

ELEMENTARY FIRST ORDER DYNAMIC EQUATIONS ON TIME SCALES



Svetlin G. Georgiev
Sorbonne University

First Order Dynamic Equations on Time Scales

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (<https://LibreTexts.org>) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by [NICE CXone Expert](#) and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptations contact info@LibreTexts.org or visit our main website at <https://LibreTexts.org>.

This text was compiled on 03/30/2026

TABLE OF CONTENTS

[Licensing](#)

1: Chapters

- [1.1: Introduction](#)
- [1.2: Linear First Order Dynamic Equations](#)
- [1.3: Bernoulli Dynamic Equations](#)
- [1.4: Riccati Dynamic Equations](#)
- [1.5: Dynamic Equations for Families of Curves](#)
- [1.6: Exact Equations](#)
- [1.7: Mathematical Modelling with First Order Dynamic Equations](#)

[Index](#)

[Glossary](#)

[Detailed Licensing](#)

Licensing

A detailed breakdown of this resource's licensing can be found in [Back Matter/Detailed Licensing](#).

CHAPTER OVERVIEW

1: Chapters

- 1.1: Introduction
- 1.2: Linear First Order Dynamic Equations
- 1.3: Bernoulli Dynamic Equations
- 1.4: Riccati Dynamic Equations
- 1.5: Dynamic Equations for Families of Curves
- 1.6: Exact Equations
- 1.7: Mathematical Modelling with First Order Dynamic Equations

1: Chapters is shared under a [GNU General Public License 3.0](#) license and was authored, remixed, and/or curated by LibreTexts.

1.1: Introduction

Suppose that $\mathbb{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ , respectively. Let $t_0 \in \mathbb{T}$ and $I \subset \mathbb{T}$.

Definition:

A dynamic equation (DE) is a relation that contains one independent variable $t \in \mathbb{T}$, the dependent variable y and some of its delta derivatives $y^\Delta, y^{\Delta^2}, \dots, y^{\Delta^n}$.

✓ Example

The equation

$$y^\Delta + \sin_1(t, t_0)yy^\Delta + y^2 = 0$$

is a DE.

✓ Example

The equation

$$y^{\Delta^2} + e_1(t, t_0)y^\sigma + \frac{1}{1 + (\cosh_3(t, t_0))^4}y^\Delta + y^3 = \frac{1}{1 + t^2}, \quad t \in \mathbb{T},$$

is a DE provided that $1 - 9\mu^2 \neq 0$.

✓ Example

The equation

$$(1 + \cosh_1(t, t_0))^3 y^{\Delta^4} (y^\sigma)^3 - y^2 = \frac{1}{1 + 10(\sin_1(t, t_0))^8}, \quad t \in \mathbb{T},$$

is a DE provided that $1 - \mu^2 \neq 0$.

Definition

The order of a dynamic equation is defined as the order of the highest delta derivative in the equation.

✓ Example

The equation

$$yy^{\Delta^3} - 3y^2 + \frac{1}{1 + (e_1(t, t_0))^4}y^{\Delta^2}y^\sigma = \frac{1}{1 + t^8}, \quad t \in \mathbb{T},$$

is a third order dynamic equation provided that $1 \in \mathcal{R}$.

✓ Example

The equation

$$y^{\Delta^4} - y^\sigma y^{\Delta^2} + y^\Delta - \sin_1(t, t_0)yy^\sigma = \frac{1}{1 + t^2}, \quad t \in \mathbb{T},$$

is a fourth order dynamic equation.

✓ Example

The equation

$$y^{\Delta^2} - t^3 y^\sigma + \sin_5(t, t_0) y^\Delta = y^2, \quad t \in \mathbb{T},$$

is a second order dynamic equation.

? Exercise

Determine the order of the following DEs.

1. $(y^\Delta)^2 - e_2(t, t_0)y = 0, \quad 2 \in \mathcal{R}, \quad t \in \mathbb{T},$
2. $y^{\Delta^3} - e_4(t, t_0)y^{\Delta^2} = \sin_3(t, t_0), \quad 4 \in \mathcal{R}, \quad t \in \mathbb{T},$
3. $(y^{\Delta^2})^3 = (\sin_{-1}(t, t_0))^2, \quad t \in \mathbb{T},$
4. $y^{\Delta^5} - e_3(t, t_0)y^{\Delta^2} = \cos_1(t, t_0), \quad t \in \mathbb{T},$
5. $y^{\Delta^{11}} - e_2(t, t_0)y^\Delta - \cos_3(t, t_0)y = \tanh_4(t, t_0), \quad 2 \in \mathcal{R}, \quad 1 - 16\mu^2 \neq 0, \quad t \in \mathbb{T}.$

In general, an n th order DE can be written in the form

$$F(t, y, y^\Delta, \dots, y^{\Delta^n}) = 0, \quad (1.1.1)$$

where F is a given function.

Definition

A functional relation between the real dependent variable y and the time scale independent variable t in some interval $I \subset \mathbb{T}$ that satisfies the given DE is said to be a solution of the DE. The general solution of an n th order DE depends on n constants $c_1, c_2, \dots, c_n$ and the independent time scale variable t . Sometimes, the solutions of a DE are said to be delta integral curves for the corresponding DE.

Obviously, the general solution of an n th order DE is written in the following form

$$\phi(t, y, c_1, c_2, \dots, c_n) = 0.$$

✓ Example

Consider the equation

$$y^\Delta = y, \quad t \in \mathbb{T}.$$

We will show that

$$y(t) = e_1(t, t_0), \quad t \in \mathbb{T},$$

is its solution. Really, we have

$$\begin{aligned} y^\Delta(t) &= e_1(t, t_0) \\ &= y(t), \quad t \in \mathbb{T}. \end{aligned}$$

✓ Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the equation

$$y^{\Delta^2} + \frac{3}{1+4t} y^\Delta = \frac{3}{1+4t} \sin_1(t, t_0) + \cos_1(t, t_0), \quad t \in \mathbb{T}.$$

We will prove that the function

$$y(t) = \frac{1}{1+t} - \cos_1(t, t_0), \quad t \in \mathbb{T},$$

is its solution. Here

$$\sigma(t) = 2t, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned} y^\Delta(t) &= -\frac{1}{(1+t)(1+\sigma(t))} + \sin_1(t, t_0) \\ &= -\frac{1}{(1+t)(1+2t)} + \sin_1(t, t_0), \\ y^{\Delta^2}(t) &= \frac{2(\sigma(t)+t)+3}{(1+t)(1+2t)(1+\sigma(t))(1+2\sigma(t))} + \cos_1(t, t_0) \\ &= \frac{2(2t+t)+3}{(1+t)(1+2t)(1+2t)(1+4t)} + \cos_1(t, t_0) \\ &= \frac{3(2t+1)}{(1+t)(1+2t)^2(1+4t)} + \cos_1(t, t_0) \\ &= \frac{3}{(1+t)(1+2t)(1+4t)} + \cos_1(t, t_0), \quad t \in \mathbb{T}. \end{aligned}$$

Hence, we arrive at

$$y^{\Delta^2}(t) + \frac{3}{1+4t}y^\Delta(t) \tag{1.1.2}$$

$$\begin{aligned} &= \frac{3}{(1+t)(1+2t)(1+4t)} + \cos_1(t, t_0) + \frac{3}{1+4t} \left(-\frac{1}{(1+t)(1+2t)} + \sin_1(t, t_0) \right) \\ &= \frac{3}{(1+t)(1+2t)(1+4t)} + \cos_1(t, t_0) - \frac{3}{(1+t)(1+2t)(1+4t)} + \frac{3}{1+4t} \sin_1(t, t_0) \\ &= \frac{3}{1+4t} \sin_1(t, t_0) + \cos_1(t, t_0), \quad t \in \mathbb{T}. \end{aligned} \tag{1.1.3}$$

✓ Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Consider the equation

$$y^{\Delta^3} + t^2 y^{\Delta^2} - 4t y^\Delta - \frac{1}{t^2} y = 48t^2 - 21t + 56 - \frac{5}{t} + \frac{6}{t^2}, \quad t \in \mathbb{T}.$$

We will prove that the function

$$y(t) = t^3 - 4t^2 + 5t + 6, \quad t \in \mathbb{T},$$

is its solution. Here

$$\sigma(t) = 3t, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned} y^\Delta(t) &= (\sigma(t))^2 + t\sigma(t) + t^2 - 4(\sigma(t)+t) + 5 \\ &= (3t)^2 + t(3t) + t^2 - 4(3t+t) + 5 \\ &= 9t^2 + 3t^2 + t^2 - 16t + 5 \\ &= 13t^2 - 16t + 5, \\ y^{\Delta^2}(t) &= 13(\sigma(t)+t) - 16 \\ &= 13(3t+t) - 16 \\ &= 52t - 16, \\ y^{\Delta^3}(t) &= 52, \quad t \in \mathbb{T}. \end{aligned}$$

Hence,

$$\begin{aligned}
 & y^{\Delta^3}(t) + t^2 y^{\Delta^2}(t) - 4t y^{\Delta}(t) - \frac{1}{t^2} y(t) \\
 &= 52 + t^2(52t - 16) - 4t(13t^2 - 16t + 5) - \frac{1}{t^2}(t^3 - 4t^2 + 5t + 6) \\
 &= 52 + 52t^3 - 16t^2 - 52t^3 + 64t^2 - 20t - t + 4 - \frac{5}{t} + \frac{6}{t^2} \\
 &= 48t^2 - 21t + 56 - \frac{5}{t} + \frac{6}{t^2}, \quad t \in \mathbb{T}.
 \end{aligned} \tag{1.1.4}$$

? Exercise

Let $\mathbb{T} = 2\mathbb{Z}$. Prove that

$$y(t) = t^4 - 3t^2, \quad t \in \mathbb{T},$$

is a solution to the equation

$$y^{\Delta^4} + t y^{\Delta^3} - 2y^{\Delta^2} + t y^{\Delta} - 4y = 12t^3 + 22t^2 - 6t - 76, \quad t \in \mathbb{T}.$$

Definition

If a solution can be obtained for some values of the constants $c_1, c_2, \dots, c_n$, then it is called a particular solution. Otherwise, it is said to be a singular solution.

✓ Example

Consider the equation

$$y^{\Delta^2} - 3y^{\Delta} + 2y = 0, \quad t \in \mathbb{T}.$$

Note that

$$y(t) = c_1 e_1(t, t_0) + c_2 e_2(t, t_0), \quad t \in \mathbb{T}, \quad c_1, c_2 \in \mathbb{R},$$

is a general solution of the considered equation. Really, we have

$$y^{\Delta}(t) = c_1 e_1(t, t_0) + 2c_2 e_2(t, t_0),$$

$$y^{\Delta^2}(t) = c_1 e_1(t, t_0) + 4c_2 e_2(t, t_0), \quad t \in \mathbb{T}.$$

Hence,

$$\begin{aligned}
 & y^{\Delta^2}(t) + 3y^{\Delta}(t) + 2y(t) \\
 &= c_1 e_1(t, t_0) + 4c_2 e_2(t, t_0) - 3(c_1 e_1(t, t_0) + 2c_2 e_2(t, t_0)) + 2(c_1 e_1(t, t_0) + c_2 e_2(t, t_0)) \\
 &= c_1 e_1(t, t_0) + 4c_2 e_2(t, t_0) - 3c_1 e_1(t, t_0) - 6c_2 e_2(t, t_0) + 2c_1 e_1(t, t_0) + 2c_2 e_2(t, t_0) \\
 &= 0, \quad t \in \mathbb{T}.
 \end{aligned}$$

The functions

$$y_1(t) = 7e_1(t, t_0) \quad \text{and} \quad y_2(t) = -3e_2(t, t_0), \quad t \in \mathbb{T},$$

are particular solutions of the considered equation.

✓ Example

Let $\mathbb{T} = \mathbb{R}$. Consider the equation

$$(y^\Delta)^2 - ty^\Delta + y = 0, \quad t \in \mathbb{T}.$$

Here

$$\sigma(t) = 2t, \quad t \in \mathbb{T}.$$

The general solution of the considered equation is given by

$$y(t) = ct^2 - c^2, \quad t \in \mathbb{T}, \quad c \in \mathbb{R}. \quad (1.1.5)$$

Really, we have

$$y^\Delta(t) = c, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned} (y^\Delta(t))^2 - ty^\Delta(t) + y(t) &= c^2 - ct + ct - c^2 \\ &= 0, \quad t \in \mathbb{T}. \end{aligned}$$

On the other hand, we will show that the function

$$y(t) = \frac{2}{9}t^2, \quad t \in \mathbb{T}, \quad (1.1.6)$$

is a solution of the given equation. Indeed, we have

$$\begin{aligned} y^\Delta(t) &= \frac{2}{9}(\sigma(t) + t) \\ &= \frac{2}{9}(2t + t) \\ &= \frac{2}{3}t, \quad t \in \mathbb{T}. \end{aligned}$$

Hence,

$$\begin{aligned} (y^\Delta(t))^2 - ty^\Delta(t) + y(t) &= \left(\frac{2}{3}t\right)^2 - t\left(\frac{2}{3}t\right) + \frac{2}{9}t^2 \\ &= \frac{4}{9}t^2 - \frac{2}{3}t^2 + \frac{2}{9}t^2 \\ &= \frac{4-6+2}{9}t^2 \\ &= 0, \quad t \in \mathbb{T}. \end{aligned}$$

We can not get (1.1.6) by (1.1.5) for any constant $c \in \mathbb{R}$. Therefore (1.1.6) is a singular solution of the considered equation.

