MATH IN PLAIN SIGHT



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CHAPTER 1: LOGIC

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Section 1.1: Introduction to Propositional Logic

COR:

- Logic Statements
- Simple Negation Compound: conjuntion, disjunction, conditional (including converse, inverse, and contrapositive), biconditional
- Symbolic form and notation
- Analysis of valid and invalid arguments, which may include: Determining validity of arguments (e.g., using truth tables
- Drawing a valid conclusion from a given set of premises
- Classic syllogisms and fallacies (e.g., Transitive Reasoning/Law of Syllogism, Affirming the Conclusion/Fallacy of the Converse)

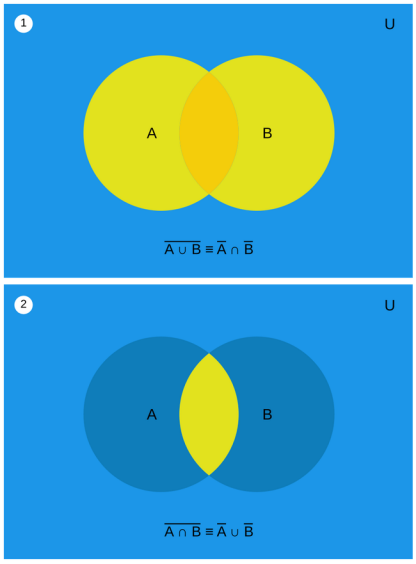


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Answers

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Section 2.1: Set Theory

Learning Objectives

- Define sets
- Write sets three different ways
- Define the empty set
- Find the cardinality of a set
- Classify sets as finite or infinite
- Decide if two sets are equal or equivalent

Introduction

Imagine organizing your music collection. You might group songs by genre such as rock, jazz, or classical, by decade such as the 80's, 90's, or 2000's, or by mood such as study, workout, or relaxation. In mathematics, we do something very similar. We organize objects into groups called sets.

Set theory is the branch of mathematics that studies these collections and the relationships between them. It provides a precise language for describing groups, connections, and patterns, and it serves as a foundation for much of modern mathematics. One reason it is important is that it allows us to communicate with precision. Instead of vague terms like many or most, we can describe exactly which elements belong to a group and how groups relate to one another.

Set theory also helps reveal connections between problems that may appear unrelated. Questions about overlapping classes, probability, or number systems often share the same underlying structure. By using sets, we can recognize and analyze these similarities more clearly. At the same time, studying sets develops important critical thinking skills such as classifying information, analyzing relationships, and reasoning carefully about inclusion and overlap.

These ideas appear in many real world contexts. Search engines use set operations to find relevant results, businesses analyze overlapping customer groups, social networks suggest connections based on shared relationships, and medical professionals identify patterns by examining combinations of symptoms.

At its core, a set is a well defined collection of distinct objects called elements. Although this definition may sound abstract, sets are everywhere, including the students in a class, the books on a shelf, or the courses you are taking this semester. As you begin studying set theory, you will see that it is more than just a mathematical topic. It is a powerful way to organize information and think clearly about the world.

Definition: Set

A **set** is a well-defined collection of distinct objects. The objects in a set can be anything: numbers, letters, people, books, or even other sets. Sets are typically denoted by capital letters A, B, C , etc. and the objects are listed within braces { }.

Examples:

- $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- $B = \{\text{apple, banana, pear}\}$

Definition: Epsilon, ϵ

The Greek letter, **Epsilon** (∈), is the mathematical symbol meaning "is an element of" or "belongs to." Its opposite is ∉, meaning "is not an element of."

Examples:

- $5 \in A$ (read as "5 is an element of A")
- $4 \notin A$ (read as "4 is not an element of A")

Example #2.1.1 🤔

Let A = {1, 2, 3, 4, 5}, B = {1, 2, 3, 4, 5, ...}, and C = all land animals. Decide whether each of the statements is true or false.

a. $1 \in A$
b. $1 \in B$
c. $6 \in A$
d. $6 \notin B$
e. $10{,}000 \in B$
f. cat $\in C$
g. fish $\in C$
h. whale $\notin C$
i. dog $\notin C$
j. $7 \in C$

✅ **Solution:**

a. True
b. True
c. False
d. False
e. True
f. True
g. False
h. True
i. False
j. False

Section 2.1: Set Theory [*In-Class Exercises* - Part 1 of 4]

1. Let A = the set of U.S. states, B = {5, 10, 15, 20, 25, ...}, and C = the set of U.S. Presidents. Decide whether each of the statements is true or false.

   a. $15 \in B$
   b. California $\notin A$
   c. $30 \notin B$
   d. George Washington $\in C$
   e. Utah $\in A$
   f. $100 \in B$
   g. $51 \in B$
   h. Miami $\in A$
   i. $25 \in C$
   j. Abraham Lincoln $\in A$
   k. $0 \notin B$

l. Taylor Swift $\in C$

Answers available at the end of this section.

Definition: Finite Set

A **finite set** is a set that contains a specific, countable number of elements. You can list all elements and determine the exact size of the set.

Examples:

- {a, b, c, d} has 4 elements
- The set of students in this classroom

Definition: Infinite Set

An **infinite set** is a set that contains an unlimited number of elements. The elements continue without end.

Example:

- The set of all natural numbers {1, 2, 3, 4, ...}
- The set of all points on a line

Definition: Variable

A **variable** is a symbol (usually a letter like x, y, or n) that represents an unknown number or a quantity that can change.

Three Ways to Represent Sets

1. **List/Roster Method**
   - Lists all elements explicitly within braces.
   - Format: {element #1, element #2, element #3, ...}
   - Examples:
     - {1, 2, 3, 4, 5}
     - {red, blue, green, yellow}
     - {2, 4, 6, 8, ...}
2. **Descriptive Method**
   - Uses words to describe the set without mathematical notation.
   - Examples:
     - "The set of all positive integers less than 6"
     - "The set of primary colors"
     - "The set of all US state capitals"
3. **Set Builder Notation**
   - Uses mathematical notation with conditions/rules to define set membership.
   - Format: {x | condition(s)}
   - Examples:
     - $\{x \mid x \geq 5\}$ = "the set of numbers greater than or equal to 5"
     - $\{x \in \mathbb{N} \mid x < 10\}$ = "the set of natural numbers less than 10"
     - $\{x \mid x = 2n, n \in \mathbb{Z}\}$ = "the set of all even integers"

Note: The symbols above $\mathbb{N}$ and $\mathbb{Z}$ are called blackboard bold style letters, because mathematicians writing on blackboards (chalk boards) would trace over letters multiple times to make them stand out, creating a "bold" effect. These symbols/letters became

standard for describing particular number sets. Here are most of the symbols used in that style and their meanings:

- The symbol $\mathbb{N}$ represents the set of all natural numbers or {1, 2, 3, 4, ...}.
- The symbol $\mathbb{W}$ represents the set of all whole numbers or {0, 1, 2, 3, 4, ...}
- The symbol $\mathbb{Z}$ represents the set of all integers or {..., −3, −2, −1, 0, 1, 2, 3, ...}.
- The symbol $\mathbb{Q}$ represents the set of all rational numbers.
- The symbol $\mathbb{R}$ represents the set of all real numbers.
- The symbol $\mathbb{C}$ represents the set of all complex numbers.

(For this text, we will only be working with $\mathbb{N}$, the set of all natural numbers)

When To Use Each Method

Method	Best for	Limitations
List/Roster	Small finite sets, concrete examples	Impractical for large/infinite sets
Descriptive	General communication, non-mathematical contexts	Can be ambiguous
Set-Builder	Mathematical sets, infinite sets, sets with clear rules	Requires mathematical background

Same Set, Three Different Ways

Here are the three different methods all representing the same exact set:

- **Roster:** {1, 2, 3, 4, 5, 6, 7, 8, 9}
- **Set-Builder:** $\{x \in \mathbb{N} \mid 1 \le x \le 9\}$
- **Descriptive:** "The set of natural numbers from 1 to 9"

Example #2.1.2 🤔

Rewrite each set the same in two other ways.

a. {1, 2, 3, 4, 5}
b. $\{x \in \mathbb{N} \mid 6 \le x \le 9\}$
c. "The set of natural numbers from 7 to 11"
d. {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}

✅ **Solution:**

a. Set-Builder: $\{x \in \mathbb{N} \mid 1 \le x \le 5\}$ and Descriptive: "The set of natural numbers from 1 to 5"
b. Roster: {6, 7, 8, 9} and Descriptive: "The set of natural numbers from 6 to 9"
c. Roster: {7, 8, 9, 10, 11} and Set-Builder: $\{x \in \mathbb{N} \mid 7 \le x \le 11\}$
d. Set-Builder: $\{x \in \mathbb{N} \mid 1 \le x \le 19$ and x is odd$\}$ and Descriptive: "The set of odd natural numbers from 1 to 19"

Section 2.1: Set Theory [*In-Class Exercises* - Part 2 of 4]

2. Rewrite each set the same in two other ways.

a. "The set of natural numbers from 3 to 7"
b. $\{x \in \mathbb{N} \mid 5 \le x \le 10\}$

c. {2, 4, 6, 8, 10}
d. {$x \in \mathbb{N} \mid 1 \le x \le 11$ and x is odd}

Answers available at the end of this section.

Definition: Empty Set (or Null Set)

An empty set (or Null Set) is a set with no elements denoted by { } or the symbol Ø.

Definition: Cardinal Number (or Cardinality) of a Set

Cardinal numbers describe "how many" elements are in a set - they represent the size or **cardinality** of a set.

The notation for cardinality is $|A|$ or $n(A)$.

Examples:

- The set A = {*a*, *b*, *c*} has cardinal number 3. So, $|A| = 3$. (Alternate notation: $n(A) = 3$)
- The set B = {red, blue} has cardinal number 2. So, $|B| = 2$. (Alternate notation: $n(B) = 2$)
- The set C = { } has cardinal number 0. So, $|C| = 0$. (Alternate notation: $n(C) = 0$)

Example #2.1.3 🤔

Find the cardinal number of each set.

a. *A* = {3, 6, 9, 12, 15, 18, 21}
b. {*x* | *x* is any month of the year}
c. *E* = Ø
d. The set of all U.S. states.

✅ **Solution:**

a. $|A| = 7$
b. 12
c. $|E| = 0$
d. 50

Section 2.1: Set Theory [*In-Class Exercises* - Part 3 of 4]

3. Find the cardinal number of each set.

a. *A* = {1, 2, 3, 4, 5, 6, 7}
b. {*λ* | *λ* is any letter of the English alphabet}
c. *Z* = {0}
d. *Z* = Ø
e. *C* = {fork, knife, spoon}
f. S = {Ã, Ä, Å, Á, Æ}
g. *A* = {1, 2, 3, 4, 5, … , 100}

Answers available at the end of this section.

Definition: Equivalent Sets

Two sets are **equivalent** if they contain exactly the same number of elements, regardless of what those elements actually are.

Examples:

- $A = \{1, 2, 3\}$ and $B = \{\text{red, blue, green}\}$ are equivalent (both have 3 elements)
- $C = \{a, b\}$ and $D = \{x, y\}$ are equivalent (both have 2 elements)

Definition: Equal Sets

Two sets are **equal** if they contain exactly the same elements. Every element in the first set must also be in the second set, and every element in the second set must also be in the first set. Equal sets are denoted using the equality symbol (=).

Since equal sets contain exactly the same elements, they must also be equivalent. Therefore, every pair of equal sets is also a pair of equivalent sets.

Examples:

- $A = \{1, 2, 3\}$ and $B = \{1, 2, 3\}$ are equal sets, so $A = B$. (A and B are also equivalent.)
- $C = \{a, b, c\}$ and $D = \{c, a, b\}$ are equal sets, so $C = D$. (The order of elements in a set does not matter. C and D are also equivalent.)
- $E = \{\text{apple, banana}\}$ and $F = \{\text{apple, banana, pear}\}$ are not equal sets because they do not contain exactly the same elements. Therefore, $E \neq F$.

Key Idea: Every pair of equal sets is also equivalent, but not every pair of equivalent sets is equal.

Example #2.1.4 🤔

Determine whether each pair of sets is equal, equivalent, both (equal and equivalent), or neither.

a. {1, 16, 19, 91, 96} and {16, 96, 1, 91, 19}
b. {do, re, mi} and {baseball, football, basketball}
c. {four} and {f, o, u, r}
d. $\{x \mid x \text{ is any month of the year}\}$ and $\{y \mid y \in \mathbb{N} \text{ and } y < 13\}$
e. {5, 7, 9, 11, 13} and {5, 7, 9, 11, 13, ...}
f. {even natural numbers less than 10} and {2, 4, 6, 8}
g. {1, 2, 10, 20} and {2, 1, 20, 11}
h. $\{y \mid y \text{ is any odd natural number}\}$ and {1, 3, 5, 7, 9, ...}

✅ **Solution:**

a. Both equal and equivalent; each set has 5 elements and the elements in both sets are the same.
b. Equivalent; each set has 3 elements, but the elements in both sets are not the same.
c. Neither; first set has 1 element and the second set has 4 elements; the elements in both sets are not the same.
d. Equivalent; each set has 12 elements, but the elements in both sets are not the same; {Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec} & {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
e. Neither; first set has 5 elements and the second set is infinite; the elements in both sets are not the same.
f. Both equal and equivalent; each set has 4 elements and the elements in both sets are the same.
g. Equivalent; each set has 4 elements, but the elements in both sets are not the same.
h. Both equal and equivalent; each set has an infinite number of elements and the elements in both sets are the same.

Section 2.1: Set Theory [*In-Class Exercises* - Part 4 of 4]

4. Determine whether each pair of sets is equal, equivalent, or neither.

   a. {1, 2, 3, 4, 5} and {6, 7, 8, 9, 10}
   b. {3, 7, 11, 13} and {13, 3, 7, 11}
   c. {5, 10, 15, 20, 25} and {5, 10, 15, 20, 25, ...}
   d. {two} and {t, w, o}
   e. {odd natural numbers less than 10} and {1, 3, 5, 7, 9}
   f. {y | y is any even natural number} and {2, 4, 6, 8, 10, ...}
   g. {x | x is any day in the month of October} and { y | $y \in \mathbb{N}$ and $y < 32$}
   h. {dog, cat, mouse} and {moose, fox, squirrel, pig}

Answers available at the end of this section.

Answers

Section 2.1: Set Theory [*In-Class Exercises* - Part 1 of 4]

a. True
b. False
c. False
d. True
e. True
f. True
g. False
h. False
i. False
j. False
k. True
l. False

Section 2.1: Set Theory [*In-Class Exercises* - Part 2 of 4]

a. Roster: {3, 4, 5, 6, 7} and Set-Builder: {$x \in \mathbb{N} \mid 3 \leq x \leq 7$}
b. Roster: {5, 6, 7, 8, 9, 10} and Descriptive: "The set of natural numbers from 5 to 10"
c. Set-Builder: {$x \in \mathbb{N} \mid 2 \leq x \leq 10$ and x is even} and Descriptive: "The set of even natural numbers from 2 to 10"
d. Roster: {1, 3, 5, 7, 9, 11} and Descriptive: "The set of odd natural numbers from 1 to 11"

Section 2.1: Set Theory [*In-Class Exercises* - Part 3 of 4]

a. 7
b. 26
c. 1
d. 0
e. 3
f. 5
g. 100

Section 2.1: Set Theory [*In-Class Exercises* - Part 4 of 4]

a. equivalent
b. equal
c. neither
d. neither
e. equal
f. equal
g. equivalent
h. neither

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Section 2.1.H: Homework

Section 2.1: Set Theory - Homework Exercises

(Section 2.1) - Homework Exercise #1

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, e, i, o, u\}$, and $C = \{1, 2, 3, 4, 5, ...\}$. Decide whether each of the statements is true or false.

a. $6 \in A$
b. $6 \in C$
c. $7 \notin A$
d. $7 \notin C$
e. $3 \notin A$
f. $e \in B$
g. $a \in A$
h. $y \notin B$

Answer available at the end of this section.

(Section 2.1) - Homework Exercise #2

Let D = {Arizona, Mississippi, Texas}, E = {Colorado, Florida, New Jersey, Utah}, and F = {Arizona, Colorado, Iowa, Ohio, Wisconsin}. Decide whether each of the statements is true or false.

a. Texas $\in D$
b. Arizona $\in E$
c. Alaska $\notin F$
d. Hawaii $\in D$
e. Iowa $\notin F$
f. Utah $\in D$
g. Mississippi $\in D$
h. Colorado $\notin E$

Answer available at the end of this section.

(Section 2.1) - Homework Exercise #3

Rewrite each set the same in two other ways.

a. {11, 12, 13, 14}
b. "The set of natural numbers from 19 to 91"
c. $\{x \in \mathbb{N} \mid 99 \leq x \leq 103\}$
d. {2, 4, 6, 8, ... , 30}

Answer available at the end of this section.

(Section 2.1) - Homework Exercise #4

Find the cardinal number of each set.

a. $R = \{8, 18, 28, 38, 48\}$
b. $C = \{0\}$
c. $B = \{1{,}000{,}000{,}000\}$
d. The set of all letters in the alphabet
e. $S = \{6, 7, 8, 9, \ldots, 32\}$
f. $P = \emptyset$
g. $\{x \mid x \text{ is any day of the week}\}$
h. $K = \{1, 11, 111, 1111, \ldots, 111111111\}$
i. $\{z \mid z \text{ is any date in a calendar non leap year}\}$
j. The set of all dogs that speak English
k. $\{w \mid w \in \mathbb{N} \text{ and } w < 13\}$
l. $J = \{\ \}$

Answer available at the end of this section.

(Section 2.1) - Homework Exercise #5

Determine whether each pair of sets is equal, equivalent, both (equal and equivalent), or neither.

a. {dog, cat} and {7, 8}
b. $\{x \mid x \text{ is any day in the month of November}\}$ and $\{y \mid y \in \mathbb{N} \text{ and } y < 31\}$
c. $\{b \mid b \in \mathbb{N} \text{ and } b > 0\}$ and {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
d. {9, 99, 999} and {△, ○, □}
e. {c, f, k, m, p, q, s, t, v, w, y} and {y, f, k, w, p, s, v, c, q, t, m}
f. $\{c \mid c \text{ is any natural number that contains a zero and } c \leq 100\}$ and {10, 20, 30, ... , 100}
g. {Monday, Tuesday, Wednesday, Thursday, Friday} and {$, &, #, @, !}
h. {even natural numbers greater than 0} and {2, 4, 6, 8, ...}

Answer available at the end of this section.

Answers

(Section 2.1) - Homework Exercise #1

a. False
b. True
c. True
d. False
e. False
f. True
g. False
h. True

(Section 2.1) - Homework Exercise #2

a. True
b. False
c. True
d. False
e. False
f. False
g. True
h. False

(Section 2.1) - Homework Exercise #3

a. Set-Builder: $\{x \in \mathbb{N} \mid 11 \leq x \leq 14\}$ and Descriptive: "The set of natural numbers from 11 to 14"
b. Roster: {19, 20, 21, 22, ... , 91} and Set-Builder: $\{x \in \mathbb{N} \mid 19 \leq x \leq 91\}$
c. Roster: {99, 100, 101, 102, 103} and Descriptive: "The set of natural numbers from 99 to 103"

d. Set-Builder: $\{x \in \mathbb{N} \mid 2 \leq x \leq 30 \text{ and } x \text{ is even}\}$ and Descriptive: "The set of even natural numbers from 2 to 30"

(Section 2.1) - Homework Exercise #4

a. 5
b. 1
c. 1
d. 26
e. 27
f. 0
g. 7
h. 9
i. 365
j. 0
k. 12
l. 0

(Section 2.1) - Homework Exercise #5

a. Equivalent; each set has 2 elements, but the elements in both sets are not the same.
b. Both equal and equivalent; each set has 30 elements and the elements in both sets are the same.
c. Neither; first set is infinite and the second set has 10 elements; the elements in both sets are not the same.
d. Equivalent; each set has 3 elements, but the elements in both sets are not the same.
e. Both equal and equivalent; each set has 11 elements and the elements in both sets are the same.
f. Both equal and equivalent; each set has 10 elements and the elements in both sets are the same.
g. Equivalent; each set has 5 elements, but the elements in both sets are not the same.
h. Both equal and equivalent; each set has an infinite number of elements and the elements in both sets are the same.

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Section 2.2: Subsets

Learning Objectives

- Define the complement of a set
- Find all subsets of a set
- Use subset notation

Before we discuss anything further regarding sets, we need to define the universal set.

Definition: Universal Set, U

A **Universal Set** (denoted by U) is the set that contains all elements under consideration in a particular mathematical discussion or problem.

- Key point: It's not "everything in existence"; it's "everything relevant to our current problem."

Every mathematical discussion about sets needs boundaries. Without defining our universal set first, we're like:

- A teacher talking about "students" without specifying which school
- A chef discussing "ingredients" without mentioning which kitchen
- A librarian organizing "books" without identifying which library

Imagine you're having a conversation about students at Orange Coast College. We would define the universal set:

$$U = \{\text{All students at OCC}\}$$

Everything you discuss (freshmen, math majors, athletes, etc.) comes from this universe. You wouldn't suddenly start talking about students from other colleges, because they are outside your "universe".

The universal set changes based on what we're discussing. Let's look at another situation.



In 2022, McDonald's teamed up with street-wear brand Cactus Plant Flea Market to create one of the most popular Happy Meal toy collections ever that was aimed at adults. The collection consisted of four figurines to collect: Grimace, Cactus Buddy, the Hamburglar, and Birdie. Here, we would define the universal set as:

$$U = \{\text{Grimace, Cactus Buddy, Hamburglar, Birdie}\}$$

Once we define our universal set U, several important relationships automatically become clear. We can now define those set relationships which are also referred to as set inclusions.

Imagine you're organizing your music collection. You have:

- All your songs
- Just your jazz songs
- Just your workout songs
- Songs that are both jazz AND good for working out

These collections have natural relationships. Your jazz songs are *part of* your entire collection, and your jazz workout songs are *part of* your jazz collection. In mathematics, we call these relationships **subsets**.

Definition: Subset, $\subseteq$

Set A is a **subset** of set B if every element in A is also in B.

Symbolically: $A \subseteq B$ (read as "A is a subset of B ")

(Consequently, if set A is not a subset of set B, then A $\not\subseteq B$.)

Example #2.2.1 🤔

Let A = {red, white} and B = {red, white, blue}. Decide whether the statement $A \subseteq B$ is true or false.

✅ **Solution:**

Since the elements red and white from set A are also contained in set B, then A is a subset of B.

Thus, the statement $A \subseteq B$ is true.

Example #2.2.2 🤔

Let A = {red, white} and B = {red, white, blue}. Decide whether the statement $B \subseteq A$ is true or false.

✅ **Solution:**

Since there exists at least one element in set B (blue) that is NOT contained in set A, then B is not a subset of A.

Thus, the statement $B \subseteq A$ is false. (*Note: $B \not\subseteq A$ is true*).

Example #2.2.3 🤔

Let A = {Cities in the United States} and B = {Cities in California}. Decide whether the statement $A \subseteq B$ is true or false.

✅ **Solution:**

Since there is at least one city in the United States (i.e., Chicago) that is NOT in California, then A is not a subset of B.

Thus, the statement $A \subseteq B$ is false. (*Note: $A \not\subseteq B$ is true*).

Example #2.2.4 🤔

Let A = {Cities in the United States} and B = {Cities in California}. Decide whether the statement $B \subseteq A$ is true or false.

✅ **Solution:**

Since every city in California is also a city in the United States, then B is a subset of A.

Thus, the statement $B \subseteq A$ is true.

Example #2.2.5 🤔

Let A = {English majors at our college} and B = {All students at our college}. Decide whether the statement $A \subseteq B$ is true or false.

✅ **Solution:**

Since every English major at the college is a student at the college, then A is a subset of B.

Thus, the statement $A \subseteq B$ is true.

Example #2.2.6 🤔

Let A = {Legislative, Executive, Judicial} and B = {The three branches of U.S. government}. Decide whether the statement $A \subseteq B$ is true or false.

✅ **Solution:**

Since the Legislative, Executive, and Judicial branches are the three branches of U.S. government, then A is a subset of B

Thus, the statement $A \subseteq B$ is true.

(*Since set A and set B are exactly the same elements, $A = B$ is also true.*)

Example #2.2.7 🤔

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Decide whether the statement $A \subseteq B$ is true or false.

✅ **Solution:**

Since 1 is an element of B and 2 is and element of B, (or equivalently $1 \in B$ and $2 \in B$), then A is a subset of B.

Thus, the statement $A \subseteq B$ is true.

Example #2.2.8 🤔

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Decide whether the statement $B \subseteq A$ is true or false.

✅ **Solution:**

Since 3 is NOT an element of A, (or equivalently $3 \notin A$), then B is not a subset of A.

Thus, the statement $B \subseteq A$ is false. (*Note: $B \not\subseteq A$ is true*).

Section 2.2: Subsets [*In-Class Exercises* - Part 1 of 3]

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$. Decide whether the statement $A \subseteq B$ is true or false.
2. Let C = {circle, triangle, rhombus} and D = {circle, triangle, hexagon}. Decide whether the statement $C \subseteq D$ is true or false.
3. Let E = {a, b, c, x, y} and F = {a, b, c}. Decide whether the statement $E \not\subseteq F$ is true or false.
4. In Major League Baseball (MLB), there are 30 teams among 6 divisions. The American League West division contains the following five teams: Angels, Astros, Athletics, Mariners and Rangers. Let M = {All 30 MLB Baseball teams} and W = {Five American League West teams}. Which statement is true, $M \subseteq W$ or $W \subseteq M$?
5. Let S = {All Star Wars movies/films} and F = {All science fiction/fantasy movies/films}. Which statement is true, $F \subseteq S$ or $S \subseteq F$?

Answers available at the end of this section.

Now, let's look at another mathematical example, where both C and D are the exact same sets. Let $C = \{1, 2, 3\}$ and $D = \{1, 2, 3\}$. Is $C \subseteq D$? Let's check.

- Is $1 \in$ D? Yes ✓
- Is $2 \in$ D? Yes ✓
- Is $3 \in$ D? Yes ✓

Thus, $C \subseteq D$ is true! This feels different from our first mathematical example above.

So, let's compare the two mathematical examples

- Earlier: $A = \{1, 2\}$ and $B = \{1, 2, 3\}$
- Here $A \subseteq B$, but A is clearly "smaller" than B (as it relates to cardinality)
- Now: $C = \{1, 2, 3\}$ and $D = \{1, 2, 3\}$
- Here $C \subseteq D$, but C and D are identical!

This observation led mathematicians to create a more specific term. When we want to say that A is a subset of B, but A is definitely not equal to B, meaning that B has *at least one* element that A doesn't have, then we call this a **proper subset**.

Definition: Proper Subset, $\subset$

A is a **proper subset** of B (written $A \subset B$) when:

1. $A \subseteq B$ (A is a subset of B), AND
2. $A \neq B$ (A is not equal to B)

Symbolically: $A \subset B$ (read as "A is a proper subset of B ")

(Consequently, if set A is a not a proper subset of set B, then A $\not\subset B$.)

So, in other words, for $A \subset B$, means that there is at least one element in set B (the set on the right) that is not in set A (the set on the left). The key difference is that a subset ($\subseteq$) allows for the possibility that $A = B$, whereas a proper subset ($\subset$), has A "strictly smaller" than B.

Example #2.2.9 🤔

Let A = {red, white} and B = {red, white, blue}. Decide whether the statement $A \subset B$ is true or false.

✅ **Solution:**

Check the following conditions:

1. Is $A \subseteq B$? Yes, every element of set A is also contained in set B. ✅
2. Is $A \neq B$? Yes, set A is not equal to set B. ✅

Since both conditions are satisfied, then A is a proper subset of B.

Thus, the statement $A \subset B$ is true.

Example #2.2.10 🤔

Let A = {red, white} and B = {red, white, blue}. Decide whether the statement $B \subset A$ is true or false.

✅ **Solution:**

Check the following conditions:

1. Is $B \subseteq A$? No, every element of set B is NOT contained in set A. ❌
2. Is $A \neq B$? Yes, set A is not equal to set B. ✅ (*Note: There is no need to check for the second condition, if the first one fails.*)

Since at least one of the conditions are NOT satisfied, then B is not a proper subset of A.

Thus, the statement $B \subset A$ is false. (*Note: $B \not\subset A$ is true*).

Example #2.2.11 🤔

Let A = {Cities in the United States} and B = {Cities in California}. Decide whether the statement $A \subset B$ is true or false.

✅ **Solution:**

Check the following conditions:

1. Is $A \subseteq B$? No, every element of set A is NOT contained in set B. ❌
2. Is $A \neq B$? Yes, set A is not equal to set B. ✅ (*Note: There is no need to check for the second condition, if the first one fails.*)

Since at least one of the conditions are NOT satisfied, then A is not a proper subset of B.

Thus, the statement $A \subset B$ is false. (*Note: $A \not\subset B$ is true*).

Example #2.2.12 🤔

Let A = {Cities in the United States} and B = {Cities in California}. Decide whether the statement $B \subset A$ is true or false.

✅ **Solution:**

Check the following conditions:

1. Is $B \subseteq A$? Yes, every element of set B is also contained in set A. ✅
2. Is $A \neq B$? Yes, set A is not equal to set B. ✅

Since both conditions are satisfied, then B is a proper subset of A.

Thus, the statement $B \subset A$ is true.

Example #2.2.13 🤔

Let A = {English majors at our college} and B = {All students at our college}. Decide whether the statement $A \subset B$ is true or false.

✅ **Solution:**

Check the following conditions:

1. Is $A \subseteq B$? Yes, every element of set A is also contained in set B. ✅
2. Is $A \neq B$? Yes, set A is not equal to set B. ✅

Since both conditions are satisfied, then A is a proper subset of B.

Thus, the statement $A \subset B$ is true.

Example #2.2.14 🤔

Let A = {Legislative, Executive, Judicial} and B = {The three branches of U.S. government}. Decide whether the statement $A \subset B$ is true or false.

✅ **Solution:**

Check the following conditions:

1. Is $A \subseteq B$? Yes, every element of set A is also contained in set B. ✅

2. Is $A \neq B$? No, set A is equal to set B. ❌

Since at least one of the conditions are NOT satisfied, then A is not a proper subset of B.

Thus, the statement $A \subset B$ is false. (*Note: $A \not\subset B$ is true*).

(Think of a proper subset as if is was a strictly less than, but not equal to, sign (<). So, is the set {1, 2, 3} < {1, 2, 3}? No.)

Example #2.2.15 🤔

Let A = {1, 2} and B = {1, 2, 3}. Decide whether the statement $A \subset B$ is true or false.

✅ **Solution:**

Check the following conditions:

1. Is $A \subseteq B$? Yes, every element of set A is also contained in set B. ✅
2. Is $A \neq B$? Yes, set A is not equal to set B. ✅

Since both conditions are satisfied, then A is a proper subset of B.

Thus, the statement $A \subset B$ is true.

(Think of a proper subset as if is was a strictly less than, but not equal to, sign (<). So, is the set {1, 2} < {1, 2, 3}? Yes.)

Example #2.2.16 🤔

Let A = {1, 2} and B = {1, 2, 3}. Decide whether the statement $B \subset A$ is true or false.

✅ **Solution:**

Check the following conditions:

1. Is $B \subseteq A$? No, every element of set B is NOT contained in set A. ❌
2. Is $A \neq B$? Yes, set A is not equal to set B. ✅ (*Note: There is no need to check for the second condition, if the first one fails.*)

Since at least one of the conditions are NOT satisfied, then A is not a proper subset of B.

Thus, the statement $B \subset A$ is false. (*Note: $B \not\subset A$ is true*).

(Think of a proper subset as if is was a strictly less than, but not equal to, sign (<). So, is the set {1, 2, 3} < {1, 2}? No.)

Example #2.2.17 🤔

Let A = {x, y, z} and B = {x, y, z}. Decide whether the statement $A \subset B$ is true or false.

✅ **Solution**

Check the following conditions:

1. Is $A \subseteq B$? Yes, every element of set A is also contained in set B. ✅
2. Is $A \neq B$? No, set A is equal to set B. ❌

Since at least one of the conditions are NOT satisfied, then A is not a proper subset of B.

Thus, the statement $A \subset B$ is false. (*Note: $A \not\subset B$ is true*).

(Think of a proper subset as if it was a strictly less than, but not equal to, sign (<). So, is the set {x, y, z} < {x, y, z}? No.)

Section 2.2: Subsets [*In-Class Exercises* - Part 2 of 3]

6. Let A = {1, 2, 3, 4} and B = {1, 2, 3, 4, 5}. Decide whether the statement $A \subset B$ is true or false.
7. Let C = {circle, triangle, hexagon} and D = {circle, triangle, octagon}. Decide whether the statement $D \subset C$ is true or false.
8. Let E = {a, b, c} and F = {a, b, c, x, y}. Decide whether the statement $E \not\subset F$ is true or false.
9. Let G = {2, 4, 6, 8, 10} and H = {2, 4, 6, 8, 10}. Decide whether the statement $G \subset H$ is true or false.
10. In Major League Baseball (MLB), there are 30 teams among 6 divisions. The National League West division contains the following five teams: Diamondbacks, Dodgers, Giants, Padres and Rockies. Let M = {All 30 MLB Baseball teams} and W = {Five National League West teams}. Which statement is true, $M \subset W$ or $W \subset M$ or neither?
11. Let S = {All planets in our solar system} and P = {Mercury, Venus, Earth}. Which statement is true, $P \subset S$ or $S \subset P$ or neither.

Answers available at the end of this section.

Properties of Subsets and Proper Subsets

1. **Every set is a subset of itself**
   - {1, 2, 3} $\subseteq$ {1, 2, 3} ✓
   - But {1, 2, 3} $\not\subset$ {1, 2, 3} (A set is not a proper subset of itself)
2. **The empty set is a subset of every set**
   - $\emptyset \subseteq$ {1, 2, 3} ✓
   - $\emptyset \subseteq$ {cats, dogs} ✓
   - $\emptyset \subseteq \emptyset$ ✓
     - *Why?* Because there are no elements in $\emptyset$ to violate the subset condition!
3. **Transitivity**
   - If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
     - A = {Calculus students}
     - B = {Math majors}
     - C = {All students}
     - $A \subseteq B \subseteq C$, so $A \subseteq C$

Common Mistakes to Avoid

- **Mistake 1: Confusing $\in$ and $\subseteq$**
  - ❌ Incorrect: {2} $\in$ {1, 2, 3}
  - ✅ Correct: {2} $\subseteq$ {1, 2, 3} or $2 \in$ {1, 2, 3}
- **Mistake 2: Forgetting the Empty Set**
  - When listing all subsets of {1, 2}, don't forget: $\emptyset$, {1}, {2}, {1, 2}
- **Mistake 3: Thinking Proper Subsets Can Be Equal**
  - If $A \subset B$, then $A \neq B$ (they cannot be equal)

Conclusion

Understanding subsets helps us:

- Organize information logically

- Understand hierarchical relationships
- Make precise statements about collections of objects
- Build a foundation for more advanced mathematical concepts

The distinction between subsets ($\subseteq$) and proper subsets ($\subset$) might seem subtle, but it's crucial for mathematical precision - just like the difference between "less than or equal to" ($\leq$) and "less than" ($<$).

Remember: Every proper subset is a subset, but not every subset is a proper subset!

Example #2.2.18 🤔

Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{1, 2, 3, 4, 5\}$, and $D = \{1, 2, 3, 4, 5, ...\}$. Decide whether each of the statements is true or false.

a. $A \subseteq B$
b. $B \subseteq A$
c. $A \subset B$
d. $B \subset A$
e. $B \subseteq C$
f. $C \subseteq B$
g. $B \subset C$
h. $C \subset B$
i. $B \subseteq D$
j. $D \subseteq B$
k. $B \subset D$
l. $D \subset B$

✅ **Solution:**

a. True
b. False
c. True
d. False
e. True
f. True
g. False
h. False
i. True
j. False
k. True
l. False

Example #2.2.19 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2\}$, $B = \{3, 4\}$, and $C = \{1, 2, 3, 4\}$. Decide whether each of the statements is true or false.

a. $A \subseteq B$
b. $B \subseteq A$
c. $A \subset B$
d. $B \subset A$
e. $A \subseteq C$
f. $C \subseteq A$
g. $A \subset C$
h. $C \subset A$

i. $A \subseteq U$
j. $U \subseteq A$
k. $U \subset A$
l. $A \subset U$

✅ **Solution:**

a. False
b. False
c. False
d. False
e. True
f. False
g. True
h. False
i. True
j. False
k. False
l. True

Section 2.2: Subsets [*In-Class Exercises* - Part 3 of 3]

12. Let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{3, 4, 5\}$, $B = \{1, 3, 6\}$, $C = \{3, 4, 5, 6\}$, and $D = \{1, 3, 6\}$. Decide whether each of the statements is true or false.
    a. $A \subseteq C$
    b. $C \subseteq A$
    c. $A \subset C$
    d. $C \subset A$
    e. $B \subseteq D$
    f. $D \subseteq B$
    g. $B \subset D$
    h. $D \subset B$
    i. $A \subseteq B$
    j. $B \subset A$
    k. $A \subset U$
    l. $U \subset A$

Answers available at the end of this section.

Answers

Section 2.2: Subsets [*In-Class Exercises* - Part 1 of 3]

1. True
2. False
3. True
4. $W \subseteq M$
5. $S \subseteq F$

Section 2.2: Subsets [*In-Class Exercises* - Part 2 of 3]

6. True
7. False
8. False
9. False
10. $W \subset M$
11. $P \subset S$

Section 2.2: Subsets [*In-Class Exercises* - Part 3 of 3]

a. True

b. False
c. True
d. False
e. True
f. True
g. False
h. False
i. False
j. False
k. True
l. False

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Section 2.2.H: Homework

Section 2.2: Subsets - Homework Exercises

(Section 2.2) - Homework Exercise #1

Let $A = \{1, 3, 7\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{3, 5, 9\}$, $D = \{1, 3, 5, 7, 9, ...\}$, and $E = \{3, 5, 9\}$. Decide whether each of the statements is true or false.

a. $A \subseteq B$
b. $B \subseteq A$
c. $A \subset B$
d. $B \subset A$
e. $A \subseteq C$
f. $C \subseteq A$
g. $A \subset C$
h. $C \subset A$
i. $B \subseteq C$
j. $C \subseteq B$
k. $B \subset C$
l. $C \subset B$
m. $C \subseteq D$
n. $D \subseteq C$
o. $C \subset D$
p. $D \subset C$
q. $C \subseteq E$
r. $E \subseteq C$
s. $E \subset C$
t. $C \subset E$

Answer available at the end of this section.

(Section 2.2) - Homework Exercise #2

Let U = the set of all letters in the English alphabet, $A = \{a, b, c\}$, $B = \{a, b, c, d\}$, $C = \{a, b, c\}$, $D = \{a, e, i, o, u, y\}$, and $E = \{a, e, i, o, u\}$. Decide whether each of the statements is true or false.

a. $A \subseteq B$
b. $B \subseteq A$
c. $A \subset B$
d. $B \subset A$
e. $A \subseteq C$
f. $C \subseteq A$
g. $A \subset C$
h. $C \subset A$
i. $E \subseteq D$
j. $E \subset D$
k. $D \subset E$
l. $D \subseteq E$
m. $D \subseteq U$
n. $D \subset U$

o. $U \subseteq D$
p. $U \subset D$

Answer available at the end of this section.

(Section 2.2) - Homework Exercise #3

Let $A = \{x \in \mathbb{N} \mid x \text{ is odd and } 0 < x < 10\}$, $B = \{3, 5, 7,\}$, $C = \{2, 4, 6, 8\}$, $D = \{3, 5, 7, 9\}$, and $E = \{1, 3, 5, 7, 9, 11\}$. Decide whether each of the statements is true or false.

a. $A \subseteq B$
b. $A \subset B$
c. $B \subseteq A$
d. $B \subset A$
e. $A \subseteq C$
f. $A \subset C$
g. $C \subseteq A$
h. $C \subset A$
i. $A \subseteq D$
j. $A \subset D$
k. $D \subseteq A$
l. $D \subset A$
m. $A \subseteq E$
n. $A \subset E$
o. $E \subseteq A$
p. $E \subset A$

Answer available at the end of this section.

Answers

(Section 2.2) - Homework Exercise #1

a. True
b. False
c. True
d. False
e. False
f. False
g. False
h. False
i. False
j. True
k. False
l. True
m. True
n. False
o. True
p. False
q. True
r. True
s. False
t. False

(Section 2.2) - Homework Exercise #2

a. True
b. False
c. True
d. False
e. True
f. True
g. False
h. False

i. True
j. True
k. False
l. False
m. True
n. True
o. False
p. False

(Section 2.2) - Homework Exercise #3

a. False
b. False
c. True
d. True
e. False
f. False
g. False
h. False
i. False
j. False
k. True
l. True
m. True
n. True
o. False
p. False

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Section 2.3: Operations on Sets

Learning Objectives

- Find the union of a set
- Find the intersection of a set
- Find the complement of a set
- Find the set difference of a set
- Order of operations on sets
- Introduce and apply De Morgan's Laws

Set Operations: Like Arithmetic for Collections

Just as we have operations for numbers (addition, subtraction, multiplication, division), we have operations for sets. These operations let us combine, compare, and manipulate collections of objects in systematic ways.

The Parallel to Number Operations

With numbers:	With sets:
3 + 5 = 8 (combining quantities)	We can combine sets (union)
10 - 4 = 6 (finding the difference)	We can find differences (difference and complement)
3 × 4 = 12 (repeated addition)	We can find what's shared (intersection)

The 4 basic operations on sets, union, intersection, complement, and set difference, are given below along with several examples for each.

Definition: Union, $\cup$

The set $A \cup B$ contains all elements that are in set A, or in set B, or in both.

Symbolically: $A \cup B$ (read as "A or B" or "A union B")

Think of it as: Adding or combining collections together

Example #2.3.1 🤔

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Find $A \cup B$.

✅ **Solution:**

$A \cup B = \{1, 2, 3, 4, 5\}$,

because the elements 1, 2, 3, 4, 5 are in set A, or set B, or in both sets A and B. Notice: We don't list 3 twice, even though it appears in both sets.

Example #2.3.2 🤔

Students taking a math class are represented by M. Students taking an English class are represented by E. If $M = \{$Alice, Bob, Diego$\}$ and $E = \{$Carlos, Diego, Emily$\}$. Find $M \cup E$.

✅ **Solution:**

$M \cup E = \{$Alice, Bob, Carlos, Diego, Emily$\}$,

because Alice, Bob, Carlos, Diego, Emily are in set M, or set E, or in both sets M and E.

Example #2.3.3 🤔

Let $C = \{@, \#, \$, \%\}$ and $D = \{\&, @, \$\}$. Find $C \cup D$.

✅ **Solution:**

$C \cup D = \{@, \#, \$, \%, \&\}$,

because the elements @, #, $, %, & are in set C, or set D, or in both sets C and D.

Example #2.3.4 🤔

Breakfast foods are represented by B. Lunch foods are represented by L. If $B = \{$eggs, toast, cereal$\}$ and $L = \{$sandwich, salad, soup$\}$. Find $B \cup L$.

✅ **Solution:**

$B \cup L = \{$eggs, toast, cereal, sandwich, salad, soup$\}$,

because eggs, toast, cereal, sandwich, salad, soup are in set B, or set L, or in both sets B & L.

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 1 of 5]

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$. Find $A \cup B$.
2. Let $C = \{$circle, triangle, rhombus$\}$ and $D = \{$triangle, rectangle, hexagon$\}$. Find $C \cup D$.
3. Let $E = \{$a, b, c, x, y$\}$ and $F = \{$d, e, x, y, z$\}$. Find $E \cup F$.
4. Let $A = \{$Broncos, Chargers, Raiders$\}$ and $N = \{$Chiefs, Seahawks$\}$. Find $A \cup N$.

Answers available at the end of this section.

Definition: Intersection, ∩

The set $A \cap B$ contains only elements that are in both sets A and B.

Symbolically: $A \cap B$ (read as "A and B" or "A intersection B")

Think of it as: Finding what's shared between collections

Example #2.3.5 🤔

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Find $A \cap B$.

✅ **Solution:**

$A \cap B = \{3\}$, because the element 3 is in both sets A and B.

Example #2.3.6 🤔

Students taking a math class are represented by M. Students taking an English class are represented by E. If $M = \{$Alice, Bob, Diego$\}$ and $E = \{$Carlos, Diego, Emily$\}$. Find $M \cap E$.

✅ **Solution:**

$M \cap E = \{$Diego$\}$, because Diego is taking both a math and an English class.

Example #2.3.7 🤔

Let $C = \{$@, #, $, %$\}$ and $D = \{$&, @, $$\}$. Find $C \cap D$.

✅ **Solution:**

$C \cap D = \{$@, $$\}$, because the elements @ and $ are in both sets C and D.

Example #2.3.8 🤔

Breakfast foods are represented by B. Lunch foods are represented by L. If $B = \{$eggs, toast, cereal$\}$ and $L = \{$sandwich, salad, soup$\}$. Find $B \cap L$.

✅ **Solution:**

$B \cap L = \{\ \}$, because there is nothing in common with either set.

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 2 of 5]

5. Let $A = \{1, 2, 3, 4, 5, 6, 9, 10\}$ and $B = \{2, 3, 5, 7, 8, 9\}$. Find $A \cap B$.
6. Let $C = \{$bird, cat, goat, pig$\}$ and $D = \{$dog, cat, pig, walrus$\}$. Find $C \cap D$.
7. Let $X = \{$a, b, c, f, h, k, m, p, t$\}$ and $Y = \{$d, e, g, j, n, o, r, s, z$\}$. Find $X \cap Y$.
8. Let P represent the set of all U.S. Presidents by last name. Let S represent all 50 U.S. states. Find $P \cap S$.

Answers available at the end of this section.

Definition: Complement, $'$

The set A' is the set of all elements in the universal set, U, that are not in A'.

Symbolically: A' (read as "A complement")

Think of it as: Everything that's NOT in the set

Example #2.3.9 🤔

Let $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 2, 3, 4\}$. Find A'.

✅ **Solution:**

$$A' = \{5, 6\},$$

because the elements 5, 6 are in the universe set, U, but not in set A. (Think of collecting Happy Meal toys from McDonalds when you were younger. There was a set of 6 different toys numbered from #1 to #6. You have collected toys #1, #2, #3, and #4. Which toys are you missing to complete your set? You are only missing toys #5 and #6 to complete your collection.)

Example #2.3.10 🤔

Let $U = \{a, b, c, d, e, f, g, h\}$ and $B = \{b, c, d, f, g\}$. Find B'.

✅ **Solution:**

$$B' = \{a, e, h\},$$

because the elements a, e, h are in the universe set, U, but not in set B.

Example #2.3.11 🤔

Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{1, 2, 3\}$. Find A'.

✅ **Solution:**

$$A' = \{4, 5\},$$

because the elements 4, 5 are in the universe set, U, but not in set A.

Example #2.3.12 🤔

Let $U = \{\text{President, Vice-President, Speaker of the House}\}$. Find U'.

✅ **Solution:**

$$U' = \{\ \},$$

because there are no elements outside the universe set, U. Every object under consideration is included in the universal set.

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 3 of 5]

9. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$. Find A'.
10. Let $U = \{\text{apple, banana, cherry, date}\}$ and $B = \{\text{banana, cherry}\}$. Find B'.
11. Let $U = \{\text{All cities in Orange County, California}\}$, $A = \{\text{Anaheim, Costa Mesa, Huntington Beach}\}$, $B = \{\text{Placentia, Atwood, Brea}\}$. Find U'.
12. Let $U = \{\alpha, \beta, \delta, \epsilon, \Sigma, \mu, \Omega\}$ and $G = \{\alpha, \beta, \epsilon, \Sigma, \mu, \Omega\}$. Find G'.

Answers available at the end of this section.

Definition: Set Difference, –

The set $A - B$ is the set of all elements that are in A but not in B.

Symbolically: $A - B$ (read as "A minus B")

Think of it as: Removing elements from a collection

Example #2.3.13 🤔

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Find $A - B$.

✅ **Solution:**

$$A - B = \{1, 2\}.$$

Why? First off, we are trying to find which elements of A that are not in B. The elements 3, 4 are in both sets. The elements 1, 2 are in A, but not in B. The elements 5, 6 have nothing to with the operation, because the operation only considers the elements in the first set, A. (Think of starting with set A. Now remove any matching elements that are also in set B. Remove elements 3, 4 from set A. Whatever element(s) left over, if any, is the answer.)

Example #2.3.14 🤔

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Find $B - A$.

✅ **Solution:**

$$B - A = \{5, 6\},$$

because the elements 5, 6 are in set B, but not in set A. The elements 3, 4 are in both sets A and B.

Example #2.3.15 🤔

Let $C = \{\text{orange, blue, purple, green, red, black, white}\}$ and $D = \{\text{blue, purple, green, red, black, white, yellow}\}$. Find $C - D$.

✅ **Solution:**

$$C - D = \{\text{orange}\},$$

because the element orange is in set C, but not in set D. The elements blue, purple, green, red, black, white are in both sets C and D.

Example #2.3.16 🤔

Let $E = \{a, b, c, d\}$ and $F = \{x, y, z\}$. Find $E - F$.

✅ **Solution:**

$$E - F = \{a, b, c, d\},$$

because the elements a, b, c, d are in set E, but not in set F. There are no common elements in both sets E and F.

Example #2.3.17 🤔

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $T = \{1, 3, 5, 7, 9\}$. Find $T - S$.

✅ **Solution:**

$$T - S = \{\ \},$$

because there are no elements that are in set T, but not in set S. The entire set T is already in S, so this creates the empty set solution for the set difference.

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 4 of 5]

13. Let $A = \{a, b, c, d, e, f\}$ and $B = \{d, e, f, g, h\}$. Find $A - B$.
14. Let $A = \{a, b, c, d, e, f\}$ and $B = \{d, e, f, g, h\}$. Find $B - A$.
15. Let $C = \{3, 5, 7, 9\}$ and $D = \{1, 3, 4, 6, 9\}$. Find $C - D$.
16. Let $E = \{\text{cat, deer, pig}\}$ and $F = \{\text{cat, deer, dog, goat, pig}\}$. Find $F - E$.
17. Let $P = \{1, 2, 3, 4\}$ and $Q = \{5, 6, 7, 8\}$. Find $Q - P$.
18. Let $J = \{\spadesuit, \heartsuit\}$ and $K = \{\diamondsuit, \clubsuit\}$. Find $J - K$.

Answers available at the end of this section.

Comparing Operations: A Summary Table

Operation	Symbol	Like Number Operation	Result
Union	$A \cup B$	Addition (+)	Everything from both sets
Intersection	$A \cap B$	Finding common factors	Only what's in both sets
Complement	A'	Negative (-)	Everything NOT in the set
Difference	$A - B$	Subtraction (–)	What's in A but not in B

Order of Operations (Like PEMDAS for Sets)

When working with set expressions that involve multiple operations, we follow a specific order of operations, similar to how PEMDAS guides us with numerical expressions.

- **First, we always evaluate anything inside parentheses or brackets**, as these explicitly tell us what to compute first. For example, in $(A \cup B) \cap C$, we must find $A \cup B$ before intersecting with C.
- **Second, we evaluate complements**, which act like exponents or negation in arithmetic. So, in an expression like $A' \cap B$, we find A' before performing the intersection.
- **Third, we perform intersections (∩) before unions (∪)**, much like how multiplication comes before addition in PEMDAS. This means $A \cup B \cap C$ is interpreted as $A \cup (B \cap C)$, not $(A \cup B) \cap C$.
- **Finally, we evaluate unions and differences from left to right** when they appear at the same level. This hierarchy exists because intersection is considered a "stronger" operation than union, binding elements more tightly together, just as multiplication binds more tightly than addition. For instance, the expression $A \cup B' \cap C \cup D$ would be evaluated as: first find B', then compute $B' \cap C$, then evaluate $A \cup (B' \cap C)$, and finally compute that result $\cup D$. Understanding this order is crucial because $A \cup B \cap C$ and $(A \cup B) \cap C$ produce completely different results—the first gives you A combined with the overlap of B and C, while the second gives you only the overlap between the union of A and B with C.

When in doubt, **use parentheses to make your intentions clear**, as they override all other rules and ensure your expression is evaluated exactly as you intend, eliminating any ambiguity in complex set operations.

Example #2.3.18 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A \cup B \cap C$.

✅ **Solution:**

- Step 1 - Intersection before union: $B \cap C = \{4, 5\}$
- Step 2 - Now union: $A \cup \{4, 5\}$

$A \cup B \cap C = \{1, 2, 3, 4, 5\}$

Example #2.3.19 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $(A \cup B) \cap C$.

✅ **Solution:**

- Step 1 - Parentheses first: $A \cup B = \{1, 2, 3, 4, 5\}$
- Step 2 - Now intersection: $\{1, 2, 3, 4, 5\} \cap C$

$(A \cup B) \cap C = \{4, 5\}$

Notice: From examples #2.3.18 & #2.3.19, $A \cup B \cap C \neq (A \cup B) \cap C$.

Example #2.3.20 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A' \cap B$.

✅ **Solution:**

- Step 1 - Complement first: $A' = \{5, 6, 7, 8\}$
- Step 2 - Now intersection: $\{5, 6, 7, 8\} \cap B$

$A' \cap B = \{5\}$

Example #2.3.21 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. $A \cap B'$.

✅ **Solution:**

- Step 1 - Complement first: $B' = \{1, 2, 6, 7, 8\}$
- Step 2 - Now intersection: $A \cap \{1, 2, 6, 7, 8\}$

$A \cap B' = \{1, 2\}$

Note: From examples #2.3.20 & #2.3.21, $A \cap B' \neq (A' \cap B)$.

Example #2.3.22 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A \cup B' \cap C$.

✅ **Solution:**

- Step 1 - Complement first: $B' = \{1, 2, 6, 7, 8\}$
- Step 2 - Intersection before union: $B' \cap C = \{6, 7\}$
- Step 3 - Now union: $A \cup \{6, 7\}$

$A \cup B' \cap C = \{1, 2, 3, 4, 6, 7\}$

Example #2.3.23 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $(A \cup B)' \cap C$.

✅ **Solution:**

- Step 1 - Parentheses first: $A \cup B = \{1, 2, 3, 4, 5\}$
- Step 2 - Complement: $(A \cup B)' = \{6, 7, 8\}$
- Step 3 - Intersection: $\{6, 7, 8\} \cap C$

$(A \cup B)' \cap C = \{6, 7\}$

Notice: From examples #2.3.22 & #2.3.23, $A \cup B' \cap C \neq (A \cup B)' \cap C$.

Example #2.3.24 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A \cup (B' \cap C)$.

✅ **Solution:**

- Step 1 - Parentheses first, complement inside: $B' = \{1, 2, 6, 7, 8\}$
- Step 2 - Still in parentheses, intersection: $B' \cap C = \{6, 7\}$
- Step 3 - Now union: $A \cup \{6, 7\}$

$A \cup (B' \cap C) = \{1, 2, 3, 4, 6, 7\}$

Notice: This is the same result in Example #2.3.22.

Example #2.3.25 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A \cap B \cup C$.

✅ **Solution:**

- Step 1: Intersection first: $A \cap B = \{3, 4\}$
- Step 2: Union: $\{3, 4\} \cup C$

$A \cap B \cup C = \{3, 4, 5, 6, 7\}$

Example #2.3.26 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A \cap (B \cup C)$.

✅ **Solution:**

- Step 1 - Parentheses first: $B \cup C = \{3, 4, 5, 6, 7\}$
- Step 2 - Union: $A \cap \{3, 4, 5, 6, 7\}$

$A \cap (B \cup C) = \{3, 4\}$

Notice: From examples #2.3.25 & #2.3.26, $A \cap B \cup C \neq A \cap (B \cup C)$.

Example #2.3.27 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A \cap C \cup B$.

✅ **Solution:**

- Step 1 - Intersection first: $A \cap C = \{4\}$
- Step 2 - Union: $\{4\} \cup B$

$A \cap C \cup B = \{3, 4, 5\}$

Example #2.3.28 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $(A \cup B)'$.

✅ **Solution:**

- Step 1 - Parentheses first: $A \cup B = \{1, 2, 3, 4, 5\}$
- Step 2 - Complement: $\{1, 2, 3, 4, 5\}'$

$(A \cup B)' = \{6, 7, 8\}$

Example #2.3.29 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A' \cap B'$.

✅ **Solution:**

- Step 1 - Complements first: $A' = \{5, 6, 7, 8\}$ & $B' = \{1, 2, 6, 7, 8\}$
- Step 2 - Intersection: $\{5, 6, 7, 8\} \cap \{1, 2, 6, 7, 8\}$

$A' \cap B' = \{6, 7, 8\}$

Notice: From Examples #2.3.28 & #2.3.29, $(A \cup B)' = A' \cap B'$.

Example #2.3.30 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $(A \cap B)'$.

✅ **Solution:**

- Step 1 - Parentheses first: $A \cap B = \{3, 4\}$
- Step 2 - Complement: $\{3, 4\}'$

$(A \cap B)' = \{1, 2, 5, 6, 7, 8\}$

Example #2.3.31 🤔

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$. Find $A' \cup B'$.

✅ **Solution:**

- Step 1 - Complements first: $A' = \{5, 6, 7, 8\}$ & $B' = \{1, 2, 6, 7, 8\}$
- Step 2 - Union: $\{5, 6, 7, 8\} \cup \{1, 2, 6, 7, 8\}$

$A' \cup B' = \{1, 2, 5, 6, 7, 8\}$

Notice: From Examples #2.3.30 & #2.3.31, $(A \cap B)' = A' \cup B'$.

Notice that Example #2.3.28 and Example #2.3.29 produced the same result {6, 7, 8}. Also, notice that Example #2.3.30 and Example #2.3.31 produced the same result {1, 2, 5, 6, 7, 8} as well. This was not coincidence. In fact, regardless of what the contents are in set A or set B, the outcomes would always be the same for $(A \cup B)' = A' \cap B'$ as well as $(A \cap B)' = A' \cup B'$. These are called **De Morgan's Laws**. De Morgan's Laws are two fundamental rules in set theory (and logic) that describe how complement interacts with union and intersection. They're named after mathematician Augustus De Morgan (1806–1871), a British mathematician and logician who made significant contributions to mathematics, particularly in the fields of logic, algebra, and set theory.

These laws show us how to "distribute" the complement operation across unions and intersections.

Definition: De Morgan's Laws

For any two finite sets, A and B, the following equations are always true:

- $(A \cup B)' = A' \cap B'$ (De Morgan's Law of Union)

- $(A \cap B)' = A' \cup B'$ (De Morgan's Law of Intersection)

These laws were published in 1847 in his work on formal logic

De Morgan's Laws tell us:

- Complementing a union gives you an intersection of complements
- Complementing an intersection gives you a union of complements
- The operation "flips" and the sets become their complements

De Morgan's Laws are essential tools in mathematics and computer science for several important reasons.

First, they help us simplify complex set expressions by providing alternative forms that may be easier to work with or understand. What might look complicated on one side of the equation can often be expressed more simply using the other side. Second, these laws are fundamental in logic and programming, where they appear constantly in computer science applications: the principle that NOT (A OR B) equals (NOT A) AND (NOT B), and that NOT (A AND B) equals (NOT A) OR (NOT B), forms the basis of how computers process logical operations, how programming languages handle conditional statements, how database queries are optimized, and how search engines filter results. Third, De Morgan's Laws give us alternative ways to think about how to solve problems. When one approach to a problem seems difficult or unclear, these laws allow us to re-frame the question in a different but equivalent way, often revealing a simpler path to the solution. Whether you're simplifying a mathematical proof, writing efficient code, designing digital circuits, or just trying to think clearly about logical relationships, De Morgan's Laws provide powerful techniques for transforming complex expressions into more manageable forms.

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 5 of 5]

For Exercises #19 through #25, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $A = \{1, 3, 4, 7\}$, $B = \{1, 5, 6\}$, $C = \{1, 10, 11\}$, $D = \{1, 3, 5, 7, 9, 11\}$, $E = \{2, 4, 6, 8, 10\}$, $F = \{3, 6, 9\}$.

19. Find $C' \cap D$.
20. Find $(A \cup B)'$.
21. Find $(D - C)'$.
22. Find $(A \cap B \cup F)'$.
23. Find $(B \cup D \cap F)'$.
24. Find $((B' \cap E) \cup A)'$.
25. Find $((A' \cup E) \cup C')'$.

Answers available at the end of this section.

Answers

Section 2.3: Operations on Sets [*In-Class Exercises*- Part 1 of 5]

1. {1, 2, 3, 4, 5, 6, 7}
2. {circle, triangle, rhombus, rectangle, hexagon}
3. {a, b, c, d, e, x, y, z}
4. {Broncos, Chargers, Raiders, Chiefs, Seahawks}

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 2 of 5]

5. {2, 3, 5, 9}
6. {cat, pig}
7. { }
8. {Washington}

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 3 of 5]

9. {2, 4, 6, 8, 10}
10. {apple, date}
11. { }
12. $\{\delta\}$

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 4 of 5]

13. {a, b, c}

14. {g, h}
15. {5, 7}
16. {dog, goat}
17. {5, 6, 7, 8}
18. {♠, ♥}

Section 2.3: Operations on Sets [*In-Class Exercises* - Part 5 of 5]

19. {3, 5, 7, 9}
20. {2, 8, 9, 10, 11}
21. {1, 2, 4, 6, 8, 10, 11}
22. {2, 4, 5, 7, 8, 10, 11}
23. {2, 4, 7, 8, 10, 11}
24. {5, 6, 9, 11}
25. {1}

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Section 2.3.H: Homework

Section 2.3: Operations on Sets - Homework Exercises

(Section 2.3) - Homework Exercise #1

Let $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, c\}$, $B = \{c, d\}$, $C = \{a, c, e, g\}$, and $D = \{g, h\}$. List the elements of the following sets. Make sure you use the notation of sets { }.

a. $A \cup B$
b. $A \cap B$
c. A'
d. $A - B$
e. $B - A$
f. $A \cup D$
g. $A \cap D$
h. D'
i. $B \cap C$
j. $B \cup C$
k. B'
l. $B - C$
m. C'
n. $C - B$
o. $C \cap D$
p. U'

Answer available at the end of this section.

(Section 2.3) - Homework Exercise #2

Let $U = \{\text{red, blue, green, yellow, orange, purple}\}$, $X = \{\text{green, yellow, purple}\}$, $Y = \{\text{red, green, orange}\}$, and $Z = \{\text{blue, green, orange}\}$. List the elements of the following sets. Make sure you use the notation of sets { }.

a. $X \cup Y$
b. $Y \cap Z$
c. Z'
d. $Z - X$
e. $Z \cap X$
f. $X \cup Z$
g. $Y - Z$
h. X'

Answer available at the end of this section.

(Section 2.3) - Homework Exercise #3

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{4, 8, 12\}$, $B = \{5, 6, 7, 8, 9, 10\}$, and $C = \{2, 4, 6, 8, 10, 12\}$. List the elements of the following sets. Make sure you use the notation of sets { }.

a. $A \cup B'$

b. $A' \cap B$
c. $A' \cap C'$
d. $A' \cup B'$
e. $A \cap C \cup B$
f. $A \cup C \cap B$
g. $C' \cap (A \cup B)$
h. $A' \cup (B' \cup C')$

Answer available at the end of this section.

Answers

(Section 2.3) - Homework Exercise #1

a. {a, b, c, d}
b. {c}
c. {d, e, f, g, h}
d. {a, b}
e. {d}
f. {a, b, c, g, h}
g. { }
h. {a, b, c, d, e, f}
i. {c}
j. {a, c, d, e, g}
k. {a, b, e, f, g, h}
l. {d}
m. {b, d, f, h}
n. {a, e, g}
o. {g}
p. { }

(Section 2.3) - Homework Exercise #2

a. {red, green, yellow, orange, purple}
b. {green, orange}
c. {red, yellow, purple}
d. {blue, orange}
e. {green}
f. {blue, green, yellow, orange, purple}
g. {red}
h. {red, blue, orange}

(Section 2.3) - Homework Exercise #3

a. {1, 2, 3, 4, 8, 11, 12}
b. {5, 6, 7, 9, 10}
c. {1, 3, 5, 7, 9, 11}; Use De Morgan's Law: $(A \cup C)' = A' \cap C'$
d. {1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12}; Use De Morgan's Law: $(A \cap B)' = A' \cup B'$
e. {4, 5, 6, 7, 8, 9, 10, 12}
f. {4, 6, 8, 10, 12}
g. {5, 7, 9}
h. {1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12}

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Section 2.4: Drawing Basic Venn Diagrams for Two Sets

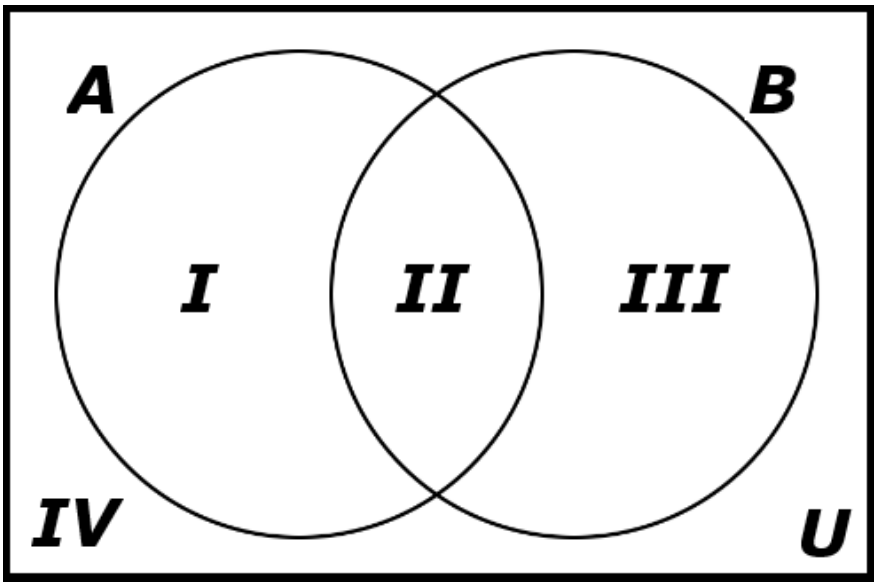
Learning Objectives

- Use Venn diagrams to illustrate operations on two sets

The Structure of a Two-Circle Venn Diagram

A two-circle Venn diagram consists of:

- A rectangle representing the universal set U
- Two circles (or ovals) representing sets A and B
- The circles overlap to show elements that belong to both sets
- Labels identifying each set and sometimes the universal set



This creates four distinct regions:

Region I: Inside A only (not in B)
Region II : Inside both A and B (the overlap; intersection)
Region III : Inside B only (not in A)
Region IV : Outside both circles (in U but not in A or B)

Shading regions in a Venn diagram is like highlighting text. It draws attention to the specific parts we're interested in. When we perform a set operation (like "A and B" or "A or B "), shading shows us exactly which elements satisfy that condition.

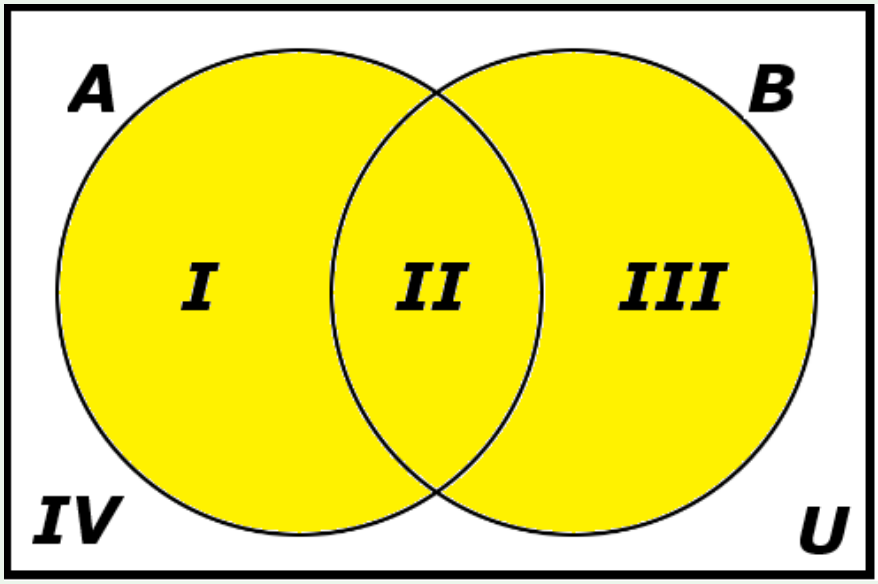
Think of shading as answering the question: "Which elements belong to this description?" The shaded region contains all elements that meet our criteria. Below are four Venn diagrams displaying each of the 4 basic operations on sets that we saw in Section 2.3.

Operation #1: Union $A \cup B$ - "OR"

The union, $A \cup B$, consists of all elements that belong to set A, set B, or both sets.

To represent the union on a Venn diagram, shade every region that is inside either circle, including the overlapping region.

Therefore, shade Regions I, II, and III.

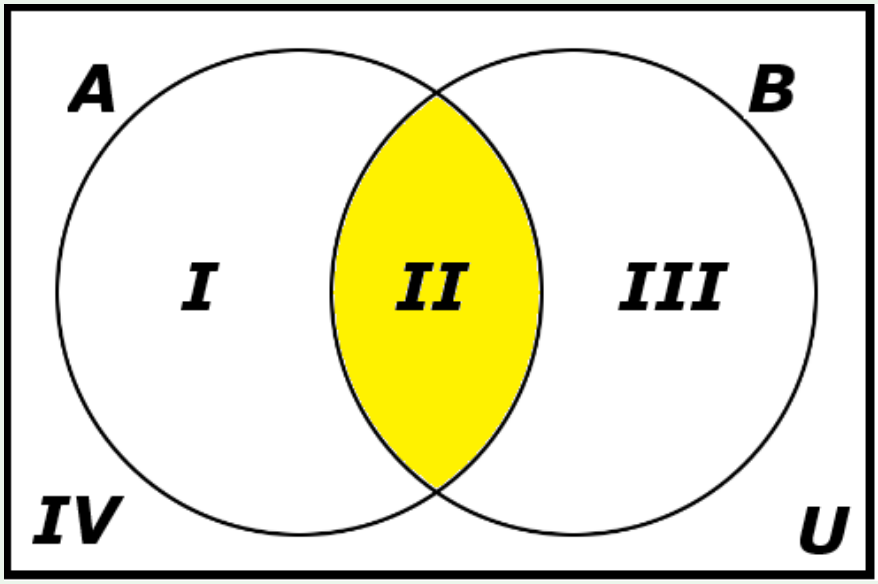


Operation #2: Intersection $A \cap B$ - "AND"

The intersection, $A \cap B$, consists of all elements that belong to **both** sets A and B.

To represent the intersection on a Venn diagram, shade only the region where the two sets overlap.

Therefore, shade Region II only.

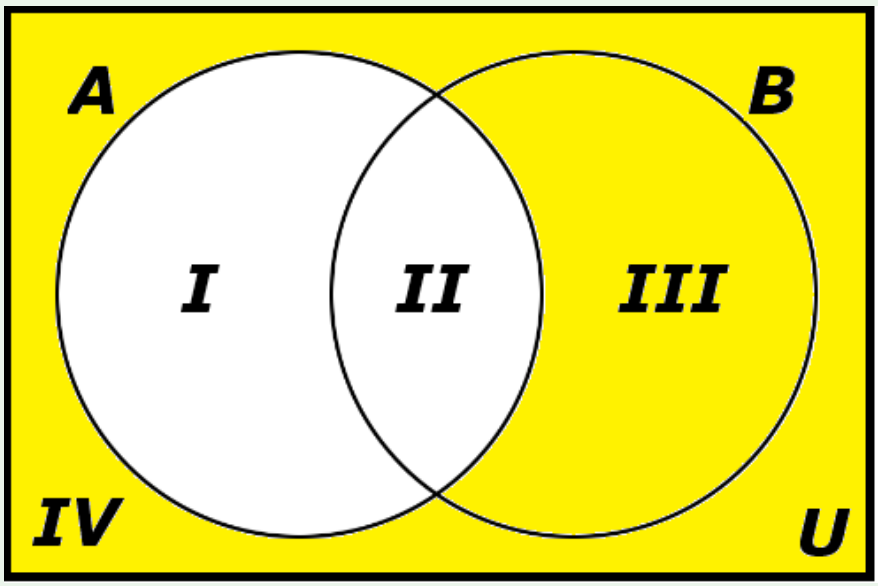


Operation #3: Complement A' - "NOT"

The complement of A, denoted by A', consists of all elements in the universal set that do not belong to set A.

To represent the complement of A on a Venn diagram, shade every region that lies outside of set A.

Therefore, shade Regions III and IV.

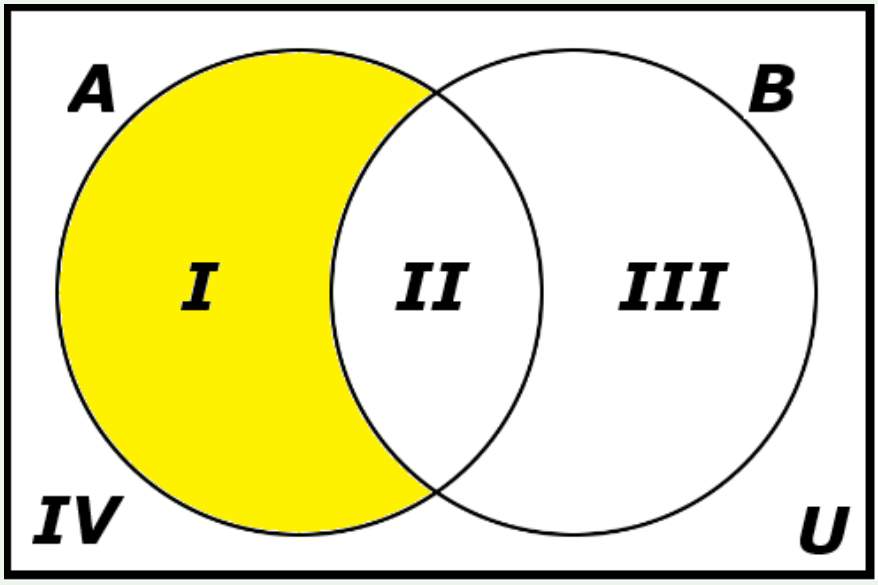


Operation #4: Set Difference $A - B$ - "SUBTRACT"

The set difference $A - B$ consists of all elements that belong to set A but do not belong to set B.

To represent $A - B$ on a Venn diagram, shade the portion of set A that does not overlap with set B.

Therefore, shade Region I only.



So, now, we know the basics, let's look at shading sets involving multiple operations.

How to Shade Using Multiple Operations

When we shade sets that involve multiple operations, we can use diagonals lines in one direction / / / to represent the first set. Then, we can use diagonals lines in the other direction \ \ \ to represent the second set. You can also use different colored pencils to represent the sets as well. For all examples in this text, we will use a light gray shade. If we need to shade a region again that is already a light gray shade, then it will be a darker shade of gray. The final solution will always be in yellow (or a yellow highlighter). We ignore everything previously in any shade of gray.

Example #2.4.1 🤔

Draw a Venn diagram to illustrate the set $(A \cup B)'$. Use a yellow highlighter to represent the solution.

✅ **Solution:**

So, based off order of operations, parentheses first, which means shade $A \cup B$, then apply the complement.

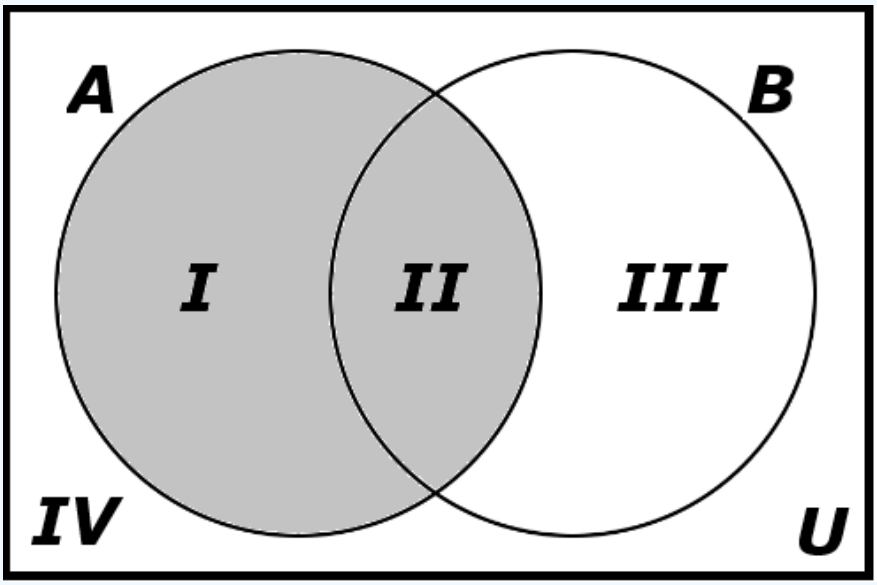
Step 1: Lightly shade all elements in set A, which includes regions I and II.

Step 2: Next, shade all elements in set B, which includes regions II and III. Notice that region II is shaded twice, resulting in a darker tone.

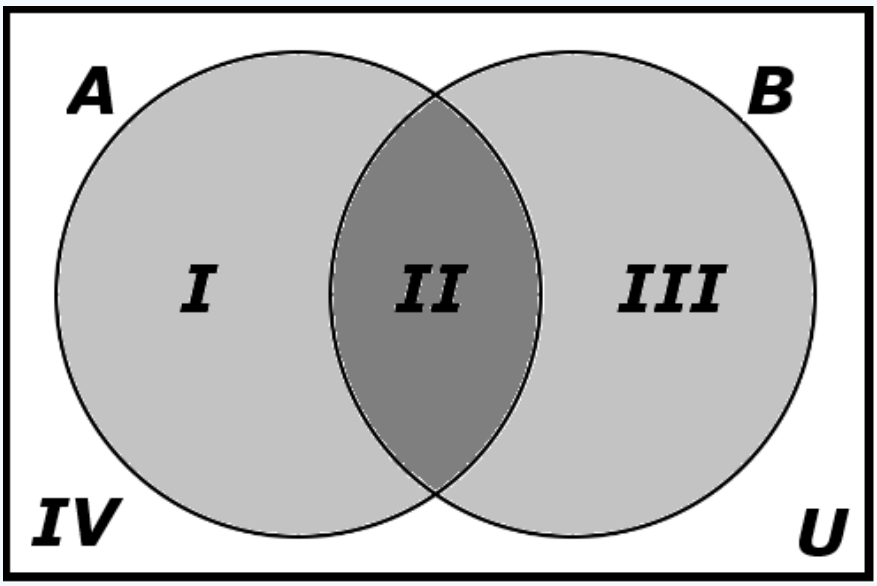
Step 3: The union, $A \cup B$ is represented by everything that is shaded (either lightly or darkly). This includes regions I, II, and III. (Note: If you recognize that $A \cup B$ is shading regions I, II, & III, you can just shade all three regions to begin with and skip Steps 1 & 2).

Step 4: The complement $(A \cup B)'$ consists of all elements outside the union. Therefore, shade the remaining region (region IV) in a contrasting color, such as yellow, to indicate the solution.

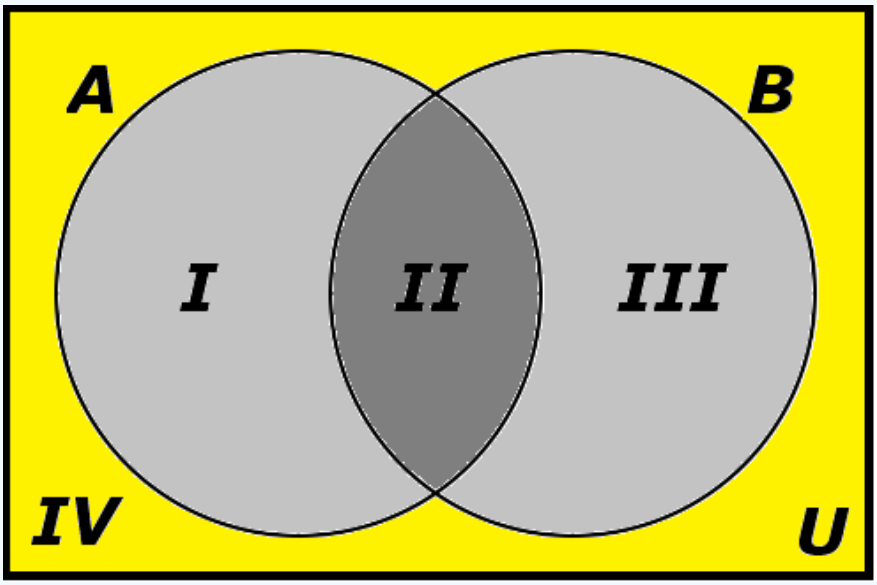
Step 5: Final representation: Remove any gray shading and leave only the complement highlighted. The solution is the set of elements in region IV.



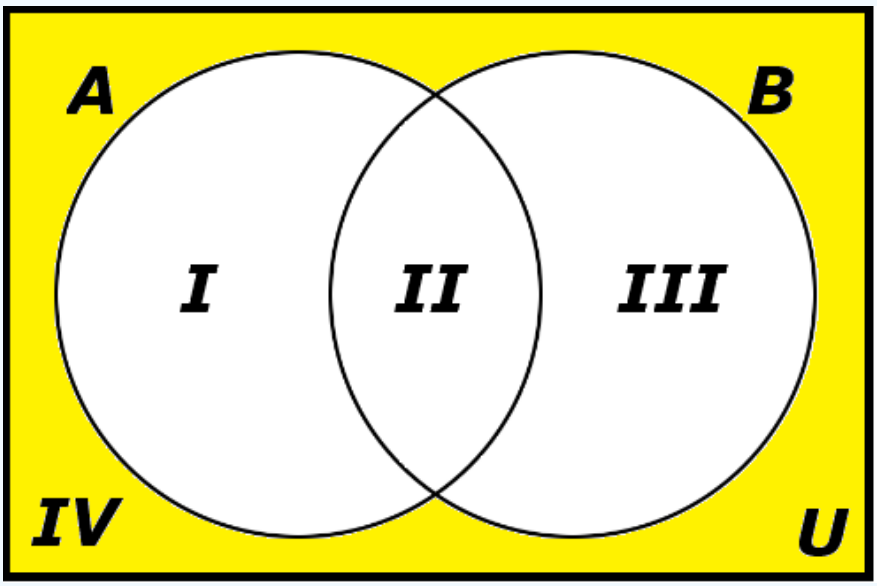
Step 1



Steps 2 & 3



Step 4



Step 5

Example #2.4.2 🤔

Draw a Venn diagram to illustrate the set $(A \cap B)'$. Use a yellow highlighter to represent the solution.

✅ **Solution:**

So, based off order of operations, parentheses first, which means shade $A \cap B$, then apply the complement.

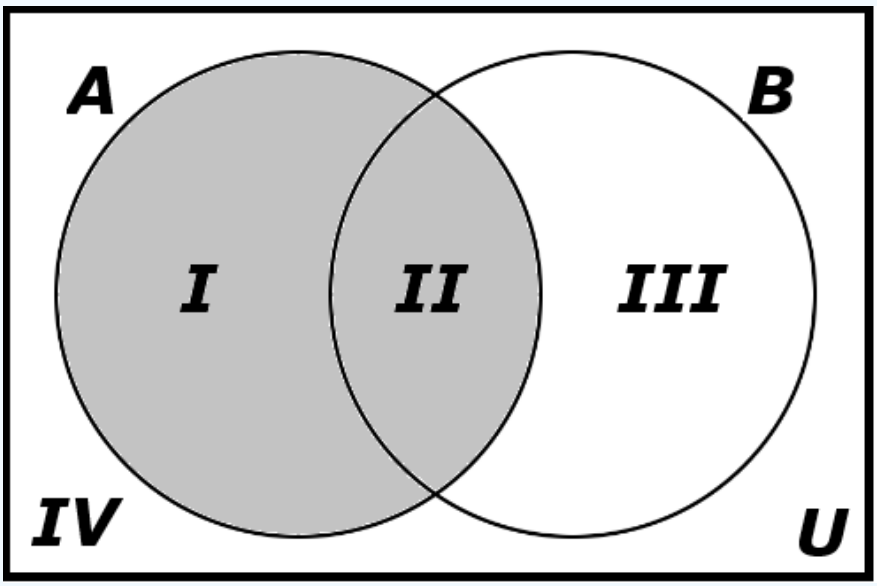
Step 1: Lightly shade all elements in set A, which includes regions I and II.

Step 2: Next, shade all elements in set B, which includes regions II and III. Notice that region II is shaded twice, resulting in a darker tone.

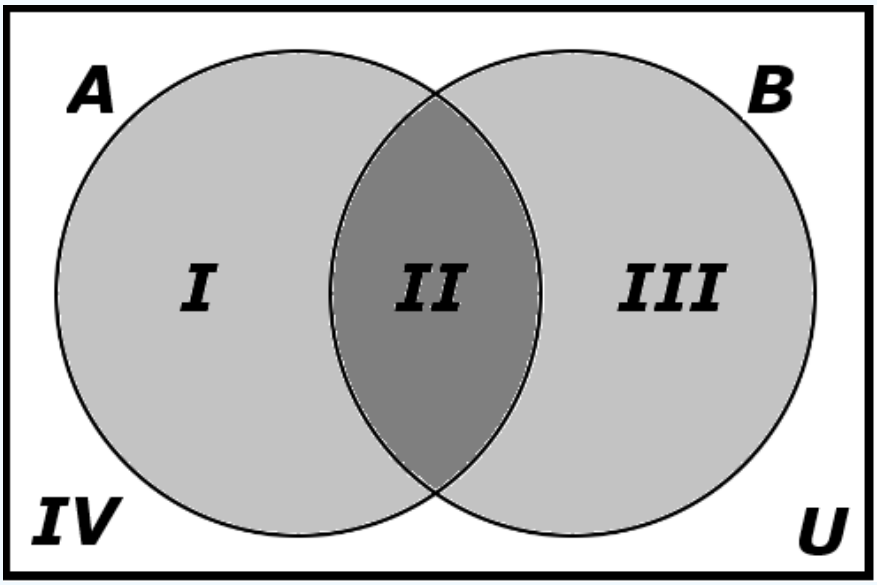
Step 3: The intersection, $A \cap B$ is represented by everything that is shaded darkly (or twice). This is region II. (Note: If you recognize that $A \cap B$ is shading region II, you can just shade that regions to begin with and skip Steps 1 & 2).

Step 4: The complement $(A \cap B)'$ consists of all elements outside the intersection. Therefore, shade the remaining regions (region I, III, & IV) in a contrasting color, such as yellow, to indicate the solution.

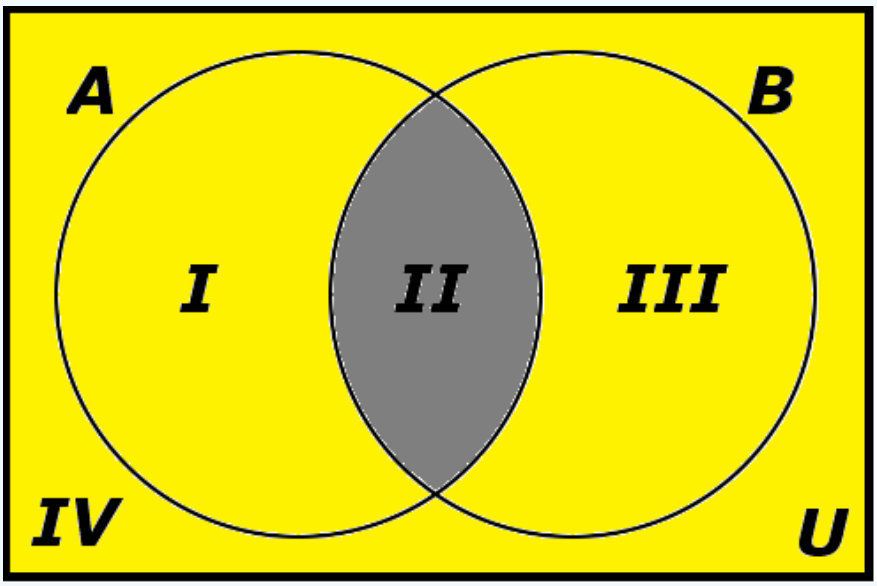
Step 5: Final representation: Remove any gray shading and leave only the complement highlighted. The solution is the set of elements in regions I, III, & V.



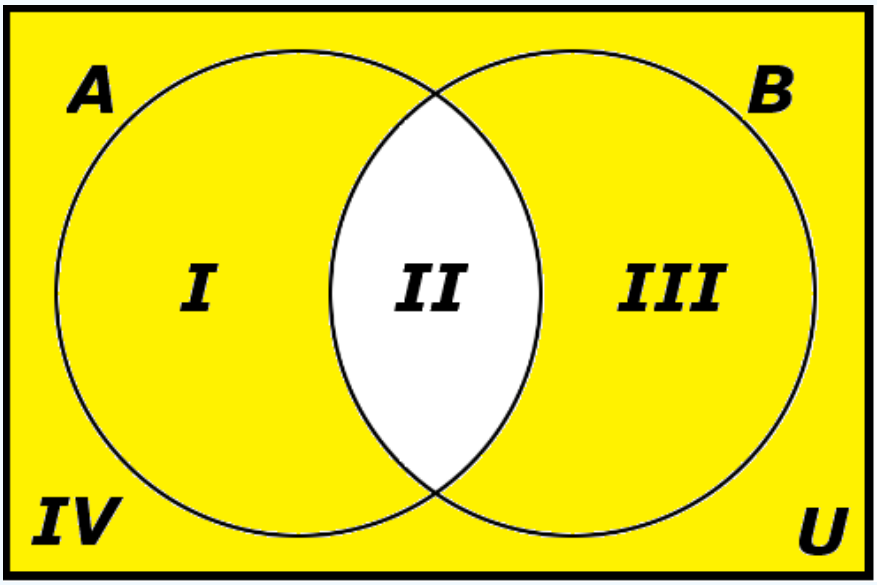
Step 1



Steps 2 & 3



Step 4



Step 5

Example #2.4.3 🤔

Draw a Venn diagram to illustrate the set $A' \cup B'$. Use a yellow highlighter to represent the solution.

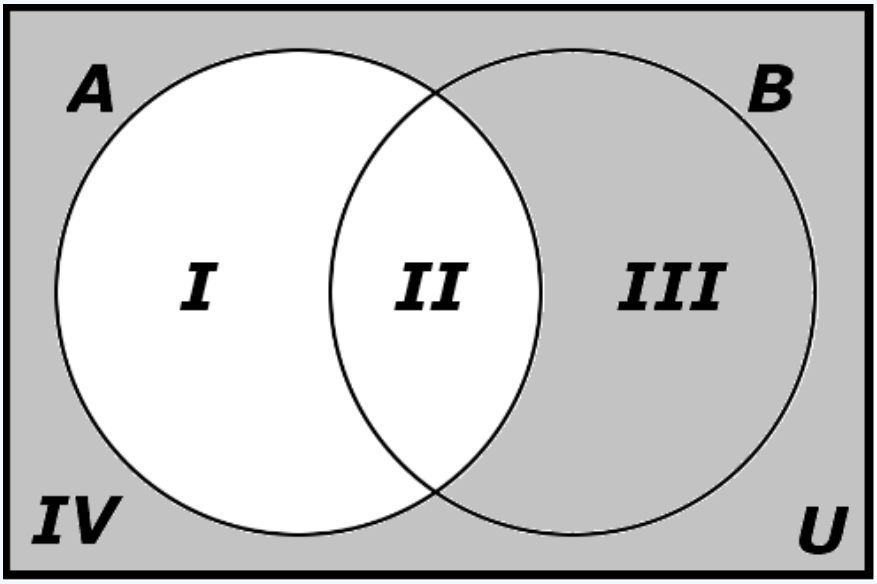
✅ **Solution:**

So, based off order of operations, A' first, B' second, then apply the union.

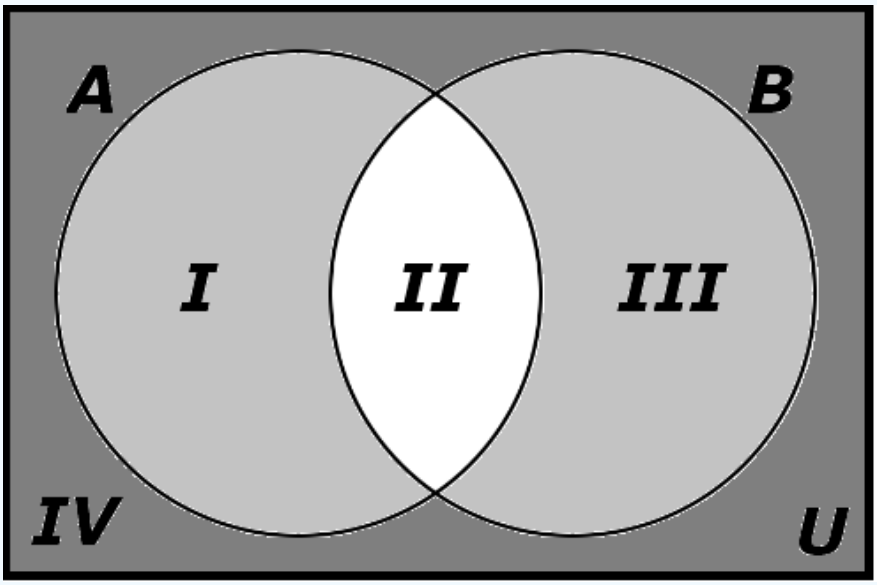
Step 1: Lightly shade all elements in set A', which includes regions III and IV.

Step 2: Next, shade all elements in set B', which includes regions I and IV. Notice that region IV is shaded twice, resulting in a darker tone.

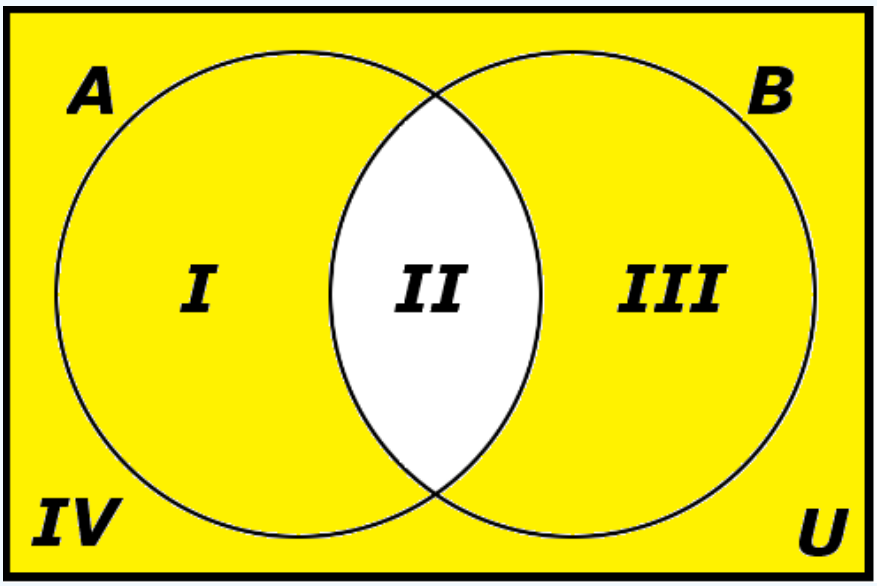
Step 3: The union $A' \cup B'$ consists of all elements shaded in A' or B' or both A' and B'. Then, $A' \cup B'$ is represented by everything that is shaded either lightly or darkly. Your final representation is regions I, III, & IV in a contrasting color, such as yellow, to indicate the solution.



Step 1



Step 2



Step 3

Example #2.4.4 🤔

Draw a Venn diagram to illustrate the set $A' \cap B'$. Use a yellow highlighter to represent the solution.

✅ **Solution:**

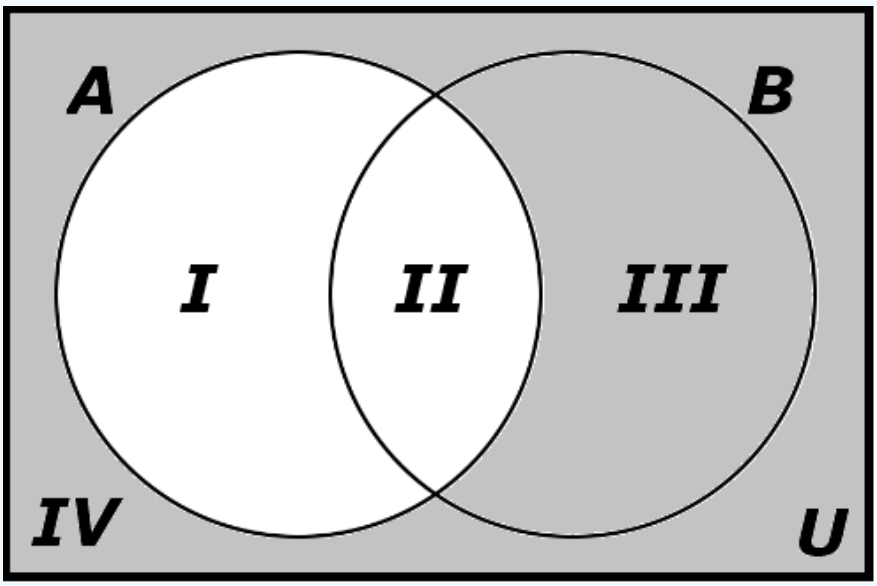
So, based off order of operations, A' first, B' second, then apply the intersection.

Step 1: Lightly shade all elements in set A', which includes regions III and IV.

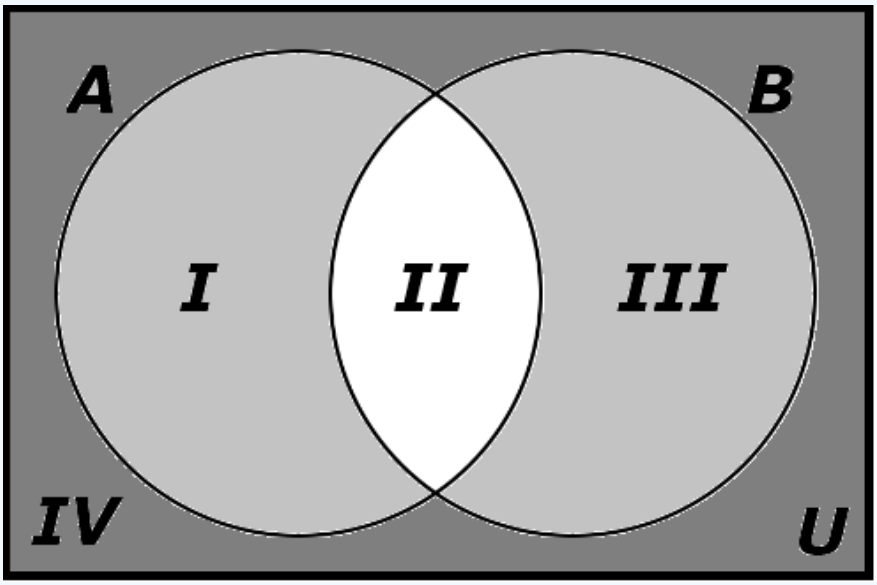
Step 2: Next, shade all elements in set B', which includes regions I and IV. Notice that region IV is shaded twice, resulting in a darker tone.

Step 3: The intersection $A' \cap B'$ consists of all elements shaded in both A' and B'. Then, $A' \cap B'$ is represented by everything that is shaded darkly. Therefore, shade region IV in a contrasting color, such as yellow, to indicate the solution.

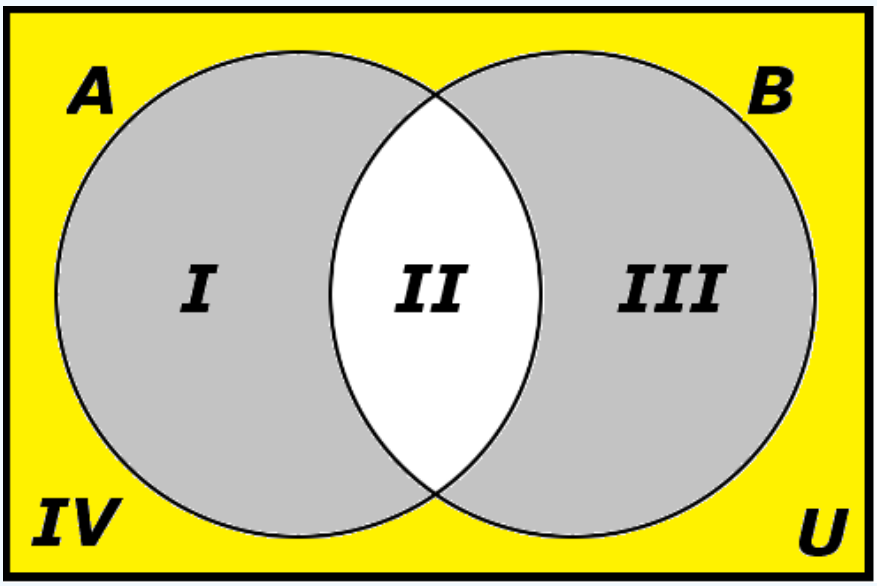
Step 4: Final representation: Remove any additional gray shading and leave only the highlighted yellow solution. The solution is the set of elements in region IV.



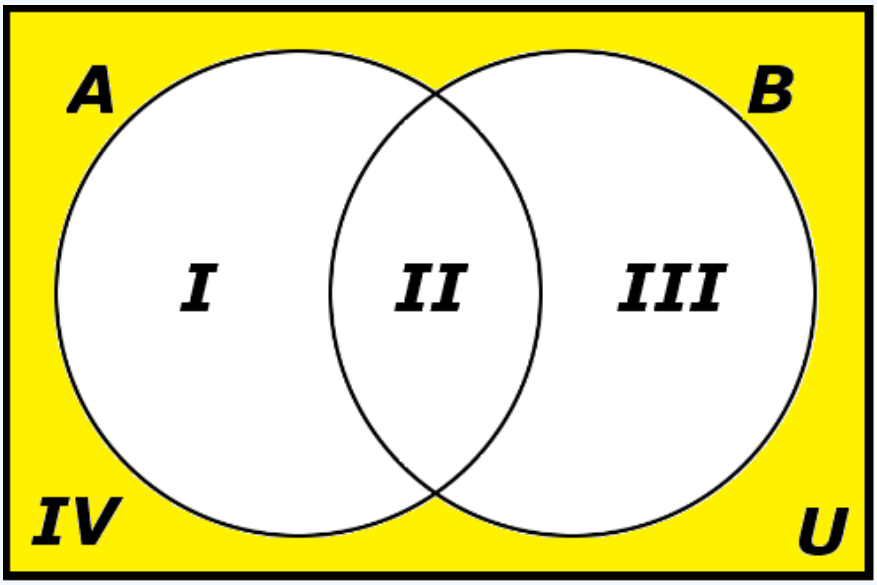
Step 1



Step 2



Step 3



Step 4

As you worked through the previous examples, you may have noticed something interesting. Let's compare the shaded regions from the previous four examples. If you examine the shaded regions carefully, you'll see that the pairs in Examples #2.1.1 & #2.1.4 and the pairs in Examples #2.1.2 & #2.1.3 produce identical shading patterns. This is not a coincidence or an error. They are identical, because those examples model DeMorgan's Laws from Section 2.2. So, here are the law's again:

Definition: De Morgan's Laws

For any two finite sets, A and B, the following equations are always true:

- $(A \cup B)' = A' \cap B'$ (De Morgan's Law of Union)
- $(A \cap B)' = A' \cup B'$ (De Morgan's Law of Intersection)

Example #2.4.5 🤔

Draw a Venn diagram to illustrate the set $A \cup B'$. Use a yellow highlighter to represent the solution.

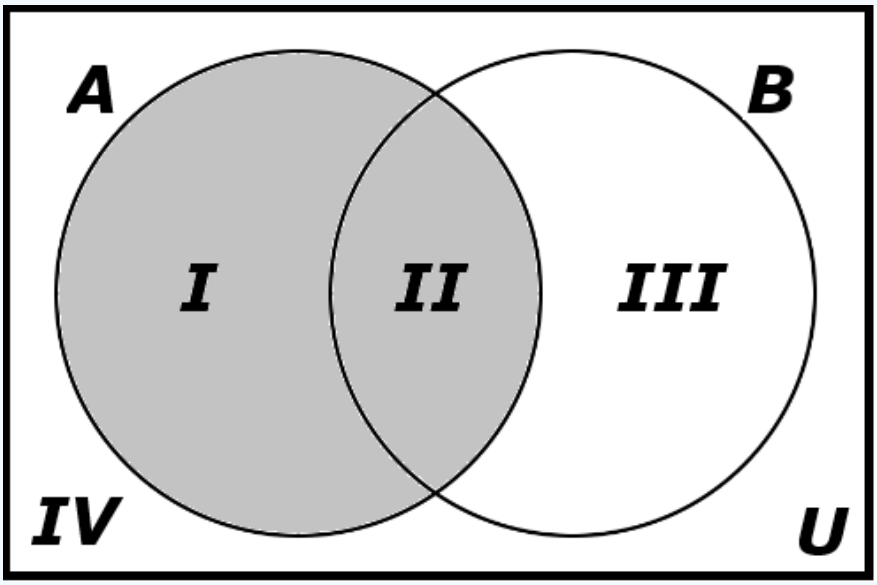
✅ **Solution:**

So, based off order of operations, A first, B' second, then apply the union.

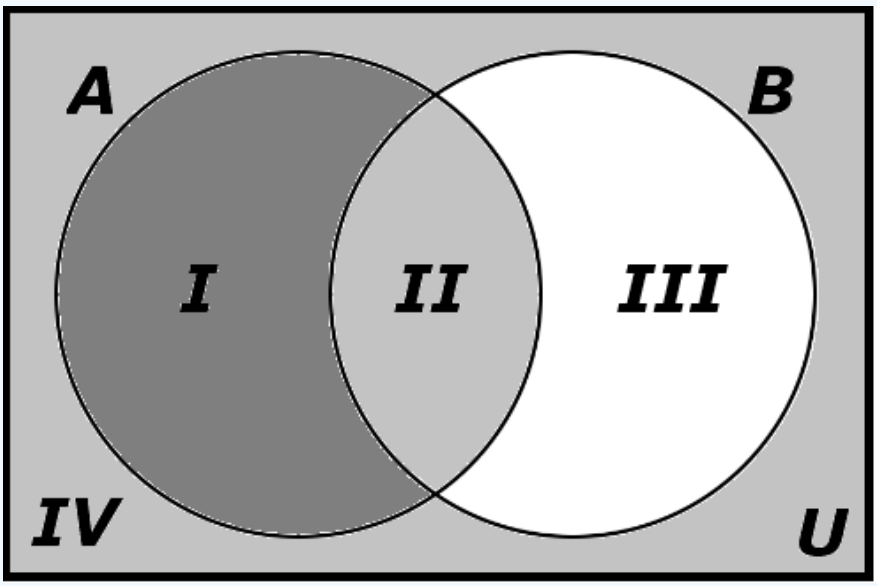
Step 1: Lightly shade all elements in set A, which includes regions I and II.

Step 2: Next, shade all elements in set B', which includes regions I and IV. Notice that region I is shaded twice, resulting in a darker tone.

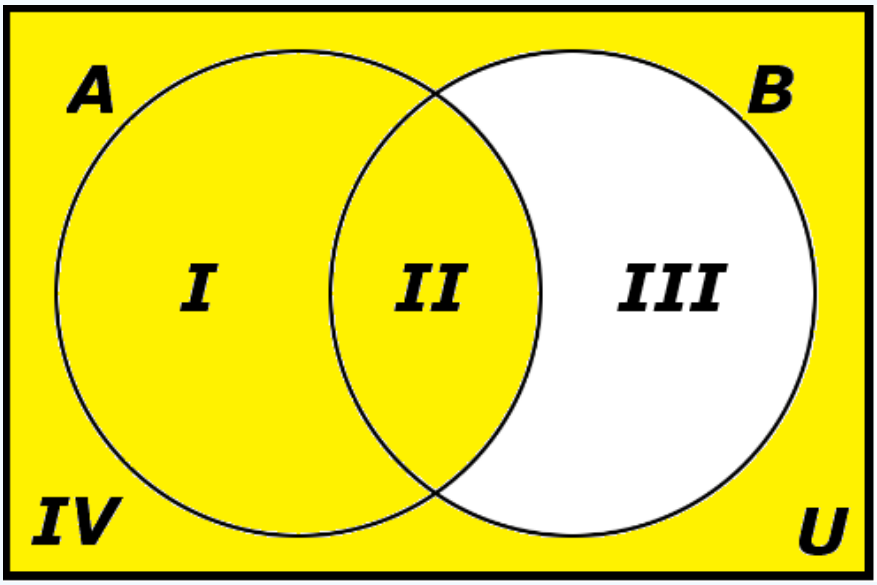
Step 3: The union, $A \cup B'$ is represented by everything that is shaded either lightly or darkly. Your final representation is regions I, II, & IV in a contrasting color, such as yellow, to indicate the solution.



Step 1



Step 2



Step 3

Example #2.4.6 🤔

Draw a Venn diagram to illustrate the set $A \cap B'$. Use a yellow highlighter to represent the solution.

✅ **Solution:**

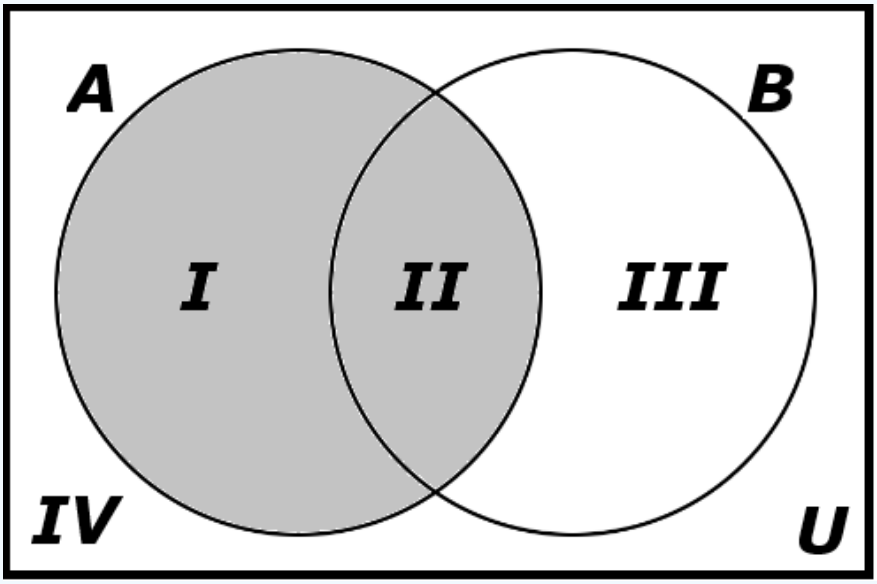
So, based off order of operations, A first, B' second, then apply the intersection.

Step 1: Lightly shade all elements in set A, which includes regions I and II.

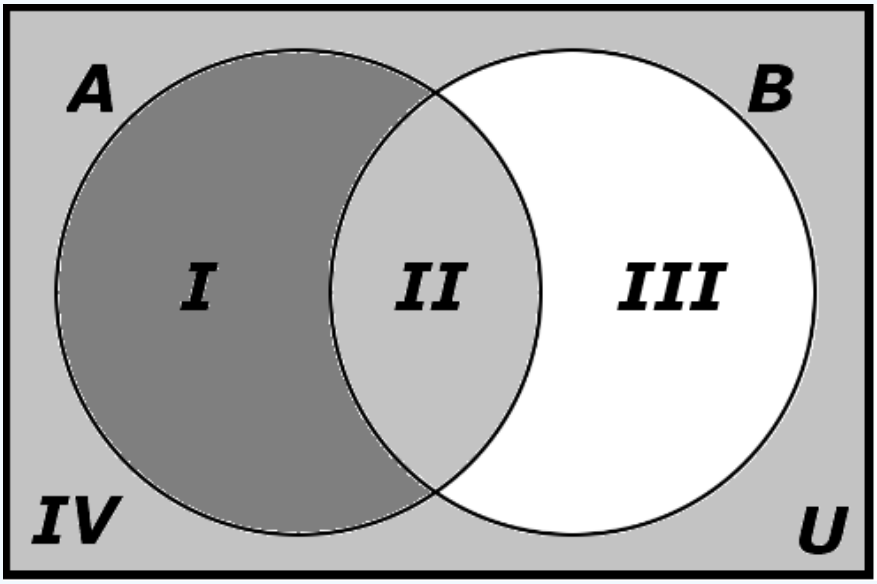
Step 2: Next, shade all elements in set B', which includes regions I and IV. Notice that region I is shaded twice, resulting in a darker tone.

Step 3: The intersection, $A \cap B'$ is represented by everything that is shaded darkly. Therefore, shade region I in a contrasting color, such as yellow, to indicate the solution.

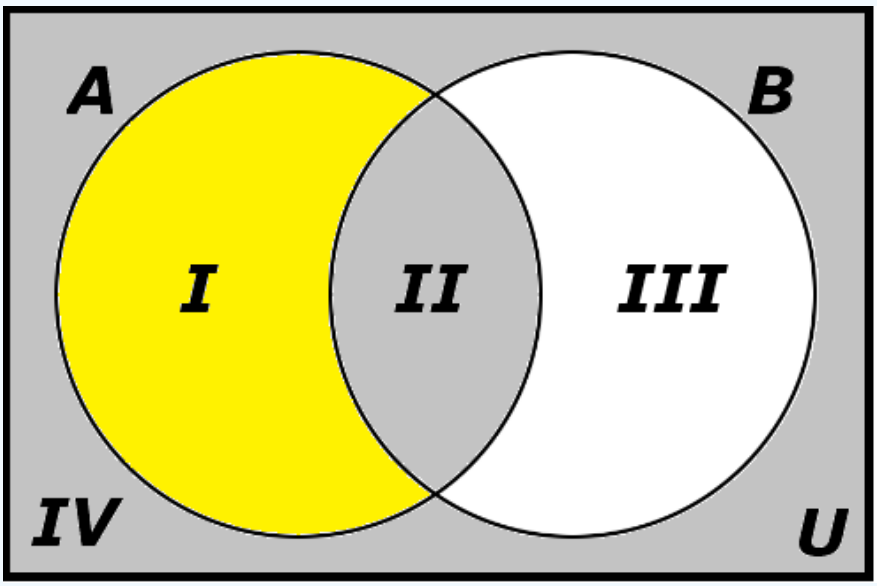
Step 4: Final representation: Remove any additional gray shading and leave only the highlighted yellow solution. The solution is the set of elements in region I.



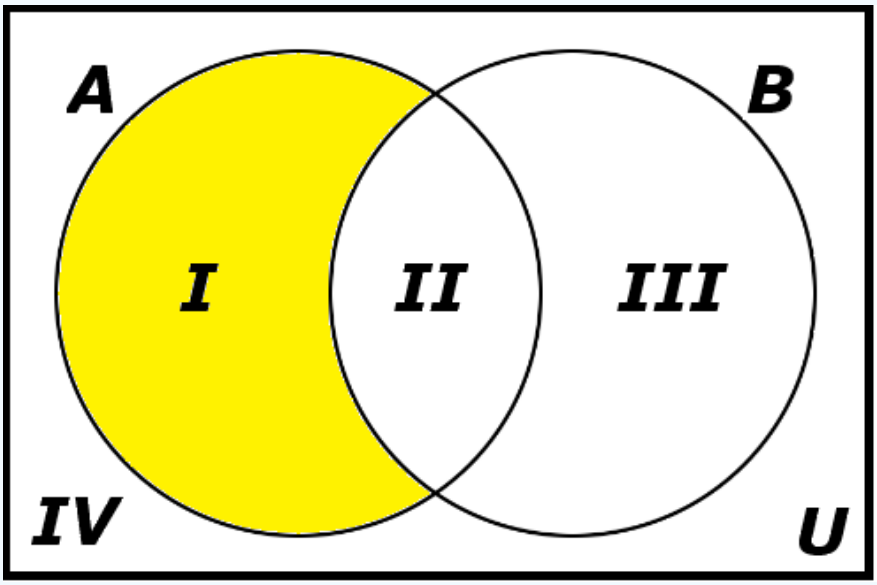
Step 1



Step 2



Step 3



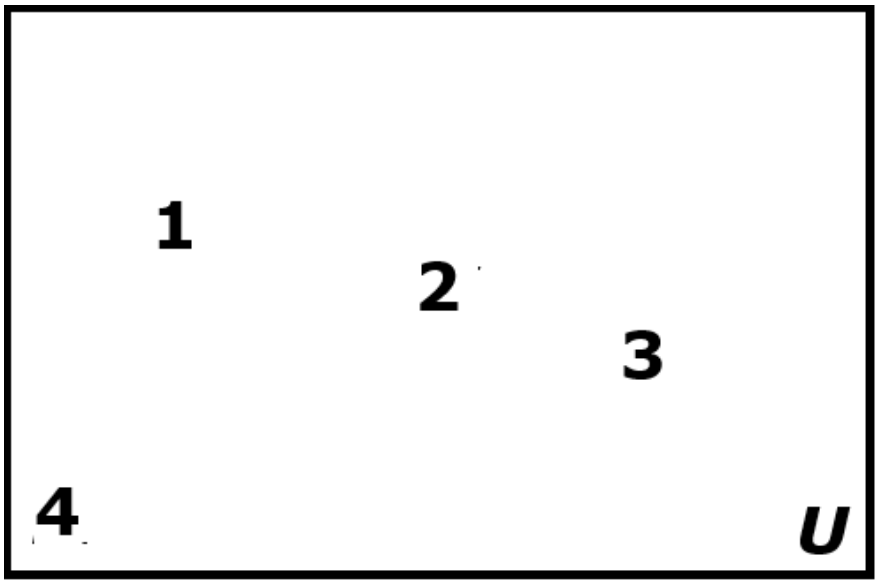
Step 4

So, now that we have seen several examples of shading, let's look at using actual elements in sets to confirm the shading examples from above.

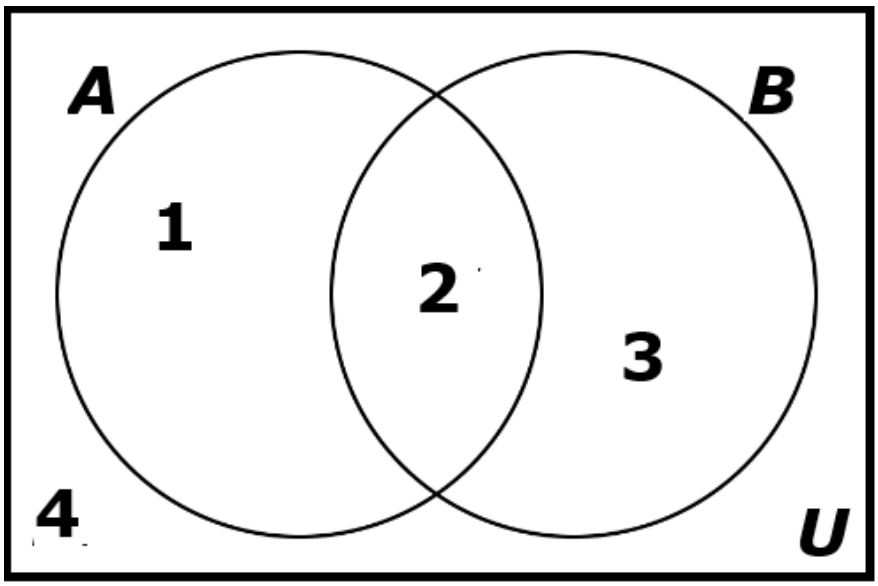
Think of the circles as rings that we would toss onto the rectangle. For example, lets say we have the following sets:

$U = \{1, 2, 3, 4\}$,
$A = \{1, 2\}$, and $B = \{2, 3\}$.

We place the elements 1, 2, 3, & 4 in the rectangle.

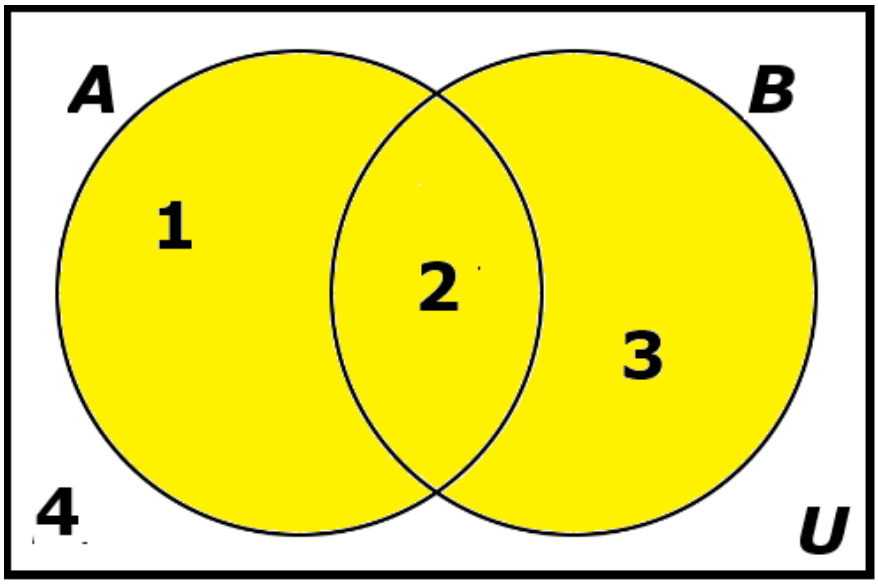


We then place labeled rings A and B on the appropriate elements. The Venn diagram would look like as follows:

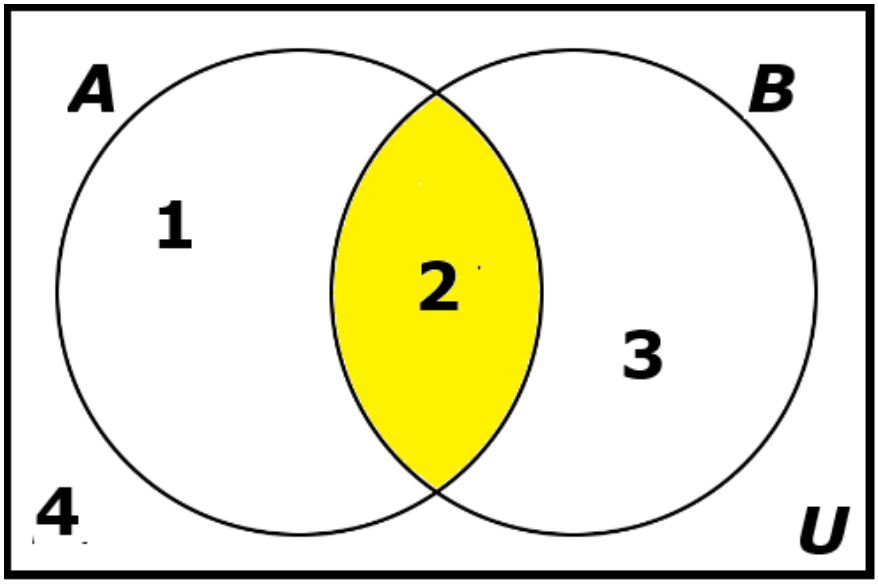


- The element 1 is in region I. (A , not B)
- The element 2 is in region II. (Both A and B, intersection)
- The element 3 is in region III. (In B, not A)
- The element 4 is in region IV. (Outside the union A or B)

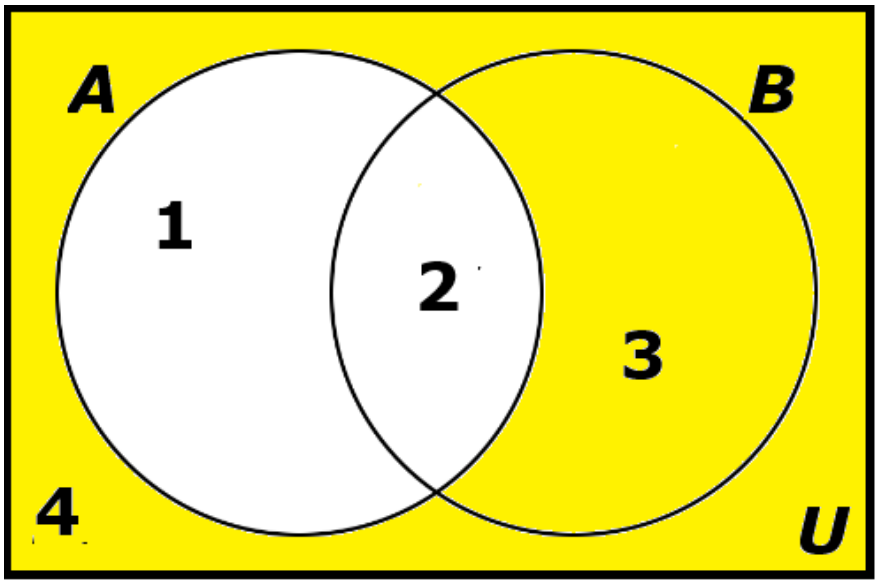
From Section 2.2, here are the 4 basic operations:



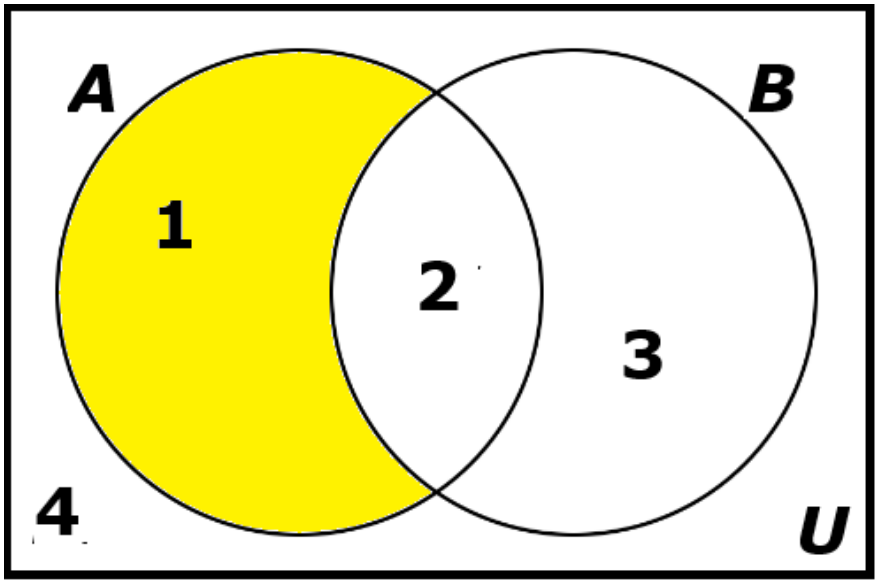
$A \cup B = \{1, 2, 3\}$



$A \cap B = \{2\}$

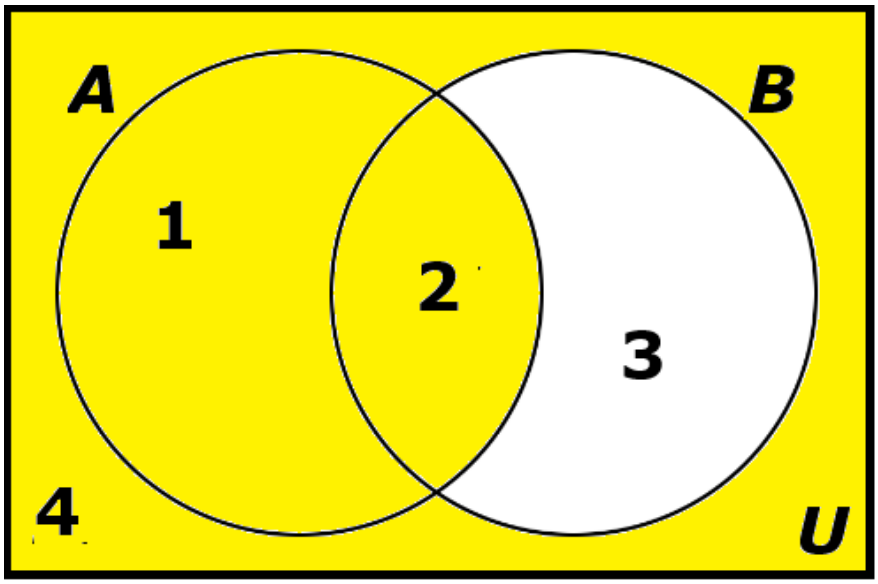


$A' = \{3, 4\}$



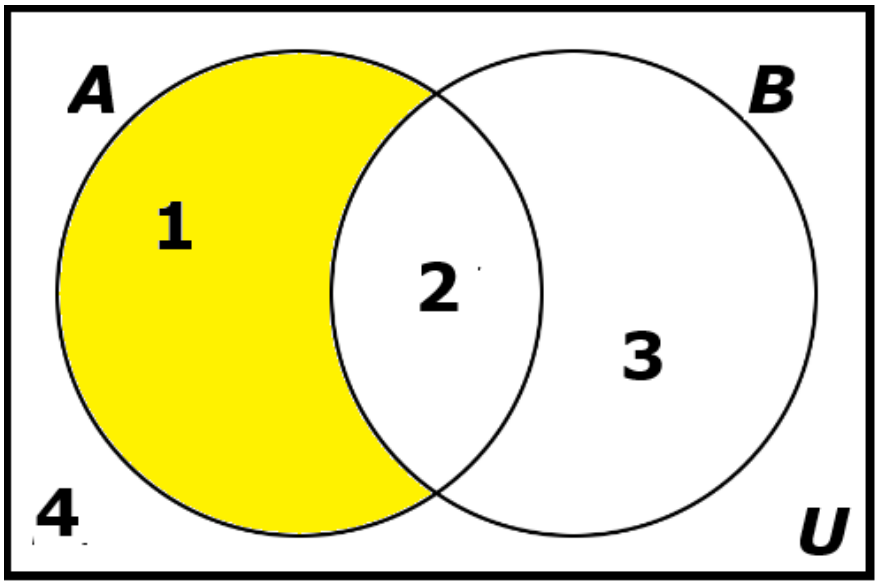
$A - B = \{1\}$

Again, from the previous section, if you wanted to find $A \cup B'$, we know $A = \{1, 2\}$ and $B' = \{1, 4\}$, so $A \cup B' = \{1, 2, 4\}$. In Example #2.4.5, we saw the Venn diagram to illustrate $A \cup B'$ was regions I, II, & IV.



$$A \cup B' = \{1, 2, 4\}$$

Again from the previous section, If you wanted to find $A \cap B'$, we know $A = \{1, 2\}$ and $B' = \{1, 4\}$, so $A \cap B' = \{1\}$. In Example #2.4.6, we saw the Venn diagram to illustrate $A \cap B'$ was region I.



$$A \cap B' = \{1\}$$

Section 2.4: Drawing Basic Venn Diagrams for Two Sets [*In-Class Exercises*]

(These exercises are the same examples from above.)

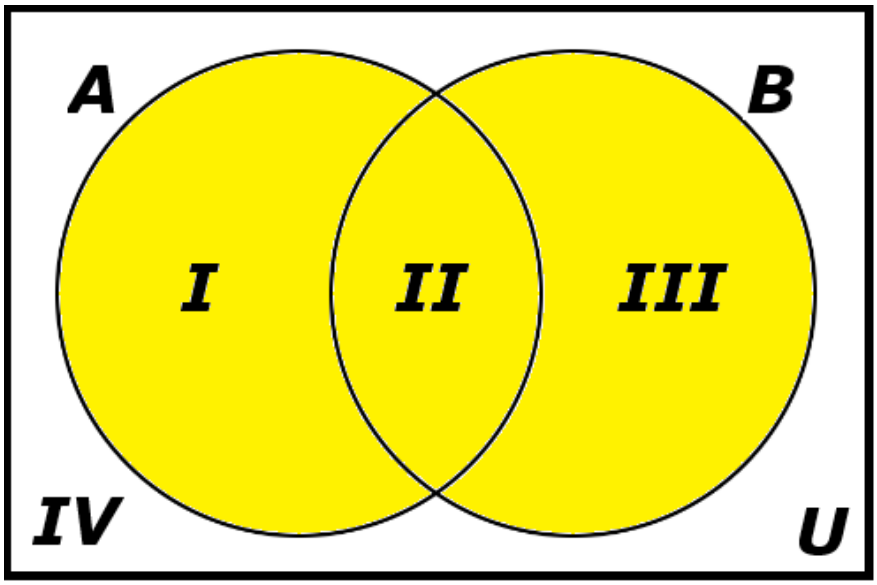
1. Draw a Venn diagram to illustrate the set $A \cup B$. Use a yellow highlighter to represent the solution.
2. Draw a Venn diagram to illustrate the set $A \cap B$. Use a yellow highlighter to represent the solution.
3. Draw a Venn diagram to illustrate the set A'. Use a yellow highlighter to represent the solution.
4. Draw a Venn diagram to illustrate the set $A - B$. Use a yellow highlighter to represent the solution.
5. Draw a Venn diagram to illustrate the set $(A \cup B)'$. Use a yellow highlighter to represent the solution.
6. Draw a Venn diagram to illustrate the set $(A \cap B)'$. Use a yellow highlighter to represent the solution.

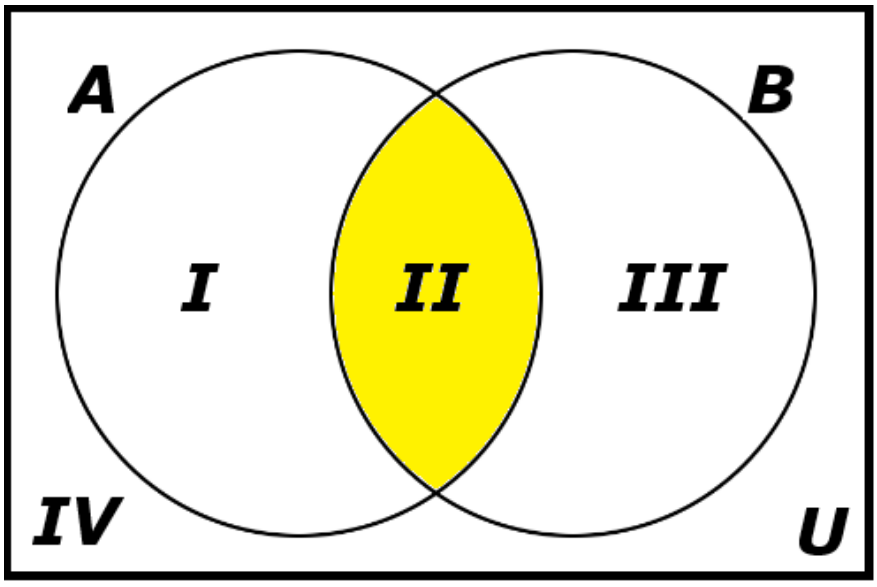
7. Draw a Venn diagram to illustrate the set $A' \cup B'$. Use a yellow highlighter to represent the solution.
8. Draw a Venn diagram to illustrate the set $A' \cap B'$. Use a yellow highlighter to represent the solution.
9. Draw a Venn diagram to illustrate the set $A \cup B'$. Use a yellow highlighter to represent the solution.
10. Draw a Venn diagram to illustrate the set $A \cap B'$. Use a yellow highlighter to represent the solution.

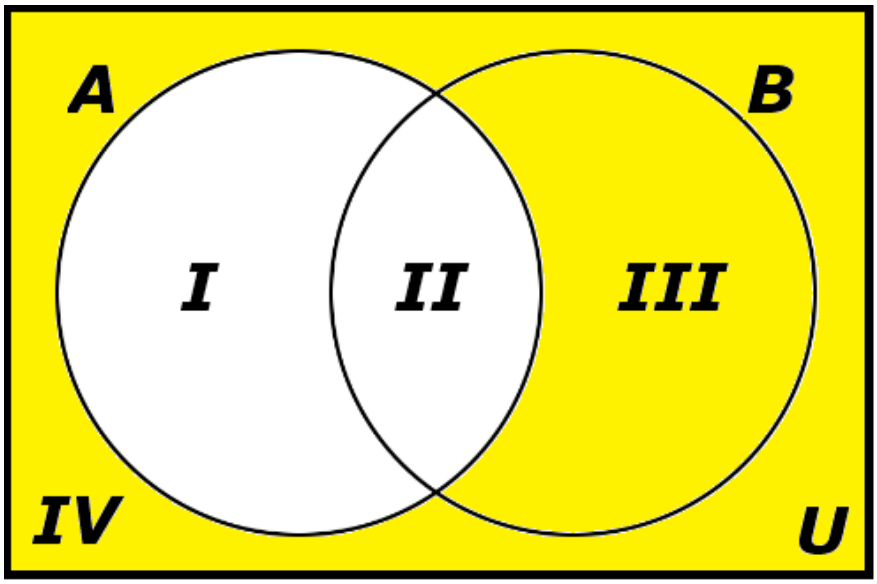
Answers available at the end of this section.

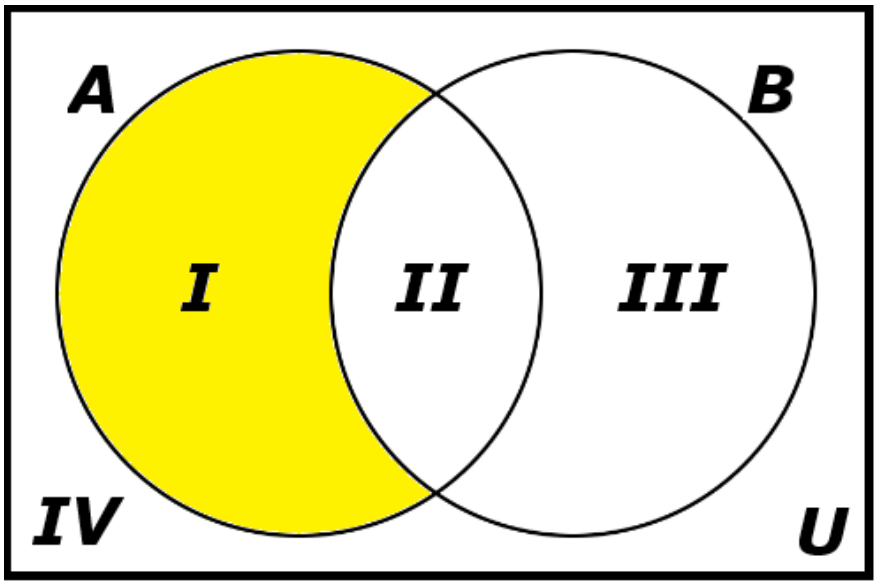
Answers

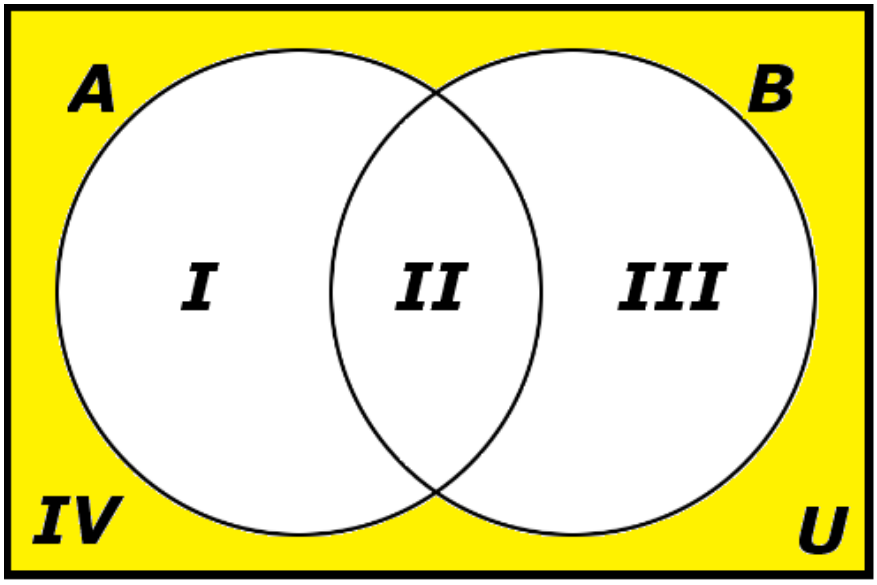
Section 2.4: Drawing Basic Venn Diagrams for Two Sets [*In-Class Exercises*]

1. 

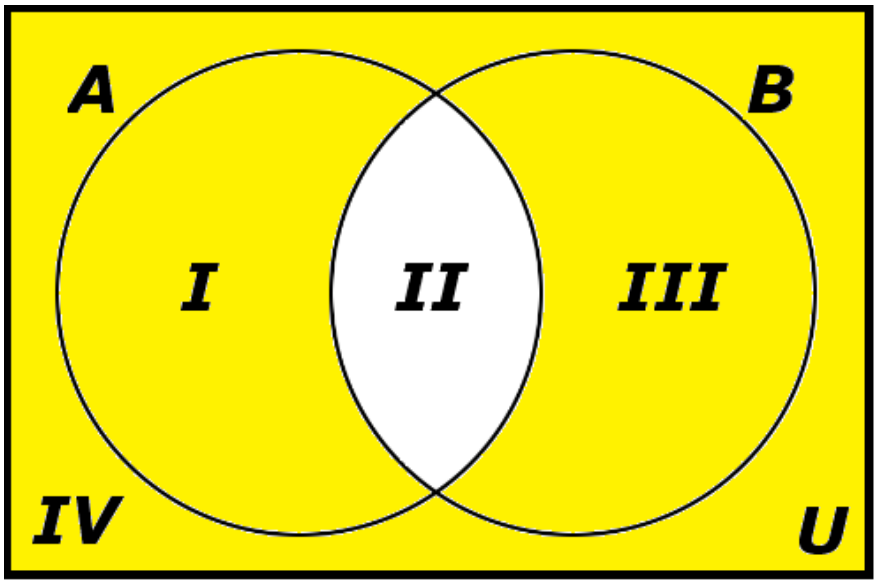
2. 

3. 

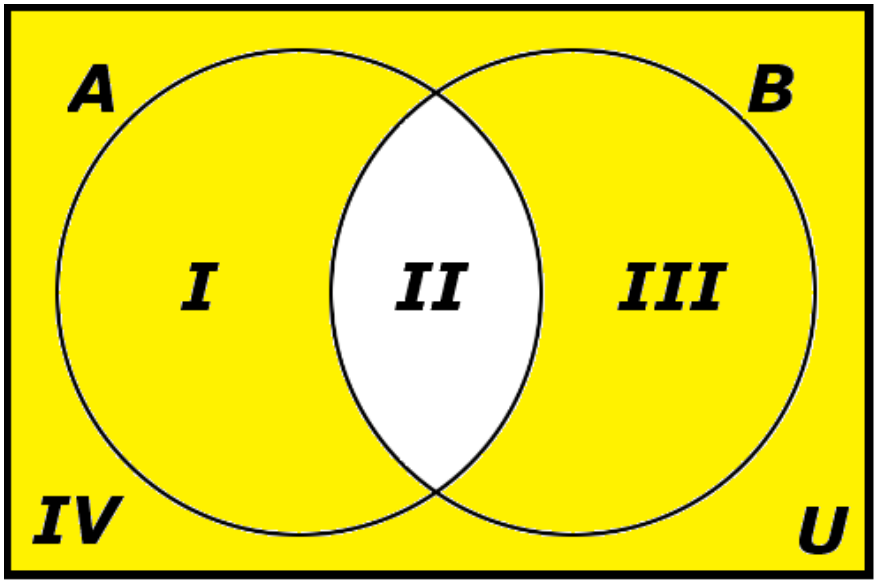
4. 

5. 

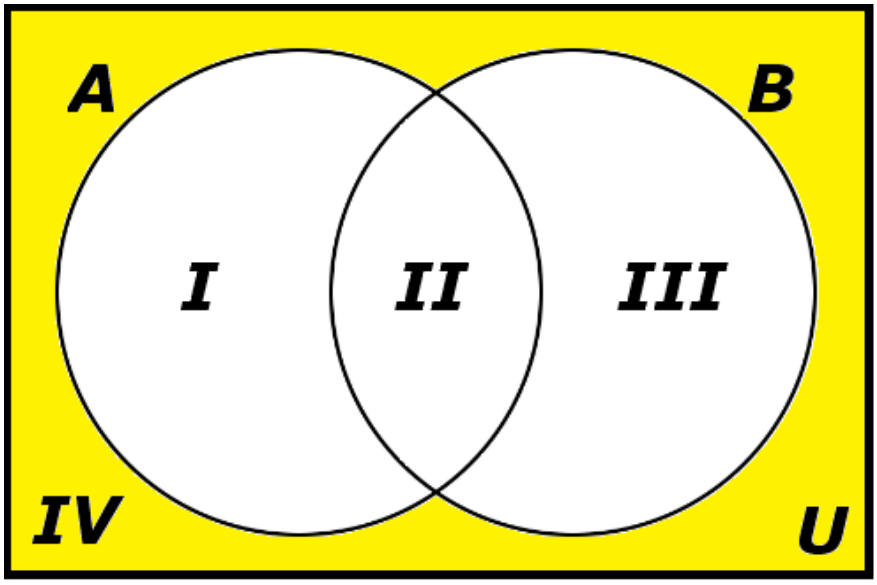
6.



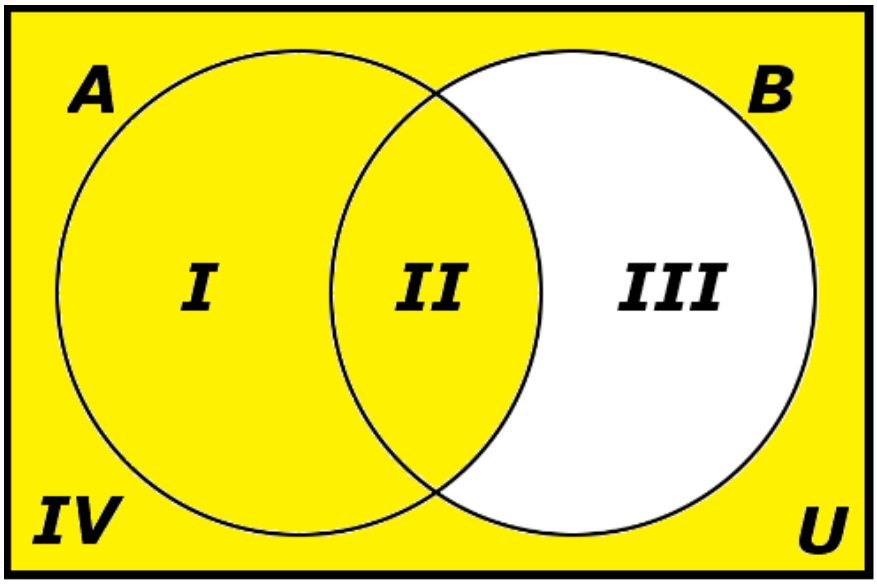
7.



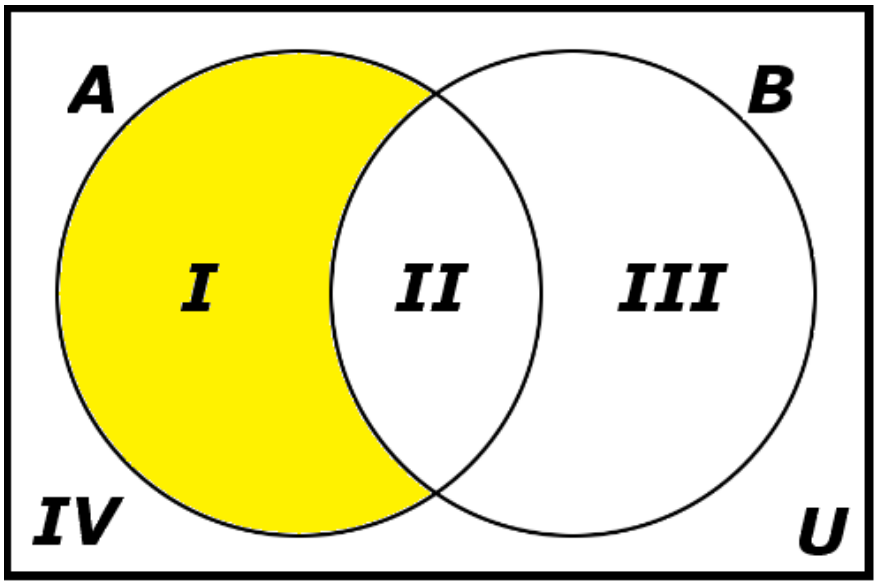
8.



9.



10.



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Section 2.4.H: Homework

Section 2.4: Drawing Basic Venn Diagrams for Two Sets - Homework Exercises

(Section 2.4) - Homework Exercise #1

Draw a Venn diagram to illustrate the set $A' \cup B$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

(Section 2.4) - Homework Exercise #2

Draw a Venn diagram to illustrate the set $A' \cap B$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

(Section 2.4) - Homework Exercise #3

Draw a Venn diagram to illustrate the set $B - A$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

(Section 2.4) - Homework Exercise #4

Draw a Venn diagram to illustrate the set $A' - B$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

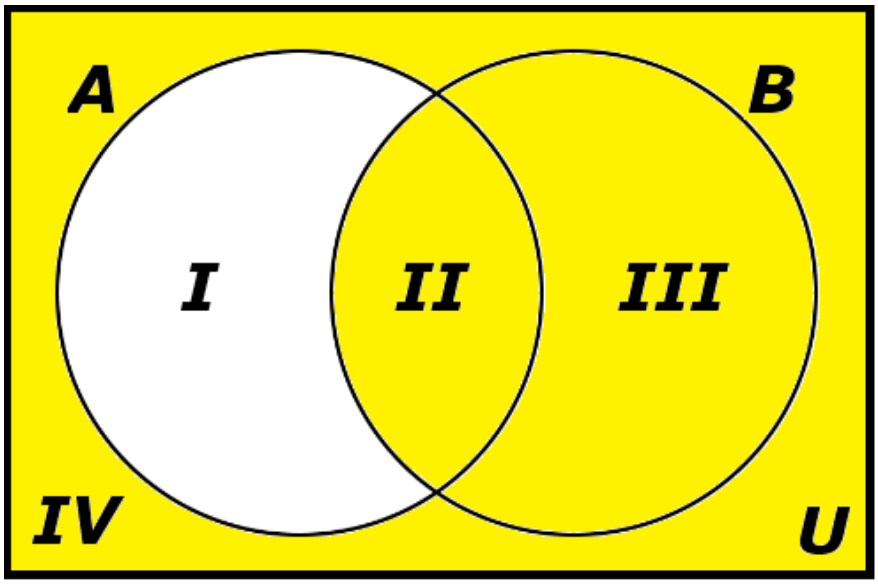
(Section 2.4) - Homework Exercise #5

Draw a Venn diagram to illustrate the set $B' - A'$. Use a yellow highlighter to represent the solution.

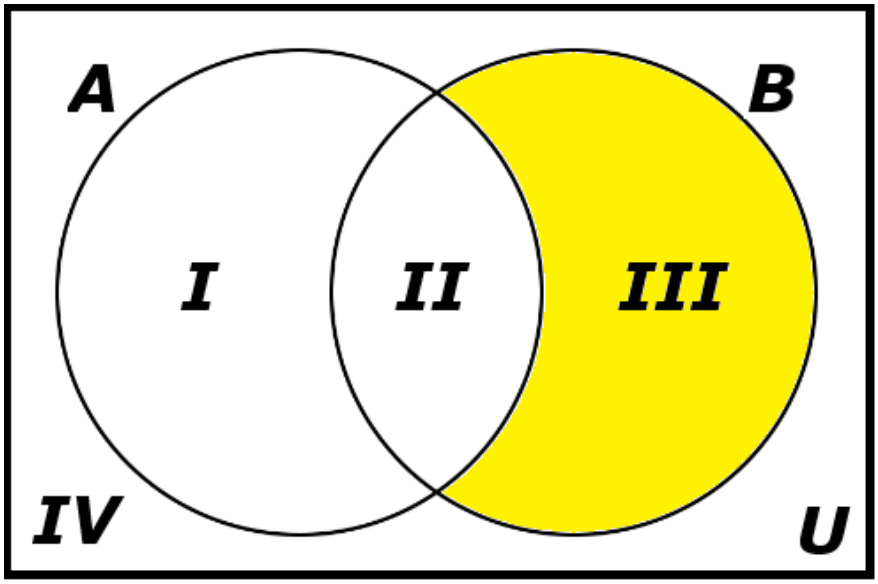
Answer available at the end of this section.

Answers

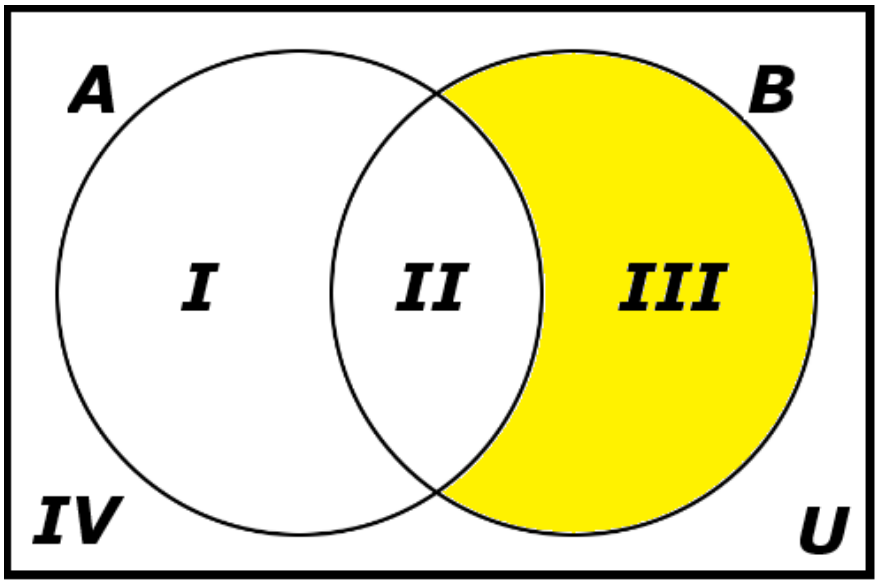
(Section 2.4) - Homework Exercise #1



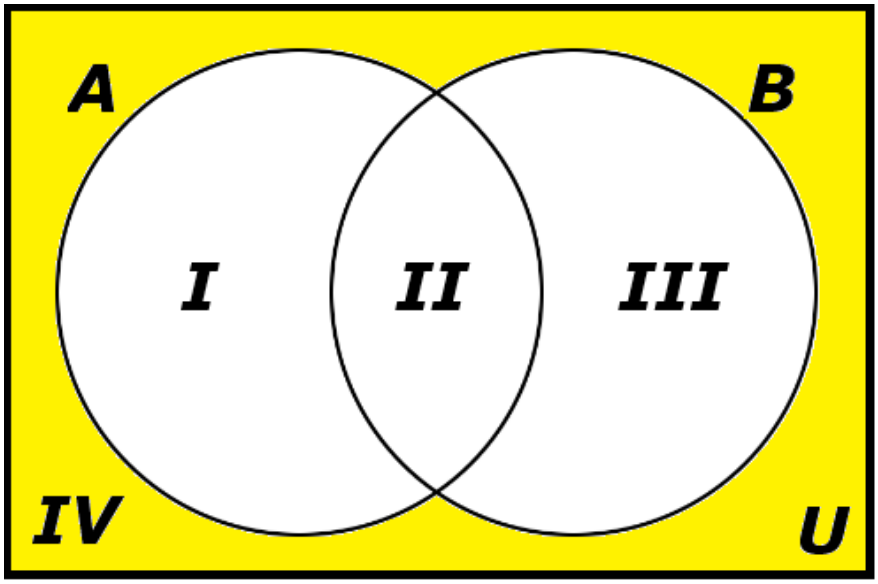
(Section 2.4) - Homework Exercise #2



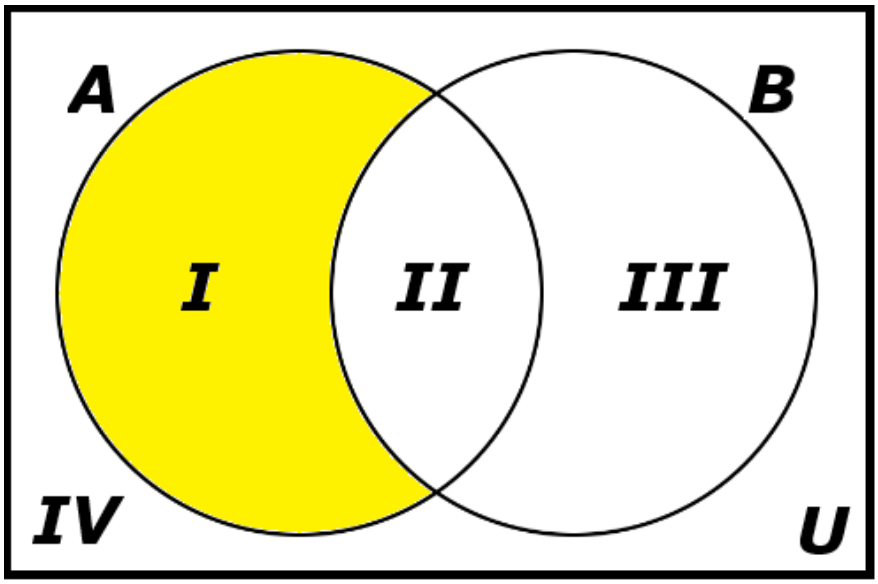
(Section 2.4) - Homework Exercise #3



(Section 2.4) - Homework Exercise #4



(Section 2.4) - Homework Exercise #5



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Section 2.5: Drawing Basic Venn Diagrams for Three Sets

Learning Objectives

- Use Venn diagrams to illustrate operations on three sets

In the previous section (Section 2.4), we learned how to shade Venn diagrams with two sets, mastering the four fundamental operations: union (OR), intersection (AND), complement (NOT), and set difference (BUT NOT). We saw how these operations create specific shading patterns that help us visualize which elements satisfy certain conditions.

