of edges is always one less than the number of vertices. In fact, any connected graph in which the number of edges is one less than the number of vertices is guaranteed to be a tree. Some examples are given in Figure **3.11.10**.

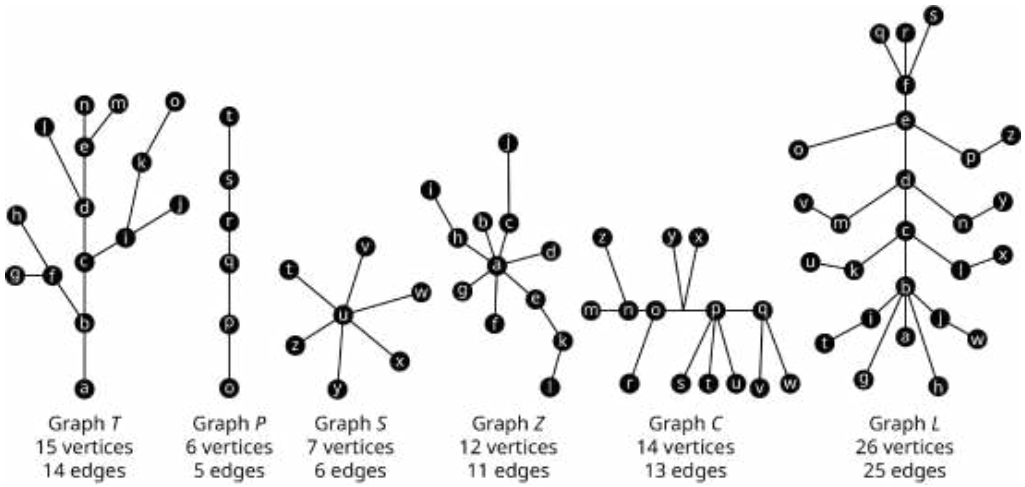


Figure **3.11.10**: Number of Vertices and Edges in Trees vs. Other Graphs

FORMULA

The number of edges in a tree graph with n vertices is $n-1$.

A connected graph with n vertices and $n-1$ edges is a tree graph.

Example 3.11.3: Exploring Characteristics of Trees

Use Graphs *I* and *J* in Figure **3.11.11** to answer each question.

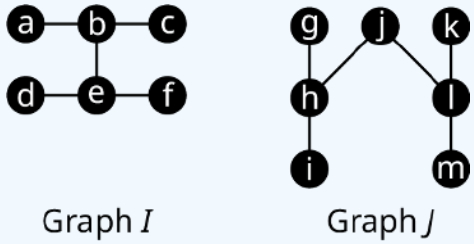


Figure **3.11.11** Graphs *I* and *J*

1. Which vertices are in each of the components that remain when edge *be* is removed from Graph *I*?
2. Determine the number of edges and the number of vertices in Graph *J*. Explain how this confirms that Graph *J* is a tree.
3. What kind of cycle is created if edge *im* is added to Graph *J*?

Answer

1. When edge *be* is removed, there are two components that remain. One component includes vertices *a*, *b*, and *c*. The other component includes vertices *d*, *e*, and *f*.
2. There are seven vertices and six edges in Graph *J*. This confirms that Graph *J* is a tree because the number of edges is one less than the number of vertices.
3. The pentagon (*i*, *h*, *j*, *l*, *m*) is created when edge *im* is added to Graph *J*.

Your Turn 3.11.3

Use Graphs *I* and *J* in Figure 3.11.11 to answer each question.

Which vertices are in each of the components that remain when edge *jl* is removed from Graph *J*?

Determine the number of edges and the number of vertices in Graph *I*. Explain how this confirms that Graph *I* is a tree.

What kind of cycle is created if edge *cf* is added to Graph *I*?

Who Knew?: Graph Theory in the Movies

In the 1997 film *Good Will Hunting*, the main character, Will, played by Matt Damon, solves what is supposed to be an exceptionally difficult graph theory problem, "Draw all the homeomorphically irreducible trees of size $n = 10$n=10." That sounds terrifying! But don't panic. Watch this great Numberphile video to see why this is actually a problem you can do at home!

Video



The Problem in *Good Will Hunting* by Numberphile

Spanning Trees

Suppose that you planned to set up your own computer network with four devices. One option is to use a "mesh topology" like the one in Figure 3.11.12, in which each device is connected directly to every other device in the network.

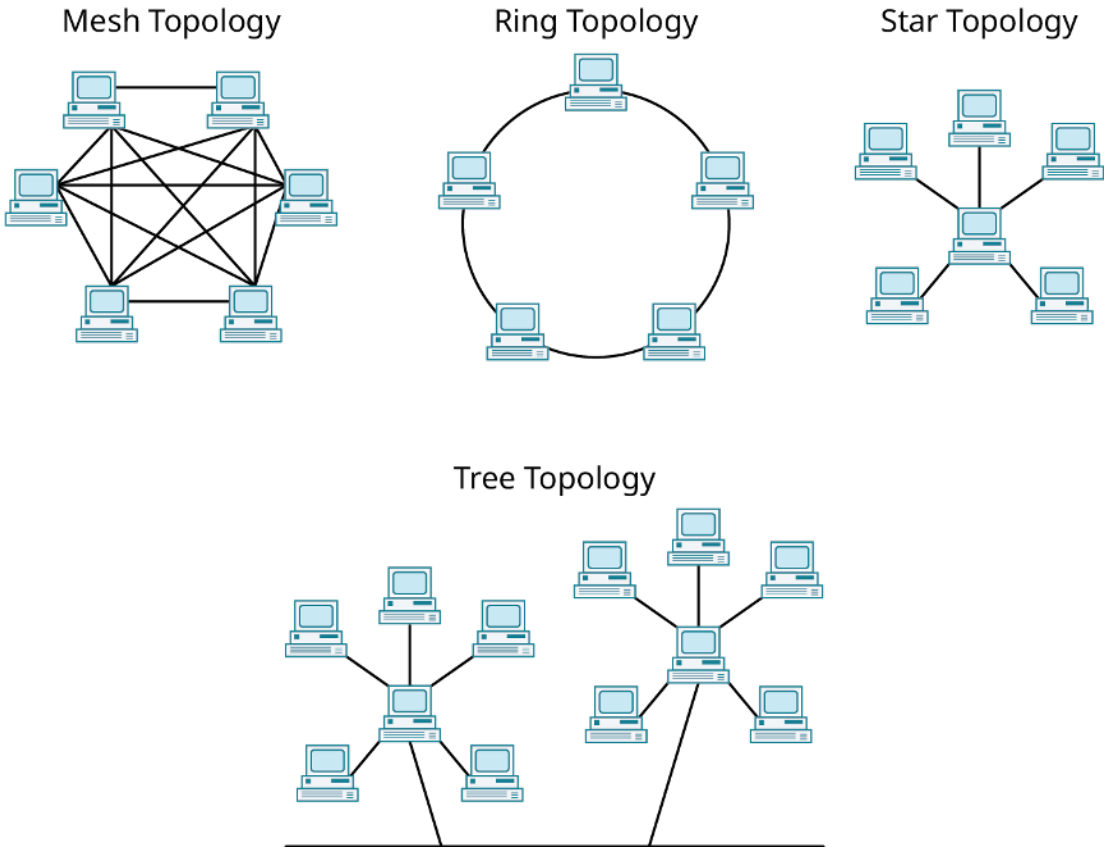


Figure **3.11.12**: Common Network Configurations

The mesh topology for four devices could be represented by the complete Graph A_1 in Figure **3.11.13** where the vertices represent the devices, and the edges represent network connections. However, the devices could be networked using fewer connections. Graphs A_2, A_3, and A_4 of Figure **3.11.13** show configurations in which three of the six edges have been removed. Each of the Graphs A_2, A_3 and A_4 in Figure 12.242 is a tree because it is connected and contains no cycles. Since Graphs A_2, A_3 and A_4 are also subgraphs of Graph A_1 that include every vertex of the original graph, they are also known as spanning trees.

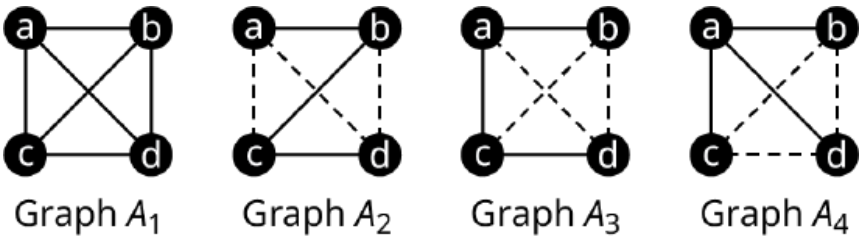


Figure **3.11.13**: Network Configurations for Four Devices

By definition, spanning trees must span the whole graph by visiting all the vertices. Since spanning trees are subgraphs, they may only have edges between vertices that were adjacent in the original graph. Since spanning trees are trees, they are connected and they are acyclic. So, when deciding whether a graph is a spanning tree, check the following characteristics:

- All vertices are included.
- No vertices are adjacent that were not adjacent in the original graph.
- The graph is connected.
- There are no cycles.

Example 3.11.49: Identifying Spanning Trees

Use Figure 3.11.14 to determine which of graphs M_1, M_2, M_3, and M_4, are spanning trees of Q.

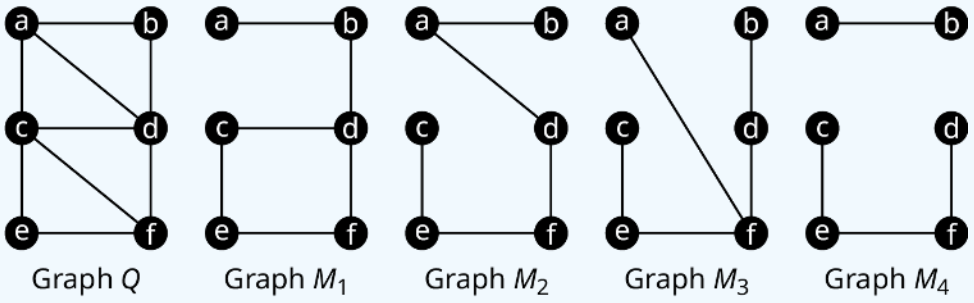


Figure 3.11.14 Graphs Q, M_1, M_2, M_3, and M_4

Answer

1. Graph M_1 is not a spanning tree of Graph Q because it has a cycle (c, d, f, e).
2. Graph M_2 is a spanning tree of Graph Q because it has all the original vertices, no vertices are adjacent in M_2 that weren't adjacent in Graph Q, Graph M_2 is connected and it contains no cycles.
3. Graph M_3 is not a spanning tree of Graph Q because vertices a and f are adjacent in Graph M_3 but not in Graph Q.
4. Graph M_4 is not a spanning tree of Graph Q because it is not connected.

So, only graph M_2 is a spanning tree of Graph Q.

Your Turn 3.11.4

Use the given figure for the following exercises.

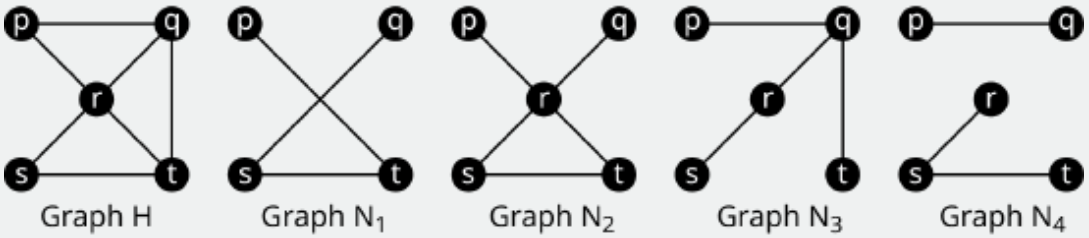


Figure 3.11.15

1. Since sq is not an edge in Graph H, Graph N_1 cannot be a spanning tree of H.
   a. True
   b. False
2. Graph N_2 is a spanning tree of Graph H.
   a. True
   b. False
3. Graph N_3 is a spanning tree of Graph H.
   a. True
   b. False
4. Since there is no path between p and t in Graph N_4, it cannot be a spanning tree of any graph.
   a. True
   b. False

Constructing a Spanning Tree Using Paths

Suppose that you wanted to find a spanning tree within a graph. One approach is to find paths within the graph. You can start at any vertex, go any direction, and create a path through the graph stopping only when you can't continue without backtracking as shown in Figure **3.11.16**.

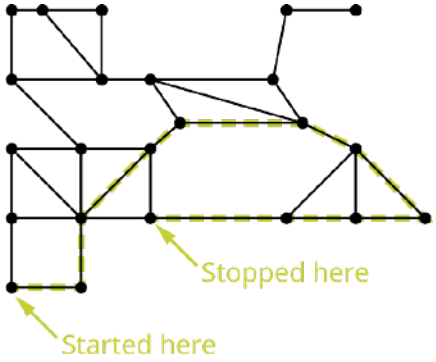


Figure **3.11.16**: First Phase to Construct a Spanning Tree

Once you have stopped, pick a vertex along the path you drew as a starting point for another path. Make sure to visit only vertices you have not visited before as shown in Figure **3.11.17**.

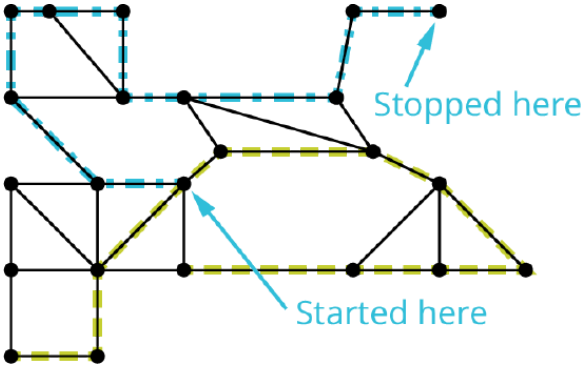


Figure **3.11.17**: Intermediate Phase to Construct a Spanning Tree

Repeat this process until all vertices have been visited as shown in Figure **3.11.18**.

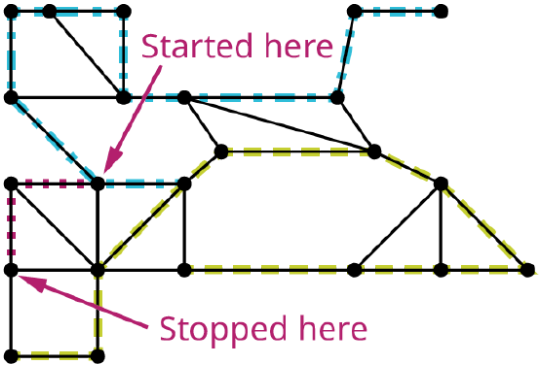


Figure **3.11.18**: Final Phase to Construct a Spanning Tree

The end result is a tree that spans the entire graph as shown in Figure **3.11.19**.

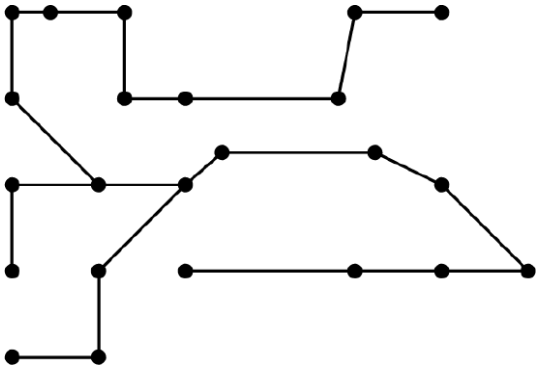


Figure 3.11.19: The Resulting Spanning Tree

Notice that this subgraph is a tree because it is connected and acyclic. It also visits every vertex of the original graph, so it is a spanning tree. However, it is not the only spanning tree for this graph. By making different turns, we could create any number of distinct spanning trees.

✔ Example 3.11.5: Constructing Spanning Trees

Construct two distinct spanning trees for the graph in Figure ✔ 3.11.20.

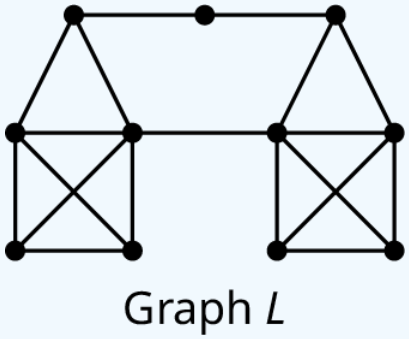


Figure ✔ 3.11.20 Graph *L*

Answer

Two possible solutions are given in Figure ✔ 3.11.21 and Figure ✔ 3.11.22.

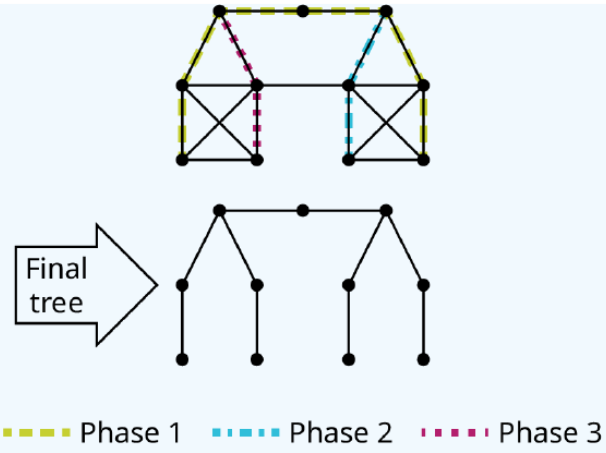


Figure ✔ **3.11.21** First Spanning Tree for Graph *L*

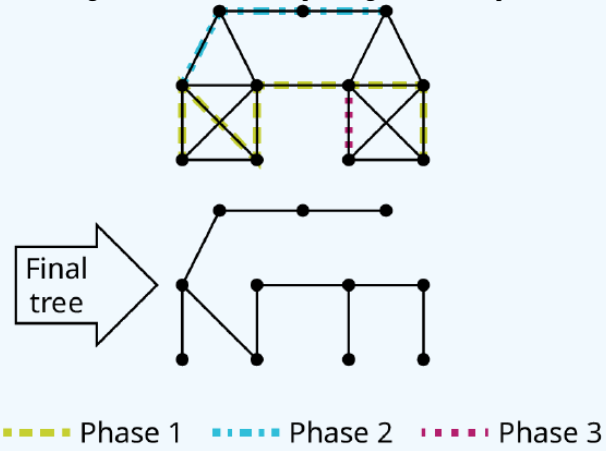


Figure ✔ **3.11.22**: Second Spanning Tree for Graph *L*

Your Turn 3.11.5

Construct three distinct spanning trees for Graph *J*.

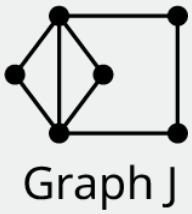


Figure **3.11.23**: Graph *J*

Revealing Spanning Trees

Another approach to finding a spanning tree in a connected graph involves removing unwanted edges to reveal a spanning tree. Consider Graph *D* in Figure **3.11.24**.

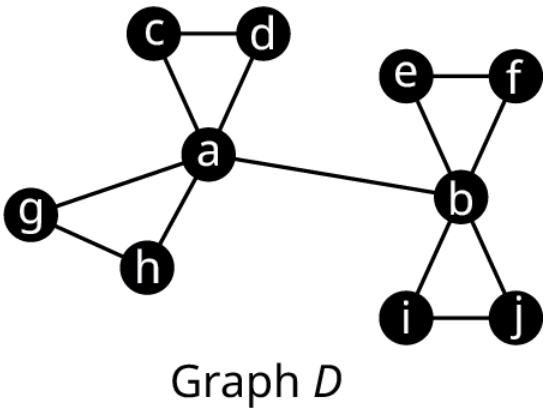


Figure **3.11.24**: Graph *D*

Graph *D* has 10 vertices. A spanning tree of Graph *D* must have 9 edges, because the number of edges is one less than the number of vertices in any tree. Graph *D* has 13 edges so 4 need to be removed. To determine which 4 edges to remove, remember that trees do not have cycles. There are four triangles in Graph *D* that we need to break up. We can accomplish this by removing 1 edge from each of the triangles. There are many ways this can be done. Two of these ways are shown in Figure **3.11.25**.

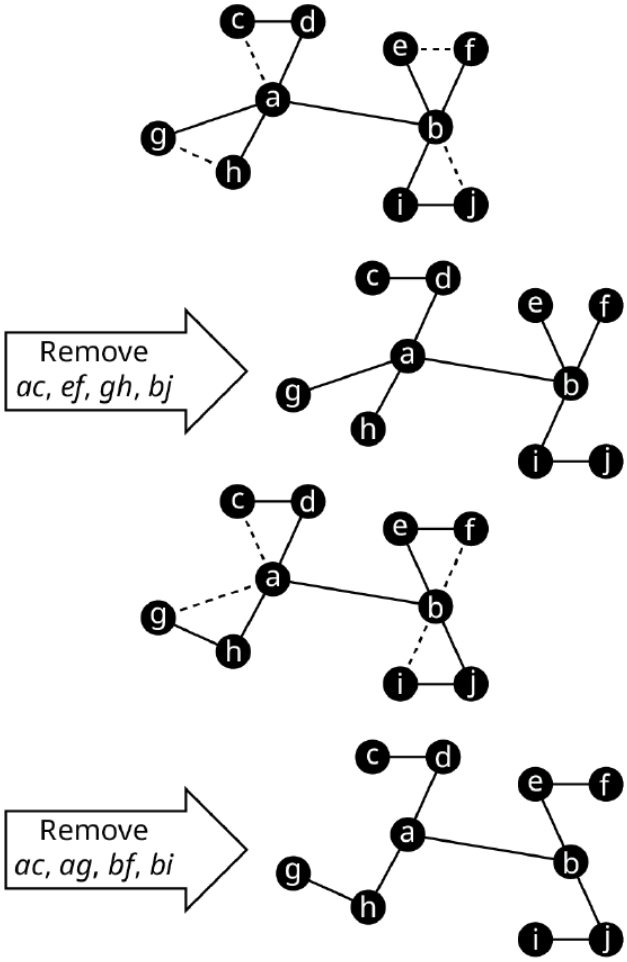


Figure **3.11.25**: Removing Four Edges from Graph *D*

https://math.libretexts.org/@go/page/214470

Video



Spanning Trees in Graph Theory

Example 3.11.6: Removing Edges to Find Spanning Trees

Use the graph in Figure 12.255 to answer each question.

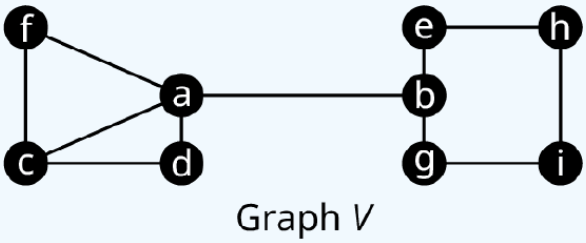


Figure 3.11.26 Graph V

1. Determine the number of edges that must be removed to reveal a spanning tree.
2. Name all the undirected cycles in Graph V.
3. Find two distinct spanning trees of Graph V.

Answer

1. Graph V has nine vertices so a spanning tree for the graph must have 8 edges. Since Graph V has 11 edges, 3 edges must be removed to reveal a spanning tree.
2. (a, c, d), (a, c, f), (a, d, c, f), and (b, e, h, i, g)
3. To find the first spanning tree, remove edge ac, which will break up both of the triangles, remove edge cf , which will break up the quadrilateral, and remove be, which will break up the pentagon, to give us the spanning tree shown in Figure 3.11.27.



Figure ✔ **3.11.27** Spanning Tree Formed Removing *ac*, *cf*, and *be*

https://math.libretexts.org/@go/page/214470

To find another spanning tree, remove ad, which will break up (a, c, d) and (a, d, c, f), remove af to break up (a, c, f), and remove hi to break up (b, e, h, i, g). This will give us the spanning tree in Figure ✔ 3.11.28.

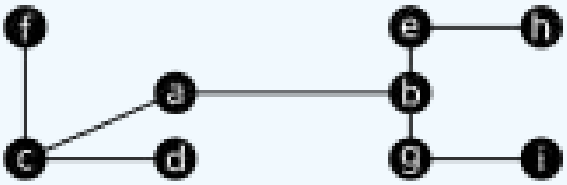


Figure ✔ 3.11.28: Spanning Tree Formed Removing *ad*, *af*, and *hi*

Your Turn 3.11.6

Name three edges that you could remove from Graph *V* in Figure 12.316 to form a third spanning tree, different from those in the solution to Example 12.50 Exercise 3.

Who Knew?: Chains of Affection

Here is a strange question to ask in a math class: Have you ever dated your ex's new partner's ex? Research suggests that your answer is probably no. When researchers Peter S. Bearman, James Moody, and Katherine Stovel attempted to compare the structure of heterosexual romantic networks at a typical midwestern high school to simulated networks, they found something surprising. The actual social networks were more like spanning trees than other possible models because there were very few short cycles. In particular, there were almost no four-cycles.

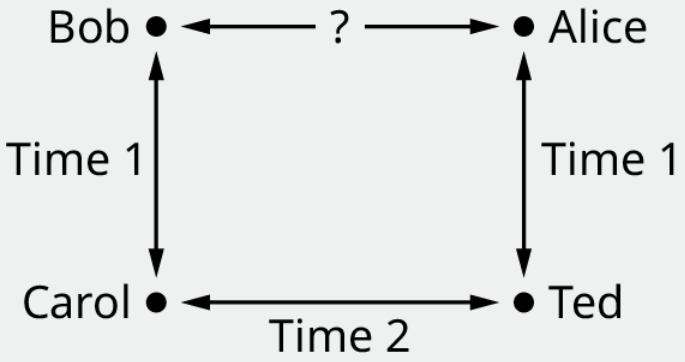


Figure 3.11.29: Chains of Affection

"…the prohibition against dating (from a female perspective) one's old boyfriend's current girlfriend's old boyfriend – accounts for the structure of the romantic network at [the highschool]."

In their article "Chains of Affection: The Structure of Adolescent Romantic and Sexual Networks," the researchers went on to explain the implications for the transmission of sexually transmitted diseases. In particular, social structures based on tree graphs are less dense and more likely to fragment. This information can impact social policies on disease prevention. (Peter S. Bearman, James Moody, and Katherine Stovel, "Chains of Affection: The Structure of Adolescent Romantic and Sexual Networks," *American Journal of Sociology* Volume 110, Number 1, pp. 44-91, 2004)

Kruskal's Algorithm

In many applications of spanning trees, the graphs are weighted and we want to find the spanning tree of least possible weight. For example, the graph might represent a computer network, and the weights might represent the cost involved in connecting two devices. So, finding a spanning tree with the lowest possible total weight, or minimum spanning tree, means saving money! The method that we will use to find a minimum spanning tree of a weighted graph is called Kruskal's algorithm. The steps for Kruskal's algorithm are:

Step 1: Choose any edge with the minimum weight of all edges.

Step 2: Choose another edge of minimum weight from the remaining edges. The second edge does not have to be connected to the first edge.

Step 3: Choose another edge of minimum weight from the remaining edges, but do not select any edge that creates a cycle in the subgraph you are creating.

Step 4: Repeat step 3 until all the vertices of the original graph are included and you have a spanning tree.

Video



Use Kruskal's Algorithm to Find Minimum Spanning Trees in Graph Theory

Example 3.11.7: Using Kruskal's Algorithm

A computer network will be set up with six devices. The vertices in the graph in Figure 3.11.30 represent the devices, and the edges represent the cost of a connection. Find the network configuration that will cost the least. What is the total cost?

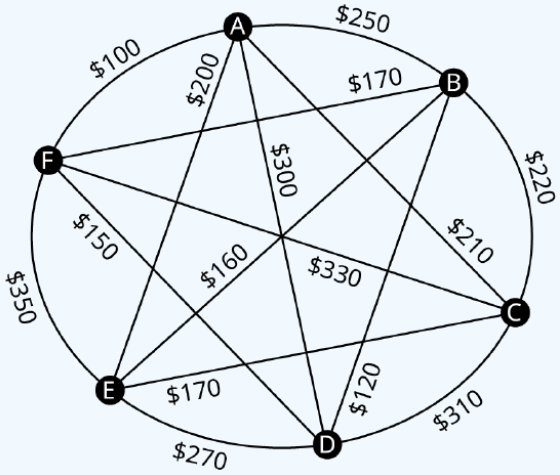


Figure 3.11.30 Graph of Network Connection Costs

Answer

A minimum spanning tree will correspond to the network configuration of least cost. We will use Kruskal's algorithm to find one. Since the graph has six vertices, the spanning tree will have six vertices and five edges.

Step 1: Choose an edge of least weight. We have sorted the weights into numerical order. The least is \$100. The only edge of this weight is edge *AF* as shown in Figure 3.11.31

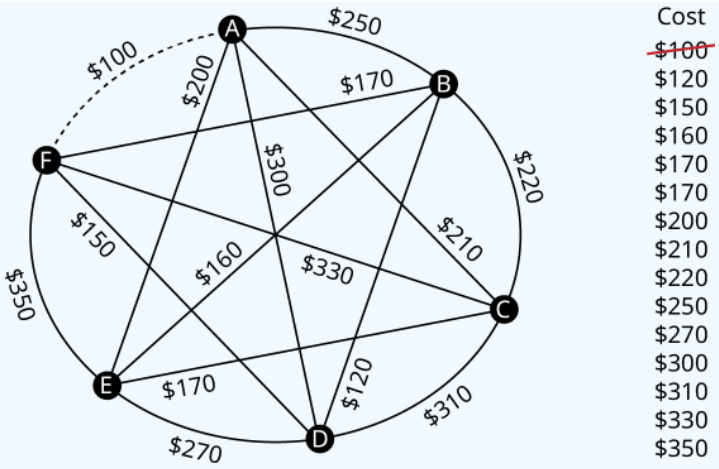


Figure ✔ **3.11.31** Step 1 Select Edge *AF*

https://math.libretexts.org/@go/page/214470

Step 2:

Choose the edge of least weight of the remaining edges, which is

BD

with \$120. Notice that the two selected edges do not need to be adjacent to each other as shown in Figure ✔ **3.11.32**

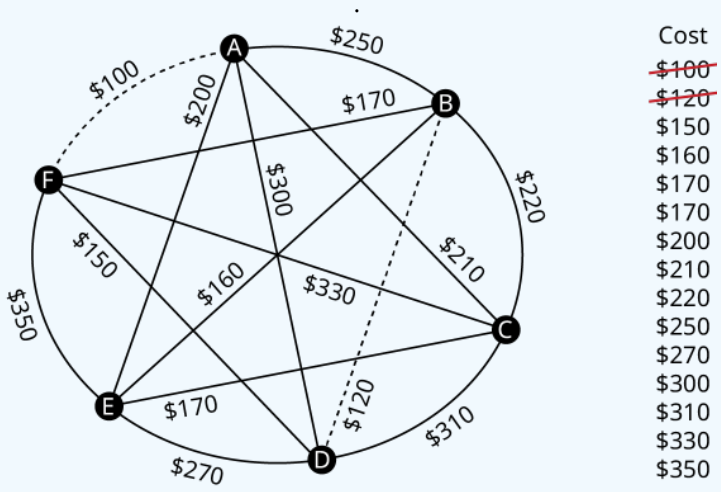


Figure ✔ **3.11.32**: Step 2 Select Edge *BD*

Step 3: Select the lowest weight edge of the remaining edges, as long as it does not result in a cycle. We select *DF* with \$150 since it does not form a cycle as shown in Figure ✔ **3.11.33**.

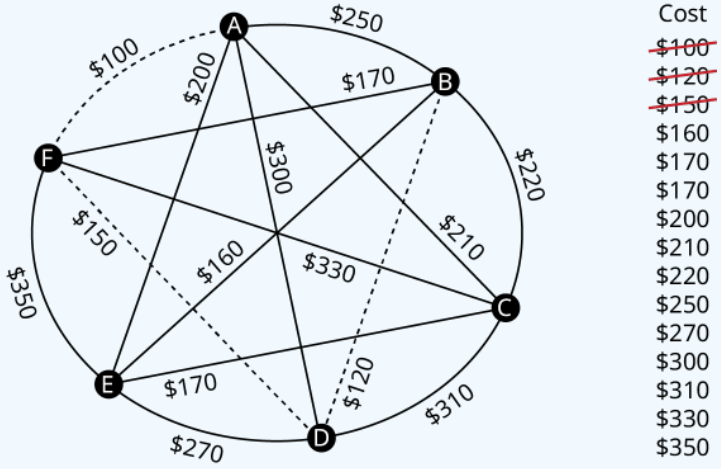


Figure ✔ **3.11.33**: Step 3 Select Edge *DF*

Repeat Step 3: Select the lowest weight edge of the remaining edges, which is *BE* with \$160 and it does not form a cycle as shown in Figure ✔ **3.11.34**. This gives us four edges so we only need to repeat step 3 once more to get the fifth edge.

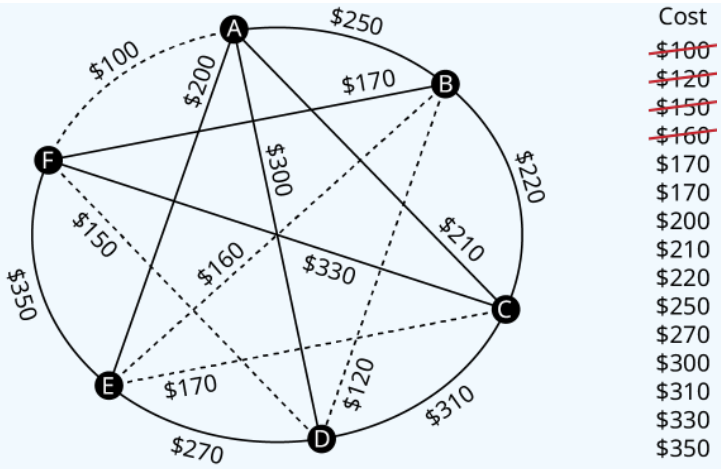


Figure ✔ **3.11.34**: Repeat Step 3 Select Edge *DF*

Repeat Step 3: The lowest weight of the remaining edges is $170. Both *BF* and *CE* have a weight of $170, but *BF* would create cycle (*b*, *d*, *f*) and there cannot be a cycle in a spanning tree as shown in Figure ✔ **3.11.35**.

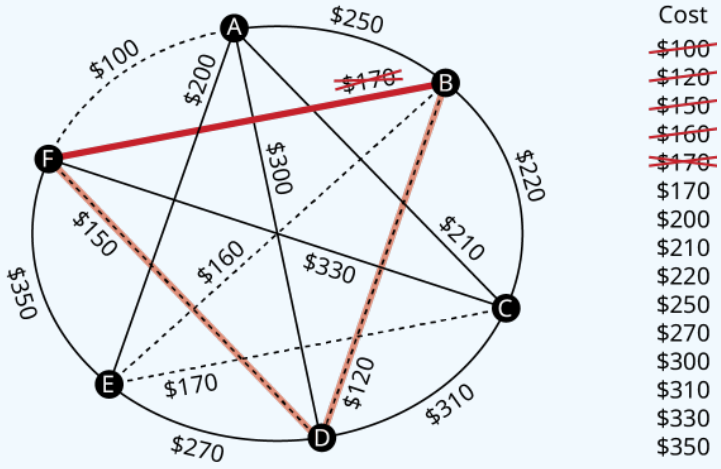


Figure ✔ **3.11.35**: Repeat Step 3 Do Not Select Edge *BF*

So, we will select *CE*, which will complete the spanning tree as shown in Figure ✔ **3.11.36**.

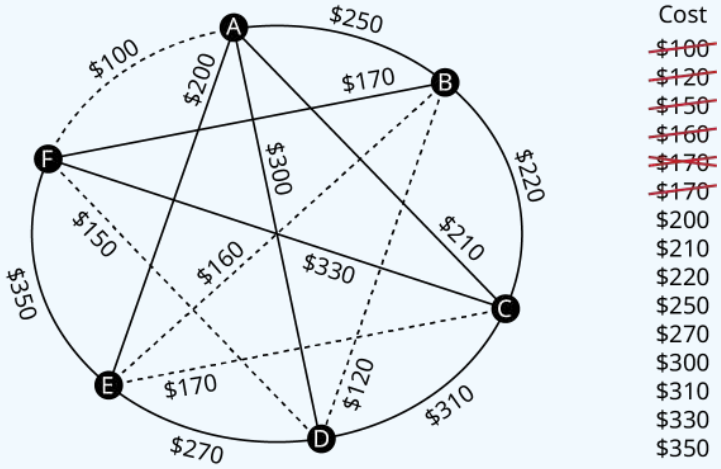


Figure ✔ **3.11.36**: Repeat Step 3 Select Edge *CE*

The minimum spanning tree is shown in Figure ✔ 3.11.37. This is the configuration of the network of least cost. The spanning tree has a total weight of $\$100 + \$120 + \$150 + \$160 + \$170 = \$700$, which is the total cost of this network configuration.

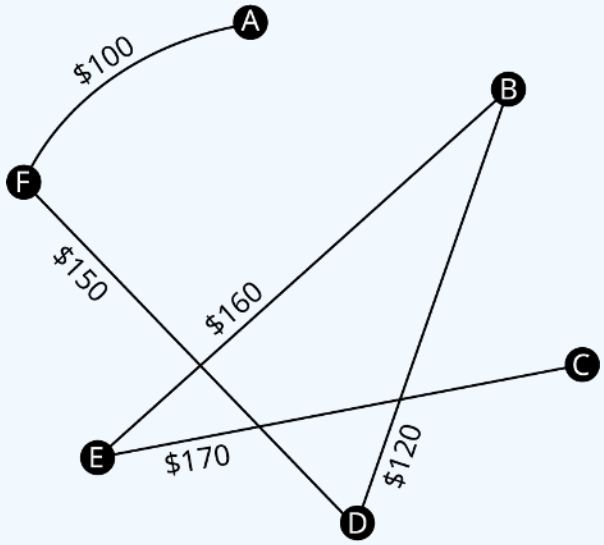


Figure ✔ 3.11.37: Final Minimum Spanning Tree

Your Turn 3.11.7

Find a minimum spanning tree for the weighted graph. Give its total weight.

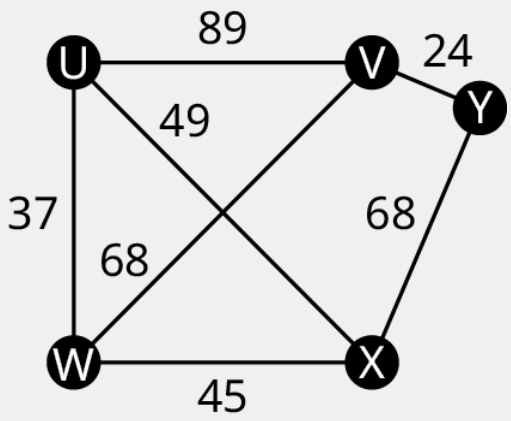


Figure 3.11.38: Weighted Graph

Check Your Understanding

Which of the following statements are true and which are false

1. The number of cycles in a spanning tree is one less than the number of vertices.
2. A spanning tree contains no triangles.
3. A spanning tree includes every vertex of the original graph.There is a unique path between each pair of vertices in a spanning tree.A spanning tree must be connected.Kruskal's algorithm is a method for finding all the different spanning trees in a given graph.
4. Only graphs that are trees have spanning trees.A minimum spanning tree of a given graph can be found using Kruskal's algorithm.
5. A minimum spanning tree of a given graph is the subgraph, which is a tree, includes every vertex of the original graph, and which has the least weight of all spanning trees.

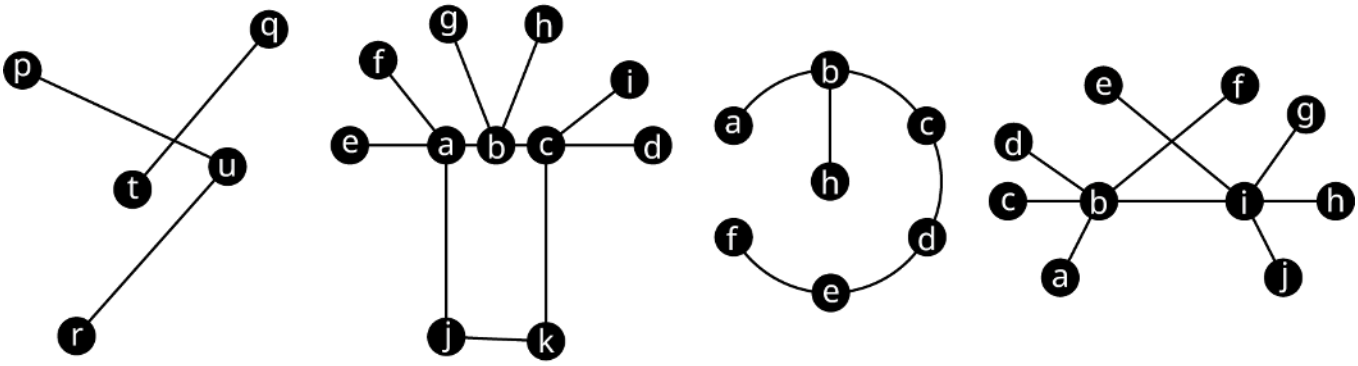
6. If a graph contains any cut edges, they must be included in any spanning tree.

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3.11.1: Exercises

For the following exercises, refer to the figure shown.



Exercise 3.11.1.1

Which graphs, if any, are trees?

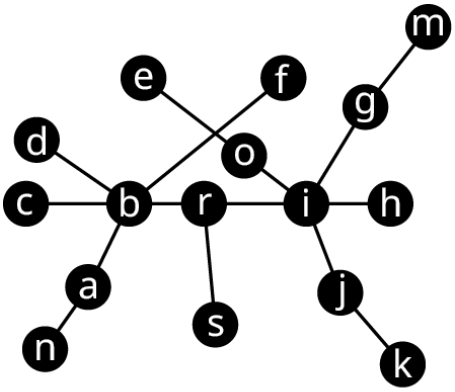
Exercise 3.11.1.2

Which graphs, if any, are not trees because they are not connected?

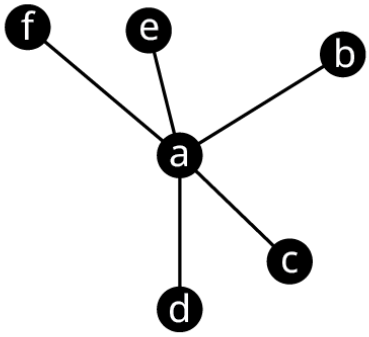
Exercise 3.11.1.3

Which graphs, if any, are not trees because they contain a cycle?

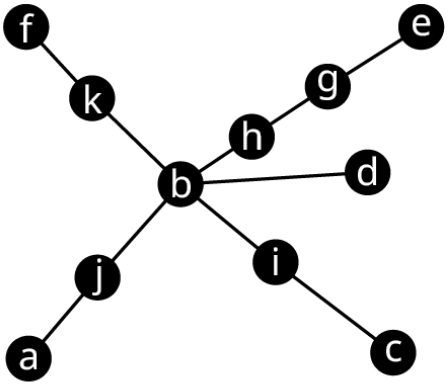
For the following exercises, refer to the figure shown. Identify any graphs that fit the given description.



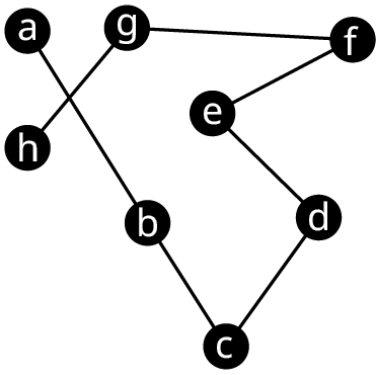
Graph T_1



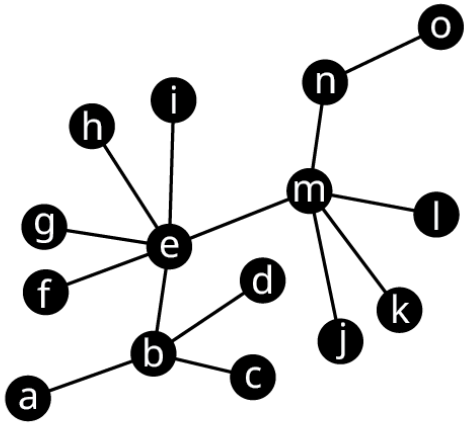
Graph T_2



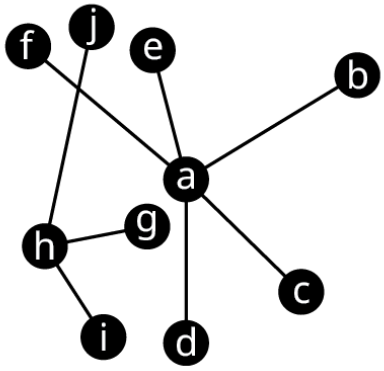
Graph T_3



Graph T_4



Graph T_5



Graph T_6

? Exercise 3.11.1.4

Tree graph

? Exercise 3.11.1.5

Star graph

? Exercise 3.11.1.6

Star like graph

? Exercise 3.11.1.7

Line graph (or path graph)

? Exercise 3.11.1.8

Lobster graph

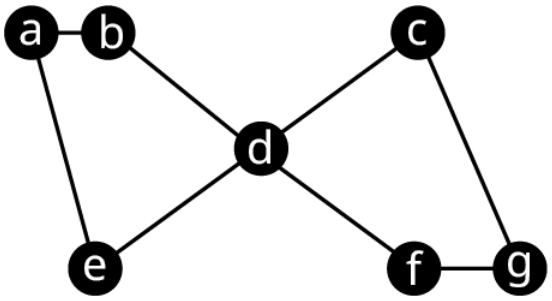
Exercise 3.11.1.9

Caterpillar graph

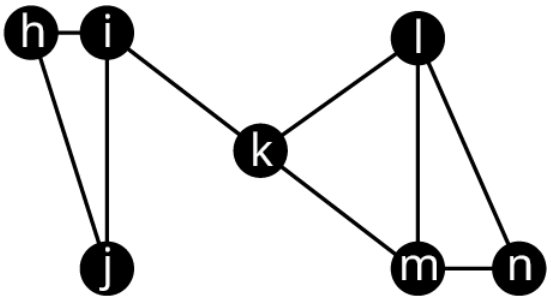
Exercise 3.11.1.10

Forest graph

For the following exercises, use the figure shown to answer the questions.



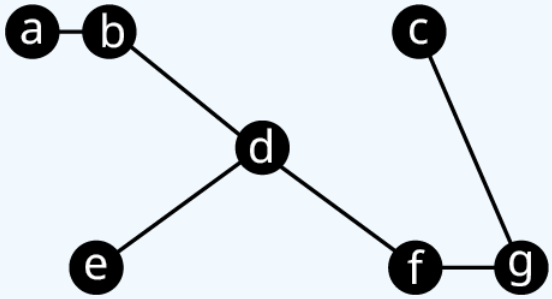
Graph *H*



Graph *Q*

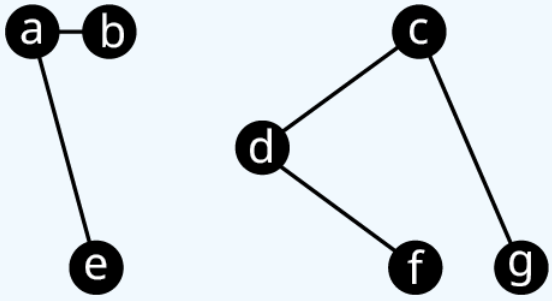
Exercise 3.11.1.11

Determine whether Graph H_1 is a spanning tree of Graph *H*. If not, explain how you know.



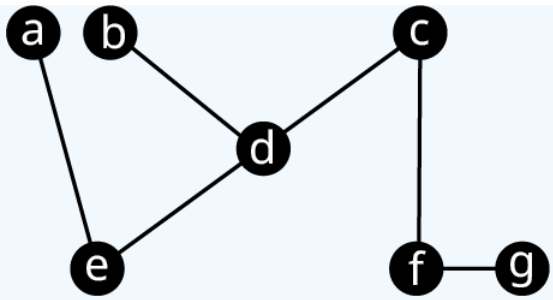
Exercise 3.11.1.12

Determine whether Graph H_2 is a spanning tree of Graph *H*. If not, explain how you know.



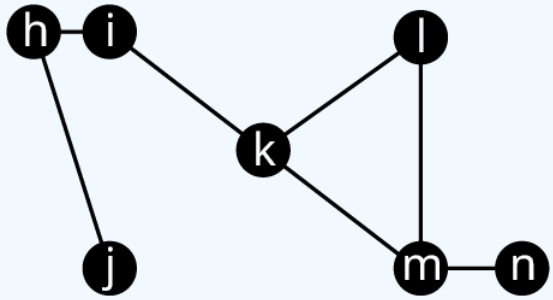
Exercise 3.11.1.13

Determine whether Graph H_3 is a spanning tree of Graph *H*. If not, explain how you know.



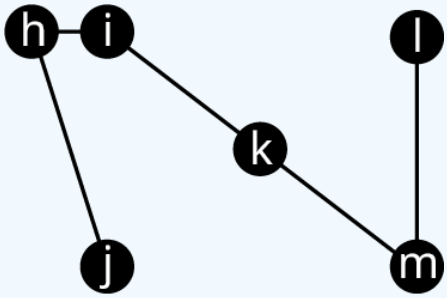
Exercise 3.11.1.14

Determine whether Graph Q_1 is a spanning tree of Graph Q. If not, explain how you know.



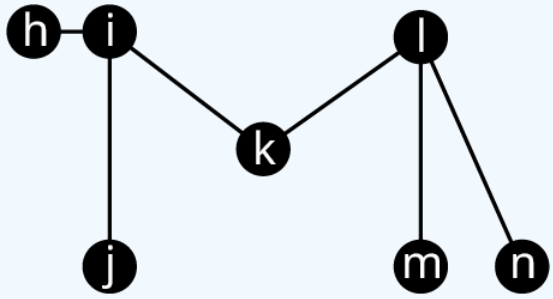
Exercise 3.11.1.15

Determine whether Graph Q_2 is a spanning tree of Graph Q. If not, explain how you know.

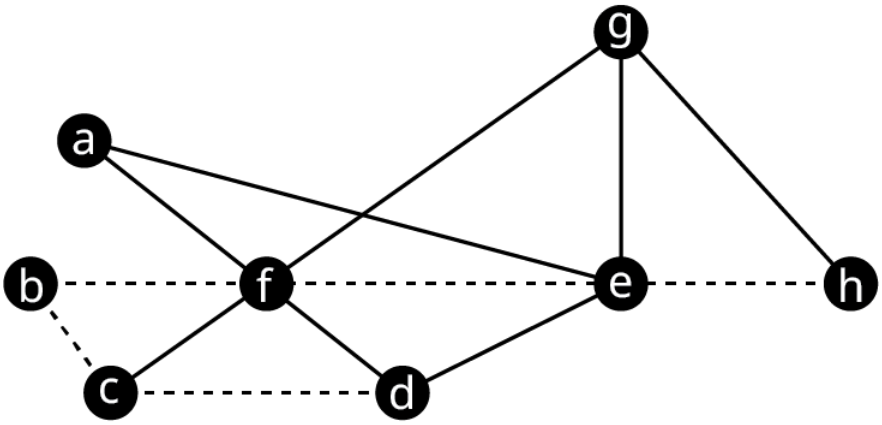


Exercise 3.11.1.16

Determine whether Graph Q_3 is a spanning tree of Graph Q. If not, explain how you know.



For the following exercises, a student has been asked to construct a spanning tree for Graph O, as shown in the figure. The dashed lines show the first step that the student took, creating a path from vertex h to vertex d.



Exercise 3.11.1.17

How many more edges must be included with the dashed edges to create a spanning tree?

Exercise 3.11.1.18

List three unused (solid) edges from Graph *O* that cannot be used to complete the spanning tree.

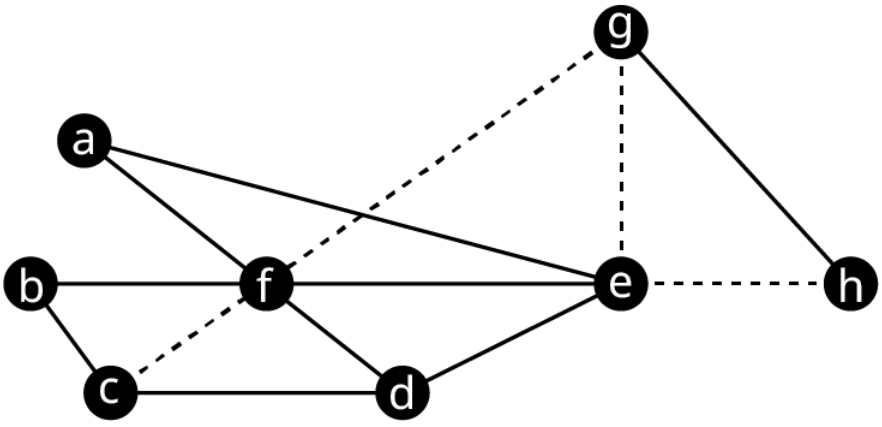
Exercise 3.11.1.19

Give an example of a set of edges that do not have *e* as an endpoint, which would complete the spanning tree.

Exercise 3.11.1.20

Give an example of a set of edges that do not have *f* as an endpoint, which would complete the spanning tree.

For the following exercises, a student has been asked to construct a spanning tree for Graph *O*, as shown in the figure. The dashed lines show the first step that the student took, creating a path from vertex *c* to vertex *h*.



Exercise 3.11.1.21

How many more edges must be included with the dashed edges to create a spanning tree?

Exercise 3.11.1.22

List two unused edges from Graph *O* that cannot be used to complete the spanning tree.

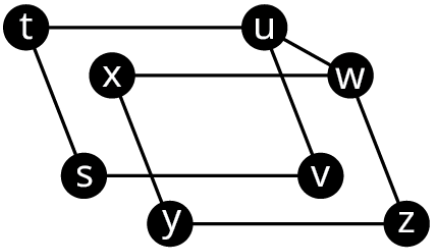
Exercise 3.11.1.23

Give an example of a set of edges that do not have *f* as an endpoint, which would complete the spanning tree.

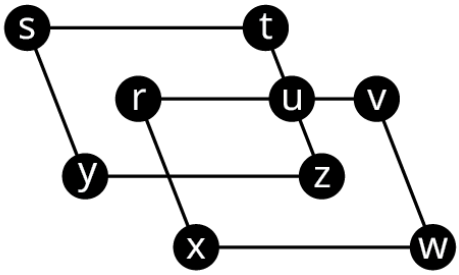
Exercise 3.11.1.24

Give an example of a set of edges that have neither *c* nor *e* as an endpoint, which would complete the spanning tree.

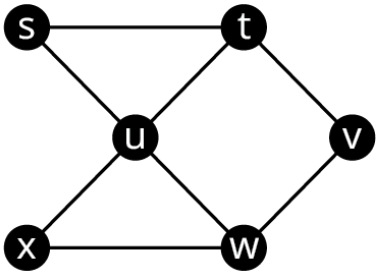
For the following exercises, use Graphs *A*, *B*, and *C*.



Graph *A*



Graph *B*



Graph *C*

Exercise 3.11.1.25

How many edges must be removed from Graph *A* to create a spanning tree?

Exercise 3.11.1.26

How many edges must be removed from Graph *B* to create a spanning tree?

Exercise 3.11.1.27

How many edges must be removed from Graph *C* to create a spanning tree?

Exercise 3.11.1.28

Identify all the distinct cyclic subgraphs of Graph *A*.

Exercise 3.11.1.29

Identify all the cyclic subgraphs of Graph *B*.

Exercise 3.11.1.30

Identify all the cyclic subgraphs of Graph *C*.

Exercise 3.11.1.31

Draw four spanning trees of Graph *A* each of which includes edges *vs*, *uv*, *wz* and *xy*.

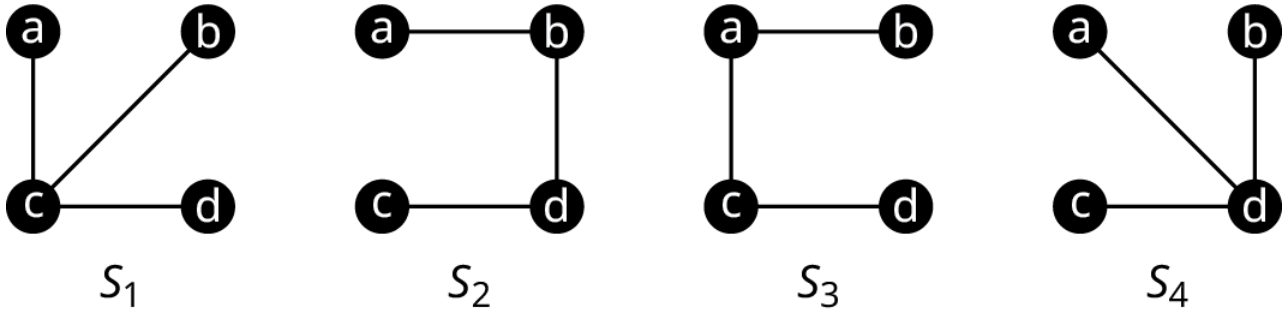
Exercise 3.11.1.32

Draw four spanning trees of Graph B which includes edge ut, but not ur.

Exercise 3.11.1.33

Draw four spanning trees of Graph C that each have only one edge with an endpoint at vertex u.

For the following exercises, use the figure shown. Draw a graph that fits the given description.



Exercise 3.11.1.34

S_1, S_2, S_3 and S_4 are all spanning trees.

Exercise 3.11.1.35

S_1 and S_2 are spanning trees, but S_4 is not.

Exercise 3.11.1.36

S_3 and S_4 are spanning trees but S_1 is not.

Exercise 3.11.1.37

S_2 and S_3 are spanning trees but S_1 is not.

Exercise 3.11.1.38

S_2 and S_3 are spanning trees but S_4 is not.

Exercise 3.11.1.39

S_1, S_2, and S_3 are spanning trees but S_4 is not.