? Exercise

Let $\mathbb{T} = \mathbb{R}$. Consider the equation

$$ty + (t+1)y^\Delta = 0, \quad t \in \mathbb{T}.$$

1. Prove that the general solution of the considered equation is given by

$$y(t) = c(t+1)e_{-1}(t, 0), \quad t \in \mathbb{T}, \quad c \in \mathbb{R}.$$

2. Prove that $t = -1$ is a singular solution of the considered equation.

Definition

The function

$$y = \phi(t), \quad t \in I,$$

is said to be an explicit solution of the equation (1.1.1) provided

$$F(t, \phi(t), \phi^\Delta(t), \phi^{\Delta^2}(t), \dots, \phi^{\Delta^n}(t)) = 0, \quad t \in I. \quad (1.1.7)$$

Definition

A relation of the form

$$\psi(t, y) = 0, \quad t \in I, \quad (1.1.8)$$

is said to be an implicit solution of the equation (1.1.1) provided that it determines one or more functions $y = \phi(t)$, $t \in I$, which satisfy (1.1.7).

Sometimes it is difficult to solve the equation (1.1.8). In these cases, if $y(\mathbb{T}) = \widetilde{\mathbb{T}}$ is a time scale with forward jump operator and delta differentiation operator σ and $\widetilde{\Delta}$, respectively, and the function ψ is enough times completely delta differentiable, then

$$\psi_{t_1}^\Delta(t, y(t)) + \psi_{t_2}^{\widetilde{\Delta}}(\sigma(t), y(t))y^\Delta(t) = 0, \quad t \in I.$$

From here, we find

$$y^\Delta(t) = -\frac{\psi_{t_1}^\Delta(t, y(t))}{\psi_{t_2}^{\widetilde{\Delta}}(\sigma(t), y(t))}, \quad t \in I,$$

provided that

$$\psi_{t_2}^{\widetilde{\Delta}}(\sigma(t), y(t)) \neq 0, \quad t \in I.$$

From here, we express $y^{\Delta^2}(t)$, $y^{\Delta^3}(t)$, ..., $y^{\Delta^n}(t)$, $t \in I$. Then, we check if

$$F(t, y(t), y^\Delta(t), \dots, y^{\Delta^n}(t)) = 0, \quad t \in I.$$

Definition

A DE is said to be linear if it is linear in y and its delta derivatives. Otherwise, it is said to be nonlinear.

Example

The DE

$$y^{\Delta^2} + 2y^\Delta + \sin_3(t, t_0)y = 1, \quad t \in \mathbb{T},$$

is a linear DE.

Example

The DE

$$(y^\Delta)^3 + y^{\Delta^3} + \cos_{-5}(t, t_0) = 0, \quad t \in \mathbb{T},$$

is a nonlinear DE.

Example

The DE

$$y^\Delta y^{\Delta^2} + \sinh_{-3}(t, t_0)y = 0, \quad t \in I,$$

is a nonlinear DE provided that $1 - 9\mu^2 \neq 0$ on $\mathbb{T}$.

? Exercise

Classify each of the following DEs as linear or nonlinear.

1. $y^\Delta + \cos_1(t, t_0)y = \sin_3(t, t_0), \quad t \in \mathbb{T}.$
2. $(y^\Delta)^4 - (e_4(t, t_0))^2 = 0, \quad t \in \mathbb{T}.$
3. $y^\Delta y^{\Delta^2} = t, \quad t \in \mathbb{T}.$

The general form of n th order linear DEs is as follows.

$$p_0(t)y^{\Delta^n} + p_1(t)y^{\Delta^{n-1}} + \cdots + p_n(t)y = f(t), \quad t \in \mathbb{T}, \quad (1.1.9)$$

where $p_j, f : \mathbb{T} \rightarrow \mathbb{R}, j \in \{1, 2, \dots, n\}$, are given functions.

Definition

The equation (1.1.9) is said to be homogeneous if $f \equiv 0$ on $\mathbb{T}$. Otherwise, the equation (1.1.9) is said to be nonhomogeneous.

✓ Example

The DE

$$y^{\Delta^2} + y^\Delta = \cos_1(t, t_0), \quad t \in \mathbb{T},$$

is nonhomogeneous.

✓ Example

The DE

$$y^{\Delta^3} + \sin_2(t, t_0)y^{\Delta^2} + y^\Delta = 0, \quad t \in \mathbb{T},$$

is a homogeneous DE.

✓ Example

The DE

$$y^{\Delta^4} + y^\Delta = t^2, \quad t \in \mathbb{T},$$

is a nonhomogeneous DE.

? Exercise

Classify each of the following DEs as homogeneous or nonhomogeneous.

1. $y^\Delta + t^2 y = t, \quad t \in \mathbb{T}.$
2. $y^{\Delta^4} + y = 0, \quad t \in \mathbb{T}.$
3. $y^{\Delta^2} - y^\Delta = 1, \quad t \in \mathbb{T}.$

Now, we assume that the equation (1.1.1) can be solved explicitly. More precisely, suppose

$$y^{\Delta^n} = f(t, y, y^\Delta, \dots, y^{\Delta^{n-1}}), \quad t \in \mathbb{T}, \quad (1.1.10)$$

where $f : \mathbb{T} \rightarrow \mathbb{R}$ is a given function.

Definition

By initial conditions for the DE (1.1.10) we mean conditions of the form

$$\begin{aligned} y(t_0) &= y_0, \\ y^\Delta(t_0) &= y_1, \\ &\vdots \\ y^{\Delta^{n-1}}(t_0) &= y_{n-1}, \end{aligned} \tag{1.1.11}$$

where $y_j \in \mathbb{R}, j \in \{0, 1, \dots, n-1\}$, are given constants.

Definition

A problem consisting of the DE (1.1.10) together with the initial conditions (1.1.11) is said to be initial value problem (IVP).

Definition

Observe that the initial conditions (1.1.11) are prescribed at a point t_0 . In many problems, these conditions are prescribed into two or more points. These conditions are called boundary conditions. A problem consisting of the DE (1.1.10) together with boundary conditions is said to be boundary value problem (BVP).

Definition

A problem consisting of the DE (1.1.10) together with the initial conditions (1.1.11) and boundary conditions is said to be initial boundary value problem (IBVP).

Example

Let $\mathbb{T} = \mathbb{Z}$. The problem

$$\begin{aligned} y^{\Delta^3} - 3y &= 0, \quad t \in \mathbb{T}, \\ y(0) &= 0, \\ y^\Delta(0) &= -1, \\ y^{\Delta^2}(0) &= 2, \end{aligned}$$

is an IVP.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. The problem

$$\begin{aligned} y^{\Delta^2} - 4y^\Delta + ty &= t^2, \quad t \in \mathbb{T}, \\ y(1) - 3y^\Delta(32) &= 0 \end{aligned}$$

is a BVP.

Example

Let $\mathbb{T} = 4^{\mathbb{N}_0}$. The problem

$$\begin{aligned}y^{\Delta^2} + ty^{\Delta} - t^2y &= \sin_1(t, 1), \quad t \in \mathbb{T}, \\y(1) &= 1, \\y^{\Delta}(1) &= -3, \\3y(1) - 5y(256) &= 0,\end{aligned}$$

is an IBVP.

The main questions for IVPs, BVPs and IBVPs are as follows.

1. Existence of solutions.
2. Uniqueness of solutions.
3. Continuous dependence on initial conditions and parameters.
4. The interval under which the solutions are defined.
5. Extension of solutions.
6. Upper and lower bounds of the unknown solutions and existence of minimal and maximal solutions.
7. Periodicity of the solutions.
8. Behaviour of the solutions.
9. Stability of the solutions.
10. Oscillations of solutions.

This page titled [1.1: Introduction](#) is shared under a [GNU General Public License 3.0](#) license and was authored, remixed, and/or curated by [Svetlin G. Georgiev](#).

1.2: Linear First Order Dynamic Equations

Let $p, q \in \mathcal{C}_{rd}(\mathbb{T})$, $p \in \mathcal{R}$. Consider the IVP

$$y^\Delta = p(t)y + q(t), \quad t \in \mathbb{T}^\kappa, \quad (1.2.1)$$

$$y(t_0) = y_0, \quad (1.2.2)$$

where $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$.

Definition

Equations of the form (1.2.1) are said to be linear first order dynamic equations.

Theorem

Suppose that

$$1 + \mu(t)p(t) \neq 0, \quad t \in \mathbb{T}.$$

Then the problem (1.2.1), (1.2.2) has unique solution represented in the form

$$y(t) = y_0 e_p(t, t_0) + \int_{t_0}^t q(s) e_{\ominus p}(\sigma(s), t) \Delta s, \quad s, t \in \mathbb{T}^\kappa. \quad (1.2.3)$$

Proof

Let $t \in \mathbb{T}^\kappa$. We multiply both sides of the equation (1.2.1) with $e_{\ominus p}(\sigma(t), t_0)$ and we get

$$y^\Delta(t) e_{\ominus p}(\sigma(t), t_0) = p(t) y(t) e_{\ominus p}(\sigma(t), t_0) + q(t) e_{\ominus p}(\sigma(t), t_0). \quad (1.2.4)$$

Note that

$$\begin{aligned} p(t) e_{\ominus p}(\sigma(t), t_0) &= p(t) (1 + \mu(t)(\ominus p)(t)) e_{\ominus p}(t, t_0) \\ &= p(t) \left(1 - \frac{\mu(t)p(t)}{1 + \mu(t)p(t)} \right) e_{\ominus p}(t, t_0) \\ &= \frac{p(t)}{1 + \mu(t)p(t)} e_{\ominus p}(t, t_0) \\ &= -(\ominus p)(t) e_{\ominus p}(t, t_0) \\ &= -e_{\ominus p}^\Delta(t, t_0), \quad t \in \mathbb{T}^\kappa. \end{aligned}$$

Then (1.2.4) takes the form

$$(y^\Delta(t)) e_{\ominus p}(\sigma(t), t_0) + (e_{\ominus p}^\Delta(t, t_0)) y(t) = q(t) e_{\ominus p}(\sigma(t), t_0), \quad t \in \mathbb{T}^\kappa,$$

or

$$(y e_{\ominus p}(\cdot, t_0))^\Delta(t) = q(t) e_{\ominus p}(\sigma(t), t_0), \quad t \in \mathbb{T}^\kappa.$$

We integrate the last equality from t_0 to t and we get

$$\int_{t_0}^t (y e_{\ominus p}(\cdot, t_0))^\Delta(s) \Delta s = \int_{t_0}^t q(s) e_{\ominus p}(\sigma(s), t_0) \Delta s, \quad t \in \mathbb{T}^\kappa,$$

or

$$y(t) e_{\ominus p}(t, t_0) - y(t_0) e_{\ominus p}(t_0, t_0) = \int_{t_0}^t q(s) e_{\ominus p}(\sigma(s), t_0) \Delta s, \quad t \in \mathbb{T}^\kappa,$$

or

$$y(t) e_{\ominus p}(t, t_0) = y_0 + \int_{t_0}^t q(s) e_{\ominus p}(\sigma(s), t_0) \Delta s, \quad t \in \mathbb{T}^\kappa,$$

or

$$\begin{aligned} y(t) &= y_0 e_p(t, t_0) + \int_{t_0}^t q(s) e_{\Theta p}(\sigma(s), t_0) e_{\Theta p}(t_0, t) \Delta s \\ &= y_0 e_p(t, t_0) + \int_{t_0}^t q(s) e_{\Theta p}(\sigma(s), t) \Delta s, \quad t \in \mathbb{T}^\kappa. \end{aligned}$$

We differentiate with respect to t the last equality and we get

$$\begin{aligned} y^\Delta(t) &= y_0 p(t) e_p(t, t_0) + p(t) \int_{t_0}^t e_{\Theta p}(\sigma(s), t) q(s) \Delta s + e_{\Theta p}(\sigma(t), \sigma(t)) q(t) \\ &= p(t) \left(y_0 e_p(t, t_0) + \int_{t_0}^t e_{\Theta p}(\sigma(s), t) q(s) \Delta s \right) + q(t) \\ &= p(t) y(t) + q(t), \quad t \in \mathbb{T}^\kappa. \end{aligned}$$

Moreover, we have

$$\begin{aligned} y(t_0) &= y_0 e_p(t_0, t_0) + \int_{t_0}^{t_0} e_{\Theta p}(\sigma(s), t) q(s) \Delta s \\ &= y_0. \end{aligned}$$

Assume that the IVP (1.2.1), (1.2.2) has two solutions y_1 and y_2 . We set

$$v = y_1 - y_2.$$

Then v is a solution of the IVP

$$\begin{aligned} v^\Delta &= p(t)v, \\ v(t_0) &= 0. \end{aligned}$$

Hence and (1.2.3), we get that

$$v = 0 \quad \text{on} \quad \mathbb{T},$$

i.e.,

$$y_1 = y_2 \quad \text{on} \quad \mathbb{T}.$$

This completes the proof.