Now, we're ready to take the next step. The same operations and shading principles apply when we work with three sets, however, the diagrams become more complex, because there are more regions to consider.

From Two Sets to Three Sets

The good news is that the logic doesn't change. The concepts of union, intersection, and complement work exactly the same way, whether we have two sets, three sets, or even more. What does change is:

- **More circles to consider:** Instead of two overlapping circles, we now have three
- **More regions to track:** Three circles create eight distinct regions instead of four
- **More complex combinations:** We can now explore relationships like "*A* and *B*, but not *C*" or "*A* or *B* or *C*"

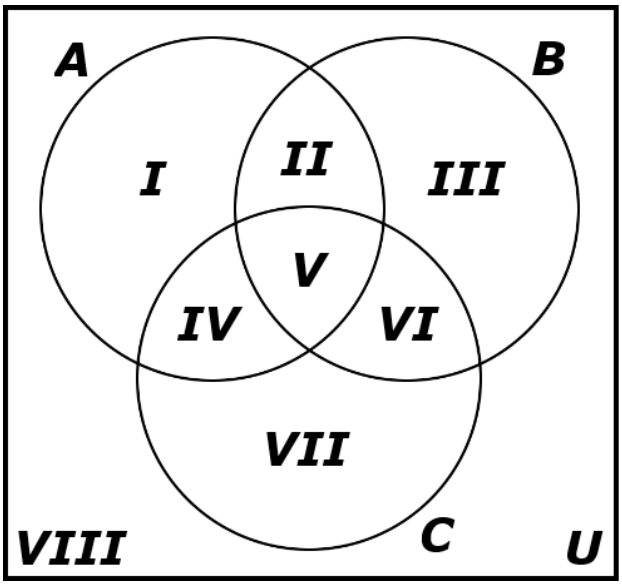
However, the fundamental approach remains the same:

- **Union (OR)** still means "shade everything in at least one of the specified sets"
- **Intersection (AND)** still means "shade only where all specified sets overlap"
- **Complement (NOT)** still means "shade everything except the specified set"
- **Set Difference (BUT NOT)** still means "shade everything in one set, but not the other set"

The Structure of a Three-Circle Venn Diagram

A three-circle Venn diagram consists of:

- A rectangle representing the universal set *U*
- Three circles (or ovals) representing sets *A, B* and *C*
- The circles overlap to show elements that belong to either one of a trio of two sets exclusively or all three sets.
- Labels identifying each set and sometimes the universal set



This creates eight distinct regions:

Region I: Inside *A* only (not in *B* and/or *C*)
Region II: Inside both *A* and *B* only (not in *C*)
Region III: Inside *B* only (not in *A* and/or *C*)
Region IV: Inside both *A* and *C* only (not in *B*)
Region V: Inside *A*, *B,* and *C* (the center; intersection of all three sets)
Region VI: Inside both *B* and *C* only (not in *A*)
Region VII: Inside *C* only (not in *A* and/or *B*)
Region VIII: Outside all three circles (in *U* but not in either *A* or *B* or *C*)

What We'll Explore

In this section, we'll revisit the types of set operations we practiced with two sets and see how they extend to three-set Venn diagrams. We'll work through examples that parallel what we've already learned, allowing you to build confidence by applying familiar concepts in a slightly more complex context.

By the end of this section, you'll be able to:

- Shade any combination of three sets using unions, intersections, complements, and differences
- Interpret complex expressions like $(A \cup B)' \cap C$ or $A \cap (B \cup C)'$
- Recognize how three-set problems connect to real-world situations with multiple overlapping categories

Let's begin by applying our shading skills to three-circle Venn diagrams, starting with the simplest cases and gradually building to more sophisticated combinations.

How to Shade Using Multiple Operations Involving Three Sets

When we shade any combination of three sets using unions, intersections, complements, and differences, we can use diagonals lines in one direction / / / to represent the first two sets at once. Recall, we are now familiar with operations on two sets, such as $A \cup B$, $A \cap B$, $A' \cap B$, $A \cap B'$, etc. or Then, we can use diagonals lines in the other direction \ \ \ to represent the third or last set. You can also use different colored pencils to represent the sets as well. For all examples in this text, we will use a light gray shade. If we need to shade a region again that is already a light gray shade, then it will be a darker shade of gray. The final solution will always be in yellow (or a yellow highlighter). We ignore everything previously in any shade of gray.

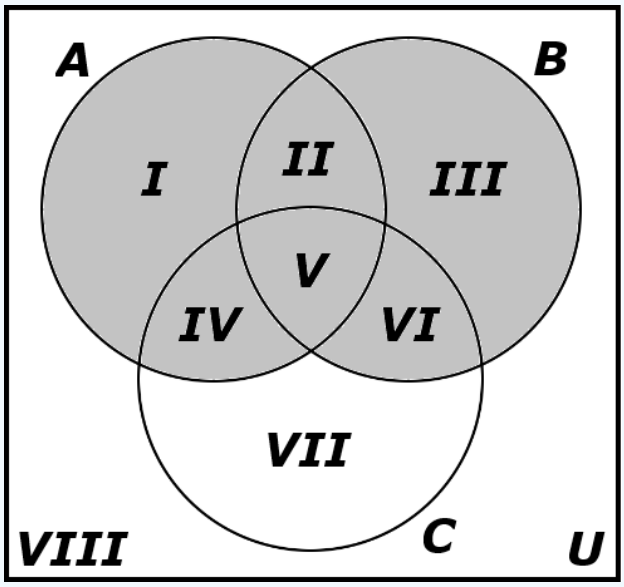
Example #2.5.1 🤔

Draw a Venn diagram to illustrate the set $(A \cup B) \cap C$. Use a yellow highlighter to represent the solution.

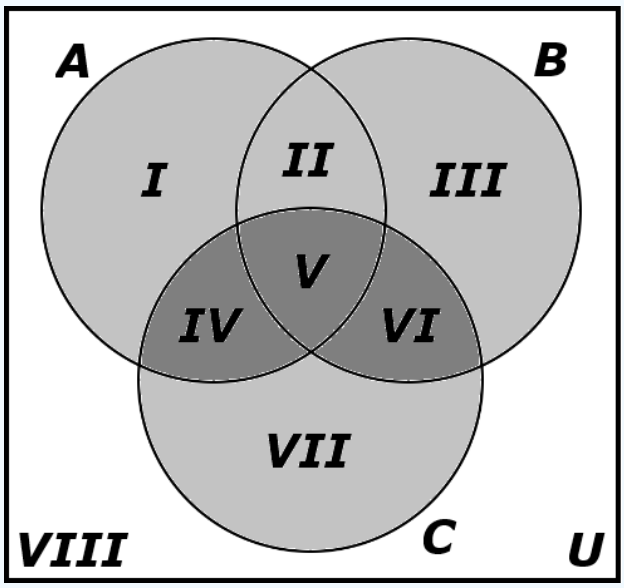
✅ **Solution:**

So, based off order of operations, parentheses first, which means shade $A \cup B$, then apply the intersection of C.

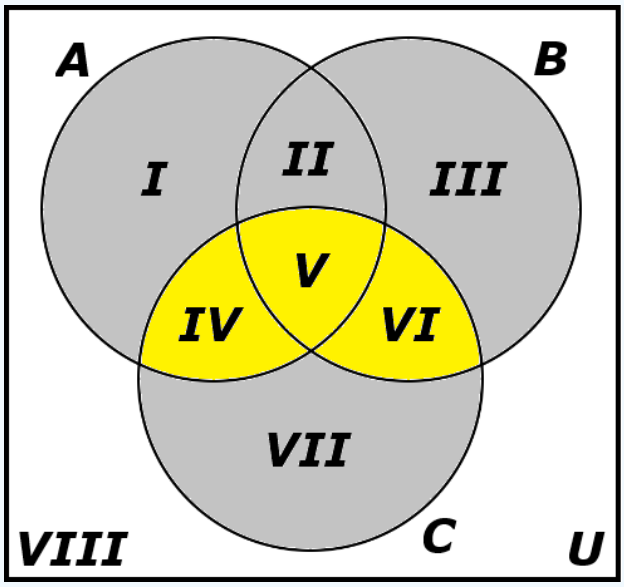
- **Step 1:** Since, we are already familiar with the union, $A \cup B$, we simply shade (lightly gray) the regions that occupy the set. This includes regions I, II, III, IV & V. (Note: We are not shading A and B separately like we did earlier in Section 2.4. We are treating $A \cup B$ as one set).
- **Step 2:** Next, shade all elements in set C, which includes regions IV, V, VI, and VII. Notice that regions IV, V, and VI is shaded twice, resulting in a darker tone.
- **Step 3:** The intersection of $A \cup B$ and C, consists of all elements shaded in both $A \cup B$ and C. Then, $(A \cup B) \cap C$ is represented by everything that is shaded darkly. Therefore, shade regions IV, V, and VI in a contrasting color, such as yellow, to indicate the solution.
- **Step 4:** Final representation: Remove any additional gray shading and leave only the highlighted yellow solution. The solution is the set of elements in regions IV, V, and VI.



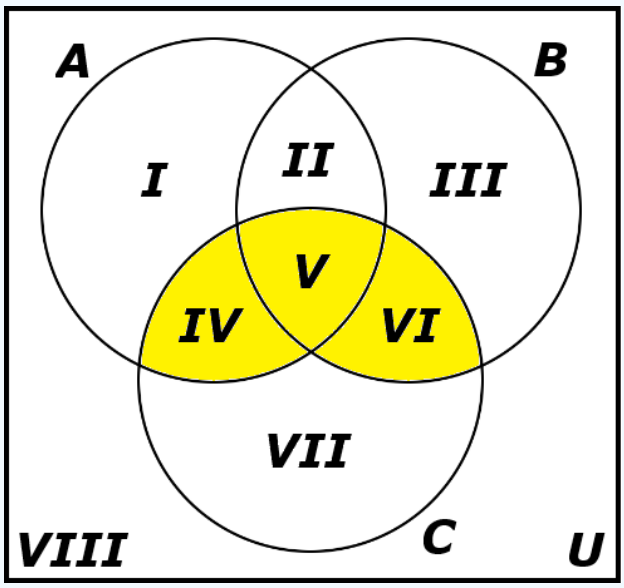
Step 1



Step 2



Step 3



Step 4

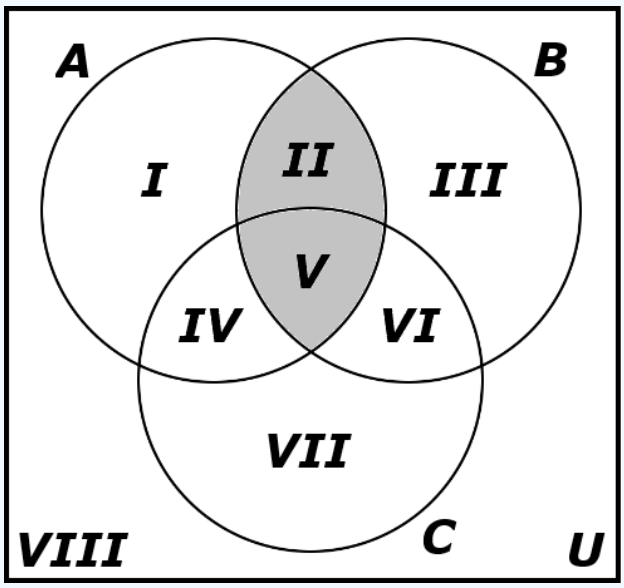
Example #2.5.2 🤔

Draw a Venn diagram to illustrate the set $(A \cap B) \cup C$. Use a yellow highlighter to represent the solution.

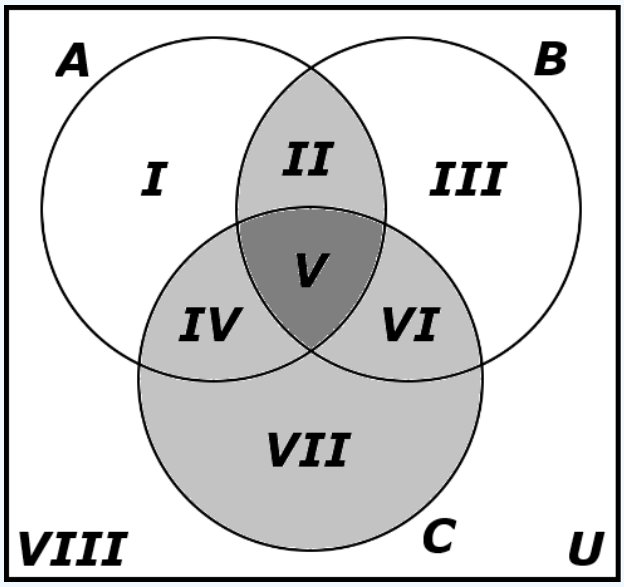
✅ **Solution:**

So, based off order of operations, parentheses first, which means shade $A \cap B$, then apply the union of C.

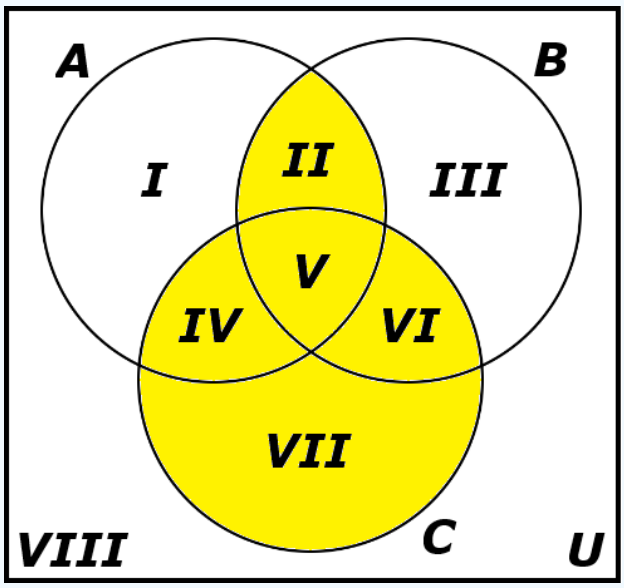
- **Step 1:** Since, we are already familiar with the union, $A \cap B$, we simply shade (lightly gray) the regions that occupy the set. This includes regions II & V. (Note: We are not shading A and B separately like we did earlier in Section 2.4. We are treating $A \cap B$ as one set).
- **Step 2:** Next, shade all elements in set C, which includes regions IV, V, VI, and VII. Notice that region V is shaded twice, resulting in a darker tone.
- **Step 3:** The union of $A \cap B$ and C, consists of all elements shaded in both $A \cap B$, C, or both. Then, $(A \cap B) \cup C$ is represented by everything that is shaded either lightly or darkly. Your final representation is regions II, IV, V, VI, and VII in a contrasting color, such as yellow, to indicate the solution.



Step 1



Step 2



Step 3

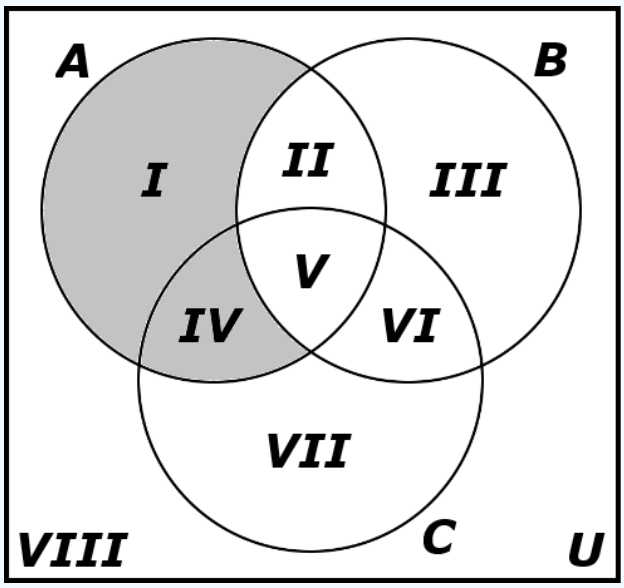
Example #2.5.3 🤔

Draw a Venn diagram to illustrate the set $(A \cap B') \cap C$. Use a yellow highlighter to represent the solution.

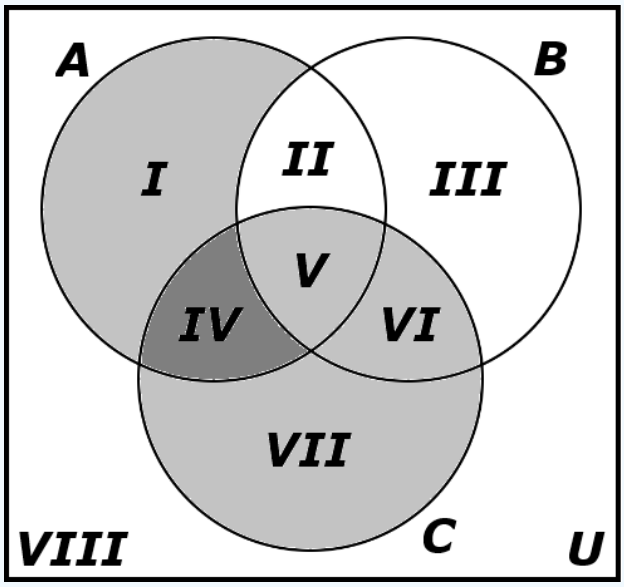
✅ **Solution:**

So, based off order of operations, parentheses first, which means shade $A \cap B'$, then apply the intersection of C.

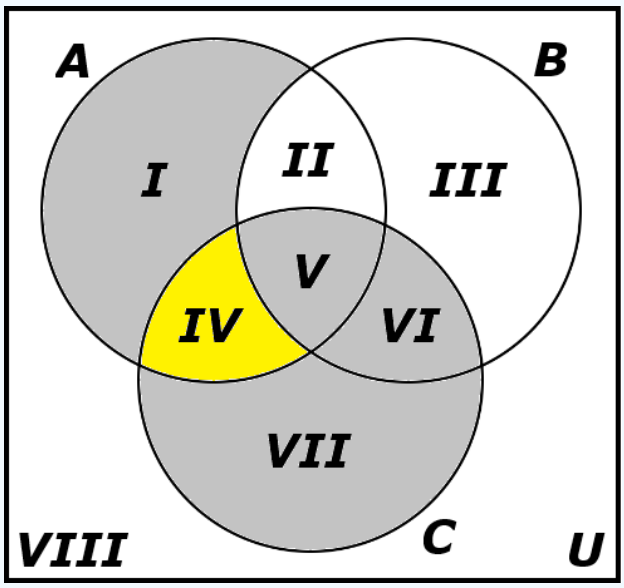
- **Step 1:** Since, we are already familiar with the intersection, $A \cap B'$, we simply shade (lightly gray) the regions that occupy the set. This includes regions I & IV. (Note: If you are not familiar with this set, then it is best to shade *A* and *B*' separately like we did earlier in Section 2.4. Then, treat $A \cap B'$ as one set and lightly shade that entire set).
- **Step 2:** Next, shade all elements in set C, which includes regions IV, V, VI, and VII. Notice that region IV is shaded twice, resulting in a darker tone.
- **Step 3:** The intersection of $A \cap B'$ and C, consists of all elements shaded in both $A \cap B'$ and C. Then, $(A \cap B') \cap C$ is represented by everything that is shaded darkly. Therefore, shade region IV in a contrasting color, such as yellow, to indicate the solution.
- **Step 4:** Final representation: Remove any additional gray shading and leave only the highlighted yellow solution. The solution is the set of elements in region IV.



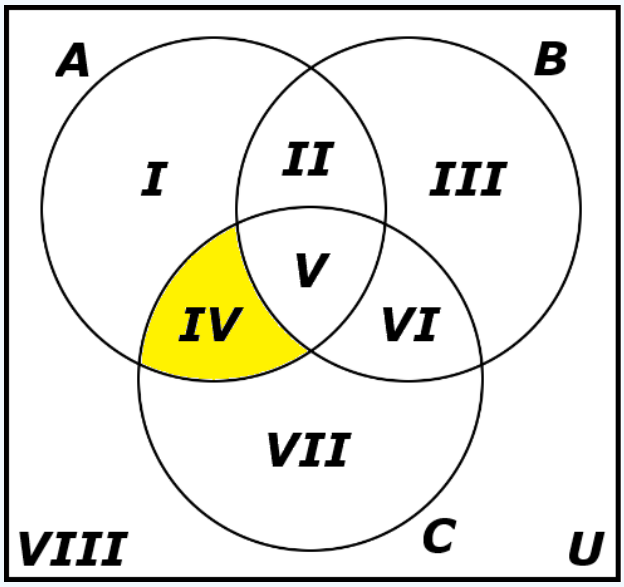
Step 1



Step 2



Step 3



Step 4

Example #2.5.4 🤔

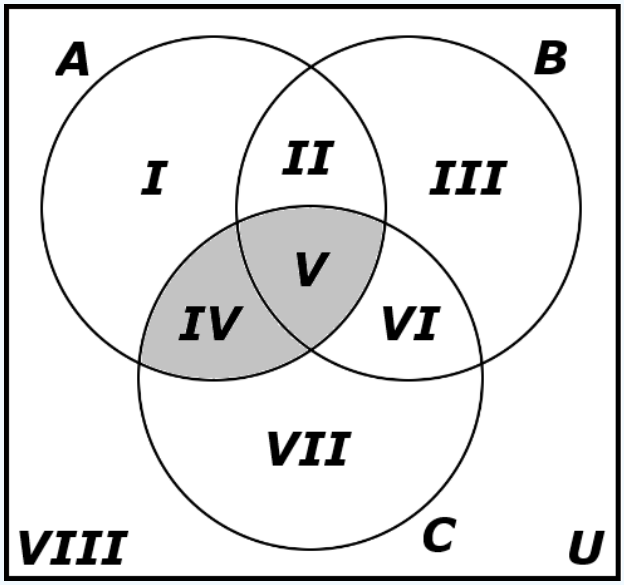
Draw a Venn diagram to illustrate the set $(A \cap C) \cup B'$. Use a yellow highlighter to represent the solution.

✅ **Solution:**

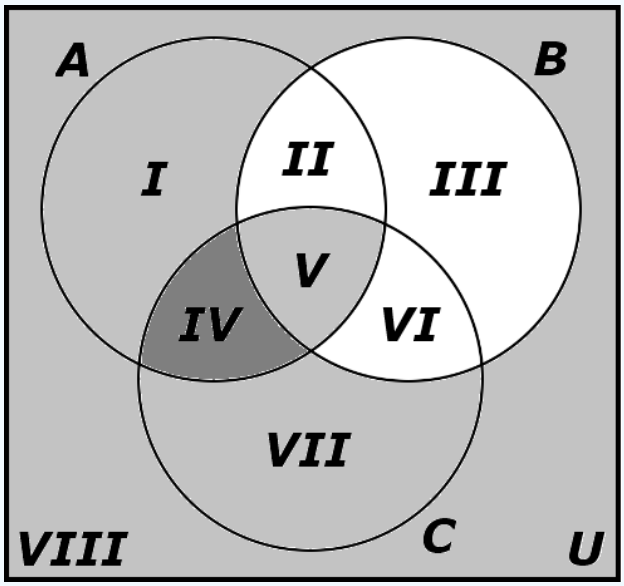
So, based off order of operations, parentheses first, which means shade $A \cap C$, then apply the union of B'.

- **Step 1:** Since, we are already familiar with the intersection, $A \cap C$, we simply shade (lightly gray) the regions that occupy the set. This includes regions IV & V. (Note: We are not shading A and C separately like we did earlier in Section 2.4. We are treating $A \cap C$ as one set).
- **Step 2:** Next, shade all elements in set B', which includes regions I, IV, VII, and VIII. Notice that region IV is shaded twice, resulting in a darker tone.

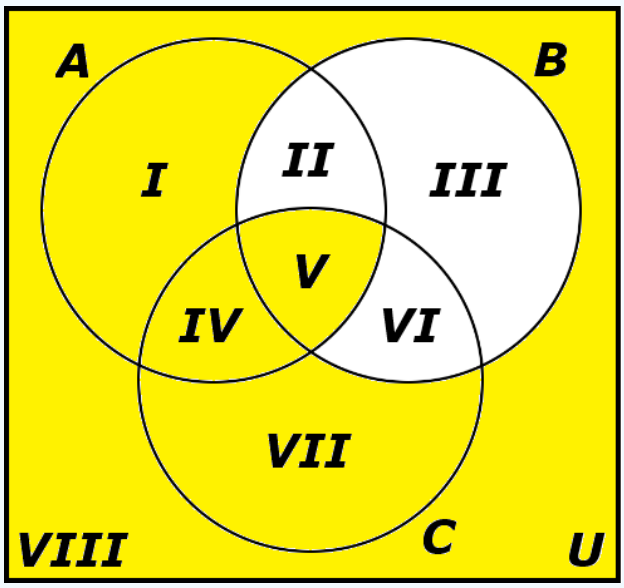
- **Step 3:** The union of $A \cap C$ and B', consists of all elements shaded in $A \cap C$, or B', or both. Then, $(A \cap C) \cup B'$ is represented by everything that is shaded either lightly or darkly. Your final representation is regions I, IV, V, VII, and VIII in a contrasting color, such as yellow, to indicate the solution.



Step 1



Step 2



Step 3

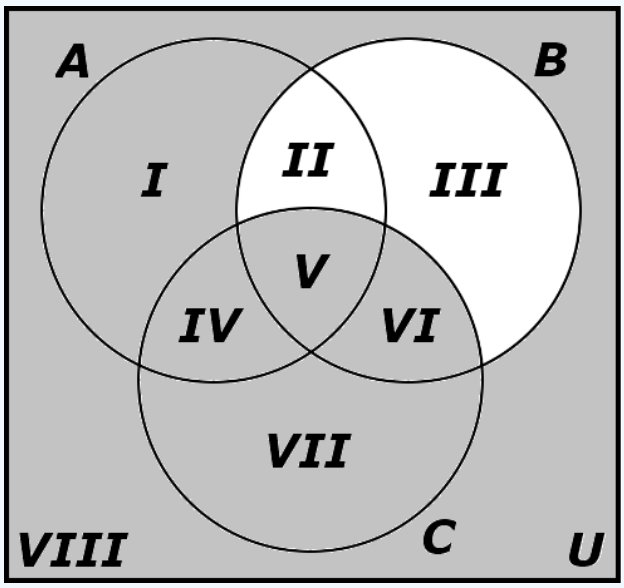
Example #2.5.5 🤔

Draw a Venn diagram to illustrate the set $(B' \cup C) - A'$. Use a yellow highlighter to represent the solution.

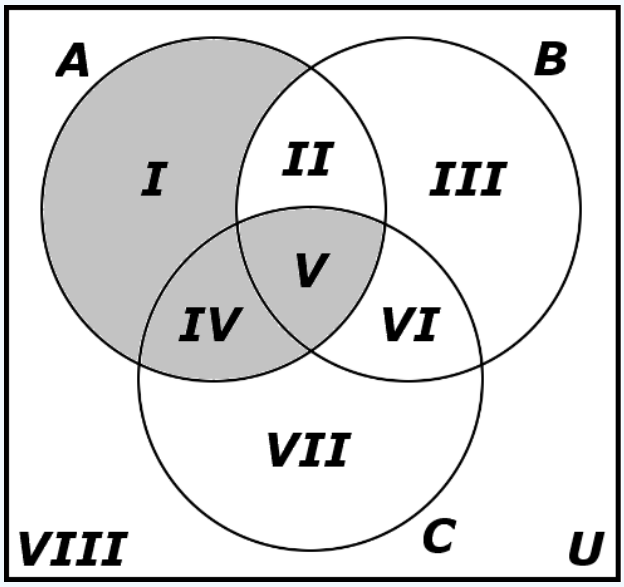
✅ **Solution:**

So, based off order of operations, parentheses first, which means shade $B' \cup C$, then apply the set difference of A'.

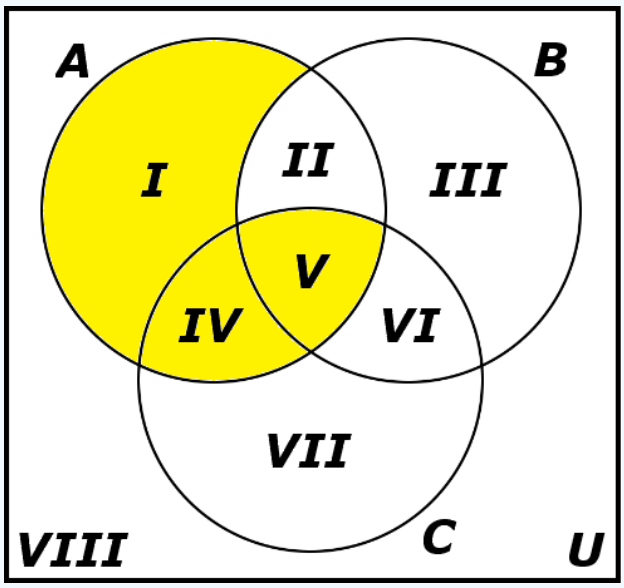
- **Step 1:** Since, we are already familiar with the union, $B' \cup C$, we simply shade (lightly gray) the regions that occupy the set. This includes regions I, IV, V, VI, VII, & VIII. (Note: If you are not familiar with this set, then it is best to shade B' and C separately like we did earlier in Section 2.4. Then, treat $B' \cup C$ as one set and lightly shade that entire set).
- **Step 2:** The set difference of $B' \cup C$ and A', consists of all elements shaded in $B' \cup C$, but not A'. Then, $(B' \cup C) - A'$ removes regions VI, VII, and VIII, leaving only regions I, IV, & V.
- **Step 3:** Your final representation is regions I, IV, & V in a contrasting color, such as yellow, to indicate the solution.



Step 1



Step 2



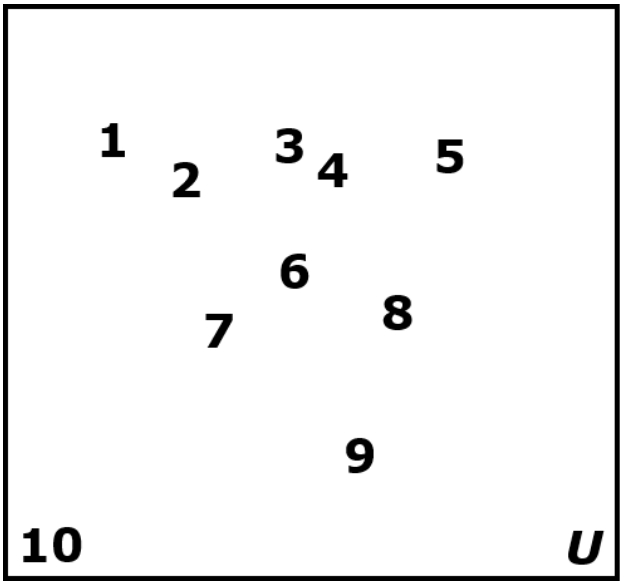
Step 3

So, now that we have seen several examples of shading, let's look at using actual elements in sets to confirm the shading examples from above.

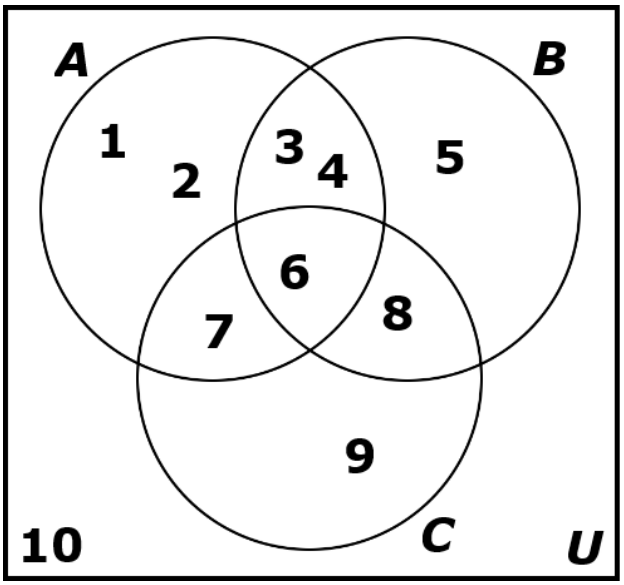
Think of the circles as rings that we would toss onto the rectangle. For example, lets say we have the following sets:

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$A = \{1, 2, 3, 4, 6, 7\}$, $B = \{3, 4, 5, 6, 8\}$, $C = \{6, 7, 8, 9\}$,
$A \cap B = \{3, 4, 6\}$, $A \cap C = \{6, 7\}$, $B \cap C = \{6, 8\}$, and $A \cap B \cap C = \{6\}$.

We place the elements 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 in the rectangle.

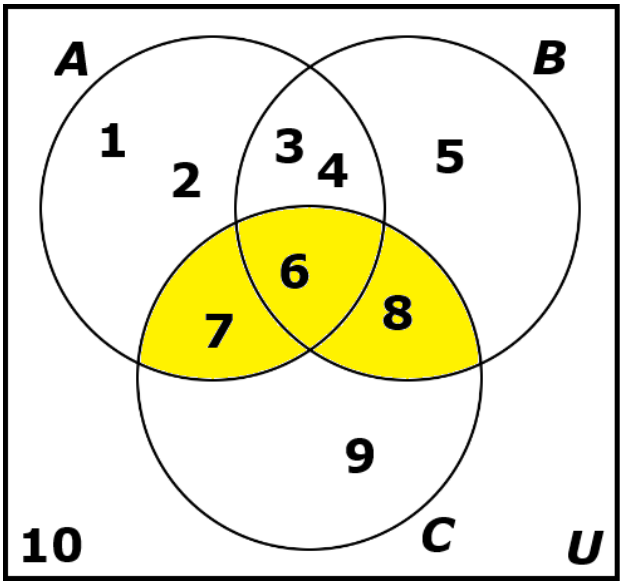


We then place labeled rings A, B, and C on the appropriate elements. The Venn diagram would look like as follows:

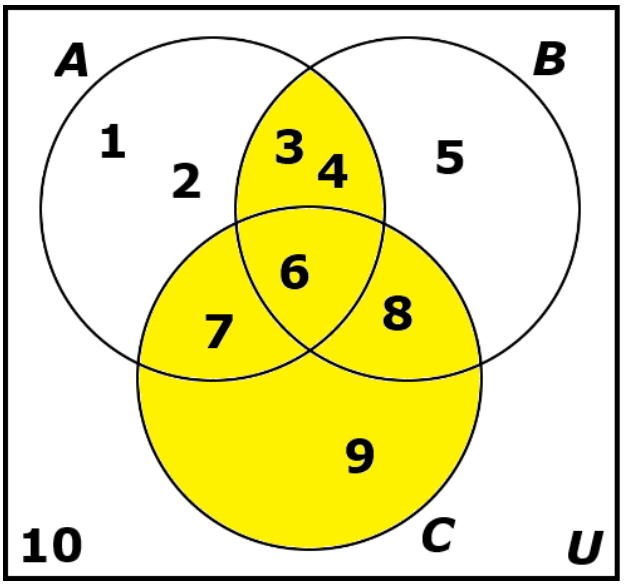


- The elements 1, 2 are in region I. (A, not B, not C)
- The elements 3, 4 are in region II. (Both A and B, not C)
- The element 5 is in region III. (In B, not A, not C)
- The element 7 is in region IV. (Both A and C, not B)
- The element 6 is in region V. (All in A, B, and C)
- The element 8 is in region VI. (Both B and C, not A)
- The element 9 is in region VII. (C, not A, not B)
- The element 10 is in region VIII. (Outside the union A or B or C)

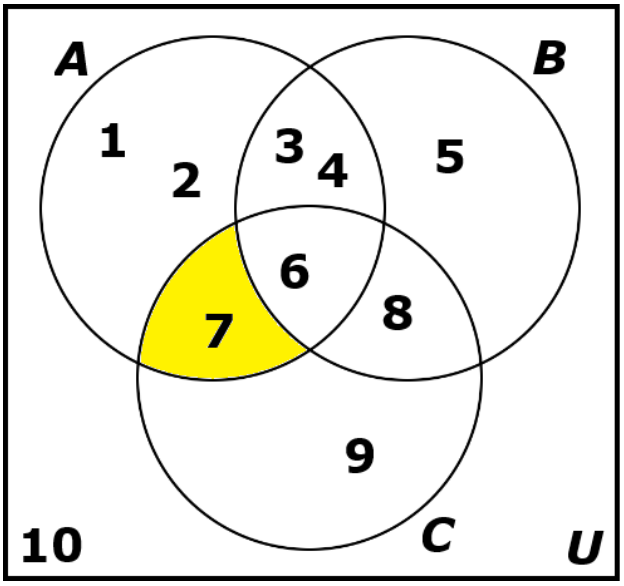
From Section 2.2 using sets, lets look at the examples from above:



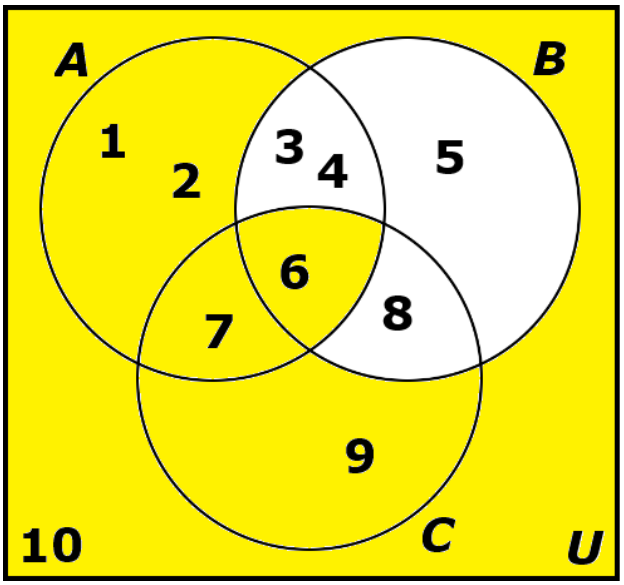
$$(A \cup B) \cap C = \{6, 7, 8\}$$



$(A \cap B) \cup C = \{3, 4, 6, 7, 8, 9\}$



$(A \cap B') \cap C = \{7\}$



$$(A \cap C) \cup B' = \{1, 2, 6, 7, 9, 10\}$$

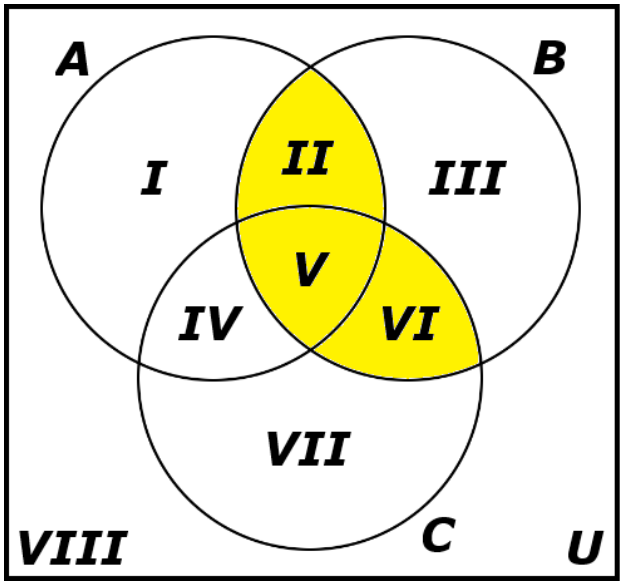
Section 2.5: Drawing Basic Venn Diagrams for Three Sets [*In-Class Exercises*]

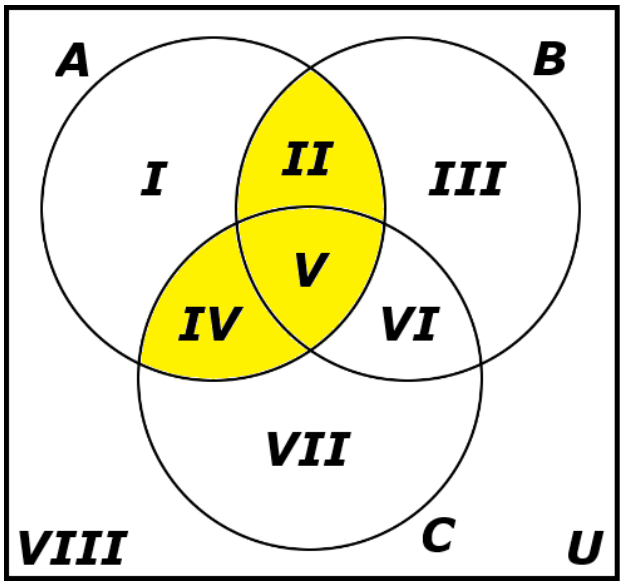
1. Draw a Venn diagram to illustrate the set $(A \cup C) \cap B$. Use a yellow highlighter to represent the solution.
2. Draw a Venn diagram to illustrate the set $(B \cup C) \cap A$. Use a yellow highlighter to represent the solution.
3. Draw a Venn diagram to illustrate the set $(A \cap B) \cup C'$. Use a yellow highlighter to represent the solution.
4. Draw a Venn diagram to illustrate the set $(A \cap B') \cup C'$. Use a yellow highlighter to represent the solution.
5. Draw a Venn diagram to illustrate the set $(A \cap C)' \cup B$. Use a yellow highlighter to represent the solution.

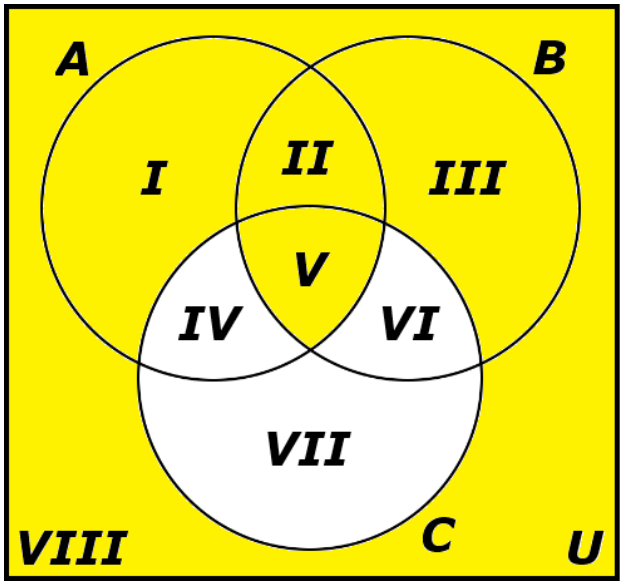
Answers available at the end of this section.

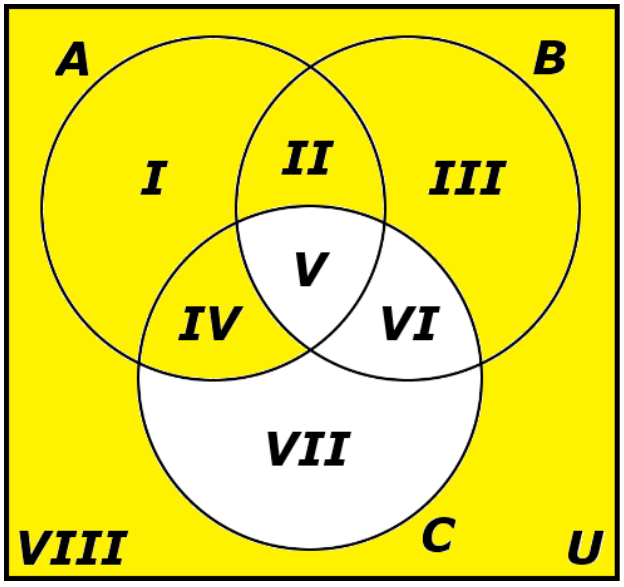
Answers

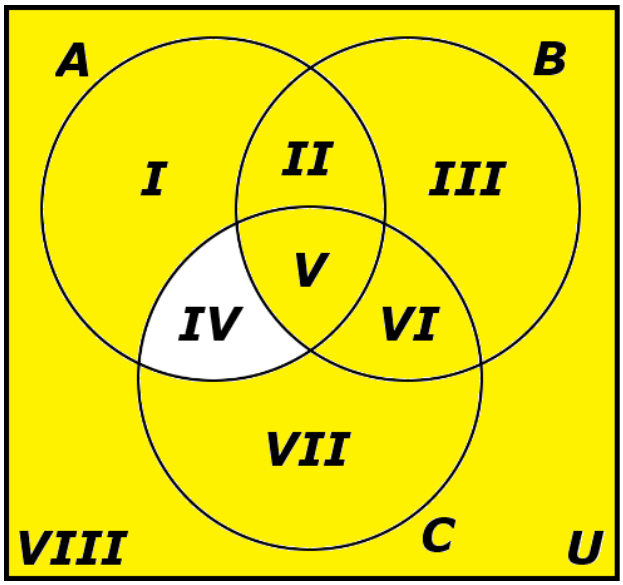
Section 2.5: Drawing Basic Venn Diagrams for Three Sets [*In-Class Exercises*]

1. 

2. 
A
B
I
II
III
V
IV
VI
VII
VIII
C
U

3. 
A
B
I
II
III
V
IV
VI
VII
VIII
C
U

4. 
A
B
I
II
III
V
IV
VI
VII
VIII
C
U

5. 

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Section 2.5.H: Homework

Section 2.5: Drawing Basic Venn Diagrams for Three Sets - Homework Exercises

(Section 2.5) - Homework Exercise #1

Draw a Venn diagram to illustrate the set $(A \cap B) \cap C$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

(Section 2.5) - Homework Exercise #2

Draw a Venn diagram to illustrate the set $(B \cup C) \cap A$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

(Section 2.5) - Homework Exercise #3

Draw a Venn diagram to illustrate the set $(A \cup B) \cap C'$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

(Section 2.5) - Homework Exercise #4

Draw a Venn diagram to illustrate the set $(A \cap C') \cup B$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

(Section 2.5) - Homework Exercise #5

Draw a Venn diagram to illustrate the set $(A - B) \cap C$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

(Section 2.5) - Homework Exercise #6

Draw a Venn diagram to illustrate the set $(A' \cup B') \cup C'$. Use a yellow highlighter to represent the solution.

Answer available at the end of this section.

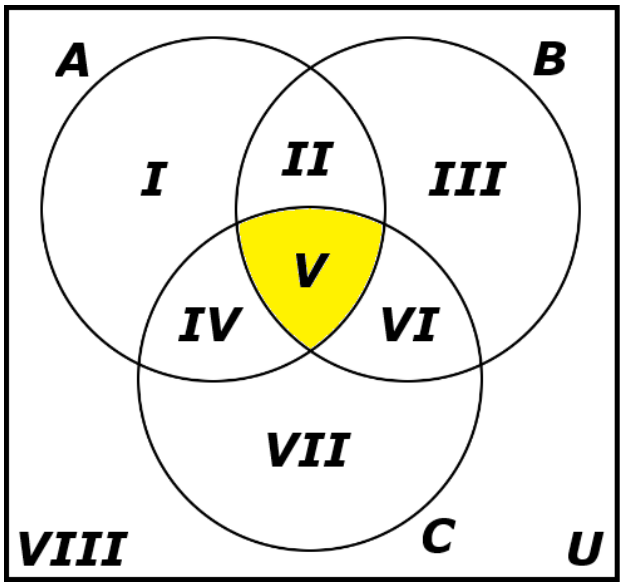
(Section 2.5) - Homework Exercise #7

Draw a Venn diagram to illustrate the set $(A \cup C') \cup B'$. Use a yellow highlighter to represent the solution.

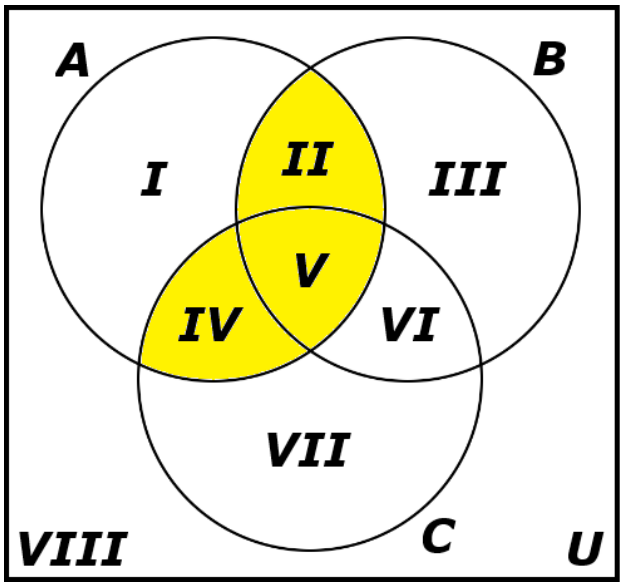
Answer available at the end of this section.

Answers

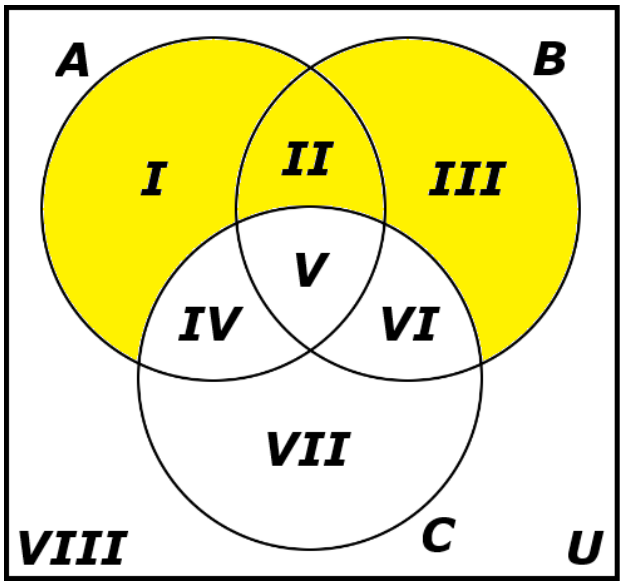
(Section 2.5) - Homework Exercise #1



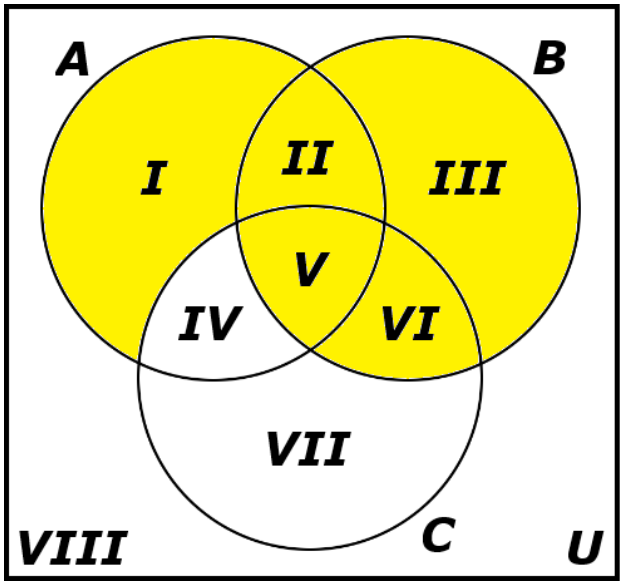
(Section 2.5) - Homework Exercise #2



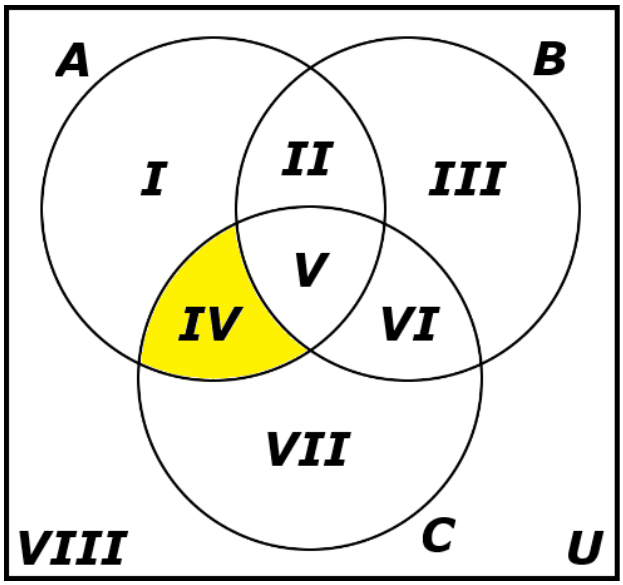
(Section 2.5) - Homework Exercise #3



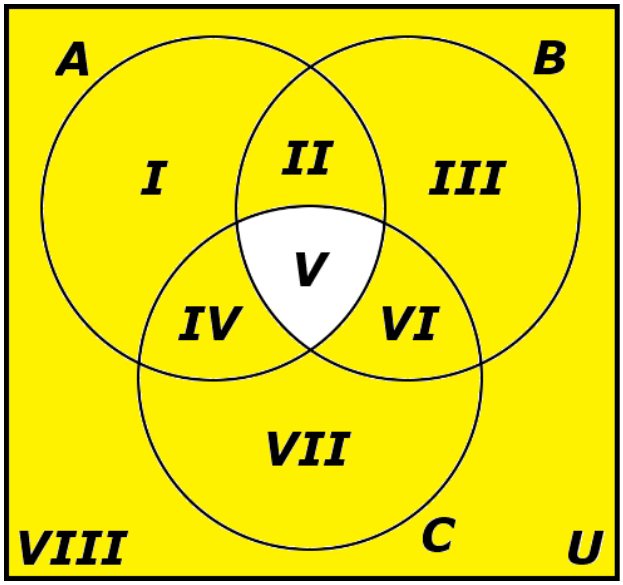
(Section 2.5) - Homework Exercise #4



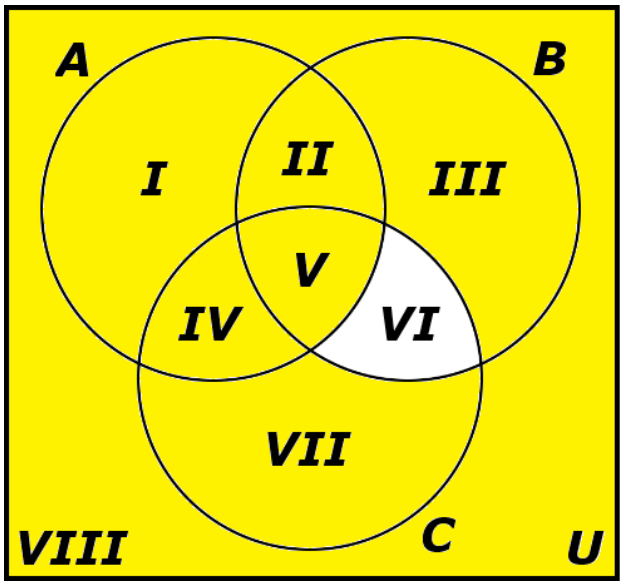
(Section 2.5) - Homework Exercise #5



(Section 2.5) - Homework Exercise #6



(Section 2.5) - Homework Exercise #7



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Section 2.6: Applications Using Two-Circle Venn Diagrams

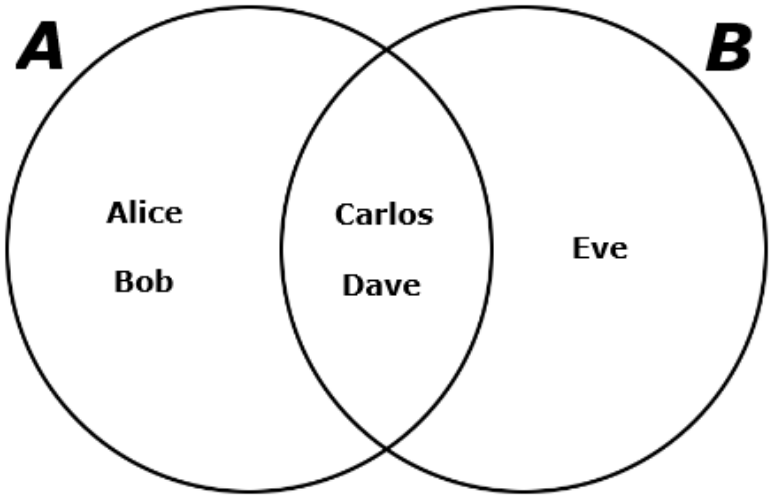
Learning Objectives

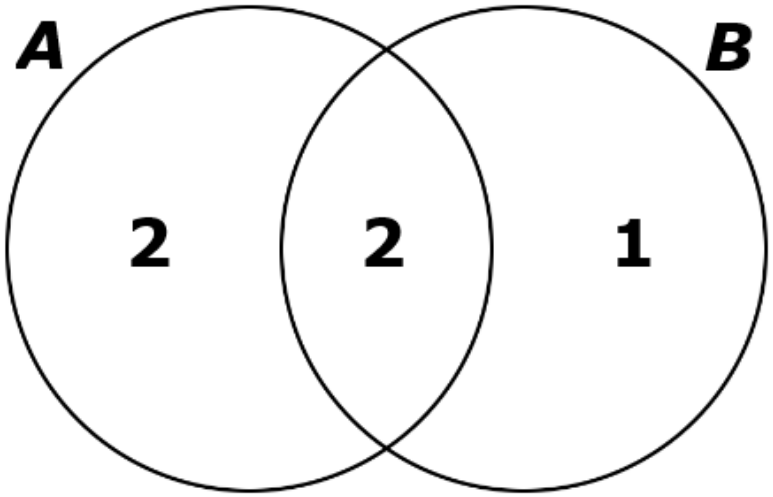
- Solve problems using two-circle Venn Diagrams

In the previous two sections, we used Venn diagrams to illustrate operations on two or three sets by shading regions that correspond to unions, intersections, set difference, and complements. We also examined examples where individual elements were placed in their respective regions. This approach is particularly useful for small sets because it allows us to see exactly which items belong to each group. However, when working with large sets, listing every element becomes cumbersome and impractical. Instead, we focus on the size of each set or its cardinality. A Venn diagram using cardinal numbers represents how many elements fall into each region rather than identifying them individually.

For instance:

If set A = {Alice, Bob, Carlos, Dave} and set B = {Carlos, Dave, Eve}, then a Venn diagram with elements would show names in each region: Alice and Bob in A only, Eve in B only, and Carol and Dave in the overlap. Yet, if we were to look at only quantities, a Venn diagram with cardinal numbers would show: 2 in A only (Alice and Bob), 1 in B only (Eve), and 2 in the overlap (Carlos and Dave).





This shift from elements to cardinal numbers allows us to handle large sets efficiently while still visualizing relationships among sets.

Types of Two-Circle Problems

Two-circle Venn diagram problems can be classified into several categories based on the type of information provided and the solution method required. Understanding these problem types enables us to select the appropriate strategy and avoid common errors.

Type 1: Direct Information Problems

In these problems, the number of elements in each region of the diagram is stated explicitly. The given information corresponds directly to the four distinct regions: elements in only set *A*, elements in only set *B*, elements in both sets, and elements in neither set. These problems require minimal calculation, as we can transfer the information directly into the appropriate regions of the diagram.

Type 2: Working from the Center Out

In these problems, we are given information about specific regions, such as "both *A* and *B* " and "*A* " and "*B* ". Another type would be both "*A* and *B* " and "at least *A* or *B*" along with either "*A* " or "*B* ". In each of these situations you must work backwards to determine the totals for each of the missing regions, since the "*A* only" and/or "*B* only" may not be given. This requires subtracting specific regions as well as adding other regions together.

Type 3: Solving for the Intersection

These problems provide the total number of elements in each set and the total number of elements in at least one set (the union). Usually "both *A* and *B* " is not given. So, we will introduce what is called the principle of inclusion-exclusion, expressed by the formula, $|A \cup B| = |A| + |B| - |A \cap B|$, to calculate the number of elements in the intersection.

The distinction between these three problem types is crucial, as each requires a different approach to reach the solution. In the following sections, we will examine the different types in detail and develop systematic methods for solving each.

The Problem-Solving Process

When approaching these types of problems with Venn diagrams that involve cardinality, we follow a systematic four-step process that works for virtually any scenario.

- **Step 1: Identify what type of problem we are working with**— carefully reading the problem to determine what groups or categories we're working with and what our universal set should be.
- **Step 2: Draw and label our Venn diagram**—creating circles for each set, clearly labeling them, and noting what the universal set represents.
- **Step 3: Fill in the information strategically**—based off the type of problem we are working with, we place the cardinal numbers in the different regions of the Venn diagram. Start with the given information first. Most problems, we always start with the innermost regions (the intersections of all sets) and work our way outward to the individual set regions, using subtraction to avoid double-counting.
- **Step 4: We answer the question(s)**—use the completed Venn diagram to extract the specific information requested, whether that's counting elements in a particular region, finding unions or intersections, or identifying complements. This methodical approach ensures we don't miss information or make counting errors, and it works whether we're dealing with two sets, three sets, or even more complex scenarios.

Example #1 (Type 1: Direct Information, Easy)

There are 30 college students.

- 10 students are enrolled in Math only
- 12 students are enrolled in English only
- 5 students are enrolled in both Math and English
- The rest of the students are not enrolled in either course.

Question: Draw a Venn diagram for the above information.

Step 1: Identify what makes this a "Direct Information" problem.

Notice that the problem provides the count for each distinct region of the Venn diagram:

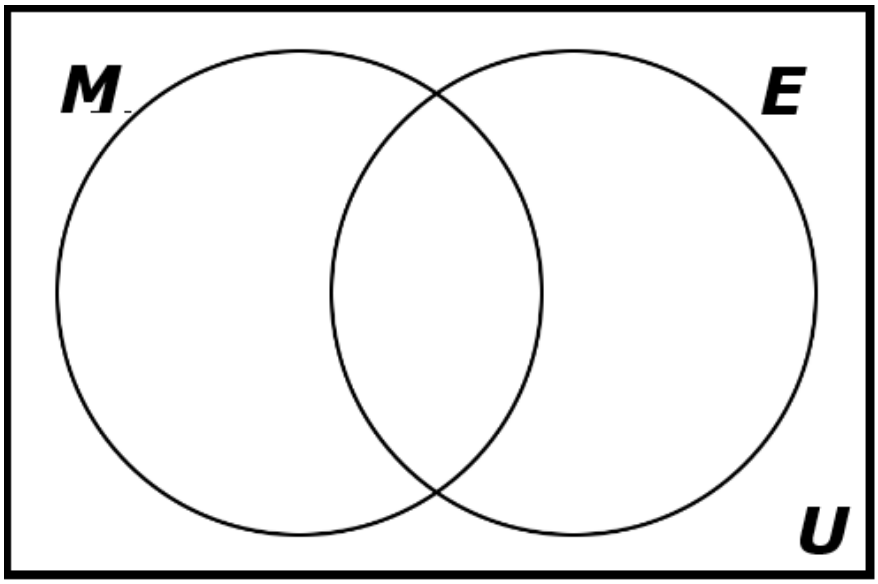
- "Both Math and English" → center region; intersection
- "Math only" → Math region excluding the intersection
- "English only" → English region excluding the intersection

Each piece of information corresponds to exactly one region in our diagram. We do not need to subtract or calculate; we simply place each number in its corresponding location.

Step 2: Draw the diagram.

Let M = Math, E = English

We draw two overlapping circles inside a rectangle. The rectangle represents all 30 college students, the left circle is college students enrolled in Math, M, and the right circle is college students enrolled in English, E. There is one overlapping region that represents college students that are enrolled in both a Math and an English course, $M \cap E$.

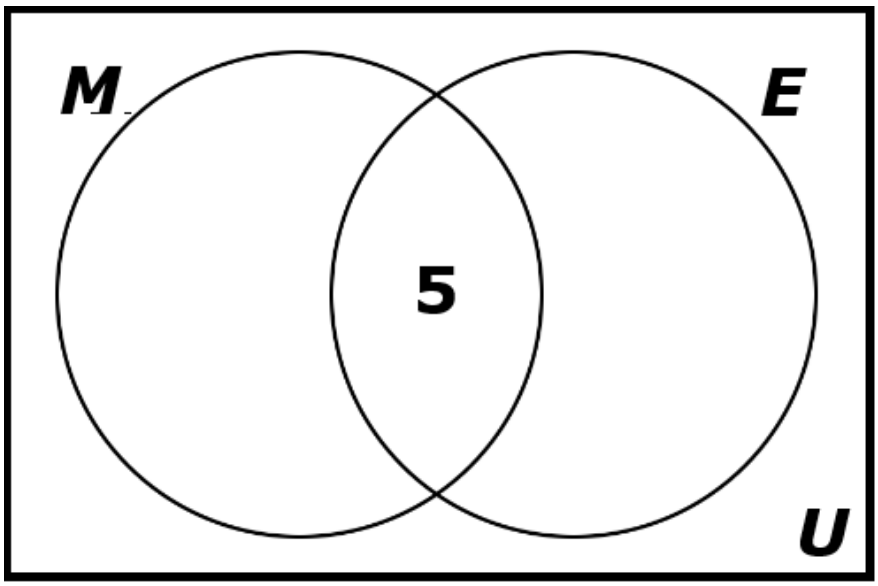


Step 3: Fill in the information strategically.

Since each piece of information corresponds to exactly one region in our diagram, we just place each number in its corresponding location.

Center:

Start with the intersection, if applicable. We know 5 students are enrolled in both courses, so we write "5" in the center overlapping region of the two circles. (For simplicity, let's use a shortcut notation, $ME = 5$, instead of $|M \cap E| = 5$, when discussing the cardinal numbers). So, $ME = 5$.

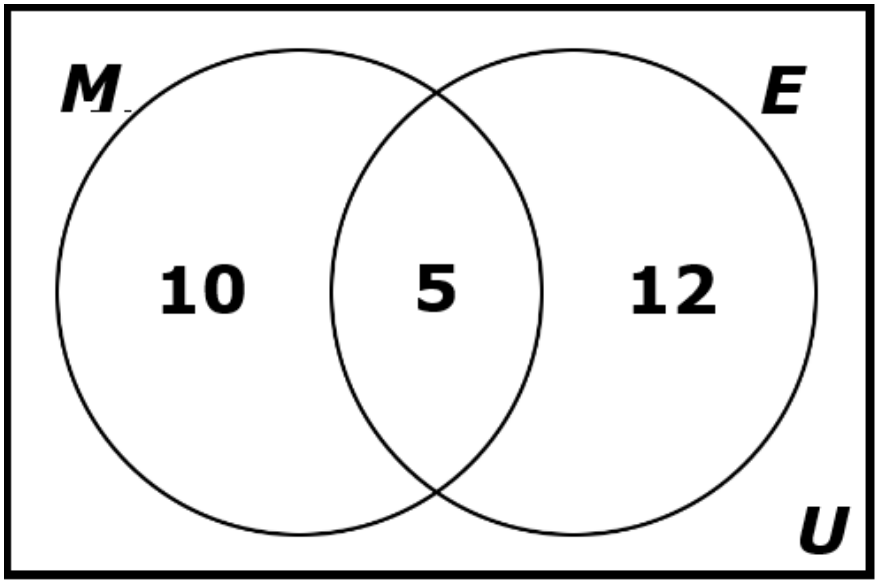


Single sets only:

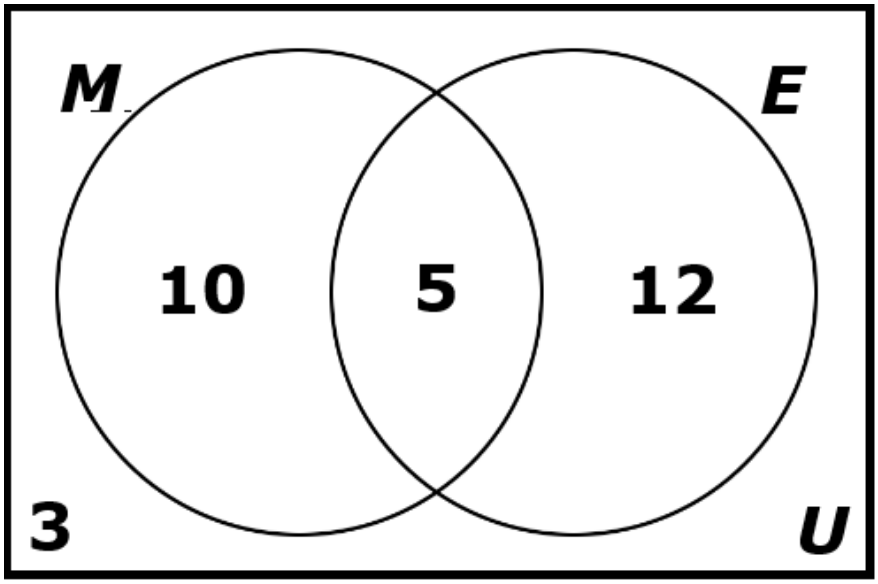
There are 10 college students enrolled in Math only. We write "10" in the non-overlapping region of Math only. (Shortcut notation, use M only = 10).

There are 12 college students enrolled in English only. We write "12" in the non-overlapping region of English only. (Shortcut

notation, use E only = 12).



Finally, we need to find how many college students are not enrolled in either course. We add up all the college students we've accounted for in the three of four regions of the Venn diagram: 10 + 5 + 12 = 27 college students are enrolled in at least one course of Math or English, or both. Since there are 30 college students, then there are 30 – 27 = 3 college students that are not enrolled in either course.. We write "3" outside the two circles, but inside the rectangle.



Step 4: Answer the questions.

Now we can answer all types of questions, such as:

- How many college students are enrolled in Math? 10 + 5 = 15
- How many college students are enrolled in at least one of the courses mentioned? 10 + 5 + 12 = 27
- How many college students are not enrolled in any of the courses mentioned? 3

Example #2 (Type 2: Working from the Center Out, Moderately Easy)

A survey of 80 high school students revealed their social media platform usage. The survey found that:

- 56 students use Instagram
- 78 students use either Instagram or Snapchat
- 43 students use both Instagram and Snapchat

Question: Draw a Venn diagram for the above information.

Step 1: Identify what makes this a "Working from the Center Out" problem.

Notice that the problem provides the full count in one set along with the union and intersection:

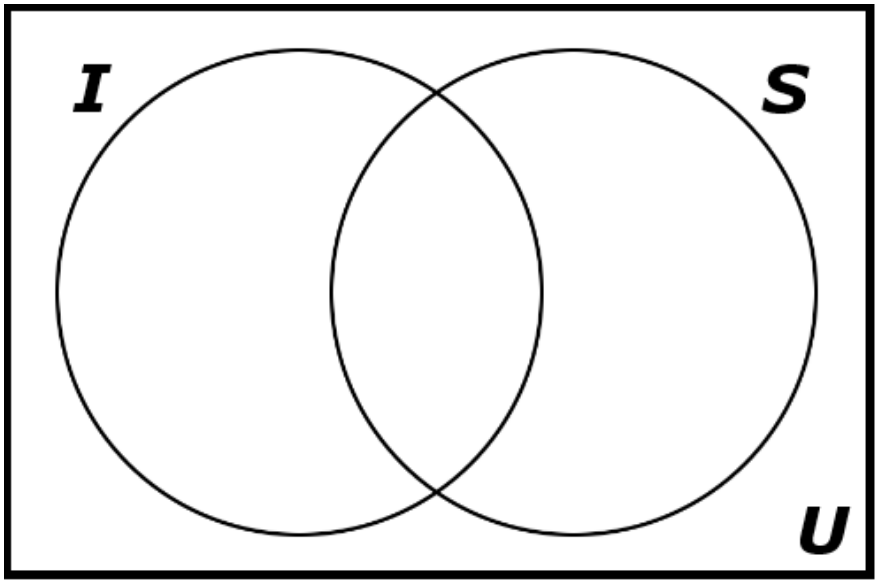
- "Both Instagram and Snapchat" → center region; intersection
- "Either Instagram and Snapchat" → all three regions created by the two circles (sets); union
- "Instagram" → Instagram region including the intersection

Each piece of information involves the intersection. Here, we need to subtract the known set total from the union set total. That difference is to be placed in the "only" region of the other set.

Step 2: Draw the diagram.

Let I = Instagram, S = Snapchat

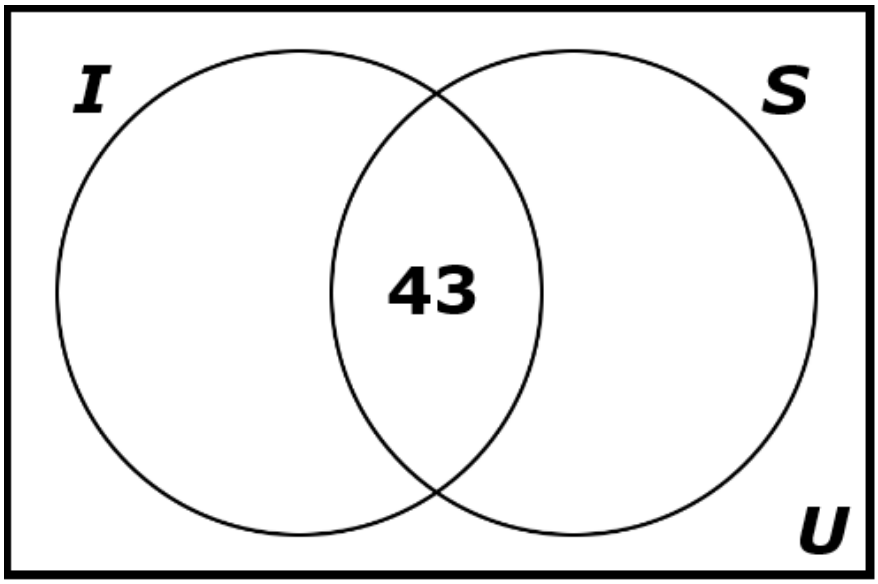
We draw two overlapping circles inside a rectangle. The rectangle represents all 80 high school students surveyed, the left circle is Instagram users, I and the right circle is Snapchat users, S. There is one overlapping region that represents both Instagram and Snapchat users, $I \cap S$.



Step 3: Fill in the information strategically.

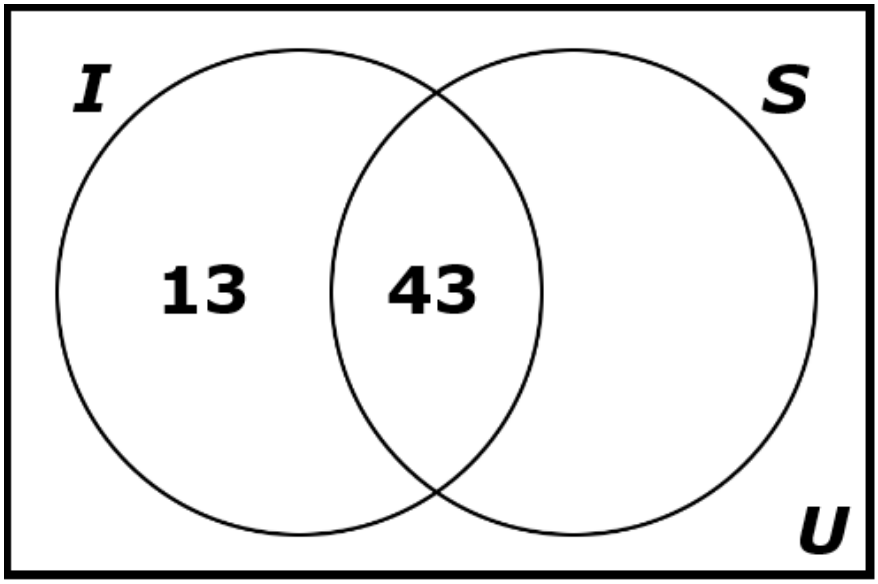
Center:

Start with the intersection. We know 43 students use both platforms, so we write "43" in the the center overlapping region of the two circles. (Shortcut notation, use $IS = 43$).

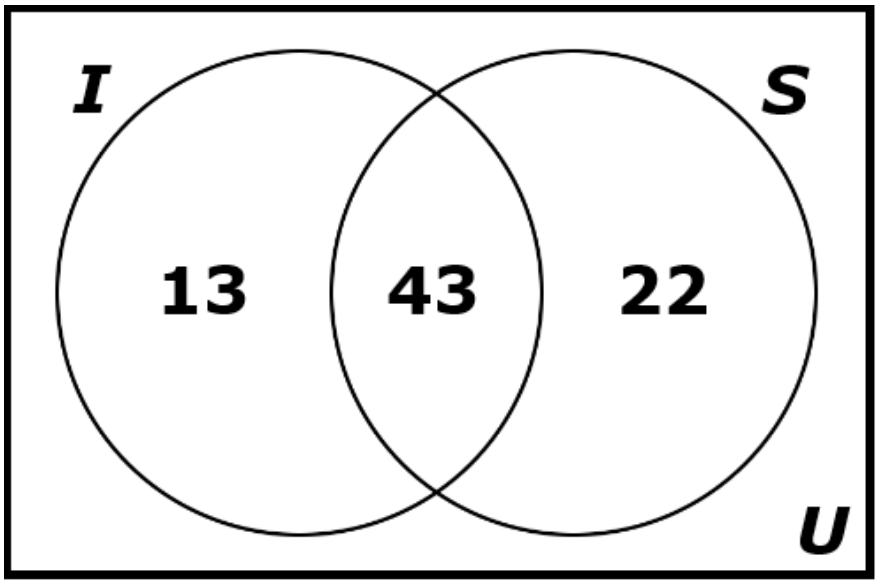


Single sets only:

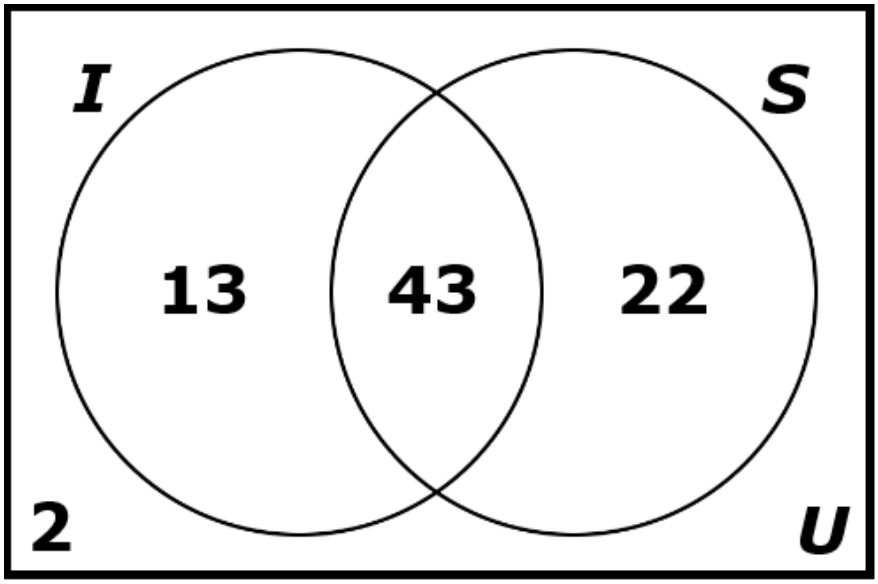
There are 56 students that use Instagram and 43 of them also use Snapchat, then 56 – 43 = 13 students use only Instagram. We write "13" in the non-overlapping region of Instagram only. (Shortcut notation, use I only = 13).



There are 78 students that use either Instagram or Snapchat or both. This is the total of all three regions made by the circles or the union, $I \cup S$. So, then 78 – 56 = 22 students use only Snapchat. We write "22" in the non-overlapping region of Snapchat only. (Shortcut notation, use S only = 22).



Finally, we need to find how many students do not use either platform. Since we know the union adds up to 78 and there are 80 students, then there are 80 – 78 = 2 students that do not use either platform. We write "2" outside the two circles, but inside the rectangle.



Step 4: Answer the questions.

Now we can answer all types of questions, such as:

- How many students use only Instagram? 13
- How many students use Snapchat? 43 + 22 = 65
- How many students use neither social media platform? 2

Example #3 (Type 2: Working from the Center Out, Moderately Easy)

Imagine you're analyzing data from a survey of 130 adults about their streaming service subscriptions. The survey found that:

- 65 adults subscribe to Netflix
- 78 adults subscribe to Hulu
- 30 adults subscribe to both Netflix and Hulu.

Question: Draw a Venn diagram for the above information.

Step 1: Identify what makes this a "Working from the Center Out" problem.

Notice that the problem provides the full count in each set along with the intersection:

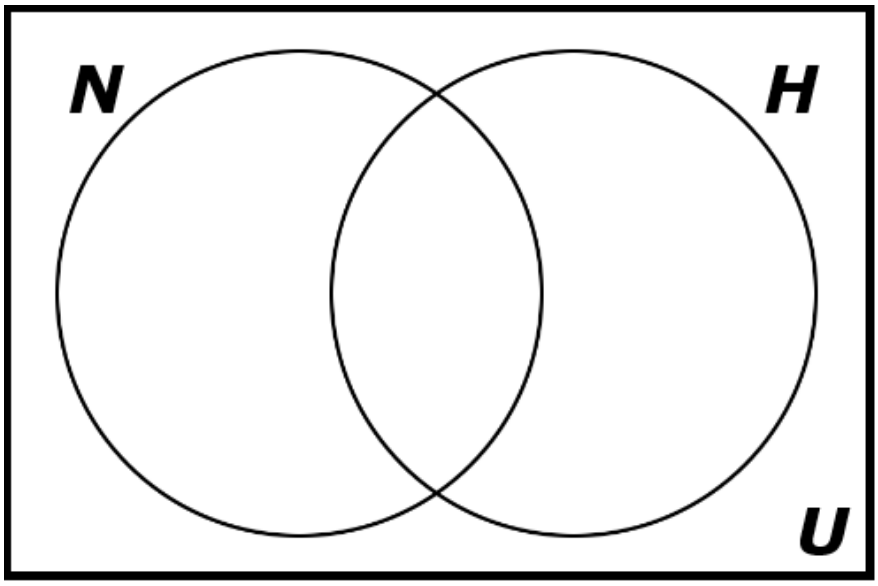
- "Both Netflix and Hulu" → center region; intersection
- "Netflix only" → Netflix region including the intersection
- "Hulu" → Hulu region including the intersection

Each piece of information involves the intersection. Here, we need to subtract the inclusion from each set total. Those differences are to be placed in the "only" regions of each set.

Step 2: Draw the diagram.

Let N = Netflix, H = Hulu

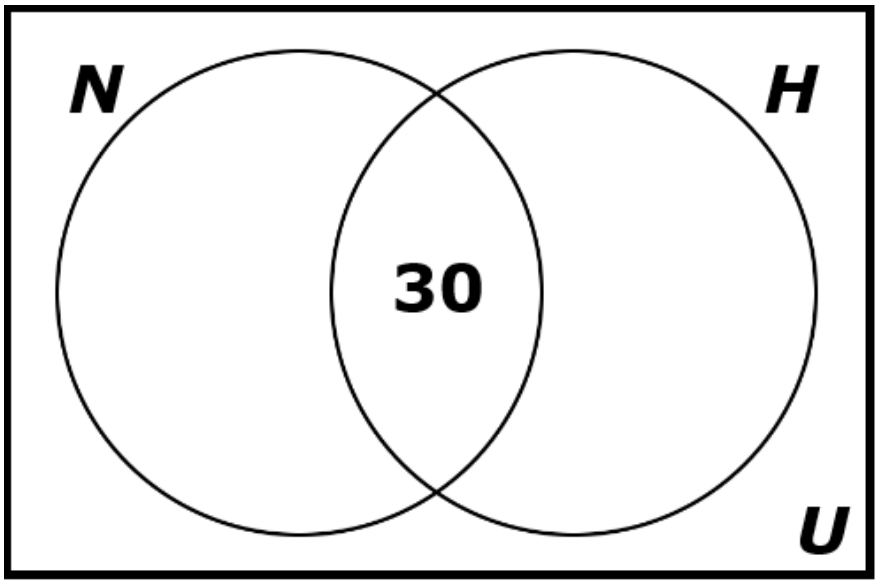
We draw two overlapping circles inside a rectangle. The rectangle represents all 100 subscribers, the left circle is Netflix subscribers, N, and the right circle is Hulu subscribers, H. There is one overlapping region that represents subscribers to both Netflix and Hulu, $N \cap H$.



Step 3: Fill in the information strategically.

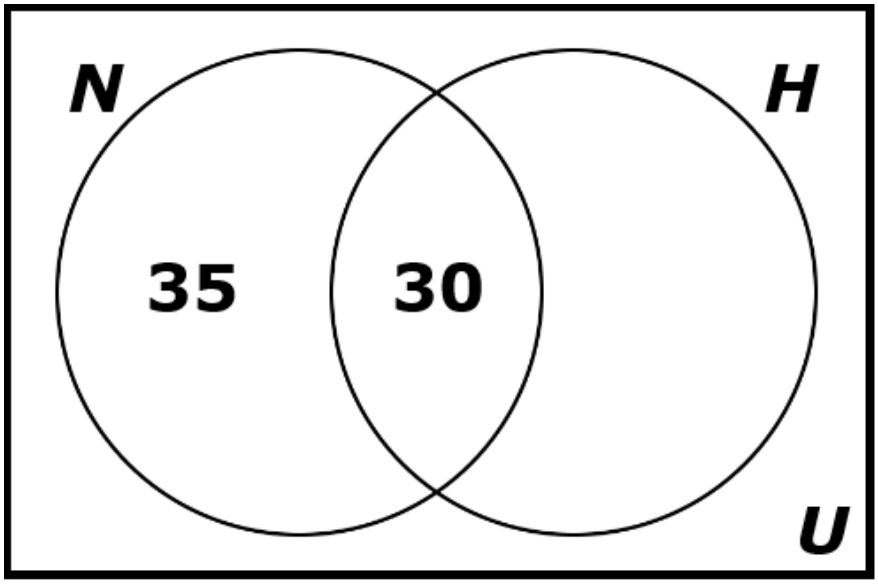
Center:

Here's where the method becomes critical, we must start with the intersection. We know 30 adults subscribe to both services, so we write "30" in the the center overlapping region of the two circles. (Shortcut notation, use NH = 30).

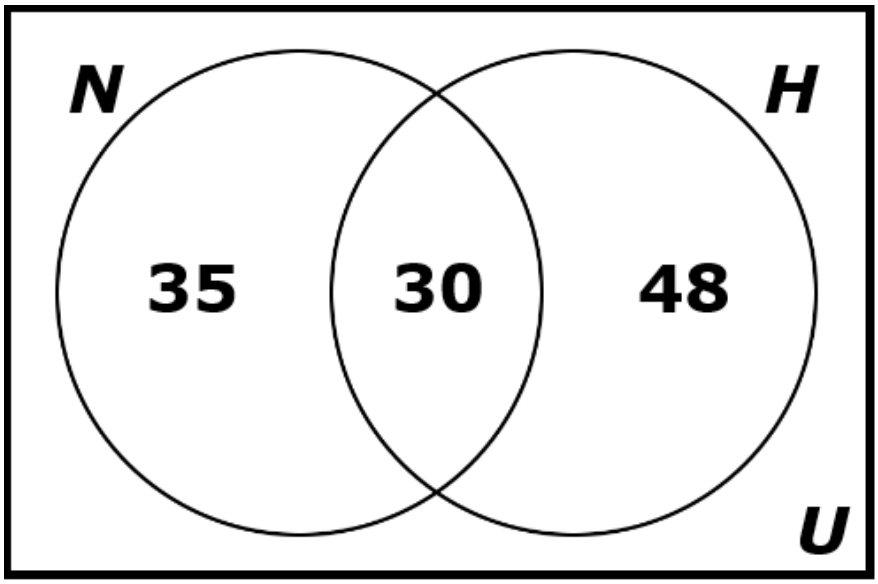


Single sets only:

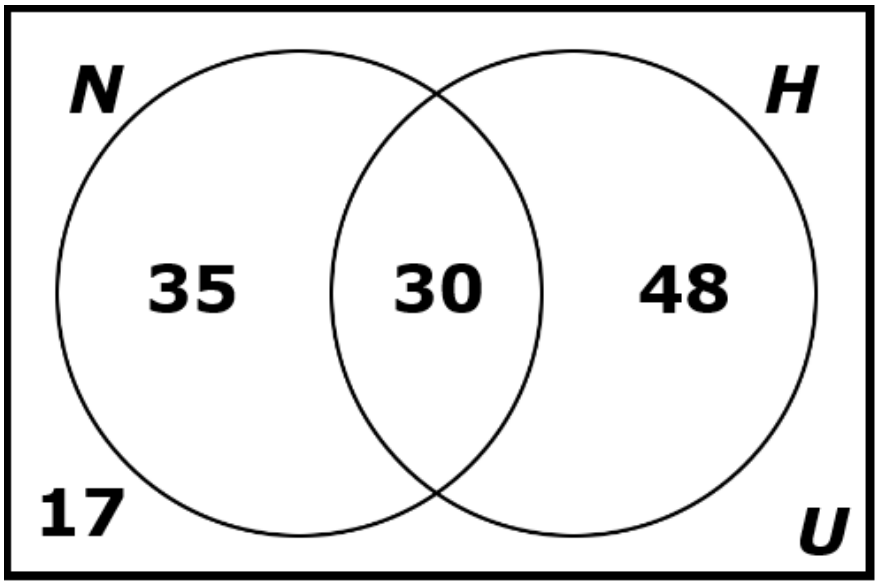
There are 65 adults that subscribed to Netflix and 30 of those also subscribe to Hulu, then 65 – 30 = 35 adults subscribe to only Netflix. We write "35" in the non-overlapping region of Netflix only. (Shortcut notation, use N only = 35).



There are 78 adults that subscribed to Hulu and 30 of those also subscribe to Netflix, then 78 – 30 = 48 adults subscribe to only Hulu. We write "48" in the non-overlapping region of Hulu only. (Shortcut notation, use H only = 48.



Finally, we need to find how many adults subscribe to neither service. We add up all the subscribers we've accounted for in the three of four regions of the Venn diagram: 35 + 30 + 48 = 113 adults subscribed to at least one service of Netflix or Hulu, or both. Since there are 130 subscribers in total, then there are 130 – 113 = 17 adults that are not subscribed to either service. We write "17" outside the two circles, but inside the rectangle.



Step 4: Answer the questions.

Now we can answer all types of questions, such as:

- How many adults are subscribed to Netflix only? 35
- How many adults are subscribed to at least one of the streaming services mentioned? 35 + 30 + 48 = 113
- How many adults are not subscribed to either one of the streaming services mentioned? 17

Example #4 (Type 3: Solving for the Intersection, Moderate)

A college bookstore surveyed 150 students about their textbook purchasing habits for the semester. The survey found that:

- 95 students bought a textbook from the physical bookstore
- 78 students bought a textbook from the online bookstore
- 127 students bought textbooks from at least one of these sources (either the physical bookstore, online, or both)

Question: Draw a Venn diagram for the above information. How many students bought textbooks from both the physical bookstore AND online?

Step 1: Identify what makes this a "Solving for the Intersection" problem.

Notice that the problem does not provide any information involving the intersection:

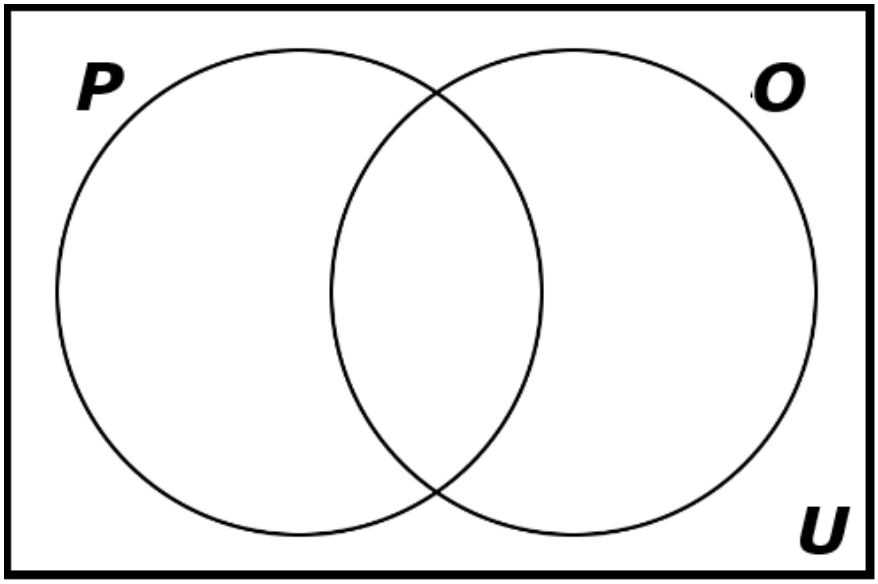
- "Either physical bookstore or online bookstore or both" → all three regions created by the two circles (sets); union
- "physical bookstore" → physical region including the intersection
- "online bookstore" → online region including the intersection

Since we do not have the intersection, we will use the union formula to find the value of the center region. Then we work backwards in a similar fashion as in the Type 2 examples.

Step 2: Draw the diagram.

Let P = physical bookstore, O = online bookstore

We draw two overlapping circles inside a rectangle. The rectangle represents all 150 students surveyed, the left circle is students who purchased from the physical bookstore, P, and the right circle is students who purchased from the online bookstore O. There is one overlapping region that represents students who purchased from the both the physical bookstore and the online bookstore, $P \cap O$.



Step 3: Fill in the information strategically.

Center:

Start with the intersection, if applicable. So, here is the first case where the intersection is not given. Move on to single sets only.

Single sets only:

We know that at least 127 students purchased either from the physical bookstore, online bookstore, or both. This means the 3

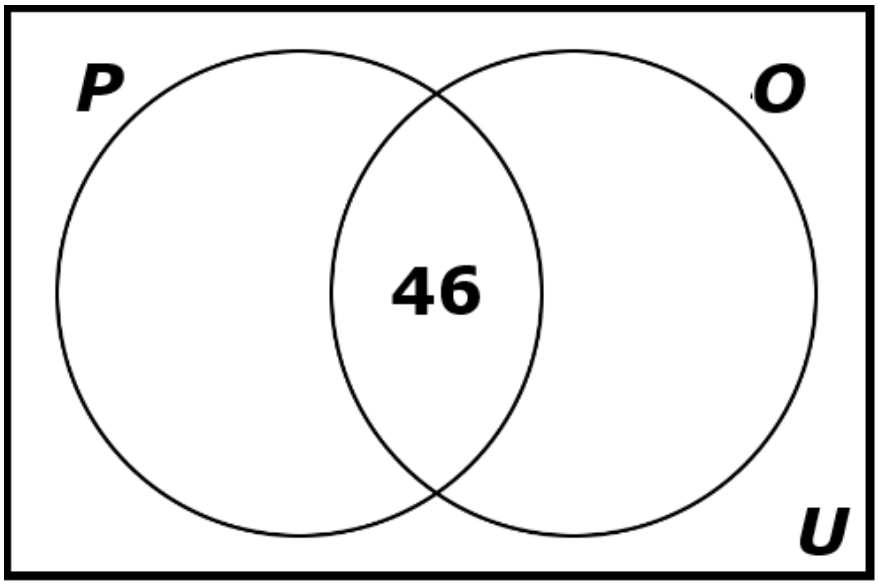
regions made by the two circles is equal to 127. Since there are 150 students total, 150 – 127 = 23 students did not purchase from either bookstore. We write "23" outside both circles but inside the rectangle. So, now, we are stuck. We know that $|P| = 95$, $|O| = 78$, and $|P \cup O| = 127$. Since $|P|$ contains the intersection $|P \cap O|$ and $|O|$ also contains the intersection $|P \cap O|$, that means there is an extra $|P \cap O|$ that is being counted. So, that means $|P \cup O| = |P| + |O| - |P \cap O|$. This is also clled 'The Union Formula'. So, using this formula, we have:

$$|P \cup O| = |P| + |O| - |P \cap O|$$

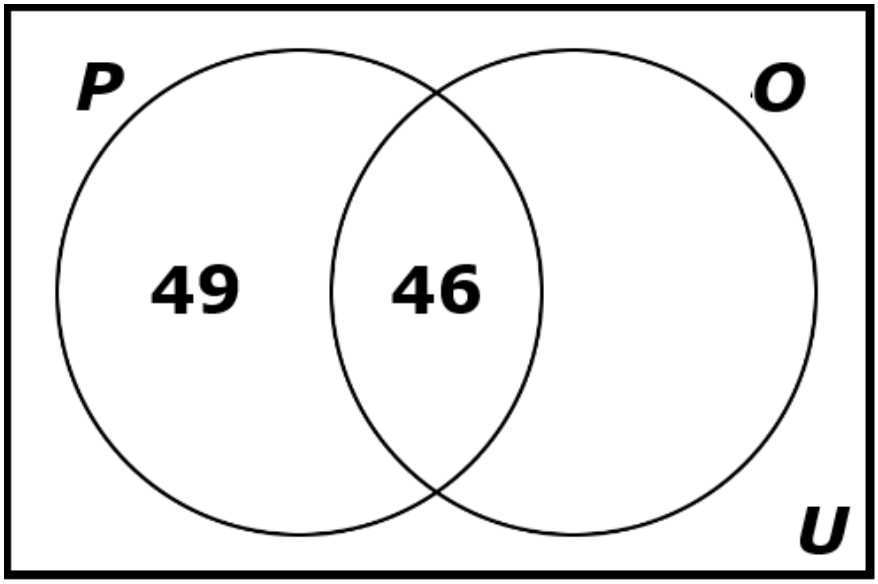
$$127 = 95 + 78 - |P \cap O|$$

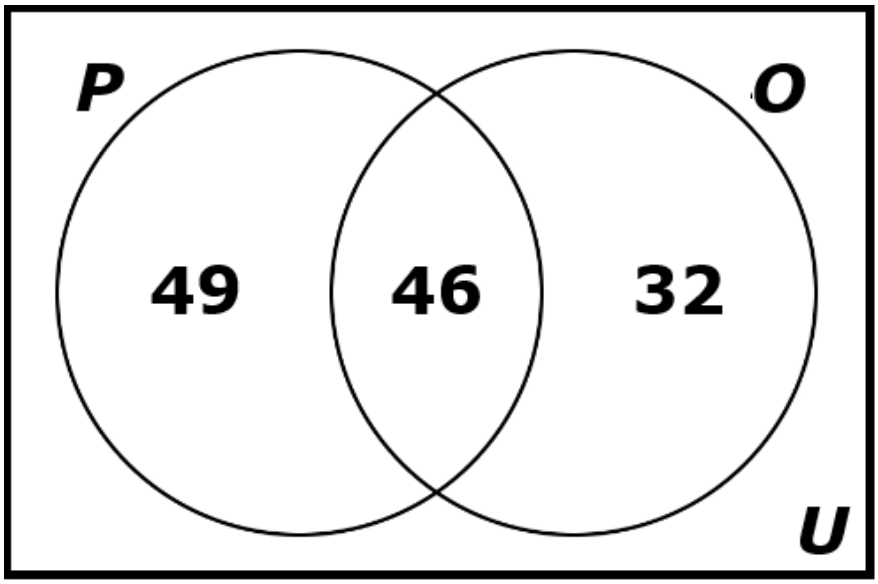
$$127 = 173 - |P \cap O|$$

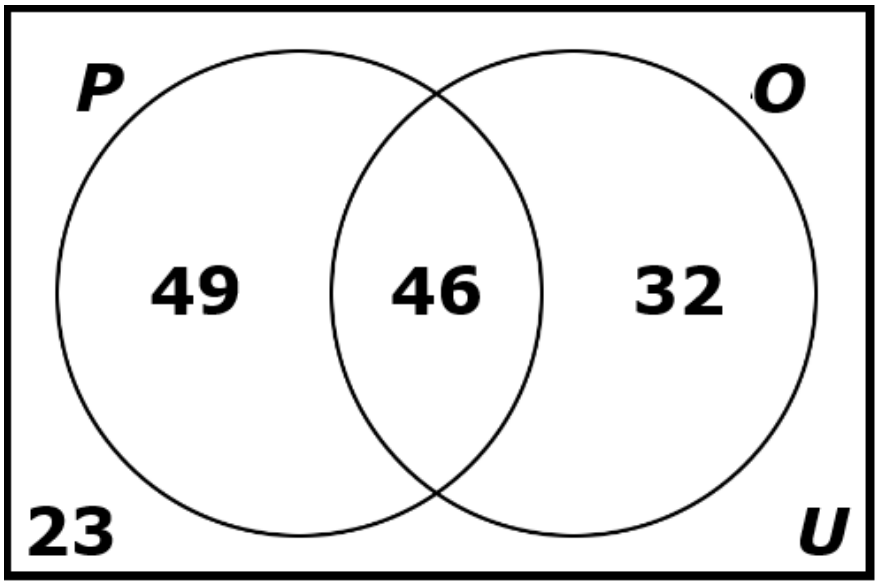
$$|P \cap O| = 46$$



Thus, we know 46 students purchased from both bookstores, so we write "46" in the overlapping region. Now, if 95 students total purchased from the physical bookstore, and 46 purchased from both bookstores, then 95 – 46 = 49 students purchased from the physical bookstore only (exclusively). We write "49" in the physical bookstore-only region. Next, if 78 students total purchased from the online bookstore, and 46 purchased from both bookstores, then 78 – 46 = 32 students purchased from the online bookstore only (exclusively). We write "32" in the online bookstore-only region.







Step 4: Answer the questions.

Now we can easily read off our answers:

- 49 students purchased from only the physical bookstore
- 32 students purchased from only the online bookstore
- 46 students purchased from both bookstores
- 23 students purchased from neither bookstore.

Common Pitfalls and How to Avoid Them

The most frequent mistake students make is double-counting, meaning placing the total number who belong to a set in the set's entire region rather than subtracting out the overlaps. Remember, when a problem says "65 adults subscribe to Netflix," that includes adults who also subscribe to Hulu. You must subtract the intersection to find the Netflix-only region. Another common error is working in the wrong order, trying to fill in outer regions before inner ones leads to confusion and errors. Always start with the innermost intersection and work outward. A third pitfall is forgetting the universal set. Students sometimes forget to account

for elements that belong to none of the sets, leaving them outside all circles but still within the universe. Finally, watch out for ambiguous language. Phrases such as "at least one," "exactly two," "only *A*," and "neither" all mean different things and correspond to different regions of your diagram.

Problem-Solving Strategies

When you encounter a Venn diagram problem, try these strategies to make the process smoother.

- Create a checklist: Before you start filling in numbers, list out every piece of information given in the problem and check them off as you use them. This ensures you don't miss anything.
- Use variables for unknowns: If you don't know a value, assign it a variable like x and write an equation based on the total. This turns the problem into a basic algebra equation that you can solve.
- Verify your work: After filling in all regions, add up all the numbers and make sure they equal your universal set total. If they don't, you've made an error somewhere.
- Draw it larger: Give yourself plenty of space to write numbers clearly in each region cramped diagrams lead to mistakes.
- Label everything: Don't rely on memory; write clear labels for what each circle represents and what the universe contains.

Example #2.6.1 🤔

A survey of 20 students were asked about their favorite subjects:

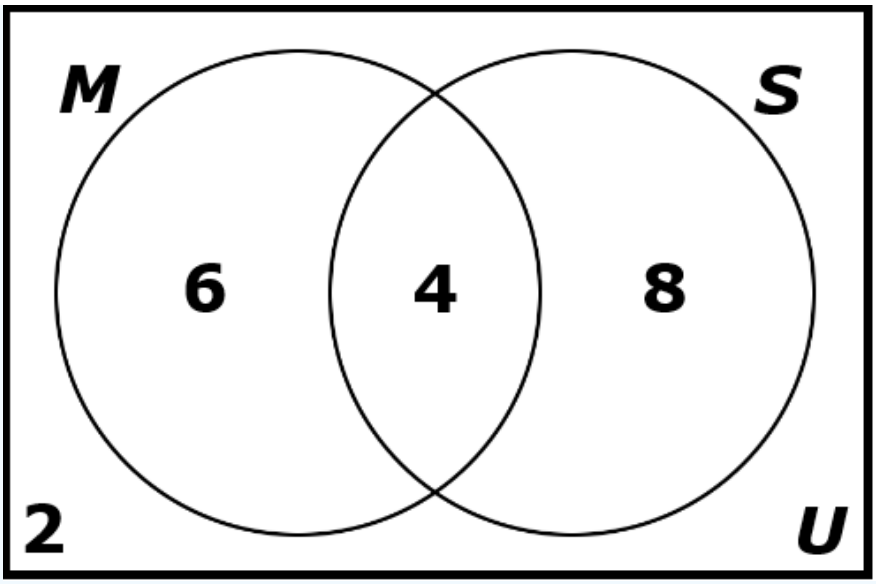
- 6 students like Math only
- 8 students like Science only
- 4 students like both Math and Science

Draw a Venn diagram to answer the following questions:

a. How many students like Math?
b. How many students like Science?
c. How many students liked neither Math or Science?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 1, "direct information" problem, because the numbers given in the problem can be placed directly into one or more regions of the Venn diagram without any additional calculations.
- **Step 2: Draw**
  - Draw and label two overlapping circles; one circle for Math and one circle for Science.
- **Step 3: Fill**
  - In the overlap (both) = 4
  - Math only = 6
  - Science only = 8
  - Add up everyone who liked at least one subject = 6 (Math only) + 8 (Science only) + 4 (both Math and Science) = 18
  - Total surveyed = 20
  - Neither = 20 – 18 = 2
- **Step 4: Answer**
  a. How many students like Math? Answer: 6 + 4 = 10
  b. How many students like Science? Answer: 8 + 4 = 12
  c. How many students liked neither Math or Science? Answer: 2



Example #2.6.2 🤔

A survey of 50 people were asked about pet ownership:

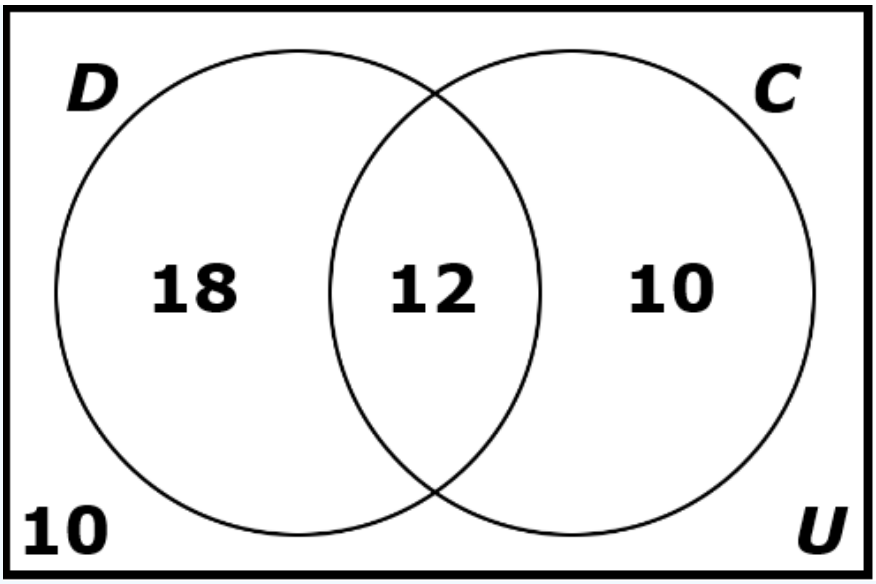
- 30 people have a dog
- 22 people have a cat
- 12 people have both a dog and a cat

Draw a Venn diagram to answer the following questions:

a. How many people own a dog, but do not own a cat?
b. How many people do not own either a dog or cat?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 2, "working from the center out" problem, because the count of both dog and cat are given.
- **Step 2: Draw**
  - Draw and label two overlapping circles; one circle for dogs and one circle for cats.
- **Step 3: Fill**
  - In the overlap (both) = 12
  - Dog only = 30 – 12 = 18
  - Cat only = 22 – 12 = 10
  - Add up everyone who owns a pet = 18 (dog only) + 10 (cat only) + 12 (both) = 40
  - Total surveyed = 50
  - Neither = 50 – 40 = 10
- **Step 4: Answer**
  a. How many people own a dog, but do not own a cat? Answer: 18
  b. How many people do not own either a dog or cat? Answer: 10



Example #2.6.3 🤔

A cafeteria surveyed 100 customers about their breakfast orders:

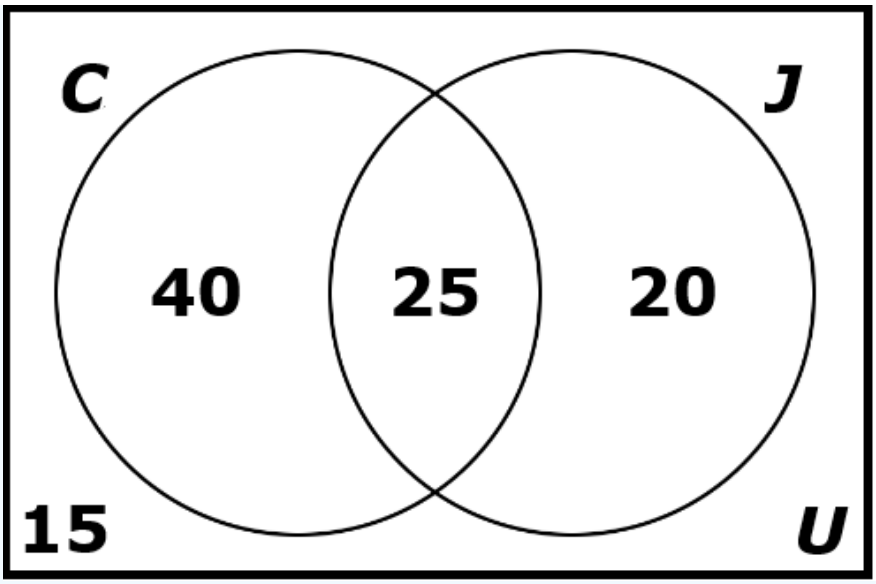
- 65 ordered coffee
- 45 ordered juice
- 25 ordered both coffee and juice

Draw a Venn diagram to answer the following questions:

a. How many customers only ordered coffee?
b. How many customers only ordered juice?
c. How many customers ordered at least one beverage?
d. How many customers ordered neither beverage?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 2, "working from the center out" problem, because the count of both coffee and juice are given.
- **Step 2: Draw**
  - Draw and label two overlapping circles; one circle for coffee and one circle for juice.
- **Step 3: Fill**
  - In the overlap (both) = 25
  - Coffee only: 65 – 25 = 40
  - Juice only: 45 – 25 = 20
  - Add up everyone who ordered a beverage = 40 (coffee only) + 20 (juice only) + 25 (both) = 85
  - Total surveyed = 100
  - Neither = 100 – 85 = 15
- **Step 4: Answer**
  a. How many customers only ordered coffee? Answer: 40
  b. How many customers only ordered juice? Answer: 20
  c. How many customers ordered at least one beverage? Answer: 85
  d. How many customers ordered neither beverage? Answer: 15



Example #2.6.4 🤔

A survey of 120 moviegoers found:

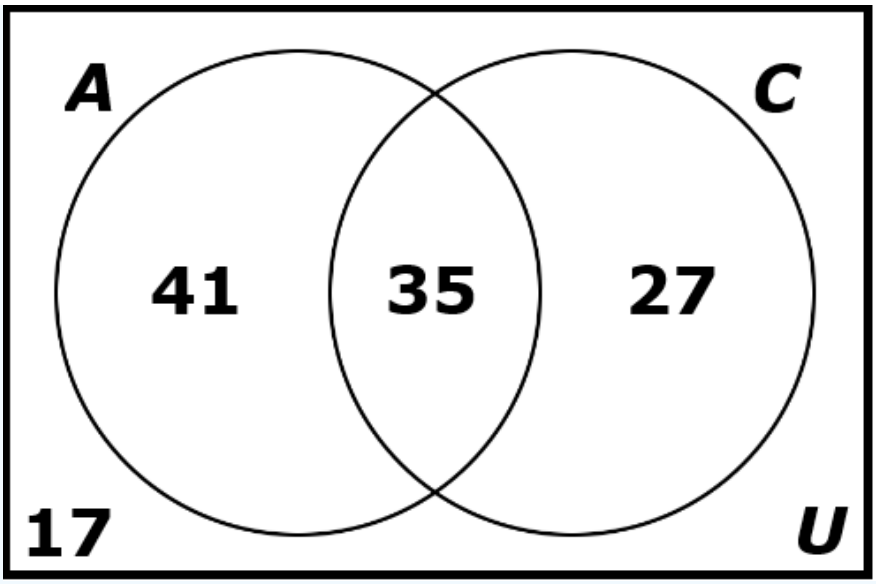
- 76 people like action movies
- 35 people like both action and comedies
- 103 people like at least one of these genres

Draw a Venn diagram to answer the following questions:

a. How many moviegoers liked action movies only?
b. How many moviegoers liked comedy movies?
c. How many moviegoers liked neither genre?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 2, "working from the center out" problem, because the count of both action and comedies are given.
- **Step 2: Draw**
  - Draw and label two overlapping circles; one circle for action and one circle for comedies.
- **Step 3: Fill**
  - In the overlap (both): 35
  - Action only: 76 – 35 = 41
  - At least like one movie = 103
  - Comedy only = 103 – 76 = 27
  - Total surveyed = 120
  - Neither = 120 – 103 = 17
- **Step 4: Answer**
  a. How many moviegoers liked action movies only? 41
  b. How many moviegoers liked comedy movies? Answer: 35 + 27 = 62
  c. How many moviegoers liked neither genre? Answer: 17



Example #2.6.5 🤔

A survey of 250 music listeners found:

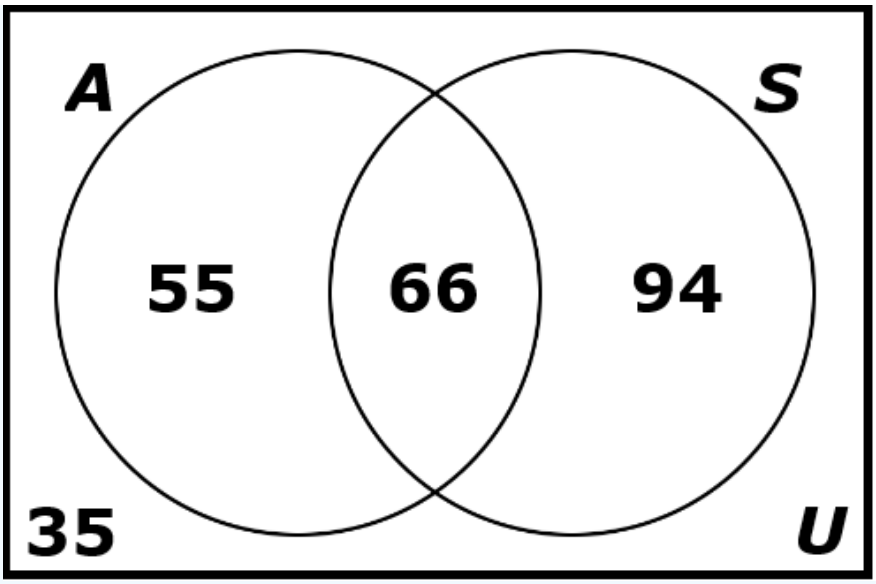
- 215 use Apple Music or Spotify or both
- 160 use Spotify
- 66 use both Apple Music and Spotify

Draw a Venn diagram to answer the following questions:

a. How many music listeners use Spotify only?
b. How many music listeners use Apple Music?
c. How many music listeners do not use Spotify or Apple Music?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 2, "working from the center out" problem, because the count of both Spotify and Apple Music are given.
- **Step 2: Draw**
  - Draw and label two overlapping circles; one circle for Spotify and one circle for Apple Music.
- **Step 3: Fill**
  - In the overlap (both): 66
  - Spotify only: 160 – 66 = 94
  - At least use one platform = 215
  - Apple Music only = 215 – 160 = 55
  - Total surveyed = 250
  - Neither = 250 – 215 = 35
- **Step 4: Answer**
  a. How many music listeners use Spotify only? Answer: 94
  b. How many music listeners use Apple Music? Answer: 66 + 55 = 121
  c. How many music listeners do not use Spotify or Apple Music? 35



Example #2.6.6 🤔

Out of 200 college students:

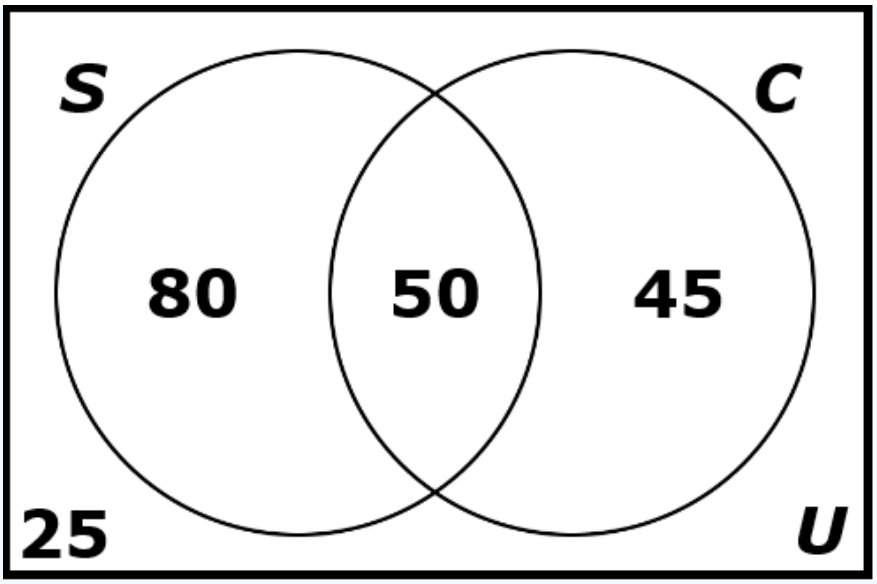
- 130 participate in sports
- 95 participate in clubs
- 175 participate in at least one activity

Draw a Venn diagram to answer the following questions:

a. How many college students do not participate in either sports or clubs?
b. How many college students participate in both sports and clubs?
c. How many college students participate in exactly one activity?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 3, "solving for the intersection" problem, because the count for both sports and clubs are not given.
- **Step 2: Draw**
  - Draw and label two overlapping circles; one circle for sports and one clubs for comedies.
- **Step 3: Fill**
  - At least participate in one activity = 175
  - Total sports and totals clubs = 130 + 95 = 225 (This total is counting the intersection twice).
  - In the overlap (both sports and clubs) = 225 – 175 = 50
  - Sports only = 130 – 50 = 80
  - Action only = 95 – 50 = 45
  - Neither = 200 – 175 = 25
- **Step 4: Answer**
  a. How many college students do not participate in either sports or clubs? Answer: 25
  b. How many college students participate in both sports and clubs? Answer: 50
  c. How many college students participate in exactly one activity? Answer: 80 + 45 = 125



Section 2.6: Drawing Basic Venn Diagrams for Two Sets [*In-Class Exercises*]

For Exercises #1 through #6, draw a Venn diagram to answer all questions:

1. In a class of 25 students,
    - 10 speak only Spanish,
    - 8 speak only French, and
    - 3 speak both Spanish and French.

    a. How many total students speak Spanish?
    b. How many total students speak French?
    c. How many total students speak neither Spanish nor French?

2. Among 40 readers,
    - 13 read only mystery novels,
    - 9 read only science fiction novels, and
    - 6 read both genres.

    a. How many total readers like science fiction?
    b. How many total readers like mystery novels?
    c. How many total readers like neither genre?
    d. How many readers like at least one genre?

3. Among 60 gym members,
    - 35 use the elliptical machine,
    - 42 use the treadmill, and
    - 26 use both an elliptical machine and treadmill.

    a. How many members use only the elliptical machine?
    b. How many members use only the treadmill?
    c. How many members use exactly one of the machines?
    d. How many members do not use either machine?

4. A survey of 100 people was conducted. The people were asked whether or not they watched the final season of the series Stranger Things. The results were as follows:
    - 72 people are currently in high school,

- 79 people watched the final season of Stranger Things, and
- 65 high schoolers watched the final season of Stranger Things.

a. How many people watched the final season of Stranger Things that were not high schoolers?
b. How many high schoolers did not watch the final season of Stranger Things?
c. How many people were not high schoolers and also did not watch the final season of Stranger Things?

5. In a college of survey of 200 students,
   - 110 are majoring in Business,
   - 80 are majoring in Psychology, and
   - 170 students are majoring in at least one of these two fields.

   a. How many students are a double major in both Business and Psychology?
   b. How many students are strictly Business majors?
   c. How many students are strictly Psychology majors?

6. In 2026, a survey regarding cell-phone applications was conducted. Out of 120 people,
   - 90 use Tik-Tok,
   - 2 use MySpace, and
   - 91 use at least one platform.

   a. How many students use both platforms?
   b. How many students use only Tik-Tok?
   c. How many students use only MySpace?
   d. What percentage of students do not use either platform?

Answers available at the end of this section.

Answers

Section 2.6: Drawing Basic Venn Diagrams for Two Sets [*In-Class Exercises*]

1. a. 13
   b. 11
   c. 4
2. a. 15
   b. 19
   c. 12
   d. 28
3. a. 9
   b. 16
   c. 25
   d. 9
4. a. 14
   b. 7
   c. 14
5. a. 20
   b. 90
   c. 60
6. a. 1
   b. 89
   c. 1
   d. $\frac{29}{120} = 0.24166666666667 \approx 24.2\%$

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Section 2.6.H: Homework

Section 2.6: Applications Using Two-Circle Venn Diagrams - Homework Exercises

(Section 2.6) - Homework Exercise #1

A coffee shop surveyed 200 customers about their preferences for two add-ins: cream and sugar.

- 45 customers use both cream and sugar
- 60 customers use only cream
- 55 customers use only sugar
- The remaining customers use neither

Draw a Venn diagram to answer the following questions:

a. How many customers use at least one add-in?
b. How many customers use cream in total?
c. How many customers use neither cream nor sugar?
d. How many customers use sugar?

Answer available at the end of this section.

(Section 2.6) - Homework Exercise #2

A community center tracked 150 volunteers and their participation in two programs, tutoring and meal delivery.

- 36 volunteers participate in both programs
- 62 volunteers participate only in tutoring
- 51 volunteers participate only in meal delivery
- The rest participate in neither program

Draw a Venn diagram to answer the following questions:

a. How many volunteers participate in at least one program?
b. How many volunteers participate in tutoring in total?
c. How many volunteers participate in exactly one program?
d. How many volunteers participate in neither program?

Answer available at the end of this section.

(Section 2.6) - Homework Exercise #3

At a grocery store,

- 25 shoppers bought both produce and dairy products
- 68 shoppers bought produce
- 95 shoppers bought produce or dairy products

Draw a Venn diagram to answer the following questions:

a. How many shoppers bought dairy products?
b. How many shoppers bought produce, but not dairy products?

Answer available at the end of this section.

(Section 2.6) - Homework Exercise #4

At a pet store,

- 8 customers bought both dog food and cat food
- 27 customers bought dog food
- 41 customers bought dog food or cat food
- 11 customers bought neither

Draw a Venn diagram to answer the following questions:

a. How many customers bought cat food?
b. How many customers bought only dog food?
c. How many customers in all were in the pet store?

Answer available at the end of this section.

(Section 2.6) - Homework Exercise #5

A survey of 50 households examined subscriptions to two magazines: Home & Garden and Cooking Monthly.

- 25 households subscribe to Home & Garden
- 23 households subscribe to Cooking Monthly
- 12 households subscribe to both magazines

Draw a Venn diagram to answer the following questions:

a. How many households subscribe to at least one magazine?
b. How many households subscribe to Home & Garden but not Cooking Monthly?
c. How many households subscribe to exactly one magazine?
d. How many households subscribe to neither magazine?
e. What percentage of households subscribe to both magazines?

Answer available at the end of this section.

(Section 2.6) - Homework Exercise #6

At a coffee shop,

- 35 customers use milk
- 17 customers use sweetener
- 12 customers use both milk and sweetener in their coffee
- 5 customers use neither

Draw a Venn diagram to answer the following questions:

a. How many customers use sweetener, but not milk?
b. How many customers in total were there?

Answer available at the end of this section.

(Section 2.6) - Homework Exercise #7

A market research study examined 100 consumers' use of two food delivery apps: DoorDash and Uber Eats.

- 42 consumers use DoorDash
- 52 consumers use Uber Eats
- 78 consumers use at least one of these apps

Draw a Venn diagram to answer the following questions:

a. How many consumers use both apps?
b. How many consumers use only DoorDash?
c. How many consumers use only Uber Eats?
d. How many consumers use neither app?
e. What percentage of consumers use both apps?

Answer available at the end of this section.

(Section 2.6) - Homework Exercise #8

A survey of 67 smartphone users found:

- 45 users have a camera with night mode
- 31 users have wireless charging capability
- 61 users have at least one of these features

Draw a Venn diagram to answer the following questions:

a. How many users have both features?
b. How many users have only night mode?
c. How many users have only wireless charging?
d. How many users have neither feature?

Answer available at the end of this section.

(Section 2.6) - Homework Exercise #9

In a group of international business professionals:

- 58% speak Spanish
- 65% speak Mandarin
- 15% speak neither language
- There are 400 professionals in the group

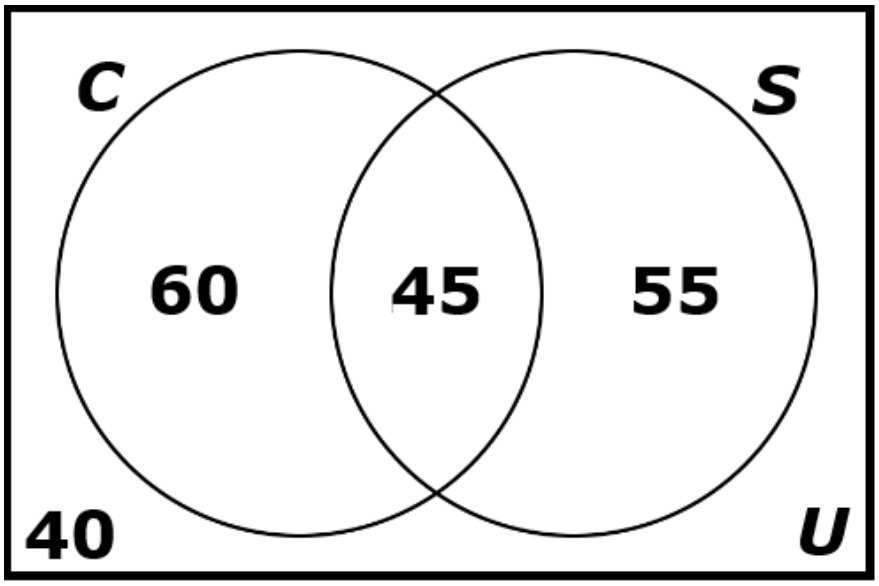
Draw a Venn diagram to answer the following questions:

a. What percentage speak both languages?
b. How many professionals speak both languages?
c. How many professionals speak only Spanish?
d. How many professionals speak exactly one language?

Answer available at the end of this section.

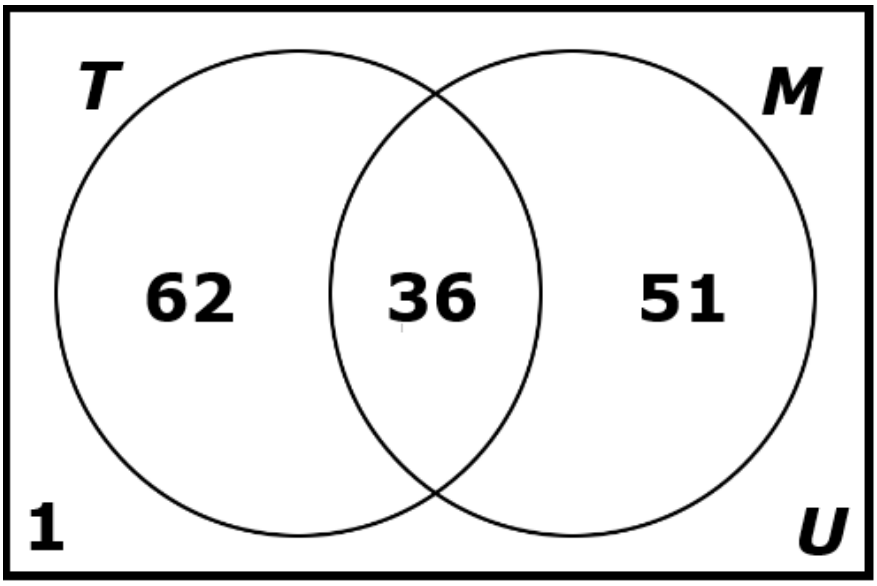
Answers

(Section 2.6) - Homework Exercise #1



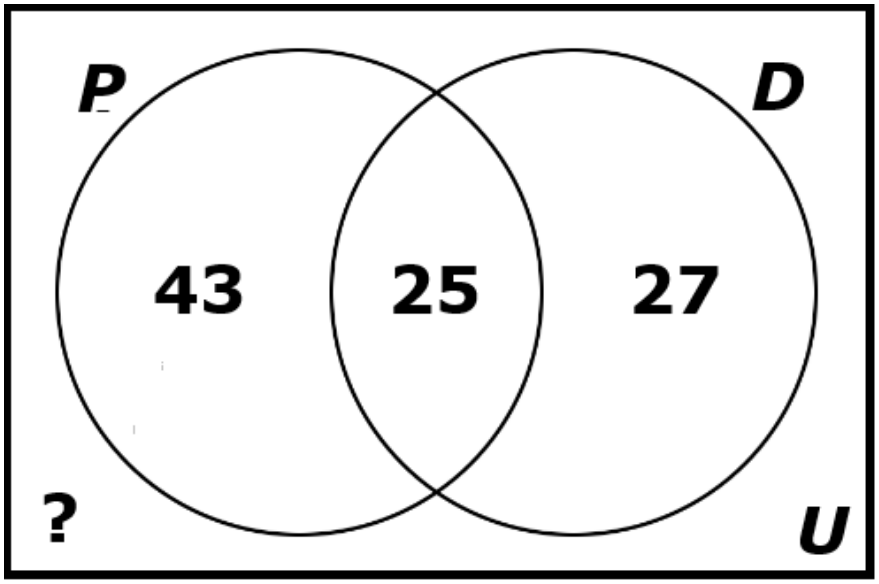
a. 160
b. 105
c. 40
d. 100

(Section 2.6) - Homework Exercise #2



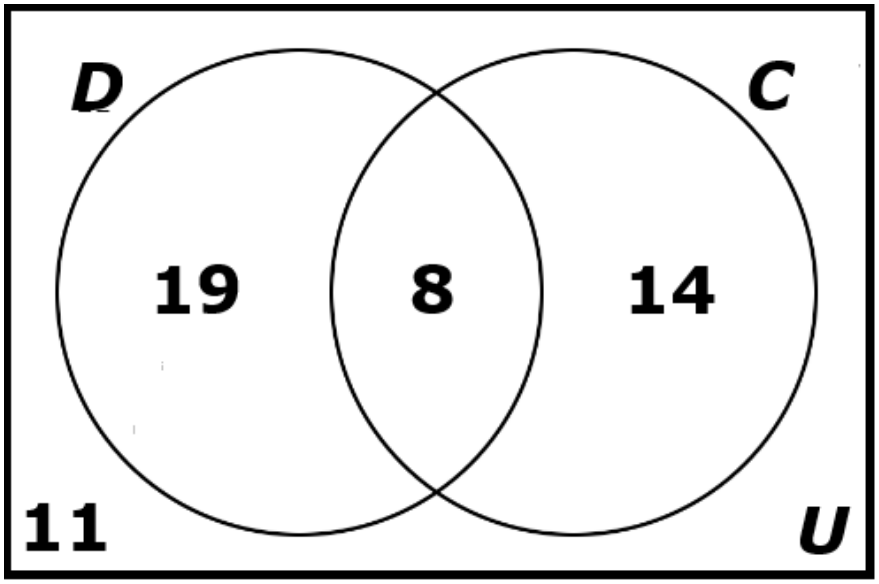
a. 149
b. 98
c. 113
d. 1

(Section 2.6) - Homework Exercise #3



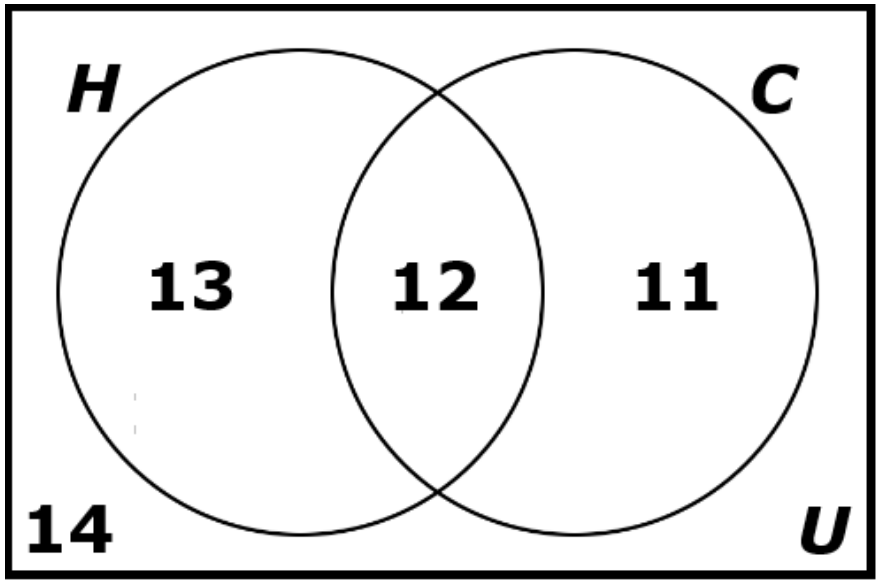
a. 52
b. 43

(Section 2.6) - Homework Exercise #4



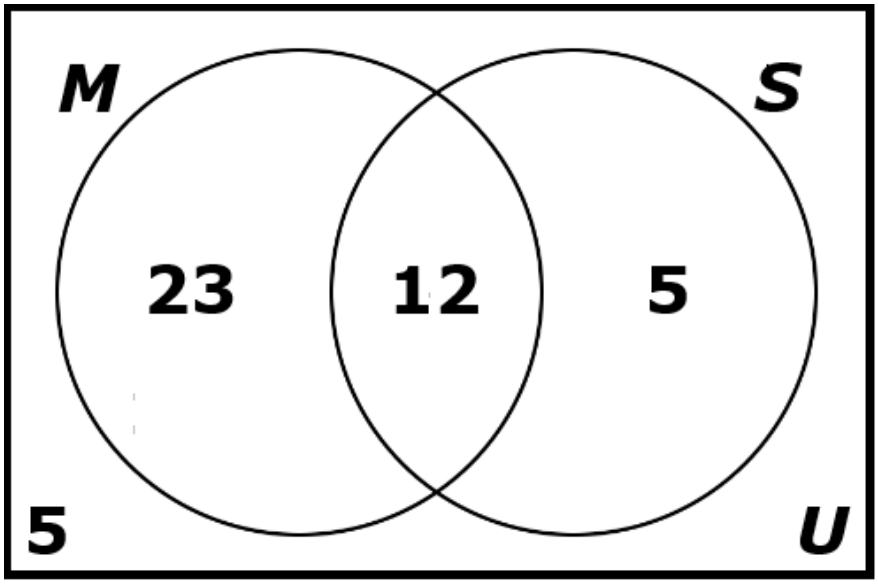
a. 22
b. 19
c. 52

(Section 2.6) - Homework Exercise #5



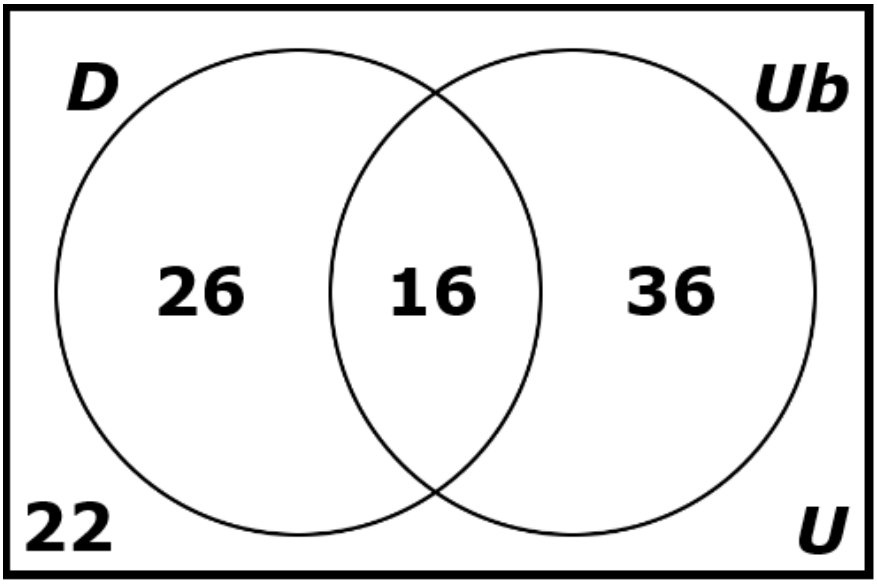
a. 36
b. 13
c. 24
d. 14
e. 12/50 = 24%

(Section 2.6) - Homework Exercise #6



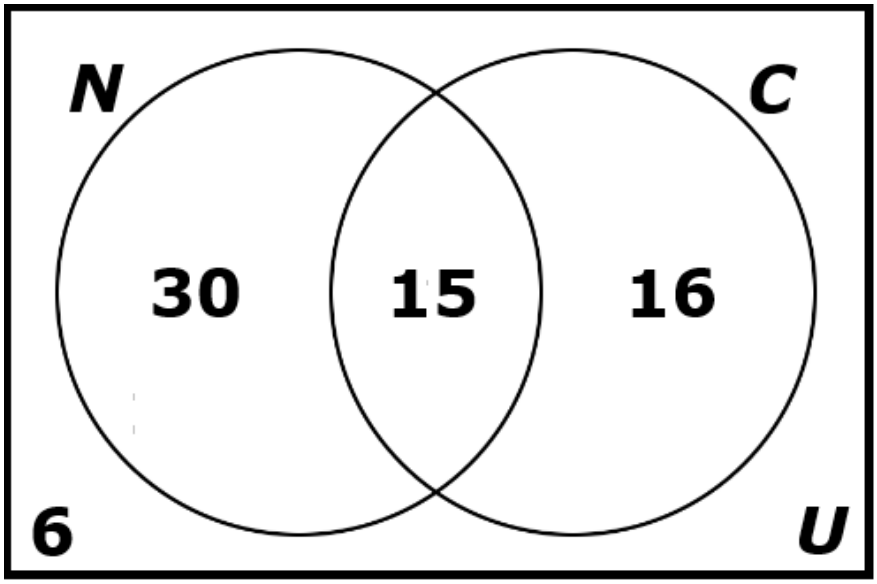
a. 5
b. 45

(Section 2.6) - Homework Exercise #7



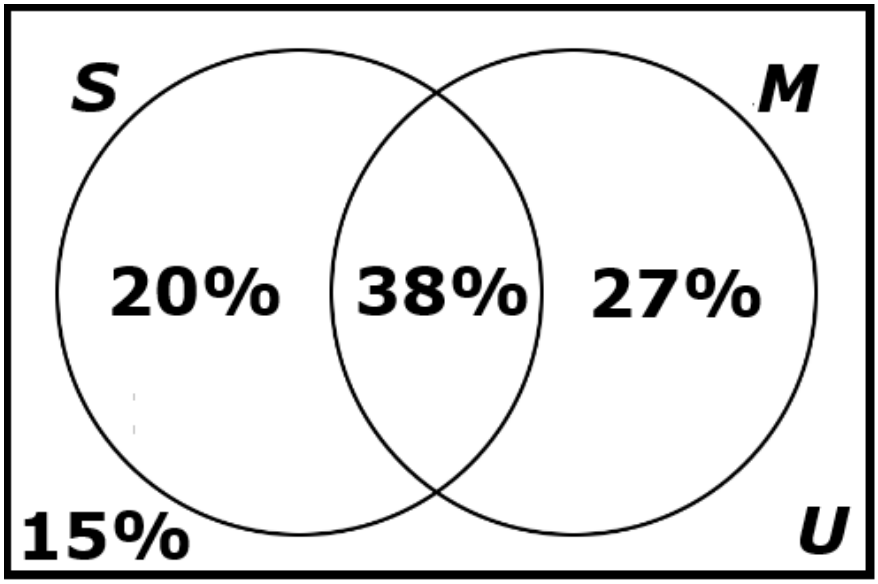
a. 16
b. 26
c. 36
d. 22
e. 16/100 = 16%

(Section 2.6) - Homework Exercise #8



a. 15
b. 30
c. 16
d. 6

(Section 2.6) - Homework Exercise #9



a. 38%
b. 38% of 400 = 152
c. 20% of 400 = 80
d. 47% of 400 = 188

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Section 2.7: Applications Using Three-Circle Venn Diagrams

Learning Objectives

- Solve problems using three-circle Venn Diagrams

In this section, we extend our study of Venn diagrams to problems involving three sets. While the fundamental principles remain the same, three-circle diagrams introduce additional complexity due to the increased number of regions and intersections. Understanding how to systematically approach these problems is essential for accurate problem-solving.

Types of Three-Circle Problems

Three-circle Venn diagram problems generally fall into two categories based on the type of information provided and the solution strategy required.

Type 1: Direct Information Problems

In these problems, the number of elements in each region of the diagram is given explicitly. The information provided directly corresponds to the distinct regions of the Venn diagram, allowing us to fill in each section without additional calculations.

Type 2: Working from the Center Out

In these problems, we are given the total number of elements in each set and the totals for various intersections (including those with all three sets). We must use the principle of inclusion-exclusion to determine the number of elements in each distinct region, beginning with the innermost region (the intersection of all three sets) and systematically working outward to the individual set regions.

The distinction between these two problem types is crucial, as each requires a different approach to reach the solution. In the following sections, we will examine both types in detail and develop systematic methods for solving each.

Example #1 (Type 1: Direct Information, Easy)

A community center surveyed 200 residents about their participation in three recreation programs: swimming, tennis, and yoga. The survey results showed:

- 18 residents participate in all three programs
- 22 residents participate in swimming and tennis only (not yoga)
- 15 residents participate in swimming and yoga only (not tennis)
- 12 residents participate in tennis and yoga only (not swimming)
- 35 residents participate in only swimming
- 28 residents participate in only tennis
- 20 residents participate in only yoga
- The remaining residents participate in none of these programs

Question: Draw a Venn diagram for the above information.

Step 1: Identify what makes this a "Direct Information" problem.

Notice that the problem provides the count for each distinct region of the Venn diagram:

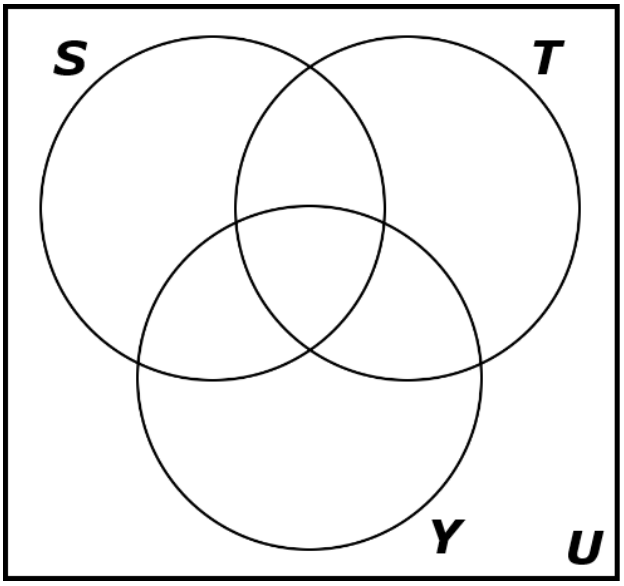
- "All three programs" → center region; intersection
- "Swimming and tennis only" → two-set intersection excluding the third set
- "Only swimming" → single-set region with no overlaps

Each piece of information corresponds to exactly one region in our diagram. We do not need to subtract or calculate; we simply place each number in its corresponding location.

Step 2: Draw the diagram.

Let *S* = swimming, *T* = tennis, *Y* = yoga

We draw three overlapping circles inside a square. The square represents all 200 residents, the top left circle is residents that participate in swimming (*S*), the top right circle is residents that participate in tennis (*T*), and the bottom circle is residents that participate in yoga (*Y*). There are three overlapping regions that represent residents that participate in exactly 2 programs (*ST*, *SY*, & *TY*) and the overlapping region in the middle represents residents that participate in all 3 programs (*STY*).

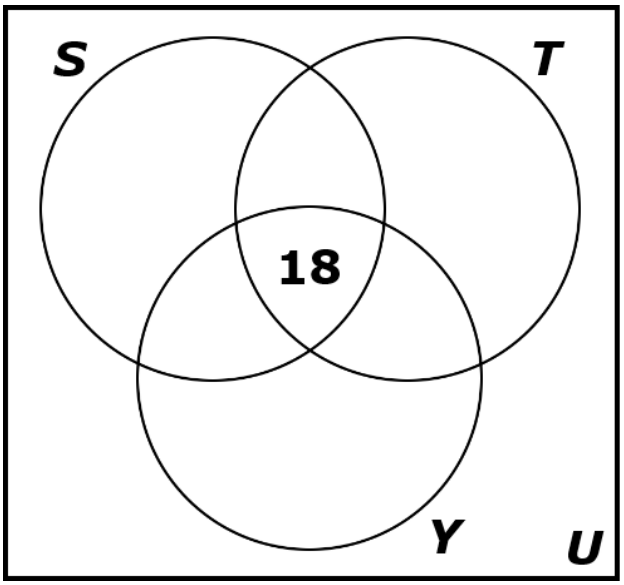


Step 3: Fill in the information strategically.

Since each piece of information corresponds to exactly one region in our diagram, we just place each number in its corresponding location.

Center:

Start with the intersection, if applicable. (All of the problems that we will see will be able to start with the intersection). We know 18 residents participate in all three programs, so we write "18" in the center overlapping region of the three circles. (Just like in Section 2.6, for simplicity, let's use a shortcut notation, $STY = 18$, instead of $|S \cap T \cap Y| = 18$, when discussing the cardinal numbers). So, use *STY* = 18.

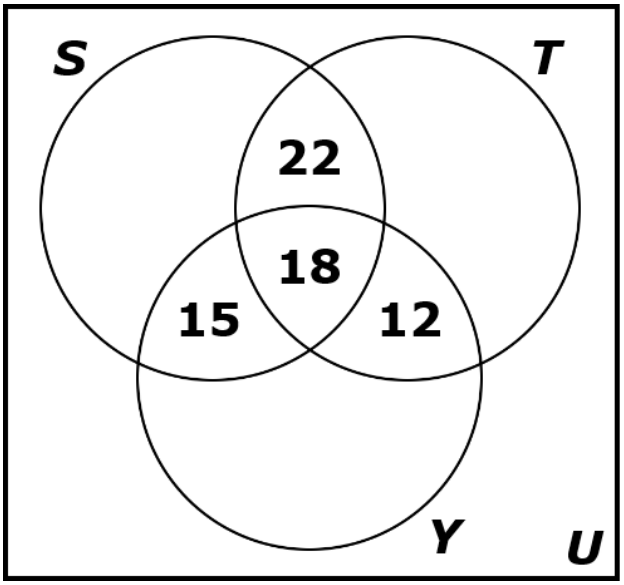


Two-set intersections only:

There are 22 residents that participate in swimming and tennis only (not yoga). We write "22" in the overlapping region of swimming and tennis only. (Shortcut notation, use *ST* only = 22).

There are 15 residents that participate in swimming and yoga only (not tennis). We write "15" in the overlapping region of swimming and yoga only. (Shortcut notation, use *SY* only = 15).

There are 12 residents that participate in tennis and yoga only (not swimming). We write "12" in the overlapping region of tennis and yoga only. (Shortcut notation, use *TY* only = 12).



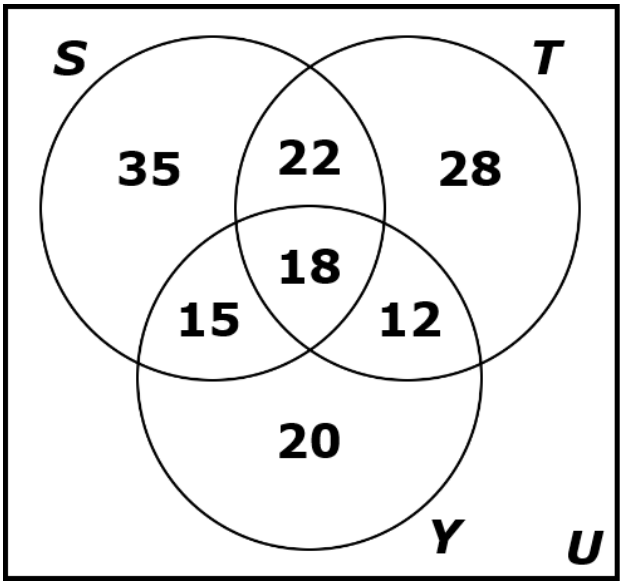
Single sets only:

There are 35 residents that participate in swimming only. We write "35" in the non-overlapping region of swimming only. (Shortcut notation, use *S* only = 35).

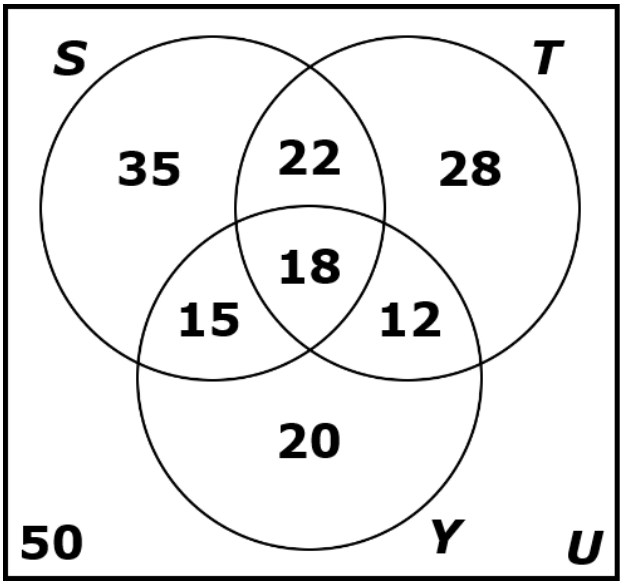
There are 28 residents that participate in tennis only. We write "28" in the non-overlapping region of tennis only. (Shortcut

notation, use *T* only = 28).

There are 20 residents that participate in yoga only. We write "20" in the non-overlapping region of yoga only. (Shortcut notation, use *Y* only = 20).



Finally, we need to find how many residents do not participate in any of the three programs. We add up all the residents we've accounted for in the seven regions of the Venn diagram: 35 + 22 + 28 + 15 + 18 + 12 + 20 = 150 residents that participate in at least one program. Since there are 200 residents in total, then there are 200 – 150 = 50 residents that did not participate in any of the three programs. We write "50" outside the three circles, but inside the square.



Step 4: Answer the questions.

Now we can answer all types of questions, such as:

- How many residents participate in at least one program? 35 + 22 + 28 + 15 + 18 + 12 + 20 = 150
- How many residents participate in exactly one program? 35 + 28 + 20 = 83
- How many residents participate in exactly two programs? 22 + 15 + 12 = 49
- How many residents participate in Yoga, but not in Tennis? 15 + 20 = 35

- How many residents did not participate in any of the three programs? 50

Example #2 (Type 2: Working from the Center Out, Moderately Easy)

A survey of 100 students asked about their participation in three clubs:

- 46 students are in drama
- 43 students are in debate
- 32 students are in music
- 22 students are in drama and debate
- 18 students are in drama and music
- 20 students are in debate and music
- 15 students are in drama, debate, and music (all three)

Question: Draw a Venn diagram for the above information.

Step 1: Identify what makes this a "Working from the Center Out" problem.

Notice that none of the information uses the word "only". This tells us that:

- The 46 students that are in drama implies that some students are in drama only and some students are in drama and maybe at least one other club. The same logic will apply to the 43 students in debate as well as the 32 students are in music.
- The 22 students that are in drama and debate implies that some students are in drama and debate only and some students may be in all three clubs. The same logic will apply to the 18 students in drama and music as well as the 20 students in debate and music.

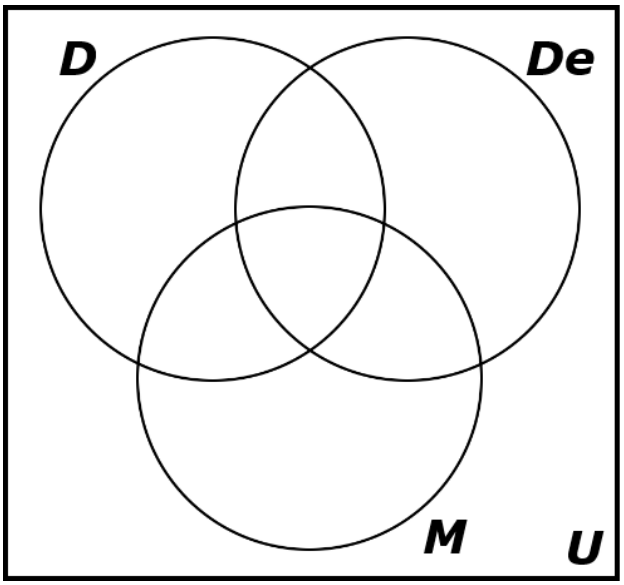
The given numbers represent totals that overlap, not distinct regions. We must calculate each distinct region by subtracting.

Step 2: Draw the diagram.

Let D = drama, De = debate, M = music. (The use of a single letter when defining sets is common for simplicity, but it is not a formal requirement. So, you can label a set as De if you wish, as long as it is clearly defined within the context of your work).

We draw three overlapping circles inside a square. The square represents all 100 students, the top left circle is students that participate in drama (D), the top right circle is students that participate in debate (De), and the bottom circle is students that participate in music (M). There are three overlapping regions that represent students that participate in exactly 2 clubs (DDe, DM, & DeM) and the overlapping region in the middle represents students that participate in all 3 clubs ($DDeM$)

Two-set Venn diagram with highlighted region II.