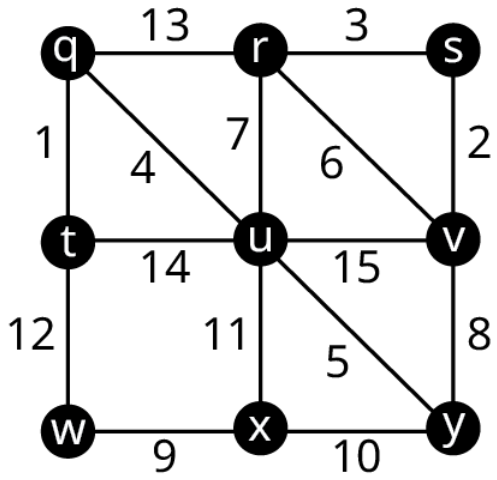
Exercise 3.11.1.40

S_2, S_3, and S_4 are spanning trees but S_1 is not.

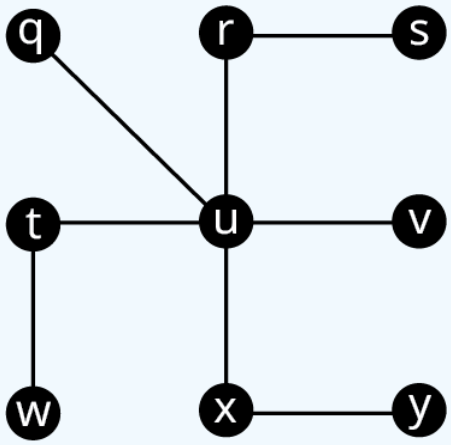
Exercise 3.11.1.41

S_1 and S_4 are spanning trees but S_2 and S_3 are not.

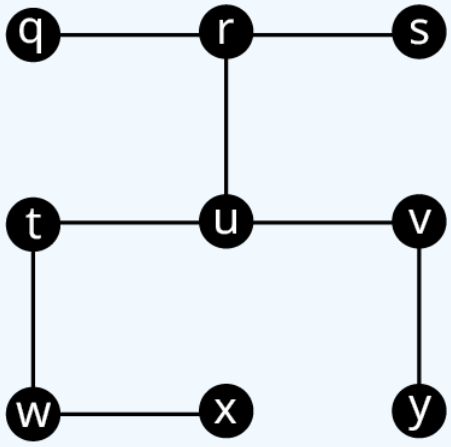
For the following exercises, use the figure shown to find the weight of the given spanning tree.



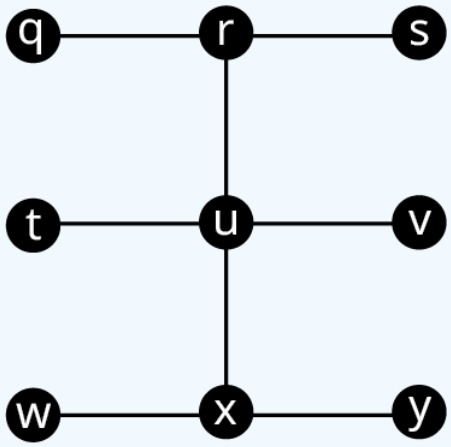
? Exercise 3.11.1.42



? Exercise 3.11.1.43



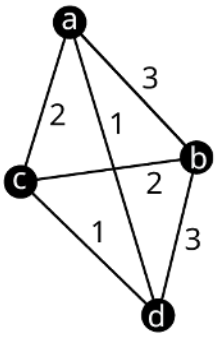
Exercise 3.11.1.44



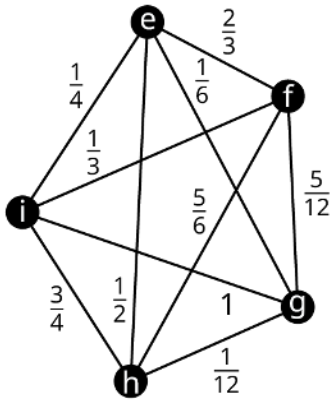
Exercise 3.11.1.45

Use Kruskal's algorithm to draw a minimum spanning tree for Graph *Z* in the provided figure. Find its weight.

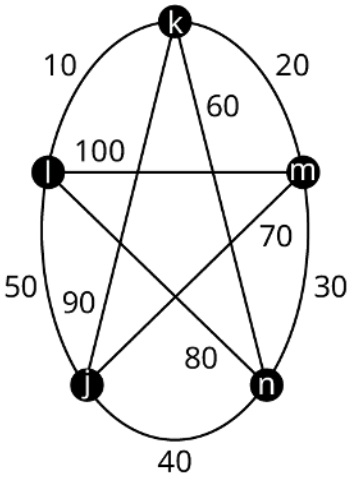
For the following exercises, draw a minimum spanning tree for the given graph, and calculate its weight.



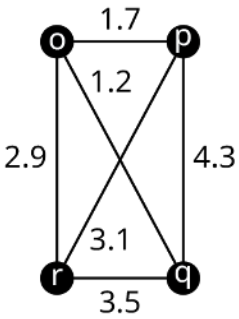
Graph *A*



Graph *B*



Graph *C*



Graph *D*

Exercise 3.11.1.46

Graph *A*

Exercise 3.11.1.47

Graph *C*

Exercise 3.11.1.48

Graph *B*

Exercise 3.11.1.49

Graph *D*

For the following exercises, draw a weighted graph to represent the given information. Then use the graph to find a minimum spanning tree and give its weight. Explain what the weight represents in the given scenario.

Exercise 3.11.1.50

City planners are tasked with building roadways to connect locations *A*, *B*, *C*, and *D*. The cost to build the roadways between any given pair of locations is given in the table.

	A	*B*	*C*	*D*
A	-	125	320	275
B	125	-	110	540
C	320	110	-	1,010
D	275	540	1,010	-

Exercise 3.11.1.51

In a video game, the goal is to visit five different lands, *V*, *W*, *X*, *Y* and *Z*, without losing all your lives. The paths between the lands are rated for danger, 1 being lowest and 10 being highest. Once a path has been traversed successfully, it is free from danger. The ratings are given in the table.

	V	*W*	*X*	*Y*	*Z*
V	-	2	4	9	10
W	2	-	6	8	No Path
X	4	6	-	7	No Path
Y	9	8	7	-	5
Z	10	No Path	No Path	5	-

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CHAPTER OVERVIEW

4: Tessellations

4.1: Tessellations

4.1.1: Exercises

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4.1: Tessellations

Learning Objectives

1. Apply translations, rotations, and reflections.
2. Determine if a shape tessellates.

The illustration shown above (Figure **4.1.1**) is an unusual pattern called a Penrose tiling. Notice that there are two types of shapes used throughout the pattern: smaller green parallelograms and larger blue parallelograms. What's interesting about this design is that although it uses only two shapes over and over, there is no repeating pattern.

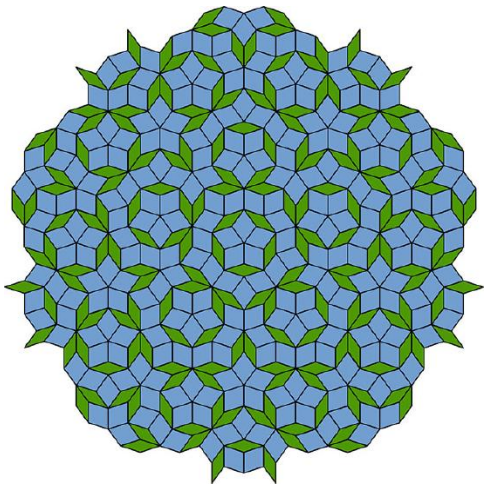


Figure **4.1.1**: Penrose tiling represents one type of tessellation. (credit: "Penrose Tiling" by Inductiveload/Wikimedia Commons, Public Domain)

In this section, we will focus on patterns that do repeat. Repeated patterns are found in architecture, fabric, floor tiles, wall patterns, rug patterns, and many unexpected places as well. It may be a simple hexagon-shaped floor tile, or a complex pattern composed of several different motifs. These two-dimensional designs are called regular (or periodic) tessellations. There are countless designs that may be classified as regular tessellations, and they all have one thing in common—their patterns repeat and cover the plane.

We will explore how tessellations are created and experiment with making some of our own as well. The topic of tessellations belongs to a field in mathematics called transformational geometry, which is a study of the ways objects can be moved while retaining the same shape and size. These movements are termed *rigid motions* and *symmetries*.

Who Knew?: M. C. Escher

A good place to start the study of tessellations is with the work of M. C. Escher. The Dutch graphic artist was famous for the dimensional illusions he created in his woodcuts and lithographs, and that theme is carried out in many of his tessellations as well. Escher became obsessed with the idea of the “regular division of the plane.” He sought ways to divide the plane with shapes that would fit snugly next to each other with no gaps or overlaps, represent beautiful patterns, and could be repeated infinitely to fill the plane. He experimented with practically every geometric shape imaginable and found the ones that would produce a regular division of the plane. The idea is similar to dividing a number by one of its factors. When a number divides another number evenly, there are no remainders, like there are no gaps when a shape divides or fills the plane.

Escher went far beyond geometric shapes, beyond triangles and polygons, beyond irregular polygons, and used other shapes like figures, faces, animals, fish, and practically any type of object to achieve his goal; and he did achieve it, beautifully, and left it for the ages to appreciate.

Video



M.C. Escher: How to Create a Tessellation



The Mathematical Art of M.C. Escher

Tessellation Properties and Transformations

A regular tessellation means that the pattern is made up of congruent regular polygons, same size and shape, including *some type of movement*; that is, some type of transformation or symmetry. Here we consider the rigid motions of translations, rotations, reflections, or glide reflections. A plane of tessellations has the following properties:

- Patterns are repeated and fill the plane.
- There are no gaps or overlaps. Shapes must fit together perfectly. (It was Escher who determined that a proper tessellation could have no gaps and no overlaps.)
- Shapes are combined using a transformation.
- All the shapes are joined at a vertex. In other words, if you were to draw a circle around a vertex, it would include a corner of each shape touching at that vertex.
- For a tessellation of regular congruent polygons, the sum of the measures of the interior angles that meet at a vertex equals $360°$.

In Figure 4.1.2, the tessellation is made up of squares. There are four squares meeting at a vertex. An interior angle of a square is $90°$ and the sum of four interior angles is $360°$. In Figure 4.1.3 , the tessellation is made up of regular hexagons. There are three hexagons meeting at each vertex. The interior angle of a hexagon is $120°$, and the sum of three interior angles is $360°$. Both tessellations will fill the plane, there are no gaps, the sum of the interior angle meeting at the vertex is $360°$, and both are achieved by translation transformations. These tessellations work because all the properties of a tessellation are present.

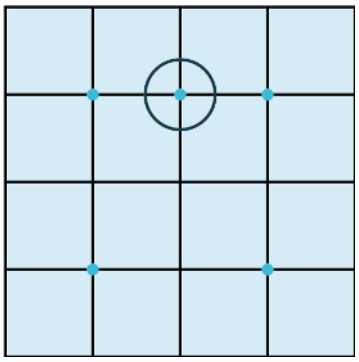


Figure 4.1.2: Tessellation – Squares

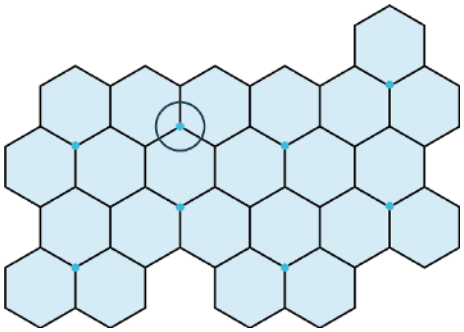


Figure 4.1.3: Tessellation – Hexagons

The movements or rigid motions of the shapes that define tessellations are classified as translations, rotations, reflections, or glide reflections. Let's first define these movements and then look at some examples showing how these transformations are revealed.

Translation

A translation is a movement that shifts the shape vertically, horizontally, or on the diagonal. Consider the trapezoid semantics ABCDFigure 10.104. We have translated it 3 units to the right and 3 units up. That means every corner is moved by the number of units and in the direction specified. Mathematicians will indicate this movement with a vector, an arrow that is drawn to illustrate the criteria and the magnitude of the translation. The location of the translated trapezoid is marked with the vertices, $A'B'C'D'$, but it is still the exact same shape and size as the original trapezoid semanticsABCDABCD.

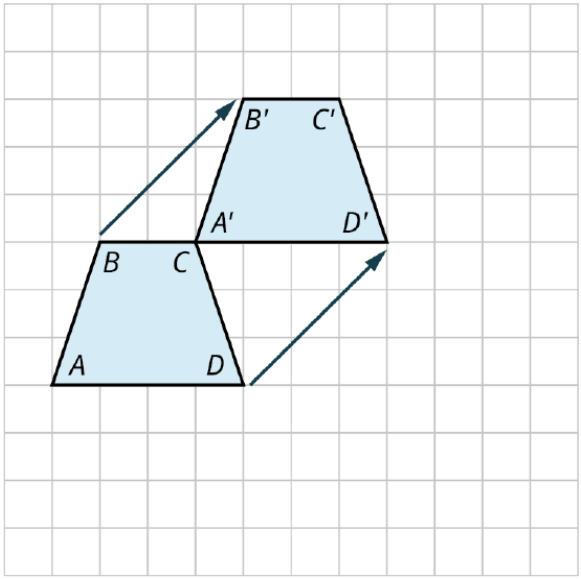


Figure 4.1.4: Translation

Example 4.1.1: Creating a Translation

Suppose you have a hexagon on a grid as in Figure 10.105. Translate the hexagon 5 units to the right and 3 units up.

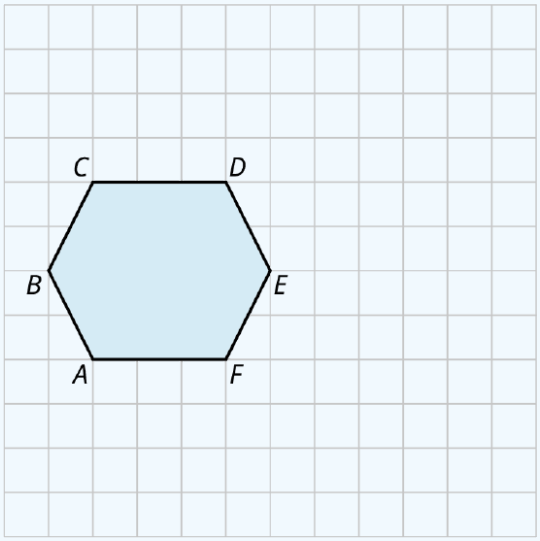
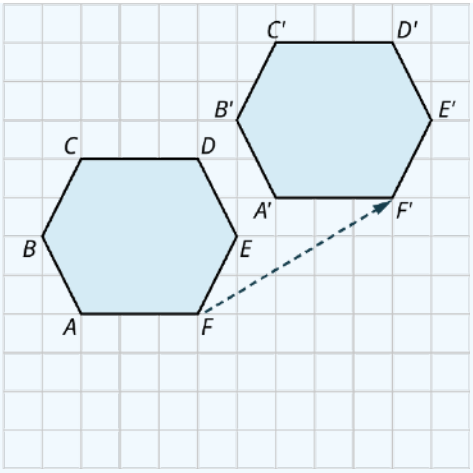


Figure 4.1.5

Answer

The best way to do this is by translating the individual points A, B, C, D, E, F. Once translated, the points become A', B', C', D', E', F'.

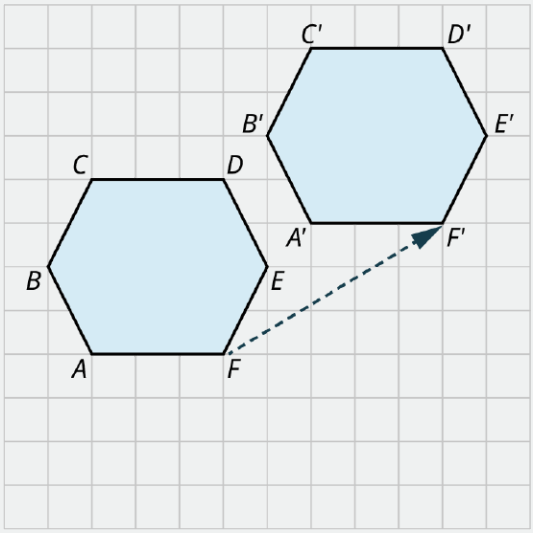


Figure

✔ 4.1.6

Your Turn 4.1.1

Translate the hexagon with points A', B', C', D', E', F' 6 units down.



Figure

📌 4.1.7

Rotation

The rotation transformation occurs when you rotate a shape about a point and at a predetermined angle. In Figure 4.1.8, the triangle is rotated around the rotation point by 90°, and then translated 7 units up and 4 units over to the right. That means that each corner is translated to the new location by the same number of units and in the same direction.

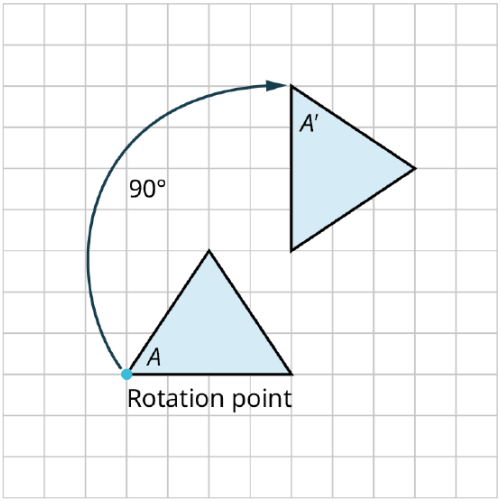


Figure 4.1.8: Rotation

We can see that semantics$\Delta A\Delta A$ is mapped to $\Delta A'\Delta A'$ by a rotation of $90°$ up and to the right. If rotated again by $90°$, the triangle would be upside down.

✔ Example 4.1.2: Applying a Rotation

Figure ✔ 4.1.9 illustrates a tessellation begun with an equilateral triangle. Explain how this pattern is produced.

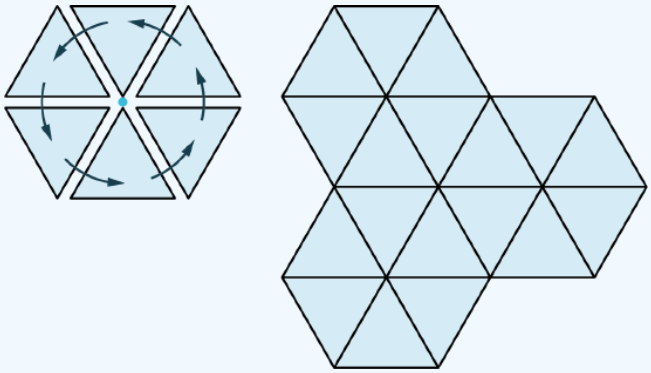


Figure ✔ 4.1.9

Answer

A rotation to the right or to the left around the vertex by $60°$, six times, produces the hexagonal shape. The sixth rotation brings the triangle back to its original position. Then, a reflection up and another one on the diagonal will reproduce the pattern. When a shape returns to its original position by a rotation, we say that it has rotational symmetry.

Your Turn 4.1.2

Starting with the triangle in the figure shown, explain how the pattern on the right was achieved.

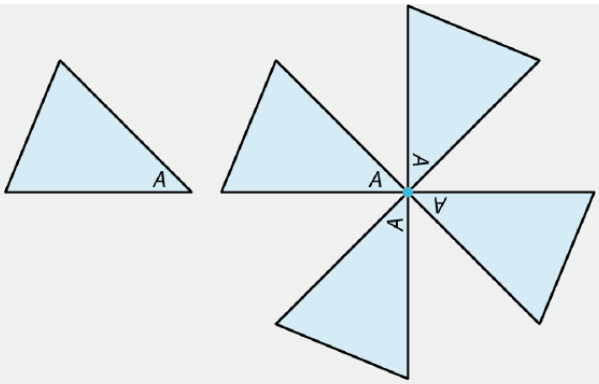


Figure 4.1.10

Reflection

A reflection is the third transformation. A shape is reflected about a line and the new shape becomes a mirror image. You can reflect the shape vertically, horizontally, or on the diagonal. There are two shapes in Figure **4.1.11**. The quadrilateral is reflected horizontally; the arrow shape is reflected vertically.

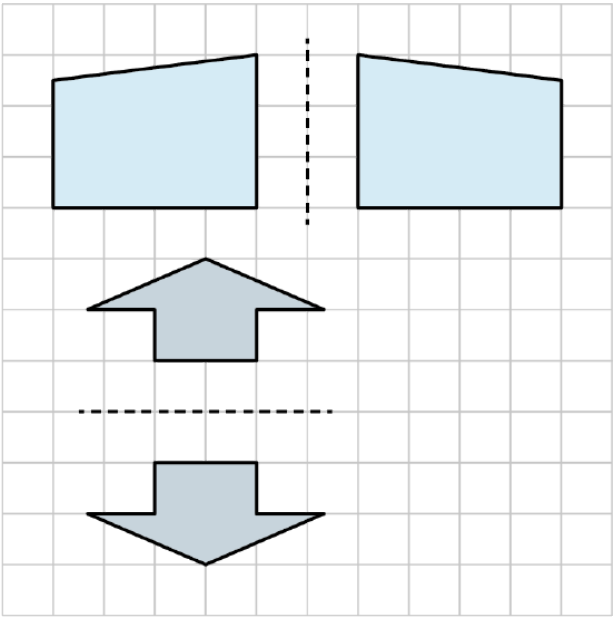


Figure **4.1.11**: Reflection

Glide Reflection

The glide reflection is the fourth transformation. It is a combination of a reflection and a translation. This can occur by first reflecting the shape and then gliding or translating it to its new location, or by translating first and then reflecting. The example in Figure **4.1.12** shows a trapezoid, which is reflected over the dashed line, so it appears upside down. Then, we shifted the shape horizontally by 6 units to the right. Whether we use the glide first or the reflection first, the end result is the same in most cases. However, the tessellation shown in the next example can only be achieved by a reflection first and then a translation.

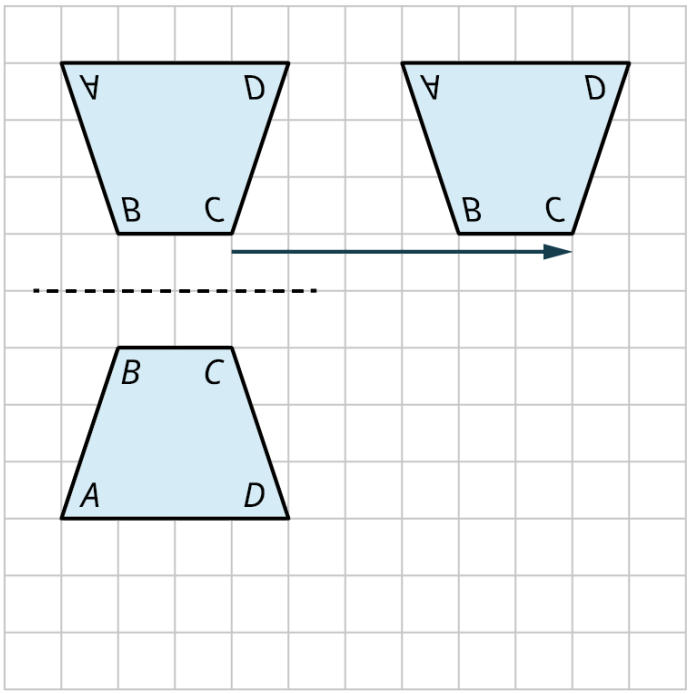


Figure 4.1.12: Glide Reflection

Example 4.1.3: Applying the Glide Reflection

An obtuse triangle is reflected about the dashed line, and the two shapes are joined together. How does the tessellation shown in Figure 4.1.13 materialize?

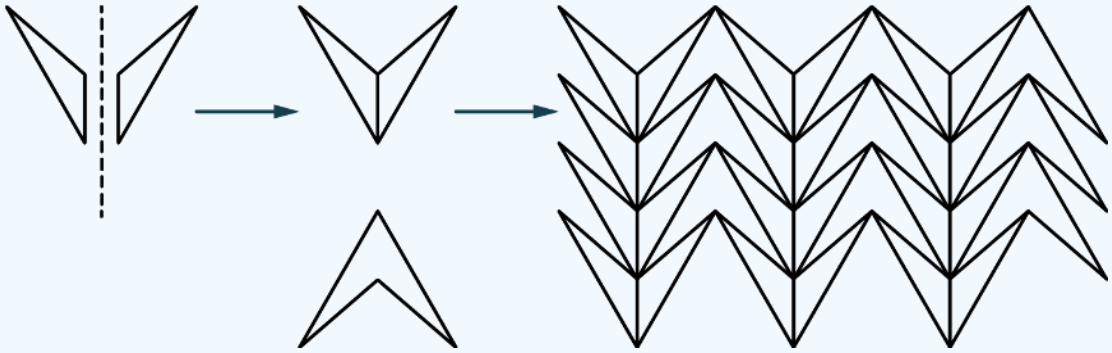


Figure 4.1.13

Answer

The new shape is reflected horizontally and joined with the original shape. It is then translated vertically and horizontally to make up the tessellation. Notice the blank spaces next to the vertical pattern. These areas are made up of the exact original shape rotated 180°, but with no line up the center. These rotated shapes are translated horizontally and vertically, and thus, the plane is tessellated with no gaps. This is an example of a glide reflection where the order of the transformations matters.

Your Turn 4.1.3

Explain how this tessellation of equilateral triangles could be produced.

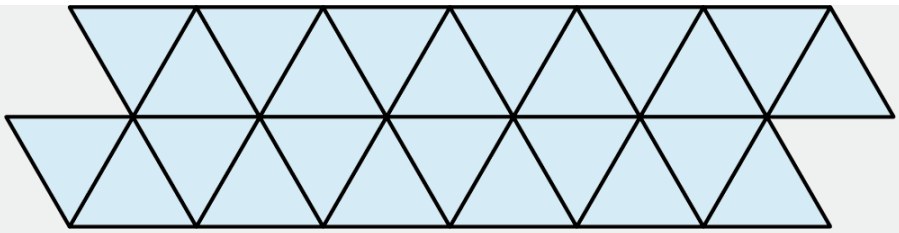


Figure 4.1.14

Example 4.1.4: Applying More Than One Tessellation

Show how this tessellation (Figure 4.1.15) can be achieved.

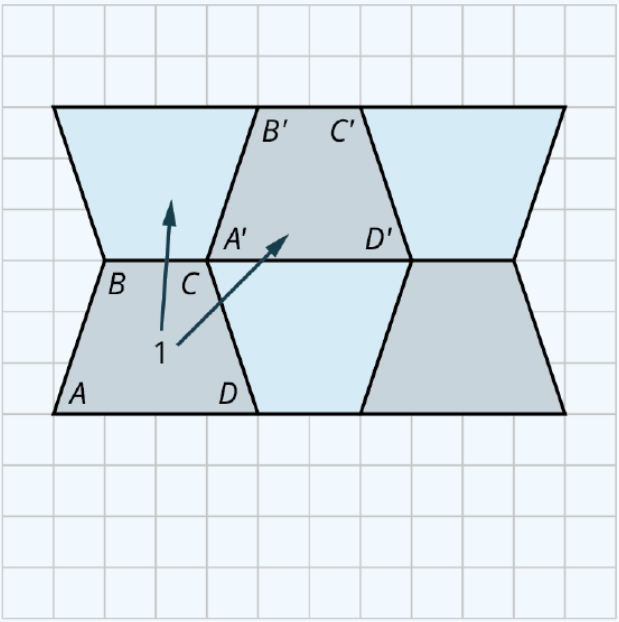


Figure 4.1.15

Answer

This is a tessellation that has one color on the front of the trapezoid and a different color on the back. There is a translation on the diagonal, and a reflection vertically. These are two separate transformations resulting in two new placements of the trapezoid. We can call this a combination of two transformations or a glide reflection.

Your Turn 4.1.4

How does this tessellation of the squares come about?

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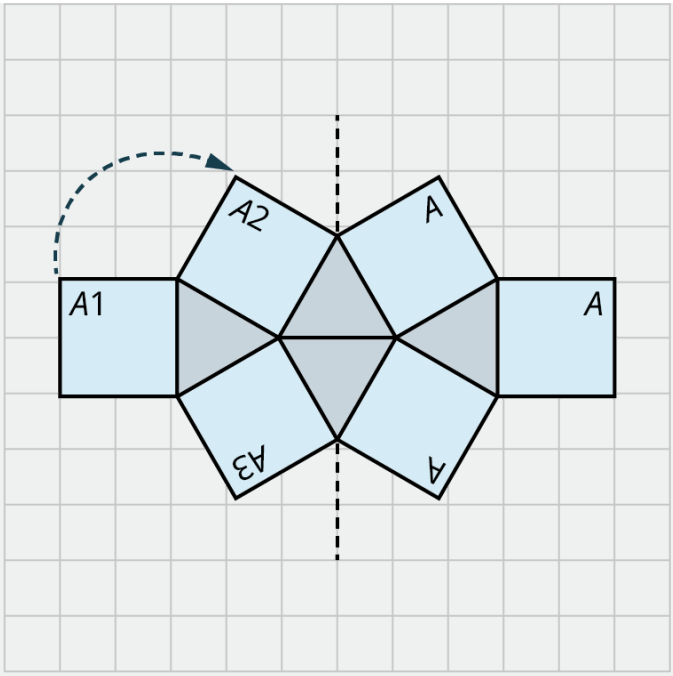


Figure 4.1.16

Interior Angles

The sum of the interior angles of a tessellation is $360°$. In Figure 4.1.17, the tessellation is made of six triangles formed into the shape of a hexagon. Each angle inside a triangle equals $60°$, and the six vertices meet the sum of those interior angles, $6\,(60°) = 360°$.

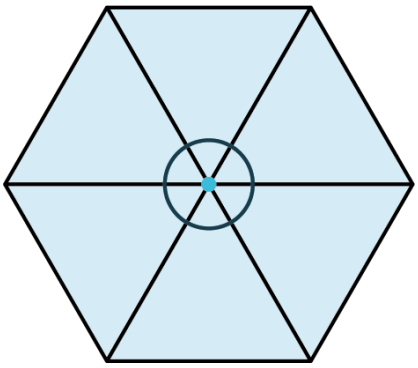


Figure 4.1.17: Interior Angles at the Vertex of Triangles

In Figure 4.1.18, the tessellation is made up of trapezoids, such that two of the interior angles of each trapezoid equals $75°$ and the other two angles equal $105°$. Thus, the sum of the interior angles where the vertices of four trapezoids meet equals $105° + 75° + 75° + 105° = 360°$.

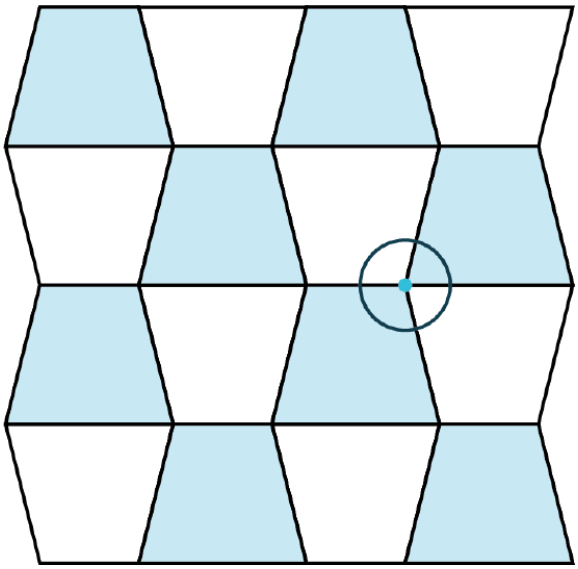


Figure 4.1.18: Interior Angles at the Vertex of Trapezoids

These tessellations illustrate the property that the shapes meet at a vertex where the interior angles sum to $360°$.

Tessellating Shapes

We might think that all regular polygons will tessellate the plane by themselves. We have seen that squares do and hexagons do. The pattern of squares in Figure 4.1.19 is a translation of the shape horizontally and vertically. The hexagonal pattern in Figure 4.1.20, is translated horizontally, and then on the diagonal, either to the right or to the left. This particular pattern can also be formed by rotations. Both tessellations are made up of congruent shapes and each shape fits in perfectly as the pattern repeats.

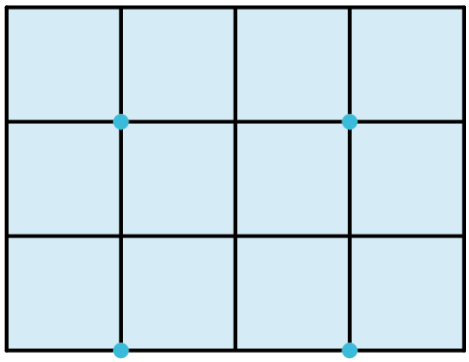


Figure 4.1.19: Translation Horizontally and Vertically

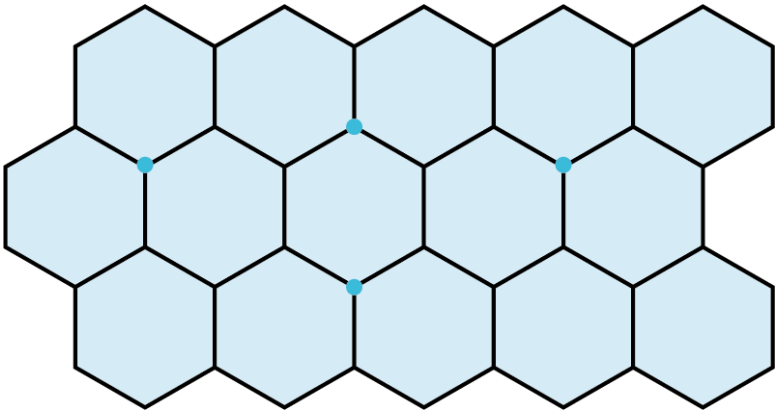


Figure 4.1.20: Translation Horizontally and Slide Diagonally

We have also seen that equilateral triangles will tessellate the plane without gaps or overlaps, as shown in Figure 4.1.21. The pattern is made by a reflection and a translation. The darker side is the face of the triangle and the lighter side is the back of the triangle,

shown by the reflection. Each triangle is reflected and then translated on the diagonal.

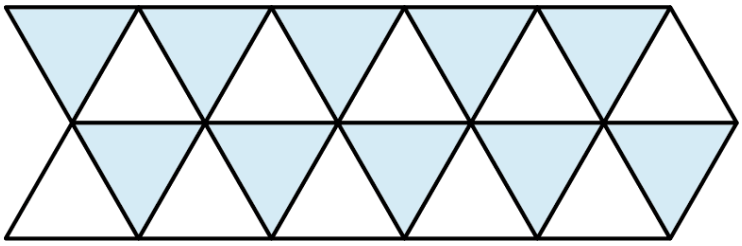


Figure 4.1.21: Reflection and Glide Translation

Escher experimented with all regular polygons and found that only the ones mentioned, the equilateral triangle, the square, and the hexagon, will tessellate the plane by themselves. Let's try a few other regular polygons to observe what Escher found.

Example 4.1.5: Tessellating the Plane

Do regular pentagons tessellate the plane by themselves (Figure 4.1.22)?

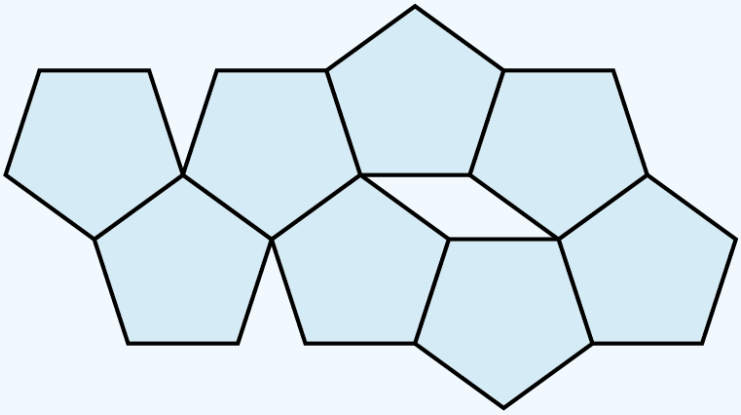


Figure 4.1.22

Answer

We can see that regular pentagons do not tessellate the plane by themselves. There is a gap, a gap in the shape of a parallelogram. We conclude that regular pentagons will not tessellate the plane by themselves.

Your Turn 4.1.5

Do regular heptagons tessellate the plane by themselves?

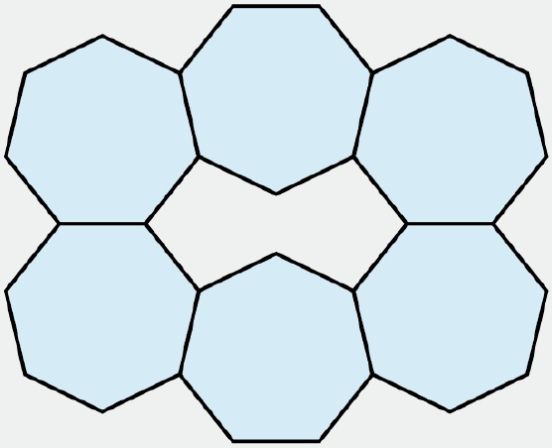


Figure 4.1.23

Example 4.1.6: Tessellating Octagons

Do regular octagons tessellate the plane by themselves (Figure 4.1.24)?

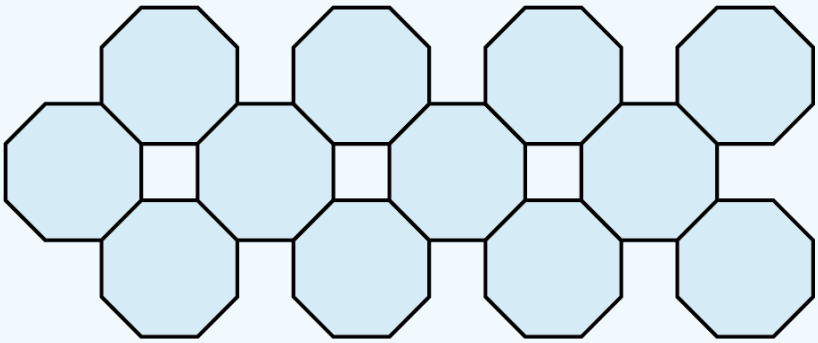


Figure 4.1.24

Answer

Again, we see that regular octagons do not tessellate the plane by themselves. The gaps, however, are squares. So, two regular polygons, an octagon and a square, do tessellate the plane.

Your Turn 4.1.6

Do regular dodecagons (12-sided regular polygons) tessellate the plane by themselves?

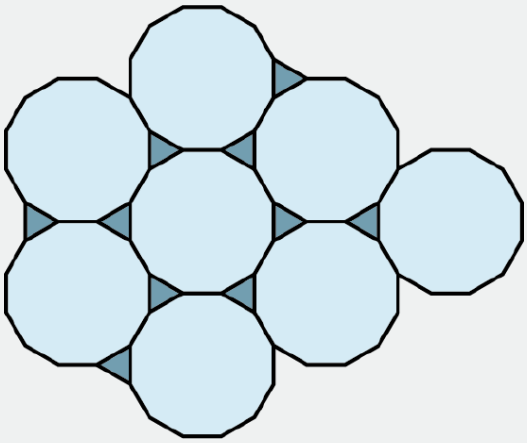


Figure 4.1.25

Just because regular pentagons do not tessellate the plane by themselves does not mean that there are no pentagons that tessellate the plane, as we see in Figure 4.1.26.

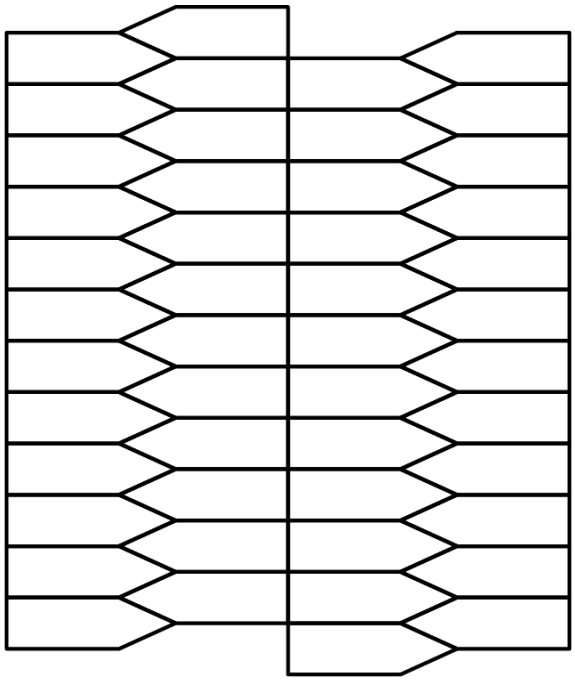


Figure **4.1.26**: Tessellation of Pentagons

Another example of an irregular polygon that tessellates the plane is by using the obtuse irregular triangle from a previous example. What transformations should be performed to produce the tessellation shown in Figure **4.1.27**?

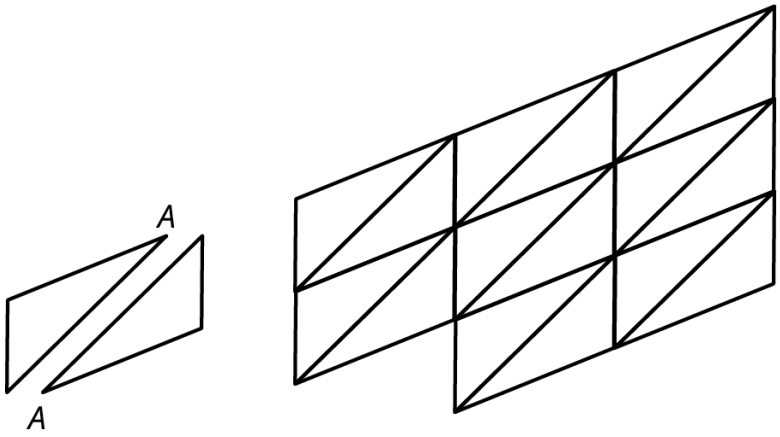


Figure 10.127: Tessellating with Obtuse Irregular Triangles

First, the triangle is reflected over the tip at point ***A***, and then translated to the right and joined with the original triangle to form a parallelogram. The parallelogram is then translated on the diagonal and to the right and to the left.

Naming

A tessellation of squares is named by choosing a vertex and then counting the number of sides of each shape touching the vertex. Each square in the tessellation shown in Figure **4.1.28** has four sides, so starting with square ***A***, the first number is 4, moving around counterclockwise to the next square meeting the vertex, square ***B***, we have another 4, square ***C*** adds another 4, and finally square ***D*** adds a fourth 4. So, we would name this tessellation a 4.4.4.4.

The hexagon tessellation, shown in Figure **4.1.29** has six sides to the shape and three hexagons meet at the vertex. Thus, we would name this a 6.6.6. The triangle tessellation, shown in Figure **4.1.30** has six triangles meeting the vertex. Each triangle has three sides. Thus, we name this a 3.3.3.3.3.3.

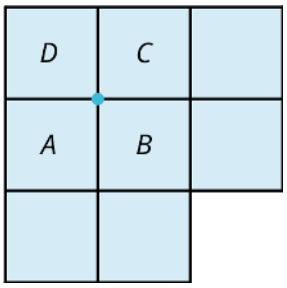


Figure 4.1.28: 4.4.4.4

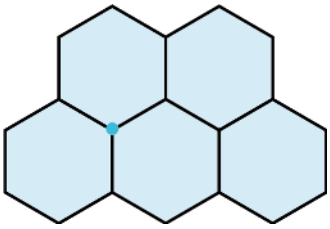


Figure 4.1.29: 6.6.6

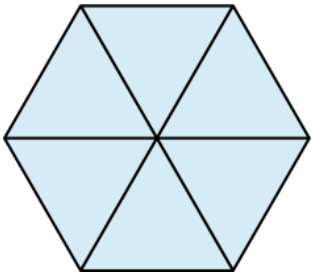


Figure 4.1.30: 3.3.3.3.3.3

✔ Example 4.1.7: Creating Your Own Tessellation

Create a tessellation using two colors and two shapes.

Answer

We used a parallelogram and an isosceles triangle. The parallelogram is reflected vertically and horizontally so that only every other corner touches. The triangles are reflected vertically and horizontally and then translated over the parallelogram. The result is alternating vertical columns of parallelograms and then triangles (Figure 10.131).

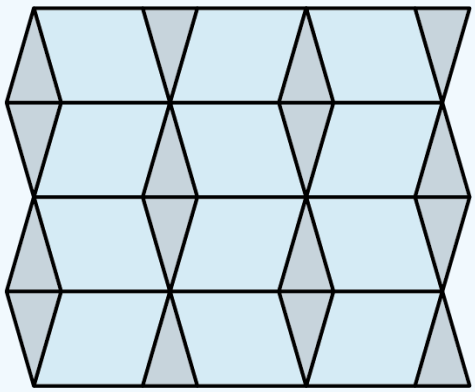


Figure ✔ 4.1.31

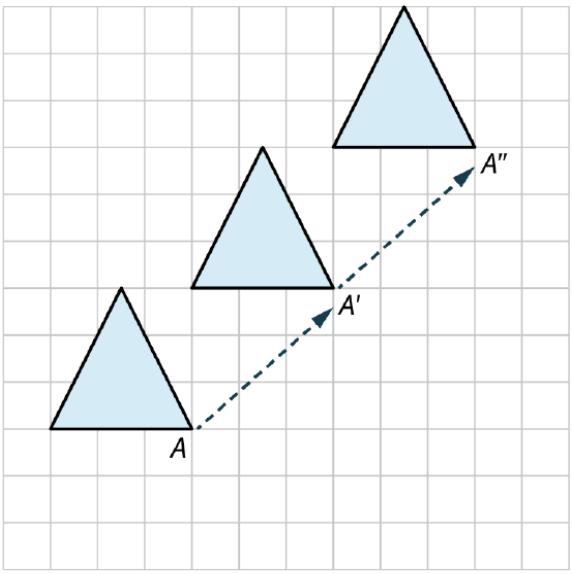
Your Turn 4.1.7

Create a tessellation using polygons, regular or irregular.

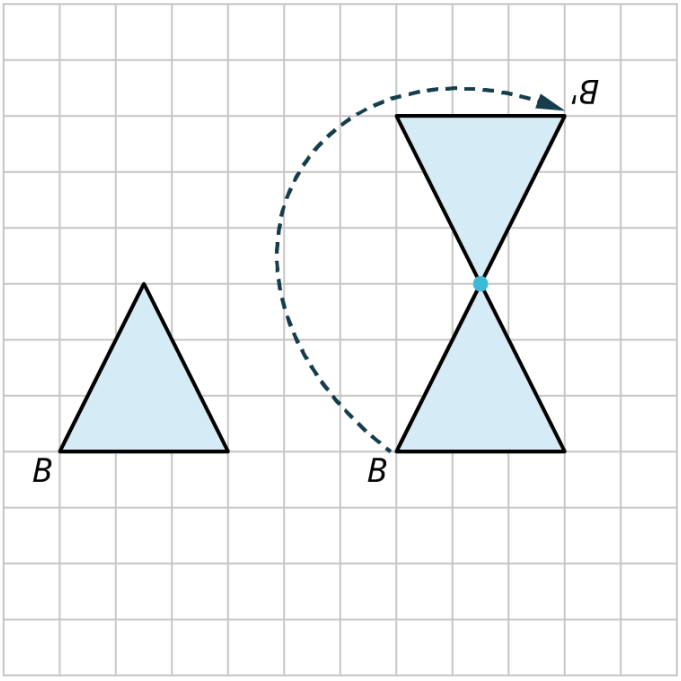
Check Your Understanding

1. What are the properties of repeated patterns that let them be classified as tessellations?

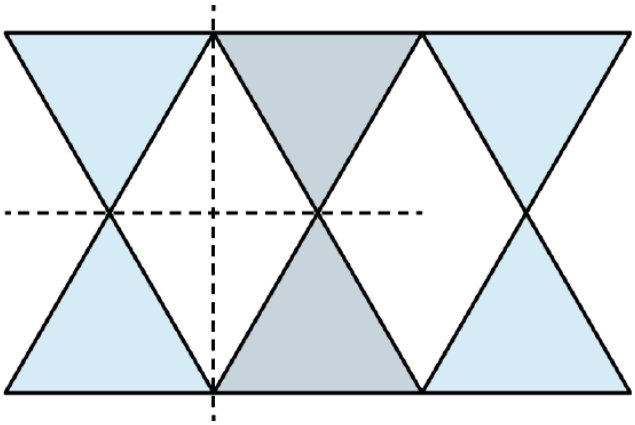
2. Explain how the using the transformation of a translation is applied to the movement of this shape starting with point ***A***.



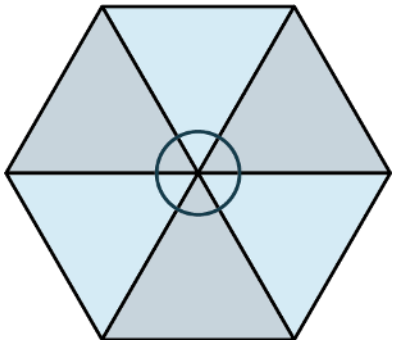
3. Starting with the triangle with vertex ***B***, describe how the transformation in this drawing is achieved.



4. Starting with a triangle with a darker face and a lighter back, describe how this pattern came about.



5. Name the tessellation in the figure shown.



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4.1.1: Exercises

Exercise 4.1.1.1

What type of movements are used to change the orientation and placement of a shape?

Exercise 4.1.1.2

What is the name of the motion that renders a shape upside down?

Exercise 4.1.1.3

What do we call the motion that moves a shape to the right or left or on the diagonal?

Exercise 4.1.1.4

If you are going to tessellate the plane with a regular polygon, what is the sum of the interior angles that surround a vertex?

Exercise 4.1.1.5

Does a regular heptagon tesselate the plane by itself?

Exercise 4.1.1.6

What are the only regular polygons that will tessellate the plane by themselves?

Exercise 4.1.1.7

What is the transformation called that revolves a shape about a point to a new position?

Exercise 4.1.1.8

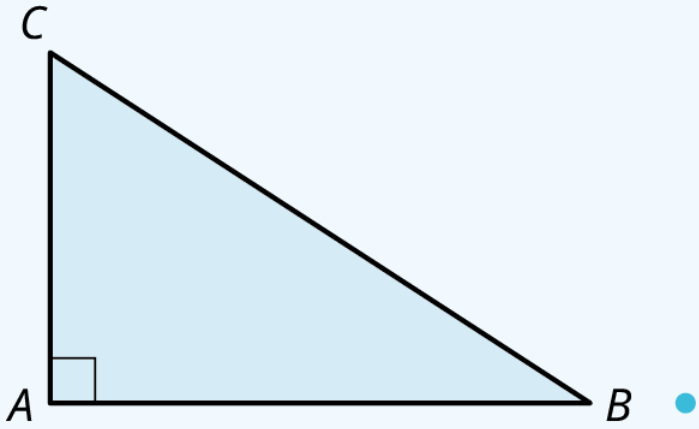
Transformational geometry is a study of what?

Exercise 4.1.1.9

Describe how to achieve a rotation transformation.

Exercise 4.1.1.10

Construct a $90°$ rotation of the triangle shown.



Exercise 4.1.1.11

Shapes can be rotated around a point of rotation or a _____.

Exercise 4.1.1.12

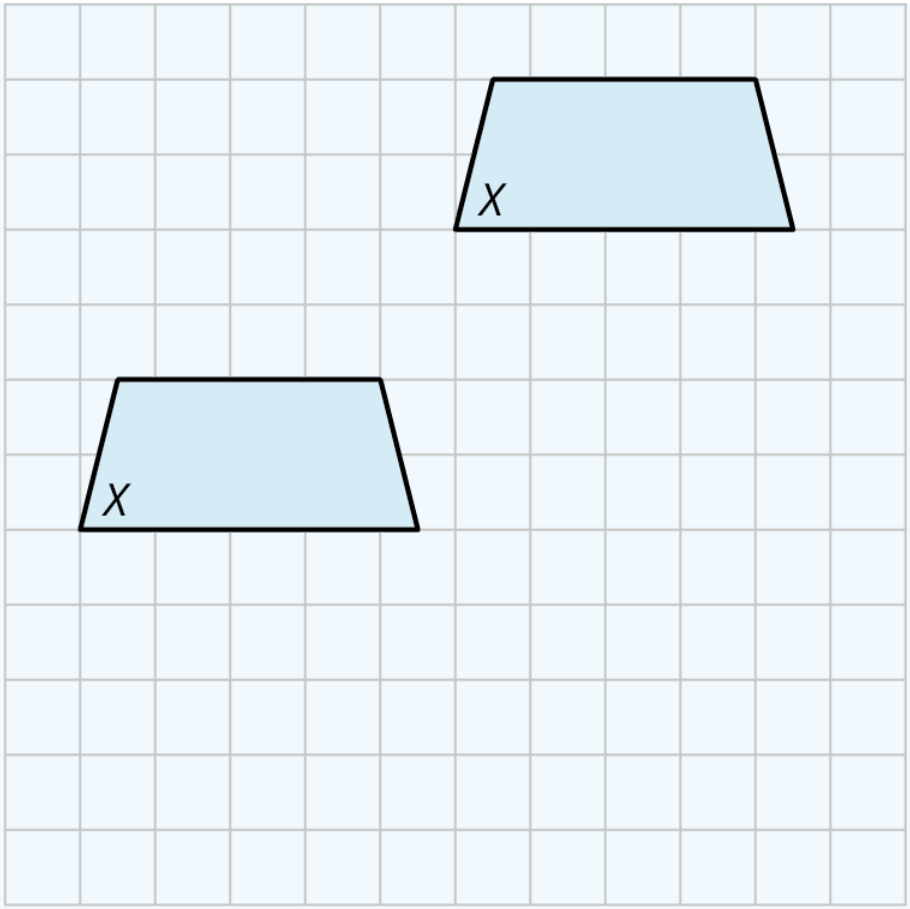
What is the name of the transformation that involves a reflection and a translation?

Exercise 4.1.1.13

What can a tessellation *not* have between shapes?

Exercise 4.1.1.14

Describe the transformation shown.

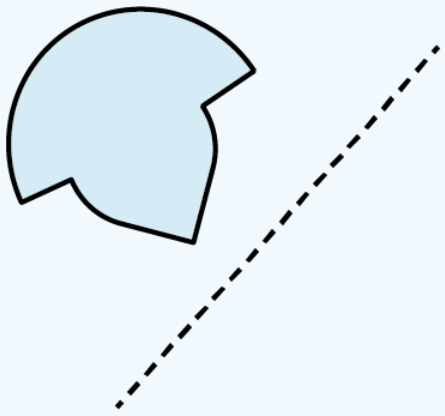


Exercise 4.1.1.15

What do we call a transformation that produces a mirror image?

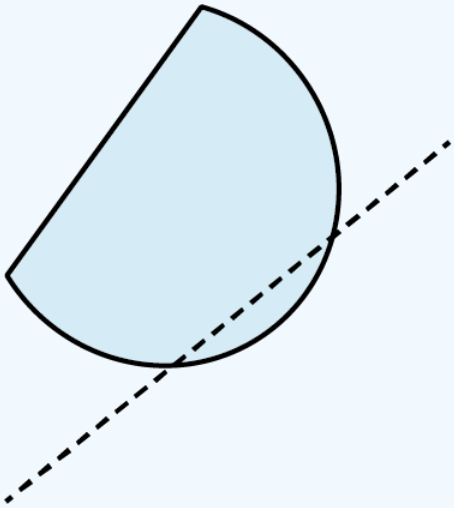
Exercise 4.1.1.16

Sketch the reflection of the shape about the dashed line.



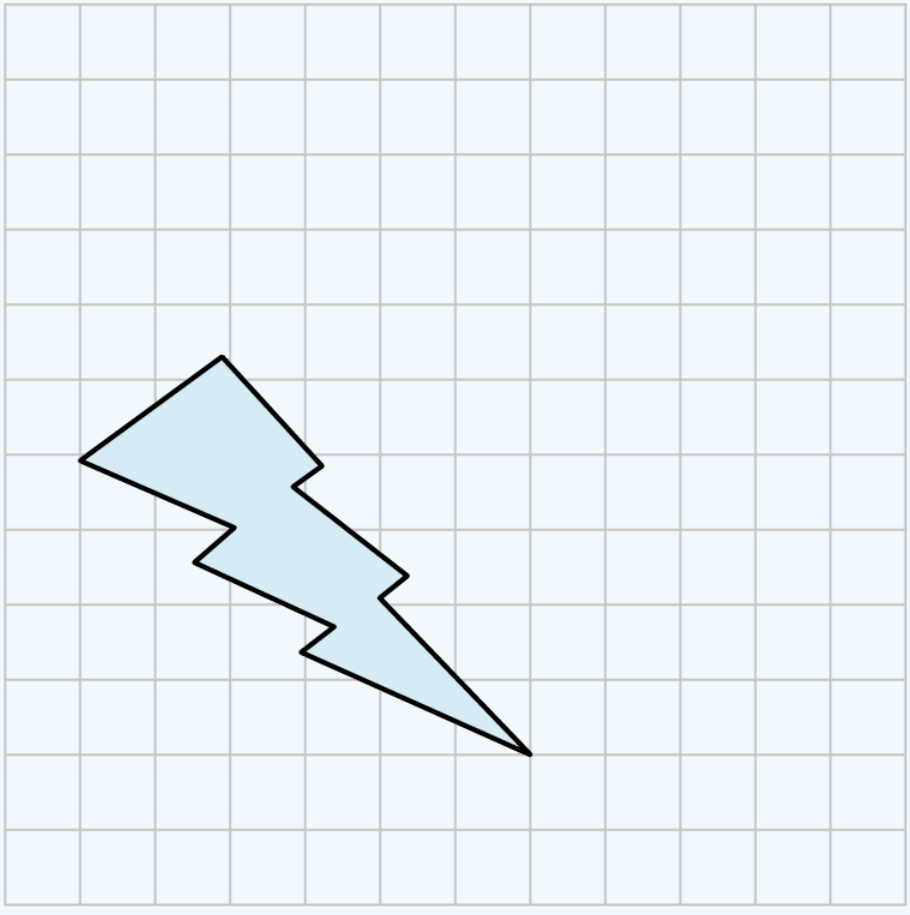
Exercise 4.1.1.17

Sketch the reflection of the shape about the dashed line.