✓ Example

Let $\mathbb{T} = \mathbb{Z}$. Consider the following IVP

$$\begin{aligned} y^\Delta &= ty + t^2 - 1, \quad t \geq 0, \quad t \in \mathbb{T}, \\ y(0) &= 2. \end{aligned}$$

Here

$$\begin{aligned} y_0 &= 2, \\ \sigma(t) &= t + 1, \\ \mu(t) &= 1, \\ p(t) &= t, \\ q(t) &= t^2 - 1, \quad t \in \mathbb{T}, \end{aligned}$$

and

$$\begin{aligned}
 e_p(t, 0) &= (1 + p(t))^t \\
 &= (1 + t)^t, \\
 e_p(t, \sigma(s)) &= (1 + p(t))^{t - \sigma(s)} \\
 &= (1 + t)^{t - \sigma(s)}, \quad s \in [0, t]_{\mathbb{T}}, \quad t \in \mathbb{T}.
 \end{aligned}$$

Then, applying (1.2.3), we arrive at

$$\begin{aligned}
 y(t) &= 2(1 + t)^t + \int_0^t (1 + t)^{t - \sigma(s)} (s^2 - 1) \Delta s \\
 &= 2(1 + t)^t + \sum_{s=0}^{t-1} (1 + t)^{t - s - 1} (s^2 - 1), \quad t \geq 0, \quad t \in \mathbb{T}.
 \end{aligned}$$

✓ Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the IVP

$$\begin{aligned}
 y^\Delta &= e_2(t, 1)y + \sin_3(t, 1), \quad t \in \mathbb{T}, \quad t \geq 1, \\
 y(1) &= 3.
 \end{aligned}$$

Here

$$\begin{aligned}
 y_0 &= 3, \\
 \sigma(t) &= 2t, \\
 \mu(t) &= t, \\
 p(t) &= e_2(t, 1), \\
 q(t) &= \sin_3(t, 1), \quad t \geq 1, \quad t \in \mathbb{T}.
 \end{aligned}$$

Then

$$\begin{aligned}
 e_2(t, 1) &= \prod_{s \in [1, t]_{\mathbb{T}}} (1 + 2s), \\
 e_{3i}(t, 1) &= \prod_{s \in [1, t]_{\mathbb{T}}} (1 + 3is), \\
 e_{-3i}(t, 1) &= \prod_{s \in [1, t]_{\mathbb{T}}} (1 - 3is), \\
 \sin_3(t, 1) &= \frac{1}{2i} \left(\prod_{s \in [1, t]_{\mathbb{T}}} (1 + 3is) - \prod_{s \in [1, t]_{\mathbb{T}}} (1 - 3is) \right), \quad t \in \mathbb{T}, \quad t \geq 1,
 \end{aligned}$$

and

$$\begin{aligned}
 e_p(t, 1) &= \prod_{s \in [1, t]_{\mathbb{T}}} \left(1 + s \prod_{\tau \in [1, s]_{\mathbb{T}}} (1 + 2\tau) \right), \\
 e_p(t, \sigma(s)) &= \prod_{\tau \in [\sigma(s), t]_{\mathbb{T}}} \left(1 + \tau \prod_{y \in [1, \tau]_{\mathbb{T}}} (1 + 2y) \right).
 \end{aligned}$$

Therefore, applying (1.2.3), we find

$$\begin{aligned}
 y(t) &= 3 \prod_{s \in [1, t]_{\mathbb{T}}} \left(1 + s \prod_{\tau \in [1, s]_{\mathbb{T}}} (1 + 2\tau) \right) \\
 &\quad + \frac{1}{2i} \int_1^t \prod_{\tau \in [\sigma(s), t]_{\mathbb{T}}} \left(1 + \tau \prod_{y \in [1, \tau]_{\mathbb{T}}} (1 + 2y) \right) \\
 &\quad \left(\prod_{\tau \in [1, s]_{\mathbb{T}}} (1 + 3i\tau) - \prod_{\tau \in [1, s]_{\mathbb{T}}} (1 - 3i\tau) \right) \\
 &= 3 \prod_{s \in [1, t]_{\mathbb{T}}} \left(1 + s \prod_{\tau \in [1, s]_{\mathbb{T}}} (1 + 2\tau) \right) \\
 &\quad + \frac{1}{2i} \sum_{s=1}^{\frac{t}{2}} s \left(\prod_{\tau \in [\sigma(s), t]_{\mathbb{T}}} \left(1 + \tau \prod_{y \in [1, \tau]_{\mathbb{T}}} (1 + 2y) \right) \right. \\
 &\quad \left. \left(\prod_{\tau \in [1, s]_{\mathbb{T}}} (1 + 3i\tau) - \prod_{\tau \in [1, s]_{\mathbb{T}}} (1 - 3i\tau) \right) \right),
 \end{aligned}$$

$t \in \mathbb{T}, \quad t \geq 1..$

? Exercise

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Solve the following IVP

$$\begin{aligned}
 y^\Delta &= \frac{1}{1+t} y - e_2(t, 1) - t^2, \quad t \in \mathbb{T}, \quad t \geq 1, \\
 y(1) &= 4.
 \end{aligned}$$

Now, we consider the following IVP

$$y^\Delta(t) = -p(t)y^\sigma(t) + q(t), \quad t \in \mathbb{T}, \quad (1.2.5)$$

$$y(t_0) = y_0, \quad (1.2.6)$$

where $p, q \in \mathcal{C}_{rd}(\mathbb{T})$, $p \in \mathcal{R}$, $y_0 \in \mathbb{R}$.

Definition

Equations of the form (1.2.5) are said to be linear first order dynamic equations.

Theorem

The problem (1.2.5), (1.2.6) has a unique solution represented in the form

$$y(t) = y_0 e_{\ominus p}(t, t_0) + \int_{t_0}^t e_{\ominus p}(t, s) q(s) \Delta s, \quad t \in \mathbb{T}^\kappa. \quad (1.2.7)$$

Proof

We multiply the equation (1.2.5) with $e_p(t, t_0)$, $t \in \mathbb{T}^\kappa$ and we find

$$y^\Delta(t) e_p(t, t_0) + p(t) e_p(t, t_0) y(\sigma(t)) = q(t) e_p(t, t_0), \quad t \in \mathbb{T}^\kappa,$$

or

$$(y e_p(\cdot, t_0))^\Delta(t) = e_p(t, t_0) q(t), \quad t \in \mathbb{T}^\kappa.$$

Now, we integrate the last equation from t_0 to t and using the initial condition (1.2.6), we arrive at

$$\int_{t_0}^t (ye_p(\cdot, t_0))^{\Delta}(s) \Delta s = \int_{t_0}^t e_p(s, t_0) q(s) \Delta s, \quad t \in \mathbb{T}^{\kappa},$$

whereupon

$$y(t)e_p(t, t_0) - y(t_0) = \int_{t_0}^t e_p(s, t_0) q(s) \Delta s, \quad t \in \mathbb{T}^{\kappa},$$

or

$$y(t)e_p(t, t_0) = y_0 + \int_{t_0}^t e_p(s, t_0) q(s) \Delta s, \quad t \in \mathbb{T},$$

and

$$\begin{aligned} y(t) &= y_0 e_{\Theta p}(t, t_0) + \int_{t_0}^t e_{\Theta p}(t_0, s) e_{\Theta p}(t, t_0) q(s) \Delta s \\ &= y_0 e_{\Theta p}(t, t_0) + \int_{t_0}^t e_{\Theta p}(t, s) q(s) \Delta s, \quad t \in \mathbb{T}^{\kappa}. \end{aligned}$$

Now, we consider (1.2.7). Observe that

$$\begin{aligned} y^{\sigma}(t) &= y_0 e_{\Theta p}(\sigma(t), t_0) + \int_{t_0}^{\sigma(t)} e_{\Theta p}(\sigma(t), s) q(s) \Delta s \\ &= y_0 (1 + (\Theta p)(t) \mu(t)) e_{\Theta p}(t, t_0) \\ &\quad + \int_{t_0}^t e_{\Theta p}(\sigma(t), s) q(s) \Delta s + \int_t^{\sigma(t)} e_{\Theta p}(\sigma(t), s) q(s) \Delta s \\ &= \frac{y_0}{1 + \mu(t)p(t)} e_{\Theta p}(t, t_0) + \int_{t_0}^t e_{\Theta p}(\sigma(t), t) e_{\Theta p}(t, s) q(s) \Delta s \\ &\quad + \mu(t) e_{\Theta p}(\sigma(t), t) q(t) \\ &= \frac{y_0}{1 + \mu(t)p(t)} e_{\Theta p}(t, t_0) + \frac{1}{1 + \mu(t)p(t)} \int_{t_0}^t e_{\Theta p}(t, s) q(s) \Delta s \\ &\quad + \frac{\mu(t)q(t)}{1 + \mu(t)p(t)} \\ &= \frac{1}{1 + \mu(t)p(t)} \left(y_0 e_{\Theta p}(t, t_0) + \int_{t_0}^t e_{\Theta p}(t, s) q(s) \Delta s + \mu(t)q(t) \right), \quad t \in \mathbb{T}^{\kappa}. \end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
 y^\Delta(t) &= y_0(\Theta p)(t)e_{\Theta p}(t, t_0) + e_{\Theta p}(\sigma(t), t)q(t) \\
 &\quad + (\Theta p)(t) \int_{t_0}^t e_{\Theta p}(t, s)q(s)\Delta s \\
 &= y_0(\Theta p)(t)e_{\Theta p}(t, t_0) + \frac{q(t)}{1 + \mu(t)p(t)} \\
 &\quad + (\Theta p)(t) \int_{t_0}^t e_{\Theta p}(t, s)q(s)\Delta s \\
 &= (\Theta p)(t) \left(y_0 e_{\Theta p}(t, t_0) + \int_{t_0}^t e_{\Theta p}(t, s)q(s)\Delta s \right) \\
 &\quad + \frac{q(t)}{1 + \mu(t)p(t)} \\
 &= -\frac{p(t)}{1 + \mu(t)p(t)} \left(y_0 e_{\Theta p}(t, t_0) + \int_{t_0}^t e_{\Theta p}(t, s)q(s)\Delta s + \mu(t)q(t) \right) \\
 &\quad + \frac{\mu(t)p(t)q(t)}{1 + \mu(t)p(t)} + \frac{q(t)}{1 + \mu(t)p(t)} \\
 &= -p(t)y^\sigma(t) + q(t), \quad t \in \mathbb{T}^\kappa.
 \end{aligned}$$

We put $t = t_0$ in (1.2.7) and we find

$$y(t_0) = y_0.$$

Now, we assume that the IVP (1.2.5), (1.2.6) has two solutions y_1 and y_2 . Let

$$v = y_1 - y_2.$$

Then, v satisfies the IVP

$$\begin{aligned}
 v^\Delta &= -p(t)v^\sigma, \quad t \in \mathbb{T}^\kappa, \\
 v(t_0) &= 0.
 \end{aligned} \tag{1.2.8}$$

Applying (1.2.7), we find

$$v(t) = 0, \quad t \in \mathbb{T}^\kappa,$$

from where

$$y_1 \equiv y_2 \quad \text{on } \mathbb{T}.$$

This completes the proof.

✓ Example

Let $\mathbb{T} = \mathbb{Z}$. Consider the IVP

$$\begin{aligned}
 y^\Delta &= -t^2 y^\sigma + t + 2, \quad t \in \mathbb{T}, \\
 y(t_0) &= -1.
 \end{aligned}$$

Here

$$\begin{aligned}
 \sigma(t) &= t + 1, \\
 \mu(t) &= 1, \\
 p(t) &= t^2, \\
 q(t) &= t + 2, \quad t \in \mathbb{T}.
 \end{aligned}$$

Then

$$\begin{aligned}(\Theta p)(t) &= -\frac{p(t)}{1 + \mu(t)p(t)} \\ &= -\frac{t^2}{1 + t^2}, \quad t \in \mathbb{T},\end{aligned}$$

and

$$\begin{aligned}e_{\Theta p}(t, s) &= \left(1 - \frac{t^2}{1 + t^2}\right)^{t-s} \\ &= \frac{1}{(1 + t^2)^{t-s}}, \\ e_{\Theta p}(t, 0) &= \frac{1}{(1 + t^2)^t}, \quad t, s \in \mathbb{T}, \quad s \leq t.\end{aligned}$$

Hence, applying (1.2.7), we find

$$\begin{aligned}y(t) &= -\frac{1}{(1 + t^2)^t} + \int_0^t \frac{1}{(1 + t^2)^{t-s}} (s + 2) \Delta s \\ &= -\frac{1}{(1 + t^2)^t} + \sum_{s=0}^{t-1} \frac{s + 2}{(1 + t^2)^{t-s}}, \quad t \in \mathbb{T}.\end{aligned}$$

✓ Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Consider the IVP

$$\begin{aligned}y^\Delta &= ty^\sigma + 1, \quad t \in \mathbb{T}, \\ y(1) &= 1.\end{aligned}$$

Here

$$\begin{aligned}\sigma(t) &= 3t, \\ \mu(t) &= 2t, \\ p(t) &= -t, \\ q(t) &= 1, \quad t \in \mathbb{T}^\kappa.\end{aligned}$$

We have

$$\begin{aligned}(\Theta p)(t) &= -\frac{-t}{1 + 2t(-t)} \\ &= \frac{t}{1 - 2t^2}, \quad t \in \mathbb{T}^\kappa,\end{aligned}$$

and

$$\begin{aligned}e_{\Theta p}(t, s) &= \prod_{\tau \in [s, t)_{\mathbb{T}}} \left(1 + \frac{2\tau \tau^2}{1 - 2\tau \tau^2}\right) \\ &= \prod_{\tau \in [s, t)_{\mathbb{T}}} \frac{1}{1 - 2\tau \tau^2}, \\ e_{\Theta p}(t, 1) &= \prod_{\tau \in [1, t)_{\mathbb{T}}} \frac{1}{1 - 2\tau \tau^2}, \quad s, t \in \mathbb{T}, \quad s \leq t.\end{aligned}$$

Now, applying (1.2.7), we find

$$\begin{aligned}
 y(t) &= \prod_{\tau \in [1, t)_{\mathbb{T}}} \frac{1}{1 - 2\tau u^2} + \int_1^t \prod_{\tau \in [s, t)_{\mathbb{T}}} \frac{1}{1 - 2\tau u^2} \Delta\tau u \\
 &= \prod_{\tau \in [1, t)_{\mathbb{T}}} \frac{1}{1 - 2\tau u^2} + \sum_{s=1}^{\frac{t}{2}} \prod_{\tau \in [s, t)_{\mathbb{T}}} \frac{1}{1 - 2\tau u^2}, \quad t \in \mathbb{T}.
 \end{aligned}$$

? Exercise

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Solve the following IVP

$$\begin{aligned}
 y^\Delta &= \frac{1}{1 + t^2} y^\sigma - t, \quad t \in \mathbb{T}^\kappa, \\
 y(1) &= 7.
 \end{aligned}$$

This page titled [1.2: Linear First Order Dynamic Equations](#) is shared under a [GNU General Public License 3.0](#) license and was authored, remixed, and/or curated by [Svetlin G. Georgiev](#).