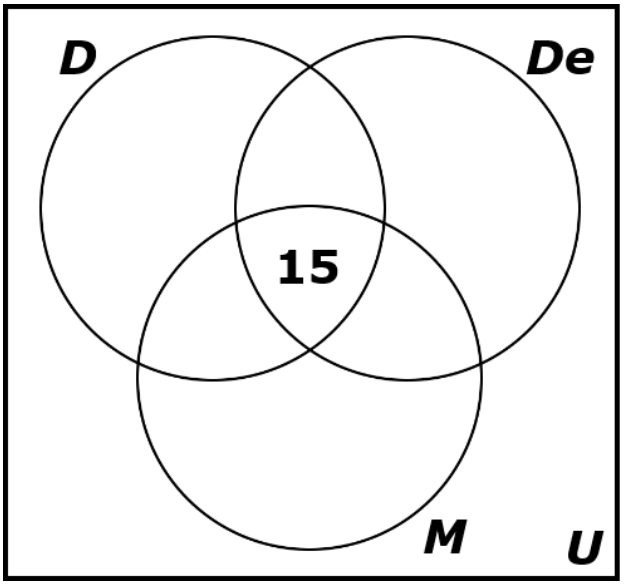


Step 3: Fill in the information strategically.

Since, most of the given numbers represent totals that overlap and not distinct regions, then we must calculate each distinct region by subtracting.

Center:

Start with the intersection, if applicable. (All of the problems that we will see in this text will be able to start with the intersection). We know 15 students participate in all three clubs, so we write "15" in the center overlapping region of the three circles. (Shortcut notation, use *DDeM* = 15).

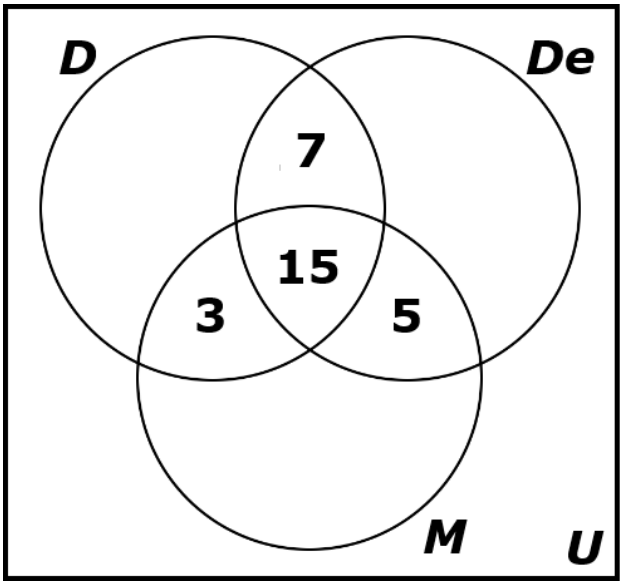


Two-set intersections only:

There are 22 students that participate in drama and debate, but 15 of those students participate in all three clubs, then there are 22 – 15 = 7 students that participate in drama and debate only. We write "7" in the overlapping region of drama and debate only. (Shortcut notation, use *DDe* = 22).

There are 18 students that participate in drama and music, but 15 of those students participate in all three clubs, then there are 18 – 15 = 3 students that participate in drama and music only. We write "3" in the overlapping region of drama and music only. (Shortcut notation, use *DM* = 18).

There are 20 students that participate in debate and Music, but 15 of those students participate in all three clubs, then there are 20 – 15 = 5 students that participate in debate and music only. We write "5" in the overlapping region of debate and music only.(Shortcut notation, use *DeM* = 20).

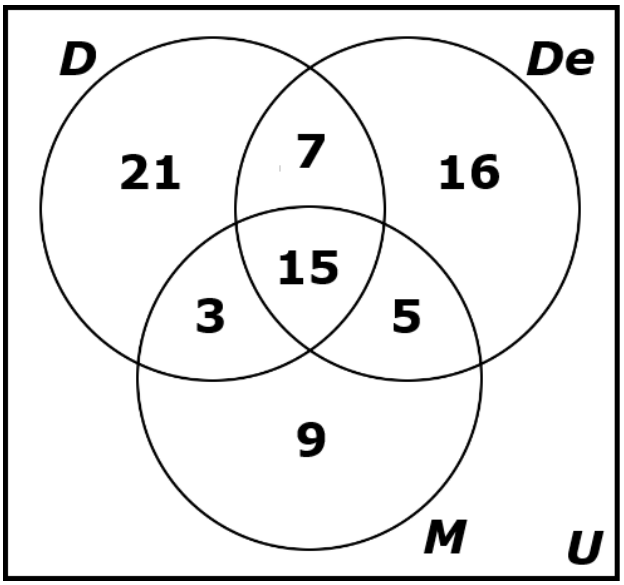


Single sets only:

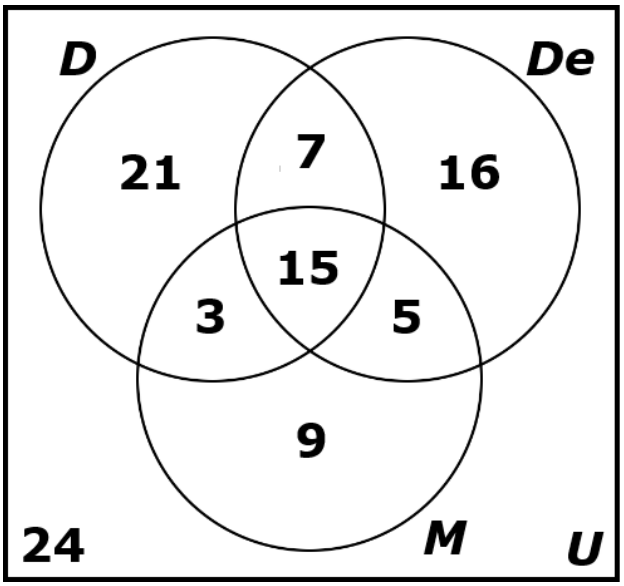
There are 46 students that participate in drama, but 15 of those students participate in all three clubs, 7 of those students also participate in debate and not music, and 3 also participate in music and not debate, then that means there are 46 – 15 – 7 – 3 = 21 students that participate in drama only. We write "21" in the non-overlapping region of drama. (Shortcut notation, use *D* = 46).

There are 43 students that participate in debate, but 15 of those students participate in all three clubs, 7 of those students also participate in drama and not music, and 5 also participate in music and not drama, then that means there are 43 – 15 – 7 – 5 = 16 students that participate in debate only. We write "16" in the non-overlapping region of debate.(Shortcut notation, use *De* = 43).

There are 32 students that participate in music but 15 of those students participate in all three clubs, 3 of those students also participate in drama and not debate, and 5 also participate in debate and not drama, then that means there are 32 – 15 – 3 – 5 = 9 students that participate in music only. We write "9" in the non-overlapping region of music. (Shortcut notation, use *M* = 32).



Finally, we need to find how many students did not participate in any of the three clubs. We add up all the students we've accounted for in the seven regions of the Venn diagram: 21 + 7 + 16 + 3 + 15 + 5 + 9 = 76 students that participate in at least one club. Since there are 100 students in total, then there are 100 – 76 = 24 students that did not participate in any of the three clubs. We write "24" outside the three circles, but inside the square.



Step 4: Answer the questions.

Now we can answer all types of questions, such as:

- How many students participate in at least one club? 21 + 7 + 16 + 3 + 15 + 5 + 9 = 76
- How many students participate in exactly one club? 21 + 16 + 9 = 46
- How many students participate in exactly two clubs? 7 + 3 + 5 = 15
- How many students participate in drama, but not debate? 21 + 3 = 24
- How many students do not participate in any club? 24

Example #2.7.1 🤔

A diner surveyed 80 customers about their breakfast orders:

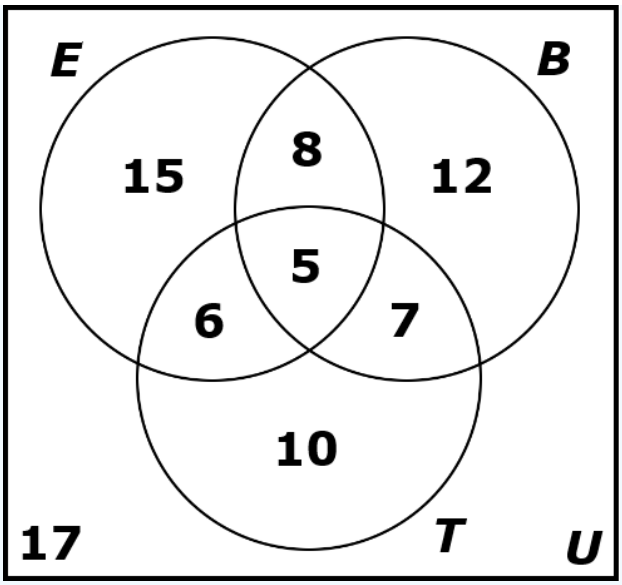
- 5 ordered eggs, bacon, and toast (all three)
- 8 ordered eggs and bacon only
- 6 ordered eggs and toast only
- 7 ordered bacon and toast only
- 15 ordered only eggs
- 12 ordered only bacon
- 10 ordered only toast
- The rest ordered none of these items

Draw a Venn diagram to answer the following questions:

a. How many ordered at least one of these items?
b. How many ordered eggs in total?
c. How many ordered exactly one item?
d. How many ordered exactly two items?
e. How many ordered none of the three items?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 1, "direct information" problem , because the numbers given in the problem can be placed directly into one or more regions of the Venn diagram without any additional calculations.
- **Step 2: Draw**
  - Draw and label three overlapping circles; one circle for eggs, one circle for bacon, and one circle for toast.
- **Step 3: Fill**
  - Eggs, bacon, and toast (all three) = 5
  - Eggs and bacon only = 8
  - Eggs and toast only = 6
  - Bacon and toast only = 7
  - Eggs only = 15
  - Bacon only = 12
  - Toast only = 10
  - Add up everyone who ordered at least one item = 10 + 12 + 15 + 7 + 6 + 8 + 5 = 63
  - Total surveyed = 80
  - Neither any of the three items = 80 − 63 = 17
- **Step 4: Answer**
  a. How many ordered at least one of these items? Answer: 63
  b. How many ordered eggs in total? Answer: 15 + 8 + 5 + 6 = 34
  c. How many ordered exactly one item? Answer: 15 + 12 + 10 = 37
  d. How many ordered exactly two items? Answer: 8 + 6 + 7 = 21
  e. How many ordered none of the three items? Answer: 17



Example #2.7.2 🤔

A coffee shop tracked 300 orders during the morning rush:

- 86 orders only coffee
- 54 orders only a pastry
- 29 orders only a breakfast sandwich
- 31 orders both coffee and a pastry only
- 24 orders both coffee and a sandwich only
- 16 orders both a pastry and a sandwich only
- 40 orders included all three items

Draw a Venn diagram to answer the following questions:

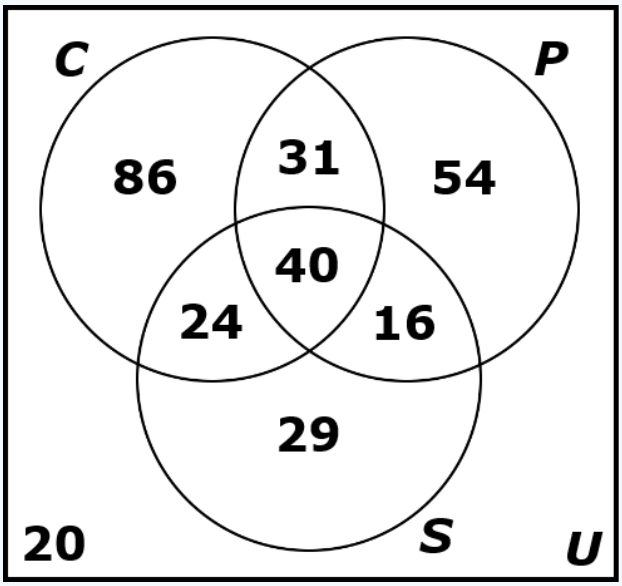
a. How many orders had coffee and a pastry, but not a sandwich?
b. How many orders had coffee?
c. How many orders had exactly two items?
d. How many orders had at least one item?
e. How many orders had none of these items?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 1, "direct information" problem , because the numbers given in the problem can be placed directly into one or more regions of the Venn diagram without any additional calculations
- **Step 2: Draw**
  - Draw and label three overlapping circles; one circle for coffee, one circle for pastry, and one circle for breakfast sandwich.
- **Step 3: Fill**
  - Coffee only = 86
  - Pastry only = 54
  - Breakfast sandwich only = 29
  - Coffee and pastry only = 31
  - Coffee and breakfast sandwich only = 24
  - Pastry and breakfast sandwich only = 16
  - Coffee, pastry, and breakfast sandwich (all three) = 40

- Add up everyone who ordered at least one item = 86 + 54 + 29 + 31 + 24 + 16 + 40 = 280
- Total orders = 300
- Neither any of the three items = 300 – 280 = 20

- **Step 4: Answer**
  a. How many orders had coffee and a pastry, but not a sandwich? Answer: 31
  b. How many orders had coffee? Answer: 86 + 31 + 40 + 24 = 181
  c. How many orders had exactly two items? Answer: 31 + 24 + 16 = 71
  d. How many orders had at least one item? Answer: 280
  e. How many orders had none of these items? Answer: 20



Example #2.7.3 🤔

A local fitness center surveyed 500 members about which facilities they use:

- 280 use the weight room
- 220 use the pool
- 190 use the group fitness classes
- 110 use both weight room and pool
- 95 use both weight room and classes
- 85 use both pool and classes
- 20 members use all three facilities

Draw a Venn diagram to answer the following questions:

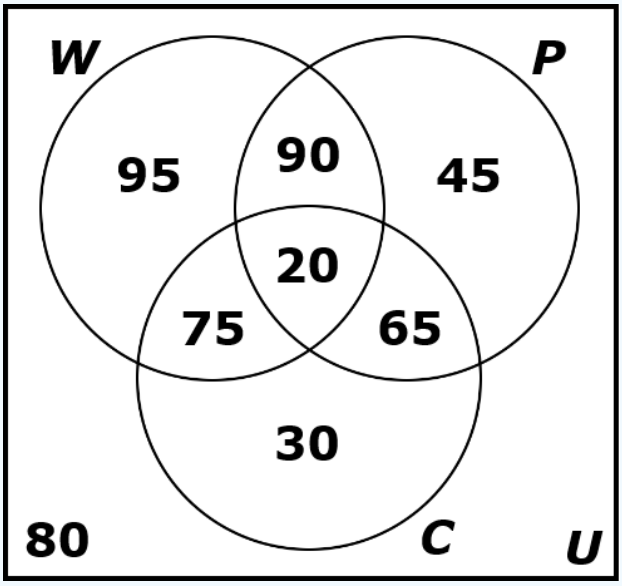
a. How many members do not use any of the three facilities?
b. How many members use exactly one of the facilities?
c. How many members use the pool, but not the group fitness classes?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 2, "working from the center out" problem. Because a majority of the information is not using the word only.
- **Step 2: Draw**
  - Draw and label three overlapping circles; one circle for weight room, one circle for pool, and one circle for classes.
- **Step 3: Fill**

- Weight room, pool, and classes (all three) = 20
- Pool and group fitness classes only = 85 – 20 = 65
- Weight room and classes only = 95 – 20 = 75
- Weight room and pool only = 110 – 20 = 90
- Classes only = 190 – 20 – 65 – 75 = 30
- Pool only = 220 – 20 – 65 – 90 = 45
- Weight room only = 280 – 20 – 75 – 90 = 95
- Add up everyone who used at least one facility = 20 + 65 + 75 + 90 + 30 + 45 + 95 = 420
- Total members = 500
- Neither any of the three facilities = 500 – 420 = 80

- **Step 4: Answer**
  a. How many members do not use any of the three facilities? Answer: 80
  b. How many members use exactly one of the facilities? Answer: 30 + 45 + 95 = 170
  c. How many members use the pool, but not the group fitness classes? Answer: 45 + 90 = 135



Example #2.7.4 🤔

A mechanic shop serviced 250 vehicles in a month:

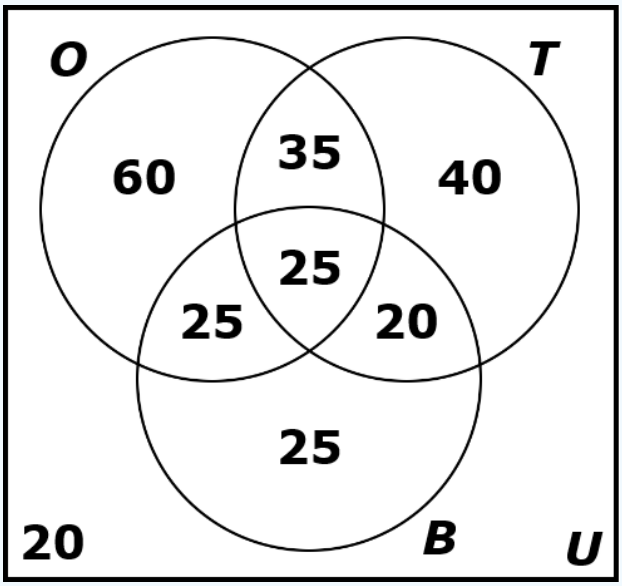
- 145 vehicles needed oil changes
- 120 vehicles needed tire service
- 95 vehicles needed brake work
- 60 vehicles needed both oil changes and tire service
- 50 vehicles needed both oil changes and brake work
- 45 vehicles needed both tire service and brake work
- 25 vehicles needed all three services

Draw a Venn diagram to answer the following questions:

a. How many vehicles needed oil changes and tire service, but not brake work?
b. How many vehicles needed only brake work?
c. How many vehicles needed exactly one service?
d. How many vehicles needed exactly two services?
e. How many vehicles needed none of these services?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 2, "working from the center out" problem. Because a majority of the information is not using the word only.
- **Step 2: Draw**
  - Draw and label three overlapping circles; one circle for oil changes, one circle for tire service, and one circle for brake work.
- **Step 3: Fill**
  - Weight room, pool, and classes (all three) = 25
  - Tire service and break work only = 45 – 25 = 20
  - Oil changes and break work only = 50 – 25 = 25
  - Oil changes and Tire service only = 60 – 25 = 35
  - Break work only = 95 – 25 – 20 – 25 = 25
  - Tire service only = 120 – 25 – 20 – 35 = 40
  - Oil changes only = 145 – 25 – 25 – 35 = 60
  - Add up vehicles who had at least one service = 25 + 20 + 25 + 35 + 25 + 40 + 60 = 230
  - Total members = 250
  - Neither any of the three services = 250 – 230 = 20
- **Step 4: Answer**
  a. How many vehicles needed oil changes and tire service, but not brake work? Answer: 35
  b. How many vehicles needed only brake work? Answer: 25
  c. How many vehicles needed exactly one service? Answer: 60 + 40 + 25 = 125
  d. How many vehicles needed exactly two services? Answer: 35 + 25 + 20 = 80
  e. How many vehicles needed none of these services? Answer: 20



Example #2.7.5 🤔

A farmers market has 200 vendors. The market organizer tracked what they sell:

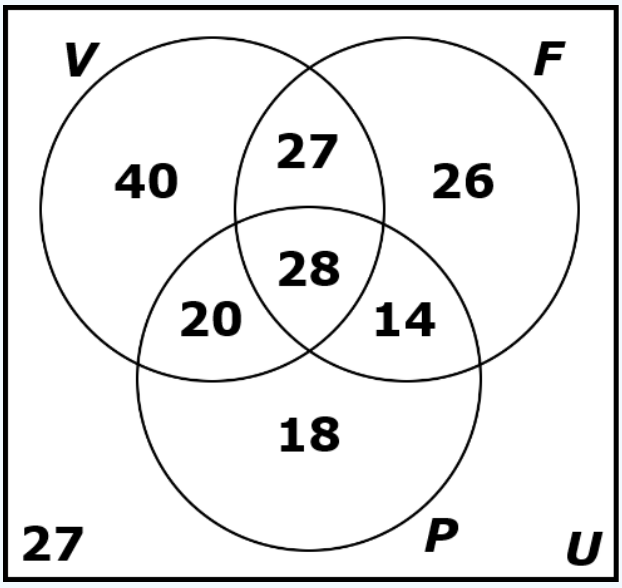
- 115 vendors sell vegetables
- 95 vendors sell fruits
- 80 vendors sell prepared foods
- 55 vendors sell both vegetables and fruits
- 48 vendors sell both vegetables and prepared foods
- 42 vendors sell both fruits and prepared foods
- 28 vendors sell all three types

Draw a Venn diagram to answer the following questions:

a. How many vendors sell vegetables and prepared foods, but not fruits?
b. How many vendors sell only prepared foods?
c. How many vendors sell exactly two types of products?
d. How many vendors sell products from at least one category?
e. How many vendors sell other items (flowers, crafts, etc.)?

✅ **Solution:**

- **Step 1: Identify**
  - This is a Type 2, "working from the center out" problem. Because a majority of the information is not using the word only.
- **Step 2: Draw**
  - Draw and label three overlapping circles; one circle for vegetables, one circle for fruits, and one circle for prepared foods.
- **Step 3: Fill**
  - Vegetables, fruits, prepared foods (all three) = 28
  - Fruits and prepared foods only = 42 – 28 = 14
  - Vegetables and prepared foods only = 48 – 28 = 20
  - Vegetables and fruits only = 55 – 28 = 27
  - Prepared foods only = 80 – 28 – 20 – 14 = 18
  - Fruits only = 95 – 28 – 27 – 14 = 26
  - Vegetables only = 115 – 28 – 27 – 20 = 40
  - Add up vendors who sold at least one category = 28 + 14 + 20 + 27 + 18 + 26 + 40 = 173
  - Total vendors = 200
  - Neither any of the three categories = 200 – 173 = 27
- **Step 4: Answer**
  a. How many vendors sell vegetables and prepared foods, but not fruits? Answer: 20
  b. How many vendors sell only prepared foods? Answer: 18
  c. How many vendors sell exactly two types of products? Answer: 27 + 20 + 14 = 61
  d. How many vendors sell products from at least one category? Answer: 173
  e. How many vendors sell other items (flowers, crafts, etc.)? Answer: 27



Section 2.7: Drawing Basic Venn Diagrams for Three Sets [*In-Class Exercises*]

For Exercises #1 through #4, draw a Venn diagram to answer all questions:

1. At IICON (Interactive Innovation Conference) in Las Vegas, 250 gamers were asked which console they own and regularly play their video games on. Here are the results of the survey:
    - 26 own only an XBOX series,
    - 52 own only a Playstation 5,
    - 43 own only a PC,
    - 36 own both an XBOX series and a Playstation 5, but not a PC,
    - 32 own both an XBOX series and a PC, but not a Playstation 5,
    - 53 own both a Playstation 5 and a PC, but not an XBOX, and
    - 7 own all three consoles.

    a. How many people own none of the three consoles?
    b. How many people own an XBOX series?
    c. How many people own exactly one console?
    d. How many people own a PC, but not an XBOX series?

2. In a recent survey, 180 streaming viewers were asked if they had watched any of the three series: The Boys, Breaking Bad, and Game of Thrones. Here are the results:
    - 12 watched only The Boys,
    - 16 watched only Breaking Bad,
    - 11 watched only Game of Thrones,
    - 24 watched The Boys and Breaking Bad, but not Game of Thrones,
    - 25 watched The Boys and Game of Thrones, but not Breaking Bad,
    - 31 watched Breaking Bad and Game of Thrones, but not The Boys, and
    - 50 watched all three series.

    a. How many viewers watched Game of Thrones?
    b. How many viewers watched The Boys, but not Breaking Bad?
    c. How many viewers watched exactly two of the three series?
    d. How many viewers didn't watch any of the three series?
    e. How many viewers didn't watch The Boys?

3. Out of 100 people asked about foods:
    - 60 like pizza,
    - 55 like tacos,
    - 45 like sushi,
    - 30 like both pizza and tacos,
    - 25 like both pizza and sushi,
    - 20 like both tacos and sushi, and
    - 12 like all three foods.

    a. How many people liked none of the three foods?
    b. How many people only liked pizza?
    c. How many people liked exactly one of the food items?
    d. How many people liked tacos, but not Sushi?

4. A toy expo surveyed 150 collectors:
    - 90 collect LEGO,
    - 70 collect Funko Pops,
    - 55 collect Pokémon cards,
    - 40 collect both LEGO and Funko Pops,
    - 35 collect both LEGO and Pokémon cards,
    - 30 collect both Funko Pops and Pokémon cards, and
    - 18 collect all three.

a. How many collectors did not collect any of the three toys/items?
b. How many collectors did not collect LEGO?
c. How many collectors collected Funko Pops only?
d. How many collectors collected Funko Pops and LEGO only?

Answers available at the end of this section.

Answers

Section 2.7: Drawing Basic Venn Diagrams for Three Sets [*In-Class Exercises*]

1. a. 1
   b. 101
   c. 121
   d. 96
2. a. 117
   b. 37
   c. 80
   d. 11
   e. 69
3. a. 3
   b. 17
   c. 46
   d. 35
4. a. 22
   b. 60
   c. 18
   d. 22

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Section 2.7.H: Homework

Section 2.7: Applications Using Three-Circle Venn Diagrams - Homework Exercises

(Section 2.7) - Homework Exercise #1

A gym tracked 500 members' use of three equipment types: treadmills, weight machines, and exercise bikes.

- 35 members used all three types
- 45 members used only treadmills and weight machines (not bikes)
- 40 members used only treadmills and bikes (not weight machines)
- 38 members used only weight machines and bikes (not treadmills)
- 95 members used only treadmills
- 85 members used only weight machines
- 72 members used only bikes
- The remaining members used none of these equipment types

Draw a Venn diagram to answer the following questions:

a. How many members used at least one equipment type?
b. How many members used treadmills?
c. How many members used exactly one equipment type?
d. How many members used exactly two equipment types?
e. How many members used none of these equipment types?

Answer available at the end of this section.

(Section 2.7) - Homework Exercise #2

A sandwich shop tracked 280 lunch orders for three extras: cheese, lettuce, and tomato.

- 16 sandwiches had all three extras
- 26 sandwiches had only cheese and lettuce (no tomato)
- 22 sandwiches had only cheese and tomato (no lettuce)
- 18 sandwiches had only lettuce and tomato (no cheese)
- 48 sandwiches had only cheese
- 40 sandwiches had only lettuce
- 35 sandwiches had only tomato
- The remaining sandwiches had none of these extras

Draw a Venn diagram to answer the following questions:

a. How many sandwiches had at least one extra?
b. How many sandwiches had cheese?
c. How many sandwiches had exactly two extras?
d. How many sandwiches were plain (had no extras)?

Answer available at the end of this section.

(Section 2.7) - Homework Exercise #3

At a hotel with 400 rooms,

- 220 rooms have ocean views
- 180 rooms have balconies
- 160 rooms have kitchenettes
- 85 rooms have both ocean views and balconies
- 75 rooms have both ocean views and kitchenettes
- 68 rooms have both balconies and kitchenettes
- 45 rooms have all three features

Draw a Venn diagram to answer the following questions:

a. How many rooms have none of these features?
b. How many rooms have ocean views and kitchenettes, but not balconies?
c. How many rooms have ocean views and balconies, but not kitchenettes?
d. How many rooms have ocean views, but not kitchenettes?
e. How many rooms have at least one of these features?

Answer available at the end of this section.

(Section 2.7) - Homework Exercise #4

A medical clinic saw 350 patients in one week. They tracked three symptoms: fever, cough, and fatigue.

- 180 patients had fever
- 165 patients had cough
- 140 patients had fatigue
- 85 patients had both fever and cough
- 75 patients had both fever and fatigue
- 70 patients had both cough and fatigue
- 45 patients had all three symptoms

Draw a Venn diagram to answer the following questions:

a. How many patients had at least one symptom?
b. How many patients had none of these symptoms?
c. How many patients had exactly two symptoms?
d. How many patients had only a cough?
e. What percentage of patients had all three symptoms?

Answer available at the end of this section.

(Section 2.7) - Homework Exercise #5

Among smartphone users,

- 280 users have 5G capability
- 320 users have wireless charging
- 250 users have water resistance
- 95 users have all three features
- 140 users have both 5G and wireless charging
- 125 users have both 5G and water resistance
- 135 users have both wireless charging and water resistance

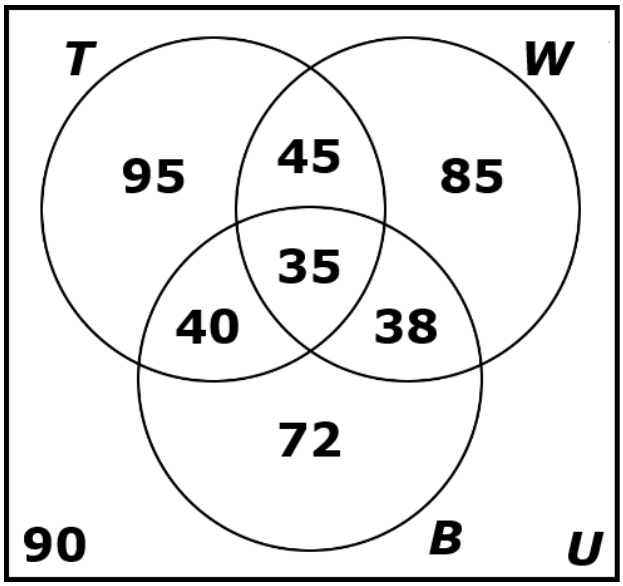
Draw a Venn diagram to answer the following questions:

a. How many users have at least one of these features?
b. How many users have 5G, but not water resistance?
c. How many users have water resistance, but not 5G?
d. How many users have only wireless charging?

Answer available at the end of this section.

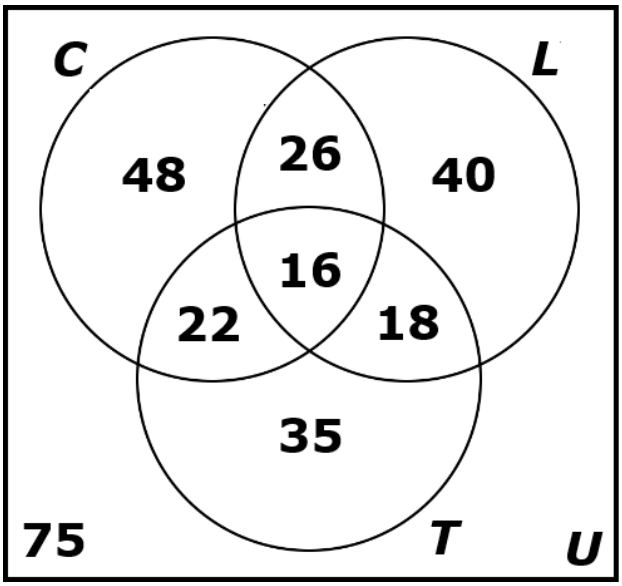
Answers

(Section 2.7) - Homework Exercise #1



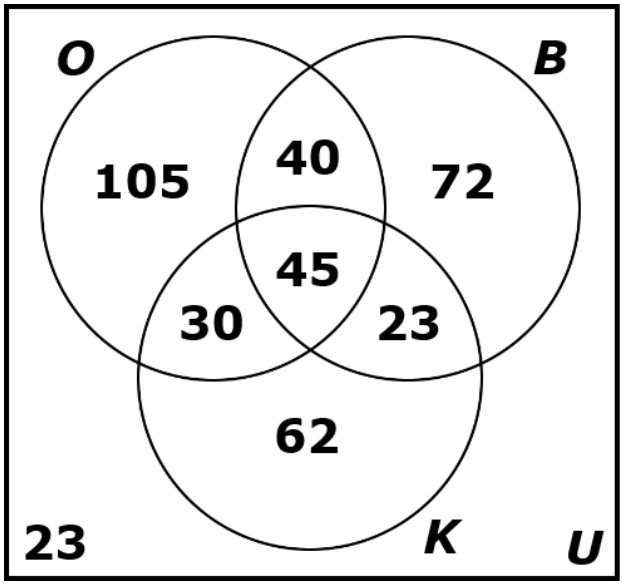
a. 410
b. 215
c. 252
d. 123
e. 90

(Section 2.7) - Homework Exercise #2



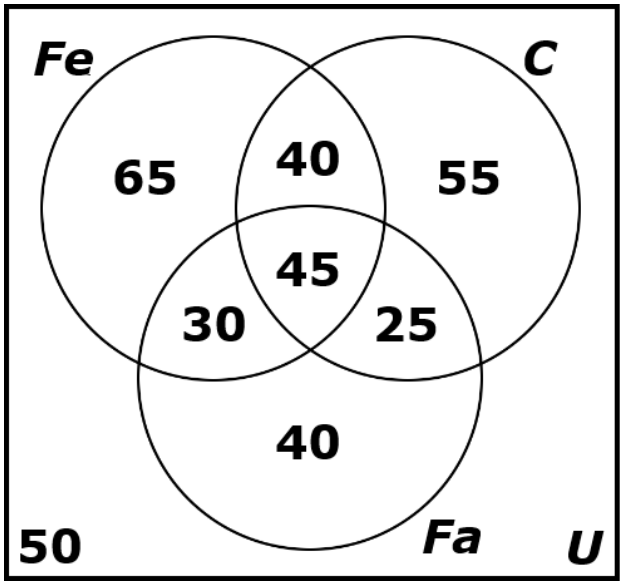
a. 205
b. 112
c. 66
d. 75

(Section 2.7) - Homework Exercise #3



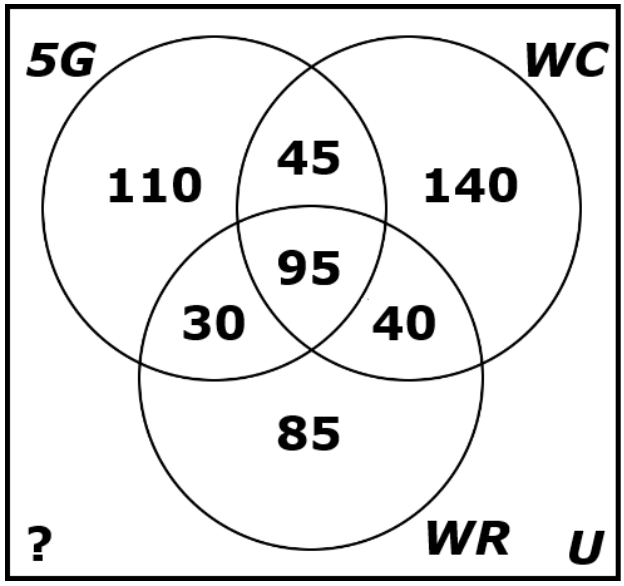
a. 23
b. 30
c. 40
d. 145
e. 377

(Section 2.7) - Homework Exercise #4



a. 300
b. 50
c. 95
d. 55
e. 45/350 = 12.86%

(Section 2.7) - Homework Exercise #5



a. 545
b. 155
c. 125
d. 140

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CHAPTER 3: MEASUREMENT

Section 3.1: U.S. Customary System - Length

Section 3.1.H: Homework

Section 3.2: U.S. Customary System - Area

Section 3.2.H: Homework

Section 3.3: U.S. Customary System - Volume

Section 3.3.H: Homework

Section 3.4: U.S. Customary System - Capacity

Section 3.4.H: Homework

Section 3.5: U.S. Customary System - Weight

Section 3.5.H: Homework

Section 3.6: The Metric System - Length, Weight, and Capacity

Section 3.6.H: Homework

Section 3.7: The Metric System - Area and Volume

Section 3.7.H: Homework

Section 3.8: Conversions Between U.S. Customary and Metric Systems

Section 3.8.H: Homework

Section 3.9: Measures of Temperature

Section 3.9.H: Homework

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Section 3.1: U.S. Customary System - Length

Learning Objectives

- Convert measurements of length in the U.S. Customary System

Introduction

Different systems of measurement have been used at different times and in different parts of the world. Sometimes, these measurement systems share the same unit names, but assign them different values. Currently, in the United States, we use a combination of two different systems: the U.S. Customary System and the Metric System.

The U.S. Customary System grew out of old English units brought over during colonial times. Later, in 1824, England standardized these measurements into the Imperial System, creating small but lasting differences between English and Imperial units that affect us today. Meanwhile, the French Revolution introduced the Metric System, a simple decimal-based system that evolved into today's International System of Units (SI). now the global standard for science, trade, and most everyday measurements outside the United States.

Many people today, including books and websites, refer to the U.S. Customary System as the English System of Measurement. Although these two systems share many similarities, they are not exactly the same. The English System usually refers to the traditional units of measurement that were historically used in England (like inches, feet, yards, rods, furlongs, gallons, etc.) before England switched to the Metric System. Whereas, the U.S. system, established in 1826, was based on older English units, that became standardized based uniquely on the needs of colonial America. Some units are the same in both systems, but others are not. For example, in the U.S. system, 1 gallon = 128 fluid ounces; whereas, 1 Imperial (English) gallon = 160 fluid ounces. Pint and quart sizes are different in the U.S. and English systems as well. Length and mass are mostly the same (inch, foot, yard, pound), but small variations exist in history.

While the United States continued using its customary system, Europe adopted the International System of Units (SI), commonly known as the Metric System. Unlike the U.S. system, which requires memorizing irregular conversions like 12 inches = 1 foot or 16 ounces = 1 pound, the Metric System is based on powers of ten, making length, volume, and mass easy to convert by simple multiplication or division. Its logic, consistency, and simplicity have made it the near-universal standard, especially in science, trade, and everyday life worldwide outside the United States.

The Metric System was created in France in the late 1700s, when measurement systems varied widely between regions, towns, and professions. This inconsistency made trade, science, and daily life confusing. In 1790, the French National Assembly, during the French Revolution, asked the French Academy of Sciences to develop a rational and universal system of measurement that would be fair to all citizens and reliable for science, engineering, and commerce.

The Academy proposed a system based on two main ideas:

- Natural constants: Units were tied to measurable features of the natural world. For example, the meter was defined as one ten-millionth of the distance from the North Pole to the equator along a line through Paris.
- Decimal structure: Units were divided or multiplied by powers of ten. Because calculations were done by hand, this made arithmetic much simpler, faster, and more consistent than using irregular conversions.

The French government officially adopted the new system in 1795, introducing the base units of meter for length, the liter for capacity, and the gram for mass. The system then spread across Europe and eventually worldwide, forming the foundation of today's International System of Units (SI).

The United States is the primary user of the U.S Customary System of Measurement. The only other countries that still use this system to a lesser extent are Myanmar (Burma) and Liberia. While most of the world uses the Metric System, these three nations haven't fully transitioned.

Summary of Key Measurement Systems

System	Origins & Key Dates	Current Use in the U.S.
English Units	Medieval blend of Roman, Anglo-Saxon, Celtic units	Foundation of both later systems
Imperial System (UK)	Established by British law in 1824 to standardize English units	Used in the UK; influenced, but not used, in the U.S.
U.S. Customary System	Rooted in colonial English units; standardized mid-1800s	Still dominant in daily U.S. life
Metric / SI System	Developed post-French Revolution (late 1700s); SI in 1960	Global standard; widely used in U.S. science and trade

If you live in the United States, you are most familiar with the U.S. Customary System through everyday units of length like inches, feet, yards, and miles; units of capacity like cups, pints, quarts, and gallons; and units of mass like ounces and pounds. If you live almost anywhere else, you are most familiar with the base units of meter, liter, and gram, and some of the more common scaling prefixes like kilo- as in kilogram, centi- as in centimeter.

Because of today's global trade, it is more common than ever to need to convert between different systems of measure. For example, you might be ordering a notebook off Amazon and need one that is 8 inches (in) wide, but the listing shows the size in centimeters (cm). In that case, you can use the conversion factor 1 in = 2.54 cm to find that this notebook should be about 20 cm wide. Because it is important to understand both the U.S. and Metric systems, we will first study conversions within the U.S. system, then conversions within the Metric System, and finally conversions between the two systems.

U.S. Customary System - Length

As stated earlier, the measurement system we use in the United States is known as the U.S. Customary System. In this system, there are many different units for measuring **length**, which describes how long something is or the distance from one end to the other.

The four most common units of length in the U.S. Customary System listed from smallest to largest are listed below.

U.S. Customary Common Units for Length: Smallest to Largest

[*Smallest*] **Inches** → **Feet** → **Yards** → **Miles** [*Largest*]

Reversing the order, and listing the four most common units of length in the U.S. Customary System from largest to smallest are listed below.

U.S. Customary Common Units for Length: Largest to Smallest

[*Largest*] **Miles** → **Yards** → **Feet** → **Inches** [*Smallest*]

Abbreviations for Common Units of Length (U.S. Customary System)

Name	Abbreviation
inch	in
foot	ft
yard	yd
mile	mi

Conversion Factors for Common Units of Length (U.S. Customary System)

Conversion Factors
1 foot (ft) = 12 inches (in)
1 yard (yd) = 3 feet (ft)
1 mile (mi) = 5,280 feet (ft)

In order to preform calculations in any system of measurement, the measurement units need to be the same. That is, feet add with feet, inches add with inches, but feet DO NOT add with inches. For example, in order to add 2 ft to 18 in, we need both measurement to either be in (a) feet, or (b) both measurement to be in inches. Ultimately, it does not matter if we convert everything to feet, or to inches, we just need to pick one unit of measurement for our calculations and stick to it. So, then, how do we convert between feet and inches? We use Unit Analysis, of course!

What is Unit Analysis?

Unit Analysis, or Dimensional Analysis, refers to a special type of mathematical calculation used to convert between different units of measurement. The basis for the calculation is the fact that when you multiply any number by 1, the number stays the same. (If you are interested, the number 1 is called a Multiplicative Identity for this reason.) This special form of the number 1, is called a conversion factor, and has the following two properties:

- It is written as a fraction with a numerator and denominator.
- The numerator equals the denominator, so it is equal to 1.

So, in Unit Analysis, if we wanted to add 2 feet to 18 inches, and express our answer in inches, we would decide to perform the addition process by using inches. Therefore, we need on to convert 2 feet to an equivalent number of inches. We start with the units we do not want to keep, in this case feet; then decide what units we want to end up with, which in this case are inches. Then, we look for a conversion factor relating feet to inches that can be expressed as a form of the number 1, such as 1 ft = 12 in from the table above. The goal of Unit Analysis is to multiply by the conversion factor, which is a form of the number 1, so that the units in the numerator and denominator match and cancel out. So, we take 2 ft and multiply by 12 in/1 ft to obtain 24 in. Now that 2 feet has been expressed as 24 inches, we can preform the original calculation of 2 ft + 18 in = 24 in + 18 in = 42 in.

How to Perform Unit Analysis

- **Step 1:** Determine your starting unit.
- **Step 2:** Determine your desired unit.
- **Step 3:** Find the conversion factor needed.
- **Step 4:** Multiply by the conversion factor.

Example #3.1.1 🤔

Convert using unit analysis: 33.6 feet to yards

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 33.6 feet. Write this as a fraction over 1.

$$\frac{33.6\text{ ft}}{1}$$

- **Step 2**: Determine our desired units, which is yards.

$$\frac{33.6\text{ ft}}{1} \longrightarrow \text{yd}$$

- **Step 3**: In order to convert from feet to yards, we need to identify a conversion factor that relates feet to yards. In the table above, we find that

$$1\text{ yard} = 3\text{ feet}$$

- **Step 4**: Since feet and yards are directly related by a single conversion factor, we only need to multiply by one fraction to produce our result. As a fraction equivalence, this conversion factor can be written as either $\frac{1\text{ yd}}{3\text{ ft}}$ or $\frac{3\text{ ft}}{1\text{ yd}}$. Now, we need to cancel the feet (starting units), so we select whichever conversion factor has feet (starting units) in its denominator to eliminate the feet (starting units) from the 33.6 ft on the left side. Using $\frac{1\text{ yd}}{3\text{ ft}}$ will make the starting units of feet cancel out.

$$\frac{33.6\ \cancel{\text{ft}}}{1} \times \frac{1\text{ yd}}{3\ \cancel{\text{ft}}}$$

Now, carry out the calculation.

$$= \frac{33.6}{1} \times \frac{1\text{ yd}}{3}$$

$$= 11.2\text{ yd}$$

So, 33.6 feet is equivalent to 11.2 yards.

Example #3.1.2 🤔

Convert using unit analysis: 27.5 yards to feet

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 27.5 yards. Write this as a fraction over 1.

$$\frac{27.5\text{ yd}}{1}$$

- **Step 2**: Determine our desired units, which is feet.

$$\frac{27.5\text{ yd}}{1} \longrightarrow \text{ft}$$

- **Step 3**: In order to convert from yards to feet, we need to identify a conversion factor that relates yards to feet. In the table above, we find that

1 yard = 3 feet

- **Step 4**: Since yards and feet are directly related by a single conversion factor, we only need to multiply by one fraction to produce our result. As a fraction equivalence, this conversion factor can be written as either $\frac{1\text{ yd}}{3\text{ ft}}$ or $\frac{3\text{ ft}}{1\text{ yd}}$. Now, we need to cancel the yards (starting units), so we select whichever conversion factor has yards (starting units) in its denominator to eliminate the yards (starting units) from the 27.5 yd on the left side. Using $\frac{3\text{ ft}}{1\text{ yd}}$ will make the starting units of yards cancel out.

$$\frac{27.5\ \cancel{\text{yd}}}{1} \times \frac{3\text{ ft}}{1\ \cancel{\text{yd}}}$$

Now, carry out the calculation.

$$= \frac{27.5}{1} \times \frac{3\text{ ft}}{1}$$
$$= 82.5\text{ ft}$$

So, 27.5 yards is equivalent to 82.5 feet.

Example #3.1.3 🤔

Convert using unit analysis: 17 miles to feet

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 17 miles. Write this as a fraction over 1.

$$\frac{17\text{ mi}}{1}$$

- **Step 2**: Determine our desired units, which is feet.

$$\frac{17\text{ mi}}{1} \longrightarrow \text{ ft}$$

- **Step 3**: In order to convert from miles to feet, we need to identify a conversion factor that relates miles to feet. In the table above, we find that

1 mile = 5,280 feet

- **Step 4**: Since miles and feet are directly related by a single conversion factor, we only need to multiply by one fraction to produce our result. As a fraction equivalence, this conversion factor can be written as either $\frac{1\text{ mi}}{5{,}280\text{ ft}}$ or $\frac{5{,}280\text{ ft}}{1\text{ mi}}$. Now, we need to cancel the miles (starting units), so we select whichever conversion factor has miles (starting units) in its denominator to eliminate the miles (starting units) from the 17 miles on the left side. Using $\frac{5{,}280\text{ ft}}{1\text{ mi}}$ will make the starting units of miles cancel out.

$$\frac{17\ \cancel{\text{mi}}}{1} \times \frac{5{,}280\text{ ft}}{1\ \cancel{\text{mi}}}$$

Now, carry out the calculation.

$$= \frac{17}{1} \times \frac{5{,}280\text{ ft}}{1}$$
$$= 89{,}760\text{ ft}$$

So, 17 miles is equivalent to 89,760 feet.

Example #3.1.4 🤔

Convert 866,780 inches to miles using unit analysis. Round your answer to the nearest hundredths place (or two decimal places).

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 866,780 inches. Write this as a fraction over 1.

$$\frac{866{,}780 \text{ in}}{1}$$

- **Step 2**: Determine our desired units, which is miles.

$$\frac{866{,}780 \text{ in}}{1} \longrightarrow \text{mi}$$

- **Step 3**: In order to convert from inches to miles, we need to identify a conversion factor that relates inches to miles. However, in the table above, there is not a single conversion factor that relates inches and miles. So, we need to use multiple conversion factors. Looking at the table above, we can go from

$$\text{in} \longrightarrow \text{ft} \longrightarrow \text{mi}$$

So, we can use the following conversion factors:

1 inch = 12 feet

5,280 feet = 1 mile

- **Step 4**: Since inches and feet are directly related by a one conversion factor, and since feet are directly related by a second conversion factor, we need to multiply by two fractions to produce our result. As a fraction equivalence, this can be written as either $\frac{1 \text{ ft}}{12 \text{ in}}$ or $\frac{12 \text{ in}}{1 \text{ ft}}$ and $\frac{1 \text{ mi}}{5{,}280 \text{ ft}}$ or $\frac{5{,}280 \text{ ft}}{1 \text{ mi}}$. Now, we need to cancel the inches (starting units), so we select whichever conversion factor has inches (starting units) in its denominator to eliminate the inches (starting units) from the 866,780 inches on the left side. Using $\frac{1 \text{ ft}}{12 \text{ in}}$ will make the starting units of yards cancel out. We then do a similar process to eliminate the next set of units which is feet. So, using $\frac{1 \text{ mi}}{5{,}280 \text{ ft}}$ will make the next set of units cancel out.

$$\frac{866{,}780 \ \cancel{\text{in}}}{1} \times \frac{1 \ \cancel{\text{ft}}}{12 \ \cancel{\text{in}}} \times \frac{1 \text{ mi}}{5{,}280 \ \cancel{\text{ft}}}$$

Now, carry out the calculation.

$$= \frac{866{,}780}{1} \times \frac{1}{12} \times \frac{1 \text{ mi}}{5{,}280}$$

$$= 13.6802398989899 \text{ mi}$$

So, 866,780 inches is equivalent to 13.68 miles.

Less Common Traditional Units of Length Still in Use Today

Although largely replaced by modern units of length, rod, furlong, and chain continue to be used in specialized fields because of tradition and practical applications.

- **Rod**
  - Land acquisition: Pipeline easement offers are often quoted in price per rod.
  - Canoeing: Maps for recreational canoeing use rods to measure portages (overland carries between waterways).
  - Surveying: A rod remains a convenient unit, especially since 1 acre = 160 square rods.

- **Furlong**
  - Horse racing: The standard unit for race lengths in many English-speaking countries.
  - Agriculture: Sometimes used to measure the length of fields and distances on farms.
- **Chain**
  - Surveying: Widely used in Canada and the U.S., since historical land measurement systems were based on it.
  - Agriculture: Used in field measurement, for example in the calibration of equipment wheels.
  - Wildland firefighting: Used to measure the rate of spread of wildfires.
  - Railways: Some railway systems (e.g., the New York City Subway) were designed using the 100-foot engineer's chain, and the concept of chainage still refers to distances along lines.

Abbreviations for Uncommon Units of Length (U.S. Customary System)

Name	Abbreviation
rod	rd or rdm
furlong	fur
chain	ch

Conversion Factors for Uncommon Units of Length (U.S. Customary System)

Conversion Factors
1 rod (rd) = 16.5 feet (ft)
4 rods (rd) = 1 chain (ch)
1 furlong (fur) = 660 feet (ft)
8 furlong (fur) = 1 mile (mi)

Example #3.1.5 🤔

In horse racing, The Belmont Stakes is traditionally the longest track in the Triple Crown, run at a distance of 1.5 miles. It is also referred to as "The Test of the Champion" because of the added stamina required to win. Convert this distance to furlongs using unit analysis.

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 1.5 miles. Write this as a fraction over 1.

$$\frac{1.5\text{ mi}}{1}$$

- **Step 2**: Determine our desired units, which is furlongs.

$$\frac{1.5\text{ mi}}{1} \longrightarrow \text{fur}$$

- **Step 3**: In order to convert from miles to furlong, we need to identify a conversion factor that relates miles to furlongs. In the table above, we find that

$$1\text{ mile} = 8\text{ furlongs}$$

- **Step 4**: Using the similar methods from Exercises #3.1.1-#3.13 above, we multiply by the following conversion factor and the units cancel out accordingly.

$$\frac{1.5 \ \cancel{\text{mi}}}{1} \times \frac{8 \text{ fur}}{1 \ \cancel{\text{mi}}}$$

Now, carry out the calculation.

$$= \frac{1.5}{1} \times \frac{8 \text{ fur}}{1}$$

$$= 12 \text{ fur}$$

So, 1.5 miles is equivalent to 12 furlongs.

Example #3.1.6 🤔

Convert 400 feet to rods using unit analysis. Round your answer to the nearest hundredths place (or two decimal places).

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 400 feet. Write this as a fraction over 1.

$$\frac{400 \text{ ft}}{1}$$

- **Step 2**: Determine our desired units, which is rods.

$$\frac{400 \text{ ft}}{1} \longrightarrow \text{rd}$$

- **Step 3**: In order to convert from feet to rods, we need to identify a conversion factor that relates feet to rods. In the table above, we find that

1 rod = 16.5 feet

- **Step 4**: Using the similar methods from Exercises #3.1.1-#3.13 above, we multiply by the following conversion factor and the units cancel out accordingly.

$$\frac{400 \ \cancel{\text{ft}}}{1} \times \frac{1 \text{ rd}}{16.5 \ \cancel{\text{ft}}}$$

Now, carry out the calculation.

$$= \frac{400}{1} \times \frac{1 \text{ rd}}{16.5}$$

$$= 24.242424242424 \text{ rd}$$

So, 400 feet is equivalent to 24.24 furlongs.

Example #3.1.7 🤔

How many inches are there in 1 chain? Convert using unit analysis. Round your answer to the nearest hundred*ths* place (or two decimal places).

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 1 chain. Write this as a fraction over 1.

$$\frac{1 \text{ ch}}{1}$$

- **Step 2**: Determine our desired units, which is inches.

$$\frac{1 \text{ ch}}{1} \longrightarrow \text{in}$$

- **Step 3**: In order to convert from chains to miles, we need to identify a conversion factor that relates chains to miles. However, in the table above, there is not a single conversion factor that relates chains and miles. So, we need to use multiple conversion factors. Looking at the tables above, we can go from

$$\text{ch} \longrightarrow \text{rd} \longrightarrow \text{ft} \longrightarrow \text{in}$$

So, we can use the following conversion factors:

1 chain = 4 rods

1 rod = 16.5 feet

1 foot = 12 inches

- **Step 4**: Using the similar method from Exercise #3.1.4 above, we multiply by the following conversion factors and the units cancel out accordingly.

$$\frac{1 \cancel{\text{ ch}}}{1} \times \frac{4 \cancel{\text{ rd}}}{1 \cancel{\text{ ch}}} \times \frac{16.5 \cancel{\text{ ft}}}{1 \cancel{\text{ rd}}} \times \frac{12 \text{ in}}{1 \cancel{\text{ ft}}}$$

Now, carry out the calculation.

$$= \frac{1}{1} \text{ ch} \times \frac{4 \text{ rd}}{1 \text{ ch}} \times \frac{16.5 \text{ ft}}{1 \text{ rd}} \times \frac{12 \text{ in}}{1 \text{ ft}}$$

$$= 792 \text{ in}$$

So, 1 chain is equivalent to 792 inches.

Section 3.1: U.S. Customary System - Length [*In-Class Exercises*]

1. Convert 20 feet to inches using unit analysis.
2. The Statue of Liberty is 3,661 inches tall from the ground to the tip of the torch. Convert this measure to feet using unit analysis.
3. Out of the 58 counties in the state of California, Orange County, is the 12th smallest in size. It is also the smallest county in Southern California by area. The dimensions are roughly 40 miles long from north-to-south and 25 miles wide from east-to-west. Convert both dimensions to feet using unit analysis.
4. Convert using unit analysis: 93 feet to yards
5. Convert 1,000,000 inches to miles using unit analysis. Round your answer to the nearest hundred*ths* place (or two decimal places).
6. The second leg of the Triple Crown of Thoroughbred (3-year olds) horse racing is called The Preakness Stakes. It is the shortest of the three races with a track distance $1\frac{3}{8}$ or 1.1875 miles. How many furlongs is the track? Round your answer to the nearest hundred*ths* place (or two decimal places).

Answers available at the end of this section.

Answers

Section 3.1: U.S. Customary System - Length [*In-Class Exercises*]

1. 240 in
2. 305.1 ft
3. 211,200 ft, 132,000 ft
4. 31 yd
5. 15.78 mi
6. 9.50 fur

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Section 3.1.H: Homework

Section 3.1: U.S. Customary System - Length - Homework Exercises

For exercises 1 through 11, use unit analysis to convert to the desired units. Round to the nearest hundredths place where necessary.

(Section 3.1) - Homework Exercise #1

Convert 156 inches to feet.

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #2

The Boston Marathon is a 26.2 mile-long race that is held annually on Patriots' Day (the third Monday in April). It is a point-to-point course that runs from Hopkinton, Massachusetts, to Copley Square in Boston. It is known for its challenging, hilly terrain, particularly the Newton Hills and the famous "Heartbreak Hill". Convert the distance in feet.

Answer available at the end of this section.

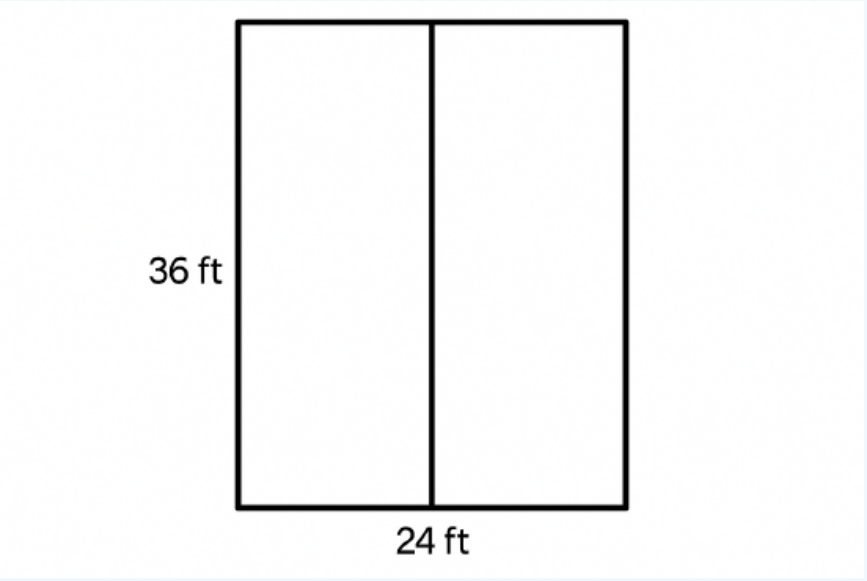
(Section 3.1) - Homework Exercise #3

Convert 2,160 inches to yards.

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #4

A farmer plans to install fencing around a rectangular area measuring 36 feet by 24 feet.



The farmer also wants to add a fence down the longest length of the field, dividing it into two equal sections; one for sheep and one for chickens. See the figure provided. The fencing company sells fencing by the yard. How many yards of fencing does the farmer need in total?

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #5

A farmer wants to know how much piping is needed for a long irrigation ditch. The irrigation line is 9,900 feet long. Convert this to miles and then to yards.

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #6

A surveyor records distances in inches on a blueprint but needs the real-world distance in miles. A measured distance on the blueprint is 18 inches, where the scale is 1 inch = 200 feet. Convert this measured blueprint distance into miles.

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #7

A 350-ft rope is cut into equal pieces, each 2 yards long. How many pieces are formed?

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #8

Convert 8,675,309 inches to miles.

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #9

A historical unit (also called an obsolete unit) is a type of measurement that was widely used during a particular time period or within a specific culture but is no longer part of modern, standardized measurement systems. These units often survive only in historical writings, older technical documents, or a few specialized trades. Before standard measurement systems existed, societies created units based on familiar, everyday references—especially body parts. Examples include:

- Cubit — the distance from the elbow to the tip of the middle finger, typically about 18 inches.
- Hand — the width of the palm excluding the thumb, standardized at 4 inches.
- Finger — the width of a finger laid flat, approximately 0.75 inches.
- Span — the distance from the tip of the thumb to the tip of the pinky when the hand is fully stretched, usually about 9 inches.

Convert 1 mile into fingers.

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #10

A survey crew is mapping a rectangular plot that measures:

- 12 chains along one side
- 8 chains along the other

A field note might say: “Set iron pin. Ran 12 ch. N 45°E to creek. Ran 8 ch. S 45°E along creek bank.” This gives a precise, standardized measurement long before tape measures or GPS. What would be the dimensions in feet?

Answer available at the end of this section.

(Section 3.1) - Homework Exercise #11

In horse racing, The Kentucky Derby is the first leg of the Triple Crown. It is frequently referred to as "The Most Exciting Two Minutes in Sports" because the race is typically completed in slightly over two minutes by three-year-old Thoroughbreds. The best time ever was 1:59.40, set by Secretariat in 1973. The track distance is $1\frac{1}{4}$ miles long. How many furlongs is this?

Answer available at the end of this section.

Answers

(Section 3.1) - Homework Exercise #1
13 ft

(Section 3.1) - Homework Exercise #2
138,336 ft

(Section 3.1) - Homework Exercise #3
60 yd

(Section 3.1) - Homework Exercise #4
52 yd

(Section 3.1) - Homework Exercise #5
1.875 miles, 3,300 yards

(Section 3.1) - Homework Exercise #6
0.682 miles

(Section 3.1) - Homework Exercise #7

58.33; 58 full pieces, with leftover rope.

(Section 3.1) - Homework Exercise #8

136.92 mi

(Section 3.1) - Homework Exercise #9

84,480 fingers

(Section 3.1) - Homework Exercise #10

792 ft along one side, 528 ft along the other

(Section 3.1) - Homework Exercise #11

10 fur

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Section 3.2: U.S. Customary System - Area

Learning Objectives

- Convert measurements of area in the U.S. Customary System

U.S. Customary System - Area

In measurement, **area** describes how much flat surface a shape covers. It tells us the space inside the boundary of a two-dimensional figure and is mostly measured in square units. In the U.S. Customary System, there many different units for measuring area.

The five most common units of area in the U.S. Customary System listed from smallest to largest are listed below.

U.S. Customary Common Units for Area: Smallest to Largest

[*Smallest*] **Square Inches** → **Square Feet** → **Square Yards** → **Acres** → **Square Miles** [*Largest*]

Reversing the order, and listing the five most common units of area in the U.S. Customary System from largest to smallest are listed below.

U.S. Customary Common Units for Area: Largest to Smallest

[*Largest*] **Square Miles** → **Acres** → **Square Yards** → **Square Feet** → **Square Inches** [*Smallest*]

Abbreviations for Common Units of Area (U.S. Customary System)

Name	Abbreviation
square inch	in^2
square foot	ft^2
square yard	yd^2
square mile	mi^2
acre	ac

Notice that the first four area units in the table use the same abbreviations as their corresponding length units, but with a superscript 2 (^{2}) added. This superscript indicates that these units measure square area, not linear distance.

When converting between square units in the U.S. customary system, such as in^2, ft^2, yd^2, and mi^2, we follow the same

unit-analysis process used for length (Section 3.1). The only difference is that the conversion factor must also be squared. Because of this, it isn't necessary to memorize separate conversion formulas for each of the squared units.

For example, we know 1 ft =12 in. If we square both sides, we get $(1\text{ ft})^2 = (12\text{ in})^2$ or $1\text{ ft}^2 = 144\text{ in}^2$. This is correct, but it is also the kind of fact that can be easy to forget. Memorizing many squared conversions, such as $1\text{ yd}^2 = 9\text{ ft}^2$, or $1\text{ mi}^2 = 27{,}878{,}400\text{ ft}^2$ can quickly become overwhelming.

Instead, it is far simpler and more reliable to use unit analysis, square only the conversion factor you need, and let the units cancel naturally. This reduces memorization and avoids common mistakes.

The acre is an exception. Because an acre is used only for area and is not based directly on squaring a length unit, it does not use the "^{2}" symbol. Therefore, you only need to remember one essential conversion factor for area involving acres:

Conversion Factor for Acre (U.S. Customary System)

Conversion Factor
1 acre (ac) = 43,560 square feet (ft^2)

The acre is used because it is a convenient and historically established unit for measuring moderate to large areas of land, particularly property, farms, ranches, and other land parcels. The acre originated as an old English unit of measurement. Historically, it represented approximately the amount of land that a team of oxen could plow in one day. Around the year 1300, the acre was standardized as a unit of measurement. Square feet work well for smaller areas, such as a house or a room. But using square feet for a large property produces very large numbers. For example, a typical $\frac{1}{4}$-acre residential lot is: $\frac{1}{4}(43{,}560)$ square feet = 10,890 square feet. It is much easier to say "a quarter-acre lot" than "10,890 square feet of land." A square mile is much too large for most individual properties. A 1-acre property is only 1/640 of a square mile, so acres provide a useful middle-sized unit between square feet (or square yards) and square miles.

How to Perform Unit Analysis for Area (2-Dimensions)

- **Step 1:** Determine your starting unit.
- **Step 2:** Determine your desired unit.
- **Step 3:** Find the conversion factor needed for length.
- **Step 4:** Square both values and the units in the conversion factor
- **Step 5:** Multiply by the squared conversion factor.

Note: If you are performing the Unit Analysis for Area on the unit, acre, leave off Step 4, and use the conversion factor: 1 acre = 43,560 ft^2.

Example #3.2.1 🤔

Convert using unit analysis: 100 square feet to square inches

---

✅ **Solution:**

- **Step 1**: Determine your starting units, which is 100 square feet. Write this as a fraction over 1.

$$\frac{100\text{ ft}^2}{1}$$

- **Step 2**: Determine your desired units, which is square inches.

$$\frac{100\text{ ft}^2}{1} \longrightarrow \text{in}^2$$

- **Step 3**: From Section 3.1, the conversion factor that relates feet to inches is

$$1 \text{ foot} = 12 \text{ inches}$$

- **Step 4**: Now, square both sides in the conversion factor. Here we will need to square both the values and the units in both the numerator and denominator.

$$1 \text{ ft} = 12 \text{ in}$$
$$(1 \text{ ft})^2 = (12 \text{ in})^2$$
$$1 \text{ ft}^2 = 144 \text{ in}^2$$

- **Step 5**: Using the similar methods from Exercises #3.1.1-#3.1.3 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{100 \cancel{\text{ ft}^2}}{1} \times \frac{144 \text{ in}^2}{1 \cancel{\text{ ft}^2}}$$

Now, carry out the calculation.

$$= \frac{100}{1} \times \frac{144 \text{ in}^2}{1}$$
$$= \frac{14{,}400 \text{ in}^2}{1}$$

So, 100 square feet is equivalent to 14,440 square inches.

Example #3.2.2 🤔

Convert using unit analysis: 70,000,000 ft^2 to mi^2

✅ **Solution:**

- **Step 1**: Determine your starting units, which is 70,000,000 square feet. Write this as a fraction over 1.

$$\frac{70{,}000{,}000 \text{ ft}^2}{1}$$

- **Step 2**: Determine your desired units, which is square miles.

$$\frac{70{,}000{,}000 \text{ ft}^2}{1} \longrightarrow \text{mi}^2$$

- **Step 3**: From Section 3.1, the conversion factor that relates feet to miles is

$$1 \text{ mile} = 5{,}280 \text{ feet}$$

- **Step 4**: Now, square both sides in the conversion factor. Here we will need to square both the values and the units in both the numerator and denominator.

$$1 \text{ mi} = 5{,}280 \text{ ft}$$
$$(1 \text{ mi})^2 = (5{,}280 \text{ ft})^2$$
$$1 \text{ mi}^2 = 27{,}878{,}400 \text{ ft}^2$$

- **Step 5**: Using the similar methods from Exercises #3.1.1-#3.1.3 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{70,000,000\ \cancel{\text{ft}^2}}{1} \times \frac{1\ \text{mi}^2}{27,878,400\ \cancel{\text{ft}^2}}$$

Now, carry out the calculation.

$$= \frac{70,000,000}{1} \times \frac{1\ \text{mi}^2}{27,878,400}$$

$$= 2.5109044995\ \text{mi}^2$$

So, 70,000,000 ft^2 is equivalent to 2.51 mi^2.

Example #3.2.3 🤔

A standard football field has an area of 6,400 square yards (approximately 6399.6 to 6400 sq yds), which is calculated by multiplying its total length of 120 yards (including the end zones) by its width of 53.33. Convert the area to square inches using unit analysis.

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 6,400 square yards. Write this as a fraction over 1.

$$\frac{6{,}400\ \text{yd}^2}{1}$$

- **Step 2**: Determine your desired units, which is square inches.

$$\frac{6{,}400\ \text{yd}^2}{1} \longrightarrow \text{in}^2$$

- **Step 3**: From Section 3.1, there is not a single conversion factor for length that relates yards and inches. (Unless you already knew that 1 yard = 36 inches). So, we need to use multiple conversion factors. Looking at the table from Section 3.1, we can go from

$$\text{yd} \longrightarrow \text{ft} \longrightarrow \text{in}$$

So, we can use the following conversion factors.

$$1\ \text{yard} = 3\ \text{feet}$$

$$1\ \text{foot} = 12\ \text{inches}$$

- **Step 4**: Now, square both sides in each conversion factor. Here we will need to square both the values and the units in both the numerator and denominator of each.

$$(1\ \text{yd})^2 = (3\ \text{ft})^2\ (1\ \text{ft})^2 = (12\ \text{in})^2$$

$$1\ \text{yd}^2 = 9\ \text{ft}^2\ 1\ \text{ft}^2 = 144\ \text{in}^2$$

- **Step 5**: Using the similar method from Exercise #3.1.4 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{6{,}400\ \cancel{\text{yd}^2}}{1} \times \frac{9\ \cancel{\text{ft}^2}}{1\ \cancel{\text{yd}^2}} \times \frac{144\ \text{in}^2}{1\ \cancel{\text{ft}^2}}$$

Now, carry out the calculation.

$$= \frac{6{,}400}{1} \times \frac{9}{1} \times \frac{144 \text{ in}^2}{1}$$

$$= 8{,}294{,}400 \text{ in}^2$$

So, 6,400 square yards is equivalent to 8,294,400 square inches.

Example #3.2.4 🤔

On September 9th, 2025, the Denver Broncos NFL football organization agreed with the state of Colorado to purchase a 58-acre site of Burnham Yard near downtown Denver, as well with Denver Water to purchase 25 acres of land. The combined 83 acres of land is to be used to build by 2031 a new state of the art stadium with a retractable roof along with a dynamic mixed-use district that will have restaurants, retail outlets, hotel and housing options, office spaces, entertainment venues, dynamic public spaces and much more. How many total square feet will this combined lot of land have?

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 83 acres. Write this as a fraction over 1.

$$\frac{83 \text{ ac}}{1}$$

- **Step 2**: Determine our desired units, which is square inches.

$$\frac{83 \text{ ac}}{1} \longrightarrow \text{ft}^2$$

- **Step 3**: From the table above, the conversion factor that relates acres to square feet is

$$1 \text{ acre} = 43{,}560 \text{ square feet}$$

- **Step 4**: Notice, since 1 acre is equivalent to 43,560 square feet, we do not need to square both the values and the units in both the numerator and denominator. So, we can skip this step.

- **Step 5**: Using the similar methods from Exercises #3.1.1-#3.1.3 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{83 \ \cancel{\text{ac}}}{1} \times \frac{43{,}560 \text{ ft}^2}{1 \ \cancel{\text{ac}}}$$

Now, carry out the calculation.

$$= \frac{83}{1} \times \frac{43{,}560 \text{ ft}^2}{1}$$

$$= 3{,}615{,}480 \text{ ft}^2$$

So, 6,400 square yards is equivalent to 8,294,400 square inches.

Section 3.2: U.S. Customary System - Area [*In-Class Exercises*]

1. A standard bedroom typically measures between 10 by 10 feet (100 ft^2) and 12 by 12 feet (144 ft^2) averaging around 132 ft^2 in the U.S.. Convert 132 ft^2 to square inches using unit analysis.
2. A small parking lot has an area of 8,100 ft^2. Using unit analysis, what is the area in square yards?

3. San Bernardino County, California, is the largest county in the United States by area, covering 20,105 square miles. It is larger than nine U.S. states and is often considered the largest in the contiguous U.S.. Convert this area to square feet using unit analysis.
4. A carpet installer orders 12 yd^2 of carpet. Use unit analysis to find out how many square inches of carpet this is?
5. A farmer has a small pasture that is 2.3 acres. Using unit analysis, how many square feet is the pasture?

Answers available at the end of this section.

Answers

Section 3.2: U.S. Customary System - Area [*In-Class Exercises*]

1. 19,008 in^2
2. 900 yd^2
3. 560,634,592,000 ft^2
4. 15,552 in^2
5. 100,188 ft^2

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Section 3.2.H: Homework

Section 3.2: U.S. Customary System - Area - Homework Exercises

For exercises 1 through 13, use unit analysis to convert to the desired units. Round to the nearest hundredths place where necessary.

(Section 3.2) - Homework Exercise #1

A blueprint shows a rectangular room with an actual area of 120 ft^2. How many square inches is this?

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #2

Out of the 58 counties in the state of California, Orange County, is the 12th smallest in size. It is also the smallest county in Southern California by area which covers approximately 948 square miles. Convert this area to square feet using unit analysis.

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #3

A student calculates the paintable area of a poster board as 3,456 in^2. How many square feet is this?

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #4

Sitka, Alaska, is often cited as the largest city by land area alone at approximately 2,870 square miles. Convert this area to square yards.

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #5

A homeowner is planting flowers in a garden measuring 18 yd^2. How many square feet of soil does the homeowner need?

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #6

A rooftop solar array covers 315 ft^2. What is the total area in square inches?

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #7

How many acres is 1 million square feet?

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #8

A shop receives a roll of window tint film measuring 12,960 in^2. Convert the area to square yards.

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #9

A research plot for testing new seeds measures 32 yd^2. Convert the area to square inches.

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #10

A parking lot expansion will add 2,950 yd^2 of paved surface. How many acres is this?

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #11

A protected area for wildlife spans 2.16 mi^2. How many acres does this represent?

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #12

How many square inches is in one acre?

Answer available at the end of this section.

(Section 3.2) - Homework Exercise #13

The smallest country in the world is Vatican City. This city-state is completely enclosed within Rome, Italy. It covers an area of about 0.19 square miles and serves as the global headquarters of the Roman Catholic Church and the residence of the Pope. What is the area of Vatican City in square feet?

Answer available at the end of this section.

Answers

(Section 3.2) - Homework Exercise #1

17,280 in^2

(Section 3.2) - Homework Exercise #2

26,428,723,200 ft^2

(Section 3.2) - Homework Exercise #3

24 ft^2

(Section 3.2) - Homework Exercise #4

8,890,112,000 yd^2

(Section 3.2) - Homework Exercise #5

162 ft^2

(Section 3.2) - Homework Exercise #6

45,360 in^2

(Section 3.2) - Homework Exercise #7

22.96 ac

(Section 3.2) - Homework Exercise #8

10 yd^2

(Section 3.2) - Homework Exercise #9

41,472 in^2

(Section 3.2) - Homework Exercise #10

0.61 ac

(Section 3.2) - Homework Exercise #11

1,382.4 ac

(Section 3.2) - Homework Exercise #12

6,272,640 in^2

(Section 3.2) - Homework Exercise #13

5,296,896 ft^2

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Section 3.3: U.S. Customary System - Volume

Learning Objectives

- Convert measurements of volume in the U.S. Customary System

U.S. Customary System - Volume

In measurement, **volume** describes how much space an object or substance occupies. It tells us the three-dimensional space inside a solid shape or container and is always measured in cubic units, such as cubic inches (in^3), cubic feet (ft^3), or cubic yards (yd^3). A related term, capacity, refers to the amount of space inside a container that can be filled, usually with a liquid. While there is a slight distinction between the two, some textbooks and students use the terms interchangeably. For clarity and consistency, this text will use the term volume when measuring the space inside a solid object and capacity when measuring the space inside a container filled with liquid. Think of it in this way: Volume is geometric. Capacity is practical.

We'll start with volume. In the U.S. System, there are many different units for measuring volume. The three most common units of volume in the U.S. Customary System listed from smallest to largest are listed below.

U.S. Customary Common Units for Volume: Smallest to Largest

[*Smallest*] **Cubic Inches** → **Cubic Feet** → **Cubic Yards** [*Largest*]

Reversing the order, and listing the three most common units of area in the U.S. Customary System from largest to smallest are listed below.

U.S. Customary Common Units for Volume: Largest to Smallest

[*Largest*] **Cubic Yards** → **Cubic Feet** → **Cubic Inches** [*Smallest*]

Abbreviations for Common Units of Volume (U.S. Customary System)

Name	Abbreviation
cubic inch	in^3
cubic foot	ft^3
cubic yard	yd^3

Notice that the area units in the table use some of the same abbreviations as their corresponding length units, but with a superscript 3 (^{3}) added. This superscript indicates that these units measure cubic volume, not linear distance.

When converting between cubic units in the U.S. customary system, such as in^3, ft^3, and yd^3, we follow the same unit-analysis

process used for length (Section 3.1). The only difference is that the conversion factor must also be cubed. Because of this, it isn't necessary to memorize separate conversion formulas for each of the cubed units.

For example, we know 1 ft =12 in. If we cube both sides, we get $(1\text{ ft})^3 = (12\text{ in})^3$ or $1\text{ ft}^3 = 1{,}728\text{ in}^3$. This is correct, but it is also the kind of fact that can be easy to forget. As we said in the previous section, memorizing cube conversions, such as $1\text{ ft}^3 = 1{,}728\text{ in}^3$ or $1\text{ yd}^3 = 27\text{ ft}^3$ can quickly become overwhelming.

Instead, it is far simpler and more reliable to use unit analysis, cube only the conversion factor you need, and let the units cancel naturally. This reduces memorization and avoids common mistakes.

You may notice that the cubic mile is not listed as common, yet it does exist. It is simply the volume of a cube that measures 1 mile on each edge. Mathematically:
$1\text{ mi}^3=(5280\text{ ft})^3=147{,}197{,}952{,}000\text{ ft}^3$. That's 147 billion cubic feet! So, the unit is real, but it's just enormous. You'd only need cubic miles for massive volumes like oceans, atmosphere, continental ice sheets, giant lakes, Earth's crust or mantle calculations. For daily life, cubic miles are way too big to be useful

How to Perform Unit Analysis for Volume (3-Dimensions)

- **Step 1:** Determine your starting unit.
- **Step 2:** Determine your desired unit.
- **Step 3:** Find the conversion factor needed for length.
- **Step 4:** Cube both values and the units in the conversion factor
- **Step 5:** Multiply by the cubed conversion factor.

Note: This is the same procedure for area, with the change of squared to cubed.

Example #3.3.1 🤔

Convert using unit analysis: 10 cubic feet to cubic inches

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 10 cubic feet. Write this as a fraction over 1.

$$\frac{10\text{ ft}^3}{1}$$

- **Step 2**: Determine your desired units, which is cubic inches.

$$\frac{10\text{ ft}^3}{1} \longrightarrow \text{in}^3$$

- **Step 3**: From Section 3.1, the conversion factor that relates feet to inches is

$$1\text{ foot} = 12\text{ inches}$$

- **Step 4**: Now, cube both sides in the conversion factor. Here we will need to cube both the values and the units in both the numerator and denominator.

$$1\text{ ft} = 12\text{ in}$$
$$1^3\text{ft}^3 = 12^3\text{ in}^3$$
$$1\text{ ft}^3 = 1{,}728\text{ in}^3$$

- **Step 5**: Using the similar methods from Exercises #3.1.1-#3.13 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{10\ \cancel{\text{ft}^3}}{1} \times \frac{1{,}728\ \text{in}^3}{1\ \cancel{\text{ft}^3}}$$

Now, carry out the calculation.

$$= \frac{10}{1} \times \frac{1{,}728\ \text{in}^3}{1}$$

$$= \frac{17{,}280\ \text{in}^3}{1}$$

So, 10 cubic feet is equivalent to 17,280 cubic inches.

Example #3.3.2 🤔

Convert using unit analysis: 15,726 ft^3 to yd^3

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 15,726 cubic feet. Write this as a fraction over 1.

$$\frac{15{,}726\ \text{ft}^3}{1}$$

- **Step 2**: Determine our desired units, which is cubic yards.

$$\frac{15{,}726\ \text{ft}^3}{1} \longrightarrow \text{yd}^3$$

- **Step 3**: From Section 3.1, the conversion factor that relates feet to inches is

$$1\ \text{yard} = 3\ \text{feet}$$

- **Step 4**: Now, cube both sides in the conversion factor. Here we will need to cube both the values and the units in both the numerator and denominator of each.

$$1\ \text{yd} = 3\ \text{ft}$$
$$1^3\text{yd}^3 = 3^3\ \text{ft}^3$$
$$1\ \text{yd}^3 = 27\ \text{ft}^3$$

- **Step 5**: Using the similar methods from Exercises #3.1.1-#3.13 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{15{,}726\ \cancel{\text{ft}^3}}{1} \times \frac{1\ \text{yd}^3}{27\ \cancel{\text{ft}^3}}$$

Now, carry out the calculation.

$$= \frac{15{,}726}{1} \times \frac{1\ \text{yd}^3}{27}$$

$$= 582.44444444\ \text{yd}^3$$

So, 15,726 yd^3 is equivalent to 582.44 yd^3.

Example #3.3.3 🤔

Landscape material, mulch, gravel, cement, and sand are sold in cubic yards. Sammy wants to make a Bocci Ball court in his backyard. He wants the court to have the dimensions of 50 ft by 10 ft by 3 in filled with sand. A local landscaping supplier states that based off his dimensions he would need 4.63 cubic yards. How many cubic inches would this be equivalent to?

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 4.63 cubic yards. Write this as a fraction over 1.

$$\frac{4.63\ \text{yd}^3}{1}$$

- **Step 2**: Determine your desired units, which is cubic inches.

$$\frac{4.63\ \text{yd}^3}{1} \longrightarrow \text{in}^3$$

- **Step 3**: From Section 3.1, there is not a single conversion factor for length that relates yards and inches. (Unless you already knew that 1 yard = 36 inches). So, we need to use multiple conversion factors. Looking at the table from Section 3.1, we can go from

$$\text{yd} \longrightarrow \text{ft} \longrightarrow \text{in}$$

So, we can use the following conversion factors.

$$1\ \text{yard} = 3\ \text{feet}$$

$$1\ \text{foot} = 12\ \text{inches}$$

- **Step 4**: Now, cube both sides in each conversion factor. Here we will need to cube both the values and the units in both the numerator and denominator of each.

$$(1\ \text{yd})^3 = (3\ \text{ft})^3\ (1\ \text{ft})^3 = (12\ \text{in})^3$$

$$1^3\ \text{yd}^3 = 3^3\ \text{ft}^3\ 1^3\ \text{ft}^3 = 12^2\ \text{in}^3$$

$$1\ \text{yd}^3 = 27\ \text{ft}^3\ 1\ \text{ft}^3 = 1{,}728\ \text{in}^3$$

- **Step 5**: Using the similar methods from Exercise #3.1.4 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{4.63\ \cancel{\text{yd}^3}}{1} \times \frac{27\ \cancel{\text{ft}^3}}{1\ \cancel{\text{yd}^3}} \times \frac{1{,}728\ \text{in}^3}{1\ \cancel{\text{ft}^3}}$$

Now, carry out the calculation.

$$= \frac{4.63}{1} \times \frac{27}{1} \times \frac{1{,}728\ \text{in}^3}{1}$$

$$= 216{,}017.28\ \text{in}^3$$

So, 4.63 cubic yards is equivalent to 216,017.28 cubic inches.

Section 3.3: U.S. Customary System - Volume [*In-Class Exercises*]

1. A storage box has a volume of 8,640 in^3. Using unit analysis, how many cubic feet is this?
2. A sandbox holds 2 yd^3 of sand. How many cubic inches is this?

Answers available at the end of this section.

Answers

Section 3.3: U.S. Customary System - Volume [*In-Class Exercises*]

1. 5 ft^3
2. 93,312 in^3

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Section 3.3.H: Homework

Section 3.3: U.S. Customary System - Volume - Homework Exercises

For exercises 1 through 13, use unit analysis to convert to the desired units. Round to the nearest hundredths place where necessary.

(Section 3.3) - Homework Exercise #1

Convert 100 ft^3 to cubic inches.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #2

A pile of soil measures 270 ft^3. Convert to yd^3.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #3

An engine block occupies 3,456 in^3. Convert to cubic feet.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #4

A concrete pour requires 6 yd^3. How many cubic feet is the pour?

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #5

A rectangular trench is 30 ft long, 2 ft wide, and 3 ft deep. Find the volume in cubic yards. (Note: $V = l \cdot w \cdot h$).

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #6

A shipping crate has a volume of 3.5 ft^3. Convert this volume to cubic inches.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #7

A concrete slab occupies 405 ft^3. How many cubic yards of concrete are required?

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #8

A landscaping company delivers 4.8 yd^3 of mulch. Convert this to cubic feet.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #9

A grain bin holds 1.2 yd^3 of grain. Convert this volume to cubic inches.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #10

A warehouse section has 1,080 ft^3 of usable space. Convert this to cubic yards.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #11

A storage container has a volume of 1.5 yd^3. Convert this volume to cubic inches.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #12

A fish tank has a volume of 25,920 in^3. Convert this to cubic feet, then to cubic yards.

Answer available at the end of this section.

(Section 3.3) - Homework Exercise #13

A concrete truck delivers 7.25 yd^3 of concrete. How many cubic inches is this?

Answer available at the end of this section.

Answers

(Section 3.3) - Homework Exercise #1

172,800 in^3

(Section 3.3) - Homework Exercise #2

10 yd^3

(Section 3.3) - Homework Exercise #3

2 ft^3

(Section 3.3) - Homework Exercise #4

162 ft^3

(Section 3.3) - Homework Exercise #5

6.67 yd^3

(Section 3.3) - Homework Exercise #6

6,048 in^3

(Section 3.3) - Homework Exercise #7

15 yd^3

(Section 3.3) - Homework Exercise #8

129.6 ft^3

(Section 3.3) - Homework Exercise #9

55,987.2 in^3

(Section 3.3) - Homework Exercise #10

40 yd^3

(Section 3.3) - Homework Exercise #11

69,984 in^3

(Section 3.3) - Homework Exercise #12

0.556 yd^3

(Section 3.3) - Homework Exercise #13

338,256 in^3

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Section 3.4: U.S. Customary System - Capacity

Learning Objectives

- Convert measurements of capacity in the U.S. Customary System

U.S. Customary System - Capacity

Thus far, volume has been treated as a geometric quantity measured in cubic units. This approach is appropriate when working with solid shapes and known dimensions. In practice, however, liquids are usually measured by **capacity** rather than by calculating cubic space. Capacity measurements describe how much liquid a container can hold and are expressed using units such as fluid ounces, cups, quarts, and gallons. This section introduces capacity and explains how these units are used and related to volume.

The most common units of capacity in the U.S. Customary System listed from smallest to largest are listed below.

U.S. Customary Common Units for Capacity: Smallest to Largest

[*Smallest*] **Teaspoons** → **Tablespoons** → **Fluid Ounces** → **Cups** → **Pints** → **Quarts** → **Gallons** → **Barrels** [*Largest*]

Reversing the order, and listing the most common units of area in the U.S. Customary System from largest to smallest are listed below.

U.S. Customary Common Units for Capacity: Largest to Smallest

[*Largest*] **Barrels** → **Gallons** → **Quarts** → **Pints** → **Cups** → **Fluid Ounces** → **Tablespoons** → **Teaspoons** [*Smallest*]

Abbreviations for Common Units of Capacity (U.S. Customary System)

Name	Abbreviation
teaspoon	tsp
tablespoon	tbsp
fluid ounce	fl oz
cup	c
pint	pt
quart	qt
gallon	gal
barrel	bbl

Conversion Factors for Common Units of Capacity (U.S. Customary System)

Conversion Factors
1 tablespoon = 3 teaspoons
1 fluid ounce = 2 tablespoons
1 cup = 8 fluid ounces
1 pint = 2 cups
1 quart = 2 pints
1 gallon = 4 quarts
1 oil barrel = 42 gallons
1 beer barrel = 31 gallons

When converting units of capacity in the U.S. customary system, we can use the same exact approach as we did in Section 3.1, where we used unit analysis and converted from one length of measurement to another.

How to Perform Unit Analysis for Capacity

- **Step 1:** Determine your starting unit.
- **Step 2:** Determine your desired unit.
- **Step 3:** Find the conversion factor needed for length.
- **Step 4:** Multiply by the conversion factor.

Example #3.4.1 🤔

Convert using unit analysis: 12 pints to cups

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 12 pints. Write this as a fraction over 1.

$$\frac{12 \text{ pt}}{1}$$

- **Step 2**: Determine our desired units, which is cups.

$$\frac{12 \text{ pt}}{1} \longrightarrow \text{c}$$

- **Step 3**: In order to convert from pints to cups, we need to identify a conversion factor that relates pints to cups. In the table above, we find that

1 pint = 2 cups

- **Step 4**: Using the similar methods from Exercises #3.1.1-#3.1.3 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{12 \cancel{\text{ pt}}}{1} \times \frac{2 \text{ c}}{1 \cancel{\text{ pt}}}$$

Now, carry out the calculation.

$$= \frac{12}{1} \times \frac{2 \text{ c}}{1}$$

$$= 24 \text{ c}$$

So, 12 pints is equivalent to 24 cups.

Example #3.4.2 🤔

Convert using unit analysis: 200 tbsp to qt

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 200 tablespoons. Write this as a fraction over 1.

$$\frac{200 \text{ tbsp}}{1}$$

- **Step 2**: Determine our desired units, which is quarts.

$$\frac{200 \text{ tbsp}}{1} \longrightarrow \text{qt}$$

- **Step 3**: In order to convert from tablespoons to quarts, we need to identify a conversion factor that relates tablespoons to quarts. However, in the table above, there is not a single conversion factor that relates tablespoons and quarts. So, we need to use multiple conversion factors. Looking at the table above, we can go from

$$\text{tbsp} \longrightarrow \text{fl oz} \longrightarrow \text{c} \longrightarrow \text{pt} \longrightarrow \text{qt}$$

So, we can use the following conversion factors:

2 tablespoons = 1 fluid ounce

8 fluid ounces = 1 cup

2 cups = 1 pint

2 pints = 1 quart

- **Step 4**: Using the similar method from Exercise #3.1.4 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{200 \text{ tbsp}}{1} \times \frac{1 \text{ fl oz}}{2 \text{ tbsp}} \times \frac{1 \text{ c}}{8 \text{ fl oz}} \times \frac{1 \text{ pt}}{2 \text{ c}} \times \frac{1 \text{ qt}}{2 \text{ pt}}$$

Now, carry out the calculation.

$$= \frac{200}{1} \times \frac{1}{2} \times \frac{1}{8} \times \frac{1}{2} \times \frac{1 \text{ qt}}{2}$$

$$= 3.125 \text{ qt}$$

So, 200 tablespoons is equivalent to 3.125 quarts.

Example #3.4.3 🤔

A keg is a pressurized metal barrel with a valve at one end, used to store and dispense beer or other carbonated drinks. A keg is equivalent to one-half of a U.S. beer barrel. An owner of a brewery wants to confirm how many $6 pints of beer that he can pour from a single standard size keg of craft beer. He also wants to know how much revenue he will make off that single keg. Help him answer his questions.

✅ **Solution:**

- **Step 1**: Determine our starting units, which is $frac12$ or 0.5 of a beer barrel. Using the decimal, write this as a fraction over 1.

$$\frac{0.5 \text{ bbl}}{1}$$

- **Step 2**: Second, we determine our desired units, which is pints.

$$\frac{0.5 \text{ bbl}}{1} \longrightarrow \text{pt}$$

- **Step 3**: In order to convert from beer barrels to pints, we need to identify a conversion factor that relates beer barrels to pints. However, in the table above, there is not a single conversion factor that relates barrels and pints. So, we need to use multiple conversion factors. Looking at the table above, we can go from

$$\text{bbl} \longrightarrow \text{gal} \longrightarrow \text{qt} \longrightarrow \text{pt}$$

So, we can use the following conversion factors:

1 beer barrel = 31 gallons

1 gallon = 4 quarts

1 quart = 2 pints

- **Step 4**: Using the similar method from Exercise #3.1.4 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{0.5 \text{ bbl}}{1} \times \frac{31 \text{ gal}}{1 \text{ bbl}} \times \frac{4 \text{ qt}}{1 \text{ gal}} \times \frac{2 \text{ pt}}{1 \text{ qt}}$$

Now, carry out the calculation.

$$= \frac{0.5}{1} \times \frac{31}{1} \times \frac{4}{1} \times \frac{2 \text{ pt}}{1}$$

$$= 124 \text{ pt}$$

So, 1 standard sized keg (or half U.S. beer barrel) is is equivalent to 124 cups. At $6 a pint, 124 x $6 = $744.

We have worked with volume in cubic units such as cubic inches and cubic feet, and now we consider capacity units like gallons and ounces. Since both volume and capacity measure space, they can be connected through conversion factors. Using unit analysis, we can convert between these units just as we would with any other measurement. Here is a list of a few of those conversion factors:

Conversion Factors between Volume and Capacity Units (U.S. Customary System)

Conversion Factors
1 cubic foot (ft^3) ≈ 7.481 gallons (gal)
1 gallon (gal) ≈ 231 cubic inches (in^3)

Example #3.4.4 🤔

Tim "The Tool Man" Taylor wants to build his own fish tank for his peppermint angelfish. Due to their specialized deep-reef habitat, they require advanced care and large tanks recommended at 55 to 70+ gallons. How many cubic feet of space would he need, if he decides to have a 75 gallon tank? Convert using unit analysis.

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 75 gallons. Write this as a fraction over 1.

$$\frac{75 \text{ gal}}{1}$$

- **Step 2**: Determine your desired units, which is cubic feet.

$$\frac{75 \text{ gal}}{1} \longrightarrow \text{ft}^3$$

- **Step 3**: From the table above the conversion factor that relates cubic feet to gallons is

$$1 \text{ cubic foot} \approx 7.481 \text{ gallons}$$

- **Step 4**: Now, there is no need to cube both sides in the conversion factor, because it is a direct conversion from cubic feet to gallons.

- **Step 5**: Using the similar methods from Exercises #3.1.1-#3.1.3 from Section 3.1, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{75 \text{ } \cancel{\text{gal}}}{1} \times \frac{1 \text{ ft}^3}{7.481 \text{ } \cancel{\text{gal}}}$$

Now, carry out the calculation.

$$= \frac{75}{1} \times \frac{1 \text{ ft}^3}{7.481}$$

$$= 10.02539767 \text{ ft}^3$$

So, 75 gallons is approximately 10.03 cubic feet.

Section 3.4: U.S. Customary System - Capacity [*In-Class Exercises*]

1. At Baskin-Robbins, take home fresh-packed of ice cream are sold in two sizes:

   - 12 oz @ $10.99
   - 24 oz @ $15.99

   Using unit analysis, how many tablespoons are there in the larger size? (*Note: Ice cream is measured by volume, not weight, using fluid ounces. It is labeled by volume because it is considered a frozen liquid, which can contain varying amounts of air (overrun). As a result, a 16 fl oz container of ice cream will not always weigh 16 oz or 1 pound*).

2. A typical pitcher used in restaurants for soda or beer is roughly 64 fluid ounces. Using unit analysis, how many pints is this?
3. A large mixing bowl holds 14 cups of batter. Convert this to quarts.
4. Convert 1 cubic yards to gallons. Round to the nearest gallon. (Note: 1 ft^3 ≈ 7.481 gal)

Answers available at the end of this section.

Answers

Section 3.4: U.S. Customary System - Capacity [*In-Class Exercises*]

1. 48 oz
2. 4 pt
3. 3.5 qt
4. ≈202 gal

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Section 3.4.H: Homework

Section 3.4: U.S. Customary System - Capacity - Homework Exercises

For exercises 1 through 19, use unit analysis to convert to the desired units. Round to the nearest hundredths place where necessary.

(Section 3.4) - Homework Exercise #1

A baker needs 6 tablespoons of vanilla extract, yet she only has a teaspoon available. How many teaspoons of vanilla extract then would she need?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #2

Convert 224 tablespoons to cups.

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #3

A swimming pool holds 15,000 gallons of water. How many quarts is this?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #4

A restaurant uses 8 gallons 3 quarts of cooking oil per week. How many pints is this?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #5

The largest container of milk sold in any grocery stores is 1 gallon. How many ounces is that?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #6

A drink machine dispenses 640 fl oz of soda in one hour. How many gallons is this?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #7

A recipe calls for 24 fluid ounces of milk. How many cups is this?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #8

A cooler contains 1.75 gallons of water. How many fluid ounces is this?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #9

A soup kitchen prepares 60 cups of soup. How many gallons is this?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #10

A brewery produces 6 beer barrels in one batch. How many gallons of beer is this?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #11

A music festival orders 2 beer barrels. How many fluid ounces of beer is that?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #12

An oil storage facility receives 15 oil barrels. How many gallons is this?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #13

Convert 3 oil barrels to quarts.

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #14

How many more teaspoons are there in an oil barrel over a beer barrel?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #15

A storage tank holds 150 gallons of water. What is the volume of the tank in cubic feet?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #16

A bathtub has dimensions 5 ft × 2.5 ft × 1.5 ft, giving a volume of 18.75 ft^3. Jeremy wants to determine how many gallons of water the bathtub can hold. He knows that 1 ft^3 ≈ 7.481 gallons. However, after doing his calculations, he gets an answer of 2.51 gallons. Jeremy realizes this cannot be correct because a bathtub clearly holds much more than a few 1-gallon containers. He is unsure where he made his mistake. What is the correct number of gallons the bathtub can hold, and where did Jeremy go wrong?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #17

A lab container has a volume of 924 in^3. How many gallons does it hold?

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #18

A landscaping company is filling a decorative pond that has a volume of 1.8 cubic yards. How many gallons of water does it hold? Round to the nearest whole number.

Answer available at the end of this section.

(Section 3.4) - Homework Exercise #19

A minim is a very small U.S. customary unit of liquid capacity used mainly in medicine and pharmacy. The conversion factor is the following:

- 1 minim = $\frac{1}{480}$ fluid ounce

That makes it one of the smallest capacity units in the system. Today, minims appear mostly in medical and pharmaceutical contexts, older prescriptions, and precise dosing references. They are not used in everyday cooking or commerce.

Using the conversion factor above, answer the following questions:

a. A pharmacuetical company has a liquid medication dose requirement of 120 minims. Convert this to fluid ounces.
b. A patient must take 960 minims of liquid over a day. Convert this amount to cups.

Answer available at the end of this section.

Answers

(Section 3.4) - Homework Exercise #1

18 tsp

(Section 3.4) - Homework Exercise #2

14 c

(Section 3.4) - Homework Exercise #3

60,000 qt

(Section 3.4) - Homework Exercise #4

70 pt (8 gallons = 64 pints; 3 quarts = 6 pints)

(Section 3.4) - Homework Exercise #5

128 oz

(Section 3.4) - Homework Exercise #6

5 gal

(Section 3.4) - Homework Exercise #7

3 c

(Section 3.4) - Homework Exercise #8

224 oz

(Section 3.4) - Homework Exercise #9

3.75 gal

(Section 3.4) - Homework Exercise #10

186 gal

(Section 3.4) - Homework Exercise #11

7,936 fl oz

(Section 3.4) - Homework Exercise #12

630 gal

(Section 3.4) - Homework Exercise #13

504 qt

(Section 3.4) - Homework Exercise #14

8,448 teaspoons

(Section 3.4) - Homework Exercise #15

20.05 ft^3

(Section 3.4) - Homework Exercise #16

140.27 gal

(Section 3.4) - Homework Exercise #17

4 gal

(Section 3.4) - Homework Exercise #18

364 gal

(Section 3.4) - Homework Exercise #19

a. 0.25 oz
b. 0.25 c

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Section 3.5: U.S. Customary System - Weight

Learning Objectives

- Convert measurements of weight in the U.S. Customary System

U.S. Customary System - Weight

In measurement, **weight** refers to the gravitational force acting on an object, while mass describes the amount of matter the object contains. Although these terms represent distinct physical concepts, they are often used interchangeably in everyday contexts and in many unit-conversion problems. For the purposes of this text, we will focus on the units of measurement and their conversion factors, rather than emphasizing the technical distinction between weight and mass.

The three most common units of weight in the U.S. Customary System listed from smallest to largest are listed below.

U.S. Customary Common Units for Weight: Smallest to Largest

[*Smallest*] **Ounces** → **Pounds** → **Tons** [*Largest*]

Reversing the order, and listing the three most common units of area in the U.S. Customary System from largest to smallest are listed below.

U.S. Customary Common Units for Weight: Largest to Smallest

[*Largest*] **Tons** → **Pounds** → **Ounces** [*Smallest*]

Abbreviations for Common Units of Weight (U.S. Customary System)

Name	Abbreviation
ounce	oz
pound	lb
ton	T

Note: *"lb" comes from the Latin word libra, which was a Roman unit of weight. Over time, English adopted the word "pound" (from Old English pund), but kept the Latin abbreviation.*

Conversion Factors for Common Units of Weight (U.S. Customary System)

Conversion Factors
1 pound (lb) = 16 ounces (oz)
1 ton (T) = 2,000 pounds (lb)

There is often confusion when discussing ounces in measurement. In the previous section, fluid ounces were introduced, and in this section, ounces are introduced. The confusion arises because both units share the same name, both are commonly used in everyday life, and both frequently appear in contexts involving food and drinks.
However, these two units measure different quantities. A fluid ounce (fl oz) measures volume (capacity), while an ounce (oz) measures weight. As a result, the two units are not interchangeable. A fluid ounce of a substance does not necessarily weigh one ounce.
For example:

- 1 fl oz of water weighs about 1 oz (approximately)
- 1 fl oz of oil weighs less than 1 oz
- 1 fl oz of syrup weighs more than 1 oz

This difference occurs because liquids have different densities. Density describes how much mass (or weight) is contained within a given amount of space. Some liquids, such as motor oil or vegetable oil, are more dense, meaning they are heavier for their size, while other liquids, such as water, are less dense, meaning they are lighter for the same volume. Because of these density differences, equal volumes of different liquids can have different weights.

When converting units of capacity in the U.S. customary system, we can use the same exact approach as we did in Section 3.1 & Section 3.4, where we used unit analysis on length and capacity conversions.

How to Perform Unit Analysis for Weight

- **Step 1:** Determine your starting unit.
- **Step 2:** Determine your desired unit.
- **Step 3:** Find the conversion factor needed for length.
- **Step 4:** Multiply by the conversion factor.

Example #3.5.1 🤔

Convert using unit analysis: 44 pounds to ounces

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 44 ounces. Write this as a fraction over 1.

$$\frac{44 \text{ lb}}{1}$$

- **Step 2**: Determine our desired units, which is ounces.

$$\frac{44 \text{ lb}}{1} \longrightarrow \text{oz}$$

- **Step 3**: In order to convert from pounds to ounces, we need to identify a conversion factor that relates pounds to ounces. In the table above, we find that

1 pound = 16 ounces

- **Step 4**: Using the similar methods from Section 3.1 and Section 3.4, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{44 \text{ lb}}{1} \times \frac{16 \text{ oz}}{1 \text{ lb}}$$

Now, carry out the calculation.

$$= \frac{44}{1} \times \frac{16 \text{ oz}}{1}$$

$$= 704 \text{ oz}$$

So, 44 pounds is equivalent to 704 ounces.

Example #3.5.2 🤔

The curb weight of a 2025 Toyota Tundra varies by trim. One of the higher trims like the i-FORCE MAX has a curb weight of 5,710 pounds. Convert this weight to tons using unit analysis.

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 5,710 pounds. Write this as a fraction over 1.

$$\frac{5{,}710 \text{ lb}}{1}$$

- **Step 2**: Determine our desired units, which is tons.

$$\frac{5{,}710 \text{ lb}}{1} \longrightarrow \text{T}$$

- **Step 3**: In order to convert from pounds to tons, we need to identify a conversion factor that relates pounds to ounces. In the table above, we find that

1 ton = 2,000 pounds

- **Step 4**: Using the similar methods from Section 3.1 and Section 3.4, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{5{,}710 \text{ lb}}{1} \times \frac{1 \text{ T}}{2{,}000 \text{ lb}}$$

Now, carry out the calculation.

$$= \frac{5{,}710}{1} \times \frac{1 \text{ T}}{2{,}000}$$

$$= 2.855 \text{ T}$$

So, 5,710 pounds is equivalent to 2.855 tons.

Example #3.5.3 🤔

A recycling center collected 96,000 ounces of aluminum cans. How many tons is this? Convert using unit analysis.

✅ **Solution:**

- **Step 1**: Determine our starting units, which is 96,000 ounces. Using the decimal, write this as a fraction over 1.

$$\frac{96{,}000 \text{ oz}}{1}$$

- **Step 2**: Second, we determine our desired units, which is tons.

$$\frac{96{,}000 \text{ oz}}{1} \longrightarrow \text{T}$$

- **Step 3**: In order to convert from ounces to tons, we need to identify a conversion factor that relates ounces to tons. However, in the table above, there is not a single conversion factor that relates ounces to tons. So, we need to use multiple conversion factors. Looking at the table above, we can go from

$$\text{oz} \longrightarrow \text{lb} \longrightarrow \text{T}$$

So, we can use the following conversion factors:

16 ounces = 1 pound

2,000 pounds = 1 ton

- **Step 4**: Using the similar methods from Section 3.1 and Section 3.4, we multiply by this conversion factor and the units cancel out accordingly.

$$\frac{96{,}000 \ \cancel{\text{oz}}}{1} \times \frac{1 \ \cancel{\text{lb}}}{16 \ \cancel{\text{oz}}} \times \frac{1 \ \text{T}}{2{,}000 \ \cancel{\text{lb}}}$$

Now, carry out the calculation.

$$= \frac{96{,}000}{1} \times \frac{1}{16} \times \frac{1}{2{,}000}$$

$$= 3 \text{ T}$$

So, 96,000 pounds pounds is equivalent to 3 tons.

Section 3.5: U.S. Customary System - Weight [*In-Class Exercises*]

1. A package of ground beef weighs 3.75 lb. Convert this to ounces using unit analysis.
2. A shipment of gravel weighs 5,500 lb. How many tons is this? Use unit analysis.
3. How many ounces are in 2 tons? Use unit analysis.

Answers available at the end of this section.

Answers

Section 3.5: U.S. Customary System - Weight [*In-Class Exercises*]

1. 60 oz
2. 2.75 T
3. 64,000 oz

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Section 3.5.H: Homework

Section 3.5: U.S. Customary System - Weight - Homework Exercises

For exercises 1 through 10, use unit analysis to convert to the desired units. Round to the nearest hundredths place where necessary.

(Section 3.5) - Homework Exercise #1

A bag of coffee weighs 40 oz. How many pounds is this?

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #2

A package of ground beef weighs 3.75 lb. Convert this to ounces.

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #3

A dump truck carries 7.5 tons of soil. Convert this to pounds.

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #4

A bakery uses 128 oz of flour in one batch. How many pounds is this?

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #5

A farmer harvests 18,500 lb of corn. Convert this to tons.

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #6

A cargo container holds 0.25 tons of material. How many ounces is this?

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #7

A scrap metal dealer buys copper wire. In one week, he purchased: Monday - 8,000 oz, Tuesday - 12,500 oz, Wednesday - 9,750 oz, Thursday - 11,200 oz, Friday - 14,550 oz. What was his total purchase for the week in tons?

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #8

A warehouse stores packages that each weigh 12 ounces. If the warehouse can hold a maximum of 15 tons, what is the maximum number of packages it can store?

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #9

There are two other smaller weight units used in the U.S. Customary System, called the grain (gr) and the dram (dr). These units are not as common, but they are still used in medicine, science, jewelry, and precision measurements today. The conversion factors are the following:

- 16 drams = 1 ounce
- 7,000 grains = 1 pound

Using the conversion factors above, answer the following questions:

a. A compound at a local pharmacy requires 8 drams of a substance. Convert to ounces.
b. A shipment of ammunition (bullets) weighs 35,000 grains. How many pounds is this?
c. A preparation uses 3 drams of a substance. How many grains is this?

Answer available at the end of this section.

(Section 3.5) - Homework Exercise #10

Another weight unit that is widely used, but not as uncommon for the everyday person in the U.S. Customary System is called the hundredweight (cwt). It is a weight unit between pounds and tons that is used in shipping, construction, agriculture, and industry. The conversion factor is the following:

- 1 hundredweight = 100 pounds

Many commodities are bought and sold in units of 100 pounds, not 2,000 pounds (or 1 ton). Hundredweight is commonly used for grain, livestock, produce, fertilizer, steel, and industrial materials. The loads on pallets in tons would give fractions which are harder to work with. Hundredweight avoids decimals and keeps numbers manageable.

Using the conversion factor above, answer the following questions:

a. A farmer sells 18 cwt of wheat. How many pounds is this?
b. A pallet weighs 2,350 lb. Convert this weight to hundredweight.
c. A recycling center collects 48,000 oz of aluminum. How many hundredweight is this?

Answer available at the end of this section.

Answers

(Section 3.5) - Homework Exercise #1

2.5 lb

(Section 3.5) - Homework Exercise #2

60 oz

(Section 3.5) - Homework Exercise #3

15,000 lb

(Section 3.5) - Homework Exercise #4

8 lb

(Section 3.5) - Homework Exercise #5

9.25 T

(Section 3.5) - Homework Exercise #6

8,000 oz

(Section 3.5) - Homework Exercise #7

1.75 T

(Section 3.5) - Homework Exercise #8

40,000 packages (15 tons = 480,000 ounces)

(Section 3.5) - Homework Exercise #9

a. 0.5 oz
b. 5 lb
c. 82.03 gr

(Section 3.5) - Homework Exercise #10

a. 1,800 lb
b. 23.5 cwt
c. 30 cwt

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Section 3.6: The Metric System - Length, Weight, and Capacity

Learning Objectives

- Convert measurements of length, weight, and capacity in the metric system

Introduction

The metric system is a method of measuring length, weight, and capacity based on powers of ten. Its strength lies in simplicity: units are converted by multiplying or dividing by ten, making calculations much easier than with systems like the U.S. customary system, which relies on memorizing conversions such as 12 inches in a foot or 16 ounces in a pound. Today, the metric system is used nearly worldwide, especially in science, because it is logical, consistent, and universal.

The Birth of the Metric System

The metric system was created in France in the late 1700s, when measurement systems varied widely between regions, towns, and professions. This inconsistency made trade, science, and daily life confusing. In 1790, the French National Assembly, during the French Revolution, asked the French Academy of Sciences to develop a rational and universal system of measurement that would be fair to all citizens and reliable for science, engineering, and commerce.

The Academy proposed a system based on two main ideas:

- Natural constants: Units were tied to measurable features of the natural world. For example, the meter was defined as one ten-millionth of the distance from the North Pole to the equator along a line through Paris.
- Decimal structure: Units were divided or multiplied by powers of ten. Because calculations were done by hand, this made arithmetic much simpler, faster, and more consistent than using irregular conversions.

The French government officially adopted the new system in 1795, introducing the meter for length, the gram for weight, and the liter for capacity. The system then spread across Europe and eventually worldwide, forming the foundation of today's International System of Units (SI).

Why Base Units on Nature?

Before the metric system, many units were based on the human body. For example, the "foot," the "cubit" (the length of a forearm), and the "span" (the width of a hand) were widely used. The problem was that these measurements varied from place to place and even from person to person, making them unreliable for trade or science.

The French reformers believed the new system had to be different. It needed to be natural, meaning based on the physical world instead of the human body. It also had to be universal, so it could be used by people everywhere, and unchanging, so it would remain the same over time. For this reason, the meter was tied to the size of the Earth itself.

This decision reflected the Enlightenment values of the time—reason, science, and universality over tradition. Over time, the definition of the meter was refined: first to a metal bar kept in Paris, then to the wavelength of light, and today to the speed of light.

The Origin of Metric Prefixes

Along with the base units, the French Academy of Sciences created a system of prefixes. These prefixes were used to show multiples and fractions of the base units. This was revolutionary because it replaced confusing local measurements and fractions with a system that scaled neatly by powers of ten.

The prefix system gave the metric system three key advantages:

- **Simplicity**: All conversions are based on powers of ten, making hand calculations much easier.
- **Universality**: The same prefixes can be applied to length, weight, and capacity.
- **Scalability**: Scientists can measure everything from the size of galaxies to the diameter of atoms using a single, consistent system.

The prefixes were chosen from Latin and Greek, the languages most commonly used in European science at the time.

The metric system is built around a small number of base units, each measuring a fundamental quantity.

Common Base Quantities, Units, and Abbreviations for Length, Weight, & Capacity (Metric System)

Quantity	Base Unit	Abbreviation
Length	meter	m
Weight	gram	g
Capacity	liter	L

From these base units, larger and smaller units are formed using prefixes.

The Metric Ladder

Prefix	Meaning	Power of 10
kilo–	thousand	10^3
hecto–	hundred	10^2
deka–	ten	10^1
(base unit)	one	10^0
deci–	tenth	10^{-1}
centi–	hundredth	10^{-2}
milli–	thousandth	10^{-3}

Think of the prefixes as a ladder. Each step on the ladder represents a factor of 10. The most commonly used metric prefixes are kilo-, centi-, and milli-. While the remaining prefixes are used less frequently, it is still important to understand them in order to accurately convert between different metric measurements.

Metric Units of Length

Prefix:	kilo–	hecto–	deka–	meter	deci–	centi–	milli–
Meaning:	Thousands	Hundreds	Tens	**UNIT**	Ten*ths*	Hundred*ths*	Thousand*ths*
Conversion:	1,000 meters	100 meters	10 meters	**1 METER**	$\frac{1}{10}$ of a meter	$\frac{1}{100}$ of a meter	$\frac{1}{1,000}$ of a meter
Symbol:	km	hm	dam	**m**	dm	cm	mm

So, to get any particular measurement other than a base unit, simply connect any prefix to the base unit. For example, use the prefix kilo– and the base unit meter. See the animation below:



Length in the metric system is measured using the **meter (m)** as the base unit. One meter represents the standard unit of length, and metric prefixes are applied to the meter to indicate larger or smaller measurements by multiplying or dividing by powers of ten. For example, a kilometer (km) equals 1,000 meters, a hectometer (hm) equals 100 meters, and a dekameter (dam) equals 10 meters. Similarly, smaller units such as the decimeter (dm), centimeter (cm), and millimeter (mm) represent one-tenth, one-hundredth, and one-thousandth of a meter, respectively. This consistent base-10 structure makes conversions within the metric system straightforward and efficient.

This same base-10 system applies to weight and capacity in the metric system. Weight is measured using the gram (g) as the base unit, and metric prefixes are applied in the same way to represent larger or smaller amounts. For example, a kilogram (kg) equals 1,000 grams, a hectogram (hg) equals 100 grams, and a milligram (mg) equals one-thousandth of a gram. Because these units follow the same power-of-ten structure as metric length, conversions between grams are performed using the same decimal-moving process.

Similarly, capacity is measured using the liter (L) as the base unit. Larger and smaller volumes are expressed by applying metric prefixes to the liter. A kiloliter (kL) equals 1,000 liters, while a milliliter (mL) equals one-thousandth of a liter. This consistency across length, weight, and capacity allows the same conversion techniques—such as the metric ladder or unit analysis—to be used regardless of the type of measurement. Here are similar tables to represent weight and capacity.

Metric Units of Weight

Prefix:	kilo–	hecto–	deka–	gram	deci–	centi–	milli–
Meaning:	Thousands	Hundreds	Tens	**UNIT**	Ten*ths*	Hundred*ths*	Thousand*ths*
Conversion:	1,000 grams	100 grams	10 grams	**1 GRAM**	$\frac{1}{10}$ of a gram	$\frac{1}{100}$ of a gram	$\frac{1}{1,000}$ of a gram
Symbol:	kg	hg	dag	**g**	dg	cg	mg

Metric Units of Capacity

Prefix:	kilo–	hecto–	deka–	liter	deci–	centi–	milli–
Meaning:	Thousands	Hundreds	Tens	**UNIT**	Ten*ths*	Hundred*ths*	Thousand*ths*
Conversion:	1,000 liters	100 liters	10 liters	**1 LITER**	$\frac{1}{10}$ of a liter	$\frac{1}{100}$ of a liter	$\frac{1}{1,000}$ of a liter
Symbol:	kL	hL	daL	**L**	dL	cL	mL

So, how do we convert fro one metric measurement to another? We could use unit analysis, but that will be a struggle to memorize every possible combination that can occur. Instead, we will approach this using the ladder approach.

Converting Metric Measurements Using a Left-to-Right Metric Ladder

The metric ladder is arranged from left to right, with larger units on the left and smaller units on the right:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

To convert from one metric measurement to another, apply the following steps:

- **Step 1:** Determine your starting unit.
- **Step 2:** Determine your desired unit.
- **Step 3:** Determine the direction of movement and count the number of steps.
  - If you move to the right, you are moving to smaller units.
  - If you move to the left, you are moving to larger units.
- **Step 4:** Move the decimal point.
  - Move the decimal point one place for each step.
  - Move the decimal to the right if you moved right. (Add zeros, if necessary).
  - Move the decimal to the left if you moved left. (Add zeros, if necessary).
- **Step 5:** Write the final answer with the correct unit.

The “King Henry” Mnemonic for Metric Prefixes

The King Henry mnemonic is a common memory aid used to help students remember the order of metric prefixes from largest to smallest. Because the metric system is based on powers of ten, keeping these prefixes in the correct sequence is essential for converting between metric units. A widely used version of the mnemonic is:

$$\text{King Henry Died By Drinking Chocolate Milk}$$

Each word corresponds to a metric prefix, listed in order from largest to smallest:

The "King Henry" Mnemonic for Metric Prefixes

Mnemonic Word	Metric Prefix	Meaning
King	kilo– (k)	1000
Henry	hecto– (h)	100
Died	deka– (da)	10
By	base unit (m, g, L)	1
Drinking	deci– (d)	0.1
Chocolate	centi– (c)	0.01
Milk	milli– (m)	0.001

The mnemonic helps students visualize the metric ladder:

$$\text{king} \longleftrightarrow \text{henry} \longleftrightarrow \text{died} \longleftrightarrow \text{by} \longleftrightarrow \text{drinking} \longleftrightarrow \text{chocolate} \longleftrightarrow \text{milk}$$

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

Moving to the right on the mnemonic means moving to smaller units, while moving to the left means moving to larger units. Each step represents a change by a factor of 10. The only challenge is between deka- and deci-, since they both begin with the letter d. One way to remember is that deci- sounds like decimal, so that is forced to go after the base unit.

Here are some other mnemonic that you could also use:

King Henry Drives By Drinking Chocolate Milk

King Henry Delivers Big Delicious Chocolate Muffins

King Henry Drove By Drinking Cold Milk

King Henry Designs Big Drinking Chocolate Milk

or you can design one on your own, such as:

King Henry Drives By Downtown Costa Mesa

The specific words in a mnemonic are not as important as the order they represent. Students are encouraged to use any version that helps them remember the sequence of metric prefixes, even if the sentence isn't grammatically correct.

Example #3.6.1 🤔

Convert 3.6 decimeters to centimeters.

✅ **Solution:**

If you haven't memorized the metric ladder, then recreate one:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is meters (the base unit).
- **Step 2**: Determine our desired units, which is centimeters.

- **Step 3**: Determine the direction of movement and count the number of steps. Going from meters (the base unit) to centimeters is 2 steps to the right.
- **Step 4**: Move the decimal point one place for each step. So, move the decimal point 2 places to the right. (Add zeros, if necessary).

$$3.60$$

$$3.\underset{\rightarrow}{60}.$$

2 places to the right

$$360.$$

- **Step 5**: Write the final answer with the correct unit.

So, 3.6 meters (m) is equivalent to 360 centimeters (cm).

Example #3.6.2 🤔

Convert 5,000 decigrams to hectograms.

✅ **Solution:**

If you haven't memorized the metric ladder, then recreate one:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is decigrams.
- **Step 2**: Determine our desired units, which is hectograms.
- **Step 3**: Determine the direction of movement and count the number of steps. Going from decigrams to hectograms is 3 steps to the left.
- **Step 4**: Move the decimal point one place for each step. So, move the decimal point 3 places to the left.

$$5000.$$

$$5.\underset{\leftarrow}{000}.$$

3 places to the left

$$5.$$

- **Step 5**: Write the final answer with the correct unit.

So, 5,000 decigrams (dm) is equivalent to 5 hectograms (hm).

Example #3.6.3 🤔

A walking path in a park is 0.85 km long. Convert the length to meters.

✅ **Solution:**

If you haven't memorized the metric ladder, then recreate one:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is kilometers.
- **Step 2**: Determine our desired units, which is meters (the base unit).

- **Step 3**: Determine the direction of movement and count the number of steps. Going from kilometers to meters (the base unit) is 3 steps to the right.
- **Step 4**: Move the decimal point one place for each step. So, move the decimal point 3 places to the right.

$$0.85$$

$$0.\underset{\longrightarrow}{850}.$$

3 places to the right

$$850.$$

- **Step 5**: Write the final answer with the correct unit.

So, 0.85 kilometers (km) is equivalent to 850 meters (m).

Example #3.6.4 🤔

Convert 123 milliliters to kiloliters.

✅ **Solution:**

If you haven't memorized the metric ladder, then recreate one:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is milliliters.
- **Step 2**: Determine our desired units, which is kiloliters.
- **Step 3**: Determine the direction of movement and count the number of steps. Going from milliliters to kiloliters is 6 steps to the left.
- **Step 4**: Move the decimal point one place for each step. So, move the decimal point 6 places to the left.

$$123.$$

$$0.\underset{\longleftarrow}{000123}.$$

6 places to the left

$$0.000123$$

- **Step 5**: Write the final answer with the correct unit.

So, 123 milliliters (mL) is equivalent to 5 kiloliters (kL).

Example #3.6.5 🤔

Convert 0.05123 dag to dg.

✅ **Solution:**

If you haven't memorized the metric ladder, then recreate one:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is dekagrams.
- **Step 2**: Determine our desired units, which is decigrams.
- **Step 3**: Determine the direction of movement and count the number of steps. Going from dekagrams to decigrams is 2 steps to the right.

- **Step 4**: Move the decimal point one place for each step. So, move the decimal point 2 places to the right.

$$0.05123$$

$$0.\overset{\rightarrow}{05}.123$$

$$2 \text{ places to the right}$$

$$5.123$$

- **Step 5**: Write the final answer with the correct unit.

So, 0.05123 dam is equivalent to 5.123 dg.

Section 3.6: The Metric System - Length, Weight, and Capacity [*In-Class Exercises*]

1. The Statue of Liberty is 93 meters tall (including pedestal). Convert this measure to centimeters.
2. The Eiffel Tower is about 330 m tall. Convert this to kilometers.
3. A typical bowling ball weighs 7.26 kg. Convert this to grams.
4. A smartphone has a mass of 180 g. Convert this to hectograms.
5. The typical large soda bottles that are sold in grocery stores are 2 liter bottles. Convert this to milliliters.
6. A coffee cup holds 350 mL of coffee. Convert this to deciliters.
7. A package weighs 3.5 dag. Convert this to milligrams.
8. A rain barrel collects 35 hL of water. Convert this to deciliters.

Answers available at the end of this section.

Answers

Section 3.6: The Metric System - Length, Weight, and Capacity [*In-Class Exercises*]

1. 9,300 cm
2. 0.33 km
3. 7,260 g
4. 1.8 hg
5. 2,000 mL
6. 3.5 dL
7. 35,000 mg
8. 35,000 dL

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Section 3.6.H: Homework

Section 3.6: The Metric System - Length, Weight, and Capacity - Homework Exercises

For exercises 1 through 15, use unit analysis to convert to the desired units. Round to the nearest hundredths place where necessary.

(Section 3.6) - Homework Exercise #1

An Olympic pool is 50 m long. Convert this to millimeters.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #2

A city block is 8 dam long. Convert this to decimeters.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #3

The Space Needle in Seattle, Washington is approximately 184 m tall. Convert this to centimeters.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #4

An adult African elephant can weigh about 6000 kg. Convert this to grams.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #5

A certain car fuel tank holds 55 L of gasoline. Convert this measure to daL.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #6

The CN Tower in Toronto, Canada is 553 m tall. Convert this to kilometers.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #7

A water bottle contains 750 mL. Convert this to deciliters.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #8

A dumbbell has a mass of 45 hg. Convert this to kilograms.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #9

An Olympic swimming pool holds about 2,500,000 L of water. Convert this to mL.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #10

A recipe uses 80 dg of spice. Convert this to grams.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #11

A tank holds 6 hL. Convert this to centiliters.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #12

A parcel has a mass of 2.4 kg. Convert this to hectograms.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #13

A liquid sample measures 62 dL. Convert this to liters.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #14

U.S. Route 6 (US 6), known as the Grand Army of the Republic Highway, is a historic east-west transcontinental highway 

stretching roughly 5,193 km (3,200 miles) long from Bishop, California to Provincetown, Massachusetts. Convert this distance to meters.Image of a U.S. Highway Route 6 sign.

Answer available at the end of this section.

(Section 3.6) - Homework Exercise #15

A block of cheese has a mass of 32 dag. Convert this to grams.

Answer available at the end of this section.

Answers

(Section 3.6) - Homework Exercise #1
50,000 mm

(Section 3.6) - Homework Exercise #2
1.56 dm

(Section 3.6) - Homework Exercise #3
18,400 cm

(Section 3.6) - Homework Exercise #4
6,000,000 g

(Section 3.6) - Homework Exercise #5
5.5 daL

(Section 3.6) - Homework Exercise #6
0.553 m

(Section 3.6) - Homework Exercise #7
7.5 dL

(Section 3.6) - Homework Exercise #8
4.5 kg

(Section 3.6) - Homework Exercise #9
2,500,000,000,000 mL

(Section 3.6) - Homework Exercise #10
8 g

(Section 3.6) - Homework Exercise #11

60,000 cL

(Section 3.6) - Homework Exercise #12

24 hg

(Section 3.6) - Homework Exercise #13

6.2 L

(Section 3.6) - Homework Exercise #14

5,193,000 m

(Section 3.6) - Homework Exercise #15

320 g

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Section 3.7: The Metric System - Area and Volume

Learning Objectives

- Convert measurements of area and volume in the metric system

In the previous section, we worked with one-dimensional measurements in the metric system, such as length (meters), weight/mass (grams), and capacity (liters). These measurements describe quantities along a single dimension. However, many real-world situations require us to measure space, not just length.

To measure two-dimensional space, we use area, which describes the amount of surface a shape covers. In the metric system, area is expressed in square units, such as square meters (m^2), square centimeters (cm^2), and square kilometers (km^2). The term "square" indicates that both the length and width are being measured, resulting in units that are multiplied by themselves.

To measure three-dimensional space, we use volume, which describes the amount of space an object occupies. Volume is measured in cubic units, such as cubic meters (m^3), cubic centimeters (cm^3), and cubic kilometers (km3). The term "cubic" reflects that length, width, and height are all being considered.

One of the advantages of the metric system is that conversions between units follow consistent patterns based on powers of ten. When working with area and volume, these patterns extend naturally from one-dimensional measurements, but with an important difference:

- Area conversions involve squaring the conversion factor.
- Volume conversions involve cubing the conversion factor.

Understanding how these units relate will allow us to convert between different metric measurements of area and volume efficiently and accurately.

Converting Metric Measurements of Lentgth, Area, and Volume

Start with the metric ladder:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

To convert from any dimension metric measurement to another dimension metric measurement, respectively, apply the following steps:

- **Step 1:** Determine your starting unit.
- **Step 2:** Determine your desired unit.
- **Step 3:** Determine the direction of movement and count the number of steps. (Ignore dimensions here).
  - If you move to the right, you are moving to smaller units.
  - If you move to the left, you are moving to larger units.
- **Step 4:** Move the decimal point.
  - If you are working with **length (one-dimension)**, move the decimal point **one** places for each step. (x1) **
  - If you are working with **area (two-dimension)**, move the decimal point **two** places for each step. (x2)
  - If you are working with **volume (three-dimension)**, move the decimal point **three** places for each step. (x3)
    - Move the decimal to the right if you moved right. (Add zeros, if necessary).
    - Move the decimal to the left if you moved left. (Add zeros, if necessary).

- **Step 5:** Write the final answer with the correct unit.

** This is the same step for length from the previous section.

Let's see how similar the process is to the examples from Section 3.6.

Example #3.7.1 🤔

a. Convert 12 m to centimeters (or cm).
b. Convert 12 m^2 to square centimeters or (cm^2).
c. Convert 12 m^3 to cubic centimeters or (cm^3).

✅ **Solution:**

a. Start with the metric ladder:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is meters (the base unit).
- **Step 2**: Determine our desired units, which is centimeters.
- **Step 3**: Determine the direction of movement and count the number of steps. (Ignore dimensions here). Going from meters (the base unit) to centimeters is 2 steps to the right.
- **Step 4**: Since we are working with **length (one-dimension)**, move the decimal point **one** place for each step. So, move the decimal point 2 places to the right. (Add zeros, if necessary).

$$12$$

$$12.\underset{\text{2 places to the right}}{\underset{\longrightarrow}{00}}.$$

$$1{,}200.$$

- **Step 5**: Write the final answer with the correct unit.

So, 12 meters (m) is equivalent to 1,200 centimeters (cm).

✅ **Solution:**

b. Start with the metric ladder:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is square meters (the base unit).
- **Step 2**: Determine our desired units, which is square centimeters.
- **Step 3**: Determine the direction of movement and count the number of steps. (Ignore dimensions here). Going from meters (the base unit) to centimeters is 2 steps to the right.
- **Step 4**: Since we are working with **area (two-dimension)**, move the decimal point **two** places for each step. So, move the decimal point 2 x 2 = 4 places to the right. (Add zeros, if necessary).

$$12$$

$$12.\underset{\text{4 places to the right}}{\underset{\longrightarrow}{0000}}.$$

$$120{,}000.$$

- **Step 5**: Write the final answer with the correct unit.

So, 12 square meters (m^2) is equivalent to 120,000 square centimeters (cm^2).

✅ **Solution:**

c. Start with the metric ladder:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is cubic meters (the base unit).
- **Step 2**: Determine our desired units, which is cubic centimeters.
- **Step 3**: Determine the direction of movement and count the number of steps. (Ignore dimensions here). Going from meters (the base unit) to centimeters is 2 steps to the right.
- **Step 4**: Since we are working with **volume (three-dimension)**, move the decimal point **three** places for each step. So, move the decimal point 3 x 2 = 6 places to the right. (Add zeros, if necessary).

$$12$$

$$12.000000.$$

6 places to the right

$$12{,}000{,}000.$$

- **Step 5**: Write the final answer with the correct unit.

So, 12 cubic meters (m^3) is equivalent to 12,000,000 cubic centimeters (cm^3).

Example #3.7.2 🤔

Convert 35,000 mm^2 to m^2.

✅ **Solution:**

Start with the metric ladder:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is square millimeters (the base unit).
- **Step 2**: Determine our desired units, which is square meters.
- **Step 3**: Determine the direction of movement and count the number of steps. (Ignore dimensions here). Going from millimeters (the base unit) to meters is 3 steps to the left.
- **Step 4**: Since we are working with **area (two-dimension)**, move the decimal point **two** places for each step. So, move the decimal point 2 x 3 = 6 places to the left. (Add zeros, if necessary).

$$35{,}000.$$

$$0.035000.$$

6 places to the left

$$0.035$$

- **Step 5**: Write the final answer with the correct unit.

So, 35,000 square millimeters (mm^2) is equivalent to 0.035 square meters (m^2).

Example #3.7.3 🤔

Convert 1 km^3 to m^3.

✅ **Solution:**

Start with the metric ladder:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is cubic kilometers (the base unit).
- **Step 2**: Determine our desired units, which is cubic meters.
- **Step 3**: Determine the direction of movement and count the number of steps. (Ignore dimensions here). Going from kilometers (the base unit) to meters is 3 steps to the right.
- **Step 4**: Since we are working with **volume (three-dimension)**, move the decimal point **three** places for each step. So, move the decimal point 3 x 3 = 9 places to the right. (Add zeros, if necessary).

$$1$$

$$1.\underrightarrow{000000000}.$$

$$\text{9 places to the right}$$

$$1{,}000{,}000{,}000.$$

- **Step 5**: Write the final answer with the correct unit.

So, 1 cubic kilometer (km^3) is equivalent to 1,000,000,000 cubic meters (m^3).

Example #3.7.4 🤔

Convert 0.25 square dekameters to square millimeters.

✅ **Solution:**

Start with the metric ladder:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is square dekameters (the base unit).
- **Step 2**: Determine our desired units, which is square millimeters.
- **Step 3**: Determine the direction of movement and count the number of steps. (Ignore dimensions here). Going from dekameters (the base unit) to millimeters is 4 steps to the right.
- **Step 4**: Since we are working with **area (two-dimension)**, move the decimal point **two** places for each step. So, move the decimal point 2 x 4 = 8 places to the right. (Add zeros, if necessary).

$$0.25$$

$$0.\underrightarrow{25000000}.$$

$$\text{8 places to the right}$$

$$25{,}000{,}000.$$

- **Step 5**: Write the final answer with the correct unit.

So, 0.25 square dekameters (dam^2) is equivalent to 25,000,000 square meters (mm^2).

Example #3.7.5 🤔

Convert 275,000 dm^3 to m^3.

✅ **Solution:**

Start with the metric ladder:

$$\text{kilo} \longleftrightarrow \text{hecto} \longleftrightarrow \text{deka} \longleftrightarrow \text{base unit} \longleftrightarrow \text{deci} \longleftrightarrow \text{centi} \longleftrightarrow \text{milli}$$

- **Step 1**: Determine our starting units, which is cubic decimeters (the base unit).
- **Step 2**: Determine our desired units, which is cubic meters.
- **Step 3**: Determine the direction of movement and count the number of steps. (Ignore dimensions here). Going from decimeters (the base unit) to meters is 1 step to the left.
- **Step 4**: Since we are working with **volume (three-dimension)**, move the decimal point **three** places for each step. So, move the decimal point 3 x 1 = 3 places to the left. (Add zeros, if necessary).

$$275{,}000.$$

$$275.000.$$

$\longleftarrow$ 3 places to the left

$$275$$

- **Step 5**: Write the final answer with the correct unit.

So, 275,000 cubic decimeters (dm^3) is equivalent to 275 cubic meters (m^3).

Section 3.7: The Metric System - Area and Volume [*In-Class Exercises*]

1. Convert 19 m to centimeters (or cm).
2. Convert 19 m^2 to centimeters (or cm^2).
3. Convert 19 m^3 to centimeters (or cm^3).
4. Convert 775,000 mm^2 to m^2.
5. Convert 0.8675309 hm^3 to dm^3.

Answers available at the end of this section.

Answers

Section 3.7: The Metric System - Area and Volume [*In-Class Exercises*]

1. 1,900 cm
2. 190,000 cm^2
3. 19,000,000 cm^3
4. 0.775 m^2
5. 867,530,900 dm^3

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Section 3.7.H: Homework

Section 3.7: The Metric System - Area and Volume - Homework Exercises

For exercises 1 through 14, use unit analysis to convert to the desired units. Round to the nearest hundredths place where necessary.

(Section 3.7) - Homework Exercise #1

Convert 2.6 m^2 to mm^2.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #2

Convert 66,000 cubic centimeters to cubic meters.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #3

A contractor needs to cover a floor that is 45 m^2. How many square centimeters is this?

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #4

A small park has an area of 2.5 km^2. Convert this to square meters.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #5

A construction site requires 0.85 m^3 of concrete. Convert this to cubic centimeters.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #6

A factory produces 750,000 mm^2. Convert this to square meters.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #7

A shipping container has a volume of 12,000 cm^3. Convert this to cubic decimeters.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #8

A room has an area of 85,000 dm^2. Convert this to square meters.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #9

A large warehouse has a volume of 50,000 m^3. Convert this to cubic hectometers.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #10

A package has a volume of 0.45 m^3. Convert this to cubic millimeters.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #11

The Statue of Liberty in New York is approximately 8,000 m^3 to 10,000 m^3. Convert this approximation to cm^3.

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #12

Knott's Berry Farm in Buena Park, California, covers approximately 57 acres, which converts to roughly 0.23 square kilometers. How many square meters is this?

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #13

Disneyland in Anaheim, California, covers approximately 500 acres, which converts to roughly 2.02 square kilometers. How many square dekameters is this?

Answer available at the end of this section.

(Section 3.7) - Homework Exercise #14

The Salton Sea is California's largest inland lake. It is located in the southeastern desert and was formed accidentally in 1905 when the Colorado River breached irrigation canals. The total volume of the Salton Sea is approximately 7,400,000,000 m^3. Convert this volume to cubic kilometers.

Answer available at the end of this section.

Answers

(Section 3.7) - Homework Exercise #1

2,600,000 mm^2

(Section 3.7) - Homework Exercise #2

0.06 m^3

(Section 3.7) - Homework Exercise #3

450,000 cm^2

(Section 3.7) - Homework Exercise #4

2,500,000 m^2

(Section 3.7) - Homework Exercise #5

850,000 cm^3

(Section 3.7) - Homework Exercise #6

0.75 m^2

(Section 3.7) - Homework Exercise #7

12 dm^3

(Section 3.7) - Homework Exercise #8

850 m^2

(Section 3.7) - Homework Exercise #9

0.05 hm^3

(Section 3.7) - Homework Exercise #10

450,000,000 mm^3

(Section 3.7) - Homework Exercise #11

8,000,000,000 cm^3 to 10,000,000,000 cm^3

(Section 3.7) - Homework Exercise #12

230,000,000 km^3

(Section 3.7) - Homework Exercise #13

20,200 dam^3

(Section 3.7) - Homework Exercise #14

7.4 km^3

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Section 3.8: Conversions Between U.S. Customary and Metric Systems

Learning Objectives

- Convert measurements between the U.S. customary system and the metric system

Up to this point, we have worked within individual systems of measurement, using familiar units such as inches, feet, and pounds in the U.S. customary system, and meters, grams, and liters in the metric system. However, in many real-world situations, it is necessary to move between these two systems.

Converting between U.S. customary and metric units is different from converting within the metric system. While metric conversions often follow predictable patterns based on powers of ten, conversions between systems require the use of conversion factors, specific relationships that relate one unit to another. For example, we use facts such as 1 inch ≈ 2.54 centimeters or 1 mile ≈ 1.609 kilometers.

These conversions are especially important in fields such as science, engineering, medicine, and international trade, where the metric system is widely used, even in countries that commonly use U.S. customary units. Being able to convert accurately ensures clear communication and avoids costly errors.

In this section, we will learn how to use conversion factors to move between systems efficiently, reinforcing the idea that understanding relationships between units is key to solving measurement problems.

We begin by examining a list of some of the most common conversion factors between the two systems.

Common English to Metric Equivalents

Length	Mass (Weight)	Capacity (Volume)
1 in = 2.54 cm	1 kg ≈ 2.2 lb	1 pt ≈ 473.2 mL
1 mi ≈ 1.61 km		1 gal ≈ 3.785 L

There are many other conversion equivalents, such as 1 ft = 0.3048 m, 1 yd = 0.9144 m, 1 oz ≈ 28.35 g, 1 fl oz ≈ 29.57 m, and 1 q ≈ 0.946 L. Not to mention the reciprocal of those equivalents, such as 1 m ≈ 3.2808 ft, 1 m ≈ 1.0936 yd, 1 g ≈ 0.03527 oz, 1 mL ≈ 0.033818 fl oz, 1 L ≈ 1.05708 q. Each of these conversions can be completed using unit analysis in a single step.

However, memorizing all possible conversion factors is neither practical nor necessary. In many situations, not every conversion factor is provided. For example, converting between less common units such as barrels and liters.

Instead, we rely on a small set of familiar relationships and use unit analysis and the ladder method to build the desired conversion step by step. Of course, knowing additional conversion factors can make the process more efficient, but understanding the method is far more important than memorization.

Conversion Cases Between Systems

Here are the conversion cases between the two systems and the method(s) to use.

1. Direct Conversion (One Step)
   - Example: in → cm or cm → in
   - **Method**: unit analysis
2. Within-System First, Then Convert
   - Example: m → in
   - Convert within metric first, then to U.S.
   - **Method**: ladder method → unit analysis
3. Convert First, Then Within-System
   - Example: in → m
   - Convert to metric first, then adjust units
   - **Method**: unit analysis → ladder method
4. Multi-Step Within U.S. Customary
   - Example: ft → cm
   - Convert within U.S. first, then to metric
   - **Method**: unit analysis → unit analysis
5. Multi-Step Within Both Systems (Less Common but Possible)
   - Example: yd → km
   - Convert within U.S. → convert to metric → adjust metric units
   - **Method**: unit analysis → unit analysis → ladder method

For conversions that are not direct when converting from metric to U.S., just reverse the steps and methods for 2. through 5.

Example #3.8.1 🤔

Convert 20 inches to centimeters.

✅ **Solution:**

$$\frac{20\text{ in}}{1} \longrightarrow \text{cm}$$

We are converting from U.S. customary to metric. There is a **direct conversion** between inches and centimeters. Use the conversion factor of inches to centimeters:

$$1\text{ in} = 2.54\text{ cm}$$

So, apply the unit analysis:

$$\frac{20\ \cancel{\text{in}}}{1} \times \frac{2.54\text{ cm}}{1\ \cancel{\text{in}}}$$

$$= \frac{20}{1} \times \frac{2.54\text{ cm}}{1}$$

$$= 50.8\text{ cm}$$

So, 20 inches is equivalent to 50.8 centimeters.

Example #3.8.2 🤔

Convert 50 kilometers to miles.

✅ **Solution:**

$$\frac{50\text{ km}}{1} \longrightarrow \text{mi}$$

We are converting from metric to U.S. customary. There is a **direct conversion** between miles and kilometers. Use the conversion factor of miles to kilometers:

$$1\text{ mi} \approx 1.61\text{ km}$$

So, apply the unit analysis:

$$\frac{50\ \cancel{\text{km}}}{1} \times \frac{1\text{ mi}}{1.61\ \cancel{\text{km}}}$$

$$= \frac{50}{1} \times \frac{1\text{ mi}}{1.61}$$

$$= 31.06\text{ mi}$$

So, 50 kilometers is equivalent to 31.06 miles.

Example #3.8.3 🤔

Convert 100 inches to meters.

✅ **Solution:**

$$\frac{100\text{ in}}{1} \longrightarrow \text{m}$$

We are converting from U.S. customary to metric units. There is **no direct conversion** between inches and meters. Then, the path we need is in → cm → m, so we proceed in steps. Use the conversion factor of inches to centimeters:

$$1\text{ in} = 2.54\text{ cm}$$

Apply the unit analysis:

$$\frac{100\ \cancel{\text{in}}}{1} \times \frac{2.54\text{ cm}}{1\ \cancel{\text{cm}}}$$

$$= \frac{100}{1} \times \frac{2.54\text{ cm}}{1}$$

$$= 254\text{ cm}$$

Now, convert centimeters to meters using the ladder method:

$$254.$$

$$2.54.$$

$$\leftarrow$$

2 places to the left

$$2.54\text{ m}$$

So, 100 inches is equivalent to 2.54 meters.

Example #3.8.4 🤔

Convert 5 kilograms to ounces.

✅ **Solution:**

$$\frac{5\text{ kg}}{1} \longrightarrow \text{oz}$$

We are converting from metric to U.S. customary units.There is **no direct conversion** between kilograms and ounces. Then, the path we need is kg → lb → oz, so we proceed in steps.

First, use the conversion factors of kilograms to pounds and pounds to ounces:

$$1\text{ kg} = 2.2\text{ lb}$$

$$1\text{ lb} = 16\text{ oz}$$

Now, apply the unit analysis:

$$\frac{5\ \cancel{\text{kg}}}{1} \times \frac{2.2\ \cancel{\text{lb}}}{1\ \cancel{\text{kg}}} \times \frac{16\text{ oz}}{1\ \cancel{\text{lb}}}$$

$$= \frac{5}{1} \times \frac{2.2}{1} \times \frac{16\text{ oz}}{1}$$

$$= 176\text{ oz}$$

So, 5 kilograms is equivalent to 176 ounces.

Example #3.8.5 🤔

Convert 12.5 quarts to milliliters.

✅ **Solution:**

$$\frac{12.5\text{ qt}}{1} \longrightarrow \text{mL}$$

We are converting from U.S. customary to metric units. There is **no direct conversion** between quarts and milliliters. Then, the path we need is qt → pt → mL, so we proceed in steps.

First, use the conversion factors of quarts to pints and pints to milliliters:

$$1\text{ qt} = 2\text{ pt}$$

$$1\text{ pt} = 473.2\text{ mL}$$

Now, apply the unit analysis:

$$\frac{12.5\ \cancel{\text{qt}}}{1} \times \frac{2\ \cancel{\text{pt}}}{1\ \cancel{\text{qt}}} \times \frac{473.2\text{ mL}}{1\ \cancel{\text{pt}}}$$

$$= \frac{12.5}{1} \times \frac{2}{1} \times \frac{473.2}{1}$$

$$= 11{,}830\text{ mL}$$

So, 12.5 quarts is equivalent to 11,830 milliliters.

Now, that we can convert linear measurements from one system to another, we will extend these ideas to area and volume, where units are squared and cubed, respectively.

Example #3.8.6 🤔

Convert 20 square inches to square centimeters.

✅ **Solution:**

$$\frac{20\ \text{in}^2}{1} \longrightarrow \quad \text{cm}^2$$

We are converting from U.S. customary to metric units. There is a **direct conversion** between inches and centimeters. Use the conversion factor of inches to centimeters and square the relationship:

$$1\ \text{in} = 2.54\ \text{cm}$$
$$(1\ \text{in})^2 = (2.54\ \text{cm})^2$$
$$1\ \text{in}^2 = 6.4516\ \text{cm}^2$$

Now, apply the unit analysis:

$$\frac{20\ \cancel{\text{in}^2}}{1} \times \frac{6.4516\ \text{cm}^2}{1\ \cancel{\text{in}^2}}$$

Now, carry out the calculation:

$$= \frac{20}{1} \times \frac{6.4516\ \text{cm}^2}{1}$$
$$= 129.032\ \text{cm}^2$$

So, 20 square inches is equivalent to 129.032 square centimeters.

Example #3.8.7 🤔

Convert 7 cubic meters to cubic inches.

✅ **Solution:**

$$\frac{7\ \text{m}^3}{1} \longrightarrow \quad \text{in}^3$$

We are converting from metric to U.S. customary units. There is **no direct conversion** between cubic meters and cubic inches. Then, the path we need is $m^3 \rightarrow cm^3 \rightarrow in^3$, so we proceed in steps.

First, convert cubic meters to cubic centimeters using the ladder method:

$$7.$$
$$7.000000.$$
$$\text{6 places to the right}$$
$$7{,}000{,}000\ \text{cm}^3$$

Next, use the conversion factor of inches to centimeters and cube the relationship:

$$1 \text{ in} = 2.54 \text{ cm}$$
$$(1 \text{ in})^3 = (2.54 \text{ cm})^3$$
$$1 \text{ in}^3 = 16.387064 \text{ cm}^3$$

Now, apply the unit analysis:

$$\frac{7{,}000{,}000 \ \cancel{\text{cm}^3}}{1} \times \frac{1 \text{ in}^3}{16.387064 \ \cancel{\text{cm}^3}}$$

Now, carry out the calculation:

$$= \frac{7{,}000{,}000}{1} \times \frac{1 \text{ in}^3}{16.387064}$$
$$= 427{,}166.208663 \text{ in}^3$$

So, 7 cubic meters is approximately 427,166.21 cubic inches.

Now that we have worked with both volume and capacity, we can connect these ideas within the metric system, just as we did in Section 3.4. One of the most important relationships is that one cubic centimeter is exactly equal to one milliliter. This shows that geometric volume and liquid capacity are directly linked. Extending this idea to larger units, one cubic meter is exactly equal to 1,000 liters. Because of these relationships, converting between metric volume and capacity is straightforward and does not require complex conversion factors. Using unit analysis and the ladder method when needed, we can move easily between cubic units and liquid measurements, reinforcing the consistency and simplicity of the metric system.

Metric Equivalents for Capacity and Volume

Capacity (Volume)
1 cm^3 = 1 mL
1 m^3 = 1,000 L

This works because both units measure the same physical idea, volume (space), but in slightly different contexts. A cubic centimeter (cm^3) describes volume geometrically, meaning the space inside a cube that is 1 cm on each side. A milliliter (mL), on the other hand, is a unit of liquid capacity, commonly used for measuring fluids like water, medicine, or beverages.

The connection comes from how the metric system is designed. If you take a cube that is 1 cm × 1 cm × 1 cm, the space inside that cube holds exactly 1 milliliter of liquid. Because the metric system is based on powers of 10 and consistent relationships, this equivalence is exact, not an approximation.

Example #3.8.8 🤔

Convert 35 cubic centimeters to milliliters.

✅ **Solution:**

$$\frac{35 \text{ cm}^3}{1} \longrightarrow \text{mL}$$

We are converting from U.S. customary to metric units. There is a **direct conversion** between cubic centimeters and milliliters.

$$1\ \text{cm}^3 = 1\ \text{mL}$$

So, apply the unit analysis:

$$\frac{35\ \cancel{\text{cm}^3}}{1} \times \frac{1\ \text{mL}}{1\ \cancel{\text{cm}^3}}$$

Now, carry out the calculation:

$$= \frac{35}{1} \times \frac{1\ \text{ml}}{1}$$

$$= 35\ \text{mL}$$

So, 35 cubic centimeters is equivalent to 35 milliliters.

NOTE: Since this is a 1-to-1 relationship, there is no need to use unit analysis. Simply replace cm^3 with mL, or vice versa.

Example #3.8.9 🤔

Convert 3 hectoliters to cubic meters.

✅ **Solution:**

$$\frac{3\ \text{hL}}{1} \longrightarrow \quad \text{m}^3$$

There is **no direct conversion** between hectoliters and cubic meters. Then, the path we need is hL → L → m^3, so we proceed in steps.

First, convert hectoliters to liters using the ladder method:

$$3.$$

$$3.\underset{\rightarrow}{00}.$$

$$\text{2 places to the right}$$

$$300\ \text{L}$$

Next, use the conversion liters to cubic meters:

$$1{,}000\ \text{L} = 1\ \text{m}^3$$

Now, apply the unit analysis:

$$\frac{300\ \cancel{\text{L}}}{1} \times \frac{1\ \text{m}^3}{1{,}000\ \cancel{\text{L}}}$$

Now, carry out the calculation:

$$= \frac{300}{1} \times \frac{1\ \text{m}^3}{1{,}000}$$

$$= 0.3\ \text{m}^3$$

So, 30 hectoliters is equivalent to 0.3 cubic meters.

Section 3.8: Conversions Between U.S. Customary and Metric Systems - Length [*In-Class Exercises*]

1. The average healthy weight for most adult domestic cats is roughly 8–12 pounds, though this varies significantly by breed, sex, and frame. While 10 pounds is often cited as a typical average, smaller breeds may be healthy at 6–8 lbs, while larger breeds like Maine Coons can be healthy at 15–25 lbs. If a Maine Coon cat weighed 25 pounds, what is its weight in kilograms?
2. Convert 250 in^3 to cm^3.
3. Convert 1.2 m^3 to gallons.
4. Convert 60 ounces to centiliters.
5. Convert 276 cubic centimeters to milliliters.
6. Convert 2 m^3 to ft^3.
7. Convert 5,500 deciliters to cubic meters.

Answers available at the end of this section.

Answers

Section 3.8: Conversions Between U.S. Customary and Metric Systems - Length [*In-Class Exercises*]

1. ≈11.36 kg
2. 4096.77 cm^3
3. ≈317.04 gal
4. ≈177.45 cL
5. 276 mL
6. 70.629 ft^3
7. 0.55 m^3

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Section 3.8.H: Homework

Section 3.8: Conversions Between U.S. Customary and Metric Systems - Homework Exercises

For exercises 1 through 19, to convert to the desired units. Round to the nearest hundredths place where necessary.

(Section 3.8) - Homework Exercise #1

A swimming pool is 50 meters long. A coach wants to describe it in feet. What is the length in feet?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #2

A piece of fabric is 2.5 meters long. A designer needs the length in inches. How many inches long is the fabric?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #3

A running track is 400 meters long. Convert to miles.

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #4

A fish tank holds 20 gallons of water. Convert to liters.

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #5

Convert 5 yards to centimeters.

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #6

A hiking trail is 8 miles long. A visitor from Europe wants to know the distance in kilometers. How many kilometers is the trail?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #7

A solar panel has an area of 1.8 m^2. What is its area in square feet?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #8

A bottle contains 750 mL of juice. How many fluid ounces is that?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #9

A rectangular garden measures 100,000 cm^2. What is the area in square inches?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #10

A storage box measures 9 ft^3. What is the volume in cubic meters?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #11

A soccer field measures 110 yards by 70 yards. What is the area in square meters?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #12

As of late April 2026, gas prices in Quebec are among the highest in Canada. The highest price was at $1.95 per liter. How much is the price per gallon?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #13

An aquarium has a volume of 1,200 cm^3. Convert to milliliters.

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #14

How many milliliters are there in 671 cubic centimeters?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #15

How many cubic centimeters are there in 56,789 L?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #16

How many kiloliters are there in 157,000,000 cubic centimeters?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #17

Convert 1200 dekaliters to cubic decimeters.

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #18

The term “cc” (cubic centimeter) comes from early metric system usage in the 19th and early 20th centuries, when scientists and physicians commonly described liquid volumes using geometric units like cubic centimeters. Because 1 cm^3 = 1 mL, the abbreviation “cc” became widely used in medicine, especially for injections and syringe measurements. However, over time, concerns grew about patient safety, particularly that handwritten “cc” could be misread as “00,” “u” (units), or other markings, leading to dosing errors. As a result, modern medical standards and organizations began recommending the exclusive use of milliliters (mL) instead of “cc.” Today, “mL” is preferred because it is clearer, standardized, and reduces the risk of misinterpretation in clinical settings.

So, if a doctor orders 5 cc of a medication, how many milliliters should be given?

Answer available at the end of this section.

(Section 3.8) - Homework Exercise #19

How many 2-liters bottles can be used to fill up a cubic meter container?

Answer available at the end of this section.

Answers

(Section 3.8) - Homework Exercise #1
164.04 ft

(Section 3.8) - Homework Exercise #2
98.43 in

(Section 3.8) - Homework Exercise #3
0.25 mi

(Section 3.8) - Homework Exercise #4
75.7 L

(Section 3.8) - Homework Exercise #5
457.2 cm

(Section 3.8) - Homework Exercise #6
12.88 km

(Section 3.8) - Homework Exercise #7
19.375 ft^2

(Section 3.8) - Homework Exercise #8
25.36 fl oz

(Section 3.8) - Homework Exercise #9
15,500.03 in^2

(Section 3.8) - Homework Exercise #10
0.255 m^3

(Section 3.8) - Homework Exercise #11
6438.18 m^2

(Section 3.8) - Homework Exercise #12
$7.38 per gallon

(Section 3.8) - Homework Exercise #13
1,200 mL

(Section 3.8) - Homework Exercise #14
671 mL

(Section 3.8) - Homework Exercise #15
56,789,000 cm^3

(Section 3.8) - Homework Exercise #16
157 kL

(Section 3.8) - Homework Exercise #17
12,000 dm^3

(Section 3.8) - Homework Exercise #18
5 mL

(Section 3.8) - Homework Exercise #19
500 2-L bottles

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Section 3.9: Measures of Temperature

Learning Objectives

- Convert measurements of temperature in different systems

Up until this point, we have worked with both the metric system and the U.S. customary system, converting between units of length, mass, capacity, area, and volume. Many of these conversions involved multiplying or dividing by known conversion factors. While the metric system often follows patterns based on powers of ten, U.S. customary conversions required memorizing relationships between units.

Temperature, however, is different from all the measurements we have studied so far. Instead of measuring size or quantity, temperature measures the degree of heat or cold. Because of this, temperature scales are based on reference points, not just unit size, and conversions between them cannot be done using multiplication or division alone.

There are three common temperature scales:

- Celsius (°C)
- Fahrenheit (°F)
- Kelvin (K).

The Celsius scale, which is considered metric, is widely used and is based on the freezing point (0°C) and boiling point (100°C) of water.

The Fahrenheit scale, used primarily in the United States, places the freezing point at 32°F and the boiling point at 212°F.

The Kelvin scale is used globally in scientific settings and begins at absolute zero, the lowest and coldest possible temperature, where molecular motion ceases. Kelvin is designed to measure absolute thermodynamic temperature, not the relative temperatures (like daily weather) often measured in Fahrenheit. Measures in Kelvin does not use "degrees" or the degree symbol like Celsius and Fahrenheit does.

To convert between these scales, we must use formulas that account for both differences in scale and differences in starting points. This makes temperature conversion a unique and fitting conclusion to our study of measurement, highlighting that not all quantities can be converted in the same way.