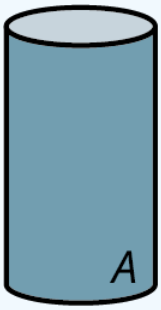
Exercise 4.1.1.18

Sketch the translation of the shape 3 units to the right and 3 units vertically.



Exercise 4.1.1.19

Rotate the shape $45°$ about the rotation point using point A as your guide.



Exercise 4.1.1.20

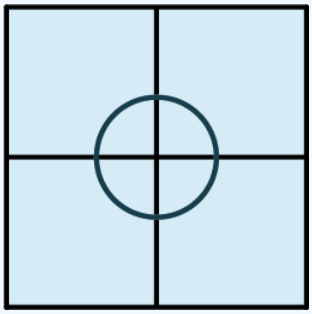
Do regular pentagons tessellate the plain by themselves?

Exercise 4.1.1.21

What do regular tessellations have in common?

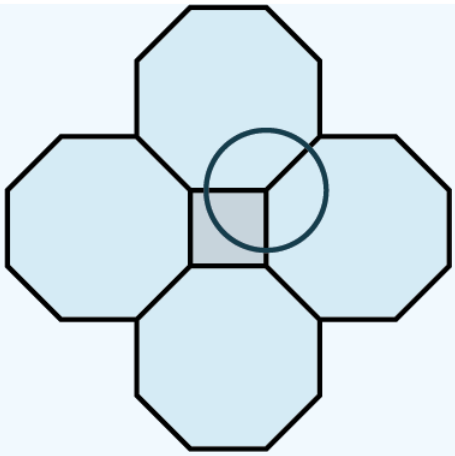
Exercise 4.1.1.22

How would we name a tessellation of squares as shown in the figure?



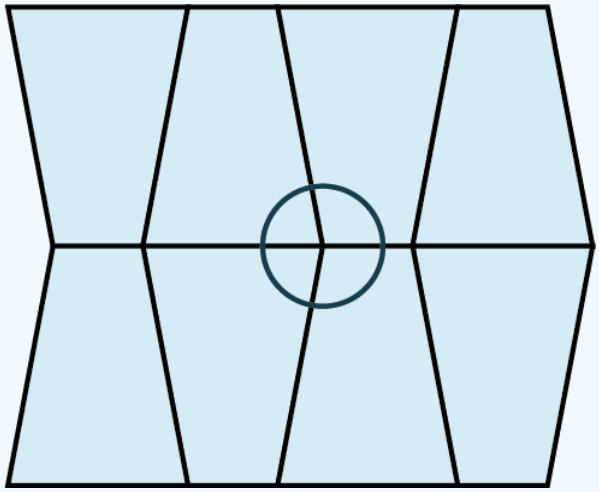
Exercise 4.1.1.23

How do we name a tessellation of octagons and squares as shown in the figure?



Exercise 4.1.1.24

How would we name a tessellation of trapezoids as shown in the figure?



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CHAPTER OVERVIEW

5: Fractals

Fractals are mathematical sets, usually obtained through recursion, that exhibit interesting dimensional properties.

5.1: Fractals
5.2: Iterated Fractals
5.3: Fractal Dimension
5.4: Complex Numbers
5.5: Complex Recursive Sequences
5.6: Exercises
5.6.1: Answers to Selected Exercises

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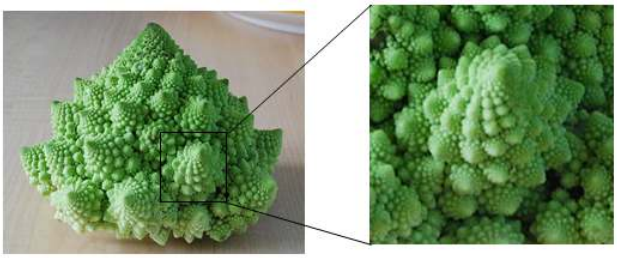
5.1: Fractals

Fractals are mathematical sets, usually obtained through recursion, that exhibit interesting dimensional properties. We'll explore what that sentence means through the rest of the chapter. For now, we can begin with the idea of self-similarity, a characteristic of most fractals.

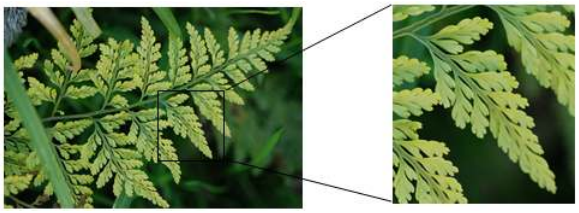
Self-similarity

A shape is **self-similar** when it looks essentially the same from a distance as it does closer up.

Self-similarity can often be found in nature. In the Romanesco broccoli pictured below[1], if we zoom in on part of the image, the piece remaining looks similar to the whole.



Likewise, in the fern frond below[2], one piece of the frond looks similar to the whole.



Similarly, if we zoom in on the coastline of Portugal[3], each zoom reveals previously hidden detail, and the coastline, while not identical to the view from further way, does exhibit similar characteristics.



[1] en.Wikipedia.org/wiki/File:Ca...ractal_AVM.JPG

[2] http://www.flickr.com/photos/cjewel/3261398909/

[3] Openstreetmap.org, CC-BY-SA

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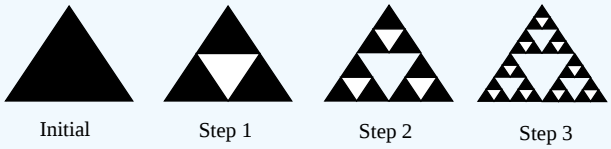
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5.2: Iterated Fractals

This self-similar behavior can be replicated through recursion: repeating a process over and over.

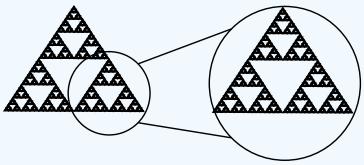
Example 1

Suppose that we start with a filled-in triangle. We connect the midpoints of each side and remove the middle triangle. We then repeat this process.



Initial Step 1 Step 2 Step 3

If we repeat this process, the shape that emerges is called the Sierpinski gasket. Notice that it exhibits self-similarity – any piece of the gasket will look identical to the whole. In fact, we can say that the Sierpinski gasket contains three copies of itself, each half as tall and wide as the original. Of course, each of those copies also contains three copies of itself.



We can construct other fractals using a similar approach. To formalize this a bit, we're going to introduce the idea of initiators and generators.

Initiators and Generators

- An **initiator** is a starting shape
- A **generator** is an arranged collection of scaled copies of the initiator

To generate fractals from initiators and generators, we follow a simple rule:

Fractal Generation Rule

At each step, replace every copy of the initiator with a scaled copy of the generator, rotating as necessary

This process is easiest to understand through example.

Example 2

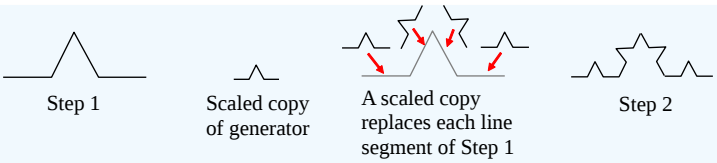
Use the initiator and generator shown to create the iterated fractal.



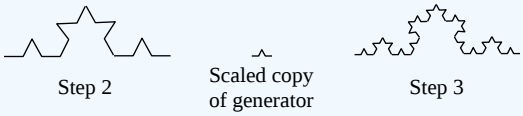
initiator generator

Solution

This tells us to, at each step, replace each line segment with the spiked shape shown in the generator. Notice that the generator itself is made up of 4 copies of the initiator. In step 1, the single line segment in the initiator is replaced with the generator. For step 2, each of the four line segments of step 1 is replaced with a scaled copy of the generator:



This process is repeated to form Step 3. Again, each line segment is replaced with a scaled copy of the generator.



Notice that since Step 0 only had 1 line segment, Step 1 only required one copy of Step 0.

Since Step 1 had 4 line segments, Step 2 required 4 copies of the generator.

Step 2 then had 16 line segments, so Step 3 required 16 copies of the generator.

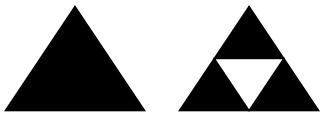
Step 4, then, would require $16 \times 4 = 64$ copies of the generator.

The shape resulting from iterating this process is called the **Koch curve**, named for Helge von Koch who first explored it in 1904.



Koch curve

Notice that the Sierpinski gasket can also be described using the initiator-generator approach



Initiator Generator

Example 3

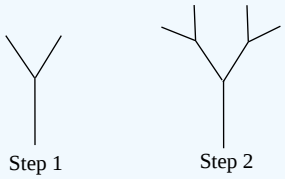
Use the initiator and generator below, however only iterate on the "branches." Sketch several steps of the iteration.



initiator generator

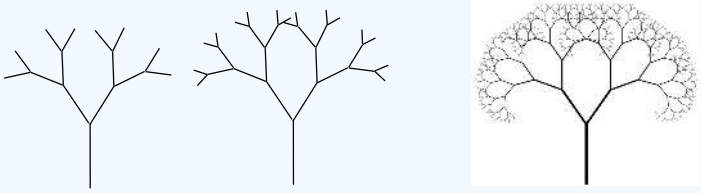
Solution

We begin by replacing the initiator with the generator. We then replace each "branch" of Step 1 with a scaled copy of the generator to create Step 2.



Step 1 Step 2

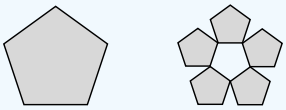
We can repeat this process to create later steps. Repeating this process can create intricate tree shapes[1].



Step 3 Step 4 Final shape

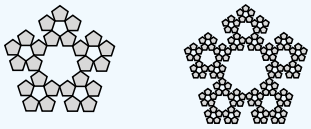
Try it Now 1

Use the initiator and generator shown to produce the next two stages



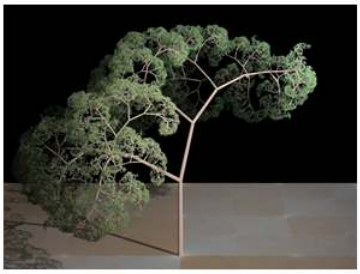
Initiator Generator

Answer



Step 2 Step 3

Using iteration processes like those above can create a variety of beautiful images evocative of nature[2][3].





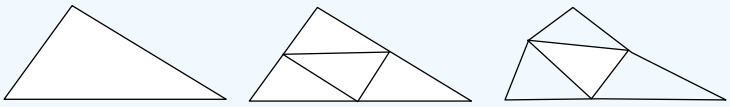
More natural shapes can be created by adding in randomness to the steps.

Example 4

Create a variation on the Sierpinski gasket by randomly skewing the corner points each time an iteration is made.

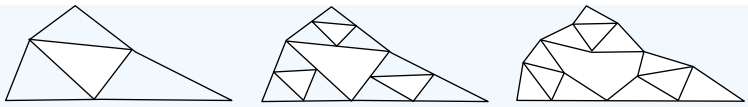
Solution

Suppose we start with the triangle below. We begin, as before, by removing the middle triangle. We then add in some randomness.



Step 0 Step 1 Step 1 with randomness

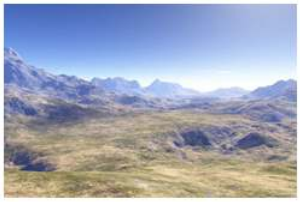
We then repeat this process.



Step 1 with randomness | Step 2 | Step 2 with randomness

Continuing this process can create mountain-like structures.

The landscape below was created using fractals, then colored and textured.



[1] http://www.flickr.com/photos/visualarts/5436068969/

[2] en.Wikipedia.org/wiki/File:Fr...e_b_-_2%29.jpg

[3] en.Wikipedia.org/wiki/File:Ba...-_4_states.PNG

[4] en.Wikipedia.org/wiki/File:Fr...lLandscape.jpg

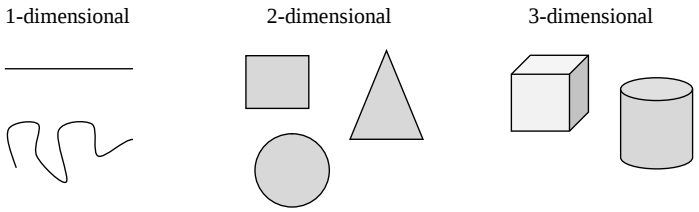
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5.3: Fractal Dimension

In addition to visual self-similarity, fractals exhibit other interesting properties. For example, notice that each step of the Sierpinski gasket iteration removes one quarter of the remaining area. If this process is continued indefinitely, we would end up essentially removing all the area, meaning we started with a 2-dimensional area, and somehow end up with something less than that, but seemingly more than just a 1-dimensional line.

To explore this idea, we need to discuss dimension. Something like a line is 1-dimensional; it only has length. Any curve is 1-dimensional. Things like boxes and circles are 2-dimensional, since they have length and width, describing an area. Objects like boxes and cylinders have length, width, and height, describing a volume, and are 3-dimensional.

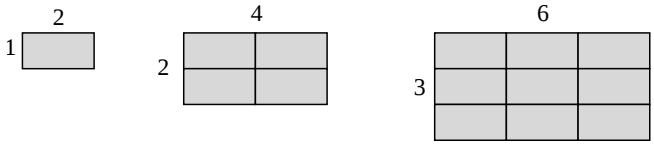


Certain rules apply for scaling objects, related to their dimension.

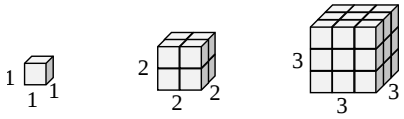
If I had a line with length 1, and wanted scale its length by 2, I would need two copies of the original line. If I had a line of length 1, and wanted to scale its length by 3, I would need three copies of the original.



If I had a rectangle with length 2 and height 1, and wanted to scale its length and width by 2, I would need four copies of the original rectangle. If I wanted to scale the length and width by 3, I would need nine copies of the original rectangle.



If I had a cubical box with sides of length 1, and wanted to scale its length, its width, and its height by 2, I would need eight copies of the original cube. If I wanted to scale the length, width, and height by 3, I would need 27 copies of the original cube.



Notice that in the 1-dimensional case, copies needed = scale.

- In the 2-dimensional case, copies needed $=$ scale^{2}.
- In the 3-dimensional case, copies needed $=$ scale^{3}.

From these examples, we might infer a pattern.

Scaling-Dimension Relation

To scale a D-dimensional shape by a scaling factor S, the number of copies C of the original shape needed will be given by:

Copies $=$ Scale ^{Dimension}, or $C = S^{D}$

Example 5

Use the scaling-dimension relation to determine the dimension of the Sierpinski gasket.

Solution

Suppose we define the original gasket to have side length 1. The larger gasket shown is twice as wide and twice as tall, so has been scaled by a factor of 2.



1 2

Notice that to construct the larger gasket, 3 copies of the original gasket are needed.

Using the scaling-dimension relation $C = S^D$, we obtain the equation $3 = 2^D$

since $2^1 = 2$ and $2^2 = 4$, we can immediately see that D is somewhere between 1 and 2; the gasket is more than a 1 -dimensional shape, but we've taken away so much area its now less than 2-dimensional.

Solving the equation $3 = 2^D$ requires logarithms. If you studied logarithms earlier, you may recall how to solve this equation (if not, just skip to the box below and use that formula):

$3 = 2^D$	Take the logarithm of both sides
$\log(3) = \log(2^D)$	Use the exponent property of logs
$\log(3) = D\log(2)$	Divide by $\log(2)$
$D = \frac{\log(3)}{\log(2)} \approx 1.585$	The dimension of the gasket is about 1.585

Scaling-Dimension Relation, to find Dimension

To find the dimension D of a fractal, determine the scaling factor S and the number of copies C of the original shape needed, then use the formula

$$D = \frac{\log(C)}{\log(S)}$$

Try it Now 2

Determine the fractal dimension of the fractal produced using the initiator and generator



Initiator Generator

Answer



1 3

Scaling the fractal by a factor of 3 requires 5 copies of the original. $D = \frac{\log(5)}{\log(3)} \approx 1.465$

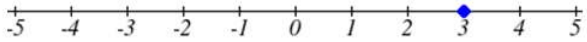
We will now turn our attention to another type of fractal, defined by a different type of recursion. To understand this type, we are first going to need to discuss complex numbers.

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5.4: Complex Numbers

The numbers you are most familiar with are called **real numbers**. These include numbers like 4, 275, -200, 10.7, ½, π, and so forth. All these real numbers can be plotted on a number line. For example, if we wanted to show the number 3, we plot a point:



To solve certain problems like $x^2 = -4$, it became necessary to introduce **imaginary numbers**.

Imaginary Number i

The imaginary number i is defined to be $i = \sqrt{-1}$.

Any real multiple of i, like $5i$, is also an imaginary number.

Example 6

Simplify $\sqrt{-9}$.

Solution

We can separate $\sqrt{-9}$ as $\sqrt{9}\sqrt{-1}$. We can take the square root of 9, and write the square root of -1 as i

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$$

Complex Number

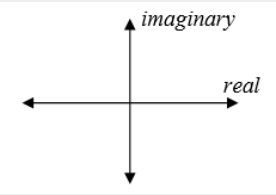
A **complex number** is a number $z = a + bi$, where a and b are real numbers

- a is the real part of the complex number
- b is the imaginary part of the complex number

To plot a complex number like $3 - 4i$, we need more than just a number line since there are two components to the number. To plot this number, we need two number lines, crossed to form a complex plane.

Complex Plane

In the **complex plane**, the horizontal axis is the real axis and the vertical axis is the imaginary axis.

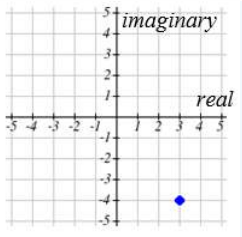


Example 7

Plot the number $3 - 4i$ on the complex plane.

Solution

The real part of this number is 3, and the imaginary part is -4. To plot this, we draw a point 3 units to the right of the origin in the horizontal direction and 4 units down in the vertical direction.



Because this is analogous to the Cartesian coordinate system for plotting points, we can think about plotting our complex number $z = a + bi$ as if we were plotting the point (a, b) in Cartesian coordinates. Sometimes people write complex numbers as $z = x + yi$ to highlight this relation.

Arithmetic on Complex Numbers

Before we dive into the more complicated uses of complex numbers, let's make sure we remember the basic arithmetic involved. To add or subtract complex numbers, we simply add the like terms, combining the real parts and combining the imaginary parts.

Example 8

Add $3 - 4i$ and $2 + 5i$.

Solution

Adding $(3 - 4i) + (2 + 5i)$, we add the real parts and the imaginary parts

$3 + 2 - 4i + 5i$

$5 + i$

Try it Now 3

Subtract $2 + 5i$ from $3 - 4i$.

Answer

$(3 - 4i) - (2 + 5i) = 1 - 9i$

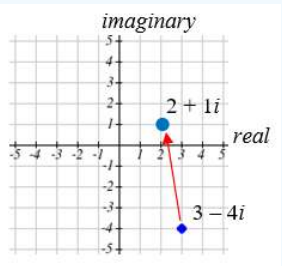
When we add complex numbers, we can visualize the addition as a shift, or translation, of a point in the complex plane.

Example 9

Visualize the addition $3 - 4i$ and $-1 + 5i$.

Solution

The initial point is $3 - 4i$. When we add $-1 + 3i$, we add -1 to the real part, moving the point 1 units to the left, and we add 5 to the imaginary part, moving the point 5 units vertically. This shifts the point $3 - 4i$ to $2 + 1i$.



We can also multiply complex numbers by a real number, or multiply two complex numbers.

Example 10

Multiply: $4(2+5i)$

Solution

To multiply the complex number by a real number, we simply distribute as we would when multiplying polynomials.

$$\begin{aligned} &4(2+5i) && \text{Distribute} \\ &= 4\cdot 2 + 4\cdot 5i && \text{Simplify} \\ &= 8+20i \end{aligned}$$

Example 11

Multiply: $(2+5i)(4+i)$

Solution

To multiply the complex number by a complex number, we simply distribute as we would when multiplying polynomials.

$$\begin{aligned} &(2+5i)(4+i) && \text{Expand} \\ &= 8+20i+2i+5i^2 && \text{Since } i=\sqrt{-1}, i^2=-1 \\ &= 8+20i+2i+5(-1) && \text{Simplify} \\ &= 3+22i \end{aligned}$$

Try it Now 4

Multiply $3-4i$ and $2+3i$.

Answer

Multiply $(3-4i)(2+3i) = 6+9i-8i-12i^2 = 6+i-12(-1) = 18+i$

To understand the effect of multiplication visually, we'll explore three examples.

Example 12

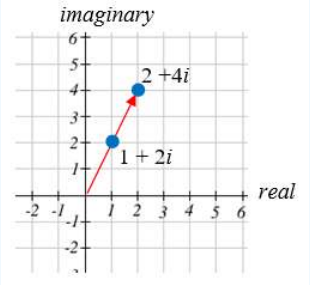
Visualize the product $2(1+2i)$

Solution

Multiplying we'd get

$2\cdot 1 + 2\cdot 2i = 2+4i$

Notice both the real and imaginary parts have been scaled by 2. Visually, this will stretch the point outwards, away from the origin.



Example 13

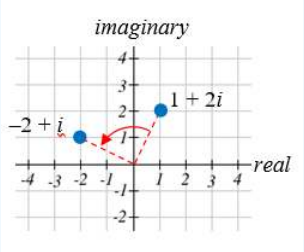
Visualize the product $i(1+2i)$

Solution

Multiplying, we'd get

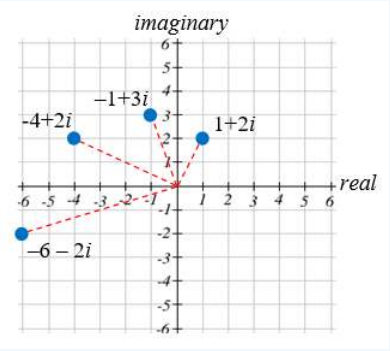
$i\cdot 1 + i\cdot 2i = i+2i^2 = i+2(-1) = -2+i$

In this case, the distance from the origin has not changed, but the point has been rotated about the origin, 90° counter-clockwise.



Example 14

Visualize the result of multiplying $1+2i$ by $1+i$. Then show the result of multiplying by $1+i$ again.



Solution

Multiplying $1+2i$ by $1+i$,

$$(1+2i)(1+i) = 1+i+2i+2i^2 = 1+3i+2(-1) = -1+3i$$

Multiplying by $1+i$ again,

$$(-1+3i)(1+i) = -1-i+3i+3i^2 = -1+2i+3(-1) = -4+2i$$

If we multiplied by $1+i$ again, we'd get $-6-2i$. Plotting these numbers in the complex plane, you may notice that each point gets both further from the origin, and rotates counterclockwise, in this case by 45°.

In general, multiplication by a complex number can be thought of as a **scaling**, changing the distance from the origin, combined with a **rotation** about the origin.

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5.5: Complex Recursive Sequences

We will now explore recursively defined sequences of complex numbers.

Recursive Sequence

A **recursive relationship** is a formula which relates the next value, z_{n+1}, in a sequence to the previous value, z_n. In addition to the formula, we need an initial value, z_0.

The sequence of values produced is the **recursive sequence**.

Example 15

Given the recursive relationship $z_{n+1} = z_n + 2, \quad z_0 = 4$, generate several terms of the recursive sequence.

Solution

We are given the starting value, $z_0 = 4$. The recursive formula holds for any value of n, so if $n = 0$, then $z_{n+1} = z_n + 2$ would tell us $z_{0+1} = z_0 + 2$, or more simply, $z_1 = z_0 + 2$

Notice this defines z_1 in terms of the known z_0, so we can compute the value:
$z_1 = z_0 + 2 = 4 + 2 = 6$

Now letting $n = 1$, the formula tells us $z_{1+1} = z_1 + 2$, or $z_2 = z_1 + 2$. Again, the formula gives the next value in the sequence in terms of the previous value.

(z_{2}=z_{1}+2=6+2=8\)

Continuing,

$z_3 = z_2 + 2 = 8 + 2 = 10$

$z_4 = z_3 + 2 = 10 + 2 = 12$

The previous example generated a basic linear sequence of real numbers. The same process can be used with complex numbers.

Example 16

Given the recursive relationship $z_{n+1} = z_n \cdot i + (1 - i), \quad z_0 = 4$, generate several terms of the recursive sequence.

Solution

We are given $z_0 = 4$. Using the recursive formula:

$z_1 = z_0 \cdot i + (1 - i) = 4 \cdot i + (1 - i) = 1 + 3i$

$z_2 = z_1 \cdot i + (1 - i) = (1 + 3i) \cdot i + (1 - i) = i + 3i^2 + (1 - i) = i - 3 + (1 - i) = -2$

$z_3 = z_2 \cdot i + (1 - i) = (-2) \cdot i + (1 - i) = -2i + (1 - i) = 1 - 3i$

$z_4 = z_3 \cdot i + (1 - i) = (1 - 3i) \cdot i + (1 - i) = i - 3i^2 + (1 - i) = i + 3 + (1 - i) = 4$

$z_5 = z_4 \cdot i + (1 - i) = 4 \cdot i + (1 - i) = 1 + 3i$

Notice this sequence is exhibiting an interesting pattern – it began to repeat itself.

Mandelbrot Set

The Mandelbrot Set is a set of numbers defined based on recursive sequences

Mandelbrot Set

For any complex number c, define the sequence $z_{n+1} = z_n^2 + c, \quad z_0 = 0$

If this sequence always stays close to the origin (within 2 units), then the number c is part of the **Mandelbrot Set**. If the sequence gets far from the origin, then the number c is not part of the set.

Example 17

Determine if $c = 1 + i$ is part of the Mandelbrot set.

Solution

We start with $z_0 = 0$. We continue, omitting some detail of the calculations

$z_1 = z_0^2 + 1 + i = 0 + 1 + i = 1 + i$

$z_2 = z_1^2 + 1 + i = (1 + i)^2 + 1 + i = 1 + 3i$

$z_3 = z_2^2 + 1 + i = (1 + 3i)^2 + 1 + i = -7 + 7i$

$z_4 = z_3^2 + 1 + i = (-7 + 7i)^2 + 1 + i = 1 - 97i$

We can already see that these values are getting quite large. It does not appear that $c = 1 + i$ is part of the Mandelbrot set.

Example 18

Determine if $c = 0.5i$ is part of the Mandelbrot set.

Solution

We start with $z_0 = 0$. We continue, omitting some detail of the calculations

$z_1 = z_0^2 + 0.5i = 0 + 0.5i = 0.5i$

$z_2 = z_1^2 + 0.5i = (0.5i)^2 + 0.5i = -0.25 + 0.5i$

$z_3 = z_2^2 + 0.5i = (-0.25 + 0.5i)^2 + 0.5i = -0.1875 + 0.25i$

$z_4 = z_3^2 + 0.5i = (-0.1875 + 0.25i)^2 + 0.5i = -0.02734 + 0.40625i$

While not definitive with this few iterations, it does appear that this value is remaining small, suggesting that $0.5i$ is part of the Mandelbrot set.

Try it Now 5

Determine if $c = 0.4 + 0.3i$ is part of the Mandelbrot set.

Answer

$z_1 = z_0^2 + 0.4 + 0.3i = 0 + 0.4 + 0.3i = 0.4 + 0.3i$

$z_2 = z_1^2 + 0.4 + 0.3i = (0.4 + 0.3i)^2 + 0.4 + 0.3i = 0.47 + 0.54i$

$z_3 = z_2^2 + 0.5i = (-0.25 + 0.5i)^2 + 0.5i = -0.1875 + 0.25i$

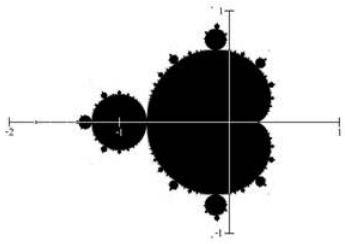
$z_4 = z_3^2 + 0.5i = (-0.1875 + 0.25i)^2 + 0.5i = -0.02734 + 0.40625i$

If all complex numbers are tested, and we plot each number that is in the Mandelbrot set on the complex plane, we obtain the shape to the right[1].

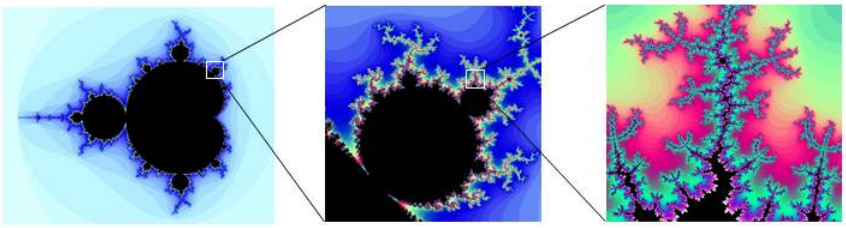
The boundary of this shape exhibits quasi-self-similarity, in that portions look very similar to the whole.

In addition to coloring the Mandelbrot set itself black, it is common to the color the points in the complex plane surrounding the set. To create a meaningful coloring, often people count the number of iterations of the recursive sequence that are required for a

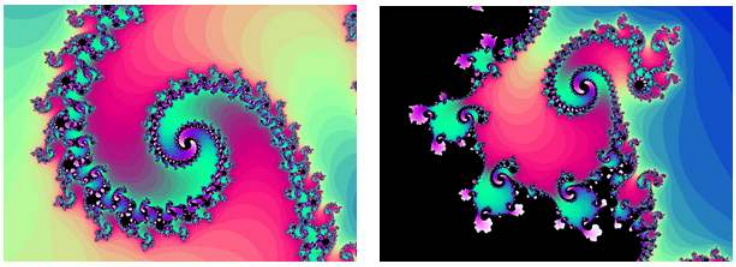
point to get further than 2 units away from the origin. For example, using $c = 1 + i$ above, the sequence was distance 2 from the origin after only two recursions.



For some other numbers, it may take tens or hundreds of iterations for the sequence to get far from the origin. Numbers that get big fast are colored one shade, while colors that are slow to grow are colored another shade. For example, in the image below[2], light blue is used for numbers that get large quickly, while darker shades are used for numbers that grow more slowly. Greens, reds, and purples can be seen when we zoom in – those are used for numbers that grow very slowly.



The Mandelbrot set, for having such a simple definition, exhibits immense complexity. Zooming in on other portions of the set yields fascinating swirling shapes.



[1] en.Wikipedia.org/wiki/File:Mandelset_hires.png

[2] This series was generated using Scott's Mandelbrot Set Explorer

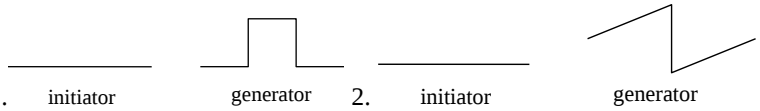
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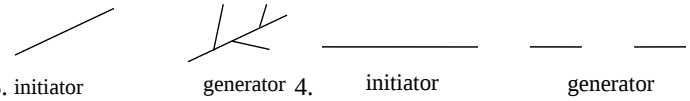
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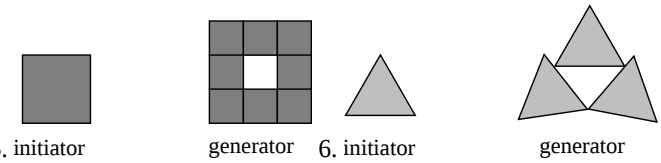
5.6: Exercises

Iterated Fractals

Using the initiator and generator shown, draw the next two stages of the iterated fractal.







7. Create your own version of Sierpinski gasket with added randomness.

8. Create a version of the branching tree fractal from example #3 with added randomness.

Fractal Dimension

9. Determine the fractal dimension of the Koch curve.

10. Determine the fractal dimension of the curve generated in exercise #1

11. Determine the fractal dimension of the Sierpinski carpet generated in exercise #5

12. Determine the fractal dimension of the Cantor set generated in exercise #4

Complex Numbers

13. Plot each number in the complex plane: a) 4 b) $-3i$ c) $-2+3i$ d) $2+i$

14. Plot each number in the complex plane: a) -2 b) $4i$ c) $1+2i$ d) $-1-i$

15. Compute: a) $(2+3i)+(3-4i)$ b) $(3-5i)-(-2-i)$

16. Compute: a) $(1-i)+(2+4i)$ b) $(-2-3i)-(4-2i)$

17. Multiply: a) $3(2+4i)$ b) $(2i)(-1-5i)$ c) $(2-4i)(1+3i)$

18. Multiply: a) $2(-1+3i)$ b) $(3i)(2-6i)$ c) $(1-i)(2+5i)$

19. Plot the number $2+3i$. Does multiplying by $1-i$ move the point closer to or further from the origin? Does it rotate the point, and if so which direction?

20. Plot the number $2+3i$. Does multiplying by $0.75+0.5i$ move the point closer to or further from the origin? Does it rotate the point, and if so which direction?

Recursive Sequences

21. Given the recursive relationship$z_{n+1}=iz_n+1, \quad z_0=2$, generate the next 3 terms of the recursive sequence.

22. Given the recursive relationship$z_{n+1}=2z_n+i, \quad z_0=3-2i$, generate the next 3 terms of the recursive sequence.

23. Using $c=-0.25$, calculate the first 4 terms of the Mandelbrot sequence.

24. Using $c=1-i$, calculate the first 4 terms of the Mandelbrot sequence.

For a given value of c, the Mandelbrot sequence can be described as *escaping* (growing large), a *attracted* (it approaches a fixed value), or *periodic* (it jumps between several fixed values). A periodic cycle is typically described the number if values it jumps between; a 2-cycle jumps between 2 values, and a 4-cycle jumps between 4 values.

For questions 25 – 30, you'll want to use a calculator that can compute with complex numbers, or use an online calculator which can compute a Mandelbrot sequence. For each value of c, examine the Mandelbrot sequence and determine if the value appears to be escaping, attracted, or periodic?

25. $c = -0.5 + 0.25i$. 26. $c = 0.25 + 0.25i$.

27. $c = -1.2$. 28. $c = i$.

29. $c = 0.5 + 0.25i$. 30. $c = -0.5 + 0.5i$.

31. $c = -0.12 + 0.75i$. 32. $c = -0.5 + 0.5i$.

Exploration

The Julia Set for c is another fractal, related to the Mandelbrot set. The Julia Set for c uses the recursive sequence: $z_{n+1} = z_n^2 + c$, $z_0 = d$, where c is constant for any particular Julia set, and d is the number being tested. A value d is part of the Julia Set for c if the sequence does not grow large.

For example, the Julia Set for -2 would be defined by $z_{n+1} = z_n^2 - 2, \quad z_0 = d$. We then pick values for d, and test each to determine if it is part of the Julia Set for -2. If so, we color black the point in the complex plane corresponding with the number d. If not, we can color the point d based on how fast it grows, like we did with the Mandelbrot Set.

For questions 33-34, you will probably want to use the online calculator again.

33. Determine which of these numbers are in the Julia Set at $c = -0.12i + 0.75i$

a) $0.25i$ b) 0.1 c) $0.25 + 0.25i$

34. Determine which of these numbers are in the Julia Set at $c = -0.75$

a) $0.5i$ b) 1 c) $0.5 - 0.25i$

You can find many images online of various Julia Sets[1].

35. Explain why no point with initial distance from the origin greater than 2 will be part of the Mandelbrot sequence

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[1] For example, www.jcu.edu/math/faculty/spitz/juliaset/juliaset.htm,

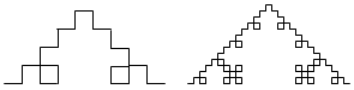
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5.6.1: Answers to Selected Exercises

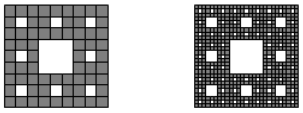
1.



3.



5.



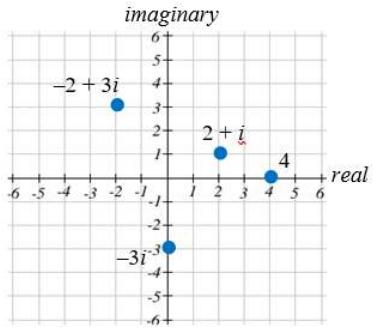
9. Four copies of the Koch curve are needed to create a curve scaled by 3.

$D = \frac{\log(4)}{\log(3)} \approx 1.262$



11. Eight copies of the shape are needed to make a copy scaled by 3. $D = \frac{\log(8)}{\log(3)} \approx 1.893$

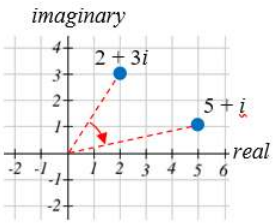
13.



15. a) $5 - i$ b) $5 - 4i$

17. a) $6 + 12i$ b) $10 - 2i$ c) $14 + 2i$

19. $(2 + 3i)(1 - i) = 5 + i$. It appears that multiplying by $1 - i$ both scaled the number away from the origin, and rotated it clockwise about 45°.



21.

$z_1 = iz_0 + 1 = i(2) + 1 = 1 + 2i$

$z_2 = iz_1 + 1 = i(1 + 2i) + 1 = i - 2 + 1 = -1 + i$

$z_3 = iz_2 + 1 = i(-1 + i) + 1 = -i - 1 + 1 = -i$

23.

$z_0 = 0$

$z_1 = z_0^2 - 0.25 = 0 - 0.25 = -0.25$

$z_2 = z_1^2 - 0.25 = (-0.25)^2 - 0.25 = -0.1875$

$z_3 = z_2^2 - 0.25 = (-0.1875)^2 - 0.25 = -0.21484$

$z_4 = z_3^2 - 0.25 = (-0.21484)^2 - 0.25 = -0.20384$

25. attracted, to approximately $-0.37766 + 0.14242i$

27. periodic 2-cycle 29. Escaping 31. periodic 3-cycle

33. a) Yes, periodic 3-cycle b) Yes, periodic 3-cycle c) No

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CHAPTER OVERVIEW

6: Cryptography

6.1: Cryptography
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6.1: Cryptography

When people need to secretly store or communicate messages, they turn to cryptography. Cryptography involves using techniques to obscure a message so outsiders cannot read the message. It is typically split into two steps: encryption, in which the message is obscured, and decryption, in which the original message is recovered from the obscured form.

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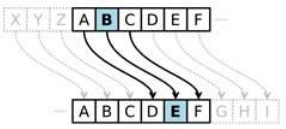
6.2: Substitution Ciphers

One simple encryption method is called a **substitution cipher**.

Substitution Cipher

A substitution cipher replaces each letter in the message with a different letter, following some established mapping.

A simple example of a substitution cipher is called the **Caesar cipher**, sometimes called a shift cipher. In this approach, each letter is replaced with a letter some fixed number of positions later in the alphabet. For example, if we use a shift of 3, then the letter A would be replaced with D, the letter 3 positions later in the alphabet. The entire mapping would look like: [1]



Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**

Maps to: **DEFGHIJKLMNOPQRSTUVWXYZABC**

Example 1

Use the Caesar cipher with shift of 3 to encrypt the message: “We ride at noon”

Solution

We use the mapping above to replace each letter. W gets replaced with Z, and so forth, giving the encrypted message: ZH ULGH DW QRRQ.

Notice that the length of the words could give an important clue to the cipher shift used. If we saw a single letter in the encrypted message, we would assume it must be an encrypted A or I, since those are the only single letters than form valid English words.

To obscure the message, the letters are often rearranged into equal sized blocks. The message ZH ULGH DW QRRQ could be written in blocks of three characters as

ZHU LGH DWQ RRQ.

Example 2

Decrypt the message GZD KNK YDX MFW JXA if it was encrypted using a shift cipher with shift of 5.

Solution

We start by writing out the character mapping by shifting the alphabet, with A mapping to F, five characters later in the alphabet.

Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**

Maps to: **FGHIJKLMNOPQRSTUVWXYZABCDE**

We now work backwards to decrypt the message. The first letter G is mapped to by B, so B is the first character of the original message. Continuing, our decrypted message is

BUY FIF TYS HAR ESA.

Removing spaces we get BUYFIFTYSHARESA. In this case, it appears an extra character was added to the end to make the groups of three come out even, and that the original message was “Buy fifty shares.”

Try it Now 1

Decrypt the message BNW MVX WNH if it was encrypted using a shift cipher with shift 9 (mapping A to J).

Answer

SEND MONEY



Notice that in both the ciphers above, the extra part of the alphabet wraps around to the beginning. Because of this, a handy version of the shift cipher is a cipher disc, such as the Alberti cipher disk shown here[2] from the 1400s. In a cipher disc, the inner wheel could be turned to change the cipher shift. This same approach is used for "secret decoder rings."

The security of a cryptographic method is very important to the person relying on their message being kept secret. The security depends on two factors:

1. The security of the method being used
2. The security of the encryption key used

In the case of a shift cipher, the method is "a shift cipher is used." The encryption key is the specific amount of shift used.

Suppose an army is using a shift cipher to send their messages, and one of their officers is captured by their enemy. It is likely the method and encryption key could become compromised. It is relatively hard to change encryption methods, but relatively easy to change encryption keys.

During World War II, the Germans' Enigma encryption machines were captured, but having details on the encryption method only slightly helped the Allies, since the encryption keys were still unknown and hard to discover. Ultimately, the security of a message cannot rely on the method being kept secret; it needs to rely on the key being kept secret.

Encryption Security

The security of any encryption method should depend only on the encryption key being difficult to discover. It is not safe to rely on the encryption method (algorithm) being kept secret.

With that in mind, let's analyze the security of the Caesar cipher.

Example 3

Suppose you intercept a message, and you know the sender is using a Caesar cipher, but do not know the shift being used. The message begins EQZP. How hard would it be to decrypt this message?

Solution

Since there are only 25 possible shifts, we would only have to try 25 different possibilities to see which one produces results that make sense. While that would be tedious, one person could easily do this by hand in a few minutes. A modern computer could try all possibilities in under a second.

Shift	Message	Shift	Message	Shift	Message	Shift	Message
1	DPYO	7	XJSI	13	RDMC	19	LXGW
2	COXN	8	WIRH	14	QCLB	20	KWFV
3	BNWM	9	VHQG	15	PBKA	21	JVEU
4	AMVL	10	UGPF	16	OAJZ	22	IUDT
5	ZLUK	11	TFOE	17	NZIY	23	HTCS
6	YKTJ	12	SEND	18	MYHX	24	GSBR
						25	FRAQ

In this case, a shift of 12 (A mapping to M) decrypts EQZP to SEND. Because of this ease of trying all possible encryption keys, the Caesar cipher is not a very secure encryption method.

Brute Force Attack

A brute force attack is a method for breaking encryption by trying all possible encryption keys.

To make a brute force attack harder, we could make a more complex substitution cipher by using something other than a shift of the alphabet. By choosing a random mapping, we could get a more secure cipher, with the tradeoff that the encryption key is harder to describe; the key would now be the entire mapping, rather than just the shift amount.

Example 4

Use the substitution mapping below to encrypt the message “March 12 0300”

Original: $mathrm{ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789}$

Maps to: **2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7P**

Solution

Using the mapping, the message would encrypt to 62SQT ZN Y1YY

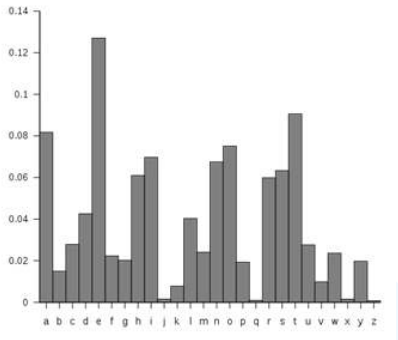
Try it Now 2

Use the substitution mapping from Example 4 to decrypt the message C2SVX2VP

Answer

PARTY AT 9

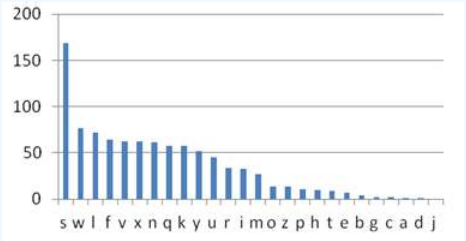
While there were only 25 possible shift cipher keys (35 if we had included numbers), there are about 10^{40} possible substitution ciphers[3]. That’s much more than a trillion trillions. It would be essentially impossible, even with supercomputers, to try every possible combination. Having a huge number of possible encryption keys is one important part of key security.



Unfortunately, this cipher is still not secure, because of a technique called frequency analysis, discovered by Arab mathematician Al-Kindi in the 9th century. English and other languages have certain letters than show up more often in writing than others.[4] For example, the letter E shows up the most frequently in English. The chart to the right shows the typical distribution of characters.

Example 5

The chart to the right shows the frequency of different characters in some encrypted text. What can you deduce about the mapping?



Solution

Because of the high frequency of the letter S in the encrypted text, it is very likely that the substitution maps E to S. Since W is the second most frequent character, it likely that T or A maps to W. Because C, A, D, and J show up rarely in the encrypted text, it is likely they are mapped to from J, Q, X, and Z.

In addition to looking at individual letters, certain pairs of letters show up more frequently, such as the pair “th.” By analyzing how often different letters and letter pairs show up an encrypted message, the substitution mapping used can be deduced[5].

[1] en.Wikipedia.org/w/index.php?title=File:Caesar3.svg&page=1. PD

[2] en.Wikipedia.org/wiki/File:Alberti_cipher_disk.JPG

[3] There are 35 choices for what *A* maps to, then 34 choices for what *B* maps to, and so on, so the total number of possibilities is 35*34*33*…*2*1 = 35! = about 10^{40}

[4] en.Wikipedia.org/w/index.php?title=File:English_letter_frequency_(alphabetic).svg&page=1 PD

[5] For an example of how this is done, see en.Wikipedia.org/wiki/Frequency_analysis

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6.3: Transposition Ciphers

Another approach to cryptography is **transposition cipher.**

Transposition Ciphers

A transposition cipher is one in which the order of characters is changed to obscure the message.

An early version of a transposition cipher was a Scytale[1], in which paper was wrapped around a stick and the message was written. Once unwrapped, the message would be unreadable until the message was wrapped around a same-sized stick again.



One modern transposition cipher is done by writing the message in rows, then forming the encrypted message from the text in the columns.

Example 6

Encrypt the message “Meet at First and Pine at midnight” using rows 8 characters long.

Solution

We write the message in rows of 8 characters each. Nonsense characters are added to the end to complete the last row.

MEETATFI

RSTANDPI

NEATMIDN

IGHTPXNR

We could then encode the message by recording down the columns. The first column, reading down, would be MRNI. All together, the encoded message would be MRNI ESEG ETAH TATT ANMP TDIX FPDN IINR. The spaces would be removed or repositioned to hide the size of table used, since that is the encryption key in this message.

Example 7

Decrypt the message CEE IAI MNL NOG LTR VMH NW using the method above with a table with rows of 5 characters.

Solution

Since there are total of 20 characters and each row should have 5 characters, then there will be $20/5 = 4$ rows.

We start writing, putting the first 4 letters, CEEI, down the first column.

CALLM

EINTH

EMORN

INGVW

We can now read the message: CALL ME IN THE MORNING VW. The VW is likely nonsense characters used to fill out the message.

More complex versions of this rows-and-column based transposition cipher can be created by specifying an order in which the columns should be recorded. For example, the method could specify that after writing the message out in rows that you should record the third column, then the fourth, then the first, then the fifth, then the second. This adds additional complexity that would make it harder to make a brute-force attack.

To make the encryption key easier to remember, a word could be used. For example, if the key word was “MONEY”, it would specify that rows should have 5 characters each. The order of the letters in the alphabet would dictate which order to read the columns in. Since E, the 4th letter in the word, is the earliest letter in the alphabet from the word MONEY, the 4th column would be used first, followed by the 1st column (M), the 3rd column (N), the 2nd column (O), and the 5th column (Y).

Example 8

Encrypt the message BUY SOME MILK AND EGGS using a transposition cipher with key word MONEY.

Solution

Writing out the message in rows of 5 characters:

BUYSO
MEMIL
KANDE
GGSPK

We now record the columns in order 4 1 3 2 5:

SIDP BMKG YMNS UEAG OLEK

As before, we’d then remove or reposition the spaces to conceal evidence of the encryption key.

Try it Now 3

Encrypt the message “Fortify the embassy” using a transposition cipher with key word HELP

Answer

HELP gives column order 2 1 3 4.

FORT

IFYT

HEEM

BASS

YPLR

Encrypted text: OFE APF IHB YRY ESL TTM SR

To decrypt a keyword-based transposition cipher, we’d reverse the process. In the example above, the keyword MONEY tells us to begin with the 4th column, so we’d start by writing SIDP down the 4th column, then continue to the 1st column, 3rd column, etc.

Example 9

Decrypt the message RHA VTN USR EDE AIE RIK ATS OQR using a row-and-column transposition cipher with keyword PRIZED.

Solution

The keyword PRIZED tells us to use rows with 6 characters. Since D comes first in the alphabet, we start with 6th column. Since E is next in the alphabet, we’d follow with the 5th column. Continuing, the word PRIZED tells us the message was recorded with the columns in order 4 5 3 6 2 1.

For the decryption, we set up a table with 6 characters in each row. Since the beginning of the encrypted message came from the last column, we start writing the encrypted message down the last column.

					R
					H
					A
					V

The 5th column was the second one the encrypted message was read from, so is the next one we write to.

				T	R
				N	H
				U	A
				S	V

Continuing, we can fill out the rest of the message.

A	I	R	S	T	R
I	K	E	O	N	H
E	A	D	Q	U	A
R	T	E	R	S	V

Reading across the rows gives our decrypted message: AIRSTRIKEONHEADQUARTERSV

Unfortunately, since the transposition cipher does not change the frequency of individual letters, it is still susceptible to frequency analysis, though the transposition does eliminate information from letter pairs.

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[1] en.Wikipedia.org/wiki/File:Skytala%26EmptyStrip-Shaded.png

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6.4: Advanced shared symmetric-key methods

Both the substitution and transposition methods discussed so far are shared **symmetric-key** methods, meaning that both sender and receiver would have to have agreed upon the same secret encryption key before any methods could be sent.

All of the methods so far have been susceptible to frequency analysis since each letter is always mapped to the same encrypted character. More advanced methods get around this weakness. For example, the Enigma machines used in World War II had wheels that rotated. Each wheel was a substitution cipher, but the rotation would cause the substitution used to shift after each character.

For a simplified example, in the initial setup, the wheel might provide the mapping

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789**
- Maps to: **2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7P**

After the first character is encrypted, the wheel rotates, shifting the mapping one space, resulting in a new shifted mapping:

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789**
- Maps to: **P2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7**

Using this approach, no letter gets encrypted as the same character over and over.

Example 10

Encrypt the message “See me”. Use a basic Caesar cipher with shift 3 as the initial substitution, but shift the substitution one place after each character.

Solution

The initial mapping is

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**
- Maps to: **DEFGHIJKLMNOPQRSTUVWXYZABC**

This would map the first letter, S to V. We would then shift the mapping by one.

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**
- Now maps to: **EFGHIJKLMNOPQRSTUVWXYZABCD**

Now the next letter, E, will map to I. Again we shift the cipher

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**
- Now maps to: **FGHIJKLMNOPQRSTUVWXYZABCDE**

The next letter, E, now maps to J. Continuing this process, the final message would be VIJSL.

Notice that frequency analysis is much less useful now, since the character E has been mapped to three different characters due to the shifting of the substitution mapping.

Try it Now 4

Decrypt the message KIQRV if it was encrypted using a basic Caesar cipher with shift 3 as the initial substitution, but shifting the substitution one place after each character.

Answer

The initial mapping was:

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**
- Now maps to: **DEFGHIJKLMNOPQRSTUVWXYZABC**

Using this, we can see the first character of the encrypted message, K, can be decrypted to the letter H. We now shift the mapping by one character.

- Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ**

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ
- Now maps to: EFGHIJKLMNOPQRSTUVWXYZABCD

The second character in the message, I, can be decrypted to the letter E. Continuing this process of shifting and decrypting, KIQRV decrypts to HELLO.



The actual Engima machines used in WWII were more complex. Each wheel consisted of a complex substitution cipher, and multiple wheels were used in a chain[1]. The specific wheels used, order of the wheels, and starting position of the wheels formed the encryption key. While captured Engima devices provided the Allied forces details on the encryption method, the keys still had to be broken to decrypt messages.

These code breaking efforts led to the development of some of the first electronic computers by Alan Turing at Bletchley Park in the United Kingdom. This is generally considered the beginnings of modern computing[2].

In the 1970s, the U.S. government had a competition and ultimately approved an algorithm deemed DES (Data Encryption Standard) to be used for encrypting government data. It became the standard encryption algorithm used. This method used a combination of multiple substitution and transposition steps, along with other steps in which the encryption key is mixed with the message. This method uses an encryption key with length 56 bits, meaning there are 2^{56} possible keys.

This number of keys make a brute force attack extremely difficult and costly, but not impossible. In 1998, a team was able to find the decryption key for a message in 2 days, using about $250,000 worth of hardware. However, the price and time will go down as computer power increases.

From 1997 to 2001 the government held another competition, ultimately adopting a new method, deemed AES (Advanced Encryption Standard). This method uses encryption keys with 128, 192, or 256 bits, providing up to 2^{256} possible keys, making brute force attacks essentially impossible.

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[1] http://en.wikipedia.org/wiki/File:En...abet_rings.jpg

[2] For a good overview, see http://www.youtube.com/watch?v=5nK_ft0Lf1s

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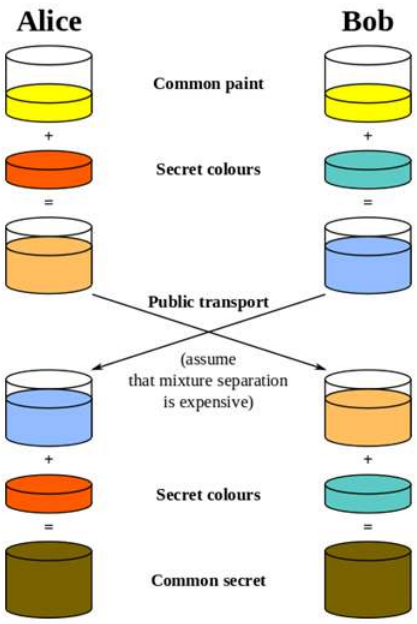
6.5: Public Key Cryptography

Suppose that you are connecting to your bank's website. It is possible that someone could intercept any communication between you and your bank, so you'll want to encrypt the communication. The problem is that all the encryption methods we've discussed require than both parties have already agreed on a shared secret encryption key. How can you and your bank agree on a key if you haven't already?

This becomes the goal of public key cryptography – to provide a way for two parties to agree on a key without a snooping third party being able to determine the key. The method relies on a one-way function; something that is easy to do one way, but hard to reverse. We will explore the Diffie-Hellman-Merkle key exchange method.

As an example, let's consider mixing paint. It's easy to mix paint to make a new color, but much harder to separate a mixed paint into the two original colors used.[1][2]

Using this analogy, Alice and Bob publically agree on a common starter color. Each then mixes in some of their own secret color. They then exchange their mixed colors.



Since separating colors is hard, even if a snooper were to obtain these mixed colors, it would be hard to obtain the original secret colors.

Once they have exchanged their mixed colors, Alice and Bob both add their secret color to the mix they obtained from the other person. In doing so, both Alice and Bob now have the same common secret color, since it contains a mix of the original common color, Alice's secret color, and Bob's secret color.

They now have a common secret color they can use as their encryption key, even though neither Alice nor Bob knows the other's secret color.

Likewise, there is no way for a snooper to obtain the common secret color without separating one of the mixed colors.

To get this process to work for computer communication, we need to have the process result in a share common number to act as the common secret encryption key. For this, we need a numerical one-way function.

Modular arithmetic

If you think back to doing division with whole numbers, you may remember finding the whole number result and the remainder after division.

Modulus [3]

The **modulus** is another name for the remainder after division.

For example, 17 mod 5 = 2, since if we divide 17 by 5, we get 3 with remainder 2.

Modular arithmetic is sometimes called clock arithmetic, since analog clocks wrap around times past 12, meaning they work on a modulus of 12. If the hour hand of a clock currently points to 8, then in 5 hours it will point to 1. While $8 + 5 = 13$, the clock wraps around after 12, so all times can be thought of as modulus 12. Mathematically, 13 mod 12 = 1.

Example 11

Compute: a) 10 mod 3 b) 15 mod 5 c) 2^7 mod 5

Solution

a) Since 10 divided by 3 is 3 with remainder 1,10 mod $3 = 1$

b) Since 15 divided by 5 is 3 with no remainder, 15 mod $5 = 0$

c) $2^7 = 128.128$ divide by 5 is 25 with remainder 3, so 2^7 mod $5 = 3$

Try it Now 5

Compute: a) 23 mod 7 b) 15 mod 7 c) 2034 mod 7

Answer

a) 2 b) 1 c) 4

Recall that when we divide 17 by 5, we could represent the result as 3 remainder 2, as the mixed number $3\frac{2}{5}$, or as the decimal 3.4. Notice that the modulus, 2, is the same as the numerator of the fractional part of the mixed number, and that the decimal part 0.4 is equivalent to the fraction . We can use these conversions to calculate the modulus of not-too-huge numbers on a standard calculator.

Modulus on a Standard Calculator

To calculate a mod n on a standard calculator

1. Divide a by n
2. Subtract the whole part of the resulting quantity
3. Multiply by n to obtain the modulus

Example 12

Calculate 31345 mod 419

Solution

$31345/419 = 74.8090692$	Now subtract 74 to get just the decimal remainder
$74.8090692 - 74 = 0.8090692$	Multiply this by 419 to get the modulus
$0.8090692 \times 419 = 339$	This tells us 0.8090692 was equivalent to $\frac{339}{419}$

In the text above, only a portion of the decimal value was written down. In practice, you should try to avoid writing down the intermediary steps, and instead allow your calculator to retain as many decimal values as it can.

The one-way function

When you use a prime number p as a modulus, you can find a special number called a generator, g, so that g^n mod p will result in all the values from 1 to $p-1$

n	3^n	3^nmod7
1	3	3
2	9	2
3	27	6
4	81	4
5	243	5
6	729	1

In the table to the top, notice that when we give values of n from 1 to 6, we get out all values from 1 to 6. This means 3 is a generator when 7 is the modulus.