1.3: Bernoulli Dynamic Equations

Consider the following IVP

$$y^\Delta = -p(t)y^\sigma + q(t)yy^\sigma, \quad t \in \mathbb{T}^\kappa, \quad (1.3.1)$$

$$y(t_0) = y_0, \quad (1.3.2)$$

where $p, q \in \mathcal{C}_{rd}(\mathbb{T})$, $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$.

Definition

Equations of the form (1.3.1) are said to be Bernoulli equations.

Suppose that $y_0 \neq 0$, $y(t)y^\sigma(t) \neq 0$, $t \in \mathbb{T}^\kappa$, and y is a solution of the IVP (1.3.1), (1.3.2). Then

$$\frac{y^\Delta(t)}{y(t)y^\sigma(t)} = -p(t)\frac{1}{y(t)} + q(t), \quad t \in \mathbb{T}^\kappa.$$

We set

$$v(t) = \frac{1}{y(t)}, \quad t \in \mathbb{T}^\kappa.$$

Then

$$v^\Delta(t) = -\frac{y^\Delta(t)}{y(t)y^\sigma(t)}, \quad t \in \mathbb{T}^\kappa,$$

and

$$\begin{aligned} v(t_0) &= \frac{1}{y(t_0)} \\ &= \frac{1}{y_0}. \end{aligned}$$

Hence, we get the following IVP

$$\begin{aligned} -v^\Delta(t) &= -p(t)v(t) + q(t), \quad t \in \mathbb{T}^\kappa, \\ v(t_0) &= \frac{1}{y_0}, \end{aligned}$$

or

$$\begin{aligned} v^\Delta(t) &= p(t)v(t) - q(t), \quad t \in \mathbb{T}^\kappa, \\ v(t_0) &= \frac{1}{y_0}. \end{aligned}$$

Applying the formulae for the solution of linear first order dynamic equation, we find

$$v(t) = e_p(t, t_0) \frac{1}{y_0} - \int_{t_0}^t e_p(t, \sigma(\tau)) q(\tau) \Delta\tau, \quad t \in \mathbb{T}^\kappa,$$

and then

$$\begin{aligned} y(t) &= \frac{1}{e_p(t, t_0) \frac{1}{y_0} - \int_{t_0}^t e_p(t, \sigma(\tau)) q(\tau) \Delta\tau} \\ &= \frac{y_0}{e_p(t, t_0) - y_0 \int_{t_0}^t e_p(t, \sigma(\tau)) q(\tau) \Delta\tau}, \quad t \in \mathbb{T}^\kappa. \end{aligned} \quad (1.3.3)$$

✓ Example

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Consider the following IVP

$$\begin{aligned} y^\Delta &= -(t+1)y^\sigma + (t^2-t)yy^\sigma, \quad t \in \mathbb{T}, \\ y(1) &= \frac{1}{3}. \end{aligned}$$

Here

$$\begin{aligned} \sigma(t) &= 3t, \\ \mu(t) &= 2t, \\ p(t) &= t+1, \\ q(t) &= t^2-t, \quad t \in \mathbb{T}, \\ t_0 &= 1, \\ y_0 &= \frac{1}{3}. \end{aligned}$$

Then

$$\begin{aligned} e_p(t, t_0) &= \prod_{s \in [1, t)_{\mathbb{T}}} (1 + 2s(s+1)), \\ e_p(t, \sigma(\tau)) &= \prod_{s \in [2\tau, t)_{\mathbb{T}}} (1 + 2s(s+1)), \quad t, \tau \in \mathbb{T}, \quad \tau < t. \end{aligned}$$

Hence, a solution of the considered IVP is as follows.

$$\begin{aligned} y(t) &= \frac{1}{3 \prod_{s \in [1, t)_{\mathbb{T}}} (1 + 2s(s+1)) - \int_1^t \left(\prod_{s \in [2\tau, t)_{\mathbb{T}}} (1 + 2s(s+1)) \right) (\tau^2 - \tau) \Delta \tau} \\ &= \frac{1}{3 \prod_{s \in [1, t)_{\mathbb{T}}} (1 + 2s(s+1)) - \prod_{\tau \in [1, t)_{\mathbb{T}}} 2\tau^2(\tau-1) \left(\prod_{s \in [2\tau, t)_{\mathbb{T}}} (1 + 2s(s+1)) \right)}, \quad t \in \mathbb{T}, \quad t > 1. \end{aligned} \tag{1.3.4}$$

? Exercise

Let $\mathbb{T} = 11^{\mathbb{N}_0}$. Find a solution of the following IVP

$$\begin{aligned} y^\Delta &= 2y^\sigma + t^2yy^\sigma, \quad t \in \mathbb{T}, \\ y(1) &= 3. \end{aligned}$$

Now, we consider the IVP

$$y^\Delta = p(t)y + q(t)yy^\sigma, \quad t \in \mathbb{T}^\kappa, \tag{1.3.5}$$

subject to the initial conditions

$$y(t_0) = y_0, \tag{1.3.6}$$

where $p, q \in \mathcal{C}_{rd}(\mathbb{T})$, $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$.

Definition

Equations of the form (1.3.5) are said to be Bernoulli equations.

Suppose that $y_0 \neq 0$ and $y(t)y^\sigma(t) \neq 0$, $t \in \mathbb{T}^\kappa$, and y is a solution of the considered IVP. We divide both sides of the equation (1.3.5) by yy^σ and we find

$$\frac{y^\Delta(t)}{y(t)y^\sigma(t)} = \frac{p(t)}{y^\sigma(t)} + q(t), \quad t \in \mathbb{T}^\kappa.$$

Set

$$z(t) = \frac{1}{y(t)}, \quad t \in \mathbb{T}^\kappa.$$

Then

$$z^\Delta(t) = -\frac{y^\Delta(t)}{y(t)y^\sigma(t)}, \quad t \in \mathbb{T}^\kappa,$$

and

$$\begin{aligned} z(t_0) &= \frac{1}{y(t_0)} \\ &= \frac{1}{y_0}. \end{aligned}$$

Thus, we get the IVP

$$\begin{aligned} -z^\Delta(t) &= p(t)z^\sigma(t) + q(t), \quad t \in \mathbb{T}^\kappa, \\ z(t_0) &= \frac{1}{y_0}, \end{aligned}$$

or

$$\begin{aligned} z^\Delta(t) &= -p(t)z^\sigma(t) - q(t), \quad t \in \mathbb{T}^\kappa, \\ z(t_0) &= \frac{1}{y_0}. \end{aligned}$$

We get a solution of the last IVP

$$z(t) = \frac{1}{y_0} e_{\ominus p}(t, t_0) + \int_{t_0}^t e_{\ominus p}(t, s) q(s) \Delta s, \quad t \in \mathbb{T}^\kappa.$$

Then, a solution of the considered IVP is given by

$$y(t) = \frac{1}{\frac{1}{y_0} e_{\ominus p}(t, t_0) + \int_{t_0}^t e_{\ominus p}(t, s) q(s) \Delta s}, \quad t \in \mathbb{T}^\kappa,$$

or

$$y(t) = \frac{y_0}{e_{\ominus p}(t, t_0) + y_0 \int_{t_0}^t e_{\ominus p}(t, s) q(s) \Delta s}, \quad t \in \mathbb{T}^\kappa.$$

✓ Example

Let $\mathbb{T} = 3\mathbb{Z}$. Consider the following IVP

$$\begin{aligned} y^\Delta(t) &= 2(t+1)y + 3yy^\sigma, \quad t \in \mathbb{T}, \quad t > 0, \\ y(0) &= 2. \end{aligned}$$

Here

$$\begin{aligned}\sigma(t) &= t + 1, \\ \mu(t) &= 1, \\ p(t) &= 2t + 2, \\ q(t) &= 3, \quad t \in \mathbb{T}, \\ t_0 &= 0, \\ y_0 &= 2.\end{aligned}$$

We have

$$\begin{aligned}1 + 3(2t + 2) &= 6t + 7 \\ &\neq 0, \quad t \in \mathbb{T},\end{aligned}$$

i.e., $-p \in \mathcal{R}$. Next,

$$\begin{aligned}(\Theta p)(t) &= -\frac{p(t)}{1 + \mu(t)p(t)} \\ &= -\frac{2(t+1)}{6t+7}, \quad t \in \mathbb{T},\end{aligned}$$

and

$$\begin{aligned}e_{\Theta p}(t, t_0) &= \prod_{\tau \in [0, t)_{\mathbb{T}}} \left(\frac{1}{6\tau + 7} \right)^{\frac{1}{3}}, \\ e_{\Theta p}(t, s) &= \prod_{\tau \in [s, t)_{\mathbb{T}}} \left(\frac{1}{6\tau + 7} \right)^{\frac{t-\tau}{3}}, \quad s, t \in \mathbb{T}, \quad s < t.\end{aligned}$$

Hence,

$$y(t) = \frac{y_0}{e_{\Theta p}(t, t_0) + y_0 \int_{t_0}^t e_{\Theta p}(t, s) q(s) \Delta s}, \quad t \in \mathbb{T}^{\kappa}.$$

? Exercise

Let $\mathbb{T} = \mathbb{T}^{\mathbf{N}_0}$. Find a solution of the following IVP

$$\begin{aligned}y^{\Delta} &= ty + yy^{\sigma}, \quad t \in \mathbb{T}, \\ y(1) &= -\frac{1}{4}.\end{aligned}$$

This page titled [1.3: Bernoulli Dynamic Equations](#) is shared under a [GNU General Public License 3.0](#) license and was authored, remixed, and/or curated by [Svetlin G. Georgiev](#).

1.4: Riccati Dynamic Equations

Consider the equation

$$y^\Delta = p(t)y + q(t)y^\sigma + r(t)yy^\sigma + f(t), \quad t \in \mathbb{T}^\kappa, \quad (1.4.1)$$

subject to the initial condition

$$y(t_0) = y_0,$$

where $p, q, r, f \in \mathcal{C}_{rd}(\mathbb{T})$, $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$.

Definition

Equations of the form (1.4.1) are said to be Riccati dynamic equations.

Suppose that $x \in \mathcal{C}_{rd}^1(\mathbb{T})$ is its particular solution. Then

$$x^\Delta(t) = p(t)x(t) + q(t)x^\sigma(t) + r(t)x(t)x^\sigma(t) + f(t), \quad t \in \mathbb{T}^\kappa. \quad (1.4.2)$$

Let

$$y(t) = z(t) + x(t), \quad t \in \mathbb{T}^\kappa.$$

Then

$$\begin{aligned} y^\Delta(t) &= z^\Delta(t) + x^\Delta(t), \\ y^\sigma(t) &= z^\sigma(t) + x^\sigma(t), \quad t \in \mathbb{T}^\kappa. \end{aligned}$$

We substitute into (1.4.1) and we find

$$\begin{aligned} z^\Delta(t) + x^\Delta(t) &= p(t)(z(t) + x(t)) + q(t)(z^\sigma(t) + x^\sigma(t)) \\ &\quad + r(t)(z(t) + x(t))(z^\sigma(t) + x^\sigma(t)) + f(t) \\ &= p(t)z(t) + p(t)x(t) + q(t)z^\sigma(t) + q(t)x^\sigma(t) \\ &\quad + r(t)z(t)z^\sigma(t) + r(t)z(t)x^\sigma(t) + r(t)x(t)z^\sigma(t) + r(t)x(t)x^\sigma(t) + f(t) \\ &= p(t)x(t) + q(t)x^\sigma(t) + r(t)x(t)x^\sigma(t) + f(t) \\ &\quad + (p(t) + r(t)x^\sigma(t))z(t) + (q(t) + r(t)x(t))z^\sigma(t) + r(t)z(t)z^\sigma(t), \quad t \in \mathbb{T}^\kappa, \end{aligned}$$

from where, using (1.4.2), we arrive at the equation

$$\begin{aligned} z^\Delta(t) &= (p(t) + r(t)x^\sigma(t))z(t) + (q(t) + r(t)x(t))z^\sigma(t) \\ &\quad + r(t)z(t)z^\sigma(t), \quad t \in \mathbb{T}^\kappa, \end{aligned}$$

which is a Bernoulli dynamic equation.

Therefore, if we know a particular solution of the equation (1.4.1) we can find other its solutions reducing it to a Bernoulli dynamic equation.