Converting Celsius to Fahrenheit

To convert a given measure in Celsius to Fahrenheit, use the formula

$$F = \frac{9}{5}C + 32$$

where C is the given measure in Celsius and F is the temperature in Fahrenheit.

Using algebra, we can solve the equation for C in terms of F, allowing us to derive a formula for converting in the opposite direction.

Converting Fahrenheit to Celsius

To convert a given measure in Fahrenheit to Celsius, use the formula

$$C = \frac{5}{9}(F - 32)$$

where F is the temperature in Fahrenheit and C is the given measure in Celsius

Although Kelvin is not commonly used in everyday situations, the same types of formulas can still be applied to convert between temperature scales.

Conversions Involving Kelvin

Here are the standard temperature conversion formulas between Kelvin and Celsius, and Fahrenheit:

$$K = C + 273.15$$

$$C = K - 273.15$$

where K is the temperature in Kelvin and C is the given measure in Celsius.

Here are the standard temperature conversion formulas between Kelvin and Fahrenheit:

$$F = 1.8K - 459.67$$

$$K = \frac{5}{9}(F - 32) + 273.15$$

where K is the temperature in Kelvin and F is the given measure in Fahrenheit.

Example #3.9.1 🤔

Convert 95°F to Celsius.

✅ **Solution:**

Use the formula converting Farenheit to Celsius:

$$C = \frac{5}{9}(F - 32)$$

Let $F = 95$ and solve for C.

$$C = \frac{5}{9}(95 - 32)$$

$$C = \frac{5}{9}(63)$$

$$\boxed{C = 35}$$

So, 95°F is equivalent 35°C.

Example #3.9.2 🤔

Convert 20°C to Farenheit.

✅ **Solution:**

Use the formula converting Celsius to Farenheit:

$$F = \frac{9}{5}C + 32$$

Let $C = 20$ and solve for F.

$$F = \frac{9}{5}(20) + 32$$

$$F = 36 + 32$$

$$\boxed{F = 20}$$

So, 20°C is equivalent 68°F.

Example #3.9.3 🤔

Convert 46°F to Celsius.

✅ **Solution:**

Use the formula converting Farenheit to Celsius:

$$C = \frac{5}{9}(F - 32)$$

Let $F = 46$ and solve for C.

$$C = \frac{5}{9}(46 - 32)$$

$$C = \frac{5}{9}(14)$$

$$C = 7.7777777778$$

$$\boxed{C \approx 7.8}$$

So, 46°F is equivalent 7.8°C.

Example #3.9.4 🤔

Convert -8°C to Farenheit.

✅ **Solution:**

Use the formula converting Celsius to Farenheit:

$$F = \frac{9}{5}C + 32$$

Let $C = -8$ and solve for F.

$$F = \frac{9}{5}(-8) + 32$$

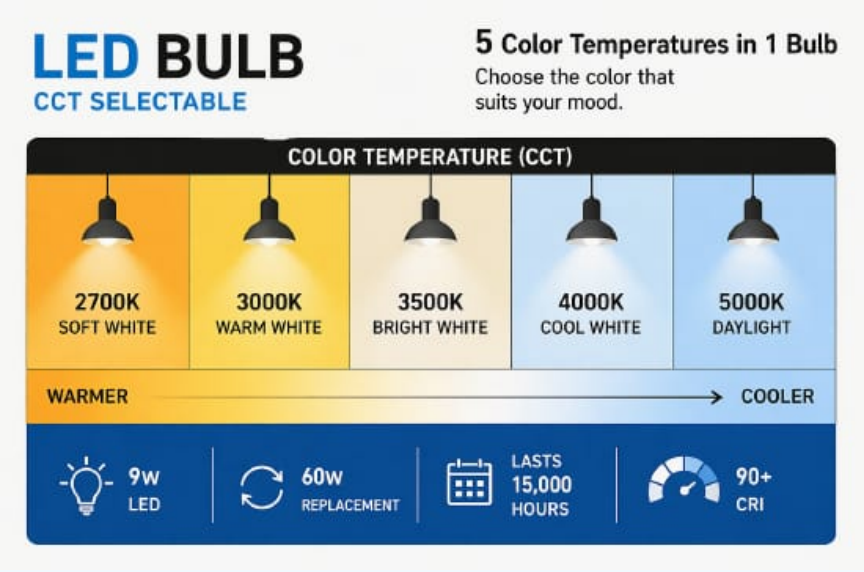
$$F = -14.4 + 32$$

$$\boxed{F = 17.6}$$

So, -8°C is equivalent 17.6°F.

Example #3.9.5 🤔

Color temperature describes the color of light emitted by an object, and it is measured in



Kelvin (K) rather than Celsius or Fahrenheit. Although it uses the word temperature, it does not directly measure how hot or cold something feels. Instead, it refers to the color that an ideal heated object would emit at a given temperature.

As an object is heated, it changes color in a predictable way:

- At lower temperatures, it glows reddish
- As the temperature increases, it becomes orange, yellow, and white
- At very high temperatures, it appears bluish

This relationship comes from physics and is based on how a “blackbody” radiates light.

As you can see in the figure to the right, some LED light bulbs offer a range of color temperatures, from warm amber tones to bright white. Convert the 4,000K that is labeled as "cool white" to Farenheit.

✅ **Solution:**

Use the formula converting Kelvin to Farenheit:

$$F = 1.8K - 459.67$$

Let $K = 4{,}000$ and solve for F.

$$F = 1.8K - 459.67$$

$$F = 1.8(4{,}000) - 459.67$$

$$F = 6{,}740.33$$

$$\boxed{F \approx 6{,}740}$$

So, 4,000K is equivalent 6,740°F.

Note: Although Kelvin can be converted to Celsius or Fahrenheit, the temperatures involved are extremely high, so color temperature is almost always expressed in Kelvin rather than everyday temperature scales.

Example #3.9.6 🤔

A common room temperature in science is approximately 80°F. What is the temperature in Kelvin?

✅ **Solution:**

Use the formula converting Kelvin to Farenheit:

$$K = \frac{5}{9}(F - 32) + 273.15$$

Let $F = 80$ and solve for K.

$$K = \frac{5}{9}(80 - 32) + 273.15$$

$$K = \frac{5}{9}(48) + 273.15$$

$$K = 26.66666666667 + 273.15$$

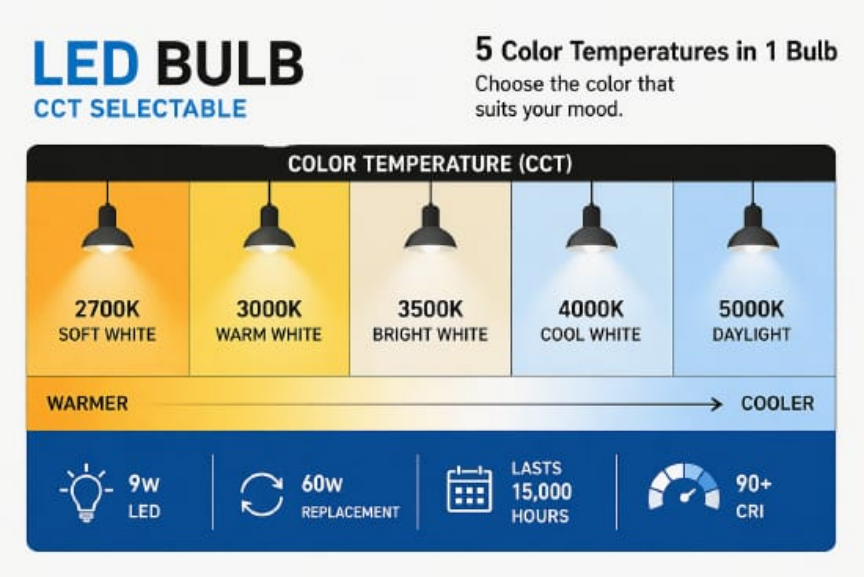
$$K = 299.816666667$$

$$\boxed{K \approx 300}$$

So, 4,000°F is equivalent 300K.

Section 3.9: Measures of Temperature [*In-Class Exercises*]

1. The highest reliably recorded air temperature on Earth occurred in Furnace Creek, Death Valley National Park on July 10th, 1913 at 134.1°F. Convert this temperature to Celsius. Round to the nearest tenths place.



2. Greenland is largely covered by an ice sheet (about 80% of it), making it one of the coldest and iciest places on Earth. Iceland, on the other hand, has green valleys, grasslands, and a milder climate, especially along the coasts. In a typical summer, Nuuk, the capital of Greenland, has an average temperature of 6.1°C. Convert this temperature to Farenheit. Round to the nearest tenths place.
3. Refer to the figure to the right. Convert the 3,000K that is labeled as "warm white" to Farenheit. Round to the nearest whole number.

Answers available at the end of this section.

Answers

Section 3.9: Measures of Temperature [*In-Class Exercises*]

1. 56.7°C
2. 43.0°F
3. 4,940K

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Section 3.9.H: Homework

Section 3.9: Measures of Temperature - Homework Exercises

For exercises 1 through 12, convert to the desired units. Round to the nearest tenths place where necessary.

(Section 3.9) - Homework Exercise #1

Convert 25°C to Fahrenheit.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #2

Convert 104°F to Celsius.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #3

Convert 50°C to Fahrenheit.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #4

Convert 81°F to Celsius.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #5

Convert 16°C to Fahrenheit.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #6

Convert 0°F to Celsius.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #7

The coldest recorded temperature on Earth was –89.2°C, measured at Vostok Station in Antarctica on July 21, 1983. Convert this temperature to Fahrenheit.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #8

The background temperature of space is approximately 2.7 Kelvin. Because space is nearly a perfect vacuum, it does not have a temperature in the usual sense of particle motion. Instead, the value 2.7K represents the temperature of the cosmic background radiation that fills space.

Convert this temperature to both Fahrenheit and Celsius.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #9

For LED light bulbs, the color label "daylight" is 5,000K. Convert this temperature to Celsius

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #10

Absolute zero in Kelvin is 0K. Convert this temperature to both Fahrenheit and Celsius.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #11

The freezing point for the Fahrenheit scale is 32°F and for the Celsus scale is 0°C. Convert either temperature to Kelvin.

Answer available at the end of this section.

(Section 3.9) - Homework Exercise #12

The boiling point for the Fahrenheit scale is 212°F and for the Celsus scale is 100°C. Convert either temperature to Kelvin.

Answer available at the end of this section.

Answers

(Section 3.9) - Homework Exercise #1

77°F

(Section 3.9) - Homework Exercise #2

40°C

(Section 3.9) - Homework Exercise #3

122°F

(Section 3.9) - Homework Exercise #4
27.2°C

(Section 3.9) - Homework Exercise #5
60.8°F

(Section 3.9) - Homework Exercise #6
−17.8°F

(Section 3.9) - Homework Exercise #7
−128.6°F

(Section 3.9) - Homework Exercise #8
−454.8°F and −270.5°C

(Section 3.9) - Homework Exercise #9
4,726.9°C

(Section 3.9) - Homework Exercise #10
−459.7°F and −273.1°C

(Section 3.9) - Homework Exercise #11
273.2K

(Section 3.9) - Homework Exercise #12
373.2K

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CHAPTER 4: FINANCE

Section 4.1: Percents

Section 4.1.H: Homework

Section 4.2: Simple Interest

Section 4.2.H: Homework

Section 4.3: Compound Interest

Section 4.3.H: Homework

Section 4.4: Effective Yield

Section 4.4.H: Homework

Section 4.5: Annuities

Section 4.5.H: Homework

Section 4.6: Installment Buying

Section 4.6.H: Homework

Section 4.7: Average Daily Balance

Section 4.7.H: Homework

Section 4.8: Student Loans, Home Buying, and Amortization

Section 4.8.H: Homework

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Section 4.1: Percents

Learning Objectives

- Convert decimals to percents
- Convert percents to decimals
- Solve application problems involving percents

Definition: Percents

Percent means "per hundred" or "out of 100." It's a way to express a fraction or ratio as parts of 100. So, $1\% = \frac{1}{100}$.

To Convert a Decimal to a Percent

- **Method 1:** Multiply the decimal by 100, then and add the percent sign (%). For example:
  - 0.75 → 0.75 × 100 = 75 → 75%
  - 0.4 → 0.4 × 100 = 40 → 40%
  - 0.09 → 0.09 × = 9 → 9%
- **Method 2:** Move the decimal point two places to the right, adding zeros if needed, then add the percent sign (%). For example:
  - 0.75 → move decimal point two places right → 75 → 75%
  - 0.4 → move decimal point two places right → 40 → 40%
  - 0.09 → move decimal point two places right → 9 → 9%

Example #4.1.1 🤔

Express each decimal as a percent.

a. 0.56
b. 0.056
c. 0.27
d. 0.8
e. 0.08
f. 8
g. 1.2
h. 1

✅ **Solution:**

- **Method 1:** Multiply the decimal by 100, then add the percent sign (%).
- **Method 2:** Move the decimal point two places to the right, adding zeros if needed, then add the percent sign (%).

a. 0.56 → 0.56 × 100 = 56 → 56% or 0.56 → move decimal two places right → 56 → 56%
b. 0.056 → 0.056 × 100 = 5.6 → 5.6% or 0.056 → move decimal two places right → 5.6 → 5.6%
c. 0.27 → 0.27 × 100 = 27 → 27% or 0.27 → move decimal two places right → 27 → 27%

d. 0.8 → 0.8 × 100 = 80 → 80% or 0.8 → move decimal two places right, add 1 zero → 80 → 80%
e. 0.08 → 0.08 × 100 = 8 → 8% or 0.08 → move decimal two places right → 8 → 8%
f. 8 → 8 × 100 = 800 → 800% or 8 → move decimal two places right, add 2 zeros → 800 → 800%
g. 1.2 → 1.2 × 100 = 120 → 120% or 1.2 → move decimal two places right, add 1 zero → 120 → 120%
h. 1 → 1 × 100 = 100 → 100% or 1 → move decimal two places right, add 2 zeros → 100 → 100%

To Convert a Percent to a Decimal

- **Method 1:** Drop the percent sign, then divide by 100. For example:
  - 25% → 25 ÷ 100 = 0.25
  - 70% → 70 ÷ 100 = 0.70 (or 0.7)
  - 3% → 3 ÷ 100 = 0.03
- **Method 2:** Drop the percent sign, then move the decimal point two places to the left, adding zeros if needed. For example:
  - 25% → move decimal point two places left → 0.25
  - 70% → move decimal point two places left → 0.70
  - 3% → move decimal point two places left → 0.03

Example #4.1.2 🤔

Express each decimal as a percent.

a. 47%
b. 90%
c. 9%
d. 7.75%
e. 0.01%
f. 150%
g. $6\frac{1}{4}$%
h. $2\frac{3}{8}$%

✅ **Solution:**

- **Method 1:** Drop the percent sign, then divide by 100.
- **Method 2:** Drop the percent sign, then move the decimal point two places to the left, adding zeros if needed.

a. 47% → 47 ÷ 100 = 0.47 or 47% → 47 → move decimal two places left → 0.47
b. 90% → 90 ÷ 100 = 0.90 (or 0.9) or 90% → 90 → move decimal two places left → 0.90 (or 0.9)
c. 9% → 9 ÷ 100 = 0.09 or 9% → 9 → move decimal two places left, add 1 zero → 0.09%
d. 7.75% → 7.75 ÷ 100 = 0.0775 or 7.75% → 7.75 → move decimal two places left, add 1 zero → 0.0775
e. 0.01% → 0.01 ÷ 100 = 0.0001 or 0.01% → 0.01 → move decimal two places left, add 2 zeros → 0.0001
f. 150% → 150 ÷ 100 = 1.5 or 150% → 150 → move decimal two places left → 1.50 (or 1.5)
g. $6\frac{1}{4}$% → 6.25 ÷ 100 = 0.0625 or $6\frac{1}{4}$% = 6.25% → 6.25 → move decimal two places left, add 1 zero → 0.0625
h. $2\frac{3}{8}$% → 2.375 ÷ 100 = 0.02375 or $2\frac{3}{8}$% = 2.375% → 2.375 → move decimal two places left, add 1 zero → 0.02375

Tax calculations represent one of the most frequent and practical uses of percent in daily life. One type of tax is a state sales tax. Sales tax is a consumption tax that affects nearly every purchase we make in our daily lives. When you buy clothing, electronics, restaurant meals, or most other goods and services, you pay not only the listed price but also an additional percentage called sales tax. This tax is collected by the business at the time of purchase and then forwarded to government authorities, making retailers essentially tax collectors for the state and local governments.

The way sales tax works is straightforward: it's calculated as a percentage of your purchase price and added to your total bill. For example, if you buy a $50 item in an area with 8% sales tax, you'll pay an additional $4 in tax ($50 × 0.08 = $4), making your total cost $54. The tax rate you pay depends on where you shop, as it combines state sales tax set by your state government with

any additional local taxes imposed by counties, cities, or special districts. This is why you might notice different tax rates when shopping in different cities or counties. For example, as of 2025, in Orange County, CA., the city of Costa Mesa has a 7.75% sales tax rate, whereas the city of Fountain Valley has an 8.75% sales tax rate and the city of Santa Ana has a 9.25% sales tax rate. (California's base state sales tax rate is 7.25%, which includes a 6% base rate plus a mandatory 1.25% local rate. So, Costa Mesa added an additional 0.5% tax rate, while Fountain Valley added and additional 1.5% tax rate, and Santa Ana added an additional 2.0% tax rate).

Sales tax serves as a crucial source of revenue for government services that benefit the community. The money collected helps fund public schools, maintain roads and infrastructure, support police and fire departments, and provide various local services. While most goods and services are subject to sales tax, many states exempt essential items like groceries and prescription medications to reduce the tax burden on basic necessities. Understanding sales tax helps you budget more accurately and become a more informed consumer and citizen.

How to Calculate Sales Tax

Tax Amount = Purchase Price × Tax Rate
Total Cost = Purchase Price + Tax Amount

- **Step 1:** Convert the tax rate from percent to decimal.
- **Step 2:** Multiply the purchase price by the decimal tax rate. This gives you the tax amount in dollars.
- **Step 3:** Add the tax amount to the original price. This gives you the total amount you'll pay.

Alternative Method: The Multiplier Approach:

Instead of calculating tax separately, you can find the total in one step:

- Total Cost = Purchase Price × (1 + Tax Rate)

Example #4.1.3 🤔

Find the sales tax and the total cost on a 65” television that costs $1,699.99. The tax rate is 8%.

✅ **Solution:**

- **Step 1:** Convert percent to decimal: 8% → 0.08
- **Step 2:** Calculate tax amount: $1,699.99 × 0.08 = $135.9992
  - Rounding to the nearest cent, we have $136.00
- **Step 3:** Find total cost: $1,699.99 + $136.00 = $1,835.99

So, the tax is $136 and the total cost is $1,835.99.

Example #4.1.4 🤔

Find the sales tax and total cost on a new 2024 Toyota Tundra Limited Edition that costs $61,711. Assume a tax rate of 7.75%.

✅ **Solution:**

- **Step 1:** Convert percent to decimal: 7.75% → 0.0775
- **Step 2:** Calculate tax amount: $61,711 × 0.0775 = $4,782.6025
  - Rounding to the nearest cent, we have $4,782.60
- **Step 3:** Find total cost: $61,711 + $4,782.61 = $66,493.60

So, the tax is $4,782.60 and the total cost is $66,493.60.

Example #4.1.5 🤔

Find the total cost including tax of a Samsung Bespoke Ultra Capacity Front Load Washer and Electric Dryer Set that costs $2,098.00. Assume the tax rate is 9.25%.

✅ **Solution:**

So, since we are not asked to calculate tax, we can use the the alternate method to calculate the total.

We know 9.25% written as a decimal is 0.0925.

Now, use the alternate method.

$$\begin{aligned}\text{Total Cost} &= \text{Purchase Price} \times (1 + \text{Tax Rate})\\ \text{Total Cost} &= \$2,098.00 \times (1 + 0.0925)\\ &= \$2.098.00 \times (1.0925)\\ &= \$2,292.065\end{aligned}$$

So, the total cost rounded to the nearest cent is $2,292.07.

Another practical application of percent is a **down payment**. A down payment is an upfront payment made when purchasing expensive items, typically expressed as a percentage of the total purchase price. Down payments are commonly required when making major purchases that involve financing. Whether you're buying a home, purchasing a car, investing in major appliances or furniture for your living space, or acquiring business equipment for a company, lenders typically expect you to pay a percentage of the total cost upfront. This initial payment demonstrates your financial commitment to the purchase and reduces the amount you need to borrow, making it a standard practice across many industries and types of significant investments.

How to Calculate a Down Payment

Down Payment Amount = Purchase Price × Down Payment Rate

Loan Amount/Balance = Purchase Price – Down Payment

Example #4.1.6 🤔

If someone wanted to put a 10% down payment on a used boat that cost $7,000, what is the down payment and the balance after the down payment?

✅ **Solution:**

We know 10% written as a decimal is 0.10.

So,

$$\begin{aligned}\text{Down Payment Amount} &= \text{Purchase Price} \times \text{Down Payment Rate}\\ &= \$7,000 \times (0.10)\\ &= \$700\end{aligned}$$

and

$$\begin{aligned}\text{Loan Amount/Balance} &= \text{Purchase Price} - \text{Down Payment}\\ &= \$7,000 - \$700\\ &= \$6,300\end{aligned}$$

So, the down payment is $700 and the balance is $6,300.

Example #4.1.7 🤔

A couple wants to put a 5% down payment on a new home that has a purchase price of $1,200,000. What is the down payment and the balance after the down payment?

✅ **Solution:**

We know 5% written as a decimal is 0.05.

So,

$$\begin{aligned}\text{Down Payment Amount} &= \text{Purchase Price} \times \text{Down Payment Rate} \\ &= \$1,200,000 \times (0.05) \\ &= \$60,000\end{aligned}$$

and

$$\begin{aligned}\text{Loan Amount/Balance} &= \text{Purchase Price} - \text{Down Payment} \\ &= \$1,200,000 - \$60,000 \\ &= \$1,140,000\end{aligned}$$

So, the down payment is $60,000 and the balance is $1,140,000.

Section 4.1: Percents [*In-Class Exercises*]

1. Express each decimal as a percent.
   a. 0.61
   b. 0.7
   c. 0.07
   d. 4
   e. 1.72
   f. 0.12345
   g. 1
   h. 0.0001
2. Express each percent as a decimal.
   a. 60%
   b. 2%
   c. 8.25%
   d. 0.001%
   e. 125%
   f. 17.9%
   g. $3\frac{1}{8}$%
   h. $12\frac{3}{4}$%
3. A Black Friday sale has the Nintendo Switch 2 on sale for $429.99. If your city sales tax rate is 8.75%, what is the sales tax on the purchase of a brand new Nintendo Switch 2?
4. Find the total cost including tax of a Jackery HomePower 3000 Solar Generator + 2 SolarSaga 200W Portable Solar Panels that costs $1,599.99. Assume the tax rate is 10.25%.
5. Private Mortgage Insurance (PMI) is a required, non-owner-protective fee for conventional loans with less than a 20% down payment, typically costing 0.46%–1.5% of the loan amount annually. It protects the lender if the buyer defaults. A 1,400 square foot, 3 bedroom, 2 bath condo in Huntington Beach, CA is selling for $775,000. What would one have to pay as a down payment to avoid paying PMI? What would be the balance after the down payment?

Answers available at the end of this section.

Answers

Section 4.1: Percents [*In-Class Exercises*]

1. a. 61%
   b. 70%
   c. 7%
   d. 400%
   e. 172%
   f. 12.345%
   g. 100%
   h. 0.01%
2. a. 0.60
   b. 0.02
   c. 0.0825
   d. 0.00001
   e. 1.25
   f. 0.179
   g. 0.03125
   h. 0.1275
3. $37.62
4. $1,763.99
5. $155,000; $620,000

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Section 4.1.H: Homework

Section 4.1: Percents - Homework Exercises

(Section 4.1) - Homework Exercise #1

1. Express each decimal as a percent.

    a. 0.03
    b. 0.1
    c. 0.27
    d. 8
    e. 0.00025
    f. 2.59
    g. 1
    h. 0.98765

Answer available at the end of this section.

(Section 4.1) - Homework Exercise #2

2. Express each percent as a decimal.

    a. 55%
    b. 4.5%
    c. 0.025%
    d. $5\frac{1}{4}$%
    e. 150%
    f. 8%
    g. 1.3%
    h. $12\frac{7}{8}$%

Answer available at the end of this section.

(Section 4.1) - Homework Exercise #3

As of 2026, the most expensive consumer-grade and prosumer drones often exceed $15,000–$20,000, with specialized industrial systems costing even more. Top contenders include the DJI Inspire 3 (approx. $16,000+) and professional mapping hexacopters like the Freefly Alta 8 ($17,495+), which offer high-end, interchangeable cameras for cinema production. What is the tax on a Freely Alta 8, if the sale price was $17,495? Assume the sales tax rate is 8.75%.

Answer available at the end of this section.

(Section 4.1) - Homework Exercise #4

A newlywed couple decides to go to Maggiano's Little Italy in Costa Mesa, CA. on Valentine's Day. The bill for dinner cost $114.00 before tax.

a. Find the total cost of the dinner including tax. The tax rate in Costa Mesa is 7.75%.
b. Many people rely on the suggested tip amounts printed at the bottom of a receipt. However, those suggested percentages are not always calculated the same way. Sometimes they're based on the pre-tax total, and other times on the post-tax total. In some cases, the listed tip amounts have even been rounded incorrectly or calculated inaccurately. So the question becomes: What should the correct tip amounts be when calculating 18%, 20%, and 22% based on the price *before* tax? What would be the total cost be including tax and tip for each rate?
c. In part b, what should the correct tip amounts be when calculating 18%, 20%, and 22% based on the price *after* tax? What would be the total cost be including tax and tip for each rate?

Answer available at the end of this section.

(Section 4.1) - Homework Exercise #5

Private Mortgage Insurance (PMI) is a required, non-owner-protective fee for conventional loans with less than a 20% down payment, typically costing 0.46%–1.5% of the loan amount annually. It protects the lender if the buyer defaults. A 2,650 square foot, 4 bedroom, 3.5 bath beach-front house in Newport Beach, CA on the Balboa Peninsula was just listed on Realtor.com. The asking price is $10,995,000. What would one have to pay as a down payment to avoid paying PMI? What would be the balance after the down payment?

Answer available at the end of this section.

Answer available at the end of this section.

(Section 4.1) - Homework Exercise #6

Car salespeople typically earn a commission ranging from 15% to 40% of the front-end gross profit on a vehicle sale, with 20% to 30% (often around 25%) being the most common rate. Rather than a percentage of the total car price, they are paid based on the profit margin, plus potential bonuses for financing, warranties, and reaching volume targets. Based on early 2026 model year pricing, the gross profit built into the MSRP for a 2026 Dodge Charger is approximately 7%. At Premier Chrysler Dodge Jeep Ram in Buena Park, a new 2026 Dodge Charger 4-Door R/T With Navigation & AWD has an MSRP of $61,200. What is the gross profit on this vehicle if using a 7% rate? Based off that amount, what would be the earned commission for a car saleswomen, if she was to earn a 25% commission on the sale of the 2026 Dodge Charger?

Answer available at the end of this section.

Answers

(Section 4.1) - Homework Exercise #1

a. 3%
b. 10%
c. 27%
d. 800%
e. 0.025%
f. 259%
g. 100%
h. 98.765%

(Section 4.1) - Homework Exercise #2

a. 0.55
b. 0.045
c. 0.00025
d. 0.0525
e. 1.5
f. 0.08
g. 0.013
h. 0.12875

(Section 4.1) - Homework Exercise #3

Sales tax = $1,530.81

(Section 4.1) - Homework Exercise #4

a. Tax = $8.84; Total cost without tip = $122.84
b. 18% tip on $114 = $20.52; 20% tip on $114 = $22.80; 22% tip on $114 = $25.08;
   - For 18%: $114 + $8.84 + $20.52 = $143.36
   - For 20%: $114 + $8.84 + $22.80 = $145.64
   - For 22%: $114 + $8.84 + $25.08 = $147.92
c. 18% tip on $122.84 = $22.11; 20% tip on $122.84 = $24.57; 22% tip on $122.84 = $27.02;
   - For 18%: $114 + $8.84 + $22.11 = $144.95
   - For 20%: $114 + $8.84 + $24.57 = $147.41
   - For 22%: $114 + $8.84 + $27.02 = $149.86

(Section 4.1) - Homework Exercise #5

Down payment = $2,199,000; Balance after payment = $8,796,000

(Section 4.1) - Homework Exercise #5

(Section 4.1) - Homework Exercise #6

Gross profit = $4,284; Commission = $1,071

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Section 4.2: Simple Interest

Learning Objectives

- Compute simple interest and future value
- Compute principal, rate, or time

Simple interest is a method of calculating interest charges or earnings based solely on the principal amount, without compounding. This fundamental concept serves as the foundation for understanding more complex financial calculations. Applications of simple interest include add-on automobile loans, short-term personal loans, certain government bonds, and some certificate of deposit arrangements. Understanding simple interest calculations enables students to evaluate financial options, make informed borrowing and investment decisions, and develop quantitative reasoning skills applicable to personal finance management.

History of Simple Interest

The idea of simple interest dates back nearly 4,000 years to the early civilizations of the Middle East and Mesopotamia, where borrowing and lending formed a core part of economic life. In fact, some of the oldest surviving written documents are interest-bearing loan agreements, highlighting how essential these financial practices were to early societies. Its importance was so widely recognized that the Code of Hammurabi, written around that time, outlined specific rules governing acceptable interest rates. These provisions reveal not only how common lending had become but also the need to regulate it to prevent potential abuses.

While simple interest dominated early financial practices, compound interest (to be introduced in Section 4.3) was also known in ancient times, though it was more thoroughly analyzed during medieval periods. Historical evidence shows that compound interest practices can be traced back 4,400 years to Sumer (the world's earliest known civilization), with these customs being passed down to the Babylonians. However, simple interest remained the more straightforward and commonly used method for most basic lending arrangements.

The mathematical understanding of interest continued to evolve over centuries. A significant milestone occurred in 1683 when Jacob Bernoulli made the first "discovery" of the mathematical constant e while studying compound interest, which further distinguished the mathematical complexity between simple and compound interest calculations.

This historical foundation explains why simple interest remains fundamental to financial education today. It represents humanity's earliest systematic approach to quantifying the cost of borrowing and the reward for lending money.

Simple Interest Formula

Simple interest is the interest calculated only on the principal amount of a loan or investment, remaining constant over the entire term of the financial agreement given by the formula:

$$I = P \cdot r \cdot t$$

where:

- I = Interest earned or paid
- P = Principal (initial amount)
- r = Annual interest rate (expressed as a decimal)
- t = Time period (in years)

The total amount A accumulated or owed is given by:

$$A = P + I$$

Simple interest differs from compound interest in that it does not include interest earned on previously accumulated interest. This characteristic makes simple interest calculations more straightforward but typically less favorable for long-term investments compared to compound interest scenarios.

Example #4.2.1 🤔

Suppose you borrow \$1,000 at 8% simple interest for 5 years. Find the total simple interest paid and the total amount of the loan that needs to be paid back.

✅ **Solution:**

Start with the simple interest formula.

$$I = P \cdot r \cdot t$$

We know that $P = \$1{,}000;\ r = 0.08;\ t = 5$, so

$$I = (\$1{,}000) \cdot (0.08) \cdot (5)$$
$$I = \$400$$

Now, find the total amount to be paid back.

$$A = P + I$$
$$A = (\$1{,}000) + (\$400)$$
$$A = \$1{,}400$$

So, the interest is \$400 and the total amount to be paid back is \$1,400.

Let's focus on Example #4.2.1 and look at a year-by-year process to further understand to see what is really happening.

Year 1:

- Interest earned = \$1,000 × 0.08 × 1 = \$80
- Total amount = \$1,000 + \$80 = \$1,080

Year 2:

- Interest earned = \$1,000 × 0.08 × 1 = \$80 (still calculated on original \$1,000)
- Total amount = \$1,080 + \$80 = \$1,160

Year 3:

- Interest earned = \$1,000 × 0.08 × 1 = \$80 (still on original \$1,000)
- Total amount = \$1,160 + \$80 = \$1,240

Year 4:

- Interest earned = \$1,000 × 0.08 × 1 = \$80 (still on original \$1,000)
- Total amount = \$1,240 + \$80 = \$1,320

Year 5:

- Interest earned = $1,000 × 0.08 × 1 = $80 (still on original $1,000)
- Total amount = $1,320 + $80 = $1,400

Notice that you earn exactly $80 every single year, regardless of how much total money has accumulated. Here is a table showing the same calculations:

Growth of a $1,000 investment at an annual simple interest rate of 8% over 5 years

Year	Principal	Interest	Amount After t Years
0	$1,000	$0	$1,000
1	$1,000	($1,000)·(0.08)·(1) = $80	$1,080
2	$1,000	($1,000)·(0.08)·(1) = $80	$1,160
3	$1,000	($1,000)·(0.08)·(1) = $80	$1,240
4	$1,000	($1,000)·(0.08)·(1) = $80	$1,320
5	$1,000	($1,000)·(0.08)·(1) = $80	$1,400

So, we get the same result of $1,400 when we use the table as we did by using the formula.

Example #4.2.2 🤔

Suppose you invest $5,000 at 4.25% simple interest for 3 years. Find the simple interest made off the investment.

✅ **Solution:**

Start with the simple interest formula.

$$I = P \cdot r \cdot t$$

We know that $P = \$5{,}000;\ r = 0.0425;\ t = 3$, so

$$I = (\$5{,}000) \cdot (0.0425) \cdot (3)$$
$$I = \$637.50$$

So, the interest made is $637.50.

Example #4.2.3 🤔

Find the simple interest for $7,000 at 5% for 35 weeks. (Assume a 52-week year).

✅ **Solution:**

Start with the simple interest formula.

$$I = P \cdot r \cdot t$$

When using the interest formula, the time t is in years. This time period is in weeks, so we need to divide 35 by the number of weeks in a year which is 52. Thus, $t = \frac{35}{52}$. Using $P = \$7{,}000;\ r = 0.05;\ t = (\frac{35}{52})$, we have

$$I = (\$7{,}000) \cdot (0.05) \cdot \left(\frac{35}{52}\right)$$
$$I = \$235.57692307692$$

So, rounding to the nearest cent, the interest is $235.58.

Example #4.2.4 🤔

Find the simple interest earned on an investment of $3,000 at 2.5% for 72 days.

✅ **Solution:**

Start with the simple interest formula.

$$I = P \cdot r \cdot t$$

When using the interest formula, the time t is in years. This time period is in days, so we need to divide 72 by the number of days in a year which is 365. Thus, $t = \frac{72}{365}$. Using $P = \$3{,}000;\ r = 0.025;\ t = (\frac{72}{365})$, we have

$$I = (\$3{,}000) \cdot (0.025) \cdot \left(\frac{72}{365}\right)$$
$$I = \$14.7945205479$$

So, the interest earned is $14.79.

Example #4.2.5 🤔

Find the future value and the amount of simple interest earned on a loan of $12,000 at 5.3% for 11 months.

✅ **Solution:**

Start with the simple interest formula.

$$I = P \cdot r \cdot t$$

When using the interest formula, the time t is in years. This time period is in months, so we need to divide 11 by the number of months in a year which is 12. Thus, $t = \frac{11}{12}$. Using $P = \$12{,}000;\ r = 0.053;\ t = (\frac{11}{12})$, we have

$$I = (\$12{,}000) \cdot (0.053) \cdot \left(\frac{11}{12}\right)$$
$$I = \$583$$

Now, find the total amount to be paid back.

$$A = P + I$$
$$A = (\$12{,}000) + (\$583)$$
$$A = \$12{,}583$$

So, the interest is $583 and the total amount to be paid back is $\$12{,}583$.

Example #4.2.6 🤔

Suppose you invest $1,000 at a 6.5% simple interest rate. Approximately, how long would it take you to double your investment?

✅ **Solution:**

Start with the simple interest formula.

$$I = P \cdot r \cdot t$$

So, $P = \$1{,}000$. Doubling the investment, then the future value is $A = \$2{,}000$. Since, $A = P + I$, then, $I = \$1{,}000$. We also know that know that $r = 0.065$. Now, we are solving for *t*, so

$$\begin{aligned} (\$1{,}000) &= (\$1{,}000) \cdot (0.065) \cdot (t) \\ (1{,}000) &= 65 \cdot t \\ \left(\frac{1{,}000}{65}\right) &= t \\ 15.384615384615 &= t \end{aligned}$$

So, the time would be roughly 15.4 years.

Example #4.2.7 🤔

How much would you need to invest at a 3.5% simple interest rate to accumulate an interest amount of $500 in 5 years?

✅ **Solution:**

Start with the simple interest formula.

$$I = P \cdot r \cdot t$$

We know that $I = \$500;\ r = 0.035;\ t = 5$. Now, we are solving for *P*, so

$$\begin{aligned} (\$500) &= (P) \cdot (0.035) \cdot (5) \\ (500) &= P \cdot 0.175 \\ \left(\frac{500}{0.175}\right) &= P \\ 2{,}857.1428571429 &= P \end{aligned}$$

So, the principal would approximately be $2,857.14.

Section 4.2: Simple Interest [*In-Class Exercises*]

1. Suppose you invest $10,000 at 6% simple interest for 2 years. Find the total simple interest earned as well as the future value.
2. Suppose you borrow $25,000 at 10.25% simple interest for 3 years. Find the total simple interest paid and the total amount of the loan that needs to be paid back.
3. Find the simple interest for $7,000 at 5% for 20 weeks. (Assume a 52-week year).
4. Find the simple interest earned on an investment of $2,000 at 5.5% for 90 days. (Assume a 365-day year).
5. Find the amount of simple interest earned on a loan of $7,000 at $8\frac{1}{4}$% for 7 months.
6. Suppose you invest $500 at a 7.5% simple interest rate. Approximately, how long would it take you to double your investment.
7. How much would you need to invest at a 4.43% simple interest rate to accumulate an interest amount of $1,000 in 2 years.

Answers available at the end of this section.

Answers

Section 4.2: Simple Interest [*In-Class Exercises*]

1. I = $1,200; A = $11,200
2. I = $7,687.50; A = $32,687.50
3. I = $134.62
4. I = $27.12
5. I = $336.88
6. $t \approx$13.33 years
7. $P \approx$ $11,286.68

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Section 4.2.H: Homework

Section 4.2: Simple Interest - Homework Exercises

(Section 4.2) - Homework Exercise #1

You decide to invest $4,500 at 6% simple interest for 3 years. How much interest is earned? What is the future value?

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #2

A savings certificate pays 4.8% simple interest. If $2,000 is invested for 15 months, find the interest earned as well as the future value.

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #3

A small business places $18,500 into a simple-interest account earning 3.25% for 2 years. How much interest is earned? What is the total amount after 2 years?

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #4

You borrow $800 from your local bank at 9% simple interest for 26 weeks. How much interest will you pay? How much must be repaid? (Assume a 52-week year).

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #5

A short-term loan of $3,400 charges 7.5% simple interest for 120 days. Find the interest and total amount due. (Assume a 365-day year).

Answer available at the end of this section.

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #6

If $216 of interest is earned at 9% simple interest over 4 years, what was the original principal?

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #7

An employer advances an employee $500 from next month's paycheck. The advance carries 5% simple interest, and repayment occurs in 8 weeks. Find the interest and the total amount paid back. (Hint: Do not round t in your calculations).

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #8

A student owes $900 to a friend who charges 7% simple interest. If it takes 14 months to repay the loan, how much interest is paid? How much total is repaid?

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #9

You deposit an amount of money that earns 5% simple interest with the goal of earning exactly $600 in interest. If you can deposit $4,000, how long will it take to earn the $600?

Answer available at the end of this section.

(Section 4.2) - Homework Exercise #10

You invest $3,500 at 4% simple interest for 210 days. How much interest did you earn?

Answer available at the end of this section.

Answers

(Section 4.2) - Homework Exercise #1
Interest = $810; Future Value = $5,310

(Section 4.2) - Homework Exercise #2
Interest = $120; Future Value = $2,120

(Section 4.2) - Homework Exercise #3
Interest = $1,202.50; Future Value = $19,702.50

(Section 4.2) - Homework Exercise #4
Interest = $36; Total Repaid = $836

(Section 4.2) - Homework Exercise #5
Interest = $83.84; Total Due = $3,483.84

(Section 4.2) - Homework Exercise #5

(Section 4.2) - Homework Exercise #6
Principal = $600

(Section 4.2) - Homework Exercise #7
Interest ≈ $3.85; Amount Paid Back ≈ $503.85

(Section 4.2) - Homework Exercise #8
Interest = $73.50; Total Repaid = $973.50

(Section 4.2) - Homework Exercise #9
Time = 3 years

(Section 4.2) - Homework Exercise #10

Interest = $80.55

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Section 4.3: Compound Interest

Learning Objectives

- Compute compound interest and future value

Compound interest is a fundamental concept in finance and mathematics that describes the process in which interest is earned not only on the original principal but also on previously accumulated interest. With simple interest, interest is calculated only on the original amount invested or borrowed. As a result, the same amount of interest is added during each time period. With compound interest, interest is calculated on both the original amount and any interest that has already been earned. As a result, the amount of interest added during each time period becomes larger over time. This repeated reinvestment of earned interest back into the principal is what gives compound interest its powerful long-term effect. Because each compounding period adds to the base on which future interest is calculated, even modest interest rates can cause significant growth when given enough time.

The frequency of compounding plays a major role in how quickly an investment or loan grows. Interest can be compounded annually, semiannually, quarterly, monthly, daily, or even continuously. The more frequent the compounding, the faster the balance grows, because interest is being added more often and begins earning interest itself. For instance, an investment compounded monthly will grow faster than the same investment compounded annually, even if both have the same nominal (stated) interest rate. This is why understanding compounding periods helps consumers compare financial products accurately.

Compound Interest Formula

Compound interest is interest calculated on both the initial principal (the original amount of money) and the accumulated interest from previous periods. In simple terms, you earn interest on your interest. The compound interest formula is as follows:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where:

- A = Future value (the accumulated amount)
- P = Principal (initial amount)
- r = Nominal annual interest rate (written as a decimal)
- n = Number of compounding periods per year
- t = Time period (in years)

Compounding terms:

- Annually (once a year), $n = 1$
- Semiannually (twice per year), $n = 2$
- Triennially (three times per year), $n = 3$
- Quarterly (four times per year), $n = 4$
- Monthly (twelve times per year), $n = 12$
- Bimonthly (twice per month, 24 times per year), $n = 24$
- Weekly (52 times per year), $n = 52$
- Biweekly (twice per week, 104 times per year), $n = 104$

- Daily (365 times per year), $n = 365$

This formula reflects how the interest rate is divided across each compounding period and how the total number of periods amplifies growth over time.

Compound interest plays a central role in long-term financial planning because it steadily increases the value of savings and investments as interest accumulates on both the principal and previously earned interest. Accounts such as retirement funds, savings accounts, bonds, and reinvested dividends all benefit from this compounding effect, which allows balances to grow more rapidly the longer the money remains invested. In contrast, compound interest can be challenging for borrowers, as unpaid interest becomes part of the principal and causes debt to grow more quickly. Credit cards illustrate this especially well, since they typically compound interest daily, leading to rising balances when only minimum payments are made.

Time is one of the most influential elements in compounding. When money is invested early, even small contributions can grow significantly because they have more years available to accumulate interest. This idea reflects the time value of money, the principle that money available now has more earning potential than the same amount received later. The sooner compounding begins, the larger the eventual return.

Banks use compound interest on both sides of their operations. For deposits such as savings accounts and certificates of deposit, banks offer compound interest to make their products attractive to customers. As account balances grow, customers are encouraged to keep their funds in the institution, and competitive rates help banks bring in new clients. Deposited money does not simply sit in the bank; it becomes part of the bank's lending supply. Banks may lend the depositor's funds at higher interest rates than they pay out, earning a profit known as the spread. For example, if a customer deposits $10,000 at a 3% interest rate, the bank may lend that same amount of $10,000 at a 7% rate and profit from the difference. The bank would essentially pay you $300 in interest as they receive $700 in interest for your money that you invested. The bank would profit the difference of $400.

On the borrowing side, banks apply compound interest to loans, including mortgages and credit cards, to increase revenue and manage risk. When interest compounds, unpaid amounts are added to the principal, causing the balance to grow if payments are postponed. This structure compensates banks for the risk of nonpayment and rewards them for longer repayment periods. Because compound interest accelerates the growth of debt, it provides an incentive for borrowers to make timely payments. Minimum payment structures, particularly on credit cards, often reduce the principal very slowly, allowing interest charges to continue building and extending the life of the debt.

By understanding how compound interest affects savings, investments, and borrowing, individuals can make more informed choices about their financial strategies. Recognizing how time, interest rates, and compounding frequency influence growth can help people plan effectively, avoid unnecessary debt, and take advantage of opportunities to build long-term wealth.

Example #4.3.1

Suppose you invest $1,000 at an 8% interest rate for 5 years compounded semiannually. Find the compound amount and the interest amount earned.

✅ **Solution:**

Start with the compound interest formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

We know that $P = \$1{,}000;\ r = 0.08;\ t = 5;\ n = 2$, so

$$A = \$1{,}000\left(1 + \frac{0.08}{2}\right)^{2 \cdot 5}$$
$$A = \$1{,}000(1 + 0.04)^{10}$$
$$A = \$1{,}000(1.04)^{10}$$
$$A = \$1{,}000(1.4802442849)$$
$$A = \$1{,}480.2442849$$

$$\boxed{A \approx \$1{,}480.24}$$

Now, find the total interest earned.

$$A = P + I$$
$$I = A - P$$
$$I = \$1{,}480.24 - \$1{,}000$$

$$\boxed{I = \$480.24}$$

So, after 5 years, the future value is $1,480.24 and the interest earned is $480.24.

Let's focus on Example #4.3.1 and look at within a year-by-year process to further understand to see what is really happening.

Year 0, (at the end of 6 months):

- Interest earned = $1,000 × 0.08 × $\frac{1}{2}$ = $40
- Total amount = $1,000 + $40 = $1,040

Year 1, (at the end of 12 months)

- Interest earned = $1,040 × 0.08 × $\frac{1}{2}$ = $41.60 (calculated on $1,040, not the original $1,000)
- Total amount = $1,040 + $41.60 = $1,081.60

Year 1: (at the end of 18 months)

- Interest earned = $1,081.60 × 0.08 × $\frac{1}{2}$ = $43.26 (calculated on $1,081.60, not $1040)
- Total amount = $1,081.60 + $43.26 = $1,124.86

Year 2, (at the end of 24 months)

- Interest earned = $1,124.86 × 0.08 × $\frac{1}{2}$ = $44.99 (calculated on $1,124.86, not $1,081.60)
- Total amount = $1,124.86 + $44.99 = $1,169.85

Year 2: (at the end of 30 months)

- Interest earned = $1,169.85 × 0.08 × $\frac{1}{2}$ = $46.79 (calculated on $1,169.85, not $1,124.86)
- Total amount = $1,169.85 + $46.79 = $1,216.64

Year 3, (at the end of 36 months)

- Interest earned = $1,216.64 × 0.08 × $\frac{1}{2}$ = $48.67 (calculated on $1,216.64, not $1,169.85)
- Total amount = $1,216.64 + $48.67 = $1,265.31

Year 3: (at the end of 42 months)

- Interest earned = $1,265.31 × 0.08 × $\frac{1}{2}$ = $50.61 (calculated on $1,265.31, not $1,216.64)
- Total amount = $1,265.31 + $50.61 = $1,315.92

Year 4, (at the end of 48 months)

- Interest earned = \$1,315.92 × 0.08 × $\frac{1}{2}$ = \$52.64 (calculated on \$1,315.92, not \$1,265.31)
- Total amount = \$1,315.92 + \$52.64 = \$1,368.56

Year 4: (at the end of 54 months)

- Interest earned = \$1,368.56 × 0.08 × $\frac{1}{2}$ = \$54.74 (calculated on \$1,368.56, not \$1,315.92)
- Total amount = \$1,368.56 + \$54.74 = \$1,423.30

Year 5: (at the end of 60 months)

- Interest earned = \$1,423.30 × 0.08 × $\frac{1}{2}$ = \$56.93 (calculated on \$1,423.30, not the original \$1,368.56)
- Total amount = \$1,423.30 + \$56.93 = \$1,480.23

Notice that the interest earned each period of six months increases. Your total grows by an exponential amount each six months. Here is a table showing the same calculations:

Growth of a \$1,000 investment compounded semiannually at an annual interest rate of 8% over 5 years

Year	Principal	Simple Interest	Amount After *t* Years
0	\$1,000	\$0	\$1,000
$\frac{1}{2}$	\$1,000	(\$1,000)·(0.08)·($\frac{1}{2}$) = \$40	\$1,040
1	\$1,040	(\$1,040)·(0.08)·($\frac{1}{2}$) = \$41.60	\$1,081.60
$1\frac{1}{2}$	\$1,081.60	(\$1,081.60)·(0.08)·($\frac{1}{2}$) = \$43.26	\$1,124.86
2	\$1,124.86	(\$1,124.86)·(0.08)·($\frac{1}{2}$) = \$44.99	\$1,169.85
$2\frac{1}{2}$	\$1,169.85	(\$1,169.85)·(0.08)·($\frac{1}{2}$) = \$46.79	\$1,216.64
3	\$1,216.64	(\$1,216.64)·(0.08)·($\frac{1}{2}$) = \$48.67	\$1,265.31
$3\frac{1}{2}$	\$1,265.31	(\$1,265.31)·(0.08)·($\frac{1}{2}$) = \$50.61	\$1,315.92
4	\$1,315.92	(\$1,315.92)·(0.08)·($\frac{1}{2}$) = \$52.64	\$1,368.56
$4\frac{1}{2}$	\$1,368.56	(\$1,368.56)·(0.08)·($\frac{1}{2}$) = \$54.74	\$1,423.30
5	\$1,423.30	(\$1,423.30)·(0.08)·($\frac{1}{2}$) = \$56.93	\$1,480.23

So, we essentially get the same result of \$1,480.24 when we use the table as we did by using the formula. The results differ by one cent because of rounding error. When the account balance is rounded to the nearest cent after each compounding period, small differences can accumulate over time. The compound interest formula, on the other hand, uses the exact values throughout the calculation and rounds only the final result.

Example #4.3.2

Suppose you invest $20,000 at an 6% interest rate for 20 years compounded monthly. Find the compound amount and the interest amount earned.

✅ **Solution:**

Start with the compound interest formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

We know that $P = \$20{,}000;\ r = 0.06;\ t = 20;\ n = 12$, so

$$\begin{aligned} A &= \$20{,}000\left(1 + \frac{0.06}{12}\right)^{12 \cdot 20} \\ A &= \$20{,}000(1 + 0.005)^{240} \\ A &= \$20{,}000(1.005)^{240} \\ A &= \$20{,}000(3.3102044758074) \\ A &= \$66{,}204.089516149 \end{aligned}$$

$$\boxed{A \approx \$66{,}204.09}$$

Now, find the total interest earned.

$$\begin{aligned} A &= P + I \\ I &= A - P \\ I &= I = \$66{,}204.09 - \$20{,}000 \end{aligned}$$

$$\boxed{I = \$46{,}204.09}$$

So, after 20 years, the future value is $66,204.09 and the interest earned is $46,204.09.

Example #4.3.3

Jordan wants to start building an emergency fund, so he deposits $5,000 into a high-yield savings account offered by his local credit union. The account earns 4.3% annual interest, and the credit union compounds interest daily. Jordan does not plan to add any additional deposits; he simply wants to let the money grow for 8 years. How much will Jordan have after 8 years? How much of that amount is interest earned?

✅ **Solution:**

Start with the compound interest formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

We know that $P = \$5{,}000;\ r = 0.043;\ t = 8;\ n = 360$, so

$$A = \$5{,}000\left(1 + \frac{0.043}{365}\right)^{365\cdot 8}$$
$$A = \$5{,}000(1 + 0.00011780822)^{2{,}920}$$
$$A = \$5{,}000(1.00011780822)^{2{,}920}$$
$$A = \$5{,}000(1.41055005547)$$
$$A = \$7{,}052.75027734$$

$$\boxed{A \approx \$7{,}052.75}$$

Now, find the total interest earned.

$$A = P + I$$
$$I = A - P$$
$$I = I = \$7{,}052.75 - \$5{,}000$$

$$\boxed{I = \$2{,}052.75}$$

So, after 8 years, the future value is $7,052.75 and the interest earned is $2,052.75.

Note: Avoid rounding too early when performing compound-interest calculations. For instance, if the value 0.00011780822 is rounded prematurely to 0.00012, the resulting future value becomes $7,064.04, which differs by $11.29 from the more precise calculation. To ensure accuracy, use as many digits from your calculator as possible at each step rather than rounding too soon.

Section 4.3: Compound Interest [*In-Class Exercises*]

1. A parent deposits $8,000 into an education fund earning 5% compounded annually for 12 years. Find the future value and interest earned.
2. A small business places $25,000 into a savings account earning 4.2% compounded quarterly for 5 years. Find the future value and interest earned.
3. A mother just gave birth to her daughter, Sad Sally. So, the mother decided to invest $5000 into a savings account at Wells Fargo and leave that money in there to invest until she turns 18 for a future college fund. The current "standard interest rate" in a savings account compounded daily at Wells Fargo is: Current Wells Fargo rates(opens in new window). What is the future value and why is Sally so sad? How much of the amount is interest?
4. So, in Exercise #3, Sad Sally's mother decided to scroll down on the webpage from the link above and realized she should invest in a special fixed rate CD (Certificate of Deposit) instead. What is the future value on this short term investment? (Note: You will have to look at the current "Special Interest Rate" and term (length) of the CD. Rates and terms are subject to change possibly on a daily basis. In the fine print on this web page it says the "interest rate is compounded daily for a 365-day period.")
5. Sad Sally's mother has chosen a different option. This time, she would rather have $18,000 for a college fund in 18 years from now for her new baby daughter when she turns 18. She wants to invest in a certificate of deposit (CD) that will earn 6% compounded daily. How much should she deposit today in order to get $18,000 in 18 years from today? How much interest will be made on the investment?

Answers available at the end of this section.

Answers

Section 4.3: Compound Interest [*In-Class Exercises*]

1. A = $14,366.85; I = $6,366.85
2. A = $30,808.20; I = $5,808.20
3. If r = 0.01%, then A = $5,009.01; I = $9.01; otherwise answers will slightly vary.
4. *Answers will vary*

5. P = \$6,113.26; I = \$11,886.74

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Section 4.3.H: Homework

Section 4.3: Compound Interest - Homework Exercises

(Section 4.3) - Homework Exercise #1

A government bond pays 3.6% interest compounded semiannually, and Mai-Ann wants to invest $15,000 for 7 years. What is the future value? How much interest is earned?

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #2

Parvin opens an auto-savings account with $4,000, earning 4% interest compounded monthly for 3 years. Find the future value along with the interest earned.

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #3

Mansour invests $12,000 into a college savings fund earning 4.8% compounded quarterly for 10 years. Find the future value and interest earned.

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #4

A high-yield savings account pays 3.8% interest, compounded daily, and you deposit $6,500 for 10 years. Find the future value along with the interest earned.

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #5

Anton receives a $3,200 bonus from work and decides to invest it in a savings account earning 3.9% annual interest compounded monthly. If he leaves the money untouched for 6 years, what will the account be worth at the end of that time?

Answer available at the end of this section.

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #6

A credit union investment earns 4.1% compounded daily for amounts of $5,000 to $9,999. Huy wants to invest $9,500 investment over a period of 9 years. Find the future value as well as the interest earned.

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #7

An unusual investment of $6,767.67 is invested at a 6.7% interest rate for 6.7 years, that is compounded every other week. Find the future value of the investment.

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #8

Thanh decides to invest $10,000 into a 5-year certificate of deposit (CD) that earns 4% interest compounded quarterly. Once that accrues, she will reinvest the full amount into a new account earning 6% interest compounded annually for an additional 8 years. Find the value of both investments.

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #9

Kasara wants $10,000 in 5 years in an account earning 4.5% compounded quarterly. How much should she deposit today?

Answer available at the end of this section.

(Section 4.3) - Homework Exercise #10

Phillippe plans to purchase a new car in three years and would like to have at least $5,000 available for a down payment. If a certificate of deposit (CD) earns 3.7% annual interest compounded daily, how much money must Phillippe invest today in order to have $5,000 in the account three years from now?

Answer available at the end of this section.

Answers

(Section 4.3) - Homework Exercise #1
Future Value = $19,255.74; Interest = $4,255.74

(Section 4.3) - Homework Exercise #2
Future Value = $4,509.09; Interest = $509.09

(Section 4.3) - Homework Exercise #3
Future Value = $19,337.56; Interest = $7,337.56

(Section 4.3) - Homework Exercise #4
Future Value = $9,504.66; Interest = $3,004.66

(Section 4.3) - Homework Exercise #5
Future Value = $4,042.13

(Section 4.3) - Homework Exercise #5

(Section 4.3) - Homework Exercise #6

Future Value = $13,739.45; Interest = $4,239.45

(Section 4.3) - Homework Exercise #7

Future Value = $10,596.03

(Section 4.3) - Homework Exercise #8

Future Value of 1st investment = $12,201.90; Future Value of 2nd investment = $19,447.97

(Section 4.3) - Homework Exercise #9

Principal = $7995.20

(Section 4.3) - Homework Exercise #10

Principal = $4,474.72

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Section 4.4: Effective Yield

Learning Objectives

- Compute effective yield

In the previous section, we saw when money is invested or borrowed, that interest can be calculated in different ways. Because different accounts use different compounding periods (such as annually, quarterly, monthly, or daily), comparing them directly can be confusing.

Effective yield, also called **annual yield**, **effective rate**, **effective annual rate** (**EAR**), or **annual percentage yield** (**APY**) provides a solution to this problem. It represents the actual annual percentage increase in an investment after accounting for compounding. In other words, the effective yield tells you how much your money truly grows in one year, regardless of how many times interest is added.

For example, an account that advertises a 5% annual interest rate compounded monthly does not actually grow by exactly 5% per year. Because interest is earned on previously earned interest throughout the year, the real growth is slightly higher. Effective yield captures this true rate of return. Understanding effective yield helps consumers compare financial products on a fair basis. Whether evaluating savings accounts, credit card rates, or loan offers, effective yield provides a single, meaningful number that reflects how interest really works over time.

In this section, we will explore how effective yield is calculated, why compounding matters, and how this concept can be used to make informed financial decisions.

Effective Yield Formula

The **effective yield** is the simple annual interest rate that would produce the same future value in one year as a given compound interest rate. It provides a way to compare different investments on an equal basis by converting any compounding schedule into an equivalent single-year growth rate. The formula for the effective yield is:

$$E = \left(1 + \frac{r}{n}\right)^n - 1$$

where:

- E = Effective yield (the accumulated amount)
- r = Nominal annual interest rate (written as a decimal)
- n = Number of compounding periods per year

Example #4.4.1 🤔

An investment grows at 8% nominal interest, compounded twice per year. What is the effective annual yield?

✅ **Solution:**

Start with the effective yield formula.

$$E = \left(1 + \frac{r}{n}\right)^n - 1$$

We know that $r = 0.08$ and $n = 2$, so

$$E = \left(1 + \frac{0.08}{2}\right)^2 - 1$$
$$E = (1 + 0.04)^2 - 1$$
$$E = (1.04)^2 - 1$$
$$E = 1.0816 - 1$$
$$E = 0.0816$$

$$\boxed{E = 8.16\%}$$

So, the effective annual yield is 8.16%.

Now, here is what the value of 8.16% means:

- If you invest \$100 at a 8% interest rate, compounded semiannually, for one year, then the future value would be \$108.16 and the interest is \$8.16, which is exactly 8.16% of \$100.
- If you invest \$1,000 at a 8% interest rate, compounded semiannually, for one year, then the future value would be \$1081.60 and the interest is \$81.60, which is exactly 8.16% of \$1,000.
- If you invest \$10,000 at a 8% interest rate, compounded semiannually, for one year, then the future value would be \$10,816.00 and the interest is \$816.00, which is exactly 8.16% of \$1,000.

Example #4.4.2 🤔

Joseph is comparing savings accounts and finds one that advertises a 6% nominal annual interest rate compounded quarterly. What is the account's effective annual yield?

✅ **Solution:**

Start with the effective yield.

$$E = \left(1 + \frac{r}{n}\right)^n - 1$$

We know that $r = 0.06$ and $n = 4$, so

$$E = \left(1 + \frac{0.06}{4}\right)^4 - 1$$
$$E = (1 + 0.015)^4 - 1$$
$$E = (1.015)^4 - 1$$
$$E = 1.061363550625 - 1$$
$$E = 0.061363550625$$

$$\boxed{E \approx 0.06136 = 6.14\%}$$

So, the effective yield is 6.136% or 6.14%, depending on how much you decide to round.

Now, here is what the value of 6.14% (or 6.136%) means:

- If you invest \$100 at a 6% interest rate, compounded quarterly, for one year, then the future value would be \$106.14 and the interest is \$6.14, which is exactly 6.14% of \$100. (Effective Yield = 6.14%, if you round to the nearest hundredth of a percent).

- If you invest $1,000 at a 6% interest rate, compounded quarterly, for one year, then the future value would be $1061.36 and the interest is $61.36, which is exactly 6.136% of $1,000. (Effective Yield = 6.136%, if you round to the nearest thousandth of a percent).

Example #4.4.3 🤔

Two investment funds advertise:

Fund A: 7.1% compounded semiannually
Fund B: 7% compounded daily

Compute the effective annual rate for each. Which fund provides the higher actual return?

✅ **Solution:**

Start with the effective yield.

$$E = \left(1 + \frac{r}{n}\right)^n - 1$$

For Fund A, we know that $r = 0.071$ and $n = 2$, so

$$\begin{aligned} E &= \left(1 + \frac{0.071}{2}\right)^2 - 1 \\ E &= (1 + 0.0355)^2 - 1 \\ E &= (1.0355)^2 - 1 \\ E &= 1.07226025 - 1 \\ E &= 0.07226025 \end{aligned}$$

$$\boxed{E \approx 0.0722 = 7.22\%}$$

So, the effective yield for Fund A is 7.22%.

For Fund B, we know that $r = 0.07$ and $n = 360$, so

$$\begin{aligned} E &= \left(1 + \frac{0.07}{360}\right)^{360} - 1 \\ E &= (1 + 0.0001944444444)^{360} - 1 \\ E &= (1.0001944444444)^{360} - 1 \\ E &= 1.0725008832111 - 1 \\ E &= 0.0725008832111 \end{aligned}$$

$$\boxed{E \approx 0.0725 = 7.25\%}$$

So, the effective yield for Fund B is 7.25%.

Thus, Fund B at an effective yield of 7.25% has a slightly higher return than Fund A at an effective yield of 7.22%.

The table below shows how the same nominal rate produces different effective yields depending on how often interest is compounded.

Comparison of effective interest yields for the same nominal interest rate under different compounding frequencies

Nominal Rate	Annually	Semi-Annually	Quarterly	Monthly	Daily	Million Times Per Year
2%	2%	2.01%	2.0151%	2.0184%	2.0201%	2.0201%
3%	3%	3.0225%	3.0339%	3.0416%	3.0453%	3.0455%
4%	4%	4.04%	4.060%	4.0742%	4.0808%	4.0811%
5%	5%	5.0625%	5.0945%	5.1162%	5.1267%	5.1271%
6%	6%	6.09%	6.1364%	6.1678%	6.1831%	6.1837%
7%	7%	7.1225%	7.1859%	7.2290%	7.2501%	7.2508%
8%	8%	8.16%	8.2432%	8.3000%	8.3277%	8.3287%
9%	9%	9.2025%	9.3083%	9.3807%	9.4162%	9.4174%
10%	10%	10.25%	10.3813%	10.4713%	10.5156%	10.5171%

When you examine the table above of effective annual yields for different compounding frequencies, an important pattern becomes clear. For a fixed nominal interest rate, the effective yield increases as the number of compounding periods per year increases. However, this increase does not occur at a constant or proportional rate. For example, moving from semiannual to quarterly compounding does not double the gain, nor does monthly compounding produce a dramatically larger jump. Instead, each additional increase in compounding frequency results in a smaller improvement in the effective yield. The growth slows down because the effect of compounding more frequently has diminishing returns. In fact, even if interest were compounded a million or even a billion times per year, the effective yield would approach a limiting value and show almost no change. This illustrates a concept called "continuous compounding" that represents the theoretical maximum yield for any given nominal rate. This concept is usually introduced in a college algebra or pre-calculus course in relation to exponential growth and decay.

Section 4.4: Effective Yield [*In-Class Exercises*]

1. Richard notices that his bank pays 3.9% interest compounded quarterly. Compute the effective yield.
2. A certificate of deposit (CD) offers 5.2% compounded monthly, while a bond fund advertises a simple annual return of 5.3% (no compounding). Compute the effective annual rate for each. Which fund provides the higher actual return?
3. Sad Sally is back looking at that Wells Fargo savings rate of 0.01%. Compounding daily, what is the APY? Round to the nearest *hundred-thousandths* of a percent.

Answers available at the end of this section.

Answers

Section 4.4: Effective Yield [*In-Class Exercises*]

1. $E = 3.96\%$
2. CD: $E = 5.33\%$; Bond fund: $E = 5.3\%$; Thus, the CD provides a slightly higher actual return by 0.03% per year
3. APY is also the Effective Yield; $E = 0.01\%$; (actual value: $E = 0.010000049862764$)

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Section 4.4.H: Homework

Section 4.4: Effective Yield - Homework Exercises

(Section 4.4) - Homework Exercise #1

Frank takes out a loan that charges 12% interest compounded monthly. Find the effective yield.

Answer available at the end of this section.

(Section 4.4) - Homework Exercise #2

Joseph discovers that his bank offers a savings account with an advertised annual interest rate of 2.75% compounded daily. Find the effective annual yield.

Answer available at the end of this section.

(Section 4.4) - Homework Exercise #3

Suppose a nominal rate of 5.7% is compounded weekly. Find the effective rate.

Answer available at the end of this section.

(Section 4.4) - Homework Exercise #4

Trevor is comparing two types of savings accounts:

- Account A: 4.8% compounded quarterly
- Account B: 4.6% compounded daily

Compute both effective annual rates. Which account truly earns more?

Answer available at the end of this section.

(Section 4.4) - Homework Exercise #5

Paul is comparing two car loans:

- Car Loan A: 6.5% APR, compounded semiannually
- Car Loan B: 6.4% APR, compounded monthly

Compute both effective rates. Which loan is actually cheaper?

Answer available at the end of this section.

Answer available at the end of this section.

(Section 4.4) - Homework Exercise #6

Vu deposits $1,000 into an account earning 3.4% compounded quarterly.

a. Compute the effective annual rate. Round to the nearest *thousandths* of a percent.
b. Compute the future value after one year using the effective rate. (Hint: Use the simple interest formula from Section 4.2)

Answer available at the end of this section.

(Section 4.4) - Homework Exercise #7

A credit union is offering a special rate of 4.53% for a short term certificate of deposit (CD) with a minimum opening balance of at least $5,000, yet not to exceed $100,000. Compute the effective rate if the interest is compounded daily.

Answer available at the end of this section.

Answers

(Section 4.4) - Homework Exercise #1
E = 12.68%

(Section 4.4) - Homework Exercise #2
E = 2.79%

(Section 4.4) - Homework Exercise #3
E = 5.86%

(Section 4.4) - Homework Exercise #4
Account A = 4.89%; Account B = 4.71%; Thus, Account A provides a better actual return by 0.18%

(Section 4.4) - Homework Exercise #5
Car Loan A = 6.61%; Car Loan B = 6.59%; Thus, Car Loan A provides a better actual return by 0.02%

(Section 4.4) - Homework Exercise #5

(Section 4.4) - Homework Exercise #6
a. E = 3.444%
b. $1,034.44

(Section 4.4) - Homework Exercise #7
E = 4.63%

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Section 4.5: Annuities

Learning Objectives

- Compute the future value of an annuity

Annuities arise in many real-world situations, including retirement income plans, pension benefits, mortgage payments, and lottery prize distributions. In general, an **annuity** is a sequence of equal payments made at regular intervals over a specified period of time. Some annuities are created through financial contracts, while others occur naturally as part of everyday financial transactions. Because annuities involve predictable streams of cash flows, they are important applications of mathematics in finance and economics.

For example, many retirees receive regular income from Social Security or employer-sponsored pension plans. These programs typically provide fixed monthly payments for the remainder of the recipient's life, making them common examples of annuities. Another example occurs when lottery winners choose to receive their prize as a series of payments rather than a lump sum. Because the payments are made at regular intervals according to a predetermined schedule, they also constitute an annuity.

Although several types of annuities exist including fixed, variable, indexed, and immediate annuities, this text focuses on fixed annuities.

The **future value of an annuity** is the total amount accumulated in the account after the final payment has been made, including all interest earned. Because each payment is deposited at a different time, each payment earns a different amount of interest. Earlier payments earn interest over many periods, while later payments earn interest over fewer periods.

Future Value of an Annuity

The **future value of an annuity** formula is as follows:

$$A = \frac{R\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

where:

- A = Future value (the accumulated amount)
- R = Regular periodic payment at the end of each period
- r = Annual interest rate (expressed as a decimal)
- n = Number of payments made per year
- t = Term of annuity (in years)

This formula accounts for the compound interest earned on each payment. Note that the number of payments made per year also equals the number of times the interest each time is compounded per year.

Example #4.5.1 🤔

Let's look back at a Section 4.3 example involving compound interest:

Suppose you invest $5,000 at an 8% interest rate for 5 years compounded semiannually. Find the future value.

Using the compound interest formula from Section 4.3, we have:

$$A = \$5{,}000\left(1+\frac{0.08}{2}\right)^{2\cdot 5}$$
$$A = \$7{,}401.22$$

So, after 5 years, the future value is $\$7{,}401.22$. The interest earned is $\$2{,}401.22$.

Here, this is a one time lump sum of $5,000 invested over 5 years compounded twice a year. But, what if you don't have $5,000 saved up to invest? Instead, let's invest $500 twice a year for 5 years. After 5 years, you will have deposited $5,000 equal to the principal in the previous example. However, investing $500 every 6 months may be more manageable than a one time deposit of $5,000, especially for a student or anyone on a fixed income. Now, we ask,

What amount will accumulate if we deposit $500 semiannually for the next 5 years at an interest rate of 8%?

✅ **Solution:**

Start with the future value of an annuity interest formula.

$$A = \frac{R\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

We know that $R = \$500;\ r = 0.08;\ t = 5;\ n = 2$, so

$$A = \frac{500\left[\left(1+\frac{0.08}{2}\right)^{2\cdot 5}-1\right]}{\frac{0.08}{2}}$$

$$A = \frac{500\left[(1.04)^{10}-1\right]}{0.04}$$

$$A = \frac{500\left[0.48024428491834\right]}{0.04}$$

$$A = \$6{,}003.0535614796$$

$$\boxed{A \approx \$6{,}003.05}$$

So, after 5 years, the future value is $\$6{,}003.05$.

The interest earned is

$$I = \text{Future Value} - (\text{Total Number of Payments}) \cdot (\text{Periodic Payment})$$
$$I = \$6{,}003.05 - (2\cdot 5)\cdot(\$500)$$
$$I = \$6{,}003.05 - \$5{,}000$$

$$\boxed{I = \$1{,}003.05}$$

The interest earned is $\$1{,}003.05$.

Here is a table showing the same calculations:

Growth of an investment account with semiannual $500 deposits earning 8% annual simple interest over 5 years

Year	Beginning Balance	Simple Interest (8% per year; 4% per period)	Deposit (at end of period)	Ending Balance after t Years (Beginning Balance + Interest + Deposit)
0	$0	$0	$0	$0
$\frac{1}{2}$	$0	($0)·(0.08)·$(\frac{1}{2})$ = $0	$500	$500
1	$500	($500)·(0.08)·$(\frac{1}{2})$ = $20	$500	$1,020
$1\frac{1}{2}$	$1,020	($1,020)·(0.08)·$(\frac{1}{2})$ = $40.80	$500	$1,560.80
2	$1,560.80	($1,560.80)·(0.08)·$(\frac{1}{2})$ = $62.43	$500	$2,123.23
$2\frac{1}{2}$	$2,123.23	($2,123.23)·(0.08)·$(\frac{1}{2})$ = $84.93	$500	$2,708.16
3	$2,708.16	($2,708.16)·(0.08)·$(\frac{1}{2})$ = $108.33	$500	$3,316.49
$3\frac{1}{2}$	$3,316.49	($3,316.49)·(0.08)·$(\frac{1}{2})$ = $132.66	$500	$3,949.15
4	$3,949.15	($3,949.15)·(0.08)·$(\frac{1}{2})$ = $157.97	$500	$4,607.12
$4\frac{1}{2}$	$4,607.12	($4,607.11)·(0.08)·$(\frac{1}{2})$ = $184.28	$500	$5,291.40
5	$5,291.40	($5,291.40)·(0.08)·$(\frac{1}{2})$ = $211.66	$500	$6,003.06

We basically get the same result of $6,003.05 (off by one cent due to rounding error) when we use the table as we did by using the formula.

Let's compare the two different ways of investing the same total amount of money:

1. A one-time lump sum of $5,000 invested for 5 years compounded semiannually.
2. A periodic annuity: $500 twice per year for 5 years (for a total of $500 × 10 = $5,000).

When you invest $5,000 as a single lump sum, the entire amount remains in the account for all 5 years. Since interest is compounded semiannually, the investment earns interest 10 times, and each interest amount becomes part of the principal. This allows the account to earn interest on both the original investment and previously earned interest, resulting in the greatest possible growth.

With a periodic annuity, you invest $500 every 6 months. Although the total amount invested is still $5,000, each deposit remains in the account for a different length of time. The first deposit earns interest for the full 5 years, while later deposits have less time to grow. The final deposit is invested for only 6 months. As a result, the later deposits earn less interest, causing the annuity to accumulate less interest than the lump-sum investment.

In summary, a lump-sum investment benefits from having the entire amount invested for the full term, allowing it to earn the

maximum amount of compound interest. A periodic annuity offers greater flexibility because deposits are made over time, but the money has less time to grow. As a result, the future value of the lump-sum investment is $7,401.22, while the future value of the periodic annuity is $6,003.05. In this example, the annuity accumulates $1,398.17 less than the lump-sum investment.

Example #4.5.2 🤔

Sheldon dreams of purchasing a vacation cabin in eight years. To build a down payment, he deposits $600 quarterly into an investment account earning 7% annual interest, compounded quarterly. How much will Sheldon have at the end of 8 years? How much interest did the account accumulate?

✅ **Solution:**

Start with the future value of an annuity formula.

$$A = \frac{R\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

We know that $R = \$600;\ r = 0.07\ t = 8;\ n = 4$, so

$$A = \frac{600\left[\left(1+\frac{0.07}{4}\right)^{4\cdot 8}-1\right]}{\frac{0.07}{4}}$$

$$A = \frac{600\left[(1.0175)^{32}-1\right]}{0.0175}$$

$$A = \frac{600\left[0.74221349218035\right]}{0.0175}$$

$$A = \$25{,}447.319731898$$

$$\boxed{A \approx \$25{,}447.32}$$

So, after 8 years, the future value is $\$25{,}447.32$.

The interest earned is

$$I = \text{Future Value} - (\text{Total Number of Payments}) \cdot (\text{Periodic Payment})$$

$$I = \$25{,}447.32 - (4 \cdot 8) \cdot (\$600)$$

$$I = \$25{,}447.32 - \$19{,}200$$

$$\boxed{I = \$6{,}247.32}$$

The interest accumulated is $\$6{,}247.32$.

Example #4.5.3 🤔

Anna & Tom are planning their wedding and want to avoid wedding loans. They decide to save for three and a half years by depositing $250 at the end of each month into an account earning 5.2% interest, compounded monthly. How much will they have at the end of the term? How much money did they save in the process?

✅ **Solution:**

Start with the future value of an annuity formula.

$$A = \frac{R\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

We know that $R = \$250;\ r = 0.052\ t = 3.5;\ n = 12$, so

$$A = \frac{250\left[\left(1+\frac{0.052}{12}\right)^{12\cdot 3.5}-1\right]}{\frac{0.052}{12}}$$

$$A = \frac{250\left[(1.0043333333333)^{42}-1\right]}{0.0043333333333}$$

$$A = \frac{250\left[0.1991426008935\right]}{0.0043333333333}$$

$$A = \$11{,}488.996205482$$

$$\boxed{A \approx \$11{,}489.00}$$

So, after 3.5 years, the future value is $\$11{,}489.00$.

The interest earned is

$$I = \text{Future Value} - (\text{Total Number of Payments}) \cdot (\text{Periodic Payment})$$

$$I = \$11{,}489.00 - (12 \cdot 3.5) \cdot (\$250)$$

$$I = \$11{,}489.00 - \$10{,}500$$

$$\boxed{I = \$989}$$

The interest, which is the money saved on the wedding is $\$989$.

Example #4.5.4 🤔

Sandy wants to take a special vacation in five years. She plans to save money regularly so she doesn't have to use credit cards. Her bank offers a savings account that earns 6% annual interest, compounded monthly. She decides to deposit $150 at the end of every month. What is the future value? How much interest was made?

✅ **Solution:**

Start with the future value of an annuity formula.

$$A = \frac{R\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

We know that $R = \$150;\ r = 0.06\ t = 5;\ n = 12$, so

$$A = \frac{150\left[\left(1+\frac{0.06}{12}\right)^{12\cdot 5}-1\right]}{\frac{0.06}{12}}$$

$$A = \frac{150\left[(1.005)^{60}-1\right]}{0.005}$$

$$A = \frac{150\left[0.34885015254932\right]}{0.005}$$

$$A = \$10{,}465.504576479$$

$$\boxed{A \approx \$10{,}465.50}$$

So, after 5 years, the future value is $\$10{,}465.50$.

The interest earned is

$$I = \text{Future Value} - (\text{Total Number of Payments}) \cdot (\text{Periodic Payment})$$

$$I = \$10{,}465.50 - (12 \cdot 5) \cdot (\$150)$$

$$I = \$10{,}465.50 - \$9{,}000$$

$$\boxed{I = \$1{,}465.50}$$

The interest earned is $\$1{,}465.50$.

Example #4.5.5 🤔

T.Q. wants to buy a reliable used car in four years. Based on current prices, he estimates he will need about $12,000 to comfortably purchase a car and cover registration, taxes, and insurance. T.Q. decides to set up a savings plan where he contributes $200 at the end of each month into a savings account that earns 4.8% annual interest, compounded monthly. What is the future value? How much interest was made? Will T.Q. fall short of his estimate?

✅ **Solution:**

Start with the future value of an annuity formula.

$$A = \frac{R\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

We know that $R = \$200;\ r = 0.048\ t = 4;\ n = 12$, so

$$A = \frac{200\left[\left(1+\frac{0.048}{12}\right)^{12\cdot 4}-1\right]}{\frac{0.048}{12}}$$

$$A = \frac{200\left[(1.004)^{48}-1\right]}{0.004}$$

$$A = \frac{200\left[0.2112065613749\right]}{0.004}$$

$$A = \$10{,}560.328068746$$

$$\boxed{A \approx \$10{,}560.33}$$

So, after 4 years, the future value is $\$10{,}560.33$.

The interest earned is

$$I = \text{Future Value} - (\text{Total Number of Payments}) \cdot (\text{Periodic Payment})$$

$$I = \$10{,}560.33 - (12 \cdot 4) \cdot (\$200)$$

$$I = \$10{,}560.33 - \$9{,}600$$

$$\boxed{I = \$960.33}$$

The interest earned is $\$960.33$.

Yes, Johnny will fall short of his estimate.

Example #4.5.6 🤔

Refer to Example #4.55. Since T.Q. fell short of his $12,000 goal, what would the actual periodic payment T.Q. needs to make in order to make that goal?

✅ **Solution:**

To actually know the exact payment for a given future value, you would use what is called a **sinking fund**. A sinking fund is a long-term savings plan in which regular, equal deposits are made into an account in order to accumulate a specific amount of money in the future. In this example, we want to set up a sinking fund with future value A = $12,000. To find the required monthly payment, we solve for R in the formula for the future value of an annuity formula.

$$A = \frac{R\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$$

Solving for R,

$$R = \frac{A\frac{r}{n}}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}$$

We know that $A = \$12{,}000;\ r = 0.048\ t = 4;\ n = 12$, so

$$R = \frac{\$12{,}000\frac{0.048}{12}}{\left[\left(1+\frac{0.048}{12}\right)^{12\cdot 4}-1\right]}$$

$$R = \frac{48}{\left[(1.004)^{48}-1\right]}$$

$$R = \frac{48}{0.21120656137492}$$

$$R = \$227.26566678387$$

$$\boxed{R \approx \$227.27}$$

So, the required periodic payment is $\$227.27$.

Note: There will be 48 scheduled payments of $\$227.27$. This translates to $\$10{,}908.96$ which means $\$12{,}000 - \$10{,}908.96 = \$1{,}101.04$ of interest made on the annuity.

Section 4.5: Annuities [*In-Class Exercises*]

1. Syed wants to deposit $200 quarterly into an investment account earning 5% annual interest, compounded quarterly for 5 years.
   a. What is the future value?
   b. How much of the final amount is the student's own contributions?
   c. How much interest was earned?
2. Leo wants a high-performance laptop for his graphic design classes. He knows he will need about $2,000 in three years.

   He notes that if he earns no interest on his savings, then he will need to save $2,000 ÷ 36 = $55.56 per month. He hopes that if he deposits $50 at the end of each month into an account earning 6% annually, compounded monthly, he would have the $2,000 needed for purchase. Will Leo have the $2,000?
3. Sad Sally's mother thought at one point to deposit $100 at the end of each month into a Wells Fargo savings at a rate of 0.01% compounded monthly for 18 years. After she did the calculations, she realized not to do so.
   a. What would the future value be in 18 years?
   b. How much of the final amount is her own contributions?
   c. How much interest was earned?
4. A 25-year old decides that her goal is to retire at age 60 with at least $1,000,000 in savings. The company investment annuity offers 7.25% annual returns, compounded monthly.
   a. What amount will she need to invest each month?
   b. Compare the amount earned from interest to the amount she deposited.

Answers available at the end of this section.

Answers

Section 4.5: Annuities [*In-Class Exercises*]

1. a. A = $4,512.60
   b. Total contributions = $4,000
   c. I = $512.60
2. A = $1,966.81; Leo comes up only $33.19 short. So, we suggest to Leo to make an extra $33.19 payment on the final payment of $50 to get exactly $2,000.
3. a. A = $21,619.36
   b. Total contributions = $21,600
   c. I = $19.36
4. a. R = $523.01
   b. I = $780,335.80; Total amount in deposits made = $523.01 x 420 = $219,664.20; Investment is more than tripled over the 40 years.

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Section 4.5.H: Homework

Section 4.5: Annuities - Homework Exercises

(Section 4.5) - Homework Exercise #1

Carly is planning a big family vacation in five years. They save and are going to deposit $600 semiannually in an account that earns 6.2% interest, compounded semiannually. What is the future value?

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #2

Naoko wants to redesign her living room in three years. She deposits $130 per month, at the end of each month, into an account earning 4.9% interest, compounded monthly.

a. What will her savings grow to in 3 years?
b. How much interest will she earn?

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #3

Sonia plans to enroll in a professional certification program in two years. She saves $140 per month, depositing at the end of each month, in an account earning 5.1% interest, compounded monthly.

a. What is the future value of her savings?
b. How much interest did she earn?

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #4

Andy plans to landscape his backyard in four years. He deposits $500 at the end of each quarter into an investment account earning 6.4% interest, compounded quarterly.

a. What is the future value after four years?
b. How much interest did he earn?

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #5

Bobby wants to start a college fund for his nephew. He deposits $2,000 at the end of each year into an account earning 6.7% interest, compounded annually, for 15 years.

a. What is the future value of the college fund?

b. How much interest did it earn?

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #6

Debbie wants to supplement her pension with additional savings. She begins at age 35 and invests $250 each month, deposited at the end of the month, into a fund earning 6.8% annual interest, compounded monthly. She plans to save for 30 years.

a. Calculate the future value of Debbie's savings after 30 years.
b. Determine how much interest she accumulated.
c. Compare the amount earned from interest to the amount she deposited.

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #7

Doug, age 45, dreams of buying a retirement vacation home near Sequoia National Park at age 65. To prepare, he deposits $600 per month into an investment account earning 7.2% annual interest, compounded monthly, for 20 years.

a. How much money will Doug have saved at retirement?
b. How much total interest has he earned?
c. If his target amount is $250,000, will he reach it?

Answer available at the end of this section.

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #8

A Roth IRA is a type of retirement savings account funded with after-tax income that allows you to withdraw from the account in retirement tax-free. Linda, age 25, wants to start saving early. She deposits $150 at the end of every month into a Roth IRA earning 8% annually, compounded monthly. She continues this plan for 40 years.

a. What is the future value of her Roth IRA after 40 years?
b. How much of the final amount is her own contributions?
c. How much comes from interest/growth?

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #9

Mariana, age 30, wants to have $500,000 saved by age 65. Her investment account earns 6.9% annual interest, compounded monthly. She has 35 years to save. Round your answer to the nearest dollar.

a. What monthly deposit must Mariana make to reach her goal of $500,000?
b. How much total money will she contribute over 35 years?
c. How much of the final balance will come from interest?

Answer available at the end of this section.

(Section 4.5) - Homework Exercise #10

Art, age 50, wants to retire at age 65 with $300,000 saved in one of his investment accounts. His account earns 6.3% annual interest, compounded monthly, and he has 15 years to save. Round your answer to the nearest dollar.

a. What monthly deposit must Art make to reach his goal of $300,000?
b. How much will he contribute in total over 15 years?
c. How much of his final balance will come from interest?

Answer available at the end of this section.

Answers

(Section 4.5) - Homework Exercise #1

A = $6910.09

(Section 4.5) - Homework Exercise #2

a. A = $5,030.44
b. I = $350.44

(Section 4.5) - Homework Exercise #3

a. A = $3,529.45
b. I = $169.45

(Section 4.5) - Homework Exercise #4

a. A = $9,035.55
b. I = $1,035.55

(Section 4.5) - Homework Exercise #5

a. A = $49,111.86
b. I = $19,111.86

(Section 4.5) - Homework Exercise #6

a. A = $293,225.82
b. I = $203,225.82
c. Deposited = $90,000, so a little over double was made in interest.

(Section 4.5) - Homework Exercise #7

a. A = $320,257.40
b. I = $176,257.40
c. Yes, with an extra $70,257.40. (Doug earns more interest than he contributed; a sign of strong compounding over 20 years).

(Section 4.5) - Homework Exercise #7

(Section 4.5) - Homework Exercise #8

a. A = $523,651.17
b. Total Contributions = $72,000
c. I = $451,651.17

(Section 4.5) - Homework Exercise #9

a. R = $284.30
b. Total Contributions = $119,405
c. I = $380,595

(Section 4.5) - Homework Exercise #10

a. R = $1,005.45
b. Total Contributions = $180,981
c. I = $119,019

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Section 4.6: Installment Buying

Learning Objectives

- Find amount financed, total installment price, and finance charge for a fixed installment loan.

Now that we've spent time studying annuities, you've seen how powerful it can be when you make regular payments into an account and let interest work for you. In every annuity example, whether it was saving for retirement, a vacation, a car, or a college fund, we assumed you had time on your side. Time to contribute. Time to grow. Time to wait. But in real life, there are many situations where people don't have that luxury. Sometimes, you need the money now, not years from now. This is where we naturally transition from annuities to installment loans.

There are two main types of installment loans commonly used: fixed installment loans and amortized loans. A **fixed installment loan** uses what's called precomputed interest. With this method, the lender calculates all of the interest up front, usually using simple interest, then adds that interest to the amount financed. The total is then divided by the number of months in the loan term, which creates a set of equal monthly payments. The major disadvantage of a fixed installment loan is that you pay the full amount of precomputed interest even if you pay the loan off early. In other words, making extra payments or paying off the balance ahead of schedule does not save you any interest. You end up paying the same total cost as if you had taken the entire loan term to repay it. Fixed installment loans include personal loans, appliance loans, furniture loans, and some types of consolidation loans.

In contrast, **amortized loans** are the type used for most car loans, student loans, and mortgages. Instead of computing all the interest upfront, an amortized loan calculates interest month by month based on the remaining balance. Because each payment reduces the principal, the amount of interest charged gradually decreases over time, and more of each payment goes toward the principal as the loan progresses. This structure gives borrowers a major advantage over precomputed-interest loans where paying the loan off early saves money. Since interest is only charged on the outstanding balance, making extra payments, whether occasionally or regularly, reduces the principal faster, which in turn reduces the total interest paid over the life of the loan. This makes amortized loans far more consumer-friendly, especially for people who plan to pay extra or hope to pay off their loan ahead of schedule. We will focus more on amortized loans later in this chapter in Section 4.9.

Every fixed installment loan includes three key financial ideas:

1. Amount Financed
2. Total Installment Price
3. Finance Charge

Understanding these helps borrowers evaluate whether a loan is affordable and whether it is a good financial decision.

Amount Financed

The **amount financed** is the actual amount of money you borrow from the lender; i.e., the principal. The amount financed consists of the cash price of the item, minus any down payment.

The formula for amount financed is:

$$\text{Amount Financed} = \text{Cash Price} - \text{Down Payment}$$

Total Installment Price

The **total installment price** is the total amount you will pay over the entire life of the loan, including both:

- the amount financed (principal), and
- the finance charge (the cost of borrowing money)

The formula for the total installment price is:

$$\text{Total Installment Price} = \text{Amount Financed} + \text{Finance Charge}$$

This value tells you the true cost of the loan.

Finance Charge

The **finance charge** is the cost of borrowing money. It is the difference between:

- what you actually borrowed, and
- what you end up paying back in total

The formulas for the finance charge assuming a simple interest, fixed installment loan are:

$$\text{Finance Charge} = I = P \cdot r \cdot t$$

$$\text{Finance Charge} = \text{Total Installment Price} - \text{Amount Financed}$$

Many people pay attention only to the monthly payment and overlook the total amount they will end up paying. The finance charge reveals the real cost of borrowing and helps consumers compare loans more effectively. The amount financed is the portion of the purchase you actually borrow. It is the number the lender uses to calculate interest. Borrowers often confuse this with the total cost of the loan, but the amount financed is only the size of the loan itself, not the full amount you will pay back. The total installment price, on the other hand, shows the true cost of the loan. It answers the question: “How much money will I spend in total by the time the loan is fully paid off?” Students are often surprised to see how much larger the total installment price is compared to the original amount borrowed. The difference between those two numbers is the finance charge. The finance charge represents the cost of borrowing money. It includes interest, administrative fees, lender risk, and other loan-related expenses. Essentially, it is the lender’s profit and the borrower’s cost for not paying the full price upfront. This is why it is so important to pay attention to interest rates. For example, many department stores, such as Kohl’s, offer their own store credit cards and advertise deals like “30% off today’s purchase when you use our card.” These promotions often hide extremely high interest rates. Over the past decade, the annual percentage rate (APR) on a Kohl's charge card has typically ranged from 29% to 34%, far higher than the interest rates we usually use in classroom examples. Notice the contrast: In our earlier annuity problems, interest rates ranged from 0.01% to around 9%. But in real life, when you borrow money, the interest rates are usually much higher, and when you invest, the interest rates are usually much lower. Understanding this difference is key to making smart financial decisions.

Charge Cards vs. Credit Cards

Store charge cards, also known as closed-loop retail cards, typically have much higher interest rates than general open-loop credit cards (i.e., Visa, Mastercard, Discover). Closed-loop means that the card can only be used at that retailer. Here are some current rates for some popular cards as of March 2026:

- Kohl’s Credit Card; APR: 30.74%
- Target Red Card; APR: around 25–30%
- Macy’s Store Card; APR: mid-to-high 20% range; Macy’s appears among top retail cards
- Best Buy Store Card; APR: high 20s
- Amazon Store Card; APR: 18.74%–27.49% (variable) for the Prime Visa (open-loop)

- Costco Anywhere Visa: APR: 18.99%–27.74% (variable) (open-loop)
- Ulta Beauty Rewards Card: APR: mid-high 20% range
- Kroger Rewards Mastercard: APR: varies between 25%–30%

Visa, Mastercard, and Discover do not set their own interest rates. The card issuers (banks like Chase, Citi, Discover, Capital One, etc.) set the APR, not the networks.

Credit card networks (Visa, Mastercard, Discover) do not issue most cards and do not control APRs, except Discover and American Express when acting as both network and issuer. However, we can report the typical APR ranges for cards issued on each network, based on aggregated data about major credit card issuers. These ranges represent what typical consumer credit cards carry on each network, based on card issuer data.

- Visa Credit Cards - Issued by banks such as Chase, Bank of America, Citi, Wells Fargo, etc. Based on nationwide averages, Visa cards usually fall in the range of 15% to 24% APR depending on credit quality.
- Mastercard Credit Cards - Issued by banks such as Citi, Capital One, Bank of America, etc. Based on nationwide averages, Mastercard cards fall in the range of 16% to 26% depending on credit quality. Capital One Mastercard often has higher low-end APRs, around 22.94%, one of the highest among issuers.
- Discover Credit Cards - Discover is both a network and the issuer, so its interest rates are directly published. Discover has the lowest average APR range among major issuers at 13.85% – 22.70% APR. This means Discover cards tend to offer lower rates, especially for customers with good credit.

Example #4.6.1 🤔

Juan has decided to purchase a 3-piece set bedroom set that includes a queen bed frame with storage and two nightstands that costs $1,048.98. He will finance through the retailer's financial services at a rate of 21.99%. He will put a down payment of $200 and finance for 6 months.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

✅ **Solution:**

a. Amount Financed:

$$\text{Amount Financed} = \text{Cash Price} - \text{Down Payment}$$
$$\text{Amount Financed} = (\$1{,}048.98) - (\$200)$$
$$\boxed{\text{Amount Financed} = \$848.98}$$

---

b. Finance Charge:

$$I = P \cdot r \cdot t$$

We know that $P = \$848.98;\ r = 0.2199;\ t = 0.5, (6 \text{ months} = 0.5 \text{ year})$, so

$$I = (\$848.98) \cdot (0.2199) \cdot (0.5)$$
$$I = \$93.345351$$
$$\boxed{\text{Finance Charge} = \$93.35}$$

---

c. Total Installment Price:

$$\text{Total Installment Price} = \text{Amount Financed} + \text{Finance Charge}$$
$$\text{Total Installment Price} = (\$848.98) + (\$93.35)$$
$$\boxed{\text{Total Installment Price} = \$942.33}$$

---

d. Monthly Payment:

$$\text{Monthly Payment} = \frac{\text{Total Installment Price}}{\text{Number of Months}}$$
$$\text{Monthly Payment} = \frac{(\$942.33)}{(6)}$$
$$\text{Monthly Payment} = 157.055$$
$$\boxed{\text{Monthly Payment} = \$157.06}$$

Example #4.6.2 🤔

Refer to Example 4.6.1. Juan noticed that the retailer is offering 0% financing (also referred to as "same as cash") for 6 months on all purchases of $500 or more. This means that no interest is charged during the first 6 months if the balance is paid in full before the promotional period ends. Assuming these new conditions and the same down payment of $200, answer the following questions.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

✅ **Solution:**

a. Amount Financed:

$$\text{Amount Financed} = \text{Cash Price} - \text{Down Payment}$$
$$\text{Amount Financed} = (\$1{,}048.98) - (\$200)$$
$$\boxed{\text{Amount Financed} = \$848.98}$$

---

b. Finance Charge:

$$I = P \cdot r \cdot t$$

We know that $P = \$848.98;\ r = 0;\ t = 0.5$, (6 months $= 0.5$ year), so

$$I = (\$848.98) \cdot (0) \cdot (0.5)$$
$$\boxed{\text{Finance Charge} = \$0}$$

---

c. Total Installment Price:

$$\text{Total Installment Price} = \text{Amount Financed} + \text{Finance Charge}$$
$$\text{Total Installment Price} = (\$848.98) + (\$0)$$
$$\boxed{\text{Total Installment Price} = \$848.98}$$

---

d. Monthly Payment:

$$\text{Monthly Payment} = \frac{\text{Total Installment Price}}{\text{Number of Months}}$$

$$\text{Monthly Payment} = \frac{(\$848.98)}{(6)}$$

$$\text{Monthly Payment} = 141.49666666667$$

$$\boxed{\text{Monthly Payment} = \$141.50}$$

Note: The monthly payment is $141.50 for the first five months and $141.48 for the final month. This occurs because the total installment price divided by the number of payments does not always produce a value that rounds exactly to the nearest cent. As a result, the final payment is sometimes adjusted slightly so that the total of all payments equals the exact amount owed.

Let's compare the results from the two previous examples:

- Example #4.6.1: APR = 21.99%; Finance Charge = $93.35; Monthly Payment = $157.06 for 6 months; Total Installment Price = $942.33
- Example #4.6.2: APR = 0%; Finance Charge = $0; Monthly Payment = $141.50 for 6 months; Total Installment Price = $849.00

These examples illustrate an important lesson in consumer finance: the monthly payment does not tell the whole story. Although the monthly payments in the two examples are relatively close, the total cost of borrowing is very different. With the 0% financing offer, no interest is charged, so the total installment price is the same as the cash price. In contrast, the loan in Example #4.6.1 results in an additional $93.35 in finance charges.

However, lower borrowing costs often come with less flexibility. A borrower who cannot afford the required monthly payment may need a longer repayment period. For example, extending the loan term to 12 months with the same APR of 21.99% reduces the monthly payment to $86.31, but increases the finance charge to $186.69. This demonstrates an important trade-off: longer loan terms generally produce lower monthly payments but higher overall borrowing costs.

Ultimately, the finance charge and total installment price reveal the true cost of a loan, while the monthly payment reflects its affordability.

0% Deferred Interest (Most Store Cards)

Many retail stores advertise “same as cash” 0% financing on purchases over a certain amount, but most of these deals use a system called deferred interest. Example #4.6.2 is based on this model, which can be risky if the borrower is not careful. Here’s how deferred interest works:

- You truly pay 0% interest only if you pay off the entire balance before the promotional period ends.
- If you fail to do so, even by just $1, the lender adds all the interest that would have been charged from day one.
- If you miss or skip a payment, the promotional terms are canceled immediately, and the lender charges all accumulated interest plus a late fee, usually between $25–$40.

In other words, the “deferred interest” is the interest that would have been applied from the beginning at the lender’s standard APR, often between 25%–30% or higher. Missing a payment or failing to pay the full balance by the deadline means this entire amount is suddenly added to your loan. This can leave you owing months of back interest all at once. Late or missed payments can also damage your credit score. A reported late payment often lowers a credit score by 50–150 points, which may cause higher interest rates on future loans or even lead to being denied new credit.

Example #4.6.3 🤔

Leila buys an engagement ring from Kay Jewelers for $3,200.00. She puts $500 down and finances the remainder through Kay’s financing program at 24.99% APR for 36 months.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

✅ **Solution:**

a. Amount Financed:

$$\text{Amount Financed} = \text{Cash Price} - \text{Down Payment}$$
$$\text{Amount Financed} = (\$3{,}200.00) - (\$500)$$
$$\boxed{\text{Amount Financed} = \$2{,}700.00}$$

---

b. Finance Charge:

$$I = P \cdot r \cdot t$$

We know that $P = \$2{,}700.00;\ r = 0.2499;\ t = 3$, $(36 \text{ months} = 3 \text{ year})$, so

$$I = (\$2{,}700.00) \cdot (0.2499) \cdot (3)$$
$$\boxed{\text{Finance Charge} = \$2{,}024.19}$$

---

c. Total Installment Price:

$$\text{Total Installment Price} = \text{Amount Financed} + \text{Finance Charge}$$
$$\text{Total Installment Price} = (\$2{,}700.00) + (\$2{,}024.19)$$
$$\boxed{\text{Total Installment Price} = \$4{,}724.19}$$

---

d. Monthly Payment:

$$\text{Monthly Payment} = \frac{\text{Total Installment Price}}{\text{Number of Months}}$$
$$\text{Monthly Payment} = \frac{(\$4{,}724.19)}{(36)}$$
$$\text{Monthly Payment} = 131.2275$$
$$\boxed{\text{Monthly Payment} = \$131.23}$$

Example #4.6.4 🤔

Renata buys a 3-seat leather power-reclining home theater set for $4,999.00. She uses the store's financing program at 25.74% APR, with no down payment, over 72 months.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

✅ **Solution:**

a. Amount Financed:

$$\text{Amount Financed} = \text{Cash Price} - \text{Down Payment}$$
$$\text{Amount Financed} = (\$4{,}999.00) - (\$0)$$
$$\boxed{\text{Amount Financed} = \$4{,}999.00}$$

---

b. Finance Charge:

$$I = P \cdot r \cdot t$$

We know that $P = \$4{,}999.00;\ r = 0.2574;\ t = 6$, (12 months = 1 year), so

$$I = (\$4{,}999.00) \cdot (0.2574) \cdot (6)$$
$$I = \$7{,}720.4556$$
$$\boxed{\text{Finance Charge} = \$7{,}720.46}$$

---

c. Total Installment Price:

$$\text{Total Installment Price} = \text{Amount Financed} + \text{Finance Charge}$$
$$\text{Total Installment Price} = (\$4{,}999.00) + (\$7{,}720.46)$$
$$\boxed{\text{Total Installment Price} = \$12{,}719.46}$$

---

d. Monthly Payment:

$$\text{Monthly Payment} = \frac{\text{Total Installment Price}}{\text{Number of Months}}$$
$$\text{Monthly Payment} = \frac{(\$12{,}719.46)}{(72)}$$
$$\text{Monthly Payment} = 176.65916666667$$
$$\boxed{\text{Monthly Payment} = \$176.66}$$

The last two examples illustrate one of the many important lessons in consumer finance: A longer loan with interest can cost much more overall than a shorter loan. It also shows why stretching payments over more months reduces the monthly payment, but increases the total cost.

Section 4.6: Installment Buying [*In-Class Exercises*]

For Exercises 1 & 2:

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

1. Amanda purchases a MacBook Air M2 at the Apple Store for $1,099.00. She uses the Apple Card Monthly Installments program at 21.24% APR (typical for lower-credit tiers). She puts $100 down and finances for 12 months.

2. Chris buys a premium 6-piece solid-wood bedroom set from a local furniture store for $6,899.99. He puts $900 down, and the store finances the balance at 24.99% APR for 60 months.

Answers available at the end of this section.

Answers

Section 4.6: Installment Buying [*In-Class Exercises*]

1. a. Amount Financed = $999.00
   b. Finance Charge = $212.19
   c. Total Installment Price = $1211.19
   d. Monthly Payment = $100.93
2. a. Amount Financed = $5999.99
   b. Finance Charge = $7,496.99
   c. Total Installment Price = $13,496.98
   d. Monthly Payment = $224.95

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Section 4.6.H: Homework

Section 4.6: Installment Buying - Homework Exercises

For Homework Exercises #1-#10 do not apply sales tax to any purchase.

(Section 4.6) - Homework Exercise #1

J.P. purchases a Samsung 75″ Crystal UHD TV from Best Buy for $899.99. He takes advantage of Best Buy's store credit program, which carries an APR of 29.74% for regular financing. He puts no money down and finances for 18 months.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #2

Janet buys school uniforms and shoes at Macy's totaling $520.00. She uses her Macy's Store Card, which carries an APR of 28.49%. She plans to pay off the balance using a structured 6-month fixed-installment plan.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #3

Thuy buys a Sealy Posturepedic queen mattress for $1,399.00 at Mattress Firm. She puts $250 down, and financing is offered at 22.99% APR for 24 months.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #4

Carol needs a new refrigerator urgently. He buys a Whirlpool 25 cu. ft. stainless refrigerator for $1,499.99 at Lowe's. Using his Lowe's Advantage Card, he finances at 26.99% APR for 24 months, with no down payment.

a. Find the amount financed.

b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #5

Tara buys a new Mac Studio, a Studio Display, an Ergonomic Chair, and a Standing Desk, totaling $6,300 from a tech/home-office retailer. She finances at 21.99% APR for 48 months, with $300 down.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #6

Valerie buys a complete living-room package (sectional sofa, coffee table, rug, console, lamps) costing $4,450.00. She finances through the store's lender at 22.99% APR, with $450 down, over 60 months.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #7

Bob purchases a full Samsung kitchen appliance package (fridge, oven, microwave, dishwasher) for $5,299.00 at Lowe's. Using their store-financing program at 26.99% APR, he finances for 48 months with no down payment.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #8

Heather wants to buy a BELLEZE Kinsley 5-piece dining set priced at $1,279.99. She finances through the store's partner bank at an APR of 23.99%, putting $150 down, and chooses a 12-month repayment plan.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #9

Roger finances a full backyard renovation through Home Depot's project-loan program. He borrows $15,000 at 9.99% APR for 84 months.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

(Section 4.6) - Homework Exercise #10

Tyler buys a $9,800 diamond engagement ring from Jared. He puts $1,200 down, and finances the remainder at 23.99% APR for 60 months.

a. Find the amount financed.
b. Find the finance charge.
c. Find the total installment price.
d. Find the monthly payment.

Answer available at the end of this section.

Answers

(Section 4.6) - Homework Exercise #1

a. Amount Financed = $899.99
b. Finance Charge = $401.49
c. Total Installment Price = $1,301.48
d. Monthly Payment = $72.30

(Section 4.6) - Homework Exercise #2

a. Amount Financed = $520.00
b. Finance Charge = $74.07
c. Total Installment Price = $594.07
d. Monthly Payment = $99.01

(Section 4.6) - Homework Exercise #3

a. Amount Financed = $1,149.00
b. Finance Charge = $528.31
c. Total Installment Price = $1,677.31
d. Monthly Payment = $69.89

(Section 4.6) - Homework Exercise #4

a. Amount Financed = $1,499.99
b. Finance Charge = $809.69
c. Total Installment Price = $2,309.68
d. Monthly Payment = $96.24

(Section 4.6) - Homework Exercise #5

a. Amount Financed = $6,000
b. Finance Charge = $5,277.60

c. Total Installment Price = $11,277.60
d. Monthly Payment = $234.95

(Section 4.6) - Homework Exercise #6

a. Amount Financed = $4,000.00
b. Finance Charge = $4598.00
c. Total Installment Price = $8,598.00
d. Monthly Payment = $143.30

(Section 4.6) - Homework Exercise #7

a. Amount Financed = $5,299.00
b. Finance Charge = $5,720.80
c. Total Installment Price = $11,019.80
d. Monthly Payment = $229.58

(Section 4.6) - Homework Exercise #7

(Section 4.6) - Homework Exercise #8

a. Amount Financed = $1,129.99
b. Finance Charge = $271.08
c. Total Installment Price = $1,401.07
d. Monthly Payment = $116.76

(Section 4.6) - Homework Exercise #9

a. Amount Financed = $15,000.00
b. Finance Charge = $10,489.50
c. Total Installment Price = $25,489.50
d. Monthly Payment = $303.45

(Section 4.6) - Homework Exercise #10

a. Amount Financed = $8,600.00
b. Finance Charge = $10,315.70
c. Total Installment Price = $18,915.70
d. Monthly Payment = $315.26

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Section 4.7: Average Daily Balance

Learning Objectives

- Calculate the average daily balance

The **Average Daily Balance (ADB)** method is the most widely used way credit card companies calculate interest on revolving balances. Under this method, interest is based on your balance each day of the billing cycle, not just the amount owed on the statement date. Every day, the lender records your current balance, and at the end of the cycle, these daily balances are added together and divided by the number of days to find the average. What many people don't realize is that interest is calculated daily, regardless of when the payment due date falls. This means that making a payment earlier in the billing cycle immediately reduces the daily balances used in the calculation, lowering your interest. Paying later, even if it's on or before the due date, keeps your daily balance higher for more days, increasing the interest charged. Because purchases and payments affect the balance on different days, the ADB method highlights why when you pay can be just as important as how much you pay when trying to reduce interest costs.

How to Calculate the Average Daily Balance

The formula for **Average Daily Balance** or **ADB**:

$$\text{ADB} = \frac{\text{Sum of all daily balances in the billing cycle}}{\text{Number of days in the cycle}}$$

To calculate the **Average Daily Balance** or **ADB**:

- **Step 1**: Find the balance after each transaction.
- **Step 2**: Find the number of blocks of days for each balance and verify the sum of days in the cycle.
- **Step 3**: Multiply those blocked days by each respective balance.
- **Step 4**: Find the sum of all those products.
- **Step 5**: Divide the sum by the number of days in the cycle or month.

How to Calculate the Finance Charge and New Balance

To calculate the **Finance Charge** (or **FC** for short) simply use the simple interest formula:

$$I = P \cdot r \cdot t$$

and replace P with the average daily balance, ADB

$$I = \text{ADB} \cdot r \cdot t \tag{4.7.1}$$

To find the **New Balance** (or **NB** for short) for the next cycle or month by adding the finance charge to the ending balance:

$$\text{New Balance} = \text{Ending Balance} + \text{Finance Charge}$$

Let's look at a very straightforward example of how the Average Daily Balance (ADB) method works. Suppose your billing cycle runs from April 1st through April 30th. Here is your credit card statement

A Sample Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/-	Amount	Balance
Apr. 1	Apr. 1	Previous Balance from March	**(+)**	$1,000.00	$1,000.00
Apr. 14	Apr. 16	Payment Made (Thank you)	**(–)**	($500.00)	$500.00
Apr. 30	Apr. 30	Ending Balance for April			**$500.00**

Let's understand the first two columns in the statement. The **transaction date** is the day you actually make a purchase, withdrawal, or payment. The **post date** (also called the "settlement date") is the day your financial institution finishes processing the transaction and officially applies it to your account. For purchases (not shown above), most institutions use what is called a same-day impact or same-day posting effect. In the example above, the payment made on April 14th was not posted to the account until two days later on April 16th. The result is that the payment does not reduce your daily balance until the day it is officially posted. Because interest under the (ADB) method is calculated using each day's balance, this timing difference benefits the financial institution. Purchases raise your balance immediately. Payments reduce your balance only when they post. This gap results in higher daily balances, which increases the ADB, and therefore increases the interest you are charged for the billing cycle.

Now, at the start of the cycle, your balance is exactly $1,000. You make a $500 payment on April 14th, and the credit card company posts (credits) it two days later on April 16. This means:

- For the first 15 days of the cycle (April 1–15), your balance was $1,000. Think day 1 = $1,000, day 2 = $1,000, … , day 15 = $1,000 (15 days at $1,000 each day)
- For the last 15 days (April 16–30), your balance was $500. Think day 16 = $500, day 17 = $500, … , day 30 = $500 (15 days at $500 each day)

To calculate the ADB, the lender uses the formula(s) above and takes the sum of all daily balances and divides by the number of days in the cycle:

$$\text{ADB} = \frac{(15 \times \$1{,}000) + (15 \times \$500)}{30}$$

$$\text{ADB} = \frac{(\$15{,}000) + (\$7{,}500)}{30}$$

$$\text{ADB} = \frac{\$22{,}500}{30}$$

$$\boxed{\text{ADB} = \$750}$$

So your Average Daily Balance is $750, even though your balance at the end of the cycle is only $500. Most people assume interest should be based on the amount they currently owe (in this case, $500). But, with the ADB method, interest is calculated on the average of all daily balances throughout the cycle, not just the ending balance. Because the balance was higher earlier in the cycle, the interest ends up higher as well. Now imagine you waited until April 30th, the last day of the cycle, to make your $500 payment. Since your balance would have been $1,000 for all 30 days, your Average Daily Balance would be:

$$\text{ADB} = \frac{(30 \times \$1{,}000)}{30}$$

$$\text{ADB} = \frac{(\$30{,}000)}{30}$$

$$\boxed{\text{ADB} = \$1{,}000}$$

In other words, paying on the last day does nothing to reduce the ADB, and your interest charge will be based on the full $1,000. Even though the payment due date might be April 30th, the interest calculation doesn't care about the due date at all. Interest is calculated daily, so:

- Paying earlier reduces your ADB and lowers your interest.
- Paying later, even on time, keeps your balance higher for more days and increases interest.
- Two people who both pay $500 by the due date might be charged very different interest amounts depending on when the payment was made.

This is why the ADB method encourages making payments as early as possible. The longer a balance remains unpaid, the more days it contributes to the average daily balance, resulting in higher interest charges. By understanding how interest is calculated, credit card holders can make informed decisions about when to make payments and better estimate the amount of interest they may owe.

Example #4.7.1 🤔

A credit card statement from February 1st to February 28th is given below. (To keep the calculations simplified, let's assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Feb. 1	Feb. 1	Previous Balance from January	**(+)**	$1,000.00	$1,000.00
Feb. 9	Feb. 9	Target	**(+)**	$75.00	$1,075.00
Feb. 14	Feb. 14	Maggiano's Italian Restaurant	**(+)**	$125.00	$1,200.00
Feb. 24	Feb. 26	Payment Made (Thank you)	**(–)**	($200.00)	$1,000.00
Feb. 28	**Feb. 28**	**Ending Balance for February**			**$1,000.00**
		Finance Charge at APR 19.9% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of March 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 19.9%.
c. Find the new balance on March 1st.

✅ **Solution:**

a. Average daily balance (ADB)

It may be easier to use a table for the first four steps. Start off with a column listing the post dates and always ignore the transaction dates.

- **Step 1**: Find the balance after each transaction.
- **Step 2**: Find the number of blocks of days for each balance and verify the sum of days in the cycle.
- **Step 3**: Multiply those blocked days by each respective balance.
- **Step 4**: Find the sum of all those products.

Calculations for the sum of the balances in the month of February

Post Date	Balance (Step 1)	Days (Step 2)	Calculations (Steps 3 & 4)
Feb. 1	$1,000.00	(9 – 1) = 8	($1,000.00)(8) = $8,000.00
Feb. 9	$1,075.00	(14 – 9) = 5	($1,075.00)(5) = $5,375.00
Feb. 14	$1,200.00	(26 – 14) = 12	($1,200.00)(12) = $14,400.00
Feb. 26	$1,000.00	(28 – 26) **+ 1*** = 3	($1,000.00)(3) = $3,000.00
		28 ✔	**Sum = $30,775.00**

***** *"Note that the difference 28 – 26 = 2 represents the number of days from Feb. 26 to Feb. 28, but not including Feb. 28. Since the balance of $1,000 carries over into Feb. 28, we must add one to the number of days in the last 'block' to obtain the correct count."*

- **Step 5**: Divide the sum by the number of days in the cycle or month.

$$\text{ADB} = \frac{\$30{,}775.00}{28}$$

$$\text{ADB} = \$1{,}099.1071428571$$

$$\boxed{\text{ADB} = \$1{,}099.11}$$

To perhaps clear up how the days work, here are the blocks of days for each respective balance:

- Block 1 @ $1,000: February 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th (8 total days)
- Block 2 @ $1,075: February 9th, 10th, 11th, 12th, 13th (5 total days)
- Block 3 @ $1,200: February 14th, 15th, 16th, 17th, 18th, 19th, 20th, 21st, 22nd, 23rd, 24th, 25th (12 total days)
- Block 4 @ $1,000: February 26th, 27th, 28th (3 total days)

The number of days match; 8 + 5 + 12 + 3 = 28 days and February has 28 days.

Here is what the ADB looks like using an equation format:

$$\text{ADB} = \frac{\text{Sum of all daily balances in the billing cycle}}{\text{Number of days in the cycle}}$$

$$\text{ADB} = \frac{(8 \times \$1{,}000.00) + (5 \times \$1{,}075.00) + (12 \times \$1{,}200.00) + (3 \times \$1{,}000.00)}{28}$$

$$\text{ADB} = \frac{(\$8{,}000.00) + (\$5{,}375.00) + (\$14{,}400.00) + (\$3{,}000.00)}{28}$$

$$\text{ADB} = \frac{\$30{,}775.00}{28}$$

$$\text{ADB} = \$1{,}099.1071428571$$

$$\boxed{\text{ADB} = \$1{,}099.11}$$

---

b. Finance Charge

$$I = \text{ADB} \cdot r \cdot t$$

We know that $\text{ADB} = \$1{,}099.11;\ r = 0.199;\ t = \frac{1}{12}$, so

$$I = (\$1{,}099.11) \cdot (0.199) \cdot \left(\frac{1}{12}\right)$$

$$I = \$18.2269075$$

$$\boxed{\text{Finance Charge} = \$18.23}$$

---

c. New Balance

$$\text{New Balance} = \text{Ending Balance} + \text{Finance Charge}$$
$$\text{New Balance} = (\$1{,}000.00) + (\$18.23)$$

$$\boxed{\text{New Balance} = \$1{,}018.23}$$

So, now the statement would have the missing information filled in for the ADB, finance charge, and the new balance.

Complete Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Feb. 1	Feb. 1	Previous Balance from January	**(+)**	$1,000.00	$1,000.00
Feb. 9	Feb. 9	Target	**(+)**	$75.00	$1,075.00
Feb. 14	Feb. 14	Maggiano's Italian Restaurant	**(+)**	$125.00	$1,200.00
Feb. 24	Feb. 26	Payment Made (Thank you)	**(–)**	($200.00)	$1,000.00
Feb. 28	Feb. 28	Ending Balance for February			**$1,000.00**
		Finance Charge at APR 19.9% (based on ADB =$1,099.11)	**(+)**	$18.23	$1,018.23
		New Balance as of March 1st			

Example #4.7.2 🤔

Elliot's VISA credit card statement along with the transactions from May 1st to May 31st is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Elliot's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
May 1	May 1	Previous Balance from April	**(+)**	$4,352.79	$4,352.79
May 3	May 3	Walmart	**(+)**	$62.34	$4,415.13
May 6	May 6	Amazon Shopping	**(+)**	$19.99	$4,435.12
May 8	May 10	VANS Shoes - RETURN/CREDIT	**(–)**	($53.86)	$4,381.26
May 20	May 20	Del Taco	**(+)**	$14.13	$4,395.39
May 27	May 30	Payment Made (Thank you)	**(–)**	($4000.00)	$395.39
May 31	**May 31**	**Ending Balance for May**			**$395.39**
		Finance Charge at APR 22.99% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of June 1st			NB = ????

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 22.99%.
c. Find the new balance on June 1st.

✅ **Solution:**

a. Average daily balance (ADB)

Use a table for the first four steps. Start off with a column listing the post dates and always ignore the transaction dates.

- **Step 1**: Find the balance after each transaction.
- **Step 2**: Find the number of blocks of days for each balance and verify the sum of days in the cycle.
- **Step 3**: Multiply those blocked days by each respective balance.
- **Step 4**: Find the sum of all those products.

Calculations for the sum of the balances in the month of May

Post Date	Balance (Step 1)	Days (Step 2)	Calculations (Steps 3 & 4)
May 1	$4,352.79	(3 – 1) = 2	($4,352.79)(2) = $8,705.58
May 3	$4,415.13	(6 – 3) = 3	($4,415.13)(3) = $13,245.39
May 6	$4,435.12	(10 – 6) = 4	($4,435.12)(4) = $17,740.48
May 10	$4,381.26	(20 – 10) = 10	($4,381.26)(10) = $43,812.60
May 20	$4,395.39	(30 – 20) = 10	($4,395.39)(10) = $43,953.90
May 30	$395.39	(31 – 30) **+ 1*** = 2	($395.39)(2) = $790.78
		31 ✔	**Sum = $128,248.73**

***** *"Note that the difference 31 – 30 = 1 represents the number of days from May 30 to May 31, but not including May 31. Since the balance of $790.78 carries over into May 31, we must add one to the number of days in the last 'block' to obtain the correct count."*

- **Step 5**: Divide the sum by the number of days in the cycle or month.

$$\text{ADB} = \frac{\$128{,}248.73}{31}$$

$$\text{ADB} = \$4{,}137.055804516$$

$$\boxed{\text{ADB} = \$4{,}137.06}$$

To perhaps clear up how the days work, here are the blocks of days for each respective balance:

- Block 1 @ $4,352.79: May 1st, 2nd (2 total days)
- Block 2 @ $4,415.13: May 3rd, 4th, 5th (3 total days)
- Block 3 @ $4,435.12: May 6th, 7th, 8th, 9th, (4 total days)
- Block 4 @ $4,381.26: May 10th, 11th, 12th, 13th, 14th, 15th, 16th, 17th, 18th, 19th (10 total days)
- Block 5 @ $4,395.39: May 20th, 21st, 22nd, 23rd, 24th, 25th, 26th, 27th, 28th, 29th (10 total days)
- Block 6 @ $395.39: May 30th, 31st (2 total days)

The number of days match; 2 + 3 + 4 + 10 + 10 + 2 = 31 days and May has 31 days.

Here is what the ADB looks like using an equation format:

$$\text{ADB} = \frac{\text{Sum of all daily balances in the billing cycle}}{\text{Number of days in the cycle}}$$

$$\text{ADB} = \frac{(2 \times \$4{,}352.79) + (3 \times \$4{,}415.13) + (4 \times \$4{,}435.12) + (10 \times \$4{,}381.26) + (10 \times \$4{,}395.39) + (2 \times \$395.39)}{31}$$

$$\text{ADB} = \frac{(\$8{,}705.58)+(\$13{,}245.39)+(\$17{,}740.48)+(\$43{,}812.60)+(\$43{,}953.90)+(\$790.78)}{31}$$

$$\text{ADB} = \frac{\$128{,}248.73}{31}$$

$$\text{ADB} = \$4{,}137.055804516$$

$$\boxed{\text{ADB} = \$4{,}137.06}$$

---

b. Finance Charge

$$I = \text{ADB} \cdot r \cdot t$$

We know that $\text{ADB} = \$4{,}137.06;\ r = 0.2299;\ t = \frac{1}{12}$, so

$$I = (\$4{,}137.06) \cdot (0.2299) \cdot \left(\frac{1}{12}\right)$$

$$I = \$79.2591745$$

$$\boxed{\text{Finance Charge} = \$79.26}$$

---

c. New Balance

$$\text{New Balance} = \text{Ending Balance} + \text{Finance Charge}$$
$$\text{New Balance} = (\$395.39) + (\$79.26)$$

$$\boxed{\text{New Balance} = \$474.65}$$

Example #4.7.3 🤔

Kadynce's Discover credit card statement from November 1st to November 30th is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Kadynce's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Nov. 1	Nov. 1	Previous Balance from October	**(+)**	$2,618.02	$2,618.02
Nov. 2	Nov. 4	Payment Made (Thank you)	**(–)**	($2,000.00)	$618.02
Nov. 25	Nov. 25	Hot Topic	**(+)**	$23.54	$641.56
Nov. 26	Nov. 26	NETFLIX	**(+)**	$26.93	$668.49
Nov. 27	Nov. 27	Regal Cinema	**(+)**	$34.00	$702.49
Nov. 30	**Nov. 30**	**Ending Balance for November**			**$702.49**
		Finance Charge at APR 17.99% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of December 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 17.99%.
c. Find the new balance on December 1st.

✅ **Solution:**

a. Average daily balance (ADB)

Use a table for the first four steps. Start off with a column listing the post dates and always ignore the transaction dates.

- **Step 1**: Find the balance after each transaction.
- **Step 2**: Find the number of blocks of days for each balance and verify the sum of days in the cycle.
- **Step 3**: Multiply those blocked days by each respective balance.
- **Step 4**: Find the sum of all those products.

Calculations for the sum of the balances in the month of November

Post Date	Balance (Step 1)	Days (Step 2)	Calculations (Steps 3 & 4)
Nov. 1	$2,618.02	(4 – 1) = 3	($2,618.02)(3) = $7,854.06
Nov. 4	$618.02	(25 – 4) = 21	($618.02)(21) = $12,978.42
Nov. 25	$641.56	(26 – 25) = 1	($641.56)(1) = $641.56
Nov. 26	$668.49	(27 – 26) = 1	($668.49)(1) = $668.49
Nov. 27	$702.49	(30 – 27) **+ 1*** = 4	($702.49)(4) = $2,809.96
		30 ✔	**Sum = $24,952.49**

***** *"Note that the difference 30 – 27 = 3 represents the number of days from Nov. 27 to Nov. 30, but not including Nov. 30. Since the balance of $702.49 carries over into Nov. 30, we must add one to the number of days in the last 'block' to obtain the correct count."*

- **Step 5**: Divide the sum by the number of days in the cycle or month.

$$\text{ADB} = \frac{\$24{,}952.49}{30}$$

$$\text{ADB} = \$831.74966666667$$

$$\boxed{\text{ADB} = \$831.75}$$

To perhaps clear up how the days work, here are the blocks of days for each respective balance:

- Block 1 @ $2,618.02: Nov. 1st, 2nd, 3rd (3 total days)
- Block 2 @ $618.02: Nov. 4th, 5th ,6th, 7th, 8th, 9th, 10th, 11th, 12th, 13th, 14th, 15th, 16th, 17th, 18th, 19th, 20th, 21th, 22th, 23th, 24th, (21 total days)
- Block 3 @ $641.56: Nov. 25th (1 total day)
- Block 4 @ $668.49: Nov. 26th (1 total day)
- Block 5 @ $702.49: Nov. 27th, 28th, 29th, 30th (4 total days)

The number of days match; 3 + 21 + 1 + 1 + 4 = 30 days and November has 30 days.

Here is what the ADB looks like using an equation format:

$$\text{ADB} = \frac{\text{Sum of all daily balances in the billing cycle}}{\text{Number of days in the cycle}}$$

$$\text{ADB} = \frac{(3 \times \$2{,}618.02) + (21 \times \$618.02) + (1 \times \$641.56) + (1 \times \$668.49) + (4 \times \$702.49)}{30}$$

$$\text{ADB} = \frac{(\$7{,}854.06) + (\$12{,}978.42) + (\$641.56) + (\$668.49) + (\$2{,}809.96)}{30}$$

$$\text{ADB} = \frac{\$24{,}952.49}{30}$$

$$\text{ADB} = \$831.74966666667$$

$$\boxed{\text{ADB} = \$831.75}$$

---

b. Finance Charge

$$I = \text{ADB} \cdot r \cdot t$$

We know that $\text{ADB} = \$831.75;\ r = 0.1799;\ t = \frac{1}{12}$, so

$$I = (\$831.75) \cdot (0.1799) \cdot \left(\frac{1}{12}\right)$$

$$I = \$12.46931875$$

$$\boxed{\text{Finance Charge} = \$12.47}$$

---

c. New Balance

$$\text{New Balance} = \text{Ending Balance} + \text{Finance Charge}$$
$$\text{New Balance} = (\$702.49) + (\$12.47)$$

$$\boxed{\text{New Balance} = \$714.96}$$

In comparing these two credit-card statements for Example #4.7.2 and Example #4.7.3, the impact of payment timing becomes very clear. Elliot's $4,000 payment was posted at the end of the billing cycle, so his balance remained above $4,300 for almost the entire month. As a result, his average daily balance (ADB) stayed extremely high at $4,137.06, leading to a finance charge of $79.26.

By contrast, Kadynce's $2,000 payment was posted very early in the month, dramatically reducing her balance for nearly the full billing cycle. Her ADB fell to $831.75, and her finance charge was only $12.47. Even though her ending balance was still $702.49 and Elliot's was much lower at $395.39, the earlier payment in the cycle saved her more than $60 in finance charges compared to Elliot.

This side-by-side comparison demonstrates why making larger payments earlier in the cycle is so powerful. It lowers your balance for more days of the month, reduces the ADB, and significantly cuts the finance charge. Over time, these savings add up and make a major difference in how quickly a debt is paid down.

Section 4.7: Average Daily Balance [*In-Class Exercises*]

1. Jorge's credit card statement from June 1st to June 30th is given below. (To keep the calculations simplified, let's assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Jorge's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Jun. 1	Jun. 1	Previous Balance from May		$484.53	$484.53
Jun. 8	Jun. 8	Albertsons	**(+)**	$52.51	$537.04
Jun. 13	Jun. 13	Kroger	**(+)**	$63.79	$600.83
Jun. 19	Jun. 21	Payment Made (Thank you)	**(–)**	($300.00)	$300.83
Jun. 24	Jun. 24	Vons	**(+)**	$74.04	$374.87
Jun. 30	**Jun. 30**	**Ending Balance for June**			**$374.87**
		Finance Charge at APR 24.00% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of July 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 24.00%.
c. Find the new balance on July 1st.

2. MacKenzie's MasterCard credit card statement from January 1st to January 31st is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

MacKenzie's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Jan. 1	Jan. 1	Previous Balance from December	**(+)**	$784.51	$784.51
Jan. 3	Jan. 3	Regal Cinema	**(+)**	$68.79	$853.30
Jan. 11	Jan. 11	Hot Topic	**(+)**	$218.11	$1,071.41
Jan. 26	Jan. 28	Payment Made (Thank you)	**(–)**	($800.00)	$271.41
Jan. 31	**Jan. 31**	**Ending Balance for November**			**$271.41**
		Finance Charge at APR 18.00% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of Feb. 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 18.00%.
c. Find the new balance on February 1st.

3. Let's take a look at Exercise #2 and see what happens if the payment of $800 was made much earlier in the payment cycle. So, instead of the $800 payment being made on January 28th, we will make it on January 5th, and see how much of an impact that makes, keeping everything else the same with regards to purchases. (Again, assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

MacKenzie's Credit Card Statement with Earlier Payment

Transaction Date	Post Date	Merchant ID / Description	+/-	Amount	Balance
Jan. 1	Jan. 1	Previous Balance from December	**(+)**	$784.51	$784.51
Jan. 3	Jan. 3	Regal Cinema	**(+)**	$68.79	$853.30
Jan. 3	Jan. 5	Payment Made (Thank you)	**(–)**	($800.00)	$53.30
Jan. 11	Jan. 11	Hot Topic	**(+)**	$218.11	$271.41
Jan. 31	**Jan. 31**	**Ending Balance for November**			**$271.41**
		Finance Charge at APR 18.00% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of Februray 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 18.00%.
c. Find the new balance on February 1st.
d. Compare the results for both Exercise #2 and Exercise #3.

Answers available at the end of this section.

Answers

Section 4.7: Average Daily Balance [*In-Class Exercises*]

1. a. ADB = $480.34
   b. FC = $9.61
   c. NB = $384.48
2. a. DB = $893.39
   b. FC = $13.40
   c. NB = $284.81
3. a. ADB = $299.89
   b. FC = $4.50
   c. NB = $275.91
   d. When the large payment is made earlier in the billing cycle, the Average Daily Balance (ADB) drops dramatically. In this case, from $893.39 down to $299.89. Notice that the ending balance is identical in both scenarios ($271.41), yet the timing of the payment makes a major difference. Because the ADB is so much lower, the finance charge also falls sharply from $13.40 to $4.50, a savings of $8.90 for the month. This happens entirely because the payment was applied 23 days earlier, reducing the balance for most of the billing cycle. This comparison shows why making larger payments earlier in the cycle is so beneficial. The earlier the payment is posted, the longer your balance stays lower and the more your finance charges shrink. Over time, especially with large balances, this strategy makes a much bigger impact and helps you pay off debt faster and cheaper.

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Section 4.7.H: Homework

Section 4.7: Average Daily Balance - Homework Exercises

(Section 4.7) - Homework Exercise #1

Stanislav's credit card statement from March 1st to March 31st is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Stanislav's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Mar. 1	Mar. 1	Previous Balance from February	**(+)**	$1,284.20	$1,284.20
Mar. 3	Mar. 5	Payment Made (Thank you)	**(–)**	($500.00)	$784.20
Mar. 10	Mar. 10	Adam's Electronics	**(+)**	$42.18	$826.38
Mar. 18	Mar. 18	Shell Gas	**(+)**	$63.55	$889.93
Mar. 25	Mar. 25	Trader Joe's	**(+)**	$89.14	$979.07
Mar. 31	**Mar. 31**	**Ending Balance for June**			**$979.07**
		Finance Charge at APR 19.99% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of April 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 19.99%.
c. Find the new balance on April 1st.

Answer available at the end of this section.

(Section 4.7) - Homework Exercise #2

Kevin's credit card statement from May 1st to May 31st is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Kevin's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/-	Amount	Balance
May 1	May 1	Previous Balance from May	**(+)**	$2,320.00	$2,320.00
May 4	May 4	Wahoo's Fish Tacos	**(+)**	$76.88	$2,396.88
May 8	May 9	Payment Made (Thank you)	**(–)**	($600.00)	$1,796.88
May 15	May 15	Surfside Sports	**(+)**	$119.75	$1,916.63
May 21	May 23	Payment Made (Thank you)	**(–)**	($300.00)	$1,616.63
May 30	May 30	TK Burger	**(+)**	$14.29	$1,630.92
May 31	**May 31**	**Ending Balance for June**			**$1,630.92**
		Finance Charge at APR 22.49% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of July 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 22.49%.
c. Find the new balance on July 1st.

Answer available at the end of this section.

(Section 4.7) - Homework Exercise #3

Dale's credit card statement from June 1st to June 30th is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Dale's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Jun. 1	Jun. 1	Previous Balance from May	**(+)**	$657.21	$657.21
Jun. 2	Jun. 2	Costco Warehouse	**(+)**	$252.51	$909.72
Jun. 4	Jun. 4	Sam's Club	**(+)**	$154.13	$1,063.85
Jun. 8	Jun. 8	Smart & Final	**(+)**	$98.76	$1,162.61
Jun. 13	Jun. 15	Payment Made (Thank you)	**(–)**	($100.00)	$1,062.61
Jun. 23	Jun. 23	WinCo Foods	**(+)**	$130.01	$1,192.62
Jun. 30	**Jun. 30**	**Ending Balance for June**			**$1,192.62**
		Finance Charge at APR 23.99% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of July 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 23.99%.
c. Find the new balance on July 1st.

Answer available at the end of this section.

(Section 4.7) - Homework Exercise #4

Jamie's credit card statement from February 1st to February 28th is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Jamie's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Feb. 1	Feb. 1	Previous Balance from January	**(+)**	$1,102.40	$1,102.40
Feb. 7	Feb. 8	Apple iCloud Storage	**(+)**	$9.99	$1,112.39
Feb. 22	Feb. 24	Payment Made (Thank you)	**(–)**	($300.00)	$812.39
Feb. 25	Feb. 25	Pearson Publishing	**(+)**	$205.84	$1,018.23
Feb. 28	**Feb. 28**	**Ending Balance for February**			**$1,018.23**
		Finance Charge at APR 18.24% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of March 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 18.24%.

c. Find the new balance on March 1st.

Answer available at the end of this section.

(Section 4.7) - Homework Exercise #5

Riki's credit card statement from October 1st to October 31st is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Tim's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/–	Amount	Balance
Oct. 1	Oct. 1	Previous Balance from September	**(+)**	$484.53	$484.53
Oct. 2	Oct. 2	YouTubeTV - Sunday Ticket (October)	**(+)**	$125.00	$609.53
Oct. 3	Oct. 5	Payment Made (Thank you)	**(–)**	($250.00)	$359.53
Oct. 17	Oct. 17	Haunted Harvest	**(+)**	$45.00	$404.53
Oct. 22	Oct. 22	Disneyland Resort - Oogie Boogie Bash	**(+)**	$199.00	$603.53
Oct. 24	Oct. 24	Universal Studios Hollywood - Halloween Horror Nights	**(+)**	$77.00	$680.53
Oct. 30	Oct. 30	Six Flags - Knott's Halloween Haunt	**(+)**	$64.00	$744.53
Oct. 31	**Oct. 31**	**Ending Balance for October**			**$744.53**
		Finance Charge at APR 21.49% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of November 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 21.49%.
c. Find the new balance on November 1st.

Answer available at the end of this section.

(Section 4.7) - Homework Exercise #6

Eduardo's credit card statement from August 1st to August 31st is given below. (Assume that the payment cycle starts at the beginning of the month and ends on the last day of the month).

Eduardo's Credit Card Statement

Transaction Date	Post Date	Merchant ID / Description	+/-	Amount	Balance
Aug. 1	Aug. 1	Previous Balance from July	**(+)**	$21,287.09	$21,287.09
Aug. 5	Aug. 5	Darya Fine Persian Cuisine	**(+)**	$123.79	$21,410.88
Aug. 7	Aug. 7	Neiman Marcus	**(+)**	$643.24	$22,054.12
Aug. 10	Aug. 10	Segerstrom Center for the Arts	**(+)**	$455.37	$22,509.49
Aug. 16	Aug. 16	Williams-Sonoma	**(+)**	$184.88	$22,694.37
Aug. 21	Aug. 21	AnQi Bistro	**(+)**	$111.56	$22,805.93
Aug. 23	Aug. 23	Saks Fifth Avenue	**(+)**	$1,321.65	$24,127.58
Aug. 24	Aug. 24	Gucci	**(+)**	$2,175.33	$26,302.91
Aug. 25	Aug. 27	Payment Made (Thank you)	**(–)**	($10,000.00)	$16,302.91
Aug. 31	**Aug. 31**	**Ending Balance for August**			**$16,302.91**
		Finance Charge at APR 15.74% (based on ADB =????)	**(+)**	FC = ????	NB = ????
		New Balance as of September 1st			

a. Find the average daily balance.
b. Find the finance charge for the month. The APR is 15.74%.
c. Find the new balance on September 1st.

Answer available at the end of this section.

Answers

(Section 4.7) - Homework Exercise #1
a) ADB = $927.48; **b)** FC = $15.45; **c)** NB = $994.52.

(Section 4.7) - Homework Exercise #2
a) ADB = $1,923.77; **b)** FC = $36.05; **c)** NB = $1,666.97.

(Section 4.7) - Homework Exercise #3
a) ADB = $1,097.07; **b)** FC = $21.93; **c)** NB = $1,214.55.

(Section 4.7) - Homework Exercise #4
a) ADB = $1,085.73; **b)** FC = $16.50; **c)** NB = $1,034.73.

(Section 4.7) - Homework Exercise #5
a) ADB = $497.72; **b)** FC = $8.91; **c)** NB = $753.44.

(Section 4.7) - Homework Exercise #6
a) ADB = $21,704.00; **b)** FC = $284.68; **c)** NB = $16,587.59.

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Section 4.8: Student Loans, Home Buying, and Amortization

Learning Objectives

- Compute daily, monthly, and entire interest on a student loan
- Understand subsidized and unsubsidized loans
- Understand the difference between capitalizing and not capitalizing interest
- Compute monthly payments on a student loan
- Find a monthly mortgage payment
- Find the total interest on a home loan

Now that we've explored how simple and compound interest work and how basic loans grow over time, we can apply those ideas to a real-life situation many college students face, which are student loans. Student loans use the same mathematical principles we've already studied like percent rates, interest growth, and repayment schedules, but they introduce new layers such as deferment, capitalization, and long-term repayment plans. Understanding how the balance changes from month to month, how interest adds up while you're in school, and the different repayment options affect the total amount you'll pay, helps you do the math in order to make smarter financial choices for your future.

Types of Student Loans

There are three main types of student loans:

1. **Federal Student Loans** - Provided by the U.S. Department of Education, these loans usually have lower interest rates, fixed APR, repayment flexibility, and no credit check for undergraduates. Federal loans include Direct Subsidized, Direct Unsubsidized, and PLUS loans. Interest on federal loans is typically simple interest while in school, but becomes capitalized (i.e., compounded) after the borrower graduates.
2. **Private Student Loans** - Provided by banks and financial institutions, these loans may have variable or fixed interest rates, rates based on credit score, and fewer repayment protections. Private loans often use compounding interest, which grows faster than federal student loans.
3. **Institutional Student Loans** - The least popular loan, provided by some colleges. Typically, these loans follow the same interest and repayment concepts, but each college sets its own terms. They're usually smaller, used for emergencies or to fill financial gaps, and they don't offer the federal protections you get with government loans.

Federal student loans are by far the most common type of student loan, so we will primarily focus onn then in this section. As mentioned above, federal student loans fall into three major categories: Direct Subsidized Loans, Direct Unsubsidized Loans, and Direct PLUS Loans. Although these loans share the same federal rules for interest rates and repayment programs, they differ in who can receive them, how interest works, and when interest is capitalized. All of this will be discussed later in this section.

Timeline of the Federal Student Loan Process

A student planning to enroll in a college or university must first complete the FAFSA (Free Application for Federal Student Aid) to apply for federal student loans. The FAFSA serves as the gateway to federal financial aid programs, including federal student loans. The application becomes available each year for the upcoming academic year, and students must provide information about their family's financial situation. To remain eligible for aid, students must renew the FAFSA each year they are enrolled in college.

After the Department of Education reviews the FAFSA and determines the student's eligibility and loan amount, the loan typically follows the sequence below:

1. **Loan Disbursement:** The loan funds are sent directly to the student's school to pay tuition, fees, and other educational expenses. If funds remain after these charges are paid, the excess is refunded to the student.
2. **While the Student Is in School:** Most federal student loans do not require payments while the student is enrolled at least half-time. However, depending on the type of loan, interest may continue to accrue. Students may choose to make interest payments while in school to reduce the overall cost of the loan.
3. **Student Leaves School:** Repayment preparation begins when the student graduates, withdraws from school, or drops below half-time enrollment.
4. **Grace Period:** Most federal student loans provide a grace period, typically six months, before repayment begins. During this time, interest may continue to accrue on some loans.
5. **Repayment Begins:** The borrower selects a repayment plan, which is often structured as an amortized loan with fixed monthly payments. If accrued interest was not paid while the student was in school or during the grace period, it may be added to the principal balance. Regular monthly payments then begin.
6. **Pause Options (If Needed):** Borrowers experiencing financial difficulties may qualify for deferment or forbearance, which temporarily pauses or reduces payments. Interest often continues to accrue during these periods.
7. **Possible Loan Forgiveness:** Certain borrowers may qualify for loan forgiveness programs, such as Public Service Loan Forgiveness (PSLF), Teacher Loan Forgiveness, or Income-Driven Repayment (IDR) forgiveness.
8. **Loan Paid Off or Forgiven:** The loan ends when the balance is fully repaid or when any remaining balance is officially forgiven under an approved forgiveness program.

As part of the second phase described above, borrowers have the option to make interest payments while in school. This means you can either:

- pay off the interest each month while you are in school and only owe the amount you borrowed (principal) six months after school is finished, or
- you can simply add on the interest for each month while you are in school to the amount you borrowed (principal) and pay it all back (principal and interest) six months after school is finished

The second option is known as "capitalizing" the interest that accrues during the borrower's time in school:

Capitalized Interest vs. Not Capitalized Interest

When interest is **capitalized**:

- No interest is paid while the borrower is still in college.
- The principal balance continues to grow while in college.
- The resulting payments will be larger.

When interest is **not capitalized**:

- Interest only is paid each month while the borrower is still in college.
- The principal balance doesn't grow while the borrower is still in college.
- The resulting payments will be smaller.

Note that these apply only to unsubsidized loans. Remember, for subsidized loans, there is "no interest" while the borrower is in college.

Types of Federal Loans

As stated earlier, federal loans include:

1. **Direct Subsidized** - Direct Subsidized Loans are federal loans for undergraduate students with demonstrated financial need. Students must complete the FAFSA to determine eligibility. Your school decides how much you can borrow based on your need and the school's cost of attendance.

- **How the interest works:**
  - The U.S. Department of Education pays the interest:
    - while you are enrolled at least half-time
    - during the 6-month grace period after leaving school
    - during approved deferment periods
- **Capitalization**
  - Because the government covers the interest, no interest is added to your balance during these times.
- Subsidized loans generally cost less over time because they avoid interest buildup during school.

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2. **Direct Unsubsidized Loans** - Direct Unsubsidized Loans are available to both undergraduate and graduate students. Unlike subsidized loans, they are not based on financial need. Students must still complete the FAFSA, and the school determines how much they can borrow based on the cost of attendance and other aid received.
   - **How the interest works**
     - Interest begins accruing as soon as the loan is disbursed.
     - This interest continues to grow:
       - while you're in school
       - during the grace period
       - during deferment or forbearance
       - the borrower either pays the interest each month, or adds it to the principal (i.e., capitalizes the interest)
     - **Capitalization**
       - If you do not pay the interest while in school, the unpaid interest capitalizes:
         - New Principal = Original Principal + Unpaid Interest
   - After capitalization, interest is charged on a growing balance, making unsubsidized loans more expensive over time.

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3. **Direct PLUS Loans** - Direct PLUS Loans come in two forms, Parent PLUS Loans, borrowed by parents of dependent undergraduates or Grad PLUS Loans, borrowed by graduate and professional students.
   - Borrowers may take out up to the full cost of attendance, minus other aid. PLUS loans require:
     - a credit check
     - FAFSA completion
     - no financial-need requirement
   - **How the interest works**
     - Interest begins accruing as soon as the loan is disbursed.
   - **Capitalization**
     - Interest that accumulates while:
       - in school (for Grad PLUS),
       - in deferment,
       - or when payments are postponed, will capitalize if unpaid.

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*Interest Rates for Direct Loans First Disbursed on or After July 1, 2026, and Before July 1, 2027**

Loan Type	Borrower Type	Fixed Interest Rate (Award year: July 1st, 2026 — June 30th, 2027*)
Federal Direct Stafford Subsidized/ Unsubsidized Loans	Undergraduate	6.52%
Federal Direct Stafford Unsubsidized Loans	Graduate or Professional	8.07%
Direct PLUS Unsubsidized Loans	Parents of undergrads. Graduate or Professional Students	9.07%

** Federal loan interest rates can change from year to year, but your own loan's rate is locked in when you borrow it and stays the same until the loan is fully repaid.*

Interest rates on federal student loans are set by federal law, not the U.S. Department of Education. Interest on federal student loans is simple interest.The interest accrues only on the principal balance, not on previously accrued interest. While the student is still in school, the amount of interest that accrues each month remains constant. (Note: Some private loans may use compound interest, which increases your interest charges. Daily interest is the amount of interest that your student loan adds every single day. It is based on your current loan balance, your interest rate, and the fact that interest grows day by day, not once a month.

How to Calculate Daily & Monthly Interest

To calculate the **Daily Interest** simply use the simple interest formula and let $t = \frac{1}{365.25}$:

$$\text{Daily Interest Amount} = P \cdot r \cdot t$$

$$\text{Daily Interest Amount} = P \cdot r \cdot \frac{1}{365.25}$$

$$\text{Daily Interest Amount} = \frac{P \cdot r}{365.25},$$

where P = principal and r is the annual interest rate.

To calculate the **Monthly Interest** multiply the daily interest above by the actual number of days in that particular month:

$$\text{Monthly Interest Amount} = \text{Daily Interest Amount} \cdot \text{Days in a Month}$$

Note: When calculating daily interest using the simple interest formula, the time variable *t* is often expressed as a fraction of a 365- or 365.25-day year. Under Banker's Rule, however, interest is calculated using the exact number of days elapsed but assumes a 360-day year. As a result, Banker's Rule generally produces slightly more interest than calculations based on a 365- or 365.25-day year.

Example #4.8.1 🤔

Manny borrows $7,000 through a unsubsidized loan at a 6.5% interest to cover one-time college expenses.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month.
c. Assume Manny graduates in 3 years and 9 months after the loan is acquired and payments begin 6 months after graduation. How much interest will accrue before regular payments begin? (Assume all months are 30 days long.)

✅ **Solution:**

a. Let $P = \$7{,}000$ and $r = 0.065$. So,

$$\text{Daily Interest Amount} = \frac{P \cdot r}{365.25}$$

$$\text{Daily Interest Amount} = \frac{\$7{,}000 \cdot 0.065}{365.25}$$

$$\text{Daily Interest Amount} = \$1.2457221081451$$

$$\boxed{\text{Daily Interest Amount} = \$1.25}$$

---

b. So,

$$\text{Monthly Interest Amount} = \text{Daily Interest Amount} \cdot \text{Days in a Month}$$
$$\text{Monthly Interest Amount} = (\$1.25) \cdot (30)$$
$$\boxed{\text{Monthly Interest Amount} = \$37.50}$$

---

c. Interest will accrue for 3 years and 9 months, followed by another 6 months, for a total of 4 years and 3 months, or 51 months. Using 30-day months for all months, then the total interest accrued is

$$\text{Total Interest} = \text{Monthly Interest Amount} \cdot \text{Total Number of Months}$$
$$\text{Total Interest} = (\$37.50) \cdot (51)$$
$$\boxed{\text{Total Interest Amount} = \$1{,}912.50}$$

Example #4.8.2 🤔

Back in the award year of 7/1/22–6/30/23, the fixed interest rates for Federal Direct Stafford Loans was 4.99%. Moe took out a Direct Unsubsidized Loan as a graduate during that time period. He borrowed $13,500.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month.
c. Let's assume that Moe will graduate in only 2 years and 3 months after the loan is acquired and payments begin 6 months after graduation. How much interest will accrue before regular payments begin? (Assume all months are 30 days long.)

✅ **Solution:**

a. Let $P = \$13{,}500$ and $r = 0.0499$. So,

$$\text{Daily Interest Amount} = \frac{P \cdot r}{365.25}$$
$$\text{Daily Interest Amount} = \frac{\$13{,}500 \cdot 0.0499}{365.25}$$
$$\text{Daily Interest Amount} = \$1.8443531827515$$
$$\boxed{\text{Daily Interest Amount} = \$1.84}$$

---

b. So,

$$\text{Monthly Interest Amount} = \text{Daily Interest Amount} \cdot \text{Days in a Month}$$
$$\text{Monthly Interest Amount} = (\$1.84) \cdot (30)$$
$$\boxed{\text{Monthly Interest Amount} = \$55.20}$$

---

c. Interest will accrue for 2 years and 3 months, followed by another 6 months, for a total of 2 years and 9 months, or 33 months. Using 30-day months for all months, then the total interest accrued is

$$\text{Total Interest} = \text{Monthly Interest Amount} \cdot \text{Total Number of Months}$$
$$\text{Total Interest} = (\$55.20) \cdot (33)$$
$$\boxed{\text{Total Interest} = \$1{,}821.60}$$

Example #4.8.3 🤔

Jack has decided to take out a Direct PLUS Unsubsidized Loan that has an annual fixed rate of 8.94% He was able to take out the max at the time which was $20,500.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month
c. Jack is going to take his time to graduate. It is going to take him 4 years and 6 months after the loan is acquired and he somehow negotiated payments to begin 9 months after graduation. How much interest will accrue before regular payments begin? (Assume all months are 30 days long.)

✅ **Solution:**

a. Let $P = \$20{,}500$ and $r = 0.0894$. So,

$$\text{Daily Interest Amount} = \frac{P \cdot r}{365.25}$$

$$\text{Daily Interest Amount} = \frac{\$20{,}500 \cdot 0.0894}{365.25}$$

$$\text{Daily Interest Amount} = \$5.017659137577$$

$$\boxed{\text{Daily Interest Amount} = \$5.02}$$

---

b. So,

$$\text{Monthly Interest Amount} = \text{Daily Interest Amount} \cdot \text{Days in a Month}$$

$$\text{Monthly Interest Amount} = (\$5.02) \cdot (30)$$

$$\boxed{\text{Monthly Interest Amount} = \$150.60}$$

---

c. Interest will accrue for 4 years and 6 months, followed by another 9 months, for a total of 5 years and 3 months, or 63 months. Using 30-day months for all months, then the total interest accrued is

$$\text{Total Interest} = \text{Monthly Interest Amount} \cdot \text{Total Number of Months}$$

$$\text{Total Interest} = (\$150.60) \cdot (63)$$

$$\boxed{\text{Total Interest} = \$9{,}487.80}$$

Section 4.8: Student Loans, Home Buying, and Amortization [*In-Class Exercises* - Part 1 of 3]

1. Suppose an undergraduate student was able to take a Federal Direct Stafford Loan out for $10,000 with the rate for this current award year. (See current interest rates above or Current Student Aid Rates).
   a. Find the daily interest.
   b. Find the interest that accrues in a 30-day month.
   c. Assume the student graduates in 4 years after the loan is acquired and payments begin 6 months after graduation. How much interest will accrue before regular payments begin? (Assume all months are 30 days long.)
2. The Federal Direct Stafford Loan program actually began in 1965 as the Guaranteed Student Loan Program. Senator Robert T. Stafford from Vermont played a major role in expanding federal student aid and shaped the policies that made college more affordable. His work on education legislation, especially around student loans, led Congress to honor him by renaming the federal loan program to the Stafford Loan in 1988. Prior to 2006, the loans used variable rates. Since 2006, the loans involve fixed rates. The lowest fixed federal student loan rate in history was set for the 7/1/20–6/30/

21 award year at 2.75%, reflecting the unusually low interest-rate environment during the CoVID-19 pandemic. Suppose someone back then was able to take a Stafford Loan out for the same $10,000 that the student borrowed in Exercise #1 with that low rate of 2.75%.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month.
c. Assume that someone graduates in 4 years after the loan is acquired and payments begin 6 months after graduation. How much interest will accrue before regular payments begin? (Assume all months are 30 days long.)
d. Compare part c. with the interest in Exercise #1c.

Answers available at the end of this section.

Fees for Federal Student Loans

Most federal student loans have loan fees that are a percentage of the total loan amount. The loan fee is deducted proportionately from each loan disbursement you receive while enrolled in school. This means the money you receive will be less than the amount you actually borrow. You're responsible for repaying the entire amount you borrowed and not just the amount you received. The current loan rate is 1.057% for Direct Subsidized Loans and Direct Unsubsidized Loans and 4.228% for Direct PLUS Loans on each disbursement, respectively. So, for Example #4.8.1, the loan fee on the $7,000 loan is $73.99. This means the government keeps $73.99 as a loan fee and the student (Manny) receives $6,926.01 total for the year. Federal loans are usually paid out in two disbursements (fall and spring), so the student receives about $3,463.01 per disbursement (half of $6,926.01.) Even though the student receives only $6,926.01, they still owe the full $7,000, plus any interest that accrues while they are in school.

Repayment

So, in the timeline above at the beginning of this section, let's now focus on bullet point #5 where it says "Repayment Starts". At this point, the student has completed school, withdrew, or dropped below half-time student status. After the grace period, (usually 6 months), the loan enters repayment, and the borrower begins making monthly payments. Federal student loans use a monthly installment system, similar to car loans or mortgages. The borrower can choose a standard repayment plan, such as a 10-year payoff plan that has fixed monthly payments and usually the lowest total interest cost. Graduated repayment plans have payments that start lower and increase every two years. This is good for borrowers whose income rises over time. If the debt was too high or the monthly payments are two high, there are extended repayment plans up to 25 years that create lower monthly payments, yet higher total interest. The chart below shows the maximum repayment period for a Direct Consolidation Loan or Federal Family Education Loan (FFEL) Consolidation Loan under the Standard Repayment Plan depending on total education loan indebtedness.

Standard Repayment Plans

Total Education Loan Indebtedness	Repayment Period
$0 — $7,499.99	10 years
$7,500 — $9999.99	12 years
$10,000 — $19999.99	15 years
$20,000 — $39999.99	20 years
$40,000 — $59999.99	25 years
$60,000 and up	30 years

Your interest rate does NOT change when you enter repayment or when you choose a repayment plan. Your rate is still determined only by the year your loan was disbursed and remains fixed for the life of that loan. The only things that change are:

- Your monthly payment amount—for example, the 10-year plan has a higher monthly payment than a 20-year plan.
- The total interest you pay over time—longer repayment periods have greater finance charges.

It's also important to note that federal student loans have no prepayment penalties, meaning borrowers may pay extra at any time. Making additional payments reduces the total interest paid over the life of the loan and shortens the overall repayment period. This type of loan structure is known as an amortized loan.

Definition: Amortization

Amortization is the process of paying off a loan through regular, scheduled payments that cover both principal (the amount borrowed) and interest (the cost of borrowing). Most consumer loans, including federal student loans, car loans, and home mortgages, are amortized loans. This means the debt is designed to be fully paid off over a specific period of time, such as 10, 15, or 20 years.

What Does "Amortized" Mean?

The word amortize means to gradually reduce and eventually eliminate a debt over time. In an amortized loan, each payment reduces the loan balance until the amount owed eventually reaches zero. An amortized loan requires payments to be made on a fixed schedule, such as monthly payments for federal student loans. Each payment is divided into two parts: interest, which is the cost of borrowing money, and principal, which is the portion used to reduce the amount still owed. Although the total payment amount remains the same under a fixed-payment plan, the amounts applied to interest and principal change over time.