This gives us our one-way function. While it is easy to compute the value of g^n mod p when we know n, it is difficult to find the exponent n to obtain a specific value.

For example, suppose we use $p = 23$ and $g = 5$. If I pick n to be 6, I can fairly easily calculate 5^6 mod $23 = 15625$ mod $23 = 8$

If someone else were to tell you $5^n \bmod 23 = 7$, it is much harder to find n. In this particular case, we'd have to try 22 different values for n until we found one that worked – there is no known easier way to find n other than brute-force guessing.

While trying 22 values would not take too long, when used in practice much larger values for p are used, typically with well over 500 digits. Trying all possibilities would be essentially impossible.

The key exchange

Before we can begin the key exchange process, we need a couple more important facts about modular arithmetic.

Modular Exponentialtion Rule

$(a^b \bmod n) = (a \bmod n)^b \bmod n$

Example 13

Compute 12^5 mod 7 using the exponentiation rule.

Solution

Evaluated directly: $12^5 = 248,832$, so 12^5 mod $7 = 248,823$ mod $7 = 3$

Using the rule above, 12^5 mod $7 = (12 \bmod 7)^5 \bmod 7 = 5^5 \bmod 7 = 3125 \bmod 7 = 3$

You may remember a basic exponent rule from algebra:

$$(a^b)^c = a^{bc} = a^{cb} = (a^c)^b$$

For example:

$$64^2 = (4^3)^2 = 4^6 = (4^2)^3 = 16^3$$

We can combine the modular exponentiation rule with the algebra exponent rule to define the modular exponent power rule.

Modular Exponent Power Rule

$$(a^b \bmod n)^c \bmod n = (a^{bc} \bmod n) = (a^c \bmod n)^b \bmod n$$

Example 14

Verify the rule above if $a = 3, b = 4, c = 5$, and $n = 7$

Solution

$(3^4 \bmod 7)^5 \bmod 7 = (81 \bmod 7)^5 \bmod 7 = 4^5 \bmod 7 = 1024 \bmod 7 = 2$

$(3^5 \bmod 7)^4 \bmod 7 = (243 \bmod 7)^4 \bmod 7 = 5^4 \bmod 7 = 625 \bmod 7 = 2$, the same result.

Try it Now 6

Use the modular exponent rule to calculate $10000 \bmod 7$, by noting $10000 = 10^4$

Answer

$10000 \bmod 7 = 10^4 \bmod 7 = (10 \bmod 7)^4 \bmod 7 = 3^4 \bmod 7 = 81 \bmod 7 = 4$

This provides us the basis for our key exchange. While it will be easier to understand in the following example, here's the process:

1. Alice and Bob agree publically on values a prime p and generator g.
2. Alice picks some secret number a, while Bob picks some secret number b.
3. Alice computes $A = g^a \bmod p$ and sends it to Bob.
4. Bob computes $B = g^b$ mod p and sends it to Alice.

5. Alice computes $B^a \bmod p$, which is $(g^b \bmod p)^a \bmod p$.
6. Bob computes $A^b \bmod p$, which is $(g^a \bmod p)^b \bmod p$.

The modular exponent power rule tells us $(g^a \bmod p)^b \bmod p = (g^b \bmod p)^a \bmod p$, so Alice and Bob will arrive at the same shared value to use as a key, even though neither knows the other's secret number, and no eavesdropper can determine this value knowing only g, p, A, and B

Example 15

Alice and Bob publically share a generator and prime modulus. In this case, we'll use 3 as the generator and 17 as the prime.

Solution

	Alice		**Bob**
Alice and Bob publically share a generator and prime modulus.	$g = 3, p = 17$	Common info	$g = 3, p = 17$
Each then secretly picks a number n of their own.	$n = 8$	secret number	$n = 6$
Each calculates $g^n \bmod p$	$3^8 \bmod 17 = 16$		$3^6 \bmod 17 = 15$
They then exchange these resulting values.	$A = 16$ $B = 15$		$B = 15$ $A = 16$
Each then raises the value they received to the power of their secret $n \bmod p$	$B^n \bmod p =$ $15^8 \bmod 17 = 1$	mix in secret number	$A^n \bmod p =$ $16^6 \bmod 17 = 1$
The result is the shared secret key.	1	shared secret key	1

The shared secrets come out the same because of the modular exponent power rule

$(a^b \bmod n)^c \bmod n = (a^c \bmod n)^b \bmod n$. Alice computed $(3^6 \bmod 17)^8 \bmod 17$ while Bob computed $(3^8 \bmod 17)^6 \bmod 17$, which the rule says will give the same results.

Notice that even if a snooper were to obtain both exchanged values $A = 16$ and $B = 15$, there is no way they could obtain the shared secret key from these without having at least one of Alice or Bob's secret numbers. There is no easy way to obtain the secret numbers from the shared values, since the function was a one-way function.

Using this approach, Alice and Bob can now use the shared secret key obtained as the key for a standard encryption algorithm like DES or AES.

Try it Now 7

Suppose you are doing a key exchange with Kylie using generator 5 and prime 23. Your secret number is 2. What number do you send to Kylie? If Kylie sends you the value 8, determine the shared secret key.

Answer

To compute the number we'd send to Kylie, we raise the generator to the power of our secret number modulus the prime: $5^2 \bmod 23 = 25 \bmod 23 = 2$.

If Kylie sends us the value 8, we determine the shared secret by raising her number to the power of our secret number modulus the prime: $8^2 \bmod 23 = 64 \bmod 23 = 18.18$ would be the shared secret.

RSA

There are several other public-key methods used, including RSA, which is very commonly used. RSA involves distributing a public encryption key, which anyone can use to encrypt messages to you, but which can only be decrypted using a separate private key. You can think of this as sending an open padlock to someone – they can lock up information, but no one can unlock it without the key you kept secret.

RSA's security relies on the difficulty of factoring large numbers. For example, it's easy to calculate that 53 times 59 is 3127, but given the number 12,317 that is a product of two primes, it's much harder to find the numbers that multiply to give that number. It's exponentially harder when the primes each have 100 or more digits. Suppose we find two primes p and q and multiply them to get $n = pq$. This number will be very hard to factor. If we also know p and q, there are shortcuts to find two numbers e and d so that $m^{ed} \bmod n = m \bmod n$ for all numbers m. Without knowing the factorization of n, finding these values is very hard.

To use RSA, we generate two primes p and q and multiply them to get $n = pq$. since we know the factorization, we can easily find e and d so $m^{ed} = m \bmod n$. Now, we lock away p
q, and d. We then send the values e and n out publically. To encrypt a message m, the sender computes $S = m^e \bmod n$. As we saw earlier, the modulus is a one-way function which makes the original message very hard to recover from S. However, we have our private key d we can use to decrypt the message. When we receive the secret message S, we compute $S^d \bmod n = (m^e)^d \bmod n = m^{ed} \bmod n = m \bmod n$, recovering the original message[4].

Example 16

Suppose that Alice has computed $n = 3127, e = 3$, and $d = 2011$. Show how Bob would encrypt the message 50 and how Alice would then decrypt it.

Solution

Bob would only know his message, $m = 50$ and Alice's public key: $n = 3127$ and $e = 3$. He would encrypt the message by computing $m^e \bmod n : 50^3 \bmod 3127 = 3047$

Alice can then decrypt this message using her private key d by computing $S^d \bmod n$:

$3047^{2011} \bmod 3127 = 50$

This method differs from Diffie-Hellman-Merkle because no exchange process is needed; Bob could send Alice an encrypted message using Bob's public key without having to communicate with Alice beforehand to determine a shared secret key. This is especially handy for applications like encrypting email, where both parties might not be online at the same time to perform a Diffie-Hellman-Merkle style key exchange.

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[1] en.Wikipedia.org/w/index.php?...nge.svg&page=1

[2] For a video overview of this process, see http://www.youtube.com/watch?v=YEBfamv-_do

[3] Sometime, instead of seeing 17 mod 5 = 2, you'll see 17 ≡ 2 (mod 5). The ≡ symbol means "congruent to" and means that 17 and 2 are equivalent, after you consider the modulus 5.

[4] Many details have been left out, including how *e* and *d* are determined, and why this all works. For a bit more detail, see http://www.youtube.com/watch?v=wXB-V_Keiu8, or http://doctrina.org/How-RSA-Works-With-Examples.html

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6.6: Exercises

Substitution ciphers

In the questions below, if it specifies an alphabetic cipher, then the original map used letters only: ABCDEFGHIJKLMNOPQRSTUVWXYZ. If it specifies an alphanumeric cipher, then the original map used letters and numbers: ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789

1. Encrypt the message “SEND SUPPLIES” using an alphabetic Caesar cipher with shift 7 (mapping A to H).

2. Encrypt the message “CANCEL CONTRACT” using an alphanumeric Caesar cipher with shift 16 (mapping A to Q).

3. Decrypt the message “2R1 ONO 5SN OXM O“ if it was encrypted using an alphanumeric Ceasar cipher with shift 10 (mapping A to K).

4. Decrypt the message “RJJY NSAJ SNHJ“ if it was encrypted using an alphabetic Ceasar cipher with shift 5 (mapping A to F).

For questions 5-8 use this substitution mapping:

Original: **ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789**
Maps to: **HLCO2BQF5WRTZN1G4D8IJ6SUVK3A0X9YME7P**

5. Use the substitution mapping to encrypt the message “DEAR DIARY”

6. Use the substitution mapping to encrypt the message “ATTACK AT SUNRISE”

7. Use the substitution mapping to decrypt the message “Z2DQ 2D1N”

8. Use the substitution mapping to decrypt the message “Z22 IHI3 YX3”

Transposition ciphers

9. Encrypt the message “Meet in the library at ten” using a tabular transposition cipher with rows of length 5 characters.

10. Encrypt the message “Fly surveillance over the northern county” using a tabular transposition cipher with rows of length 8 characters.

11. Decrypt the message “THE VHI NIE SAN SHT STI MQA DAN SDR S“ if it was encrypted using a tabular transposition cipher with rows of length 7 characters.

12. Decrypt the message “DOLR UTIR INON KVEY AZ“ if it was encrypted using a tabular transposition cipher with rows of length 6 characters.

13. Encrypt the message “Buy twenty million” using a tabular transposition cipher with the encryption keyword “RENT”.

14. Encrypt the message “Attack from the northeast” using a tabular transposition cipher with the encryption keyword “POWER”.

15. Decrypt the message “RYL OEN ONI TPM IEE YTE YDH WEA HRM S” if it was encrypted using a tabular transposition cipher with the encryption keyword “READING”.

16. Decrypt the message “UYH SRT ABV HLN SEE L” if it was encrypted using a tabular transposition cipher with the encryption keyword “MAIL”.

Shifting substitution ciphers

17. Encrypt the message “SEND SUPPLIES” using an alphabetic Caesar cipher that starts with shift 7 (mapping A to H), and shifts one additional space after each character is encoded.

18. Encrypt the message “CANCEL CONTRACT” using an alphabetic Caesar cipher that starts with shift 5 (mapping A to F), and shifts one additional space after each character is encoded.

Modular arithmetic

19. Compute

a. $15 \bmod 4$
b. $10 \bmod 5$
c. $257 \bmod 11$

20. Compute

a. $20 \bmod 4$
b. $14 \bmod 3$
c. $86 \bmod 13$

21. Determine if 4 is a generator modulus 11

22. Determine if 2 is a generator modulus 13

23. Use the modular exponent rule to calculate $157^{10} \bmod 5$

24. Use the modular exponent rule to calculate $133^{8} \bmod 6$

Diffie-Hellman-Merkle key exchange

25. Suppose you are doing a key exchange with Marc using generator 5 and prime 23. Your secret number is 7. Marc sends you the value 3. Determine the shared secret key.

26. Suppose you are doing a key exchange with Jen using generator 5 and prime 23. Your secret number is 4. Jen sends you the value 8. Determine the shared secret key.

RSA

27. Suppose that Alice has computed $n = 33, e = 7,$ and $d = 3$. Show how Bob would encrypt the message 5 and how Alice would then decrypt it.

28. Suppose that Alice has computed $n = 55, e = 7,$ and $d = 13$. Show how Bob would encrypt the message 8 and how Alice would then decrypt it.

Extensions

29. To further obscure a message, sometimes the usual alphabet characters are replaced with other symbols. Design a new set of symbols, and use it to encode a message. Exchange with a friend and see if they can decode your message.

30.To make an encryption harder to break, sometimes multiple substitution and transposition ciphers are used in sequence. For example, a method might specify that the first letter of the encryption keyword be used to determine the initial shift for a Caesar cipher (perhaps with a rotating cipher), and also be used for a transposition cipher. Design your own sequence of encryption steps and encrypt a message. Exchange with a friend and see if they can follow your process to decrypt the message.

31. When using large primes, computing values like $67^{24} \bmod 83$ can be difficult on a calculator without using additional tricks, since 67^{24} is a huge number. We will explore an approach used.

a. Notice that $67^2 \bmod 83$ is fairly easy to calculate: $67^2 \bmod 83 = 4489 \bmod 83 = 7$. Since $67^4 \bmod 83 = (67^2)^2 \bmod 83$ can be rewritten using the modular exponent rule as $(67^2 \bmod 83)^2 \bmod 83$, this is also easy to evaluate: $67^4 \bmod 83 = (67^2 \bmod 83)^2 \bmod 83 = 7^2 \bmod 83 = 49$ This process can be continued to find $67^8 \bmod 83$ as $(6^4)^2 \bmod 83$. Find this value, then find $67^{16} \bmod 83$ and $67^{32} \bmod 83$
b. There is a rule that $(ab) \bmod n = (a \bmod n)(b \bmod n) \bmod n$ Noting that $17000 = 170 \times 100$, calculate 17000 mod 83 using the rule above.
c. Note that $67^5 = 67^4 67$. Use this, along with the rule from above and the results from part a to compute $67^5 \bmod 83$
d. Note that $67^7 = 67^{4+2+1} = 67^4 67^2 67^1$. Compute $67^7 \bmod 83$.
e. Write 67^{24} as a product of powers of 67, and use this to compute $67^{24} \bmod 83$

32. Use the process from the previous question to evaluate $23^{34} \bmod 37$.

33. To encrypt text messages with RSA, the words are first converted into a string of numbers, and then encrypted. Several characters are usually combined together to produce a message number smaller than the modulus, but approximately the same size. Look up an ASCII table to convert the message “SCALE THE WALLS” to numbers, then encrypt it using the RSA public key $n = 10823, e = 5$. Since ASCII characters are two digits, pair up characters to form four-digit numbers before encoding. For example A is 65 and B is 66, so the character pair AB could be treated as the number 6566 and encrypted as 10148

34. Explore approaches to steganography that don’t require specialized software. Attempt to hide a message using one of these techniques, and see if a fellow student can detect the message.

35. When you visit a secure website, your web browser will report that the site’s identity has been verified by a third party, called a certificate authority. This is meant to assure you that you are visiting the actual company’s website. Research how these certificates work.



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6.6.1: Answers to Selected Exercises

1. ZLU KZB WWS PLZ 3. SHRED EVIDENCE

5. O2H DO5 HDV 7. MERGER ON

9. MNI YNE TBA AEH RTA TEA TAI LRE A

11. THE STASH IS HIDDEN AT MARVINS QNS

13. UEM IYN IOB WYL TTL N

15. HIRE THIRTY NEW EMPLOYEES MONDAY

17. ZMW NDG CDA YVK

19. a) 3 b) 0 c) 4

21. We test out all n from 1 to 10

n	4^n	$4^n \bmod 11$
1	4	4
2	16	5
3	64	9
4	256	3
5	1024	1
6	4096	4
7	16384	5
8	65536	9
9	262144	3
10	1048576	1

Since we have repeats, and not all values from 1 to 10 are produced (for example, there is no n is $4^n \bmod 11 = 7$), 4 is *not* a generator $\bmod 11$.

23. $157^{10} \bmod 5 = (157 \bmod 5)^{10} \bmod 5 = 2^{10} \bmod 5 = 1024 \bmod 5 = 4$

25. $3^7 \bmod 23 = 2$

27. Bob would send $5^7 \bmod 33 = 14$. Alice would decrypt it as $14^3 \bmod 33 = 5$

31.

a. $67^8 \bmod 83 = \left(67^4 \bmod 83\right)^2 \bmod 83 = 49^2 \bmod 83 = 2401 \bmod 83 = 77$

$67^{16} \bmod 83 = \left(67^8 \bmod 83\right)^2 \bmod 83 = 77^2 \bmod 83 = 5929 \bmod 83 = 36$

b. $17000 \bmod 83 = (100 \bmod 83)*(170 \bmod 83) \bmod 83 = (17)(4) \bmod 83 = 68$

c. $67^5 \bmod 83 = \left(67^4 \bmod 83\right)(67 \bmod 83) \bmod 83 = (49)(67) \bmod 83 = 3283 \bmod 83 = 46$

d. $67^7 \bmod 83 = \left(67^4 \bmod 83\right)\left(67^2 \bmod 83\right)(67 \bmod 83) \bmod 83 = (49)(7)(67) \bmod 83 = 22981 \bmod 83 = 73$

e. $67^{24} = 67^{16}67^8$ so $67^{24} \bmod 83 = \left(67^{16} \bmod 83\right)\left(67^8 \bmod 83\right) \bmod 83 = (77)(36) \bmod 83 = 2272 \bmod 83 = 33$

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CHAPTER OVERVIEW

7: Voting and Apportionment

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7.6.1: Exercises

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7.1: Introduction

Suppose a friend asked you, “When did you last vote?” What would your answer be? Maybe you would tell your friend that the last time you voted was during the last presidential election, or perhaps you would tell your friend that you prefer not to vote. When thinking about voting, presidential campaigns or advertisements for reelections may come to mind, but you can cast your vote in many ways. Have you liked a post, followed a creator, friended a stranger, or clicked a heart online today? In the digital age, it's possible to vote several times a day. Voting systems are not only the machines that drive every democracy on Earth, but they are also the engines driving social media and many other aspects of life. A deeper understanding of these voting systems will enhance your ability to successfully engage with the world in which we live.



Figure **7.1.1** Voters cast their ballots in one of the world’s many democracies. (credit: “Governor Votes Early” by Maryland GovPics/Flickr, CC BY 2.0)

In this chapter, you will become one of the founders of the new democratic country of Imaginaria. You have a great responsibility to the people of this fledgling democracy. You have been tasked with writing the portion of the constitution that lays out voting procedures. In preparation for this important task, you will explore the various types of voting systems, from school board elections to Twitter wars. You will see how these types are alike, how they differ, and how they might be applied in Imaginaria. Most importantly, you will learn about the mathematically inherent advantages and disadvantages of various voting systems so that you can make informed choices to better the lives of the Imaginarians.

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7.2: Voting Methods



Figure 7.2.1: President Barack Obama votes in the 2012 election. (credit: Pete Souza/White House, Public Domain)

Learning Objectives

1. Apply plurality voting to determine a winner.
2. Apply runoff voting to determine a winner.
3. Apply ranked-choice voting to determine a winner.
4. Apply Borda count voting to determine a winner.
5. Apply pairwise comparison and Condorcet voting to determine a winner.
6. Apply approval voting to determine a winner.
7. Compare and contrast voting methods to identify flaws.

Today is the day that you begin your quest to collaborate on the constitution of Imaginaria! Let's begin by thinking about the selection of a leader who can serve as president. It seems straightforward; if the majority of citizens prefer a particular candidate, that candidate should win. But not all votes are decided by a simple majority. Why not? What are the options?

Majority versus Plurality Voting

When an election involves only two options, a simple majority is a reasonable way to determine a winner. A majority is a number equaling more than half, or greater than 50 percent of the total.

Let's take a look at the outcomes of U.S. presidential elections to understand more. Table 11.1 displays the results of the 2000 U.S. presidential election. Like most presidential elections, this election involved more than two options. If that is the case, is it possible that no single candidate will receive more than half of the votes cast?

Table 11.1 Results of the Popular Vote for the 2000 U.S. Presidential Election (source: https://www.fec.gov/introduction-cam...president2000/)

Candidate (Party Label)	Popular Vote Total
Al Gore (Democrat)	50,999,897
George W. Bush (Republican)	50,456,002
Ralph Nader (Green)	2,882,955
Patrick J. Buchanan (Reform/Independent)	448,895
Harry Browne (Libertarian)	384,431
Howard Phillips (Constitution)	98,020
Other	134,900
Total:	**105,405,100**

Example 7.2.1: Majority of Popular Vote in the 2000 U.S. Presidential Election

Refer back to Table 11.1. Did any single candidate secure the majority of popular votes?

Answer

Step 1: Calculate 50 percent of 105,405,100 by multiplying the decimal form of 50 percent, which is 0.50, by 158,394,605: $0.50\,(105{,}405{,}100) = 52{,}702{,}550$

Step 2: Determine the minimum number of votes needed to have a majority. The minimum number of votes required is the lowest counting number that is larger than 50 percent of the votes. To have a majority, an individual candidate must have more than 52,702,550; so, a majority candidate must have 52,702,551 votes or more.

Step 3: Compare the number of votes each candidate received to 52,702,551. According to the data in Table 11.1, none of the candidates secured a majority.

Your Turn 7.2.1

According to the Cook Political Report, a total of 158,394,605 votes were cast in the 2020 U.S. presidential election. Of those, 81,281,502 were cast for Joe Biden, 74,222,593 for Donald Trump, and 2,890,510 for other candidates. Although U.S. presidential elections are not determined by popular vote, did either Joe Biden or Donald Trump secure a majority of the votes?

Unlike in the 2000 U.S. presidential election, a candidate won the majority of votes in the 2020 election (see Table 11.1). It is a common occurrence for no single candidate to receive a majority of the votes in an election with more than two candidates. When this occurs, the candidate with the largest portion of the votes is said to have a **plurality**.

Example 7.2.2: Plurality of Popular Vote in the 2000 U.S. Presidential Election

Refer again to Table 11.1. In the 2000 U.S. presidential election, which candidate had a plurality of popular votes?

Answer

Al Gore secured 50,999,897 votes which was more than any other single candidate. Therefore, he had a plurality of the popular votes.

Your plans for Imaginarian elections will likely include primary elections, or preliminary elections to select candidates for a principal or general election. Table 11.2 displays the results of the 2018 U.S. Senate Republican primary for Maryland.

Table 11.2 (source: ballotpedia.org/United_State...lican_primary))

Top Four Republican Candidates	Votes	Percentage of Party Votes
Cambell, Tony	51,426	29.22%
Chaffee, Chris	42,328	24.05%
Grigorian, Christina J.	30,756	17.48%
Graziani, John R.	15,435	8.77%
Total Votes	175,981	100%

Your Turn 7.2.2

Consider the results of the 2018 U.S. Senate primary for Maryland. Determine which candidate won the primary for the Republicans (R). Did the candidate win a majority or a plurality of Republican votes?

Consider how election by plurality, not majority, is the most common method of selecting candidates for public office.

Who Knew?

The U.S. Electorial College: Winning the Presidency without a Plurality

In the 2000 U.S. presidential election, Al Gore had a plurality of the popular votes, but he did not win the election. Why? This occurred because the U.S. president and vice president are elected by electors rather than a direct vote by the citizens. The electors are part of the Electoral College, a body of people representing the states. Why was the Electoral College created? The Electoral College was created as a compromise between those authors of the U.S. Constitution who believed Congress should elect the president, and those who believed the citizens should vote directly. The popular vote was not recorded until the presidential election of 1824. Since then, only five presidents have been elected without winning a plurality of the popular vote: John Quincy Adams in 1824, Rutherford B. Hayes in 1876, Grover Cleveland in 1888, George Bush in 2000, and Donald Trump in 2016.

Runoff Voting

Has your family ever debated what to have for dinner? Suppose your family is deciding on a restaurant and exactly half of you want to have pizza but the other half want hamburgers. How do you decide when the result is a tie? You need a tiebreaker!

Will the new democracy of Imaginaria need tiebreakers? When no candidate satisfies the requirements to win the election, a runoff election, or second election, is held to determine a winner.

How would runoff voting work in Imaginaria? There are many types of **runoff voting systems**, which are voting systems that utilize a runoff election when the first round does not result in a winner. The method for implementing a runoff election can vary widely, particularly in the criteria used to determine whether a candidate will be on the ballot in the second election. For example, a two-round system is a runoff voting system in which only the top candidates advance to the runoff election. In some two-round systems, only the top two candidates are on the second ballot, or it may be any candidate who secures a certain percentage of the vote will advance. The Hare Method is another runoff voting system in which only the candidate(s) with the very least votes are eliminated. This can potentially result in several rounds of runoff elections.

Example 7.2.3: Runoff Election for Condominium Association President

A condominium association elects a new president every two years by a two-round system of voting. If none of the candidates receive a majority, the association charter states that the top two candidates will be eligible to participate in a runoff election. In a particular year, five residents were nominated. The results of the first round are given in the table below.

Candidate	Votes in First Round
Abou	18
Baiocchi	10
Campana	5
Dali	11
Eugene	4

1. Is there a winner based on the first round? Why or why not?
2. If there is a winner, who won? If there should be a runoff, who will advance to the second round?

Answer

1. A majority of 48 total votes is required to win. Begin by finding 50 percent of 48, which is calculated as follows: $0.50(48) = 24$. A majority is 25 or more. No candidate has a majority, so there is no winner based on the first round.
2. Abou and Dali advance to the second round.

Your Turn 7.2.3

The student government bylaws of a particular college require that a new president is elected annually by plurality voting. In the event of a tie, the bylaws require the candidate(s) with the fewest votes to be removed from the ballot and a runoff election to be held with the remaining candidates. This process is repeated until a single candidate receives a plurality and wins the election. The results of two voting rounds are given in the table below.

Candidate	Votes in First Round	Votes in Second Round
Ferguson	158	168
Garcia	103	104
Hearn	157	180
Isaac	123	123
Jackman	58	Eliminated
Kelly	72	74
Lim	158	180

Which two candidates tied in the second round? Does this mean there will be a third election?

If there was a winner, who won? If there should be a runoff, who must be removed from the ballot?

Steps to Determine Winner by Plurality or Majority Election with Runoff

To determine the winner by plurality or when a majority election with runoff occurs, we take these three steps:

Step 1: If a majority is required to win the election, determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes. If a majority is not required, move to Step 2.

Step 2: Count the number of votes for each candidate in the current round of voting. If a single candidate has enough votes to win a plurality, or a majority as appropriate, then you are done! Otherwise, eliminate a predetermined number of candidates based on the rules of the election. Elimination conditions may vary. For example, the rules may state that the candidate(s) with the fewest votes will be eliminated (as in the Hare method), or that only the candidates meeting a certain threshold will move on (as in a two-round system). Once the appropriate candidates are eliminated, move on to Step 3.

Step 3: Hold a runoff election. If the runoff is simulated using a list of voter's preferences, renumber the preferences to reflect the remaining number of options in such a way that the original order of preference is retained. Then repeat Step 2.

Note: The second and third steps may be repeated as many times as necessary for voting procedures that allow multiple runoffs.

Example 7.2.4: Family Dinner Night

The five members of the Chionilis family—Annette, Rene, Seema, Titus, and Galen—have decided to get takeout for dinner. They are trying to decide on a restaurant. The options are Rainbow China, Dough Boys Pizza, Taco City, or Caribbean Flavor. They will use majority election with runoffs where the restaurant with the fewest votes is eliminated in each round. The preferences of each family member are listed by first initial in the table below. An entry of 1 represents the person's first choice; 2, their second; and so on. For example, Annette's second choice is Dough Boys Pizza.

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Taco City	3	4	2	4	2
Caribbean Flavor	4	1	4	3	4

Use the information in the table to answer the following questions.

1. Which common type of runoff voting method is this?
2. List the results of each round of voting based on this information and determine which restaurant was ultimately chosen.

Answer

1. the Hare Method
2. **Step 1:** Determine the number of votes necessary to have a majority. There are five family members, so 50 percent of 5 is $0.50(5) = 2.5$. A majority is three or more votes.
   Step 2: Count the number of votes for each restaurant in the first election. In a list of voter preferences, the 1s represent the top choice of each voter, which corresponds to their vote in the first round.
   Results of Round 1:
   - Rainbow China — 2 votes
   - Dough Boys — 2 votes
   - Taco City — 0 votes
   - Caribbean Flavor — 1 vote

   No restaurant received a majority. Eliminate Taco City, which has the fewest first place votes:

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Caribbean Flavor	4	1	4	3	4

   Step 3: Hold a runoff election. In other words, hold a second round. Since we have a list of the voters' preferences with the eliminated option removed, we will renumber the preferences as first, second, and third so that we keep the original order of preference. The result is that we will count the second-place vote of any voter whose first choice was eliminated.

Options	A	R	S	T	G
Rainbow China	1	3	2	1	2
Dough Boys Pizza	2	2	1	2	1
Caribbean Flavor	3	1	3	3	3

Step 4: Repeat the process from Step 2. Count the number votes for each restaurant in the first-round election. Since we are using a list of preferences, we need to count the number of 1s received by each restaurant.
Results of Round 2:

- Rainbow China — 2 votes
- Dough Boys — 2 votes
- Caribbean Flavor — 1 vote

No single restaurant has three votes. Eliminate Caribbean Flavor, which has the fewest first place votes.

OPTIONS	A	R	S	T	G
Rainbow China	1	3	2	1	2
Dough Boys Pizza	2	2	1	2	1

Step 5: Repeat the process from Step 3. Hold another runoff election. This will be Round 3. Renumber the voters' preferences as first and second this time.

Options	A	R	S	T	G
Rainbow China	1	2	2	1	2
Dough Boys Pizza	2	1	1	2	1

Step 6: Repeat the process from Step 2 one last time. Count the number of first place votes for each remaining restaurant.

Results of Round 3:

- Rainbow China — 2 votes
- Dough Boys — 3 votes

Determine whether any one choice has a majority. Yes! Dough Boys has three votes, so it is the winner!

Your Turn 7.2.4

There are six members on the board of a Parent Teacher Association (PTA) at a local elementary school: the president (P), the vice president (V), the recording secretary (R), the liaison to the administration (L), the treasurer (T), and the chief fundraiser (C). The board must decide which equipment to purchase for the classrooms with moneys from their annual fundraisers. The preferences of the board members are shown in the table below.

	P	V	R	L	T	C
Option A	3	4	1	3	4	1
Option B	1	3	2	1	3	2
Option C	2	2	3	2	1	4
Option D	4	1	4	4	2	3

The board uses plurality method and a runoff in the event of a tie, such that the option(s) with the least votes will be eliminated in each round. Which option will be chosen?

Ranked-Choice Voting

In Example **7.2.4** and Your Turn **7.2.4**, you were given a list that ordered each voter's preferences. This ordering is called a preference ranking. A ballot in which a voter is required to give an ordering of their preferences is a ranked ballot, and any voting system in which a voter uses a ranked ballot is referred to as ranked voting.

The vote for the Academy Awards uses a ranked ballot. The table below provides an example of a ranked ballot for the 2020 Academy Award nominees for Best Director.

Candidate for Best Director	Rank top choice as 1, next choice as 2, and so on.				
Martin Scorsese, *The Irishman*	1	2	3	4	5
Todd Phillips, *Joker*	1	2	3	4	5
Sam Mendes, *1917*	1	2	3	4	5
Quentin Tarantino, *Once Upon a Time in Hollywood*	1	2	3	4	5
Bong Joon-ho, *Parasite*	1	2	3	4	5

As you decide on the voting methods that will be used in your new democracy, budget must be a consideration. You might consider a particular type of ranked voting called ranked-choice voting (RCV), which simulates a series of runoff elections without the usual time and expense involved when voters must repeatedly return to the polls, like we did in Example **7.2.4**.

The method of ranked-choice voting (RCV), also called instant runoff voting (IRV), is a version of the Hare Method, using *preference ranking* so that, if no single candidate receives a majority, the least popular selections can be eliminated and the results can be recounted, without the need for more elections.

Checkpoint

Ranked voting can be confused with ranked-choice voting, but ranked voting is a more general category which includes ranked-choice voting and several other voting methods.

As we explore examples of ranked voting, we will summarize the voters' preference rankings using a table in which the top row shows the number of ballots that ranked the options in the same order. Let's practice interpreting the information in this type of table.

Example 7.2.5: Interpreting the Sample Preference Summary

Refer to the table below containing voters' preference rankings to answer the following questions.

Number of Ballots	100	200	150	75
Option A	1	4	3	4
Option B	2	3	4	2
Option C	4	2	1	1
Option D	3	1	2	3

1. How many voters ranked the options in the following order: Option A in fourth place, Option B in second place, Option C in first place, and Option D in third place?
2. How many ballots in total were collected?
3. How many voters indicated that Option C was their first choice?

Answer

1. The column farthest to the right displays this ordering. The top entry in this column is 75; so, there were 75 voters who ranked the options in this way.
2. The sum of the top row gives the total number of ballots collected: $100 + 200 + 150 + 75 = 525$. So, there were 525 ballots collected.
3. In the Option C row, there are two entries of 1 which indicate a first choice for that option. These occur in the last two columns. The sum of the top entries in these columns is $150 + 75 = 225$. So, 225 voters indicated Option C as their first choice.

Your Turn 7.2.5

A kindergarten class votes on their favorite colors using a ranked ballot. Use the results in the following table to answer the questions.

Number of Ballots	**4**	**6**	**4**	**7**
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

How many students voted in total?

How many students voted for yellow as their favorite color?

How many students in the class indicated that blue was their favorite color while green was their least favorite color?

Now that we've covered how to read a summary of preference rankings, let's practice using the ranked-choice method to determine the winner of an election. Recall that ranked-choice voting is still the Hare Method where the candidate with the very least number of votes is eliminated each round until a majority is attained. The difference here is that the voters have completed a ranked ballot, so they don't have to visit the polls multiple times. Here are the steps for ranked-choice voting.

Steps to Determine Winner by Ranked-Choice Voting

To determine the winner when ranked-choice voting occurs, we take these three steps:

Step 1: Determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes.

Step 2: Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election and we are done! Otherwise, eliminate the candidate(s) with the fewest votes and complete Step 3.

Step 3: Reallocate the votes to the remaining candidates, and repeat Step 2.

Checkpoint

Be methodical to avoid arithmetic errors. Make sure that each time you count the number of first place votes they sum to the number of ballots.

Video



How Does Ranked-Choice Voting Work?

Example 7.2.6: Most Popular Color in Kindergarten

Let's review the kindergarten class color preferences again, and this time determine which color would be selected based on these results using the ranked-choice method.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Answer

Step 1: Determine whether any candidate received a majority. There were 21 ballots. Fifty percent of 21 is $0.50(21) = 10.5$. A majority is 11.

Step 2: Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Red: 0
- Blue: $4+4=8$
- Green: 0
- Yellow: 0
- Purple: 7
- Pink: 6

Notice that $8+7+6=21$, which is the total number of ballots. Confirming this helps to catch any arithmetic or counting errors. No candidate has a majority with 11 or more votes. We must eliminate red, green, and yellow which had the fewest votes with 0 each. The remaining votes that must be counted for Round 2 are given in the table below.

Number of Ballots	4	6	4	7
Blue	1	2	1	4
Purple	4	3	4	1
Pink	3	1	3	6

Step 3: Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1, 2, and 3 in such a way that the order of preference is retained as seen in the table below:

Number of Ballots	4	6	4	7
Blue	1	2	1	2
Purple	3	3	3	1
Pink	2	1	2	3

Step 4: Repeat the process from Step 2. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4+4=8$
- Purple: 7
- Pink: 6

Confirm that $8+7+6=21$. Great! No candidate has 11 or more votes. We must eliminate pink which had the fewest votes with 6. The remaining votes that must be counted for Round 2 are shown in the table below.

Number of Ballots	4	6	4	7
Blue	1	2	1	2
Purple	3	3	3	1

Step 5: Repeat the process from Step 3. Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1 and 2;

Number of Ballots	4	6	4	7
Blue	1	1	1	2
Purple	2	2	2	1

Step 6: Repeat the process from Step 2 one last time. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4 + 6 + 4 = 14$
- Purple: 7

Blue has a majority and wins the election!

Video

Determine Winner of Election by Ranked-Choice Method (aka Instant Runoff)

Your Turn 7.2.6

Suppose that 58 *Star Wars* fans were asked to vote for their favorite *Star Wars* character. They were given a ranked ballot, and the results are shown in the following table. Use ranked-choice voting to determine the winner.

Number of Ballots	7	6	10	8	4	5	6	7	2	3
Han Solo	3	5	1	3	2	4	2	2	1	4
Princess Leia	2	1	3	6	4	3	3	1	3	2
Luke Skywalker	6	6	6	1	5	5	5	4	6	1
Chewbacca	4	4	5	4	6	1	6	5	4	5
Yoda	1	2	4	2	3	2	1	3	2	3
R2-D2	5	3	2	5	1	6	4	6	5	6

Borda Count Voting

Ranked-choice voting is one type of ranked voting that simulates multiple runoffs based on ranked ballots. Another type of ranked voting is the **Borda count method**, which uses ranked ballots that award candidates points corresponding to the number of candidates ranked lower on each ballot.

To understand how this works, let's review the favorite colors of our kindergarten class from the table below. Let's focus on the votes represented by the first column of the preference summary.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Each student had six options. This first column tells us that four students ranked blue as their first choice, red as their second choice, pink as their third choice, purple as their fourth choice, yellow as their fifth choice, and green as their sixth choice. Blue was ranked higher than $6-1=5$ other colors. For each of the four students who completed their ballot in this way, blue would receive five points. Since there were four ballots with this ordering, blue would receive $5(4)=20$ points from the first column. To determine the total points for each candidate, we have to find the sum of the points they received in each column.

To determine the winner of a contest using the Borda count method, we must compare total number of points earned by each candidate. The candidate with the most points is the winner. Each row of the preference summary corresponds to a single candidate. To find the number of points received by a particular candidate in the preference summary, or their **Borda score**, we will need to focus on the row in which that candidate appears.

Before we practice determining the winner of a Borda count election, let's examine how to find the Borda score for a single candidate.

Example 7.2.7: Most Popular Color in Kindergarten Revisited

Let's review the ballots from the kindergarten class again, as shown in the table below. This time, let's determine the Borda score received by the color purple.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Answer

Step 1: Find the number of points received by the candidate in each column.

Column 1: $4\times(6-4)=4\times2=8$

Column 2: $6\times(6-3)=6\times3=18$

Column 3: $4\times(6-4)=4\times2=8$

Column 4: $7\times(6-1)=7\times5=35$

Step 2: Find the sum of the points received in each column. This is the total number of points received by this candidate: $8+18+8+35=69$

This process can also be combined into one step as shown here.

$$\begin{aligned}&4\times(6-4)+6\times(6-3)+4\times(6-4)+7\times(6-1)\\&=4\times2+6\times3+4\times2+7\times5\\&=8+18+8+35\\&=69\end{aligned}$$

Purple received 69 points in this election.

Checkpoint

When calculating a Borda score in one step, be careful to use the correct order of operations. Perform the subtraction inside each pair of parentheses first, then perform each multiplication, and then perform each addition.

Your Turn 7.2.7

Consider the color preferences of the kindergarten class once more. Using the Borda count method, determine the total number of points the color blue received.

Now let's determine the winner of an election by comparing the Borda scores for each of the candidates.

Example 7.2.8: Determine the Winner by Two Ranked Voting Methods

Use the table below, which displays a sample preference summary, to answer the questions that follow.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Use the ranked-choice voting method to determine the winner of the election.
2. Use the Borda count method to determine the winner of the election.

Answer

1. **Step 1:** Determine the number of votes needed to achieve a majority. The number of ballots is $95+90+110+115=410$. Fifty percent of 410 is $0.50(410)=205$. So, 206 votes or more is a majority.

   Step 2: Count the number of first place votes for each candidate.

   - Option A has $110+115=225$
   - Option B has 0
   - Option C has 90
   - Option D has 95

   Since Option A has a majority, Option A is the winner by the ranked-choice method.

2. The Borda scores would be:
   - Option A: $95(4-4)+90(4-4)+110(4-1)+115(4-1)=675$
   - Option B: $95(4-2)+90(4-2)+110(4-2)+115(4-2)=820$
   - Option C: $95(4-3)+90(4-1)+110(4-3)+115(4-4)=475$
   - Option D: $95(4-1)+90(4-3)+110(4-4)+115(4-3)=490$

   Since Option B has a Borda score of 820 points, Option B is the winner by the Borda count method.

Your Turn 7.2.8

Answer the following questions using the table below, which summarizes Imaginarian voter preferences.

Number of Ballots	**12**	19	**27**	**29**	**31**	**21**
Candidate A	1	1	2	2	3	3
Candidate B	2	3	1	3	1	2
Candidate C	3	2	3	1	2	1

Use the ranked-choice voting method to determine the winner of the election.

Use the Borda count method to determine the winner of the election.

Compare the results of the two methods. Did the same candidate win? What observations can you make about the results?

The Borda count method may seem too complicated to even consider using for Imaginaria, but each voting method has its own pros and cons. The Borda count method, for example, favors compromise candidates over divisive candidates. A **compromise candidate** is not the first choice of most of the voters, but is more acceptable to the population as a whole than the other candidates. A **divisive candidate** is simultaneously the first choice of a large portion of the voters and the last choice of another large portion of the voters.

In Example 11.8, Candidate A was ranked first by 225 voters, but was ranked last by 185 voters. No voters ranked Candidate A as second or third. It appears that, although Candidate A had the majority of first place votes, there was a significant minority who strongly disliked them. Candidate A was a divisive candidate. Candidate B, on the other hand, was the second choice of every voter, making Candidate B a good compromise. The Borda count method chose Candidate B, a compromise candidate, that was more acceptable to the population as a whole. This scenario is cited by both opponents and proponents of the Borda count method.

Video



Determine Winner of Election by Borda Count Method

Pairwise Comparison and Condorcet Voting

We have discussed two kinds of ranked voting methods so far: ranked-choice and Borda count. A third type of ranked voting is the **pairwise comparison method**, in which the candidates receive a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie. If one candidate earns more points than the others, then that candidate wins. This method is one of several **Condorcet voting methods**, which are methods in which candidates are ranked and then compared pairwise to each other, a candidate having to beat all others in order to win. These methods vary in the way candidates are scored, and there is not always a clear winner. A candidate who wins each possible pairing is known as a **Condorcet candidate**. These terms are named after the Marquis de Condorcet, a French philosopher and mathematician who preferred the pairwise comparison method to the Hare method and made public arguments in its favor.

People in Mathematics: Marquis de Condorcet

Condorcet voting methods are named for the Marquis de Condorcet, a French philosopher and mathematician known for, among other accomplishments, writing "*Sur l'admission des femmes au droit de Cité*" ("On the Admission of Women to the Rights of Citizenship"), in 1789, the first published essay on the political rights of women.

For more details visit this Web site.

If you include a Condorcet voting method in the constitution of Imaginaria, the election supervisors may want to use a pairwise comparison matrix like the one in Figure 11.3. It's a tool used to list the number of wins associated with each pairing of two candidates. Each candidate will receive a point for each win and a half a point for each tie. Each pairing is listed twice, once for the number of wins of a candidate over a particular challenger and once for the number of wins of the challenger over that candidate.

Opponent / Runner	A	B	C
A wins	--	A over B	A over C
B wins	B over A	--	B over C
C wins	C over A	C over B	--

Figure 7.2.2: Pairwise Comparison Matrix for Three Candidates

Steps to Determine a Winner by Pairwise Comparison Method Using a Matrix

To determine the winner when the pairwise comparison method is used, we take these three steps:

Step 1: On the matrix, indicate a losing matchup by crossing out a box, ☒, and tie match ups by drawing a slash through the box, \.

Step 2: Award each candidate 1 point for a win, half a point for a tie, and 0 points for a loss.

Step 3: Identify the winner, which is the candidate with the most points.

Video

Determine Winner of Election by Using the Pairwise Comparison Method

Before you decide on the pairwise comparison method for Imaginaria, review what's involved in constructing a pairwise comparison matrix from a summary of ranked ballots. Then we can use the matrix to determine the winner of the election. Does the winner using the Borda method still win?

Example 7.2.9: Construct and Use a Pairwise Comparison Matrix

Consider the summary of ranked ballots shown in the table below. Determine the winner of an election using the pairwise comparison method.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Construct a pairwise comparison matrix for the sample summary of ranked ballots in the table above.
2. Use the pairwise comparison method to determine a winner.

3. Recall that in Example 11.8, Candidate A won by the ranked-ballot method, and Candidate B won by the Hare method. Did the same candidate win using the pairwise comparison method?
4. Is the winner a Condorcet candidate?

Answer

There are four candidates on the ballots. We will need a row and a column for each candidate in addition to the headings, so we will draw a five by five matrix.

Opponent / Runner	Option A	Option B	Option C	Option D
Option A wins	--	A over B	A over C	A over D
Option B wins	B over A	--	B over C	B over D
Option C wins	C over A	C over B	--	C over D
Option D wins	D over A	D over B	D over C	--

Figure ✔ **7.2.3** Pairwise Comparison Matrix for Four Candidates

1. **Step 1:** Refer to Figure 11.4 to determine the values that belong in each cell.
   - A over B: A is preferred to B in columns 3 and 4. So, A scores $110 + 115 = 225$ points.
   - A over C: A is preferred to C in columns 3 and 4. So, A scores 225 points again.
   - A over D: Similarly, A scores 225 points.
   - B over A: B is preferred to A in columns 1 and 2. So, B scores $95 + 90 = 185$ points.

   Step 2: Continuing in this way, we complete the pairwise comparison matrix, as shown in Figure 11.5.

Runner \ Opponent	Option A	Option B	Option C	Option D
Option A wins	--	AB 225	AC 225	AD 225
Option B wins	BA 185	--	BC 320	BD 315
Option C wins	CA 185	CB 90	--	CD 200
Option D wins	DA 185	DB 95	DC 210	--

Figure ✔ 7.2.4: Pairwise Comparison Matrix for Four Candidates with Vote Counts

2. **Step 1:** Losing pairings are crossed off with an $\boxed{\times}$. In the event of a tie, we will draw a slash, $\boxed{\backslash}$.

   Step 2: Determine the number of points for each candidate by analyzing their row of wins. Each win is 1 point, each loss, $\boxed{\times}$, is 0 points, and each tie, $\boxed{\backslash}$, is half a point. Construct an additional column for each candidate's points.

Runner \ Opponent	Option A	Option B	Option C	Option D	Points
Option A wins	--	AB 225	AC 225	AD 225	3
Option B wins	~~BA 185~~	--	BC 320	BD 315	2
Option C wins	~~CA 185~~	~~CB 90~~	--	~~CD 200~~	0
Option D wins	~~DA 185~~	~~DB 95~~	DC 210	--	1

Figure ✔ **7.2.5**: Pairwise Comparison Matrix for Four Candidates with Pairwise Winners and Points Column Added

Step 3: The winner by the pairwise comparison method is Option A with 3 points.

3. Option A was not the winner by the Hare method.
4. The winner, Option A, is a Condorcet candidate because Option A won each pairwise comparison.

Checkpoint

Notice that the pairwise vote totals are not used to determine the points. Vote totals are only used to determine a win or a loss. Avoid the common error of adding the values in each row to get the points.

Your Turn 7.2.9

According to *Variety* magazine, there were 8,469 eligible Oscar voters in 2020. To make our matrix easier to work with, we've rounded this number up to 8,700. We'd never do this in a real voting situation. Suppose that the voter preferences for the ballot for Best Director had been as shown in the table below.

Number of Ballots	2,400	2,100	1,900	1,200	1,100
Martin Scorsese, *The Irishman*	3	1	4	5	2
Todd Phillips, *Joker*	1	4	5	2	4
Sam Mendes, *1917*	5	5	3	4	5
Quentin Tarantino, *Once Upon a Time in Hollywood*	4	2	1	3	3
Bong Joon-ho, *Parasite*	2	3	2	1	1

Refer to the table to answer each question.

Construct a pairwise comparison matrix for the Best Director ballots.

Who is the “Best Director” according to the pairwise comparison method?

Is the winner a Condorcet candidate?

Three Key Questions

Before you decide if you want to use the pairwise comparison method for Imaginarian elections, let’s consider three questions that might affect your decision.

I. Is there always a winner?
II. If there is a winner, is the winner always a Condorcet candidate?
III. If there is a Condorcet candidate, does that candidate always win?

Let’s think about why these questions might be important to you if you chose the pairwise comparison method. First, if no candidate meets the criteria to win an election, you will need a backup plan such as a runoff election. Second, if the winner is not a Condorcet candidate, then there is at least one candidate who beat the winner in a pairwise matchup and the supporters of that candidate might question the validity of the election. Finally, if there is a Condorcet candidate who beat every other candidate in a pairwise matchup, it is reasonable to conclude that it would be unfair for anyone else to win. The rest of the examples in this section should illustrate these key concepts.

Example 7.2.10: Rock, Paper, Scissors by Pairwise Comparison

Suppose that three people are playing the game Rock, Paper, Scissors. On the count of three, each person shows a hand signal for rock, paper, or scissors. Each hand signal beats another hand signal. The group keeps having a tie because Person A always picks rock, Person B always picks paper which beats rock, and Person C always picks scissors which beats paper, and is beaten by rock! This leads to a disagreement about which choice is best. They decide to use the pairwise comparison method determine the winner. Their preference rankings are given in the following table.

Voters	A	B	C
Rock (R)	1	3	2
Paper (P)	2	1	3
Scissors (S)	3	2	1

Answer

Construct the comparison matrix:

Opponent / Runner	Rock (R)	Paper (P)	Scissors (S)	Points
Rock wins	--	RP 2	RS 1	1
Paper wins	PR 1	--	PS 2	1
Scissors wins	SR 2	SP 1	--	1

Figure ✔ **7.2.6** Rock, Paper, Scissors Pairwise Comparison Matrix

There is a tie! There is no winner.

Example **7.2.10** illustrates the answer to the first key question. The pairwise comparison method does not always result in a winner. For example, much like the game of Rock, Paper, Scissors, it is possible for a cyclic pattern to emerge in which each candidate beats the next until the last candidate who beats the first.



Figure **7.2.7**: Rock, Paper, Scissors Cyclic Outcome

Your Turn 7.2.10

A pairwise comparison matrix is given. Determine the winner by the pairwise comparison method. If there is not a winner, explain why. If there is a winner, tell whether the winner is a Condorcet candidate.

Runner \ Opponent	P	Q	R	S	T	Z
P wins	--	PQ 1	PR 4	PS 6	PT 4	PZ 5
Q wins	QP 5	--	QR 6	QS 4	QT 1	QZ 2
R wins	RP 2	RQ 0	--	RS 5	RT 4	RZ 1
S wins	SP 0	SQ 2	SR 1	--	ST 6	SZ 5
T wins	TP 2	TQ 5	TR 2	TS 0	--	TZ 4
Z wins	ZP 1	ZQ 4	ZR 5	ZS 1	ZT 2	--

Figure 7.2.8: A Pairwise Comparison Matrix

Now, you have the answer to the second key question. The pairwise comparison matrix in YOUR TURN 11.10 is an example of a scenario where a winner is not a Condorcet candidate.

The answer to the third question is not as clear. If there is a Condorcet candidate, does that candidate win? So far, we have not come across a contradictory example where the Condorcet candidate didn't win, but we cannot know with certainty that it is not possible by looking at examples. Instead, we will need to use some reasoning. Let's review some particular cases of elections with a certain number of candidates, and then we will try to generalize the scenario to an election with n candidates.

Example 7.2.11: Does the Condorcet Candidate Win?

1. Suppose there is an election with five candidates—A, B, C, D, and E—and that Candidate C is a Condorcet candidate. How many points did Candidate C win?
2. What is the greatest number of points that any one of the other candidates could win?
3. Is it possible for Candidate C to lose or tie?

Answer

1. In any pairwise election with five candidates, each candidate must compete against four other candidates. It follows that the most points a single candidate can win is four points, which would occur if the candidate won every matchup. As a Condorcet candidate, Candidate C won all the pairwise matchups against Candidates A, B, D, and E, earning four points.
2. The rest of the candidates lost to Candidate C. The most points a particular candidate could win if they won matchups with each of the other three candidates is three points.
3. Since Candidate C has four points and the rest of the candidates have three points or less, Candidate C is the winner. Therefore, it is not possible for Candidate C to tie or lose.

Your Turn 7.2.11

Suppose there is an election with 26 candidates, A through Z, and that Candidate C is a Condorcet candidate.

How many points did Candidate C win?

What is the greatest number of points that any one of the other candidates could win?

Is it possible for Candidate C to lose or tie?

Let's consider a general case where there are n candidates. One of the candidates is a Condorcet candidate. Since the Condorcet candidate wins all matchups, the Condorcet candidate wins $n-1$ points. Since each of the other candidates lost to the Condorcet candidate, the most a single candidate could win is $n-2$. Since the Condorcet candidate won $n-1$ points and each other candidate won $n-2$ points or fewer, the Condorcet candidate is the winner. You have your answer to the third key question! If there is a Condorcet candidate, that candidate is always the winner.

Approval Voting

The last type of voting system you will consider for your budding democracy is an approval voting system. In this system, each voter may approve any number of candidates without rank or preference for one over another (among the approved candidates), and the candidate approved by the most voters wins. This voting system has aspects in common with plurality voting and Condorcet voting methods, but it has characteristics that distinguish it from both. An approval voting ballot lists the candidates and provides the option to approve or not approve each candidate.

The term "approval voting" was not used until the 1970s Brams, Steven J.; Fishburn, Peter C. (2007), *Approval Voting*, Springer-Verlag, p. xv, ISBN 978-0-387-49895-9, although its use has been documented as early as the 13th century (Brams, Steven J. (April 1, 2006). *The Normative Turn in Public Choice* (PDF) (Speech). Presidential Address to Public Choice Society. New Orleans, Louisiana.) Approval voting has the appeal of being simpler than ranked voting methods. It also allows an individual voter to support more than one candidate equally. This has appeal for those who do not want a split vote among a few mainstream candidates to lead to the election of a fringe candidate. It also has appeal for those who want an underdog to have a chance of success because voters will not worry about wasting their vote on a candidate who is not believed likely to win.

Example 7.2.12: Rock, Paper, Scissors, Lizard, Spock

Suppose that Person A/B/C were just about to give up on their game of Rock, Paper, Scissors when they were joined by Person D who reminded them that their updated version, Rock, Paper, Scissors, Lizard, Spock was a far superior game with the added rules that Lizard eats Paper, Paper disproves Spock, Spock vaporizes Rock, Rock crushes Lizard, Lizard poisons Spock, Spock smashes Scissors, and Scissors decapitates Lizard.

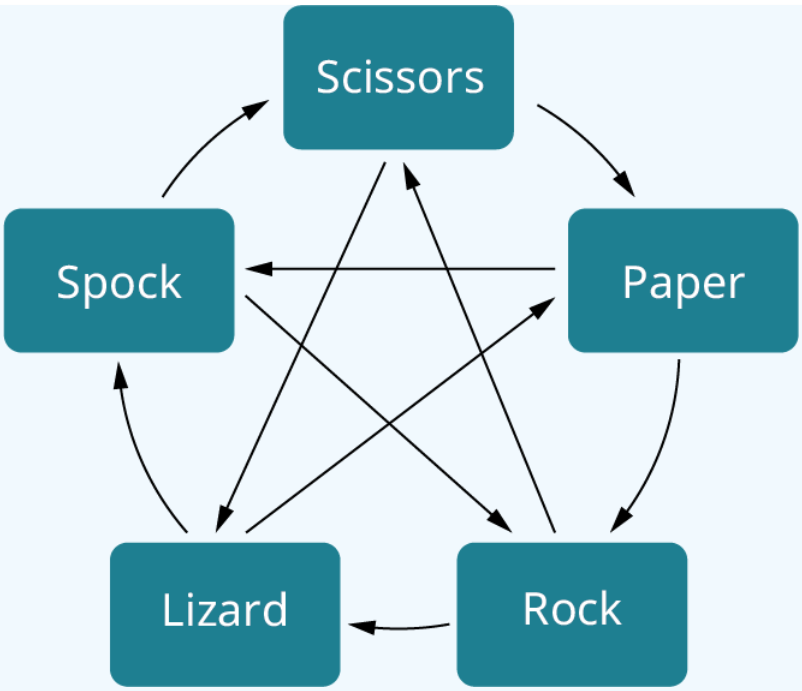


Figure ✔ 7.2.9 Rock, Paper, Scissors, Lizard, Spock Dominance

Person D encourages their friends to hold a new election. This time, for the sake of simplicity, the group decides to use approval voting to determine the best move in the game. The summary of approval ballots for Rock, Paper, Scissors, Lizard, Spock is given in the table below.

Voters	A	B	C	D
Rock	Yes	No	No	No
Paper	No	Yes	No	No
Scissors	No	No	Yes	No
Lizard	No	No	No	Yes
Spock	Yes	Yes	Yes	Yes

Answer

Count the number of approval votes for each candidate by counting the number of "Yes" votes in each row of the table.

- Rock: 1
- Paper: 1
- Scissors: 1
- Lizard: 1
- Spock: 4

Spock is the winning candidate, approved by four voters!

Your Turn 7.2.12

The Chionilis family is trying to decide on a restaurant again, but now they don't want to deal with multiple runoffs or even ranking. They will use the approval voting method shown in the following table. Each family member will approve their top two choices. Rainbow China won when multiple runoffs were used. Find the winner for tonight's dinner.

Options	A	R	S	T	G	E	M	D
Rainbow China	Yes	No	No	Yes	No	No	Yes	Yes
Dough Boys Pizza	Yes	Yes	Yes	Yes	No	Yes	No	Yes
Taco City	No	No	Yes	No	Yes	Yes	No	No
Caribbean Flavor	No	Yes	No	No	Yes	No	Yes	No

Example 7.2.13: The Chionilis Family Is Hungry Again!

The eight members of the Chionilis family—Annette, Rene, Seema, Titus, Galen, Elena, Max and Demitri—have another decision to make. Approval voting worked out nicely the last time. They are going to use it again, but this time, Annette, Rene, Seema, and Galen are feeling a little indecisive. They can't narrow their choice down to two. They will approve their three top choices, but the other family members will only approve two. These choices are reflected in the following table. Determine the restaurant that will be chosen.