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the equation

$$y^\Delta = -t(t+1)y^\sigma + \frac{1}{2}ty + tyy^\sigma - 4t^5 + 4t^4 + \frac{7}{2}t^3 + 3t, \quad t \in \mathbb{T}.$$

Firstly, we will show that

$$x(t) = t^2, \quad t \in \mathbb{T},$$

is its particular solution. Really, we have

$$\begin{aligned}
 \sigma(t) &= 2t, \\
 x^\sigma(t) &= (\sigma(t))^2 \\
 &= (2t)^2 \\
 &= 4t^2, \\
 x^\Delta(t) &= \sigma(t) + t \\
 &= 3t, \quad t \in \mathbb{T}.
 \end{aligned}$$

Then

$$\begin{aligned}
 &-(t^2 + t)x^\sigma(t) + \frac{1}{2}tx(t) + tx(t)x^\sigma(t) - 4t^5 + 4t^4 + \frac{7}{2}t^3 + 3t \\
 &= -(t^2 + t)(4t^2) + \frac{1}{2}t(t^2) + t(t^2) + t(t^2)(4t^2) - 4t^5 + 4t^4 + \frac{7}{2}t^3 + 3t \\
 &= -4t^4 - 4t^3 + \frac{1}{2}t^3 + 4t^5 - 4t^5 + 4t^4 + \frac{7}{2}t^3 + 3t \\
 &= 3t \\
 &= x^\Delta(t), \quad t \in \mathbb{T}.
 \end{aligned}$$

Now, we set

$$y(t) = z(t) + t^2, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned}
 y^\Delta(t) &= z^\Delta(t) + 3t, \\
 y^\sigma(t) &= z^\sigma(t) + (\sigma(t))^2 \\
 &= z^\sigma(t) + (2t)^2 \\
 &= z^\sigma(t) + 4t^2, \quad t \in \mathbb{T}.
 \end{aligned}$$

We substitute in the given equation and we arrive at

$$\begin{aligned}
 z^\Delta(t) + 3t &= -(t^2 + t)(z^\sigma(t) + 4t^2) + \frac{1}{2}t(z(t) + t^2) \\
 &\quad + t(z(t) + t^2)(z^\sigma(t) + 4t^2) - 4t^5 + 4t^4 + \frac{7}{2}t^3 + 3t \\
 &= -(t^2 + t)z^\sigma(t) - 4t^4 - 4t^3 + \frac{1}{2}tz(t) + \frac{1}{2}t^3 \\
 &\quad + tz(t)z^\sigma(t) + 4t^3z(t) + t^3z^\sigma(t) + 4t^5 \\
 &\quad - 4t^5 + 4t^4 + \frac{7}{2}t^3 + 3t \\
 &= (t^3 - t^2 - t)z^\sigma(t) + \left(4t^3 + \frac{1}{2}t\right)z(t) + tz(t)z^\sigma(t) + 3t, \quad t \in \mathbb{T},
 \end{aligned}$$

whereupon

$$z^\Delta(t) = (t^3 - t^2 - t)z^\sigma(t) + \left(4t^3 + \frac{1}{2}t\right)z(t) + tz(t)z^\sigma(t), \quad t \in \mathbb{T}.$$

Suppose that $z(t)z^\sigma(t) \neq 0$, $t \in \mathbb{T}$. Then

$$\frac{z^\Delta(t)}{z(t)z^\sigma(t)} = (t^3 - t^2 - t)\frac{1}{z(t)} + \left(4t^3 + \frac{1}{2}t\right)\frac{1}{z^\sigma(t)} + t, \quad t \in \mathbb{T}.$$

Set

$$v(t) = \frac{1}{z(t)}, \quad t \in \mathbb{T}.$$

Hence,

$$v^{\Delta}(t) = -\frac{z^{\Delta}(t)}{z(t)z^{\sigma}(t)},$$

$$v^{\sigma}(t) = \frac{1}{z^{\sigma}(t)}, \quad t \in \mathbb{T}.$$

Then, we get the equation

$$-v^{\Delta}(t) = (t^3 - t^2 - t)v(t) + \left(4t^3 + \frac{1}{2}t\right)v^{\sigma}(t) + t, \quad t \in \mathbb{T},$$

or

$$\begin{aligned} v^{\Delta}(t) &= -(t^3 - t^2 - t)v(t) - \left(4t^3 + \frac{1}{2}t\right)v^{\sigma}(t) - t \\ &= -(t^3 - t^2 - t)v(t) - \left(4t^3 + \frac{1}{2}t\right)(v(t) + tv^{\Delta}(t)) - t \\ &= -(t^3 - t^2 - t)v(t) - \left(4t^3 + \frac{1}{2}t\right)v(t) - t^2\left(4t^2 + \frac{1}{2}\right)v^{\Delta}(t) - t \\ &= -\left(5t^3 - t^2 - \frac{1}{2}t\right)v(t) - t^2\left(4t^2 + \frac{1}{2}\right)v^{\Delta}(t) - t, \quad t \in \mathbb{T}, \end{aligned}$$

whereupon

$$\left(1 + t^2\left(4t^2 + \frac{1}{2}\right)\right)v^{\Delta}(t) = -\left(5t^3 - t^2 - \frac{1}{2}t\right)v(t) - t, \quad t \in \mathbb{T},$$

or

$$v^{\Delta}(t) = -\frac{5t^3 - t^2 - \frac{1}{2}t}{1 + t^2\left(4t^2 + \frac{1}{2}\right)}v(t) - \frac{t}{1 + t^2\left(4t^2 + \frac{1}{2}\right)}, \quad t \in \mathbb{T}.$$

Set

$$f_1(t) = -\frac{5t^3 - t^2 - \frac{1}{2}t}{1 + t^2\left(4t^2 + \frac{1}{2}\right)},$$

$$f_2(t) = -\frac{t}{1 + t^2\left(4t^2 + \frac{1}{2}\right)}, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned} e_{f_1}(t, 1) &= \prod_{s=1}^t \left(1 - s \frac{5s^3 - s^2 - \frac{1}{2}s}{1 + s^2\left(4s^2 + \frac{1}{2}\right)}\right) \\ &= \prod_{s=1}^t \left(\frac{1 + 4s^4 + \frac{1}{2}s^2 - 5s^4 + s^3 + \frac{1}{2}s^2}{1 + s^2\left(4s^2 + \frac{1}{2}\right)}\right) \\ &= \prod_{s=1}^t \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2\left(4s^2 + \frac{1}{2}\right)}\right), \quad t \in \mathbb{T}, \end{aligned}$$

provided that $f_1 \in \mathcal{R}$, and

$$e_{f_1}(t, \sigma(\tau)) = \prod_{s=2\tau}^t \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2\left(4s^2 + \frac{1}{2}\right)}\right), \quad t, \tau \in \mathbb{T}, \quad \tau < t,$$

provided that $f_1 \in \mathcal{R}$ Therefore

$$\begin{aligned}
 v(t) &= c e_{f_1}(t, 1) + \int_1^t e_{f_1}(t, \sigma(\tau)) f_2(\tau) \Delta \tau \\
 &= c \prod_{s=1}^{\frac{t}{2}} \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2 (4s^2 + \frac{1}{2})} \right) \\
 &\quad + \int_1^t \left(\prod_{s=2\tau}^{\frac{t}{2}} \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2 (4s^2 + \frac{1}{2})} \right) \right) \left(-\frac{\tau}{1 + \tau^2 (4\tau^2 + \frac{1}{2})} \right) \Delta \tau \\
 &= c \prod_{s=1}^{\frac{t}{2}} \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2 (4s^2 + \frac{1}{2})} \right) \\
 &\quad + \sum_{\tau=1}^{\frac{t}{2}} \tau \left(\prod_{s=2\tau}^{\frac{t}{2}} \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2 (4s^2 + \frac{1}{2})} \right) \right) \left(-\frac{\tau}{1 + \tau^2 (4\tau^2 + \frac{1}{2})} \right),
 \end{aligned}$$

provided that $f_1 \in \mathcal{R}$, $t \in \mathbb{T}$, $t > 1$. Here c is a nonzero constant. Therefore

$$z(t) = \frac{1}{c \prod_{s=1}^{\frac{t}{2}} \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2 (4s^2 + \frac{1}{2})} \right) + \sum_{\tau=1}^{\frac{t}{2}} \tau \left(\prod_{s=2\tau}^{\frac{t}{2}} \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2 (4s^2 + \frac{1}{2})} \right) \right) \left(-\frac{\tau}{1 + \tau^2 (4\tau^2 + \frac{1}{2})} \right)}, \quad t \in \mathbb{T}, \quad t > 1,$$

provided that $f_1 \in \mathcal{R}$, and

$$\begin{aligned}
 y(t) &= t^2 + \frac{1}{c \prod_{s=1}^{\frac{t}{2}} \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2 (4s^2 + \frac{1}{2})} \right) + \sum_{\tau=1}^{\frac{t}{2}} \tau \left(\prod_{s=2\tau}^{\frac{t}{2}} \left(\frac{-s^4 + s^3 + s^2 + 1}{1 + s^2 (4s^2 + \frac{1}{2})} \right) \right) \left(-\frac{\tau}{1 + \tau^2 (4\tau^2 + \frac{1}{2})} \right)}, \\
 &\quad t \in \mathbb{T}, \quad t > 1,
 \end{aligned}$$

provided that $f_1 \in \mathcal{R}$, are solutions of the considered Riccati equation.

? Exercise

Let $\mathbb{T} = 3^{\mathbb{N}_0}$. Find solutions of the equation

$$y^\Delta = 2y - 6y^\sigma - yy^\sigma + \frac{4}{(1+t)(1+3t)}, \quad t \in \mathbb{T},$$

if

$$x(t) = \frac{1}{1+t}, \quad t \in \mathbb{T},$$

is its solution.

This page titled [1.4: Riccati Dynamic Equations](#) is shared under a [GNU General Public License 3.0](#) license and was authored, remixed, and/or curated by [Svetlin G. Georgiev](#).

1.5: Dynamic Equations for Families of Curves

If $f(t, y)$ is defined and continuous in a domain D , then the solutions to the any dynamic equation

$$y^\Delta = f(t, y) \quad (1.5.1)$$

form a family of curves that covers D .

Conversely, if a given family of curves covers D simply, then it is possible to find a first order dynamic equation such that the curves in the family are the solutions of the dynamic equation. To see this, let $(\bar{t}, \bar{y}) \in D$ is an arbitrary point and $y = \phi(t)$ is the curve in the family that goes through this point. We define the function f by setting

$$f(\bar{t}, \bar{y}) = \phi^\Delta(\bar{t}),$$

then clearly each curve in the family is a solution of the dynamic equation (1.5.1).

✓ Example

Consider the family of concentric circles

$$t^2 + y^2 = r^2, \quad t \in \mathbb{T}, \quad t < r.$$

We differentiate both. sides of the last equation with respect to t and we get

$$\begin{aligned} 0 &= \sigma(t) + t + y^\Delta y + y^\sigma y \\ &= \sigma(t) + t + y^\Delta (y + y^\sigma), \quad t \in \mathbb{T}, \quad t < r, \end{aligned}$$

whereupon

$$y^\Delta + \frac{\sigma(t) + t}{y + y^\sigma} = 0, \quad t \in \mathbb{T}, \quad t < r,$$

is the dynamic equation that the given family of concentric circles satisfied. Note that the functions

$$y(t) = \pm \sqrt{r^2 - t^2}, \quad t \in \mathbb{T}, \quad t < r,$$

are solutions of the obtained equation. Really, we have

$$\begin{aligned} y^\Delta(t) &= \lim_{s \rightarrow \sigma(t)} \frac{y(\sigma(t)) - y(s)}{\sigma(t) - s} \\ &= \pm \lim_{s \rightarrow \sigma(t)} \frac{\sqrt{r^2 - (\sigma(t))^2} - \sqrt{r^2 - s^2}}{\sigma(t) - s} \\ &= \pm \lim_{s \rightarrow \sigma(t)} \frac{(\sqrt{r^2 - (\sigma(t))^2} - \sqrt{r^2 - s^2})(\sqrt{r^2 - (\sigma(t))^2} + \sqrt{r^2 - s^2})}{(\sigma(t) - s)(\sqrt{r^2 - (\sigma(t))^2} + \sqrt{r^2 - s^2})} \\ &= \pm \lim_{s \rightarrow \sigma(t)} \frac{s^2 - (\sigma(t))^2}{(\sigma(t) - s)(\sqrt{r^2 - (\sigma(t))^2} + \sqrt{r^2 - s^2})} \\ &= \mp \lim_{s \rightarrow \sigma(t)} \frac{\sigma(t) + s}{\sqrt{r^2 - (\sigma(t))^2} + \sqrt{r^2 - s^2}} \\ &= \mp \frac{\sigma(t) + t}{\sqrt{r^2 - (\sigma(t))^2} + \sqrt{r^2 - t^2}} \\ &= - \frac{\sigma(t) + t}{y^\sigma(t) + y(t)}, \quad t \in \mathbb{T}, \quad t < r. \end{aligned}$$

✓ Example

Now, we will find a dynamic equation

$$y^\Delta = f(y)$$

satisfied by the curve

$$y(t) = 4e_2(t, t_0) + 3, \quad t \in \mathbb{T}, \quad (1.5.2)$$

where $t_0 \in \mathbb{T}$ is fixed. We compute the derivative of y and we find

$$y^\Delta(t) = 8e_2(t, t_0), \quad t \in \mathbb{T}.$$

From here, we find

$$e_2(t, t_0) = \frac{y^\Delta(t)}{8}, \quad t \in \mathbb{T}.$$

We substitute the last expression into (1.5.2) and we arrive at

$$y(t) = 4 \left(\frac{y^\Delta(t)}{8} \right) + 3, \quad t \in \mathbb{T},$$

or

$$y^\Delta(t) = 2y(t) - 3, \quad t \in \mathbb{T}.$$

? Exercise

Find a dynamic equation of the form

$$y^\Delta = f(y)$$

satisfied by the curve

$$y(t) = 7e_3(t, t_0) - 4, \quad t \in \mathbb{T},$$

where t_0 is fixed.

This page titled [1.5: Dynamic Equations for Families of Curves](#) is shared under a [GNU General Public License 3.0](#) license and was authored, remixed, and/or curated by [Svetlin G. Georgiev](#).