How Amortization Works Over Time

At the beginning of an amortized loan, the loan balance is at its highest. Since interest is calculated based on the remaining balance, a larger portion of each payment goes toward interest and only a small portion goes toward reducing the principal. As payments are made, the loan balance gradually decreases. Because the balance is smaller, less interest is charged each month. As a result, a larger portion of each payment begins to reduce the principal. Near the end of the loan, most of each payment is applied to the principal, while only a small amount is applied to interest. This gradual shift from paying mostly interest to paying mostly principal is the defining characteristic of an amortized loan.

Why Amortization Matters for Students

Understanding amortization is important because it affects both the total cost of borrowing and the benefits of making extra payments. The length of the repayment period has a significant impact on the total interest paid over the life of the loan. Shorter repayment plans, such as a 10-year repayment plan, pay down the principal more quickly and generally result in lower total interest charges. Longer repayment plans, such as those lasting 20 to 25 years, typically have lower monthly payments but result in substantially more interest being paid over time because the principal remains unpaid for a longer period. Amortization also affects the impact of extra payments. Since interest is calculated using the current loan balance, any additional payment that reduces the principal lowers future interest charges. As a result, making extra payments can shorten the repayment period, reduce the total interest paid, and save a significant amount of money over the life of the loan. This is why prepayment, or paying more than the required minimum payment, can be such a powerful strategy. Federal student loans generally allow borrowers to make extra payments at any time without incurring prepayment penalties.

Understanding amortization gives borrowers the financial literacy needed to manage student loans confidently and make smart repayment choices. Now let's look at the mathematics involved.

Amortization Formula

The fixed periodic payment for an amortized loan is calculated with the standard amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

where:

- R = Regular periodic payment at the end of each period
- P = Principal (amount borrowed)
- r = Annual interest rate (expressed as a decimal)
- n = Number of payments made per year
- t = Term of loan (in years)

Example #4.8.4 🤔

In Example #4.8.1, Manny borrowed $7,000 through an unsubsidized federal student loan at an annual interest rate of 6.5% to cover one-time college expenses. We determined that the total interest that accrued while Manny was in school was $1,912.50. Because this is an unsubsidized loan, Manny must either

- pay the interest while in school, or
- allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume Manny finishes school, his 6-month grace period ends, and it is now time to begin repayment. Answer the following questions.

a. How much does Manny need to pay back if he capitalizes the interest?
b. Find the monthly payment on the loan if the term is 10 years.
c. What is the total amount Manny pays back under the 10-year plan?
d. Suppose Manny did not capitalize the interest. Find the monthly payment on the 10-year loan.
e. What is the total amount Manny repays if he pays interest during school?
f. Compare parts c) and e). How much money did Manny save by not capitalizing his interest?

✅ **Solution:**

a. Capitalizing interest means adding the accrued interest to the original principal:

$$P_{NEW} = P_{ORIG} + I_C$$
$$P_{NEW} = \$7{,}000 + \$1{,}912.50$$
$$\boxed{P_{NEW} = \$8{,}912.50}$$

This becomes the principal for the 10-year repayment plan.

---

b. Use the amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Let $P = \$8{,}912.50$, $r = 0.065$, and $t = 10$. Then

$$R = \frac{8{,}912.50 \cdot \frac{0.065}{12}}{1 - \left(1 + \frac{0.065}{12}\right)^{-(12)(10)}}$$

$$R = \frac{48.276041666667}{1 - (1.00541666666667)^{-120}}$$

$$R = \frac{48.276041666667}{1 - (0.52296229318374)}$$

$$R = \frac{48.276041666667}{0.47703770681626}$$

$$R = \$101.19963469813$$

$$\boxed{R = \$101.20}$$

---

c. Because Manny capitalized interest, he prepaid none of the interest while in school:

$$\text{Total Amount Paid} = \text{Prepaid Interest} + 120 \cdot (\text{Monthly Payment})$$
$$\text{Total Amount Paid} = \$0 + 120(\$101.20)$$
$$\text{Total Amount Paid} = \$0 + \$12{,}144.00$$

$$\boxed{\text{Total Amount Paid} = \$12{,}144.00}$$

---

d. If Manny pays the interest while in school, then none of that interest is added to the loan balance. Thus, the principal for repayment is $P = \$7{,}000.00$.

Use the amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Let $P = \$7{,}000.00$, $r = 0.065$, and $t = 10$. Then

$$R = \frac{7{,}000.00 \cdot \frac{0.065}{12}}{1 - \left(1 + \frac{0.065}{12}\right)^{-(12)(10)}}$$

$$R = \frac{37.916666666667}{1 - (1.00541666666667)^{-120}}$$

$$R = \frac{37.916666666667}{1 - (0.52296229318374)}$$

$$R = \frac{37.916666666667}{0.47703770681626}$$

$$R = \$79.483584054019$$

$$\boxed{R = \$79.48}$$

---

e. What is the total amount Manny repays if he pays interest during school?

Here, the interest was prepaid:

$$\text{Total Amount Paid} = \text{Prepaid Interest} + 120 \cdot (\text{Monthly Payment})$$

Since Manny is not capitalizing the interest, then he is prepaying the interest while he is in school. Thus, the prepaid interest is $1,912.50, since the interest was NOT added on to the principal in part d. So,

$$\text{Total Amount Paid} = \$1{,}912.50 + 120(\$79.48)$$
$$\text{Total Amount Paid} = \$1{,}912.50 + \$9{,}537.60$$
$$\boxed{\text{Total Amount Paid} = \$11{,}450.10}$$

---

f. Compare parts (c) and (e). How much money did Manny save by paying the interest while in school?

- In part (c) when interest was **capitalized**
  - Interest owed for 51 months = $1,912.50
  - Interest paid prior to loan payment = $0
  - P = $7,000 + $1,912.50 = $8,912.50
  - Monthly Payment = $101.20
  - Total amount paid = $12,144.00
- In part (e) when interest was **NOT capitalized**
  - Monthly interest owed for 51 months = $1,912.50
  - Interest paid prior to loan payment = $1,912.50
  - P = $7,000 + $0 = $7,000
  - Monthly Payment = $79.48
  - Total amount paid = $11,450.10

Manny pays an extra $12,144.00 – $11,450.10 = $693.90 in interest over 10 years because he decided to capitalize the interest on his unsubsidized loan by having the interest added to the principal amount. Thus, he would save $693.90 if he did not capitalize the interest.

Example #4.8.5 🤔

In Example #4.8.2, Moe took out a Direct Unsubsidized Loan as a graduate borrowing $13,500 with an interest rate of 4.99%. We determined that the total interest that accrued while Moe was in school was $1,821.60. Because this is an unsubsidized loan, Moe must either

- pay the interest while in school, or
- allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume Moe finishes school, his 6-month grace period ends, and it is now time to begin repayment. Answer the following questions.

a. How much does Moe need to pay back if he capitalizes the interest?
b. Find the monthly payment on the loan if the term is 10 years. (Based off the amount, Moe was offered the choice of a 10-year, 12-year, 15-year, or 20-year; he chose the 10-year plan).
c. What is the total amount Moe pays back under the 10-year plan?
d. Suppose Moe did not capitalize the interest. Find the monthly payment on the 10-year loan.
e. What is the total amount Moe repays if he pays interest during school?
f. Compare parts c) and e). How much money did Moe save by not capitalizing his interest?

✅ **Solution:**

a. Capitalizing interest means adding the accrued interest to the original principal:

$$P_{NEW} = P_{ORIG} + I_C$$
$$P_{NEW} = \$13{,}500 + \$1{,}821.60$$

$$\boxed{P_{NEW} = \$15{,}321.60}$$

This becomes the principal for the 10-year repayment plan.

---

b. Use the amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Let $P = \$15{,}321.60$, $r = 0.0499$, and $t = 10$. Then

$$R = \frac{15{,}321.60 \cdot \frac{0.0499}{12}}{1 - \left(1 + \frac{0.0499}{12}\right)^{-(12)(10)}}$$

$$R = \frac{63.71232}{1 - (1.0041583333333)^{-120}}$$

$$R = \frac{63.71232}{1 - (0.60776598567667)}$$

$$R = \frac{63.71232}{0.39223401432333}$$

$$R = \$162.43445920904$$

$$\boxed{R = \$162.43}$$

---

c. Because Moe capitalized interest, he prepaid none of the interest while in school:

$$\text{Total Amount Paid} = \text{Prepaid Interest} + 120 \cdot (\text{Monthly Payment})$$

$$\text{Total Amount Paid} = \$0 + 120(\$162.43)$$

$$\text{Total Amount Paid} = \$0 + \$19{,}491.60$$

$$\boxed{\text{Total Amount Paid} = \$19{,}491.60}$$

---

d. If Moe pays the interest while in school, then none of that interest is added to the loan balance. Thus, the principal for repayment is $P = \$13{,}500.00$.

Use the amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Let $P = \$13{,}500.00$, $r = 0.0499$, and $t = 10$. Then

$$R = \frac{13{,}500.00 \cdot \frac{0.0499}{12}}{1 - \left(1 + \frac{0.0499}{12}\right)^{-(12)(10)}}$$

$$R = \frac{56.1375}{1 - (1.0041583333333)^{-120}}$$

$$R = \frac{56.1375}{1 - (0.60776598567667)}$$

$$R = \frac{56.1375}{0.39223401432333}$$

$$R = \$143.12246758315$$

$$\boxed{R = \$143.12}$$

---

e. What is the total amount Moe repays if he pays interest during school?

Here, the interest was prepaid:

$$\text{Total Amount Paid} = \text{Prepaid Interest} + 120 \cdot (\text{Monthly Payment})$$

Since Manny is not capitalizing the interest, then he is prepaying the interest while he is in school. Thus, the prepaid interest is $1{,}912.50$, since the interest was NOT added on to the principal in part d. So,

$$\text{Total Amount Paid} = \$1{,}821.60 + 120(\$143.12)$$

$$\text{Total Amount Paid} = \$1{,}821.60 + \$17{,}174.40$$

$$\boxed{\text{Total Amount Paid} = \$18{,}996.00}$$

---

f. Compare parts (c) and (e). How much money did Moe save by paying the interest while in school?

- In part (c) when interest was **capitalized**
  - Interest owed for 33 months = $1,821.60
  - Interest paid prior to loan payment = $0
  - P = $13,500 + $1,912.50 = $15,321.60
  - Monthly Payment = $162.43
  - Total amount paid = $19,491.60
- In part (e) when interest was **NOT capitalized**
  - Monthly interest owed for 33 months = $1,821.60
  - Interest paid prior to loan payment = $1,821.60
  - P = $13,500 + $0 = $13,500
  - Monthly Payment = $143.12
  - Total amount paid = $18,996.00

Moe pays an extra $19,491.60 – $18,996.00 = $495.60 in interest over 10 years because he decided to capitalize the interest on his unsubsidized loan by having the interest added to the principal amount. Thus, he would save $495.60 if he did not capitalize the interest.

Example #4.8.6 🤔

In Example #4.8.3, Jack took out a Direct PLUS Unsubsidized Loan borrowing $20,500 with an interest rate of 8.94%. We determined that the total interest that accrued while Jack was in school was $9,487.80. Because this is an unsubsidized loan, Jack must either

- pay the interest while in school, or

- allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume Jack finishes school, his 6-month grace period ends, and it is now time to begin repayment. Answer the following questions.

a. How much does Jack need to pay back if he capitalizes the interest?
b. Find the monthly payment on the loan if the term is 20 years. (Based off the amount, Jack was offered the choice of a 10-year, 12-year, 15-year, or 20-year; he chose the 20-year plan).
c. What is the total amount Jack pays back under the 20-year plan?
d. Suppose Jack did not capitalize the interest. Find the monthly payment on the 20-year loan.
e. What is the total amount Jack repays if he pays interest during school?
f. Compare parts c) and e). How much money did Jack save by not capitalizing his interest?

✅ **Solution:**

a. Capitalizing interest means adding the accrued interest to the original principal:

$$P_{NEW} = P_{ORIG} + I_C$$
$$P_{NEW} = \$20{,}500 + \$9{,}487.80$$
$$\boxed{P_{NEW} = \$29{,}987.80}$$

This becomes the principal for the 20-year repayment plan.

---

b. Use the amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Let $P = \$29{,}987.80$, $r = 0.0894$, and $t = 20$. Then

$$R = \frac{29{,}987.80 \cdot \frac{0.0894}{12}}{1 - \left(1 + \frac{0.0894}{12}\right)^{-(12)(20)}}$$
$$R = \frac{223.40911}{1 - (1.00745)^{-240}}$$
$$R = \frac{223.40911}{1 - (0.16840683404992)}$$
$$R = \frac{223.40911}{0.83159316595008}$$
$$R = \$268.65193119373$$
$$\boxed{R = \$268.65}$$

---

c. Because Jack capitalized interest, he prepaid none of the interest while in school:

$$\text{Total Amount Paid} = \text{Prepaid Interest} + 240 \cdot (\text{Monthly Payment})$$
$$\text{Total Amount Paid} = \$0 + 240(\$268.65)$$
$$\text{Total Amount Paid} = \$0 + \$64{,}476.00$$

$$\boxed{\text{Total Amount Paid} = \$64{,}476.00}$$

---

d. If Jack pays the interest while in school, then none of that interest is added to the loan balance. Thus, the principal for repayment is $P = \$20{,}500.00$.

Use the amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Let $P = \$20{,}500.00$, $r = 0.0894$, and $t = 20$. Then

$$R = \frac{20{,}500.00 \cdot \frac{0.0894}{12}}{1 - \left(1 + \frac{0.0894}{12}\right)^{-(12)(20)}}$$

$$R = \frac{152.725}{1 - (1.00745)^{-240}}$$

$$R = \frac{152.725}{1 - (0.16840683404992)}$$

$$R = \frac{152.725}{0.83159316595008}$$

$$R = \$183.65350540792$$

$$\boxed{R = \$183.65}$$

---

e. What is the total amount Jack repays if he pays interest during school?

Here, the interest was prepaid:

$$\text{Total Amount Paid} = \text{Prepaid Interest} + 240 \cdot (\text{Monthly Payment})$$

Since Jack is not capitalizing the interest, then he is prepaying the interest while he is in school. Thus, the prepaid interest is $\$9{,}487.80$, since the interest was NOT added on to the principal in part d. So,

$$\text{Total Amount Paid} = \$9{,}487.80 + 240(\$183.65)$$

$$\text{Total Amount Paid} = \$9{,}487.80 + \$53{,}563.80$$

$$\boxed{\text{Total Amount Paid} = \$53{,}563.80}$$

---

f. Compare parts (c) and (e). How much money did Moe save by paying the interest while in school?

- In part (c) when interest was **capitalized**
  - Interest owed for 63 months = $9,487.80
  - Interest paid prior to loan payment = $0
  - P = $20,500 + $9,487.80 = $29,987.80
  - Monthly Payment = $268.65
  - Total amount paid = $64,476.00
- In part (e) when interest was **NOT capitalized**
  - Monthly interest owed for 63 months = $9,487.80
  - Interest paid prior to loan payment = $9,487.80
  - P = $20,500 + $0 = $20,500
  - Monthly Payment = $183.65
  - Total amount paid = $53,563.80

Jack pays an extra $64,476.00 – $53,563.80 = $10,912.20 in interest over 20 years because he decided to capitalize the interest on his unsubsidized loan by having the interest added to the principal amount. Thus, he would save $10,912.20 if he did not capitalize the interest.

Section 4.8: Student Loans, Home Buying, and Amortization [*In-Class Exercises* - Part 2 of 3]

3. In Exercise #1, the student borrowed $10,000 through an unsubsidized federal student loan at an annual interest rate of 6.39% to cover one-time college expenses. We determined that the total interest that accrued while she was in school was $2,835.00. Because this is an unsubsidized loan, she must either

    - pay the interest while in school, or
    - allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume the student finishes school, her 6-month grace period ends, and it is now time to begin repayment. Answer the following questions.

a. How much does the student need to pay back if she capitalizes the interest?
b. Find the monthly payment on the loan if the term is 10 years. (Based off the amount, she was offered the choice of a 10-year, 12-year, or 15-year; she chose the 10-year plan).
c. What is the total amount she pays back under the 10-year plan?
d. Suppose the student did not capitalize the interest. Find the monthly payment on the 10-year loan.
e. What is the total amount the student repays if she pays interest during school?
f. Compare parts c) and e). How much money did the student save by not capitalizing her interest?

4. Suppose that the low interest-rate environment during the CoVID-19 pandemic of 2.75% was used for the same $10,000 that the student borrowed in Exercise #1. We determined that the total interest that accrued while she was in school was $1,215.00. Because this is an unsubsidized loan, she must either

    - pay the interest while in school, or
    - allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume the student finishes school, her 6-month grace period ends, and it is now time to begin repayment. Answer the following questions.

a. How much does the student need to pay back if she capitalizes the interest?
b. Find the monthly payment on the loan if the term is 10 years.
c. What is the total amount she pays back under the 10-year plan?
d. Suppose the student did not capitalize the interest. Find the monthly payment on the 10-year loan.
e. What is the total amount the student repays if she pays interest during school?
f. Compare parts c) and e). How much money did the student save by not capitalizing her interest?

Answers available at the end of this section.

Purchasing a Home

Buying a home is one of the biggest financial decisions most people will make in their lifetime. Unlike smaller purchases that can be paid for upfront, the cost of a home is usually far greater than what a typical buyer can afford in cash. To make homeownership possible, buyers rely on a mortgage, a long-term loan specifically designed to finance real estate.

A mortgage is a loan issued by a bank or lending institution that allows a buyer to purchase a home over an extended period of time, often 15, 20, or 30 years. Thirty years is by far the most popular repayment period for mortgage home loans. The home itself serves as collateral, meaning the lender can claim the property if the borrower fails to meet the repayment terms. In return, the borrower makes regular monthly payments until the loan is paid in full.

For fixed-rate mortgages, by far the most common type, this means the borrower pays the same amount each month, even though the loan balance gradually decreases over time. Early in the loan, most of the payment goes toward interest, while later payments go mostly toward principal, the amount originally borrowed. This gradual pay-down process is an amortization, which was the type of loans we saw earlier used for students.

The amortization formula used for long-term student loans is the same formula used for a mortgage.

Example #4.8.7 🤔

A homebuyer purchases a 2-bedroom, 1-bath, 920-sqft. condo in Newport Beach, CA for $875,000 and makes a 20% down payment. The remaining amount is financed with a 30-year fixed-rate mortgage at 6.25% APR.

a. Find the down payment.
b. Find the amount of the loan (mortgage).
c. Compute the monthly mortgage payment.
d. Determine the total amount paid over the life of the loan.
e. How much total interest will the borrower pay?

✅ **Solution:**

a. Down payment:

$$\text{Down Payment} = \text{Purchase Price} \cdot \text{Down Payment Rate}$$
$$\text{Down Payment} = (\$875{,}000) \cdot (0.20)$$
$$\boxed{\text{Down Payment} = \$175{,}000}$$

---

b. Loan amount:

$$\text{Loan Amount} = \text{Purchase Price–Down Payment}$$
$$\text{Loan Amount} = (\$875{,}000)–(\$175{,}000)$$
$$\boxed{\text{Loan Amount} = \$700{,}000}$$

---

c. Monthly mortgage payment.

Use the amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Let $P = \$700{,}000$, $r = 0.0625$, and $t = 30$. Then

$$R = \frac{700{,}000 \cdot \frac{0.0625}{12}}{1 - \left(1 + \frac{0.0625}{12}\right)^{-(12)(30)}}$$

$$R = \frac{3{,}645.8333333333}{1 - (1.0052083333333)^{-360}}$$

$$R = \frac{3,645.8333333333}{1 - (0.15410299895366)}$$

$$R = \frac{3,645.8333333333}{0.84589700104634}$$

$$R = \$4310.0204029847$$

$$\boxed{R = \$4{,}310.02}$$

---

d. Total amount paid:

$$\text{Total Amount Paid} = \text{Down Payment} + 360 \cdot (\text{Monthly Payment})$$
$$\text{Total Amount Paid} = \$175{,}000 + 360 \cdot (\$4{,}310.02)$$
$$\text{Total Amount Paid} = \$175{,}000 + \$1{,}551{,}607.20$$

$$\boxed{\text{Total Amount Paid} = \$1{,}726{,}607.20}$$

---

e. Total Interest:

$$\text{Total Interest} = \text{Total Amount Paid–Purchase Price}$$
$$\text{Total Interest} = (\$1{,}726{,}607.20)–(\$875{,}000)$$

$$\boxed{\text{Total Interest} = \$851{,}607.20}$$

To understand how amortization works, let's examine the month-to-month process used in Example #4.7.8. At the end of each month, the lender calculates the simple interest by multiplying the loan's current balance by the monthly interest rate. This interest amount is then subtracted from the fixed monthly payment. The remainder becomes the principal paid for that month. That principal portion is deducted from the loan balance, producing a new, lower balance for the next month. This cycle of, interest calculation, subtracting from the payment, reducing the balance, repeats every month throughout the life of the loan. When these calculations are carried out for all payments in the loan term, the result is called an amortization schedule. A partial amortization schedule for Example #4.7.8 is shown below.

Amortization (Partial) Schedule for Example #4.8.7

Month	Payment	Interest	Principal	Balance
0	$0	$0	$0	$700,000
1	$4310.02	$I = P \cdot r \cdot t$ = ($700,000)(0.0625)($\frac{1}{12}$) = $3,645.83	$4310.02 – $3,645.83 = $664.19	$700,000 – $664.19 = $699,335.81
2	$4310.02	$I = P \cdot r \cdot t$ = ($699,335.81)(0.0625)($\frac{1}{12}$) = $3,642.37	$4310.02 – $3,642.37 = $667.65	$699,335.81 – $667.65 = $698,668.16
3	$4310.02	$I = P \cdot r \cdot t$ = ($698,668.16)(0.0625)($\frac{1}{12}$) = $3,638.90	$4310.02 – $3,638.90 = $671.12	$698,668.16 – $671.12 = $697,997.04
4	$4310.02	$I = P \cdot r \cdot t$ = ($697,997.04)(0.0625)($\frac{1}{12}$) = $3,635.40	$4310.02 – $3,635.40 = $674.62	$697,997.04 – $674.62 = $697,322.42
5	$4310.02	$I = P \cdot r \cdot t$ = ($697,322.42)(0.0625)($\frac{1}{12}$) = $3,631.89	$4310.02 – $3,631.89 = $678.13	$697,322.42 – $678.13 = $696,644.29
6	$4310.02	$I = P \cdot r \cdot t$ = ($696,644.29)(0.0625)($\frac{1}{12}$) = $3,628.36	$4310.02 – $3,628.36 = $681.66	$696,644.29 – $681.66 = $695,962.63
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
355	$4310.02	$I = P \cdot r \cdot t$ = ($25,395.08)(0.0625)($\frac{1}{12}$) = $132.27	$4310.02 – $132.27 = $4,177.75	$25,395.08 – $4,177.75 = $21,217.33
356	$4310.02	$I = P \cdot r \cdot t$ = ($21,217.33)(0.0625)($\frac{1}{12}$) = $110.51	$4310.02 – $110.51 = $4,199.51	$21,217.33 – $4,199.51 = $17,017.82
357	$4310.02	$I = P \cdot r \cdot t$ = ($17,017.82)(0.0625)($\frac{1}{12}$) = $88.63	$4310.02 – $88.63 = $4,221.39	$17,017.82 – $4,221.39 = $12,796.43
358	$4310.02	$I = P \cdot r \cdot t$ = ($12,796.43)(0.0625)($\frac{1}{12}$) = $66.65	$4310.02 – $66.65 = $4,243.37	$12,796.43 – $4,243.37 = $8,553.06
359	$4310.02	$I = P \cdot r \cdot t$ = ($8,553.06)(0.0625)($\frac{1}{12}$) = $44.55	$4310.02 – $44.55 = $4,265.47	$8,553.06 – $4,265.47 = $4,287.59

360	$4310.02	$I = P \cdot r \cdot t$ = ($4,287.59)(0.0625)($\frac{1}{12}$) = $22.33	$4310.02 – $22.33 = $4,287.59	$4,287.59 – $4,287.59 = $0.00

For a full 360-month schedule, there are several amortization calculators that can be found online. Here is one: Amortization Calculator(opens in new window)

Example #4.8.8 🤔

A newlywed couple decides to purchase a new home. On Realtor.com, they found a 3-bedroom, 1.5-bath, 1,330-sqft. townhouse in American Fork, UT. The house is selling for $369,000 and the couple is putting 10% down. The remaining amount is financed with a 30-year fixed-rate mortgage at a 5.75% annual percentage rate.

a. Find the down payment.
b. Find the amount of the loan (mortgage).
c. Compute the monthly mortgage payment.
d. Determine the total amount paid over the life of the loan.
e. How much total interest will the couple pay?

✅ **Solution:**

a. Down payment:

$$\text{Down Payment} = \text{Purchase Price} \cdot \text{Down Payment Rate}$$
$$\text{Down Payment} = (\$369{,}000) \cdot (0.10)$$
$$\boxed{\text{Down Payment} = \$36{,}900}$$

---

b. Loan amount:

$$\text{Loan Amount} = \text{Purchase Price–Down Payment}$$
$$\text{Loan Amount} = (\$369{,}000)–(\$36{,}900)$$
$$\boxed{\text{Loan Amount} = \$332{,}100}$$

---

c. Monthly mortgage payment.

Use the amortization formula:

$$R = \frac{P \cdot \frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Let $P = \$332{,}100$, $r = 0.0575$, and $t = 30$. Then

$$R = \frac{332{,}100 \cdot \frac{0.0575}{12}}{1 - \left(1 + \frac{0.0575}{12}\right)^{-(12)(30)}}$$

$$R = \frac{1{,}591.3125}{1 - (1.0047916666667)^{-360}}$$

$$R = \frac{1,591.3125}{1 - (0.17890857777924)}$$

$$R = \frac{1,591.3125}{0.82109142222076}$$

$$R = \$4310.0204029847$$

$$\boxed{R = \$1{,}938.05}$$

---

d. Total amount paid:

$$\text{Total Amount Paid} = \text{Down Payment} + 360 \cdot (\text{Monthly Payment})$$
$$\text{Total Amount Paid} = \$36{,}900 + 360 \cdot (\$1{,}938.05)$$
$$\text{Total Amount Paid} = \$36{,}900 + \$697{,}698$$

$$\boxed{\text{Total Amount Paid} = \$734{,}598}$$

---

e. Total Interest:

$$\text{Total Interest} = \text{Total Amount Paid–Purchase Price}$$
$$\text{Total Interest} = (\$734{,}598)–(\$369{,}000)$$

$$\boxed{\text{Total Interest} = \$365{,}598}$$

The interest is almost the purchase price of the house.

Section 4.8: Student Loans, Home Buying, and Amortization [*In-Class Exercises* - Part 3 of 3]

5. A 30-acre parcel in Jackpot, Nevada is listed for $189,997. The average construction cost for a budget-level build in Nevada ranges from $130 to $250 per square foot. An investor plans to build a 3,000 square foot home and chooses the low end of the construction-cost range, resulting in an estimated building cost of:

   $$3{,}000 \text{ sq ft} \times \$130 = \$390{,}000$$

   To cover both the land purchase and construction costs, the investor obtains a $600,000 loan with 10% down, financed as a 30-year fixed-rate mortgage at an annual interest rate of 5.5%.

   a. Find the down payment.
   b. Find the amount of the loan (mortgage).
   c. Compute the monthly mortgage payment.
   d. Determine the total amount paid over the life of the loan.
   e. How much total interest will the investor pay?

6. A couple has decided to refinance their current home. They qualify for a 20-year fixed-rate mortgage at 5%, and the remaining balance on their existing loan is $450,000.

a. Compute the monthly mortgage payment.
b. Determine the total amount paid over the life of the loan.
c. How much total interest will the borrower pay?

Answers available at the end of this section.

Answers

Section 4.8: Student Loans, Home Buying, and Amortization [*In-Class Exercises* - Part 1 of 3]

1. a. For a 6.39% rate; Daily Interest = $1.75
   b. Monthly Interest (30 days) = $52.50
   c. Total Interest = $2,835.00
2. a. Daily Interest = $0.75
   b. Monthly Interest (30 days) = $22.50
   c. Total Interest = $1,215.00
   d. The student would have paid $1,620 more with the higher rate.

Section 4.8: Student Loans, Home Buying, and Amortization [*In-Class Exercises* - Part 2 of 3]

3. a. For a 6.39% rate; P_{NEW} = $12,835
   b. Monthly Payment = $145.02
   c. Total Amount = $17,402.40
   d. Monthly Payment = $112.99
   e. Total Amount = $16,393.80
   f. The student would save $17,402.40 – $16,393.80 = $1,008.60 if she did not capitalize the interest.
4. a. For a 2.75% rate; P_{NEW} = $11,215
   b. Monthly Payment = $107.00
   c. Total Amount = $12,840.00
   d. Monthly Payment = $95.41
   e. Total Amount = $12,664.20
   f. The student would save $12,840.00 – $12,664.20 = $175.80 if she did not capitalize the interest.

Section 4.8: Student Loans, Home Buying, and Amortization [*In-Class Exercises* - Part 3 of 3]

5. a. Down = $60,000
   b. Amount of loan = $540,000
   c. Monthly payment = $3,066.06
   d. Total amount = $1,103,781.60 ($1,163,781.60 including down payment)
   e. Total interest = $563,781.60
6. a. Monthly payment = $2,969.80
   b. Total amount = $712,752.00
   c. Total interest = $262,752.00

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Section 4.8.H: Homework

Section 4.8 - Student Loans, Home Buying, and Amortization - Homework Exercises

(Section 4.8) - Homework Exercise #1

A student takes out a $6,500 Direct Unsubsidized Loan at an annual percentage rate of 5.50% during the 2023–2024 award year.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month.
c. Assume the student graduates in 3.5 years after the loan is acquired and payments begin 6 months after graduation. How much interest will accrue before regular payments begin? (Since, there is no actual start date, just use 30-day months for all months.

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #2

Mia takes out a $12,000 Direct Unsubsidized Loan at a fixed APR of 4.99%.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month.
c. Assume the student graduates in 3 years, 9 months after the loan is acquired and payments begin 6 months after graduation. How much interest will accrue before regular payments begin? (Since, there is no actual start date, just use 30-day months for all months.)

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #3

Vincent takes out a $9,500 Direct Unsubsidized Loan at a fixed APR of 6.39% during the award year of 2025–2026. Vincent also applied for a 12-month grace/deferment extension.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month.
c. Assume the student graduates in 4 years after the loan is acquired and payments begin 12 months after graduation. How much interest will accrue before regular payments begin? (Since, there is no actual start date, just use 30-day months for all months.)

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #4

Jules has decided to take out a Direct PLUS Unsubsidized Loan that has an annual fixed rate of 8.94%. He is borrowing $15,000.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month.
c. Assume the student graduates in 2 years, 6 months after the loan is acquired and payments begin 6 months after graduation. How much interest will accrue before regular payments begin? (Since, there is no actual start date, just use 30-day months for all months.)

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #5

Zed has taken out a graduate Direct PLUS Unsubsidized Loan of $24,000 that has an annual fixed rate of 7.94%.

a. Find the daily interest.
b. Find the interest that accrues in a 30-day month.
c. Assume the student graduates in 2 years, 9 months after the loan is acquired and payments begin 9 months after graduation. How much interest will accrue before regular payments begin? (Since, there is no actual start date, just use 30-day months for all months.)

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #6

Answer each of the following questions with the choices of subsidized, unsubsidized, capitalized, or not capitalized.

a. When interest is covered by government while the borrower is still in school, a student loan is called:
b. When interest on a student loan is paid during college by the borrower, we say that the interest is:.
c. When interest on a student loan is not paid during college by the borrower, but rather after the borrower graduates, we say that the interest is:
d. When interest is covered by the borrower while the borrower is still in school, a student loan is called:

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #7

In Homework Exercise #1, a student took out a $6,500 Direct Unsubsidized Loan at an annual percentage rate of 5.50% and the total interest was $1,411.20. Because this is an unsubsidized loan, the student must either

- pay the interest while in school, or
- allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume the student finishes school, his 6-month grace period ends, and it is now time to begin repayment. Answer the following questions.

a. How much does the student need to pay back if he capitalizes the interest?
b. Find the monthly payment on the loan if the term is 10 years.
c. What is the total amount the student pays back under the 10-year plan?
d. Suppose the student did not capitalize the interest. Find the monthly payment on the 10-year loan.
e. What is the total amount the student repays if he pays interest during school?
f. Compare parts c) and e). How much money did the student save by not capitalizing his interest?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #8

In Homework Exercise #2, Mia borrowed a $12,000 Direct Unsubsidized Loan at a fixed APR of 4.99% and the total interest was $2,509.20. Because this is an unsubsidized loan, Mia must either

- pay the interest while in school, or
- allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume Mia finishes school, her 6-month grace period ends, and it is now time to begin repayment. Answer the following questions.

a. How much does Mia need to pay back if she capitalizes the interest?
b. Find the monthly payment on the loan if the term is 15 years.
c. What is the total amount Mia pays back under the 15-year plan?
d. Suppose Mia did not capitalize the interest. Find the monthly payment on the 15-year loan.
e. What is the total amount Mia repays if she pays interest during school?
f. Compare parts c) and e). How much money did Mia save by not capitalizing her interest?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #9

In Homework Exercise #3, Vincent took out a $9,500 Direct Unsubsidized Loan at a fixed APR of 6.39% and also applied for a 12-month grace/deferment extension. The total interest was $2,988.00. Because this is an unsubsidized loan, Vincent must either

- pay the interest while in school, or
- allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume Vincent finishes school, his 12-month grace period ends, and it is now time to begin repayment. He opts for a shorter term (10-year) than the standard 15-year usually offered. Answer the following questions.

a. How much does Vincent need to pay back if he capitalizes the interest?
b. Find the monthly payment on the loan if the term is 10 years.
c. What is the total amount Vincent pays back under the 10-year plan?
d. Suppose Vincent did not capitalize the interest. Find the monthly payment on the 10-year loan.
e. What is the total amount Vincent repays if he pays interest during school?
f. Compare parts c) and e). How much money did Vincent save by not capitalizing his interest?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #10

In Homework Exercise #4, Jules took out a $15,000 Direct PLUS Unsubsidized Loan that has an annual fixed rate of 8.94%. The total interest was $3,963.60. Because this is an unsubsidized loan, Jules must either

- pay the interest while in school, or
- allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume Jules finishes school, his 6-month grace period ends, and it is now time to begin repayment. He luckily retrieves his stolen wallet back from Pumpkin and Honey Bunny at the cafe. Answer the following questions.

a. How much does Jules need to pay back if he capitalizes the interest?
b. Find the monthly payment on the loan if the term is 15 years.
c. What is the total amount Jules pays back under the 15-year plan?
d. Suppose Jules did not capitalize the interest. Find the monthly payment on the 15-year loan.
e. What is the total amount Jules repays if he pays interest during school?
f. Compare parts c) and e). How much money did Jules save by not capitalizing his interest?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #11

In Homework Exercise #5, Zed had taken out a graduate Direct PLUS Unsubsidized Loan of $24,000 with has an annual fixed rate of 7.94% that accrued a total interest of $6,577.20. Since this is an unsubsidized loan, Zed must either

- pay the interest while in school, or
- allow the interest to capitalize, meaning it will be added to the loan balance when he enters repayment.

Assume Zed finishes school, his 9-month grace period ends, and it is now time to begin repayment. Answer the following questions.

a. How much does Zed need to pay back if he capitalizes the interest?
b. Find the monthly payment on the loan if the term is 20 years.
c. What is the total amount Zed pays back under the 20-year plan?
d. Suppose Jules did not capitalize the interest. Find the monthly payment on the 20-year loan.
e. What is the total amount Zed repays if he pays interest during school?
f. Compare parts c) and e). How much money did Zed save by not capitalizing his interest?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #12

Mitchell plans to purchase a downtown Las Vegas condo listed at $325,000. He places a 10% down payment, and finances the remainder with a 30-year fixed-rate mortgage at 6.1%. Property taxes and HOA fees will be added later, but for now, Mitchell wants to estimate the principal-and-interest portion of the monthly payment.

a. Find the down payment.
b. Find the amount of the loan (mortgage).
c. Compute the monthly mortgage payment.
d. Determine the total amount paid over the life of the loan.
e. How much total interest will Mitchell pay?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #13

Tatyana, a first-time homebuyer, purchases a home for $289,000 using an FHA loan with a required 3.5% down payment. The mortgage carries a 30-year fixed rate of 5.9%.

a. Find the down payment.
b. Find the amount of the loan (mortgage).
c. Compute the monthly mortgage payment.
d. Determine the total amount paid over the life of the loan.
e. How much total interest will Tatyana pay?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #14

Ramon wants to buy a cabin near Lake Tahoe for $540,000. Because this is a second home, his lender requires a 25% down payment. He secures a 30-year fixed mortgage at 6.35% for the remaining amount.

a. Find the down payment.
b. Find the amount of the loan (mortgage).
c. Compute the monthly mortgage payment.
d. Determine the total amount paid over the life of the loan.
e. How much total interest will Tab pay?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #15

Sunny takes out a $980,000 mortgage with a 30-year amortization schedule at 5.85% interest.

a. Compute the monthly mortgage payment.
b. Determine the total amount paid over the life of the loan.
c. How much total interest will Marti pay?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #16

In mortgage financing, points are up-front fees paid by the borrower to the lender at closing. Points are expressed as a percentage of the loan amount, and they serve two different purposes depending on the type of point. One of the main types is called "Discount Points". Discount points are fees paid at closing in exchange for a reduced interest rate on the mortgage. This is sometimes called “buying down the rate.” Each point typically reduces the interest rate by 0.25% or more. The actual discount varies by lender and market conditions. This is good for borrowers who plan to stay in the home for many years. Discount points also reduce the monthly payment, to save money in total interest over the long term.

A buyer finances $510,000 at a 30-year fixed rate of 5.7%. To lower their interest rate, the buyer chooses to pay two discount points upfront (each point = 1% of the loan amount). After paying points, the lender reduces the rate to 5.3%.

a. Compute the cost of points. (Multiply 2 by 1% of the loan).

b. Calculate the monthly payment before buying points. (Amortization formula at 5.7%).
c. Determine the total amount paid over the life of the loan before points.
d. Calculate the monthly payment after buying points. (Amortization formula at 5.3%).
e. Determine the total amount paid over the life of the loan after points. (Be sure to include part (a)).
f. How much money would you save each month in the monthly payment?
g. How much money would you save in all by buying the points? (Subtract parts (e) from part (c)).

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #17

According to Realtor.com, a 4-bedroom, 4.5-bath, 8,171 sqft, 0.37-acre house on Coast Hwy. in Laguna Beach that overlooks the ocean is selling for $19,995,000. A buyer, Phoebe, is interested. She is considering putting $200,000 down and finance the remainder with a 30-year jumbo fixed-rate mortgage at 6.6%.

a. Find the amount of the loan (mortgage).
b. Compute the monthly mortgage payment.
c. Determine the total amount paid over the life of the loan.
d. How much total interest would Phoebe pay?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #18

Anthony currently owes $312,500 on a 30-year mortgage taken out 8 years ago. Interest rates have dropped, and he has decided to refinance the remaining balance into a 20-year fixed-rate loan at 4.9%.

a. Compute the monthly mortgage payment.
b. Determine the total amount paid over the life of the loan.
c. How much total interest will Anthony pay?

Answer available at the end of this section.

(Section 4.8) - Homework Exercise #19

A family is relocating to Reno, Nevada and purchases a newly built home priced at $715,000. They choose a 15-year fixed-rate mortgage at 4.75% and put 15% down.

a. Find the down payment.
b. Find the amount of the loan (mortgage).
c. Compute the monthly mortgage payment.
d. Determine the total amount paid over the life of the loan.
e. How much total interest will the borrower pay?

Answer available at the end of this section.

Answers

(Section 4.8) - Homework Exercise #1

a. Daily Interest = $0.98

b. Monthly Interest (30 days) = $29.40
c. Total interest = $1,411.20

(Section 4.8) - Homework Exercise #2

a. Daily Interest = $1.64
b. Monthly Interest (30 days) = $49.20
c. Total interest = $2,509.20

(Section 4.8) - Homework Exercise #3

a. Daily Interest = $1.66
b. Monthly Interest (30 days) = $49.80
c. Total interest = $2,988.00

(Section 4.8) - Homework Exercise #4

a. Daily Interest = $3.67
b. Monthly Interest (30 days) = $110.10
c. Total interest = $3,963.60

(Section 4.8) - Homework Exercise #5

a. Daily Interest = $5.22
b. Monthly Interest (30 days) = $156.60
c. Total interest = $6,577.20

(Section 4.8) - Homework Exercise #6

a. subsidized
b. not capitalized
c. capitalized
d. unsubsidized

(Section 4.8) - Homework Exercise #7

a. For a 5.50% rate; P_{NEW} = $7,911.20
b. Monthly Payment = $85.86
c. Total Amount = $10,303.20
d. Monthly Payment = $70.54
e. Total Amount = $9,876.00
f. The student would save $10,303.20 – $9,876.00 = $427.20 if the student did not capitalize the interest.

(Section 4.8) - Homework Exercise #8

a. For a 4.99% rate; P_{NEW} = $14,509.20
b. Monthly Payment = $114.66
c. Total Amount = $20,638.80
d. Monthly Payment = $94.83
e. Total Amount = $19,578.60
f. The student would save $20,638.80 – $19,578.60 = $1,060.20 if Mia did not capitalize the interest.

(Section 4.8) - Homework Exercise #9

a. For a 6.39% rate; P_{NEW} = $12,488.00
b. Monthly Payment = $141.10
c. Total Amount = $16,932.00
d. Monthly Payment = $107.34
e. Total Amount = $15,868.80
f. The student would save $16,932.00 – $15,868.80 = $1,063.20 if Vincent did not capitalize the interest.

(Section 4.8) - Homework Exercise #10

a. For an 8.94% rate; P_{NEW} = $18,963.60
b. Monthly Payment = $191.67
c. Total Amount = $34,500.60
d. Monthly Payment = $151.61
e. Total Amount = $31,253.40
f. The student would save $34,500.60 – $31,253.40 = $3,247.20 if Jules did not capitalize the interest.

(Section 4.8) - Homework Exercise #11

a. For an 7.94% rate; P_{NEW} = $30,577.20
b. Monthly Payment = $254.62
c. Total Amount = $61,108.80
d. Monthly Payment = $199.85
e. Total Amount = $54,541.20
f. The student would save $61,108.80 – $54,541.20 = $6,567.60 if Zed did not capitalize the interest.

(Section 4.8) - Homework Exercise #12

a. Down payment = $32,500
b. Amount of loan = $292,500
c. Monthly payment = $1,772.53
d. Total amount = $638,110.80 ($670,610.80 including down payment)
e. Total interest = $345,610.80

(Section 4.8) - Homework Exercise #13

a. Down payment = $10,115
b. Amount of loan = $278,885

c. Monthly payment = $1,654.17
d. Total amount = $595,501.20 ($605,616.20 including down payment)
e. Total interest = $316,616.20

(Section 4.8) - Homework Exercise #14

a. Down payment = $135,000
b. Amount of loan = $405,000
c. Monthly payment = $2,520.05
d. Total amount = $907,218.00 ($1,042,218 including down payment)
e. Total interest = $502,218.00

(Section 4.8) - Homework Exercise #15

a. Monthly payment = $5,781.42
b. Total amount = $2,081,311.20
c. Total interest = $1,101,311.20

(Section 4.8) - Homework Exercise #16

a. Cost of two points = $10,200
b. Monthly payment = $2,960.04
c. Total amount of loan at 5.7% = $1,065,614.40
d. Monthly payment = $2,832.05
e. Total amount of loan at 5.3% = $1,029,738.00
f. Monthly savings = $127.99
g. Savings = $35,876.40

(Section 4.8) - Homework Exercise #17

a. Amount of loan = $19,795,000
b. Monthly payment = $126,422.51
c. Total amount = $45,512,103.60 ($45,712,103.60 including down payment)
d. Total interest = $25,717,103.60

(Section 4.8) - Homework Exercise #18

a. Monthly payment = $2,045.14
b. Total amount = $490,833.60
c. Total interest = $178,333.60

(Section 4.8) - Homework Exercise #19

a. Down payment = $107,250
b. Amount of loan = $607,750
c. Monthly payment = $4,727.27
d. Total amount = $850,908.60 ($958,158.60 including down payment)
e. Total interest = $243,158.60

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CHAPTER 5: PROBABILITY

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Section 5.1: Probability Basics

Learning Objectives

- Understand basic terms such as experiment, outcome, event, and sample space
- Compute theoretical probabilities
- Compute empirical probabilities
- The possible values a probability can take on

Introduction to Probability: Understanding the Foundation

Probability is the branch of mathematics that deals with uncertainty and chance. The probability of an event is a numerical measure representing how likely the event is to happen.

Every day, we encounter situations where we cannot predict exactly what will happen, but we can measure how likely different possibilities are. Whether we're wondering if it will rain, if our favorite team will win, or if we'll hit a red light on the way to work, we're thinking about probability. To study probability mathematically, we first need to understand four fundamental concepts that help us organize and analyze uncertain situations: experiments, outcomes, events, and sample spaces.

Definition: Experiment

An **experiment** in probability is any process or activity that produces uncertain results.

Probability experiments have unpredictable outcomes each time they're performed. Common examples of probability experiments include flipping a coin, rolling a die, drawing a card from a deck, or selecting a student randomly from a class. Even everyday activities like checking tomorrow's weather, choosing a route to work, or picking a movie to watch can be viewed as probability experiments. The key characteristic of any experiment is that we know what outcomes are possible, but we cannot predict with certainty which specific outcome will occur on any given trial, unless there is only on possible outcome.

Definition: Outcome

The **outcomes** of an experiment are the individual results that can occur when the experiment is performed.

Each outcome represents one specific way the experiment can end. For example, when flipping a coin, there are exactly two possible outcomes: heads or tails. When rolling a standard six-sided die, the possible outcomes are 1, 2, 3, 4, 5, or 6. When drawing a single card from a standard deck, there are 52 possible outcomes (one for each card). It's important to note that outcomes should be defined clearly and specifically, and they should be mutually exclusive, meaning that only one outcome can occur at a time. Additionally, the outcomes should be equally likely when we're dealing with fair or unbiased experiments.

Definition: Event

An **event** is a collection of one or more outcomes that we're interested in studying.

While an outcome is a single specific result, an event can include multiple outcomes that share some common characteristic. For instance, when rolling a die, "getting an even number" is an event that includes three outcomes: 2, 4, and 6. Similarly, "drawing a red card" from a deck is an event that includes 26 different outcomes (all hearts and diamonds). Events can be simple (containing just one outcome) or compound (containing multiple outcomes). We can also have events that include no outcomes (impossible events) or events that include all possible outcomes (certain events). Understanding events is crucial because real-world probability problems typically involve compound events.

Definition: Sample Space

The **sample space** is the complete set of all possible outcomes for a particular experiment.

It represents every single thing that could happen when the experiment is performed, with no possibilities left out. For a coin flip, the sample space is {heads, tails}. For rolling a die, the sample space is {1, 2, 3, 4, 5, 6}. For drawing a card from a standard deck, the sample space contains all 52 cards. The sample space serves as the foundation for calculating probabilities because it defines the "universe" of possibilities for that experiment. When we calculate the probability of an event, we're essentially comparing how many outcomes in the sample space belong to that event versus the total number of outcomes in the sample space.

Understanding experiments, outcomes, events, and sample spaces now gives us the vocabulary and conceptual tools needed to tackle more complex probability problems and make sense of uncertainty in our world. When we want to find the probability of an event, we first identify our experiment and determine its sample space. Then we identify which specific outcomes from the sample space make up the event we're interested in. After that, we use a basic probability formula to find the actual probability. This systematic approach allows us to move from informal thinking about chance to precise mathematical calculations. If we assume that all outcomes are equally
likely, then the "**theoretical probability**" of an event is defined by the following formula.

Definition: Theoretical Probability

Theoretical probability is the probability calculated based on mathematical reasoning and the known structure of an experiment, assuming all outcomes are equally likely. It represents what we expect to happen in theory, without actually performing the experiment. The formula for theoretical probability is:

$$P(\text{Event } E) = \frac{\text{Number of Outcomes in } E}{\text{Number of Possible Outcomes in Sample Space}}$$

Another type of probability is called an empirical probability.

Definition: Empirical Probability

Empirical probability (also called **experimental probability**) is the probability calculated based on actual experimental data or observed results. It represents what actually happens when an experiment is performed multiple times. The formula for empirical probability is:

$$P(\text{Event } E) = \frac{\text{Number of Times Event } E \text{ Occurred}}{\text{Total Number of Trials}}$$

The Fundamental Range

In probability theory, every probability value must fall within a specific, well-defined range. All probabilities are expressed as numbers between 0 and 1, inclusive. This means that for any event E, we have $0 \leq P(E) \leq 1$. This constraint is not arbitrary and it reflects the logical boundaries of what probability represents and provides a universal scale for measuring uncertainty.

Interpreting Values 0, 0.5, and 1

Probability = 0 (Impossible Events)

When $P(E) = 0$, the event is impossible and will never occur under the given conditions. This represents absolute certainty that something will not happen. For example, when rolling a standard six-sided die, the probability of rolling a 7 is 0/6 = 0, because 7 is not among the possible outcomes {1, 2, 3, 4, 5, 6}.

Probability = 0.5 (Equally Likely)

When $P(E) = 0.5$, the event E has an equal chance of occurring or not occurring. This represents maximum uncertainty. The classic example is flipping a fair coin. The probability of obtaining heads is 1/2 = 0.5. This means that over many trials, we expect heads to occur about half the time.

Probability = 1 (Certain Events)

When $P(E) = 1$, the event E is certain and will definitely occur every time the experiment is performed. This represents absolute certainty that something will happen. For instance, when rolling a standard die, the probability of rolling an odd or even number is 6/6 = 1, since every possible outcome satisfies this condition.

Interpreting Values Between 0 and 1

Low Probability (Close to 0)

Values near 0 indicate unlikely events. For example:

- $P(E) = 0.1$ means the event occurs about 10% of the time (1 in 10 trials)
- $P(E) = 0.05$ means the event occurs about 5% of the time (1 in 20 trials)
- These events are improbable but not impossible

Moderate Probability (Around 0.5)

Values near 0.5 indicate moderate uncertainty:

- $P(E) = 0.4$ means the event occurs about 40% of the time (4 in 10 trials)
- $P(E) = 0.6$ means the event occurs about 60% of the time (6 in 10 trials)
- These events have reasonable chances of occurring or not occurring

High Probability (Close to 1)

- $P(E) = 0.9$ means the event occurs about 90% of the time (9 in 10 trials)
- $P(E) = 0.95$ means the event occurs about 95% of the time (19 in 20 trials)
- These events are highly probable but not guaranteed

Common Misconceptions to Avoid

- **Probabilities Cannot Exceed 1** - If your calculation yields a probability greater than 1, you've made an error. Check your counting of favorable outcomes and total outcomes.

- **Probabilities Cannot Be Negative** - Negative probabilities have no meaning in classical probability theory. If you get a negative result, review your calculation method.

Multiple Ways to Express Probabilities - Fractions, Decimals, and Percentages

The same probability can be expressed in different formats, all representing the same likelihood:

- **Fraction:** 1/4
- **Decimal:** 0.25
- **Percentage:** 25%

All three expressions represent the same probability and fall within our required range of 0 to 1, inclusive (when using decimal form).

Example #5.1.1 🤔

Decide whether or not each given number could represent a probability.

a. 0.56
b. 84%
c. $\frac{7}{9}$
d. $\frac{5}{4}$
e. 0
f. -0.48
g. 62%
h. 2
i. $-\frac{1}{3}$
j. 110%
k. $\frac{5}{5}$
l. 1.79
m. -25

✅ **Solution:**

So, we are looking for numbers that are between 0 and 1 inclusive.

a. Yes, 0.56 could represent a probability; 0.56 is between 0 and 1 inclusive.
b. Yes, 84% could represent a probability; $84\% = 0.84$.
c. Yes, $\frac{7}{9}$ could represent a probability; $\frac{7}{9} \approx 0.778$; 0.778 is between 0 and 1 inclusive.
d. No, $\frac{5}{4}$ cannot represent a probability; $\frac{5}{4} = 1.25$; 1.25 is larger than 1.
e. Yes, 0 could represent a probability; 0 is between 0 and 1 inclusive.
f. No, -0.48 cannot represent a probability; negative numbers cannot represent probabilities.
g. Yes, 62% could represent a probability; $62\% = 0.62$.
h. No, 2 cannot represent a probability; 2 is larger than 1.
i. No, $-\frac{1}{3}$ cannot represent a probability; negative numbers cannot represent probabilities.
j. No, 110% cannot represent a probability; $110\% = 1.1$; 1.1 is larger than 1.
k. Yes, $\frac{5}{5}$ could represent a probability; $\frac{5}{5} = 1$; 1 is between 0 and 1 inclusive.
l. No, 1.79 cannot represent a probability; 1.79 is larger than 1.
m. No, -25 cannot represent a probability; negative numbers cannot represent probabilities.

Example #5.1.2 🤔

A single six-sided die is rolled. Determine the probability of each of the following outcomes.

a. a '6'
b. odd number
c. a number less than '3'
d. a number greater than or equal to '4'
e. a number less than '7'
f. a number greater than '9'

✅ **Solution:**

There are 6 outcomes in the sample space, which is $\{1, 2, 3, 4, 5, 6\}$.

a. $\frac{1}{6}$; there is one outcome '6' from six total outcomes.
b. $\frac{3}{6} = \frac{1}{2}$; there are three odd outcomes '1', '3', and '5' from the six total outcomes.
c. $\frac{2}{6} = \frac{1}{3}$; there are two outcomes '1', and '2' from the six total outcomes.
d. $\frac{3}{6} = \frac{1}{2}$; there are three outcomes '4', '5', and '6' from the six total outcomes.
e. $\frac{6}{6} = 1$; all six total outcomes are already less than 7.
f. $\frac{0}{6} = 0$; there are no outcomes greater than 9 from the six total outcomes.

Example #5.1.3 🤔

The Weather Channel App says that the chance of rain in your area is 30%. What is the probability of rain? Express your answer as a percentage and a decimal.

✅ **Solution:**

0.3 or 30%

Example #5.1.4 🤔

According to the American Veterinary Medical Association (AVMA), as of 2024, the proportion of U.S. households who owned a dog is 0.455. What is the probability that a randomly selected house owns a dog?

✅ **Solution:**

0.455 or 45.5%

Example #5.1.5 🤔

A marble is drawn randomly from a jar that contains 6 red marbles, 2 white marbles, and 5 blue marbles. Find the probability of each given event.

a. A red marble is drawn
b. A white marble is drawn
c. A green marble is drawn
d. A blue marble is drawn
e. A red or blue marble is drawn

✅ **Solution:**

a. $\frac{6}{13}$; there are 6 red marbles from a total of 13 marbles.
b. $\frac{2}{13}$; there are 2 white marbles from a total of 13 marbles.
c. $\frac{0}{13} = 0$; there are no green marbles from a total of 13 marbles.

d. $\frac{5}{13}$; there are 5 blue marbles from a total of 13 marbles.
e. $\frac{11}{13}$; there are 6 red marbles and 5 blue marbles. So there are 11 red or blue marbles from a total of 13 marbles.

Example #5.1.6 🤔

In a survey, 160 people indicated they prefer cats, 225 indicated they prefer dogs, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.

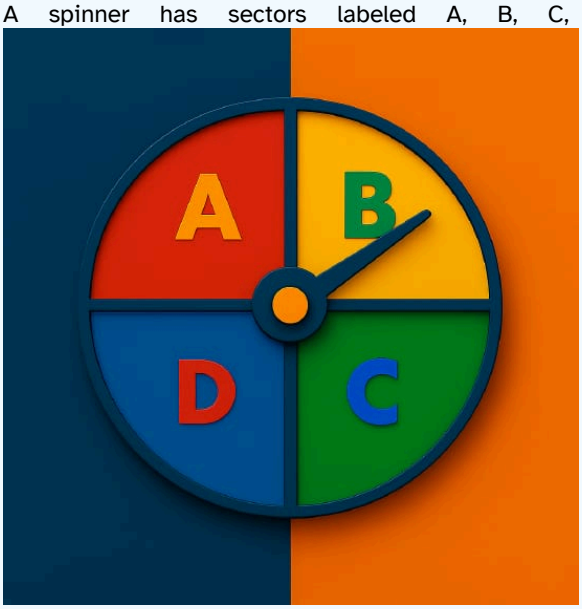
✅ **Solution:**

First, we need to add all of the people in the survey, $160 + 225 + 40 = 425$.

So, the probability of a person preferring cats is $\frac{160}{425} = 0.37647058823 \approx 0.376$.

Example #5.1.7 🤔

A spinner has sectors labeled A, B, C, and D with equal probability. The spinner gets spun.



a. What is the probability that the spinner lands on C?
b. What is the probability that the spinner does not land on B?
c. What is the probability that the spinner lands on A or D?

✅ **Solution:**

Since there are four sectors all with equal probability, then the probability 1/4 represents each sector.

a. $\frac{1}{4}$; sector C is one of a total of 4 sectors.
b. $\frac{3}{4}$; there are 3 sectors A, C, and D, other than B.
c. $\frac{2}{4} = \frac{1}{2}$; sectors A and D are two of a total of 4 sectors.

Section 5.1: Probability Basics [*In-Class Exercises*]

1. Decide whether or not each given number could represent a probability.
   a. 0.23

b. $\frac{3}{4}$
c. -0.65
d. 120%
e. $\frac{3}{2}$
f. 48
g. 62%
h. 1.5
i. 0
j. 0.5
k. 1
l. 0.15
m. $\frac{4}{4}$
n. $-\frac{2}{5}$

2. A single six-sided die is rolled. Determine the probability of each of the following outcomes.
   a. a '1'
   b. an even number
   c. a number less than '2'
   d. a number greater than or equal to '5'
   e. a number less than '1'
   f. a number greater than '0'
3. In a class of 30 students, 12 are biology majors. If one student is chosen at random, what is the probability the student is a biology major?
4. A fair coin is tossed. What is the probability of getting tails?
5. A jar contains 5 red marbles, 7 blue marbles, and 8 green marbles.
   a. What is the probability of selecting a blue marble?
   b. What is the probability of selecting a green marble?
   c. What is the probability of selecting a red or a green marble?
   d. What is the probability of selecting a purple marble?
   e. What is the probability of selecting a red or a blue marble?
6. A spinner has sectors labeled A, B, C, D, and E with equal probability. The spinner is spun.
   a. What is the probability that the spinner lands on B?
   b. What is the probability that the spinner does not land on D?
   c. What is the probability that the spinner lands on A or E?
7. A study finds that 1 out of every 20 phone batteries fails within the first year. If you choose a phone at random, what is the probability its battery will not fail in the first year?

Answers available at the end of this section.

Answers

Section 5.1: Probability Basics [*In-Class Exercises*]

1. a. Yes
   b. Yes
   c. No
   d. No
   e. No
   f. No
   g. Yes
   h. No
   i. Yes
   j. Yes
   k. Yes
   l. Yes
   m. Yes
   n. No
2. a. $\frac{1}{6}$
   b. $\frac{3}{6} = \frac{1}{2}$
   c. $\frac{1}{6}$
   d. $\frac{2}{6} = \frac{1}{3}$
   e. $\frac{0}{6} = 0$

   f. $\frac{6}{6} = 1$
3. $\frac{12}{30} = \frac{2}{5}$
4. $\frac{1}{2}$
5. a. $\frac{7}{20}$
   b. $\frac{8}{20} = \frac{2}{5}$
   c. $\frac{13}{20}$
   d. $\frac{0}{20} = 0$
   e. $\frac{12}{20} = \frac{3}{5}$.
6. a. $\frac{1}{5}$
   b. $\frac{4}{5}$
   c. $\frac{2}{5}$
7. $\frac{19}{20}$

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Section 5.1.H: Homework

Section 5.1: Probability Basics - Homework Exercises

(Section 5.1) - Homework Exercise #1

Decide whether or not each given number could represent a probability.

a. -0.24
b. 3
c. 0
d. 49%
e. $\frac{3}{7}$
f. $\frac{7}{3}$
g. 22.3%
h. 0.73
i. $-\frac{2}{5}$
j. 125%
k. 1
l. $\frac{6}{6}$
m. 1.41
n. 0%
o. -30
p. -13.4%
q. 100%

Answer available at the end of this section.

(Section 5.1) - Homework Exercise #2

A single six-sided die is rolled. Determine the probability of each of the following outcomes.

a. a '5'
b. an odd number greater than 1
c. a number less than '5'
d. a number greater than or equal to '4'
e. a number greater than '6'
f. a number less than '8'

Answer available at the end of this section.

(Section 5.1) - Homework Exercise #3

The cooperative storytelling game, Dungeons & Dragons, that was widely popular in the 80's, contains a set of seven different polyhedral dice, one of them being the crucial D20. The D20 is a 20-sided die called an icosahedron and has outcomes from 1 through 20, inclusive. Each outcome is equally likely to occur. Suppose a D20 is rolled. Determine the probability of each of the following outcomes.

a. an '11'

b. an even number
c. a number greater than '15'
d. a number less than or equal to '8'
e. a number greater than '1'
f. a two-digit number

Answer available at the end of this section.

(Section 5.1) - Homework Exercise #4

A class has 32 students, 7 of whom are freshmen, 11 of whom are sophomores, 14 of whom are juniors, and no seniors. One student is chosen at random.

a. What is the probability the student is a freshman?
b. What is the probability the student is a sophomore?
c. What is the probability the student is a junior?
d. What is the probability the student is a senior?
e. What is the probability the student is not a freshman?

Answer available at the end of this section.

(Section 5.1) - Homework Exercise #5

Miscarriage is the most common complication of pregnancy in the United States, occurring in 15–20% of clinically-recognized pregnancies, or 750,000–1,000,000 cases annually, according to the NIH. If a women is pregnant, what is the probability range the pregnancy results in a miscarriage?

Answer available at the end of this section.

(Section 5.1) - Homework Exercise #6

In 2013, Candy Land was redesigned to have a spinner instead of cards, where the spinner included all outcomes that were previously on the cards. This change only lasted one year before the game reverted back to cards. The spinner had the following 12 colored sectors: single blue, double blue, single green, double green, single orange, double orange, single purple, double purple, single red, double red, single yellow, and double yellow, each with equal probability. (There are also little sectors of different candies on the spinner that we are ignoring, so we are focusing on color spaces only). The spinner is spun.

a. What is the probability that the spinner lands on a single green?
b. What is the probability that the spinner lands on a double red?
c. What is the probability that the spinner lands on a single color?
d. What is the probability that the spinner lands on double color?
e. What is the probability that the spinner lands on a blue color?
f. What is the probability that the spinner lands on an orange or purple color?
g. What is the probability that the spinner lands on a single yellow or a double blue?
h. What is the probability that the spinner does not land on a purple color?
i. What is the probability that the spinner lands on a pink color?

Answer available at the end of this section.

(Section 5.1) - Homework Exercise #7

A jar contains 12 white marbles, 13 black marbles, and 9 pink marbles. A marble is drawn at random.

a. What is the probability of selecting a black marble?
b. What is the probability of not selecting a pink marble?
c. What is the probability of selecting a blue marble?
d. What is the probability of selecting a white marble?
e. What is the probability of selecting a black or a pink marble?

Answer available at the end of this section.

Answers

(Section 5.1) - Homework Exercise #1

a. No
b. No
c. Yes
d. Yes
e. Yes
f. No
g. Yes
h. Yes
i. No
j. No
k. Yes
l. Yes
m. No
n. Yes
o. No
p. No
q. Yes

(Section 5.1) - Homework Exercise #2

a. $\frac{1}{6}$
b. $\frac{2}{6} = \frac{1}{3}$
c. $\frac{4}{6} = \frac{2}{3}$
d. $\frac{3}{6} = \frac{1}{2}$
e. $\frac{0}{6} = 0$
f. $\frac{6}{6} = 1$

(Section 5.1) - Homework Exercise #3

a. $\frac{1}{20}$
b. $\frac{10}{20} = \frac{1}{2}$
c. $\frac{5}{20} = \frac{1}{4}$
d. $\frac{8}{20} = \frac{2}{5}$
e. $\frac{19}{20}$
f. $\frac{11}{20}$

(Section 5.1) - Homework Exercise #4

a. $\frac{7}{32}$
b. $\frac{11}{32}$
c. $\frac{14}{32} = \frac{7}{16}$
d. $\frac{0}{32} = 0$
e. $\frac{25}{32}$

(Section 5.1) - Homework Exercise #5

0.15 to 0.20 or 15% to 20%

(Section 5.1) - Homework Exercise #6

a. $\frac{1}{12}$
b. $\frac{1}{12}$
c. $\frac{6}{12} = \frac{1}{2}$
d. $\frac{6}{12} = \frac{1}{2}$
e. $\frac{2}{12} = \frac{1}{6}$
f. $\frac{4}{12} = \frac{1}{3}$
g. $\frac{2}{12} = \frac{1}{6}$
h. $\frac{10}{12} = \frac{5}{6}$
i. $\frac{0}{12} = 0$

(Section 5.1) - Homework Exercise #7

a. $\frac{13}{34}$
b. $\frac{25}{34}$
c. $\frac{0}{34} = 0$

d. $\frac{12}{34} = \frac{6}{17}$
e. $\frac{22}{34} = \frac{11}{17}$

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Section 5.2: The Fundamental Counting Principle

Learning Objectives

- Use the fundamental counting principle

The **Fundamental Counting Principle** (also called the **Multiplication Principle**) is a counting method that helps us determine the total number of ways multiple events can occur.

To illustrate this principle, suppose you want to buy an 1-scoop ice cream cone on a hot summer day. The ice cream shop offers 3 flavors (Vanilla, Chocolate, and Strawberry), and 2 cone types (Waffle cone and Sugar cone). You would like to know how many different 1-scoop ice cream cone combinations can be made.

Here is what we know:

- 3 flavors: Vanilla, Chocolate, and Strawberry
- 2 cone types: Waffle cone and Sugar cone
- Two independent events: choosing flavor + choosing cone type

Thus, each of the 3 flavors can be paired with any of the 2 cone types as follows:

- Vanilla gives you: 2 combinations (Vanilla + Waffle cone; Vanilla + Sugar cone)
- Chocolate gives you: 2 combinations (Chocolate + Waffle cone; Chocolate + Sugar cone)
- Strawberry gives you: 2 combinations (Strawberry + Waffle cone; Strawberry + Sugar cone)

Therefore, there are $2 + 2 + 2 = 6$ combinations. This is exactly the same as $3 \cdot 2 = 6$. Extending this idea leads us to the Fundamental Counting Principle.

Definition: The Fundamental Counting Principle

If one event can occur in m ways and a second event can occur in n ways, then both events together can occur in $m \cdot n$ ways.

This rule can extend to any number of events:

If there are k events that can occur in $n_1, n_2, n_3, \ldots, n_k$ ways respectively, then all events together can occur in $n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_k$ ways.

Note: In the 1-scoop ice cream cone example, it is possible to generate the answer of 6 different ways through several different methods in addition to the Fundamental Counting Principle. We could have used a systematic table listing all flavor-cone pairs, or drawn a tree diagram showing all possible branches from each choice. Tree diagrams will be discussed later in Section 5.4.

However, as we consider problems with more events and more possibilities, these alternative methods become increasingly cumbersome and time-consuming. For instance, imagine if the ice cream shop offered 10 flavors, 4 cone types, 5 toppings, and 3 sizes—that would result in $10 \cdot 4 \cdot 5 \cdot 3 = 600$ combinations! Creating a table with 600 entries or drawing a tree diagram with 600 branches would be impractical and prone to errors.

The Fundamental Counting Principle provides a quicker, easier, and more efficient method that scales beautifully with complexity. Rather than laboriously listing every possibility, we can simply identify the number of choices for each independent event and

multiply them together. This makes the Fundamental Counting Principle not just a mathematical concept, but an essential problem-solving tool for tackling real-world counting scenarios efficiently and accurately.

Example #5.2.1 🤔

You are dressing for a job interview. You open your closet and see 4 dress shirts and 3 dress pants. How many different outfits can you make?

(*Note: Assume that all shirts are distinct from one another and all pants are distinct from one another, unless otherwise noted. Apply this assumption for all similar problems in this section.*)

✅ **Solution:**

Using the Fundamental Counting Principle, there are 4 different shirts and 3 different pants.

Therefore, there are $4 \cdot 3 = 12$ different outfits that can be made.

Example #5.2.2 🤔

A new restaurant has opened with a menu offering a choice of 3 salads, 8 main dishes, and 7 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

✅ **Solution:**

Using the Fundamental Counting Principle, there are 3 salads, 8 main dishes, and 7 desserts.

Therefore, there are $3 \cdot 8 \cdot 7 = 168$ different meals that can be made.

Example #5.2.3 🤔

A particular video game allows users to create their own avatar (character). There are 8 skin-tone colors, 12 hair styles, 6 eye colors, 15 outfit options, and 5 accessories. How many different avatars can be created if exactly one option from each category must be selected?

✅ **Solution:**

Using the Fundamental Counting Principle, there are 8 skin-tone colors choices, 12 hair style choices, 6 eye color choices 15 outfit option choices, 5 accessory choices.

Therefore, there are $8 \cdot 12 \cdot 6 \cdot 15 \cdot 5 = 43{,}200$ different video game characters that can be made.

The three examples presented above represent basic applications of the Fundamental Counting Principle where all choices are independent and unrestricted. As we progress to more advanced applications, we encounter problems that incorporate constraints, limitations, and specific selection criteria.

These enhanced scenarios better reflect real-world situations where complete freedom of choice is rare. In the problems that follow, you will learn to navigate restrictions such as prohibited first digits, mandatory selections from distinct categories, and varying numbers of available options. The fundamental approach remains unchanged:

- **Step 1:** Determine how many events need to occur
- **Step 2:** Determine the number of possibilities per event
- **Step 3:** Use the Fundamental Counting Principle (multiply through) to determine the total number of outcomes

If you are having trouble with Step 1, it may help to draw an empty slot for each event, as shown in Example #5.2.4 below.

Example #5.2.4 🤔

From 1980 to present, California has used a standard 7-character license plate format for



passenger vehicles: N-LLL-NNN (1 number, 3 letters, 3 numbers). An example is shown below to the right. Assuming no restrictions apply and characters may be repeated, how many different license plates are possible?
Note: The digits 0–9 are available for the number positions (10 choices each) and the letters A–Z are available for the letter positions (26 choices).

✅ **Solution:**

So, the best strategy here is to start out with a blank license plate. We will have seven empty "slots".

For the first slot, we need a number. There are 10 different digits to choose from, so put 10 in the first slot.

$$\underline{10}$$

For the next three slots, we need a letter in each slot. There are 26 different numbers to choose from, so put 26 in each of the next three slots.

$$\underline{10}\ \underline{26}\ \underline{26}\ \underline{26}$$

For the last three slots, we need a number in each slot. There are 10 different digits to choose from, so put 10 in each of the last three slots.

$$\underline{10}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{10}\ \underline{10}\ \underline{10}$$

Using the Fundamental Counting Principle, we now multiply all those choices in each of their respective slots.

Therefore, there are $10 \cdot 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$ license plates that can be made.

Example #5.2.5 🤔

You received a new ATM card in the mail. You need to select a passcode consisting of exactly 4 digits, where each digit can be chosen from {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. Assuming that digits can be repeated (for example, 1111 or 2002 are valid passcodes), how many different passcodes are possible?

✅ **Solution:**

Using the Fundamental Counting Principle, there are 10 different choices of digits for the first slot, 10 different digits for the second slot, 10 different digits for the third slot, and 10 different digits for the fourth slot.

Therefore, there are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ passcodes that can be made.

Example #5.2.6 🤔

You received a new ATM card in the mail. You need to select a passcode consisting of exactly 4 digits, where each digit can be chosen from {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. Assuming that digits cannot be repeated (for example, 1234 or 5792 are valid passcodes, while 1123 or 3455 are not valid passcodes), how many different passcodes are possible?

✅ **Solution:**

Using the Fundamental Counting Principle, there are 10 different digits for the first slot, 9 different digits for the second slot, 8 different digits for the third slot, and 7 different digits for the fourth slot.

Therefore, there are $10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$ passcodes that can be made.

Example #5.2.7 🤔

A phone number consists of 7 digits, where each digit can be chosen from {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. However, the first digit cannot be 0 or 1. If repetition of digits is allowed, how many different 7-digit phone numbers are possible?

✅ **Solution:**

Using the Fundamental Counting Principle, there are 8 different digits for the first slot and 10 different digits for each of the second through seventh slots.

Therefore, there are $8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000$ seven-digit phone numbers that can be made.

Example #5.2.8 🤔

A test is divided into two parts. The first part has 6 multiple-choice questions where each question can be answered as A, B, C, or D. The second part has 4 True/False questions, where each question can be answered as either True or False. How many different answer keys could be created for this entire test?

✅ **Solution:**

Using the Fundamental Counting Principle, there are 4 choices for each of the 6 multiple-choice questions and 2 choices for each of the 4 true/false questions.

Therefore, there are $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 4^6 \cdot 2^4 = 4{,}096 \cdot 16 = 65{,}536$ different ways to make an answer key.

Section 5.2: The Fundamental Counting Principle [*In-Class Exercises*]

1. A laptop can be configured with one of each of the following to choose from: 3 processor types, 2 RAM sizes, 3 storage sizes, 2 graphics options, and 4 color options. How many different laptops can be made?
2. When ordering online from Yogurtland, they usually offer 12 different flavors of yogurt and 24 different toppings. How many different orders could you make from one flavor and one topping?
3. A quiz has 6 true/false questions. How many different ways can a student answer all 6 questions?
4. Arizona's standard passenger license plates feature a six-character "alphabet soup" format. The six characters can be any number 0-9 or any letter from A-Z, with the exceptions of I, O, Q, and U to avoid confusion. How many different six-character license plates can be made?

5. A social security number contains 9 digits using any digit from 0-9 inclusive. How many different social security numbers can be made?
6. An ATM at a particular bank requires a 6-digit numeric pin. How many 6-digit numeric pins can be made?
7. An ATM at a particular bank requires a 6-digit numeric pin. How many 6-digit numeric pins can be made, if no digits can be repeated?
8. At BJ's Restaurant & Brewhouse, you can customize your pizza with the following toppings.
    - **Meat**: Cupping Pepperoni, Italian Sausage, Smoked Bacon, Chicken, Smoked Ham, Housemade Meatballs, & Anchovies
    - **Veggie**: Mushrooms, Black Olives, Jalapenos, Pineapple, Green Bell Peppers, Fresh Basil, Roasted Garlic, Fire Roasted Red Peppers, Onions, Red Onions, & Green Onions

If a pizza was ordered with a choice of one meat topping and one veggie topping, how many different ways can you order a pizza?

9. As of 2026, in the NFL, there have been 60 Super Bowls that have been played. One of 16 NFC (National Football Conference) teams always plays one of the 16 AFC (American Football Conference) teams in the Super Bowls. There have been several NFL team pairings that have faced each other in the Super Bowl more than once. In fact, there have been nine different pairings that have occurred more that once. They are:
    - Dallas Cowboys vs. Pittsburgh Steelers in Super Bowls 10, 13, & 30
    - Miami Dolphins vs. Washington Redskins in Super Bowls 7 & 17
    - Cincinnati Bengals vs. San Francisco 49ers in Super Bowls 16 & 23
    - Buffalo Bills vs. Dallas Cowboys in Super Bowls 27 and 28
    - New England Patriots vs. New York Giants in Super Bowls 42 and 46
    - New England Patriots vs. Philadelphia Eagles in Super Bowls 39 & 52
    - New England Patriots vs. St. Louis/Los Angeles Rams in Super Bowls 36 and 53
    - Kansas City Chiefs vs. San Francisco 49ers in Super Bowls 54 and 58
    - New England Patriots vs. Seattle Seahawks in Super Bowls 49 and 60

How many different pairings of an NFC team and an AFC team are possible?

10. California State Department of Motor Vehicles offers personalized “California Museums Snoopy” license plates and each character can be a digit from 0–9 or a letter A–Z. How many different 6-character license plates are possible if there are no restrictions?



Answers available at the end of this section.

Answers

Section 5.2: The Fundamental Counting Principle [*In-Class Exercises*]

1. $3 \cdot 2 \cdot 3 \cdot 2 \cdot 4 = 144$ different laptops
2. $12 \cdot 24 = 288$ different orders
3. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$ different ways
4. $32 \cdot 32 \cdot 32 \cdot 32 \cdot 32 \cdot 32 = 32^6 = 1,073,741,824$ different license plates
5. $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^9 = 1,000,000,000$ different social security numbers
6. $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^6 = 1,000,000$ different pins
7. $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151,200$ different pins
8. $7 \cdot 11 = 77$ different pizzas
9. $16 \cdot 16 = 256$ different pairings
10. $36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 36^6 = 2{,}176{,}782{,}336$ different license plates

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Section 5.2.H: Homework

Section 5.2: The Fundamental Counting Principle - Homework Exercises

(Section 5.2) - Homework Exercise #1

A restaurant offers a dinner special where you choose one appetizer from 4 options, one main course from 6 options, and one dessert from 3 options. How many different dinner combinations are possible?

Answer available at the end of this section.

(Section 5.2) - Homework Exercise #2

Maria is getting dressed for school. She has 5 different shirts, 4 different pairs of pants, 3 different pairs of shoes. How many different outfits can she create?

Answer available at the end of this section.

(Section 5.2) - Homework Exercise #3

A state issues license plates with the format: 3 letters followed by 3 digits. Letters can be any of the 26 letters in the alphabet. Digits can be any number from 0 to 9 inclusive and repetition is allowed. How many different license plates are possible?

Answer available at the end of this section.

(Section 5.2) - Homework Exercise #4

A quiz has 8 true/false questions. How many different ways can a student answer all 8 questions?

Answer available at the end of this section.

(Section 5.2) - Homework Exercise #5

The Baskin-Robbins "31 Flavors" concept represents a different ice cream flavor for each day of the month, a slogan originally created by founders Burt Baskin and Irv Robbins. While the slogan is still used, the company now offers more than 1,300 flavors in its library. Let's say a certain Baskin-Robbins shop offers 31 different flavors of ice cream. Three friends each order a single scoop of ice cream.

a. How many different orders are possible?
b. How many different orders are possible, if each friend ordered a different flavor?

Answer available at the end of this section.

(Section 5.2) - Homework Exercise #6

A local phone number has 10 digits, including the area code. The first and fourth digits cannot be a 0 or 1, but all other digits can be any number from 0 to 9. How many different phone numbers are possible?

Answer available at the end of this section.

(Section 5.2) - Homework Exercise #7

The restaurant, Bagel Me, in Anaheim Hills, California offers the following bagel and cream cheese selections:

- **Bagel Selections**: garlic, onion, cheese, cheese onion, jalapeño, cheese jalapeño, cheese everything, everything, sesame seed, glazer, multigrain, salt, egg, poppyseed, plain, blueberry, very berry, cinnamon raisin, energy bar, pumpkin (seasonal), chocolate chip.
- **Cream Cheese Selections**: plain, veggie, chive, strawberry, jalapeño onion, jalapeño cheddar, lox spread, honey-almond.

If a customer ordered any one bagel with one type of cream cheese, how many bagels with cream cheese can be made?

Answer available at the end of this section.

(Section 5.2) - Homework Exercise #8

A particular bank account password requires an 8-digit password. It requires at least one lowercase letter, one uppercase letter, one numeric digit, and one symbol. How many 8-digit passwords can be made, if you decided to use exactly any two lowercase letters for the first two digits, any two uppercase letters for the next two digits, any two numeric digits for the next two digits, and any two of these following seven symbols, (!, @, #, $, %, &, ?), for the last two digits?

Answer available at the end of this section.

(Section 5.2) - Homework Exercise #9

You're creating a 4-character code with these rules:

- Position 1: Must be a vowel (A, E, I, O, U)
- Position 2: Must be a consonant (21 consonants)
- Position 3: Must be an even digit (0, 2, 4, 6, 8)
- Position 4: Must be an odd digit (1, 3, 5, 7, 9)

How many codes are possible?

Answer available at the end of this section.

Answers

(Section 5.2) - Homework Exercise #1

$4 \cdot 6 \cdot 3 = 72$ different dinners

(Section 5.2) - Homework Exercise #2

$5 \cdot 4 \cdot 3 = 60$ different outfits

(Section 5.2) - Homework Exercise #3

$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ license plates

(Section 5.2) - Homework Exercise #4

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8 = 256$ different ways

(Section 5.2) - Homework Exercise #5

a. $31 \cdot 31 \cdot 31 = 31^3 = 29{,}791$ different ways
b. $31 \cdot 30 \cdot 29 = 26{,}970$ different ways

(Section 5.2) - Homework Exercise #6

$8 \cdot 10 \cdot 10 \cdot 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot = 8^2 \cdot 10^8 = 6{,}400{,}000{,}000$ different phone numbers

(Section 5.2) - Homework Exercise #7

$21 \cdot 8 = 168$ different bagels

(Section 5.2) - Homework Exercise #8

$26 \cdot 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 7 \cdot 7 = 26^4 \cdot 10^2 \cdot 7^2 = 2{,}239{,}182{,}400$ passwords

(Section 5.2) - Homework Exercise #9

$5 \cdot 21 \cdot 5 \cdot 5 = 2{,}625$ codes

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Section 5.3: Permutations

Learning Objectives

- Calculate the value of factorial expressions
- Find the number of permutations of *n* objects
- Find the number of permutations of *n* objects taken *r* at a time
- Find the number of permutations when some objects are alike

Previously, we studied the Fundamental Counting Principle, which allows us to find the total number of outcomes of multiple independent events by multiplying the choices at each step. For example, 3 shirt choices and 2 pants choices give us 3 · 2 = 6 possible outfits. In this example, the order of the choices is not significant.

Now we explore situations where *order matters*, like arranging students in a line or letters in a word. A **permutation** is an arrangement of objects where changing the order creates a different result. In this section, we will study three ways of counting permutations, each with a distinct formula or method.Prior to that, we need to understand what is called factorial notation.

Introduction to Factorial Notation

When solving permutation problems, you'll notice a specific multiplication pattern that appears repeatedly. Let's examine this pattern by looking in detail at our previous example of the Fundamental Counting Principle.

Example: How many ways can you arrange the letters A, B, C, D?

Solution:

- 1st position: We can choose any of the 4 letters (A, B, C, or D) → 4 choices
- 2nd position: After placing one letter, we have 3 remaining letters to choose from → 3 choices
- 3rd position: After placing two letters, we have 2 remaining letters to choose from → 2 choices
- 4th position: After placing three letters, only 1 letter remains → 1 choice

By applying the Fundamental Counting Principle and multiplying the number of available choices at each step, we find that the total number of possible arrangements equals 4 · 3 · 2 · 1 = 24 different ways.

You can see that this calculation follows a distinct multiplication pattern where we start with the total number of objects (4) and multiply by each consecutive smaller integer until we reach 1. Because this pattern of multiplying a number by every positive integer smaller than itself down to 1 occurs so frequently in mathematics, it has been given a special name and symbol. This pattern is called "factorial" and is written with an exclamation point (!). Below is the definition of the factorial notation.

Definition: Permutation Type #1 - Factorial Notation

Factorial notation $n!$ represents the product of all positive integers from n down to 1, expressed by the formula:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

- "$n!$" is pronounced "n factorial."
- So, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$, and is read: "five factorial."
- Also, we define $0! = 1$.

Example #5.3.1 🤔

How many ways can 8 different books be arranged on a shelf?

✅ **Solution:**

Note that the order of each arrangement is significant. So, *ORDER MATTERS* here. Also, since, **all** books are to be arranged, we call this a "basic" or "full" permutation. Here we use the formula $n!$ where $n = 8$. Thus, there are $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$ different book arrangements.

Example #5.3.2 🤔

You have 10 favorite songs on your Spotify playlist. How many ways can you play all 10 songs in order?

✅ **Solution:**

Note that the order of each arrangement is significant. So, *ORDER MATTERS* here. Also, since, **all** songs are to be arranged, we call this a "basic" or "full" permutation. Here we use the formula $n!$ where $n = 10$. Thus, there are $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}628{,}800$ different song orders.

Example #5.3.3 🤔

How many different ways can 6 different colored pencils be arranged in a pencil case?

✅ **Solution:**

Note that the order of each arrangement is significant. So, *ORDER MATTERS* here. Also, since, **all** pencils are to be arranged, we call this a "basic" or "full" permutation. Here we use the formula $n!$ where $n = 6$. Thus, there are $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ different colored pencil arrangements.

Example #5.3.4 🤔

How many distinct arrangements of the letters in the word "FLORIDA" are possible?

✅ **Solution:**

Note that the order of each arrangement is significant. So, *ORDER MATTERS* here and notice that all 7 letters are different. Also, since, **all** letters are to be arranged, we call this a "basic" or "full" permutation. Here we use the formula $n!$ where $n = 7$. Thus, there are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$ different 7-letter distinct arrangements.

Examples #5.3.1 thru #5.3.4 each represent a "basic" or "full" permutation, where all the objects are distinct and each is being arranged in a specific order. Here is a quick summary with the examples and the formula.

Basic or Full Permutations

- An ordered arrangement of ***ALL*** distinct objects.
- Example: How many ways can you arrange 5 books on a shelf?
- Example: How many 5-letter words can be made from the letters in the word TEXAS?
- Uses the formula: $n!$

Now, what if we wanted to only arrange a few of the distinct objects? Let's look at the following example:

Example: How many ways can you choose a president and a vice-president from a group of 4 different people? Assume that no candidate can fill both offices.

Solution: So, in this case, we are not using all four individuals, since we are only choosing two from the group of four. To better understand this further, let's give the following names for the group: Amy, Bill, Cassidy, and Daphne. Without loss of generality, assume we choose the president first. There are four candidates for president, and three remaining for vice president, for a total of $4 \cdot 3 = 12$ different ways the two offices may be filled.

Distinct Arrangements of Amy, Bill, Cassidy, & Daphne

Amy & **B**ill	**B**ill & **A**my	**C**assidy & **A**my	**D**aphne & **A**
Amy & **C**assidy	**B**ill & **C**assidy	**C**assidy & **B**ill	**D**aphne & **B**
Amy & **D**aphne	**B**ill & **D**aphne	**C**assidy & **D**aphne	**D**aphne & **C**a

AB	**BA**	**CA**	**DA**
AC	**BC**	**CB**	**DB**
AD	**BD**	**CD**	**DC**

for simplicity,

Since, it is implied that both **AB** and **BA** are two completely different choices, even though they are both the same set of people, this means that the *ORDER MATTERS* here. (In the next section, we will focus on situations where order does not matter.)

So, what if the above problem had either more people, more positions, or both? Rather than counting all possible permutations individually, we may instead use the following formula. This formula gives the number of **"partial"** permutations of *r* objects chosen from *n* objects.

Definition: Permutation Type #2 - Partial Permutation

The number of distinct arrangements of r distinct objects taken from a larger set of n objects, where $r < n$, is denoted by ${}_nP_r$ and is given by the formula:

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Example #5.3.5 🤔

Referring to the example above, how many ways can you choose a president and a vice-president from a group of 4 different people?

✅ **Solution:**

So, *ORDER MATTERS* here, because the titles president and vice-president are different positions. Also, since, only 2 of the 4 people are to be selected, we call this a "partial" permutation.

Thus, we will use the "partial" permutation formula

$$ {}_nP_r = \frac{n!}{(n-r)!}, $$

where $n = 4$ and $r = 2$.

$$ {}_4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{24}{2} = 12 $$

Thus, there are 12 different ways you can choose a president and a vice-president.

✅ **Alternative Solution:**

Here are some other ways to apply and calculate using the same formula from above:

$$ {}_4P_2 = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 4 \cdot 3 = 12 $$

or

$$ {}_4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2!}{2!} = \frac{4 \cdot 3 \cdot \cancel{2!}}{\cancel{2!}} = 4 \cdot 3 = 12 $$

or using a calculator function to get directly to the solution

$$ {}_4P_2 = 12 $$

or lastly, we can use the Fundamental Counting Principle here; we can think of 2 'slots' where we can place any of the 4 people. The first slot would have 4 choices and the second slot would then only have 3 choices to choose from. Applying the Fundamental Counting Principle from the previous section, we have

$$ 4 \cdot 3 = 12 $$

Using the Fundamental Counting Principle is by far the easiest of any of the above methods, except maybe for the direct result from a calculator function. For the purposes of this section, we will use the partial permutation formula wherever applicable, in order to emphasize that we are counting permutations (as opposed to combinations, discussed in Section 5.4.)

Example #5.3.6 🤔

There are 7 different books on a desk. How many ways can you arrange 3 of those 7 books on a shelf?

✅ **Solution:**

So, *ORDER MATTERS* here. Also, since, only 3 out of the 7 books are to be arranged, we call this a 'partial' permutation.

Thus, we will use the "partial" permutation formula

$$ {}_nP_r = \frac{n!}{(n-r)!}, $$

where $n = 7$ and $r = 3$,

$$ {}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{5{,}040}{24} = 210 $$

There are 210 different ways the books can be arranged.

Example #5.3.7 🤔

A committee of 9 people needs to elect a president, a vice-president, a secretary, and a treasurer from among its members. How many ways can this be done?

✅ **Solution:**

So, *ORDER MATTERS* here, because the titles president, vice-president, secretary, and treasurer are different positions. Also, since, only 4 of the 9 people are to be selected, we call this a 'partial' permutation.

Thus, we will use the "partial" permutation formula

$$ {}_nP_r = \frac{n!}{(n-r)!}, $$

where $n = 9$ and $r = 4$,

$$ {}_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{362{,}880}{120} = 3{,}024 $$

There are $3{,}024$ different ways you can choose a president, vice-president, secretary, and a treasurer.

Example #5.3.8 🤔

A photographer wants to arrange 3 people from a group of 8 for a portrait. How many different arrangements from left to right are possible?

✅ **Solution:**

So, *ORDER MATTERS* here. Also, since, only 3 out of the 8 people are to be arranged, we call this a 'partial' permutation.

Thus, we will use the "partial" permutation formula

$$ {}_nP_r = \frac{n!}{(n-r)!}, $$

where $n = 8$ and $r = 3$,

$$ {}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{40{,}320}{120} = 336 $$

There are 336 different ways the group can be arranged for a portrait.

Example #5.3.9 🤔

In a race with 10 runners, how many different ways can 1st, 2nd, and 3rd place be awarded?

✅ **Solution:**

So, *ORDER MATTERS* here. Also, since, only 3 out of the 10 runners are to be arranged, we call this a 'partial' permutation.

Thus, we will use the "partial" permutation formula

$$ {}_nP_r = \frac{n!}{(n-r)!}, $$

where $n = 10$ and $r = 3$,

$$_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{3{,}628{,}800}{5{,}040} = 720$$

There are 720 different ways that 1st, 2nd, and 3rd place can be awarded.

Examples #5.3.5 thru #5.3.9 each represent a 'partial' permutation, where all the objects are distinct and some but not all objects are arranged in a specific order. Here is a quick summary with the examples and the formula.

Partial Permutations

- An ordered arrangement of ***SOME BUT NOT ALL*** distinct objects.*
- Example: How many ways can you arrange 2 from the 5 books on a shelf?
- Example: How many ways can you create a 4-digit passcode with no repetition of digits.
- Uses the formula: $_nP_r = \frac{n!}{(n-r)!}$

* You could use an ordered arrangement of ALL distinct objects here. Since $n = r$, it would revert back to the Full Permutations formula.

Note that, up to now, we have assumed that all the objects we choose from are distinct from each other. Now, let us consider what happens when some of these objects are identical with each other.

For the previous two types of permutations that we have discussed, notice that any type of arrangements that were involved always had distinct objects, items, people, etc. For our last type of permutation, we will be looking at the case where we arrange all objects with some objects that are identical.

Example: How many distinct arrangements are possible using the letters in "OHIO"? Here we are using all 4 letters, however they are not distinct. There are two repeat letters being used. The letter O is used twice, while the letters H and I are only used once. So, let's look at all of the permutations if the last letter was a Z (or the word "OHIZ".

Distinct Arrangements of O, H, I, & Z

OHIZ	OIZH	HIZO	IZOH
ZHIO	ZIOH	HIOZ	IOZH
OHZI	OZHI	HZIO	IZHO
ZHOI	ZOHI	HOIZ	IOHZ
OIHZ	OZIH	HZOI	IHOZ
ZIHO	ZOIH	HOZI	IHZO

Now, let's replace the letter Z back with the letter O.

All Arrangements of O, H, I, & O

OHIO	OIOH	HIOO	IOOH
OHIO	OIOH	HIOO	IOOH
OHOI	OOHI	HOIO	IOHO
OHOI	OOHI	HOIO	IOHO
OIHO	OOIH	HOOI	IHOO
OIHO	OOIH	HOOI	IHOO

As you can see in the table above, some words are exactly the same, because the order in which the two Os occur is not significant. These identical words are not to be counted as distinct permutations. So, writing only the distinct words in the table above, we get

Distinct Arrangements of O, H, I, & O

OHIO	OIOH	HIOO	IOOH
OHOI	OOHI	HOIO	IOHO
OIHO	OOIH	HOOI	IHOO

Thus, there are only 12 different 4-letter distinct arrangements.

What if the word had more letters and multiple sets of letters were identical, for example the word "MISSISSIPPI"? As before, there is a formula that can be used to directly compute the total number of permutations without actually listing them as illustrated above. This formula gives the number of distinct permutations of *n* objects among which some are identical with others; i.e., number of **"full permutation with identical objects"**.

Definition: Permutation Type #3 - Full Permutations with Identical Objects

The number of distinct arrangements of n objects of which k_1 objects are alike, k_2 objects are alike, etc., is given by the formula:

$$\frac{n!}{k_1! \cdot k_2! \cdot k_3! \cdot \ \cdots \ \cdot k_p!}$$

Example #5.3.10 🤔

How many distinct arrangements of the letters in the word "OHIO" are possible?

✅ **Solution:**

So, *ORDER MATTERS* here. Also, since, **all** letters are to be arranged and we have repeat or identical letters, we call this a "full permutation with identical objects". Here we need to do a letter count:

4 letters total: There are 2 **O**'s, 1 **H**, and 1 **I**.

Thus, we will use the "full permutation with identical objects" formula

$$\frac{n!}{k_1! \cdot k_2! \cdot k_3! \cdot \ \cdots \ \cdot k_p!},$$

where $n = 4, k_1 = 2, k_2 = 1, k_3 = 1$

$$\frac{4!}{2! \cdot 1! \cdot 1!} = \frac{24}{2 \cdot 1 \cdot 1} = \frac{24}{2} = 12$$

There are 12 different 4-letter distinct arrangements using all the letters in the word "OHIO".

Example #5.3.11 🤔

How many distinct arrangements of the letters in the word "MISSISSIPPI" are possible?

✅ **Solution:**

So, *ORDER MATTERS* here. Also, since, **all** letters are to be arranged and we have repeat or identical letters, we call this a "full permutation with identical objects". Here we need to do a letter count:

11 letters total: There is 1 **M**, 4 **I**'s, 4 **S**'s, and 2 **P**'s.

Thus, we will use the "full permutation with identical objects" formula

$$\frac{n!}{k_1! \cdot k_2! \cdot k_3! \cdot \ \cdots \ \cdot k_p!},$$

where $n = 11, k_1 = 1, k_2 = 4, k_3 = 4, k_4 = 2$

$$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = \frac{39{,}916{,}800}{1 \cdot 24 \cdot 24 \cdot 2} = \frac{39{,}916{,}800}{1{,}152} = 34{,}650$$

There are 34,650 different 11-letter distinct arrangements using all the letters in the word "MISSISSIPPI".

Example #5.3.12 🤔

How many distinct arrangements of the letters in the word "CALIFORNIA" are possible?

✅ **Solution:**

So, *ORDER MATTERS* here. Also, since, **all** letters are to be arranged and we have repeat or identical letters, we call this a "full permutation with identical objects". Here we need to do a letter count:

10 letters total: There is 1 **C**, 2 **A**'s, 1 **L**, 2 **I**'s, 1 F, 1 **O**, 1 **R**, and 1 **N**'s. (To make our calculations easier, we will ignore any letters that occur only once, since this amounts to deleting the extraneous factors of 1! from the expression).

Thus, we will use the "full permutation with identical objects" formula

$$\frac{n!}{k_1! \cdot k_2! \cdot k_3! \cdot \ \cdots \ \cdot k_p!},$$

where $n = 10, k_1 = 2, k_2 = 2$

$$\frac{10!}{2! \cdot 2!} = \frac{3{,}628{,}800}{2 \cdot 2} = \frac{3{,}628{,}800}{4} = 907{,}200$$

There are 907,200 different 10-letter distinct arrangements using all the letters in the word "CALIFORNIA".

Example #5.3.13 🤔

How many distinct arrangements of the letters in the word "ABIBLIOPHOBIA" are possible?

✅ **Solution:**

So, *ORDER MATTERS* here. Also, since, **all** letters are to be arranged and we have repeat or identical letters, we call this a "full permutation with identical objects". Here we need to do a letter count:

13 letters total: There is 2 **A**'s, 3 **B**'s, 3 **I**'s, 1 L, 2 **O**'s, 1 **P**, and 1 **H**. (To make our calculations easier, we are going to only focus on the duplicate letters, since the single letters will yield a $1!$ in which will not matter in the calculations).

Thus, we will use the "full permutation with identical objects" formula

$$\frac{n!}{k_1! \cdot k_2! \cdot k_3! \cdot \ \cdots \ \cdot k_p!},$$

where $n = 13, k_1 = 2, k_2 = 3, k_3 = 3, k_4 = 2$

$$\frac{13!}{2! \cdot 3! \cdot 3! \cdot 2!} = \frac{6{,}227{,}020{,}800}{2 \cdot 6 \cdot 6 \cdot 2} = \frac{6{,}227{,}020{,}800}{144} = 43{,}243{,}200$$

There are 43,243,200 different 13-letter distinct arrangements using all the letters in the word "ABIBLIOPHOBIA".

Example #5.3.14 🤔

How many ways can you arrange 8 marbles in a row if 3 are red, 3 are blue, and 2 are green?

✅ **Solution:**

Think as the colored marbles as letters; i.e., RRRBBBGG. So, *ORDER MATTERS* here. Also, since, **all** marbles are to be arranged and we have repeat or identical marbles, we call this a "full permutation with identical objects". Here we need to do a marble count:

8 marbles total: There are 3 **reds**, 3 **blues**, and 2 **greens**.

Thus, we will use the "full permutation with identical objects" formula

$$\frac{n!}{k_1! \cdot k_2! \cdot k_3! \cdot \ \cdots \ \cdot k_p!},$$

where $n = 8, k_1 = 3, k_2 = 3, k_3 = 2$

$$\frac{8!}{3! \cdot 3! \cdot 2!} = \frac{40{,}320}{6 \cdot 6 \cdot 2} = \frac{40{,}320}{72} = 560$$

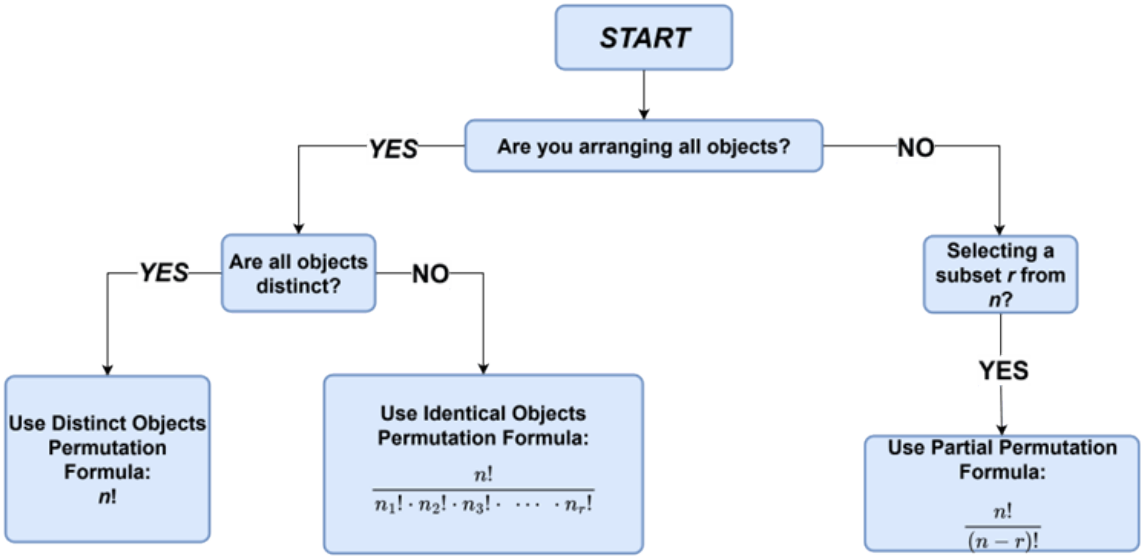
There are 560 different marble arrangements.

Examples #5.3.10 thru #5.3.14 each represent 'full' permutations with 'identical objects'. Here is a quick summary with the examples and the formula.

Full Permutations with Identical Objects

- An ordered arrangement of ***ALL*** objects some of which are identical with each other.
- Example: How many 5-letter words can be made from the letters in the word APPLE?
- Example: How many 7-letter words can be made from the letters in the word BALLOON?
- Example: A parking lot has 7 spaces in a row. If 4 identical red cars and 3 identical blue cars need to be parked, how many different arrangements are possible?
- Uses the formula: $\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \ \cdots \ \cdot n_p!}$

Which Permutation Formula Should You Use? Here is a flow chart to help you distinguish which type of permutation to use.



Section 5.3: Permutations [*In-Class Exercises*]

1. A family of 5 buys Monster Jam tickets. The 5 seats are all in a row. They want to know how many different ways they could sit. How many different seating arrangements are possible?
2. A student has 8 alphabet magnets, all different letters. How many different 8-letter arrangements can they make?
3. A group of 9 parents from a PTA organization needs to elect a President, Vice President, Secretary, and Treasurer. No parent can hold more than one position. How many different ways can these four positions be filled?
4. An escape room requires players to press three colored buttons in a specific order. The buttons are Red, Blue, Green, Yellow, and Purple. No button can be used twice. How many different 3 button sequences can be made?
5. Rita has 13 different ornaments and wants to arrange 4 of them on her mantle. How many ways can she do so?
6. In the 2026 Winter Olympics held in Milano, Italy, there were 12 different men's hockey teams in the tournament. In how many ways can three teams finish with the Gold, Silver, and Bronze medals?
7. Find the number of distinguishable permutations of the letters in the word WYOMING.
8. Find the number of distinguishable permutations of the letters in the word HAWAII.
9. Find the number of distinguishable permutations of the letters in the word SENSELESSNESSES. (Meaning several acts, situations, or qualities of being senseless).

Answers available at the end of this section.

Answers

Section 5.3: Permutations [*In-Class Exercises*]

1. $5! = 120$
2. $8!=40{,}320$
3. ${}_9P_4 = 3{,}024$
4. ${}_5P_3 = 60$
5. ${}_{13}P_4 = 17{,}160$
6. ${}_{12}P_3 = 1{,}320$
7. $7! = 5{,}040$
8. $\frac{6!}{2!2!} = \frac{720}{2 \cdot 2} = \frac{720}{4} = 180$
9. $\frac{15!}{7! \cdot 5! \cdot 2!} = \frac{1{,}307{,}674{,}368{,}000}{5{,}040 \cdot 120 \cdot 2} = \frac{1{,}307{,}674{,}368{,}000}{1{,}209{,}600} = 1{,}081{,}080$

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Section 5.3.H: Homework

Section 5.3: Permutations - Homework Exercises

(Section 5.3) - Homework Exercise #1

You have 5 different books and want to arrange them on a shelf. How many different arrangements are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #2

A baseball team has 9 players, and each player must be assigned a position in the batting lineup. How many different batting orders can be created using the 9 players?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #3

Six runners compete in a race. Assume there are no ties. How many different possible finish orders are there?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #4

A locker requires a 3-number combination where each number is from 1 to 40 and no number can be repeated. How many different combinations are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #5

A class of 25 students needs to elect a President, Vice President, Secretary, and Treasurer. No student can hold more than one position. How many different ways can these four positions be filled?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #6

A baseball team has 15 players, but only 9 can be in the starting lineup. The order in the lineup matters. How many different batting orders are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #7

Eight runners compete in a race. How many different ways can they finish 1st, 2nd, and 3rd place? (Assume no ties)

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #8

How many 5-letter "words" (they don't need to be real words) can be formed from the 26 letters of the alphabet, if no letter can be used more than once?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #9

How many distinct arrangements of the letters in the word "MATH" are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #10

How many distinct arrangements of the letters in the word "UNCOPYRIGHTABLE" are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #11

How many distinct arrangements of the letters in the word "GREEN" are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #12

How many distinct arrangements of the letters in the word "BANANA" are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #13

How many distinct arrangements of the letters in the word "BOOKKEEPER" are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #14

How many distinct arrangements of the letters in the word "POSSESSES" are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #15

The word "SUPERCALIFRAGILISTICEXPIALIDOCIOUS" was popularized by the 1964 Disney film Mary Poppins, where it appears in a famous song of the same name sung by Julie Andrews (Mary Poppins) and Dick Van Dyke (Bert). How many distinct arrangements of the letters in the word "SUPERCALIFRAGILISTICEXPIALIDOCIOUS" are possible?

Answer available at the end of this section.

(Section 5.3) - Homework Exercise #16

Pneumonoultramicroscopicsilicovolcanoconiosis is a lung disease caused by inhaling very fine silica dust. It was actually coined in the 1930s partly as a curiosity and to be an extremely long medical term. Many dictionaries list it as the longest English word appearing in common dictionaries.

How many distinct arrangements of the letters in the 45-letter word "PNEUMONOULTRAMICROSCOPICSILICOVOLCANOCONIOSIS" are possible? Give your answer in calculator ready form.

Answer available at the end of this section.

Answers

(Section 5.3) - Homework Exercise #1
5! = 120 different arrangements

(Section 5.3) - Homework Exercise #2
9! = 362,880 different batting orders

(Section 5.3) - Homework Exercise #3
6! = 720 different finishes

(Section 5.3) - Homework Exercise #4
${}_{40}P_3 =$ 59,280 different ways

(Section 5.3) - Homework Exercise #5
${}_{25}P_4 =$ 303,600 different ways

(Section 5.3) - Homework Exercise #6
${}_{15}P_9 =$ 1,816,214,400 different orders

(Section 5.3) - Homework Exercise #7
${}_{8}P_3 =$ 336 different finishes

(Section 5.3) - Homework Exercise #8
${}_{26}P_5 =$ 7,893,600 different words

(Section 5.3) - Homework Exercise #9
4! = 24 different arrangements

(Section 5.3) - Homework Exercise #10
15! = 1,307,674,368,000 different arrangements

(Section 5.3) - Homework Exercise #11
$\frac{5!}{2!} = \frac{120}{2} =$ 60 different arrangements

(Section 5.3) - Homework Exercise #12
$\frac{6!}{3!\cdot 2!} = \frac{720}{6\cdot 2} = \frac{720}{12} =$ 60 different arrangements

(Section 5.3) - Homework Exercise #13
$\frac{10!}{3!\cdot 2!\cdot 2!} = \frac{3,628,800}{6\cdot 2\cdot 2} = \frac{3,628,800}{24} =$ 151,200 different arrangements

(Section 5.3) - Homework Exercise #14
$\frac{9!}{5!\cdot 2!} = \frac{362,880}{120\cdot 2} = \frac{362,880}{240} =$ 1,512 different arrangements

(Section 5.3) - Homework Exercise #15
$\frac{34!}{7!\cdot (3!)^4\cdot (2!)^5} \approx 1.41 \times 10^{30}$ different arrangements

(Section 5.3) - Homework Exercise #16
$\frac{45!}{9!\cdot (6!)^2\cdot (4!)^2\cdot 3!\cdot (2!)^5}$ different arrangements

(There are 9 O's, 6 C's, 6 I's, 4 N's, 4 S's, 3 L's, 2 A's, 2 M's, 2 P's, 2 R's, 2 U's, 1 E, 1 T, and 1 V).

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Section 5.4: Combinations

Learning Objectives

- Distinguish between a combination and a permutation.
- Find the number of combinations of *n* objects taken *r* at a time.
- Use the combination rule in conjunction with the fundamental counting principle.

In the previous section on permutations, we developed systematic methods for counting arrangements where order matters. We learned to calculate the number of ways to arrange objects in sequences, select and arrange items from larger groups, and solve counting problems involving rankings, orderings, and positions. The key characteristic of these counting problems was that changing the order of objects creates different, distinguishable outcomes, whether we were arranging books on a shelf, determining race finishing orders, or finding the number of ways a president and vice president can be selected from a group of people.

However, when considering counting problems that arise from real-world scenarios, you may encounter situations where the order of the selections does not matter. Consider the difference between these two questions:

- "In how many ways can we select and arrange 3 students from a class of 20 to serve as president, vice-president, and secretary?"
- "In how many ways can we select 3 students from a class of 20 to serve on a committee?"

In the second question, we don't care about the order in which we select the students or how we arrange them, we simply want to know how many different groups of 3 students we can form. If we used the formula for ${}_{20}P_3$, we would be counting each group of 3 students multiple times (once for each possible arrangement), leading to an incorrect and inflated count.

This brings us to an entirely different category of counting problems: situations where **order does not matter**. In these scenarios, we're interested in selecting or choosing objects from a group, but we don't care about the sequence or arrangement of our selections. Whether we choose students A, B, and C or students C, A, and B for our committee, we end up with the same committee. This type of selection problem requires a different mathematical approach than the arrangement problems we solved with permutations. A choice of some objects from a larger group, regardless of order, is known as a "**combination**". While counting permutations answers the question, "How many ways can we arrange?", counting combinations answers the question, "How many ways can we choose?" This distinction is crucial for solving counting problems accurately and, by extension, for calculating probabilities correctly in scenarios involving selections rather than arrangements.

Just as permutations enhanced our ability to solve probability problems involving arrangements, combinations will enable us to tackle probability questions involving selections. Many real-world probability scenarios involve choosing items from groups without regard to order:

- Selecting lottery numbers
- Choosing committee members
- Drawing cards for a poker hand
- Picking items for quality control testing
- Forming teams or groups

By mastering combination techniques, we'll be able to calculate probabilities for these selection-based scenarios accurately and efficiently.

In this section, we will learn to distinguish clearly between situations requiring permutations (where order matters) and those requiring combinations (where order doesn't matter). We'll develop the combination formula, explore its relationship to

permutations, and apply combination counting techniques to solve both counting problems and probability questions.

In the previous section, we saw an example where there were 12 ways to choose a President and a Vice-President from a group of 4 different people named Amy, Bill, Cassidy, and Daphne.

Now, let's pose a different scenario: How many ways can you choose two secretary positions from a group of 4 different people? So, if we use the abbreviations of the previous four different people from before, Amy, Bill, Cassidy, & Daphne, and we look at those 12 choices we obtained from the previous section, then we again have the following:

AB	**BA**	**CA**	**DA**
AC	**BC**	**CB**	**DB**
AD	**BD**	**CD**	**DC**

Observe that both **AB** and **BA** appear in our results. However, since we are selecting two secretaries without assigning specific roles (different from our President and Vice-President example where positions mattered), these represent the same selection. We are double-counting this outcome. Since "*ORDER DOES NOT MATTER*" in combinations, we must remove all such duplicate selections, which reduces our total count to six combinations:

AB	**AC**	**BC**	**AD**
BD	**CD**		

What if the above problem had more people or more of a selection for the secretary position or both? Luckily, there is a formula that can be used to directly compute the total number of permutations without actually listing them as illustrated above.

Definition: Combinations

The number of ways to select r objects from a group of n objects, where the order of selection does not matter, is denoted by ${}_nC_r$ and is given by the formula:

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!} $$

Example #5.4.1 🤔

How many ways can you choose two secretary positions from a group of 4 different people?

✅ **Solution:**

Note that, *ORDER DOES NOT MATTER* here, because the secretary positions do not have ranks.

Thus, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 4$ and $r = 2$.

$$ {}_4C_2 = \frac{4!}{(4-2)! \cdot 2!} = \frac{4!}{2! \cdot 2!} = \frac{24}{2 \cdot 2} = \frac{24}{4} = 6 $$

There are 6 different ways.

✅ **Alternative Solution:**

You can also use a calculator function to get directly to the solution

$$ {}_4C_2 = 6 $$

Example #5.4.2 🤔

In how many ways can a sorority of 20 members select three members to serve on a committee?

✅ **Solution:**

Note that, *ORDER DOES NOT MATTER* here, because the sorority members do not have ranks.

Thus, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 20$ and $r = 3$.

$$ {}_{20}C_3 = \frac{20!}{(20-3)! \cdot 3!} = \frac{20!}{17! \cdot 3!} = \frac{20 \cdot 19 \cdot 18 \cdot 17!}{17! \cdot 6} = \frac{20 \cdot 19 \cdot 18 \cdot \cancel{17!}}{\cancel{17!} \cdot 6} = \frac{20 \cdot 19 \cdot 18}{6} = 1{,}140 $$

There are $1{,}140$ different ways.

Note: Observe that $20!$ was not expanded out entirely in our work. The reason is that $20!$ produces a number too large for standard calculators, which would display the result in scientific notation. Since $20! = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot \cdots \cdot 3 \cdot 2 \cdot 1$, we can rewrite it as $20! = 20 \cdot 19 \cdot 18 \cdot 17!$. We expand only to $17!$ because this will cancel directly with the $17!$ in the denominator. The key strategy is to expand factorials just enough to create terms that will cancel with factorials in the denominator.

✅ **Alternative Solution:**

You can also use a calculator function to get directly to the solution

$$ {}_{20}C_3 = 1{,}140 $$

Example #5.4.3 🤔

A textbook search committee is considering seven books for possible adoption. The committee has decided to select three of the seven for further consideration. In how many ways can they do so?

✅ **Solution:**

Note that, *ORDER DOES NOT MATTER* here, because choosing the three books do not have any ranks to them.

Thus, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 7$ and $r = 3$.

$$ {}_7C_3 = \frac{7!}{(7-3)! \cdot 3!} = \frac{7!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 6} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!} \cdot 6} = \frac{7 \cdot 6 \cdot 5}{6} = 35 $$

There are 35 different ways.

✅ **Alternative Solution:**

You can also use a calculator function to get directly to the solution

$$ {}_7C_3 = 35 $$

Example #5.4.4 🤔

A hand of five-card draw poker consists of an unordered arrangement of five cards from a standard deck of 52 cards. How many five-card draw poker hands are possible?

✅ **Solution:**

Note that, *ORDER DOES NOT MATTER* here. The order in which you receive your cards or arrange them in your hand does not matter because the hand's value is based solely on the combination of cards you hold, not the sequence in which they were dealt.

Thus, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 52$ and $r = 5$.

$$ {}_{52}C_5 = \frac{52!}{(52-5)! \cdot 5!} = \frac{52!}{47! \cdot 5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{47! \cdot 120} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!} \cdot 120} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{120} = \frac{311{,}875{,}200}{120} = 2{,}59 $$

There are $2{,}598{,}960$ different ways.

✅ **Alternative Solution:**

You can also use a calculator function to get directly to the solution

$$ {}_{52}C_5 = 2{,}598{,}960 $$

Example #5.4.5 🤔

In 1988, the California Lottery was initiated. There were 49 lotto balls, each numbered from 1 to 49. In order to play, you must choose 6 numbers. If you pick all 6 winning numbers, you win the jackpot. In how many ways can someone choose 6 numbers from the 49?

✅ **Solution:**

Note that, *ORDER DOES NOT MATTER* here. The order in which you choose your numbers does not matter, because the lottery is based solely on the combination of numbers you choose, not the sequence in which they were chosen.

Thus, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 49$ and $r = 6$.

$$ {}_{49}C_6 = \frac{49!}{(49-6)! \cdot 6!} = \frac{49!}{43! \cdot 6!} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43!}{43! \cdot 720} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot \cancel{43!}}{\cancel{43!} \cdot 720} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{720} = \frac{10,068,}{7} $$

There are $13{,}983{,}816$ different ways.

✅ **Alternative Solution:**

You can also use a calculator function to get directly to the solution

$$ {}_{49}C_6 = 13{,}983{,}816 $$

Example #5.4.6 🤔

Refer to Example #5.4.5. In 1990, the California Lottery changed its main Lotto game from a 6/49 format to a 6/53 format. This change reduced the likelihood of winning the jackpot each week, resulting in fewer winners and allowing jackpots to grow larger in subsequent drawings. The change created 53 lotto balls, each numbered from 1 to 53, choosing 6 numbers for each play. In how many ways can someone choose 6 numbers from the 53?

✅ **Solution:**

Note that, *ORDER DOES NOT MATTER* here. The order in which you choose your numbers does not matter, because the lottery is based solely on the combination of numbers you choose, not the sequence in which they were chosen.

Thus, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 53$ and $r = 6$.

$$ {}_{53}C_6 = \frac{53!}{(53-6)! \cdot 6!} = \frac{53!}{47! \cdot 6!} = \frac{53 \cdot 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{47! \cdot 720} = \frac{53 \cdot 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!} \cdot 720} = \frac{53 \cdot 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{720} = \frac{16,529,}{7} $$

There are $22{,}957{,}480$ different ways.

✅ **Alternative Solution:**

You can also use a calculator function to get directly to the solution

$$ {}_{53}C_6 = 22{,}957{,}480 $$

Example #5.4.7 🤔

How many ways can 4 cars and 5 trucks be selected from 6 cars and 8 trucks to be tested for a safety inspection?

✅ **Solution:**

Note that, *ORDER DOES NOT MATTER* here, because there is no mention of any ranks.

Now, in this situation there are two parts. We need to select 4 cars from a total of 6 and then we need to select 5 trucks from a total of 8 trucks.

For the cars, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 6$ and $r = 4$.

$$ {}_6C_4 = \frac{6!}{(6-4)! \cdot 4!} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5 \cdot 4!}{2 \cdot 4!} = \frac{6 \cdot 5 \cdot \cancel{4!}}{2 \cdot \cancel{4!}} = \frac{6 \cdot 5}{2} = 15 $$

For the trucks, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 8$ and $r = 5$.

$$ {}_8C_5 = \frac{8!}{(8-5)! \cdot 5!} = \frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 6} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!} \cdot 6} = \frac{8 \cdot 7 \cdot 6}{6} = 56 $$

Since we need to select cars AND trucks, we multiply using the Fundamental Counting Principle from Section 5.2:

$$ = {}_6C_4 \cdot {}_8C_5 $$

$$ = 15 \cdot 56 $$

$$ = 840 $$

There are 840 different ways.

✅ **Alternative Solution:**

You can also use a calculator function to get directly to the solution

$$ {}_6C_4 \cdot {}_8C_5 = 840 $$

Example #5.4.8 🤔

There is a group of 6 men and 8 women. How many ways can you select two men and two women from that group?

✅ **Solution:**

Note that, *ORDER DOES NOT MATTER* here, because there is no mention of any ranks.

Now, in this situation there are two parts. We need to select 2 men from a group of 6 men and then we need to select 2 women from a group of 8 women.

For the men, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 6$ and $r = 2$.

$$ {}_6C_2 = \frac{6!}{(6-2)! \cdot 2!} = \frac{6!}{4! \cdot 2!} = \frac{6 \cdot 5 \cdot 4!}{4! \cdot 2} = \frac{6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!} \cdot 2} = \frac{6 \cdot 5}{2} = 15 $$

For the women, we will use the formula for combinations

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!}, $$

where $n = 8$ and $r = 2$.

$$ {}_8C_2 = \frac{8!}{(8-2)! \cdot 2!} = \frac{8!}{6! \cdot 2!} = \frac{8 \cdot 7 \cdot 6!}{6! \cdot 2} = \frac{8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!} \cdot 2} = \frac{8 \cdot 7}{2} = 28 $$

Since we need to select men AND women, we multiply using the Fundamental Counting Principle from Section 5.2:

$$ = {}_6C_2 \cdot {}_8C_2 $$

$$ = 15 \cdot 28 $$

$$ = 420 $$

There are 420 different ways.

✅ **Alternative Solution:**

You can also use a calculator function to get directly to the solution

$$ {}_6C_2 \cdot {}_8C_2 = 420 $$

Section 5.4: Combinations [*In-Class Exercises*]

1. Out of 7 volunteers, a teacher must choose 2 to sit on a discussion panel. How many panels can be formed?
2. A student government has 12 members. They need to choose a 3-person subcommittee. How many different 3-person committees can be formed?
3. An ice cream shop offers 10 toppings, and you want 4 different toppings on a sundae. In how many ways can you choose 4 toppings?
4. A class has 18 students. The teacher randomly chooses 2 students to pair up for an example demonstration. How many possible student pairs are there?
5. A group has 9 movies available and wants to choose 3 to show during a weekend marathon. They don't care about the order they're shown, just which movies are chosen. How many sets of 3 movies can be selected?
6. A counselor wants to randomly choose 5 students from a group of 30 to participate in a survey interview. How many selections of 5 students are possible?
7. A grocery store display has 7 types of fruit and 5 types of vegetables. A shopper wants to pick 2 fruits and 3 vegetables. How many different produce selections are possible?
8. At BJ's Restaurant & Brewhouse, you can customize your pizza with the following toppings.
   - **Meat**: Cupping Pepperoni, Italian Sausage, Smoked Bacon, Chicken, Smoked Ham, Housemade Meatballs, & Anchovies
   - **Veggie**: Mushrooms, Black Olives, Jalapenos, Pineapple, Green Bell Peppers, Fresh Basil, Roasted Garlic, Fire Roasted Red Peppers, Onions, Red Onions, & Green Onions

   If we were to get a pizza with three different meat toppings and three different vegetable toppings, how many different ways can you order a pizza?
9. The mathematics division at a local community college includes 5 calculus instructors, 4 statistics instructors, and 3 trigonometry instructors. The Vice President of Instruction wants to select 2 instructors from each subject area to speak at a high-school recruitment event. How many different groups of instructors can be formed?

10. **Challenge.** In the game Bingo, a BINGO card has 5 rows and 5 columns, thus providing 25 total spaces. The columns

B	I	N	G	O
6	20	33	48	74
15	26	37	53	63
2	17	FREE	56	69
11	28	42	49	61
3	23	38	60	73

are labeled from left to right with the letters: 'B', 'I', 'N', 'G', 'O'. Each column contains 5 distinct numbers from 15 different numbers. With one exception, the center space in the 'N' column is a "free" space, so the "N" column will only contain 4 distinct numbers. So, each space in the 'B' column contains a number from 1 to 15, each space in the 'I' column contains a number from 16 to 30, each space in the 'N' column contains a number from 31 to 45, each space in the 'G' column contains a number from 46 to 60, and each space in the 'O' column contains a number from 61 to 75. How many different BINGO cards can be made under these conditions.

Answers available at the end of this section.

Answers

Section 5.4: Combinations [*In-Class Exercises*]

1. ${}_7C_2 = 21$
2. ${}_{12}C_3 = 220$
3. ${}_{10}C_4 = 210$
4. ${}_{18}C_2 = 153$
5. ${}_9C_3 = 84$
6. ${}_{30}C_5 = 142{,}506$
7. ${}_7C_2 \cdot {}_5C_3 = 210$
8. ${}_7C_3 \cdot {}_{11}C_3 = 5{,}775$
9. ${}_5C_2 \cdot {}_4C_2 \cdot {}_3C_2 = 180$
10. $(5! \cdot ({}_{15}C_5))^4 \cdot (4! \cdot ({}_{15}C_4)) = 552{,}446{,}474{,}061{,}128{,}648{,}601{,}600{,}000 \approx 5.52 \times 10^{26}$.

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Section 5.4.H: Homework

Section 5.4: Combinations - Homework Exercises

(Section 5.4) - Homework Exercise #1

A club has 8 members. They need to select 3 members to form a planning committee. How many different committees are possible?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #2

A pizza shop offers 10 different toppings. You want to order a pizza with exactly 4 toppings. How many different 4-topping pizzas can you create?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #3

A basketball team has 12 players. The coach needs to choose 2 players to be team captains. How many different pairs of captains are possible?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #4

You have 15 books on your shelf and are going on vacation. You can only pack 5 books. How many different sets of 5 books could you choose?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #5

A company has 8 men and 7 women. They need to form a 5-person committee with exactly 3 men and 2 women. How many different committees are possible?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #6

An ice cream shop has 8 fruit flavors and 6 chocolate-based flavors. You want to choose 3 scoops with 2 fruit flavors and 1 chocolate-based flavor. How many combinations are possible?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #7

A debate team has 10 members, including their captain Sarah. They need to choose 4 people to compete in a tournament, and Sarah must be one of them. How many different 4-person teams are possible?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #8

A jury pool consists of 18 people: 10 women and 8 men. How many juries can be formed with 7 women and 5 men?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #9

A school has 25 freshmen, 30 sophomores, 28 juniors, and 22 seniors. They need to select a 6-person student council with at least one student from each class. How many different councils are possible if they select exactly 2 freshmen, 1 sophomore, 2 juniors, and 1 senior?

Answer available at the end of this section.

(Section 5.4) - Homework Exercise #10

Powerball is a two-drum lottery game, meaning winning numbers are drawn from two separate machines. Powerball is an American lottery game that is played in 45 States, Washington D.C., Puerto Rico, and the US Virgin Islands. (As of 2026, not played in Nevada, Utah, Alabama, Alaska, and Hawaii). The current version of Powerball is as follows: It costs $2.00 to play. You are to pick five numbers numbered from 1–69 and one number ("The Powerball") numbered 1–26. Every Wednesday and Saturday night at 10:59 p.m. Eastern Time, the five white balls out of a drum with 69 balls and the one red "Powerball" out of a drum with 26 red balls will be selected and revealed. The selection process is random and each of the 6 balls chosen has the same likelihood of being drawn. (On November 7, 2022, Powerball produced the largest lottery jackpot in history; the $2.04 billion jackpot was won by someone in Altadena, California.)

Here are the payouts along with the odds of winning based on the winning combinations:

Match	Prize	Odds
	Grand Prize	1 in 292,201,338.00
	$1,000,000	1 in 11,688,053.52
	$50,000	1 in 913,129.18
	$100	1 in 36,525.17
	$100	1 in 14,494.11
	$7	1 in 579.76
	$7	1 in 701.33
	$4	1 in 91.98
	$4	1 in 38.32

The overall odds of winning a prize are 1 in 24.87.
The odds presented here are based on a $2 play (rounded to two decimal places).

a. In how many ways can someone choose 5 white balls and 1 red Powerball?
b. In how many ways can a player win the Grand Prize by matching all five white balls and the red Powerball?

Answer available at the end of this section.

Answers

(Section 5.4) - Homework Exercise #1

${}_8C_3 = 56$ different committees

(Section 5.4) - Homework Exercise #2

${}_{10}C_4 = 210$ different pizzas

(Section 5.4) - Homework Exercise #3

${}_{12}C_2 = 66$ different pairs

(Section 5.4) - Homework Exercise #4

${}_{15}C_5 = 3{,}003$ different sets of books

(Section 5.4) - Homework Exercise #5

$({}_8C_3) \cdot ({}_7C_2) = 56 \cdot 21 = 1{,}176$ different committees

(Section 5.4) - Homework Exercise #6

$({}_8C_2) \cdot ({}_6C_1) = 28 \cdot 6 = 168$ different combinations

(Section 5.4) - Homework Exercise #7

${}_1C_1 \cdot {}_9C_3 = 1 \cdot 84 = 84$ different teams

(Section 5.4) - Homework Exercise #8

$({}_{10}C_7) \cdot ({}_8C_5) = 120 \cdot 56 = 6{,}720$ different juries

(Section 5.4) - Homework Exercise #9

$({}_{25}C_2) \cdot ({}_{30}C_1) \cdot ({}_{28}C_2) \cdot ({}_{22}C_1) = 300 \cdot 30 \cdot 378 \cdot 22 = 74{,}844{,}000$ different councils

(Section 5.4) - Homework Exercise #10

a. $({}_{69}C_5) \cdot ({}_{26}C_1) = 11{,}238{,}513 \cdot 26 = 292{,}201{,}338$ different choices
b. $({}_5C_5) \cdot ({}_1C_1) = 1 \cdot 1 = 1$ way to win

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Section 5.5: Tree Diagrams

Learning Objectives

- Use tree diagrams to find sample spaces and compute probabilities.

At the beginning of this section, we've learned how to calculate probabilities for single events. This works perfectly when one thing happens, such as, rolling a die, drawing a card, or spinning a spinner once. But what happens when we have more than one event? For instance, flip a coin twice, draw two marbles from an urn, choose a shirt and then choose pants. Suddenly, keeping track of all the possibilities becomes confusing. How many outcomes are there? What's the probability of each? Did we miss any? Tree diagrams solve this problem by giving us a visual way to organize all possible outcomes when multiple events happen in sequence. They help us see every possibility, calculate probabilities systematically, and avoid making mistakes. Think of tree diagrams as a road map for probability; they show us every path we can take and help us calculate the probability of ending up at each destination. In this section, we'll learn how to create and use tree diagrams to solve multi-stage probability problems with confidence and clarity.

Definition: Tree Diagram

A **tree diagram** is a visual tool used to systematically list all possible outcomes of a probability experiment or sequence of events. It gets its name from its branching structure that resembles an upside-down tree.

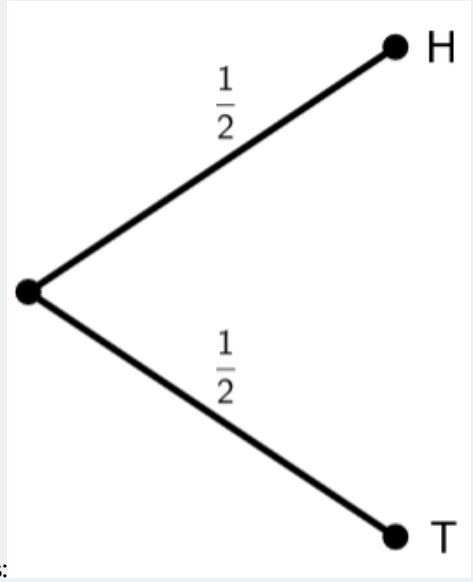
Key Features of a tree diagram:

- **Branches** represent different possible outcomes or choices
- **Nodes** (junction points) represent decision points or stages
- **Paths** from start to end represent complete sequences of outcomes
- **Probabilities** are often labeled on branches

Advantages of Using Tree Diagrams

1. **Visual Organization**
   - Makes complex probability problems easier to understand
   - Shows all possible outcomes systematically
   - Prevents missing outcomes
2. **Probability Calculation**
   - Multiply probabilities along each path
   - Add probabilities of different paths leading to same outcome
   - Clear visualization of dependent vs. independent events
3. **Sequential Events**
   - Perfect for multi-stage experiments
   - Shows how earlier events affect later probabilities
   - Demonstrates "with replacement" vs. "without replacement"
4. **Educational Value**
   - Helps students understand probability concepts
   - Builds logical thinking skills
   - Foundation for more advanced probability theory

One-Stage Tree Diagram for Flipping A Coin

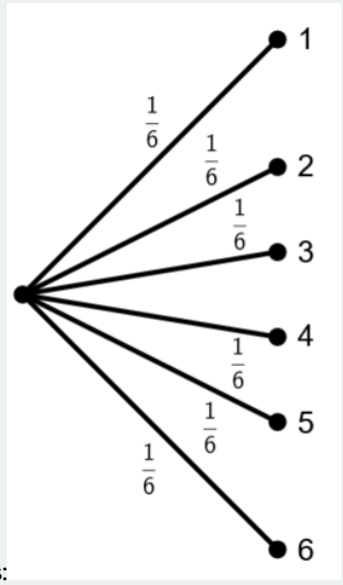


For a single flip, there are two branches:

Heads (H) and Tails (T), each with a 50% or $\frac{1}{2}$ probability.

- **For one coin flip:**
  1. Start with a single point (node).
  2. Draw two branches from this point. Label one branch "H" (Heads) and the other "T" (Tails).
  3. Label each branch with its probability: $\frac{1}{2}$
  4. At the end of each branch, represents a possible outcome. The possible outcomes are H and T. Verify that the probabilities of all possible outcomes sum to 1: $\left(\frac{1}{2}+\frac{1}{2}=1\right)$

One-Stage Tree Diagram for Rolling A Single Die



For a single roll, there are six branches:

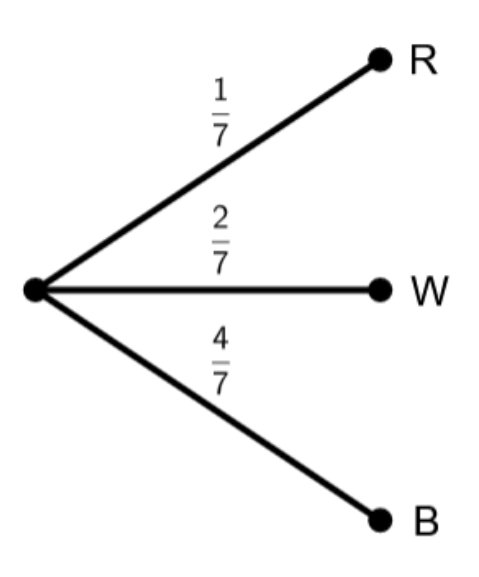
1, 2, 3, 4, 5, and 6, each with a $16\frac{2}{3}\%$ or $\frac{1}{6}$ probability.

- **For one coin flip:**
  1. Start with a single point (node).
  2. Draw six branches from this point. Label each branch "1", "2", "3", "4", "5", and "6", respectively.
  3. Label each branch with its probability: $\frac{1}{6}$
  4. Verify that the probabilities of all possible outcomes sum to 1: $\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=1\right)$

The two tree diagrams presented above demonstrate experiments with equally likely outcomes. It is important to note, however, that equal likelihood is not a universal characteristic of all probability experiments. We will now explore several examples that illustrate situations with unequal probabilities.

Example #5.5.1 🤔

Draw one marble from an urn containing one red, two white, and four blue marbles. Draw a tree diagram to represent the experiment.



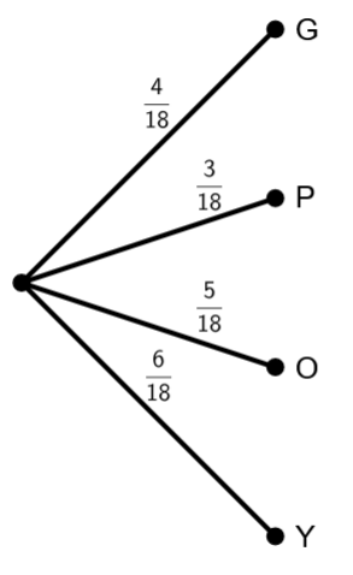
✅ **Solution:**

- **For one draw:**
  1. Start with a single point (node).
  2. Draw three branches from this point. Label one branch "R" (Red), one branch "W" (White), and the other "B" (Blue).
  3. Label the "R" branch $\frac{1}{7}$, the "W" branch $\frac{2}{7}$, and the "R" branch $\frac{4}{7}$.
  4. Verify that the probabilities of all possible outcomes sum to 1: $\left(\frac{1}{7}+\frac{2}{7}+\frac{4}{7}=1\right)$

Example #5.5.2 🤔

Draw one marble from an urn containing four green, three purple, five orange, and six yellow marbles. Draw a tree diagram to represent the experiment.

✅ **Solution:**

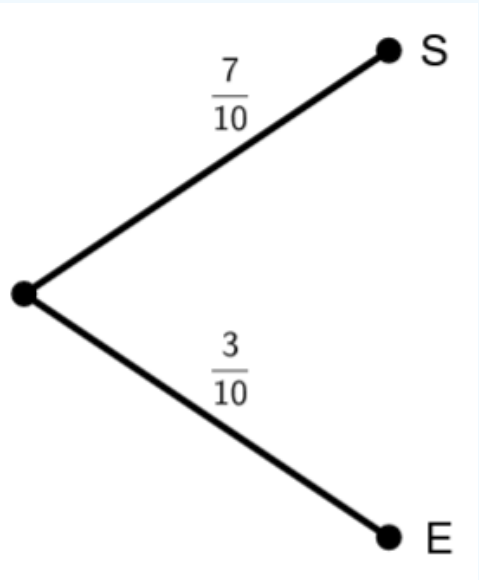


- **For one draw:**
  1. Start with a single point (node).
  2. Draw four branches from this point. Label one branch "G" (Green), one branch "P" (Purple), one branch "O" (Orange), and the other "Y" (Yellow).
  3. Label the "G" branch $\frac{4}{18}$, the "P" branch $\frac{3}{18}$, the "O" branch $\frac{5}{18}$, and the "Y" branch $\frac{6}{18}$.
  4. Verify that the probabilities of all possible outcomes sum to 1: $\left(\frac{4}{18}+\frac{3}{18}+\frac{5}{18}+\frac{6}{18}=1\right)$

Example #5.5.3 🤔

You order a package online. The store offers two shipping options: standard shipping (free) and express shipping ($15). Last month's records determined the 70% of customers choose the standard shipping option, while 30% choose the express shipping option. Draw a tree diagram to represent the shipping options.

✅ **Solution:**



- **For one choice:**

1. Start with a single point (node).
2. Draw two branches from this point. Label one branch "S" (standard) and the other "E" (express).
3. Label the "S" branch $\frac{7}{10}$, and the "E" branch $\frac{3}{10}$.
4. Verify that the probabilities of all possible outcomes sum to 1: $\left(\frac{7}{10}+\frac{3}{10}=1\right)$

With replacement vs. without replacement

Recalling the earlier examples involving the random selection of a single marble, we might ask what happens when two marbles are drawn instead. As before, we can represent the possible outcomes using a tree diagram. Because the experiment involves two successive selections, this type of diagram is called a **two-stage tree diagram**.

To determine the probabilities associated with the second draw, we must know whether the first marble is returned to the urn before the second draw. This leads to two different situations: with replacement and without replacement.

When a selection is made **with replacement**, the item that was drawn is returned to the original group before the next selection occurs. As a result, every item remains available for each draw. This means that the total number of possible outcomes stays the same, and the probabilities remain unchanged from one draw to the next.

When a selection is made **without replacement**, the item that was drawn is not returned before the next selection occurs. As a result, the selected item can no longer be chosen again. This causes the total number of available outcomes to decrease after each draw, and the probabilities change from one selection to the next.

Here is a summary:

Features of With/Without Replacement

Feature	With Replacement	Without Replacement
Item returned?	Yes	No
Total outcomes change?	No	Yes
Probabilities stay the same?	Yes	No
Common example	Coin flips / Die rolls	Marble draws / Card draws

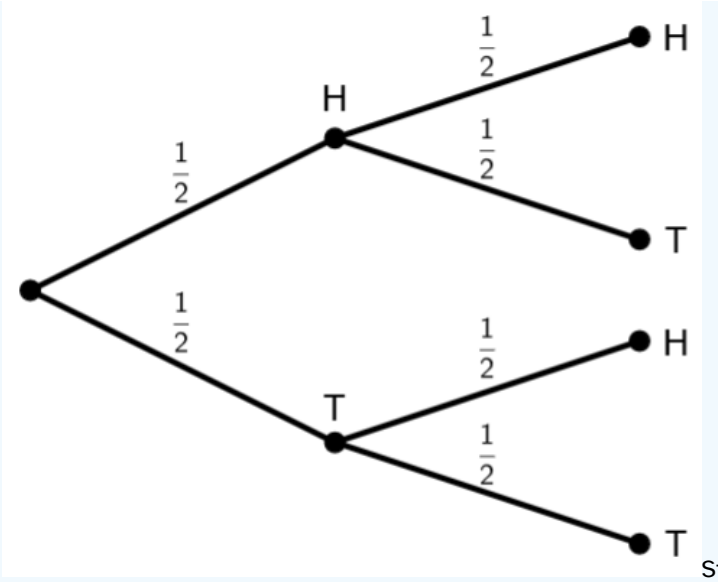
Example #5.5.4 🤔

Flip a fair coin twice. Draw a tree diagram to represent the experiment.

✅ **Solution:**

For a single flip, there are two branches: Heads (H) and Tails (T), each with a 50% or $\frac{1}{2}$ probability. This is exactly the same case in a one stage tree diagram.

- **First flip:**



1. Start with a single point (node).
2. Draw two branches from this point. Label one branch "H" (Heads) and the other "T" (Tails).
3. Label each branch with its probability: $\frac{1}{2}$
4. The possible outcomes are H and T. The sum of the probabilities must equal 1, $\left(\frac{1}{2}+\frac{1}{2}=1\right)$.

- **Second flip:**
  - Second flip is **with** replacement, because the first outcome can occur again.
    1. Go to the node at the end of the first branch from the one-stage tree diagram.
    2. Draw two branches from this point. Label one branch "H" (Heads) and the other "T" (Tails).
    3. Label each branch with its probability: $\frac{1}{2}$
    4. Now, go to the node at the end of the next branch from the one-stage tree diagram.
    5. Draw two branches from this point. Label one branch "H" (Heads) and the other "T" (Tails).
    6. Label each branch with its probability: $\frac{1}{2}$
    7. There should be four possible outcomes, HH, HT, TH, and TT. The sum of the probabilities must equal 1 for each set of branches from any of the three nodes, which is each $\frac{1}{2}+\frac{1}{2}=1$.

From the two-stage tree diagram above, we can ask several follow up questions:

a. **What is the probability of getting two heads?** To find the probability of a specific path, you multiply the probabilities along the branches. Here, we are looking for the outcome HH. Follow the branch that gets us HH. As you get to HH, the probabilities that are along the branches are $\frac{1}{2}$ and $\frac{1}{2}$. Now, multiply those probabilities. Thus, the probability P(HH) = $\frac{1}{2} \cdot \frac{1}{2}=\frac{1}{4}$.

b. **What is the probability of getting exactly one head and exactly one tail?** This is considered an "either/or" scenario. To find the probability of an "either/or" scenario, multiply the probabilities along each branch, then add the probabilities of the relevant paths together. For this problem, we want either HT or TH. Follow the branch for HT multiplying those probabilities, P(HT) = $\frac{1}{2} \cdot \frac{1}{2}=\frac{1}{4}$, and the branch for TH multiplying those probabilities, P(TH) = $\frac{1}{2} \cdot \frac{1}{2}=\frac{1}{4}$, then add their probabilities. Thus, the probability getting exactly one head and exactly one tail = P(HT) + P(TH) = $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

c. **What is the probability of getting at least one head?** Here, we will cut to the chase. At least one head means, we want HT, TH, or HH. So, the probability of getting at least one head = P(HH) + P(HT)+ P(TH) = $\frac{1}{2} \cdot \frac{1}{2}+\frac{1}{2} \cdot \frac{1}{2}+\frac{1}{2} \cdot \frac{1}{2}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$.

d. **What is the probability of getting at most one head?** At most one head means (0 or 1 head), we want HT, TH, or TT. So, the probability of getting at most one head = P(HT) + P(TH)+ P(TT) = $\frac{1}{2} \cdot \frac{1}{2}+\frac{1}{2} \cdot \frac{1}{2}+\frac{1}{2} \cdot \frac{1}{2}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$.

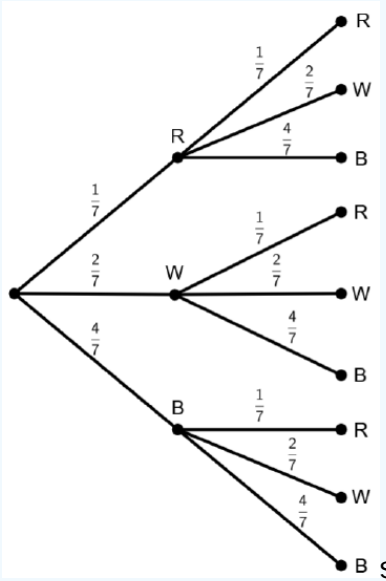
The principles used for constructing two-stage tree diagrams extend naturally to multi-stage diagrams involving three, four, or more stages. However, as the number of stages increases, these diagrams become significantly more complex and cumbersome to construct. For problems involving numerous stages, alternative probability methods and advanced techniques provide more efficient solution strategies. These advanced methods and techniques are typically covered in higher-level mathematics courses such as statistics, combinatorics, or advanced probability theory.

Example #5.5.5 🤔

Draw two marbles **with** replacement from an urn containing one red, two white, and four blue marbles. Draw a tree diagram to represent the experiment.

✅ **Solution:**

- **First draw:**



1. Start with a single point (node).
2. Draw two branches from this point. Label one branch "R" (Red), one branch "W" (White), and the other "B" (Blue).
3. Label the "R" branch $\frac{1}{2}$, the "W" branch $\frac{2}{7}$, and the "B" branch $\frac{4}{7}$.
4. The possible outcomes are R, W, and B. The sum of the probabilities must equal 1, $\left(\frac{1}{7}+\frac{2}{7}+\frac{4}{7}=1\right)$.

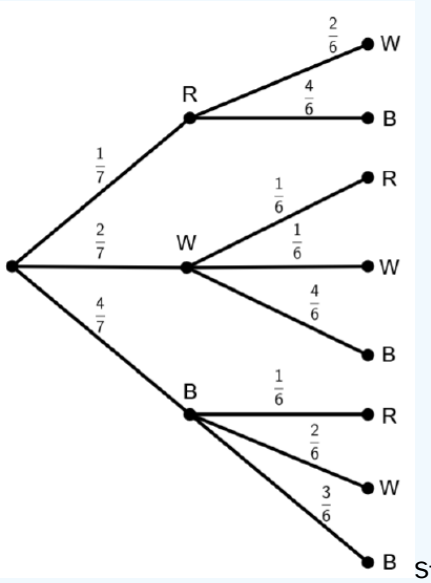
- **Second draw:**
  - Second draw is **with** replacement, which means the first draw is returned to the urn.
    1. Go to the node at the end of the first branch (R) from the one-stage tree diagram.
    2. Draw three branches from this point.
    3. Label one branch "R" (Red), another branch "W" (White), and another branch "B" (Blue).
    4. Label each branch with its appropriate probability: $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}$, respectively.
    5. Now, go to the second node at the end of the next branch (W) from the one-stage tree diagram.
    6. Draw three branches from this point.
    7. Label one branch "R" (Red), another branch "W" (White), and another branch "B" (Blue).
    8. Label each branch with its appropriate probability: $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}$, respectively.
    9. Now, go to the third node at the end of the last branch (B) from the one-stage tree diagram.
    10. Draw three branches from this point. Label one branch "R" (Red), another branch "W" (White), and another branch "B" (Blue).
    11. Label each branch with its appropriate probability: $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}$, respectively.
    12. There should be a total of nine possible outcomes, RR, RW, RB, WR, WW, WB, BR, BW, and BB. The sum of the probabilities must equal 1 for each branch that connects any two nodes.

Example #5.5.6 🤔

Draw two marbles **without** replacement from an urn containing one red, two white, and four blue marbles. Draw a tree diagram to represent the experiment.

✅ **Solution:**

- **First draw:**

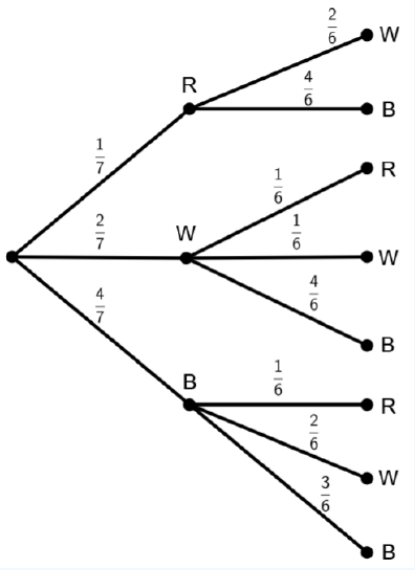


1. Start with a single point (node).
2. Draw three branches from this point. Label one branch "R" (Red), one branch "W" (White), and the other "B" (Blue).
3. Label the "R" branch $\frac{1}{2}$, the "W" branch $\frac{2}{7}$, and the "B" branch $\frac{4}{7}$.
4. The possible outcomes are R, W, and B. The sum of the probabilities must equal 1, $\left(\frac{1}{7}+\frac{2}{7}+\frac{4}{7}=1\right)$.

- **Second draw:**
  - Second draw is **without** replacement, which means the first draw is NOT returned to the urn. This means the sample space on the second draw has changed from the first.
    1. Go to the node at the end of the first branch (R) from the one-stage tree diagram.
    2. Draw only two branches from this point. Label one branch "W" (White) and another branch "B" (Blue). There is no possible outcome of a red on the second draw, because the only red marble was removed on the first draw.
    3. Label each branch with its appropriate probability: $\frac{2}{6}, \frac{4}{6}$, respectively.
    4. Now, go to the second node at the end of the next branch (W) from the one-stage tree diagram.
    5. Draw three branches from this point.
    6. Label one branch "R" (Red), another branch "W" (White), and another branch "B" (Blue).
    7. Label each branch with its appropriate probability: $\frac{1}{6}, \frac{1}{6}, \frac{4}{6}$ respectively.
    8. Now, go to the third node at the end of the last branch (B) from the one-stage tree diagram.
    9. Draw three branches from this point. Label one branch "R" (Red), another branch "W" (White), and another branch "B" (Blue).
    10. Label each branch with its appropriate probability: $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}$ respectively.
    11. There should be a total of eight possible outcomes, RW, RB, WR, WW, WB, BR, BW, and BB. Compared to Example #5.5.5, the probabilities of obtaining these individual outcomes are different, because this experiment was done **without** replacement. Again, the sum of the probabilities must equal 1 for each branch that connects any two nodes.

Example #5.5.7 🤔

In Example #5.5.6, when two marbles were drawn **without** replacement, answer the following questions.



a. What is the probability of drawing both blue marbles?
b. What is the probability of drawing a red and a blue marble?
c. What is the probability of drawing at least one blue marble?
d. What is the probability of drawing both red marbles?

✅ **Solution:**

First draw:

a. $P(\text{BB}) = \frac{4}{7} \cdot \frac{3}{6} = \frac{4 \cdot 3}{7 \cdot 6} = \frac{2}{7}$
b. $P(\text{RB}) + P(\text{BR}) = \frac{1}{7} \cdot \frac{4}{6} + \frac{4}{7} \cdot \frac{1}{6} = \frac{1 \cdot 4}{7 \cdot 6} + \frac{4 \cdot 1}{7 \cdot 6} = \frac{2}{21} + \frac{2}{21} = \frac{4}{21}$
c. $P(\text{RB}) + P(\text{WB}) + P(\text{BR}) + P(\text{BW}) + P(\text{BB}) = \frac{1}{7} \cdot \frac{4}{6} + \frac{2}{7} \cdot \frac{4}{6} + \frac{4}{7} \cdot \frac{1}{6} + \frac{4}{7} \cdot \frac{2}{6} + \frac{4}{7} \cdot \frac{3}{6} = \frac{4}{42} + \frac{8}{42} + \frac{4}{42} + \frac{8}{42} + \frac{12}{42} = \frac{36}{42} = \frac{6}{7}$
d. $P(\text{RR}) = 0$

Section 5.5: Tree Diagrams [*In-Class Exercises*]

1. An urn containing eight orange marbles, three purple marbles, and four gray marbles. A marble is drawn at random. Draw a tree diagram to represent the experiment.
2. A standard deck of 52 cards contains the following 4 suits: 13 diamonds, 13 hearts, 13 clubs, and 13 spades. A card is drawn at random and the suit is observed. Draw a tree diagram to represent the experiment.
3. A 4-sided die with outcomes 1, 2, 3, and 4 is rolled. Draw a tree diagram to represent the experiment.



4. Draw two coins **with** replacement from a coffee can that contains five pennies, four nickels, and three dimes. Draw a tree diagram to represent the experiment.
   a. What is the probability of drawing both pennies?

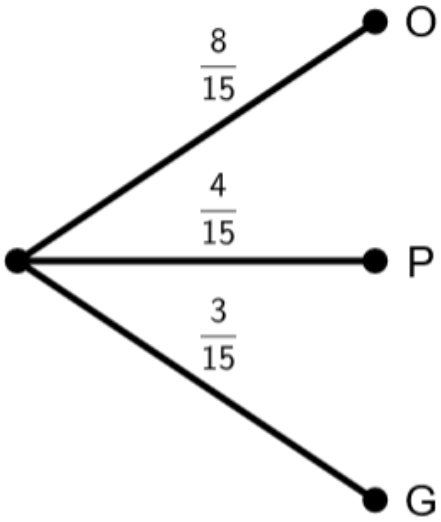
b. What is the probability of drawing a dime on the first draw and a nickel on the second?
c. What is the probability of drawing a dime and a nickel?
d. What is the probability of drawing at least one dime?

5. Draw two marbles **without** replacement from an urn containing eight orange marbles, three purple marbles, and four gray marbles. Draw a tree diagram to represent the experiment.
   a. What is the probability of drawing both purple marbles?
   b. What is the probability of drawing an orange and a purple marble?
   c. What is the probability of drawing at least one orange marble?
   d. What is the probability of not drawing both gray marbles?

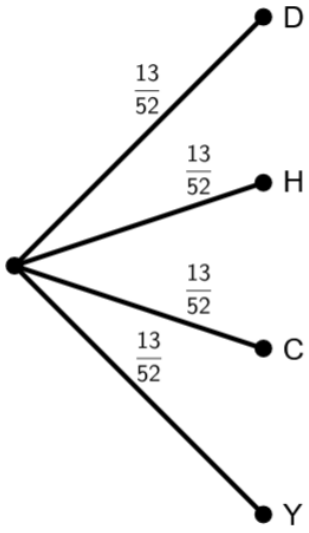
Answers available at the end of this section.

Answers

Section 5.5: Tree Diagrams [*In-Class Exercises*]

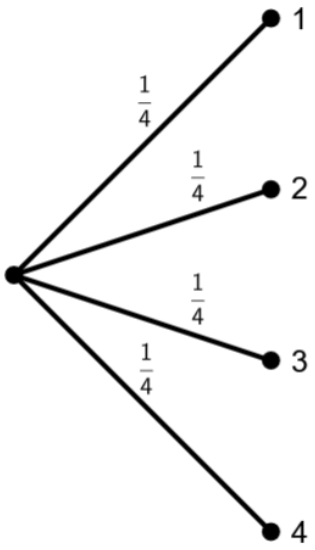


1.

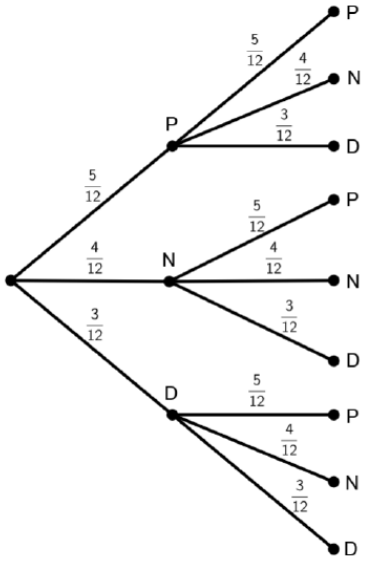


2.

3.

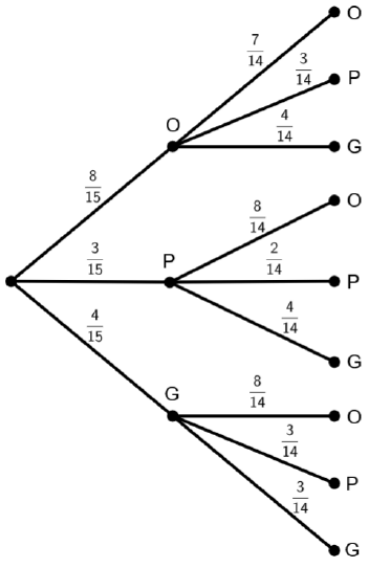


4.



a. $\frac{25}{144}$
b. $\frac{12}{144} = \frac{1}{12}$
c. $\frac{24}{144} = \frac{1}{6}$
d. $\frac{63}{144} = \frac{7}{16}$

5.



a. $\frac{6}{210} = \frac{1}{35}$
b. $\frac{48}{210} = \frac{8}{35}$
c. $\frac{168}{210} = \frac{4}{5}$
d. $1 - \frac{12}{210} = \frac{198}{210} = \frac{33}{35}$

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Section 5.5.H: Homework

Section 5.5: Tree Diagrams - Homework Exercises

(Section 5.5) - Homework Exercise #1

A bag contains 2 red marbles and 3 blue marbles. Draw two marbles **with** replacement.

a. Draw a tree diagram to represent the experiment.
b. What is the probability of drawing both red marbles?
c. What is the probability of drawing exactly one red marble?