Options	A	R	S	T	G	E	M	D
Rainbow China	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes
Dough Boys Pizza	Yes	Yes	Yes	Yes	No	Yes	No	Yes
Taco City	Yes	No	Yes	No	Yes	Yes	No	No
Caribbean Flavor	No	Yes	No	No	Yes	No	Yes	No

Answer

Count the number of approval votes for each restaurant by counting the number of “Yes” votes in each row.

- Rainbow China: 7
- Dough Boys Pizza: 6
- Taco City: 4
- Caribbean Flavor: 3

This time, Rainbow China won!

Your Turn 7.2.13

What would be the outcome of the election if every member of the Chionilis family approved their top three choices from the table below?

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Taco City	3	4	2	4	2
Caribbean Flavor	4	1	4	3	4

Compare and Contrast Voting Methods to Identify Flaws

Wow! We have covered a lot of options for the voting methods. Now, you need to decide which one is best for Imaginaria. Imaginarians might consider characteristics of certain voting systems desirable and others undesirable. In some cases, voters may consider these undesirable traits to be flaws in a voting system that are significant enough to motivate them to reject that system. If you are feeling a bit overwhelmed by this decision, maybe it would help to read about the experiences of others who have faced similar questions.

Consider the 2000 U.S. presidential election in which Green Party candidate Ralph Nader and Reform Party candidate Pat Buchanan were on the ballet running against the mainstream candidates, Democrat Al Gore and Republican George W. Bush. The voting results for Florida are given in Table 11.3.

Table 11.3 Florida Results in the 2000 U.S. Presidential Election (source: https://www.fec.gov/resources/cms-co...ltsbyState.pdf)

Candidate	Party	Votes	Percentage
(G) George W. Bush	Republican	2,912,790	48.85%
(A) Al Gore	Democrat	2,912,253	48.84%
(R) Ralph Nader	Green	97,488	1.63%
(P) Pat Buchanan	Reform	17,484	0.29%
(H) Harry Brown	Libertarian	16,415	0.28%
(O) 7 Other Candidates	Other	6,680	0.11%
	Total	5,963,110	

In more than one state, Buchanan was able to split the Republican vote enough to allow Gore to win that state. Nader split the Democrat vote in Florida and New Hampshire by enough votes to prevent Gore from winning those states. Had Gore won either state, he would have had enough electoral votes to win the election. Instead, Bush won. This is an example of a flaw in the plurality system of voting: the spoiler.

A spoiler is a less popular candidate who takes votes from a more popular candidate with similar positions, swinging the race to another candidate with vastly different views that they would not support. This encourages voters not to vote for the candidate that they perceive to be the best, but instead for the candidate they can live with who they perceive to have a better chance of winning. Some voters may prefer a method such as approval voting, which does not have this trait in common with plurality voting.

Example 7.2.14: The Spoiler Controversy

Because the vote counts for George W. Bush and Al Gore differed by only 537 votes, many Democrats blamed Ralph Nader and the Green Party for their loss. Let's consider how the election results might have differed if the approval voting method had been used.

Use Table 11.3 and the following assumptions to extrapolate the results of an approval method election:

- 100 percent of Pat Buchanan supporters would approve George W. Bush.
- 100 percent of Ralph Nader supporters would approve Al Gore.
- 72 percent of Libertarians would approve George W. Bush.
- 28 percent of Libertarians would approve Al Gore (as was roughly the known percentage at the time according to the Cato Institute).
- 50 percent of the supporters of other candidates would approve George Bush while 50 percent would approve Al Gore.

Answer

Step 1: Create a summary of approval ballots based on the given assumptions. For the Libertarian candidate, 72 percent of 16,415 of the votes is $0.72(16,415) = 11,819$ and 28 percent is $0.28(16,415) = 4,596$. For the other candidates, 50 percent of

the votes is $0.50(6,680) = 3,340$.

Number of Votes	2,912,790 (G)	2,912,253 (A)	97,488 (R)	17,484 (B)	11,819 (72% H)	4,596 (28% H)	3,340 (50% O)	3,340 (50% O)
(G) George W. Bush	Yes	No	No	Yes	Yes	No	No	Yes
(A) Al Gore	No	Yes	Yes	No	No	Yes	Yes	No
(R) Ralph Nader	No	No	Yes	No	No	No	No	No
(P) Pat Buchanan	No	No	No	Yes	No	No	No	No
(H) Harry Brown	No	No	No	No	Yes	Yes	No	No
(O) 7 Other Candidates	No	No	No	No	No	No	Yes	Yes

Step 2: Count the number of approval votes for each candidate.

- George W. Bush: $2,912,790 + 17,484 + 11,819 + 3340 = 2,945,433$
- Al Gore: $2,912,253 + 97,488 + 4,596 + 3,340 = 3,017,677$
- Ralph Nader: $97,488$
- Pat Buchanan: $17,484$
- Harry Brown: $11,819 + 4,596 = 16,415$
- Other Candidates: $6,680$

In this scenario, Al Gore is the winner.

Your Turn 7.2.14

Extrapolate the results of an approval method election using Table 11.3, the assumptions from Example 11.14, and the additional assumptions that the supporters of Al Gore would all approve Ralph Nader and that half of the supporters of George Bush would approve Pat Buchanan while half would approve Harry Brown.

The results in Example 11.14 and Your Turn 11.14 highlight one of the characteristics of approval voting. Ralph Nader moved up from a distant third place finish to a close second place finish when Al Gore's supporters approved him on their ballots. In this way, fringe candidates have a better chance of winning, which some voters consider a flaw but others consider a benefit.

Another aspect of approval voting systems that is a concern to many voters is that candidates in approval elections might encourage their loyal supporters to approve them and only them to avoid giving support to any other candidate. If this occurred, the election in effect becomes a traditional plurality election. This is a flaw that cannot occur in an instant runoff system since all candidates are ranked.

Example 7.2.15: Three Habitable Planets

In the future, humans have explored distant solar systems and found three habitable planets which could be colonized. Since it will take all available resources to colonize one planet, humans must agree on the planet. Planet A has the most comfortable climate and most plentiful resources, but it is the farthest from Earth making travel to the planet a challenge. Planet B is half the distance but will require more resources to make comfortable. Planet C is the least suitable of the three and terraforming

will be required, but it is close enough to make travel between Earth and Planet C possible on a more regular basis. The table below provides the voter preferences for the colonization of each planet.

Percentage of Voters	45%	15%	40%
Planet A	1	3	3
Planet B	2	1	2
Planet C	3	2	1

If the entire population were able to vote, determine the winning planet using each of the methods listed below.

1. Plurality
2. Ranked-choice method
3. Borda count

Answer

1. The plurality method only considers the top choice of each voter. By this system, Planet A has 45 percent of the vote, Planet B has 15 percent of the vote, and Planet C has 40 percent of the vote. Planet A wins.
2. Using either instant runoff or a two-round system, Planet B with only 15 percent of the vote will be eliminated in the first round. In Round 2, the 15 percent that voted for Planet B would vote for their second choice, Planet C. This leaves Planet A with 45 percent and Planet C with 55 percent. Planet C has a majority and wins the election.
3. To find the Borda score for each candidate, imagine there are exactly 100 voters. Then the summary of ranked ballots looks like:

Out of 100 Voters	45	15	40
Planet A	1	3	3
Planet B	2	1	2
Planet C	3	2	1

The Borda score for each candidate is as follows:
Planet A: $(3-1)(45)+(3-3)(15)+(3-3)(40)=90$
Planet B: $(3-2)(45)+(3-1)(15)+(3-2)(40)=115$
Planet C: $(3-3)(45)+(3-2)(15)+(3-1)(40)=95$
Planet B wins.

Your Turn 7.2.15

The juniors at a high school in Central Florida are voting for a theme park to visit for an end of the year field trip. The options are Disney's four theme parks: Animal Kingdom, Magic Kingdom, EPCOT Center, or Hollywood Studios.

Percentage of Voters	35%	25%	30%
Animal Kingdom	1	3	4
Magic Kingdom	2	1	3
Epcot Center	3	4	1
Hollywood Studios	4	2	2

Determine the outcome of the vote by each of the given voting methods.

1. Plurality
2. Ranked-choice method
3. Borda count

The election in Example **7.2.15** involves a scenario in which there are two extreme candidates, Planet A and Planet B, and a moderate candidate, Planet C. The supporters of the extreme candidates prefer the moderate candidate to the other extremist ones. This makes Planet C a compromise candidate. In this case, both the plurality method and ranked-choice voting resulted in the election of one of the extreme candidates, but the Borda count method elected the compromise candidate in this scenario. Depending on a person's perspective, this may be perceived as a flaw in either ranked-choice and plurality systems, or the Borda count method.

In Fairness in Voting Methods, we will analyze the fairness of each voting system in greater detail using objective measures of fairness.

Tech Check: Voting Calculators

It is possible to create Excel spreadsheets that complete the calculations necessary to determine the winner of an election by various voting methods. In some cases, this work has already been done and posted online. As you practice applying the various voting methods that could be used in Imaginaria, quick Internet search will lead to sites such as Ms. Hearn Math with free specialty calculators.

These sites can be a great way to check your results!

Check Your Understanding

1. Name three voting methods that use a ranked ballot.
2. Determine whether the following statement is true or false: The same ranked ballots may result in a different winner depending on which voting method is used.
3. Determine whether the following statement is true or false. A majority candidate is always a Condorcet candidate.
4. The _____ method is a system of voting using ranked ballots in which each candidate is awarded points corresponding to the number of candidates ranked lower on each ballot.
5. The _____ method is a system of voting using ranked ballots (or multiple elections) in which each candidate receives a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie.
6. The _____ method is a runoff voting system in which only the candidate(s) with the very least votes are eliminated.
7. Explain the differences between two-round voting and ranked-choice voting.

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7.2.1: Exercises

For the following exercises, identify the winning candidate based on the described voter profile, if possible. If it is not possible, state so. Explain your reasoning.

Exercise 7.2.1.1

In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, D 112.

Exercise 7.2.1.2

In the first round of a ranked-choice election with three candidates—A, B, and C. Candidate A received 55 first place rankings; Candidate B received 25; and Candidate C received 30.

Exercise 7.2.1.3

The pairwise matchup points for each candidate were: A 1, B $1\frac{1}{2}$, D $\frac{1}{2}$.

Exercise 7.2.1.4

In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16.

Exercise 7.2.1.5

There is a pairwise comparison election with candidates A, B, and C. Candidate A had the most first choice rankings, Candidate B has the highest Borda score, and Candidate C is a Condorcet candidate.

Exercise 7.2.1.6

In the first round of a ranked-choice election with three candidates—A, B, and C—Candidate A received 20 first place rankings, Candidate B received 25, and Candidate C received 30.

For the following exercises, use the table.

O'Malley	De La Fuente	Clinton	Sanders	Other
110,227	67,331	17,174,432	13,245,671	322,276

Popular Vote in the 2016 U.S. Democratic Presidential Primary (source: Federal Election Commission, *Federal Elections 2016 Report*)

Exercise 7.2.1.7

Calculate the number of votes required to have a majority of the popular vote in the 2016 U.S. Democratic Presidential Primary.

Exercise 7.2.1.8

Which candidate had a plurality? Did this candidate have a majority?

For the following exercises, use the given table.

Candidate	Votes
Bush	281,189
Trump	13,783,037
Cruz	7,455,780
Rubio	3,354,067
Carson	822,242
Kasich	4,198,498
Other	337,714

Popular Vote in the 2016 U.S. Republican Presidential Primary (source: Federal Election Commission, *Federal Elections 2016 Report*)

Exercise 7.2.1.9

Calculate the number of votes required to have a majority of the popular vote in the 2016 U.S. Republican Presidential Primary.

Exercise 7.2.1.10

Which candidate had a plurality? Did this candidate have a majority?

For the following exercises, use Table 11.4 and Table 11.5.

Exercise 7.2.1.11

Suppose the Republican Primary in 2016 was a two-round system. Would there be a second round? Why or why not? If so, which candidates would advance to the second round?

Exercise 7.2.1.12

Suppose the Democratic Primary in 2016 was a two-round system. Would there be a second round? Why or why not? If so, which candidates would advance to the second round?

Exercise 7.2.1.13

Suppose the Democratic Primary in 2016 used the Hare method. Would there be a second round? Why or why not?

Exercise 7.2.1.14

Suppose the Republican Primary in 2016 used the Hare method. Would there be a second round? Why or why not?

For the following exercises, use the following table and the Hare method.

Options	A	B	C	D	E
Candidate 1	1	3	3	1	3
Candidate 2	2	1	1	2	4
Candidate 3	3	4	2	4	1
Candidate 4	4	2	4	3	2

Exercise 7.2.1.15

How many votes are needed to win by the Hare method?

Exercise 7.2.1.16

How many votes does each candidate receive in Round 1?

Exercise 7.2.1.17

Which candidates advance to Round 2?

Exercise 7.2.1.18

How many votes does each remaining candidate receive in Round 2?

Exercise 7.2.1.19

Will there be a third round? Why or why not?

Exercise 7.2.1.20

Which candidate wins the election?

For the following exercises, use the sample summary of ranked ballots in the given table.

Number of Ballots	10	20	15	5
Option A	1	4	3	4
Option B	2	3	4	2
Option C	4	2	1	3
Option D	3	1	2	1

Sample Summary of Ranked Ballots

Exercise 7.2.1.21

How many votes were recorded, and how many are required to have a majority?

Exercise 7.2.1.22

How many voters indicated that Option A was their first choice?

Exercise 7.2.1.23

How many voters indicated that Option B was their first choice?

Exercise 7.2.1.24

How many voters indicated that Option A was their last choice?

Exercise 7.2.1.25

How many voters indicated that Option B was their last choice?

Exercise 7.2.1.26

Use ranked-choice voting to determine the two candidates in the final round and the number of votes they each receive in that round.

Exercise 7.2.1.27

Is there a winning candidate? If so, which candidate? Justify your answer.

Suppose that 55 Star Wars fans were asked to vote for their favorite new Star Wars character. They were given a ranked ballot, and the results are shown in the table. Use this table and ranked-choice voting for the following exercises.

Number Of Ballots	7	6	10	8	4	5	6	7	2
Finn	3	5	1	3	2	4	2	2	1
Rey	2	1	3	2	4	3	3	1	3
Poe	1	6	4	1	5	5	5	4	2
BB8	4	2	2	4	6	1	6	5	4
Rose	6	4	5	6	3	2	1	3	6
Kylo	5	3	6	5	1	6	4	6	5

Favorite New Heroes in Star Wars Sequels Ballot Preferences

Exercise 7.2.1.28

How many votes does each candidate get on the first round of voting?

Exercise 7.2.1.29

How many votes are required to get a majority?

Exercise 7.2.1.30

Which candidates remain in the final round, and how many votes do they have?

Exercise 7.2.1.31

Who is the winner of the election?

Refer to Table 11.6 for the following exercises.

Exercise 7.2.1.32

Find the Borda score for each candidate.

Exercise 7.2.1.33

Compare your results from question 32 to those from question 26. Compare the winner and the second-place candidate using the Borda count method to those using the ranked-choice method. Are they the same?

Refer to Table 11.7 for the following exercises.

Exercise 7.2.1.34

Find the Borda score for each candidate.

Exercise 7.2.1.35

Compare your results from question 34 to those from question 30. Compare the winner and the second-place candidate using the Borda count method to those using the ranked-choice method. Are they the same?

For the following exercises, use the table below.

Number of Ballots	100	80	110	105	55
Option A	1	1	4	4	2
Option B	2	2	2	3	1
Option C	4	4	1	1	4
Option D	3	3	3	2	3

Exercise 7.2.1.36

Do any candidates appear to be divisive candidates? Justify your answer.

Exercise 7.2.1.37

Do any candidates appear to be compromise candidates? Justify your answer.

Exercise 7.2.1.38

How many votes are required for a majority?

Exercise 7.2.1.39

Which candidate is eliminated first by the ranked-choice method?

Exercise 7.2.1.40

Which candidate is eliminated second by the ranked-choice method?

Exercise 7.2.1.41

Which candidate is the winner by the ranked-choice method?

Exercise 7.2.1.42

What are the Borda scores for each candidate?

Exercise 7.2.1.43

Which candidate is the winner by the Borda count method?

Exercise 7.2.1.44

Which method resulted in a win for the compromise candidate: ranked-choice voting or the Borda count method or both?

Use the pairwise comparison matrix in the given figure for the following exercises.

Runner \ Opponent	Q	R	S	T
Q wins	--	QR 3	QS 2	QT 1
R wins	RQ 1	--	RS 3	RT 2
S wins	SQ 2	SR 1	--	ST 3
T wins	TQ 3	TR 2	TS 1	--

Pairwise Comparison Matrix for Candidates Q, R, S and T

Exercise 7.2.1.45

Analyze the pairwise comparison matrix. Display the pairings in a table and indicate the winner of each matchup by marking an [x] through the losing matchups and a single slash | through the ties.

Exercise 7.2.1.46

Calculate the points received by each candidate in the pairwise comparison matrix.

Exercise 7.2.1.47

Determine whether there is a winner of the pairwise comparison election represented by the matrix. If there is a winner, determine whether the winner is a Condorcet candidate.

Use the pairwise comparison matrix in the given figure for the following exercises.

Opponent / Runner	U	V	W	X	Y
U wins	--	UV 1	UW 3	UX 3	UY 4
V wins	VU 5	--	VW 6	VX 4	VY 1
W wins	WU 3	WV 0	--	WX 5	WY 4
X wins	XU 3	XV 2	XW 1	--	XY 6
Y wins	YU 2	YV 5	YW 2	YX 0	--

Pairwise Comparison Matrix for Candidates U, V, W, X, and Y

Exercise 7.2.1.48

Analyze the pairwise comparison matrix. Display the pairings in a table and indicate the winner of each matchup.

Exercise 7.2.1.49

Calculate the points received by each candidate in the pairwise comparison matrix.

Exercise 7.2.1.50

Determine whether there is a winner of the pairwise comparison election represented by the matrix. If there is a winner, determine whether the winner is a Condorcet candidate and explain your reasoning.

Exercise 7.2.1.51

In J.K. Rowling's *Harry Potter* series, Albus Dumbledore was the headmaster of Hogwarts for many years. Imagine that an election is to be held to find his successor. Severus Snape, the head of Slytherin House, will be running against the heads of Gryffindor and Ravenclaw, Minerva McGonagall and Filius Flitwick. Use the preference rankings for each candidate in the following table to construct a pairwise comparison matrix.

Percentage of Vote	25%	40%	35%
(S) Snape	1	3	3
(M) McGonagall	3	1	2
(F) Flitwick	2	2	1

Exercise 7.2.1.52

Analyze the pairwise comparison matrix you constructed for question 51. Display the pairings in a table and indicate the winner of each matchup.

Exercise 7.2.1.53

Use the pairwise comparison matrix from questions 51 and 52 to find the number of points earned by each candidate. Who is the winner by the pairwise comparison method?

Exercise 7.2.1.54

Is the winner of the Hogwarts headmaster election a Condorcet candidate? Explain how you know.

Exercise 7.2.1.55

The women of *The Big Bang Theory* decide to hold their own approval voting election to determine the best option in Rock, Paper, Scissors, Lizard, Spock. Use the summary of their approval ballots in the table below to determine the number of votes for each candidate. Determine the winner, or state that there is none.

Voters	Penny	Bernadette	Amy
Rock	Yes	No	No
Paper	Yes	Yes	No
Scissors	Yes	Yes	Yes
Lizard	No	No	No
Spock	Yes	No	Yes

For the following exercises, use the table below.

Percentage of Vote	40%	35%	25%
Candidate A	1	3	2
Candidate B	2	1	3
Candidate C	3	2	1

? Exercise 7.2.1.56

Which candidate is the winner by the ranked-choice method?

? Exercise 7.2.1.57

Suppose that they used the approval method and each voter approved their top two choices. Which candidate is the winner by the approval method?

? Exercise 7.2.1.58

Which candidate is the winner by the Borda count method?

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7.3: Fairness in Voting Methods



Figure **7.3.1**: Citizens strive to ensure their voting system is fair. (credit: "Governor Votes Early" by Maryland GovPics/Flickr, CC BY 2.0)

Learning Objectives

1. Compare and contrast fairness of voting using majority criterion.
2. Compare and contrast fairness of voting using head-to-head criterion.
3. Compare and contrast fairness of voting using monotonicity criterion.
4. Compare and contrast fairness of voting using irrelevant alternatives criterion.
5. Apply Arrow's Impossibility Theorem when evaluating voting fairness.

Now that we've covered a variety of voting methods and discussed their differences and similarities, you might be leaning toward one method over another. You will need to convince the other founders of Imaginaria that your preference will be the best for the country. Before your collaborators approve the inclusion of a voting method in the constitution, they will want to know that the voting method is a fair method. In this section, we will formally define the characteristics of a fair system. We will analyze each voting previously discussed to determine which characteristics of fairness they have, and which they do not. In order to guarantee one ideal, we must often sacrifice others.

The Majority Criterion

One of the most fundamental concepts in voting is the idea that most voters should be in favor of a candidate for a candidate to win, and that a candidate should not win without majority support. This concept is known as the majority criterion.

With respect to the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—we will explore two important questions:

1. Which of these voting systems satisfy the majority criterion and which do not?
2. Is it always "fair" for a voting system to satisfy the majority criterion?

Keep in mind that this criterion only applies when one of the candidates has a majority. So, the examples we will analyze will be based on scenarios in which a single candidate has more than 50 percent of the vote.

Example 7.3.1: Roommates Choose Fast Food

It's final exams week and seven college students are hungry. They must get food, but from which drive thru? Their preferences are listed in the table below. The majority have listed McDonald's as their top choice. Let's calculate what the results of the election will be using various voting methods.

Voters	A	B	C	D	E	F	G
(M) McDonald's	1	1	1	1	5	5	5
(B) Burger King	2	2	2	2	2	2	2
(T) Taco Bell	4	4	3	4	3	3	1
(O) Pollo Tropical	3	5	4	3	4	1	3
(I) Pizza Hut	5	3	5	5	1	4	4

1. Which restaurant is the winner using the plurality voting method?
2. Which restaurant is the winner using the ranked-choice voting method?
3. Does the majority criterion apply? If so, for which of voting method(s), if any, did the majority criterion fail?

Answer

1. For plurality voting, we only need to count the first-place votes for each candidate. In this case, McDonald's has four first place votes, which is a majority and wins the election automatically.
2. For ranked-choice voting, McDonald's also wins because it has a majority at the end of Round 1.
3. The majority criterion does apply because one candidate had a majority of the first-place votes. The majority criterion did not fail by either method, because in each case the majority candidate won.

Your Turn 7.3.1

Use the information in the table to answer the following questions.

Voters	L	M	N	O	P
Option A	2	2	3	2	1
Option B	1	4	1	4	2
Option C	3	1	4	1	3
Option D	4	1	2	3	4

Which option is the winner when using the plurality voting method?

Which option is the winner when using the ranked-choice voting method?

Does the majority criterion apply? If so, for which voting method(s), if any, did the majority criterion fail?

From Example 11.16, it appears that the plurality and ranked-choice voting methods satisfy the majority criterion. In general, the majority candidate always wins in a plurality election because the candidate that has more than half of the votes has more votes

than any other candidate. The same is true for ranked-choice voting; and there will never be a need for a second round when there is a majority candidate. Let's examine how some of the other voting methods stand up to the majority criterion.

Example 7.3.2: Roommates Choose Fast Food

Those seven college students are hungry again! Their preferences haven't changed, as shown below. Let's calculate if the results change when we use different voting methods.

VOTERS	A	B	C	D	E	F	G
(M) McDonald's	1	1	1	1	5	5	5
(B) Burger King	2	2	2	2	2	2	2
(T) Taco Bell	4	4	3	4	3	3	1
(O) Pollo Tropical	3	5	4	3	4	1	3
(I) Pizza Hut	5	3	5	5	1	4	4

1. Which restaurant is the winner using the pairwise comparison voting method?
2. Which restaurant is the winner using the Borda count voting method?
3. Does the majority criterion apply? If so, for which voting method(s), if any, did the majority criterion fail?

Answer

1. For pairwise comparison, notice that McDonald's is a Condorcet candidate because it wins every pairwise comparison. So, McDonald's is the winner.
2. For the Borda count, we must calculate the Borda score for each candidate: McDonald's is 16, Burger King is 21, Taco Bell is 13, Pollo Tropical is 12, Pizza Hut is 8. The winner is Burger King!
3. Yes, the majority criterion applies because McDonald's has the majority of first place votes. The majority criterion only fails using the Borda method.

Your Turn 7.3.2

Use the information in the following table to find the winner using each of the voting methods in parts 1 and 2, then answer the question in part 3.

Voters	L	M	N	O	P
Option A	2	2	3	2	2
Option B	1	4	1	4	1
Option C	3	1	4	1	3
Option D	4	3	2	3	4

Pairwise comparison

Borda count

Does the majority criterion apply? If so, for which voting method(s) did the majority criterion fail?

Example 11.17 demonstrates a concept that we also saw in Borda Count Voting—the Borda method frequently favors the compromise candidate over the divisive candidate. This can happen even when the divisive candidate has a majority, as it did in this example. Although a majority of the voters were in favor of McDonald's, a significant minority was strongly opposed to McDonald's, ranking it last. Since the Borda score includes all rankings, this strong opposition has an impact on the outcome of the election.

Pairwise comparison will always satisfy the majority criterion because the candidate with the majority of first-place votes wins each pairwise matchup. While it is possible for the majority candidate to win by the Borda count method, it is not guaranteed. So, the Borda method fails the majority criterion. A summary of each voting method as it relates to the majority criterion is found in the following table.

Voting Method	Majority Criterion
Plurality	Satisfies
Ranked-choice	Satisfies
Pairwise comparison	Satisfies
Borda count	Violates

If you prefer the Borda method, you might argue that its failure to satisfy the majority criterion is actually one of its strengths. As we saw in Example 11.16 and Example 11.17, the majority have the power to vote for their own benefit at the expense of the minority. While four students were very enthusiastic about McDonald's, three students were strongly opposed to McDonald's. It is reasonable to say that the better option would be Burger King, the compromise candidate, which everyone ranked highly and no one strongly opposed. The Borda method is designed to favor a candidate that is acceptable to the population as a whole. In this way, the Borda method avoids a downfall of strict majority rule known as the tyranny of the majority, which occurs when a minority of a population is treated unfairly because their situation is different from the situation of the majority.

The people of Imaginaria should know that the power of the majority to vote their will has serious implications for other groups. For example, according to the UCLA School of Law Williams Institute, the LGBTQ+ community in the United States makes up approximately 4.5 percent of the population. When elections occur that include issues that affect the LGBTQ+ community, members of the LGBTQ+ community depend on the 95.5 percent of the population who do not identify as LGBTQ+ to consider their perspectives when voting on issues such as same-sex marriage, the use of public restrooms by transgender people, and adoption by same-sex couples.

People in Mathematics: Robert Dahl

In his book, *Democracy and Its Critics*, Robert Dahl wrote, "If a majority is not entitled to do so, then it is thereby deprived of its rights; but if a majority is entitled to do so, then it can deprive the minority of its rights." Dahl was a renowned political theorist, but he is also considered to be a mathematician since his work utilizes ideas from an area of mathematics known as Game Theory (Mathematics Genealogy Project, NDSU Department of Mathematics with the American Mathematical Society).

Who Knew?: Three Branches of Government

Concerns about the consequences of majority rule are not new. In 1788, John Adams warned of the consequences of majority rule and he argued for three branches of government as a way to temper them. In the early 1800s, a young French aristocrat named Alexis de Tocqueville toured the United States and wrote *Democracy in America*, which focused on the impact of democracy on political and civil societies. He observed, even then, the dominance of the white majority over the Indigenous people and enslaved people, which was perpetuated by majority rule.

Video



Separation of Powers and Checks and Balances

Head-to-Head Criterion

Another fairness criterion you must consider as you select a voting method for Imaginaria is the Condorcet criterion, also known as the head-to-head criterion. An election method satisfies the Condorcet criterion provided that the Condorcet candidate wins the election whenever a Condorcet candidate exists. A Condorcet method is any voting method that satisfies the Condorcet criterion.

Recall from Three Key Questions that not every election has a Condorcet candidate; the Condorcet criterion will not apply to every election. Also recall that a Condorcet candidate cannot lose an election by pairwise comparison. So, the pairwise comparison voting method is said to satisfy the Condorcet criterion.

✔ Example 7.3.3: Spending Tax Refund

A survey asked a random sample of 100 people in the United States to rank their priorities for spending their tax refund. The options were (V) go on vacation, (S) put into savings, (D) pay off debt, or (T) other. The pairwise comparison matrix for the results is in Figure 11.12. Determine whether the Condorcet criterion applies.

Opponent / Runner	V	S	D	T
V wins	--	~~VS 33~~	~~VD 37~~	VT 100
S wins	SV 67	--	~~SD 36~~	ST 100
D wins	DV 63	DS 64	--	DT 96
T wins	~~TV 0~~	~~TS 0~~	~~TD 4~~	--

Figure ✔ 7.3.2 Pairwise Comparison Matrix for Tax Refund Spending

Answer

The Condorcet criterion only applies when there is a Condorcet candidate. “Pay off debt” (D) is a Condorcet candidate because D wins every matchup. Yes, the Condorcet criterion applies to this election.

Your Turn 7.3.3

Determine whether the Condorcet criterion applies based on the summary of ranked ballots given in the table below.

Votes	3	2
Option A	1	3
Option B	2	1
Option C	3	2

Example 7.3.4: Spending Tax Refund

Let's return to the survey about tax refund spending from Example 11.18. We know that the Condorcet criterion applies because Option D, "Pay off debt," is a Condorcet candidate, which wins every pairwise match up.

Use the information in the ballot summary from the table below to find the winner and determine whether the Condorcet criterion is satisfied in this election when each of the following voting methods are used.

Votes	33	32	31	4
On a vacation (V)	1	3	3	2
Put into savings (S)	3	1	2	1
Pay off debt (D)	2	2	1	4
Other (T)	4	4	4	3

1. Plurality
2. Ranked-choice voting
3. Borda count

Answer

1. V wins 33 first place votes; S, 36; D, 31; and T, 0. So candidate S, "Put into savings," has a plurality and wins. Since the Condorcet candidate D didn't win, the Condorcet criterion is violated.
2. Use the steps outlined in Ranked-Choice Voting for determining the winner of an election by ranked-choice voting, the application of the Hare method in which instant runoffs are used.

   Step 1: The number of votes needed to achieve a majority is 51.

   Step 2: As illustrated in part 1, no candidate has a majority of first-place votes; so the candidate with the fewest votes, T, must be eliminated.

   Step 3: Reallocate votes to the remaining candidates for the second round:

Votes	33	32	31	4
On a vacation (V)	1	3	3	2
Put into savings (S)	3	1	2	1
Pay off debt (D)	2	2	1	3

   Step 4: Repeat the process from Step 2. Count the first-place votes for each candidate: V has 33 votes, S has 36 votes, D has 31. votes. Eliminate candidate D, "Pay off debt," for the third round.

Step 5: Repeat the process from Step 3. Reallocate votes to the remaining candidates for the third round:

Votes	33	32	31	4
On a vacation (V)	1	2	2	2
Put into savings (S)	2	1	1	1

Step 6: Repeat the process from Step 2 one last time. Count the first-place votes for each candidate: V has 33, S has 67. Candidate S, "Put into savings," has a majority and wins. Since the Condorcet candidate D candidate D, "Pay off debt," didn't win, the Condorcet criterion is violated.

3. Calculate the Borda score for each candidate.

   V: $33(4-1)+32(4-3)+31(4-3)+4(4-2)=170$

   S: $33(4-3)+32(4-1)+31(4-2)+4(4-1)=203$

   D: $33(4-2)+32(4-2)+31(4-1)+4(4-4)=223$

   T: $33(4-4)+32(4-4)+31(4-4)+4(4-3)=4$

Candidate D, "Pay off debt," has the highest Borda score and wins. Since D was the Condorcet candidate, this election satisfies the Condorcet criterion.

Your Turn 7.3.41

Use the summary of ranked ballots below to find the winner and determine whether the Condorcet criterion is satisfied when each of the following voting methods are used. Recall that Option A is a Condorcet candidate.

Votes	3	2
Option A	1	3
Option B	2	1
Option C	3	2

1. Plurality
2. Ranked-choice voting
3. Borda count

As we have seen, the plurality method, ranked-choice voting, and the Borda count method each fail the Condorcet criterion in some circumstances. Of the four main ranked voting methods we have discussed, only the pairwise comparison method satisfies the Condorcet criterion every time. A summary of each voting method as it relates to the Condorcet criterion is found in the following table.

Voting Method	Condorcet Criterion
Plurality	Violates
Ranked-choice	Violates
Pairwise comparison	Satisfies
Borda count	Violates

Monotonicity Criterion

The citizens of Imaginaria might be surprised to learn that it is possible for a voter to cause a candidate to lose by ranking that candidate higher on their ballot. Is that fair? Most voters would say, "Absolutely not!!" This is an example of a violation of the

fairness criterion called the **monotonicity criterion**, which is satisfied when no candidate is harmed by up-ranking nor helped by down-ranking, provided all other votes remain the same.

Consider a scenario in which voters are permitted a first round that is not binding, and then they may change their vote before the second round. Such a first round can be called a "straw poll." Now, let's suppose that a particular candidate won the straw poll. After that, several voters are convinced to increase their support, or up-rank, that winning candidate and no voters decrease that support. It is reasonable to expect that the winner of the first round will also win the second. Similarly, if some of the voters decide to decrease their support, or down-rank, a losing candidate, it is reasonable to expect that candidate will still lose in the second round.

You might be wondering why it's called the monotonicity criterion. In mathematics, the term monotonicity refers to the quality of always increasing or always decreasing. For example, a person's age is monotonic because it always increases, whereas a person's weight is not monotonic because it can increase or decrease. If the only changes to the votes for a particular candidate after a straw poll are in one direction, this change is considered monotonic.

If you are going to make an informed decision about which voting method to use in Imaginaria, you need to know which of the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—satisfy the monotonicity criterion.

Example 7.3.5: Favorite Dog Breed by Plurality

The local animal shelter is having a vote-by-donation charity event. For a $10 donation, an individual can complete a ranked ballot indicating their favorite large dog breed: standard poodle, golden retriever, Labrador retriever, or bulldog. Use the summary of ballots below to answer each question.

Votes	42	53	61	24
(S) Standard Poodle	1	3	2	1
(G) Golden Retriever	3	1	4	4
(L) Labrador Retriever	4	2	1	2
(B) Bulldog	2	4	3	3

1. Determine the winner of the election by plurality.
2. Suppose that the 53 voters in the second column increased their ranking of the winner by 1. Determine the winner by plurality with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part 3 is also true for plurality voting and the monotonicity criterion in general? Why or why not?

Answer

1. The number of votes for each candidate are: S 66, G 53, L 61, and B 0. The winner is the standard poodle.
2. If the 53 voters in the second column rank S as 2 and L as 3, then the number of votes for each candidate are: S with 66, G with 53, L with 61, and B with 0. The winner is still the standard poodle.
3. This election does not violate the monotonicity criterion because the winner was not hurt by up-ranking.
4. In general, increasing the ranking for a winner of a plurality election will either leave them with the same or more first place votes while leaving the other candidates with the same or fewer first place votes. So a plurality election will never violate the monotonicity criterion.

Your Turn 7.3.5

Use the Favorite Large Dog Breed Ballot Summary to answer each question.

Determine the winner of the election by Borda count.

Suppose that the 61 voters in the third column increased their ranking of the winner by 1. Determine the winner by Borda count with the new rankings.

Does this election violate the monotonicity criterion?

Do you think the result of part 3 is true for Borda count and the monotonicity criterion in general? Why or why not?

✔ Example 7.3.6: Favorite Dog Breed by Pairwise Comparison

Earlier, we discovered that the summary of ranked ballots shown in the table below results in the pairwise comparison matrix in Figure 11.13. Use this information to answer the questions.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

Runner \ Opponent	Option A	Option B	Option C	Option D	Points
Option A wins	--	AB 225	AC 225	AD 225	3
Option B wins	BA 185	--	BC 320	BD 315	2
Option C wins	CA 185	CB 90	--	CD 200	0
Option D wins	DA 185	DB 95	DC 210	--	1

Figure ✔ 7.3.3 Analyzed Pairwise Comparison Matrix for Sample Summary of Ranked Ballots

1. Determine the winner of the election by the pairwise comparison method.
2. Suppose that the 95 voters in the first column increased their ranking of the winner by 1. Determine the winner by the pairwise comparison method with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part 3 is true for the pairwise comparison method and the monotonicity criterion in general? Why or why not?

Answer

1. By the pairwise comparison method, Option A wins with three points.
2. If the 95 voters in the first column of Figure 11.13 increased their ranking of the winner by 1, then C would fall into fourth place and A would move up to third place on those ballots. This would only affect the matchup between A and C, and the

result would be that A would gain 95 votes while C would lose 95 votes. This means A would have 320 votes and C would have 90. Since A already was ahead of C, this just puts A further ahead and causes no change to the election results.

3. Since the winner A is not hurt by an up-rank, and the loser C is not helped by a down-rank, this election is fair by the monotonicity criterion.
4. Yes, the monotonicity criterion would be satisfied by the pairwise comparison method, because an up-rank of the winner can never decrease the number of pairwise wins. Similarly, a down-rank can never increase the number of pairwise wins.

Your Turn 7.3.6

Use the Favorite Large Dog Breed Ballot Summary to answer each question.

Determine the winner of the election by the ranked-choice method.

Suppose that the 24 voters in the last column of the table increased their ranking of the winner by 1. Determine the winner by the ranked-choice method with the new rankings.

Does this election violate the monotonicity criterion?

Do you think the result in question 3 is true for the ranked-choice method and the monotonicity criterion in general? Why or why not?

The last few examples illustrate that the plurality method, pairwise comparison voting, and the Borda count method each satisfy the monotonicity criterion. Of the four main ranked voting methods we have discussed, only the ranked-choice method violates the monotonicity criterion. A summary of each voting method as it relates to the Condorcet criterion is found in the table below.

Voting Method	Monotonicity Criterion
Plurality	Satisfies
Ranked-choice	Violates
Pairwise comparison	Satisfies
Borda count	Satisfies

Irrelevant Alternatives Criterion

We have covered a lot about voting fairness, but there is one more fairness criterion that you and the other Imaginarians should know. Consider this well-known anecdote that is sometimes attributed to the American philosopher Sidney Morgenbesser:

A man is told by his waiter that the dessert options this evening are blueberry pie or apple pie. The man orders the apple pie. The waiter returns and tells him that there is also a third option, cherry pie. The man says, “In that case, I would like the blueberry pie.” (*Gaming the Vote: Why Elections Aren’t Fair (and What We Can Do About It)*, William Pound stone, p. 50, ISBN 0-8090-4893-0)

This story illustrates the concept of the Irrelevant Alternatives Criterion, also known as the **Independence of Irrelevant Alternatives Criterion (IIA),** which means that the introduction or removal of a third candidate should not change or reverse the rankings of the original two candidates relative to one another. In particular, if a losing candidate is removed from the race or if a new candidate is added, the winner of the race should not change.

Example 7.3.7: Apple, Blueberry, or Cherry?

Suppose that 30 students in a class are going to vote on whether to have apple, blueberry, or cherry pie. Use the summary of ranked ballots in below to answer each question.

Number of Ballots	14	12	4
(A) Apple Pie	1	3	3
(B) Blueberry Pie	2	1	2
(C) Cherry Pie	3	2	1

1. Determine the winner of the election by plurality.
2. Which candidate would win a plurality election if cherry pie were removed from the ballot?
3. Does this election violate the IIA?

Answer

1. The number of first place votes for each candidate is: A with 14, B with 12, and C with 4. Apple pie has the most first-place votes and wins the election.
2. If cherry pie is removed from the ballot, then the four voters in the third column now rank blueberry pie as their first choice. So the four votes for C now belong to B. This means that blueberry pie has 16 votes compared to the 14 votes for apple pie. Blueberry pie now wins the plurality election.
3. Yes, the election violates the IIA because the removal of a losing candidate from the ballot changed the winner of the election.

Your Turn 7.3.7

The local animal shelter is having another vote-by-donation charity event! This time, for a $10 donation, an individual can complete a ranked ballot indicating their favorite small dog breed, miniature, from the ones provided: miniature poodle, Yorkshire terrier, or Chihuahua. Use the summary of ballots below to answer each question.

Votes	53	42	24	11
(P) Miniature Poodle	1	3	3	2
(Y) Yorkshire Terrier	2	2	1	1
(C) Chihuahua	3	1	2	3

1. Determine the winner of the election by the ranked-choice method.
2. Determine the winner of the election if poodles are removed from the ballot.
3. Does this election violate IIA?

Example 7.3.8: Best Fourth Wall Breaking Stare on The Office

The NBC sitcom *The Office* ran for nine years and has been one of the most popular streamed television shows of all time. One of the trademarks of the show was that characters would often break the fourth wall to communicate with the audience just by staring directly into the camera. In fact, there is a website dedicated to "*The Office* stares" where you can watch over 700 of these stares! Suppose that 36 fans were asked which character had the best "*The Office* stare." Use the ballot summary in below to answer each question.

Number of Ballots	9	11	7	6	3
(J) Jim Halpert (John Krasinski)	1	2	4	2	4
(P) Pam Beesly-Halpert (Jenna Fischer)	4	1	2	4	3
(D) Dwight Schrute (Rainn Wilson)	2	3	3	1	2
(M) Michael Scott (Steve Carell)	3	4	1	3	1

1. Determine the winner of the election by the pairwise comparison method.
2. Determine the winner of the election by the pairwise comparison method if Michael Scott is removed from the ballot.
3. Does this election violate IIA?

Answer

1. Construct and analyze a pairwise comparison matrix:

Runner \ Opponent	J	P	D	M	Points
J wins	--	~~JP 15~~	JD 20	JM 26	2
P wins	PJ 21	--	~~PD 18~~	~~PM 11~~	$1\frac{1}{2}$
D wins	~~DJ 16~~	~~DP 18~~	--	DM 26	$1\frac{1}{2}$
M wins	~~MJ 10~~	MP 25	~~MD 10~~	--	1

Figure ✔ **7.3.4** Pairwise Comparison Matrix for Jim, Pam, Dwight, and Michael

Jim Halpert wins with 2 points.

2. If Michael Scott is removed, the summary of ranked ballots becomes:

Number of Ballots	9	11	7	6	3
(J) Jim Halpert (John Krasinski)	1	2	3	2	3
(P) Pam Beesly-Halpert (Jenna Fischer)	3	1	1	3	2
(D) Dwight Schrute (Rainn Wilson)	2	3	2	1	1

Construct and analyze a pairwise comparison matrix:

Runner \ Opponent	J	P	D	Points
J wins	--	~~JP 15~~	JD 20	1
P wins	PJ 21	--	~~PD 18~~	$1\frac{1}{2}$
D wins	~~DJ 16~~	~~DP 18~~	--	0.5

Figure ✔ **7.3.5**: Pairwise Comparison Matrix for Jim, Pam, and Dwight

Pam wins with $1\frac{1}{2}$ points.

3. Yes, this violates the IIA, because the winning candidate was hurt by the elimination of a losing candidate.

Your Turn \(\PageIndex{8}\)

Use the initial ballot summary from Example 11.23 to answer the questions.

Determine the winner of the election by the Borda count method.

Determine the winner of the election by the Borda count method if Michael Scott is removed from the ballot.

Does this election violate IIA?

We have seen that all four of the main voting systems we are working with fail the Irrelevant Alternatives Criterion (IIA). A summary of each voting method as it relates to the IIA criterion is found in the table below.

Voting Method	Irrelevant Alternatives Criterion
Plurality	Violates
Ranked-choice	Violates
Pairwise comparison	Violates
Borda count	Violates

Who Knew?

Electronic Voting: Does Your Vote Count?

In order for an election to be fair, voting must be accessible to everyone and every vote must be counted. When hundreds of thousands to millions of votes must be collected and counted in a short period of time, deciding it can be challenging to be on counting procedures that are accurate and secure. Electronic voting machines or even Internet voting can speed up the process, but how reliable are these methods? This has been a subject for debate for years.

In a press release on August 3, 2007, California Secretary of State Debra Bowen explained the results of an extensive review of electronic voting systems in her state. She said that transparency and auditability were key. She went on to say, “I think voters and counties are the victims of a federal certification process that hasn’t done an adequate job of ensuring that the systems made available to them are secure, accurate, reliable and accessible. Congress enacted the Help America Vote Act, which pushed many counties into buying electronic systems that—as we’ve seen for some time and we saw again in the independent

UC review—were not properly reviewed or tested to ensure that they protected the integrity of the vote." Secretary Bowden subsequently ordered that voting machines must have tighter security to be used in California. (DB07:042, Secretary of State Debra Bowen Moves to Strengthen Voter Confidence in Election Security Following Top-to-Bottom Review of Voting Systems, sos.ca.gov/elections.) In some instances, the use of electronic voting in parliamentary elections has been discontinued completely for security reasons. For example, according to the National Democratic Institute, the Netherlands returned to all paper ballots and hand counting in 2006. Will you use voting machines or Internet voting in Imaginaria?

So far, every one of the voting methods we have analyzed has failed one or more of the fairness criteria in one election or another.

Voting Method	Majority Criterion	Condorcet Criterion (Head-to-Head Criterion)	Monotonicity Criterion	(Independence of) Irrelevant Alternatives Criterion
Plurality	Satisfies	Violates	Satisfies	Violates
Ranked-choice	Satisfies	Violates	Satisfies	Violates
Pairwise comparison	Satisfies	Satisfies	Violates	Violates
Borda count	Violates	Violates	Satisfies	Violates

You might be wondering if there is a voting system you could recommend for Imaginaria that satisfies all the fairness criteria. If there is one, it remains to be discovered and it is not a voting system that is based solely on preference rankings. In 1972, Harvard Professor of Economics Kenneth J. Arrow received the Nobel Prize in Economics for proving Arrow's Impossibility Theorem, which states that any voting system, either existing or yet to be created, in which the only information available is the preference rankings of the candidates, will fail to satisfy at least one of the following fairness criteria: the majority criterion, the Condorcet criterion, the monotonicity criterion, and the independence of irrelevant alternatives criterion. This theorem only applies to a specific category of voting systems—those for which the preference ranking is the only information collected. There are other types of voting systems to which the Impossibility Theorem does not apply. For example, there is a class of voting systems called Cardinal voting systems that allow for rating the candidates in some way.

"Rating" is different from "ranking" because a voter can give different candidates the same rating. Consider the five-star rating systems used by various industries, or the thumbs up/thumbs down rating system used on YouTube. Could there be a Cardinal voting system that does not violate any of the fairness criteria we have discussed? It's possible, but more research must be done in order to prove it.

People in Mathematics: Kenneth J. Arrow

In 1972, Kenneth J. Arrow, a Harvard Professor of Economics, received the Nobel Prize in Economics jointly with Sir John Hicks, another world renowned economist, for their contributions to economic theory. In particular, Professor Arrow proved mathematically that no ranked voting system meets all four of the fairness criteria discussed in this section. The statement of this fact is known as Arrow's Impossibility Theorem. Visit this site for more details on Professor Arrow

Check Your Understanding

1. Which fairness criterion is violated by all four of the main ranked voting methods presented in this chapter?
2. Which of the four main ranked voting methods presented in this chapter satisfies the Condorcet criterion?
3. Which of the four main ranked voting methods presented in this section violates the majority criterion?
4. Which of the four main ranked voting methods presented in this section violates the monotonicity criterion?
5. According to Arrow's Impossibility Theorem, which of the four main ranked voting methods presented in this chapter violate at least one of the fairness criteria?
6. Determine whether the following statement is true or false and explain your reasoning: Any ranked election that violates the majority criterion also violates the Condorcet criterion.

7. Determine whether the following statement is true or false and explain your reasoning: Any ranked election that violates the Condorcet criterion also violates the majority criterion.
8. Does Arrow's Impossibility Theorem apply to approval voting? Why or why not?
9. Determine whether each statement is true or false. Explain your reasoning.
10. Arrow's Impossibility Theorem guarantees that ranked voting systems always lead to unfair elections.
11. Approval voting is in the class of voting systems called Cardinal Voting systems.

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7.3.1: Exercises

For the following exercises, identify which fairness criteria, if any, are violated by characteristics of the described voter profile. Explain your reasoning.

Exercise 7.3.1.1

In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, D 112. The pairwise matchup points for each candidate would have been: A 1, B 3, C 1, D 1.

Exercise 7.3.1.2

In a Borda count election, the candidates have the following Borda scores: A 1245, B 1360, C 787. Candidate A received 55 percent of the first-place rankings.

Exercise 7.3.1.3

In a pairwise comparison election, the four candidates initially received the following points for winning matchups: A 2, B $1\frac{1}{2}$, $1\frac{1}{2}$, C 1, D $1\frac{1}{2}$. When candidate C dropped out of the election, the remaining candidates received: A 1, B $1\frac{1}{2}$, D $\frac{1}{2}$.

Exercise 7.3.1.4

In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16. The pairwise matchup points for the same voter profiles would have been: A 2, B 0, C 1, D 3.

Exercise 7.3.1.5

In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16. When Candidate E was added to the ballot, the Borda scores became: A 25, B 21, C 15, D 24, E 18.

Exercise 7.3.1.6

In a pairwise comparison election, Candidate C was a Condorcet candidate in a straw poll. When the actual election took place, several voters up-ranked Candidate C on their ballots, but no other changes were made to the voter preferences, and Candidate B won the election.

For the following exercises, use the table below.

Votes	49	51
Candidate A	3	1
Candidate B	1	2
Candidate C	2	3

Exercise 7.3.1.7

In a pairwise comparison election, Candidate A was in first place, Candidate B was in second place, and Candidate C was in third place. When the actual election tool place, the only changes were that several voters down-ranked Candidate B on their ballots, but the outcome remained the same.

Exercise 7.3.1.8

Determine the Borda score for each candidate and the winner of the election using the Borda count method.

Exercise 7.3.1.9

Is there a majority candidate? If so, which candidate?

Exercise 7.3.1.10

Does this Borda method election violate the majority criterion? Justify your answer.

Exercise 7.3.1.11

Is there a Condorcet candidate? If so, which candidate?

Exercise 7.3.1.12

Does this Borda method election violate the Condorcet criterion? Justify your answer.

Exercise 7.3.1.13

If Candidate C is removed from the ballot, which candidate wins by the Borda count method?

Exercise 7.3.1.14

Does this Borda count method election violate IIA? Justify your answer.

Use the table below for the following exercises.

Votes	7	9	12	15	5
Candidate A	4	4	1	1	4
Candidate B	1	1	2	2	3
Candidate C	3	2	3	4	1
Candidate D	2	3	4	3	2

Exercise 7.3.1.15

Determine Borda score for each candidate and the winner of the election using the Borda count method.

Exercise 7.3.1.16

Is there a majority candidate? If so, which candidate?

Exercise 7.3.1.17

Does this Borda method election violate the majority criterion? Justify your answer.

Exercise 7.3.1.18

Is there a Condorcet candidate? If so, which candidate?

Exercise 7.3.1.19

Does the Borda method election violate the Condorcet criterion? Justify your answer.

Exercise 7.3.1.20

Can an election that fails the majority criterion satisfy the Condorcet criterion? Why or why not?

For the following exercises, use the table below.

Number of Ballots	10	7	5	5	4
Candidate A	1	3	3	3	4
Candidate B	3	2	1	4	1
Candidate C	2	4	2	1	2
Candidate D	4	1	4	2	3

Exercise 7.3.1.21

Determine the Borda score for each candidate and the winner of the election using the Borda count method.

Exercise 7.3.1.22

Is there a majority candidate?

Exercise 7.3.1.23

Does the election violate the majority criterion? Justify your answer.

Exercise 7.3.1.24

Determine the winner by pairwise comparison.

Exercise 7.3.1.25

Is there a Condorcet candidate?

Exercise 7.3.1.26

Does the Borda election violate the Condorcet criterion? Justify your answer.

Exercise 7.3.1.27

Determine the winner by the ranked-choice method.

Exercise 7.3.1.28

Does the ranked-choice election violate the majority criterion? Justify your answer.

Exercise 7.3.1.29

Does the ranked-choice election violate the Condorcet criterion? Justify your answer.

Exercise 7.3.1.30

Can an election that fails the Condorcet criterion satisfy the majority criterion? Why or why not?

Use the table below for the following exercises.

Number of Ballots	49	48	3
Candidate A	1	3	3
Candidate B	2	1	2
Candidate C	3	2	1

Exercise 7.3.1.31

Determine the winner of the election using the plurality method.

Exercise 7.3.1.32

Determine the winner by pairwise comparison.

Exercise 7.3.1.33

Is there a Condorcet candidate?

Exercise 7.3.1.34

Does this plurality election violate the Condorcet criterion? Justify your answer.

Exercise 7.3.1.35

If Candidate C is removed from the ballot, which candidate wins by plurality?

Exercise 7.3.1.36

Does this plurality violate the IIA? Explain your reasoning.

Use the sample ballot summary below for the following exercises.

Number of Ballots	16	20	24	8
Candidate A	1	3	2	1
Candidate B	3	1	4	4
Candidate C	4	2	1	2
Candidate D	2	4	3	3

Exercise 7.3.1.37

Determine the winner of the election using the ranked-choice method.

Exercise 7.3.1.38

Determine the winner by pairwise comparison.

Exercise 7.3.1.39

Is there a Condorcet candidate?

Exercise 7.3.1.40

Does this ranked-choice election violate the Condorcet criterion? Justify your answer.

Exercise 7.3.1.41

If the four voters in the last column rank Candidate C ahead of A, which candidate wins by the ranked-choice method?

Exercise 7.3.1.42

Does this ranked-choice election violate the monotonicity criterion? Explain your reasoning.

Use the sample ballot summary below for the following exercises.

Number of Ballots	15	12	9	3
Candidate A	1	3	3	2
Candidate B	2	2	1	1
Candidate C	3	1	2	3

Exercise 7.3.1.43

Determine the winner of the election using the ranked-choice method.

Exercise 7.3.1.44

How could it be demonstrated that this ranked-choice election violates IIA?

Exercise 7.3.1.45

Determine the winner of the election by the Borda method.

Exercise 7.3.1.46

Does this Borda method election violate the IIA? Why or why not?

Exercise 7.3.1.47

Does this Borda method election violate the monotonicity criterion? Why or why not?

Use the pairwise comparison matrix in the given figure for the following exercises.

Runner \ Opponent	L	M	N	Points
L wins	--	LM 3	LN 2	$1\frac{1}{2}$
M wins	ML 1	--	MN 3	1
N wins	NL 2	NM 1	--	$\frac{1}{2}$

Exercise 7.3.1.48

Which candidate wins the pairwise election?

Exercise 7.3.1.49

Determine the winner by pairwise comparison if N were removed from the ballot.

Exercise 7.3.1.50

Determine the winner by pairwise comparison if M were removed from the ballot.

Exercise 7.3.1.51

Does this pairwise election satisfy the IIA?

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7.4: Standard Divisors, Standard Quotas, and the Apportionment Problem



Figure 7.4.1: Every person at a party gets their fair slice of the cake. (credit: “apple spice cake” by Mark Bonica/Flickr, CC BY 2.0)

Learning Objectives

1. Analyze the apportionment problem and applications to representation.
2. Evaluate applications of standard divisors.
3. Evaluate applications of standard quotas.

The Apportionment Problem

In the new democracy of Imaginaria, there are four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Each state will have representatives in the Imaginarian Legislature. You might now have an agreement on which voting method your citizens will use to elect representatives. However, before that process can even begin, you must decided on how many representatives each state will receive. This decision will present its own challenges.

When sharing your birthday cake, it’s only fair that everyone gets the same portion size, right? You were portioning the cake by dividing it up equally and giving everyone a slice. A great thing about cake is that you can slice it any way you want, but how do you apportion, or divide and distribute, items that can't be sliced? Suppose that you have a box of 16 Ring Pops™, gem-shaped lollipops on a plastic ring. You are going to share the box with four other kids. Dividing the 16 Ring Pops™ among the group of five leads to a problem; after each person in the group gets three Ring Pops™, there is still one left! Who gets the last one? The apportionment problem is how to fairly divide, or apportion, available resources that must be distributed to the recipients in whole, not fractional, parts.

The apportionment problem applies to many aspects of life, including the representatives in the Imaginarian legislature. The table below provides a short list of examples of resources that must be apportioned in whole parts, and the recipients of those resources.

Resource	Recipients
Covid-19 Vaccines	Nations around the world
Airport Terminals	Airlines
Faculty Positions at a University	Departments
Public Schools	Communities
U.S. House of Representatives Seats	States
Parliamentary Seats	Political Parties

Fair division of a resource is not necessarily equal division of the resource like when distributing cake slices. When distributing airport terminals amongst airlines, there are many factors to consider such as the size of the airline, the number and types of aircraft they have, and the demand for the service. In most cases, fairness is defined as being proportional; two quantities are proportional if they have the same relative size. In the case of the Covid-19 vaccine, the expectation would be that countries with larger populations get more doses of the vaccine. In the Imaginarian legislature, the expectation may be that the states with larger populations will receive the larger number of representatives. This concept is referred to as a part-to-part ratio.

Suppose that a supermarket has a special on pies, two for $5. The first customer purchases four pies for $10, and the second customer purchases eight pies for $20. The dollar to pie ratio for the first customer is mfrac10dollars4pies=2.5dollars per pie10dollars4pies=2.5dollars per pie and the dollar to pie ratio for the second customer is mfrac20dollars5pies=2.5dollars per pie20dollars5pies=2.5dollars per pie. So, the dollar to pie ratio is constant. Although the customers do not spend the same amount of money, the amount each spent was proportional to the number of pies purchased.

Now suppose that the supermarket changed the special to $5 for the first pie, and $2 for each additional pie. In that case, four pies would cost $\$5+3(\$2)=\$11$, while 8 pies would cost $\$5+7(\$2)=\$19$. The dollar to pie ratios would be mfrac 11dollars4pies=2.75dollars per pie11dollars4pies=2.75dollars per pie and mfrac19dollars8pies=2.375dollars per pie19dollars8pies=2.375dollars per pie, respectively. This special does not result in a constant part to part ratio. The dollars spent are not proportional to the number of pies purchased.

Video



What Is a Ratio?



What Are the Different Types of Ratios?