1.6: Exact Equations

Suppose that the graininess function μ of the time scale $\mathbb{T}$ is delta differentiable on $\mathbb{T}$.

Let in the dynamic equation

$$y^\Delta = f(t, y) \quad (1.6.1)$$

the function f to be

$$f(t, y) = -\frac{M(t, y)}{N(t, y)}, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

where $\mathbb{T}_1$ is a time scale with forward jump operator and delta differentiation operator σ_1 and Δ_1 , respectively, and $M, N \in \mathcal{C}_{rd}^1(\mathbb{T} \times \mathbb{T}_1)$. Let also, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{T}_1$ be fixed and for any $y: \mathbb{T} \rightarrow \mathbb{T}_1$ one has

$$y(\sigma(t)) = \sigma_1(y(t)), \quad t \in \mathbb{T}.$$

Then, the equation (1.6.1) takes the form

$$y^\Delta = -\frac{M(t, y)}{N(t, y)}, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

or

$$M(t, y) + N(t, y)y^\Delta = 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \quad (1.6.2)$$

Definition

The equation (1.6.2) is said to be exact if there exists a function $u \in \mathcal{C}_{rd}^1(\mathbb{T} \times \mathbb{T}_1)$ such that

$$\begin{aligned} u_t^\Delta(t, y) &= M(t, y), \\ u_y^{\Delta_1}(\sigma(t), y) &= N(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned} \quad (1.6.3)$$

The nomenclature comes from the fact that

$$\begin{aligned} M(t, y(t)) + N(t, y)y^\Delta(t) &= u_t^\Delta(t, y(t)) + u_y^{\Delta_1}(\sigma(t), y)y^\Delta(t) \\ &= \frac{\Delta u}{\Delta t}(t, y(t)), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Note that if $u \in \mathcal{C}_{rd}^1(\mathbb{T} \times \mathbb{T}_1)$ satisfies the conditions (1.6.3), using that $M, N \in \mathcal{C}_{rd}^1(\mathbb{T} \times \mathbb{T}_1)$, we conclude that $u \in \mathcal{C}_{rd}^2(\mathbb{T} \times \mathbb{T}_1)$.

The sufficient and necessary condition the equation (1.6.2) to be exact is contained in the next result.

Theorem

Let $M, N \in \mathcal{C}_{rd}^1(\mathbb{T} \times \mathbb{T}_1)$. Then, the equation (1.6.2) is exact if and only if the following condition

$$N_t^\Delta(t, y) = (1 + \mu^\Delta(t)) M_y(\sigma(t), y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \quad (1.6.4)$$

Proof

Firstly, suppose that the equation (1.6.2) is an exact equation. Then the conditions (1.6.3) hold. From (1.6.3), we have

$$u_{ty}^{\Delta\Delta_1}(t, y) = M_y^\Delta(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

whereupon

$$u_{ty}^{\Delta\Delta_1}(\sigma(t), y) = M_y^\Delta(\sigma(t), y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \quad (1.6.5)$$

Moreover, we find

$$u_{yt}^{|Delta_1\sigma\Delta}(t, y) = N_t^\Delta(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \quad (1.6.6)$$

Since

$$\begin{aligned} u_{yt}^{\Delta_1\sigma\Delta}(t, y) &= (1 + \mu^\Delta(t)) u_{yt}^{\Delta_1\Delta\sigma}(t, y) \\ &= (1 + \mu^\Delta(t)) u_{yt}^{\Delta_1\Delta}(\sigma(t), y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

using (1.6.6), we get

$$(1 + \mu^\Delta(t)) u_{yt}^{\Delta_1\Delta}(\sigma(t), y) = N_t^\Delta(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

or

$$u_{yt}^{\Delta_1\Delta}(\sigma(t), y) = \frac{1}{1 + \mu^\Delta(t)} N_t^\Delta(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Because $u \in \mathcal{C}_{rd}^2(\mathbb{T} \times \mathbb{T}_1)$, we have that

$$u_{yt}^{\Delta_1\Delta}(\sigma(t), y) = u_{ty}^{\Delta_1\Delta}(\sigma(t), y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

By this observation, applying (1.6.5) and (1.6.6), we arrive at

$$\frac{1}{1 + \mu^\Delta(t)} N_t^\Delta(t, y) = M_y^{\Delta_1}(\sigma(t), y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

i.e., we get (1.6.4).

Conversely, if M and N satisfy (1.6.4), then the equation (1.6.2) is an exact equation. To establish this, we shall exhibit a function $u \in \mathcal{C}_{rd}^1(\mathbb{T} \times \mathbb{T}_1)$ that satisfies the conditions (1.6.3). We integrate both sides of the equation

$$u_t^{\Delta_1}(t, y) = M(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

and we find

$$u(t, y) = \int_{t_0}^t M(s, y) \Delta s + g(y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

where $g = g(y)$ is an arbitrary function of y that plays the role of the constant of integration. We shall get the function g by using the equation

$$u_y^{\Delta_1}(\sigma(t), y) = N(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Indeed, we have

$$\begin{aligned} u_y^{\Delta_1}(t, y) &= \frac{\Delta_1}{\Delta_1 y} \int_{t_0}^t M(s, y) \Delta s + g^\Delta(y) \\ &= \int_{t_0}^t M_y^{\Delta_1}(s, y) \Delta s + g^\Delta(y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

and

$$\begin{aligned} u_y^{\Delta_1}(\sigma(t), y) &= \int_{t_0}^{\sigma(t)} M_y^{\Delta_1}(s, y) \Delta s + g^\Delta(y) \\ &= N(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Since

$$\begin{aligned}
 \frac{\Delta}{\Delta t} \left(N(t, y) - \int_{t_0}^{\sigma(t)} M_y^{\Delta_1}(s, y) \Delta s \right) &= N_t^{\Delta}(t, y) - \left(\int_{t_0}^t M_y^{\Delta_1}(s, y) \Delta s \right)^{\sigma \Delta} \\
 &= N_t^{\Delta}(t, y) - (1 + \mu^{\Delta}(t)) \left(\int_{t_0}^t M_y^{\Delta_1}(s, y) \Delta s \right)^{\Delta \sigma} \\
 &= N_t^{\Delta}(t, y) (- (1 + \mu^{\Delta}(t))') (M_y^{\Delta_1}(t, y))^{\sigma} \\
 &= N_t^{\Delta}(t, y) - (1 + \mu^{\Delta}(t)) M_y^{\Delta_1}(\sigma(t), y) \\
 &= 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,
 \end{aligned}$$

the function

$$N(t, y) - \int_{t_0}^{\sigma(t)} M_y^{\Delta_1}(s, y) \Delta s$$

must depend on y only. Therefore

$$g^{\Delta}(y) = N(t, y) - \int_{t_0}^{\sigma(t)} M_y^{\Delta_1}(s, y) \Delta s, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

whereupon

$$\begin{aligned}
 g(y) &= \int_{y_0}^y N(t, x) \Delta_1 x - \int_{y_0}^y \left(\int_{t_0}^{\sigma(t)} M_y^{\Delta_1}(s, x) \Delta s \right) \Delta_1 x + g(y_0) \\
 &= \int_{y_0}^y N(t, x) \Delta_1 x - \int_{t_0}^{\sigma(t)} \left(\int_{y_0}^y M_y^{\Delta_1}(s, x) \Delta_1 x \right) \Delta s + g(y_0) \\
 &= \int_{y_0}^y N(t, x) \Delta_1 x - \int_{t_0}^{\sigma(t)} (M(s, y) - M(s, y_0)) \Delta s + g(y_0) \\
 &= \int_{y_0}^y N(t, x) \Delta_1 x - \int_{t_0}^{\sigma(t)} M(s, y) \Delta s + \int_{t_0}^{\sigma(t)} M(s, y_0) \Delta s + g(y_0), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.
 \end{aligned}$$

Consequently

$$\begin{aligned}
 u(t, y) &= \int_{t_0}^t M(s, y) \Delta s + \int_{y_0}^y N(t, x) \Delta_1 x - \int_{t_0}^{\sigma(t)} M(s, y) \Delta s + \int_{t_0}^{\sigma(t)} M(s, y_0) \Delta s + g(y_0) \\
 &= \int_{y_0}^y N(t, x) \Delta_1 x + \int_{t_0}^{\sigma(t)} M(s, y_0) \Delta s - \int_t^{\sigma(t)} M(s, y) \Delta s + g(y_0) \\
 &= \int_{y_0}^y N(t, x) \Delta_1 x + \int_{t_0}^{\sigma(t)} M(s, y_0) \Delta s - \mu(t) M(t, y) + g(y_0), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,
 \end{aligned}$$

and the solution of the equation (1.6.2) is given by

$$\int_{y_0}^y N(t, x) \Delta_1 x + \int_{t_0}^{\sigma(t)} M(s, y_0) \Delta s - \mu(t) M(t, y) = C, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \quad (1.6.7)$$

where C is a constant.

Now, we will determine the function u by the equation

$$u_y^{\Delta_1}(\sigma(t), y) = N(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

We integrate the last equation from y_0 to y and we find

$$u(\sigma(t), y) = \int_{y_0}^y N(t, x) \Delta_1 x + h(t), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \quad (1.6.8)$$

where $h = h(t)$ is an arbitrary function of t that plays the role of the constant of integration. We will find this function using the equation

$$u_t^{\Delta}(t, y) = M(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

By (1.6.8), we obtain

$$u(t, y) + \mu(t)u_t^{\Delta}(t, y) = \int_{y_0}^y N(t, x) \Delta_1 x + h(t), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

or

$$u(t, y) = -\mu(t)M(t, y) + \int_{y_0}^y N(t, x) \Delta_1 x + h(t), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

We differentiate the last equation with respect to t and we arrive at

$$\begin{aligned} M(t, y) &= u_t^{\Delta}(t, y) \\ &= -\mu^{\Delta}(t)M(\sigma(t), y) - \mu(t)M_t^{\Delta}(t, y) + \frac{\Delta}{\Delta t} \int_{y_0}^y N(t, x) \Delta_1 x + h^{\Delta}(t) \\ &= -\mu^{\Delta}(t)M(\sigma(t), y) - \mu(t)M_t^{\Delta}(t, y) + \int_{y_0}^y N_t^{\Delta}(t, x) \Delta_1 x + h^{\Delta}(t), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

whereupon

$$M(t, y) + \mu(t)M_t^{\Delta}(t, y) = -\mu^{\Delta}(t)M(\sigma(t), y) + \int_{y_0}^y N_t^{\Delta}(t, x) \Delta_1 x + h^{\Delta}(t),$$

$(t, y) \in \mathbb{T} \times \mathbb{T}_1$, or

$$(1 + \mu^{\Delta}(t)) M(\sigma(t), y) = \int_{y_0}^y N_t^{\Delta}(t, x) \Delta_1 x + h^{\Delta}(t), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Because

$$\begin{aligned} &\frac{\Delta}{\Delta y} \left((1 + \mu^{\Delta}(t)) M(\sigma(t), y) - \int_{y_0}^y N_t^{\Delta}(t, x) \Delta_1 x \right) \\ &= (1 + \mu^{\Delta}(t)) M_y^{\Delta_1}(\sigma(t), y) - N_t^{\Delta}(t, y) \\ &= N_t^{\Delta}(t, y) - N_t^{\Delta}(t, y) \\ &= 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

the function

$$(1 + \mu^\Delta(t)) M(\sigma(t), y) - \int_{y_0}^y N_t^\Delta(t, x) \Delta_1 x, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

must depend on t only. Therefore

$$h^\Delta(t) = (1 + \mu^\Delta(t)) M(\sigma(t), y) - \int_{y_0}^y N_t^\Delta(t, x) \Delta_1 x, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Hence,

$$\begin{aligned} h(t) &= \int_{t_0}^t (1 + \mu^\Delta(s)) M(\sigma(s), y) \Delta s - \int_{t_0}^t \int_{y_0}^y N_t^\Delta(s, x) \Delta_1 x \Delta s + h(t_0) \\ &= \int_{t_0}^t (1 + \mu^\Delta(s)) M(\sigma(s), y) \Delta s - \int_{y_0}^y \int_{t_0}^t N_t^\Delta(s, x) \Delta s \Delta_1 x + h(t_0) \\ &= \int_{t_0}^t (1 + \mu^\Delta(s)) M(\sigma(s), y) \Delta s - \int_{y_0}^y N(t, x) \Delta_1 x + \int_{y_0}^y N(t_0, x) \Delta_1 x + h(t_0), \end{aligned}$$

$(t, y) \in \mathbb{T} \times \mathbb{T}_1$. From here, for the function u , we find

$$\begin{aligned} u(t, y) &= -\mu(t)M(t, y) + \int_{y_0}^y N(t, x) \Delta_1 x + \int_{t_0}^t (1 + \mu^\Delta(s)) M(\sigma(s), y) \Delta s \\ &\quad - \int_{y_0}^y N(t, x) \Delta_1 x + \int_{y_0}^y N(t_0, x) \Delta_1 x + h(t_0) \\ &= -\mu(t)M(t, y) + \int_{t_0}^t (1 + \mu^\Delta(s)) M(\sigma(s), y) \Delta s + \int_{y_0}^y N(t_0, x) \Delta_1 x + h(t_0), \end{aligned}$$

$(t, y) \in \mathbb{T} \times \mathbb{T}_1$, and the solution of the equation (1.6.2) is given by

$$-\mu(t)M(t, y) + \int_{t_0}^t (1 + \mu^\Delta(s)) M(\sigma(s), y) \Delta s + \int_{y_0}^y N(t_0, x) \Delta_1 x = C, \quad (1.6.9)$$

$$(t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

where C is a constant. This completes the proof.