Answer available at the end of this section.

(Section 5.5) - Homework Exercise #2

A spinner has equally likely outcomes, red, blue, and green. You spin it twice.

a. Draw a tree diagram to represent the experiment.
b. What is the probability of landing on green on both spins?
c. What is the probability of landing on red on the first spin and then landing on blue marble on the second spin?
d. What is the probability of landing on red or blue in any order?
e. What is the probability of not landing on blue?

Answer available at the end of this section.

(Section 5.5) - Homework Exercise #3

Draw two marbles **with** replacement from an urn containing two yellow, five purple, and three teal marbles.

a. Draw a tree diagram to represent the experiment.
b. What is the probability of drawing both purple marbles?
c. What is the probability of drawing a teal marble on the first draw and then a yellow marble on the second draw?
d. What is the probability of drawing a teal marble and yellow marble in any order?
e. What is the probability of drawing at least one teal marble?

Answer available at the end of this section.

(Section 5.5) - Homework Exercise #4

Draw two marbles **without** replacement from an urn containing two yellow, five purple, and three teal marbles.

a. Draw a tree diagram to represent the experiment.
b. What is the probability of drawing both purple marbles?
c. What is the probability of drawing a teal marble on the first draw and then a yellow marble on the second draw?
d. What is the probability of drawing a teal marble and yellow marble in any order?
e. What is the probability of drawing at least one teal marble?

Answer available at the end of this section.

(Section 5.5) - Homework Exercise #5

Draw two coins **without** replacement from a jar containing one gold coin, one silver coin, and eight pennies.

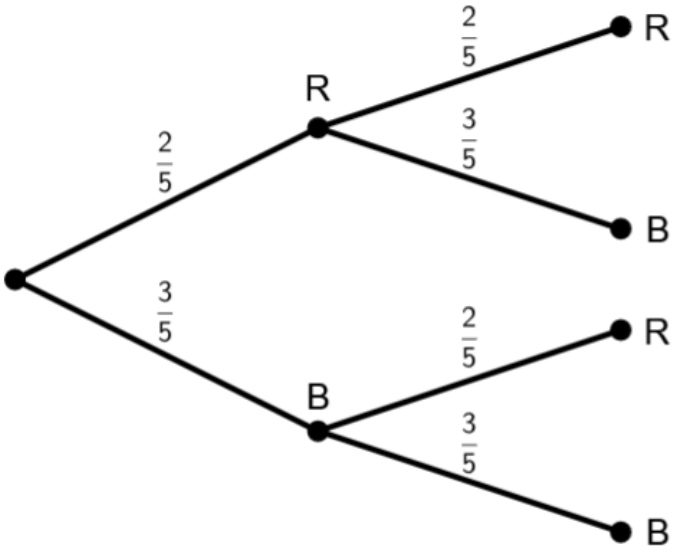
a. What is the probability of drawing both the gold and silver coin in any order?
b. What is the probability of drawing both pennies?
c. What is the probability of drawing at least one gold coin?
d. What is the probability of drawing at least one penny?

Answer available at the end of this section.

Answer available at the end of this section.

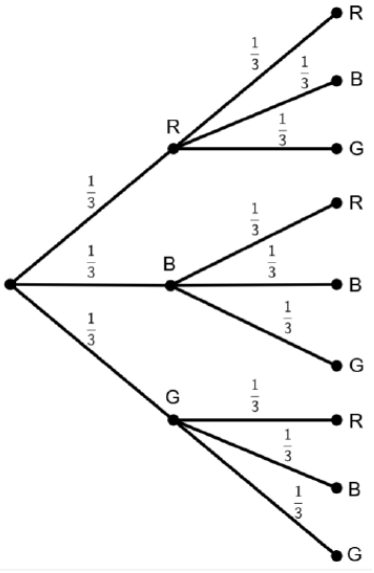
Answers

(Section 5.5) - Homework Exercise #1



a. A starting node branches into two outcomes: R with probability two-fifths and B with probability three-fifths. From each of these outcomes, two additional branches follow: R with probability two-fifths and B with probability three fifths.
b. $\frac{4}{25}$
c. $\frac{12}{25}$

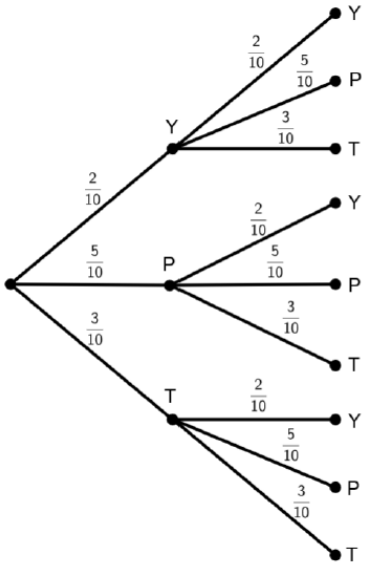
(Section 5.5) - Homework Exercise #2

a. 

A starting node branches into three outcomes: R with probability one-third, B with probability one-third, and G with probability one-third. From each of these outcomes, three additional branches follow: R with probability one-third, B with probability one-third, and G with probability one-third.

b. $\frac{1}{9}$
c. $\frac{1}{9}$
d. $\frac{2}{9}$
e. $\frac{4}{9}$

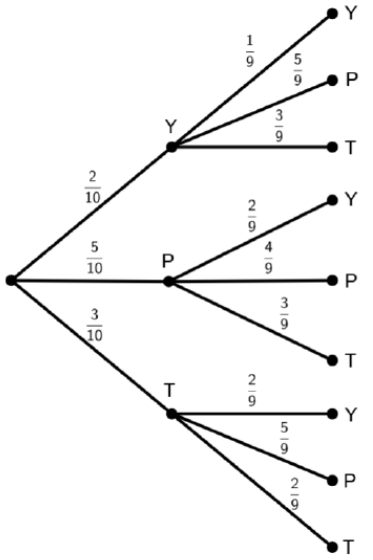
(Section 5.5) - Homework Exercise #3

a. 

A starting node branches into three outcomes: Y with probability two-tenths, P with probability five-tenths, and T with probability three-tenths. From each of these outcomes, three additional branches follow: Y with probability two-tenths, P with probability five-tenths, and T with probability three-tenths.

b. $\frac{25}{100} = \frac{1}{4}$
c. $\frac{6}{100} = \frac{3}{50}$
d. $\frac{12}{100} = \frac{3}{25}$
e. $\frac{51}{100}$

(Section 5.5) - Homework Exercise #4



a. A starting node branches into three outcomes: Y with probability two-tenths, P with probability five-tenths, and T with probability three-tenths. From the Y outcome, three branches follow: Y with probability one-ninth, P with probability five-ninths, and T with probability three-ninths. From the P outcome, three branches follow: Y with probability two-ninths, P with probability four-ninths, and T with probability three-ninths. From the T outcome, three branches follow: Y with probability two-ninths, P with probability five-ninths, and T with probability two-ninths.

b. $\frac{20}{90} = \frac{2}{9}$
c. $\frac{6}{90} = \frac{1}{15}$
d. $\frac{12}{90} = \frac{2}{15}$
e. $\frac{48}{90} = \frac{8}{15}$

(Section 5.5) - Homework Exercise #5

a. $\frac{1}{45}$
b. $\frac{56}{90} = \frac{28}{45}$
c. $\frac{18}{90} = \frac{1}{5}$
d. $\frac{88}{90} = \frac{44}{45}$

(Section 5.5) - Homework Exercise #5

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Section 5.6: More on Probability

Learning Objectives

- Use tables to find sample spaces and compute probabilities for rolling a pair of dice
- Use sample spaces and compute probabilities for drawing a card from a standard deck of cards

In Section 5.1, we explored the fundamental concepts of probability. We worked with simple scenarios like coin flips, rolling a single die, spinner wheels, and drawing colored marbles from bags. These examples helped us understand the core principles without getting bogged down in complexity. We could easily list all outcomes, count them, and calculate probabilities.

Now it's time to apply these foundational principles to situations you encounter in everyday life, specifically, dice games and card games. Whether you're playing Monopoly with family, enjoying poker with friends, or simply trying to understand the odds in a casino, dice and cards are universal tools of chance that have entertained humanity for centuries.

The good news is that all the principles you've already learned still apply. We're not learning new rules of probability, instead we're simply applying the same rules to richer, more interesting scenarios.

When analyzing the probability of rolling two dice, it is important to distinguish between the possible sums and the actual sample space of the experiment. While many applications focus on the sum of the two dice showing face up, which produces eleven possible results (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12), these eleven outcomes are not equally likely to occur. Each sum does not have a probability of 1/11 because some sums can be achieved in more ways than others. For instance, there are significantly more ways to obtain a sum of 7 than a sum of 2 or 12.

To properly analyze the probabilities associated with rolling two dice, we must first identify the complete sample space of the experiment. Each die has 6 possible outcomes (1, 2, 3, 4, 5, 6), and since the dice are rolled independently, we can apply the Fundamental Counting Principle. The first die contributes 6 possible outcomes, and the second die contributes 6 possible outcomes. Therefore, the total number of equally likely outcomes in the sample space is $6 \times 6 = 36$.

These 36 outcomes represent all possible ordered pairs (first die, second die) that can occur when rolling two distinguishable dice. To clearly illustrate this complete sample space and demonstrate why certain sums are more likely than others, we will organize these 36 outcomes in a systematic table format.

Sample Space for Rolling Two Dice

	Die #2: 1	Die #2: 2	Die #2: 3	Die #2: 4	Die #2: 5	Die #2: 6
Die #1: 1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
Die #1: 2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
Die #1: 3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
Die #1: 4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
Die #1: 5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
Die #1: 6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

How to Read the Table

- The **rows** represent the outcomes of the first die (1 through 6)
- The **columns** represent the outcomes of the second die (1 through 6)
- Each **cell** in the grid contains an ordered pair (first die, second die)

To find any specific outcome, locate the row corresponding to the first die's value and the column corresponding to the second die's value. The intersection of that row and column shows the ordered pair representing that outcome.

Example #5.6.1 🤔

A pair of single six-sided die is rolled. Answer the following questions.

a. What is the probability a sum of 7 is rolled?
b. What is the probability a sum of 3 is rolled?
c. What is the probability a sum of 8 is rolled?
d. What is the probability a sum of 13 is rolled?
e. What is the probability a sum of at least 11 is rolled?
f. What is the probability a sum of at most 5 is rolled?
g. What is the probability there is at least one '4' on either or both dies rolled?
h. What is the probability that both dice are the same?

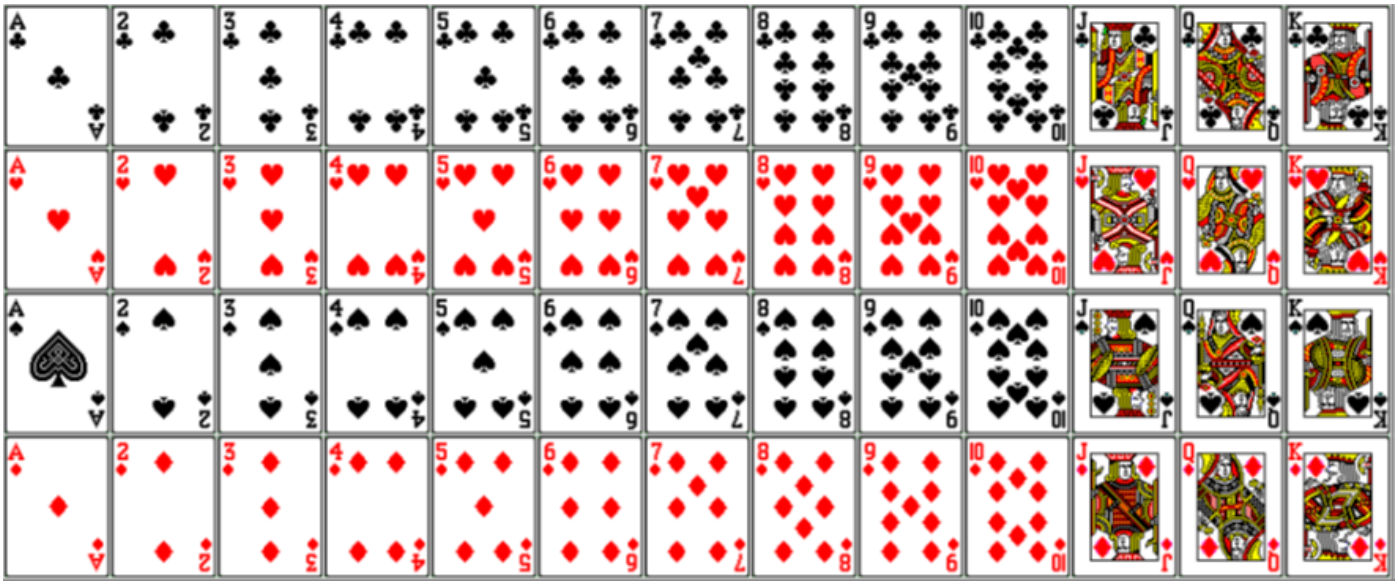
✅ **Solution:**

So, we look at the table for the outcomes of rolling two dice.

a. There are six outcomes that make a sum of 7. They are $(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)$. Thus, the probability is $\frac{6}{36} = \frac{1}{6}$.
b. There are two outcomes that make a sum of 3. They are $(1,2),(2,1)$. Thus, the probability is $\frac{2}{36} = \frac{1}{18}$.
c. There are five outcomes that make a sum of 8. They are $(2,6),(6,2),(3,5),(5,3),(4,4)$. Thus, the probability is $\frac{5}{36}$.
d. There are no (zero) outcomes that make a sum of 13. Thus, the probability is $\frac{0}{36} = 0$.
e. At least 11 means 11 or more. There are three outcomes that make a sum of at least 11. They are $(5,6),(6,5),(6,6)$. Thus, the probability is $\frac{3}{36} = \frac{1}{12}$.
f. At most 5 means 5 or less. There are ten outcomes that make a sum of at most 5. They are $(1,1),(1,2),(2,1),(1,3),(3,1),(2,2)(1,4),(4,1),(2,3),(3,2)$. Thus, the probability is $\frac{10}{36} = \frac{5}{18}$.
g. There are eleven outcomes that make at least one '4' on either or both dies. They are $(1,4),(2,4),(3,4),(5,4),(6,4),(4,1),(4,2),(4,3),(4,5),(4,6),(4,4)$. Thus, the probability is $\frac{11}{36}$.
h. There are six outcomes that make both dice the same. They are $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)$. Thus, the probability is $\frac{6}{36} = \frac{1}{6}$.

Now that we understand the basics of probability, let's explore one of the most common and useful applications: calculating probabilities using a standard deck of playing cards. Card problems appear frequently in probability because they provide a perfect example of a well-defined sample space with known outcomes.

A standard deck of playing cards contains exactly 52 different cards, making it an ideal tool for probability calculations. The deck is organized using two classification systems that work together to create each unique card. First, there are 4 different suits, which are the symbols that appear on each card: hearts (♥), diamonds (♦), clubs (♣), and spades (♠). Second, there are 13 different ranks (A, 2-10, J, Q, K), which represent the values or names of the cards. When we combine each of the 4 suits with each of the 13 ranks, we get our complete set of 52 distinct cards (4 × 13 = 52).



Example #5.6.2 🤔

For each of the following questions, draw a single card from a standard deck of playing cards.

a. What is the probability a heart is drawn?
b. What is the probability a red card is drawn?
c. What is the probability a '7' is drawn?
d. What is the probability a '5' of clubs is drawn?
e. What is the probability a '9' of moons is drawn?
f. What is the probability a '6' or a queen is drawn?
g. What is the probability an ace or spade is drawn?
h. What is the probability a '4' or a black card is drawn?

✅ **Solution:**

So, we look at the sample space for the 52 outcomes of a standard deck of playing cards.

a. There are 13 hearts. Thus, the probability is $\frac{13}{52} = \frac{1}{4}$.
b. There are 26 red cards. Thus, the probability is $\frac{26}{52} = \frac{1}{2}$.
c. There are four '7's. Thus, the probability is $\frac{4}{52} = \frac{1}{13}$.
d. There is only one '5' of clubs. Thus, the probability is $\frac{1}{52}$.
e. There are no suits called moons. Thus, the probability is $\frac{0}{52} = 0$.
f. There are four '6's and four queens. Thus, the probability is $\frac{8}{52} = \frac{2}{13}$.
g. There are 4 aces and there are 12 spades not including the ace of spades already counted. Thus, the probability is $\frac{16}{52} = \frac{4}{13}$.
h. There are four '4's and there are 24 black cards not including the '4's already counted. Thus, the probability is $\frac{28}{52} = \frac{7}{13}$.

A Brief History of Playing Cards

Ancient Beginnings

The exact origin of playing cards remains somewhat mysterious, but historians generally agree that playing cards originated in China during the Tang Dynasty (618-907 CE). These early cards were likely derived from ancient Chinese "money cards" or "domino cards" and were used for both games and fortune-telling.

From China, playing cards spread along trade routes, reaching:

- Persia and Arabia (by the 11th century), where they evolved into new forms
- Egypt (by the 12th-13th centuries), where Mamluk cards featured suits of cups, coins, swords, and polo sticks
- Europe (by the late 14th century), arriving through trade with the Islamic world and Mediterranean commerce

Evolution in Europe

When playing cards reached Europe in the 1370s, they quickly became popular among all social classes. Different regions developed their own suit systems:

- Italian suits: Cups, Coins, Swords, Batons
- German suits: Hearts, Bells, Leaves, Acorns
- Spanish suits: Cups, Coins, Swords, Clubs
- Swiss suits: Shields, Roses, Acorns, Bells

The French Standard

The playing cards we use today—the French deck—emerged in the late 15th century in France. French cardmakers made several innovations:

1. Simplified suit symbols: They created the four suits we know today:
    - Hearts (♥) - representing the clergy and love
    - Diamonds (♦) - representing merchants and wealth
    - Clubs (♣) - representing peasants and agriculture
    - Spades (♠) - representing the military and nobility
2. Standardized court cards: Kings, Queens, and Jacks (originally called Knaves)
3. Efficient design: The French suits were easier and cheaper to reproduce using stencils, giving them an economic advantage
4. Reversible face cards: By the 19th century, cards were printed with images that looked right-side-up from either direction

Why This Design Succeeded

The French deck became the international standard for several reasons:

- Economic efficiency: Simple suit symbols reduced production costs
- Clarity: The four distinct suit symbols were easy to distinguish
- Colonial spread: French and British colonization spread this deck worldwide
- Printing advances: The design worked well with mass production techniques
- Gaming versatility: The structure supported countless game variations

The English Contribution

When the French deck reached England in the late 15th century, the English made their own modifications:

- Changed the French term "Valet" to "Jack" (or "Knave")
- Added indices (the small numbers and suit symbols in the corners) in the 19th century
- Standardized the court card designs we recognize today
- Introduced the Joker around 1860 (originally for the game of Euchre)

The Anglo-American deck is essentially the French deck with English naming conventions and design refinements.

Connection to Probability Theory

Historically, card games (along with dice games) motivated the development of probability theory:

- Gerolamo Cardano (1501-1576) studied dice and card games, writing the first book on probability
- Blaise Pascal and Pierre de Fermat (1654) developed probability theory while analyzing gambling problems
- Christian Huygens (1657) published the first probability textbook, inspired by games of chance

- Modern combinatorics grew partly from analyzing card and dice probabilities

When you study card probability, you're following in the footsteps of mathematical giants who were themselves fascinated by these same questions.

Section 5.6: More on Probability [*In-Class Exercises*]

1. A pair of single six-sided die is rolled. Answer the following questions.
   a. What is the probability a sum of 2 is rolled?
   b. What is the probability a sum of 11 is rolled?
   c. What is the probability a sum of 9 is rolled?
   d. What is the probability a sum of 1 is rolled?
   e. What is the probability a sum of at least 10 is rolled?
   f. What is the probability a sum of at most 3 is rolled?
   g. What is the probability there is at least one '5' on either or both dies rolled?
   h. What is the probability that both dice are different?
   i. What is the probability that the sum is even?
   j. What is the probability that the sum is a prime number? (A prime number is a number, other than 1, that is divisible by 1 and itself).
2. For each of the following questions, draw a single card from a standard deck of playing cards.
   a. What is the probability a black card is drawn?
   b. What is the probability a diamond is drawn?
   c. What is the probability a '9' is drawn?
   d. What is the probability a '5' of spades is drawn?
   e. What is the probability a '3' or a jack is drawn?
   f. What is the probability an '8' of cups is drawn?
   g. What is the probability a king of diamonds is drawn?
   h. What is the probability a red card or a heart is drawn?
   i. What is the probability a '10' or a red card is drawn?
3. A pair of single five-sided die with outcomes {1, 2, 3, 4, 5} is rolled. Answer the following questions.
   a. What is the probability a sum of 6 is rolled?
   b. What is the probability a sum of 10 is rolled?
   c. What is the probability a sum of at least 4 is rolled?
   d. What is the probability a sum of 12 is rolled?

Answers available at the end of this section.

Answers

Section 5.6: More on Probability [*In-Class Exercises*]

1. a. $\frac{1}{36}$
   b. $\frac{2}{36} = \frac{1}{18}$
   c. $\frac{4}{36} = \frac{1}{9}$
   d. $\frac{0}{36} = 0$
   e. $\frac{6}{36} = \frac{1}{6}$
   f. $\frac{3}{36} = \frac{1}{12}$
   g. $\frac{11}{36}$
   h. $\frac{30}{36} = \frac{5}{6}$
   i. $\frac{18}{36} = \frac{1}{2}$
   j. $\frac{15}{36} = \frac{5}{12}$
2. a. $\frac{26}{52} = \frac{1}{2}$
   b. $\frac{13}{52} = \frac{1}{4}$
   c. $\frac{4}{52} = \frac{1}{13}$
   d. $\frac{1}{52}$
   e. $\frac{8}{52} = \frac{2}{13}$
   f. $\frac{0}{52} = 0$
   g. $\frac{1}{52}$
   h. $\frac{26}{52} = \frac{1}{2}$

i. $\frac{28}{52} = \frac{7}{13}$

3. a. $\frac{5}{25} = \frac{1}{5}$
   b. $\frac{1}{25}$
   c. $\frac{22}{25}$
   d. $\frac{0}{52} = 0$

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Section 5.6.H: Homework

Section 5.6: More on Probability - Homework Exercises

(Section 5.6) - Homework Exercise #1

A pair of single six-sided dice is rolled. Answer the following questions.

a. What is the probability a sum of 12 is rolled?
b. What is the probability a sum of 7 is rolled?
c. What is the probability a sum of 3 is rolled?
d. What is the probability a sum of 10 is rolled?
e. What is the probability a sum of 8 is rolled?
f. What is the probability a sum of 4 is rolled?
g. What is the probability a sum of 14 is rolled?
h. What is the probability a sum of 6 is rolled?
i. What is the probability a sum of 11 is rolled?
j. What is the probability a sum of 5 is rolled?
k. What is the probability a sum of 2 is rolled?
l. What is the probability a sum of 9 is rolled?
m. What is the probability a product of 9 is rolled? (product = the result of multiplying 2 or more numbers)
n. What is the probability a product of 6 is rolled?
o. What is the probability a product of 5 is rolled?
p. What is the probability a product of 7 is rolled?
q. What is the probability a product of 12 is rolled?
r. What is the probability a sum of at least 9 is rolled?
s. What is the probability a sum of at most 4 is rolled?
t. What is the probability a sum of at least 7 is rolled?
u. What is the probability a sum of at most 11 is rolled?
v. What is the probability there is at least one '2' on either or both dies rolled?
w. What is the probability that both dice are the same?
x. What is the probability that the sum is a single digit odd number?
y. In the casino game of craps, rolling a sum of a '7' or '11' on the come-out roll (initial roll) is an automatic win for players who bet on the "Pass Line". What is the probability of rolling a sum of '7' or '11'?

Answer available at the end of this section.

(Section 5.6) - Homework Exercise #2

For each of the following questions, draw a single card from a standard deck of playing cards.

a. What is the probability a '3' is drawn?
b. What is the probability a spade is drawn?
c. What is the probability a diamond or a club is drawn?
d. What is the probability a red '7' is drawn?
e. What is the probability a '4' of hearts is drawn?
f. What is the probability a '10' is drawn?
g. What is the probability a '2' or a queen is drawn?
h. What is the probability an ace of stars is drawn?
i. What is the probability a jack of spades is drawn?
j. What is the probability a '5' or a '6' is drawn?

k. What is the probability a '5' and a '6' is drawn?
l. What is the probability a green card is drawn?
m. What is the probability a '9' or a black card is drawn?
n. The Scots consider the Nine of Diamonds to be an unlucky card, and its nickname goes back to the 19th century. The Seven of Diamonds is known as the 'Beer Card' and originated in Denmark. The Four of Clubs known as the 'Devil's Bedpost' is derived from the design of the pips on the card, which can be imagined to be the four posts of a bed. The Ace of Spades is commonly referred to as 'The Death Card'. What is the probability that you draw an 'unlucky card', or the 'Beer Card' or the 'Devil's Bedpost' or the 'Death Card'?
o. Let's say you just found out that your deck of cards is missing the ace of spades. If you were to draw from this deck with one less card, what is the probability of drawing an ace?

Answer available at the end of this section.

(Section 5.6) - Homework Exercise #3

Pinochle is a trick-taking, Ace-Ten card game typically for two to four players and is played with a 48-card deck. It is derived from the French card game Bezique or Bésigue. Players score points by trick-taking and also by forming combinations of cards into melds. Each hand is played in three phases: bidding, melds, and tricks. The standard game today is called "Partnership Auction Pinochle." A Pinochle deck consists of two copies of each of the 9, 10, jack, queen, king, and ace cards of all four suits, for 48 cards per deck. Aces are always considered high. Pinochle follows a nonstandard card ordering. The complete ordering from highest to lowest is A, 10, K, Q, J, 9. The sample space is shown below.



Draw a single card from a deck of Pinochle cards.

a. How many cards are there in a Pinochle deck?
b. What is the probability that you draw a heart?
c. What is the probability that you draw a red card?
d. What is the probability that you draw a '9'?
e. What is the probability that you draw a '10' of diamonds?
f. What is the probability that you draw a '7'?
g. What is the probability that you draw a 'face' card?
h. What is the probability that you draw a red ace?
i. Under Pinochle rules, what is the probability that you draw a card higher than a king?

Answer available at the end of this section.

Answers

(Section 5.6) - Homework Exercise #1

a. $\frac{1}{36}$
b. $\frac{6}{36} = \frac{1}{6}$
c. $\frac{2}{36} = \frac{1}{18}$
d. $\frac{3}{36} = \frac{1}{12}$
e. $\frac{5}{36}$
f. $\frac{3}{36} = \frac{1}{12}$
g. $\frac{0}{36} = 0$
h. $\frac{5}{36}$
i. $\frac{2}{36} = \frac{1}{18}$
j. $\frac{4}{36} = \frac{1}{9}$
k. $\frac{1}{36}$
l. $\frac{4}{36} = \frac{1}{9}$
m. $\frac{1}{36}$
n. $\frac{4}{36} = \frac{1}{9}$
o. $\frac{2}{36} = \frac{1}{18}$
p. $\frac{0}{36} = 0$
q. $\frac{4}{36} = \frac{1}{9}$
r. $\frac{10}{36} = \frac{5}{18}$
s. $\frac{6}{36} = \frac{1}{6}$
t. $\frac{21}{36} = \frac{7}{12}$
u. $\frac{35}{36}$
v. $\frac{11}{36}$
w. $\frac{6}{36} = \frac{1}{6}$
x. $\frac{16}{36} = \frac{4}{9}$
y. $\frac{8}{36} = \frac{2}{9}$

(Section 5.6) - Homework Exercise #2

a. $\frac{4}{52} = \frac{1}{13}$
b. $\frac{13}{52} = \frac{1}{4}$
c. $\frac{26}{52} = \frac{1}{2}$
d. $\frac{2}{52} = \frac{1}{26}$
e. $\frac{1}{52}$
f. $\frac{0}{52} = 0$
g. $\frac{8}{52} = \frac{2}{13}$
h. $\frac{0}{52} = 0$
i. $\frac{1}{52}$
j. $\frac{8}{52} = \frac{2}{13}$
k. $\frac{0}{52} = 0$
l. $\frac{0}{52} = 0$
m. $\frac{28}{52} = \frac{7}{13}$
n. $\frac{4}{52} = \frac{1}{13}$
o. $\frac{3}{51}$

(Section 5.6) - Homework Exercise #3

a. 48
b. $\frac{12}{48} = \frac{1}{4}$
c. $\frac{24}{48} = \frac{1}{2}$
d. $\frac{8}{48} = \frac{1}{6}$
e. $\frac{2}{48} = \frac{1}{24}$
f. $\frac{0}{48} = 0$
g. $\frac{24}{48} = \frac{1}{2}$
h. $\frac{4}{48} = \frac{1}{12}$
i. $\frac{16}{48} = \frac{1}{3}$

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CHAPTER 6: STATISTICS

Section 6.1: Four Basic Methods of Sampling

Section 6.1.H: Homework

Section 6.2: Frequency Distributions

Section 6.2.H: Homework

Section 6.3: Stem-and-Leaf Plots

Section 6.3.H: Homework

Section 6.4: Pie Graphs

Section 6.4.H: Homework

Section 6.5: Bar Graphs

Section 6.5.H: Homework

Section 6.6: Line Graphs

Section 6.6.H: Homework

Section 6.7: Histograms

Section 6.7.H: Homework

Section 6.8: Measures of Center

Section 6.8.H: Homework

Section 6.9: Measures of Variation

Section 6.9.H: Homework

Section 6.10: Percentiles

Section 6.10.H: Homework

Section 6.11: Quartiles

Section 6.11.H: Homework

Section 6.12: The Normal Distribution

Section 6.12.H: Homework

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Section 6.1: Four Basic Methods of Sampling

Learning Objectives

- Define data and statistics
- Explain the difference between a population and a sample
- Describe four basic methods of sampling

What is Statistics?

Statistics is the science of learning from data. In our modern world, we are constantly surrounded by information; from election polls and medical studies to sports analytics and social media trends. Statistics provides us with the tools and methods to collect, organize, analyze, and interpret this data in meaningful ways. Rather than relying on intuition or guesswork, statistics enables us to make informed decisions, test hypotheses, identify patterns, and draw reliable conclusions even in the face of uncertainty and variability. Whether you're a scientist conducting research, a business professional analyzing market trends, a healthcare worker evaluating treatment effectiveness, or simply a citizen trying to make sense of the news, statistical thinking is essential. In essence, statistics transforms raw data into actionable knowledge, making it one of the most practical and powerful tools you can master in today's data-driven society.

Definition: Data

Data are facts, observations, or measurements that are collected for analysis. Data can be classified as either:

- Qualitative (categorical): Descriptive information (e.g., colors, names, categories)
- Quantitative (numerical): Numerical measurements (e.g., heights, weights, temperatures)

Data is typically the raw material that statisticians work with to extract meaningful information.

Definition: Statistics

Statistics is the science of collecting, organizing, analyzing, interpreting, and presenting data. It involves methods for making decisions and drawing conclusions about populations based on sample data. Statistics can also refer to numerical measures calculated from data.

Definition: Population

A **population** is the complete collection of all individuals, objects, or measurements that are of interest in a particular study. It includes every member of the group being studied. For example:

- All registered voters in a country
- Every car battery manufactured by a company
- All students enrolled at Orange Coast College

Definition: Sample

A **sample** is a subset or portion of a population that is selected for study. It's a smaller, manageable group chosen to represent the larger population. For example:

- 1,000 randomly selected voters from all registered voters
- 100 car batteries tested from a production line
- 500 students surveyed from Orange Coast College

The goal is to use sample data to make inferences about the entire population, which is often more practical and cost-effective than studying every member of the population.

There are different strategies for gathering data depending on practical constraints like cost, time, accessibility, and the nature of the population. Below are the four basic probability sampling methods where each member has an equally likely chance of being selected.

The Four Basic Sampling Methods

Simple Random Sampling (SRS)

Every member of the population has an equal and independent chance of being selected. This is like drawing names from a hat where each name has the same probability of selection.

- **Example**: Using a random number generator to select 100 student ID numbers from a list of all 5,000 students at a university.
- **Advantages**: Unbiased, straightforward
- **Disadvantages**: Requires a complete list of the population; may be impractical for large populations

Systematic Sampling

Select every k^{th} member of the population after a random starting point. The interval k is calculated by dividing the population size by the desired sample size.

- **Example**: From a list of 1,000 employees, randomly start at person #7, then select every 10th person (7, 17, 27, 37, etc.) to get 100 people.
- **Advantages**: Easy to implement, more convenient than simple random sampling
- **Disadvantages**: Can introduce bias if there's a hidden pattern in the list

Stratified Sampling

Divide the population into distinct subgroups (strata) based on specific characteristics, then randomly sample from each stratum. The strata should be homogeneous within but heterogeneous between groups.

- **Example**: Divide students by year (freshman, sophomore, junior, senior), then randomly select 25 students from each year to ensure all years are represented.
- **Advantages**: Ensures representation of all subgroups; often more precise than simple random sampling
- **Disadvantages**: Requires knowledge of population characteristics; more complex to implement

Cluster Sampling

Divide the population into clusters (groups), randomly select entire clusters, then survey all members within the selected clusters (or sample within clusters in two-stage cluster sampling).

- **Example**: Randomly select 10 schools from a district (clusters), then survey all students in those 10 schools.
- **Advantages**: Cost-effective and practical for geographically dispersed populations
- **Disadvantages**: Less precise than other methods; clusters may not represent the full population diversity

Quick Summary for The Four Basic Sampling Methods

- **Simple Random**: Choose individuals randomly
- **Systematic**: Select every n^{th} individual
- **Stratified**: Divide into groups, sample from each
- **Cluster**: Randomly choose entire groups

The Key Difference Between Stratified and Cluster Sampling

Students often have a difficult time recognizing the difference between stratified sampling and cluster sampling. The fundamental difference lies in how the groups are formed and used. In stratified sampling, you divide the population into homogeneous subgroups called strata based on important characteristics (like age, income, education level, or gender), and then you sample from every stratum. The goal is to ensure that each distinct subgroup is represented in your final sample. For example, if you're surveying college students and you know that freshman, sophomores, juniors, and seniors may have different perspectives, you create four strata and randomly select students from each class year. This guarantees that all years are represented proportionally (or equally, depending on your design). The key point is that strata are created to be internally similar, so all freshmen share similar characteristics, while remaining distinct from one another.

In cluster sampling, you also divide the population into groups called clusters, but these clusters are typically formed by convenience or natural groupings (like schools, neighborhoods, city blocks, or hospitals) rather than by shared characteristics. More importantly, you randomly select only some clusters and then survey everyone (or a sample) within those selected clusters. You don't sample from all clusters, only the chosen ones. For instance, if studying elementary school children across a state, you might randomly select 15 schools (clusters) and survey all students in those 15 schools, ignoring all other schools entirely. Unlike strata, clusters should ideally be heterogeneous—each cluster should be a mini-representation of the entire population. The main advantage of cluster sampling is efficiency: it's much cheaper and easier to survey everyone in a few locations than to travel across the entire population, even though it typically produces less precise estimates than stratified sampling.

Example #6.1.1 🤔

A researcher numbered all of the colleges willing to release grade information and chose every tenth one. Identify the sampling method used.

✅ **Solution:**

Systematic.

Why It's Systematic Sampling?

- Ordered/numbered list: All colleges are numbered in a specific sequence
- Fixed interval: Every 10th college is selected (the sampling interval $k = 10$)
- Regular pattern: The selection follows a consistent, evenly-spaced pattern throughout the list

Why It's Not Other Methods?

- Not simple random: Not every college has an equal chance; once you choose your starting point and interval, the colleges selected are predetermined
- Not stratified: Colleges aren't divided into subgroups with samples taken from each group
- Not cluster: Individual colleges are being selected, not groups of colleges

Example #6.1.2 🤔

A researcher randomly selected three states and then gathered information from all colleges willing to release grade information in those states. Identify the sampling method used.

✅ **Solution:**

Cluster sampling.

Why It's Cluster Sampling?

- Natural groupings: The states serve as clusters, geographic groupings, that contain multiple colleges
- Randomly select entire clusters: The researcher randomly selected only 3 states (not all states)
- Survey all/many members within selected clusters: All colleges willing to release information within those 3 selected states are included
- Other clusters excluded: Colleges in the states that weren't selected are completely excluded from the study

Why It's Not Other Methods?

- Not stratified: The researcher doesn't sample from all states, only the 3 selected ones. If it were stratified, they would select some colleges from every state (or every region)
- Not systematic: There's no "every *k*th" selection pattern from an ordered list
- Not simple random: Individual colleges aren't being randomly selected from the entire country; entire geographic regions (states) are selected first

Example #6.1.3 🤔

A researcher numbered all of the colleges willing to release grade information and chose several of them using a random number generator. Identify the sampling method used.

✅ **Solution:**

Simple random sampling.

Why It's Simple Random Sampling?

- Complete list: All colleges willing to release information are numbered (creating a sampling frame)
- Random number generator: A truly random mechanism is used to select the sample
- Direct individual selection: Individual colleges are randomly selected directly from the entire population, not through any systematic pattern or grouping
- Equal probability: Each college has an equal and independent chance of being selected

Why It's Not Other Methods?

- Not systematic: There's no fixed interval or "every *k*th" pattern. The random number generator selects colleges independently, not in a regular sequence
- Not stratified: Colleges aren't divided into subgroups (by region, size, type, etc.) with samples taken from each subgroup

- Not cluster: Individual colleges are being selected, not groups of colleges. Also, the researcher isn't selecting all colleges within certain randomly chosen groups.

Example #6.1.4 🤔

A researcher grouped the schools by two-year state schools, four-year state schools, two-year private schools, and four-year private schools. The researcher then decided to choose five schools from each of those groups. Identify the sampling method used.

✅ **Solution:**

Stratified sampling.

Why It's Stratified Sampling?

- Population divided into strata: Schools are divided into four distinct subgroups (strata) based on two characteristics:
  - Type of institution (state vs. private)
  - Length of program (two-year vs. four-year)
- Sample from ALL strata: The researcher selects schools from each of the four groups—no group is excluded
- Ensures representation: Every type of school (two-year state, four-year state, two-year private, four-year private) is guaranteed to be represented in the final sample
- Equal allocation: Five schools are selected from each stratum (though the problem doesn't explicitly state "randomly," it's implied in proper sampling)

Why It's Not Other Methods?

- Not simple random: Schools aren't just randomly selected from all schools, they're deliberately grouped first
- Not cluster: The researcher doesn't randomly select some groups and ignore others; samples are taken from all four groups
- Not systematic: There's no "every k^{th}" selection pattern

Example #6.1.5 🤔

You want to survey customer satisfaction from all customers of an online store. You have a list of 10,000 email addresses. You use a random number generator to pick 200 email addresses. You send the survey only to those 200 customers. Identify the sampling method used.

✅ **Solution:**

Simple random sampling.

Why It's Simple Random Sampling?

- Complete list: You have all 10,000 email addresses (a complete sampling frame)
- Random number generator: A truly random mechanism is used to select the sample
- Direct individual selection: Individual customers are randomly selected directly from the entire population of 10,000
- Equal and independent probability: Each customer has an equal chance of being selected (200/10,000 = 2%), and each selection is independent of the others
- No pattern or grouping: There's no systematic interval, no grouping into strata, and no cluster structure, just pure random selection

Why It's Not Other Methods?

- Not systematic: There's no "every *k*th" pattern or fixed interval. The random number generator selects email addresses independently and randomly
- Not stratified: Customers aren't divided into subgroups (like by purchase history, location, or demographics) before sampling
- Not cluster: Individual customers are being selected, not groups of customers

Example #6.1.6 🤔

A school district wants to understand opinions on remote learning. They know experiences may differ by grade level. They divide the student population into three strata: elementary school, middle school, and high school. From each stratum, they randomly select 100 students to survey—100 from elementary, 100 from middle school, 100 from high school. Identify the sampling method used.

✅ **Solution:**

Stratified sampling.

Why It's Stratified Sampling?

- Population divided into strata: Students are divided into three distinct, homogeneous subgroups based on grade level:
  - Elementary school
  - Middle school
  - High school
- Sample from ALL strata: Random samples are taken from each of the three groups; no group is excluded
- Characteristic-based grouping: The strata are created based on an important characteristic (grade level) that is believed to affect the variable of interest (opinions on remote learning)
- Ensures representation: Every grade level is guaranteed to be represented in the final sample of 300 students
- Equal allocation: In this case, equal sample sizes (100 from each stratum) are used, ensuring balanced representation

Why It's Not Other Methods?

- Not simple random: Students aren't just randomly selected from the entire district without consideration of grade level
- Not cluster: The district doesn't randomly select some schools and ignore others; they sample from all three grade-level groups
- Not systematic: There's no "every *k*th" selection pattern from an ordered list

Example #6.1.7 🤔

A hospital has 1,000 patients on its inpatient floor. They list them alphabetically and want to survey 100 patients. They randomly choose a starting point between 1 and 10; suppose they pick 7. Then they pick every 10th patient: patient #7, #17, #27, and so on. Identify the sampling method used.

✅ **Solution:**

Systematic sampling.

Why It's Systematic Sampling?

- Ordered list: All 1,000 patients are listed in a specific order (alphabetically)
- Sampling interval (*k*): There's a fixed interval of 10—every 10th patient is selected
- *k* = Population size ÷ Desired sample size = 1,000 ÷ 100 = 10

- Random starting point: A number is randomly chosen from the first interval (1-10), which turns out to be 7
- Regular, evenly-spaced pattern: Once the starting point is selected, the pattern is predetermined and consistent: 7, 17, 27, 37, 47, 57... continuing until 100 patients are selected

Why It's Not Other Methods?

- Not simple random: The selection isn't completely random. Once you pick patient #7 as the start, all other selections are determined by the interval pattern
- Not stratified: Patients aren't divided into subgroups with samples taken from each group
- Not cluster: Individual patients are being selected using a pattern, not selecting entire groups of patients

Section 6.1: Four Basic Methods of Sampling [*In-Class Exercises*]

For each scenario, in Exercises 1-10, identify the sampling method used: simple random, stratified, cluster, or systematic.

1. A professor wants to survey students at a college with 10,000 students. She assigns each student a number and uses a random number generator to select 200 students to survey.
2. To survey middle school students in a district, a researcher randomly selects five schools and surveys every student in those schools.
3. A store wants customer feedback. Starting with the 7th customer of the day, the manager surveys every 20th customer after that.
4. A researcher studying study habits divides students into freshmen, sophomores, juniors, and seniors. She then randomly selects 50 students from each class year.
5. A researcher divides voters into Democrats, Republicans, and Independents, then randomly surveys 100 people from each group.
6. A university randomly selects 300 student ID numbers from a list of all enrolled students.
7. Starting with the 10th name on a customer list, a company surveys every 25th customer.
8. A school counselor randomly selects 20 students from each grade level to study stress levels.
9. A city randomly selects four neighborhoods and surveys every household in those neighborhoods.
10. A public health researcher wants to study exercise habits among high school students in a large state. Instead of surveying individual students across the entire state, she randomly selects 10 high schools and surveys every student at those schools.

Answers available at the end of this section.

Answers

Section 6.1: Four Basic Methods of Sampling [*In-Class Exercises*]

1. Simple random sampling (Key idea: pure chance, no grouping or structure)
2. Cluster (Key idea: randomly choose groups, then survey everyone in those groups)
3. Systematic (Key idea: regular spacing or intervals in selection)
4. Stratified (Key idea: ensures all important groups are represented rather than risking over- or under-representation)
5. Stratified (Key idea: ensures all major groups are represented rather than risking over- or under-representation)
6. Simple random sampling (Key idea: pure chance, no grouping or structure)
7. Systematic (Key idea: regular spacing or intervals in selection)
8. Stratified (Key idea: ensures all important groups are represented rather than risking over- or under-representation)
9. Cluster (Key idea: randomly choose groups, then survey everyone in those groups)
10. Cluster (Key idea: randomly choose groups, then survey everyone in those groups).

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Section 6.1.H: Homework

Section 6.1: Four Basic Methods of Sampling - Homework Exercises

(Section 6.1) - Homework Exercise #1

The company has offices in 10 cities across the country (all with roughly the same number of employees in similar roles). You don't have the capacity to travel to every office to collect your data, so you use random sampling to select 3 offices. Identify the sampling method used.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #2

All employees of the company are listed in alphabetical order. From the first 10 numbers, you randomly select a starting point: number 6. From number 6 onward, every 10th person on the list is selected (6, 16, 26, 36, and so on), and you end up with a sample of 100 people. Identify the sampling method used.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #3

You want to select a simple random sample of 1000 employees of a social media marketing company. You assign a number to every employee in the company database from 1 to 1000, and use a random number generator to select 100 numbers. Identify the sampling method used.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #4

A student council surveys 100 students by getting random samples of 25 freshmen, 25 sophomores, 25 juniors, and 25 seniors. Identify the sampling method used.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #5

A kindergarten teacher puts her students' names in a hat and chooses 4 students without looking. Identify the sampling method used.

Answer available at the end of this section.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #6

An airline company wants to survey its customers one day, so they randomly select flights that day and survey every passenger on those flights. Identify the sampling method used.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #7

At a local technical school, five auto repair classes are randomly selected and all of the students from each class are interviewed. Identify the sampling method used.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #8

Thirty-five math majors, 33 music majors and 45 history majors are randomly selected from 251 math majors, 518 music majors and 332 history majors at the state university. Identify the sampling method used.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #9

A market researcher randomly selects 200 homeowners under 65 years of age and 200 homeowners over 65 years of age. Identify the sampling method used.

Answer available at the end of this section.

(Section 6.1) - Homework Exercise #10

An education researcher randomly selects 60 of the nation's junior colleges and interviews all of the professors at each school. Identify the sampling method used.

Answer available at the end of this section.

Answers

(Section 6.1) - Homework Exercise #1
Cluster

(Section 6.1) - Homework Exercise #2
Systematic

(Section 6.1) - Homework Exercise #3
Simple random sample

(Section 6.1) - Homework Exercise #4
Stratified

(Section 6.1) - Homework Exercise #5
Simple random sample

(Section 6.1) - Homework Exercise #5

(Section 6.1) - Homework Exercise #6
Cluster

(Section 6.1) - Homework Exercise #7
Cluster

(Section 6.1) - Homework Exercise #8
Stratified

(Section 6.1) - Homework Exercise #9
Stratified

(Section 6.1) - Homework Exercise #10
Cluster

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Section 6.2: Frequency Distributions

Learning Objectives

- Contrast descriptive and inferential statistics
- Construct an ungrouped frequency distribution for a data set
- Construct a grouped frequency distribution for a data set

Statistics can be divided into two main branches, descriptive statistics and inferential statistics, each serving different purposes in data analysis.

Definition: Descriptive Statistics

Descriptive statistics involves methods for organizing, summarizing, and presenting data in a clear and understandable way. It describes what the data shows without drawing conclusions beyond the data itself.

Common Tools & Techniques for Descriptive Statistics

- Measures of center: Mean, median, mode
- Measures of spread: Range, standard deviation, variance, interquartile range
- Graphical displays: Histograms, stem-and-leaf plots, pie charts, bar graphs, box plots
- Tables: Frequency distributions, contingency tables
- Percentages and proportions: Summary statistics

Note: Most of the tools and techniques mentioned above will be discussed further throughout the chapter.

Examples of Descriptive Statistics

- The average (mean) test score in a class is 82
- 65% of survey respondents prefer option A
- The temperature ranged from 68°F to 83°F over 20 days
- Creating a stem-and-leaf plot of customer ages
- Calculating that the median household income in a city is $55,000

Descriptive statistics only describes the data you have collected. It makes no predictions or generalizations beyond that specific data set.

Definition: Inferential Statistics

Inferential statistics involves methods for making predictions, decisions, or generalizations about a population based on sample data. It goes beyond the data to draw conclusions.

Common Tools & Techniques for Inferential Statistics

- Hypothesis testing: t-tests, chi-square tests, ANOVA
- Confidence intervals: Estimating population parameters with a margin of error
- Regression analysis: Predicting relationships between variables
- Probability theory: Foundation for making inferences
- Sampling distributions: Understanding variability in estimates
- P-values and significance levels: Determining statistical significance

Note: All of the tools and techniques mentioned above will NOT be discussed in this text. These topics are discussed in an Elementary Statistics class such as a STAT C1000 Introduction To Statistics course that can be taken at any California Community College.

Examples of Inferential Statistics

- Based on a sample of 500 voters, we estimate that 52% ± 4% of all voters support the candidate
- A clinical trial concludes that a new drug is significantly more effective than a placebo ($P < 0.05$)
- Predicting that increasing study time by 1 hour will raise test scores by 5 points on average
- Testing whether there's a significant difference in customer satisfaction between two store locations
- Using sample data to determine if a new teaching method improves student performance

Inferential statistics uses sample data to make generalizations about populations. It involves uncertainty and requires probability theory to assess reliability.

Now, that we are familiar with descriptive statistics as the broad field of organizing and summarizing data, **frequency distributions** provide the first concrete tool we need to learn. Frequency distributions are often the starting point for all data analysis. Before you can calculate a mean, create a histogram, or understand a distribution's shape, you need to know how often each value (or range of values) occurs. This makes frequency distributions a logical "first step" in the descriptive statistics toolkit.

Definition: Frequency Distributions

A **frequency distribution** is a table that organizes data by showing how often each value (or group of values) occurs.

Two Types of Frequency Distributions

- Ungrouped frequency distributions - Used for either qualitative data (categorical data) or single value data
- Grouped frequency distributions - Used for quantitative data (numerical data) where the data needs to be grouped into class intervals

Recall from the previous section that qualitative or categorical variables yields descriptive information (e.g., colors, names, categories), whereas quantitative variables yields numerical measurements (e.g., heights, weights, temperatures). Generally speaking if you have a small data set with relatively few distinct values (typically fewer than 15-20 unique values), then its best to use ungrouped frequency distributions. On the other hand, if you have a large data set or many unique values, it is best to use grouped frequency distributions where the data needs to be grouped into class intervals.

Constructing Ungrouped Frequency Distributions

- **Step 1 — Organize your data (optional, if not using tallies)**
- **Step 2 — Identify all unique values in your data sets**
- **Step 3 — Create a two-column table (without tallies) or a three-column table (with tallies):**
  - Two-column table:
    - Column 1: "Value" or "Data Value" (list each unique value)

        - Column 2: "Frequency" (count how many times each appears)
    - Three-column table:
        - Column 1: "Value" or "Data Value" (list each unique value)
        - Column 2: "Tally" (tally each observation; this is best when you did not arrange the data in order in Step 1)
        - Column 3: "Frequency" (count how many times each appears)
- **Step 4 — Count or use tally marks**
- **Step 5 — Record the count**
- **Step 6 — Check your work**

Let's see several examples for the simpler of the two, ungrouped frequency distributions.

Example #6.2.1 🤔

Consider the data set for twenty overall ratings from 1 through 10 of an exclusive Disney+ show:

4, 7, 5, 9, 10, 2, 8, 1, 7, 9, 7, 6, 1, 9, 7, 6, 4, 1, 8, 7

Construct an ungrouped frequency distribution for the data set.

✅ **Solution:**

- **Step 1 — Organize your data (optional, if not using tallies):** 1, 1, 1, 2, 4, 4, 5, 6, 6, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 10
- **Step 2 — Identify all unique values in your data sets:** 1, 2, 4, 5, 6, 7, 8, 9, 10
- **Step 3 — Create a two-column table (without tallies) or a three-column table (with tallies):**
    - Two-column table:
        - Column 1: "Value" or "Data Value" (list each unique value)
        - Column 2: "Frequency" (count how many times each appears)
    - Three-column table:
        - Column 1: "Value" or "Data Value" (list each unique value)
        - Column 2: "Tally" (tally each observation; this is best when you did not arrange the data in order in Step 1)
        - Column 3: "Frequency" (count how many times each appears)
- **Step 4 — Count or use tally marks:** Count how many times each value appears in your data.
- **Step 5 — Record the count:** Place each count in the frequency column.
- **Step 6 — Check your work:** The sum of all frequencies should equal total number of data values.

Example #6.2.1 Solution: Ungrouped Frequency Distribution

Ratings	Tally	Frequency
1	| | |	3
2	|	1
3		0
4	| |	2
5	|	1
6	| |	2
7	$\cancel{\text{\| \| \| \|}}$	5
8	| |	2
9	| | |	3
10	|	1
Total		**20** ✓

Example #6.2.2 🤔

A parking lot attendant recorded the number of cars parked each hour for 25 hours:

15, 18, 15, 20, 18, 15, 20, 18, 15, 20, 18, 20, 22, 17, 19, 18, 20, 18, 18, 17, 18, 23, 18, 20, 18

Construct an ungrouped frequency distribution for the data set.

✅ **Solution:**

- **Step 1 — Organize your data (optional, if not using tallies):** Since there are many observations, it is best to skip this step and use tallies going forward.
- **Step 2 — Identify all unique values in your data sets:** 15, 17, 18, 19, 20, 22, 23
- **Step 3 — Create a three-column table (with tallies):**
  - Three-column table:
    - Column 1: "Value" or "Data Value" (list each unique value)
    - Column 2: "Tally" (tally each observation; this is best when you did not arrange the data in order in Step 1)
    - Column 3: "Frequency" (count how many times each appears)
- **Step 4 — Count or use tally marks:** Using the unordered data set, tally each observation starting with first observation, then the second observation, and so on, until all observations are accounted for.
- **Step 5 — Record the count:** Count the tallies and place each count in the frequency column.
- **Step 6 — Check your work:** The sum of all frequencies should equal total number of data values.

Example #6.2.2 Solution: Ungrouped Frequency Distribution

No. of Cars	Tally	Frequency
15	𝍸	4
16		0
17	||	2
18	𝍸 𝍸	10
19	|	1
20	𝍸 |	6
21		0
22	|	1
23	|	1
Total		**25** ✓

Example #6.2.3 🤔

A single dice was rolled 30 times with the following results:

3, 5, 2, 6, 4, 1, 3, 5, 6, 2, 4, 3, 6, 1, 5, 3, 4, 2, 6, 5, 3, 5, 1, 6, 2, 5, 3, 4, 6, 5

Construct an ungrouped frequency distribution for the data set.

✅ **Solution:**

- **Step 1 — Organize your data (optional, if not using tallies):** Since there are many observations, it is best to skip this step and use tallies going forward.
- **Step 2 — Identify all unique values in your data sets:** 1, 2, 3, 4, 5, 6
- **Step 3 — Create a three-column table (with tallies):**
  - Three-column table:
    - Column 1: "Value" or "Data Value" (list each unique value)
    - Column 2: "Tally" (tally each observation; this is best when you did not arrange the data in order in Step 1)
    - Column 3: "Frequency" (count how many times each appears)
- **Step 4 — Count or use tally marks:** Using the unordered data set, tally each observation starting with first observation, then the second observation, and so on, until all observations are accounted for.
- **Step 5 — Record the count:** Count the tallies and place each count in the frequency column.
- **Step 6 — Check your work:** The sum of all frequencies should equal total number of data values.

Example #6.2.3 Solution: Ungrouped Frequency Distribution

Outcome	Tally	Frequency
1	|||	3
2	||||	4
3		6
4	||||	4
5		7
6		6
Total		**30** ✓

Now, let's consider a data set consisting of temperatures (in degrees Fahrenheit) recorded in 20 cities on a particular day, rounded to the nearest degree: 58, 68, 65, 79, 51, 94, 64, 80, 69, 113, 63, 70, 102, 70, 92, 87, 77, 64, 64, 72. If we were to arrange the data in ascending order, we have 51, 58, 63, 64, 64, 64, 65, 68, 69, 70, 70, 72, 77, 79, 80, 87, 92, 94, 102, 113. Notice that almost all of the values are unique, which means most values have a frequency of 1. The only observation that does not have a frequency of 1 is '64', which has a frequency of 3.

An ungrouped frequency distribution would be ineffective for this data set. Since most values are unique, the resulting distribution would consist primarily of frequencies equal to one, appearing as a little to nothing more than a vertical listing of the original data set that was initially written horizontally. Such a representation provides minimal insight into the data's structure or patterns.

When data contain many distinct values or span a wide range, it becomes necessary to organize the values into class intervals. This grouping process creates a grouped frequency distribution, which reveals patterns and trends that would otherwise remain obscured in the raw data.

To construct a grouped frequency distribution, we must first establish appropriate class intervals, unless a class width and starting point are explicitly provided. This process involves several key decisions and calculations. For all intent purposes in this text, a class width be given as well as a starting value to use for the lower limit of the first class. If you would prefer to learn the process when a class width and starting point is not provided, use the interactive pull-down menu below. Otherwise, skip to the next one.

Constructing Grouped Frequency Distributions When a Class Width and Starting Point Are Not Given

- **Step 1 — Find the Range**
  - Range = Maximum value – Minimum value
- **Step 2 — Selecting the Number of Classes**
  - The number of classes should be chosen to provide a clear picture of the distribution without excessive detail or oversimplification. Generally, 5 to 20 classes are appropriate, though 5 to 10 classes work well for most introductory applications. Several guidelines exist for determining the number of classes:
    - For small data sets (n < 50), use 5-7 classes
    - For moderate data sets (50 ≤ n ≤ 100), use 7-10 classes
    - A common rule of thumb: $k \approx \sqrt{n}$, where k is the number of classes and n is the sample size
- **Step 3 — Calculating Class Width**
  - Once the number of classes is determined, calculate the **class width** using the formula: Class Width = Range ÷ Number of Classes
  - It is standard practice to round up to a convenient whole number or decimal value, even if the result is not exact.
- **Step 4 — Determine the Lower Limits of Each Class Interval**
  - With the class width determined, we construct the class intervals beginning with the minimum value (or a convenient value slightly below it).

- Lower Class Limit: The smallest value that can belong to a class
- Beginning with the starting point, create the other lower limits of each class by adding the class width repeatedly. The class width is defined as the difference of any two consecutive lower limits. So, simply add the class width to any lower limit of a class to get the lower limit for the very next class.

- **Step 5 — Determine the Upper Limits of Each Class Interval**
  - Upper Class Limit: The largest value that can belong to a class
  - Beginning with the lower limit, add one less than the class width to obtain the upper limit. Continue to do this for the rest of the upper limits. You could also create the other upper limits of each class by adding the class width repeatedly after you get the upper class limit of the first class. The class width can also be defined as the difference of any two consecutive upper limits.
- **Step 6 — Create the Frequency Distribution**
  - Create the classes by writing the lower limit then a dash followed by the upper limit, for each respective class.
  - Place the classes in the first column.
  - Use tallies and record the frequency/counts in the last column.
  - Verify the total. The sum of all frequencies should equal total number of data values.

Example #6.2.4 🤔

Let's use the data set from above of the temperatures (in degrees Fahrenheit) recorded in 20 cities on a particular day, rounded to the nearest degree:

58, 68, 65, 79, 51, 94, 64, 80, 69, 113, 63, 70, 102, 70, 92, 87, 77, 64, 64, 72.

If we were to arrange the data in ascending order, we have

Construct a grouped frequency distribution for the data set.

✅ **Solution:**

- **Step 1 — Find the Range**
  - Range = Maximum value – Minimum value
  - Maximum temperature = 113°F
  - Minimum temperature = 51°F
  - Range = 113 – 51 = 62°F
- **Step 2 — Selecting the Number of Classes**
  - The number of classes should be chosen to provide a clear picture of the distribution without excessive detail or oversimplification. Generally, 5 to 20 classes are appropriate, though 5 to 10 classes work well for most introductory applications. Several guidelines exist for determining the number of classes:
    - For small data sets (n < 50), use 5-7 classes. For our temperature data set with n = 20, we might choose 6 or 7 classes. Let's choose 6.
- **Step 3 — Calculating Class Width**
  - Class Width = Range ÷ Number of Classes
  - Class Width = 62 ÷ 6 = 10.33...
  - Rounding ***up*** to a convenient value: Class Width = 11°F.
- **Step 4 — Determine the Lower Limits of Each Class Interval**
  - With the class width determined, we construct the class intervals beginning with the minimum value (or a convenient value slightly below it).
    - Lower Class Limit: The smallest value that can belong to a class. In this data set, either 50 or 51 (the actual lowest value in the data set) is a good choice. Here we will choose out starting point at 51.
    - Beginning with the observation of 51, we create the other lower limits of each class by adding the class width of 11 repeatedly. The class width is defined as the difference of any two consecutive lower limits. So, simply add the class width to any lower limit of a class to get the lower limit for the very next class.
      - First class: Lower limit = 51
      - Second class: Lower limit = 62
      - Third class: Lower limit = 73
      - Fourth class: Lower limit = 84

    - Fifth class: Lower limit = 95
    - Sixth class: Lower limit = 106
  - We continue creating classes until all data values (up to 113°F) are covered. Note that if there was a seventh class, the lower limit would have been 117. Since the maximum value of 113 is less than 117, there is no need for the seventh class.
- **Step 5 — Determine the Upper Limits of Each Class Interval**
  - Upper Class Limit: The largest value that can belong to a class.
  - Beginning with the lower limit of 51, add one less than the class width to obtain the upper limit. So, 51 + 11 – 1 = 61. Continue to do this for the rest of the upper limits. You could also create the other upper limits of each class by adding the class width of 11 repeatedly after you get the upper class limit of the first class. The class width can also be defined as the difference of any two consecutive upper limits.
    - First class: Upper limit = 61
    - Second class: Upper limit = 72
    - Third class: Upper limit = 83
    - Fourth class: Upper limit = 94
    - Fifth class: Upper limit = 105
    - Sixth class: Upper limit = 116
- **Step 6 — Create a three-column table (with tallies)** (Exactly the same as ungrouped frequency distributions earlier)
  - Create the classes by writing the lower limit then a dash followed by the upper limit, for each respective class.
  - Place the classes in the first column.
  - Use tallies and record the frequency/counts in the last column.
  - Verify the total. The sum of all frequencies should equal total number of data values.

Example #6.2.4 Solution: Grouped Frequency Distribution

Temperature	Tally	Frequency
51—61	||	2
62—72	𝍸 𝍸	10
73—83	|||	3
84—94	|||	3
95—105	|	1
106—116	|	1
Total		**20** ✓

By applying the guidelines above, one can see that the frequency distributions will vary depending on the desired class width. A much simpler approach would to have the lower class limit given along with the desired class width. If that was to be the case, then there would be only one correct frequency distribution made under those conditions.

Constructing Grouped Frequency Distributions When a Class Width and Starting Point Are Given

- **Step 1 — Identify the Given Information**
  - Class width
  - Starting point
  - Data range
- **Step 2 — Determine the Lower Limits of Each Class Interval**
  - Lower Class Limit: The smallest value that can belong to a class

- Beginning with the starting point, create the other lower limits of each class by adding the class width repeatedly. The class width is defined as the difference of any two consecutive lower limits. So, simply add the class width to any lower limit of a class to get the lower limit for the very next class.

- **Step 3 — Determine the Upper Limits of Each Class Interval**
  - Upper Class Limit: The largest value that can belong to a class
  - Beginning with the lower limit, add one less than the class width to obtain the upper limit. Continue to do this for the rest of the upper limits. You could also create the other upper limits of each class by adding the class width repeatedly after you get the upper class limit of the first class. The class width can also be defined as the difference of any two consecutive upper limits.
- **Step 4 — Create a three-column table (with tallies)** (Exactly the same as ungrouped frequency distributions earlier)
  - Create the classes by writing the lower limit then a dash followed by the upper limit, for each respective class.
  - Place the classes in the first column.
  - Use tallies and record the frequency/counts in the last column
  - Verify the total. The sum of all frequencies should equal total number of data values.

Example #6.2.5 🤔

Let's use the data set from above of the temperatures (in degrees Fahrenheit) recorded in 20 cities on a particular day, rounded to the nearest degree:

58, 68, 65, 79, 51, 94, 64, 80, 69, 113, 63, 70, 102, 70, 92, 87, 77, 64, 64, 72.

Construct a grouped frequency distribution with a class width of 9 and a starting point of the lowest observation in the data set.

✅ **Solution:**

- **Step 1 — Identify the Given Information**
  - Class width: 8
  - Starting point: 51 (the minimum value in our data set)
  - Data range: From 51 to 113
- **Step 2 — Determine the Lower Limits of Each Class Interval**
  - Beginning with the lower observation of 51, we create the other lower limits of each class by adding the class width of 8 repeatedly. The class width is defined as the difference of any two consecutive lower limits. So, simply add the class width to any lower limit of a class to get the lower limit for the very next class.
    - First class: Lower limit = 51
    - Second class: Lower limit = 59
    - Third class: Lower limit = 67
    - Fourth class: Lower limit = 75
    - Fifth class: Lower limit = 83
    - Sixth class: Lower limit = 91
    - Seventh class: Lower limit = 99
    - Eighth class: Lower limit = 107
  - We continue creating classes until all data values (up to 113°F) are covered. Note that if there was a ninth class, the lower limit would have been 115. Since the maximum value of 113 is less than 115, there is no need for the ninth class.
- **Step 3 — Determine the Upper Limits of Each Class Interval**
  - Beginning with the lower limit of 51, add one less than the class width to obtain the upper limit. So, 51 + 8 – 1 = 58. Continue to do this for the rest of the upper limits. You could also create the other upper limits of each class by adding the class width of 8 repeatedly after you get the upper class limit of the first class. The class width can also be defined as the difference of any two consecutive upper limits.
    - First class: Upper limit = 58
    - Second class: Upper limit = 66
    - Third class: Upper limit = 74
    - Fourth class: Upper limit = 82
    - Fifth class: Upper limit = 90

    - Sixth class: Upper limit = 98
    - Seventh class: Upper limit = 106
    - Eighth class: Upper limit = 114
- **Step 4 — Create a three-column table (with tallies)** (Exactly the same as ungrouped frequency distributions earlier)
    - Create the classes by writing the lower limit then a dash followed by the upper limit, for each respective class.
    - Place the classes in the first column.
    - Use tallies and record the frequency/counts in the last column.
    - Verify the total. The sum of all frequencies should equal total number of data values.

Example #6.2.5 Solution: Grouped Frequency Distribution

Temperature	Tally	Frequency
51—58	||	2
59—66	𝍸	5
67—74	𝍸	5
75—82	|||	3
83—90	|	1
91—98	||	2
99—106	|	1
107—114	|	1
Total		**20** ✓

The grouped frequency distribution from above reveals several patterns:

- The highest concentration of temperatures occurs in the 59 — 66°F and 67 — 74°F ranges (5 cities each)
- Temperatures are relatively spread across the range
- There are some unusually high temperatures (102°F and 113°F) in the upper classes
- The distribution shows some clustering in the middle ranges with a few unusual outliers* at both ends

* Note: An outlier is a value that is much larger or much smaller than most other observations in the data set.

Example #6.2.6 🤔

A survey recorded the number of hours 30 students spent on homework per week:

12, 4, 18, 7, 15, 22, 9, 14, 20, 11, 16, 8, 19, 13, 10, 17, 21, 6, 23, 14, 12, 15, 18, 11, 16, 9, 20, 13, 17, 10

Construct a grouped frequency distribution with a class width of 5 and a starting point of the lowest observation in the data set.

✅ **Solution:**

- **Step 1 — Identify the Given Information**
    - Class width: 5
    - Starting point: 4 (the minimum value in our data set)

- Data range: From 4 to 23

- **Step 2 — Determine the Lower Limits of Each Class Interval**
  - Beginning with the lower observation of 4, we create the other lower limits of each class by adding the class width of 5 repeatedly. The class width is defined as the difference of any two consecutive lower limits. So, simply add the class width to any lower limit of a class to get the lower limit for the very next class.
    - First class: Lower limit = 4
    - Second class: Lower limit = 9
    - Third class: Lower limit = 14
    - Fourth class: Lower limit = 19
  - We continue creating classes until all data values (up to 23) are covered. Note that if there was a fifth class, the lower limit would have been 24. Since the maximum value of 25 is less than 24, there is no need for the fifth class.
- **Step 3 — Determine the Upper Limits of Each Class Interval**
  - Beginning with the lower limit of 4, add one less than the class width to obtain the upper limit. So, 4 + 5 – 1 = 8. Continue to do this for the rest of the upper limits. You could also create the other upper limits of each class by adding the class width of 5 repeatedly after you get the upper class limit of the first class. The class width can also be defined as the difference of any two consecutive upper limits.
    - First class: Upper limit = 8
    - Second class: Upper limit = 13
    - Third class: Upper limit = 18
    - Fourth class: Upper limit = 23
- **Step 4 — Create a three-column table (with tallies)** (Exactly the same as ungrouped frequency distributions earlier)
  - Create the classes by writing the lower limit then a dash followed by the upper limit, for each respective class.
  - Place the classes in the first column.
  - Use tallies and record the frequency/counts in the last column.
  - Verify the total. The sum of all frequencies should equal total number of data values.

Example #6.2.6 Solution: Grouped Frequency Distribution

No. of Hours	Tally	Frequency
4—8	𝍸	4
9—13	𝍸 𝍸	10
14—18	𝍸 𝍸	10
19—23	𝍸	6
Total		**30** ✓

Example #6.2.7 🤔

The weights (in pounds) of 22 packages shipped by a company:

8, 15, 23, 12, 19, 27, 10, 18, 24, 14, 21, 9, 16, 25, 11, 20, 28, 13, 22, 17, 26, 15

Construct a grouped frequency distribution with a class width of 4 and a starting point of the lowest observation in the data set.

✅ **Solution:**

- **Step 1 — Identify the Given Information**
  - Class width: 4
  - Starting point: 8 (the minimum value in our data set)
  - Data range: From 8 to 28

- **Step 2 — Determine the Lower Limits of Each Class Interval**
  - Beginning with the lower observation of 8, we create the other lower limits of each class by adding the class width of 4 repeatedly. The class width is defined as the difference of any two consecutive lower limits. So, simply add the class width to any lower limit of a class to get the lower limit for the very next class.
    - First class: Lower limit = 8
    - Second class: Lower limit =12
    - Third class: Lower limit = 16
    - Fourth class: Lower limit = 20
    - Fifth class: Lower limit = 24
    - Sixth class: Lower limit = 28
  - We continue creating classes until all data values (up to 28) are covered. Note that if there was a seventh class, the lower limit would have been 32. Since the maximum value of 28 is less than 32, there is no need for the seventh class.
- **Step 3 — Determine the Upper Limits of Each Class Interval**
  - Beginning with the lower limit of 51, add one less than the class width to obtain the upper limit. So, 8 + 4 – 1 = 11. Continue to do this for the rest of the upper limits. You could also create the other upper limits of each class by adding the class width of 4 repeatedly after you get the upper class limit of the first class. The class width can also be defined as the difference of any two consecutive upper limits.
    - First class: Upper limit = 11
    - Second class: Upper limit = 15
    - Third class: Upper limit = 19
    - Fourth class: Upper limit = 23
    - Fifth class: Upper limit = 27
    - Sixth class: Upper limit = 31
- **Step 4 — Create a three-column table (with tallies)** (Exactly the same as ungrouped frequency distributions earlier)
  - Create the classes by writing the lower limit then a dash followed by the upper limit, for each respective class.
  - Place the classes in the first column.
  - Use tallies and record the frequency/counts in the last column.
  - Verify the total. The sum of all frequencies should equal total number of data values.

Example #6.2.7 Solution: Grouped Frequency Distribution

No. of Pounds	Tally	Frequency
8—11	||||	4
12—15	~~||||~~	5
16—19	||||	4
20—23	||||	4
24—27	||||	4
28—31	|	1
Total		**22** ✓

Example #6.2.8 🤔

The following data shows the prices (in dollars) of 20 textbooks:

78, 92, 65, 88, 110, 73, 85, 82, 67, 105, 89, 76, 98, 71, 85, 45, 102, 53, 91, 80

Construct a grouped frequency distribution with a class width of 9 and a starting point of the lowest observation in the data set.

✅ **Solution:**

- **Step 1 — Identify the Given Information**
  - Class width: 9
  - Starting point: 45 (the minimum value in our data set)
  - Data range: From 45 to 110
- **Step 2 — Determine the Lower Limits of Each Class Interval**
  - Beginning with the lower observation of 45, we create the other lower limits of each class by adding the class width of 9 repeatedly. The class width is defined as the difference of any two consecutive lower limits. So, simply add the class width to any lower limit of a class to get the lower limit for the very next class.
    - First class: Lower limit = 45
    - Second class: Lower limit = 54
    - Third class: Lower limit = 63
    - Fourth class: Lower limit = 72
    - Fifth class: Lower limit = 81
    - Sixth class: Lower limit = 90
    - Seventh class: Lower limit = 99
    - Eighth class: Lower limit = 108
  - We continue creating classes until all data values (up to 110) are covered. Note that if there was a ninth class, the lower limit would have been 117. Since the maximum value of 110 is less than 117, there is no need for the ninth class.
- **Step 3 — Determine the Upper Limits of Each Class Interval**
  - Beginning with the lower limit of 45, add one less than the class width to obtain the upper limit. So, 45 + 9 – 1 = 53. Continue to do this for the rest of the upper limits. You could also create the other upper limits of each class by adding the class width of 9 repeatedly after you get the upper class limit of the first class. The class width can also be defined as the difference of any two consecutive upper limits.
    - First class: Upper limit = 53
    - Second class: Upper limit = 62
    - Third class: Upper limit = 71
    - Fourth class: Upper limit = 80
    - Fifth class: Upper limit = 89
    - Sixth class: Upper limit = 98
    - Seventh class: Upper limit = 107
    - Eighth class: Upper limit = 116
- **Step 4 — Create a three-column table (with tallies)** (Exactly the same as ungrouped frequency distributions earlier)
  - Create the classes by writing the lower limit then a dash followed by the upper limit, for each respective class.
  - Place the classes in the first column.
  - Use tallies and record the frequency/counts in the last column.
  - Verify the total. The sum of all frequencies should equal total number of data values.

Example #6.2.8 Solution: Grouped Frequency Distribution

Prices	Tally	Frequency
45—53	||	2
54—62		0
63—71	|||	3
72—80	||||	4
81—89	~~||||~~	5
90—98	|||	3
99—107	||	2
108—116	|	1
Total		**20** ✓

Example #6.2.9 🤔

The number of customers served per day at a restaurant over 31 days:

78, 105, 132, 89, 115, 145, 95, 122, 152, 85, 110, 138, 92, 118, 148, 111
82, 108, 135, 98, 125, 155, 88, 112, 142, 102, 128, 158, 75, 120, 140

Construct a grouped frequency distribution with a class width of 15 and a starting point of the lowest observation in the data set.

✅ **Solution:**

- **Step 1 — Identify the Given Information**
  - Class width: 15
  - Starting point: 75 (the minimum value in our data set)
  - Data range: From 75 to 158
- **Step 2 — Determine the Lower Limits of Each Class Interval**
  - Beginning with the lower observation of 75, we create the other lower limits of each class by adding the class width of 15 repeatedly. The class width is defined as the difference of any two consecutive lower limits. So, simply add the class width to any lower limit of a class to get the lower limit for the very next class.
    - First class: Lower limit = 75
    - Second class: Lower limit = 90
    - Third class: Lower limit = 105
    - Fourth class: Lower limit = 120
    - Fifth class: Lower limit = 135
    - Sixth class: Lower limit = 150
  - We continue creating classes until all data values (up to 158) are covered. Note that if there was a seventh class, the lower limit would have been 165. Since the maximum value of 158 is less than 165, there is no need for the seventh class.
- **Step 3 — Determine the Upper Limits of Each Class Interval**

- Beginning with the lower limit of 75, add one less than the class width to obtain the upper limit. So, 75 + 15 – 1 = 89. Continue to do this for the rest of the upper limits. You could also create the other upper limits of each class by adding the class width of 15 repeatedly after you get the upper class limit of the first class. The class width can also be defined as the difference of any two consecutive upper limits.
  - First class: Upper limit = 89
  - Second class: Upper limit = 104
  - Third class: Upper limit = 119
  - Fourth class: Upper limit = 134
  - Fifth class: Upper limit = 149
  - Sixth class: Upper limit = 164
- **Step 4 — Create a three-column table (with tallies)** (Exactly the same as ungrouped frequency distributions earlier)
  - Create the classes by writing the lower limit then a dash followed by the upper limit, for each respective class.
  - Place the classes in the first column.
  - Use tallies and record the frequency/counts in the last column.
  - Verify the total. The sum of all frequencies should equal total number of data values.

Example #6.2.9 Solution: Grouped Frequency Distribution

No. of Customers	Tally	Frequency
75—89	𝍸	6
90—104	𝍷	4
105—119	𝍸 ||	7
120—134	𝍸	5
135—149	𝍸 |	6
150—164	|||	3
Total		**31** ✓

Section 6.2: Frequency Distributions [*In-Class Exercises*]

1. Twenty overall ratings from 1 through 10 of an exclusive Netflix show is given below:

   4, 2, 5, 9, 10, 2, 8, 1, 7, 9, 7, 4, 1, 9, 7, 6, 4, 1, 8, 7

   Construct an ungrouped frequency distribution for the data set.

2. The heights (in inches) of 28 adult males is recorded.

   64, 65, 66, 66, 67, 67, 68, 68, 68, 69, 69, 69, 70, 70,
   70, 71, 71, 71, 72, 72, 72, 73, 73, 73, 74, 74, 75, 76

   Construct a grouped frequency distribution for the data set.

3. A random sample of annual salaries (in thousands of dollars) of 18 parents who just had their first child are given below:

   48, 50, 53, 55, 56, 60, 60, 60, 61,
   63, 68, 70, 72, 75, 78, 80, 85, 90

   Construct a grouped frequency distribution using a class width of 7 and start the first class with the lowest value.

4. Twenty AA batteries were tested to determine how long they would last. The results, to the nearest minute, were recorded as follows:

   423, 369, 387, 411, 393, 399, 371, 377, 409, 392, 417, 431, 401, 363, 391, 405, 425, 400, 381, 399

   Construct a grouped frequency distribution using a class width of 9 and start the first class with the lowest value.

Answers available at the end of this section.

Answers

Section 6.2: Frequency Distributions [*In-Class Exercises*]

1.

Exercise #6.2.1 Solution: Ungrouped Frequency Distribution

Ratings	Tally	Frequency
1	| | |	3
2	| |	2
3		0
4	| | |	3
5	|	1
6	|	1
7	| | | |	4
8	| |	2
9	| | |	3
10	|	1
Total		**20** ✓

2. Answers may vary. Answer below is using 5 classes with a class width of 3.

Exercise #6.2.2 Solution: Grouped Frequency Distribution

Heights	Tally	Frequency
64—66	𝍸	4
67—69	𝍸 𝍸	8
70—72	𝍸 𝍸 𝍸	9
73—75	𝍸 𝍸	6
76—78	𝍸	1
Total		**28** ✓

3.

Exercise #6.2.3 Solution: Grouped Frequency Distribution

Salaries	Tally	Frequency
48—54	𝍸	3
55—61	𝍸 𝍸	6
62—68	𝍸	2
69—75	𝍸	3
76—82	𝍸	2
83—89	𝍸	1
90—96	𝍸	1
Total		**18** ✓

4.

Exercise #6.2.4 Solution: Grouped Frequency Distribution

Minutes	Tally	Frequency
363—371	| | |	3
372—380	|	1
381—389	| |	2
390—398	| | |	3
399—407	𝍸	5
408—416	| |	2
417—425	| | |	3
426—434	|	1
Total		**20** ✓

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Section 6.2.H: Homework

Section 6.2: Frequency Distributions - Homework Exercises

(Section 6.2) - Homework Exercise #1

The 2025 World Series Champion Los Angeles Dodgers scored the following runs in each of their 17 games in the post season:

10, 8, 5, 4, 2, 2, 2, 5, 3, 5, 4, 5, 6, 2, 1, 3, 5

Construct an ungrouped frequency distribution for the data set.

Answer available at the end of this section.

(Section 6.2) - Homework Exercise #2

Students rated their satisfaction with a course on a scale of 1-5:

4, 1, 3, 4, 2, 5, 3, 2, 5, 2, 3, 5, 1, 5, 4, 5, 4, 4, 5, 2, 2, 3, 4, 5

Construct an ungrouped frequency distribution for the data set.

Answer available at the end of this section.

(Section 6.2) - Homework Exercise #3

A survey asked 34 people how many cups of coffee they drink per day:

2, 1, 3, 2, 0, 0, 1, 2, 3, 1, 1, 1, 2, 0, 3, 2, 1, 2, 3, 1, 2, 0, 1, 2, 3, 2, 1, 2, 2, 2, 1, 1, 2, 0

Construct an ungrouped frequency distribution for the data set.

Answer available at the end of this section.

(Section 6.2) - Homework Exercise #4

Daily rainfall amounts (in millimeters) recorded over 25 days:

2, 8, 15, 5, 11, 18, 3, 12, 20, 7, 14, 22, 4, 10, 17, 6, 13, 19, 9, 16, 23, 1, 11, 15, 8

Construct a grouped frequency distribution.

Answer available at the end of this section.

(Section 6.2) - Homework Exercise #5

Annual salaries (in thousands of dollars) for 28 employees:

38, 52, 68, 45, 59, 75, 42, 56, 72, 48, 62, 78, 40, 54, 70, 46, 60, 76, 44, 58, 74, 50, 54, 80, 36, 53, 69, 47

Construct a grouped frequency distribution with a class width of 7 and a starting point of the lowest observation in the data set.

Answer available at the end of this section.

Answer available at the end of this section.

(Section 6.2) - Homework Exercise #6

Reaction times (in milliseconds) for 23 participants in a study:

275, 248, 215, 232, 259, 288, 225, 252, 280, 238, 265, 295, 220, 247, 272, 235, 262, 228, 255, 282, 240, 298, 268

Construct a grouped frequency distribution with a class width of 11 and a starting point of the lowest observation in the data set.

Answer available at the end of this section.

Answers

(Section 6.2) - Homework Exercise #1

Homework Exercise #1 Solution: Ungrouped Frequency Distribution

No. of Runs	Tally	Frequency
1	|	1
2	||||	4
3	||	2
4	||	2
5	𝍸	5
6	|	1
7		0
8	|	1
9		0
10	|	1
Total		**17** ✓

(Section 6.2) - Homework Exercise #2

Homework Exercise #2 Solution: Ungrouped Frequency Distribution

Ratings	Tally	Frequency
1		2
2		5
3		4
4		6
5		7
Total		**24** ✓

(Section 6.2) - Homework Exercise #3

Homework Exercise #3 Solution: Ungrouped Frequency Distribution

No. of Cups	Tally	Frequency
0		5
1		10
2		14
3		5
Total		**34** ✓

(Section 6.2) - Homework Exercise #4

Answers may vary. Answer below is using 6 classes with a class width of 4.

Homework Exercise #4 Solution: Grouped Frequency Distribution

Rainfall (in)	Tally	Frequency
1 — 4		4
5 — 8		5
9 — 12		5
13 — 16		5
17 — 20		4
21 — 24		2
Total		**25** ✓

(Section 6.2) - Homework Exercise #5

Homework Exercise #5 Solution: Grouped Frequency Distribution

Salary	Tally	Frequency
36 — 42	||||	4
43 — 49		5
50 — 56		6
57 — 63	||||	4
64 — 70	|||	3
71 — 77	||||	4
78 — 84	||	2
Total		**28** ✓

(Section 6.2) - Homework Exercise #5

(Section 6.2) - Homework Exercise #6

Homework Exercise #6 Solution: Grouped Frequency Distribution

Reaction Time	Tally	Frequency
215 — 225	|||	3
226 — 236	|||	3
237 — 247	|||	3
248 — 258	|||	3
259 — 269	||||	4
270 — 280	|||	3
281 — 291	||	2
292 — 302	||	2
Total		**23** ✓

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Section 6.3: Stem-and-Leaf Plots

Learning Objectives

- Draw a stem and leaf plot for a data set

Another type of graphical display is a **stem-and-leaf plot**. Stem-and-leaf plots offer several unique advantages that make them valuable tools for data analysis. Unlike other graphical displays such as histograms or bar charts that summarize data into bars or categories, a stem-and-leaf plot maintains all the original data points. This dual nature being both a graph and a data list makes it particularly useful for exploratory data analysis.

Definition: Stem-and-Leaf Plot

A **stem-and-leaf plot** (also called a stem-and-leaf diagram, stem-and-leaf graph, and/or stemplot) is a data display method that organizes numerical data while preserving the actual data values. It provides a quick visual representation of the distribution's shape while allowing you to see every individual data point.

Each data value is split into two parts:

- Stem: The leading digit(s)
- Leaf: The trailing digit (usually the last digit)

The stems are listed vertically, and the leaves are listed horizontally next to their corresponding stem.Edit section

Making a Stem-and-Leaf Plot

- **Step 1** — Organize your data: Arrange your data values in order from smallest to largest. This isn't strictly necessary, but it makes the process easier and ensures your final plot is organized.
- **Step 2** — Decide on stems and leaves: Determine how to split each number:
  - Stem: Usually the leading digit(s) — the tens place for two-digit numbers
  - Leaf: Usually the last digit — the ones place for two-digit numbers
- **Step 3** — List all possible stems vertically: Write the stems in a column from smallest to largest, even if some stems won't have any leaves. This shows where gaps exist in the data
- **Step 4** — Add the leaves: Go through your sorted data and place each leaf next to its corresponding stem. Write leaves in order from smallest to largest. (If any leaf or leaves are too long to your disliking, you have the option to use to lines per stem, if necessary).
- **Step 5** — Add a key: Always include a key that explains how to read the plot. This is essential for interpretation.

✅ A Stem-and-Leaf Plot is BEST used for:

✓ Small to moderate data sets (roughly 15-100 values)
✓ Initial exploratory data analysis
✓ When you need to preserve exact data values

✓ Quick manual analysis without technology
✓ Comparing two small groups
✓ Teaching introductory statistics concepts

❌ A Stem-and-Leaf Plot is NOT IDEAL for:

✗ Very large data sets (becomes cluttered and hard to read)
✗ Data with many decimal places (too complex)
✗ Formal presentations (histograms look more professional)
✗ When only the distribution shape matters (histogram is cleaner)
✗ Comparing more than two groups

The stem-and-leaf plot is a "best of both worlds" tool: it gives you the visual insight of a graph while maintaining the precision of a data list. It's particularly valuable in the early stages of data analysis when you want to understand your data quickly without losing any information. While modern technology has made more sophisticated graphs easier to create, the stem-and-leaf plot remains a practical, efficient, and instructive tool for understanding distributions.

Example #6.3.1 🤔

In the NFL, the 2013 Denver Broncos set the record for both the most points scored in a season and most passing yards in a season, at 606 and 5,572, respectively, both the all-time NFL records. In that year, the following points scored per game is the following:

49, 41, 37, 52, 51, 35, 33, 45, 28, 27, 31, 35, 51, 20, 37, 34

Draw a stem-and-leaf plot for the data.

✅ **Solution:**

- **Step 1** — Organize your data: 20, 27, 28, 31, 33, 34, 35, 35, 37, 37, 41, 45, 49, 51, 51, 52
- **Step 2** — Decide on stems and leaves: Stems are the tens place and the leaves are the ones place (last digit) of any two-digit number in the data set.
- **Step 3** — List all possible stems vertically: Write the stems in a column from smallest to largest, even if some stems won't have any leaves. This shows where gaps exist in the data.
- **Step 4** — Add the leaves: Go through your sorted data and place each leaf next to its corresponding stem. Write leaves in order from smallest to largest.

$$\begin{array}{ll} 2 & 078 \\ 3 & 1345577 \\ 4 & 159 \\ 5 & 112 \end{array}$$

- **Step 5** — Add a key: Key: 2|0 = 20

Example #6.3.2 🤔

A meteorologist recorded the high temperatures (in °F) for 20 days:

68, 72, 75, 71, 69, 74, 78, 82, 79, 76, 73, 75, 77, 80, 81, 74, 76, 78, 79, 83

Draw a stem-and-leaf plot for the data.

✅ **Solution:**

- **Step 1** — Organize your data: 68, 69, 71, 72, 73, 74, 74, 75, 75, 76, 76, 77, 78, 78, 79, 79, 80, 81, 82, 83
- **Step 2** — Decide on stems and leaves: Stems are the tens place and the leaves are the ones place (last digit) of any two-digit number in the data set.
- **Step 3** — List all possible stems v
- ertically: Write the stems in a column from smallest to largest, even if some stems won't have any leaves. This shows where gaps exist in the data.
- **Step 4** — Add the leaves: Go through your sorted data and place each leaf next to its corresponding stem. Write leaves in order from smallest to largest.

```
6  89
7  12344556678899
8  0123
```

- **Step 5** — Add a key: Key: 6|8 = 68

Example #6.3.3 🤔

The ages of 20 customers at a local movie theater was recorded. The data set is as follows:

62, 55, 19, 15, 43, 28, 27, 16, 41, 57, 21, 21, 25, 46, 48, 19, 29, 23, 40, 20

Draw a stem-and-leaf plot for the data.

✅ **Solution:**

- **Step 1** — Organize your data: 15, 16, 19, 19, 20, 21, 21, 23, 25, 26, 27, 28, 29, 40, 41, 43, 46, 48, 55, 57
- **Step 2** — Decide on stems and leaves: Stems are the tens place and the leaves are the ones place (last digit) of any two-digit number in the data set.
- **Step 3** — List all possible stems vertically: Write the stems in a column from smallest to largest, even if some stems won't have any leaves. This shows where gaps exist in the data.
- **Step 4** — Add the leaves: Go through your sorted data and place each leaf next to its corresponding stem. Write leaves in order from smallest to largest.

```
1  5699
2  01135789
4  01368
5  57
6  2
```

- **Step 5** — Add a key: Key: 1|5 = 15.

Example #6.3.4 🤔

In the 2025-2026 National Hockey League (NHL) season, the overall standings in points for all 32 teams is displayed below:

2025-2026 NHL point standings by division

Pacific Division	Pts	Central Division	Pts	Metropolitan Division	Pts	Atlantic Division	Pts
Vegas Golden Knights	95	Colorado Avalanche	121	Carolina Hurricanes	113	Buffalo Sabres	109
Edmonton Oilers	93	Dallas Stars	112	Pittsburgh Penguins	98	Tampa Bay Lightning	106
Anaheim Ducks	92	Minnesota Wild	104	Philadelphia Flyers	98	Montréal Canadiens	106

Pacific Division	Pts	Central Division	Pts	Metropolitan Division	Pts	Atlantic Division	Pts
Los Angeles Kings	90	Utah Mammoth	92	Washington Capitals	95	Boston Bruins	100
San Jose Sharks	86	St. Louis Blues	86	Columbus Blue Jackets	92	Ottawa Senators	99
Seattle Kraken	79	Nashville Predators	86	New York Islanders	91	Detroit Red Wings	92
Calgary Flames	77	Winnipeg Jets	82	New Jersey Devils	87	Florida Panthers	84
Vancouver Canucks	58	Chicago Blackhawks	72	New York Rangers	77	Toronto Maple Leafs	78

Draw a stem-and-leaf plot for the data.

✅ **Solution:**

- **Step 1** — Organize your data: 58, 72, 77, 77, 78, 79, 82, 84, 86, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 95, 95, 98, 98, 99, 100, 104, 106, 106, 109, 112, 113, 121
- **Step 2** — Decide on stems and leaves: Stems are the tens place and the leaves are the ones place (last digit) of any two or three-digit number in the data set.
- **Step 3** — List all possible stems vertically: Write the stems in a column from smallest to largest, even if some stems won't have any leaves. This shows where gaps exist in the data.
- **Step 4** — Add the leaves: Go through your sorted data and place each leaf next to its corresponding stem. Write leaves in order from smallest to largest.

```
5   8
7   27789
8   246667
9   012222355889
10  04669
11  23
12  1
```

- **Step 5** — Add a key: Key: 5|8 = 58 or 12|1 = 121.

Example #6.3.5 🤔

The GPA's (rounded to the nearest tenth) of a fundamental high school were noted:

2.8, 2.9, 2.0, 2.4, 3.4, 4.0, 3.8, 3.6, 3.8, 2.9, 2.0, 2.8, 2.7, 0.9, 4.0, 3.5, 3.5, 2.8, 2.4, 2.3, 3.5, 2.8, 2.9, 3.2, 3.5, 3.7, 4.0, 2.2

Draw a stem-and-leaf plot for the data.

✅ **Solution:**

- **Step 1** — Organize your data: 0.9, 2.0, 2.0, 2.2, 2.3, 2.4, 2.4, 2.7, 2.8, 2.8, 2.8, 2.8, 2.9, 2.9, 2.9, 3.2, 3.4, 3.5, 3.5, 3.5, 3.5, 3.6, 3.7, 3.8, 3.8, 4.0, 4.0, 4.0
- **Step 2** — Decide on stems and leaves: So, for the first time we have decimals. As long as the data doesn't have a long range of values, we can still construct a stem-and-leaf plot here. Stems, now, are the ones place and the leaves are the tenths place (last digit) of any number in the data set.
- **Step 3** — List all possible stems vertically: Write the stems in a column from smallest to largest, even if some stems won't have any leaves. This shows where gaps exist in the data.
- **Step 4** — Add the leaves: Go through your sorted data and place each leaf next to its corresponding stem. Write leaves in order from smallest to largest.

0 | 9
2 | 00234478888999
3 | 2455556788
4 | 000

- **Step 5** — Add a key: Key: 0|9 = 0.9. (This is why it is inportant to have the key here, because there are no decimal points in the stem-and-leaf plot.

Section 6.3: Stem-and-Leaf Plots [*In-Class Exercises*]

For Exercises 1 through 3, draw a stem-and-leaf plot for each data set given.

1. The following data represent the number of minutes 15 students spent studying for an exam:

   45, 62, 58, 71, 67, 74, 53, 60, 68, 72, 59, 65, 55, 70, 64

2. The following data show the number of pages read by students last week:

   102, 87, 95, 110, 124, 98, 105, 116, 89, 92, 101, 108, 117, 90, 96, 112, 99, 120

3. The following data represent sales (in dollars) at a café during their slowest hour:

   145, 152, 138, 160, 149, 176, 155, 142, 158, 151, 147, 181, 132, 154, 152, 168, 118, 175

4. According to the California Department of Conservation, the table below shows the California's largest ever recorded earthquakes since 1800, ranked by date.

California's largest ever recorded earthquakes since 1800

Magnitude	Date	Location	Damage
7.9	Jan. 9, 1857	Fort Tejon	Two killed; 220-mile surface scar
7.4	Mar. 26, 1872	Owens Valley	27 killed; three aftershocks of magnitude >6
7.8	April 18, 1906	San Francisco	Possibly 3,000 killed; 225,000 displaced
7.2	Jan. 22, 1923	Mendocino*	Damaged homes in several towns
7.1	Nov. 4, 1927	SW of Lompoc*	No major injuries; slight damage in 2 counties
6.4	March 10, 1933	SE of Long Beach	115 killed; led to new building codes for schools
7.0	May 18, 1940	El Centro	9 killed; $6 million in damage
7.3	July 21, 1952	Kern County	12 killed; 3 magnitude >6 aftershocks in 5 days

Magnitude	Date	Location	Damage
6.6	Feb. 9, 1971	San Fernando	65 killed; 2,000 injured; $505 million in damage
7.4	Nov. 8, 1980	W. of Eureka*	Injured 6; $2 million in damage
6.9	Oct. 17, 1989	Bay Area	63 killed; 3,753 hurt; up to $10 billion in damage
7.2	April 25, 1992	Cape Mendocino	356 injuries; $48.3 million in damage
7.3	June 28, 1992	Landers	One killed; 400 injured; $9.1 million in damage
6.7	Jan. 17, 1994	Northridge	57 killed; 9,000 hurt, up to $40 billion in damage
7.1	Oct. 16, 1999	Ludlow	Minimal damage due to remote location
7.1	July 5, 2019	Ridgecrest/Trona	Preceded by M6.4 quake; no fatalities

Draw a stem-and-leaf plot for the magnitude ratings.

Answers available at the end of this section.

Answers

Section 6.3: Stem-and-Leaf Plots [*In-Class Exercises*]

1.
4 | 5
5 | 3589
6 | 024578
7 | 0124

2.
8 | 79
9 | 025689
10 | 1258
11 | 0267
12 | 04

3.
11 | 8
13 | 28
14 | 2579
15 | 122458
16 | 08
17 | 56
18 | 1

4.
6 | 4679
7 | 11122334489

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Section 6.3.H: Homework

Section 6.3: Stem-and-Leaf Plots - Homework Exercises

(Section 6.3) - Homework Exercise #1

Daily rainfall amounts (in millimeters) recorded over 25 days:

2, 8, 15, 5, 11, 18, 3, 12, 20, 7, 14, 22, 4, 10, 17, 6, 13, 19, 9, 16, 23, 1, 11, 15, 8

Draw a stem-and-leaf plot for the data.

Answer available at the end of this section.

(Section 6.3) - Homework Exercise #2

Annual salaries (in thousands of dollars) for 28 employees:

38, 52, 68, 45, 59, 75, 42, 56, 72, 48, 62, 78, 40, 54, 70, 46, 60, 76, 44, 58, 74, 50, 54, 80, 36, 53, 69, 47

Answer available at the end of this section.

(Section 6.3) - Homework Exercise #3

Reaction times (in milliseconds) for 23 participants in a study:

275, 248, 215, 232, 259, 288, 225, 252, 280, 238, 265, 295, 220, 247, 272, 235, 262, 228, 255, 282, 240, 298, 268

Answer available at the end of this section.

(Section 6.3) - Homework Exercise #4

Monthly electric bills (in dollars) for 26 households:

85, 112, 95, 128, 102, 128, 89, 135, 108, 96, 122, 91, 115, 105, 130, 98, 125, 110, 93, 120, 100, 132, 107, 116, 94, 127

Draw a stem-and-leaf plot for the data.

Answer available at the end of this section.

(Section 6.3) - Homework Exercise #5

Two classes took the same math quiz. Their scores are shown below.

Class A scores: 68, 72, 75, 81, 84, 86, 88, 90, 92
Class B scores: 70, 74, 76, 79, 83, 85, 87, 89, 93

Draw a back-to-back stem-and-leaf plot for the data. Note: A back-to-back stem-and-leaf plot places two stem-and-leaf plots next to each other, sharing the same stems in the center. For example, the data of 12, 13, 24 for one set, and the data 15, 16, 27 for the second set would look like this:

$$\begin{array}{cc|c} 32 & 1 & 56 \\ 4 & 2 & 7 \end{array}$$

Answer available at the end of this section.

Answer available at the end of this section.

Answers

(Section 6.3) - Homework Exercise #1

0 | 1234567889
1 | 011234556789
2 | 023

(Section 6.3) - Homework Exercise #2

3 | 68
4 | 0245678
5 | 02344689
6 | 0289
7 | 024568
8 | 0

(Section 6.3) - Homework Exercise #3

21 | 5
22 | 058
23 | 258
24 | 078
25 | 259
26 | 258
27 | 25
28 | 028
29 | 58

(Section 6.3) - Homework Exercise #4

8 | 59
9 | 134568
10 | 02578
11 | 0256
12 | 025788
13 | 025

(Section 6.3) - Homework Exercise #5

8 | 6
52 | 7 | 0469
8641 | 8 | 3579
20 | 9 | 3

(Section 6.3) - Homework Exercise #5

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Section 6.4: Pie Graphs

Learning Objectives

- Draw pie graphs

Definition: Pie Graph

A **pie graph** (also called a pie chart) is a circular graph divided into slices, where each slice represents a category or part of the whole. The size of each slice is proportional to the percentage or frequency of that category relative to the total. Pie graphs are particularly useful for displaying how a whole is divided into parts, making it easy to see the relative proportions at a glance.

When to Use Pie Graphs

Pie graphs are most effective when you want to:

- Show how parts make up a whole (100%)
- Display categorical data with relatively few categories (typically 2-6)
- Emphasize proportions or percentages rather than exact values
- Make quick visual comparisons of relative sizes

Key Components of a Pie Graph

1. **The Circle (Pie)**: Represents the whole data set or 100%
2. **Slices (Sectors)**: Each slice represents one category, with its size proportional to its percentage of the total
3. **Labels**: Identify what each slice represents, often including the category name and percentage or frequency
4. **Title**: Describes what the pie graph represents

Advantages of Pie Graphs

- **Intuitive**: Easy to understand—everyone recognizes the concept of "pieces of a pie"
- **Shows proportions clearly**: Visual impact makes it obvious which categories are larger or smaller
- **Emphasizes the whole**: Reinforces that all categories together equal 100%
- **Aesthetically appealing**: Colorful and engaging for presentations and reports

Limitations of Pie Graphs

- **Difficult to compare similar-sized slices**: When percentages are close, it's hard to tell which is larger
- **Not suitable for many categories**: More than 5-6 slices become cluttered and hard to read
- **Poor for precise comparisons**: Bar graphs are better when exact values matter
- **Can't show trends over time**: Only displays data for one time point
- **Requires percentages or proportions**: Doesn't work well with raw counts unless converted

When NOT to use Pie Graphs

Avoid pie graphs when:

- You have more than 6-7 categories (especially if some slices are too small)
- Categories have very similar percentages (hard to distinguish)
- You need to compare multiple data sets
- Exact values are more important than proportions
- You're showing change over time

In these cases, bar graphs, line graphs, or other displays are more appropriate.

Constructing a Pie Graph

- **Step 1** — Obtain the sum of your categorical data.
- **Step 2** — Calculate the frequency and percentage for each category.
- **Step 3** — Verify that percentages sum to 100%.
- **Step 4** — Sketch the circle and mark the slices based off the percentages (or use software).
- **Step 5** — Label each slice with the category name and percentage.
- **Step 6** — Add a title and legend, if necessary.

Example #6.4.1 🤔

A college surveyed 200 students about their class standing:

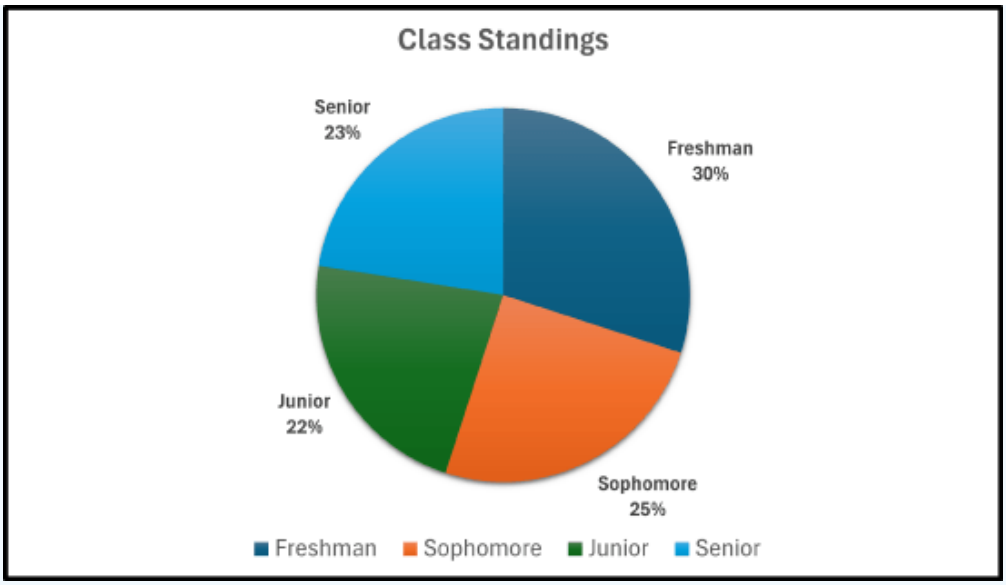
Distribution of students by class level

Classes	Frequency
Freshman	60
Sophomore	50
Junior	45
Senior	45

Create a pie graph to display this data.

✅ **Solution:**

- **Step 1 — Obtain the sum of your categorical data:** Here the sum was given. We will still verify to be sure. 60 + 50 + 45 + 45 = 200.
- **Step 2 — Calculate the frequency and percentage for each category:**
  - Freshman: $\frac{60}{200} = 30\%$
  - Sophomore: $\frac{50}{200} = 25\%$
  - Junior: $\frac{45}{200} = 22.5\%$
  - Senior: $\frac{45}{200} = 22.5\%$
- **Step 3 — Verify that percentages sum to 100%:** 30% + 25% + 22.5% + 22.5% = 100%.
- **Step 4 — Sketch the circle and mark the slices based off the percentages (or use software):** Here we used Excel software.
- **Step 5 — Label each slice with the category name and percentage:** Label each of the slices along with the percentages: Freshman at 30%, Sophomore at 25%, Junior at 22%, and Senior at 23%.
- **Step 6 — Add a title and legend, if necessary:** Add the title, "Class Standings" or some other similar title.



Example #6.4.2 🤔

A teacher recorded final grades for 40 students:

Distribution of letter grades

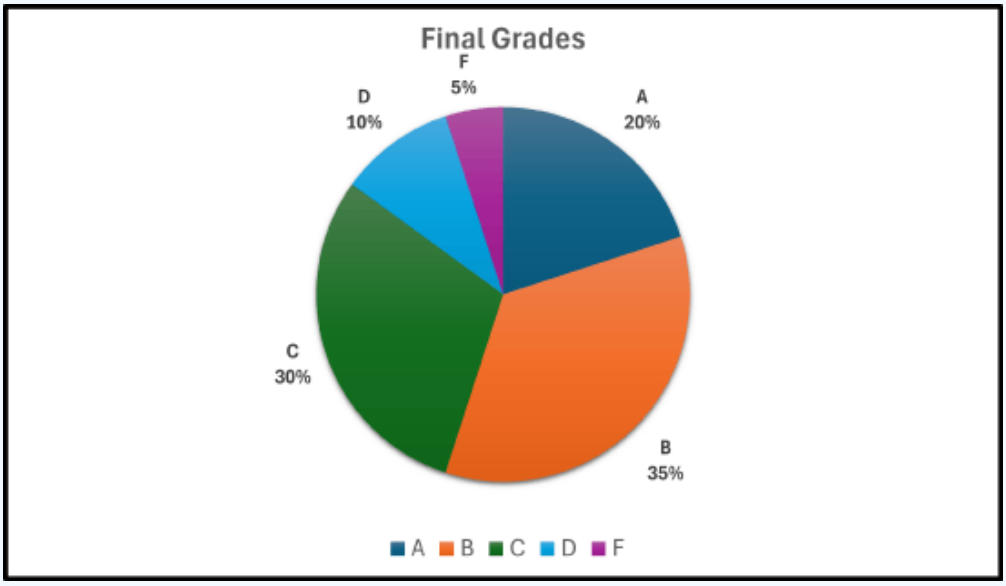
Classes	Frequency
A	8
B	14
C	12
D	4
F	2

Create a pie graph to display this data.

✅ **Solution:**

- **Step 1 — Obtain the sum of your categorical data:** Here the sum was given. We will still verify to be sure. 8 + 14 + 12 + 4 + 2 = 40.
- **Step 2 — Calculate the frequency and percentage for each category:**
  - A: $\frac{8}{40} = 20\%$
  - B: $\frac{14}{40} = 35\%$
  - C: $\frac{12}{40} = 30\%$
  - D: $\frac{4}{40} = 10\%$
  - F: $\frac{2}{40} = 5\%$
- **Step 3 — Verify that percentages sum to 100%:** 20% + 35% + 30% + 10% + 5% = 100%.
- **Step 4 — Sketch the circle and mark the slices based off the percentages (or use software):** Here we used Excel software.
- **Step 5 — Label each slice with the category name and percentage:** Label each of the slices along with the percentages: A at 20%, B at 35%, C at 30%, D at 10%, and F at 5%.

- **Step 6 — Add a title and legend, if necessary:** Add the title, "Final Grades" or some other similar title.



Example #6.4.3 🤔

A survey was conducted and asked adults their primary source of news:

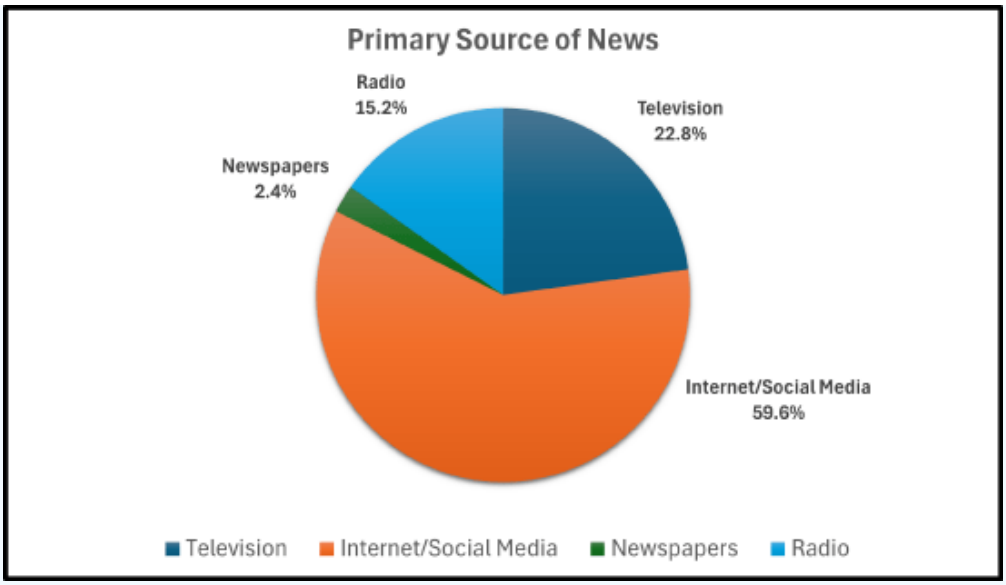
Primary news sources of respondents

News Source	Respondents
Television	57
Internet/Social Media	149
Newspapers	6
Radio	38

Create a bar graph to display this data.

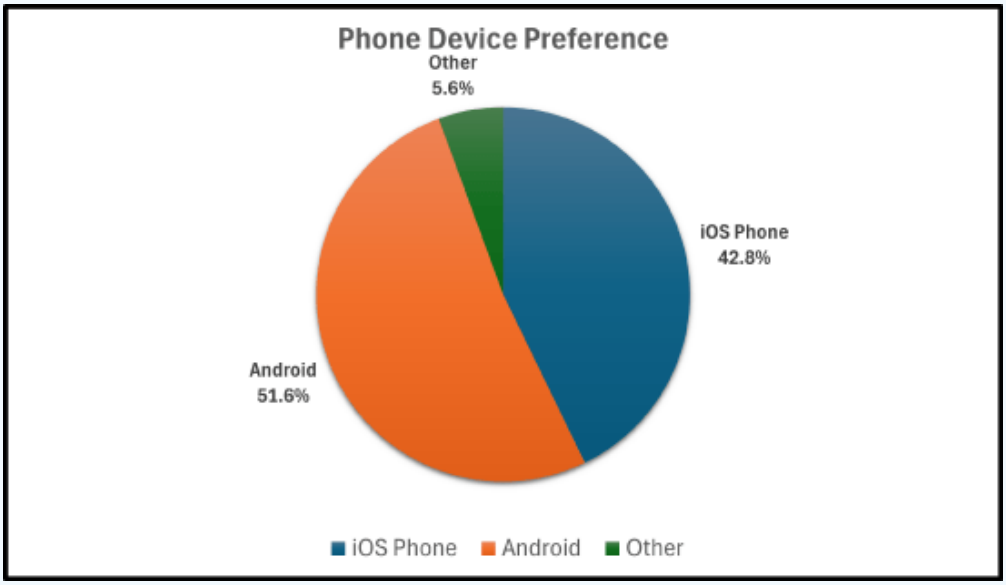
✅ **Solution:**

- **Step 1 — Obtain the sum of your categorical data:** The sum is 57 + 149 + 6 + 38 = 250.
- **Step 2 — Calculate the frequency and percentage for each category:**
  - Television: $\frac{57}{250} = 22.8\%$
  - Internet/Social Media: $\frac{149}{250} = 59.6\%$
  - Newspapers: $\frac{6}{250} = 2.4\%$
  - Radio: $\frac{38}{250} = 15.2\%$
- **Step 3 — Verify that percentages sum to 100%:** 22.8% + 59.6% + 2.4% + 15.2% = 100%.
- **Step 4 — Sketch the circle and mark the slices based off the percentages (or use software):** Here we used Excel software.
- **Step 5 — Label each slice with the category name and percentage:** Label each of the slices along with the percentages: Television at 22.8%, Internet/Social Media at 59.6%, Newspapers at 2.4%, and Radio at 15.2%
- **Step 6 — Add a title and legend, if necessary:** Add the title, "Primary Source of News" or some other similar title.



Example #6.4.4 🤔

A survey of 500 smartphone users were asked which type of phone that they own/prefer. The results are in the pie graph below:



a. How many smartphone users own an iPhone?
b. How many smartphone users do not own an Android?
c. Based off these results, how many Android users would we expect there to be in a town of 1,200 people.

✅ **Solution:**

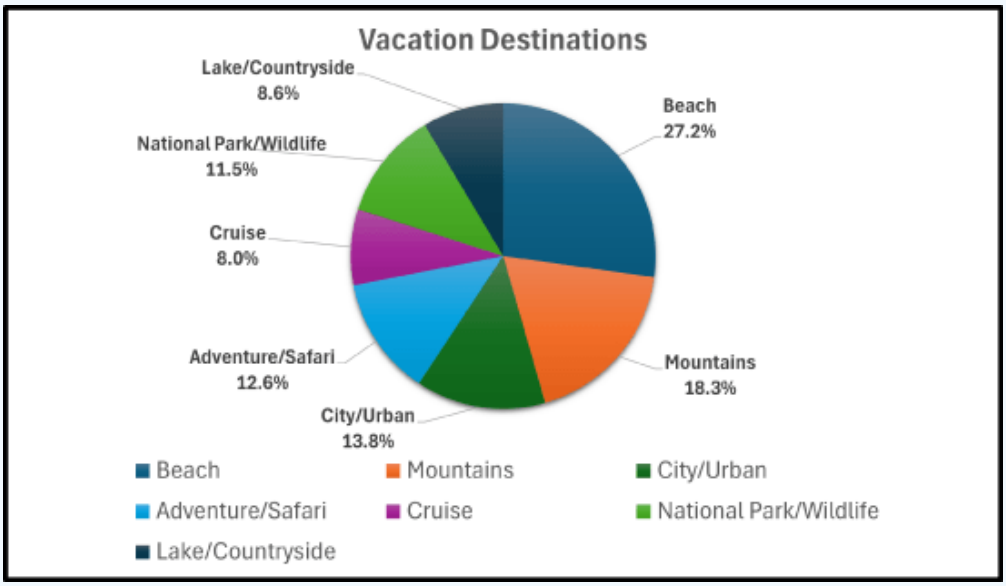
a. How many smartphone users own an iPhone?
   - Change 42.8% to a decimal, which is 0.428. Then, $(0.428) \cdot (500) = 214$. So, 214 smartphone users own an iPhone.

b. How many smartphone users do not own an Android?
   - $5.6\% + 42.8\% = 48.4\%$. Change 48.4% to a decimal, which is 0.484. Then, $(0.484) \cdot (500) = 242$. So, 242 smartphone users do not own an Android.

c. Based off these results, how many Android users would we expect there to be in a town of $1{,}200$ people.
   - Change 51.6% to a decimal, which is 0.516. Then, $(0.516) \cdot (1{,}200) = 619.2$. So, about 619 smartphone users own an Android.

Example #6.4.5 🤔

A travel agency surveyed 349 clients about preferred vacation types. The results are in the pie graph below:



a. How many clients preferred the mountains?
b. How many clients preferred lake/countryside or national parks/wildlife?
c. Which vacation type is the most popular? How many clients preferred that vacation type?
d. Which vacation type is the least popular? How many clients preferred that vacation type?

✅ **Solution:**

a. How many clients preferred the mountains?
   - Change 18.3% to a decimal, which is 0.183. Then, $(0.183) \cdot (349) = 63.867$. So, 64 clients preferred the mountains.

b. How many clients preferred lake/countryside or national parks/wildlife?
   - $8.6\% + 11.5\% = 20.1\%$. Change 20.1% to a decimal, which is 0.201. Then, $(0.201) \cdot (349) = 70.1492$. So, 70 clients preferred the lake/countryside or national parks/wildlife.

c. Which vacation type is the most popular? How many clients preferred that vacation type?
   - Change 27.2% to a decimal, which is 0.272. Then, $(0.272) \cdot (349) = 94.928$. So, about 95 clients preferred the beach.

d. Which vacation type is the least popular? How many clients preferred that vacation type?
   - Change 8% to a decimal, which is 0.08. Then, $(0.08) \cdot (349) = 27.92$. So, about 28 clients preferred a cruise.

Section 6.4: Pie Graphs [*In-Class Exercises*]

1. A student recorded how many hours they spent on weekly activities:

Hours Spent on Weekly Activities

Activity	Hours
Sleep	56

Activity	Hours
School	35
Homework	10
Sports	5
Leisure	14

Construct a pie graph for the data.

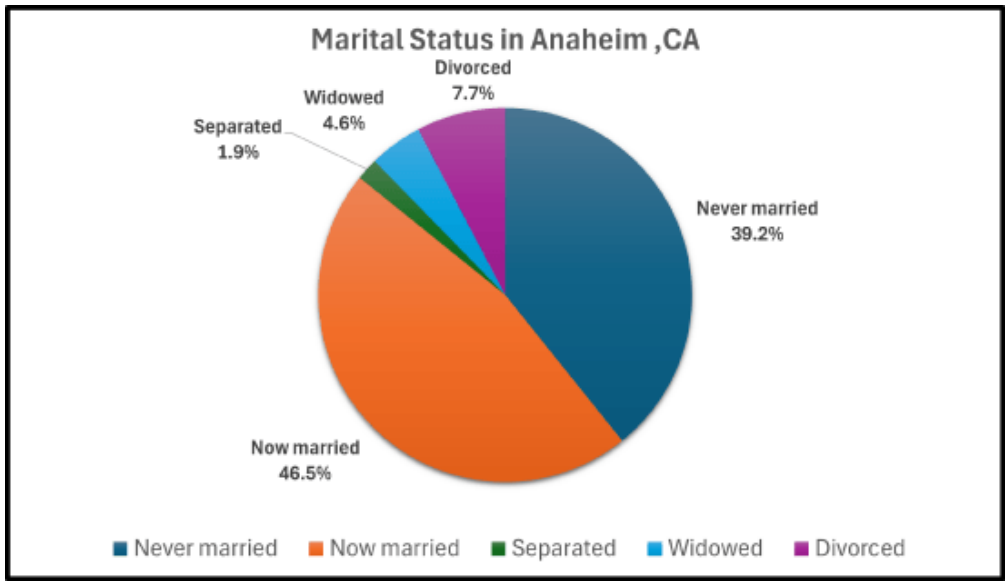
2. A survey of 100 8th-graders was conducted. They were asked what their favorite sport is. The results are given in the frequency table.

Favorite sports of 8th graders

Sport	No. of 8th Graders
Baseball	17
Basketball	26
Football	22
Hockey	15
Track	6
Volleyball	9
Wrestling	5

Construct a pie graph for the data.

3. Four states have no official minimum marriage age: California, Mississippi, New Mexico, and Oklahoma. However, all four require parental consent, court approval, or both for a minor to marry. According to city-data.com, the following pie graph shows the marital status of the population for ages 15 years and over in Anaheim, CA.



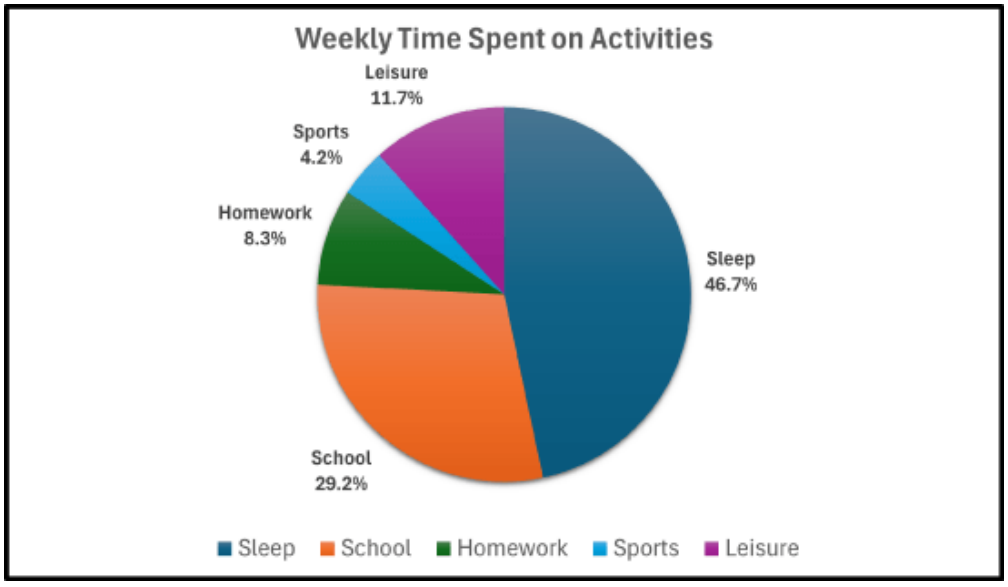
a. If the population of Anaheim is 344,000, how many people are divorced?
b. If the population of Anaheim is 344,000, how many people are divorced or separated?
c. If the population of Anaheim is 344,000, how many people were ever married?

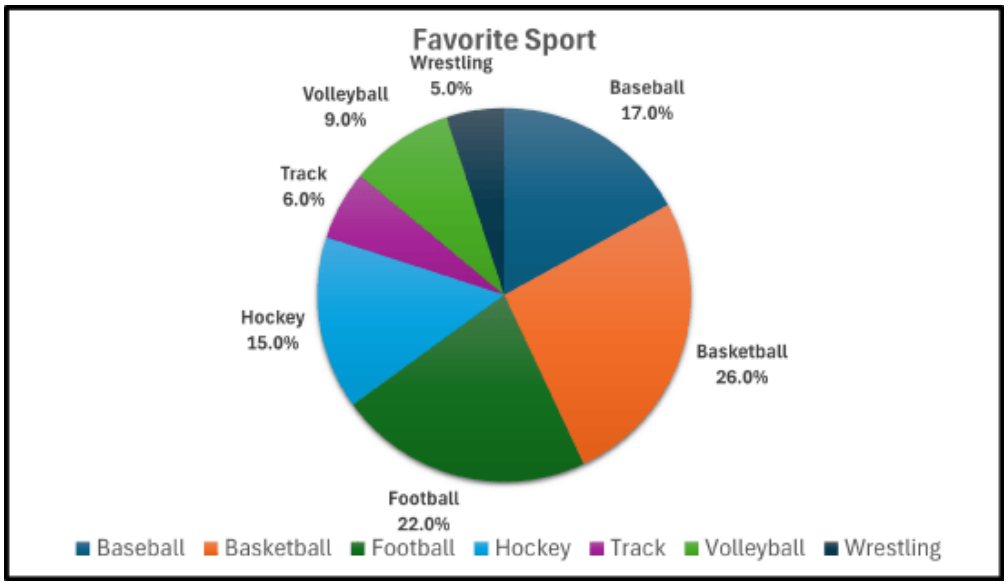
Answers available at the end of this section.

Answers

Section 6.4: Pie Graphs [*In-Class Exercises*]

1.



2. 

3.
   a. 26,488
   b. 33,024
   c. 209,152

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Section 6.4.H: Homework

Section 6.4: Pie Graphs - Homework Exercises

(Section 6.4) - Homework Exercise #1

In a blood drive with 400 donors:

Blood donors by blood type

Blood Type	Donors
O	180
A	160
B	44
AB	16

Create a pie graph to display this data.

Answer available at the end of this section.

(Section 6.4) - Homework Exercise #2

At a local outdoor mall, 200 teenagers were asked about their favorite music streaming service:

Music streaming services used by respondents

Service	Users
Spotify	88
Apple Music	56
YouTube Music	34
Other	22

Create a pie graph to display this data.

Answer available at the end of this section.

(Section 6.4) - Homework Exercise #3

A theater tracked 1,000 ticket sales for one week:

Movie tickets sold by genre

Genre	Tickets Sold
Action	350
Comedy	280
Drama	140
Horror	105
Other	125

Create a pie graph to display this data.

Answer available at the end of this section.

(Section 6.4) - Homework Exercise #4

In a neighborhood of households, pet ownership was recorded:

Households grouped by pet ownership

Type of Pet	No. of Households
Dog(s) only	47
Cat(s) only	37
Both dog(s) and cat(s)	22
Other pet(s)	11
No pet(s)	38

Create a pie graph to display this data.

Answer available at the end of this section.

(Section 6.4) - Homework Exercise #5

The weekly sales (in dollars) rounded to the nearest $5 at a snack stand were recorded:

Daily sales totals from Monday through Friday

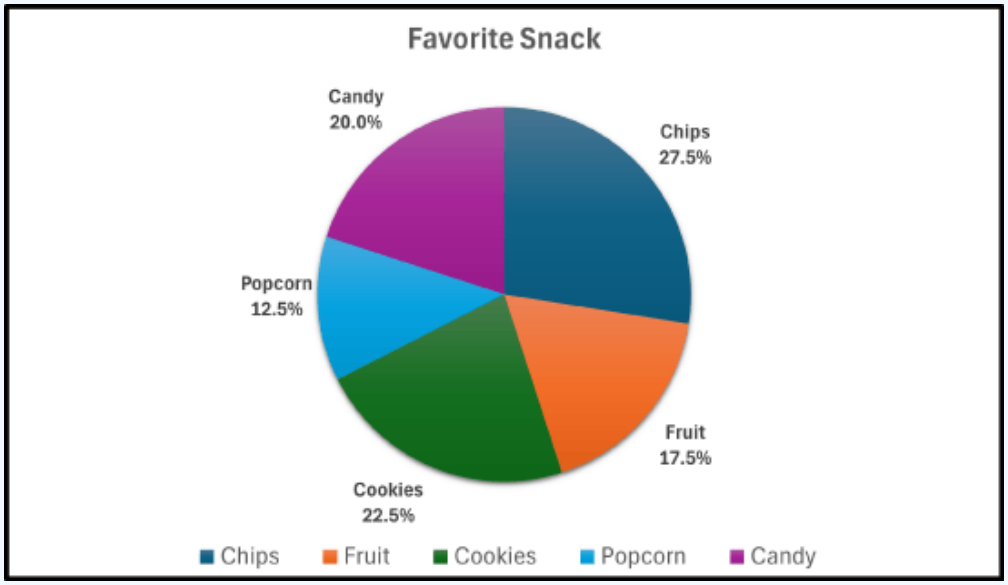
Day	Sales
Monday	$120
Tuesday	$135
Wednesday	$110
Thursday	$130
Friday	$185

Create a pie graph to display this data.

Answer available at the end of this section.

(Section 6.4) - Homework Exercise #6

The pie graph below shows a survey of 200 customers' favorite snack from the snack stand at a community sports park.

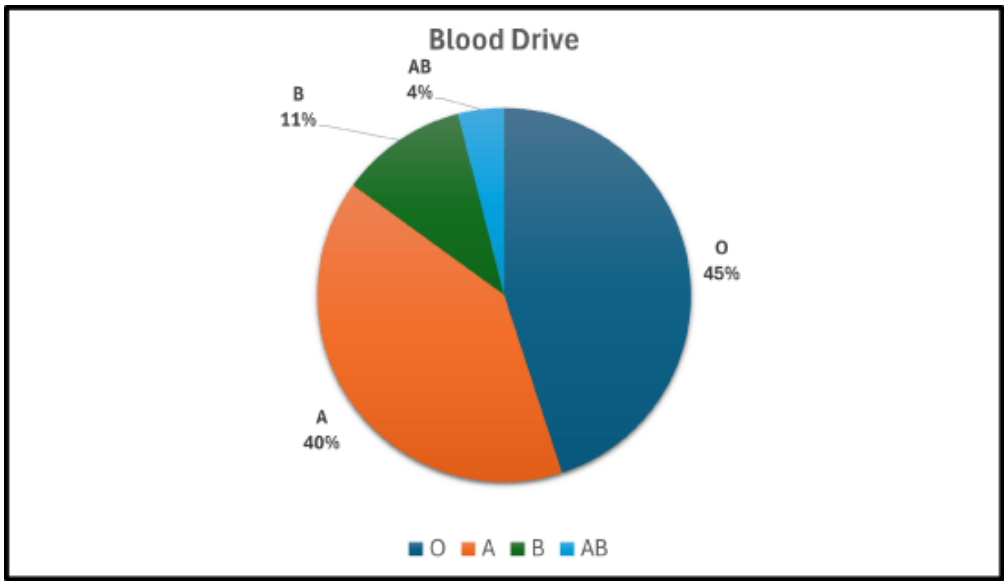


a. Which snack was the most popular?
b. Which snack was the least popular?
c. How many customers said their favorite snack was fruit?
d. How many customers said their favorite snack was cookies?
e. How many customers said their favorite snack was chips or cookies?

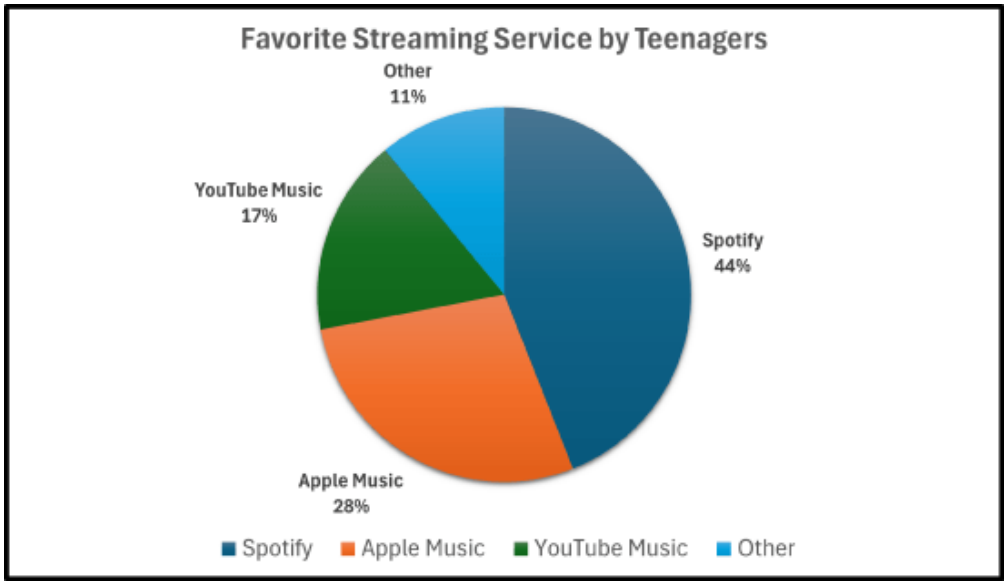
Answer available at the end of this section.

Answers

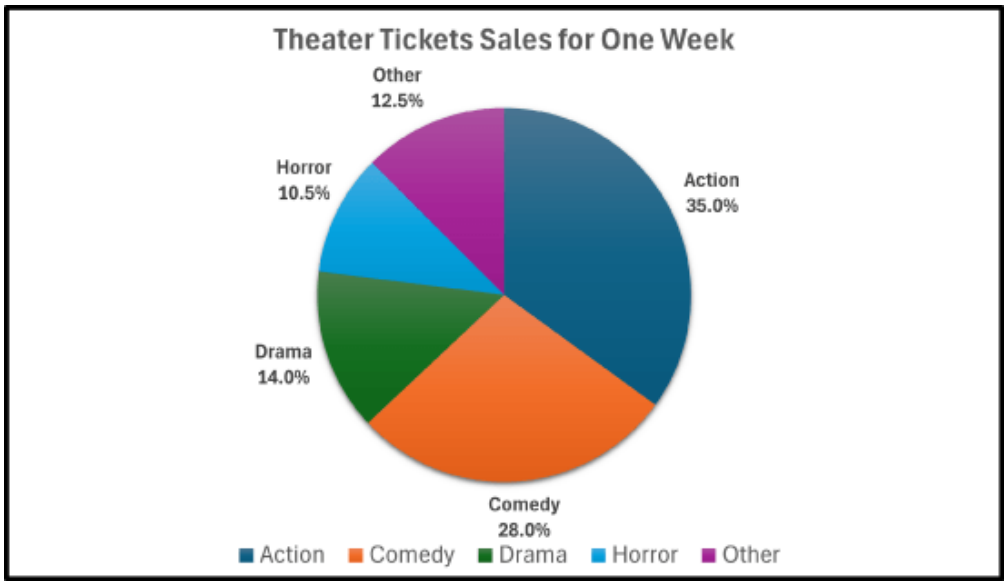
(Section 6.4) - Homework Exercise #1



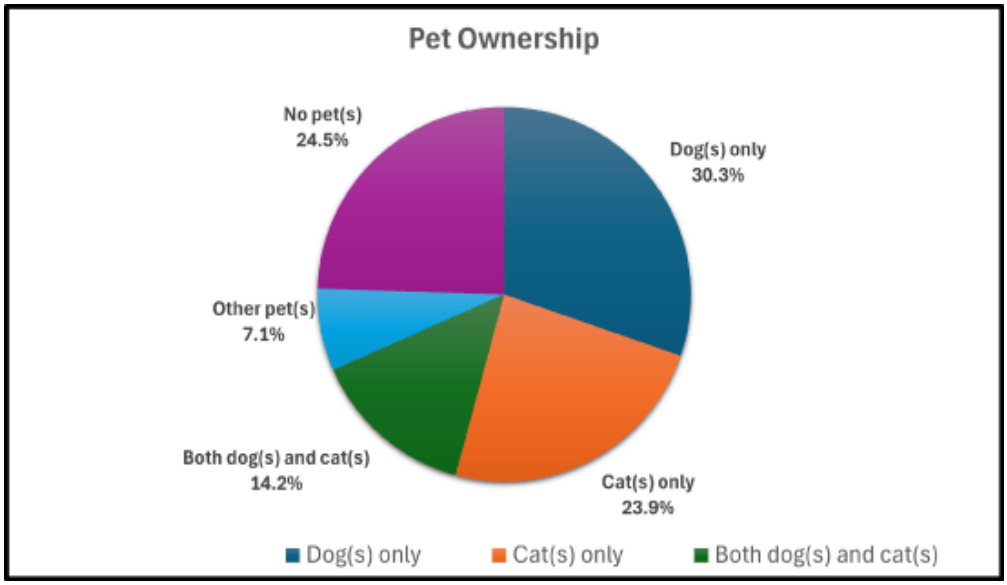
(Section 6.4) - Homework Exercise #2



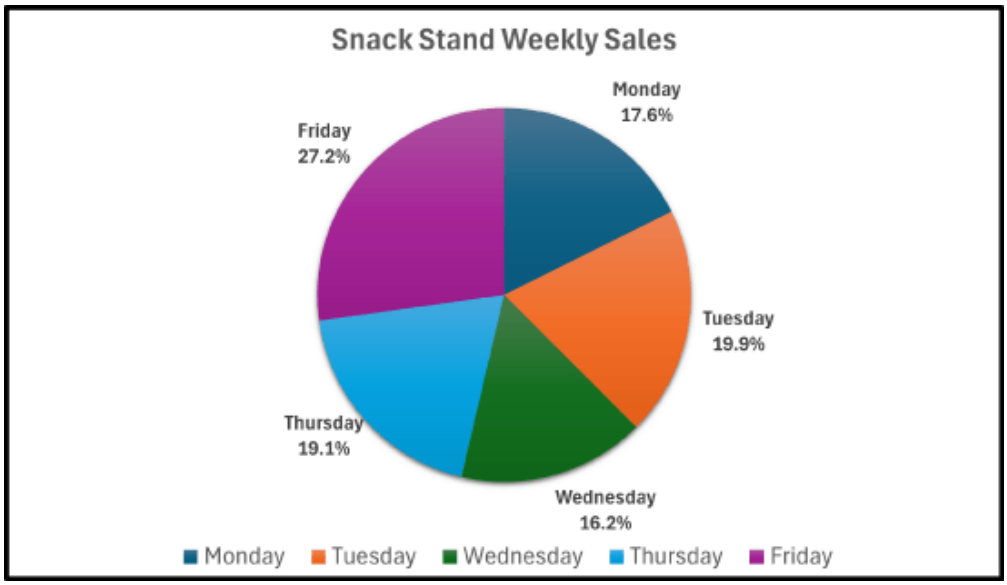
(Section 6.4) - Homework Exercise #3



(Section 6.4) - Homework Exercise #4



(Section 6.4) - Homework Exercise #5



(Section 6.4) - Homework Exercise #6

a. Chips
b. Popcorn
c. 35
d. 45
e. 100

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Section 6.5: Bar Graphs

Learning Objectives

- Draw bar graphs

Definition: Bar Graph

A **bar graph** (also called a bar chart) is a graphical display that uses rectangular bars to represent data. Each bar corresponds to a category or group, and the length or height of the bar is proportional to the value or frequency it represents. Bar graphs are one of the most versatile and widely used methods for displaying categorical data, making comparisons between groups clear and straightforward.

When to Use Bar Graphs

Bar graphs are most effective when you want to:

- Compare values across different categories or groups
- Display categorical or discrete data
- Show frequencies, counts, or measurements for distinct groups
- Make it easy to compare the sizes of different categories
- Present data where categories have no natural order or connection

Key Components of a Bar Graph

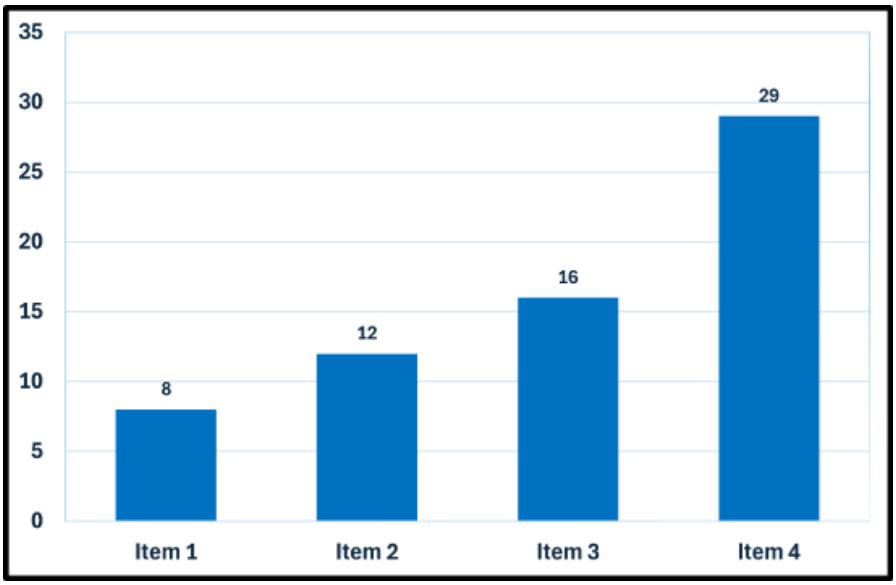
1. Horizontal Axis (x-axis): Typically shows the categories or groups being compared
2. Vertical Axis (y-axis): Represents the scale of measurement (frequency, count, percentage, amount, etc.)
3. Bars: Rectangular shapes whose heights (or lengths) represent the data values
    - Bars are separated by spaces to emphasize that categories are distinct
    - All bars should have the same width
4. Labels: Clear identification of categories and the measurement scale
5. Title: Describes what the bar graph represents
6. Scale: Consistent intervals on the measurement axis, typically starting at zero

Each category is assigned a position along one axis (usually horizontal), and a bar is drawn to a height (or length) that corresponds to its value on the other axis. The bars are separated by spaces to show that the categories are distinct and unrelated. By comparing the heights of the bars, viewers can quickly see which categories have larger or smaller values.

Types of Bar Graphs

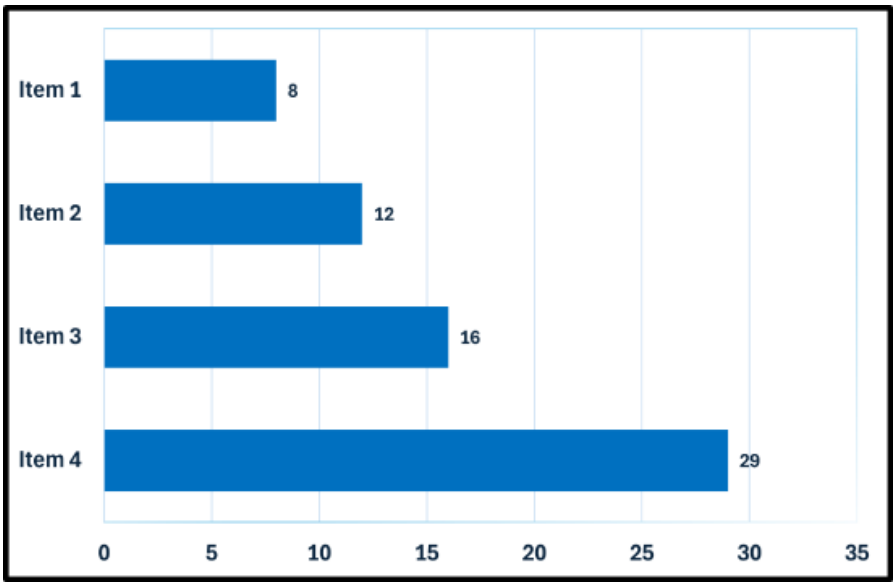
1. **Vertical Bar Graph**: Bars extend upward from the horizontal axis

- Most common type
- Categories on *x*-axis, values on *y*-axis
- Also called a column chart



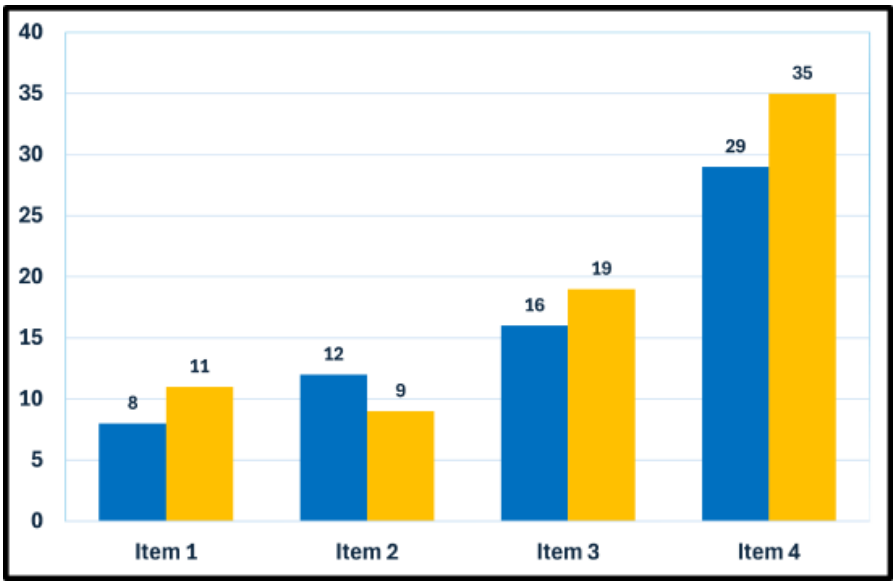
2. **Horizontal Bar Graph**: Bars extend horizontally from the vertical axis

- Categories on *y*-axis, values on *x*-axis
- Useful when category names are long
- Easier to read with many categories



3. **Side-by-Side Bar Graph**: Multiple bars for each category, placed side-by-side

- Compares subcategories within each main category
- Example: Comparing male and female enrollment across different majors
- Emphasizes comparison, not trends over time



Although bar graphs come in many forms, this section focuses only on these three most commonly used types. These examples provide the foundation students need to recognize how categorical data can be compared, grouped, or displayed over time. While other variations of bar graphs exist, such as segmented bars, paired bars, and population bars, they tend to be used in more specialized contexts or rely on the same underlying principles already demonstrated. By concentrating on the most widely used forms, this keeps the explanations clear and accessible without overwhelming learners with unnecessary variations. Students will still be well-prepared to interpret other bar graph formats they may encounter, since the core ideas remain the same.

Advantages of Bar Graphs

- **Easy to read and understand**: Simple, intuitive format that most people recognize
- **Excellent for comparisons**: Bar heights make it obvious which values are larger or smaller
- **Works with many categories**: Can display numerous categories without becoming cluttered
- **Precise values**: Bar heights allow for accurate reading of exact values
- **Flexible**: Can be vertical or horizontal, grouped or stacked
- **Shows individual values clearly**: Each category stands alone
- **No misleading connections**: Spaces between bars prevent implying relationships that don't exist

Limitations of Bar Graphs

- **Not ideal for trends over time**: Line graphs are better for showing continuous change
- **Can become cluttered**: Too many categories or groups can make the graph hard to read
- **Limited in showing relationships**: Shows comparisons but not correlations between variables
- **Scale manipulation**: Starting the y-axis above zero can exaggerate differences

When NOT to use Bar Graphs

Avoid bar graphs when:

- You want to show trends or changes over continuous time (use line graphs)
- You need to display parts of a whole (use pie graphs for small numbers of categories)
- You're showing the relationship between two quantitative variables (use scatter plots)
- Data is continuous rather than categorical

Constructing a Bar Graph

- **Step 1 — Organize your data in a table with categories and their corresponding values**
- **Step 2 —Draw and label the axes**
  - Horizontal axis (*x*-axis): List the categories with equal spacing
  - Vertical axis (*y*-axis): Show the measurement scale (frequency, count, percentage, etc.)
  - Include units where appropriate
- **Step 3 — Determine the scale for the vertical axis (*y*-axis):**
  - Find the maximum value in your data
  - Choose an upper limit that accommodates all values
  - Use consistent intervals (e.g., 0, 10, 20, 30, 40...)
  - Generally start at zero to avoid distorting comparisons
- **Step 4 — Draw the bars**
  - For each category, draw a rectangular bar
  - Height (or length) should accurately represent the value
  - Keep all bars the same width
  - Leave spaces between bars to show distinct categories
- **Step 5 — Label Clearly**
  - Mark categories on the horizontal axis (*x*-axis)
  - Ensure vertical axis (*y*-axis) scale is clear
  - Consider adding value labels on top of bars for precision
- **Step 6 — Add a descriptive title, if necessary**
  - Example: "Favorite Sports Among 100 High School Students"

Example #6.5.1 🤔

A college surveyed 200 students about their class standing:

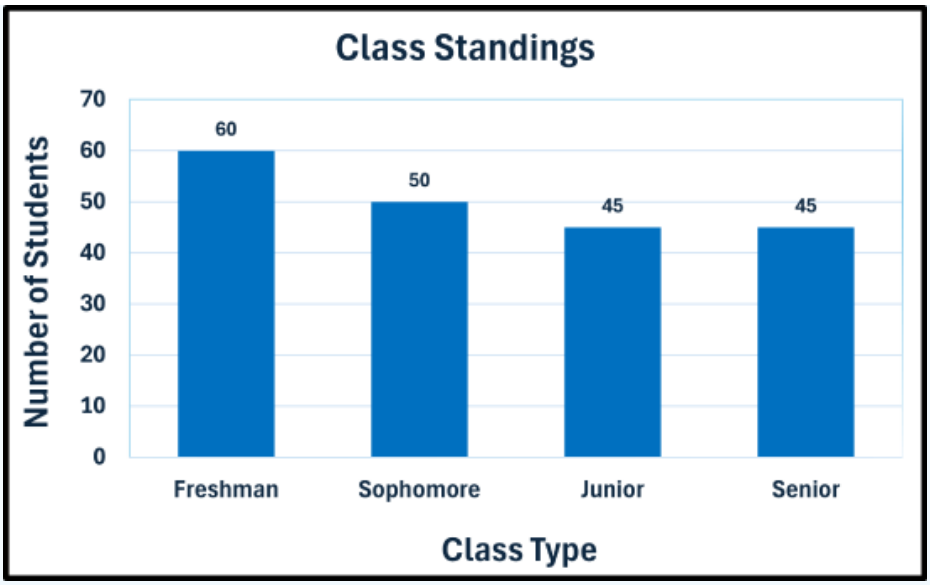
Distribution of students by class level

Classes	Frequency
Freshman	60
Sophomore	50
Junior	45
Senior	45

Create a bar graph to display this data.

✅ **Solution:**

- **Step 1 — Organize your data in a table with categories and their corresponding values.** This step is done, since data was given in a frequency distribution.
- **Step 2 — Draw and label the axes.** The horizontal axis is labeled "Classes" and the vertical axis is labeled "Frequency".
- **Step 3 — Determine the scale for the vertical axis (*y*-axis).** Since the maximum frequency is 60, then increments of 10 can be used for the tick marks on the axis.
- **Step 4 — Draw the bars.** Draw rectangular bars of equal width up to the frequency of each class.
- **Step 5 — Label clearly.** Label each of the rectangular bars: Freshman, Sophomore, Junior, and Senior.
- **Step 6 — Add a descriptive title, if necessary.** Add the title, "Class Standings" or some other similar title.



Example #6.5.2 🤔

A teacher recorded final grades for 40 students:

A, B, F, C, B, D, B, C, B, C, A, C, B, C, B, C, A, D, C, B, B, C, C, A, C, C, B, A, A, B, A, D, B, D, F, B, B, A, B, C

Create a bar graph to display this data.

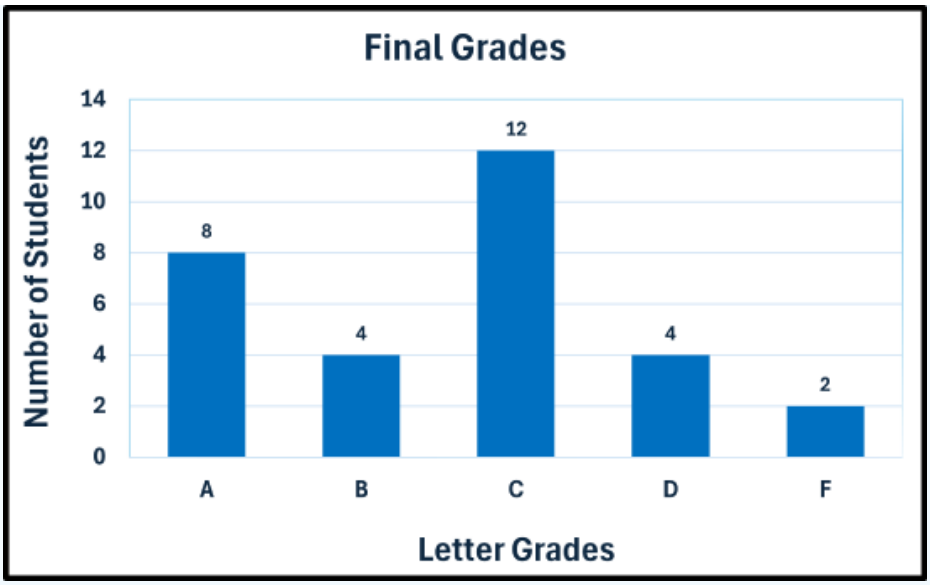
✅ **Solution:**

- **Step 1 — Organize your data in a table with categories and their corresponding values.** Create a frequency distribution for the data.

Distribution of letter grades

Grades	Frequency
A	8
B	14
C	12
D	4
F	2

- **Step 2 — Draw and label the axes.** The horizontal axis is labeled "Grades" and the vertical axis is labeled "Frequency".
- **Step 3 — Determine the scale for the vertical axis (*y*-axis).** Since the maximum frequency is 12, then increments of 2 can be used for the tick marks on the axis.
- **Step 4 — Draw the bars.** Draw rectangular bars of equal width up to the frequency of each letter grade.
- **Step 5 — Label clearly.** Label each of the rectangular bars: A, B, C, D, and F.
- **Step 6 — Add a descriptive title, if necessary.** Add the title, "Final Grades" or some other similar title.



Example #6.5.3 🤔

A survey was conducted and asked adults their primary source of news:

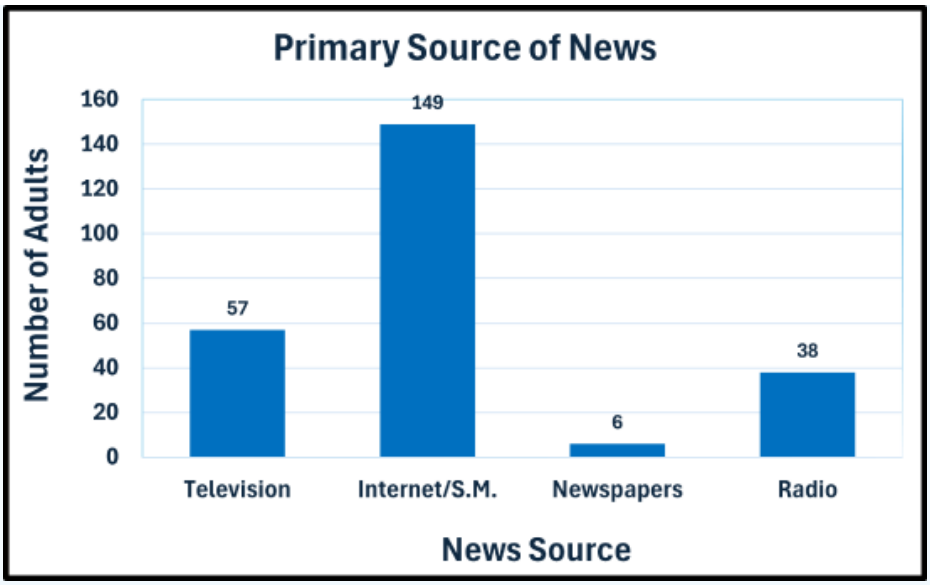
Primary news sources of respondents

News Source	Respondents
Television	57
Internet/Social Media	149
Newspapers	6
Radio	38

Create a bar graph to display this data.

✅ **Solution:**

- **Step 1 — Organize your data in a table with categories and their corresponding values.** This step is done, since data was given in a frequency distribution.
- **Step 2 — Draw and label the axes.** The horizontal axis is labeled "Grades" and the vertical axis is labeled "Frequency".
- **Step 3 — Determine the scale for the vertical axis (*y*-axis).** Since the maximum frequency is 149, then increments of 20 can be used for the tick marks on the axis.
- **Step 4 — Draw the bars.** Draw rectangular bars of equal width up to the frequency of each letter grade.
- **Step 5 — Label clearly.** Label each of the rectangular bars: Television, Internet/Social Media, Newspapers, and Radio.
- **Step 6 — Add a descriptive title, if necessary.** Add the title, "Primary Source of News" or some other similar title.



Example #6.5.4 🤔

A survey of 500 smartphone users were asked which type of phone that they own/prefer is given below:

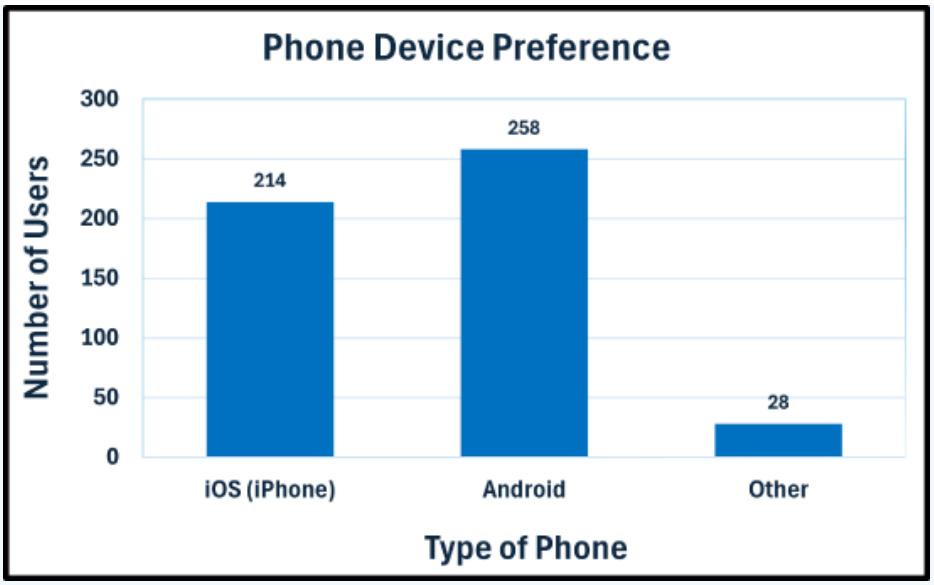
Preferred types of phones among users

Type of Phone	Users
iOS (iPhone)	214
Android	258
Other	28

Create a bar graph to display this data.