Example 7.4.1: Ratio of Faculty to Students at a College

The following table provides a comparison of the number of faculty members in each department at a particular college to the student head count in that department and the number of class sections in that department in the Spring semester. Use this information to answer the questions.

Department	Mathematics	English	History	Science
(S) Student Head Count	4800	2376	1536	2880
(C) Class Sections	120	108	48	96
(T) Total Faculty	30	27	12	24
(F) Full-Time Faculty	10	9	4	8
(P) Part-Time Faculty	20	18	8	16

1. Determine the ratios for each department: S to C, C to T, S to T, F to P
2. What are the units of the ratios that you found?
3. Which of these pairs, if any, has a constant part to part ratio? State the ratio.
4. Does it appear that the total number of faculty positions were allocated to each department based on student head count, the number of class sections, or neither? Justify your answer.

Answer

1. Divide the first quantity by the second, for each department, as shown in the table below.
2. Answers are provided in last column of the table.

Department	eMathematics	English	History	Science	Units of Ratios Found
S to C	$\frac{4800}{120} = 40$	$\frac{2376}{108} = 22$	$\frac{1536}{48} = 32$	$\frac{2880}{96} = 30$	Students per class section
C to T	$\frac{120}{30} = 4$	$\frac{108}{27} = 4$	$\frac{48}{12} = 4$	$\frac{96}{24} = 4$	Class sections per faculty member
S to T	$\frac{4800}{30} = 160$	$\frac{2376}{27} = 88$	$\frac{1536}{12} = 128$	$\frac{2880}{24} = 120$	Students per faculty member
F to P	$\frac{10}{20} = \frac{1}{2}$	$\frac{9}{18} = \frac{1}{2}$	$\frac{4}{8} = \frac{1}{2}$	$\frac{8}{16} = \frac{1}{2}$	Full-time faculty member per part-time faculty member

3. The ratio of class sections to faculty members is a constant ratio of four. The ratio of full-time faculty to part-time faculty is a constant ratio of $\frac{1}{2}$.
4. It appears that the faculty positions were allocated based on the number of class sections because there is a constant ratio of four class sections per faculty member.

Your Turn 7.4.1

The SAT is to be administered at a high school. In preparation, pencils have been distributed to each of the classrooms based on the room capacity. Use the information in the following table to answer each question.

Room Number	B	C	D	E
Room Capacity (number of student desks)	24	18	32	22
Number of Pencils	36	27	48	33

Find the part to part ratio of desks to pencils for each room. Represent it as both a reduced fraction and a decimal rounded to the nearest hundredth as needed.

Find the part to part ratio of pencils to desks for each room. Represent it as both a reduced fraction and a decimal rounded to the nearest hundredth as needed.

Have the pencils been distributed proportionally? If so, what is the constant ratio of pencils to desks? Give the units.

There are some useful relationships between quantities that are proportional to each other. When there is a constant ratio between two quantities, the one quantity can be found by multiplying the other by that ratio. Remember the supermarket special on pies, 2 pies for $5? The ratio of dollars to pies is mfracdollars2pies=2.5dollars per pie5 dollars2pies=2.5dollars per pie and the ratio of pies to dollars is $\frac{2 \text{ pies}}{5 \text{ dollars}} = 0.4 \text{ pies per}$dollar2pies5dollars=0.4pies per dollar. These two values are reciprocals of each other, $\frac{1}{2.5} = 0.4$ and $\frac{1}{0.4} = 2.5$. This means that multiplying by one has the same effect as dividing by the other. This also means that knowing either constant ratio allows us to calculate the price given the number of pies. To find the cost of 20 pies, multiply by the ratio of dollars to pies or divide by the ratio of pies to dollars.

- $20 \text{ pies} \times 2.50 \text{ dollars per pie} = 50 \text{ dollars}$50dollars
- $20 \text{ pies} \div 0.4 \text{ pies per dollar} = 50 \text{ dollars}$50dollars

These patterns are true in general.

FORMULA

Let A be a particular item and B another such that there is a constant ratio of A to B

- r- ratio of B' s to $A's = \frac{1}{\text{ratio of } A' \text{ s to } B's}$ and ratio of A' 's to $B's = \frac{1}{\text{ratio of } B's \text{ to } A's}$
- units of $A =$ (units of B) $\times$ (ratio of A 's to $B's$) $= \frac{\text{units of } B}{\text{ratio of } B' \text{ sto } A' \text{ s}}$
- units of $B =$ (units of A) $\times$ (ratio of B' 's to A' s) $= \frac{\text{units of } A}{\text{ratio of } A' \text{ 's to } B' \text{ s}}$

Example 7.4.2: Ratio of Faculty to Students at a College

Refer to the information given in Example 7.4.1 .

1. If there are 32 class sections each semester in the Fine Art department, and the same ratio is used to determine the number of faculty members, how many faculty members would you expect to see in the Fine Art department?
2. If the Health Sciences department has 6 full-time faculty members, how many part-time faculty members are in the department?

Answer

1. Multiply the number of class sections by the ratio of faculty members per class section to find the number of faculty. Since there are 4 faculty members per class, the number of faculty members in 32 classes should be $32.4 = 128$ faculty members.
2. Multiply the number of full-time faculty by the ratio of part-time to full-time to find the number of part-time. Since ratio of full-time faculty to part-time faculty at the college is $\frac{1}{2}$ or 1 full-time per 2 part time, the ratio of part-time to full-time is $\frac{2}{1} = 2$ part-time to 1 full-time; so the number of part-time faculty in a department with 6 full-time faculty should be $6.4 = 24$ part-time faculty.

Your Turn 7.4.2

Refer again to the table providing information on classroom capacities and pencil distribution.

Room Number	B	C	D	E
Room Capacity (number of student desks)	24	18	32	22
Number of Pencils	36	27	48	33

1. Determine the number of pencils that would be allocated to a classroom F with 28 desks by multiplying the number of desks by the ratio of pencils to desks.
2. Determine the number of pencils that would be allocated to a classroom F with 28 desks by dividing the number of desks by the ratio of desks to pencils.
3. Assuming pencils continued to be distributed in the same manner, if there were another classroom, G, that received 51 pencils, how many desks would we expect to find in the room?

The apportionment application that will be important to the founders of Imaginaria occurs in representative democracies in which elected persons represent a group. The United Kingdom, France, and India each have a parliament, and the United States has a Congress, just as Imaginaria will have a legislature! The citizens of a country must decide what portion of the representatives each group, such as a state or province or even a political party, will have. A larger portion of representatives means greater influence over policy.

Example 7.4.3: Ratio of U.S. Representatives to State Population

The table below contains a list of the five U.S. states with the greatest number of representatives in the U.S. House of Representatives, along with the population of that state in 2021. Use the information in the table to answer the questions.

Table https://www.britannica.com/topic/Uni...-State-1787120 [representative seats])

State	Representative Seats	State Population
(CA) California	53	39,613,000
(TX) Texas	36	29,730,300
(NY) New York	27	19,300,000
(FL) Florida	27	21,944,600
(PA) Pennsylvania	18	12,804,100

1. What is the ratio of State Population to Representative Seats for each state to the nearest hundred thousand?
2. What is the ratio of Representative Seats to State Population for each state rounded to seven decimal places?
3. What is the ratio of Representative Seats to State Population for each state rounded to six decimal places?
4. Does there appear to be a constant ratio? Justify your answer.

Answer

1. CA 700,000; TX 800,000; NY 700,000; FL 800,000; PA 700,000
2. CA 0.0000013; TX 0.0000012; NY 0.0000014; FL 0.0000012; PA 0.0000014
3. CA 0.000001; TX 0.000001; NY 0.000001; FL 0.000001; PA 0.000001
4. The ratio of State Population to Representative Seats seems to be either 700,000 or 800,000. There does appear to be a constant ratio of about 0.000001 of Representative Seats to State Population if we round off to the sixth decimal place.

Your Turn 7.4.3

Table 7.4.2 contains a list of the five U.S. states ranked sixth through tenth in the number of representatives in the U.S. House of Representatives, along with the population of that state in 2021. Use the information in the table to answer the questions.

Table https://www.britannica.com/topic/Uni...-State-1787120 [representative seats])

State	Representative Seats	State Population
(IL) Illinois	18	12,804,100
(OH) Ohio	16	11,714,600
(MI) Michigan	14	9,992,430
(GA) Georgia	14	10,830,000
(NC) North Carolina	13	10,701,000

1. What is the ratio of State Population to Representative Seats for each state to the nearest hundred thousand?
2. What is the ratio of Representative Seats to State Population for each state rounded to seven decimal places?
3. What is the ratio of Representative Seats to State Population for each state rounded to six decimal places?
4. Does there appear to be a constant ratio? Are the results similar to the top five states?

Video



Math Antics – Rounding

You might be wondering why the ratio doesn't appear to be quite the same depending on the rounding of the values. We will see that the key to this variation lies in the fractions. Just like the five children sharing 16 Ring Pops™, there are going to be leftovers and there are many methods for deciding what to do with those leftovers.

The Standard Divisor

There are two houses of congress in the United States: the Senate and the House of Representatives. Each state has two senators, but the number of representatives depends on the population of the state. The number of representative seats in the U.S. House of Representatives is currently set by law to be 435. In order to distribute the seats fairly to each state, the ratio of the population of the U.S. to the number of representative seats must be calculated. The ratio of the total population to the house size is called the standard divisor, and it is the number of members of the total population represented by one seat.

Although apportionment applies to many other scenarios, such as the pencil distribution during the SAT, the terminology of apportionment is based on the House of Representatives scenario. Thus, several government-related terms take on a more general meaning. The **states** are the recipients of the apportioned resource, the **seats** are the units of the resource being apportioned, the **house size** is the total number of seats to be apportioned, the **state population** is the measurement of the state's size, and the **total population** is the sum of the state populations.

FORMULA

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{House Size}}$$

Example 7.4.4: The Standard Divisor of the U.S. House of Representatives 2021

As of this writing, the Census.gov website U.S. Population clock showed a population of 332,693,997. There are 435 seats in the U.S. House of Representatives. Find the standard divisor rounded to the nearest tenth.

Answer

Dividing **330,147,881** people by 435 seats, there are 758,960.6 people per representative.

Your Turn 7.4.4

By the end of the first U.S. Congress in 1791, there were 13 states, 65 representative seats, and approximately 3,929,214 citizens. Find the standard divisor rounded to the nearest tenth.

Whether the standard divisor is less than, equal to, or greater than 1 depends on the ratio of the population to the number of seats.

- The standard divisor will be equal to 1 if the total population is equal to the number of seats. This would mean that each member of the population is allocated their own personal seat.
- The standard divisor will be a number between 0 and 1 when the total population is less than the number of seats. This means that each member of the population is allocated more than one seat.
- The standard divisor will be a number greater than 1when the total population is greater than the number of seats. This means that a certain number of members of the population will share 1 seat.

If the population is five children and the house consists of five pieces of candy, the standard divisor is $\frac{5 \text{ children}}{5 \text{ candies}} = 1$ child per candy meaning each child gets one candy. If the population is five children and ten pieces of candy, the standard divisor is $\frac{5 \text{ children}}{10 \text{ candies}} = 0.5$ child per candy meaning that each child gets more than one candy. If the population is five children and four pieces of candy, the standard divisor is $\frac{5 \text{ children}}{4 \text{ candies}} = 1.25$ child per candy meaning that each child gets less than one candy.

If the seats in the Imaginarian legislature are distributed to the states based on population, then the house size will be less than the population and we should expect the standard divisor to be a number greater than 1.

Example 7.4.5: School Resource Officers in Brevard County, Florida

The public schools in a certain county have been allotted 349 school resource officers to be distributed among 327 public schools attended by approximately 271,500 students.

1. Identify the states, seats, house size, state population, and total population in this apportionment scenario.
2. Describe the ratio the standard divisor represents in this scenario and calculate the standard divisor to the nearest tenth.

Answer

1. The states are the schools in that county. The seats are the school resource officers. The house size is the number of school resource officers, which is 349. The state population is the number of students in a particular school, which was not given. The total population consists of the sum of the school populations, which is 271,500.

2. The standard divisor is the ratio of the total population to the house size, which is the number of students served by each resource officer. Divide 271,500 students $\div 349$ officers $= 777.9$ students per officer.

Your Turn 7.4.5

The Hernandez family and the Higgins family went trick-or-treating together for Halloween last year. They returned with 313 pieces of candy, which they will now apportion to the families. The Hernandez family has three children and the Higgins family has four children.

1. Identify the states, seats, house size, state population, and total population in this apportionment scenario.
2. Describe the ratio the standard divisor represents in this scenario and calculate the standard divisor to four decimal places.

The Standard Quota

Once the standard divisor for the Imaginarian legislature is calculated, the next task is to determine the number of seats that each state should receive, which is referred to as the state's standard quota. Unless all the states have the same population, each state will receive a different number of seats because the quantities will be proportionate to the state populations. To determine those amounts, we will use an idea we learned earlier. Recall that, when the number of units of item A is proportionate to the number of units of item B, we have: unitsofA=units ofBratio ofB's toA'sunits ofA=units ofBratio ofB's toA's

In this case, we are trying to calculate the number of seats a state should be apportioned, the state's standard quota. So A So A would refer to seats allocated to a particular state, while B would refer to the state population. This means that the ratio of B to A is the ratio of the total population to house size, which is the standard divisor. So in apportionment terms, we have the following formula.

FORMULA

State's Standard Quota $= \frac{\text{State Population}}{\text{Standard Divisor}}$ seats

Example ✔ **7.4.4** outlined that the Census.gov website U.S. Population clock showed a population of 330,147,881, there are 435 seats in the U.S. House of Representatives, and the standard divisor was 758,960.6 people per representative. The state of California has a population of approximately 39,613,000. Use these values to determine the standard quota for California to two decimal places.

Answer

California's standard Quota $= \frac{\text{California's population}}{\text{Standard Divisor}} = \frac{39{,}613{,}000}{758{,}960.6} = \mathbf{52.19}$ representatives

Your Turn **7.4.6**

By the end of the first U.S. Congress in 1791, there were 13 states, 65 representative seats, and approximately 3,929,214 citizens. In that year, the state of Delaware had a population of approximately 59,000. people. Use this information and the standard divisor you found in Your Turn 11.27 to find Delaware's standard quota rounded to two decimal places.

✔ Example **7.4.7**: Apportionment of Laptops in a Science Department

The science department of a high school has received a grant for 34 laptops. They plan to apportion them among their six classrooms based on each classroom's student capacity. Use the values in the table below to find the standard quota for each classroom.

Room	Students
A	30
B	25
C	28
D	32
E	24
F	27

Answer

Step 1: Identify the state population, total population, and the house size. The states are the classrooms, and the state populations are listed in the table. The total population is the sum of the state populations, which is 166. The house size is the number of seats, or laptops, to be allocated, which is 34.

Step 2: Calculate the standard divisor by dividing the total population by the house size.

$$\text{Standard Divisor} = \frac{\text{Total of Room Capacities}}{\text{Number of Laptops}} = \frac{166}{34} \approx 4.88$$

students per laptop.

Step 3: Calculate the standard quota by dividing the state population by the standard divisors, as shown in the table below. Room's Standard Quota $= \frac{\text{Room Capacity}}{\text{Standard Divisor}}$

Room	Room Capacity	Room's Standard Quota
A	30	$30 \div 4.88 \approx 6.15$ laptops
B	25	$25 \div 4.88 \approx 5.12$ laptops
C	28	$28 \div 4.88 \approx 5.74$ laptops
D	32	$32 \div 4.88 \approx 6.56$ laptops
E	24	$24 \div 4.88 \approx 4.92$ laptops
F	27	$27 \div 4.88 \approx 5.53$ laptops

Step 4: Find the sum of the standard quotas. $6.15 + 5.12 + 5.74 + 6.56 + 4.91 + 5.53 = 34.01$. This is only slightly off from the number of laptops -34 - which can be caused by rounding off in previous steps. This is a good indication that the calculations were correct. If you find that the value of the sum of the standard quotas is significantly different from the house size (number of seats), it is possible that the standard divisor was calculated using too few decimal places. Calculate the standard divisor and standard quotas again but round off to a greater number of decimal places.

Your Turn 7.4.7

This year the Hernandez family and the Higgins family were joined by the Ho family for Halloween trick-or-treating. The Hernandez family has three children, the Higgins family has four children, and the Ho family has two children. This time, they collected 527 pieces of candy, which they are going to apportion based on the number of children. Find the standard quota for each family. Round all values to four decimal places. If traditional rounding methods are applied to determine the actual number whole number values of pieces of candy received by each family, do the values sum to 527?

Check Your Understanding

In the following questions, assume there is a constant ratio between units of A and units of B. Two students are having a discussion. Determine who is correct: Student 1, Student 2, both, or neither.

1. Student 1 says that the number of units of A is the product of the number of units of B times the ratio of A to B.

Student 2 says that the number of units of A is the quotient of the number of units of B divided by the ratio of A to B.

2. Student 1 says that the number of units of A is the quotient of the number of units of B divided by the ratio of B to A.

Student 2 says that the number of units of B is the quotient of the number of units of A divided by the ratio of A to B.

3.Suppose there are 110 of item A and 55 of item B.

Student 1 says the ratio of A to B is $\frac{1}{2}$ Student 2 says the ratio of B to A is $\frac{1}{0.5}$.

4. Suppose the constant ratio of A to B is 0.75 . Student 1 says the ratio of A to B is $\frac{4}{3}$ Student 2 says the ratio of B to A is $\frac{3}{4}$.

5. Student 1 says that Standard Divisor $= \frac{\text{House Size}}{\text{Total Population}}$.

Student 2 says that Standard Divisor $= \frac{\text{Sum of the State Populations}}{\text{Total Number of Seats Available}}$.

6. Student 1 says that State's Standard Divisor $= \frac{\text{Total Population}}{\text{State's Standard Quota}}$.

Student 2 says that State's Standard Quota $= \frac{\text{State Population}}{\text{Standard Divisor}}$.

7. Student 1 says that State's Standard Quota $=$ State Population $\div \frac{\text{Total Population}}{\text{House Size}}$.

Student 2 says that State's Standard Quota $= \frac{\text{State Population}}{\text{Standard Divisor}}$

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7.4.1: Exercises

For the following exercises, identify the states, the seats, and the state population (the basis for the apportionment) in the given scenarios.

Exercise 7.4.1.1

A parent has 25 pieces of candy to split among their four children. They will earn the candy based on how many minutes of chores they children did this week.

Exercise 7.4.1.2

The board of trustees of a college has recently approved the installation of 70 new emergency blue lights in five parking lots. The number of lights in each lot will be proportionate to the size of the parking lot, which is to be measured in acres.

Exercise 7.4.1.3

The reading coach at an elementary school has 52 prizes to distribute to students as a reward for time spent reading.

(The Coronavirus Crisis, by Pien Huang, Shots Health News From NPR, npr.org, November 24, 2020)

For the following exercises, use the given information to find the standard divisor to the nearest hundredth. Include the units.

Exercise 7.4.1.6

The total population is 2,235 automobiles, and the number of seats is 14 warehouses.

Exercise 7.4.1.7

The total population is 135 hospitals, and the number of seats is 200 respirators.

For the following exercises, use the given information to find the standard quota. Include the units.

Exercise 7.4.1.8

The state population is eight residents in a unit, and the standard divisor is 1.75 residents per parking space.

Exercise 7.4.1.9

The state population is 52 ICU patients each week, and the number of seats is 6.5 patients per respirator.

Exercise 7.4.1.10

The total population is 145 basketball players, the number of seats is 62 trophies, and the state population is 14 basketball players on Team Tigers.

Exercise 7.4.1.11

The total population is 12 giraffes, the number of seats is nine water troughs, and the state population is three giraffes in Enclosure C.

For the following exercises, use the table below, which shows student head count, class section, and total faculty in each of four college departments.

Department	(M) Math	(E) English	(H) History	(S) Science	(O) College Overall
(T) Student Head Count	4800	2376	1536	2880	87118
(C) Class Sections	120	108	48	96	3712
(F) Faculty Members	30	27	12	24	928

Exercise 7.4.1.12

Determine the F to T ratios for each department rounded to four decimal places as needed. What are the units?

Exercise 7.4.1.13

Determine the C to F ratios for each department rounded to four decimal places as needed. What are the units?

Exercise 7.4.1.14

What is the F to T ratio for the college overall? Include units. How does it compare to the F to T ratios for individual departments?

Exercise 7.4.1.15

What is the overall C to F ratio? Include units. How does it compare to the C to F ratios for individual departments?

Exercise 7.4.1.16

Does there appear to be a constant F to T ratio? If so, what is the ratio? If not, what implications does this have about the different departments?

Exercise 7.4.1.17

Does there appear to be a constant C to F ratio? If so, what is the ratio? If not, what implications does this have about the different departments?

Exercise 7.4.1.18

If the departments are the states, the students are the population, and the faculty members are the seats, use the College Overall column to determine the standard divisor for the apportionment of the faculty rounded to two decimal places as needed. Include the units.

Exercise 7.4.1.19

If the departments are the states, the classes are the population, and the faculty members are the seats, use the Overall College column to determine the standard divisor rounded to two decimal places as needed. Include the units.

Exercise 7.4.1.20

Use the standard divisor from question 18 to find the standard quota for each department rounded to two decimal places as needed. What are the units?

Exercise 7.4.1.21

Use the standard divisor from question 19 to find the standard quota for each department rounded to two decimal places as needed.

For the following exercises, use this information: Wakanda, the domain of the Black Panther, King T'Challa has six fortress cities. In Wakandan, the word "birnin" means "fortress city." King T'Challa has found 111 Vibranium artifacts that must be distributed among the fortress cities of Wakanda. He has decided to apportion the artifacts based on the number of residents of each birnin. The table below displays the populations of major Wakandan cities.

Fortress Cities	Birnin Djata (D)	Birnin T'Chaka (T)	Birnin Zana (Z)	Birnin S'Yan (S)	Birnin Bashenga (B)	Birnin Azzaria (A)	Total Population
Residents	26,000	57,000	27,000	18,000	64,000	45,000	237,000

Exercise 7.4.1.22

Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.

Exercise 7.4.1.23

Find the standard divisor for the apportionment of the Vibranium artifacts. Round to the nearest tenth as needed. Include the units.

Exercise 7.4.1.24

Find each birnin's standard quota for the apportionment of the Vibranium artifacts. Round to the nearest hundredth as needed. What are the units?

Exercise 7.4.1.25

Find the sum of the standard quotas. Is it reasonably close to the number of artifacts available for distribution?

For the following exercises, suppose that 6.4 million doses of COVID-19 vaccine are to be distributed among U.S. states. The vaccines will either be distributed based on the total state population or based on the number of people over 65 years old, as shown in the table below.

State	State Population	State Population Age 65+	Percentage of State Population 65+
(CA) California	39,613,000	5,669,000	14.3%
(TX) Texas	29,730,300	3,602,000	12.6%
(NY) New York	19,300,000	3,214,000	16.4%
(FL) Florida	21,944,600	4,358,000	20.5%
(PA) Pennsylvania	12,804,100	2,336,000	18.2%
(US) United States	330,151,000	52,345,000	15.8%

Exercise 7.4.1.26

Find the standard divisor for the apportionment of the vaccine doses by population using the estimate for the total U.S. population. Round to the nearest tenth as needed. Include the units.

Exercise 7.4.1.27

Find each state's standard quota for the apportionment of the vaccine doses. Round to the nearest tenth as needed. What are the units?

Exercise 7.4.1.28

Find the standard divisor for the apportionment of the vaccine doses by population age 65 and older using the estimate for the total U.S. population of people aged 65 and older. Round to the nearest tenth as needed. Include the units.

Exercise 7.4.1.29

Find each state's standard quota for the apportionment of the vaccine doses by total state population. Round to the nearest tenth as needed. What are the units?

Exercise 7.4.1.30

Compare the standard quota for each state based on the entire state population to the standard quota for each state based on the portion of the population age 65 and older. Which states would receive more doses of vaccine if the apportionment were based on the population of people age 65 and older?

Exercise 7.4.1.31

Approximately 15.8 percent of U.S. residents are age 65 and older.

a. Which of the five states listed have a percentage of residents age 65 and older greater than 15.8 percent?
b. Which of the five states listed have a percentage of residents age 65 and older less than 15.8 percent?
c. Explain the correlation.

For the following exercises, use this information: Children from five families—the Chorro family, the Eswaran family, the Javernick family, the Lahde family, and the Stolly family—joined a town-wide Easter egg hunt. When they returned with their baskets, they had 827 eggs! They decided to share their eggs among the families based on the number of children in each family, as shown in the table below.

Family	Number of Children
(C) Chorro	3
(E) Eswaran	2
(J) Javernick	4
(L) Lahde	1
(S) Stolly	5

Exercise 7.4.1.32

Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.

Exercise 7.4.1.33

Find the standard divisor for the apportionment of the Easter eggs. Round to five decimal places as needed. Include the units.

Exercise 7.4.1.34

Find each family's standard quota for the apportionment of the Easter eggs. Round to the nearest hundredth as needed. What are the units?

Exercise 7.4.1.35

Find the sum of the standard quotas from exercise 34. Is the sum reasonably close to the number of Easter eggs available for distribution?

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7.5: Apportionment Methods

Learning Objectives

1. Describe and interpret the apportionment problem.
2. Apply Hamilton's Method.
3. Describe and interpret the quota rule.
4. Apply Jefferson's Method.
5. Apply Adams's Method.
6. Apply Webster's Method.
7. Compare and contrast apportionment methods.
8. Identify and contrast flaws in various apportionment methods.

A Closer Look at the Apportionment Problem

In Standard Divisors, Standard Quotas, and the Apportionment Problem we calculated the standard divisor and the standard quotas in various apportionment scenarios. The results of those calculations routinely led to fractions and decimals of units. However, the seats in the House of Representatives, laptops in a classroom, or a variety of other resources, are indivisible, meaning they cannot be divided up into fractional parts. This leaves a decision to be made. For example, if the standard quota for the number of laptops to be distributed to a classroom is 12.44 units, how do we deal with the fractional part of 0.44? It is unclear if the classroom should receive 12 units, 13 units, or some other value. Let's try traditional rounding to the nearest whole number value.



Figure 7.5.1: Schoolchildren depend on apportionment of resources like laptops among schools and classroom. (credit: "Richmond Public Schools" by Virginia Department of Education/Flickr, CC BY 2.0)

Example 7.5.1: Installing Emergency Lights

The board of trustees of a college has recently approved the installation of 70 new emergency blue lights in three parking lots. The number of lights in each lot will be proportionate to the size of the parking lot, which isto be measured in acres. The total number of acres is 34; so the standard divisor is $\frac{34}{70} \approx 0.4857$. The standard quota for each lot is listed in the table below. Use this information to answer each question.

Lot	Acres	Lot's Standard Quota
A	15	$15 \div 0.4857 \approx 30.88$ emergency blue lights
B	9	$9 \div 0.4857 \approx 18.53$ emergency blue lights
C	10	$10 \div 0.4857 = 20.59$ emergency blue lights

1. Use traditional rounding to determine the number of lights assigned to each lot.
2. Find the sum of the values from part 1.

3. Does the sum found in part 2 equal the number of lights available?

Answer

1. If traditional rounding is used, there will be 31, 19, and 21 lights distributed to each lot, respectively.
2. The total of these values is 71.
3. No, the total from part 2 is one more than the number of lights available. In other words, one of the parking lots must get 1 fewer light than apportioned.

Your Turn 7.5.1

The science department of a high school has received a grant for 34 laptops. They plan to apportion them among their six classrooms based on each classroom's student capacity. Use the standard quotas in the table below to answer each question.

Room	Room Capacity	Room's Standard Quota
A	30	$30 \div 4.88 \approx 6.15$ laptops
B	25	$25 \div 4.88 \approx 5.12$ laptops
C	28	$28 \div 4.88 \approx 5.74$ laptops
D	32	$32 \div 4.88 \approx 6.56$ laptops
E	24	$24 \div 4.88 \approx 4.92$ laptops
F	27	$27 \div 4.88 \approx 5.53$ laptops

Use traditional rounding to determine the number of laptops assigned to each classroom.

Find the sum of the values from part 1.

Does the sum found in part 2 equal the number of laptops available?

Example 7.5.1 demonstrates that we cannot successfully apportion indivisible resources by rounding off each standard quota using traditional rounding. This leaves us with a problem. What is a fair way to distribute the fractional parts of the standard quotas? We will refer to this as the **apportionment problem**. Several methods for making this decision will be discussed.

In 2015, the U.S. Air Force had a fleet of approximately 281 A-10C Thunderbolt II aircraft. Suppose that the Air Force administration wanted to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. Use the information in the table below to answer each question.

Base Pilots (A) Alpha 13 (B) Bravo 12 (C) Charlie 5 (D) Delta 16 (E) Echo 7 (F) Foxtrot 9

1. Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.
2. Find the standard divisor for the apportionment of the aircraft. Round to four decimal places as needed. Include the units.
3. Find each Air Force base's standard quota for the apportionment of the aircraft. Round to the nearest hundredth as needed. What are the units?
4. How does this example demonstrate the apportionment problem? Will traditional rounding solve the problem?

Answer

1. The states are the bases, the seats are the aircraft, and the state populations are the pilots at a given base.
2. Standard Divisor $= \frac{\text{Total Population}}{\text{House Size}} = \frac{13+12+5+16+7+9}{27} = \frac{62}{27} \approx 2.2963$ pilots per aircraft.
3. $\text{A}\frac{13}{2.2963} \approx 5.66,\ \text{B}\frac{12}{2.2963} \approx 5.23, \text{C}\frac{5}{2.2963} \approx 2.18, \text{D}\frac{16}{2.2963} \approx 6.97, \text{E}\frac{7}{2.2963} \approx 3.05,\ \text{F}\ \frac{9}{2.2963} \approx 3.92$. The units are aircraft.
4. This example demonstrates the apportionment problem because it is not possible to send a fractional number of aircraft to an Air Force base. On the other hand, if we use traditional rounding methods to get whole numbers, the results are $5 + 5$

$+2+7+3+4=26$ aircraft will be apportioned, which is one less than the number of aircraft that were supposed to be apportioned.

Your Turn 7.5.2

The reading coach at an elementary school has 13 gift cards to distribute to their three students as a reward for time spent reading. When they calculated the standard quota for each student based on the number of minutes they student had read, the results were: 4.49 gift cards, 4.03 gift cards, and 4.48 gift cards. How does this demonstrate the apportionment problem?

Hamilton's Method of Apportionment

One of the problems encountered when standard quotas are transformed into whole numbers using traditional rounding is that it is possible for the sum of the values to be greater than the number of seats available. A reasonable way to avoid this is to always round down, even when the first decimal place is five or greater. For example, a standard quota of 12.33 and a standard quota of 12.99 would both round down to 12. This is called the lower quota.

Example 7.5.3: Lower Quota for Apportionment of Aircraft

The Air Force administration wants to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. The standard quotas for each base are listed in the table below. Use this information to answer the questions.

Base	Standard Quota
(A) Alpha	$\frac{13}{2.2963} \approx 5.66$ aircraft
(B) Bravo	$\frac{12}{2.2963} \approx 5.23$ aircraft
(C) Charlie	$\frac{5}{2.2963} \approx 2.18$ aircraft
(D) Delta	$\frac{16}{2.2963} \approx 6.97$ aircraft
(E) Echo	$\frac{7}{2.2963} \approx 3.05$ aircraft
(F) Foxtrot	$\frac{9}{2.2963} \approx 3.92$ aircraft

1. Give the lower quota for each Air Force base.
2. Find the sum of the lower quotas. By how much does this sum fall short of the actual number of aircraft?

Answer

1. Round down. The lower quota for each Air Force base is 5, 5, 2, 6, 3, 3, respectively.
2. The sum is 24. This is 3 fewer than the actual number of aircraft.

Your Turn 7.5.3

The apportionment of 70 new emergency blue lights in three parking lots is based on acreage. The standard quota for each lot is listed in the table below. Use this information to answer each question.

Lot	Acres	Lot's Standard Quota
A	15	$15 \div 0.4857 \approx 30.88$ emergency blue lights
B	9	$9 \div 0.4857 \approx 18.53$ emergency blue lights
C	10	$10 \div 0.4857 = 20.59$ emergency blue lights

Give the lower quota for each parking lot.

Find the sum of the lower quotas.

By how much does this sum fall short of the actual number of emergency lights?

If the standard quotas are all rounded down, their sum will always be less than or equal to the house size. Then, it would only remain to find a fair way to distribute any remaining seats. Alexander Hamilton, who was a general in the American Revolution, author of the Federalist Papers, and the first secretary of the treasury, took this approach to apportionment.

Steps for Hamilton's Method of Apportionment

There are five steps we follow when applying Hamilton's Method of apportionment:

1. Find the standard divisor.
2. Find each state's standard quota.
3. Give each state the state's lower quota (with each state receiving at least 1 seat).
4. Give each remaining seat one at a time to the states with the largest fractional parts of their standard quotas until no seats remain.
5. Check the solution by confirming that the sum of the modified quotas equals the house size.

Video



Hamilton Method of Apportionment

Example 7.5.4: Hawaiian School Districts

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under 19 years old, This data is provided in the table below. Using the Hamilton method, calculate how many schools would be funded in each state.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under age 19	46,310	224,230	20	16,560	38,450	325,570

Answer

Step 1: Calculate the standard divisor. Divide the total population, 325,570, by the house size, 616 seats. The standard divisor is 528.52.

Step 2: Find each state's standard quota:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616

Step 3: Find each state's lower quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Lower Quota	87	424	1	31	72	615

Step 4: Compare the sum of the states' lower quotas, 615, to the house size, 616. One seat remains to be apportioned and must be given to the state with the largest fractional part: Maui with 0.75. So, the final Hamilton quotas are as follows: Hawaii 87, Honolulu 424, Kalawao 1, Kauai 31, and Maui 73.

Step 5: Find the total to confirm the sum of the quotas equals the house size, 616. Then $87 + 424 + 1 + 31 + 73 = 616$. The apportionment is complete.

Your Turn 7.5.4

In the country of Imaginaria, there will be four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Suppose there will be 35 seats in the legislature of Imaginaria. Use Hamilton's method of apportionment to determine the number of seats in each state based on the populations in the following table.

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

Tech Check: Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Hamilton apportionment calculator.

This can be a useful tool to confirm your results!

The Quota Rule

A characteristic of an apportionment that is considered favorable is when the final quota values all either result from rounding down or rounding up from the standard quotas. The value that results from rounding down is called the lower quota, and the value that results from rounding up is called the upper quota.

As we explore more methods of apportionment, we will consider whether they satisfy the quota rule. If a scenario exists in which a particular apportionment allocates a value greater than the upper quota or less than the lower quota, then that apportionment violates the quota rule and the apportionment method that was used violates the quota rule.

Example 7.5.5: Which Apportionment Method Satisfies the Quota Rule?

Several apportionment methods have been used to allocate 125 seats to ten states and the results are shown in the table below. Determine which apportionments do not satisfy the quota rule and justify your answer.

	State A	State B	State C	State D	State E	State F	State G
Standard Quota	41.26	16.00	5.77	2.64	7.82	10.47	0.21
Lower Quota	41	16	5	2	7	10	0
Upper Quota	42	17	6	3	8	11	1
Method X	43	16	5	2	7	10	1
Method Y	41	16	6	2	8	10	1
Method Z	42	16	7	3	7	9	1

Answer

Look for states such that the number of seats allocated differs from the lower or upper quota. Method X violates the quota rule because State A receives 43 seats instead of 41 or 42. Method Z violates the quota rule because State C receives 7 seats instead of 5 or 6 and State F receives 9 instead of 10 or 11.

Your Turn 7.5.5

Apportionment Method V has been used to allocate 125 seats to ten states as shown in the table below. Determine whether the apportionment satisfies the quota rule and justify your answer.

	State A	State B	State C	State D	State E	State F	State G
Standard Quota	41.26	10.70	16.00	13.11	17.00	5.77	2.64
Lower Quota	41	10	16	13	17	5	2
Upper Quota	42	11	17	14	18	6	3
Method V	42	11	17	13	18	4	2

It is possible for an apportionment method to satisfy the quota rule in some scenarios but violate it in others. However, because the Hamilton method always begins with the lower quota and either adds one to it or keeps it the same, the final Hamilton quota will always consist of values that are either lower quota values or upper quota values. When an apportionment method has this characteristic, it is said to satisfy the quota rule. So, we can say:

The Hamilton method of apportionment satisfies the quota rule.

Although the Hamilton method of apportionment satisfies the quota rule, it can result in some unexpected outcomes, which has caused it to pass in and out of favor of the U.S. government over the years. There are several apportionment methods that have been popular alternatives, such as Jefferson's method of apportionment that the founders of Imaginaria should consider.

Jefferson's Method of Apportionment

Another approach to dealing with the fractional parts of the standard quotas is to modify the standard divisor so that the total of the resulting modified lower quotas is the necessary number of seats. This is the approach used by Jefferson.

In Jefferson's method, the change to the standard divisor is made so that the total of the modified lower quotas equals the house size. The change in the standard divisor to get the modified divisor is relatively small. There is not a formula for this. The modified divisor is found by "guess and check." It is important to remember that *increasing* the divisor *decreases* the quotas, but *decreasing* the divisor *increases* the quotas. So, if you need a larger quota, try reducing the divisor, and if you need a smaller quota, try increasing the divisor.

Example 7.5.6: Modifying a Standard Divisor

Suppose the population of a state is 50 and the standard divisor is 12.5.

1. Find the state's standard quota.
2. Increase the standard divisor by 2 units and use the modified divisor to determine the modified quota for the state.
3. Decrease the modified divisor from part 2 by 1.5 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from part 2 and the value of the modified divisor from part 3 and use it to determine the modified quota for the state.
5. Which modified quota was the largest, the modified quota from part 2, from part 3, or from part 4? Explain why.

Answer

1. The state's standard quota is $\frac{50}{12.5} = 4$.
2. The modified divisor is 14.5 . The modified quota is $\frac{50}{14.5} \approx 3.45$.
3. The modified divisor is 13 . The modified quota is $\frac{50}{13} \approx 3.85$.
4. One value between 13 and 14.5 is 13.5 . With a modified divisor of 13.5 , the modified quota is $\frac{50}{13.5} \approx 3.70$.
5. The modified quota from part 3 was the largest because the divisor was the smallest of the three. Dividing the same number by a smaller value gives a larger result.

Your Turn 7.5.6

Suppose the population of a state is 12 and the standard divisor is 0.225.

1. Find the state's standard quota.
2. Decrease the standard divisor by 0.200 units and use the modified divisor to determine the modified quota for the state.
3. Increase the modified divisor from part 2 by 0.100 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from part 2 and the value of the modified divisor from part 3 and use it to determine the modified quota for the state.
5. Which modified quota was the smallest, the modified quota from part 2, from part 3, or from part 4? Explain why.

When you use Jefferson's method, you might have to adjust the divisor several times find modified lower quotas that sum to the house size. First, guess what the divisor should be based on the sum of the lower quotas and then increase or decrease it from there based on whether the sum needs to be smaller or larger respectively. If the result still does not produce lower quotas that sum to the house size, adjust again. Keep a record of the values that didn't work to help you narrow your search.

Steps for Jefferson's Method of Apportionment

We take four steps to apply Jefferson's Method of apportionment:

- **Step 1:** Find the standard divisor.
- **Step 2:** Find each state's quota. This will be the standard quota the first time Step 2 is completed and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
- **Step 3:** Find the states' lower quotas (with each state receiving at least one seat), and their sum.
- **Step 4:** If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the lower quotas is less than the number of seats, reduce the standard divisor. If the sum of the lower quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

Example 7.5.7: Hawaiian State Representative Districts

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of five counties based on the residents under the age of 19. With the data in the table below, apply Jefferson's method to apportion the schools to the counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Answer

Step 1: The process for finding the standard divisor, standard quotas, and lower quotas is the same in the Hamilton and Jefferson methods of apportionment. We walked through the Hamilton Method in Example 11.34, and following these steps resulted in lower quotas as shown in the table below.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616
Lower Quota	87	424	1	31	72	615

Step 2: Compare the sum of the states' lower quotas, 615, to the house size, 616. Since 615 is less than 616, use a modified divisor that is less than the standard divisor of 528.52. Try 526.00.

Step 3: Find each state's modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{526.00} \approx 88.04$	$\frac{224,230}{526.00} \approx 426.29$	$\frac{20}{526.00} \approx 0.04$	$\frac{16,560}{526.00} \approx 31.48$	$\frac{38,450}{526.00} \approx 72.75$	616
Lower Quota	88	426	1	31	72	618

Step 4: The new sum of the lower quotas is 2 units greater than 616. We have overshot the goal. So, increase the divisor to a value between 526.00 and 528.52. Try 527.00.

Step 5: Repeat the process of finding the quotas. Find each state's modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526.00:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{527.00} \approx 87.87$	$\frac{224,230}{527.00} \approx 425.48$	$\frac{20}{527.00} \approx 0.04$	$\frac{16,560}{527.00} \approx 31.42$	$\frac{38,450}{527.00} \approx 72.96$	616
Lower Quota	87	425	1	31	72	616

Step 6: The new sum of the lower quotas equals the house size. The apportionment is complete.

The apportionment is: Hawaii County 87, Honolulu County 425, Kalawao County 1, Kauai 31, and Maui 72 schools.

When using Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

Your Turn 7.5.71

Let's return to the Imaginarian states of Fictionville, Pretendstead, Illusionham, and Mythbury. Suppose that there are going to be 35 seats in the legislature. This time use Jefferson's method of apportionment to determine the number of seats in each state based on the populations in the table below. How many seats would each state receive?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

Notice that, in this apportionment, Mythbury received more than the upper quota. Since this apportionment of representatives to Imaginarian states by Jefferson's method does not satisfy the quota rule, we say that:

Jefferson's method violates the quota rule.

We have discussed two apportionment methods: one that satisfies the quota rule and one that does not. Before you decide which method to use in Imaginaria, there are a couple more options to consider.

Video



Jefferson Apportionment Method

Tech Check: Apportionment Calculators

It is possible to create Excel spreadsheets that complete the calculations necessary to complete a Jefferson Apportionment. In some cases, this work has already been done and posted online. Check out websites such Ms. Hearn Math for a free Jefferson apportionment calculator.

This can be a useful tool to confirm your results!

Adams's Method of Apportionment

Adams's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, Adams used the upper quotas.

To apply Adams's Method of apportionment, there are four steps we follow:

1. Find the standard divisor.
2. Find each state's quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
3. Find the states' upper quotas and their sum.
4. If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the upper quotas is less than the number of seats, reduce the standard divisor. If the sum of the upper quotas is greater than the number of seats, increase the

standard divisor. Return to Step 2 using the modified divisor.

Example 7.5.8: Hawaiian School Districts

As in earlier examples, suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under the age of 19. Use the data in the following table and the Adams method to apportion the schools to the counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Answer

Step 1: The steps of finding the standard divisor and each state's quota are the same in the Jefferson and Adams methods. As in Example 11.37, the standard divisor is 528.52.

Step 2: Find each state's upper quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616
Upper Quota	88	425	1	32	73	619

Step 3: Compare the sum of the states' upper quotas, 619, to the house size, 616. Since 619 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00.

Step 4: Find each state's modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 534:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{534.00} \approx 86.72$	$\frac{224,230}{534.00} \approx 419.91$	$\frac{20}{534.00} \approx 0.04$	$\frac{16,560}{534.00} \approx 31.01$	$\frac{38,450}{534.00} \approx 72.00$	616
Upper Quota	88	420	1	32	72	613

Step 5: The new sum of the upper quotas is 3 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 532.00.

Step 6: Find each state's modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 532.00:

County	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{532.00} \approx 87.05$	$\frac{224,230}{532.00} \approx 421.48$	$\frac{20}{532.00} \approx 0.04$	$\frac{16,560}{532.00} \approx 31.13$	$\frac{38,450}{532.00} \approx 72.27$	616
Upper Quota	88	422	1	32	73	616

Step 7: The new sum of the upper quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 88, Honolulu County 422, Kalawao County 1, Kauai 32, and Maui 73 schools.

When using Adams's method, just as with Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

Your Turn 7.5.8

There are four states in Imaginaria: Fictionville, Pretendstead, Illusionham, and Mythbury. Assume there will be 35 seats in the legislature of Imaginaria. Use Adams's method of apportionment to determine the number of seats in each state based on the populations in the table below. How many seats would each state receive?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

In this apportionment, Mythbury received less than the state's lower quota. So, this apportionment is an example of a scenario in which the Adams's method violates the quota rule.

Adams's method of apportionment violates the quota rule.

So far, only Hamilton's method satisfies the quota rule, but there is one more apportionment method you should consider for Imaginaria.

Video



Adams Method Apportionment Calculator

Tech Check: Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Adams Method apportionment calculator.

This can be a useful tool to confirm your results!

Webster's Method of Apportionment

Webster's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did or the sum of the upper quotas as Adams did, Webster used traditional rounding.

To apply Webster's method of apportionment, there are four steps we take:

1. Find the standard divisor.
2. Find each state's quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
3. Round each state's quota to the nearest whole number and find the sum of these values.
4. If the sum of the rounded quotas equals the number of seats, the apportionment is complete. If the sum of the rounded quotas is less than the number of seats, reduce the divisor. If the sum of the rounded quotas is greater than the number of seats, increase

the divisor. Return to Step 2 using the modified divisor.

When using Webster's method, just as with Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

Example 7.5.9: Hawaiian School Districts

Use the data in the table below to apportion 616 schools to Hawaiian counties. This time, use Webster's method.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Answer

To apply Webster's method of apportionment, there are four steps we take:

Step 1: The processes of finding the standard divisor and standard quota are the same in the Jefferson, Adams, and Webster's methods. As in the previous examples, the standard divisor is 528.52.

Step 2: Find each state's rounded quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616
Rounded Quota	88	424	1	31	73	617

Step 3: Compare the sum of the states' rounded quotas, 617, to the house size, 616. Since 617 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00.

Step 4: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 534:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{534.00} \approx 86.72$	$\frac{224,230}{534.00} \approx 419.91$	$\frac{20}{534.00} \approx 0.04$	$\frac{16,560}{534.00} \approx 31.01$	$\frac{38,450}{534.00} \approx 72.00$	616
Upper Quota	87	420	1	31	72	612

Step 5: The new sum of the rounded quotas is 4 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 530.00.

Step 6: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 530.00:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{530.00} \approx 87.38$	$\frac{224,230}{530.00} \approx 423.08$	$\frac{20}{530.00} \approx 0.04$	$\frac{16,560}{530.00} \approx 31.25$	$\frac{38,450}{530.00} \approx 72.55$	616
Upper Quota	87	423	1	31	73	615

Step 7: The new sum of the rounded quotas is 1 unit less than 616. Larger quotas are needed. So, decrease the divisor to a value between 528.52 and 530.00. Try 529.50.

Step 8: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.50:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{529.50} \approx 87.46$	$\frac{224,230}{529.50} \approx 423.48$	$\frac{20}{529.50} \approx 0.04$	$\frac{16,560}{529.50} \approx 31.27$	$\frac{38,450}{529.50} \approx 72.62$	616
Upper Quota	87	423	1	31	73	615

Step 9: The new sum is still only 1 unit less than 616. Larger quotas are needed, but not much larger. So, decrease the divisor to a value between 528.52 and 529.50. Try 529.30.

Step 10: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.30:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{529.30} \approx 87.49$	$\frac{224,230}{529.30} \approx 423.63$	$\frac{20}{529.30} \approx 0.04$	$\frac{16,560}{529.30} \approx 31.29$	$\frac{38,450}{529.30} \approx 72.64$	616
Upper Quota	87	424	1	31	73	616

Step 11: The new sum of the rounded quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 87, Honolulu County 424, Kalawao County 1, Kauai 31, and Maui 73 schools.

Your Turn 7.5.9

If you use Webster's method to apportion 35 legislative seats to the 4 states of Imaginaria, Fictionville, Pretendstead, Illusionham, and Mythbury, with the populations given in the table below, what is the resulting apportionment?

	Fictionville	Pretendstead	Illusionham	Mythbury	Total
Population	71,000	117,000	211,000	1,194,000	1,593,000

So far, we know that the Hamilton method satisfies the quota rule, while the Jefferson and Adams methods do not. The apportionments in the Example and Your Turn above are both scenarios in which the Webster method satisfies the quota rule. Does it always? We have a little more work to do to find out. However, one thing is clear. Not all apportionment methods have the same results. Before you make such an important decision for Imaginaria, it's important to think about the differences in the apportionments that result from these four methods. How will the differences affect the citizens of Imaginaria?

Tech Check: Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Webster Method apportionment calculator.

This can be a useful tool to confirm your results!

Comparing Apportionment Methods

Recall that the four apportionment methods discussed in this chapter differ in two main ways:

- Whether or not a modified divisor is used
- The type of rounding of the quotas that is used

How might these differences affect Imaginarians? In the next two examples, we will compare the results when different apportionment methods are applied to the same scenario.

Example 7.5.11: Hawaiian School Districts with Different Apportionment Methods

Let's use the results from Example 7.5.5, Example 7.5.8, Example 7.5.9, and Example 7.5.10 to compare the four apportionment methods we have discussed. The following table summarizes the results of the results of the Hamilton, Jefferson, Adams and Webster methods when applied to the apportionment of 616 schools to Hawaiian counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Under 19 years old	46,310	224,230	20	16,560	38,450
Hamilton	87	424	1	31	73
Jefferson	87	425	1	31	72
Adams	88	422	1	32	73
Webster	87	424	1	31	73

1. Do any of the apportionment methods result in the same apportionment? If so, which ones?
2. Which apportionment method would the citizens of the largest county likely favor most and least? Justify your answer.
3. As a group, which apportionment method would the citizens of the other four counties likely favor most and least? Justify your answer.

Answer

1. Yes, the Hamilton and Webster methods result in the same apportionment.
2. The largest county is Honolulu. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method. They would likely favor the Adams method of apportionment least because they received the least number of seats by that method.
3. As a group, the other four counties received 192 seats by either the Hamilton or Webster method, 194 seats by the Adams method, and 191 seats by the Jefferson method. They would likely favor the Adams method the most and favor the Jefferson methods the least.

Your Turn 7.5.11

In Your Turn 7.5.5, Example 7.5.8, Example 7.5.9, and Example 7.5.10 , you apportioned 35 legislative seats among the four states of Imaginaria using the Hamilton, Jefferson, Adams, and Webster methods of apportionment. To understand how the differences in the apportionments might affect Imaginarians, answer these questions.

Which apportionment method would the citizens of the largest state likely favor most and least? Justify your answer.

As a group, which apportionment method would the citizens of the other three states likely favor most and least? Justify your answer.

The Adams method favored the smaller states and the Jefferson method favored the larger states in the previous example, but is this the case in general?

Since the Jefferson method begins with the lower quotas, any adjustment to the quotas will be an increase. As you have seen, this is accomplished by using a modified divisor that is smaller than the standard divisor. The next example compares the impact of a decreasing divisor on the modified quotas of large states to the impact of the same size decrease on small states.

Example 7.5.12: Effect of Decreasing Divisors on Modified Quotas

The following table displays the effect of reducing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states and use the table answer each question.

		Modified Quotas		
State	Population	Divisor: 10,500	Divisor: 10,000	Divisor: 9,500
A	10,000	0.95	1	1.05
B	100,000	9.52	10	10.53
C	1,000,000	95.24	100	105.26

1. When the divisor decreases from 10,500 to 10,000, how many representatives are gained by each state based on the lower quota?
2. When the divisor decreases from 10,000 to 9,500, how many representatives are gained by each state based on the lower quota?
3. Which state gains the most representatives each time the divisor is decreased?

Answer

1. Since a state must have at least one seat, State A begins with 1 seat and still has one seat. State B begins with 9 seats and increases to 10 seats. State C begins with 95 seats and increases to 100 seats. So, State A gains 0, B gains 1, and C gains 5 seats.
2. State A begins with 1 and still has 1. State B begins with 10 and still has 10. State C begins with 100 and increases to 105. So, State A gains 0, State B gains 0, and State C gains 5.
3. State C, the largest state, gains the most representatives each time the divisor is decreased.

This example demonstrates that the Jefferson method is biased toward states with larger populations because the modified divisor is smaller than the standard divisor. On the other hand, the Adams's method, which begins with the upper quotas, must increase the standard divisor in order to reduce the quotas. Once again, the effect on the number of seats is greater for the larger states, but this time they are decreased. This means that the Adams's method favors states with smaller populations.

Your Turn 7.5.12

The following table displays the effect of increasing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states and use the table to answer each question.

		Modified Quotas		
State	Population	Divisor: 11,500	Divisor: 12,000	Divisor: 12,500
A	10,000	0.87	0.83	0.8
B	100,000	8.7	8.33	8
C	1,000,000	86.96	83.33	80

When the divisor increases from 11,500 to 12,000, how many representatives are lost by each state based on the upper quota?

When the divisor increases from 12,000 to 12,500, how many representatives are lost by each state based on the upper quota?

Which state loses the most representatives each time the divisor is increased?

Flaws in Apportionment Methods

As we have seen, different apportionment methods can have the same results in some scenarios but different results in others. Citizens of states which receive fewer seats with a particular apportionment method will view the apportionment method as flawed and argue in favor of a different method. This inevitably creates debates regarding the use of one method over another. Methods that favor larger states are likely to be challenged by smaller states, methods that favor smaller states are likely to be challenged by

larger states, and methods that violate the quota rule are likely to be challenged by states of any size depending on the circumstances.

Suppose that the State of Hawaii House of Representatives had 51 representatives, each with their own district. Imagine that redistricting were underway, and the representative districts were to be apportioned to each of five counties based on population. The following table shows the apportionment that would result from the use of the Jefferson, Adams, and Webster methods of apportionment.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Population	201,500	974,600	100	72,300	167,400
Lower Quota	7	35	0	2	6
Upper Quota	8	36	1	3	7
Jefferson	7	35	1	2	6
Adams	7	34	1	3	6
Webster	7	34	1	3	6

From the table, you can see that Hawaii, Kalawao, and Maui receive the same number of seats regardless of the method used. However, citizens of Honolulu would likely reject the Adams and Webster methods arguing that they violate the quota rule. Similarly, citizens of Kauai would probably reject the Jefferson method based on the argument that it unfairly favors the larger states. This scenario demonstrates that the Adams and Webster methods violate the quota rule, but the Jefferson method also violates the quota rule at times. **The Hamilton method is the only method that satisfies the quota rule in all scenarios.** It also consistently favors neither larger nor smaller states. Unfortunately, it can have some strange and results in certain circumstances, which you will see in the next section.

Who Knew?: Gerrymandering: A Subtle Way to Impact Apportionment

In addition to your choice of voting method and your choice of apportionment method, there is another important decision to make which could potentially have a huge impact on the fairness of elections in Imaginaria—the creation of electoral districts. In example above, we imagined that there were 51 state legislators in Hawaii, each representing their own district. But how did the legislators decide on the boundaries for these districts? Typically, boundaries are drawn so that each district has approximately the same number of residents, but the percentage of residents in each district with a particular political affiliation can swing the power from one group to another. When the districts are drawn to impact the power of a political party, ethnic or racial group, or other group, this is called gerrymandering. For example, districts can be drawn so that a particular group is spread thinly across districts, increasing the likelihood that they will not have strong representation.

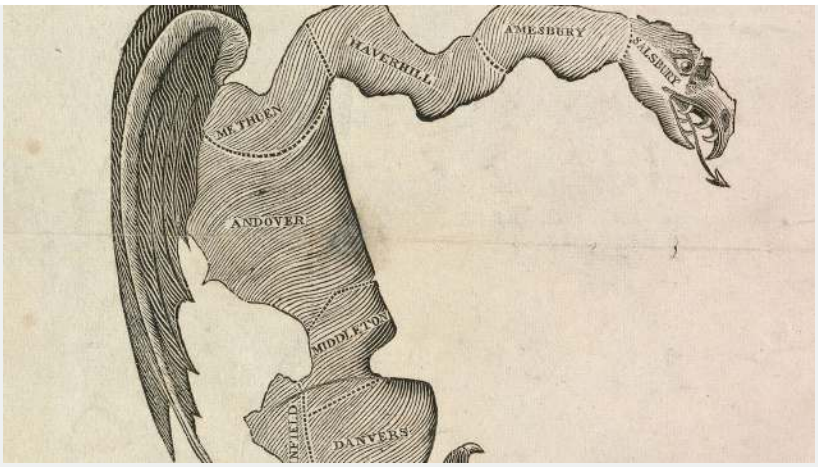


Figure 7.5.2: This cartoon map conveys the idea that the drawing of the map may impact election outcomes. (credit: "The Gerry Mander"/Wikimedia Commons, Public Domain)

"The Gerry-mander" first appeared in this cartoon-map in the Boston Gazette, March 26, 1812, and was soon reproduced in several other Massachusetts newspapers in response to election district changes initiated by governor Eldridge Gerry. Note that while the practice is named after him, Gerry was not the first to employ it.

People in Mathematics: Jonathan Mattingly

Jonathan Mattingly is a mathematician who was featured in a *Nature* article titled "The Mathematicians Who Want to Save Democracy". Mattingly is a mathematician at Duke University in North Carolina and he runs election simulations based on alternate versions of electoral districts in order to analyze the effects of gerrymandering. He has even been asked to testify as an expert witness in court. Mattingly and other mathematicians who are working on the problem will potentially have an impact on the redistricting that will occur as a result of the 2020 census. (Carrie Arnold, "The Mathematicians Who Want to Save Democracy," *Nature 546*, 200–202, 2017.)

Check Your Understanding

1. Which of the four apportionment methods discussed in this section does not use a modified divisor?
2. Which of the four apportionment methods discussed in this section satisfies the quota rule?
3. Which of the four apportionment methods discussed in this section is biased toward states with larger populations?
4. Which of the four apportionment methods discussed in this section is biased toward states with smaller populations?
5. Which of the four apportionment methods discussed in this section begin the apportionment with a state's upper quota and adjust down?
6. Which of the four apportionment methods discussed in this section begin the apportionment with a state's lower quota and adjust up?
7. Which of the four apportionment methods discussed in this section use traditional rounding?
8. Does the change from a standard divisor to a modified divisor tend to change the number of seats for larger or smaller states more?