The above proof is constructive, i.e., we can explicitly find a solution of the equation (1.6.2) by (1.6.7) and (1.6.9). In (1.6.7) and (1.6.9) the choice of t_0 and y_0 is at our disposal, except that these must be chosen so that the integrals remain proper.

✓ Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $\mathbb{T}_1 = 3^{\mathbb{N}_0}$. Consider the equation

$$\frac{3}{4}ty - \frac{y^3}{26} + (t^2 - ty^2)y^\Delta = 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Here

$$\begin{aligned} \sigma(t) &= 2t, \\ \sigma_1(y) &= 3y, \\ \mu(t) &= t, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

and

$$M(t, y) = \frac{3}{4}ty - \frac{y^3}{26},$$

$$N(t, y) = t^2 - ty^2, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

We have that μ is delta differentiable and

$$\mu^\Delta(t) = 1, \quad t \in \mathbb{T},$$

and

$$\begin{aligned} N_t^\Delta(t, y) &= \sigma(t) + t - y^2 \\ &= 2t + t - y^2 \\ &= 3t - y^2, \\ M_y^{\Delta_1}(t, y) &= \frac{3}{4}t - \frac{(\sigma_1(y))^2 + \sigma_1(y)y + y^2}{26} \\ &= \frac{3}{4}t - \frac{(3y)^3 + 3y^2 + y^2}{26} \\ &= \frac{3}{4}t - \frac{9y^2 + 4y^2}{26} \\ &= \frac{3}{4}t - \frac{y^2}{2}, \\ M_y^{\Delta_1}(\sigma(t), y) &= \frac{3}{4}\sigma(t) - \frac{y^2}{2} \\ &= \frac{3}{4}(2t) - \frac{y^2}{2} \\ &= \frac{3}{2}t - \frac{y^2}{2}, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Hence,

$$\begin{aligned} (1 + \mu^\Delta(t)) M_y^{\Delta_1}(\sigma(t), y) &= (1 + 1) \left(\frac{3}{2}t - \frac{y^2}{2} \right) \\ &= 3t - y^2 \\ &= N_t^\Delta(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Therefore the considered equation is an exact equation. We will find its general solution using the formulae (1.6.7). By (1.6.7), we get

$$\begin{aligned} C &= \int_1^y N(t, x) \Delta_1 x + \int_1^{2t} M(s, 1) \Delta s - tM(t, y) \\ &= \int_1^y (t^2 - 0tx^2) \Delta_1 x + \int_1^{2t} \left(\frac{3}{4}s - \frac{1}{26} \right) \Delta s - t \left(\frac{3}{4}ty - \frac{y^3}{26} \right) \\ &= t^2 \int_1^y \Delta_1 x - t \int_1^y x^2 \Delta_1 x + \frac{3}{4} \int_1^{2t} s \Delta s - \frac{1}{26} \int_1^{2t} \Delta s - \frac{3}{4}t^2y + t \frac{y^3}{26} \\ &= t^2(y - 1) - \frac{t}{13} x^3 \Big|_{x=1}^{x=y} + \frac{3}{4} \left(\frac{s^2}{3} \right) \Big|_{s=1}^{s=2t} - \frac{1}{26} (2t - 1) - \frac{3}{4}t^2y + t \frac{y^3}{26} \\ &= t^2(y - 1) - \frac{ty^3}{13} + \frac{t}{13} + \frac{3}{4} \left(\frac{4t^2}{3} \right) - \frac{1}{4} - \frac{t}{13} + \frac{1}{26} - \frac{3}{4}t^2y + t \frac{y^3}{26} \\ &= t^2y - t^2 - \frac{ty^3}{13} + \frac{t}{13} + t^2 - \frac{1}{4} - \frac{t}{13} + \frac{1}{26} - \frac{3}{4}t^2y + t \frac{y^3}{26} \\ &= -\frac{11}{52} - \frac{ty^3}{26} + \frac{1}{4}t^2y, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

i.e., the general solution of the considered equation is

$$-\frac{ty^3}{26} + \frac{t^2y}{4} = C, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

where C is a constant.

Now, we apply the formulae (1.6.9). We get

$$\begin{aligned} C &= -tM(t, y) + 2 \int_1^t M(2s, y) \Delta s + \int_1^y N(1, x) \Delta_1 x \\ &= -t \left(\frac{3}{4}ty - \frac{y^3}{26} \right) + 2 \int_1^t \left(\frac{3}{4}(2s)y - \frac{y^3}{26} \right) \Delta s \\ &\quad + \int_1^y (1 - x^2) \Delta_1 x \\ &= -\frac{3}{4}t^2y + t\frac{y^3}{26} + 3y \int_1^t s \Delta s - \frac{y^3}{13} \int_1^t \Delta s \\ &\quad + \int_1^y \Delta_1 x - \int_1^y x^2 \Delta_1 x \\ &= -\frac{3}{4}t^2y + \frac{ty^3}{26} + 3y \left(\frac{s^2}{3} \right) \Big|_{s=1}^{s=t} - \frac{y^3}{13}(t-1) \\ &\quad + y - 1 - \frac{1}{13}x^3 \Big|_{x=1}^{x=y} \\ &= -\frac{3}{4}t^2y + \frac{ty^3}{26} + yt^2 - [y - t\frac{y^3}{13} + \frac{y^3}{13} \\ &\quad + y - 1 - \frac{y^3}{13} + \frac{1}{13}] \\ &= -\frac{12}{13} - \frac{ty^3}{26} + \frac{1}{4}t^2y, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

whereupon we get

$$-\frac{ty^3}{26} + \frac{1}{4}t^2y = C, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

where C is a constant.

✓ Example

Let $\mathbb{T} = 2\mathbb{Z}$ and $\mathbb{T}_1 = 4^{\mathbb{N}_0}$. Consider the equation

$$2(t+1) - y^3 - 21(t+2)y^2y^\Delta = 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Here

$$\begin{aligned} \sigma(t) &= t + 2, \\ \sigma_1(y) &= 4y, \\ \mu(t) &= 2, \\ \mu^\Delta(t) &= 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

and

$$\begin{aligned} M(t, y) &= 2(t+1) - y^3, \\ N(t, y) &= -21(t+2)y^2, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Then

$$\begin{aligned}M_y^{\Delta_1}(t, y) &= -((\sigma_1(y))^2 + y\sigma_1(y) + y^2) \\&= -((4y)^2 + 4y^2 + y^2) \\&= -21y^2, \\M_y^{\Delta_1}(\sigma(t), y) &= -21y^2, \\N_t^{\Delta}(t, y) &= -21y^2, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.\end{aligned}$$

From here, we find

$$\begin{aligned}N_t^{\Delta}(t, y) &= (1 + \mu^{\Delta}(t))M_y^{\Delta_1}(\sigma(t), y) \\&= -21y^2, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.\end{aligned}$$

Therefore the considered equation is an exact equation. We will find its solution using the formulae (1.6.7). We have

$$\begin{aligned}C &= \int_1^y N(t, x) \Delta_1 x + \int_0^{\sigma(t)} M(s, 1) \Delta s - 2M(t, y) \\&= -21(t+2) \int_1^y x^2 \Delta_1 x + \int_0^{t+2} (2(s+1) - 1) \Delta s - 2(2(t+1) - y^3) \\&= -21(t+2) \frac{x^3}{21} \Big|_{x=1}^{x=y} + \int_0^{t+2} (2s+1) \Delta s - 4(t+1) + 2y^3 \\&= -(t+2)y^3 + (t+2) + 2 \int_0^{t+2} s \Delta s + \int_0^{t+2} \Delta s - 4(t+1) + 2y^3 \\&= -ty^3 - 3t - 2 + 2 \left(\frac{s^2}{2} - s \right) \Big|_{s=0}^{s=t+2} + t + 2 \\&= -ty^3 - 2t + (s^2 - 2s) \Big|_{s=0}^{s=t+2} \\&= -ty^3 - 2t + (t+2)^2 - 2(t+2) \\&= -ty^3 - 2t + t^2 + 4t + 4 - 2t - 4 \\&= t^2 - ty^3, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.\end{aligned}$$

Thus, the general solution of the considered equation is

$$t^2 - ty^3 = C, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

where C is a constant.

? Exercise

Derive the general solution of the equation in Example ??? using the formulae (1.6.9).

? Exercise

Let $\mathbb{T} = 3^{\mathbb{N}_0}$ and $\mathbb{T}_1 = 4\mathbb{Z}$. Consider the equation

$$-2\sin_2(t, 1)y - 12ty^3 + 1 + (\cos_2(3t, 1) - 27t^2(3y^2 + 12y + 16))y^{\Delta} = 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

1. Check if the considered equation is an exact equation. 2. Derive the general solution of the considered equation using the formulas (1.6.7) and (1.6.9).

Definition

For the equation (1.6.2) a nonzero function $h = h(t, y)$, $(t, y) \in \mathbb{T} \times \mathbb{T}_1$, is called an integrating factor if the equivalent equation

$$hM + hNy^\Delta = 0 \quad (1.6.10)$$

is an exact equation. \vbox{\index{Integrating Factor}}

Example

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, and $\mathbb{T}_1 = \mathbb{Z}$. Consider the following dynamic equation

$$7t - 3y^2 - 4t(2y + 1)y^\Delta = 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Here

$$\begin{aligned} \sigma(t) &= 2t, \\ \mu(t) &= 1, \\ \mu^\Delta(t) &= 1, \\ \sigma_1(y) &= y + 1, \\ \mu_1(y) &= 1, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

and

$$\begin{aligned} M(t, y) &= 7t - 3y^2, \\ N(t, y) &= -4t(2y + 1), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

We have

$$\begin{aligned} M_y^{\Delta_1}(t, y) &= -3(\sigma_1(y) + y) \\ &= -3(y + 1 + y) \\ &= -3(2y + 1), \\ M_y^{\Delta_1}(\sigma(t), y) &= -3(2y + 1), \\ N_t^\Delta(t, y) &= -4(2y + 1), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Hence,

$$\begin{aligned} (1 + \mu^\Delta(t)) M_y^{\Delta_1}(\sigma(t), y) &= -6(2y + 1) \\ &\neq -4(2y + 1) \\ &= N_t^\Delta(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Thus, the given equation is not an exact equation. We multiply the given equation with $t \in \mathbb{T}$ and we find

$$7t^2 - 3ty^2 - 4t^2(2y + 1)y^\Delta = 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \quad (1.6.11)$$

Set

$$\begin{aligned} M_1(t, y) &= 7t^2 - 3ty^2, \\ N_1(t, y) &= -4t^2(2y + 1), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

We have

$$\begin{aligned} M_{1y}^{\Delta_1}(t, y) &= -3t(\sigma_1(y) + y) \\ &= -3t(y + 1 + y) \\ &= -3t(2y + 1), \\ M_{1y}^{\Delta_1}(\sigma(t), y) &= -3\sigma(t)(2y + 1) \\ &= -6t(2y + 1), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1, \end{aligned}$$

and

$$\begin{aligned} N_{tt}^{\Delta}(t, y) &= -4(\sigma(t) + t)(2y + 1) \\ &= -4(2t + t)(2y + 1) \\ &= -12t(2y + 1), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

From here, we arrive at

$$\begin{aligned} (1 + \mu^{\Delta}(t)) M_{1y}^{\Delta_1}(\sigma(t), y) &= 2(-6t(2y + 1)) \\ &= -12t(2y + 1) \\ &= N_{tt}^{\Delta}(t, y), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Thus, the equation (1.6.11) is an exact equation and $h(t, y) = t$, $(t, y) \in \mathbb{T} \times \mathbb{T}_1$, is an integrating factor for the given equation.

Theorem

If the dynamic equation (1.6.2) has $u(t, y) = C$, $(t, y) \in \mathbb{T} \times \mathbb{T}_1$, where C is a constant, as its solution, then the equation (1.6.2) admits an integrating factor.

Proof

Note that y^{Δ} computed from (1.6.2) and

$$u_t^{\Delta}(t, y(t)) + u_y^{\Delta_1}(\sigma(t), y(t))y^{\Delta}(t) = 0, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

must be the same. We have

$$y^{\Delta}(t) = -\frac{M(t, y(t))}{N(t, y(t))}, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

and

$$y^{\Delta}(t) = -\frac{u_t^{\Delta}(t, y(t))}{u_y^{\Delta_1}(\sigma(t), y)}, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Therefore

$$\frac{M(t, y(t))}{N(t, y(t))} = \frac{u_t^{\Delta}(t, y(t))}{u_y^{\Delta_1}(t, y(t))}, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

and

$$\frac{u_t(t, y(t))}{M(t, y(t))} = \frac{u_y^{\Delta_1}(\sigma(t), y(t))}{N(t, y(t))} = h(t, y(t)), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Hence,

$$\begin{aligned} h(t, y(t)) (M(t, y(t)) + N(t, y(t))y^{\Delta}(t)) &= u_t(t, y(t)) + u_y^{\Delta_1}(t, y(t))y^{\Delta}(t) \\ &= \frac{\Delta u}{\Delta t}(t, y(t)), \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1. \end{aligned}$$

Thus, the equation (1.6.10) is an exact equation and h is an integrating factor for the equation (1.6.2). This completes the proof.

1.7: Mathematical Modelling with First Order Dynamic Equations

Mathematical modelling of a system in an application area of interest (e.g., physics, biology, engineering, economics, ...) is the translation of our knowledge or beliefs of that system into the language of mathematics. It typically involves writing down an equation (or set of equations) that describes the behaviour of the system.

The majority of real-world problems are far too complex to model in their entirety. The process of developing a mathematical model therefore involves making a number of simplifications along the way. We accept that all models are limited and, to some extent, wrong; but if the modelling process is conducted with intelligence and care, models are still undeniably useful. They enable us to understand complex phenomena, make predictions about the future and test the impact of changes to a system.