✅ **Solution:**

- **Step 1 — Organize your data in a table with categories and their corresponding values.** This step is done, since data was given in a frequency distribution.
- **Step 2 — Draw and label the axes.** The horizontal axis is labeled "Grades" and the vertical axis is labeled "Frequency".
- **Step 3 — Determine the scale for the vertical axis (*y*-axis).** Since the maximum frequency is 258, then increments of 50 can be used for the tick marks on the axis.
- **Step 4 — Draw the bars.** Draw rectangular bars of equal width up to the frequency of each letter grade.
- **Step 5 — Label clearly.** Label each of the rectangular bars: iOS (iPhone), Android, and Other.
- **Step 6 — Add a descriptive title, if necessary.** Add the title, "Phone Device Preference" or some other similar title.



Example #6.5.5 🤔

A travel agency surveyed their clients about preferred vacation types:

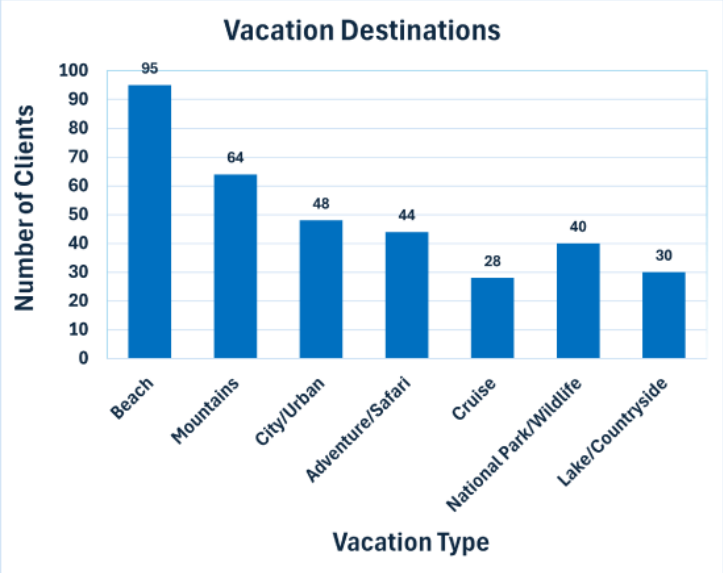
Preferred vacation types among clients

Vacation Type	Clients
Beach	95
Mountains	64
City/Urban	48
Adventure/Safari	44
Cruise	28
National Park/Wildlife	40
Lake/Countryside	30

Create a bar graph to display this data.

✅ **Solution:**

- **Step 1 — Organize your data in a table with categories and their corresponding values.** This step is done, since data was given in a frequency distribution.
- **Step 2 — Draw and label the axes.** The horizontal axis is labeled "Grades" and the vertical axis is labeled "Frequency".
- **Step 3 — Determine the scale for the vertical axis (*y*-axis).** Since the maximum frequency is 95, then increments of 10 can be used for the tick marks on the axis.
- **Step 4 — Draw the bars.** Draw rectangular bars of equal width up to the frequency of each letter grade.
- **Step 5 — Label clearly.** Label each of the rectangular bars: Beach, Mountains, City/Urnam, Adventure/Safari, Cruise, National Park/Wildfire, and Lake/Countryside.
- **Step 6 — Add a descriptive title, if necessary.** Add the title, "Vacation Destinations" or some other similar title.



Section 6.5: Bar Graphs [*In-Class Exercises*]

1. A bag of peanut M&M's were opened and the following colors with their respective quantities were present: 6 Blue, 4 Brown, 1 Green, 5 Orange, 3 Red, 2 Yellow. Construct a bar graph for the data.
2. A survey of 100 8th-graders was conducted. They were asked what their favorite sport is. The results are given in the frequency table.

   Favorite sports of 8th graders

Sport	Number of 8th Graders
Baseball	17
Basketball	26
Football	22
Hockey	15
Track	6
Volleyball	9
Wrestling	5

   Create a bar graph to display this data.

3. The table below provides the total electoral votes for the last eight U.S. Presidential elections.

   U.S. electoral vote totals 1996-2024

Year	Democrat	Republican
1996	379	159

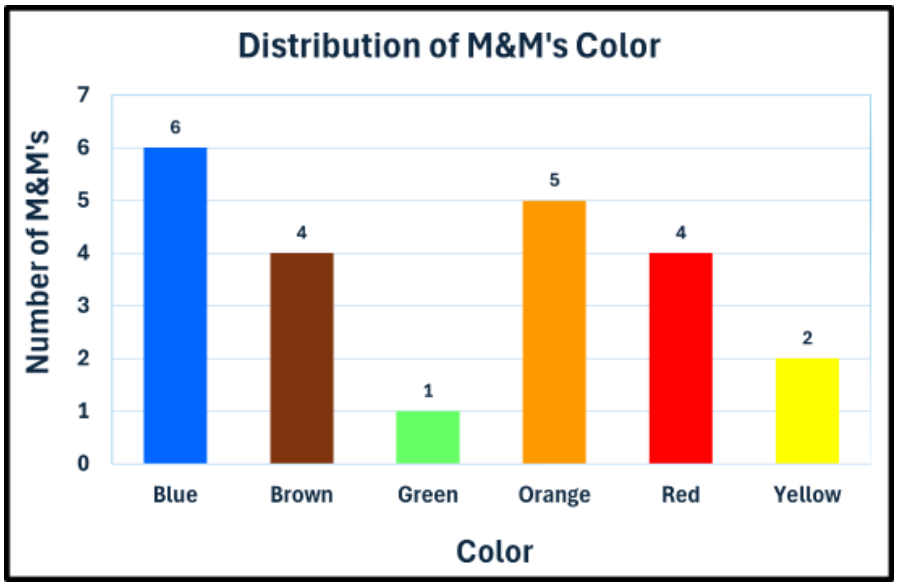
Year	Democrat	Republican
2000	266	271
2004	251	286
2008	365	173
2012	332	206
2016	227	304
2020	306	232
2024	226	312

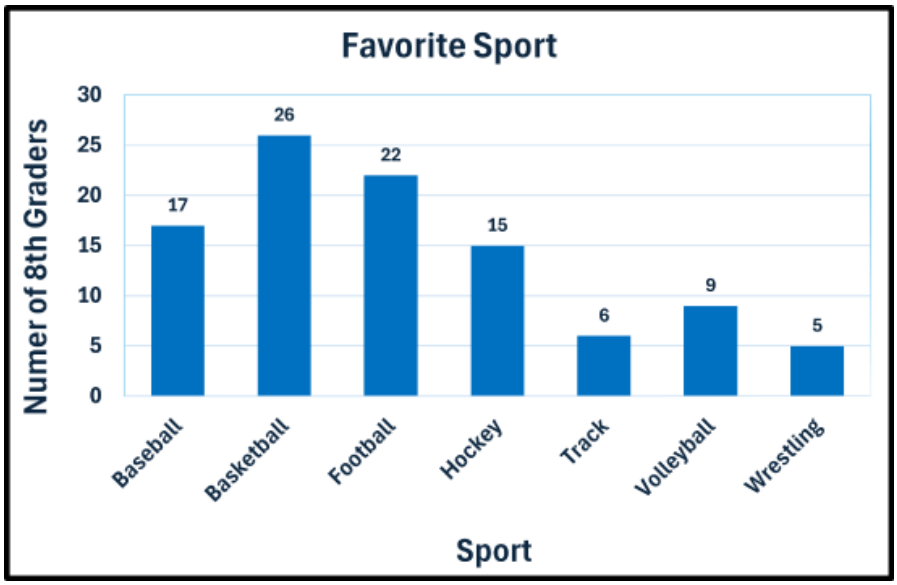
Construct a side-by-side bar graph to display this data.

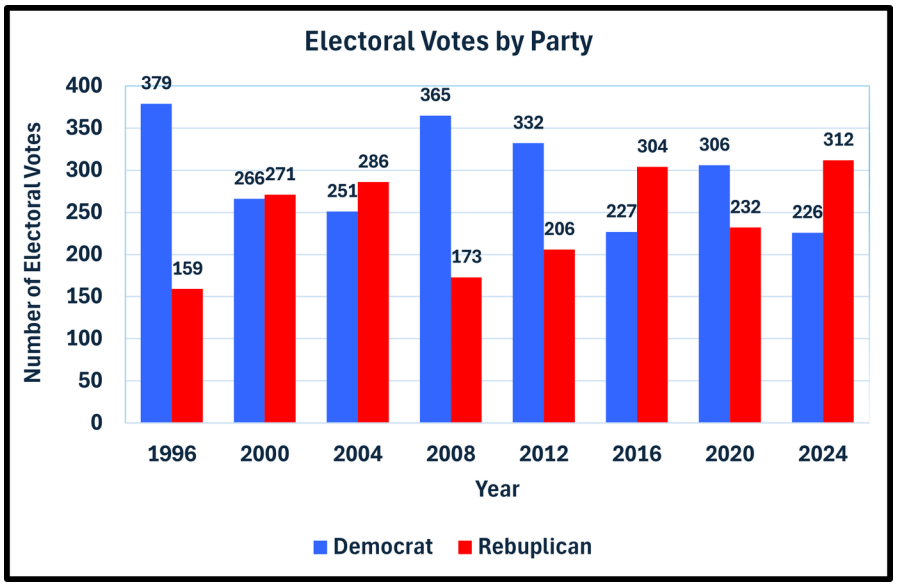
Answers available at the end of this section.

Answers

Section 6.5: Bar Graphs [*In-Class Exercises*]

1. 

2. 

3. 

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Section 6.5.H: Homework

Section 6.5: Bar Graphs - Homework Exercises

(Section 6.5) - Homework Exercise #1

In a blood drive with 400 donors:

Blood donors by blood type

Blood Type	Donors
O	180
A	160
B	44
AB	16

Create a bar graph to display this data.

Answer available at the end of this section.

(Section 6.5) - Homework Exercise #2

At a local outdoor mall, 200 teenagers were asked about their favorite music streaming service:

Music streaming services used by respondents

Service	Users
Spotify	88
Apple Music	56
YouTube Music	34
Other	22

Create a bar graph to display this data.

Answer available at the end of this section.

(Section 6.5) - Homework Exercise #3

A theater tracked 1,000 ticket sales for one week:

Movie tickets sold by genre

Genre	Tickets Sold
Action	350
Comedy	280
Drama	140
Horror	105
Other	125

Create a bar graph to display this data.

Answer available at the end of this section.

(Section 6.5) - Homework Exercise #4

In a neighborhood of households, pet ownership was recorded:

Households grouped by pet ownership

Type of Pet	No. of Households
Dog(s) only	47
Cat(s) only	37
Both dog(s) and cat(s)	22
Other pet(s)	11
No pet(s)	38

Create a bar graph to display this data.

Answer available at the end of this section.

(Section 6.5) - Homework Exercise #5

The weekly sales (in dollars) rounded to the nearest $5 at a snack stand were recorded:

Daily sales totals from Monday through Friday

Day	Sales
Monday	$120
Tuesday	$135
Wednesday	$110
Thursday	$130
Friday	$185

Create a bar graph to display this data.

Answer available at the end of this section.

(Section 6.5) - Homework Exercise #6

The table below provides the total electoral votes for the U.S. Presidential elections from 1968 through 1988.

Electoral vote totals by political party from 1968 through 1988

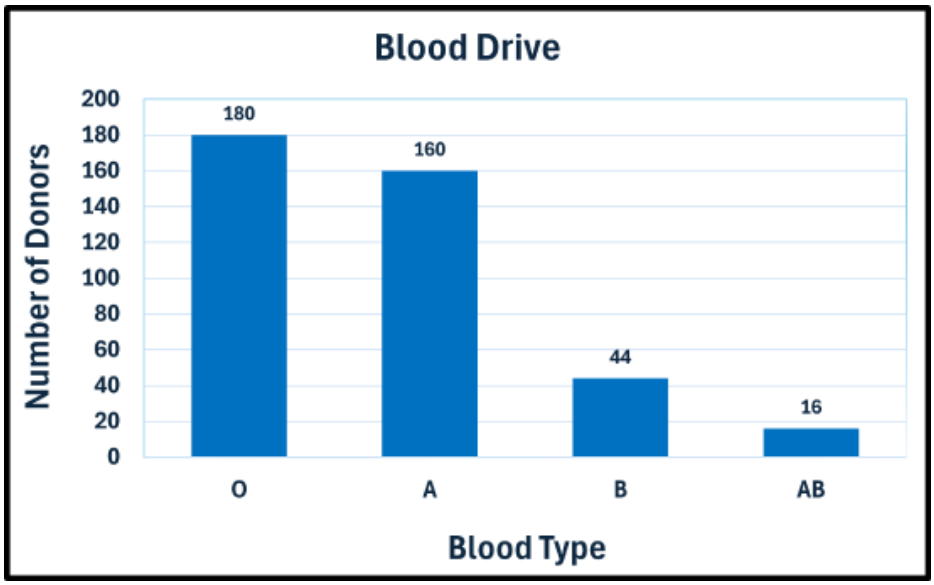
Year	Democrat	Republican	Other
1968	191	301	46
1972	17	520	1
1976	297	240	1
1980	49	489	
1984	13	525	
1988	111	426	1

Create a side-by-side bar graph to display this data.

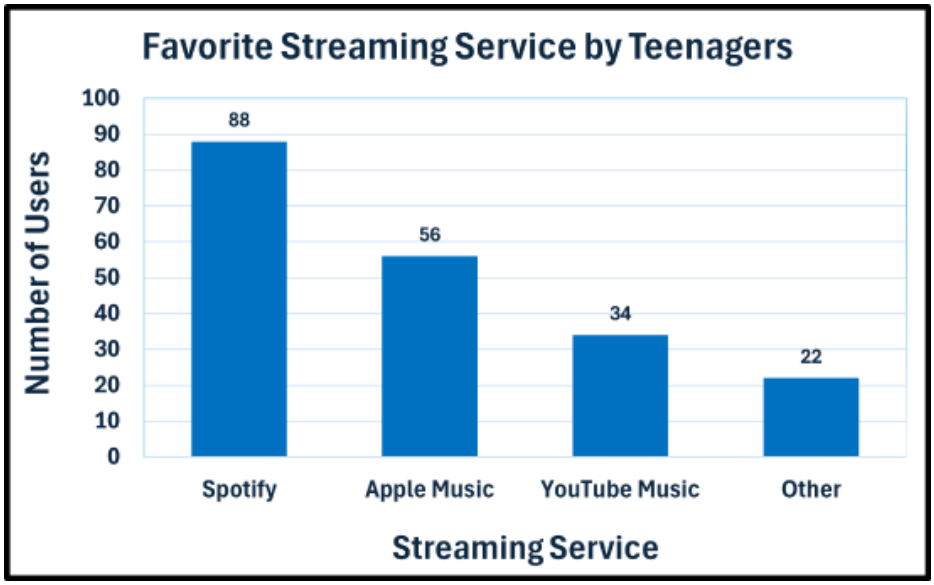
Answer available at the end of this section.

Answers

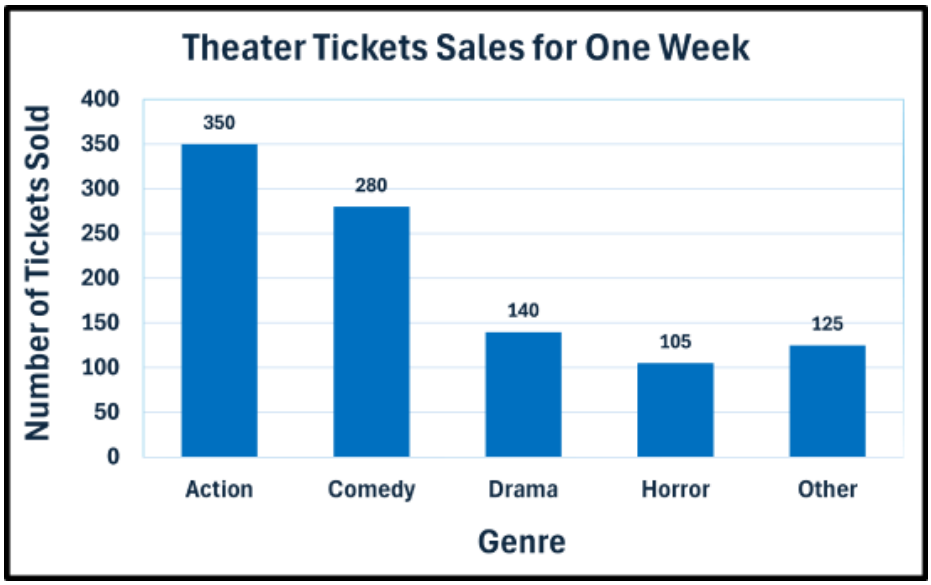
(Section 6.5) - Homework Exercise #1



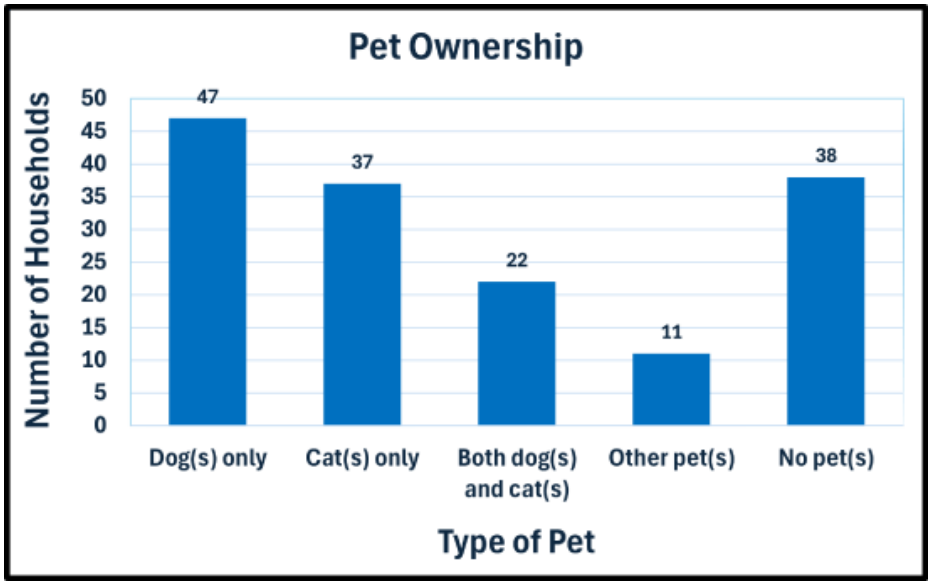
(Section 6.5) - Homework Exercise #2



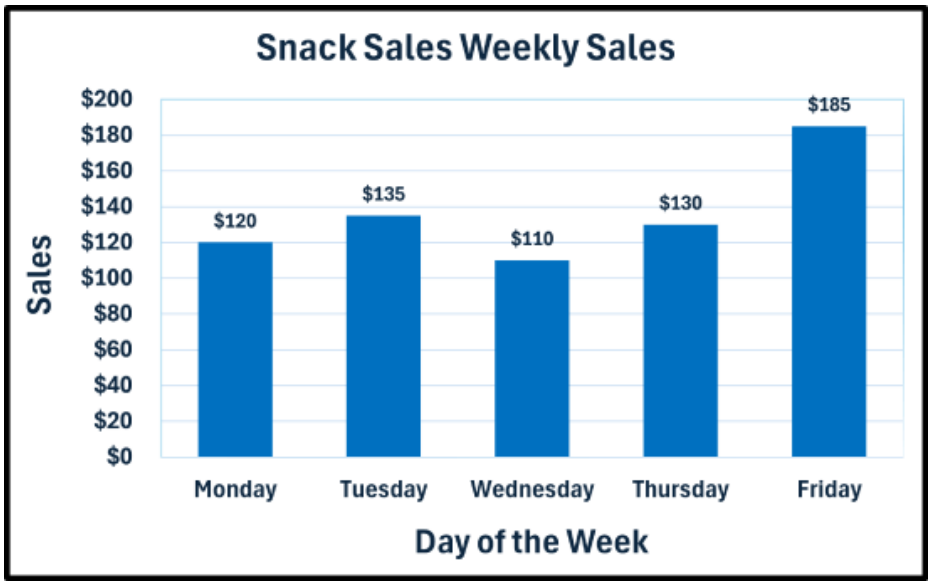
(Section 6.5) - Homework Exercise #3



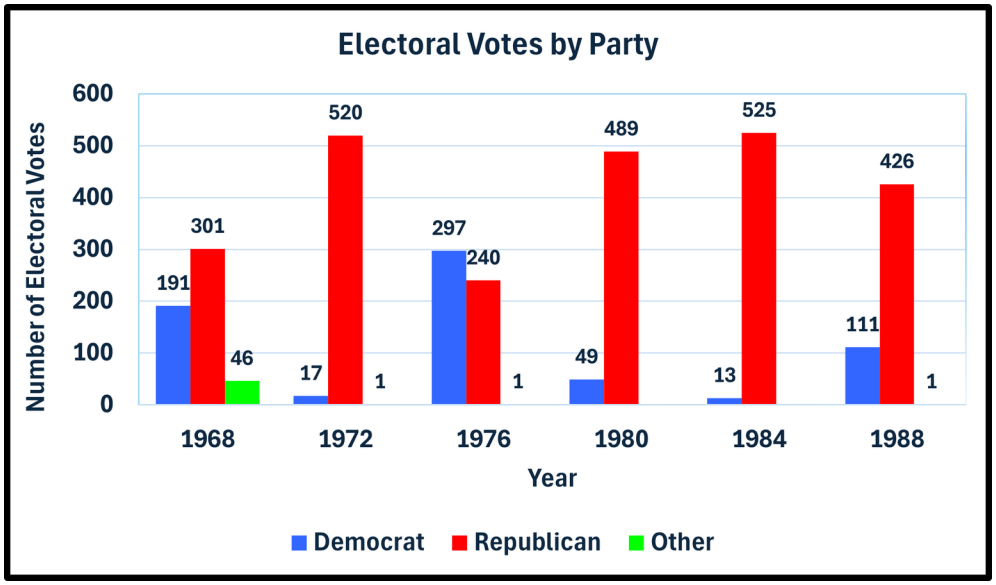
(Section 6.5) - Homework Exercise #4



(Section 6.5) - Homework Exercise #5



(Section 6.5) - Homework Exercise #6



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Section 6.6: Line Graphs

Learning Objectives

- Draw line graphs

Definition: Line Graph

A **line graph** (also called a **line chart** or **line plot**) is a type of graph that displays data as points connected by straight line segments. Line graphs are particularly effective for showing trends, patterns, and changes in data over time or across ordered categories. The connected points create a visual flow that makes it easy to see how values increase, decrease, or remain constant.

When to Use Line Graphs

Line graphs are most effective when you want to:

- Show change or trends over time (days, months, years, etc.)
- Display continuous data or data measured at regular intervals
- Compare trends between two or more groups
- Emphasize the rate of change rather than individual values
- Track progress or performance over time
- Identify patterns such as seasonal variations or cycles

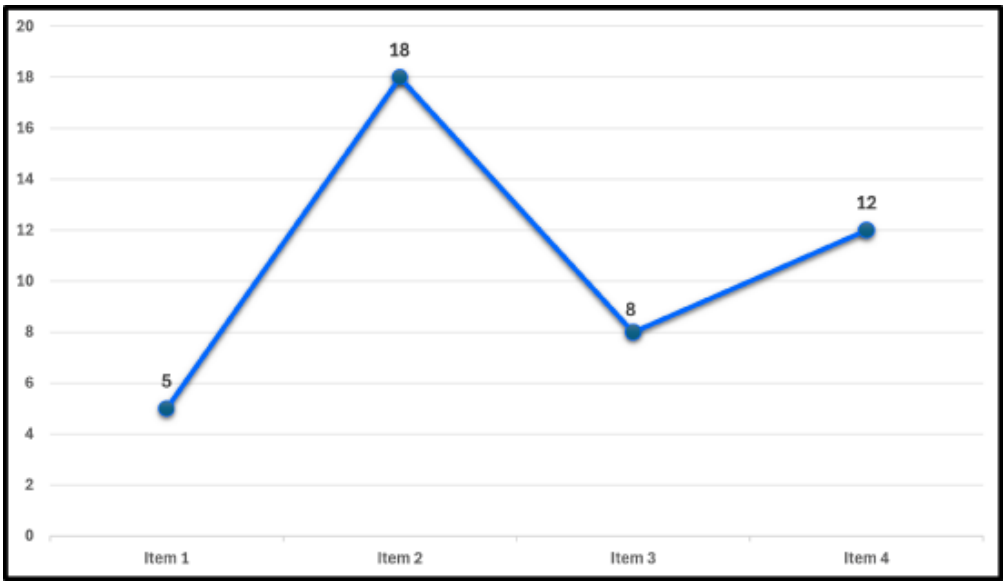
Key Components of a Line Graph

- **Horizontal Axis (*x*-axis)**: Typically represents time or an ordered sequence (years, months, days, measurement intervals)
- **Vertical Axis (*y*-axis)**: Represents the variable being measured (temperature, price, test scores, population, etc.)
- **Data Points**: Individual values plotted at their corresponding coordinates
- **Line Segments**: Straight lines connecting consecutive data points, showing the progression from one value to the next
- **Labels and Titles**: Clear axis labels with units, and a descriptive title for the entire graph
- **Scale**: Consistent intervals on both axes, with the y-axis starting typically (but not always) at zero
- **Legend** (for multiple lines): Identifies what each line represents when comparing multiple data sets

Types of Line Graphs

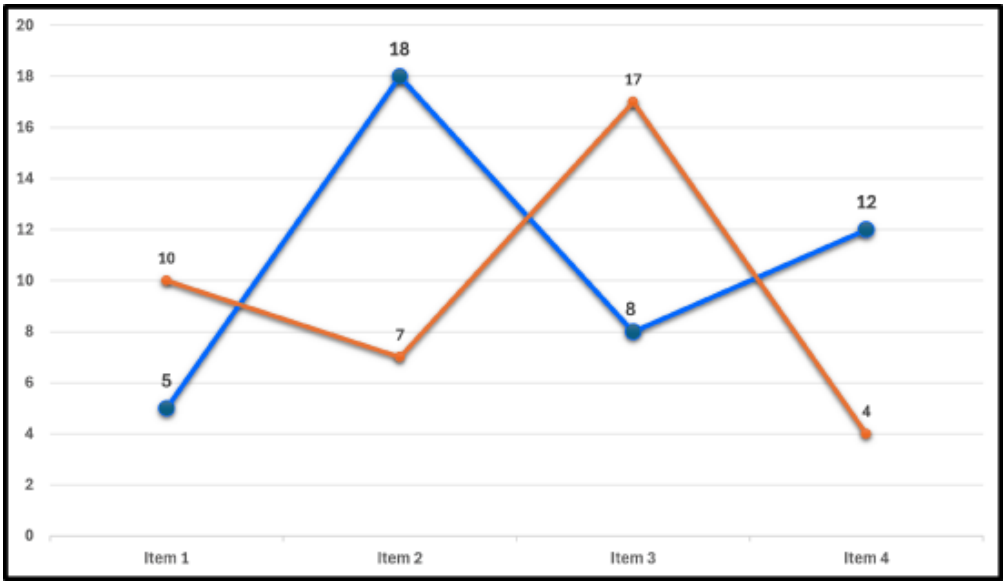
1. **Simple Line Graph**: Shows one variable over time

- Example: Daily temperature for one week
- One line tracking a single data series



2. **Multiple Line Graph**: Shows two or more variables on the same graph for comparison

- Example: Comparing sales for Product A and Product B over the same time period
- Different lines are usually distinguished by color or line style (solid, dashed, dotted)
- Requires a legend to identify each line



How Line Graphs Work

Each data point is plotted at the intersection of its *x*-coordinate (often time) and *y*-coordinate (measured value). Once all points are plotted, they are connected in order with straight lines. The resulting graph creates a visual representation of how the variable changes, with:

- Upward slopes indicating increases
- Downward slopes showing decreases
- Horizontal sections representing no change
- Steep slopes showing rapid change
- Gradual slopes indicating slow, steady change

Advantages of Line Graphs

- **Excellent for showing trends**: The connected lines make patterns immediately visible
- **Easy to see changes over time**: Increases and decreases are obvious from the line's direction
- **Multiple data sets**: Can display several lines on one graph for comparison
- **Shows rate of change**: Steep lines indicate rapid change; flat lines show stability
- **Continuous flow**: Creates a sense of progression and continuity
- **Predictions**: Trends can suggest future values (though caution is needed)
- **Identifies patterns**: Cycles, peaks, and valleys become apparent

Limitations of Line Graphs

- **Can be misleading with non-continuous data**: Connecting discrete, unrelated categories may imply relationships that don't exist
- **Cluttered with too many lines**: More than 3-4 lines become difficult to read and compare
- **Scale manipulation**: Changing the *y*-axis scale can exaggerate or minimize trends
- **Assumes linear change between points**: The straight lines suggest constant rate of change between measurements, which may not be accurate
- **Requires ordered data**: The *x*-axis must represent a logical sequence
- **Can hide individual data points**: Focus on trend may obscure specific values

When NOT to use Line Graphs

Avoid line graphs when:

- Data is not ordered or sequential (use bar graphs instead for unrelated categories)
- You're comparing unrelated categories where connections would be meaningless
- You want to emphasize individual values rather than trends
- Data points are too sparse to show meaningful trends
- The independent variable is categorical with no natural order
- You need to show parts of a whole (use pie charts)
- You're showing the relationship between two quantitative variables (use scatter plots)

Constructing a Line Graph

- **Step 1** — Obtain the the data in a table.
- **Step 2** — Set up the axes and label.
- **Step 3** — Plot the data points.
- **Step 4** — Connect the points.
- **Step 5** — Add a title and legend, if necessary.

Example #6.6.1 🤔

A small business tracked its monthly revenue (in thousands of dollars) for the first six months:

Monthly revenues from January to June

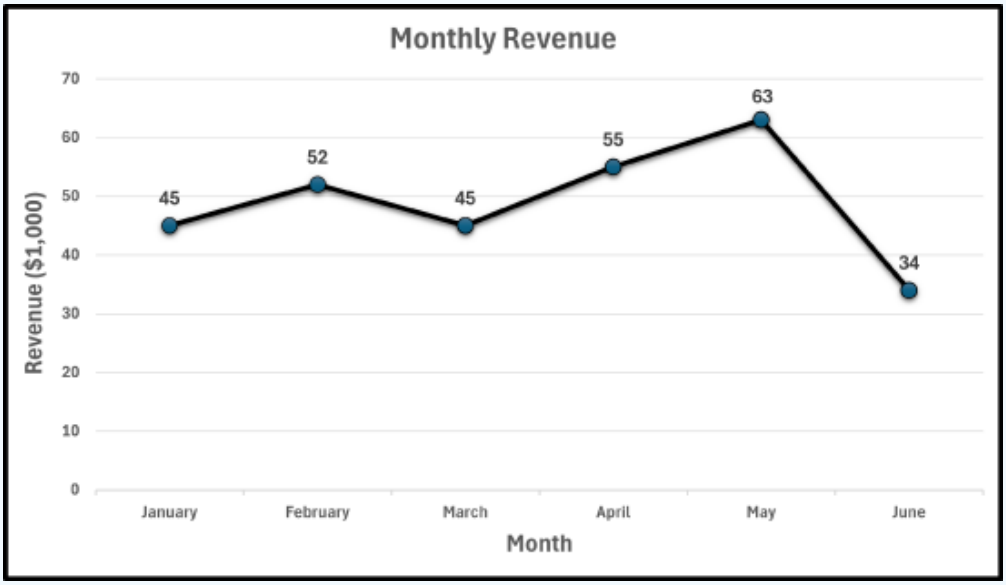
Month	Revenue
January	45
February	52

Month	Revenue
March	45
April	55
May	63
June	34

Create a line graph to display this data.

✅ **Solution:**

- **Step 1 — Obtain the data in a table:** This step is done, since the data was given in a frequency distribution.
- **Step 2 — Set up the axes and label:**
  - Label the main axes: "Months", for the horizontal axis & "Revenue ($1,000)", for the vertical axis.
  - On the horizontal axis, "Months", label the following: January, February, March, April, May, and June.
  - On the vertical axis, "Revenue ($1,000)", since the minimum frequency is 34 and the maximum frequency is 63, then increments of 5 can be used for the tick marks on the vertical axis, starting from 0 (or 30) to 65 (or 70).
- **Step 3 — Plot the data points:** On January at 45, February at 52, March at 45, April at 55, May at 63, and June at 34.
- **Step 4 — Connect the points:** Connect the points from left to right using one straight line between each set of points.
- **Step 5 — Add a title and legend, if necessary:** Add the title, "Monthly Revenue" or some other similar title.



Example #6.6.2 🤔

According to the United States Geological Survey (USGS), there were 112 earthquakes in California with a magnitude of 4.5 or higher, from the years 2014 to 2024. The following number of earthquakes with a magnitude of 4.5 or more is given in the table below.

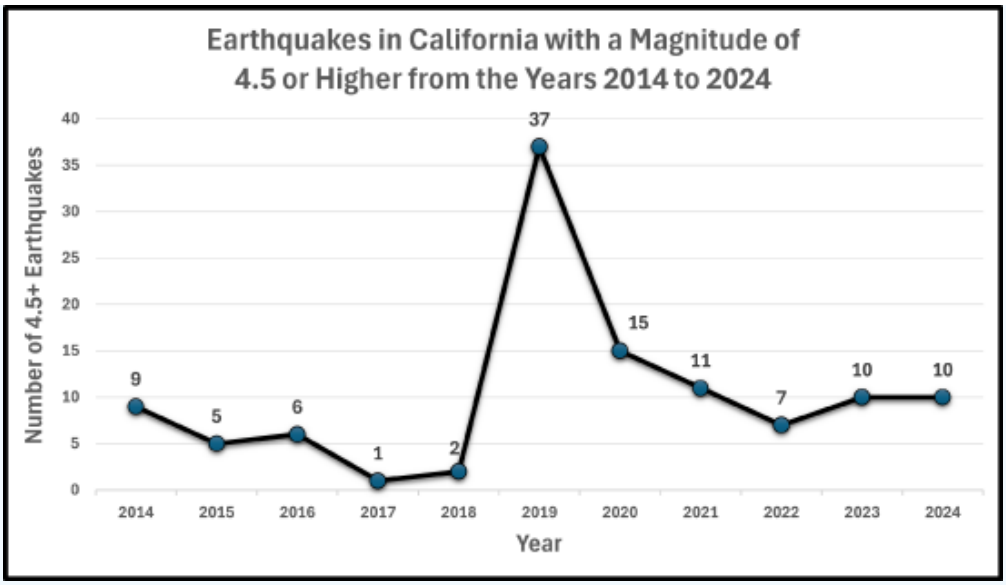
4.5+ Earthquakes by Year in California

Year	Number of 4.5+ Earthquakes
2014	9
2015	5
2016	6
2017	1
2018	2
2019	37
2020	15
2021	11
2022	7
2023	10
2024	10

Create a line graph to display this data.

✅ **Solution:**

- **Step 1 — Obtain the data in a table:** This step is done, since the data was given in a frequency distribution.
- **Step 2 — Set up the axes and label:**
  - Label the main axes: "Year", for the horizontal axis & "Number of 4.5+ Earthquakes", for the vertical axis.
  - On the horizontal axis, "Months", label the following: 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023, and 2024.
  - On the vertical axis, "Number of 4.5+ Earthquakes", since the minimum frequency is 1 and the maximum frequency is 37, then increments of 5 can be used for the tick marks on the vertical axis, starting from 0 to 40.
- **Step 3 — Plot the data points:** On 2014 at 9, 2015 at 5, 2016 at 6, 2017 at 1, 2018 at 2, 2019 at 37, 2020 at 15, 2021 at 11, 2022 at 7, 2023 at 10, and 2024 at 10.
- **Step 4 — Connect the points:** Connect the points from left to right using one straight line between each set of points.
- **Step 5 — Add a title and legend, if necessary:** Add the title, "Earthquakes in California with a Magnitude of 4.5 or Higher from the Years 2014 to 2024" or some other similar title.



Example #6.6.3 🤔

According to the U.S. Energy Information Administration (EIA), the average monthly retail price per gallon for regular gasoline in California during 2024 is given below:

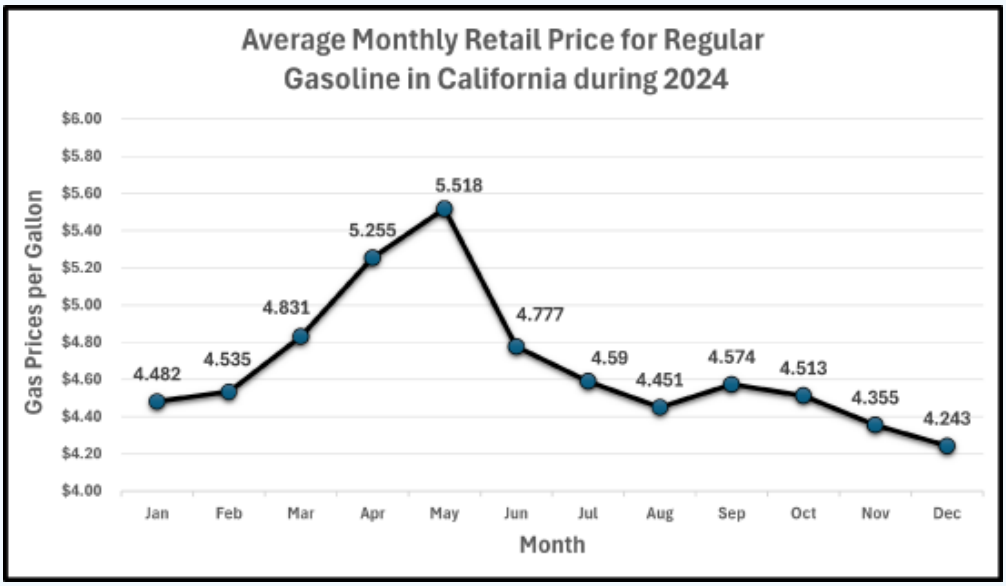
Average Monthly Gasoline Prices in California During 2024

Month	Price per Gallon (in dollars)
Jan	$4.482
Feb	$4.535
Mar	$4.831
Apr	$5.255
May	$5.518
Jun	$4.777
Jul	$4.590
Aug	$4.451
Sep	$4.574
Oct	$4.513
Nov	$4.355
Dec	$4.243

Create a line graph to display this data.

✅ **Solution:**

- **Step 1 — Obtain the data in a table:** This step is done, since the data was given in a frequency distribution.
- **Step 2 — Set up the axes and label:**
  - Label the main axes: "Month", for the horizontal axis & "Price per Gallon (in dollars)", for the vertical axis.
  - On the horizontal axis, "Months", label the following: Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec.
  - On the vertical axis, "Price per Gallon (in dollars)", since the minimum frequency is $4.243 and the maximum frequency is $5.518, then increments of $.20 can be used for the tick marks on the vertical axis, starting from $4 to $6.
- **Step 3 — Plot the data points:** On Jan at $4.482, Feb at $4.535, Mar at $4.831, Apr at $5.255, May at $5.518, Jun at $4.777, Jul at $4.590, Aug at $4.451, Sep at $4.574, Oct at $4.513, Nov at $4,355, and Dec at $4.243.
- **Step 4 — Connect the points:** Connect the points from left to right using one straight line between each set of points.
- **Step 5 — Add a title and legend, if necessary:** Add the title, "Average Monthly Retail Price for Regular Gasoline in California during 2024" or some other similar title.



Example #6.6.4 🤔

A student tracked the number of hours spent studying each week during an 8-week semester:

Study Hours by Week

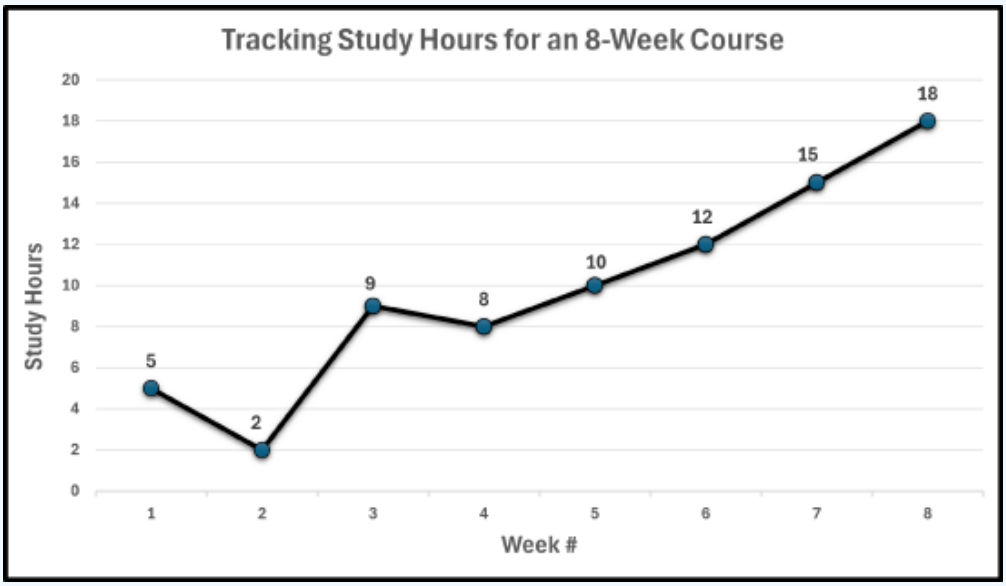
Week #	Study Hours
1	5
2	2
3	9
4	8
5	10
6	12

Week #	Study Hours
7	15
8	18

Create a line graph to display this data.

✅ **Solution:**

- **Step 1 — Obtain the data in a table:** This step is done, since the data was given in a frequency distribution.
- **Step 2 — Set up the axes and label:**
  - Label the main axes: "Week #", for the horizontal axis & "Study Hours", for the vertical axis.
  - On the horizontal axis, "Week #", label the following: 1, 2, 3, 4, 5, 6, 7, and 8.
  - On the vertical axis, "Study Hours", since the minimum frequency is 2 and the maximum frequency is 18, then increments of 2 can be used for the tick marks on the vertical axis, starting from 0 to 20.
- **Step 3 — Plot the data points:** On 1 at 5, 2 at 2, 3 at 9, 4 at 8, 5 at 10, 6 at 12, 7 at 15, and 8 at 18.
- **Step 4 — Connect the points:** Connect the points from left to right using one straight line between each set of points.
- **Step 5 — Add a title and legend, if necessary:** Add the title, "Tracking Study Hours for an 8-Week Course" or some other similar title.



Section 6.6: Line Graphs [*In-Class Exercises*]

1. The Los Angeles Rams during the 2025 regular season, scored the most points in the NFL with 518. The following total points scored by quarter (overtime excluded) for the Los Angeles Rams is given in the table below:

Points Scored by Quarter

Quarter	Points Scored
First Quarter	113
Second Quarter	144

Quarter	Points Scored
Third Quarter	124
Forth Quarter	130

Construct a line graph for the data.

2. The number of sections covered on each Math A100 Liberal Arts Math exam is shown below in the frequency table.

Number of Sections Covered on each Math A100 Liberal Arts Mathematics Exam

Exam #	No. of Sections
Exam #1	16
Exam #2	14
Exam #3	12
Exam #4	16
Exam #5	8

Construct a line graph for the data.

3. The typical monthly electricity usage (in kilowatts per hour, kWh) for a U.S. household based on publicly available summaries from the U.S. Energy Information Administration (EIA) in a city for an entire year is given in the frequency table below.

Monthly Household Electricity Usage

Date	kWh
Jan	470
Feb	430
Mar	410
Apr	390
May	420
Jun	560
Jul	640
Aug	610
Sep	520
Oct	460

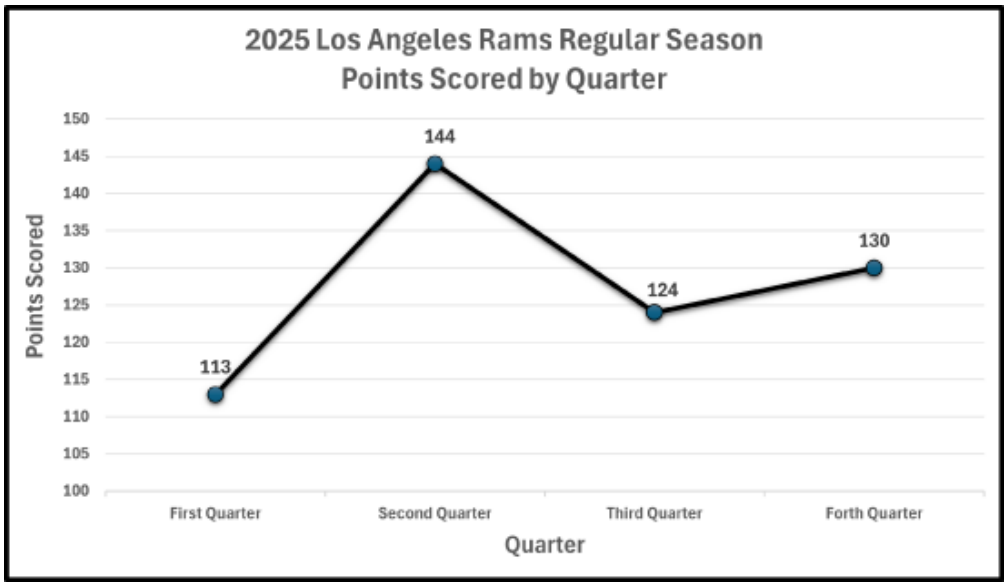
Date	kWh
Nov	430
Dec	450

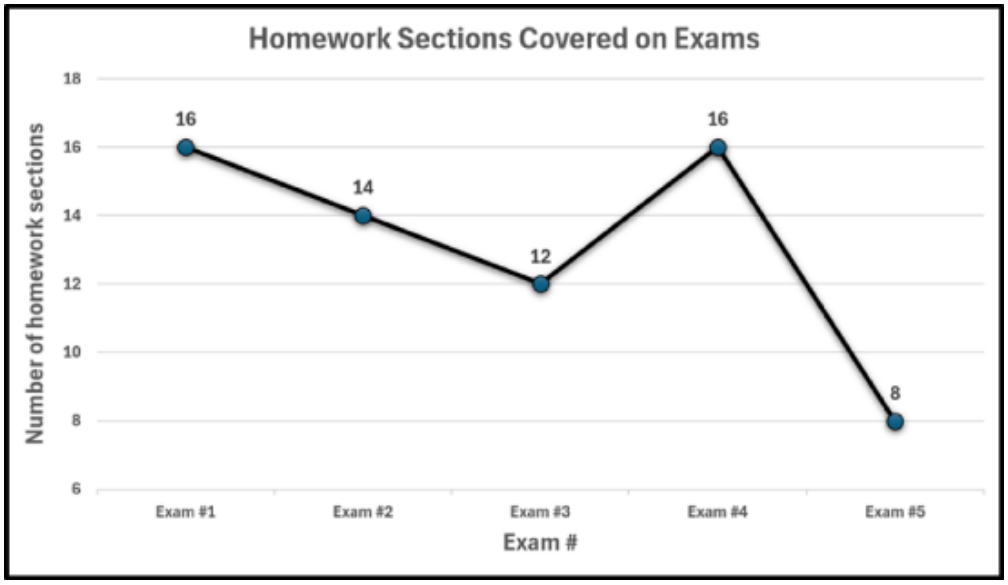
Construct a line graph for the data.

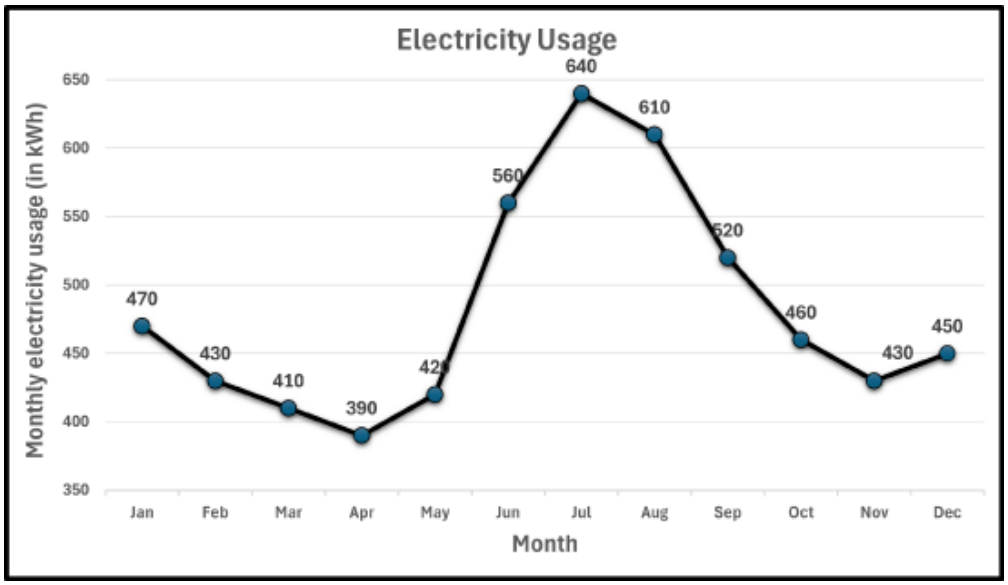
Answers available at the end of this section.

Answers

Section 6.6: Line Graphs [*In-Class Exercises*]

1. 

2. 

3. 

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Section 6.6.H: Homework

Section 6.6: Line Graphs - Homework Exercises

(Section 6.6) - Homework Exercise #1

The distance from a Wi-Fi router vs. download speed are given in the frequency distribution below:

Wi-Fi router vs. download speed

Distance (ft)	Speed (Mbps)
1	120
5	110
10	95
15	80
20	65
30	45

Construct a line graph for the data set.

Answer available at the end of this section.

(Section 6.6) - Homework Exercise #2

Strength training often leads to predictable improvements in athletic performance, especially when beginners follow a structured program. In this example, we examine how an individual's bench press strength increases as they complete more training sessions at the gym. The frequency distribution below represents the maximum amount of weight (in pounds) the person can lift after a certain number of workouts.

Bench press maximums by number of training sessions

Training Sessions	Bench Press Max (lbs)
1	95
5	105
10	120

Training Sessions	Bench Press Max (lbs)
15	135
20	145
25	155

Construct a line graph for the data.

Answer available at the end of this section.

(Section 6.6) - Homework Exercise #3

Pickleball began in 1965 on Bainbridge Island, Washington, by three friends where they invented the game during a family gathering.After gaining momentum in the Pacific Northwest, pickleball spread across the United States. By the early 2000's, community centers, retirement communities, and schools were adopting the game. In the 2021, pickleball experienced explosive growth. It was named the fastest-growing sport in the U.S. for four consecutive years (2021–2024) by the Sports & Fitness Industry Association. Here are the results in a frequency distribution table:

Pickleball revenue by year

Year	Revenue (in billions)
2021	1.3928
2022	1.3229
2023	1.4724
2024	1.6388
2025	1.8239

Construct a line graph for the revenues.

Answer available at the end of this section.

(Section 6.6) - Homework Exercise #4

The daily high temperatures (in °F) in Anaheim, CA were obtained for the first two weeks of 2026. The results are given in the frequency table below:

Daily high temperatures in Anaheim, CA

Date	High Temperature
1/1/26	62
1/2/26	64
1/3/26	63

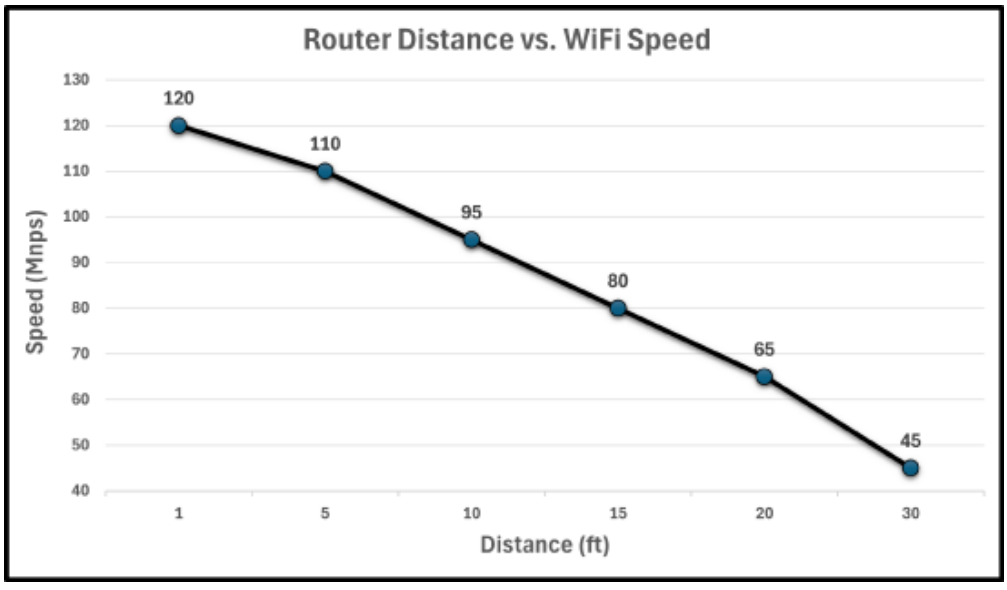
Date	High Temperature
1/4/26	61
1/5/26	60
1/6/26	59
1/7/26	58
1/8/26	60
1/9/26	62
1/10/26	65
1/11/26	66
1/12/26	64
1/13/26	63
1/14/26	62

Construct a line graph for the temperatures.

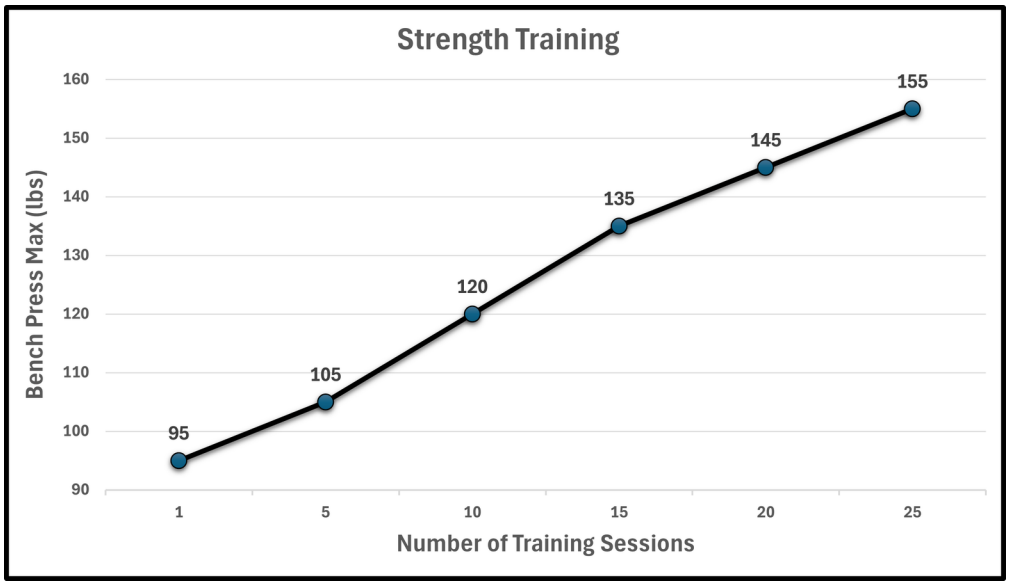
Answer available at the end of this section.

Answers

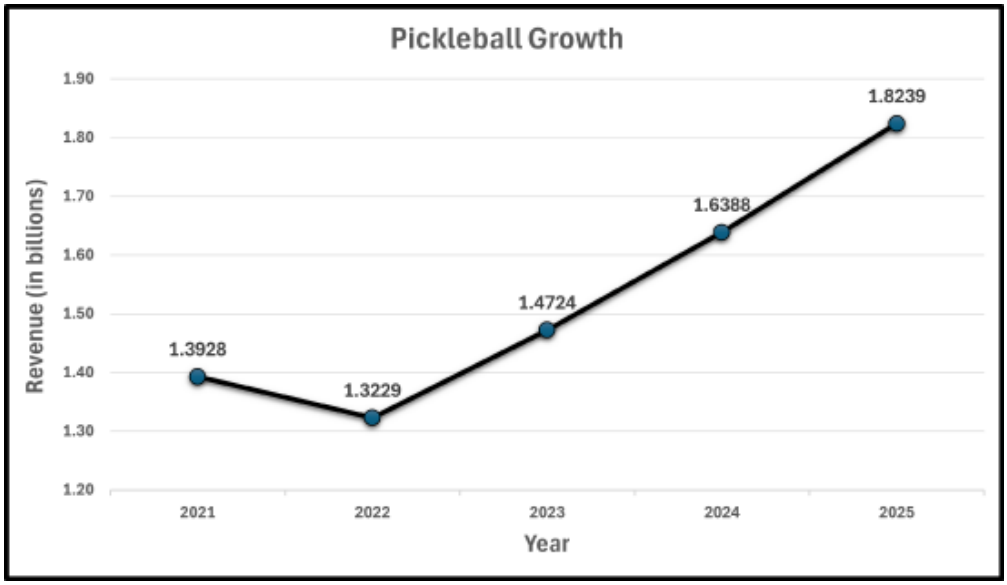
(Section 6.6) - Homework Exercise #1



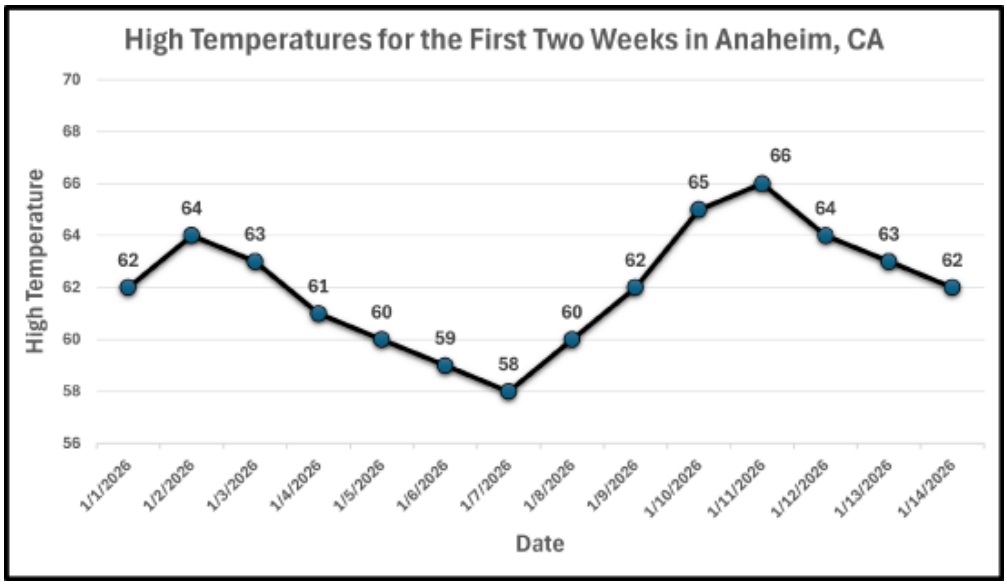
(Section 6.6) - Homework Exercise #2



(Section 6.6) - Homework Exercise #3



(Section 6.6) - Homework Exercise #4



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Section 6.7: Histograms

Learning Objectives

- Draw a histogram for a data set

Definition: Histogram

A **histogram** is a graphical display that shows the distribution of numerical data by dividing the data into intervals (called classes or bins) and displaying the frequency of observations in each interval as bars. Unlike bar graphs, histograms display continuous numerical data, and the bars touch each other to emphasize the continuous nature of the data. Histograms are essential tools for understanding the shape, center, and spread of a data distribution.

When to Use Histograms

Histograms are most effective when you want to:

- Display the distribution of continuous numerical data
- Show the shape of a data distribution
- Identify patterns such as symmetry, skewness, or uniformity
- See where data values are concentrated
- Identify gaps, clusters, or outliers in the data
- Understand the spread or variability of data
- Compare frequency distributions visually

Key Components of a Histogram

1. **Horizontal Axis (*x*-axis)**: Shows the numerical scale divided into class intervals (bins)
   - Represents the variable being measured (height, test scores, age, temperature, etc.)
   - Divided into equal-width intervals
2. **Vertical Axis (*y*-axis)**: Shows the frequency (count) or relative frequency (percentage) of observations in each interval
3. **Bars (Rectangles)**: Each bar represents one class interval
   - Height shows the frequency for that interval
   - Width represents the class interval
   - Bars touch each other (no spaces) to show continuous data
4. **Class Intervals (Bins)**: The ranges into which data is divided
   - Should have equal width
   - Should cover all data values
   - Typically use 5-20 intervals depending on data size
5. **Labels and Title**: Clear axis labels and descriptive title

Advantages of Histograms

- **Shows distribution shape**: Immediately reveals if data is symmetric, skewed, uniform, or bimodal
- **Identifies patterns**: Clusters, gaps, and outliers become visible

- **Easy to construct and interpret**: Simple visual representation
- **Shows central tendency**: Where data concentrates
- **Reveals spread**: Shows how variable or consistent the data is
- **Handles large datasets**: Can summarize hundreds or thousands of observations
- **Foundation for statistical analysis**: Helps check assumptions for further analysis

Limitations of Histograms

- **Loses individual data values**: Grouping obscures specific observations
- **Sensitive to bin width**: Different bin widths can create different impressions
- **Sensitive to bin boundaries**: Starting/ending points affect appearance
- **Less precise than raw data**: Can't calculate exact statistics from histogram alone
- **Requires numerical data**: Won't work with categorical data
- **Can be misleading**: Poor choices in bins or scale can distort interpretation

Constructing a Histogram

- **Step 1** — Obtain the data in a frequency distribution table.
- **Step 2** — Draw the histogram.
  - **Draw the axes:** Draw a horizontal line for the x-axis and a vertical line for the y-axis.
  - **Label the axes:** Label the horizontal axis with your chosen class width or bin intervals and the vertical axis with a frequency scale that accommodates your highest count.
  - **Draw the bars:** For each class/bin, draw a vertical bar with a height corresponding to its frequency. The bars should have the exact same width and should touch each other to show the continuous nature of the data.
- **Step 3** — Add a title, if necessary.

Two Types of Frequency Histograms

- Ungrouped frequency histograms - Used for either qualitative data (categorical data) or single value data
- Grouped frequency histograms - Used for quantitative data (numerical data) where the data needs to be grouped into class intervals.

Let's see several examples for both types of frequency histograms.

Example #6.7.1 🤔

The number of text messages sent in one hour by 20 teenagers is given below in a frequency distribution table:

Distribution of text messages

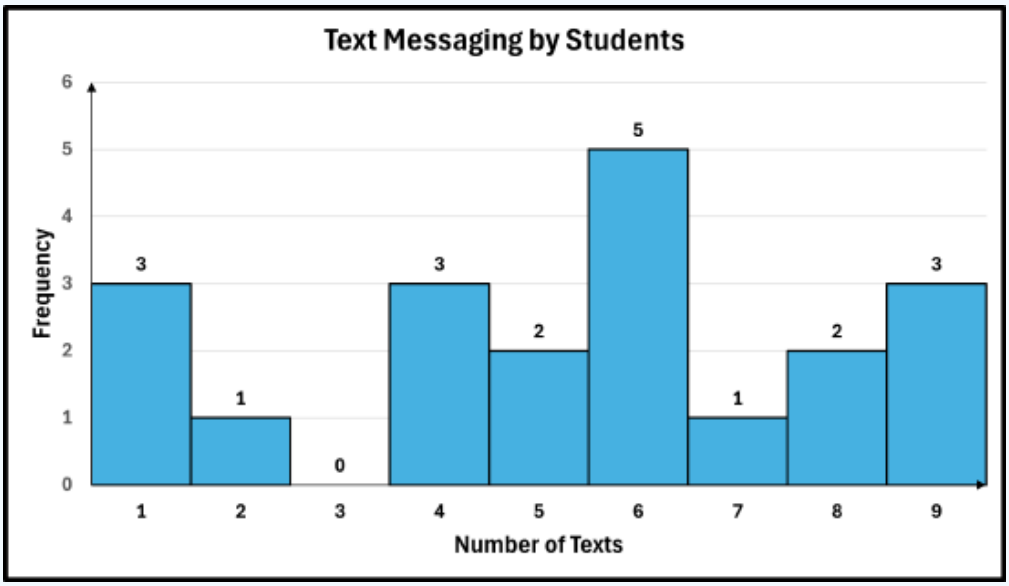
Number of Texts	Frequency
1	3
2	1
3	0
4	3
5	2
6	5

Number of Texts	Frequency
7	1
8	2
9	3

Draw an ungrouped frequency histogram for the data set.

✅ **Solution:**

- **Step 1 — Obtain the data in a frequency distribution table:** Here, the frequency distribution was given.
- **Step 2 — Draw the histogram:**
  - **Draw the axes:** Draw a horizontal line for the *x*-axis and a vertical line for the *y*-axis.
  - **Label the axes:** Label the horizontal axis with your chosen class width or bin intervals and the vertical axis with a frequency scale that accommodates your highest count. So, since the data is ungrouped, the first bin is labeled 1 and represents all data values that are equal to 1. The second bin is labeled 2 and represents all data values that are equal to 2. And so on, and so on.
  - **Draw the bars:** For each class/bin, draw a vertical bar with a height corresponding to its frequency. The bars should have the exact same width and should touch each other to show the continuous nature of the data.
- **Step 3 — Add a title, if necessary:** Add the title, "Text Messaging by Students" or some other similar title.



Example #6.7.2 🤔

The frequency distribution for twenty-five overall ratings from 1 through 10 of an exclusive Paramount Plus show is given below:

Distribution of ratings

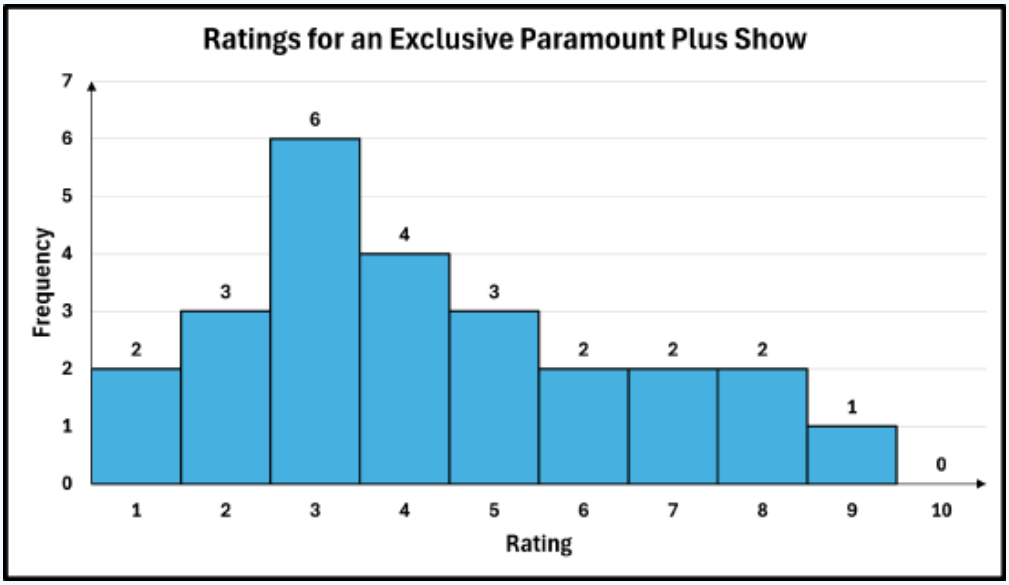
Ratings	Frequency
1	2

Ratings	Frequency
2	3
3	6
4	4
5	3
6	2
7	2
8	2
9	1
10	0

Draw an ungrouped frequency histogram for the data set.

✅ **Solution:**

- **Step 1 — Obtain the data in a frequency distribution table:** Here, the frequency distribution was given.
- **Step 2 — Draw the histogram:**
  - **Draw the axes:** Draw a horizontal line for the *x*-axis and a vertical line for the *y*-axis.
  - **Label the axes:** Label the horizontal axis with your chosen class width or bin intervals and the vertical axis with a frequency scale that accommodates your highest count. So, since the data is ungrouped, the first bin is labeled 1 and represents all data values that are equal to 1. The second bin is labeled 2 and represents all data values that are equal to 2. And so on, and so on.
  - **Draw the bars:** For each class/bin, draw a vertical bar with a height corresponding to its frequency. The bars should have the exact same width and should touch each other to show the continuous nature of the data.
- **Step 3 — Add a title, if necessary:** Add the title, "Ratings for an Exclusive Paramount Plus Show" or some other similar title.



Example #6.7.3 🤔

The heights (in inches) of 28 adult males is recorded.

64, 65, 66, 66, 67, 67, 68, 68, 68, 69, 69, 69, 70, 70,
70, 71, 71, 71, 72, 72, 72, 73, 73, 73, 74, 74, 75, 76

Draw an ungrouped frequency histogram for the data set.

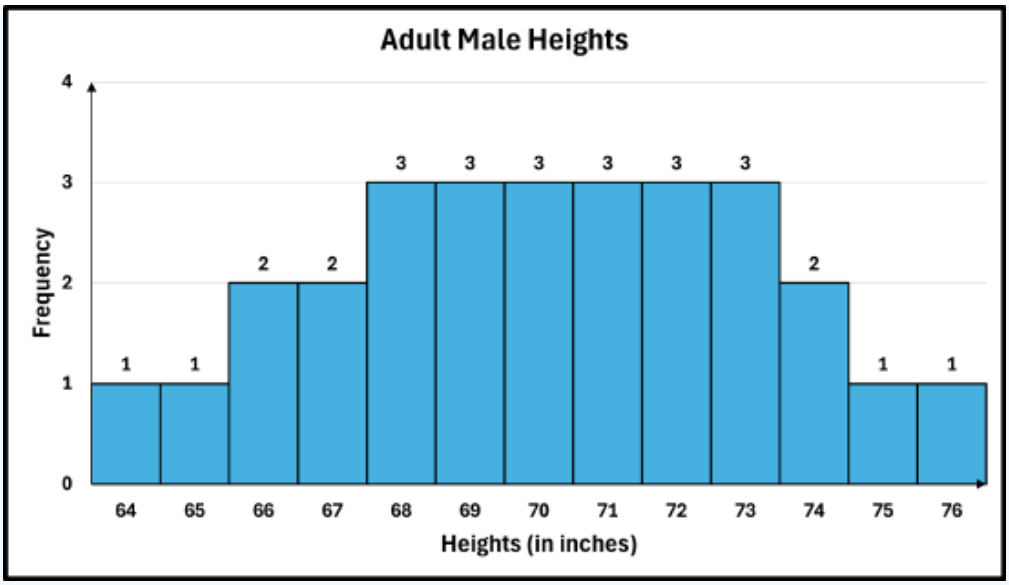
✅ **Solution:**

- **Step 1 — Obtain the data in a frequency distribution table:** A frequency distribution for the data set is given below.

Distribution of adult males' heights

Heights	Frequency
64	1
65	1
66	2
67	2
68	3
69	3
70	3
71	3
72	3
73	3
74	2
75	1
76	1

- **Step 2 — Draw the histogram:**
  - **Draw the axes:** Draw a horizontal line for the *x*-axis and a vertical line for the *y*-axis.
  - **Label the axes:** Label the horizontal axis with your chosen class width or bin intervals and the vertical axis with a frequency scale that accommodates your highest count. So, since the data is ungrouped, the first bin is labeled 64 and represents all data values that are equal to 64. The second bin is labeled 65 and represents all data values that are equal to 65. And so on, and so on.
  - **Draw the bars:** For each class/bin, draw a vertical bar with a height corresponding to its frequency. The bars should have the exact same width and should touch each other to show the continuous nature of the data.
- **Step 3 — Add a title, if necessary:** Add the title, "Text Messaging by Students" or some other similar title.



Example #6.7.4 🤔

The temperatures (in degrees Fahrenheit) recorded in 20 cities on a particular day, rounded to the nearest degree are listed below:

58, 68, 65, 79, 51, 94, 64, 80, 69, 113, 63, 70, 102, 70, 92, 87, 77, 64, 64, 72

Here is the grouped frequency distribution for the above data using a class width of 11 and a starting point of 51.

Grouped distribution of temperatures

Temperature	Frequency
51—61	2
62—72	10
73—83	3
84—94	3
95—105	1
106—116	1

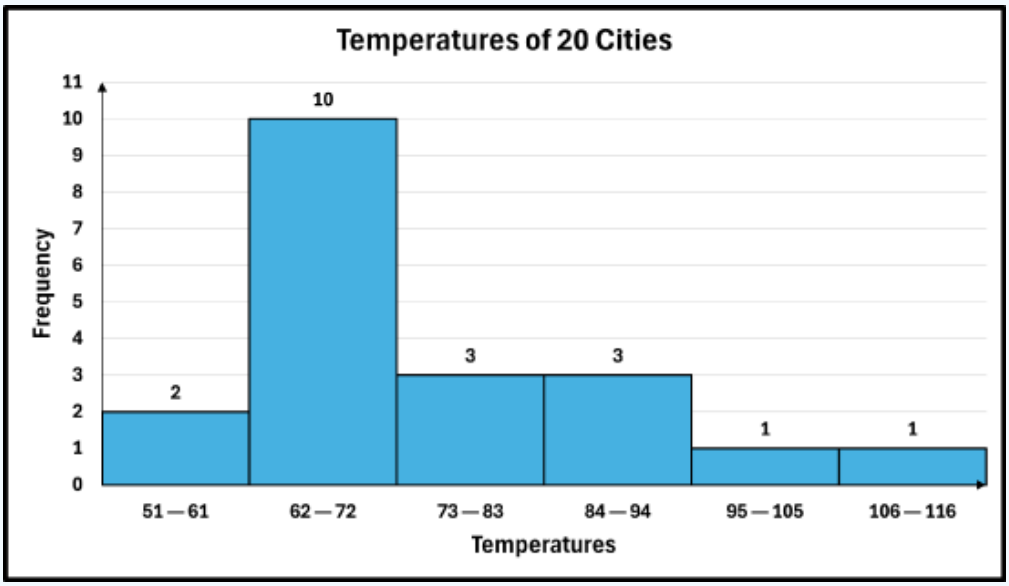
Draw an grouped frequency histogram for the data set.

✅ **Solution:**

- **Step 1 — Obtain the data in a frequency distribution table:** Here, the frequency distribution was given.
- **Step 2 — Draw the histogram:**
  - **Draw the axes:** Draw a horizontal line for the *x*-axis and a vertical line for the *y*-axis.

- **Label the axes:** Label the horizontal axis with your chosen class width or bin intervals and the vertical axis with a frequency scale that accommodates your highest count. So, since the data is grouped, the first bin is labeled 51—61 and represents all data values that are between 51 and 61 inclusive. The second bin is labeled 62—72 and represents all data values that are between 62 and 72 inclusive. The third bin is labeled 73—83 and represents all data values that are between 73 and 83 inclusive. And so on, and so on.
- **Draw the bars:** For each class/bin, draw a vertical bar with a height corresponding to its frequency. The bars should have the exact same width and should touch each other to show the continuous nature of the data.

- **Step 3 — Add a title, if necessary:** Add the title, "Temperatures of 20 Cities" or some other similar title.



Example #6.7.5 🤔

In a history class, the test scores (out of 100) for 30 students was recorded below in a grouped frequency distribution for the above data using a class width of 10 and a starting point of 60.

Grouped distribution of test scores

Ratings	Frequency
60—69	4
70—79	11
80—89	8
90—99	7

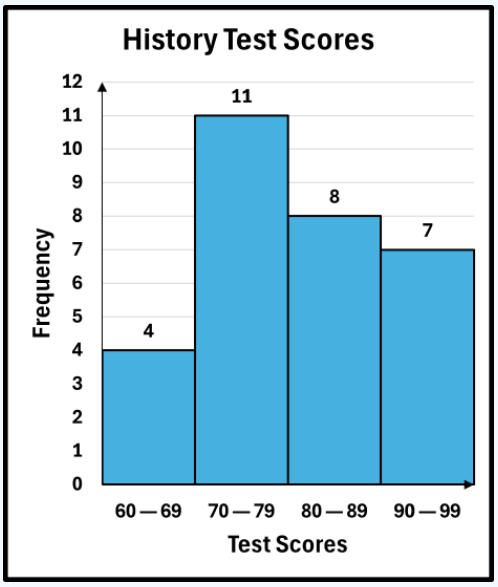
Draw a grouped frequency histogram for the data set.

✅ **Solution:**

- **Step 1 — Obtain the data in a frequency distribution table:** Here, the frequency distribution was given.
- **Step 2 — Draw the histogram:**
  - **Draw the axes:** Draw a horizontal line for the *x*-axis and a vertical line for the *y*-axis.

- **Label the axes:** Label the horizontal axis with your chosen class width or bin intervals and the vertical axis with a frequency scale that accommodates your highest count. So, since the data is grouped, the first bin is labeled 60—69 and represents all data values that are between 60 and 69 inclusive. The second bin is labeled 70—79 and represents all data values that are between 70 and 79 inclusive. And so on, and so on.
- **Draw the bars:** For each class/bin, draw a vertical bar with a height corresponding to its frequency. The bars should have the exact same width and should touch each other to show the continuous nature of the data.

- **Step 3 — Add a title, if necessary:** Add the title, "History Test Scores" or some other similar title.



Example #6.7.6 🤔

The weight loss (in pounds) over 6 months for 32 program participants were recorded:

19, 18, 25, 15, 32, 20, 28, 14, 35, 22, 16, 30, 19, 38, 24, 17,
31, 21, 36, 23, 18, 29, 20, 34, 25, 16, 33, 22, 27, 12, 35, 26

Draw a grouped frequency histogram for the above data using a class width of 4 and a starting point of the lowest observation in the data set.

✅ **Solution:**

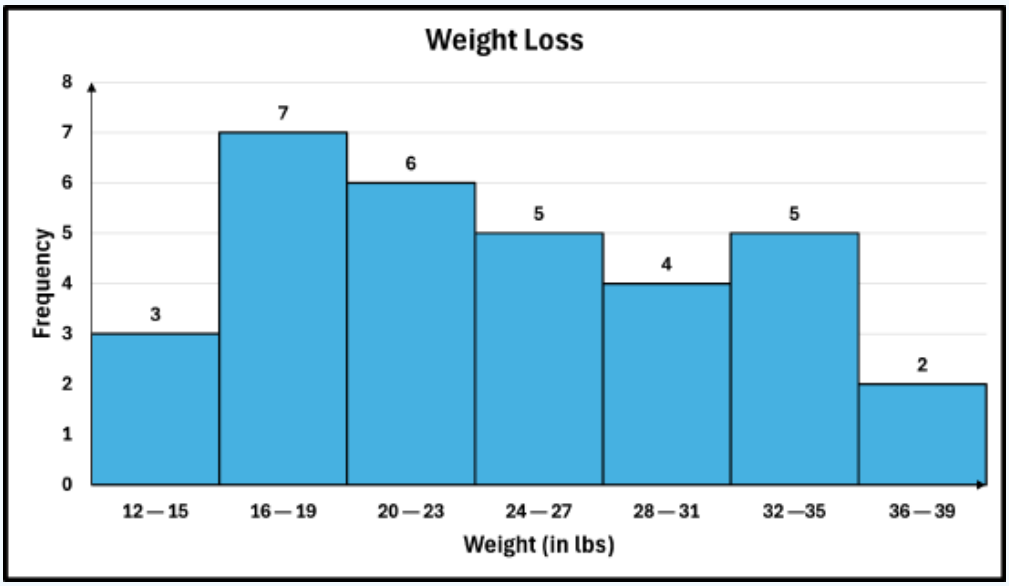
- **Step 1 — Obtain the data in a grouped frequency distribution table:** A grouped frequency distribution for the data set is given below. (See Section 6.2 for reference).

Grouped distribution of pounds lost

Weight Loss (in pounds)	Frequency
12—15	3
16—19	7
20—23	6
24—27	5

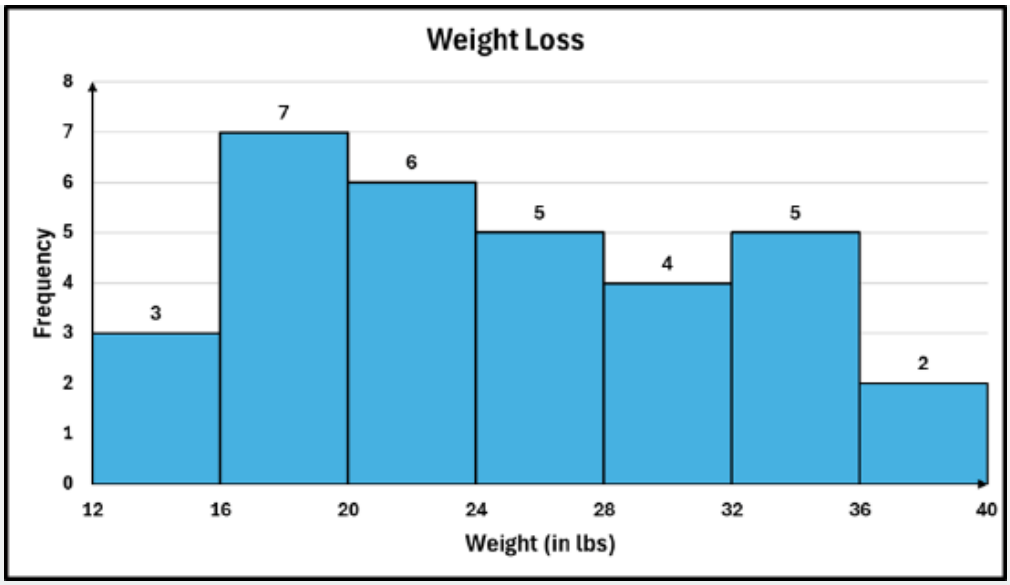
Weight Loss (in pounds)	Frequency
28—31	4
32—35	5
36—39	2

- **Step 2 — Draw the histogram:**
  - **Draw the axes:** Draw a horizontal line for the *x*-axis and a vertical line for the *y*-axis.
  - **Label the axes:** Label the horizontal axis with your chosen class width or bin intervals and the vertical axis with a frequency scale that accommodates your highest count. So, since the data is grouped, the first bin is labeled 12—15 and represents all data values that are between 12 and 15 inclusive. The second bin is labeled 16—19 and represents all data values that are between 16 and 19 inclusive. And so on, and so on.
  - **Draw the bars:** For each class/bin, draw a vertical bar with a height corresponding to its frequency. The bars should have the exact same width and should touch each other to show the continuous nature of the data.
- **Step 3 — Add a title, if necessary:** Add the title, "Weight Loss" or some other similar title.



Note: Alternate Histogram

In statistics, there is an alternative way to draw a histogram that changes only how the *x*-axis is labeled. Instead of placing a class label under each bar, the axis is marked with the class boundaries. This means each bar spans from the lower limit of a class up to, but not including, the upper limit. In this format, the sides of each bar line up exactly with the lower class limits, and every value falls into the interval that includes its lower boundary but excludes its upper boundary. Here is what the alternate histogram would look like for Example #6.7.6:



So, the bin/bar between 12 and 16, represents the first class of 12—15, the second bin/bar between 16 and 20, represents the second class of 16—19, and so on. In the data set, there are two observations of 16, both of which would be counted in the second bin, not the first bin.

Section 6.7: Histograms [*In-Class Exercises*]

1. Draw an ungrouped frequency histogram for the following data set: 1, 1, 2, 3, 4, 4, 4, 4, 5, 6, 6, 7, 7, 7, 7, 7, 7, 7, 9, 9
2. Twenty AA batteries were tested to determine how long they would last. The results, to the nearest minute, were recorded as follows:

423, 369, 387, 411, 393, 399, 371, 377, 409, 392,
417, 431, 401, 363, 391, 405, 425, 400, 381, 399

The above data is in a grouped frequency distribution below using a class width of 9 and a starting point of 363.

Grouped distribution of battery lifetimes

Minutes	Frequency
363—371	3
372—380	1
381—389	2
390—398	3
399—407	5
408—416	2
417—425	3
426—434	1

Draw a grouped frequency histogram for the above data in the frequency distribution.

3. The following data represent sales (in dollars) at a café during their slowest hour:

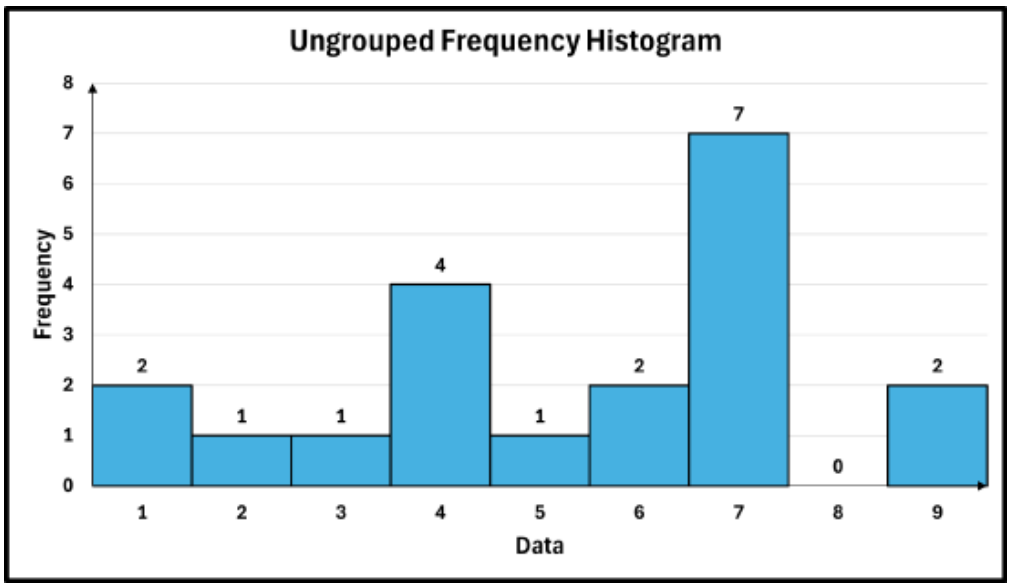
145, 152, 138, 160, 149, 176, 155, 142, 158, 151, 147, 181, 132, 154, 152, 168, 118, 175

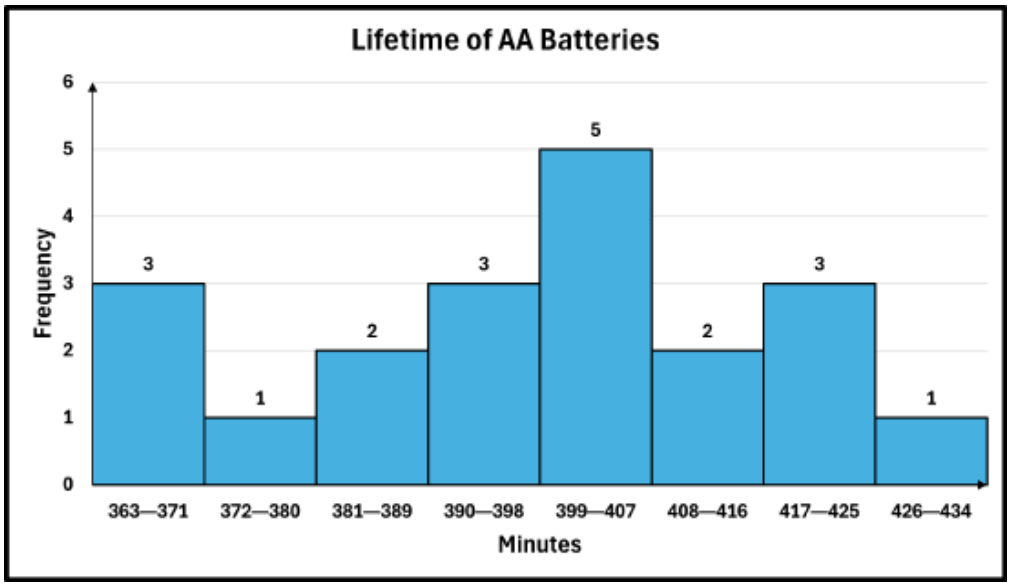
Draw a grouped frequency histogram for the above data using a class width of 7 and a starting point of the lowest observation in the data set.

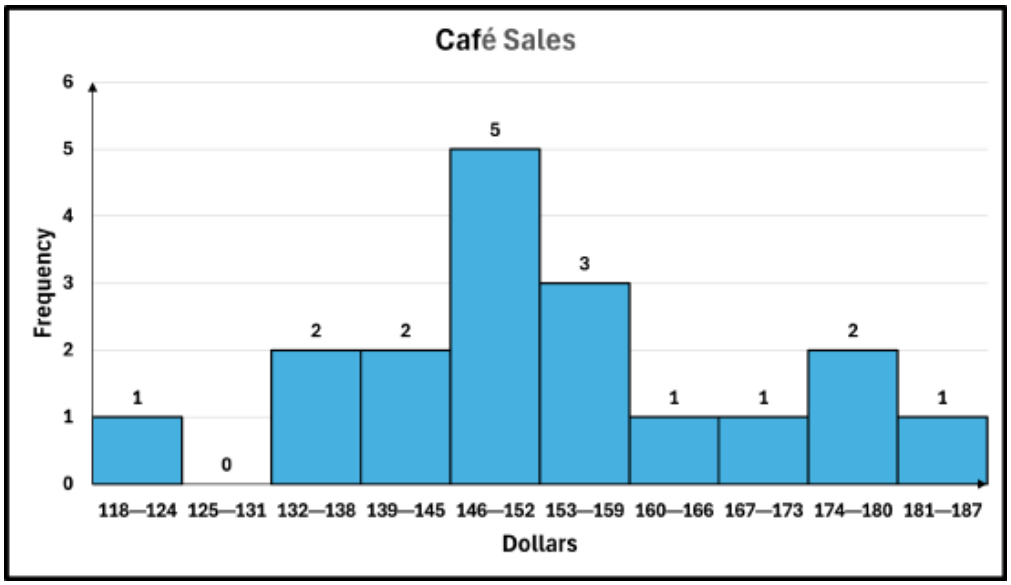
Answers available at the end of this section.

Answers

Section 6.7: Histograms [*In-Class Exercises*]

1. 

2. 

3. 

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Section 6.7.H: Homework

Section 6.7: Histograms - Homework Exercises

(Section 6.7) - Homework Exercise #1

Draw an ungrouped frequency histogram for the following data set: 15, 17, 17, 17, 17, 19, 20, 20, 21, 22, 22, 22, 23, 24, 24, 24, 24, 24, 24, 25, 26, 26

Answer available at the end of this section.

(Section 6.7) - Homework Exercise #2

A random sample of 15 households was conducted to determine the number of pets per household. The data was found to be:

0, 1, 2, 1, 3, 2, 1, 0, 1, 2, 4, 1, 0, 3, 2

Draw an ungrouped frequency histogram for the data set above.

Answer available at the end of this section.

(Section 6.7) - Homework Exercise #3

The ages (in years) of 25 student's vehicles in a high school parking lot were recorded.

2, 5, 3, 7, 4, 6, 2, 3, 5, 8, 9, 4, 3, 6, 7, 5, 5, 6, 4, 7, 7, 6, 5, 3, 8

Draw an ungrouped frequency histogram for the data set above.

Answer available at the end of this section.

(Section 6.7) - Homework Exercise #4

The monthly electric bills (in dollars) of 24 homes was obtained:

110, 105, 98, 123, 140, 136, 115, 108, 145, 160, 138, 130, 122, 118, 150, 155, 147, 132, 92, 125, 119, 142, 135, 128

Draw a grouped frequency histogram for the data above using a class width of 8 and a starting point of the lowest observation in the data set.

Answer available at the end of this section.

(Section 6.7) - Homework Exercise #5

An environmentalist is conducting a study on car mileage per gallon (MPG) for a fleet of vehicles off an assembly. The MPG of 25 vehicles were as follows:

18, 22, 25, 28, 19, 24, 26, 30, 21, 23, 27, 29, 17, 20, 31, 22, 24, 25, 28, 26, 30, 29, 18, 21, 23

Draw a grouped frequency histogram for the data above using a class width of 3 and a starting point of the lowest observation in the data set.

Answer available at the end of this section.

(Section 6.7) - Homework Exercise #6

The commute times (in minutes) of 40 employees at a major law form in Los Angeles was recorded:

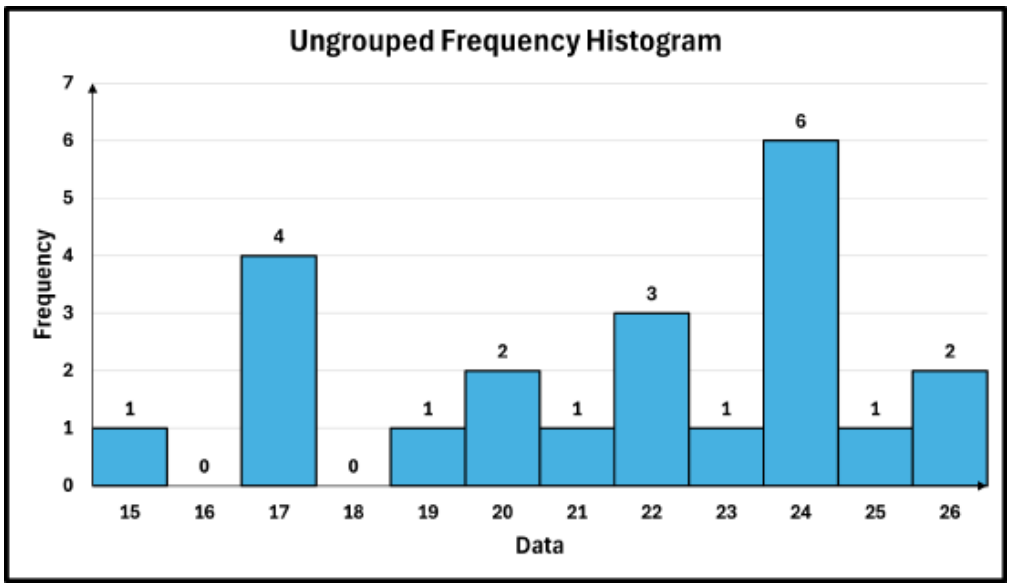
72, 25, 52, 73, 45, 32, 28, 50, 41, 36, 48, 30, 78, 28, 39, 33, 100, 55, 43, 111,
18, 105, 40, 49, 41, 59, 24, 76, 29, 37, 42, 35, 88, 34, 20, 38, 44, 71, 85, 66

Draw a grouped frequency histogram for the above data using a class width of 12 and a starting point of the lowest observation in the data set.

Answer available at the end of this section.

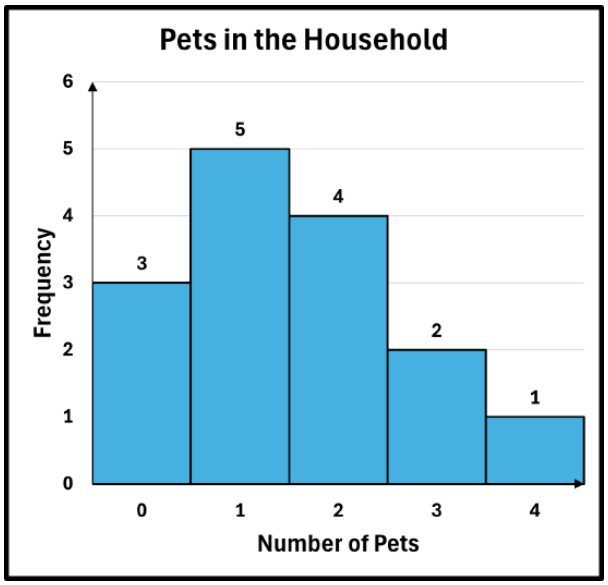
Answers

(Section 6.7) - Homework Exercise #1



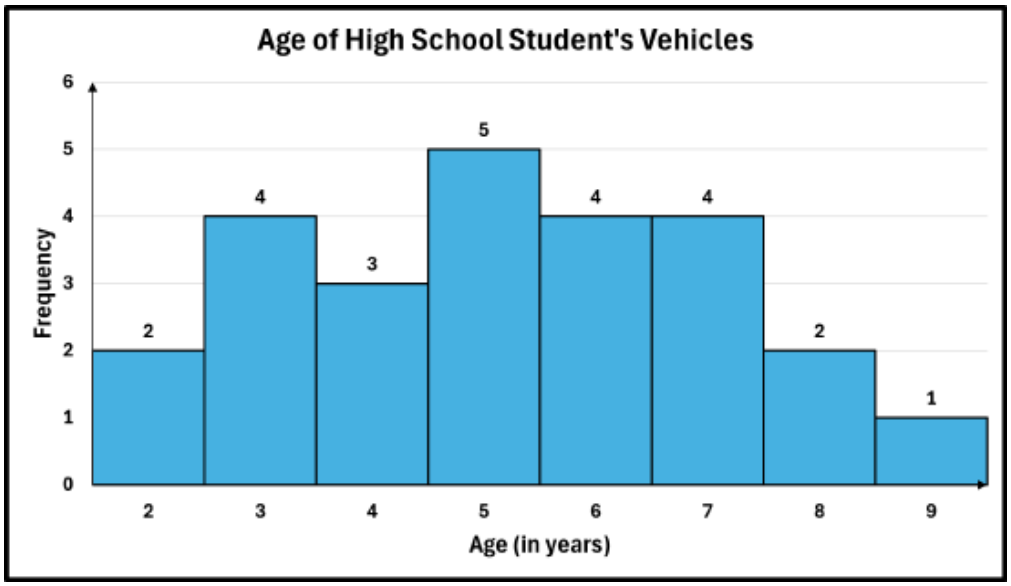
The histogram titled “Ungrouped Frequency Histogram” displays the frequency of individual data values from 15 through 26. The x-axis is labeled “Data,” and the y-axis is labeled “Frequency,” ranging from 0 to 7. Each bar represents how many times a specific value appears in the data set. The value 15 appears 1 time. The value 16 appears 0 times. The value 17 appears 4 times. The value 18 appears 0 times. The value 19 appears 1 time. The value 20 appears 2 times. The value 21 appears 1 time. The value 22 appears 3 times. The value 23 appears 1 time. The value 24 appears 6 times, which is the highest frequency in the graph. The value 25 appears 1 time, and the value 26 appears 2 times. Overall, the histogram shows that 24 is the most common value, while 16 and 18 do not appear in the data set.

(Section 6.7) - Homework Exercise #2



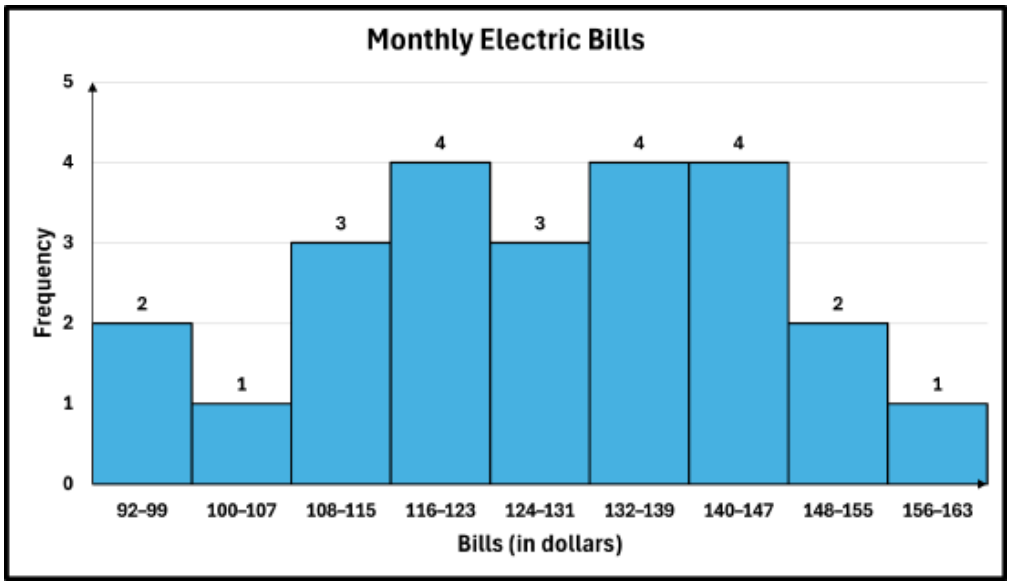
The histogram titled “Pets in the Household” shows the frequency of households based on the number of pets they own. The x-axis is labeled “Number of Pets” and includes values from 0 to 4. The y-axis is labeled “Frequency” and ranges from 0 to 6. Households with 0 pets have a frequency of 3. Households with 1 pet have the highest frequency at 5. Households with 2 pets have a frequency of 4. Households with 3 pets have a frequency of 2, and households with 4 pets have the lowest frequency at 1. Overall, the histogram shows that households with 1 or 2 pets are the most common, while households with 4 pets are the least common.

(Section 6.7) - Homework Exercise #3



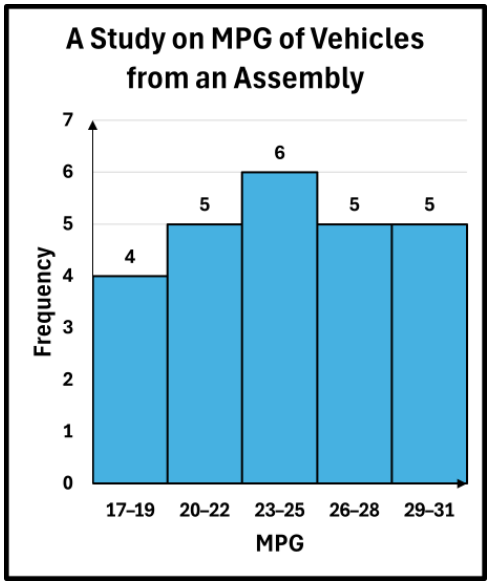
The histogram titled “Age of High School Student’s Vehicles” shows the frequency of vehicle ages owned by high school students. The x-axis is labeled “Age (in years)” and includes ages from 2 to 9 years. The y-axis is labeled “Frequency” and ranges from 0 to 6. Vehicles that are 2 years old have a frequency of 2. Vehicles that are 3 years old have a frequency of 4. Vehicles that are 4 years old have a frequency of 3. Vehicles that are 5 years old have the highest frequency at 5. Vehicles that are 6 years old have a frequency of 4, and vehicles that are 7 years old also have a frequency of 4. Vehicles that are 8 years old have a frequency of 2, while vehicles that are 9 years old have the lowest frequency at 1. Overall, the histogram shows that most student vehicles are between 3 and 7 years old, with 5-year-old vehicles appearing most often.

(Section 6.7) - Homework Exercise #4



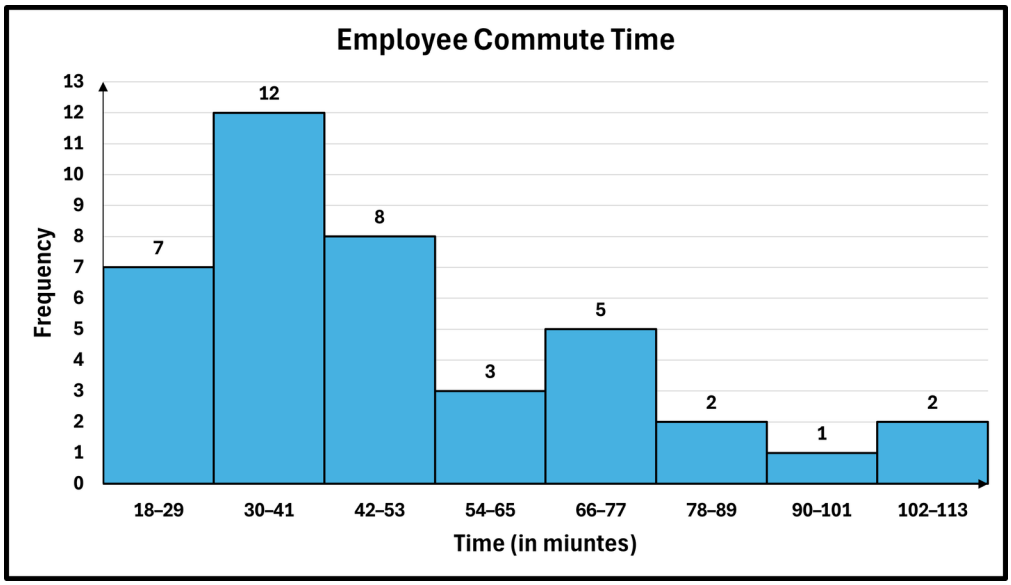
The histogram titled “Monthly Electric Bills” displays the frequency of electric bill amounts grouped into dollar intervals. The x-axis is labeled “Bills (in dollars)” and shows grouped intervals from 92–99 dollars up to 156–163 dollars. The y-axis is labeled “Frequency” and ranges from 0 to 5. The interval 92–99 dollars has a frequency of 2. The interval 100–107 dollars has a frequency of 1. The interval 108–115 dollars has a frequency of 3. The interval 116–123 dollars has a frequency of 4. The interval 124–131 dollars has a frequency of 3. The interval 132–139 dollars has a frequency of 4. The interval 140–147 dollars also has a frequency of 4. The interval 148–155 dollars has a frequency of 2, and the interval 156–163 dollars has a frequency of 1. Overall, the histogram shows that most monthly electric bills fall between 116 and 147 dollars, while the lowest and highest bill ranges occur less often.

(Section 6.7) - Homework Exercise #5



The histogram titled “A Study on MPG of Vehicles from an Assembly” displays the frequency of vehicle fuel efficiency grouped into MPG intervals. The x-axis is labeled “MPG” and includes the intervals 17–19, 20–22, 23–25, 26–28, and 29–31. The y-axis is labeled “Frequency” and ranges from 0 to 7. The interval 17–19 MPG has a frequency of 4. The interval 20–22 MPG has a frequency of 5. The interval 23–25 MPG has the highest frequency at 6. The interval 26–28 MPG has a frequency of 5, and the interval 29–31 MPG also has a frequency of 5. Overall, the histogram shows that most vehicles have fuel efficiency between 20 and 31 MPG, with the 23–25 MPG interval occurring most frequently.

(Section 6.7) - Homework Exercise #6



The histogram titled “Employee Commute Time” displays the frequency of employee commute times grouped into time intervals measured in minutes. The x-axis is labeled “Time (in minutes)” and the y-axis is labeled “Frequency,” ranging from 0 to 13. The interval 18–29 minutes has a frequency of 7 employees. The interval 30–41 minutes has the highest frequency with 12 employees. The interval 42–53 minutes has a frequency of 8. The interval 54–65 minutes has a frequency of 3. The interval 66–77 minutes has a frequency of 5. The interval 78–89 minutes has a frequency of 2. The interval 90–101 minutes has a frequency of 1. The interval 102–113 minutes has a frequency of 2. Overall, the histogram shows that shorter commute times are more common, especially between 30 and 41 minutes, while very long commute times occur less frequently.

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Section 6.8: Measures of Center

Learning Objectives

- Compute the mean of a data set
- Compute the median of a data set
- Compute the mode of a data set
- Compute the midrange of a data set
- Compare the four measures of central tendency

Measures of central tendency (sometimes referred to as Measures of Center) are statistical values that describe the center or typical value of a quantitative data set. They give us a single number that represents the "middle" or "most common" point of the data.

Measures of Central Tendency

The four measures of central tendency are called the mean, median, mode, and midrange.

Note: Most statistical textbooks indicate that there are only three measures of central tendency which are mean, median, and mode.

The first measure of central tendency, mean, is a type of average and is probably the measure of central tendency most people are familiar with. However, there are other measures of average, such as the median, mode, and midrange. The term “average” is more general and can refer to different central values depending on the context. In everyday language, “average” usually means the arithmetic mean (the same as the mean). In statistics, however, “average” could also refer to the mean, median, or mode, all of which are measures of central tendency.

For example, if someone says “the average household income,” they are referring to the mean, not a median or a mode. If they say “the average student score was the middle score,” they’re using median as the average. All means are averages, but not all averages are means. The mean is a precise, mathematical type of average.

Definition: Mean

The **mean** is a type of average. It's a way to describe the center or typical value of a set of numbers.

How to find the mean:

- **Step 1** — Add up all the values in the set.
- **Step 2** — Divide the total by how many values there are.

The mathematical formula for sample mean denoted by $\bar{x}$ is the following:

For a data set $x_1, x_2, x_3, \ldots, x_n$, the mean is calculated by the formula

$$\bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n},$$

where n is the sample size.

In statistics, the population mean μ is the average of all values in a population, while the sample mean $\bar{x}$ is the average of a sample and is used to estimate the population mean. For all intense purposes for this text, mean in general and sample mean are the same thing.

Example #6.8.1 🤔

Find the mean of the data set: 4, 8, 6, 2, 10

✅ **Solution:**

- **Step 1** — Add the numbers: 4 + 8 + 6 + 2 + 10 = 30
- **Step 2** — Divide by how many values there are which is 5: 30 ÷ 5 = 6

So, the mean is 6.

Using the mathematical formula for mean, we have

$$\bar{x} = \frac{\sum x_i}{5} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = \frac{4 + 8 + 6 + 2 + 10}{5} = \frac{30}{5} = 6$$

Example #6.8.2 🤔

The ages of the last 9 children seen by a local children's dentist was recorded:

4, 5, 7, 8, 3, 6, 2, 9, 2

Find the mean age of the children.

✅ **Solution:**

- **Step 1** — Add the numbers: 4 + 5 + 7 + 8 + 3 + 6 + 2 + 9 + 2 = 46
- **Step 2** — Divide by how many numbers there are which is 9: 46 ÷ 9≈5.1

Thus, the mean is 5.1 (or 5.11, if you want a more approximate answer).

Using the mathematical formula for mean, we have

$$\bar{x} = \frac{\sum x_i}{9} = \frac{x_1 + x_2 + x_3 + \cdots + x_9}{9} = \frac{4 + 5 + 7 + 8 + 3 + 6 + 2 + 9 + 2}{9} = \frac{46}{9} \approx 5.1$$

Example #6.8.3 🤔

Find the mean of the data set for a random sample of 10 GPA's from a particular English class:

4.0, 3.65, 2.15, 3.34, 2.87, 3.46, 2.75, 2.94, 2.66, 3.06

✅ **Solution:**

$$\bar{x} = \frac{\sum x_i}{10} = \frac{x_1 + x_2 + x_3 + \cdots + x_{10}}{10} = \frac{4.0 + 3.65 + 2.15 + 3.34 + 2.87 + 3.46 + 2.75 + 2.94 + 2.66 + 3.06}{10} = \frac{30.88}{10} = 3.088 \approx 3.09$$

So, the mean is 3.09.

Definition: Median

The **median** is a type of average. it represents the "middle" value in a data set when the numbers are arranged in order from smallest to largest.

How to find the median:

- **Step 1** — Arrange the data in ascending order
- **Step 2** — Determine if the number of values in the data set is odd or even.
- **Step 3** —
  - If the number of data points is odd, the median is the middle value. (Example: In the data set, {3, 5, 9}, the median is 5).
  - If the number of data points is even, the median is the mean of the two middle values. (Example: In the data set, {1, 3, 4, 8}, the median is $\frac{3+4}{2} = \frac{7}{2} = 3.5$.

The median is especially useful because it is less affected by extreme low and/or high values (outliers) than the mean (average). Let's say that the data set was actually {1, 3, 4, 88} instead of the above data set {1, 3, 4, 8}. The median would still be the same value of 3.5, but the mean would change from 4 to 24. In statistics, the median is considered **resistant**, if it is not strongly influenced by extreme values (outliers) or skewed data. A **resistant statistic** gives a reliable summary of the data even when there are unusually large or small values. A **non-resistant statistic** changes a lot when outliers are present.

Example #6.8.4 🤔

Find the median of the data set: 4, 8, 6, 2, 10

✅ **Solution:**

- **Step 1** — Arrange the data in ascending order: 2, 4, 6, 8, 10.
- **Step 2** — Determine if the number of values in the data set is odd or even. Since there are 5 values, the number of values in the set is odd.
- **Step 3** — An odd number of values means there is only one middle value which is the value of 6.

So, the median is 6.

Example #6.8.5 🤔

Ratings of a popular show exclusive to Paramount Plus are given in the following data set:

7, 10, 9, 9, 3, 5, 6, 9, 2, 5

Find the median of the data set above

✅ **Solution:**

- **Step 1** — Arrange the data in ascending order: 2, 3, 5, 5, 6, 7, 9, 9, 9, 10.
- **Step 2** — Determine if the number of values in the data set is odd or even. Since there are 10 values, the number of values in the set is even.
- **Step 3** — An even number of values means there are two middle values, which are 6 and 7. Now, find the mean of those two middle values 6 and 7 which is $\frac{6+7}{2} = \frac{13}{2} = 6.5$.

So, the median is 6.5.

Example #6.8.6 🤔

A random sample of 15 monthly rent costs from Costa Mesa, CA were taken from apartments.com:

$2,240 $3,290 $3,150 $3,075 $2,819
$2,925 $2,725 $2,100 $2,200 $3,210
$2,300 $3,199 $2,352 $3,982 $2,259

Find the median of the rent costs.

✅ **Solution:**

- **Step 1** — Arrange the data in ascending order: 2100, 2200, 2259, 2240, 2300, 2352, 2725, 2819, 2925, 3075, 3150, 3199, 3210, 3290, 3982.
- **Step 2** — Determine if the number of values in the data set is odd or even. Since there are 15 values, the number of values in the set is odd.
- **Step 3** — An odd number of values means there is only one middle value which is the value of 2819.

So, the median is $2,819.

Definition: Mode

The **mode** is a type of average. It represents the value that occurs most frequently in a data set.

- A data set can have:
  - No mode → if no value repeats.
  - One mode (**unimodal**) → if exactly one value occurs most often.
  - Two modes (**bimodal**) → if exactly two values tie for highest frequency.
  - More than two modes (**multimodal**) → if several values occur equally often.

Example #6.8.7 🤔

Find the mode of the following data set: 3, 7, 8, 4, 9, 6, 2, 8

✅ **Solution:**

- Arranging the data in ascending order, we have 2, 3, 4, 6, 7, 8, 8, 9.
- In this data set, there are two 8's and all other values have frequency one.

So, the mode is 8.

Example #6.8.8 🤔

Find the mode of the following data set: 5, 6, 8, 4, 8, 6, 5, 4

✅ **Solution:**

- Arranging the data in ascending order, we have 4, 4, 5, 5, 6, 6, 8, 8.
- In this data set, the values 4, 5, 6, and 8, all have frequency two and all other values have frequency one.

This data set is multimodal. The modes are 4, 5, 6, and 8.

Example #6.8.9 🤔

The following total days in each month of a non-leap year Gregorian calendar is the following:

31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31

Find the mode of days in the set.

✅ **Solution:**

- Arranging the data in ascending order, we have 28, 30, 30, 30, 30, 31, 31, 31, 31, 31, 31, 31.
- In this data set, the value 31 occurs (7) times, while 30 occurs (4) times, and 28 only once (1).

So, the mode is 31.

Example #6.8.10 🤔

he following data set is the total number of home runs each season by the Los Angeles Angels from the 2009 season to the 2024 season:

186, 155, 155, 145, 164, 172, 165, 156, 197, 175, 220, 85, 190, 190, 185, 183

Find the mode for the data set above.

✅ **Solution:**

- Arranging the data in ascending order, we have 85, 145, 155, 155, 156, 164, 165, 172, 175, 183, 185, 186, 190, 190, 197, 220.
- In this particular data set, there are two 155's and two 190's. All other values have frequency one.

This data set is bimodal. The modes are 155 and 190.

Example #6.8.11 🤔

According to science.nasa.gov, the radius of each of the 8 planets in our solar system is given below in the table:

The radius of each planet in our solar system

Mercury	Neptune	Earth	Mars	Jupiter	Saturn	Uranus	Venus
1,516	3,760	3,959	2,106	43,441	36,184	15,759	15,299

Find the mode for the data set above.

✅ **Solution:**

- You could go ahead and arrange the data in ascending order, however, a simple glance clearly shows that all of the values are different from one another. In order to qualify for a mode, you have to have at least one repetition of any value or values.

Thus, there is no mode.

Definition: Midrange

The **midrange** is a type of average. It is found by taking the mean of the highest and lowest values in a data set.

How to find the midrange:

- **Step 1** — Identify the lowest and the highest values in the data set. (Arrange the data in order, if necessary).
- **Step 2** — Add up the lowest and the highest values.
- **Step 3** — Divide the total by 2.

Or, equivalently, the formula

$$\text{Midrange} = \frac{\text{lowest value} + \text{highest value}}{2}$$

Weaknesses of Midrange

The midrange statistic has many weaknesses. Here is a few of them:

1. **Extremely sensitive to outliers** - The midrange depends only on the smallest and largest values. If either extreme is unusually high or low, the midrange can shift dramatically, even if most of the data are clustered elsewhere.
2. **Ignores most of the data** - Unlike the mean or median, the midrange does not use information from the majority of observations, making it a poor representation of where the data are actually centered.
3. **Often not a typical value** - The midrange may fall in a region where no data values exist, especially in skewed distributions.

Despite its limitations, the midrange is still taught and occasionally used for several reasons. The midrange is very easy to compute and understand, making it useful as an introductory concept when first discussing measures of center. It is also useful in controlled situations. When data are uniformly distributed and free of outliers (such as bounded physical measurements), the midrange can provide a quick approximation of the center. Ot can also emphasize the role of extremes. Studying the midrange helps students see how extreme values affect measures of center, reinforcing why the mean and median are often preferred.

Example #6.8.12 🤔

Find the midrange of the following data set: 5, 8, 19, 10, 7, 11

✅ **Solution:**

- **Step 1** — Identify the lowest and the highest values in the data set: Notice that the data is unordered. The lowest value is 5 and the highest value is 19 (not 11).
- **Step 2** — Add up the lowest and the highest values: 5 + 19 = 24
- **Step 3** — Divide the total by 2: 24 ÷ 2 = 12

So, the midrange is 12.

Or, using the formula for midrange, we have

$$\text{Midrange} = \frac{5 + 19}{2} = \frac{24}{2} = 12$$

Example #6.8.13 🤔

Find the midrange of the data set: 12, 4, 6, 4, 7, 11, 8, 10, 2, 17, 9

✅ **Solution:**

- **Step 1** — Identify the lowest and the highest values in the data set: Notice that the data is unordered. The lowest value is 2 and the highest value is 17.
- **Step 2** — Add up the lowest and the highest values: 2 + 17 = 19
- **Step 3** — Divide the total by 2: 19 ÷ 2 = 9.5

So, the midrange is 9.5.

Or, using the formula for midrange, we have

$$\text{Midrange} = \frac{2+17}{2} = \frac{19}{2} = 9.5$$

Example #6.8.14 🤔

Temperature highs over the last 20 days were recorded below:

88, 86, 92, 81, 85, 88, 87, 90, 82, 95, 92, 84, 93, 86, 91, 92, 91, 89, 87, 83

Find the midrange of the data set above.

✅ **Solution:**

- **Step 1** — Identify the lowest and the highest values in the data set: Notice that the data is unordered. The lowest value is 81 and the highest value is 95.
- **Step 2** — Add up the lowest and the highest values: 81 + 95 = 176
- **Step 3** — Divide the total by 2: 176 ÷ 2 = 88

So, the midrange is 88.

Or, using the formula for midrange, we have

$$\text{Midrange} = \frac{81+95}{2} = \frac{176}{2} = 88$$

Section 6.8: Measures of Center [*In-Class Exercises*]

1. The time, in minutes, it took 9 students to complete their first quiz in a History class was:

   5, 7, 5, 2, 9, 4, 9, 9, 10

Find the mean, median, mode, and midrange for the above data set of times.

2. Find the mean, median, mode, and midrange of the following data set: 6, 10, 5, 9, 7, 12, 5, 11, 8, 6.
3. Find the median and mode of the following data set: 5, 6, 7, 6, 7, 8, 8, 5.
4. Find the median and mode of the following data set: 13, 17, 19, 23, 24, 27, 11.
5. According to the California Department of Conservation, the table below shows the California's largest ever recorded earthquakes since 1800, ranked by date.

California's largest ever recorded earthquakes since 1800

Magnitude	Date	Location	Damage
7.9	Jan. 9, 1857	Fort Tejon	Two killed; 220-mile surface scar

Magnitude	Date	Location	Damage
7.4	Mar. 26, 1872	Owens Valley	27 killed; three aftershocks of magnitude >6
7.8	April 18, 1906	San Francisco	Possibly 3,000 killed; 225,000 displaced
7.2	Jan. 22, 1923	Mendocino*	Damaged homes in several towns
7.1	Nov. 4, 1927	SW of Lompoc*	No major injuries; slight damage in 2 counties
6.4	March 10, 1933	SE of Long Beach	115 killed; led to new building codes for schools
7.0	May 18, 1940	El Centro	9 killed; $6 million in damage
7.3	July 21, 1952	Kern County	12 killed; 3 magnitude >6 aftershocks in 5 days
6.6	Feb. 9, 1971	San Fernando	65 killed; 2,000 injured; $505 million in damage
7.4	Nov. 8, 1980	W. of Eureka*	Injured 6; $2 million in damage
6.9	Oct. 17, 1989	Bay Area	63 killed; 3,753 hurt; up to $10 billion in damage
7.2	April 25, 1992	Cape Mendocino	356 injuries; $48.3 million in damage
7.3	June 28, 1992	Landers	One killed; 400 injured; $9.1 million in damage
6.7	Jan. 17, 1994	Northridge	57 killed; 9,000 hurt, up to $40 billion in damage
7.1	Oct. 16, 1999	Ludlow	Minimal damage due to remote location
7.1	July 5, 2019	Ridgecrest/Trona	Preceded by M6.4 quake; no fatalities

Find the mean, median, mode, and midrange of the magnitude ratings.

Answers available at the end of this section.

Answers

Section 6.8: Measures of Center [*In-Class Exercises*]

1. Mean = 6.7 , Median = 7, Mode = 9, Midrange = 6.
2. Mean = 7.9 , Median = 7.5, Mode = 5 & 6, Midrange = 8.5.
3. Median = 6.5, Mode = 5, 6, 7 & 8.
4. Median = 19, Mode = None.
5. Mean = 7.15 , Median = 7.15, Mode = 7., Midrange = 7.15.

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Section 6.8.H: Homework

Section 6.8: Measures of Center - Homework Exercises

(Section 6.8) - Homework Exercise #1

The wait times (in minutes) for customers of a local family restaurant was recorded:

5, 10, 12, 15, 20, 25, 10, 15, 10, 12, 18, 20, 30, 45

Find the mean, median, mode, and midrange of wait times.

Answer available at the end of this section.

(Section 6.8) - Homework Exercise #2

A random sample of 20 household sizes were obtained:

1, 2, 2, 3, 3, 4, 4, 2, 3, 5, 4, 3, 2, 1, 6, 2, 3, 4, 2, 3

Find the mean, median, mode, and midrange of household sizes.

Answer available at the end of this section.

(Section 6.8) - Homework Exercise #3

The number of text messages sent per day by junior high school students was recorded:

15, 20, 25, 30, 35, 40, 25, 20, 30, 45, 50, 60, 25, 30, 35, 100

Find the mean, median, mode, and midrange of text messages sent.

Answer available at the end of this section.

(Section 6.8) - Homework Exercise #4

The number of steps on a particular day was recorded at a health fitness center.

11,542, 6,215, 5,379, 4,387, 12,222, 9,153, 6,264, 10,501, 7,458, 9,176, 8,327

Find the mean, median, mode, and midrange of text messages sent.

Answer available at the end of this section.

(Section 6.8) - Homework Exercise #5

Annual salaries (in thousands of dollars) for 28 employees:

38, 52, 68, 45, 59, 75, 42, 56, 72, 48, 62, 78, 40, 54, 70, 46, 60, 76, 44, 58, 74, 50, 54, 80, 36, 53, 69, 47

Find the mean, median, mode, and midrange of annual salaries.

Answer available at the end of this section.

(Section 6.8) - Homework Exercise #6

Reaction times (in milliseconds) for 23 participants in a study:

275, 248, 215, 232, 259, 288, 225, 252, 280, 238, 265, 295, 220, 247, 272, 235, 262, 228, 255, 282, 240, 298, 268

Find the mean, median, mode, and midrange of reaction times.

Answer available at the end of this section.

(Section 6.8) - Homework Exercise #7

Monthly electric bills (in dollars) for 26 households:

85, 112, 95, 128, 102, 128, 89, 135, 108, 96, 122, 91, 115, 105, 130, 98, 125, 110, 93, 120, 100, 132, 107, 116, 94, 127

Find the mean, median, mode, and midrange of monthly electric bills.

Answer available at the end of this section.

(Section 6.8) - Homework Exercise #8

The table below lists Winter Olympic Games by year and location along with the number of gold medals won by the United States at each event.

United States gold medals at the Winter Olympics from 1924 through 2026

Year	Location	No. of Gold Medals
1924	Chamonix, France	1
1928	St. Moritz, Switzerland	2
1932	Lake Placid, New York, USA	6
1936	Garmisch-Partenkirchen, Germany	1
1948	St. Moritz, Switzerland	3
1952	Oslo, Norway	4
1956	Cortina d'Ampezzo, Italy	2

Year	Location	No. of Gold Medals
1960	Squaw Valley, California, USA	3
1964	Innsbruck, Austria	1
1968	Grenoble, France	1
1972	Sapporo, Jappan	3
1976	Innsbruck, Austria	3
1980	Lake Placid, New York USA	6
1984	Sarajevo, Yugoslavia	4
1988	Calgary, Alberta, Canada	2
1992	Albertville, France	5
1994	Lillehammer, Norway	6
1998	Nagano, Japan	6
2002	Salt Lake City, Utah, USA	10
2006	Turin, Italy	9
2010	Vancouver, British Columbia, Canada	9
2014	Sochi, Russia	9
2018	Pyeongchang, South Korea	9
2022	Beijing, China	9
2026	Milan-Cortina, Italy	12

Find the mean, median, mode, and midrange of the number of gold medals won.

Answer available at the end of this section.

Answers

(Section 6.8) - Homework Exercise #1

Mean ≈ 17.6 , Median = 15 , Mode = 10 , Midrange = 25

(Section 6.8) - Homework Exercise #2

Mean = 2.95 , Median = 3 , Mode = 2 & 3 , Midrange = 3.5

(Section 6.8) - Homework Exercise #3

Mean ≈ 36.6 , Median = 30 , Mode = 25 & 30 , Midrange = 57.5

(Section 6.8) - Homework Exercise #4

Mean ≈ 8,238.5 , Median = 8,327 , Mode = None , Midrange = 8,304.5

(Section 6.8) - Homework Exercise #5

Mean ≈ 57.36 , Median = 55 , Mode = 54 , Midrange = 58

(Section 6.8) - Homework Exercise #6

Mean ≈ 255.6 , Median = 255 , Mode = None , Midrange = 256.5

(Section 6.8) - Homework Exercise #7

Mean ≈ 110.1 , Median = 109 , Mode = 128 , Midrange = 110

(Section 6.8) - Homework Exercise #8

Mean ≈ 3.34 , Median = 4 , Mode = 9 , Midrange = 6.5

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Section 6.9: Measures of Variation

Learning Objectives

- Compute the range of a data set
- Compute the variance of a data set
- Compute the standard deviation of a data set
- Interpret standard deviation

In the previous section, when we collect data, whether it's test scores, daily temperatures, or the ages of people in a group, we often start by looking at measures of center, like the mean or median. These tell us what a "typical" value looks like. But center alone never tells the whole story. Imagine two classes that both have an average test score of 80:

- In one class, almost everyone scored between 78 and 82.
- In the other, half the class scored above 95 and the other half below 65.

Same average (mean and median are both 80), totally different situations. This is where measures of variation become essential.

Measures of Variation

The main three measures of variation are called the range, variance, and standard deviation.

Note: Some statistical textbooks may include interquartile range also as a measure of variation.

What Are Measures of Variation?

Measures of variation (also called **measures of spread** or **dispersion**) describe how spread out or clustered data values are. They help answer questions like:

- Are most values close to the center?
- Are the values widely scattered?
- Are there extreme values far from the rest?

Understanding variation helps us interpret data more accurately and avoid misleading conclusions.

Why Variation Matters

Even if two datasets share the same mean or median, their variation can differ dramatically. Variation tells us:

- Consistency (e.g., are students' scores steady or all over the place?)
- Risk (e.g., does investment return vary a lot or stay predictable?)
- Reliability (e.g., is a machine producing parts with uniform size?)

In everyday life, low variation often means *predictability*, and high variation may mean *uncertainty* or *diversity*.

Definition: Range

The **range** is the simplest measure of variation. It is calculated by the formula:

$$\text{Range} = \text{Maximum value} - \text{Minimum value}$$

It shows the total span of the data but is very sensitive to outliers.

Note: Range was discussed earlier in Section 6.?. Outliers will be discussed further in Section 6.11.

Example #6.9.1 🤔

Find the range of the data set: 3, 3, 5, 7, 8, 8, 11, 12, 17

✅ **Solution:**

Make sure the data set is in ascending order, which it is. Now, use the formula:

$$\text{Range} = \text{Maximum value} - \text{Minimum value}$$

$$\text{Range} = 17 - 3$$

$$\text{Range} = 14$$

Thus, the range is 14.

Example #6.9.2 🤔

Find the range of the data set: 6, 8, 9, 17, 12, 15, 11

✅ **Solution:**

Make sure the data set is in ascending order, which it is not. Placing the data in ascending order, we have 6, 8, 9, 11, 12, 15, 17. Now, use the formula:

$$\text{Range} = \text{Maximum value} - \text{Minimum value}$$

$$\text{Range} = 17 - 6$$

$$\text{Range} = 11$$

Thus, the range is 11.

Definition: Standard Deviation

The **standard deviation** of a data set is a measure of how spread out the values are. It tells you, on average, how far the data values tend to be from the mean. A small standard deviation means the values are close to the mean (low variation). A large standard deviation means the values are more spread out (high variation).

How to find a standard deviation of a sample set:

- **Step 1** — List the data. (Ascending order is not necessary).
- **Step 2** — Find the mean.

- **Step 3** — Subtract the mean from each value in the data set.
- **Step 4** — Square each difference.
- **Step 5** — Add up all of the squared differences.
- **Step 6** — Divide by $n-1$. (Note: This value is called the variance).
- **Step 7** — Take the square root.

The mathematical formula for the steps above is the following:

For a sample data set, the sample standard deviation, s, is defined as:

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

where: x_i = each individual value, $\bar{x}$ = sample mean, and n is the sample size.

A great example to understand standard deviation and the role it takes on is looking at gas prices at different gas stations. When comparing gas prices across different gas stations in a city or region, it's common to find that each station has its own average price (mean). Some stations consistently charge a little less, while others consistently charge a little more. However, even though these mean prices differ, the standard deviations for the stations are often quite similar. This happens because the patterns of pricing tend to move together. A gas station that is usually one of the cheapest will remain one of the cheapest even when prices fluctuate. Likewise, a station that is typically more expensive will continue to be among the higher-priced stations. In other words, the whole market tends to shift up or down at the same time due to factors like wholesale prices, taxes, and supply conditions. As a result, while the overall level of prices (the mean) may differ from station to station, the amount of variation they experience (the standard deviation) tends to be similar. Prices at each station rise and fall by roughly comparable amounts over time, even if they start from different "baseline" levels.
So although one gas station may average 20 cents cheaper than another, both stations often fluctuate by roughly the same amount day to day or week to week. This shared pattern of movement explains why their standard deviations are similar even when their means are not.

Although standard deviation will become especially important later when we study the normal distribution, our focus in this section is simply on how to calculate it. For now, we treat standard deviation as a numerical summary that tells us how spread out the values in a data set are around the mean. It gives us a single number that reflects the typical distance between each data point and the average. By learning to compute standard deviation now, we build the foundation needed for understanding more advanced ideas, such as how the normal distribution uses standard deviation to describe predictable patterns in real-world data.

At this stage, however, our goal is only to become comfortable with the calculation process and to recognize standard deviation as a basic measure of variation.

Example #6.9.3 🤔

Suppose a professor records the number of hours ten students studied for an exam: 3, 4, 5, 6, 6, 6, 7, 11. Round your answer to the nearest hundredth.

✅ **Solution:**

- **Step 1 — List the data. (Ascending order is not necessary):** 3, 4, 5, 6, 6, 6, 7, 11
- **Step 2 — Find the mean:** $\bar{x} = \frac{\sum x_i}{8} = \frac{x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8}{8} = \frac{3+4+5+6+6+6+7+11}{8} = \frac{48}{8} = 6$
- **Step 3 — Subtract the mean from each value in the data set:** Here, it is easier to do this in a table.
- **Step 4 — Square each difference:** Here, this is also easier to do this in a table.

Calculating squared differences from the mean for a data set

Step 1: Data	Step 3: Subtract the mean	Step 4: Square the difference
3	3 – 6 = –3	$(-3)^2 = 9$
4	4 – 6 = –2	$(-2)^2 = 4$
5	5 – 6 = –1	$(-1)^2 = 1$
6	6 – 6 = 0	$(0)^2 = 0$
6	6 – 6 = 0	$(0)^2 = 0$
6	6 – 6 = 0	$(0)^2 = 0$
7	7 – 6 = 1	$(1)^2 = 1$
11	11 – 6 = 5	$(5)^2 = 25$

- **Step 5 — Add up all of the squared differences:** 9 + 4 + 1 + 0 + 0 + 0 + 1 + 25 = 40
- **Step 6 — Divide by *n* – 1. (Note: This result here is called the variance):** $\frac{40}{8-1} = \frac{40}{7} = 5.7142857142857$
- **Step 7 — Take the square root:** $\sqrt{5.7142857142857} = 2.3904572186688 \approx 2.39$

So, the standard deviation is 2.39.

✅ **Alternative Solution:**

Let's use the mathematical formula for standard deviation of a sample, $s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$. If we focus only on the numerator, $(x_i - \bar{x})^2$, we can create the same three column table from above, yet labeling the headers which are the actual steps 1, 3, and 4 in the table. Those steps mathematically are x_i, $(x_i - \bar{x})$, and $(x_i - \bar{x})^2$, respectively.

Calculating squared differences from the mean for a data set

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
3	3 – 6 = –3	$(-3)^2 = 9$
4	4 – 6 = –2	$(-2)^2 = 4$
5	5 – 6 = –1	$(-1)^2 = 1$
6	6 – 6 = 0	$(0)^2 = 0$
6	6 – 6 = 0	$(0)^2 = 0$
6	6 – 6 = 0	$(0)^2 = 0$
7	7 – 6 = 1	$(1)^2 = 1$
11	11 – 6 = 5	$(5)^2 = 25$

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
		Sum = 40

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

$$s = \sqrt{\frac{40}{8-1}}$$

$$s = \sqrt{\frac{40}{7}}$$

$$s = \sqrt{5.7142857142857}$$

$$s = 2.3904572186688$$

$$s \approx 2.39$$

So, the standard deviation is 2.39.

Example #6.9.4 🤔

Find the standard deviation of the following data set: 5, 7, 10, 5, 9, 9, 8, 2, 7, 8. Find the standard deviation of the data set. Round your answer to the nearest hundredth.

✅ **Solution:**

- **Step 1 — List the data. (Ascending order is not necessary):** 5, 7, 10, 5, 9, 9, 8, 2, 7, 8
- **Step 2 — Find the mean:** $\bar{x} = \frac{\sum x_i}{10} = \frac{x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}}{8} = \frac{5+7+10+5+9+8+2+7+8}{10} = \frac{70}{10} = 7$
- **Step 3 — Subtract the mean from each value in the data set:** Here, it is easier to do this in a table.
- **Step 4 — Square each difference:** Here, this is also easier to do this in a table.

Calculating squared differences from the mean for a data set

Step 1: Data	Step 3: Subtract the mean	Step 4: Square the difference
5	5 – 7 = –2	$(-2)^2 = 4$
7	7 – 7 = 0	$(0)^2 = 0$
10	10 – 7 = 3	$(3)^2 = 9$
5	5 – 7 = –2	$(-2)^2 = 4$
9	9 – 7 = 2	$(2)^2 = 4$
9	9 – 7 = 2	$(2)^2 = 4$
8	8 – 7 = 1	$(1)^2 = 1$
2	2 – 7 = –5	$(-5)^2 = 25$

Step 1: Data	Step 3: Subtract the mean	Step 4: Square the difference
7	7 – 7 = 0	$(0)^2 = 0$
8	8 – 7 = 1	$(1)^2 = 1$

- **Step 5 — Add up all of the squared differences:** 4 + 0 + 9 + 4 + 4 + 4 + 1 + 25 + 0 + 1 = 52
- **Step 6 — Divide by *n* – 1. (Note: The result here is called the variance):** $\frac{52}{10-1} = \frac{52}{9} = 5.777777777777778$
- **Step 7 — Take the square root:** $\sqrt{5.77777777} = 2.403700850309326 \approx 2.40$

So, the standard deviation is 2.40.

✅ **Alternative Solution:**

Let's use the mathematical formula for standard deviation of a sample, $s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$. If we focus only on the numerator, $(x_i - \bar{x})^2$, we can create the same three column table from above, yet labeling the headers which are the actual steps 1, 3, and 4 in the table. Those steps mathematically are x_i, $(x_i - \bar{x})$, and $(x_i - \bar{x})^2$, respectively.

Calculating squared differences from the mean for a data set

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
5	5 – 7 = –2	$(-2)^2 = 4$
7	7 – 7 = 0	$(0)^2 = 0$
10	10 – 7 = 3	$(3)^2 = 9$
5	5 – 7 = –2	$(-2)^2 = 4$
9	9 – 7 = 2	$(2)^2 = 4$
9	9 – 7 = 2	$(2)^2 = 4$
8	8 – 7 = 1	$(1)^2 = 1$
2	2 – 7 = –5	$(-5)^2 = 25$
7	7 – 7 = 0	$(0)^2 = 0$
8	8 – 7 = 1	$(1)^2 = 1$
		Sum = 52

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

$$s = \sqrt{\frac{70}{10-1}}$$

$$s = \sqrt{\frac{70}{9}}$$

$$s = \sqrt{5.777777777777778}$$

$$s = 2.403700850309326$$

$$s \approx 2.39$$

So, the standard deviation is 2.40.

Example #6.9.5 🤔

Car battery warranties typically offer 1–4 years of free replacement, often followed by a pro-rated period up to 84 months (7 years), with coverage based on purchase date and type. One brand of battery logged the last 15 batteries that came back defective under their pro-rated 84-month warranty. Here is the data on the time for each battery lasted:

51, 37, 60, 62, 55, 71, 42, 65, 68, 74, 40, 64, 68, 59, 22

Find the standard deviation of the data set. Round your answer to the nearest hundredth.

✅ **Solution:**

- **Step 1 — List the data. (Ascending order is not necessary):** 51, 37, 60, 62, 55, 71, 42, 65, 68, 74, 40, 64, 68, 59, 22
- **Step 2 — Find the mean:** $\bar{x} = \frac{\sum x_i}{15} = \frac{x_1+x_2+x_3+\cdots+x_{15}}{15} = \frac{51+37+60+62+55+71+42+65+68+74+40+64+68+59+22}{15} = \frac{838}{15} = 55.86666666666667 \approx 55.87$
- **Step 3 — Subtract the mean from each value in the data set:** Here, it is easier to do this in a table.
- **Step 4 — Square each difference:** Here, this is also easier to do this in a table.

Calculating squared differences from the mean for a data set

Step 1: Data	Step 3: Subtract the mean	Step 4: Square the difference
51	51 – 55.87 = –4.87	$(-4.87)^2$ = 23.7169
37	37 – 55.87 = –18.87	$(-18.87)^2$ = 356.0769
60	60 – 55.87 = –4.13	$(-4.13)^2$ = 17.0569
62	62 – 55.87 = 6.13	$(6.13)^2$ = 37.5769
55	55 – 55.87 = –0.87	$(-0.87)^2$ = 0.7569
71	71 – 55.87 = 15.13	$(15.13)^2$ = 228.9169
42	42 – 55.87 = –13.87	$(-13.87)^2$ = 192.3769
65	65 – 55.87 = 9.13	$(9.13)^2$ = 83.3569
68	68 – 55.87 = 12.13	$(12.13)^2$ = 147.1369
74	74 – 55.87 = 18.13	$(18.13)^2$ = 328.6969
40	40 – 55.87 = –15.87	$(-15.87)^2$ = 251.8569

Step 1: Data	Step 3: Subtract the mean	Step 4: Square the difference
64	64 – 55.87 = 8.13	$(8.13)^2$ = 66.0969
68	68 – 55.87 = 12.13	$(12.13)^2$ = 147.1369
59	59 – 55.87 = 3.13	$(3.13)^2$ = 9.7969
22	22 – 55.87 = –33.87	$(-33.87)^2$ = 1,147.1769

- **Step 5 — Add up all of the squared differences:** The sum of the last column is 3,037.7335
- **Step 6 — Divide by *n* – 1. (Note: This result here is called the variance):** $\frac{3,037.7335}{15-1} = \frac{3,037.7335}{14} = 216.98096428571$
- **Step 7 — Take the square root:** $\sqrt{216.98096428571} = 14.730273734242 \approx 14.73$

So, the standard deviation is 14.73.

✅ **Alternative Solution:**

Let's use the mathematical formula for standard deviation of a sample, $s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$. If we focus only on the numerator, $(x_i - \bar{x})^2$, we can create the same three column table from above, yet labeling the headers which are the actual steps 1, 3, and 4 in the table. Those steps mathematically are x_i, $(x_i - \bar{x})$, and $(x_i - \bar{x})^2$, respectively.

Calculating squared differences from the mean for a data set

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
51	51 – 55.87 = –4.87	$(-4.87)^2$ = 23.7169
37	37 – 55.87 = –18.87	$(-18.87)^2$ = 356.0769
60	60 – 55.87 = –4.13	$(-4.13)^2$ = 17.0569
62	62 – 55.87 = 6.13	$(6.13)^2$ = 37.5769
55	55 – 55.87 = –0.87	$(-0.87)^2$ = 0.7569
71	71 – 55.87 = 15.13	$(15.13)^2$ = 228.9169
42	42 – 55.87 = –13.87	$(-13.87)^2$ = 192.3769
65	65 – 55.87 = 9.13	$(9.13)^2$ = 83.3569
68	68 – 55.87 = 12.13	$(12.13)^2$ = 147.1369
74	74 – 55.87 = 18.13	$(18.13)^2$ = 328.6969
40	40 – 55.87 = –15.87	$(-15.87)^2$ = 251.8569
64	64 – 55.87 = 8.13	$(8.13)^2$ = 66.0969

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
68	68 – 55.87 = 12.13	$(12.13)^2$ = 147.1369
59	59 – 55.87 = 3.13	$(3.13)^2$ = 9.7969
22	22 – 55.87 = –33.87	$(-33.87)^2$ = 1,147.1769
		Sum = 3,037.7335

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

$$s = \sqrt{\frac{3,037.7335}{15-1}}$$

$$s = \sqrt{\frac{3,037.7335}{14}}$$

$$s = \sqrt{216.98096428571}$$

$$s = 14.730273734242$$

$$s \approx 14.73$$

So, the standard deviation is 14.73.

Definition: Variance

Variance of a sample data set, like standard deviation, measures how far values tend to be from the mean on average. However, variance measures the average squared distance from the mean. Because the variance uses squared distances, this makes it sensitive to large deviations. The sample variance is simply the square of the sample standard deviation, or

$$s^2 = \frac{\sum (x - \bar{x})^2}{n-1}$$

Note: This is the same formula for sample standard deviation without the square root.

Example #6.9.6 🤔

Refer to Example #6.9.3. Suppose a professor records the number of hours ten students studied for an exam: 3, 4, 5, 6, 6, 6, 7, 11. Find the variances of the data set. Round your answer to the nearest hundredth.

✅ **Solution:**

So, to find the variance of a data set, just perform the first six steps for standard deviation. Do not perform Step 7. Do not take the square root. When you are finished with Step 6, that value is the variance.

- **Step 1 — List the data. (Ascending order is not necessary):** 3, 4, 5, 6, 6, 6, 7, 11
- **Step 2 — Find the mean:** $\bar{x} = \frac{\sum x_i}{8} = \frac{x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8}{8} = \frac{3+4+5+6+6+6+7+11}{8} = \frac{48}{8} = 6$
- **Step 3 — Subtract the mean from each value in the data set:** Here, it is easier to do this in a table.
- **Step 4 — Square each difference:** Here, this is also easier to do this in a table.

Calculating squared differences from the mean for a data set

Step 1: Data	Step 3: Subtract the mean	Step 4: Square the difference
3	3 – 6 = –3	$(-3)^2 = 9$
4	4 – 6 = –2	$(-2)^2 = 4$
5	5 – 6 = –1	$(-1)^2 = 1$
6	6 – 6 = 0	$(0)^2 = 0$
6	6 – 6 = 0	$(0)^2 = 0$
6	6 – 6 = 0	$(0)^2 = 0$
7	7 – 6 = 1	$(1)^2 = 1$
11	11 – 6 = 5	$(5)^2 = 25$

- **Step 5 — Add up all of the squared differences:** 9 + 4 + 1 + 0 + 0 + 0 + 1 + 25 = 40
- **Step 6 — Divide by *n* – 1.** $\frac{40}{8-1} = \frac{40}{7} = 5.7142857142857 \approx 5.71$

So, the variance is 5.71.

✅ **Alternative Solution:**

Let's use the mathematical formula for standard deviation of a sample, $s^2 = \frac{\sum(x_i-\bar{x})^2}{n-1}$. If we focus only on the numerator, $(x_i - \bar{x})^2$, we can create the same three column table from above, yet labeling the headers which are the actual steps 1, 3, and 4 in the table. Those steps mathematically are x_i, $(x_i - \bar{x})$, and $(x_i - \bar{x})^2$, respectively.

Calculating squared differences from the mean for a data set

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
3	3 – 6 = –3	$(-3)^2 = 9$
4	4 – 6 = –2	$(-2)^2 = 4$
5	5 – 6 = –1	$(-1)^2 = 1$
6	6 – 6 = 0	$(0)^2 = 0$
6	6 – 6 = 0	$(0)^2 = 0$
6	6 – 6 = 0	$(0)^2 = 0$
7	7 – 6 = 1	$(1)^2 = 1$
11	11 – 6 = 5	$(5)^2 = 25$

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
		Sum = 40

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$$

$$s^2 = \frac{40}{8-1}$$

$$s^2 = \frac{40}{7}$$

$$s^2 = 5.7142857142857$$

$$s^2 \approx 5.71$$

So, the variance is 5.71.

Example #6.9.7 🤔

If the variance of a sample data set is known to be 100, then what is the standard deviation?

✅ **Solution:**

Since the variance is given, the standard deviation is the square root of the variance. So, $\sqrt{100} = 10$.

The standard deviation is 10.

Example #6.9.8 🤔

If the standard deviation of a sample data set is known to be 8, then what is the variance?

✅ **Solution:**

Since the standard deviation is given, the variance is the square of the standard deviation. So, $8^2 = 64$.

The variance is 8.

Example #6.9.9 🤔

If the variance of a sample data set is known to be 7, then what is the standard deviation?

✅ **Solution:**

Since the variance is given, the standard deviation is the square root of the variance. So, $\sqrt{7} = 2.6457513110646 \approx 2.65$.

The standard deviation is 2.65.

Example #6.9.10 🤔

If the standard deviation of a sample data set is known to be 1.2, then what is the variance?

✅ **Solution:**

Since the standard deviation is given, the variance is the square of the standard deviation. So, $1.2^2 = 1.44$.

The variance is 1.44.

Section 6.9: Measures of Variation [*In-Class Exercises*]

1. Find the range of the data set: 2, 5, 5, 5, 6, 8, 12, 15
2. Find the range of the data set: 5, 4, 3, 11, 12, 13
3. Find the standard deviation and variance of the following sample data set: 7, 6, 8, 5, 6, 7, 8, 9
4. Ten students flip a coin 100 times and record the number of times it lands on heads. The results for the number of heads is 44, 49, 52, 62, 53, 48, 54, 49, 46, 51. Find the standard deviation and variance of the data set. Round your answer to the nearest hundredth.
5. If the variance of a sample data set is known to be 4, then what is the standard deviation?
6. If the standard deviation of a sample data set is known to be 9, then what is the variance?

Answers available at the end of this section.

Answers

Section 6.9: Measures of Variation [*In-Class Exercises*]

1. Range = 13
2. Range = 10
3. Variance = $s^2 \approx 1.31$; Standard deviation = $s \approx 1.71$
4. Variance = $s^2 \approx 25.07$; Standard deviation = $s \approx 5.01$
5. Standard deviation = 2
6. Variance = 81

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Section 6.9.H: Homework

Section 6.9: Measures of Variance - Homework Exercises

(Section 6.9) - Homework Exercise #1

Find the range of the data set: 4, 8, 10, 12, 12, 15, 18, 19, 21, 24

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #2

Find the range of the data set:

38, 52, 68, 45, 59, 75, 42, 56, 72, 48, 62, 78, 40, 54, 70, 46, 60, 76, 44, 58, 74, 50, 54, 80, 36, 53, 69, 47

Answer available at the end of this section.

(Section 6.3) - Homework Exercise #3

Monthly electric bills (in dollars) for 26 households:

85, 112, 95, 128, 102, 128, 89, 135, 108, 96, 122, 91, 115, 105, 130, 98, 125, 110, 93, 120, 100, 132, 107, 116, 94, 127

Find the range of the data set.

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #4

The frequency distribution below shows the runs scored in each of their 17 games in the post season for the 2025 World Series Champion Los Angeles Dodgers:

Frequency distribution of runs scored in 17 games

No. of Runs	Frequency
1	1
2	4
3	2
4	2
5	5
6	1
7	0
8	1
9	0
10	1

Find the range of the runs scored.

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #5

Find the standard deviation of the following data set: 4, 4, 6, 9, 9, 11, 13. Find the standard deviation of the data set. Round your answer to the nearest hundredth.

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #6

Find the standard deviation of the following data set: 7, 11, 5, 8, 4, 17, 18, 10, 9, 8, 13. Find the standard deviation of the data set. Round your answer to the nearest hundredth.

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #7

The NOAA, National Oceanic and Atmospheric Administration, provides data, tools, and information to help people understand, prepare for and adapt to our changing climate patterns. According to the NOAA, a sample week of data in January was 59, 61, 63, 64, 62, 58, 60.

Find the standard deviation of the data set and round your answer to the nearest tenth.

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #8

The cholesterol levels (mg/dL) in a random health study was found to be:

165, 172, 181, 190, 195, 205, 210, 178, 186, 199

Find the standard deviation of the data set and round your answer to the nearest tenth.

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #9

A random sample of monthly rent prices (in dollars) for 1-bedroom apartments in a Southern California region according to Apartments.com.

$1895, $1970, $2100, $2050, $1925, $2250, $2180, $2065, $1990, $2145

Find the standard deviation of the data set and round your answer to the nearest tenth.

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #10

Two classes took the same math quiz. Their scores are shown below.

Class A scores: 68, 72, 75, 81, 84, 86, 88, 90, 92
Class B scores: 70, 74, 76, 79, 83, 85, 87, 89, 93

Find the standard deviation of both data sets. Round your answer to the nearest hundredth.

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #11

If the variance of a sample data set is known to be 121, then what is the standard deviation?

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #12

If the variance of a sample data set is known to be 30, then what is the standard deviation?

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #13

If the standard deviation of a sample data set is known to be 9, then what is the variance?

Answer available at the end of this section.

(Section 6.9) - Homework Exercise #14

If the standard deviation of a sample data set is known to be 57, then what is the variance?

Answer available at the end of this section.

Answers

(Section 6.9) - Homework Exercise #1
Range = 24 – 4 = 20

(Section 6.9) - Homework Exercise #2
Range = 80 – 36 = 44

(Section 6.3) - Homework Exercise #3
Range = 135 – 85 = 50

(Section 6.9) - Homework Exercise #4
Range = 10 – 1 = 9

(Section 6.9) - Homework Exercise #5
$s \approx 3.46$

(Section 6.9) - Homework Exercise #6
$s \approx 4.49$

(Section 6.9) - Homework Exercise #7
$s \approx 2.2^{\circ}$

(Section 6.9) - Homework Exercise #8
$s \approx 15.5$ mg/dL

(Section 6.9) - Homework Exercise #9
$s \approx \$155.5$

(Section 6.9) - Homework Exercise #10
For Class A, $s \approx 8.41$. For Class B, $s \approx 7.56$

(Section 6.9) - Homework Exercise #11
$s = 11$

(Section 6.9) - Homework Exercise #12
$s \approx 5.47$

(Section 6.9) - Homework Exercise #13
$s^2 = 81$

(Section 6.9) - Homework Exercise #14
$s^2 = 3{,}249$

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Section 6.10: Percentiles

Learning Objectives

- Find a data value corresponding to a given percentile
- Compute the percentile rank for a data value
- Use percentile rank to compare values from different data sets

What are Measures of Position?

Measures of position are statistical values that tell you where a specific data point stands relative to the rest of the dataset. Unlike measures of center (mean, median, mode) that describe the "typical" value, measures of position describe the location or rank of values within a distribution.

Measures of position transform absolute values into relative comparisons, making it easier to:

- ✓ Understand where you stand
- ✓ Compare across different scales
- ✓ Identify outliers and unusual values
- ✓ Make informed decisions based on population distributions

In this section along with the next section, we will focus primarily on percentiles and quartiles, as they are the most intuitive and widely used measures of position in statistics.

Imagine you just took the SAT and received a score of 1360 out of 1600. The score of 1360 is called a raw score. The raw percentage is 85% (1360/1600). Your first questions might be: "Is this a good score?" "How do I compare to other students?" "Did I do better than most people, or worse?" It doesn't immediately answer these questions. We need context.

Now suppose you learn that your score is at the 90th percentile. What does this tell you? The 90th percentile means that you scored better than 90% of all test-takers. It also means that only 10% of students scored higher than you. Suddenly, your score has meaning. The percentile gives you a clear picture of where you stand relative to everyone else. Percentiles transform raw data into meaningful comparisons.

Definition: Percentiles

A **percentile** is a value that indicates the percentage of data that falls at or below that value.

If your score is in the Pth percentile, you scored better than (or equal to) $P\%$ of the entire group.

The Pth percentile is a value such that at least $P\%$ of the data values are less than or equal to that value, and at least $(100 - P)\%$ are greater than or equal to that value.

Key components:

- P is a number between 0 and 100
- The percentile is a position in the ordered dataset
- It divides the data into two groups: those below and those above

How To Find Position Given a Percentile

To find the position of a data set of sample size n given a percentile:

- **Step 1** — Arrange the data in ascending order. (This isn't strictly necessary, but it makes the process easier and ensures your final plot is organized).
- **Step 2** — Calculate the position by using the following formula:

$$\text{Position} = \frac{\text{Percentile}}{100} \cdot n$$

- **Step 3** — The result should be rounded up to the nearest whole number.
- **Step 4** — Locate the value.

Note: The reason we round up is because at least $P\%$ of the data is at or below that value. Think $\leq$.

How To Find Percentiles Given Position

To find the percentile of a data set of sample size n given a position:

- **Step 1** — Arrange the data in ascending order. (This isn't strictly necessary, but it makes the process easier and ensures your final plot is organized).
- **Step 2** — Calculate the percentile by using the following formula:

$$\text{Percent of a value } x = \frac{\text{Number of data values less than } x}{n} \cdot 100$$

- **Step 3** — The result should be rounded to the nearest whole number.