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7.5.1: Exercises

For the following exercises, use the standard quotas given in the table below.

	State A	State B	State C	State D	State E	State F	Total Seats
Scenario X	17.63	26.62	10.81	16.01	13.69	15.24	100
Scenario Y	12.37	7.59	71.71	6.75	5.76	20.81	125
Scenario Z	3.53	31.56	2.95	5.12	9.84	NA	53

Exercise 7.5.1.1

Round the standard quota for each state in Scenario X using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Exercise 7.5.1.2

Round the standard quota for each state in Scenario Y using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Exercise 7.5.1.3

Round the standard quota for each state in Scenario Z using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Exercise 7.5.1.4

Find the lower quota for each state in Scenario Y. If each state is allocated its lower quota, how many seats remain to be allocated?

Exercise 7.5.1.5

Find the lower quota for each state in Scenario X. If each state is allocated its lower quota, how many seats remain to be allocated?

Exercise 7.5.1.6

Find the lower quota for each state in Scenario Z. If each state is allocated its lower quota, how many seats remain to be allocated?

Exercise 7.5.1.7

Find the upper quota for each state in Scenario X and determine how much the sum of the upper quotas exceeds the house size.

Exercise 7.5.1.8

Find the upper quota for each state in Scenario Y and determine how much the sum of the upper quotas exceeds the house size.

Exercise 7.5.1.9

Find the upper quota for each state in Scenario Z and determine how much the sum of the upper quotas exceeds the house size.

Exercise 7.5.1.10

Determine the Hamilton apportionment for Scenario Y.

Exercise 7.5.1.11

Determine the Hamilton apportionment for Scenario X.

Exercise 7.5.1.12

Determine the Hamilton apportionment for Scenario Z.

For the following exercises, use the information in the table below, which gives standard and final quotas for Methods X, Y, and Z.

	State A	State B	State C	State D	State E
Standard Quota	1.67	3.33	5.00	6.67	8.33
Method X	2	2	5	7	9
Method Y	1	3	5	7	9
Method Z	1	3	5	6	10

Exercise 7.5.1.13

Does the apportionment resulting from method X satisfy the quota rule? Why or why not?

Exercise 7.5.1.14

Does the apportionment resulting from method Z satisfy the quota rule? Why or why not?

Exercise 7.5.1.15

Does the apportionment resulting from method Y satisfy the quota rule? Why or why not?

In the movie *Black Panther*, the hero lives in the fictional country of Wakanda. Imagine that 111 Vibranium artifacts must be distributed among the fortress cities, or birnin, of Wakanda based on the population of each birnin. Use the population and standard quota information in the table below for the following exercises.

	Birnin Djata (D)	Birnin T'Chaka (T)	Birnin Zana (Z)	Birnin S'Yan (S)	Birnin Bashenga (B)	Birnin Azzaria (A)	Total
Residents	26,000	57,000	27,000	18,000	64,000	45,000	237,000
Standard Quota	12.18	26.70	12.65	8.43	29.98	21.08	111

Exercise 7.5.1.16

Modify the standard quota for each state using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Exercise 7.5.1.17

Find the standard lower quota for each state. If each state is allocated its lower quota, how many seats remain to be allocated?

Exercise 7.5.1.18

Find the standard upper quota for each state and determine how much the sum of the upper quotas exceeds the house size.

Exercise 7.5.1.19

Use the Hamilton method to apportion the artifacts.

Exercise 7.5.1.20

Find the modified lower quota for each state using a modified divisor of 2,000. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 7.5.1.21

Find the modified lower quota for each state using a modified divisor of 2,100. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 7.5.1.22

Use the Jefferson method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Exercise 7.5.1.23

Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?

Exercise 7.5.1.24

Find the modified upper quota for each state using a modified divisor of 2,250. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 7.5.1.25

Find the modified upper quota for each state using a modified divisor of 2,150. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 7.5.1.26

Use the Adams method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

? Exercise 7.5.1.27

Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?

? Exercise 7.5.1.28

Which method of apportionment, Jefferson or Adams, is a resident of Birnin T'Chaka likely to prefer? Justify your answer.

? Exercise 7.5.1.29

Use the Webster method to apportion the artifacts. Determine whether it is necessary to modify the divisor. If so,indicate the value of the modified divisor.

? Exercise 7.5.1.30

Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

? Exercise 7.5.1.31

Which of the four methods of apportionment from this section (Hamilton, Jefferson, Adams, or Webster) are the residents of Birnin S'Yan likely to prefer? Justify your answer.

Children from five families—the Chorro family, the Eswaran family, the Javernick family, the Lahde family, and the Stolly family—joined a town Easter egg hunt. When they returned with their baskets, they had 827 eggs! They decided to share their eggs amongst the families based on the number of children in each family. Use the population and standard quota information in the table below for the following exercises.

	(C) Chorro	(E) Eswaran	(J) Javernick	(L) Lahde	(S) Stolly	Total
Children	3	2	4	1	5	15
Standard Quota	155.04	103.36	206.72	103.36	258.40	827

? Exercise 7.5.1.32

Modify the standard quota for each state using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

? Exercise 7.5.1.33

Find the standard lower quota for each state. If each state is allocated its lower quota, how many seats remain to be allocated?

? Exercise 7.5.1.34

Find the standard upper quota for each state, and determine how much the sum of the upper quotas exceeds the house size.

? Exercise 7.5.1.35

Use the Hamilton method to apportion the Easter eggs.

Exercise 7.5.1.36

Find the modified lower quota for each state using a modified divisor of 0.01800. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 7.5.1.37

Find the modified lower quota for each state using a modified divisor of 0.01810. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 7.5.1.38

Use the Jefferson method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Exercise 7.5.1.39

Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?

Exercise 7.5.1.40

Find the modified upper quota for each state using a modified divisor of 0.0182. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 7.5.1.41

Find the modified upper quota for each state using a modified divisor of 0.01816. Is the sum of the modified quotas too high, too low, or equal to the house size?

Exercise 7.5.1.42

Use the Adams method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Exercise 7.5.1.43

Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?

Exercise 7.5.1.44

Use the Webster method to apportion the Easter eggs. Determine whether it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Exercise 7.5.1.45

Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

For the following exercises, use this information: Suppose that the State of Delaware received 2,000 packs of COVID-19 vaccines, with ten doses per pack. These (unopened) packs must be distributed to the three counties based on total population. Use the population information in the table below to determine how many packs of vaccine will be distributed to each county based on the given apportionment method.

	(N) New Castle	(K) Kent	(S) Sussex
Residents	558,753	180,786	234,225

? Exercise 7.5.1.46

Hamilton’s Method

? Exercise 7.5.1.47

Jefferson’s Method

? Exercise 7.5.1.48

Adams's Method

? Exercise 7.5.1.49

Webster’s Method

? Exercise 7.5.1.50

Notice that the apportionments found in questions 46, 47, 48, and 49 all satisfy the quota rule. Does this contradict the statement, “The Jefferson, Adams, and Webster methods of apportionment all violate the quota rule”? Why or why not?

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7.6: Fairness in Apportionment Methods

Learning Objectives

1. Describe and illustrate the Alabama paradox.
2. Describe and illustrate the population paradox.
3. Describe and illustrate the new-states paradox.
4. Identify ways to promote fairness in apportionment methods.

Apportionment Paradoxes

The citizens of Imaginaria will want the apportionment method to be as fair as possible. There are certain characteristics that they would reasonably expect from a fair apportionment.

- If the house size is increased, the state quotas should all increase or remain the same but never decrease.
- If one state's population is growing more rapidly than another state's population, the faster growing state should not lose a seat while a slower growing state maintains or gains a seat.
- If there is a fixed number of seats, adding a new state should not cause an existing state to gain seats while others lose them.

However, apportionment methods are known to contradict these expectations. Before you decide on the right apportionment for Imaginarians, let's explore the apportionment paradox, a situation that occurs when an apportionment method produces results that seem to contradict reasonable expectations of fairness.

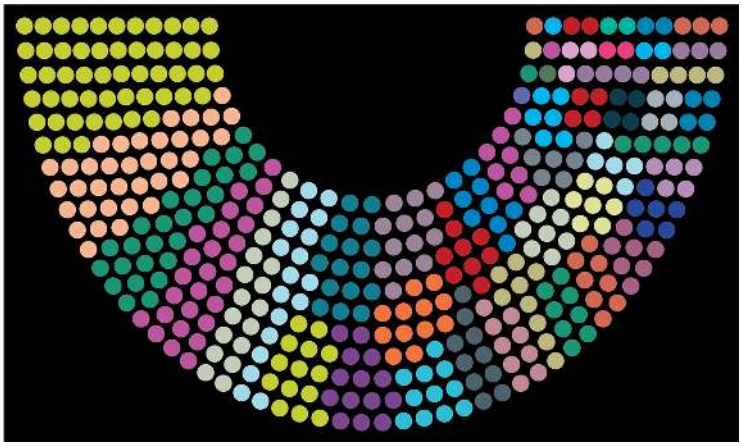


Figure **7.6.1**: In this seating chart for the House of Representatives, each color indicates representatives from a particular state.

There is a lot that the founders of Imaginaria can learn from U.S. history. The constitution of the United States requires that the seats in the House of Representatives be apportioned according to the results of the census that occurs every decade, but the number of seats and the apportionment method is not stipulated. Over the years, several different apportionment methods and house sizes have been used and scrutinized for fairness. This scrutiny has led to the discovery of several of these apportionment paradoxes.

The Alabama Paradox

At the time of the 1880 U.S. Census, the Hamilton method of apportionment had replaced the Jefferson method. The number of seats in the House of Representatives was not fixed. To achieve the fairest apportionment possible, the house sizes were chosen so that the Hamilton and Webster methods would result in the same apportionment. The chief clerk of the Census Bureau calculated the apportionments for house sizes between 275 and 350. There was a surprising result that became known as the Alabama paradox, which is said to occur when an increase in house size reduces a state's quota. Alabama would receive eight seats with a house size of 299, but only receive seven seats if the house size increased to 300. (Michael J. Caulfield (Gannon University), "Apportioning Representatives in the United States Congress - Paradoxes of Apportionment," *Convergence* (November 2010), DOI:10.4169/loci003163)

Example 7.6.1: The 1880 Alabama Quota

The 1880 census recorded the population of Alabama as 1,513,401 and that of the U.S. as 62,979,766.

1. Calculate the standard divisor and standard quota for the State of Alabama based on a house size of 299.
2. Calculate the standard divisor and standard quota for the State of Alabama based on a house size of 300.
3. Did the standard quota increase or decrease when the house size increased?
4. Consider the Hamilton method of apportionment. Explain how Alabama's final quota could be smaller with a larger standard quota.

Answer

1. The standard divisor is $62{,}979{,}766 \div 299 = 210, \text{resulting in } 634.6689$ citizens per seat. The standard quota for Alabama is $1{,}513{,}401 \div 210{,}634.7 = 7.1850$ seats.
2. The standard divisor is $62{,}979{,}766 \div 300 = 209, \text{resulting in } 932.5533$ citizens per seat. The standard quota for Alabama is $1{,}513{,}401 \div 209{,}932.6 = 7.2090$ seats.
3. The standard quota increased.
4. In each case, the state would receive the lower quota of 7 and then be awarded one more seat if the fractional part of the standard quota were high enough relative to the fractional parts of the other states' standard quotas. When the house size was 299, Alabama received one of the remaining seats after the lower quotas were distributed. When the house size was 300, Alabama did not receive one of the remaining seats after the lower quotas were distributed. It must have been the case that either the fractional part 0.2090 ranked lower amongst the other fractional parts of the state quotas than the fractional part 0.1850 did, or there were fewer remaining seats, or both.

After the 1900 census, the Census Bureau again calculated the apportionment based on various house sizes. It was determined that Colorado would receive three seats with a house size of 356, but only two seats with a house size of 357.

Your Turn 7.6.1

The 1900 census recorded the population of Colorado as 539,700 and that of the U.S. as 76,212,168.

a. Calculate the standard divisor and standard quota for the State of Colorado based on a house size of 356.
b. Calculate the standard divisor and standard quota for the State of Colorado based on a house size of 357.
c. Did the standard quota increase or decrease when the house size increased?
d. Consider the Hamilton method of apportionment. Explain how Colorado's final quota could be smaller with a larger standard quota.

Example 7.6.2: Hamilton's Method and the Alabama Paradox

Suppose that States A and B each have a population of 6, while State C has a population of 2.

1. Use the Hamilton method to apportion 10 seats.
2. Use the Hamilton method to apportion 11 seats.
3. Does this example demonstrate the Alabama paradox? If so, how?

Answer

1. **Step 1:** The total population is 14. The standard divisor is $14 \div 10 = 1.4$ individuals per seat.

   Step 2: The states' standard quotas are as follows: A $6 \div 1.4 \approx 4.29$, B $6 \div 1.4 \approx 4.29$, and C $2 \div 1.4 \approx 1.43$.

   Step 3: The states' lower quotas are as follows: A 4, B 4, and C 1.

   Step 4: The sum of the lower quotas is 9, which means there is one seat remaining to be apportioned. State C has the highest fractional part and receives the additional seat.

   Step 5: The final apportionment is as follows: A 4, B 4, and C 2, which sums to 10.

2. **Step 1:** The total population is 14. The standard divisor is $14 \div 11 \approx 1.2727$ individuals per seat.

Step 2: The states' standard quotas are: A $6 \div 1.2727 \approx 4.71$, B $6 \div 1.2727 \approx 4.71$, and C $2 \div 1.2727 \approx 1.57$.

Step 3: The states' lower quotas are: A 4, B 4, and C 1.

Step 4: The sum of the lower quotas is 9, which means there are two seats remaining to be apportioned. A and B have the highest fractional parts and receive the additional seats.

Step 5: The final apportionment is: A 5, B 5, and C 1.

3. Yes, this demonstrates the Alabama paradox because State C receives two seats if the house size is 10, but only one seat if the house size is 11.

Your Turn 7.6.2

Suppose that the founders of Imaginaria decide to have a parliament that apportions seats to four political parties based on the portion of the vote each party has earned. Also, suppose that Party A has 56.7 thousand votes, Party B has 38.5 thousand votes, Party C has 4.2 thousand votes, and Party D has 0.6 thousand votes.

a. Use the Hamilton method to apportion 323 seats.
b. Use the Hamilton method to apportion 324 seats.
c. Does this example demonstrate the Alabama paradox? If so, how?

The Population Paradox

It is important for the founders of Imaginaria to keep in mind that the populations of states change as time passes. Some populations grow and some shrink. Some populations increase by a large amount while others increase by a small amount. These changes may necessitate a reapportionment of seats, or the recalculation of state quotas due to a change in population. It would be reasonable for Imaginarians to expect that the state with a population that has grown more than others will gain a seat before the other states. Once again, this is not always the case with the Hamilton method of apportionment. The population growth rate of a state is the ratio of the change in the population to the original size of the population, often expressed as a percentage. This value is positive if the population is increasing and negative if the population is decreasing. The population paradox occurs when a state with an increasing population loses a seat while a state with a decreasing population either retains or gains seats. More generally, the population paradox occurs when a state with a higher population growth rate loses seats while a state with a lower population growth rate retains or gains seats.

Notice that the population paradox definitions has two parts. If either part applies, then the population paradox has occurred. The first part of the definition only applies when one state has a decreasing population. The second part of the definition applies in all situations, whether there is a state with a decreasing population or not. It will be easier to identify situations that involve a decreasing population. The other situations requires the calculation of a growth rate. The reason that we don't have to calculate a growth rate when one state has a decreasing population and the other has an increasing population is that increasing population has a positive growth rate which is always greater than the negative growth rate of a decreasing population.

Checkpoint

A state must lose a seat in order for the population paradox to apply. It is not enough for a state with a lower growth rate to gain a seat while a state with a higher growth rate retains the same number of seats.

Example 7.6.3: Apportionment of Respirators to Hospitals

Suppose that 18 respirators are to be apportioned to three hospitals based on their capacities. The Hamilton method is used to allocate the respirators in 2020, then to reallocate based on new capacities in 2021. The results are shown in the table below. How does this demonstrate the population paradox?

Hospital	Capacity in 2020	Respirators in 2020	Change in Capacity	Growth Rate = $\frac{\text{Change in Capacity}}{\text{Capacity in 2020}}$Change in Capacity Capacity in 2020	Capacity in 2021	Respirators in 2021
A	825	9	57	$\frac{57}{825} \approx 0.0691 = 6.91\%$	882	9
B	613	7	13	$\frac{13}{613} \approx 0.0212 = 2.12\%$	626	6
C	239	2	3	$\frac{3}{239} = 0.0126 = 1.26\%$	242	3

Answer

Hospital B lost a respirator while hospital C gained one, even though hospital B had a higher growth rate than hospital C.

Your Turn 7.6.3

Suppose that 18 respirators are to be apportioned to three hospitals based on their capacities. The respirators are allocated based on the Hamilton method in 2020, then reallocated based on new capacities in 2021. The results are shown in the table below. How does this demonstrate the population paradox?

Hospital	Capacity in 2020	Respirators in 2020	Change in Capacity	Growth Rate \ (\frac\)	Capacity in 2021	Respirators in 2021
A	237	5	6	$6 \div 237 = 0.0253 = 2.53\%$	243	6
B	889	21	69	$69 \div 889 \approx 0.0776 = 7.76\%$	958	21
C	674	16	18	$18 \div 674 \approx 0.0267 = 2.67\%$	692	15

FORMULA

The growth rate of a population can be calculated by subtracting the previous population size from the current population size, and then dividing the difference by the previous population size.

$$\text{population growth rate} = \text{mfrac}$$

sizeprevious population sizepopulation growth rate=current population size−previous population sizeprevious population size

Checkpoint

Make sure to calculate the subtraction before the division. If you are entering the values in a calculator, it is helpful to put parentheses around the subtracted terms.

Example 7.6.4: The Congress of Costaguana

The country of Costaguana has three states: Azuera with a population of 894,000; Punta Mala with a population of 696,000; and Esmeralda with a population of 215,000. There are 38 seats in the Congress of Costaguana. The apportionment of the seats is determined by Hamilton's method to be: 19 for Azuera, 15 for Punta Mala, and 4 for Esmerelda. A census reveals that the

population has grown and the seats must be reapportioned. If Azuera now has 953,000 residents, Punta Mala now has 706,000 residents, and Esmerelda now has 218,000 residents, how many seats will each state receive upon reapportionment? How is this an example of the population paradox?

Answer

The Hamilton reapportionment is: 19 for Azuera, 14 for Punta Mala, and 5 for Esmerelda. This is an example of the population paradox because Punta Mala lost a seat to Esmerelda, even though Punta Mala's population grew by 1.44 percent while Esmerelda's only grew by 1.40 percent.

Your Turn 7.6.4

The country of Elbonia has three states: Mudston with a population of 866,000; WallaWalla with a population of 626,000; and Dilberta with a population of 256,000. There are 38 seats in the Congress of Elbonia. The apportionment of the seats is determined by Hamilton's method to be: 19 for Mudston, 14 for WallaWalla, and 5 for Dilberta. A census reveals that the population has grown and the seats must be reapportioned. If Mudston now has 921,000 residents, WallaWalla now has 640,000 residents, and Dilberta now has 260,000 residents, how many seats will each state receive upon reapportionment? How is this an example of the population paradox?

The New-States Paradox

As a founder of Imaginaria, you might also consider the possibility that Imaginaria could annex nearby lands and increase the number of states. This occurred several times in the United States such as when Oklahoma became a state in 1907. The House size was increased from 386 to 391 to accommodate Oklahoma's quota of five seats. When the seats were reapportioned using Hamilton's method, New York lost a seat to Maine despite the fact that their populations had not changed. This is an example of the new-state paradox, which occurs when the addition of a new state is accompanied by an increase in seats to maintain the standard ratio of population to seats, but one of the existing states loses a seat in the resulting reapportionment.

Example 7.6.51: New State of Oscuridad

The country of San Lorenzo has grown from two states to three. The house size of the congress has been increased by eight and the seats have been reapportioned to accommodate the new state of Oscuridad. The constitution mandates the use of the Hamilton method of apportionment. Use this information and the following table to answer the questions.

State	Population (in hundreds)	Original Apportionment	Reapportionment
Clara	7,100	39	40
Velasco	9,080	51	50
Oscuridad	1,500	Not Applicable	8

1. What was the original house size?
2. What is the new house size?
3. How is this reapportionment an example of the new-states paradox?

Answer

1. $39 + 51 = 90$
2. $40 + 50 + 8 = 98$
3. The original state of Velasco lost a seat to the original state of Clara when the new state of Oscuridad was added.

Your Turn 7.6.5

The country of Narnia has grown from two states to three. The house size of the congress has been increased by five and the seats have been reapportioned to accommodate the new state of Chippingford. The constitution mandates the use of the Hamilton method of apportionment. Use this information and the following table to answer the questions.

State	Population (in hundreds)	Original Apportionment	Reapportionment
Beruna	7600	39	40
Beaversdam	9720	51	50
Chippingford	1000	Not Applicable	5

a. What was the original house size?
b. What is the new house size?
c. How is this reapportionment an example of the new-states paradox?

Example 7.6.6: The Growing Country of Gulliversia

The country of Gulliversia has two states: Lilliput with a population of 700,000 and Brobdingnag with a population of 937,000. The constitution of Gulliversia requires that the 90 congressional seats be apportioned by Hamilton's method. Lilliput has received 38 seats while Brobdingnag has received 52 seats. Recently, the island of Houyhnhnmsland with a population of 170,000 has joined the union, becoming a state of Gulliversia. When Houyhnhnmsland is included, nine additional seats must be apportioned to maintain the same ratio of seats to citizens. Use Hamilton's method to reapportion the 99 seats to the three states. How is the resulting apportionment an example of the new-states paradox?

Answer

The reapportionment gives 39 seats to Lilliput, 51 seats to Brobdingnag, and 9 seats to Houyhnhnmsland. This is an example of the new-states paradox because the original state of Brobdingnag lost a seat to the original state of Lilliput when the new state was added to the union.

Your Turn 7.6.7

The country of Neverland has two states: Neverwood with a population of 760,000 and Mermaids Lagoon with a population of 943,000. The constitution of Neverland requires that the 84 congressional seats be apportioned by Hamilton's method. Neverwood has received 37 seats while Mermaids Lagoon has received 47 seats. Recently, the island of Marooners Rock with a population of 190,000 has joined the union, becoming a state of Neverland. When Marooners Rock is included, nine additional seats must be apportioned to maintain the same ratio of seats to citizens. Use Hamilton's method to reapportion the 93 seats to the three states. How is the resulting apportionment an example of the new-states paradox?

When a new state is added, it is necessary to determine the amount that the house size must be increased to retain the original ratio of population to seats, in other words to keep the original standard divisor. To calculate the new house size, divide the new population by the original standard divisor, and round to the nearest whole number.

FORMULA

New House Size = mfracPopulationOriginal Standard Divisorrounded to the nearest whole number.New House Size=New PopulationOriginal Standard Divisorrounded to the nearest whole number.

Example 7.6.8: Oklahoma Joins the Union

Oklahoma was admitted as the 46th state on November 16, 1907. Before Oklahoma joined the union, the U.S. population was approximately 75,030,000 and the House of Representatives had 386 seats. The new state had a population of approximately 970,000. Use this information to estimate the original standard divisor to the nearest hundred, the new population, the new house size, and the number of seats Oklahoma should receive.

Answer

Step 1: Original Standard Divisor $\approx \frac{75{,}030{,}000}{386} \approx 194{,}400$Original Standard Divisor≈75,030,000386≈194,400

Step 2: New Population $\approx 75,030,000 + 970,000 = 76,000,000$

Step 3: New House Size $\approx \frac{76{,}000{,}000}{194{,}400} \approx 391$

Step 4: There are $391 - 386 = 5$ new seats to be apportioned to Oklahoma.

Your Turn 7.6.8

New Mexico was admitted as the 47th state on January 6, 1912. Before New Mexico joined the union, the U.S. population was approximately 76,000,000 and the House of Representatives had 391 seats. The new state had a population of approximately 300,000. Use this information to estimate the original standard divisor to the nearest hundred, the new population, the new house size, and the number of seats New Mexico should receive.

The Search for the Perfect Apportionment Method

The ideal apportionment method would simultaneously satisfy the following four fairness criteria.

- Satisfy the quota rule
- Avoid the Alabama paradox
- Avoid the population paradox
- Avoid the new-states paradox

We have seen that the Hamilton method allows the Alabama paradox, the population paradox, and the new-states paradox in some apportionment scenarios. Let's explore the results of the other methods of apportionment we have discussed in some of the same scenarios.

Example 7.6.9: Orange Grove and the New-States Paradox

The incorporated town of Orange Grove consists of two subdivisions: The Oaks with 1,254 residents and The Villages with 10,746 residents. A council with 100 members supervises the municipality's operations. The council votes to annex an unincorporated subdivision called The Lakes with a population of 630. They plan to increase the size of the council to maintain the ratio of seats to residents such that the new council will have 100 seats plus the number of seats given to The Lakes. Use each of the following apportionment methods and indicated number of additional seats to find the original and new apportionment and determine whether the new-state paradox occurs.

1. Jefferson's method with five additional seats.
2. Adams's method with six additional seats.
3. Webster's method with five additional seats

Answer

1. Using a modified divisor of 119, the original apportionment would have been: The Oaks 10 and The Villages 90. Using a modified divisor of 119, the new apportionment would be: The Oaks 10, The Villages 90, and The Lakes 5. The new-state paradox does not occur.
2. Using a modified divisor of 121, the original apportionment would have been: The Oaks 11 and The Villages 89. Using a modified divisor of 121, the new apportionment would be: The Oaks 11, The Villages 89, and The Lakes 6. The new-state paradox does not occur.

3. Using the standard divisor of 120, the original apportionment would have been: The Oaks 10 and The Villages 90. Using a modified divisor of 119.5, the new apportionment would be: The Oaks 10, The Villages 90, and The Lakes 5. The new-state paradox does not occur.

Your Turn 7.6.9

Suppose that in 2016, States A, B, and C had populations of 13 million, 12 million, and 112 million, respectively. In 2020, State A has grown by 1 million residents, State B has lost 1 million residents, and State C has gained 2 million residents. Compare the apportionments in 2016 to 2020 using each method given below. Which of the four methods violate(s) the population paradox in this scenario?

We have seen in our examples that neither the population paradox nor the new-states paradox occurred when using the Jefferson, Adams, and Webster methods. It turns out that, although all three of these divisor methods violate the quota rule, none of them ever causes the population paradox, new-states paradox, or even the Alabama paradox. On the other hand, the Hamilton method satisfies the quota rule, but will cause the population paradox, the new-states paradox, and the Alabama paradox in some scenarios.

In 1983, mathematicians Michel Balinski and Peyton Young proved that no method of apportionment can simultaneously satisfy all four fairness criteria.

There are other apportionment methods that satisfy different subsets of these fairness criteria. For example, the mathematicians, Balinski and Young who proved the Balinski-Young Impossibility Theorem created a method that both satisfies the quota rule and is free of the Alabama paradox. (Balinski, Michel L.; Young, H. Peyton (November 1974). “A New Method for Congressional Apportionment.” *Proceedings of the National Academy of Sciences*. 71 (11): 4602–4606.) However, no method may always follow the quota rule and simultaneously be free of the population paradox. (Balinski, Michel L.; Young, H. Peyton (September 1980). "The Theory of Apportionment" (PDF). Working Papers. International Institute for Applied Systems Analysis. WP-80-131.)

So, as you and your fellow founders of Imaginaria make the important decision about the right apportionment method for Imaginaria, do not look for a perfect apportionment method. Instead, look for an apportionment method that best meets the needs and concerns of Imaginarians.

Check Your Understanding

1. In the following questions, a scenario is given. Determine whether the scenario is an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these.
2. A city purchased five new firetrucks and apportioned them among the existing fire stations. Although your neighborhood fire station has the same proportion of the city’s firetrucks as before the new ones were purchased, it now has one fewer.
3. The school resources officers in a county were reapportioned based on the most recent census. The number of students at Chapel Run Elementary went up while the number of students at Panther Trail Elementary went down. However, Chapel Run now has one fewer resources officer while Panther Trail has one more than it did previously.
4. The standard quota for States A, B, and C are 1.19, 2.73, and 5.71 respectively. State A received 1 seat, State B received 3 seats, and State C received 4 seats.
5. A corporation that owns several hospitals purchased an additional hospital, causing the doctor to patient ratio to decrease.
6. When the city of Cocoa annexed an adjacent unincorporated community, the number of seats on the city council was increased to maintain the standard ratio of citizens to seats, but one existing community of Cocoa still lost a seat on the city council to another existing community of Cocoa when the new community was added.
7. In the following questions, a type of scenario is described. Indicate which paradox could arise in a scenario of this kind.
8. A reapportionment occurs because the populations of the states change, and the house size remains the same.
9. A reapportionment occurs because the house size increases, and the populations of the states remain the same.
10. A reapportionment occurs because an additional state is added to the union, the populations of the original states remain the same, and the house size is increased to correspond to the population of the new state.

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7.6.1: Exercises

In the following exercises, determine whether the scenario violates the Alabama paradox. Justify your answer.

Exercise 7.6.1.1

A company with an office in each of four cities must distribute 145 new Chromebooks to the four offices. It is determined that Office A will receive 42, Office B will receive 17, Office C will receive 35, and Office D will receive 51. At the last minute, it is discovered that there are 146 Chromebooks. When they are reapportioned, Office A receives 42, Office B receives 16, Office C receives 36, and Office D receives 52.

Exercise 7.6.1.2

A county with three towns has 30 garbage trucks to apportion. Attenborough receives 6 trucks, Breckenridge receives 8 trucks, and Cabbotsville receives 16 trucks, in proportion to their populations. When one additional truck is purchased, the reallocation results in 5 trucks for Attenborough, 9 for Breckenridge, and 8 for Cabbotsville though there has been no change in populations.

Exercise 7.6.1.3

A group of 60 dentists work for a company that runs five offices. The dentists have been apportioned to the offices by the number of patients. Office A receives 14 dentists, Office B receives 11, Office C receives 11, Office D receives 12, and Office E receives 12. When a new dentist joins the group, the new apportionment gives the following: A 14, B 11, C 11, D 12, and E 13.

In the following exercises, determine whether the scenario violates the population paradox. Justify your answer.

Exercise 7.6.1.4

A company with locations in three cities plans to give 200 achievement awards that shall be apportioned to the three cities by population. City A with 9,150 employees is allotted 61 awards, City B with 6,040 employees is allotted 40 awards, and City C with 14,810 employees is allotted 99 awards to distribute. Then it is discovered that the number of employees is out of date. The awards are reallocated based on the new populations: City A with 9,180; City B with 6,040; and City C with 14,930. It turns out that the apportionment remains the same.

Exercise 7.6.1.5

A soccer club must apportion soccer balls to the teams among four age brackets based on the number of teams in each bracket. Bracket U8 receives 32 balls, U12 receives 46 balls, U15 receives 29 balls, and U18 receives 25 balls. Mid-season, the balls are reapportioned. U8 has decreased by 10 percent, U12 has increased by 20 percent, U15 remains the same, and U18 has increased by 10 percent. The reapportionment gives 33 balls to U8, 46 balls to U12, 29 balls to U15, and 14 balls to U18.

Exercise 7.6.1.6

READ (Reading Education Assistance Dogs) trains and certifies therapy animals to serve in classrooms and help children develop a love of reading in a low-stress environment. Suppose that 20 therapy teams are apportioned to three schools based on their populations. When the population of School A decreases by 10 percent, the population of School B increases by 10 percent, and the population of School C remains constant, the apportionment remains the same.

In the following exercises, determine whether the scenario violates the new-states paradox. Justify your answer.

Exercise 7.6.1.7

A charity organization has 851 volunteers in Country A and 3449 volunteers in Country B. There are 43 lead organizers that must be apportioned to the two locations. Country A receives 9 while Country B receives 34. When the operations are expanded to Country C with 725 volunteers, 7 new lead organizers are added to the team. When the lead organizers are reapportioned, Country A receives 9, Country B receives 34, and Country C receives 7 lead organizers.

Exercise 7.6.1.8

A country has two states. There are 22 seats in the legislature. State A has 4 seats, and State B has 18 seats. When State C joins the union, the number of representatives is increased by five. Under the new apportionment, State A receives 3 seats, State B receives 19 seats, and State C receives 5 seats.

Exercise 7.6.1.9

In the Garunga Solar System, there are four inhabited planets. Three of the planets are members of the United Association of Garungan Planets (UAGP). The UAGP has 67 seats. Planet Angluertka has 40 seats, planet Bangluertka has 27 seats, and planet Clangluertka has 13 seats. When planet Danggluertka decides to join the UAGP, 14 seats are added to accommodate them proportionately. The new allocation of seats gives 40 seats to Angluertka, 26 seats to Bangluertka, 14 seats to Clangluertka, and 14 seats to Danggluertka.

In the following exercises, use the Hamilton method of apportionment to answer the questions.

Exercise 7.6.1.10

When the number of seats changed from 147 to 148, the standard quotas changed from A 44.24, B 17.35, C 37.12, and D 48.29 to A 45.54, B 17.47, C 37.37, and D 48.62.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.11

When the number of seats changed from 126 to 127, the standard quotas changed from A 9.57, B 29.49, C 33.89, D 28.43, and E 24.61 to A 9.65, B 29.72, C 34.16, D 38.66, and E 24.81.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.12

When the number of seats changed from 25 to 26, the standard quotas changed from A 2.21, B 5.25, C 11.27, and D 6.27 to A 2.30, B 5.46, C 11.72, and D 6.52.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.13

When the number of seats changed from 25 to 26, the standard quotas changed from A 2.43, B 5.42, and C 8.15 to A 2.46, B 5.47, and C 8.07.

a. How did the increase in seats impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.14

The house size is 18. When the population of State A increases by 11.76 percent, State B increases by 16.22 percent, and State C increases by 12.18 percent, the standard quotas change from A 2.46, B 6.00, and C 9.53 to A 2.41, B 6.25, and C 9.34.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.15

The house size is 100. When the population of State A increases by 20 percent, State B increases by 10 percent, State C increases by 30 percent, and the populations of States D, E, and F remain the same. The standard quotas change from A 12.50, B 25.00, C 9.38, D 18.75, E 12.50 and F 21.88 to A 13.91, B 25.51, C 11.30, D 17.39, E 11.59, and F 20.29.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.16

The house size is 17. When the population of State A increases by 12.73 percent, State B increases by 15.63 percent, and State C increases by 12.90 percent, the standard quotas change from A 2.53, B 5.90, and C 8.57 to A 2.51, B 5.99, and C 8.50.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.17

The house size is 38 seats. When the population of State A increases by 6.60 percent, State B increases by 1.44 percent, and State C increases by 1.40 percent, the standard quotas change from A 18.82, B 14.65, and C 4.53 to A 19.59, B 14.29, and C 4.41.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.18

The house size is 24 seats. When the population of State A increases by 28 percent, State B increases by 26 percent, and State C increases by 15 percent, the standard quotas change from A 3.38, B 6.32, and C 14.30 to A 3.63, B 6.67, and C 13.71.

a. How did the change in populations impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.19

The house size was 60. There were three states with standard quotas of A 4.18, B 15.38, and C 40.44. A fourth state was annexed, and the house size was increased to 65. The new standard quotas are A 4.17, B 15.33, C 40.32, and D 5.18.

a. How did the additional state impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.20

The house size was 50. There were three states with standard quotas of A 9.41, B 24.42, and C 16.17. A fourth state was annexed, and the house size was increased to 66. The new standard quotas are A 9.36, B 24.30, C 16.09, and D 16.25.

a. How did the additional state impact the apportionment?

b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.21

The house size was 27. There were three states with standard quotas of A 6.39, B 11.40, and C 9.21. A fourth state was annexed, and the house size was increased to 35. The new standard quotas are A 6.38, B 11.37, C 9.19, and D 8.06.

a. How did the additional state impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

Exercise 7.6.1.22

The house size was 100. There were three states with standard quotas of A 26.09, B 30.43, and C 43.48. A fourth state was annexed, and the house size was increased to 122. The new standard quotas are A 26.14, B 30.50, C 43.57, and D 21.78.

a. How did the additional state impact the apportionment?
b. Is this apportionment an example of a paradox? Justify your answer.

In the following exercises, use the information in the table below.

State	A	B	C	D	E	F	G	H	I	J	K	L	P	Q	R
Population	624	1,219	979	**3,462**	**7,470**	**4,264**	**5,300**	263	809	931	781	676	**150**	**250**	**350**
Original House Size	38			**204**				126					**50**		
Updated House Size	39			**205**				127					**51**		

Exercise 7.6.1.23

Consider States A, B, and C.

a. Determine the apportionment for States A, B, and C with the original house size using the Hamilton method.
b. Determine the apportionment for States A, B, and C with the updated house size using the Hamilton method.
c. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Exercise 7.6.1.24

Consider States D, E, F, and G.

a. Determine the apportionment for States D, E, F, and G with the original house size using the Hamilton method.
b. Determine the apportionment for States D, E, F, and G with the updated house size using the Hamilton method.
c. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Exercise 7.6.1.25

Consider States H, I, J, K, and L.

a. Determine the apportionment for States H, I, J, K, and L with the original house size using the Hamilton method.
b. Determine the apportionment for States H, I, J, K, and L with the updated house size using the Hamilton method.

c. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Exercise 7.6.1.26

Consider States P, Q, and R.

a. Determine the apportionment for States P, Q, and R with the original house size using the Hamilton method.
b. Determine the apportionment for States P, Q, and R with the updated house size using the Hamilton method.
c. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

In the following exercises, use the information in the table below.

State	Original Population	Updated Population	Population Growth Rate	House Size
A	889	958	7.76%	
B	674	692	2.67%	42
C	237	243	2.53%	
D	**12,032**	**14,124**	$d\%$	
E	**10,789**	**9,726**	$e\%$	**135**
F	**995**	**2,304**	$f\ \%$	
G	901	1,156	28.3%	
H	1,683	2,125	26.3%	24
I	3,808	4,369	14.7%	
X	**56**	**63**	$x\%$	
Y	**125**	**141**	$y\%$	**16**
Z	**182**	**213**	$z\%$	
P	6,534	6,534	$p\%$	
Q	7,832	7,810	$q\%$	40
R	13,959	13,992	$r\%$	
S	20,515	21,164	$s\%$	

Exercise 7.6.1.27

Calculate the population growth rates d, e, and f for States D, E, and F. Give answer as a percentage rounded to one decimal place.

Exercise 7.6.1.28

Calculate the population growth rates p, q, r, and s for States P, Q, R, and S. Give answer as a percentage rounded to one decimal place.

Exercise 7.6.1.29

Calculate the population growth rates x, y, and z for States X, Y, and Z. Give answer as a percentage rounded to one decimal place.

Exercise 7.6.1.30

Consider States A, B, and C.

a. Determine the Hamilton apportionment for States A, B, and C with the original population.
b. Determine the Hamilton apportionment for States A, B, and C with the updated population.
c. Does the increase in population of States A, B, and C from the original population to the updated population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Exercise 7.6.1.31

Consider States D, E, and F.

a. Determine the Hamilton apportionment for States D, E, and F with the original population.
b. Determine the Hamilton apportionment for States D, E, and F with the updated population.
c. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Exercise 7.6.1.32

Consider States G, H, and I.

a. Determine the Hamilton apportionment for States G, H, and I with the original population.
b. Determine the Hamilton apportionment for States G, H, and I with the updated population.
c. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Exercise 7.6.1.33

Consider States X, Y, and Z.

a. Determine the Hamilton apportionment for States X, Y, and Z with the original population.
b. Determine the Hamilton apportionment for States X, Y, and Z with the updated population.
c. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Exercise 7.6.1.34

Consider States P, Q, R, and S.

a. Determine the Hamilton apportionment for States P, Q, R, and S with the original population.
b. Determine the Hamilton apportionment for States P, Q, R, and S with the updated population.
c. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

In the following exercises, use the information in the table below.

State	Population	Original House Size	New House Size
A	627		
B	1,287	25	
C	973		**32**
D	**815**		
E	520		
F	1,510	50	
G	1,060		h
H	**950**		
P	1,222	100	
Q	473		
R	**225**		r
K	1,688		
L	7,912	48	
M	**1,448**		m
T	150		
U	250		
V	350	50	w
W	**450**		

? Exercise 7.6.1.35

Consider States A, B, and C.

a. Calculate the standard divisor based on the original house size.
b. Use the Hamilton method to apportion the seats.

? Exercise 7.6.1.36

Consider States E, F, and G.

a. Calculate the standard divisor based on the original house size.
b. Use the Hamilton method to apportion the seats.

? Exercise 7.6.1.37

Consider States P and Q.

a. Calculate the standard divisor based on the original house size.
b. Use the Hamilton method to apportion the seats.

Exercise 7.6.1.38

Consider States K and L.

a. Calculate the standard divisor based on the original house size.
b. Use the Hamilton method to apportion the seats.

Exercise 7.6.1.39

Suppose that States A, B, and C annex State D and increase the house size proportionately.

a. Calculate the standard divisor based on the new house size.
b. Use the Hamilton method to reapportion the seats.
c. Does the new-states paradox occur?

Exercise 7.6.1.40

Suppose that States E, F, and G annex State H and increase the house size proportionately.

a. Determine the new house size, h, that is necessary.
b. Calculate the standard divisor based on the new house size.
c. Use the Hamilton method to reapportion the seats.
d. Does the new-states paradox occur?

Exercise 7.6.1.41

Suppose that States P and Q annex State R and increase the house size proportionately.

a. Determine the new house size, r, that is necessary.
b. Calculate the standard divisor based on the new house size.
c. Use the Hamilton method to reapportion the seats.
d. Does the new-states paradox occur?

Exercise 7.6.1.42

Suppose that States K and L annex State M and increase the house size proportionately.

a. Determine the new house size, m, that is necessary.
b. Calculate the standard divisor based on the new house size.
c. Use the Hamilton method to reapportion the seats.
d. Does the new-states paradox occur?

Exercise 7.6.1.43

Suppose that States T, U, and V annex State W and increase the house size proportionately.

a. Calculate the standard divisor based on the original house size and only States T, U, and V.
b. Use the Hamilton method to apportion the seats to T, U, and V.
c. Determine the new house size when State W is annexed.
d. Calculate the standard divisor based on the new house size.
e. Use the Hamilton method to reapportion the seats.
f. Does the new-states paradox occur? (Refer to part b.)

Exercise 7.6.1.44

Suppose 24 seats are apportioned to States A, B, and C with populations of 16, 15, and 125 respectively. Then the populations of States A, B, and C change to 17, 15, and 126 respectively.

a. Demonstrate that the population paradox occurs when the Hamilton method is used.
b. Determine whether the population paradox occurs when the Webster method is used. Justify your answer.

Exercise 7.6.1.45

Suppose that 10 seats are apportioned to States A, B, and C with populations 6, 6, and 2 respectively. Then the number of seats is increased to 11. Demonstrate that the Alabama paradox occurs when the Hamilton method is used.

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CHAPTER OVERVIEW

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8.1: Introduction

Where do we find math around us? Math can be found in areas that are expected and sometimes in areas that are surprising. There are many ways that mathematical concepts, such as those in this text, are infused in the world around us. In this chapter, we will explore a sampling of five distinct areas from everyday life where math's impact plays a meaningful role.



Figure **8.1.1** Math can be found in many different areas and subjects. (credit: modification of work "Chemistry/Physics Library" by University Libraries/Flickr, CC BY 2.0)

Math's impact on art can be found in numerical relationships that are known to create or enhance beauty. The Fibonacci numbers are one mathematical example that can be found in nature such as in petal count of a rose. On a different note, a mathematical exploration can aid in making a convincing argument on how we can positively impact our environment. Whether looking at the choices of a single individual or the larger impact offered from a collaborative effort, there are measurable responses to positively address climate change. Turning to medicine, which has been a topic of global importance in recent years, we will explore how math is used to determine drug dosage rates and test the validity of a vaccine. Switching back to items of aesthetic nature, we will examine some foundational components of music which, like art, brings beauty and joy to our lives. Finally, we will explore some ways that math is used in sports to predict future performance and analyze tournaments styles.

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8.2: Math and Art

Learning Objectives

1. Identify and describe the golden ratio.
2. Identify and describe the Fibonacci sequence and its application to nature.
3. Apply the golden ratio and the Fibonacci sequence relationship.
4. Identify and compute golden rectangles.

> *Art is the expression or application of human creative skill and imagination, typically in a visual form such as painting or sculpture, producing works to be appreciated primarily for their beauty or emotional power. -Oxford Dictionary*

Art, like other disciplines, is an area that combines talent and experience with education. While not everyone considers themself skilled at creating art, there are mathematical relationships commonly found in artistic masterpieces that drive what is considered attractive to the eye. Nature is full of examples of these mathematical relationships.



Figure 8.2.1: Sunflower seeds appear in a pattern that involves the Fibonacci sequence. (credit: "Sunflower Surprise" by frankieleon/Flickr, CC BY 2.0)

Enroll in a cake decorating class and, when you learn how to create flowers out of icing, you will likely be directed as to the number of petals to use. Depending on the desired size of a rose flower, the recommendation for the number of petals to use is commonly 5, 8, or 13 petals. If learning to draw portraits, you may be surprised to learn that eyes are approximately halfway between the top of a person's head and their chin. Studying architecture, we find examples of buildings that contain golden rectangles and ratios that add to the beautifying of the design. The Parthenon (Figure 8.2.2), which was built around 400 BC, as well as modern-day structures such the Washington Monument are two examples containing these relationships. These seemingly unrelated examples and many more highlight mathematical relationships that we associate with beauty in artistic form.



Figure 8.2.2: The Parthenon in Greece demonstrates the golden ratio. (credit: "Parthénon" by Julien Maury/Flickr, Public Domain Mark 1.0)

Golden Ratio

The golden ratio, also known as the golden proportion, is a ratio aspect that can be found in beauty from nature to human anatomy as well as in golden rectangles that are commonly found in building structures. The golden ratio is expressed in nature from plants to creatures such as the starfish, honeybees, seashells, and more. It is commonly noted by the Greek letter ϕ (pronounced “fee”). $\phi = \text{mfrac}52\phi=1+52$, which has a decimal value approximately equal to 1.618.

Consider Figure 8.2.2: Note how the building is balanced in dimension and has a natural shape. The overall structure does not appear as if it is too wide or too tall in comparison to the other dimensions.

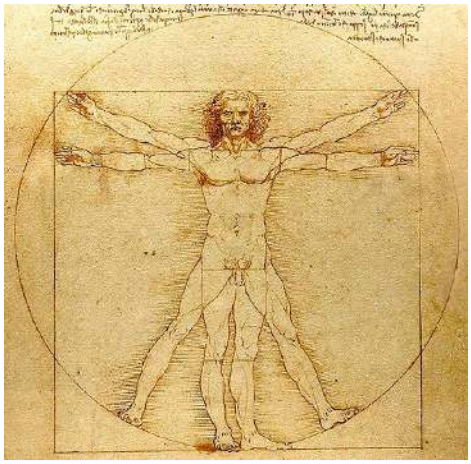


Figure 8.2.3: *Vitruvian Man* by Leonardo da Vinci (credit: "Vitruvian Man" by Leonardo da Vinci/Wikimedia Commons, Public Domain)

The golden ratio has been used by artists through the years and can be found in art dating back to 3000 BC. Leonardo da Vinci is considered one of the artists who mastered the mathematics of the golden ratio, which is prevalent in his artwork such as *Virtuvian Man* (Figure 8.2.4). This famous masterpiece highlights the golden ratio in the proportions of an ideal body shape.

The golden ratio is approximated in several physical measurements of the human body and parts exhibiting the golden ratio are simply called golden. The ratio of a person’s height to the length from their belly button to the floor is ϕ or approximately 1.618. The bones in our fingers (excluding the thumb), are golden as they form a ratio that approximates ϕ. The human face also includes several ratios and those faces that are considered attractive commonly exhibit golden ratios.

Example 8.2.1: Using Golden Ratio and a Person’s Height

If a person’s height is 5 ft 6 in, what is the approximate length from their belly button to the floor rounded to the nearest inch, assuming the ratio is golden?

Answer

Step 1: Convert the height to inches

$$\textbf{5 ft 6 in} = \textbf{66 in}$$

Step 2: Calculate the length from the belly button to the floor, L.

$$66/L = 1.618$$
$$L = 40.8 \text{ in}$$

The length from the person’s belly button to the floor would be approximately 41 in.

Your Turn 8.2.1

If a person’s height is 6 ft 2 in, what is the approximate length from their belly button to the floor rounded to the nearest inch if the ratio is golden?

Fibonacci Sequence and Application to Nature



Figure **8.2.4**: Rose petals appear in a Fibonacci spiral. (credit: “rilke4” by monchoohcnom/Flickr, Public Domain Mark 1.0)

The Fibonacci sequence can be found occurring naturally in a wide array of elements in our environment from the number of petals on a rose flower to the spirals on a pine cone to the spines on a head of lettuce and more. The Fibonacci sequence can be found in artistic renderings of nature to develop aesthetically pleasing and realistic artistic creations such as in sculptures, paintings, landscape, building design, and more. It is the sequence of numbers beginning with 1, 1, and each subsequent term is the sum of the previous two terms in the sequence (1, 1, 2, 3, 5, 8, 13, …).

The petal counts on some flowers are represented in the Fibonacci sequence. A daisy is sometimes associated with plucking petals to answer the question “They love me, they love me not.” Interestingly, a daisy found growing wild typically contains 13, 21, or 34 petals and it is noted that these numbers are part of the Fibonacci sequence. The number of petals aligns with the spirals in the flower family.

Example 8.2.2: Applying the Fibonacci Sequence to Rose Petals

Suppose you were creating a rose out of icing, assuming a Fibonacci sequence in the petals, how many petals would be in the row following a row containing 13 petals?

Answer

The number of petals on a rose is often modeled with the numbers in the Fibonacci sequence, which is 1, 1, 2, 3, 5, 8, 13,…, where the next number in the sequence is the sum of $8 + 13 = 21$. There would be 21 petals on the next row of the icing rose.

Your Turn 8.2.2

If a circular row on a pinecone contains 21 scales and models the Fibonacci sequence, approximately how many scales would be found on the next circular row?

Golden Ratio and the Fibonacci Sequence Relationship

Mathematicians for years have explored patterns and applications to the world around us and continue to do so today. One such pattern can be found in ratios of two adjacent terms of the Fibonacci sequence.

Recall that the Fibonacci sequence = 1, 1, 3, 5, 8, 13,… with 5 and 8 being one example of adjacent terms. When computing the ratio of the larger number to the preceding number such as 8/5 or 13/8, it is fascinating to find the golden ratio emerge. As larger numbers from the Fibonacci sequence are utilized in the ratio, the value more closely approaches ϕ, the golden ratio.

Example 8.2.3: Finding Golden Ratio in Adjacent Fibonacci Terms

The 24th Fibonacci number is 46,368 and the 25th is 75,025. Show that the ratio of the 25th and 24th Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.

Answer

$$\frac{75,025}{46,368} = 1.618$$

The ratio of the 25th and 24th term is approximately equal to the value of ϕ rounded to the nearest thousandth, 1.618.

Your Turn 8.2.3

The 23rd Fibonacci number is 28,657 and the 24th is 46,368. Show that the ratio of the 24th and 23rd Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.



Figure 8.2.5: The pyramids of Giza in Egypt (credit: "Giza Pyramids" by Vincent Brown/Flickr, CC BY 2.0)

Golden Rectangles

Turning our attention to man-made elements, the golden ratio can be found in architecture and artwork dating back to the ancient pyramids in Egypt (Figure 13.6) to modern-day buildings such as the UN headquarters. The ancient Greeks used golden rectangles —any rectangles where the ratio of the length to the width is the golden ratio—to create aesthetically pleasing as well as solid structures, with examples of the golden rectangle often being used multiple times in the same building such as the Parthenon, which is shown in Figure 8.2.1. Golden rectangles can be found in twentieth-century buildings as well, such as the Washington Monument.

Looking at another man-made element, artists paintings often contain golden rectangles. Well-known paintings such as Leonardo da Vinci's *The Last Supper* and the *Vitruvian Man* contain multiple golden rectangles as do many of da Vinci's masterpieces.

Whether framing a painting or designing a building, the golden rectangle has been widely utilized by artists and are considered to be the most visually pleasing rectangles.

Example 8.2.4: Finding Golden Rectangle in Frames

A frame has dimensions of 8 in by 6 in. Calculate the ratio of the sides rounded to the nearest thousandth and determine if the size approximates a golden rectangle.

Answer

8/6 = 1.333; A golden rectangle's ratio is approximately 1.618. The frame dimensions are close to a golden rectangle.

Your Turn 8.2.4

A frame has dimensions of 10 in by 8 in. Calculate the ratio of the sides rounded to the nearest thousandth and determine if the size approximates a golden rectangle.

People in Mathematics: M.C. Escher

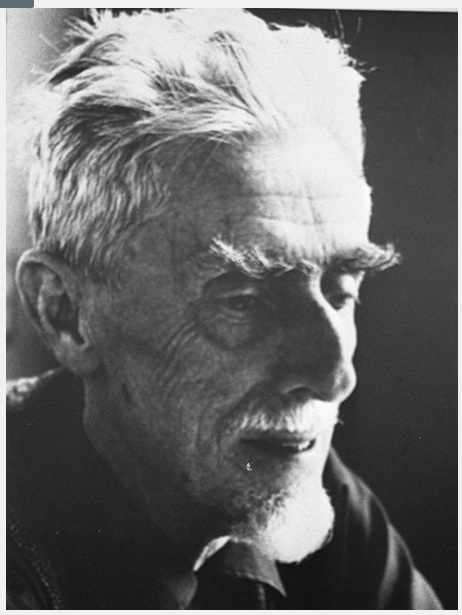


Figure 8.2.6: M.C. Escher (credit: "M.C. Escher" by Hans Peters (ANEFO)/Dutch National Archives, CC0 1.0 Public Domain)

Mauritis Cornelis Escher was a Dutch-born world-famous graphic artist and his work can be found in murals, stamps, wallpaper designs, illustrations in books, and even carpets. Over his lifetime, M.C. Escher created hundreds of lithographs and wood engravings as well as more than 2,000 sketches.

Escher's work is characterized with the infusion of geometric designs that obey most of the mathematical rules. If you study his work closely, you can see where he breaks a mathematical relationship to create famous illusions such as soldiers marching around the top of a square turret where the soldiers appear to be always going uphill but are contained on a single set of stairs in a square. Look closely and the golden ratio as well as golden rectangles abound in Escher's work.

Like many famous people, M.C. Escher did not find success in his early school years. Before finding success, Escher failed his final school exam and quit a short stint in architecture. Finding a graphic arts teacher who recognized Escher's talent, Escher completed art school and enjoyed traveling through Italy, where he found much of his inspiration for his work.

Check Your Understanding

1. What is the value of the golden ratio to the nearest thousandth?
2. What are the first 10 terms of the Fibonacci sequence?
3. What is a golden rectangle?

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8.2.1: Exercises

Exercise 8.2.1.1

A person's height is 5 ft 2 in. What is the approximate length from their belly button to the floor rounded to the nearest inch?

Exercise 8.2.1.2

A person's height is 6 ft 3 in. What is the approximate length from their belly button to the floor rounded to the nearest inch?

Exercise 8.2.1.3

A person's length from their belly button to the floor is 3 ft 11 in. What is the person's approximate height rounded to the nearest inch?

Exercise 8.2.1.4

A person's length from their belly button to the floor is 58 in. What is the person's approximate height rounded to the nearest inch?

Exercise 8.2.1.5

The spikes on a pineapple mirror the Fibonacci sequence. If a row on a pineapple contains five spikes, approximately how many spikes would be found on the next larger row of spikes?

Exercise 8.2.1.6

The leaves on a plant mirror the Fibonacci sequence. If a set of leaves on the plant contains 5 leaves, how many leaves would be found on the previous smaller set of leaves?

Exercise 8.2.1.7

The spines on a head of lettuce mirror the Fibonacci sequence. If a head of lettuce contains 13 spines, approximately how many spines would be found on the next inside layer?

Exercise 8.2.1.8

The seeds on a sunflower mirror the Fibonacci sequence. If a circular layer on the sunflower contains 55 seeds, approximately how many seeds would be found on the next larger circular layer?

Exercise 8.2.1.9

The segments on a palm frond mirror the Fibonacci sequence. If a palm frond contains 89 segments, approximately how many segments would be found on the next larger palm frond?

Exercise 8.2.1.10

The 19th term of the Fibonacci sequence is 4,181 and the 20th term is 6,765. What is the 21st term of the sequence?

Exercise 8.2.1.11

The 23rd term of the Fibonacci sequence is 28,657 and the 24th term is 46,368. What is the 22nd term of the sequence?

Exercise 8.2.1.12

The 18th term of the Fibonacci sequence is 2,584 and the 20th term is 6,765. What is the 19th term of the sequence?

Exercise 8.2.1.13

The 25th term of the Fibonacci sequence is 75,025 and the 20th term is 6,765. What is the 24th term of the sequence?

Exercise 8.2.1.14

The 10th Fibonacci number is 55 and the 11th is 89. Show that the ratio of the 11th and 10th Fibonacci numbers is approximately ϕ. Round your answer to the nearest thousandth.

Exercise 8.2.1.15

The 23rd Fibonacci number is 28,657 and the 24th is 46,368. Show that the ratio of the 24th and 23rd Fibonacci numbers is approximately ϕ. Round your answer to the nearest ten-thousandth.

Exercise 8.2.1.16

The 22nd Fibonacci number is 17,711 and the 21st is 10,946. Show that the ratio of the 22nd and 21st Fibonacci numbers is approximately ϕ. Round your answer to the nearest ten-thousandth.

Exercise 8.2.1.17

The 16th term of the Fibonacci sequence is 987. Use the approximate value of ϕ of 1.618 to estimate the 15th term. Round your answer to the nearest whole number.

Exercise 8.2.1.18

The 26th term of the Fibonacci sequence is 121,393. Use the approximate value of ϕ of 1.618 to estimate the 25th term. Round your answer to the nearest whole number.