Typically, mathematical models can be classified in a number of ways, e.g., empirical vs. mechanistic. Empirical models are often data-driven and use statistical methods to infer relations between different variables in a system; mechanistic models use prior knowledge and understanding of the mechanism that causes change in a system.

In many applications, we are given the value of a quantity at the present time (for example, the temperature of coffee in a cup, the number of people infected with a virus, the concentration of carbon dioxide in the atmosphere) and we wish to predict its value in the future. To do this, we must know how quickly the quantity is changing. Mathematically, the rate of change of this quantity is its derivative. If we can write down an equation relating this derivative to some knowledge of the mechanism that causes the change, we can calculate how the temperature changes, how the number of infected people changes, how the concentration of carbon dioxide changes. This gives rise to a differential equation—which is a mechanistic model of the quantity of interest.

✓ Modelling in Geometry

We will find a curve in the t, y -plane that passes through the point $(t_0, y_0) \in \mathbb{T} \times \mathbb{T}_1$ and whose tangent at a point (t, y) has slope

$$\frac{\sigma(t) + t}{(\sigma_1(y))^2 + y\sigma_1(y) + y^2}.$$

Since the slope of the tangent line is $\frac{\Delta_1 y}{\Delta t}$, we have

$$\frac{\Delta_1 y}{\Delta t} = \frac{\sigma(t) + t}{(\sigma_1(y))^2 + y\sigma_1(y) + y^2}. \quad (1.7.1)$$

Because the curve passes through the point (t_0, y_0) , we have the initial condition

$$y(t_0) = y_0. \quad (1.7.2)$$

Thus, we get the following initial value problem

$$\begin{aligned} ((\sigma_1(y))^2 + y\sigma_1(y) + y^2)\Delta_1 y &= (\sigma(t) + t)\Delta t, \\ y(t_0) &= y_0. \end{aligned}$$

The obtained equation is a separable equation. For its general solution, we find

$$\int ((\sigma_1(y))^2 + y\sigma_1(y) + y^2)\Delta_1 y = \int (\sigma(t) + t)\Delta t + c, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1.$$

Here c is a constant of integration that will be determined below. Hence,

$$y^3 + t^2 = c, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1$$

Now, we will find the constant c using the initial condition. We get

$$y_0^3 = t_0^2 + c$$

or

$$c = y_0^3 - t_0^2.$$

Therefore

$$y^3 = t^2 + y_0^3 - t_0^2, \quad (t, y) \in \mathbb{T} \times \mathbb{T}_1,$$

is the desired curve.

✓ Bacteria Growth

Let $N(t)$ denote the number of bacteria growing on a rate of nutrients at time t . Suppose that we know the initial value of N at time $t = t_0$. At this time, the value of N is N_0 , or

$$N(t_0) = N_0.$$

We want to know at what time the number of bacteria becomes $10N_0$.

To build a mathematical model for this problem, let us first describe how the number of bacteria changes with time using mathematics. Suppose that over the time interval $(t, t + \sigma(t))$, the number of bacteria increases by an amount

$$\delta N = N(\sigma(t)) - N(t)$$

which we write as

$$N(\sigma(t)) - N(t) = k(N, t)(\sigma(t) - t).$$

Then

$$\frac{N(\sigma(t)) - N(t)}{\sigma(t) - t} = k(N, t).$$

Thus, we get the following dynamic equation

$$N^\Delta(t) = k(N, t).$$

To make a progress, we need some information on the function $k(N, t)$. Suppose by doing experiments, we observe that the rate of change of N is proportional of N . This means that if there are, say, twice as many bacteria, then N will grow twice as rapidly. Mathematically,

$$k(N, t) = lN,$$

where l is a constant we can measure from experiments. Therefore, the mathematical problem for our bacteria growth problem is

$$N^\Delta(t) = lN(t), \quad t \in \mathbb{T}.$$

Note that

$$N(t) = e_l(t, t_0), \quad t \in \mathbb{T},$$

is a solution of the obtained equation.

We extract the answer to the question we ask. Let t_1 be the time at which we have

$$N = 10N_0.$$

Then

$$e_l(t_1, t_0) = 10N_0.$$

✓ The Newton Law of Cooling

Sir Isaac Newton FRS: 1643–1727, English physicist, mathematician, astronomer, natural philosopher, alchemist and theologian.

The Newton law of cooling states that if an object is hotter than the ambient temperature, then the rate of change of the object's temperature is proportional to the temperature difference between the object and its surrounding. Mathematically, we write

$$Y^{\Delta}(t) = k(Y^{\sigma} - A), \quad t \in \mathbb{T}, \quad t \geq t_0,$$

$$Y(t_0) = Y_0,$$

where $Y(t)$ is the object temperature, A is the ambient temperature (obviously it is a constant), t is time and k is a positive constant. This is a first order dynamic equation

$$Y^{\Delta}(t) + kY^{\sigma}(t) = kA, \quad t \in \mathbb{T}, \quad t \geq t_0. \quad (1.7.3)$$

So, the integrating factor is

$$R(t) = e_k(t, t_0), \quad t \in \mathbb{T}, \quad t \geq t_0.$$

We multiply the equation (1.7.3) by $e_k(t, t_0)$ and we get

$$Y^{\Delta}(t)e_k(t, t_0) + ke_k(t, t_0)Y^{\sigma}(t) = kAe_k(t, t_0), \quad t \in \mathbb{T}, \quad t \geq t_0,$$

or

$$(Ye_k(\cdot, t_0))^{\Delta}(t) = kAe_k(t, t_0), \quad t \in \mathbb{T}, \quad t \geq t_0.$$

By the last expression, we get

$$\begin{aligned} Y(t)e_k(t, t_0) &= Y_0 + kA \int_{t_0}^t e_k(s, t_0) \Delta s \\ &= Y_0 + Ae_k(s, t_0) \Big|_{s=t_0}^{s=t} \\ &= Y_0 + Ae_k(t, t_0) - A, \quad t \in \mathbb{T}, \quad t \geq t_0. \end{aligned}$$

Hence,

$$Y(t) = (Y_0 - A)e_{\Theta k}(t, t_0) + A, \quad t \in \mathbb{T}, \quad t \geq t_0.$$

Suppose that $\mu(t)k < 1$, $t \in \mathbb{T}$, $t \geq t_0$. Then

$$\begin{aligned} e_{\Theta k}(t, t_0) &= e^{\int_{t_0}^t \frac{1}{\mu(s)} \log(1 + \mu(s)(\Theta k)(s)) \Delta s} \\ &= e^{\int_{t_0}^t \frac{1}{\mu(s)} \log\left(\frac{1}{1 + k\mu(s)}\right) \Delta s} \\ &= e^{-\int_{t_0}^t \frac{1}{\mu(s)} \log(1 + k\mu(s)) \Delta s} \\ &\leq e^{-\int_{t_0}^t \frac{1}{\mu(s)} \left(\mu(s)k - \frac{(\mu(s))^2 k^2}{2}\right) \Delta s} \\ &= e^{-\int_{t_0}^t \left(k - \frac{\mu(s)k^2}{2}\right) \Delta s} \\ &= e^{-\int_{t_0}^t k \left(1 - \frac{\mu(s)k}{2}\right) \Delta s} \\ &\leq e^{-\frac{k}{2}(t - t_0)} \\ &\rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Thus, when $\mu(t)k < 1$, $t \in \mathbb{T}$, $t \geq t_0$, we have that $Y(t) \rightarrow A$, $t \rightarrow \infty$.

This page titled [1.7: Mathematical Modelling with First Order Dynamic Equations](#) is shared under a [GNU General Public License 3.0](#) license and was authored, remixed, and/or curated by [Svetlin G. Georgiev](#).

Index

A

accessible education

[Front Matter](#)
[InfoPage](#)

advanced calculus

[1.3: Bernoulli Dynamic Equations](#)

affordable materials

[Front Matter](#)

B

bacterial growth

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

Bernoulli dynamic equations

[1: Chapters](#)

Bernoulli equations

[1.3: Bernoulli Dynamic Equations](#)

biology

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

boundary value problem

[1.1: Introduction](#)

C

collaboration

[Front Matter](#)

comprehension

[Back Matter](#)

concepts

[Back Matter](#)

continuous dependence

[1.1: Introduction](#)

continuous functions

[1.3: Bernoulli Dynamic Equations](#)
[1.5: Dynamic Equations for Families of Curves](#)

curriculum development

[InfoPage](#)

customization

[Front Matter](#)

D

definitions

[Back Matter](#)

derivatives

[1.5: Dynamic Equations for Families of Curves](#)

difference equations

[1: Chapters](#)

differential equations

[1.1: Introduction](#)
[1.2: Linear First Order Dynamic Equations](#)
[1.3: Bernoulli Dynamic Equations](#)
[1.6: Exact Equations](#)
[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

differentiation

[1.5: Dynamic Equations for Families of Curves](#)

dynamic equation

[1.6: Exact Equations](#)

dynamic equations

[1: Chapters](#)
[1.2: Linear First Order Dynamic Equations](#)
[1.5: Dynamic Equations for Families of Curves](#)

dynamic system

[1.6: Exact Equations](#)

E

economics

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

educational organizations

[Front Matter](#)

educational technology

[InfoPage](#)

engineering

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

equations

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

exact equation

[1.6: Exact Equations](#)

examples

[1.2: Linear First Order Dynamic Equations](#)
[Back Matter](#)

exercise

[1.5: Dynamic Equations for Families of Curves](#)

existence

[1: Chapters](#)

existence of solutions

[1.1: Introduction](#)

F

family of curves

[1.5: Dynamic Equations for Families of Curves](#)

financial barriers

[Front Matter](#)

first order equations

[1.5: Dynamic Equations for Families of Curves](#)

fixed points

[1.5: Dynamic Equations for Families of Curves](#)

functions

[1.6: Exact Equations](#)

G

general solutions

[1: Chapters](#)

global education access

[Front Matter](#)

glossary

[Back Matter](#)

guidelines

[Back Matter](#)

I

images

[Back Matter](#)

initial value problem

[1.1: Introduction](#)
[1.2: Linear First Order Dynamic Equations](#)

initial value problems

[1: Chapters](#)
[1.3: Bernoulli Dynamic Equations](#)

integrating factor

[1.6: Exact Equations](#)

interactive learning

[Front Matter](#)

interactive textbooks

[InfoPage](#)

IVP

[1.2: Linear First Order Dynamic Equations](#)

L

LibreTexts

[Front Matter](#)
[InfoPage](#)

linear

[1.1: Introduction](#)
[1.2: Linear First Order Dynamic Equations](#)

linear first order dynamic equations

[1: Chapters](#)

links

[Back Matter](#)

M

mathematical analysis

[1.3: Bernoulli Dynamic Equations](#)
[1.5: Dynamic Equations for Families of Curves](#)
[1.6: Exact Equations](#)

mathematical modeling

[1.3: Bernoulli Dynamic Equations](#)

mathematical modelling

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

mathematical proof

[1.2: Linear First Order Dynamic Equations](#)
[1.6: Exact Equations](#)

N

Newton's law of cooling

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

nonlinear

[1.1: Introduction](#)

numerical methods

[1.3: Bernoulli Dynamic Equations](#)

O

OER

[InfoPage](#)

online textbooks

[InfoPage](#)

open education initiatives

[InfoPage](#)

oscillations

[1.1: Introduction](#)

P

physics

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

predictions

[1.7: Mathematical Modelling with First Order Dynamic Equations](#)

R

real analysis

[1.3: Bernoulli Dynamic Equations](#)

Riccati dynamic equations

[1: Chapters](#)

S

solution methods

[1: Chapters](#)

[1.3: Bernoulli Dynamic Equations](#)

solutions

[1.5: Dynamic Equations for Families of Curves](#)

stability

[1.1: Introduction](#)

T

terms

[Back Matter](#)

textbook

[Back Matter](#)

theorem

[1.2: Linear First Order Dynamic Equations](#)

[1.6: Exact Equations](#)

U

unique solution

[1.2: Linear First Order Dynamic Equations](#)

uniqueness

[1: Chapters](#)

uniqueness of solutions

[1.1: Introduction](#)

V

variable transformation

[1.6: Exact Equations](#)

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: Elementary First Order Dynamic Equations on Time Scales (Georgiev)

Webpages: 18

All licenses found:

- [GPL](#): 88.9% (16 pages)
- [Undeclared](#): 11.1% (2 pages)

By Page

- [Elementary First Order Dynamic Equations on Time Scales \(Georgiev\)](#) - [GPL](#)
 - [Front Matter](#) - [GPL](#)
 - [TitlePage](#) - [GPL](#)
 - [InfoPage](#) - [GPL](#)
 - [Table of Contents](#) - [Undeclared](#)
 - [Licensing](#) - [GPL](#)
 - [1: Chapters](#) - [GPL](#)
 - [1.1: Introduction](#) - [GPL](#)
 - [1.2: Linear First Order Dynamic Equations](#) - [GPL](#)
 - [1.3: Bernoulli Dynamic Equations](#) - [GPL](#)
 - [1.4: Riccati Dynamic Equations](#) - [GPL](#)
 - [1.5: Dynamic Equations for Families of Curves](#) - [GPL](#)
 - [1.6: Exact Equations](#) - [GPL](#)
 - [1.7: Mathematical Modelling with First Order Dynamic Equations](#) - [GPL](#)
 - [Back Matter](#) - [GPL](#)
 - [Index](#) - [GPL](#)
 - [Glossary](#) - [GPL](#)
 - [Detailed Licensing](#) - [Undeclared](#)