Exercise 8.2.1.19

A frame has dimensions of 20 in by 24 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

Exercise 8.2.1.20

A fence has dimensions of 75 in by 45 in. Calculate the ratio of the sides rounded to the nearest tenth and determine if the size approximates a golden rectangle.

Exercise 8.2.1.21

A frame has a length of 50 in. Calculate the width rounded to the nearest inch if the frame is to be a golden rectangle.

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8.3: Math and the Environment



Figure 8.3.1: Solar panels harness the sun's energy to power homes, businesses, and various methods of transportation. (credit: modification of work “Craters of the Moon solar array” by NPS Climate Change Response/Flickr, Public Domain Mark 1.0)

Learning Objectives

1. Compute how conserving water can positively impact climate change.
2. Discuss the history of solar energy.
3. Compute power needs for common devices in a home.
4. Explore advantages of solar power as it applies to home use.

Climate change and emissions management have been debated topics in recent years. However, more and more people are recognizing the impacts that have resulted in temperature changes and are seeking timely and effective action. The World Meteorological Organization shared in a June 2021 publication that “2021 is a make-or-break year for climate action, with the window to prevent the worst impacts of climate change—which include ever more frequent more intense droughts, floods and storms—closing rapidly.” The problem no longer belongs to a few countries or regions but rather is a worldwide concern measured with increasing temperatures leading to decreased glacier coverage and resulting rise in sea levels.

The good news is, there are small steps that each of us can do that collectively can positively impact climate change.

Making a Positive Impact on Climate Change—Water Usage

Our use of water is one element that impacts climate change. Having access to clean, potable water is critical for not only our health but also for the health of our ecosystem. About 1 out of 10 people on our planet do not have easy access to clean water to drink. As each of us conserves water, we prolong the life span of fresh water from our lakes and rivers and also reduce the impact on sewer systems and drainage in our communities. Additionally, as we conserve water, we also conserve electricity that is used to bring water to and in our homes. So, what can we do to help conserve water?

Example 8.3.1: Brushing Your Teeth (One Person’s Contribution)

Brushing your teeth with the water running continually uses about 4 gal of water. Turning the faucet off when you are not rinsing uses less than one-fourth of a gallon of water. Considering the recommendation to brush your teeth twice a day, how much water would be saved in a week if the faucet was off when not rinsing?

Answer

Leaving the water running continually:

Step 1: Calculate gallons used not with water running continually:

Brushing twice a day for 7 days using 4 gal of water for each brushing

$$(2 \text{ times a day})(7 \text{ days})(4 \text{ gal}) = 56 \text{ gal}$$

Step 2: Calculate gallons used turning the faucet off when you are not rinsing:

Brushing twice a day for 7 days using 0.25 gal of water for each brushing

$$(2 \text{ times a day})(7 \text{ days})(0.25 \text{ gal}) = 3.5 \text{ gal}$$

Step 3: Calculate savings:

$$\begin{aligned}\text{Savings} &= 56 \text{ gal} - 3.5 \text{ gal} \\ &= 52.5 \text{ gal}\end{aligned}$$

During one week, 52.5 gal of water would be saved if one person turned the faucet off except when rinsing when brushing your teeth.

Your Turn 8.3.1

Brushing your teeth with the water running continually uses about 4 gal of water. Turning the faucet off when you are not rinsing uses less than one-fourth of a gallon of water. Consider the recommendation to brush your teeth twice a day, how much water would be saved in a month (4 weeks) if the faucet was off when not rinsing?

Example 8.3.2: Brushing Your Teeth (Multiple People's Contribution – Town)

Using the data in Example 8.3.1, how much water would be saved in a month if one-fifth of a town's population of 15,000 turned the faucet off when brushing their teeth except when rinsing?

Answer

Step 1: From Example 13.5, we found that 1 person saves 52.5 gal per week.

Step 2: Calculate the population to save water:

$$\text{One-fifth of } 15{,}000 \text{ people} = 3{,}000 \text{ people}$$

Step 3: One-fifth of a town's population turning off the faucet when brushing their teeth for a month:

$$(3{,}000)(52.5 \text{ gal per week})(4 \text{ weeks}) = 630{,}000 \text{ gal}$$

During one month, 630,000 gallons of water would be saved if one-fifth of a town of 15,000 people turned the faucet off except when rinsing when brushing their teeth.

Your Turn 8.3.2

Using the data in Your Turn 8.3.1, how much water would be saved in a month if one-third of a town's population of 15,000 turned the faucet off when brushing their teeth except when rinsing?

Example 8.3.3: Brushing Your Teeth (Multiple People's Contribution – State)

Using the data in Example 13.5, how much water would be saved in a year if one-fourth of the population of the state of Minnesota, which is approximately 5.6 million people, turned the faucet off when brushing their teeth except when rinsing for a year (52 weeks)?

Answer

Step 1: From Example 13.5, we found that 1 person saves 52.5 gal per week.

Step 2: Calculate the population to save water:

$$\text{One-fourth of } 5.6 \text{ million people} = 1.4 \text{ million people}$$

Step 3: One-fourth of a town's population turning off the faucet when brushing their teeth for a month:

$$(1.4 \text{ million})(52.5 \text{ gal per week})(52 \text{ weeks}) = 3{,}822 \text{ million gal}$$

During one year, 3,822 million gal of water would be saved if one-fourth of the state of Minnesota turned the faucet off except when rinsing when brushing their teeth.

Your Turn 8.3.3

Using the data in Example 13.5, how much water would be saved in a year if one-sixth of the population of the state of Florida, which is approximately 21.6 million people, turned the faucet off when brushing their teeth except when rinsing for a year (52 weeks)?

History of Solar Energy

In the mid-1800s, Willoughby Smith discovered photoconductive responsiveness in selenium. Shortly thereafter, William Grylls Adams and Richard Evans Day discovery that selenium can produce electricity if exposed to the sun was a major breakthrough. Less than 10 years later, Charles Fritts invented the first solar cells using selenium. Jumping a mere 100 years later, Bell Labs in the United States produced the first practical photovoltaic cells in the mid-1950s and developed versions used to power satellites in the same decade.

Solar panel use has exploded in recent decades and is now used by residences, organizations, businesses, and government buildings such as the White House, space to power satellites, and various methods of transportation. One reason for the expansion is a continuing drop in cost combined with an increase in performance and durability. In the mid-1950s, the cost of a solar panel was around $300 per watt capability. Twenty years later, the cost was a third of the 1950s' cost. Currently, solar panel cost has dropped to less than $1 per watt while decreasing in size as well as increasing in longevity. The dropping price and improved performance has moved solar to a modest investment that can pay for itself in less than half the time of systems from 15 years ago.

Who Knew?: Solar Power's Age

The sun has been harnessed by humans for centuries. The earliest recorded use of tapping the sun's energy for power dates back to the seventh century BC when man focused the sun's rays through a magnifying glass to create fire. Four thousand years later, we find historical record of using mirrors to focus the sun and light torches, often for ceremonial proceedings. Use of the sun to light torches continued through the centuries and has been recorded by various cultures including the Chinese civilization in 20 AD and beyond.

In more recent years, the sun was harnessed to power ovens on ships traversing to oceans in the 1700s. At the same time, the power of the sun was utilized to power steamboats through the 1800s. Mária Telkes, a Hungarian-born American scientist, invented a widely deployed solar seawater distiller used on World War II life rafts. Soon after, she partnered with architect Eleanor Raymond to design the first modern home to be completely heated by solar power. Air warmed on rooftop collectors transferred heat to salts, which stored the heat for later use.

Although solar panels as we know them today are relatively new in history, use of the sun to harness power is much older.

Compute Power Needs for Common Home Devices

A kilowatt (kW) is 1,000 watts (W). A kilowatt-hour (kWh) is a measurement of energy use, which is the amount of energy used by a 1,000-watt device to run for an hour. Using the definition of a kilowatt-hour, to calculate how long it would take to consume 1 kWh of power, we divide 1,000 by the watts use of a device.

FORMULA

1,000/watts=time needed to use1 kW

For example, a 75 W bulb would take $\mathbf{1{,}000 \div 75 = 13.3\ hours}$ to use 1 kW of power.

FORMULA

watts/1,000=kilowatt hours

Example 8.3.4: Calculating the Kilowatt-Hours Needed to Run a Television

A 48 in plasma television uses about 200 W. How many kilowatt-hours are needed to run the television in a month if the television is one for an average of 2.5 hours a day?

Answer

Step 1: $1,000/(200$ watts $)= 5$ hours to use 1 kW

Step 2: (2.5 hours a day)(30 days) $= 75$ hours of use

Step 3: 75 hour / 5 hours per \(\mathrm{kW}=15 \mathr**m{~kW}\)**

The television will consume about 15 kW in a month.

Your Turn 8.3.4

A 40 in plasma television uses about 175 W. How many kilowatt-hours are needed to run the television in a month if the television is on for an average of 3 hours a day?

Example 8.3.5: Calculating the Cost to Run a Refrigerator

A medium-sized Energy Star–rated refrigerator uses about 575 W and runs for about 8 hours per day. What is the monthly (30 days) cost of running the refrigerator if the electric rate is 12 cents per kilowatt-hour?

Answer

Step 1: Calculate the watts per day:

$(575 \text{ W})(8 \text{ hours}) = 4{,}600 \text{ W per day}$

Step 2: Calculate the kilowatt-hours.

$(4{,}600)/(1{,}000) = 4.6 \text{ kWh}$

Step 3: Calculate the daily cost.

$(4.6 \text{ kWh})(12 \text{ cents}) = 55 \text{ cents} = \$0.55$

Step 4: Calculate the monthly cost.

$(\$0.55)(30) = \$16.50$

It would cost about $16.50 to run the refrigerator for a month.

Your Turn 8.3.5

A dorm-sized Energy Star–rated refrigerator uses about 375 W and runs for about 9 hours per day. What is the monthly cost of running the refrigerator if the electric rate is 14 cents per kilowatt-hour?

Example 8.3.6: Calculating the Kilowatt-Hours to Run an Oven

An electric oven is labeled as 4,000 W. How much would it cost to bake a cake for 30 minutes if the electric rate is 14 cents per kilowatt-hour?

Answer

Step 1: Determine the time it takes to use 1 kW of power:

$$1{,}000/(4{,}000 \text{ watts}) = 0.25 \text{ hours to use 1 kW}$$

For every 15 minutes, the oven uses 1 kW of power.

Step 2: Determine how many kilowatt-hours are needed to bake the cake for 30 minutes:

$$(30 \text{ minutes})/(15 \text{ minutes per kW}) = 2 \text{ kW}$$

Step 3: Calculate the cost of the oven usage:

$$(2 \text{ kW})(14 \text{ cents per kWh}) = 28 \text{ cents}$$

It would cost about 28 cents to bake the cake.

Your Turn 8.3.6

A toaster oven is labeled as 2,000 W. How much would it cost to warm leftovers from a meal for 15 minutes if the electric rate is 12 cents per kilowatt-hour?

Solar Advantages

There are multiple advantages that solar power can offer us today including reducing greenhouse gas and CO_2 emissions, powering vehicles, reducing water pollution, reducing strain on limited supply of other power options such as fossil fuels. We will look further at reducing greenhouse gas and CO_2 emissions.

Any gas that prevents infrared radiation from escaping Earth's atmosphere is a greenhouse gas. There are 24 currently identified greenhouse gases of which carbon dioxide is one. When measuring the impact of any of the greenhouse gases, the measurements are given in units of carbon dioxide emissions. For this reason, greenhouse gas and carbon dioxide have become interchangeable in discussions.

People in Mathematics: Charles Fritts and Mohammad M. Atalla

Charles Fritts, a New York inventor, is credited with creating the first solar cell, which he installed on a rooftop in New York City in 1884. While the solar cell was not very efficient, having a rate of conversion between 1 to 2%, this was a major step early in solar power energy. Today's solar cells have an efficiency on average of 15 to 20%, which yields a notably higher impact. Nonetheless, the work that Fritts successfully completed marked the start of solar energy through the use of photovoltaic solar panels in the United States.

Mohamed M. Atalla was an Egyptian-born scientist who moved to the United States to complete his studies, and undertook research and development at Bell Laboratories in New Jersey. Many of the early efficiency gains in solar cells were due to his development of processes for using silicon within electronic devices. Atalla's work led to the invention of silicon transistors and microchips (including his own invention of the MOSFET, the most widely used transistor in the world), and quickly increased the efficiency of solar cells.

Example 8.3.7: Calculating the Solar Power for Average Home Use in Kilowatts

If a home uses approximately 30 kW of electricity per day, what size solar system would be needed to fuel 80% of a home's needs for a month (30 days)?

Answer

$$(30 \text{ kW hours})(30 \text{ days})(0.80) = 720 \text{ kW}$$

A solar system capable of producing 720 kW a month would be needed.

Your Turn 8.3.7

A tiny home uses approximately 12 kW of electricity per day. What size solar system would be needed to fuel 80% of a home's needs for a month (30 days)?

Check Your Understanding

1. What is the relationship between 1 kW and watts?
2. An average bath uses about 35 gal of water. A water-saving showerhead uses approximately 2 gal of water per minute. If a person typically takes one bath a week, how much water is saved by replacing the baths with showers lasting 5 minutes over the course of a month?
3. How long does it take for a 30 W Google Nest audio to use 1 kW of power?
4. How long would it take for a 60 W bulb to consume 1 kW of power? Round your answer to the nearest hour.

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8.3.1: Exercises

Exercise 8.3.1.1

A typical showerhead uses 5 gal of water per minute. A water-saving showerhead uses approximately 2 gal of water per minute. How much water would one person save in a month if they take a 6-minute shower 4 times a week?

Exercise 8.3.1.2

An average toilet uses 5 gal of water per flush. A high-efficiency toilet uses about 1.25 gal per flush. How much water would a household save in a week if the toilet was flushed 8 times a day?

Exercise 8.3.1.3

When washing dishes, leaving the faucet running utilizes about 15 gal every 5 minutes, where filling the sink and turning the faucet off except to rinse the dishes uses about 5 gal for washing a dishpan-sized load of dishes. How much water is saved by not leaving the faucet running if it takes 10 minutes to wash a dishpan-sized load of dishes each day for a month containing 30 days?

Exercise 8.3.1.4

Leaving the water running when washing your hands consumes about 4 gal of water, whereas turning the water off when lathering reduces the water used to 1 gal. How much water is saved in an apartment of 6 people in an academic term of 16 weeks if each person averages washing their hands 3 times a day?

Exercise 8.3.1.5

How long would it take for an energy-saving light bulb to consume 1 kW of power if the bulb is rated at 7.5 W? Round your answer to the nearest hour.

Exercise 8.3.1.6

How long would it take for a 120 W light bulb to consume 1 kW of power? Round your answer to the nearest hour.

Exercise 8.3.1.7

A portable television uses about 80 W per hour. How many kilowatt-hours are needed to run the television during a 3-day trip if the television is run for an average of 5.5 hours a day?

Exercise 8.3.1.8

A flat iron to straighten hair is rated at 331 W. If it is used for 15 minutes a day, 5 times a week, how much would it cost a user over the course of a month if the electric rate is 13 cents per kilowatt-hour? Round your answer to the nearest cent.

Exercise 8.3.1.9

You purchase a window air conditioner for your apartment living room rated at 1,000 W. If you run the air conditioner for an average of 3 hours a day for a week, how much would it cost if the electric rate is 12 cents per kilowatt-hour?

Exercise 8.3.1.10

A cabin uses approximately 25 kW of electricity per day. What size solar system would be needed to fuel 90% of the cabin's needs for a month if the cabin is used 2 days a week?

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8.4: Math and Medicine



Figure 13.9: Shoppers wear masks during the Covid-19 pandemic. (credit: "True Covid Scene - Mask Buying" by Joey Zanotti/Flickr, CC BY 2.0)

Learning Objectives

1. Compute the mathematical factors utilized in concentrations/dosages of drugs.
2. Describe the history of validating effectiveness of a new drug.
3. Describe how mathematical modeling is used to track the spread of a virus.

The pandemic that rocked the world starting in 2020 turned attention to finding a cure for the Covid-19 strain into a world race and dominated conversations from major news channels to households around the globe. News reports decreeing the number of new cases and deaths locally as well as around the world were part of the daily news for over a year and progress on vaccines soon followed. How was a vaccine able to be found so quickly? Is the vaccine safe? Is the vaccine effective? These and other questions have been raised through communities near and far and some remain debatable. However, we can educate ourselves on the foundations of these discussions and be more equipped to analyze new information related to these questions as it becomes available.

Concentrations and Dosages of Drugs

Consider any drug and the recommended dosage varies based on several factors such as age, weight, and degree of illness of a person. Hospitals and medical dispensaries do not stock every possible needed concentration of medicines. Drugs that are delivered in liquid form for intravenous (IV) methods in particular can be easily adjusted to meet the needs of a patient. Whether administering anesthesia prior to an operation or administering a vaccine, calculation of the concentration of a drug is needed to ensure the desired amount of medicine is delivered.

The formula to determine the volume needed of a drug in liquid form is a relatively simple formula. The volume needed is calculated based on the required dosage of the drug with respect to the concentration of the drug. For drugs in liquid form, the concentration is noted as the amount of the drug per the volume of the solution that the drug is suspended in which is commonly measured in g/mL or mg/mL.

Suppose a doctor writes a prescription for 6 mg of a drug, which a nurse calculates when retrieving the needed prescription from their secure pharmaceutical storage space. On the shelves, the drug is available in liquid form as 2 mg per mL. This means that 1 mg of the drug is found in 0.5 mL of the solution. Multiplying 6 mg by 0.5 mL yields 3 mL, which is the volume of the prescription per single dose.

FORMULA

Volume needed = (medicine dosage required) (weight of drug by volume).

A common calculation for the weight of a liquid drug is measured in grams of a drug per 100 mL of solution and is also called the percentage weight by volume measurement and labeled as % w/v or simply w/v.

Checkpoint

Note that the units for a desired dose of a drug and the units for a solution containing the drug or pill form of the drug must be the same. If they are not the same, the units must first be converted to be measured in the same units.

Suppose you visit your doctor with symptoms of an upset stomach and unrelenting heartburn. One possible recourse is sodium bicarbonate, which aids in reducing stomach acid.

Example 8.4.1: Calculating the Quantity in a Mixture

How much sodium bicarbonate is there in a 250 mL solution of 1.58% w/v sodium bicarbonate?

Answer

$1.58\%\ \text{w/v} = 1.58\ \text{g}$ sodium bicarbonate in 100 mL. If there is 250 mL of the solution, we have 2.5 times as much sodium bicarbonate as in 100 mL. Thus, we multiply 1.58 by 2.5 to yield 3.95 g sodium bicarbonate in 250 mL solution.

Your Turn 8.4.1

How many milligrams of sodium chloride are there in 200 mL of a 0.9% w/v normal saline solution?

Example 8.4.2: Calculating the Quantity of Pills Needed

A doctor prescribes 25.5 mg of a drug to take orally per day and pills are available in 8.5 mg. How many pills will be needed each day?

Answer

The prescription and the pills are in the same units which means no conversions are needed. We can divide the units of the drug prescribed by the units in each pill: $25.5/8.5 = 3$. So, 3 pills will be needed each day.

Your Turn 8.4.2

How many pills would be needed for a patient who has been prescribed 25.5 mg of a drug if each pill contains 4.25 mg.

Example 8.4.3: Calculating the Drug Dose in Milligrams, Based on Patient Weight

A patient is prescribed 2 mg/kg of a drug to be delivered intramuscularly, divided into 3 doses per day. If the patient weighs 45 kg, how many milligrams of the drug should be given per dose?

Answer

Step 1: Calculate the total daily dose of the drug based on the patient's weight (measured in kilograms):

$(2\ \text{mg/kg})(45\ \text{kg}) = 90\ \text{mg}$

Step 2: Divide the total daily dose by the number of doses per day:

$90\ \text{mg}/3 = 30\ \text{mg}$

The patient should receive 30 mg of the drug in each dose.

Your Turn 8.4.3

A patient is prescribed 1.4 mg/kg of a drug to be delivered intramuscularly, divided into 2 doses per day. If the patient weighs 60 kg, how many milliliters of the drug should be given per dose?

Checkpoint

Note that the units for a patient's weight must be compatible with the units used in the medicine measurement. If they are not the same, the units must first be converted to be measured in the same units.

Example 8.4.4: Calculating the Drug Dose in Milliliters, Based on Patient Weight

A patient is prescribed 2 mg/kg of a drug to be delivered intramuscularly, divided into 3 doses per day. If the drug is available in 20 mg/mL and the patient weighs 60 kg, how many milliliters of the drug should be given per dose?

Answer

Step 1: Calculate the total daily dose of the drug (measured in milligrams) based on the patient's weight (measured in kilograms):

$$(2 \text{ mg/kg})(60 \text{ kg}) = 120 \text{ mg}$$

Step 2: Calculate the volume in each dose:

$$(120 \text{ mg daily total})/(3 \text{ doses a day}) = 40 \text{ mg per dose}$$

Step 3: Calculate the volume based on the strength of the stock:

$$(\text{prescribed dose needed})/(\text{stock dose}) = \text{volume}$$
$$(40 \text{ mg per dose})/(20 \text{ mg/mL}) = 2 \text{ mL}$$

The patient should receive 2 mL of the stock drug in each dose.

Your Turn 8.4.4

A patient is prescribed 2.5 mg/kg of a drug to be delivered intramuscularly, divided into 2 doses per day. If the drug is available in 5 mg/mL and the patient weighs 52 kg, how many milligrams of the drug should be given per dose?

Who Knew?: Math Statistics from the CDC

The Centers for Disease Control and Prevention (CDC) states that about half the U.S. population in 2019 used at least one prescription drug each month, and about 25% of people used three or more prescription drugs in a month. The resulting overall collective impact of the pharmaceutical industry in the United States exceeded $1.3 trillion a year prior to the 2020 pandemic.

Validating Effectiveness of a New Vaccine

The process to develop a new vaccine and be able to offer it to the public typically takes 10 to 15 years. In the United States, the system typically involves both public and private participation in a process. During the 1900s, several vaccines were successfully developed, including the following: polio vaccine in the 1950s and chickenpox vaccine in the 1990s. Both of these vaccines took years to be developed, tested, and available to the public. Knowing the typical timeline for a vaccine to move from development to administration, it is not surprising that some people wondered how a vaccine for Covid-19 was released in less than a year's time.

Lesser known is that research on coronavirus vaccines has been in process for approximately 10 years. Back in 2012, concern over the Middle Eastern respiratory syndrome (MERS) broke out and scientists from all over the world began working on researching coronaviruses and how to combat them. It was discovered that the foundation for the virus is a spike protein, which, when delivered as part of a vaccine, causes the human body to generate antibodies and is the platform for coronavirus vaccines.

When the Covid-19 pandemic broke out, Operation Warp Speed, fueled by the U.S. federal government and private sector, poured unprecedented human resources into applying the previous 10 years of research and development into targeting a specific vaccine for the Covid-19 strain.

People in Mathematics: Shibo Jiang

Dr. Shibo Jiang, MD, PhD, is co-director the Center for Vaccine Development at the Texas Children's Hospital and head of a virology group at the New York Blood Center. Together with his colleagues, Jiang has been working on vaccines and treatments for a range of viruses and infections including influenzas, HIV, Sars, HPV and more recently Covid-19. His work has been recognized around the world and is marked with receiving grants amounting to over $20 million from U.S. sources as well as the same from foundations in China, producing patents in the United States and China for his antiviral products to combat world concerns.

Jiang has been a voice for caution in the search for a vaccine for Covid-19, emphasizing the need for caution to ensure safety in the development and deployment of a vaccine. His work and that of his colleagues for over 10 years on other coronaviruses paved the way for the vaccines that have been shared to combat the Covid-19 pandemic.

Mathematical Modeling to Track the Spread of a Vaccine

With a large number of people receiving a Covid-19 vaccine, the concern at this time is how to create an affordable vaccine to reach people all over the world. If a world solution is not found, those without access to a vaccine will serve as incubators to variants that might be resistant to the existing vaccines.

As we work to vaccinate the world, attention continues with tracking the spread of the Covid-19 and its multiple variants. Mathematical modeling is the process of creating a representation of the behavior of a system using mathematical language. Digital mathematical modeling plays a key role in analyzing the vast amounts of data reported from a variety of sources such as hospitals and apps on cell phones.

When attempting to represent an observed quantitative data set, mathematical models can aid in finding patterns and concentrations as well as aid in predicting growth or decline of the system. Mathematical models can also be useful to determine strengths and vulnerabilities of a system, which can be helpful in arresting the spread of a virus.

The chapter on Graph Theory explores one such method of mathematical modeling using paths and circuits. Cell phones have been helpful in tracking the spread of the Covid-19 virus using apps regulated by regional government public health authorities to collect data on the network of people exposed to an individual who tests positive for the Covid-19 virus.

People in Mathematics: Gladys West

Dr. Gladys West is a mathematician and hidden figure with a rich résumé of accomplishments spanning Air Force applications and work at NASA. Born in 1930, West rose and excelled both academically and in her professional life at a time when Black women were not embraced in STEM positions. One of her many accomplishments is the Global Positioning System (GPS) used on cell phones for driving directions.



Figure 8.4.2: Gladys West (credit: "Dr. Gladys West Hall" by The US Air Force/Wikimedia Commons, Public Domain)

West began work as a human computer, someone who computes mathematical computations by hand. Considering the time and complexity of some calculations, she became involved in programming computers to crunch computations. Eventually, West created a mathematical model of Earth with detail and precision that made GPS possible, which is utilized in an array of devices from satellites to cell phones. The next time you tag a photo or obtain driving directions, you are tapping into the mathematical modeling of Earth that West developed.

Consider the following graph (Figure 8.4.3):

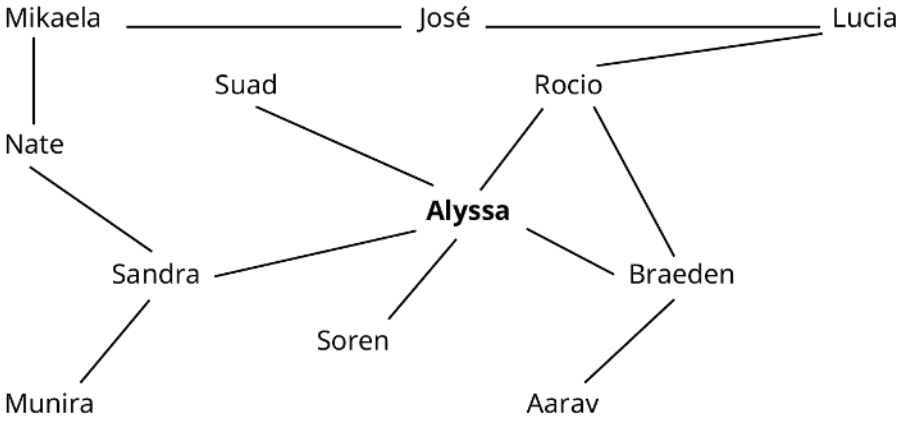


Figure 8.4.3: Contact Tracing for Math 111 Section 1

At the center of the graph, we find Alyssa, whom we will consider positive for a virus. Utilizing the technology of phone apps voluntarily installed on each phone of the individuals in the graph, tracking of the spread of the virus among the 6 individuals that Alyssa had direct contact with can be implemented, namely Suad, Rocio, Braeden, Soren, and Sandra.

Let's look at José's exposure risk as it relates to Alyssa. There are multiple paths connecting José with Alyssa. One path includes the following individuals: José to Mikaela to Nate to Sandra to Alyssa. This path contains a length of 4 units, or people, in the contact tracing line. There are 2 more paths connecting José to Alyssa. A second path of the same length consists of José to Lucia to Rocio to Braeden to Alyssa. Path 3 is the shortest and consists of José to Lucia to Rocio to Alyssa. Tracking the spread of positive cases in the line between Alyssa and José aids in monitoring the spread of the infection.

Now consider the complexity of tracking a pandemic across the nation. Graphs such as the one above are not practical to be drawn on paper but can be managed by computer programs capable of computing large volumes of data. In fact, a computer-generated mathematical model of contact tracing would look more like a sphere with paths on the exterior as well as on the interior. Mathematical modeling of contact tracing is complex and feasible through the use of technology.

✔ Example 8.4.5: Using Mathematical Modeling

For the following exercises, use the sample contact tracing graph to identify paths (Figure ✔ 8.4.4).

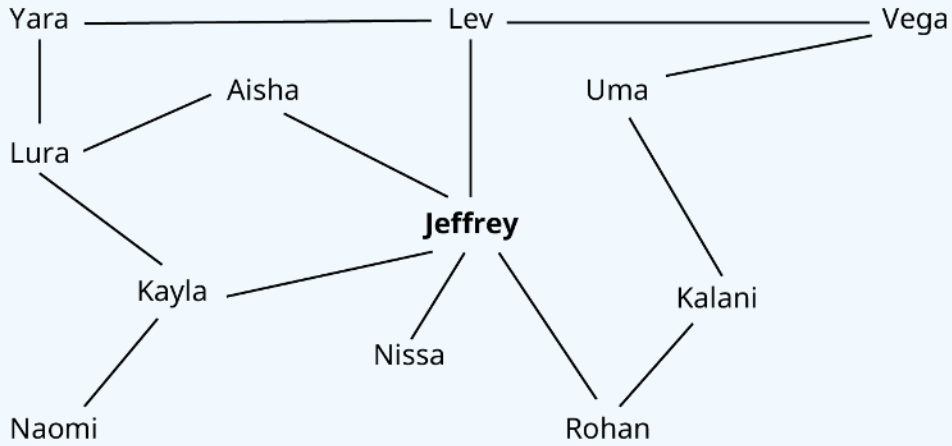


Figure ✔ 8.4.4 Contact Tracing for ECON 250 Section 1

1. How many people have a path of length 2 from Jeffrey?
2. Find 2 paths between Kayla and Rohan.
3. Find the shortest path between Yara and Kalani. State the length and people in the path.

Answer

1. 5 (Lura, Naomi, Kalani, Vega, Yara)
2. Answers will vary. Two possible answers are as follows:
   a. Kayla, Jeffrey, Rohan
   b. Kayla, Lura, Yara, Lev, Vega, Uma, Kalani, Rohan
3. Length is 4. People in path = Yara, Lev, Vega, Uma, Kalani

📌 Your Turn 8.4.5

For the following exercises, use Figure 📌 8.4.5.

List all the names of those who have a path length of three from Uma.

What is the length of the shortest path from Naomi to Vega?

Check Your Understanding

? Exercise 8.4.1

What two pieces of information are needed to calculate the volume of a prescription drug to be dispensed?

? Exercise 8.4.2

Research on coronavirus vaccines began in 2020.

a. True
b. False

Exercise 8.4.3

What is mathematical modeling, and how is it used in the world of medicine with pandemics?

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8.4.1: Exercises

Exercise 8.4.1.1

How many grams of sodium bicarbonate are contained in a 300 mL solution of 1.35% w/v sodium bicarbonate?

Exercise 8.4.1.2

How many grams of sodium bicarbonate are contained in a 175 mL solution of 1.85% w/v sodium bicarbonate?

Exercise 8.4.1.3

Using a saline solution that is 0.75% w/v, how many milligrams of sodium chloride are in 150 mL?

Exercise 8.4.1.4

Using a saline solution that is 1.25% w/v, how many milligrams of sodium chloride are in 200 mL?

Exercise 8.4.1.5

A prescription calls for a patient to receive 23 mg daily of a drug to be taken in pill form for 5 days. If the pills are available in 5.75 mg, how many pills will the patient need for the full prescription run?

Exercise 8.4.1.6

A prescription calls for a patient to receive 21 mg daily of a drug to be taken in pill form daily. If the pills are available in 3.5 mg, how many pills will the patient need each day?

Exercise 8.4.1.7

A patient is prescribed 4mg/kg of a drug to be delivered daily intramuscularly, divided into 2 doses. If the patient weighs 30 kg, how many milligrams of the drug would be needed for each dose?

Exercise 8.4.1.8

A patient is prescribed 1.5 mg/kg of a drug to be delivered intramuscularly, divided into 3 doses per day. If the drug is available in 12.5 mg/mL and the patient weighs 50 kg, how many milliliters of the drug would be given per dose?

Exercise 8.4.1.9

A patient is prescribed 0.5 mg/kg of a drug to be delivered intramuscularly, divided into 2 doses per day. If the drug is available in 2.5 mg/mL and the patient weighs 45 kg, how many milliliters of the drug would be given per dose?

Exercise 8.4.1.10

A patient is prescribed 1.5 mg/kg of a drug to be delivered intramuscularly, divided into 3 doses per day. If the drug is available in 30 mg/mL and the patient weighs 54 kg, how many milliliters of the drug would be given per dose?

For the following exercises, use the mathematical modeling graph showing contact tracing for students in a particular class.

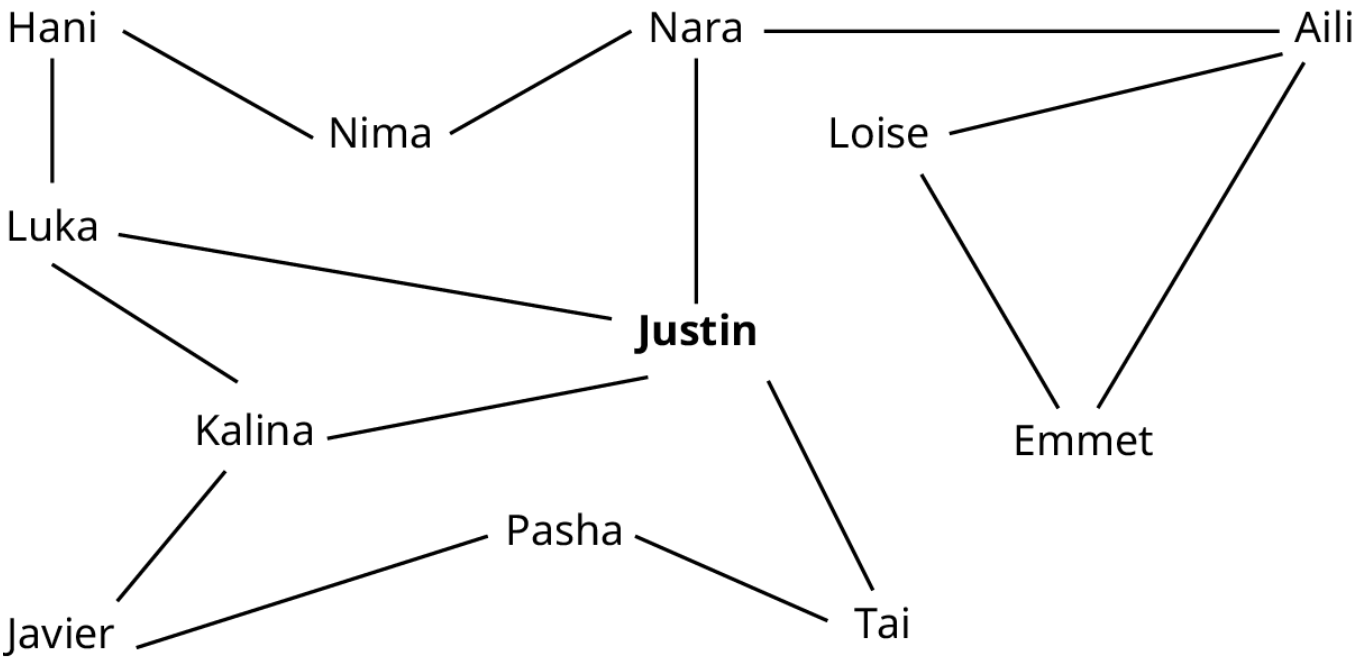


Figure 13.13: Contract Tracing for Students in a Class

? Exercise 8.4.1.11

List the people who have a length of 2 from Justin.

? Exercise 8.4.1.12

Find 2 paths of with a length of 3 from Emmet.

? Exercise 8.4.1.13

Find the shortest path from Aili to Kalina.

? Exercise 8.4.1.14

Which people does the model show as directly in contact with Nara?

? Exercise 8.4.1.15

Find the shortest path from Tai to Hani.

? Exercise 8.4.1.16

Of the 12 people in the model, how many have a path of 2 or less from Justin?

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8.5: Math and Music

Learning Objectives

1. Describe the basics of frequency related to sound.
2. Describe the basics of pitch as it relates to music.
3. Describe and evaluate musical notes, half-steps, whole steps, and octaves.
4. Describe and find frequencies of octaves.

> *"The world's most famous and popular language is music."*
>
> *- Psy, South Korean singer, rapper, songwriter, and record producer*

Imagine a world without music and many of us would struggle to fill the void. Music uplifts, inspires, heals, and generally adds dimension to virtually every aspect of our lives. But what is music? For some it is a song; for others it may be the sounds of birds or the rhythmic sound of drumming or a myriad of other sounds. Whatever you consider music, it is all around us and is an integral part of our lives. "Music can raise someone's mood, get them excited, or make them calm and relaxed. Music also—and this is important—allows us to feel nearly or possibly all emotions that we experience in our lives. The possibilities are endless" (Galindo, 2009).



Figure Figure **8.5.1**: Contact Tracing for ECON 250 Section 1 Friends sing music together around a campfire. (credit: modification of work "Fire is hot! (2)" by Chetan Sarva/Flickr, CC BY 2.0)

What music you listen to can impact your mood and emotions. In similar fashion, the music we choose can often tell those around us something about our current moods and emotions. Consider the music you may have been listening to as you today or even as you are reading this text. What cues to your mood do your music selections share? Albert Einstein is quoted as saying, "If I were not a physicist, I would probably be a musician. I often think in music. I live my daydreams in music. I see my life in terms of music." What clues do your recent music choices say about your mood or how your day is going?

Basics of Frequency as It Relates to Sound

Every sound is created by an object vibrating and these vibrations travel in waves that are captured by our ears. Some vibrations we may be able to see, such as a plucked guitar string moving, whereas other vibrations we may not be able to see, such as the sound created when we hold our breath when accidentally dropping our cell phone on a hard floor. We don't see the vibrations of our cell phone hitting the floor; however, any audible sound created in the fall is the result of vibrations in the form of sound waves, which can be pictured similarly to waves moving through the ocean.

The waves of sounds each have a frequency, or rate of vibration of sound waves, that measures the number of waves completed in a single second and are measured in hertz (Hz; one Hz is one cycle per second). Louder sounds have stronger vibrations or are created closer to our ear. The further an ear is from the source of the sound, the quieter the sound will appear.

Sounds range in frequency from 16 Hz to ultrasonic values, with humans able to hear sounds in a frequency range of about 20 Hz to 20,000 Hz. Adults lose the ability to hear the upper end of the range and typically top out in the ability to hear in a frequency of

15,000–17,000 Hz. Sounds with a frequency above 17,000 Hz are less likely to be heard by adults while still being audible to children.

While frequency plays a key role in audible sounds, so too does the sound level, which can be measured in decibels (dB), which are the units of measure for the intensity of a sound or the degree of loudness. As a sound level increases, the decibel level increases.

A person with average hearing can hear sounds down to 0 dB. Those with exceptionally good hearing can hear even quieter sounds, down to approximately –5 dB. The following table includes sample sounds with their related decibel values.

Sound	Decibels (dB)
Firecrackers	140
Take-off of military jet from aircraft carrier	130
Clap of thunder	120
Auto horn standing next to the vehicle	110
Outboard motor	100
Motorcycle	90
Noise inside a car in city traffic	80
Typical washing machine	70
Public conversation, such as at a restaurant	60
Private conversation	50
Hum of a computer with fan blowing	40
Quiet whisper	30
Swishing leaves	20
Regular breathing	10
Lowest typical sound audible by teenagers	0

Example 8.5.1: Selecting Decibel Value of Sounds

Select the most representative decibel value for each of the following sounds:

1. car wash: 25 dB, 55 dB, 85 dB
2. vacuum cleaner: 15 dB, 70 dB, 90 dB
3. ship’s engine room: 30 dB, 65 dB, 95 dB
4. approaching subway car: 70 dB, 100 dB, 120 dB

Answer

1. 85 dB
2. 70 dB
3. 95 dB
4. 100 dB

Your Turn 8.5.1

Match each of the following sounds with a corresponding decibel level: live outdoor concert, birds chirping, window air conditioner, garbage disposal.

a. 40 dB
b. 60 dB
c. 80 dB
d. 100 dB

Basics of Pitch

When considering the various sound levels the human ear can hear, the ear perceives sound both from the frequency level and the pitch of a sound. The quality of the sound is referred to as pitch, the tonal quality of a sound and how high or low the tone. Sounds with a high frequency have a high pitch, such as 900 Hz, and sounds with a low pitch have a low frequency, such as 50 Hz.

Let's take a look at frequency and pitch using a string instrument such as a guitar or piano. When a string is plucked on a guitar or a key is played on a piano, the related string vibrates at a frequency that is related to the length and thickness of the string. The frequency is measurable and has a singular value. The pitch of the note played is open for interpretation, as the pitch is a function of personal opinion.

Who Knew?: Graphing Calculators and Music

It may be interesting to note that a TI-84 and TI-Nspire graphing calculators can be utilized to tune a musical instrument by measuring the frequency of a note using a small plug-in accessory that captures the sound waves from a note and displays the corresponding frequency. Using the displayed frequency, an instrumentalist can then make the needed adjustment to perfectly tune an instrument. This method of tuning an instrument can be helpful whether a novice player or a seasoned instrumentalist because the instrument can be tuned precisely to the correct frequency.

Note Values, Half-Steps, Whole Steps, and Octaves

> *"There are not more than five musical notes, yet the combinations of these five give rise to more melodies than can ever be heard."*
>
> *- Sun Tzu, Chinese strategist*

Moving our exploration to note values, the frequency of all notes is well defined by a specific and unique frequency for each note that is measurable. We will explore keys on a keyboard to discuss notes that have the same relationships with any instrument or musical piece.

Let's look at Figure 8.5.2. The white keys are labeled with the letters A–G and the photo begins with middle C, which can be found in the middle of a keyboard. This labeling of the keys repeats across an entire keyboard and keys to the right have a higher pitch and frequency than keys to the left. Each of the keys correlates to a musical note.



Figure 8.5.2 Piano Keys and Notes (credit: modification of work "Contemplate" by Walt Stoneburner/Flickr, CC BY 2.0)

Movement up or down between any two consecutive keys (black and white) or notes constitutes a half-step. Movement of one half-step sometimes involves a sharp (#) or a flat (♭) symbol. For example, $D^{\#}$ is one half-step above D and $D^{\flat}$ is one half-step below D. Note that this is not always true as one half-step above B is C, and one half-step below F is E. In similar fashion, a whole step is movement up or down between any two half-steps on a keyboard.

Example 8.5.2: Identifying Half-Steps

Name which keys are one half-step up and one half-step down from the following:

1. D
2. E
3. G#

Answer

1. up $D^{\#}$, down $D^{\flat}$
2. up F, down $E^{\flat}$
3. up A, down G

Your Turn 8.5.2

Name which keys are one half-step up and one half-step down from the following:

a. $F^{\#}$
b. B
c. $G^{\flat}$

Example 8.5.3: Identifying Whole Steps

Name which keys are one whole step up and one whole step down from the following:

1. $F^{\#}$
2. E
3. $A^{\flat}$

Answer

1. Up $G^{\#}$, down E
2. Up $F^{\#}$, down D
3. Up $B^{\flat}$, down $G^{\flat}$

Your Turn 8.5.3

Name which keys are one whole step up and one whole step down from the following:

1. $D^{\flat}$
2. $C^{\#}$
3. E

You may have noticed that there are eight letters of the alphabet used to label notes. Selecting any one note and counting up 12 half-steps you will find that the numbering for notes begins at the same value as you started from. This collection of 12 consecutive half-notes is called an octave and is a basic foundational component in music theory.

Example 8.5.4: Listing All Notes in an Octave

List the 12 notes forming an octave, beginning with the note C.

Answer

C, $C^{\#}$, D, $D^{\#}$, E, F, $F^{\#}$, G, $G^{\#}$, A, $A^{\#}$, B

Your Turn 8.5.4

List the 12 notes forming an octave, beginning with the note G.

Frequencies of Octaves

Notes that are one octave apart have the same name and are related in frequency values. Given the frequency of any note, the frequency of same note one octave higher is doubled and this pattern continues as you move up and down the notes on a keyboard or any other musical instrument. Song writers and singers use this knowledge to change the pitch of a note up or down to align with a person's vocal range. Regardless of which C is played or sung, the pitch is the same and the frequency is related by a power or two.

Labeled keys on a keyboard are numbered for ease in identification. For example, middle C is labeled as C_4 on a full keyboard as it is the fourth C from the left in a set of eight notes. The frequency of C_4 is 262 Hz, rounded to the nearest whole number.

Example 8.5.5: Calculating the Frequency Values of Octaves

Given that the frequency of C_4 is 262 Hz, find the approximate frequency of C_6.

Answer

The frequency of each consecutive higher octave doubles. Given that the frequency of C_4 is 262 Hz, the frequency of C_5 is found by doubling the frequency of C_4, which is 524 Hz. In similar fashion, the frequency of C_6 is found by doubling the frequency of C_5, which yields 1,048 Hz.

Your Turn 8.5.5

Given that the frequency of E_4 is 330 Hz, find the approximate frequency of E_2 rounded to the nearest whole number.

People in Mathematics: David Cope

David Cope has a somewhat eclectic list of job titles ranging from author and music professor to scientist and artificial intelligence researcher. Cope combined his interests when he developed software that can analyze a piece of music and create a new and unique musical piece in the same style as the original. Some of his well-known products have been based off of the classical music of Mozart, creating what has been called Mozart's "42nd Symphony," as well as other genres including opera and a range of current music styles. Cope has also composed original musical pieces in collaboration with a computer.

We have explored some basics components of frequency, pitch, note relationships, and octaves, which are building blocks of music. It may be exciting to learn that the mathematical relationships found in music are vast and grow in complexity beyond the math commonly studied in high school.

Who Knew?: Spotify Royalty Payments

Streaming services have grown exponentially in popularity thanks in large part to customized music listening through cell phone use and devices such as Google as well as Amazon Echo and Alexa devices for home and vehicles, adding to ways that artists are paid royalties. Spotify, which was launched in 2008, typically plays artists $0.06 per time a song is streamed, with some artists receiving up to $0.84 per play amounting to over $9 billion in revenue for Spotify in 2020. Since 2014, Spotify's revenue has grown over a billion dollars a year, with roughly half of their revenue being paid out in royalties, which was good news for artists during the Covid-19 pandemic when in-person concerts and shopping were hindered.

Check Your Understanding

Exercise 8.5.1

What is the difference between frequency and pitch of a note

Exercise 8.5.2

Are higher frequencies associated with higher or lower pitch notes

Exercise 8.5.3

What frequency range can humans hear

Exercise 8.5.4

As a sound appears louder, does the decibel value increase or decrease

Exercise 8.5.5

What is the lowest decibel value that most people can hear

Exercise 8.5.6

What is the typical audible frequency range that adults can hear?

Exercise 8.5.7

How does the frequency of a note change when increased by one octave?

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8.5.1: Exercises

For the following exercises, select the most representative decibel level for each sound.

Exercise 8.5.1.1

Gas lawn mower: 70 dB, 100 dB, 120 dB

Exercise 8.5.1.2

Radio playing loud enough to sing along to at home: 20 dB, 50 dB, 70 dB

Exercise 8.5.1.3

Office noise on a floor of people working at desks: 40 dB, 60, dB, 80 dB

Exercise 8.5.1.4

Conversation in a quiet room: 60 dB, 80 dB, 100 dB

Exercise 8.5.1.5

Fast-moving train passing by when sitting in a car about 20 ft from the tracks with windows down: 40 dB, 60 dB, 80 dB

For the following exercises, state the requested note.

Exercise 8.5.1.6

What is one half-step above $G^{\#}$?

Exercise 8.5.1.7

What is one whole step down from $G^{\flat}$?

Exercise 8.5.1.8

What is three half-steps above D?

Exercise 8.5.1.9

What is two whole steps down from $A^{\#}$?

Exercise 8.5.1.10

What is two half-steps down from E?

Exercise 8.5.1.11

What is an octave above B_4?

Exercise 8.5.1.12

What is two octaves below F_5?

Exercise 8.5.1.13

What is three octaves above G_2?

Exercise 8.5.1.14

What is an octave below G_2?

Exercise 8.5.1.15

Given that the frequency of $A_4 = 440$ Hz, what is the frequency of A_1 ?

Exercise 8.5.1.16

Given that the approximate frequency of $F_4 = 349$ Hz, what is the approximate frequency of F_5, rounded to the nearest whole number?

Exercise 8.5.1.17

Given that the approximate frequency of $A_4{}^{\#} = 233$ Hz, what is the approximate frequency of $A_6{}^{\#}$, rounded to the nearest whole number?

Exercise 8.5.1.18

Given that the approximate frequency of $D_4 = 294$ Hz, what is the approximate frequency of D_2, rounded to the nearest whole number?

Exercise 8.5.1.19

Given that the approximate frequency of $G_4{}^{\#} = 349$ Hz, what is the approximate frequency of $G_3{}^{\#}$, rounded to the nearest whole number?

Exercise 8.5.1.20

Given that the frequency of $A_3 = 220$ Hz, what is the frequency of A_5 ?

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8.6: Math and Sports

Learning Objectives

1. Describe why data analytics (statistics) is crucial to advance a team's success.
2. Describe single round-robin method of tournaments.
3. Describe single-elimination method of tournaments.
4. Explore math in baseball, fantasy football, hockey, and soccer (projects at the end of the section).

Sports are big business and entertainment around the world. In the United States alone, the revenue from professional sports is projected to bring in over $77 billion, which includes admission ticket costs, merchandise, media coverage access rights, and advertising. So, whether or not you enjoy watching professional sports, you probably know someone who does. Some celebrities compete to be part of half-time shows and large companies vie for commercial spots that are costly but reach a staggering number of viewers, some who only watch the half-time shows and advertisements.



Figure 8.6.1: Fans support their team by attending games and wearing team gear. (credit: "Cheering Touchdowns" by Steven Miller/Flickr, CC BY 2.0)

Data Analytics (Statistics) Is Crucial to Advance a Team's Success

Analyzing the vast data that today's world has amassed to find patterns and to make predictions for future results has created a degree field for data analytics at many colleges, which is in high demand in places that might surprise you. One such place is in sports, where being able to analyze the available data on your team's players, potential recruits, opposing team strategies, and opposing players can be paramount to your team's success.

Hollywood turned the notion of using data analytics into a major motion picture back in 2011 with the release of *Moneyball*, starring Brad Pitt, which grossed over $110 million. The critically acclaimed movie, based on a true story as shared in a book by Michael Lewis, follows the story of a general manager for the Oakland Athletics who used data analytics to take a team comprised of relatively unheard of players to ultimately win the American League West title in a year's time. The win caught the eye of other team managers and owners, which started an avalanche of other teams digging into the data of players and teams.

In today's world of sports, a team has multiple positions utilizing data analytics from road scouts who evaluate a potential recruit's skills and potential to the ultimate position of general manager who is typically the highest-paid (non-player) employee with the exception of the coaches. Being able to understand and evaluate the available data is big business and is a highly sought after skill set. In college and professional sports, it is no longer sufficient to have a strong playbook and great players. The science to winning is in understanding the math of the data and using it to propel your team to excelling.

Single Round-Robin Tournaments

A common tournament style is single round-robin tournaments (Figure 8.6.2), where each team or opponent plays every other team or opponent, and the champion is determined by the team that wins the most games. Ties are possible and are resolved based on

league rules.

8 Team round robin

Team 1	
W	L

Team 2	
W	L

Team 3	
W	L

Team 4	
W	L

Team 5	
W	L

Team 6	
W	L

Team 7	
W	L

Team 8	
W	L

Round 1
2 vs 1
3 vs 8
4 vs 7
5 vs 6

Round 2
3 vs 4
1 vs 7
8 vs 6
2 vs 5

Round 3
6 vs 2
7 vs 8
4 vs 1
5 vs 3

Round 4
7 vs 5
8 vs 4
2 vs 3
6 vs 1

Round 5
1 vs 3
4 vs 2
5 vs 8
6 vs 7

Round 6
4 vs 6
8 vs 1
2 vs 7
3 vs 6

Round 7
7 vs 3
8 vs 2
1 vs 5
6 vs 4

Figure **8.6.2**: Single-Round Robin Tournament

An advantage of the round-robin tournament style is that no one team has the advantage of seeding, which eliminates some teams from playing against each other based on rank of their prior performance. Rather, each team plays every other team, providing equal opportunity to triumph over each team. In this sense, round-robin tournaments are deemed the fairest tournament style.

One hindrance to employing a round-robin-style tournament is the potential for the number of games involved in tournament play to determine a winner. Determining the number of games can be found easily using a formula which, as we will see, can quickly grow in the number of games required for a single round-robin tournament.

FORMULA

The number of games in a single round-robin tournament with n teams is $n(n-1)/2$.

Example 8.6.1: Calculating the Number of Games in Single Round-Robin Tournaments

Find the number of games in a single round-robin tournament for each of the following numbers of teams:

1. 4 teams
2. 8 teams
3. 20 teams

Answer

1. Using the formula with 4 teams yields $4(4-1)/2 = 6$ tournament games.
2. Using the formula with 8 teams yields $8(8-1)/2 = 28$ tournament games.
3. Using the formula with 20 teams yields $20(20-1)/2 = 190$ tournament games.

Your Turn 8.6.1

Find the number of games in a single round-robin tournament for each of the following numbers of teams:

a. 5 teams
b. 12 teams
c. 25 teams

As the examples show, single round-robin tournament play can quickly grow in the number of games required to determine a champion. As such, some tournaments elect to employ variations of single round-robin tournament play as well as other tournament styles such as elimination tournaments.

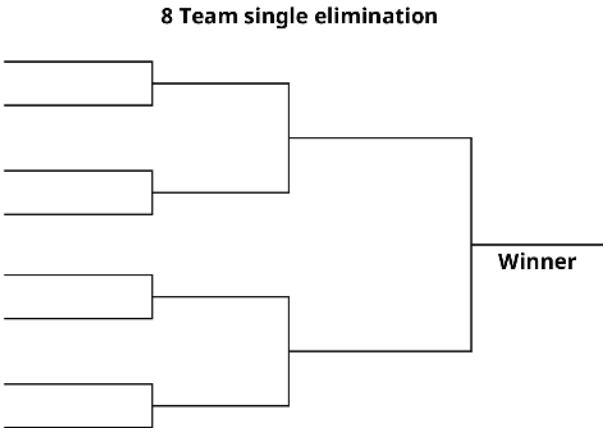


Figure 8.6.3: Single-Elimination Tournament (credit: final TK)

Single-Elimination Tournaments

When desiring a more efficient tournament style to determine a champion, one option is single-elimination tournaments (Figure 13.18), where teams are paired up and the winner advances to the next round of play. The losing team is defeated from tournament play and does not advance in the tournament, although some leagues offer consolation matches.

FORMULA

The number of games in a single-elimination tournament with n teams is $(n-1)$.

Example 8.6.2: Calculating the Number of Games in Single-Elimination Tournaments

Find the number of games in a single-elimination tournament for each of the following numbers of teams:

a. 4 teams
b. 8 teams
c. 20 teams

Answer

1. Using the formula with 4 teams yields $(4-1)=3$ tournament games.
2. Using the formula with 8 teams yields $(8-1)=7$ tournament games.
3. Using the formula with 20 teams yields $(20-1)=19$ tournament games.

Your Turn 8.6.2

Find the number of games in a single-elimination tournament for each of the following numbers of teams:

a. 5 teams
b. 12 teams

c. 25 teams

A single-elimination tournament offers an advantage over single round-robin tournament style of play in the number of games needed to complete the tournament. As you can see, in comparing the number of games in a single round-robin tournament in Example 13.22 with the number of games in single-elimination tournament as shown in Example **8.6.2**, the number of games required for single round-robin can quickly become unmanageable to schedule.

There are modifications to both the round-robin and elimination tournament styles such as double round-robin and double-elimination tournaments. Next time you observe a college or professional sporting event, see if you can determine the tournament style of play.

Who Knew?: Sports Popularity Shifts

Sport has been a popular entertainment venue for hundreds of years and the popularity of various sports shifts over time as well as in different regions of the world today. In today's world, the most popular sport is, overwhelmingly, soccer, with over 4 billion fans followed by cricket with 2.5 billion fans. It may surprise you to learn that American football doesn't rank in the top 10 most popular sports in the world today, yet table tennis ranks in sixth place and golf fills the last spot in the top 10 world sports.

In the early 1930s, baseball ranked a close second, with basketball virtually tying for third place. By the mid-1940s, hockey slightly led in first place over basketball. Ten years later, hockey remained in the number one sport, but cricket pushed basketball to third place. In the 1960s, soccer dominated in popularity. Over the next 30 years, basketball dropped in popularity and there was much movement in the popularity of sports. By the late 1990s, soccer swept to first place, where it has since continued to grow in popularity.

Check Your Understanding

1. How is data analytics used in sports today?
2. What is the most popular sport in today's world?
3. What is the formula to compute the number of games played in a single round-robin tournament with n teams assuming single round?
4. What is the formula to compute the number of games played in a single-elimination tournament?

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8.6.1: Exercises

Exercise 8.6.1.1

Briefly describe data analytics.

Exercise 8.6.1.2

Name a sport's career title that relies on analysis of data.

For the following exercises, compute how many games would be played in the style of tournament and number of teams given in each question. Assume all tournaments are single round-robin or single-elimination.

Exercise 8.6.1.3

How many games would be played with 4 teams using a round-robin tournament?

Exercise 8.6.1.4

How many games would be played with 8 teams using a round-robin tournament?

Exercise 8.6.1.5

How many games would be played with 10 teams using a round-robin tournament?

Exercise 8.6.1.6

How many games would be played with 25 teams using a round-robin tournament?

Exercise 8.6.1.7

How many games would be played with 12 teams using a round-robin tournament?

Exercise 8.6.1.8

How many games would be played with 4 teams using a single-elimination tournament?

Exercise 8.6.1.9

How many games would be played with 9 teams using a single-elimination tournament?

Exercise 8.6.1.10

How many games would be played with 15 teams using a single-elimination tournament?

Exercise 8.6.1.11

How many games would be played with 50 teams using a single-elimination tournament?

Exercise 8.6.1.12

How many games would be played with 24 teams using a single-elimination tournament?

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