Note: There are multiple valid methods for calculating percentiles, and different textbooks, calculators, and software packages may use different formulas or rules (such as rounding, interpolation, or specific position formulas). As a result, it is possible to obtain slightly different percentile values for the same data set depending on the method used. For this course, we are following the methods demonstrated above.

Percentiles are essential for:

- Relative comparison: Understanding where a value stands compared to others
- Standardized interpretation: Comparing across different scales or distributions
- Outlier identification: Spotting unusual values
- Fair assessment: Evaluating performance considering the entire distribution
- Decision-making: Setting thresholds based on population characteristics

Percentiles are used in real-world applications, such as:

- Education: SAT/ACT scores, GPA rankings, class standings
- Medicine: Growth charts (height, weight for children), blood pressure ranges
- Finance: Income brackets, wealth distribution
- Business: Salary negotiations, performance reviews
- Quality control: Manufacturing tolerances, product specifications

Example #6.10.1 🤔

The weights (in pounds) of 28 newborn babies are given below:

5.6, 5.7, 5.9, 6.0, 6.2, 6.3, 6.5, 6.7, 6.7, 6.9, 6.9, 6.9, 7.2, 7.2, 7.3, 7.4, 7.4, 7.5, 7.6, 7.7, 7.7, 7.7, 7.8, 8.0, 8.1, 8.3, 8.4, 8.7

What score is at the 80th percentile?

✅ **Solution:**

Step 1: Arrange the data in ascending order.

- Data was given in ascending order

Step 2: Calculate the position.

- Position = (Percentile/100) × n
- Position = (80/100) × 28 = 22.4
- Round up, so position = 23

Step 3: Locate the value.

- Find the 23rd value in the ordered list
- Counting from the left: 5.6, 5.7, 5.9, 6.0, 6.2, 6.3, 6.5, 6.7, 6.7, 6.9, 6.9, 6.9, 7.2, 7.2, 7.3, 7.4, 7.4, 7.5, 7.6, 7.7, 7.7, 7.7, **7.8**, 8.0, 8.1, 8.3, 8.4, 8.7
- The 23rd value is in bold
- Thus, the 80th percentile is 7.8 pounds

This means that

- A weight of 7.8 pounds is at the 80th percentile.
- 80% of newborn weights are less than 7.8 pounds.
- 20% of newborn weights are greater than 7.8 pounds.
- If a newborn's weight was 7.8 pounds, then the baby's weight is higher than 80% of the newborns in that group.

Example #6.10.2 🤔

The temperatures on a particular day across 25 states in the western U.S. was recorded:

94, 86, 75, 89, 92, 64, 78, 102, 77, 82, 69, 75, 86, 84, 81, 107, 97, 80, 83, 79, 72, 68, 74, 88, 71

Find the 28th percentile.

✅ **Solution:**

Step 1: Arrange the data in ascending order.

- 64, 68, 69, 71, 72, 74, **75**, 76, 77, 78, 79, 80, 81, 82, 83, 84, 86, 86, 88, 89, 92, 94, 97, 102, 107 (Value is in bold)

Step 2: Calculate the position.

- Position = (Percentile/100) × n
- Position = (28/100) × 25 = 7 (No rounding up needed)

Step 3: Locate the value.

- Find the 7th value in the ordered list.
- Counting from the left: 64, 68, 69, 71, 72, 74, **75**, 76, 77, 78, 79, 80, 81, 82, 83, 84, 86, 86, 88, 89, 92, 94, 97, 102, 107
- The 7th value is in bold.

- Thus, the 28th percentile is 75 degrees

This means that

- A temperature of 75 degrees is at the 28th percentile.
- 28% of temperatures are 75 degrees or below.
- 72% of temperatures are 75 degrees or above.
- If a temperature is 75, then its higher than 28% of the temperatures recorded.

Example #6.10.3 🤔

The heights (in inches) of 20 students in a math class are:

$$73, 60, 72, 61, 66, 65, 75, 68, 60, 66, 67, 70, 71, 72, 66, 68, 68, 64, 70, 66$$

Find the percentile rank for a height of 65.

✅ **Solution:**

Step 1: Arrange the data in ascending order.

- 60, 60, 61, 64, **65**, 66, 66, 66, 66, 67, 68, 68, 68, 70, 70, 71, 72, 72, 73, 75 (Value is in bold)

Step 2: Calculate the total number of values to the left of the ordered data set.

- There are 4 values to the left of 65

Step 3: Use the formula.

$$\text{Percentile for the value } x = \frac{\text{Number of data values less than } x}{n} \cdot 100$$

$$\text{Percentile for the value } 65 = \frac{\text{Number of data values less than } 65}{20} \cdot 100$$

$$\text{Percent for the value } 65 = 20\%$$

Thus, the value of 65 is the 20th percentile.

Example #6.10.4 🤔

According to the *2025 Fuel Economy Guide*, the driving range (in miles) for combined city/highway driving (55% city and 45% highway) of all 17 2025 BMW compact EV cars made is given here:

$$266, 244, 318, 295, 267, 227, 287, 268, 295, 278, 271, 253, 250, 239, 266, 262, 248$$

Find the percentile rank for a height of 278.

✅ **Solution:**

Step 1: Arrange the data in ascending order.

- 227, 239, 244, 248, 250, 253, 262, 266, 266, 267, 268, 271, **278**, 287, 295, 295, 318 (Value is in bold)

Step 2: Calculate the total number of values to the left of the ordered data set.

- There are 12 values to the left of 278

Step 3: Use the formula.

$$\text{Percentile for the value } x = \frac{\text{Number of data values less than } x}{n} \cdot 100$$

$$\text{Percentile for the value } 278 = \frac{\text{Number of data values less than } 12}{17} \cdot 100$$

$$\text{Percent for the value } 278 = 70.58823529\%$$

Thus, the value of 278 is the 71st percentile.

Section 6.10: Percentiles [*In-Class Exercises*]

1. The points scored of 11 players from the Los Angeles Clippers against the Philadelphia 76ers from the February 2nd, 2026, regular season NBA game was recorded.

   15, 29, 8, 6, 17, 21, 13, 0, 2, 2, 0

   What score is at the 85th percentile?

2. The SAT Math scores from 22 seniors from a particular high school:

   480, 500, 510, 530, 540, 550, 560, 570, 580, 590, 600, 610, 620, 630, 640, 650, 660, 670, 690, 700, 710, 720

   What score is at the 22nd percentile?

3. A random sample of 24 patient's systolic blood pressures (in mmHg) were obtained from a local hospital:

   102, 108, 110, 112, 115, 118, 120, 122, 124, 126, 128, 130, 132, 134, 136, 138, 140, 142, 145, 148, 150, 152, 155, 160

   Find the percentile rank for a blood pressure of 115.

4. Daily customer counts for a café over four weeks are given:

   48, 50, 52, 53, 55, 57, 58, 60, 62, 63, 65, 66, 68, 70, 72, 73, 75, 77, 78, 80, 82, 85, 88, 90, 92, 95, 100

   Find the percentile rank for a count of 88.

Answers available at the end of this section.

Answers

Section 6.10: Percentiles [*In-Class Exercises*]

1. 21
2. 115
3. 17th
4. 82nd

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Section 6.10.H: Homework

Section 6.10: Percentiles - Homework Exercises

(Section 6.10) - Homework Exercise #1

A statistics class of 20 students takes an exam. Here are the scores (out of 100), arranged in order:

52, 58, 63, 65, 68, 71, 73, 75, 78, 80, 82, 84, 85, 87, 89, 91, 93, 95, 97, 99

What score is at the 60th percentile?

Answer available at the end of this section.

(Section 6.10) - Homework Exercise #2

Test scores of 40 students are given below:

78, 52, 96, 63, 89, 38, 72, 96, 55, 85, 63, 32, 87, 97, 68, 46, 91, 56, 75, 88,
62, 80, 73, 97, 58, 93, 63, 100, 71, 82, 65, 88, 77, 96, 72, 89, 76, 84, 78, 95

What score is at the 65th percentile?

Answer available at the end of this section.

(Section 6.10) - Homework Exercise #3

The weekly salaries (in dollars) of sixteen government workers are given below:

887, 770, 820, 958, 1,234, 835, 1,089, 979, 829, 1,043, 861, 911, 855, 1,121, 934

What salary is at the 35th percentile?

Answer available at the end of this section.

(Section 6.10) - Homework Exercise #4

The weight (in pounds) of 20 random adults are listed below:

131, 138, 145, 149, 154, 156, 163, 166, 168, 170, 174, 175, 179, 181, 186, 194, 199, 201, 211, 222

Find the percentile rank for a weight of 199.

Answer available at the end of this section.

(Section 6.10) - Homework Exercise #5

Thirty-five numbers from 1 through 100 were randomly assigned by a random number generator and listed below:

76, 23, 88, 51, 9, 67, 32, 19, 62, 45, 7, 79, 54, 91, 38, 73, 2, 85,
59, 16, 93, 66, 28, 80, 50, 11, 76, 62, 98, 17, 71, 46, 82, 57, 37

Find the percentile rank for a value of 23.

Answer available at the end of this section.

Answer available at the end of this section.

(Section 6.10) - Homework Exercise #6

The weekly salaries (in dollars) of fifteen government workers are given below:

887, 770, 820, 958, 1,234, 835, 1,089, 979, 829, 1,043, 861, 911, 855, 1,121, 934

Find the percentile rank for a salary of 887.

Answer available at the end of this section.

Answers

(Section 6.10) - Homework Exercise #1

The 60th percentile is 84

(Section 6.10) - Homework Exercise #2

The 65th percentile is 85

(Section 6.10) - Homework Exercise #3

The 35th percentile is $861

(Section 6.10) - Homework Exercise #4

The value of 199 is the 80th percentile

(Section 6.10) - Homework Exercise #5

The value of 23 is the 20th percentile

(Section 6.10) - Homework Exercise #5

(Section 6.10) - Homework Exercise #6

The value of $887 is the 40th percentile

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Section 6.11: Quartiles

Learning Objectives

- Compute quartiles for a data set

While any percentile from 1 to 99 can be useful, certain percentiles are so important they have special names. Some of those percentiles are called quartiles.

Definition: Quartiles

A quartile is one of three values that divide a data set, ordered from lowest to highest, into four equal parts (or quarters).

The three quartiles are the following:

Q_1 - First Quartile (25th percentile):

- 25% of data falls below this value
- 75% of data falls above this value
- Also called the lower quartile

Q_2 - Second Quartile (50th percentile):

- 50% of data falls below this value
- 50% of data falls above this value
- This is the MEDIAN
- Also called the middle quartile

Q_3 - Third Quartile (75th percentile):

- 75% of data falls below this value
- 25% of data falls above this value
- Also called the upper quartile

Quartiles can quickly evaluate the center of your data (Q_2 or median), the spread of your data, as well as the shape of your distribution (symmetric vs. skewed).Edit section

How To Find Quartiles

- **Step 1** — Organize your data: Arrange your data values in order from smallest to largest. This isn't strictly necessary, but it makes the process easier and ensures your final plot is organized.

To find the quartiles of a data set:

- **Step 1** — Arrange the data in ascending order.

- **Step 2** — Find the median which is Q_2.
- **Step 3** — If the data is
  - an even size, then split the data into two halves. A lower half and an upper half.
  - an odd size, ignore the median, and split the data into two halves. A lower half and an upper half.
- **Step 4** — Find the median of the lower half which is Q_1.
- **Step 5** — Find the median of the upper half which is Q_3.

Example #6.11.1 🤔

A statistics class of 20 students takes an exam. Here are the scores (out of 100), arranged in order:

52, 58, 63, 65, 68, 71, 73, 75, 78, 80, 82, 84, 85, 87, 89, 91, 93, 95, 97, 99

Find the quartiles of the data set.

✅ **Solution:**

Step 1: Arrange the data in ascending order.

- Data was given in ascending order.

Step 2: Find the median, which is Q_2

- 52, 58, 63, 65, 68, 71, 73, 75, 78, **80**, **82**, 84, 85, 87, 89, 91, 93, 95, 97, 99 (The middle values are in bold)
- The median is $\frac{80+82}{2} = \frac{162}{2} = 81$. So, Q_2 = 81

Step 3: The data is an even size, so split the data into two halves. A lower half and an upper half.

- Lower half: 52, 58, 63, 65, 68, 71, 73, 75, 78, 80
- Upper half: 82, 84, 85, 87, 89, 91, 93, 95, 97, 99

Step 4: Find the median of the lower half.

- Lower half: 52, 58, 63, 65, **68**, **71**, 73, 75, 78, 80 (The middle values are in bold)
- The median is $\frac{68+71}{2} = \frac{139}{2} = 69.5$. So, Q_1 = 69.5

Step 5: Find the median of the upper half.

- Upper half: 82, 84, 85, 87, **89**, **91**, 93, 95, 97, 99 (The middle values are in bold)
- The median is $\frac{89+91}{2} = \frac{180}{2} = 90$. So, Q_3 = 90

Thus, Q_1 = 69.5, Q_2 = 81, & Q_3 = 90.

Example #6.11.2 🤔

The temperatures on a particular day across 25 states in the western U.S. was recorded:

94, 86, 75, 89, 92, 64, 78, 102, 77, 82, 69, 75, 86, 84, 81, 107, 97, 80, 83, 79, 72, 68, 74, 88, 71

Find the quartiles of the data set.

Step 1: Arrange the data in ascending order.

- 64, 68, 69, 71, 72, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 86, 86, 88, 89, 92, 94, 97, 102, 107

Step 2: Find the median, which is Q_2

- 64, 68, 69, 71, 72, 74, 75, 76, 77, 78, 79, 80, **81**, 82, 83, 84, 86, 86, 88, 89, 92, 94, 97, 102, 107 (The middle value is in bold)
- The median is 81. So, $Q_2 = 81$

Step 3: The data is an odd size, so ignore the median, and split the data into two halves. A lower half and an upper half.

- Lower half: 64, 68, 69, 71, 72, 74, 75, 76, 77, 78, 79, 80
- Upper half: 82, 83, 84, 86, 86, 88, 89, 92, 94, 97, 102, 107

Step 4: Find the median of the lower half.

- Lower half: 64, 68, 69, 71, 72, **74**, **75**, 76, 77, 78, 79, 80 (The middle values are in bold)
- The median is $\frac{74+75}{2} = \frac{149}{2} = 74.5$. So, $Q_1 = 74.5$

Step 5: Find the median of the upper half.

- Upper half: 82, 83, 84, 86, 86, **88**, **89**, 92, 94, 97, 102, 107 (The middle values are in bold)
- The median is $\frac{88+89}{2} = \frac{177}{2} = 88.5$. So, $Q_3 = 88.5$

Thus, $Q_1 = 74.5$, $Q_2 = 81$, & $Q_3 = 88.5$.

Example #6.11.3 🤔

The heights (in inches) of 19 students in a 5th grade science class are:

48, 63, 51, 59, 61, 50, 52, 56, 60, 54, 52, 57, 59, 52, 58, 56, 52, 59, 52

Find the quartiles of the data set.

Step 1: Arrange the data in ascending order.

- 48, 50, 51, 52, 52, 52, 52, 52, 54, 56, 56, 57, 58, 59, 59, 59, 60, 61, 63

Step 2: Find the median, which is Q_2

- 48, 50, 51, 52, 52, 52, 52, 52, 54, **56**, 56, 57, 58, 59, 59, 59, 60, 61, 63 (The middle value is in bold)
- The median is 56. So, $Q_2 = 56$

Step 3: The data is an even size, so split the data into two halves. A lower half and an upper half.

- Lower half: 48, 50, 51, 52, 52, 52, 52, 52, 54
- Upper half: 56, 57, 58, 59, 59, 59, 60, 61, 63

Step 4: Find the median of the lower half.

- Lower half: 48, 50, 51, 52, **52**, 52, 52, 52, 54 (The middle value is in bold)
- The median is 52. So, $Q_1 = 52$

Step 5: Find the median of the upper half.

- Upper half: 56, 57, 58, 59, **59**, 59, 60, 61, 63 (The middle value is in bold)
- The median is 59. So, $Q_3 = 59$

Thus, $Q_1 = 52$, $Q_2 = 56$, & $Q_3 = 59$.

Example #6.11.4 🤔

In late 2025, according to Realtor.com, the 17 housing prices (in millions) of single family homes close to Orange Coast College was obtained.

1.42, 1.34, 1.69, 1.38, 3.29, 1.59, 1.45, 1.43, 3.38, 1.89, 2.39, 2.49, 1.49, 1.56, 1.73. 1.47, 1.39

Find the quartiles of the data set.

Step 1: Arrange the data in ascending order.

- 1.34, 1.38, 1.39, 1.42, 1.43, 1.45, 1.47, 1.49, 1.56, 1.59, 1.69, 1.73, 1.89, 2.39, 2.49, 3.29, 3.38

Step 2: Find the median, which is Q_2

- 1.34, 1.38, 1.39, 1.42, 1.43, 1.45, 1.47, 1.49, **1.56**, 1.59, 1.69, 1.73, 1.89, 2.39, 2.49, 3.29, 3.38 (The middle value is in bold)
- The median is 1.56. So, $Q_2 = 1.56$

Step 3: The data is an even size, so split the data into two halves. A lower half and an upper half.

- Lower half: 1.34, 1.38, 1.39, 1.42, 1.43, 1.45, 1.47, 1.49
- Upper half: 1.59, 1.69, 1.73, 1.89, 2.39, 2.49, 3.29, 3.38

Step 4: Find the median of the lower half.

- Lower half: 1.34, 1.38, 1.39, **1.42**, **1.43**, 1.45, 1.47, 1.49 (The middle values are in bold)
- The median is $\frac{1.42+1.43}{2} = \frac{2.85}{2} = 1.425$. So, $Q_1 = 1.425$

Step 5: Find the median of the upper half.

- Upper half: 1.59, 1.69, 1.73, **1.89**, **2.39**, 2.49, 3.29, 3.38 (The middle values are in bold)
- The median is $\frac{1.89+2.39}{2} = \frac{4.28}{2} = 2.14$. So, $Q_3 = 2.14$

Thus, $Q_1 = 1.425$, $Q_2 = 1.56$, & $Q_3 = 2.14$; (All values in millions).

Example #6.11.5 🤔

A transportation researcher surveyed 22 employees at a mid-sized company to understand their daily commute times (in minutes, one-way) to help the company plan flexible work arrangements.

20, 70, 52, 15, 65, 25, 28, 40, 32, 30, 45, 38, 55, 50, 18, 85, 62, 58, 22, 42, 48, 35

Find the quartiles of the data set.

Step 1: Arrange the data in ascending order.

- 15, 18, 20, 22, 25, 28, 30, 32, 35, 38, 40, 42, 45, 48, 50, 52, 55, 58, 62, 65, 70, 85

Step 2: Find the median, which is Q_2

- 15, 18, 20, 22, 25, 28, 30, 32, 35, 38, **40**, **42**, 45, 48, 50, 52, 55, 58, 62, 65, 70, 85 (The middle values are in bold)

- The median is $\frac{40+42}{2} = \frac{82}{2} = 41$. So, Q_2 = 41

Step 3: The data is an even size, so split the data into two halves. A lower half and an upper half.

- Lower half: 15, 18, 20, 22, 25, 28, 30, 32, 35, 38, 40
- Upper half: 42, 45, 48, 50, 52, 55, 58, 62, 65, 70, 85

Step 4: Find the median of the lower half.

- 15, 18, 20, 22, 25, **28**, 30, 32, 35, 38, 40
- The median is 28. So, Q_1 = 28

Step 5: Find the median of the upper half.

- Upper half: 42, 45, 48, 50, 52, **55**, 58, 62, 65, 70, 85 (The middle value is in bold)
  The median is 55. So, Q_3 = 55

Thus, Q_1 = 28, Q_2 = 41, & Q_3 = 55.

Section 6.11: Quartiles [*In-Class Exercises*]

For Exercises 1 through 6, find the quartiles of each data set.

1. Data set: 1, 1, 2, 3, 4, 6
2. Data set: 2, 2, 3, 4, 5, 7, 9
3. Data set: 3, 4, 4, 5, 6, 7, 9, 11
4. Data set: 4, 4, 5, 7, 8, 9, 10, 14, 15
5. The owner of a supermarket recorded the number of customers who came into his store each hour in a day. The results were 7, 18, 10, 6, 11, 7, 5, 9, 12, 8, 15, 3, and 11.
6. Here are the highest temperatures ever recorded $(^\circ F)$ in 32 different U.S. states:

93 , 105 , 105 , 105 , 106 , 106 , 107 , 107 , 108 , 110 , 110 , 112 , 112 , 112 , 114 , 114
114 , 115 , 116 , 117 , 118 , 118 , 118 , 118 , 118 , 119 , 121 , 124 , 127 , 129 , 132 , 134

Answers available at the end of this section.

Answers

Section 6.11: Quartiles [*In-Class Exercises*]

1. Q_1 = 1, Q_2 = 2.5, & Q_3 = 4
2. Q_1 = 2, Q_2 = 4, & Q_3 = 7
3. Q_1 = 4, Q_2 = 5.5, & Q_3 = 8
4. Q_1 = 4.5, Q_2 = 8, & Q_3 = 12
5. Q_1 = 6.5, Q_2 = 9, & Q_3 = 11.5
6. Q_1 = 107.5, Q_2 = 114, & Q_3 = 118

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Section 6.11.H: Homework

Section 6.11: Quartiles - Homework Exercises

(Section 6.11) - Homework Exercise #1

The weight (in pounds) of 20 random adults are listed below:

131, 138, 145, 149, 154, 156, 163, 166, 168, 170, 174, 175, 179, 181, 186, 194, 199, 201, 211, 222

Find the quartiles of the data set.

Answer available at the end of this section.

(Section 6.11) - Homework Exercise #2

Test scores of 40 students are given below:

78, 52, 96, 63, 89, 38, 72, 96, 55, 85, 63, 32, 87, 97, 68, 46, 91, 56, 75, 88,
62, 80, 73, 97, 58, 93, 63, 100, 71, 82, 65, 88, 77, 96, 72, 89, 76, 84, 78, 95

Find the quartiles of the data set.

Answer available at the end of this section.

(Section 6.11) - Homework Exercise #3

Thirty-five numbers from 1 through 100 were randomly assigned by a random number generator and listed below:

76, 23, 88, 51, 9, 67, 32, 19, 62, 45, 7, 79, 54, 91, 38, 73, 2, 85,
59, 16, 93, 66, 28, 80, 50, 11, 76, 62, 98, 17, 71, 46, 82, 57, 37

Find the quartiles of the data set.

Answer available at the end of this section.

(Section 6.11) - Homework Exercise #4

The weekly salaries (in dollars) of fifteen government workers are given below:

887, 770, 820, 958, 1,234, 835, 1,089, 979, 829, 1,043, 861, 911, 855, 1,121, 934

Find the quartiles of the data set.

Answer available at the end of this section.

(Section 6.11) - Homework Exercise #5

The annual incomes (in thousands of dollars) of a sample of local households are:

22, 28, 31, 34, 35, 38, 42, 45, 47, 50, 50, 52, 55, 58, 62, 66, 70, 85

Find the quartiles of the data set.

Answer available at the end of this section.

Answer available at the end of this section.

(Section 6.11) - Homework Exercise #6

The commute times (in minutes) for employees at a company are:

40, 32, 12, 20, 30, 35, 38, 45, 22, 60, 55, 50, 15, 25, 27, 18

Find the quartiles of the data set.

Answer available at the end of this section.

(Section 6.11) - Homework Exercise #7

The following data represent the number of hours students spent studying in the library during midterms week.

2, 3, 4, 4, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, 26, 28, 30

Find the quartiles of the data set.

Answer available at the end of this section.

(Section 6.11) - Homework Exercise #8

A university surveyed commuting students and recorded their one-way commute times (in minutes)

5, 7, 8, 10, 12, 13, 15, 17, 18, 20, 22, 23, 25, 27, 28, 30, 32, 33, 35, 37, 38, 40, 42, 43, 45, 47, 48, 50, 52, 53, 55, 57, 60, 62, 65, 67, 68, 70, 72, 75, 78, 80, 85, 90, 95, 100, 105, 110, 115, 120, 130

Find the quartiles of the data set.

Answer available at the end of this section.

Answers

(Section 6.11) - Homework Exercise #1

$Q_1 = 155$, $Q_2 = 172$, & $Q_3 = 190$

(Section 6.11) - Homework Exercise #2

$Q_1 = 63$, $Q_2 = 77.5$, & $Q_3 = 89$

(Section 6.11) - Homework Exercise #3

$Q_1 = 28$, $Q_2 = 57$, & $Q_3 = 76$

(Section 6.11) - Homework Exercise #4

Q_1 = \$835, Q_2 = \$911, & Q_3 = \$1,043

(Section 6.11) - Homework Exercise #5

Q_1 = 35, Q_2 = 48.5, & Q_3 = 58

(Section 6.11) - Homework Exercise #5

(Section 6.11) - Homework Exercise #6

Q_1 = 21, Q_2 = 31, & Q_3 = 42.5

(Section 6.11) - Homework Exercise #7

Q_1 = 7, Q_2 = 11, & Q_3 = 18.5

(Section 6.11) - Homework Exercise #8

Q_1 = 25, Q_2 = 47, & Q_3 = 70

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Section 6.12: The Normal Distribution

Learning Objectives

- Identify characteristics of a data set that is normally distributed
- Apply the empirical rule
- Compute *z* scores and use them to compare values from different data sets
- Use a table and *z* scores as well as technology to find areas under the standard normal distribution

Introduction

In Section 6.7, we used histograms to display data. A histogram shows how frequently values occur within certain intervals, providing a visual representation of how the data are distributed. As the number of data values increases and the class intervals (bins) become narrower, something interesting begins to happen: the tops of the bars start to form a smoother and more consistent shape.

Rather than focusing on the individual bars, we can imagine a continuous curve that follows the overall outline of the histogram. This smooth curve represents the general pattern of the data and is called a distribution curve.

Definition: Distribution

A **distribution** shows how data are spread across different values and how frequently those values occur. It can be represented using a histogram or a smooth curve that outlines the overall pattern of the data.

A distribution gives us insight into where the data are centered and how they are spread. When we draw a smooth curve over a histogram, we are creating an idealized model of that distribution. In this model:

- The horizontal axis represents the data values
- The vertical height represents how common those values are

Unlike a histogram, the curve is continuous (it has no gaps) and idealized, meaning it represents the overall pattern rather than the exact data.

A histogram also helps us identify important features of the data, such as where values cluster, how spread out they are, whether the distribution is symmetric or skewed, and whether any extreme values (outliers) are present. As we collect more data or use smaller intervals, the histogram becomes smoother, and its shape more closely resembles the underlying distribution.

In this sense, the distribution can be thought of as the smooth curve that the histogram is approximating.

Let's examine an example that illustrates this relationship.

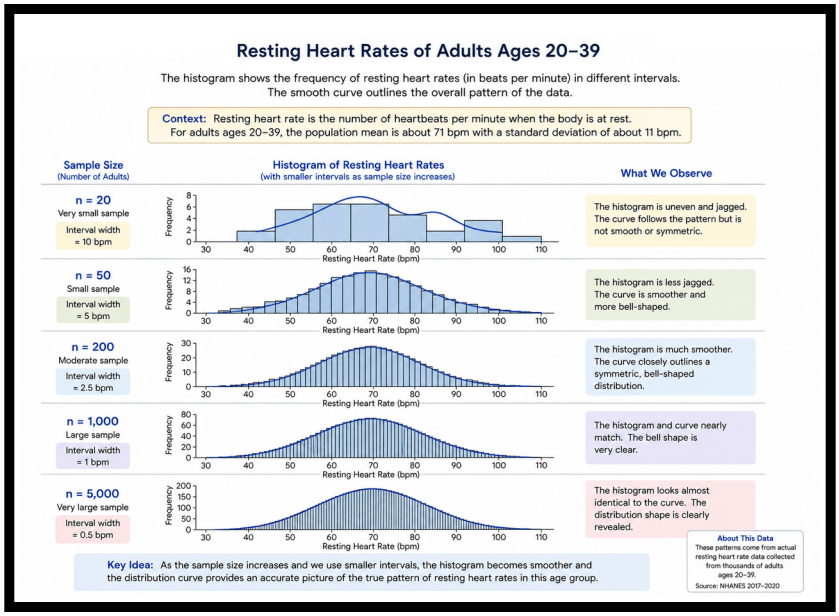
Suppose a researcher from the American Heart Association measures the resting heart rates of adults ages 20–39. If the researcher begins with a small sample of about 20 individuals, the resulting histogram may appear uneven and somewhat scattered. With only a few data values, the overall pattern of the distribution is not yet clear.

As the sample size increases to around 50 individuals, the data begin to cluster more consistently around a central value. The histogram becomes less irregular, and the general shape of the distribution starts to emerge. At this stage, the researcher also

gains a better estimate of the true mean resting heart rate for the population.

When the sample size grows to 200 or more individuals, the histogram becomes noticeably smoother. By using smaller intervals (class widths), the bars more closely follow a continuous pattern, and a smooth distribution curve can be drawn to represent the data. The shape now appears approximately symmetric and bell-shaped.

With very large samples, such as 1,000 or more individuals, the histogram and the distribution curve nearly coincide. The data clearly reveal the overall pattern of resting heart rates in this age group, and the mean and spread of the distribution can be estimated with greater confidence. See infographic below.



The shape of a histogram provides important clues about the underlying distribution of the data. When the bars form a single mound that is approximately symmetric on both sides, the distribution suggests a balanced pattern centered around a typical value. If the bars extend farther to the right, the distribution is **right-skewed**, indicating that most observations are lower with a few unusually large values. If the bars extend farther to the left, the distribution is **left-skewed**, indicating a few unusually small values. In addition, a histogram that is wide and spread out reflects greater variability in the data, while one that is narrow and concentrated indicates that the values are more consistent.

These visual patterns allow us to move from the histogram to a smoother, more general description of the data. The histogram gives us a picture of what we observe, while the distribution curve provides a mathematical model of that pattern, one that can be used to analyze data, compare values, and make predictions.

One of the most important and commonly occurring distributions is the normal distribution.

Definition: Normal Distribution

In many real-world situations, data tend to cluster around a central value, with fewer observations occurring as we move away from that center. This type of pattern is called a **normal distribution**.

A **normal distribution** is a continuous, symmetric distribution that is centered around its mean $\overline{x}$ and has a characteristic bell-shaped curve. Because of this shape, it is often referred to as a bell curve or normal curve.

Key Properties of the Normal Distribution

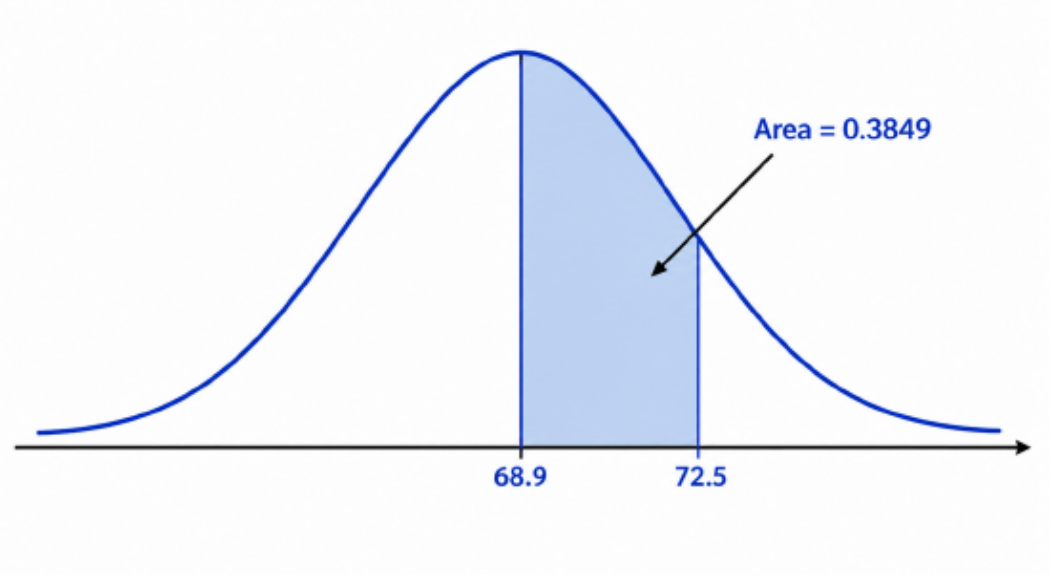
- The distribution is symmetric about its center

- The mean, median, and mode are equal and located at the center
- The graph forms a single, bell-shaped curve
- Most data values are clustered near the mean
- The frequency of values decreases as we move farther from the mean in either direction
- The curve is continuous and extends indefinitely in both directions (the tails approach, but never touch, the horizontal axis)
- The area under the curve corresponds to the proportion of data within a given interval
- The total area under the curve equals 1 (or 100%), representing all possible observations

These properties make the normal distribution a useful model for many real-world phenomena, as it captures the idea that most observations are typical, while extreme values are increasingly rare. Let's examine the properties regarding area.

Example #6.12.1 🤔

The graph below models the heights of adult men in the United States using a normal distribution. The horizontal axis represents height in inches, and the shaded region represents the proportion of men whose heights fall within a specified interval.



a. What is the mean height?
b. What percentage of adult men are between 68.9 and 72.5 inches?
c. If there are 7,536 men at a stadium concert, how many of them would you expect to be between 68.9 and 72.5 inches tall?

✅ **Solution:**

a. The mean is the value in the very center and it is also the highest point on the graph itself. So, the mean height of U.S. adult men is 68.9 inches.
b. From the properties of the normal distribution, the area under the curve represents the proportion of data within a given interval. This means that the area under the curve between 68.9 and 72.5 is 0.3849, or 38.49%. Because area under the normal curve also represents probability, this tells us that the probability of an adult man with a height between 68.9 and 72.5 inches is 0.3849, or 38.49%.
c. Since the area under the normal curve between 68.9 and 72.5 inches is 0.3849, this represents 38.49% of the men between those heights. To find how many men fall within this height range, multiply the total number of men by the percentage. So, 7,536 × 0.3849 = 2,900.6064. Rounding to the nearest whole number, we would expect approximately 2,901 men to be between 68.9 and 72.5 inches tall.

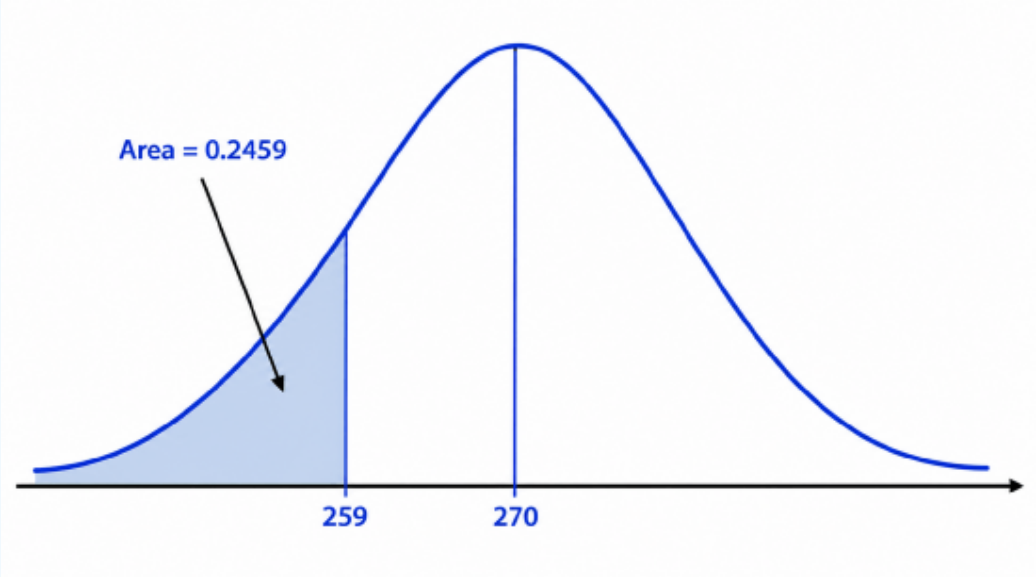
Example #6.12.2 🤔

The length of human pregnancies, measured from conception to birth, has an average of about 270 days. Extensive medical research has shown that pregnancy lengths follow an approximately normal distribution, meaning most pregnancies occur near this average, with fewer occurring much earlier or later.

In medical practice, pregnancy is typically measured from the first day of the last normal menstrual period, which leads to an expected length of about 280 days (40 weeks). This method is used to estimate a due date.

A pregnancy is considered preterm if delivery occurs before 37 weeks (259 days). Pregnancies that last between 41 and 42 weeks are referred to as late-term, while those that extend beyond 42 weeks (294 days) are classified as post-term. Both preterm and post-term pregnancies may involve increased risks for the baby and/or the mother.

The graph below models pregnancy length using a normal distribution for women in the United States. The horizontal axis represents time in days, and the shaded region represents the proportion of pregnancies that fall within a specified interval.



a. What is the mean length of a pregnancy?
b. What percentage of pregnancies are pre-term, meaning they last less than 37 weeks (259 days)?
c. If a hospital records 5,000 births, how many of those would be expected to be less than 37 weeks?
d. If a hospital records 5,000 births, how many of those would be expected to last more than 270 days?

✅ **Solution:**

a. The mean is the value in the very center and it is also the highest point on the graph itself. So, the mean length of a pregnancy is 270 days.
b. From the properties of the normal distribution, the area under the curve represents the proportion of data within a given interval. This means that the area under the curve to the left of 259 is 0.2459 or 24.59%. Because area under the normal curve also represents probability, this tells us that the probability of the length of a pregnancy less than 259 days is 0.2459 or 24.59%.
c. Since the area under the normal curve to the left of 294 is 0.2459, this represents 24.59% of all births to have a pregnancy length of 259 days or less. To determine how many pregnancies fall within this range, multiply the total number of hospital births by the percentage. So, 5,000 × 0.2459 = 1,229.5. Therefore, we would expect approximately 1,230 births to have a pregnancy length of 259 days or less.
d. Since 270 days is the mean, the area (not shown) under the normal curve greater than 270 days is 0.5, this represents 50% of all pregnancies. To determine how many pregnancies fall within this range, multiply the total number of hospital births by the percentage. So, 5,000 × 0.5 = 2,500. Therefore, we would expect approximately 2,500 births to have a pregnancy length of 270 days or more.

Now, that we understand that areas under a normal curve represent probabilities, we are ready to develop tools that allow us to interpret and estimate those probabilities more efficiently. We saw that shading a region under the curve corresponds directly to the likelihood of observing values within that interval. The question now becomes: How can we determine these areas without it being given?

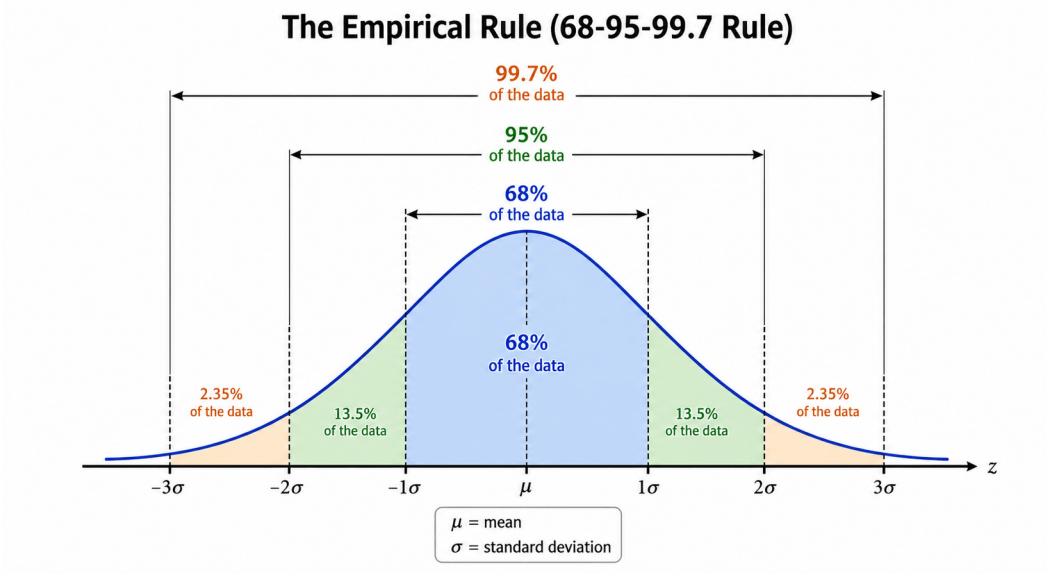
One of the most useful approximations for working with normal distributions is the Empirical Rule, sometimes called the 68–95–99.7 Rule. This rule describes how data are distributed around the mean in any normal distribution.

Emperical Rule or 68–95–99.7 Rule

If a distribution is approximately normal, then:

- About 68% of the data lie within 1 standard deviation of the mean
- About 95% of the data lie within 2 standard deviations of the mean
- About 99.7% of the data lie within 3 standard deviations of the mean

Here is a visual of how the Empirical Rule looks:



- From −1σ to 1σ: 68% of the data
- From −2σ to 2σ: 95% of the data
- From −3σ to 3σ: 99.7% of the data

A small legend at the bottom states: μ = mean σ = standard deviation The diagram illustrates the Empirical Rule for normal distributions, showing how data are distributed around the mean.

These percentages correspond directly to areas under the normal curve.

- Saying that 68% of the data fall within one standard deviation is equivalent to saying that the area under the curve between $\mu - \sigma$ and $\mu + \sigma$ is 0.68 or 68%, no matter what μ and σ are equal to.
- Saying that 95% of the data fall within one standard deviation is equivalent to saying that the area under the curve between $\mu - 2\sigma$ and $\mu + 2\sigma$ is 0.95 or 95%, no matter what μ and σ are equal to.
- Saying that 99.7% of the data fall within one standard deviation is equivalent to saying that the area under the curve between $\mu - 3\sigma$ and $\mu + 3\sigma$ is 0.997 or 99.7%, no matter what μ and σ are equal to.

Remember, that the distribution must be normal or approximately normal in order to use the percentages of 68%, 95%, and 99.7% respectively.

Example #6.12.3 🤔

The length of human pregnancies, measured from conception to birth, has a mean of approximately 270 days with a standard deviation of 16 days. Assume that pregnancy lengths are normally distributed. Use the Empirical Rule to answer the following:

a. What interval (in days) contains approximately 68% of all pregnancies lengths?
b. What interval (in days) contains approximately 95% of all pregnancies lengths?
c. What interval (in days) contains approximately 99.7% of all pregnancies lengths?

✅ **Solution:**

a. The Empirical Rule tells us that about 68% of the data lie within 1 standard deviation of the mean.So,

$$270 \pm 16 \Rightarrow (270 - 16, 270 + 16) \Rightarrow (254, 286)$$

Approximately 68% of all pregnancies last between 254 and 286 days. This represents the typical range where most pregnancy lengths fall.

---

b. The Empirical Rule tells us that about 95% of the data lie within 2 standard deviations of the mean.So,

$$270 \pm 2(16) \Rightarrow 270 \pm 32 \Rightarrow (270 - 32, 270 + 32) \Rightarrow (238, 302)$$

Approximately 95% of all pregnancies last between 238 and 302 days. Pregnancy lengths outside this range are less common.

---

c. The Empirical Rule tells us that about 99.7% of the data lie within 3 standard deviations of the mean.So,

$$270 \pm 3(16) \Rightarrow 270 \pm 48 \Rightarrow (270 - 48, 270 + 48) \Rightarrow (222, 318)$$

Approximately 99.7% of all pregnancies last between 238 and 302 days. Values outside this interval would be considered extremely unusual.

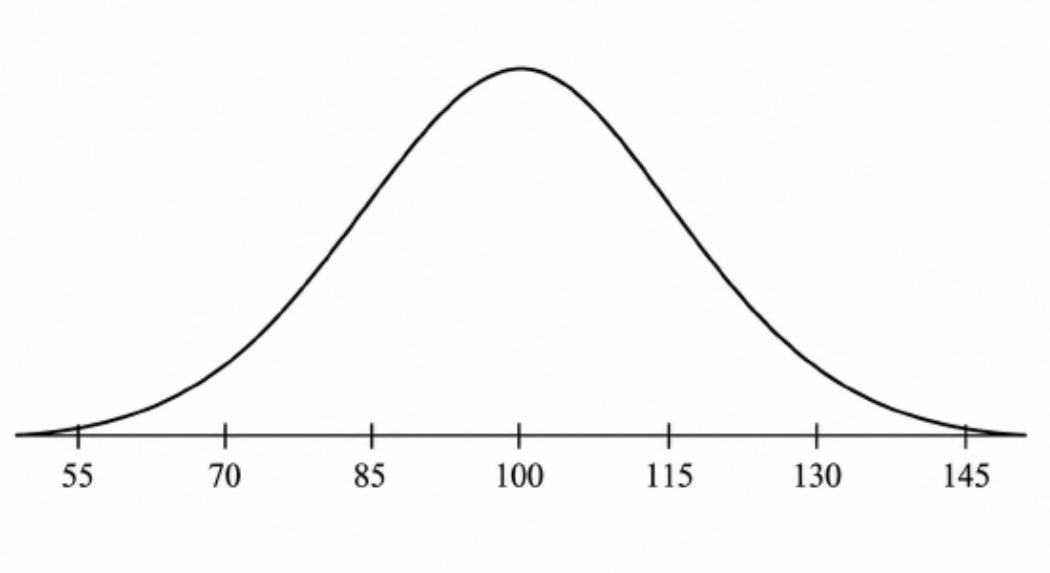
Example #6.12.4 🤔

IQ stands for Intelligence Quotient. It is a standardized score obtained from assessments designed to measure cognitive abilities such as problem-solving, memory, spatial reasoning, and processing speed, relative to a large population. An IQ test provides a general measure of intellectual ability. IQ scores are scaled so that most individuals fall within a predictable range. The distribution of IQ scores from the modern IQ test (WAIS-IV) is approximately normal, with a mean of 100 and a standard deviation of 15.

a. What approximate percentage of IQ scores fall between 70 and 130?
b. What approximate percentage of IQ scores fall between 55 and 145?
c. What approximate percentage of IQ scores fall between 85 and 115?
d. What approximate percentage of IQ scores fall below 100?

✅ **Solution:**

When a table of areas is not used, the Empirical Rule provides a quick way to estimate probabilities. It is helpful to sketch a normal curve and label seven key points on the horizontal axis: the mean, along with one, two, and three standard deviations both above and below the mean.



a. From the sketch, the values 70 and 130 are each two standard deviations from the mean, since 2(15) = 30. The Empirical Rule tells us that approximately 95% of the data lie within two standard deviations of the mean. Therefore, about 95% of all IQ scores fall between 70 and 130.

---

b. From the sketch, the values 55 and 145 are each three standard deviations from the mean, since 3(15) = 45. The Empirical Rule tells us that approximately 99.7% of the data lie within three standard deviations of the mean. Therefore, about 99.7% of all IQ scores fall between 55 and 145.

---

c. From the sketch, the values 85 and 115 are each one standard deviation from the mean, since 1(15) = 15. The Empirical Rule tells us that approximately 68% of the data lie within one standard deviation of the mean. Therefore, about 68% of all IQ scores fall between 85 and 115.

---

d. Since the mean is 100 and a normal distribution is symmetric, the values 100 and below represent half the area. Therefore, about 50% of all IQ scores fall less than 100.

From Raw Values to Standardized Values

While the Empirical Rule gives us useful benchmarks, real-world data are not always centered at 0 or scaled in convenient units. For example, heights, test scores, and prices all have different means and standard deviations. To compare values across different normal distributions, we need a way to standardize them.

This leads to the concept of the *z*-score.

The *z*-Score

A z-score tells us how many standard deviations a value is from the mean. It transforms any normal distribution into a common scale.

$$z = \frac{x - \mu}{\sigma}$$

where

- x is the original value
- μ is the mean
- σ is the standard deviation.

A z-score answers the question: How far, in standard deviation units, is this value from the mean?

- A positive *z*-score indicates the value is above the mean
- A negative *z*-score indicates the value is below the mean
- A *z*-score of 0 means the value is exactly at the mean

Example #6.12.5 🤔

Find the z-score for the value 170, when the mean is 160 and the standard deviation is 10.

✅ **Solution:**

$$\mathrm{z} = \frac{x - \mu}{\sigma}$$

$$\mathrm{z} = \frac{170 - 160}{10}$$

$$\boxed{\mathrm{z} = 1}$$

Thus, the z-score is 1. This means that the score of 160 is exactly **one** standard deviation to the right of the mean.

Example #6.12.6 🤔

Referring to the modern IQ test in Example #6.12.4, find the *z*-score for the value 80, when the mean is 100 and the standard deviation is 15.

✅ **Solution:**

$$\mathrm{z} = \frac{x - \mu}{\sigma}$$

$$\mathrm{z} = \frac{80 - 100}{15}$$

$$\boxed{\mathrm{z} \approx -1.33}$$

Thus, the z-score is –1.33. This indicates that an IQ score of 80 is approximately 1.33 standard deviations below the mean.

The Standard Normal Distribution

When we convert values to *z*-scores, we transform data from any normal distribution into a special distribution called the standard normal distribution.

To see how this works, consider Example 6.12.4. In that example, we sketched a normal curve and labeled seven key values on the horizontal axis: 55, 70, 85, 100, 115, 130, and 145. These values were determined using the mean and multiples of the standard deviation (up to three standard deviations below and above the mean).

If we convert each of these values to *z*-scores, we obtain: –3, –2, –1, 0, 1, 2, 3

These *z*-scores represent the number of standard deviations each value lies from the mean. Importantly, this pattern is always the same, regardless of the original mean μ and standard deviation σ. Any normal distribution, when standardized, will align with

these same reference points.

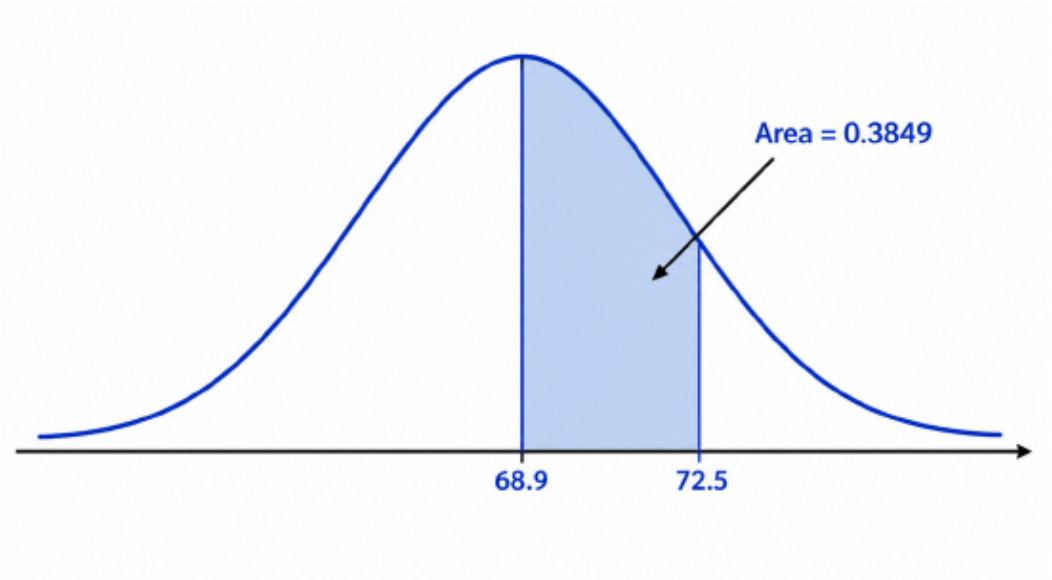
The resulting distribution is called the standard normal distribution.

Definition: Standard Normal Distribution

The **standard normal distribution** is a normal distribution with mean, $\mu = 0$, and standard deviation $\sigma = 1$.

By converting values to z-scores, we can analyze any normal distribution using a single, standardized model.

Consider the normal curve shown in Example #6.12.1.



In that example, the area of the shaded region was given as 0.3849. But where did this value come from? It is clearly not obtained from the Empirical Rule, which only provides rough estimates.

In general, finding the exact area under a curve requires techniques from calculus. While methods such as numerical approximation (for example, Simpson's Rule) can be used, they are beyond the scope of this course. Fortunately, mathematicians have already computed these areas and organized them into tables for us. These tables, known as standard normal tables, list areas under the curve corresponding to specific *z*-scores.

However, the curve in Example 6.12.1 is based on raw data and is not yet standardized. To use a standard normal table, we must first convert the original values into *z*-scores.

Let us standardize the values 68.9 and 72.5. The mean is 68.9, and suppose the standard deviation is 3.

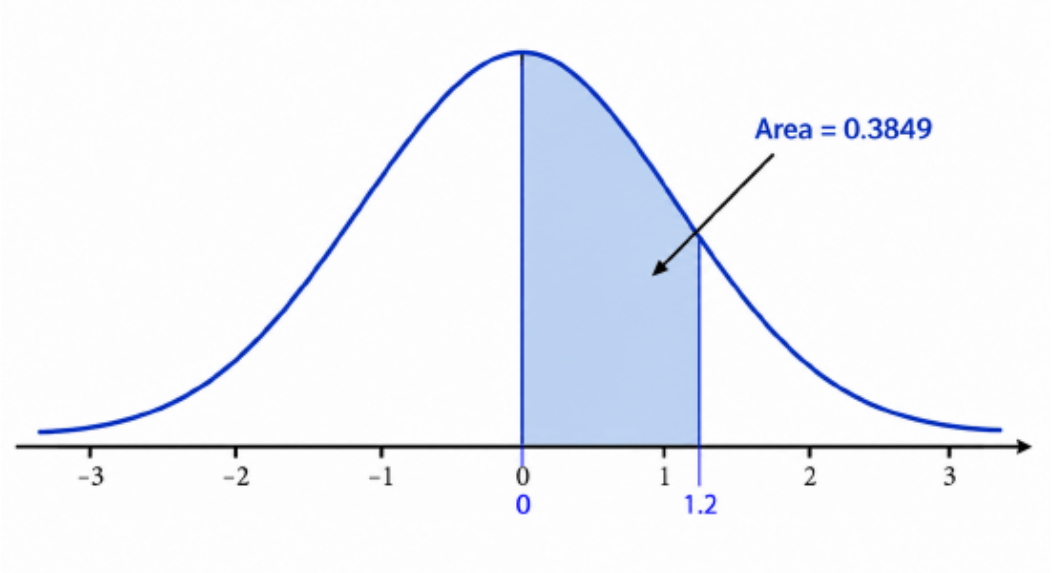
For the score of 68.9:

$$\mathrm{z} = \frac{68.9 - 68.9}{3} = 0$$

For the score of 72.5:

$$\mathrm{z} = \frac{72.5 - 68.9}{3} = 1.2$$

Now the data have been transformed into *z*-scores. Although the numerical values along the horizontal axis have changed, the shape of the curve and the area under it remain exactly the same.



With the data now standardized, we can use a standard normal table to determine the area. This is why tables and graphs of the standard normal distribution are so useful, because they allow us to find probabilities for any normal distribution once the data have been converted to *z*-scores.

USE THE SCROLL BAR TO THE FAR RIGHT END TO VIEW THE REST OF THE TABLE ⬇️

TABLE 6.1: Areas Under a Standard Normal Curve — Left Tail (P(Z < z))

Table 6.1 - Standard Normal Distribution Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

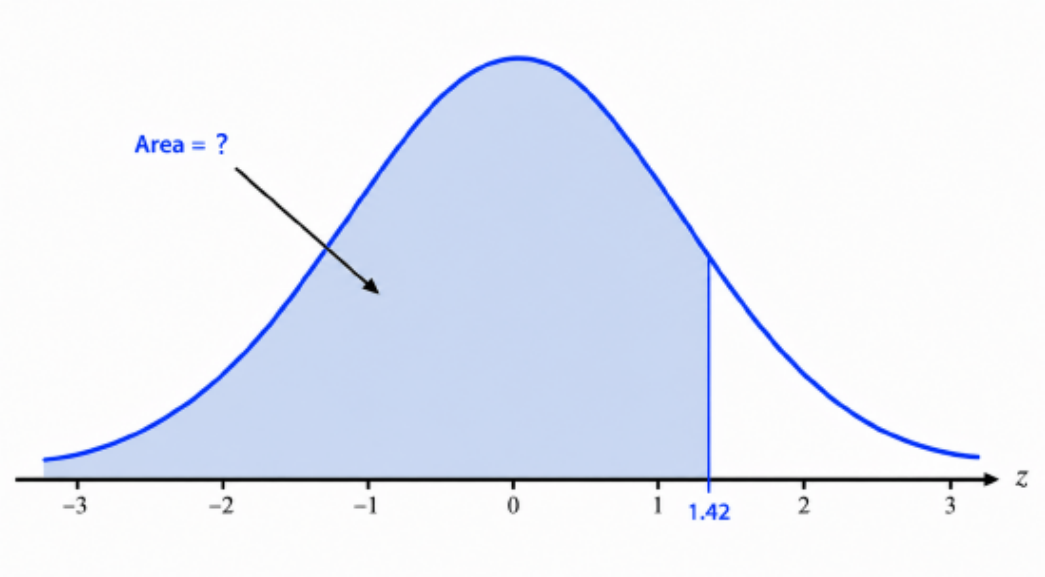
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

If you would like to print out a copy of the table, click here: Standard Normal Table (in PDF)

Example #6.12.7 🤔

Determine the area under the standard normal curve that lies to the left of $z = 1.42$.

✅ **Solution:**



Symbolically, this can be written as: $P(z < 1.42)$.

First, identify and shade the region of interest on the standard normal curve, which includes all values to the left of $z = 1.42$.

Next, refer to Table 6.1: Areas Under the Standard Normal Curve.

This table lists z-scores ranging from −3.49 to 3.49. The ones and tenths digits appear in the leftmost column, while the hundredths digits are shown across the top row. The values in the body of the table represent the corresponding areas, rounded to four decimal places.

To find the area associated with $z = 1.42$:

- Use the row 1.4 for the first two digits and tenths place.

TABLE 6.1 (Partial)

z	0.00	0.01	0.02	0.03
0.9	0.8159	0.8186	0.8212	0.8238
1.0	0.8413	0.8438	0.8461	0.8485
1.1	0.8643	0.8665	0.8686	0.8708
1.2	0.8849	0.8869	0.8888	0.8907
1.3	0.9032	0.9049	0.9066	0.9082
1.4	0.9192	0.9207	0.9222	0.9236
1.5	0.9332	0.9345	0.9357	0.9370
1.6	0.9452	0.9463	0.9474	0.9484
1.7	0.9554	0.9564	0.9573	0.9582
1.8	0.9641	0.9649	0.9656	0.9664

- Use the column 0.02 for the hundredths place.
- The intersection gives the cumulative probability 0.9222.

Other nearby table values are visible for surrounding z-scores, showing probabilities increasing as z-values increase.

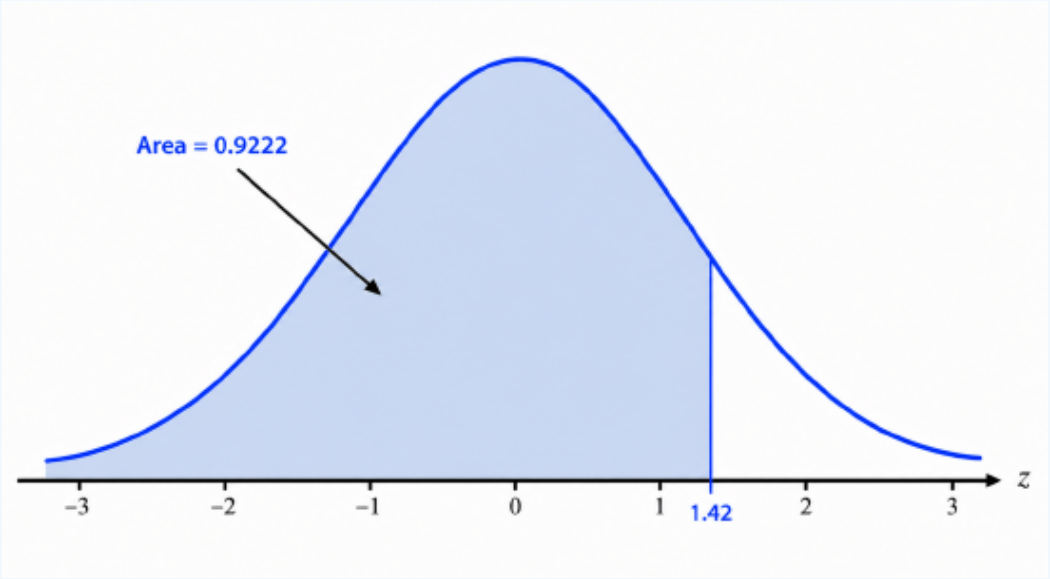
- Locate the row corresponding to 1.4 (the ones and tenths digits)
- Then move across to the column for 0.02 (the hundredths digit)

The value at this intersection is 0.9222.

NOTE: The areas displayed in the middle of Table 6.1 represents the area to the LEFT of the specified z-score.

So, since our original question was pertaining to finding the area to the **left** of the *z*-score 1.42, then the answer is that area associated which is 0.9222.

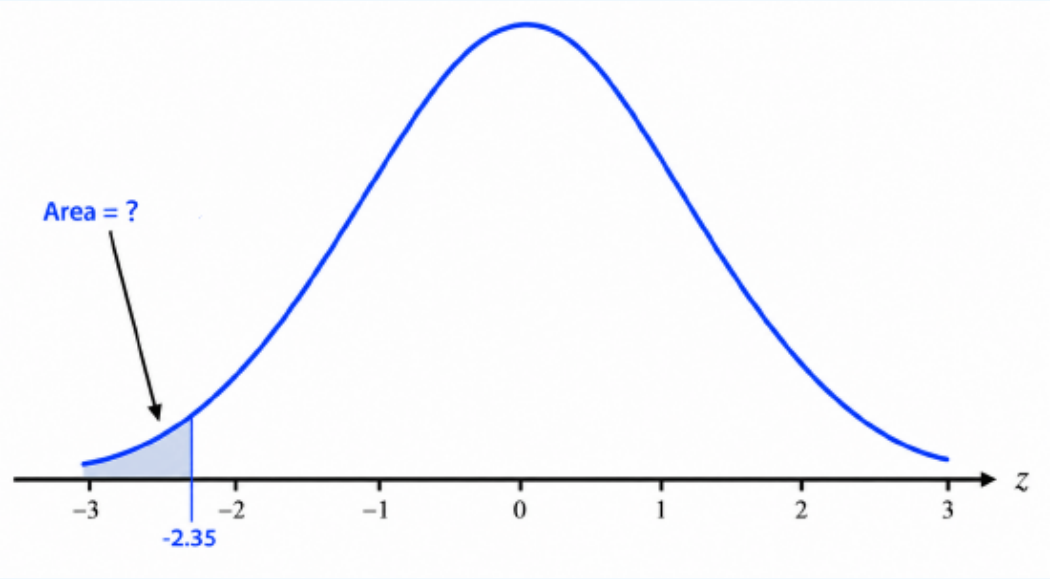
Thus, the area under the standard normal curve that lies to the left of $z = 1.42$ is 0.9222.
Symbolically, $P(z < 1.42) = 0.9222$.



Example #6.12.8 🤔

Determine the area under the standard normal curve that lies to the left of $z = -2.35$.

✅ **Solution:**



Symbolically, this can be written as: $P(z < -2.35)$.

First, identify and shade the region of interest on the standard normal curve, which includes all values to the left of $z = -2.35$.

Next, refer to Table 6.1: Areas Under the Standard Normal Curve.

This table lists z-scores ranging from −3.49 to 3.49. The ones and tenths digits appear in the leftmost column, while the hundredths digits are shown across the top row. The values in the body of the table represent the corresponding areas, rounded to four decimal places.

To find the area associated with $z = -2.35$:

TABLE 6.1 (Partial)

z	0.00	0.01	0.02	0.03	0.04	0.05
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202

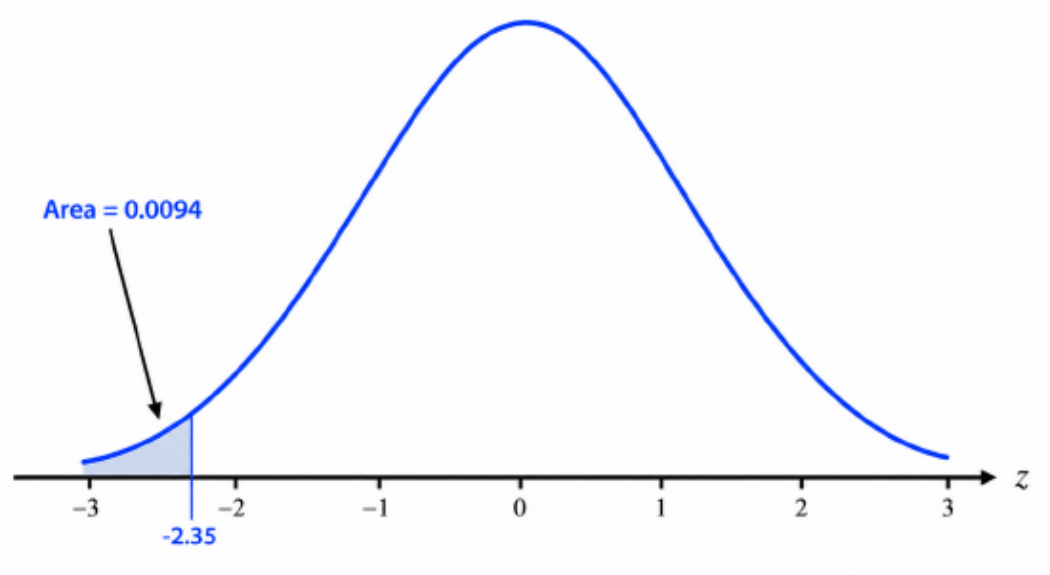
- Locate the row corresponding to −2.3 (the ones and tenths digits)
- Then move across to the column for 0.05 (the hundredths digit)

The value at this intersection is 0.0094.

NOTE: The areas displayed in the middle of Table 6.1 represents the area to the LEFT of the specified z-score.

So, since our original question was pertaining to finding the area to the **left** of the *z*-score –2.35, then the answer is that area associated which is 0.0094.

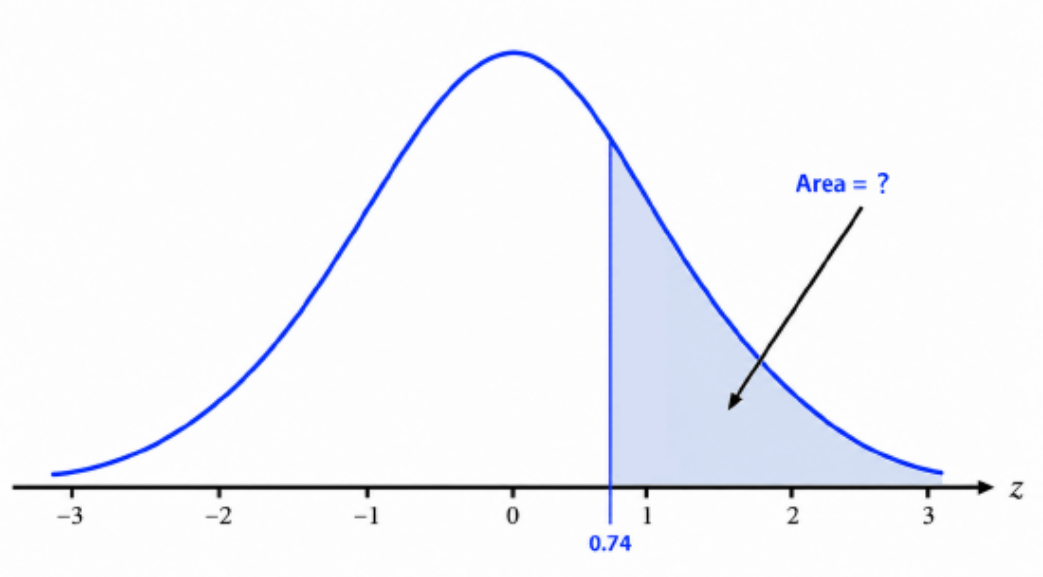
Thus, the area under the standard normal curve that lies to the left of $z = -2.35$ is 0.0094.
Symbolically, $P(z < -2.35) = 0.0094$.



Example #6.12.9 🤔

Determine the area under the standard normal curve that lies to the right of $z = 0.74$.

✅ **Solution:**



Symbolically, this can be written as: $P(z > 0.74)$.

First, identify and shade the region of interest on the standard normal curve, which includes all values to the right of $z = 0.74$.

Next, refer to Table 6.1: Areas Under the Standard Normal Curve.

This table lists z-scores ranging from −3.49 to 3.49. The ones and tenths digits appear in the leftmost column, while the hundredths digits are shown across the top row. The values in the body of the table represent the corresponding areas, rounded to four decimal places.

To find the area associated with $z = 0.74$:

TABLE 6.1 (Partial)

z	0.00	0.01	0.02	0.03	0.04	0.05
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289

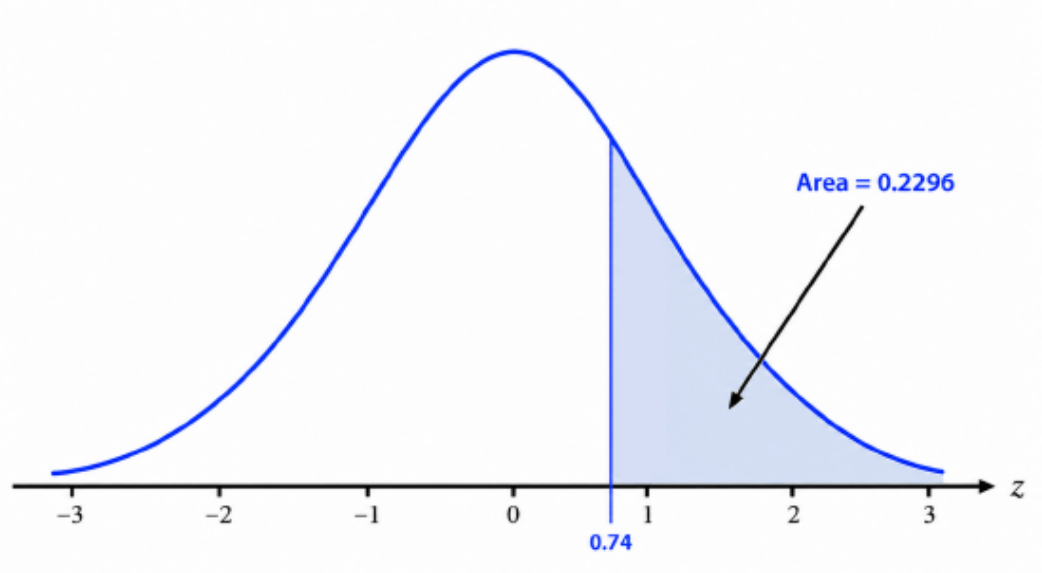
- Locate the row corresponding to 0.7 (the ones and tenths digits)
- Then move across to the column for 0.04 (the hundredths digit)

The value at this intersection is 0.7704.

NOTE: The areas displayed in the middle of Table 6.1 represents the area to the LEFT of the specified z-score.

So, since our original question was pertaining to finding the area to the **right** of the z-score 0.74, then the answer is that area associated subtracted from 1 which is 1 − 0.7704 = 0.2296.

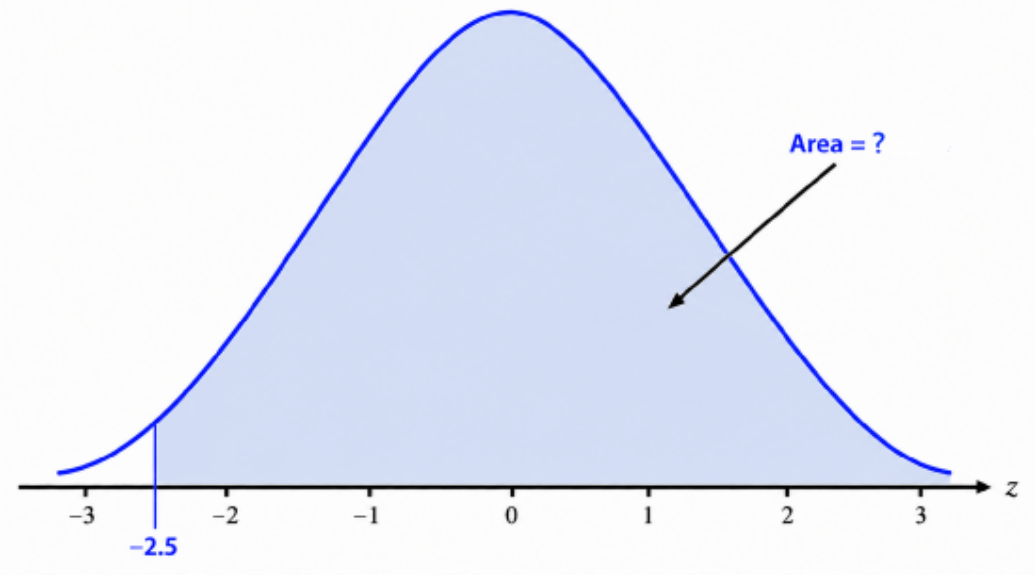
Thus, the area under the standard normal curve that lies to the right of $z = 0.74$ is 0.2296.
Symbolically, $P(z > 0.74) = 1 - P(z < 0.74) = 1 - 0.7704 = 0.2296$.



Example #6.12.10 🤔

Determine the area under the standard normal curve that lies to the right of $z = -2.5$.

✅ **Solution:**



Symbolically, this can be written as: $P(z > -2.5)$.

First, identify and shade the region of interest on the standard normal curve, which includes all values to the right of $z = -2.5$.

Next, refer to Table 6.1: Areas Under the Standard Normal Curve.

This table lists z-scores ranging from −3.49 to 3.49. The ones and tenths digits appear in the leftmost column, while the hundredths digits are shown across the top row. The values in the body of the table represent the corresponding areas, rounded to four decimal places.

To find the area associated with $z = -2.5$:

TABLE 6.1 (Partial)

z	0.00	0.01	0.02	0.03
−2.7	0.0035	0.0034	0.0033	0.0032
−2.6	0.0047	0.0045	0.0044	0.0043
−2.5	0.0062	0.0060	0.0059	0.0057
−2.4	0.0082	0.0080	0.0078	0.0075
−2.3	0.0107	0.0104	0.0102	0.0099
−2.2	0.0139	0.0136	0.0132	0.0129

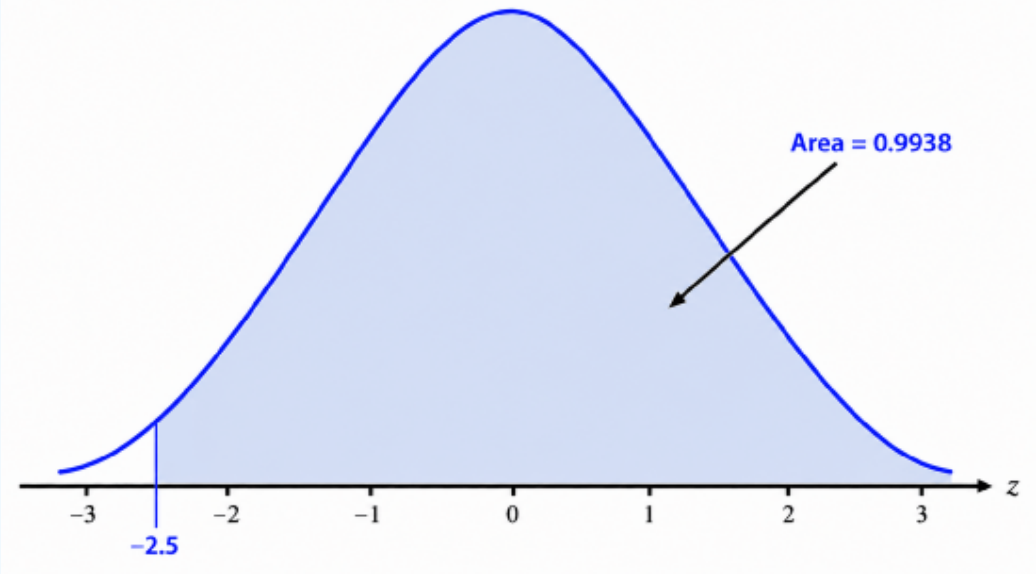
- Locate the row corresponding to –2.5 (the ones and tenths digits)
- Then move across to the column for 0.00 (the hundredths digit)

The value at this intersection is 0.9938.

NOTE: The areas displayed in the middle of Table 6.1 represents the area to the LEFT of the specified z-score.

So, since our original question was pertaining to finding the area to the **right** of the *z*-score –2.5, then the answer is that area associated subtracted from 1 which is 1 – 0.0062 = 0.9938.

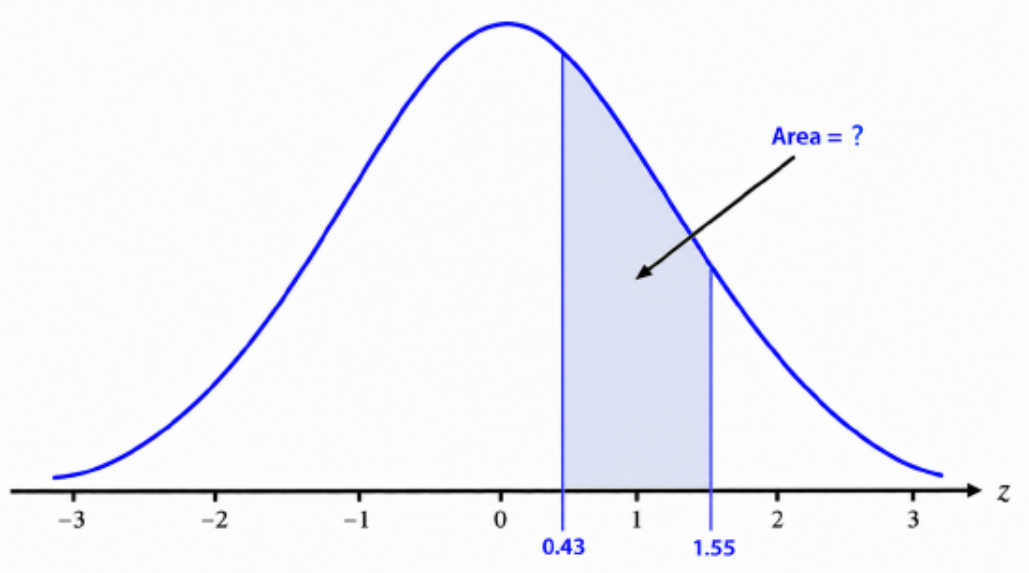
Thus, the area under the standard normal curve that lies to the right of $z = -2.5$ is 0.9938.
Symbolically, $P(z > -2.5) = 1 - P(z < -2.5) = 1 - 0.0062 = 0.9938$.



Example #6.12.11 🤔

Determine the area under the standard normal curve that lies between $z_1 = 0.43$ and $z_2 = 1.55$.

✅ **Solution:**



Symbolically, this can be written as: $P(0.43 < z < 1.55)$.

First, identify and shade the region of interest on the standard normal curve, which includes all values between $z_1 = 0.43$ and $z_2 = 1.55$.

Next, refer to Table 6.1: Areas Under the Standard Normal Curve.

This table lists z-scores ranging from −3.49 to 3.49. The ones and tenths digits appear in the leftmost column, while the hundredths digits are shown across the top row. The values in the body of the table represent the corresponding areas, rounded to four decimal places.

To find the areas associated with both $z_1 = 0.43$ and $z_2 = 1.55$:

TABLE 6.1 (Partial)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686

- Start with the larger of the two z-scores, which is $z_2 = 1.55$.
- Locate the row corresponding to 1.5 (the ones and tenths digits)
- Then move across to the column for 0.05 (the hundredths digit)

The value at this intersection is 0.9394.

- Now, take the smaller of the two z-scores, which is $z_1 = 0.43$.
- Locate the row corresponding to 0.4 (the ones and tenths digits)
- Then move across to the column for 0.03 (the hundredths digit)

TABLE 6.1 (Partial)

z	0.00	0.01	0.02	0.03	0.04
0.2	0.5793	0.5832	0.5871	0.5910	0.5948
0.3	0.6179	0.6217	0.6255	0.6293	0.6331
0.4	0.6554	0.6591	0.6628	0.6664	0.6700
0.5	0.6915	0.6950	0.6985	0.7019	0.7054
0.6	0.7257	0.7291	0.7324	0.7357	0.7389
0.7	0.7580	0.7611	0.7642	0.7673	0.7704

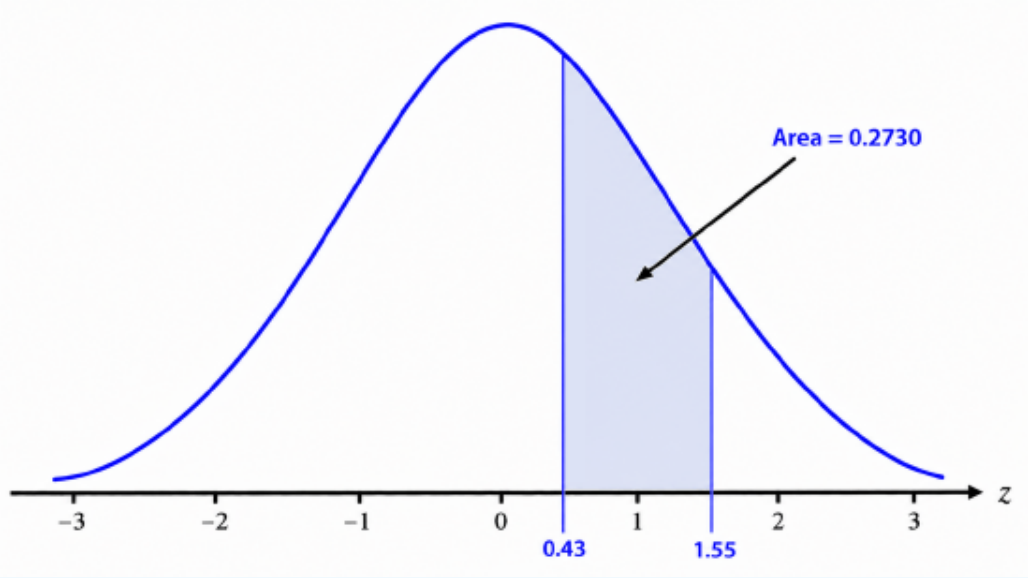
The value at this intersection is 0.6664.

NOTE: The areas displayed in the middle of Table 6.1 represents the area to the LEFT of the specified z-score.

So, since our original question was pertaining to finding the area **between** z_1 = 0.43 and z_2 = 1.55, then the answer is the difference of the corresponding cumulative areas, which is 0.9394 – 0.6664 = 0.2730.

The reason why is that the area to the left of z_2 = 1.55 is 0.9394. The area to the left of z_1 = 0.43 is 0.2730. Subtracting these values isolates the area between the two *z*-scores.

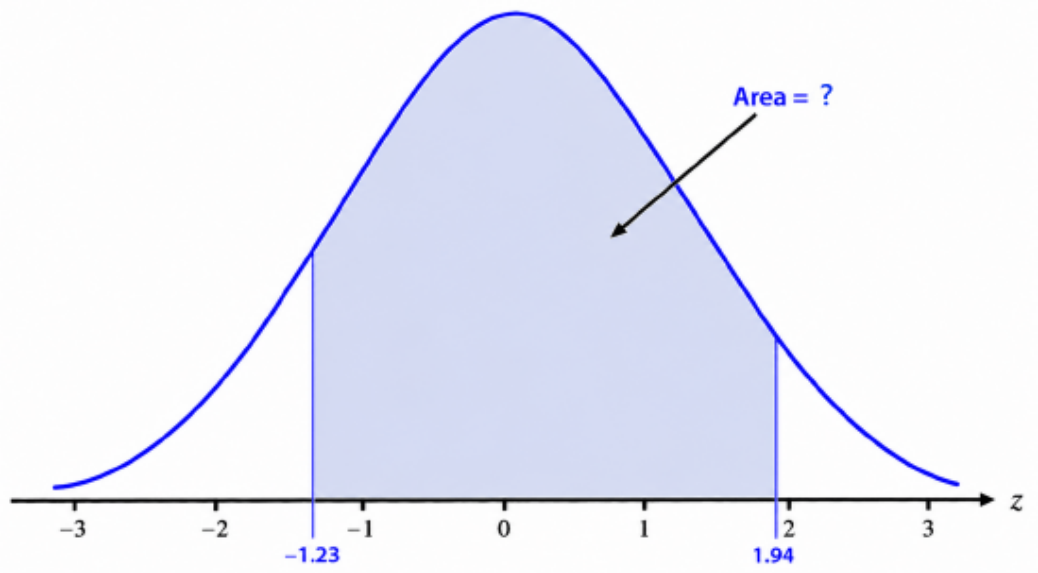
Thus, the area under the standard normal curve that lies between z_1 = 0.43 and z_2 = 1.55 is 0.2730.
Symbolically, $P(0.43 < z < 1.55) = P(z < 1.55) - P(z < 0.43) = 0.9394 - 0.6664 = 0.2730$.



Example #6.12.12 🤔

Determine the area under the standard normal curve that lies between z_1 = –1.23 and z_2 = 1.94.

✅ **Solution:**



Symbolically, this can be written as: $P(-1.23 < z < 1.94)$.

First, identify and shade the region of interest on the standard normal curve, which includes all values between $z_1 = -1.23$ and $z_2 = 1.94$.

Next, refer to Table 6.1: Areas Under the Standard Normal Curve.

This table lists z-scores ranging from −3.49 to 3.49. The ones and tenths digits appear in the leftmost column, while the hundredths digits are shown across the top row. The values in the body of the table represent the corresponding areas, rounded to four decimal places.

To find the areas associated with both $z_1 = -1.23$ and $z_2 = 1.94$:

TABLE 6.1 (Partial)

z	0.00	0.01	0.02	0.03	0.04	0.05
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878

- Start with the larger of the two z-scores, which is $z_2 = 1.94$.
- Locate the row corresponding to 1.9 (the ones and tenths digits)
- Then move across to the column for 0.04 (the hundredths digit)

The value at this intersection is 0.9738.

- Now, take the smaller of the two z-scores, which is $z_1 = -1.23$.
- Locate the row corresponding to −1.2 (the ones and tenths digits)

- Then move across to the column for 0.03 (the hundredths digit)

TABLE 6.1 (Partial)

z	0.00	0.01	0.02	0.03	0.04
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492

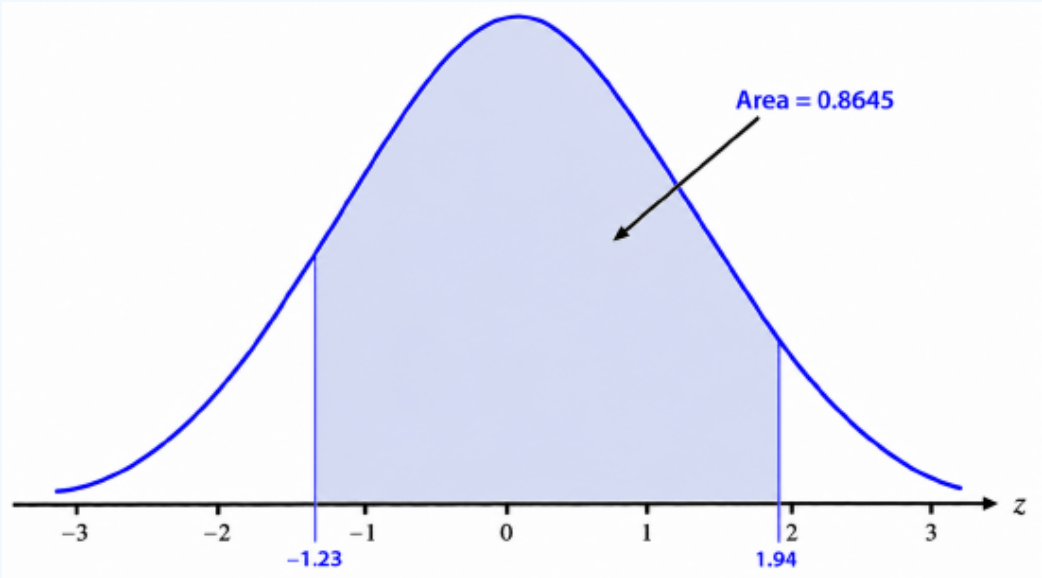
The value at this intersection is 0.1093.

NOTE: The areas displayed in the middle of Table 6.1 represents the area to the LEFT of the specified z-score.

So, since our original question was pertaining to finding the area **between** $z_1 = -1.23$ and $z_2 = 1.94$, then the answer is the difference of the corresponding cumulative areas, which is 0.9738 – 0.1093 = 0.8645.

The reason why is that the area to the left of $z_2 = 1.94$ is 0.9738. The area to the left of $z_1 = -1.23$ is 0.1093. Subtracting these values isolates the area between the two z-scores.

Thus, the area under the standard normal curve that lies between $z_1 = -1.23$ and $z_2 = 1.94$ is 0.8645.
Symbolically, $P(-1.23 < z < 1.94) = P(z < 1.94) - P(z < -1.23) = 0.9738 - 0.1093 = 0.8645$.



The Evolution of IQ Testing and Terminology

In Example #6.12.4, the term modern IQ test refers to the WAIS-IV (Wechsler Adult Intelligence Scale, Fourth Edition), a widely used assessment for measuring adult intelligence. This test replaced earlier instruments, such as the Binet–Simon Intelligence Scale, which was one of the first formal methods for evaluating cognitive ability.

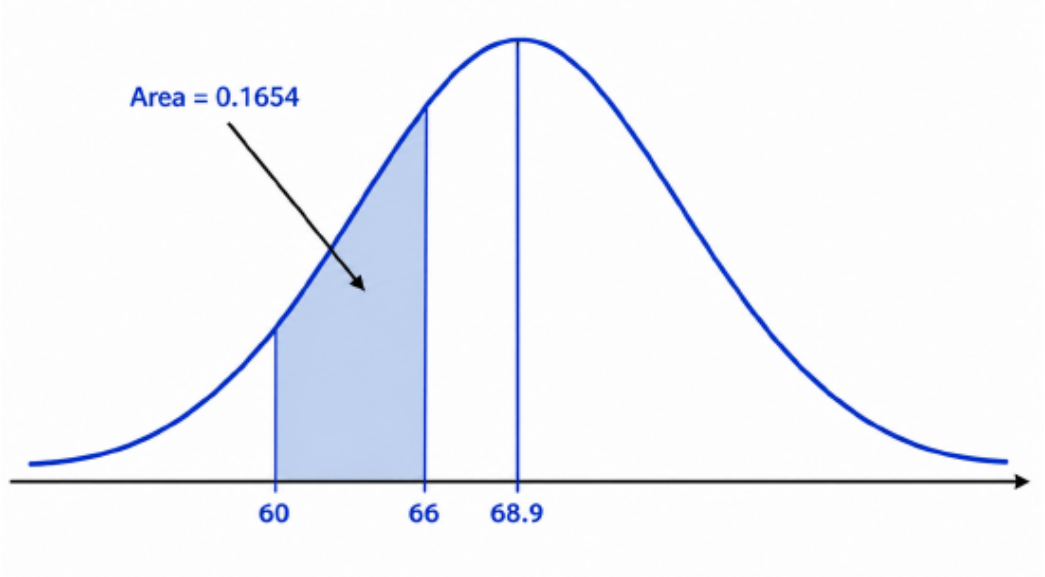
In the early 20th century, terms such as idiot, imbecile, and moron were used as formal clinical classifications to describe levels of intellectual disability based on IQ scores. Psychologists, including Henry H. Goddard, associated these labels with specific score ranges: idiot (0–24), imbecile (25–49), and moron (50–74). At the time, these categories were also linked to estimated developmental levels; for example, a "moron" was said to have the mental capacity of a child aged 7 to 10.

These classifications were used in educational and institutional settings and were also connected to broader social policies of the time, including the Buck v. Bell decision, which upheld compulsory sterilization laws.

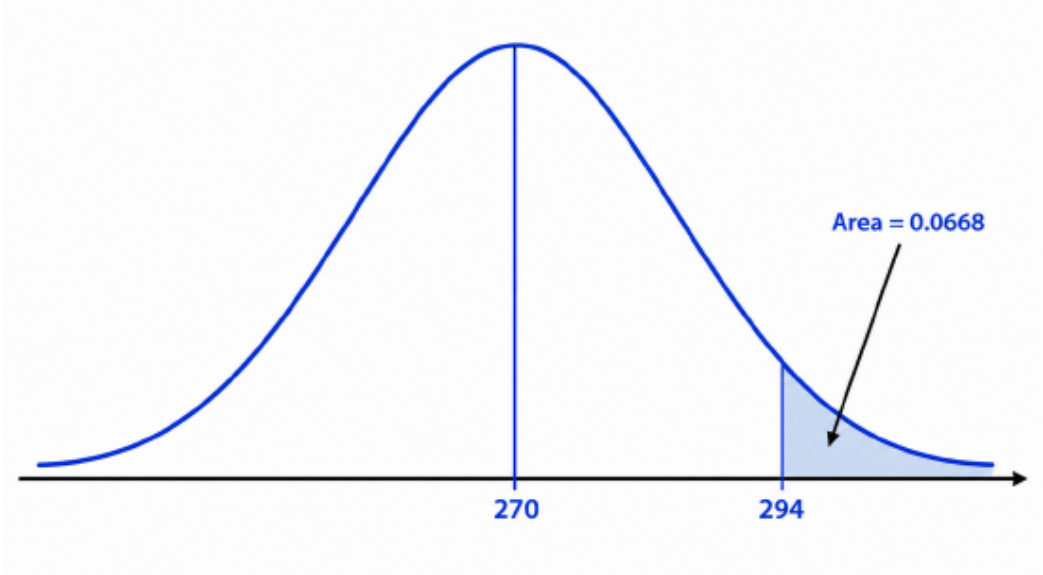
By the late 20th century, these diagnostic terms were abandoned due to their increasingly negative and inappropriate use in everyday language. Over time, they came to be recognized as harmful, stigmatizing, and inaccurate ways of describing individuals. As a result, they were replaced with more neutral and descriptive classifications, such as mild, moderate, severe, and profound intellectual disability, which remain in use today.

Section 6.12: The Normal Distribution [*In-Class Exercises*]

1. The graph below shows a normal distribution for the heights of adult men in the United States. The numbers on the horizontal axis are the heights in inches. An area of a shaded region is shown as well.



   a. What is the mean height?
   b. What percentage of adult men are between 60 and 66 inches?
   c. If there are 28,538 men in a stadium at an NFL football game, how many of them would you expect to be between 60 and 66 inches tall?
2. The length of human pregnancies, measured from conception to birth, has an average of about 270 days. Extensive medical research has shown that pregnancy lengths follow an approximately normal distribution, meaning most pregnancies occur near this average, with fewer occurring much earlier or later. In medical practice, pregnancy is typically measured from the first day of the last normal menstrual period, which leads to an expected length of about 280 days (40 weeks). This method is used to estimate a due date. A pregnancy is considered preterm if delivery occurs before 37 weeks (259 days). Pregnancies that last between 41 and 42 weeks are referred to as late-term, while those that extend beyond 42 weeks (294 days) are classified as post-term. Both preterm and post-term pregnancies may involve increased risks for the baby and/or the mother. The graph below models pregnancy length using a normal distribution for women in the United States. The horizontal axis represents time in days, and the shaded region represents the proportion of pregnancies that fall within a specified interval.

3. 

   a. What is the mean length of a pregnancy?
   b. What percentage of pregnancies are post-term, meaning they last longer than 42 weeks (294 days)?
   c. If a hospital records 5,000 births, how many of those would be expected to last more than 42 weeks?
   d. If a hospital records 5,000 births, how many of those would be expected to last more than 270 days?

4. A particular LED light bulb has a mean lifetime of 3000 hours and a standard deviation of 700 hours. Assume the lifetimes are normally distributed.
   Use the Empirical Rule to answer the following:
   a. What interval (in hours) contains approximately 68% of all light bulb lifetimes?
   b. What interval (in hours) contains approximately 95% of all light bulb lifetimes?
   c. What interval (in hours) contains approximately 99.7% of all light bulb lifetimes?
5. The SAT is designed so that scores are roughly normally distributed across test-takers. The mean score is 1,050 while the standard deviation is 210. The score of 1,050 generally represents the 49th to 50th percentile, meaning a student with this score performed roughly in the middle of all test-takers.
   a. What approximate percentage of SAT scores fall between 630 and 1,470?
   b. What approximate percentage of SAT scores fall between 840 and 1,260?
   c. What approximate percentage of SAT scores fall between 420 and 1680?
   d. What approximate percentage of SAT scores fall below 1,050?
6. Refer to Example #3. What is the z-score for an LED light bulb that lasted 3,700 hours? What is the z-score for an LED light bulb that lasted 2,000 hours?
7. An instructor teaches two sections of a statistics course: a morning class and an afternoon class. Kenny is enrolled in the morning section, while his friend Cartman is enrolled in the afternoon section. Both students take the same exam. Kenny scores 85 on the exam. In the morning class, the mean score is 72 with a standard deviation of 9. Cartman scores 88, and in the afternoon class, the mean is 78 with a standard deviation of 8. Find the z-score for each student's exam score and determine who performed better relative to their class.
8. Determine the area under the standard normal curve that lies to the left of $z = 2.12$.
9. Determine the area under the standard normal curve that lies to the left of $z = 0.48$.
10. Determine the area under the standard normal curve that lies to the left of $z = -2.65$.
11. Determine the area under the standard normal curve that lies to the right of $z = 1.63$.
12. Determine the area under the standard normal curve that lies to the right of $z = -0.6$.
13. Determine the area under the standard normal curve that lies to the right of $z = -2.49$.
14. Determine the area under the standard normal curve that lies between $z_1 = 0.67$ and $z_2 = 1.35$.
15. Determine the area under the standard normal curve that lies between $z_1 = -1.25$ and $z_2 = 0.11$.
16. Determine the area under the standard normal curve that lies between $z_1 = -1$ and $z_2 = 1$.
17. Determine the area under the standard normal curve that lies to the left of $z = -4.54$.
18. Determine the area under the standard normal curve that lies to the right of $z = -3.76$.
19. Determine the area under the standard normal curve that lies to the left of $z = 8.13$.
20. Determine the area under the standard normal curve that lies to the right of $z = 7.47$.

Answers available at the end of this section.

Answers

Section 6.12: The Normal Distribution [*In-Class Exercises*]

1. a. 68.9 inches
   b. 16.54%
   c. Approximately 4,720
2. a. 270 days
   b. 6.68%
   c. 334
   d. 2,500
3. CS
   a. (2,300, 3700)
   b. (1,600, 4,400)
   c. (900, 5,100)
4. a. 95%
   b. 68%
   c. 99.7% (Note: The maximum score on an SAT is1,600. It is possible for three standard deviations 1,680 to exceed the maximum score 1,600, because SAT scores are not perfectly normally distributed; rather, they are a truncated distribution. In a true normal distribution, values extend to infinity in both directions, but real-world tests have hard limits (ceilings and floors) that "cut off" the theoretical tails of the curve.
   d. 50%
5. 1; –1.43
6. Kenny: 1.44 ; Cartman: 1.25; Kenny performed better relative to his class, then Carman did, even though Cartman had a higher score.
7. 0.9830
8. 0.6844
9. 0.0040
10. 0.0516
11. 0.7257
12. 0.9936
13. 0.1629
14. 0.4382
15. 0.6826
16. ≈ 0
17. ≈ 1
18. ≈ 1
19. ≈ 0

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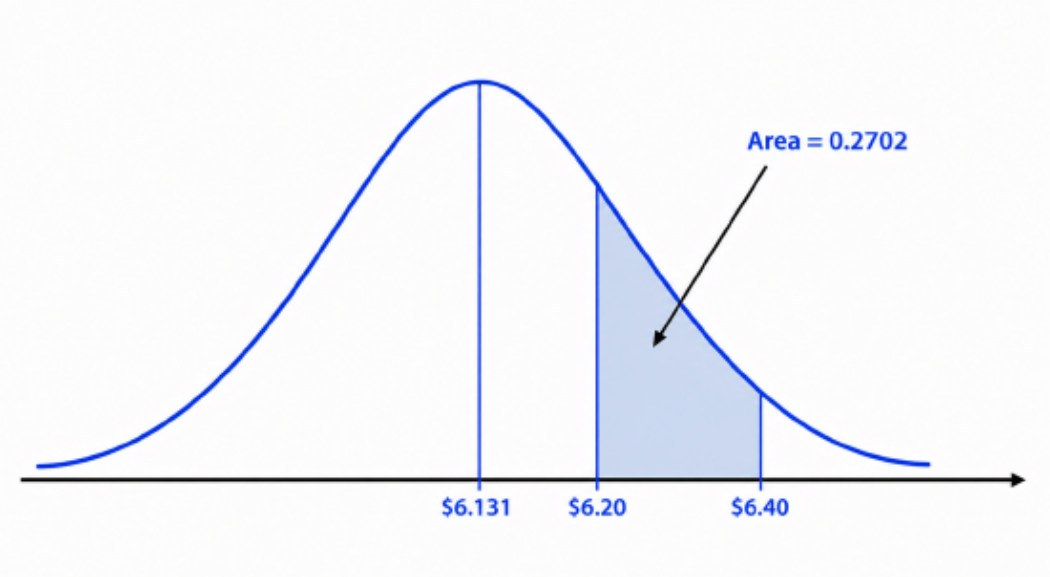
Section 6.12.H: Homework

Section 6.12: The Normal Distribution - Homework Exercises

(Section 6.12) - Homework Exercise #1

As of May 5th, 2026, the average price of regular unleaded gasoline in California was $6.131 per gallon, according to AAA, which is the highest average of all 50 states. The graph below models gas prices in California using a normal distribution. The horizontal axis represents price in dollars per gallon, and the shaded region represents the proportion of prices that fall within a specified interval.

Source: https://gasprices.aaa.com/?state=CA



a. What is the mean price?
b. What percentage of prices fall between $6.20 and $6.40?
c. If a random sample of 1,000 gas stations were chosen, how many of them would you expect to have gas prices between $6.20 and $6.40?

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #2

As of May 5th, 2026, the average price of regular unleaded gasoline in Oklahoma was $3.897 per gallon, according to AAA, which is the lowest average of all 50 states. The graph below models gas prices in Oklahoma using a normal distribution. The horizontal axis represents price in dollars per gallon, and the shaded region represents the proportion of prices that fall within a specified interval.

Source: https://gasprices.aaa.com/?state=OK



a. What is the mean price?
b. What percentage of prices fall less than $3.70?
c. If a random sample of 500 gas stations were chosen, how many of them would you expect to have gas prices less than $3.70?

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #3

The birth weights of full-term babies have a mean of 7.5 pounds and a standard deviation of 1 pound. Assume the weights are normally distributed. Use the Empirical Rule to answer the following:

a. What interval (in pounds) contains approximately 68% of all birth weights?
b. What interval (in pounds) contains approximately 95% of all birth weights?
c. What interval (in pounds) contains approximately 99.7% of all birth weights?

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #4

Commute times to work have a mean of 30 minutes and a standard deviation of 6 minutes. Assume the times are normally distributed. Use the Empirical Rule to answer the following:

a. What interval (in minutes) contains approximately 68% of all commute times?
b. What interval (in minutes) contains approximately 95% of all commute times?
c. What interval (in minutes) contains approximately 99.7% of all commute times?

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #5

Resting heart rates have a mean of 72 bpm and a standard deviation of 8 bpm. Assume the data are normally distributed. Use the Empirical Rule to answer the following:

a. What approximate percentage of values fall below the mean?
b. What approximate percentage of values fall above the mean?
c. What approximate percentage of values fall between 48 and 96?
d. What approximate percentage of values fall between 56 and 88?
e. What approximate percentage of values fall between 64 and 80?

What approximate percentage of values fall outside 2 standard deviations?

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #6

Weekly grocery spending for households in a certain area has a mean of $150 and a standard deviation of $25. Assume the data are normally distributed. Use the Empirical Rule to answer the following:

a. What approximate percentage of values fall between $125 and $175?
b. What approximate percentage of values fall between $75 and $225?
c. What approximate percentage of values fall between $100 and $200?
d. What approximate percentage of values are above $200?

What approximate percentage of values fall more than 2 standard deviations below the mean?

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #7

A rechargeable battery has a mean lifetime of 12 hours with a standard deviation of 1.5 hours. Find the z-score for a battery that lasts 15 hours.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #8

The average daily high temperature in a city is 75°F with a standard deviation of 8°F. Find the z-score for a day with a high of 60°F.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #9

Cholesterol levels have a mean of 190 mg/dL and a standard deviation of 25 mg/dL. A patient has a level of 215 mg/dL. Find the z-score. Interpret what this value means in context.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #10

Two employees work at different warehouses.

- Employee A processes 52 packages per hour. In their warehouse, the mean is 45 packages per hour with a standard deviation of 5.

- Employee B processes 60 packages per hour. In their warehouse, the mean is 50 packages per hour with a standard deviation of 8.

Find the *z*-score for each employee and determine who performed better relative to their workplace.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #11

Two athletes run in two different races of the same distance:

- Runner A finishes in 52 seconds (mean = 60, standard deviation = 5)
- Runner B finishes in 48 seconds (mean = 55, standard deviation = 3)

Find each runner's *z*-score and determine who performed better relative to their group.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #12

An instructor teaches two classes of a liberal arts math course back to back in the morning. Erik is enrolled in the first class, while his best friend Christopher is enrolled in the second class. Both students take the same final exam. Erik scores 83 on the exam. In the first class, the mean score is 85 with a standard deviation of 6. Christopher scores an 87 on his exam. In the second class, the mean is 90 with a standard deviation of 9. Find the *z*-score for each student's exam score and determine who performed better relative to their class.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #13

Determine the area under the standard normal curve that lies to the left of $z = 1.11$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #14

Determine the area under the standard normal curve that lies to the left of $z = -1.79$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #15

Determine the area under the standard normal curve that lies to the right of $z = 0.09$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #16

Determine the area under the standard normal curve that lies to the right of $z = 0.7$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #17

Determine the area under the standard normal curve that lies between $z_1 = 1.14$ and $z_2 = 1.67$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #18

Determine the area under the standard normal curve that lies between $z_1 = -1.72$ and $z_2 = 0.93$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #19

Determine the area under the standard normal curve that lies to the left of $z = 2.37$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #20

Determine the area under the standard normal curve that lies to the between $z_1 = -2$ and $z_2 = -1$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #21

Determine the area under the standard normal curve that lies to the right of $z = 3.11$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #22

Determine the area under the standard normal curve that lies to the left of $z = -6.32$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #23

Determine the area under the standard normal curve that lies to the right of $z = 7.11$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #24

Determine the area under the standard normal curve that lies to the left of $z = 3.87$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #25

Determine the area under the standard normal curve that lies to the left of $z = -5.55$.

Answer available at the end of this section.

(Section 6.12) - Homework Exercise #26

Determine the area under the standard normal curve that lies to the between $z_1 = -4$ and $z_2 = 4$.

Answer available at the end of this section.

Answers

(Section 6.12) - Homework Exercise #1

1. $6.131 or $6.13
2. 27.02%
3. Approximately 270

(Section 6.12) - Homework Exercise #2

1. $6.131 or $6.13
2. 5.03%
3. Approximately 25

(Section 6.12) - Homework Exercise #3

a. (6.5, 8.5)
b. (5.5, 9.5)
c. (4.5, 10.5)

(Section 6.12) - Homework Exercise #4

a. (24, 36)
b. (18, 42)
c. (12, 48)

(Section 6.12) - Homework Exercise #5

a. 50%
b. 50%
c. 99.7%
d. 95%
e. 68%

(Section 6.12) - Homework Exercise #6

a. 68%
b. 99.7%
c. 95%
d. 2.5%

(Section 6.12) - Homework Exercise #7

2

(Section 6.12) - Homework Exercise #8

–1.875

(Section 6.12) - Homework Exercise #9

1; The patient's cholesterol level is exactly one standard deviation above the normal level it should be

(Section 6.12) - Homework Exercise #10

Employee A: 1.4; Employee B: 1.25. Employee A performed better relative to their workplace, because their productivity is farther above the mean compared to Employee B.

(Section 6.12) - Homework Exercise #11

Runner A: –1.6; Runner B: –2.33; Since lower race times are better, a more negative z-score indicates a stronger performance relative to the group. Runner B had a stronger performance relative to their group.

(Section 6.12) - Homework Exercise #12

Erik: -0.33 ; Christopher: -0.33; Erik and Christopher each performed the same relative to each other's respective class, even though Christopher had a higher

score.

(Section 6.12) - Homework Exercise #13

0.8665

(Section 6.12) - Homework Exercise #14

0.0367

(Section 6.12) - Homework Exercise #15

0.4641

(Section 6.12) - Homework Exercise #16

0.2420

(Section 6.12) - Homework Exercise #17

0.0797

(Section 6.12) - Homework Exercise #18

0.7811

(Section 6.12) - Homework Exercise #19

0.0089

(Section 6.12) - Homework Exercise #20

0.1359

(Section 6.12) - Homework Exercise #21

0.0009

(Section 6.12) - Homework Exercise #22

≈ 0

(Section 6.12) - Homework Exercise #23

≈ 0

(Section 6.12) - Homework Exercise #24

≈ 1

(Section 6.12) - Homework Exercise #25

≈ 0

(Section 6.12) - Homework Exercise #26

≈ 1

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CHAPTER 7: VOTING METHODS

Section 7.1: Preference Tables

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Section 7.2: Plurality Method

Section 7.2.H: Homework

Section 7.3: Condorcet Criterion

Section 7.3.H: Homework

Section 7.4: Borda Count Method

Section 7.4.H: Homework

Section 7.5: Majority Criterion

Section 7.5.H: Homework

Section 7.6: Plurality-with-Elimination Method

Section 7.6.H: Homework

Section 7.7: Monotonicity Criterion

Section 7.7.H: Homework

Section 7.8: Pairwise Comparison Method

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Section 7.9: Independence of Irrelevant Alternatives (IIA) Criterion

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Section 7.10: Approval Voting

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Section 7.11: Apportionment - The Basics

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Section 7.14: Apportionment - Adams' Method

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Section 7.16: Apportionment - Huntington-Hill Method

Section 7.16.H: Homework

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Section 7.1: Preference Tables

Learning Objectives

- Interpret the information in a preference table

The Challenge of Group Decision-Making

Imagine your class needs to choose one destination for a field trip. Three options are available:

- Museum (M)
- Amusement Park (A)
- Beach (B)

Simple scenario: If everyone votes and the Museum gets 15 votes, Amusement Park gets 8 votes, and Beach gets 7 votes, the Museum wins, right? But wait, what if those 15 Museum voters represent only half the class? That means 15 other students (the majority!) preferred something else. Is this fair? Does this truly represent what the class wants?

Now imagine a more complex situation: What if different people feel differently how strongly they prefer each option? What if the Museum voters would be fine with the Beach, but the Amusement Park voters absolutely dislike the Museum? Should that matter?

This is the fundamental problem: When groups of people need to make a single collective choice, how do we fairly combine everyone's preferences into one decision?

Why Simple Majority Voting Isn't Always Enough

With Two Options: Majority voting works perfectly.

- Option A vs. Option B
- Count the votes
- Whichever gets more wins
- Clear, fair, done

With Three or More Options: Things get complicated.

- No one might get a majority (>50%)
- The "winner" might be strongly opposed by most voters
- Your vote might feel "wasted" if you don't vote for a frontrunner
- The outcome might depend on which voting method you use

Historical Example: The 2000 U.S. Presidential Election

- George W. Bush vs. Al Gore (main candidates)
- Ralph Nader (third-party candidate)
- Nader received about 97,000 votes in Florida
- Bush won Florida by only 537 votes
- Many Nader voters preferred Gore to Bush as their second choice
- Did Nader "spoil" the election by splitting the vote?

This raises questions: Would a different voting method have produced a different (perhaps "better" or "fairer") outcome?

What Are Voting Methods?

Voting methods (also called electoral systems or social choice functions) are mathematical procedures for translating individual preferences into a collective decision. The key Idea here is that different methods can produce different winners from the exact same set of voter preferences. Think of voting methods as different "recipes" for combining the same ingredients (voter preferences) into different outcomes (winners).

The Four Major Voting Methods

In this chapter, we will explore the four major voting methods that represent different philosophical approaches to fairness. They are the following:

Plurality Method (First-Past-the-Post)
Borda Count Method (Positional Voting)
Plurality with Elimination (Instant Runoff Voting / Ranked-Choice Voting)
Pairwise Comparison (Condorcet Method)

Why We Need Multiple Voting Methods

You might wonder: "If there are different methods that give different results, isn't there one correct method we should always use?" This is what we will explore throughout the chapter to determine if one such method even exists.

So, prior to discussing any type of voting methods, we must first have a full understanding of preference tables.

Preference Table

A **preference table** (also called a preference schedule or voter profile) is a systematic way to organize and display voting data when voters rank multiple candidates or options in order of preference. Rather than listing each individual ballot separately, preference tables group identical rankings together, showing how many voters shared each particular ranking pattern.

Preference tables are fundamental tools in voting theory and social choice theory, the branches of mathematics and statistics that study how groups make collective decisions. They help us analyze elections with multiple candidates, understand different voting methods, and explore questions about fairness in democratic decision-making.

Preference tables help by capturing more information. They take into account not just first choices only, but complete rankings. This allows us to:

- See second and third choices and so on
- Identify compromise options
- Apply different voting methods
- Make more informed group decisions

Basic Structure of a Preference Table

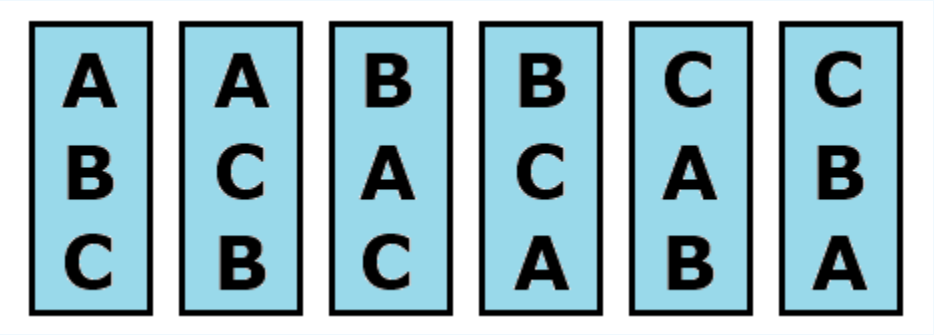
# of Voters	[#]	[#]	[#]	⋯
1st Choice	Candidate	Candidate	Candidate	⋯
2nd Choice	Candidate	Candidate	Candidate	⋯
3rd Choice	Candidate	Candidate	Candidate	⋯
⋯	⋯	⋯	⋯	⋯

How to Read a Preference Table:

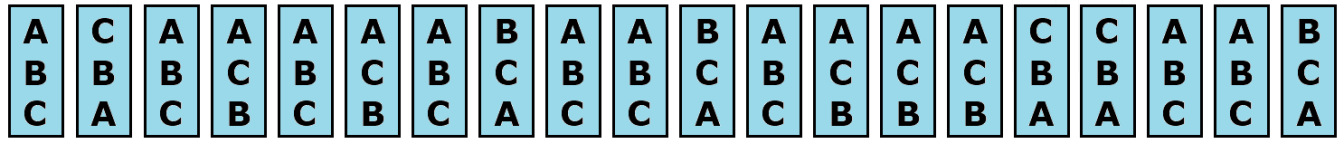
- Each column is representing a "group" of voters with identical preferences
- Reading down the column shows their ranking from favorite to least favorite
- Each voter appears in exactly one column
- The number [#] at the top of each column shows how many voters had that exact ranking
- The sum of all columns equals total number of voters

Example #7.1.1 🤔

Suppose there are candidates A, B, and C. You are asked to rank each candidate in order of preference. In this case, there are six possible ways to rank the candidates, as shown:



Now, suppose that 20 club members are to rank each candidate in order of preference. The results are as follows:

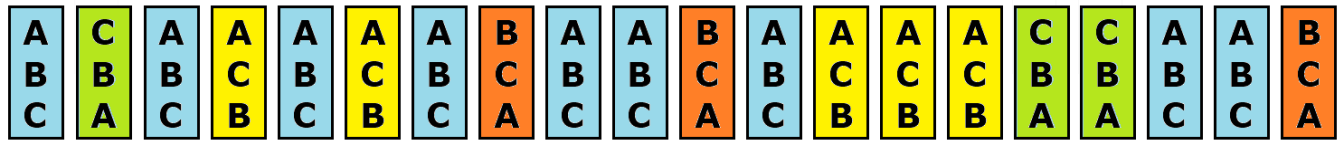


Create a preference table based on the above results.

✅ **Solution:**

To organize our ballots, we'll group together all the ones that have identical rankings. We can visualize this by assigning a color to each unique preference order:

- Blue: A > B > C (A first, B second, C third)
- Yellow: A > C > B (A first, C second, B third)
- Orange: B > C > A (B first, C second, A third)
- Green: C > B > A (C first, B second, A third)



Note: *This organization technique is not necessary required for similar problems, but helps show it is clear, visual, and educational; perfect for helping students understand the ballot-grouping concept!*

So, tallying up the votes, here are the results in a preference table:

Preference ballot showing voter rankings for candidates A, B, and C

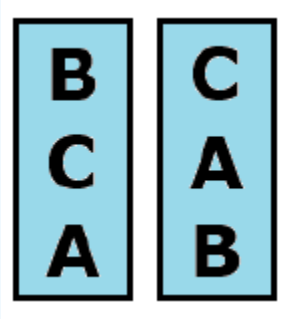
Number of Voters	9	5	3	3
First Choice	A	A	B	C
Second Choice	B	C	C	B
Third Choice	C	B	A	A

From the table:

The sum of the numbers in the top row indicates the total number of voters, 9 + 5 + 3 + 3 = 20.

- The "First Choice" row indicates:
  - There are 9 + 5 = 14 voters that picked candidate A as their first choice.
  - There are 3 voters that picked B as their first choice.
  - There are 3 voters that picked C as their first choice.
- The "Second Choice" row indicates:
  - There are 9 + 3 = 11 voters that picked candidate B as their second choice.
  - There are 5 + 3 = 8 voters that picked C as their second choice.
  - There are 0 voters that picked A as their second choice.
- The "Third Choice" row indicates:
  - There are 3 + 3 = 6 voters that picked candidate A as their third choice.
  - There are 5 voters that picked B as their third choice.
  - There are 9 voters that picked C as their third choice.

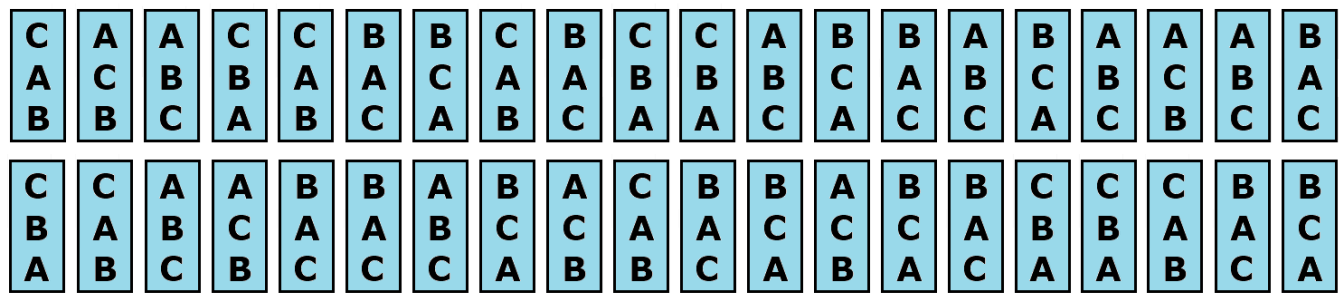
Also, note there are no voters who casts the following ranked ballots:



Thus, those possible rankings are not listed as columns (or outcomes) in the preference table above.

Example #7.1.2 🤔

A school council has 40 members and are asked to rank three candidates in order of preference. The results are as follows:



Create a preference table based on the above results.

✅ **Solution:**

So, tallying up the votes, here are the results in a preference table:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	9	7	7	6	6
First Choice	B	B	A	C	C
Second Choice	A	C	B	B	A
Third Choice	C	A	C	A	B

Example #7.1.3 🤔

Consider the following preference table for a union president from faculty at a local college:

Preference ballot showing voter rankings for candidates Amy, Don, Jen, and Rob

Number of Voters	11	10	8	6	3
First Choice	Rob	Don	Rob	Amy	Rob
Second Choice	Jen	Jen	Amy	Jen	Don
Third Choice	Don	Amy	Jen	Don	Jen
Fourth Choice	Amy	Rob	Don	Rob	Amy

a. How many total voters were there?
b. How many people voted for Rob as their first choice?
c. Which candidate has the least number of first choice votes?
d. How many people voted for Jen as their second choice?

✅ **Solution:**

a. Add the top row: 11 + 10 + 8 + 6 + 3 = 38 votes
b. Look at the first choice row and add the corresponding numbers from the top row for Rob: 11 + 8 + 3 = 22 people
c. Look at the first choice row and add the corresponding numbers from the top row for all candidates: Jen has 0 first place votes
d. Look at the second choice row and add the corresponding numbers from the top row: 11 + 10 + 6 = 27 people

Example #7.1.4 🤔

A group of fans at the John Wick Experience in Las Vegas, NV were asked to rank the four films from the John Wick Franchise. The results are in the following preference table:

Preference ballot showing voter rankings for the four John Wick movies

Number of Voters	15	8	6	
First Choice	John Wick 2	John Wick 4	John Wick 4	
Second Choice	John Wick 4	John Wick 1	John Wick 2	
Third Choice	John Wick 1	John Wick 2	John Wick 1	
Fourth Choice	John Wick 3	John Wick 3	John Wick 3	

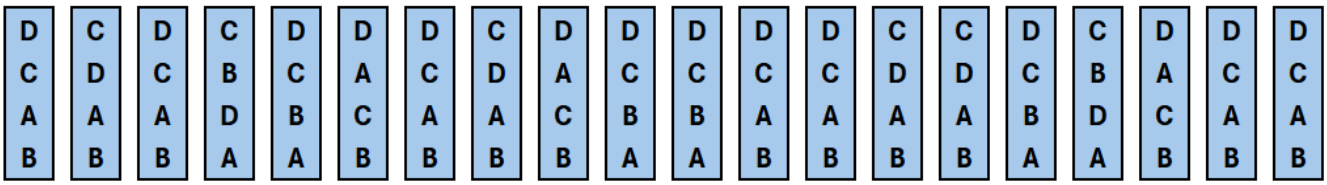
a. How many total voters were there?
b. How many people voted for John Wick 1 as their first choice?
c. Which film has the most number of first choice votes?
d. Which film has the least number of first choice votes?
e. Which film has the most number of last choice votes?
f. How many people voted for John Wick 4 as their second choice?

✅ **Solution:**

a. Add the top row: 15 + 8 + 6 + 5 + 2 = 36 votes
b. Look at the first choice row and add the corresponding numbers from the top row for John Wick 1: 5 people
c. Look at the first choice row and add the corresponding numbers from the top row for all candidates: John Wick 4 has a total of 8 + 6 + 2 = 16 votes, which is the most
d. Look at the first choice row and add the corresponding numbers from the top row for all candidates: John Wick 3 has a total of 0 votes which is the least
e. Look at the fourth choice row and add the corresponding numbers from the top row for all candidates: John Wick 3 has a total of 15 + 8 + 6 + 5 = 34 votes, which is the most
f. Look at the second choice row and add the corresponding numbers from the top row for John Wick 4: John Wick 4 has a total of 15 votes

Section 7.1: Preference Tables [*In-Class Exercises*]

1. The leader of a local Benevolent and Protective Order of Elks (BPOE) lodge is known as the Exalted Ruler. A local Elks Lodge has 20 members and are asked to rank four candidates in order of preference to be the next Exalted Ruler. The results are as follows:



Create a preference table based on the above results.

2. A group of fans after seeing a special advanced screening of Scream 7 were asked to rank the seven films from the Scream franchise. The results are in the following preference table:

Preference ballot showing voter rankings for the seven Scream movies

Number of Voters	7	5	4
First Choice	Scream 1	Scream 3	Scream 1
Second Choice	Scream 3	Scream 1	Scream 2
Third Choice	Scream 2	Scream 2	Scream 3
Fourth Choice	Scream 4	Scream 4	Scream 4
Fifth Choice	Scream 5	Scream 6	Scream 5
Sixth Choice	Scream 7	Scream 5	Scream 6
Seven Choice	Scream 6	Scream 7	Scream 7

a. How many total voters were there?
b. How many people voted for the Scream 1 as their first choice?
c. Which film has the least number of first choice votes?
d. Which film has the most number of last choice votes?
e. How many people voted for Scream 3 as their third choice?
f. Which film has the most number of second choice votes?

Answers available at the end of this section.

Answers

Section 7.1: Preference Tables [*In-Class Exercises*]

1. *Preference ballot showing voter rankings for candidates A, B, C, and D*

Number of Voters	7	4	4	3	2
First Choice	D	D	C	D	C
Second Choice	C	C	D	A	B
Third Choice	A	B	A	C	D
Fourth Choice	B	A	B	B	A

2.

a. 20
b. 11
c. Scream 4, Scream 5, Scream 6
d. Scream 6
e. 7
f. Scream 1

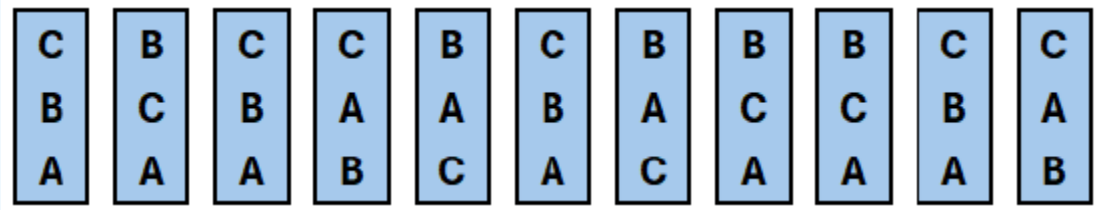
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Section 7.1.H: Homework

Section 7.1: Preference Tables - Homework Exercises

(Section 7.1) - Homework Exercise #1

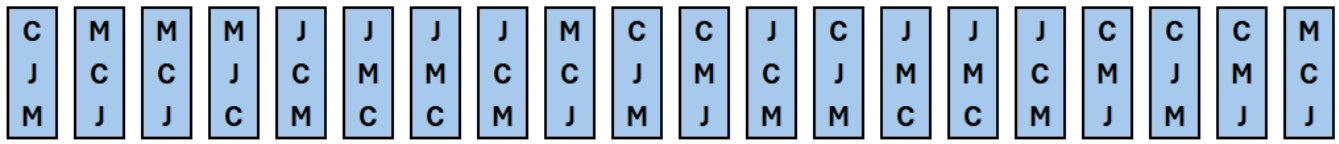
Create a preference table based on the eleven preference ballots for Candidates A, B, and C.



Answer available at the end of this section.

(Section 7.1) - Homework Exercise #2

The Los Angeles Angels polled 20 of their most tenured employees and asked them to rank their preference of a future bobble head giveaway for next season. The team wants to feature one of their most tenured players from the California Angels days that has not yet had a bobble head of them made prior. The fans were asked to rank the following players of Chuck Finley (C), Jim Fregosi (J), and Mike Witt (W). The results are below:



Answer available at the end of this section.

(Section 7.1) - Homework Exercise #3

Consider the following preference table for a union representative from math faculty at a local college:

Preference ballot showing voter rankings for six candidates

Number of Voters	12	11	10	5	
First Choice	Marguerite	Jorge	Aleksandra	Khalil	
Second Choice	Jorge	Jeffrey	Jorge	Adriana	
Third Choice	Khalil	Aleksandra	Adriana	Jorge	
Fourth Choice	Aleksandra	Adriana	Khalil	Aleksandra	
Fifth Choice	Jeffrey	Marguerite	Marguerite	Jeffrey	
Sixth Choice	Adriana	Khalil	Jeffrey	Marguerite	

a. How many total voters were there?
b. How many people voted for Aleksandra as their fourth choice?
c. Which candidate has the most number of first choice votes?
d. Which candidate has the least number of first choice votes?
e. Which candidate has the most number of last choice votes?
f. Which candidate has the least number of last choice votes?

Answer available at the end of this section.

(Section 7.1) - Homework Exercise #4

On December 19th, 2019, the final episode of the Star Wars Saga, The Rise of Skywalker, was released in theaters. An exclusive group of those that had just seen the movie were asked to rank the last three movies of the series in order of preference. The results are below:

Preference ballot showing voter rankings for the three Star Wars movies

Number of Voters	51	25	7
First Choice	The Force Awakens	The Force Awakens	The Rise of
Second Choice	The Last Jedi	The Rise of Skywalker	The Force
Third Choice	The Rise of Skywalker	The Last Jedi	The Las

a. How many total voters were there?
b. How many people listed The Rise of Skywalker as their first choice?
c. How many people listed The Force Awakens and The Rise of Skywalker in their top two?
d. Which film has the least number of first choice votes?
e. Which film has the most number of last choice votes?
f. There is one possible ranking not listed. What ranking was not listed?

Answer available at the end of this section.

(Section 7.1) - Homework Exercise #5

A PTA (Parent Teacher Association) is a formal organization of parents, teachers, and staff aimed at improving the educational experience, health, and well-being of children. There are 40 members and they are asked to rank three candidates in order of preference for the future PTA President opening. The results are as follows:

A B C	C B A	A C B	A B C	B C A	B A C	C B A	B C A	C B A	B C A	A C B	B A C	B A C	C B A	B C A	A C B	A C B	B A C	C A B	B A C
B C A	B A C	C B A	B A C	C B A	B C A	C B A	B C A	B A C	A C B	C B A	C B A	A B C	B A C	A B C	B C A	A C B	A C B	C B A	B C A

Create a preference table based on the above results.

Answer available at the end of this section.

Answers

(Section 7.1) - Homework Exercise #1

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	4	3	2	2
First Choice	C	B	C	B
Second Choice	B	C	A	A
Third Choice	A	A	B	C

(Section 7.1) - Homework Exercise #2

Preference ballot showing voter rankings for candidates C, J, and M

Number of Voters	4	4	4	4	3	1
First Choice	J	J	C	M	C	M
Second Choice	C	M	J	C	M	J
Third Choice	M	C	M	J	J	C

(Section 7.1) - Homework Exercise #3

a. 41
b. 17
c. Jorge (13)
d. Jeffrey (0)
e. Adriana and Jeffrey (12 each)
f. Jorge (0)

(Section 7.1) - Homework Exercise #4

a. 86
b. 8
c. 32
d. The Last Jedi (2)
e. The Rise of Skywalker (53)
f. In order: The Last Jedi, The Rise of Skywalker, The Force Awakens (0 each)

(Section 7.1) - Homework Exercise #5

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	10	9	9	7	4	1
First Choice	C	B	B	A	A	C
Second Choice	B	A	C	C	B	A
Third Choice	A	C	A	B	C	B

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Section 7.2: Plurality Method

Learning Objectives

- Determine the winner of an election using the plurality method

As stated in the previous section, when an election includes more than two candidates, different methods can be used to determine a winner. These methods are called **voting methods**, and each one uses a distinct rule for counting votes and declaring a result. The choice of voting method can significantly affect the outcome of an election. Here is the first of the four major voting methods.

Major Voting Method #1 - The Plurality Method

The plurality method (also called first-past-the-post or simple plurality) is the most common and straightforward voting system used around the world. It's likely the method you're most familiar with because it's used in most U.S. elections.

Each voter ranks the candidates. The candidate who receives the most first place votes wins even if they don't receive a majority (more than 50%) of the votes.

Example #7.2.1 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	11	10	8	6
First Choice	A	D	B	D
Second Choice	C	B	A	C
Third Choice	D	C	C	B
Fourth Choice	B	A	D	A

Use the plurality method to determine a winner.

✅ **Solution:**

Looking at the First Choice row,

Candidate A = 11 first place votes
Candidate B = 8 first place votes
Candidate C = 0 first place votes
Candidate D = 10 + 6 = 16 first place votes

Thus, Candidate D is the winner by the plurality method.

Example #7.2.2 🤔

A survey of 23 customers was conducted asking them to rank which protein they prefer at a local Chipotle restaurant. The results are listed below:

Preference ballot showing voter rankings for candidates Barbacoa, Carnitas, Chicken, and Steak

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas	Chicken	Steak
Second Choice	Chicken	Barbacoa	Carnitas	Barbacoa
Third Choice	Barbacoa	Chicken	Steak	Chicken
Fourth Choice	Carnitas	Steak	Barbacoa	Carnitas

Use the plurality method to determine which protein is the winner.

✅ **Solution:**

Looking at the First Choice row,

Steak = 8 + 3 = 11 first place votes
Carnitas = 7 first place votes
Chicken = 5 first place votes
Barbacoa = 0 first place votes

Thus, Steak is the winner by the plurality method.

Example #7.2.3 🤔

City council members need to choose a new location for the city's community center. Four sites are being considered:

Preference ballot showing voter rankings for Downtown, Eastside, Northside, and Westside

Number of Voters	35	30	20	15
First Choice	Downtown	Northside	Eastside	Westside
Second Choice	Northside	Eastside	Westside	Eastside
Third Choice	Eastside	Westside	Northside	Northside
Fourth Choice	Westside	Downtown	Downtown	Downtown

a. Use the plurality method to determine which new location is the winner.
b. How many voters put Westside and Eastside in their top two?
c. How many people voted?

✅ **Solution:**

a. Looking at the First Choice row,
   - Downtown = 35 first place votes
   - Northside = 30 first place votes
   - Eastside = 20 first place votes
   - Westside = 15 first place votes

Thus, Downtown is the winner by the plurality method.

b. Looking at the First Choice and Second Choice rows: 20 + 15 = 35
c. Looking at the top row: 35 + 30 + 20 + 15 = 100

Section 7.2: The Plurality Method [*In-Class Exercises*]

1. Consider the following preference table with Candidates A, B, & C:

   Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	7	6	6	5	3
First Choice	C	B	A	B	A
Second Choice	A	A	B	C	C
Third Choice	B	C	C	A	B

   Use the plurality method to determine a winner.

2. A group of 41 people is choosing where to eat lunch during their meeting. The four fast food delivery options are: Board & Brew, Naugles, T.K. Burgers, or In-N-Out Burgers. Each voter ranks the restaurants from most preferred to least preferred.

   Preference ballot showing voter rankings for candidates Board & Brew, In-N-Out, Naugles, and T.K. Burgers

Number of Voters	11	9	8	6	4	3
First Choice	In-N-Out	T.K. Burgers	Naugles	Board & Brew	T.K.Burgers	In-N-Out
Second Choice	Naugles	Naugles	T.K.Burgers	T.K. Burgers	In-N-Out	T.K. Burgers
Third Choice	T.K. Burgers	Board & Brew	In-N-Out	In-N-Out	Naugles	Board & Brew
Fourth Choice	Board & Brew	In-N-Out	Board & Brew	Naugles	Board & Brew	Naugles

   Use the plurality method to determine a winner.

Answers available at the end of this section.

Answers

Section 7.2: The Plurality Method [*In-Class Exercises*]

1. Candidate B (11 votes)
2. In-N-Out (14 votes)

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Section 7.2.H: Homework

Section 7.2: Plurality Method - Homework Exercises

(Section 7.2) - Homework Exercise #1

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	4	3	2	2
First Choice	A	B	C	C
Second Choice	B	C	B	A
Third Choice	C	A	A	B

Use the plurality method to determine a winner.

Answer available at the end of this section.

(Section 7.2) - Homework Exercise #2

Consider the following preference table with Candidates A, B, C, & D:

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	9	8	6	4	3
First Choice	D	C	A	A	C
Second Choice	B	B	C	C	D
Third Choice	A	A	A	B	A
Fourth Choice	C	D	B	A	B

Use the plurality method to determine a winner.

Answer available at the end of this section.

(Section 7.2) - Homework Exercise #3

A group of 30 office coworkers are choosing where to eat lunch during their meeting. The restaurant options are: The Habit, Chick-Fil-A, and Panera Bread. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Chick-Fil-A, The Habit, and Panera Bread

Number of Voters	11	9	6	4
First Choice	The Habit	Chick-Fil-A	Panera Bread	Chick-Fil-A
Second Choice	Panera Bread	Panera Bread	Chick-Fil-A	The Habit
Third Choice	Chick-Fil-A	The Habit	The Habit	Panera Bread

Use the plurality method to determine a winner.

Answer available at the end of this section.

(Section 7.2) - Homework Exercise #4

A class of 37 students is voting on a location for their end-of-year field trip. The four options are the Aquarium of the Pacific (A), the Butterfly Release at the Botanical Gardens (B), Disneyland (D), or the Griffith Observatory (G). The results are in the preference table below.

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	14	10	8	4	1
First Choice	A	B	D	D	B
Second Choice	G	G	B	G	D
Third Choice	B	D	A	B	G
Fourth Choice	D	A	G	A	A

Use the plurality method to determine a winner.

Answer available at the end of this section.

(Section 7.2) - Homework Exercise #5

A Los Angeles county Buffalo Wild Wings polled their customers and asked them to rank the local L.A. professional sports teams. Here are the results listed in a preference table.

Preference ballot showing voter rankings for candidates Chargers, Clippers, Dodgers, Kings, Lakers, and Rams

Number of Voters	15	13	9	9	4	3	3	1
First Choice	Dodgers	Lakers	Dodgers	Lakers	Rams	Lakers	Rams	Clippers
Second Choice	Lakers	Dodgers	Rams	Dodgers	Chargers	Rams	Chargers	Chargers
Third Choice	Rams	Kings	Lakers	Kings	Kings	Dodgers	Lakers	Kings
Fourth Choice	Kings	Rams	Clippers	Clippers	Dodgers	Kings	Clippers	Dodgers
Fifth Choice	Clippers	Chargers	Kings	Rams	Clippers	Chargers	Kings	Rams
Sixth Choice	Chargers	Clippers	Chargers	Chargers	Lakers	Clippers	Dodgers	Lakers

Use the plurality method to determine a winner.

Answer available at the end of this section.

Answers

(Section 7.2) - Homework Exercise #1
Tie between Candidate A and Candidate C (4 votes each).

(Section 7.2) - Homework Exercise #2
Candidate C (11 votes).

(Section 7.2) - Homework Exercise #3
Chick-Fil-A (13 votes).

(Section 7.2) - Homework Exercise #4
Aquarium of the Pacific (14 votes).

(Section 7.2) - Homework Exercise #5
Lakers (25 votes).

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Section 7.3: Condorcet Criterion

Learning Objectives

- Decide if an election violates the Condorcet criterion

The plurality method has been used in democratic elections for centuries because it is simple, easy to understand, and easy to administer. Despite these advantages, the method has several limitations that can produce outcomes that some voters may view as unfair.

A candidate can win without receiving a majority of the votes, especially when three or more candidates are running. The method is also vulnerable to vote splitting, where similar candidates divide support and unintentionally help another candidate win. In addition, voters may choose to vote strategically rather than for their true favorite candidate. Because the plurality method records only first-choice votes, it provides limited information about voter preferences. Finally, it can fail to elect the Condorcet winner, which is a candidate who would defeat every other candidate in a head-to-head comparison. This last issue is one that we will examine more closely in this section.

To evaluate voting methods, mathematicians use fairness criteria, which are standards used to determine whether a voting system produces reasonable and representative outcomes. Just as a thermometer or a scale can be evaluated based on its accuracy and reliability, a voting method can be evaluated according to the fairness principles it satisfies.

Fairness Criteria

A **fairness criterion** is a precise statement of a specific behavior that we expect to happen if an election is to be considered fair. (Note that “criterion” is the singular form, and “criteria” is the plural form.)

The Four Main Fairness Criteria

We'll examine the four most important and intuitive fairness criteria:

- Condorcet Criterion
- Majority Criterion
- Monotonicity Criterion
- Independence of Irrelevant Alternatives Criterion (IIA)

Fairness criteria are tests we use to evaluate whether a voting method produces fair outcomes.

The Process for Fairness

1. **Apply a voting method**: (Plurality, Borda Count, Plurality with Elimination, or Pairwise Comparison) to determine a winner
2. **Test the fairness criteria**: Check whether the winner also satisfies each of the four fairness criteria:
    - Condorcet Criterion
    - Majority Criterion
    - Monotonicity Criterion

- Independence of Irrelevant Alternatives Criterion (IIA)

3. **Evaluate the result**:
    1. **No violation**: If the voting method's winner satisfies the criterion
    2. **Violation**: If the criterion identifies a different candidate who "should" have won

The Goal: Find a voting method that satisfies all four fairness criteria for every possible election scenario.

Condorcet Criterion (Fairness Criterion #1)

If a candidate would defeat every other candidate in a head-to-head (one-on-one) comparison, called the Condorcet Criterion, that candidate should win the election. Such a candidate is called a Condorcet winner.

Satisfying Fairness Criteria

- When we can show that a voting method is guaranteed to fulfill a given fairness criterion in any possible election, we say that the method satisfies the criterion and there is no violation.
- If a voting method does not satisfy a given fairness criterion, even if it fails in only one election, then we say that the method violates the given criterion.

Example #7.3.1 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	5	4	3
First Choice	A	B	C
Second Choice	B	A	A
Third Choice	C	C	B

a. Use the plurality method to determine a winner.
b. Does the election violate the Condorcet criterion?

✅ **Solution:**

a. Looking at the First Choice row, Candidate A has 5 first place votes, Candidate B has 4 first place votes, and Candidate C has 3 first place votes. **Thus, Candidate A is the winner by the plurality method.**
b. We need to check the one-on-one scores for each possible pair of candidates. The match ups are A vs. B, A vs. C, and B vs. C.

Use a table for each match-up to record your results.

A	B
5	
	4
3	
8	**4**

So, for A vs. B,

- Who ranks higher in the first column?: A does, so 5 votes for A.
- Who ranks higher in the second column?: B does, so 4 votes for B.
- Who ranks higher in the third column?: A does, so 3 votes for A.
- Result: A beats B, 8-4

A	C
5	
4	
	3
9	**3**

So, for A vs. C,

- Who ranks higher in the first column?: A does, so 5 votes for A.
- Who ranks higher in the second column?: A does, so 4 votes for A.
- Who ranks higher in the third column?: C does, so 3 votes for C.
- Result: A beats C, 9-3

B	C
5	
4	
	3
9	**3**

So, for B vs. C,

- Who ranks higher in the first column?: B does, so 5 votes for B.
- Who ranks higher in the second column?: B does, so 4 votes for B.
- Who ranks higher in the third column?: C does, so 3 votes for C.
- Result: B beats C, 9-3

It would be simpler to summarize everything all at once in a table with all of the head-to-head match-ups from above.

A	B
5	
	4
3	
8	**4**
A wins B	

A	C
5	
4	
	3
9	**3**
A wins C	

B	C
5	
4	
	3
9	**3**
B wins C	

Here are the head-to-head records:

A: 2 wins (beat B and C)
B: 1 win (beat C only)
C: 0 wins

Thus, Candidate A beats everyone head-to-head. Candidate A is the Condorcet winner.

Since, Candidate A wins by the plurality method and is also the Condorcet winner, there is not a violation of the Condorcet criterion. (**NO**)

Note: Technically, when using the Condorcet criterion, since Candidate A beat Candidate B and then Candidate A beat Candidate C, we technically do not need to show the last head-to-head match-up, since Candidate A already won the election.

Example #7.3.2 🤔

Consider the following preference table with Candidates X, Y, & Z:

Preference ballot showing voter rankings for candidates X, Y, and Z

Number of Voters	40	35	25
First Choice	X	Z	Y
Second Choice	Z	Y	Z
Third Choice	Y	X	X

a. Use the plurality method to determine a winner.
b. Does the election violate the Condorcet criterion?

✅ **Solution:**

a. Looking at the First Choice row, Candidate X has 40 first place votes, Candidate Y has 35 first place votes, and Candidate Z has 25 first place votes. **Thus, Candidate X is the winner by the plurality method.**
b. We need to check the one-on-one scores for each possible pair of candidates. The match ups are X vs. Y, X vs. Z, and Y vs. Z.

Use a table for each match-up to record your results.

X	Y
40	
	35
	25
40	**60**
Y wins X	

X	Z
40	
	35
	25
40	**60**
Z wins X	

Y	Z
	40
	35
25	
25	**75**
Z wins Y	

Here are the head-to-head records:

X: 0 wins
Y: 1 win (beat X only)
Z: 2 wins (beat X and Y)

Thus, Candidate Z beats everyone head-to-head. Candidate Z is the Condorcet winner.

Since, Candidate X wins by the plurality method, yet Candidate Z is the Condorcet winner, there is a violation of the Condorcet criterion. (**YES**)

Example #7.3.3 🤔

A survey of 23 customers was conducted asking them to rank which protein they prefer at a local Chipotle restaurant. The results are listed below:

Preference ballot showing voter rankings for candidates Barbacoa, Carnitas, Chicken, and Steak

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas	Chicken	Steak

Number of Voters	8	7	5	3
Second Choice	Chicken	Barbacoa	Carnitas	Barbacoa
Third Choice	Barbacoa	Chicken	Steak	Chicken
Fourth Choice	Carnitas	Steak	Barbacoa	Carnitas

a. Use the plurality method to determine which protein is the winner.
b. Does the election violate the Condorcet criterion?

✅ **Solution:**

a. Looking at the First Choice row, Steak has 8 + 3 = 11 first place votes, Carnitas has 7 first place votes, Chicken has 5 first place votes, and Barbacoa has 0 first place votes. **Thus, Steak is the winner by the plurality method.**
b. We need to check the one-on-one scores for each possible pair of candidates until a winner if possible exists. We do not necessarily need to show ALL pairings. If we find a winner first, you may not need to show all head-to-head matchups. Notice here we will start witch Steak first, since they were the winner using the plurality method. This strategy sometimes reduces the number of head-to-head matchups when determining if there was a violation.

Summarizing the results from the preference table in another table with some to all of the head-to-head match-ups from above, we have

Steak	Carnitas
8	
	7
	5
3	
11	**12**
Carnitas wins Steak	

Carnitas	Chicken
	8
7	
	5
	3
7	**16**
Chicken wins Carnitas	

Chicken	Barbacoa
8	
	7
5	
	3
13	**10**
Chicken wins Barbacoa	

Chicken	Steak
	8
7	
5	
	3
12	**11**
Chicken wins Steak	

- In the first match-up, Carnitas vs. Steak, Carnitas wins. So, in the next match-up, use Carnitas again against another protein.
- In the second match-up, Carnitas vs. Chicken, Chicken wins. So, in the next match-up, use Chicken again against another protein.
- In the third match-up, Chicken vs. Barbacoa, Chicken wins again. So, in the next match-up, use Chicken again against the final protein.
- In the fourth match-up, Chicken vs. Steak, Chicken wins again. We do not need to show any more match-ups, since we have a winner. We don't need all of the head-to-head records. If possible, once a winner has been obtained, there is no need to show the rest of the other match-ups.

Thus, Chicken beats every other protein head-to-head. Chicken is the Condorcet winner.

Since, Steak wins by the plurality method, yet Chicken is the Condorcet winner, there is a violation of the Condorcet criterion. (**YES**)

Example #7.3.4 🤔

City council members need to choose a new location for the city's community center. Four sites are being considered:

Preference ballot showing voter rankings for Downtown, Eastside, Northside, and Westside

Number of Voters	35	30	20	15
First Choice	Downtown	Northside	Eastside	Westside
Second Choice	Northside	Eastside	Westside	Eastside
Third Choice	Eastside	Westside	Northside	Northside
Fourth Choice	Westside	Downtown	Downtown	Downtown

a. Use the plurality method to determine which new location is the winner.
b. Does the election violate the Condorcet criterion?

✅ **Solution:**

a. Looking at the First Choice row, Downtown has 35 first place votes, Northside has 30 first place votes, Eastside has 20 first place votes, and Westside has 15 first place votes. **Thus, Downtown is the winner by the plurality method.**
b. We need to check the one-on-one scores for each possible pair of candidates until a winner if possible exists. We do not necessarily need to show ALL pairings. If we find a winner first, you may not need to show all head-to-head matchups. Notice here we will start witch Downtown first, since they were the winner using the plurality method. This strategy sometimes reduces the number of head-to-head matchups when determining if there was a violation.

Summarizing the results from the preference table in another table with some to all of the head-to-head match-ups from above, we have

Downtown	Northside	Northside	Eastside	Northside	Westside
35		35		35	
	30	30		30	
	20		20		20
	15		15		15
35	**65**	**65**	**35**	**65**	**35**

Northside wins Downtown Northside wins Eastside Northside wins Westside

Downtown	Northside
35	
	30
	20
	15
35	**65**

Northside wins Downtown

Northside	Eastside
35	
30	
	20
	15
65	**35**

Northside wins Eastside

Northside	Westside
35	
30	
	20
	15
65	**35**

Northside wins Westside

- In the first match-up, Downtown vs. Northside, Northside wins. So, in the next match-up, use Northside again against another location.
- In the second match-up, Northside vs. Eastside, Northside wins again. So, in the next match-up, use Northside again against the final location.
- In the third match-up, Northside vs. Westside, Northside wins again. We do not need to show any more match-ups, since we have a winner. We don't need all of the head-to-head records. If possible, once a winner has been obtained, there is no need to show the rest of the other match-ups.

Thus, Northside beats every other site head-to-head. Northside is the Condorcet winner.

Since, Downtown wins by the plurality method, yet Northside is the Condorcet winner, there is a violation of the Condorcet criterion. (**YES**)

Example #7.3.5 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	7	5	2	2
First Choice	C	B	A	C
Second Choice	B	C	B	A
Third Choice	A	A	C	B

a. Use the plurality method to determine which new location is the winner.
b. Does the election violate the Condorcet criterion?

✅ **Solution:**

a. Looking at the First Choice row, Candidate A has 2 first place votes, Candidate B has 5 first place votes, and Candidate C has 7 + 2 = 9 first place votes. **Thus, C is the winner by the plurality method.**
b. We need to check the one-on-one scores for each possible pair of candidates until a winner if possible exists. We do not necessarily need to show ALL pairings. If we find a winner first, you may not need to show all head-to-head matchups. Notice here we will start witch Candidate C first, since they were the winner using the plurality method. This strategy sometimes reduces the number of head-to-head matchups when determining if there was a violation.

Summarizing the results from the preference table in another table with some to all of the head-to-head match-ups from above, we have

C	A	C	B
7		7	
5			5
	2		2
1		1	
2	**13**	**8**	**7**

C wins A C wins B

- In the first match-up, C vs. A, C wins. So, in the next match-up, use C again another candidate.
- In the second match-up, C vs. B, C wins again. We do not need to show A vs. B, since we have a winner. We don't need all of the head-to-head records. If possible, once a winner has been obtained, there is no need to show the rest of the other match-ups.

Thus, Candidate C beats everyone head-to-head. Candidate C is the winner by the Condorcet criterion.

Since, Candidate C wins by the plurality method and is also the Condorcet winner, there is not a violation of the Condorcet criterion. (**NO**)

Example #7.3.6 🤔

A survey of 13 students was conducted to determine how they ranked the Astronomy, Biology, and Chemistry labs from most to least preferred.

Preference ballot showing voter rankings for candidates Astronomy, Biology, and Chemistry

Number of Voters	6	5	2
First Choice	Astronomy	Biology	Chemistry
Second Choice	Chemistry	Astronomy	Biology
Third Choice	Biology	Chemistry	Astronomy

a. Use the plurality method to determine which new location is the winner.
b. Does the election violate the Condorcet criterion?

✅ **Solution:**

a. Looking at the First Choice row, Astronomy has 6 first place votes, Biology has 5 first place votes, and Chemistry has 2 first place votes. **Thus, Astronomy is the winner by the plurality method.**
b. We need to check the one-on-one scores for each possible pair of candidates until a winner if possible exists. We do not necessarily need to show ALL pairings. If we find a winner first, you may not need to show all head-to-head matchups. Notice here we will start witch Astronomy first, since they were the winner using the plurality method. This strategy sometimes reduces the number of head-to-head matchups when determining if there was a violation.

Summarizing the results from the preference table in another table with some to all of the head-to-head match-ups from above, we have

Astronomy	Biology	Astronomy	Chemistry	Biology	Chemistry
6		6			6
	5	5		5	
	2		2		2
6	**7**	**11**	**2**	**5**	**8**

Biology wins Astronomy Astronomy wins Chemistry Chemistry wins Biology

- In the first match-up, Astronomy vs. Biology, Biology wins. So, in the next match-up, use Biology and again the other subject.
- In the second match-up, Biology vs. Chemistry, Chemistry wins. So, in the next match-up, use Chemistry again against the Astronomy.
- In the third and final match-up, Astronomy vs. Chemistry, Astronomy wins.

Thus, there is no winner by the Condorcet criterion, because no one particular subject won out the other two.

Since, Astronomy wins by the plurality method, yet there is no Condorcet winner, there is a violation of the Condorcet criterion. (**YES**)

Does the plurality method violate the Condorcet criterion?

YES. The plurality method violates the majority criterion, because a candidate who wins all head-to-head matchups may not receive the most first-place votes. (As we saw in Examples #7.5.1a & 7.5.4).

Even if a voting method violates a fairness criterion in a particular example, this does not mean the method always fails that criterion. Rather, it shows that the method has the potential to violate the criterion under certain election scenarios. Therefore, in general, we say a method violates a criterion if there exists at least one case in which the criterion is not satisfied.

With the examples above, we saw a case where the Condorcet criterion satisfied the plurality method as well as a case where the Condorcet criterion violated the plurality method. So, this means that the plurality method is flawed and is not considered a fair

method. In the next two sections, we will look at three more major voting methods along with three additional fairness criterion, in hopes to find a voting method that is flawless.

A voting method is considered completely fair or flawless if and only if it satisfies all reasonable fairness criteria under all possible election scenarios.

Requirements for a "fair" method:

- ✓ Never violates the Condorcet criterion
- ✓ Never violates the Majority criterion
- ✓ Never violates the Monotonicity criterion
- ✓ Remains independent of irrelevant alternatives
- ✓ Never exhibits vulnerability to strategic manipulation
- ✓ Treats all voters and candidates symmetrically
- ✓ Produces consistent, predictable outcomes

In other words: A fair method must have zero structural flaws; no scenarios where it produces arguably unjust outcomes.

So, here is our initial table that with regards to the four fairness criteria:

Four Fairness Criteria Table (1/16 Complete)

Fairness Criterion → Major Voting Method ↓	Condorcet Criterion	Majority Criterion	Monotonicity Criterion	Independence of Irrelevant Alternatives Criterion
Plurality	**Violation Possible**	**?**	**?**	**?**
Borda Count	**?**	**?**	**?**	**?**
Plurality-with-Elimination	**?**	**?**	**?**	**?**
Pairwise Comparison	**?**	**?**	**?**	**?**

The table summarizes the four major voting methods. When a ✔ is present this means that the fairness criteria are always satisfied by the voting methods. If there is at least one case for a violation using a fairness criterion with a voting method, that means that particular voting method is not always fair and marked "Violation Possible". From the five examples above, there were at least three cases where there was a violation when using the plurality method, thus we place "Violation Possible" instead of a ✔.

For the next six sections, we will introduce the rest of the major voting methods as well as the rest of the fairness criterions and fill in the rest of the table above with either a ✔ or a "Violation Possible" in each cell. We are looking for a major voting method that has ✔'s across all four fairness criterion's and unfortunately, we have determined that the plurality method will not satisfy at least one of them.

Historical Note: Marquis de Condorcet

The Condorcet criterion is named after Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet (1743–1794), a French



philosopher, mathematician, and political thinker from the Enlightenment era.

Condorcet is best known today for his work on voting theory. He formulated what is now called the Condorcet method of voting, based on pairwise (head-to-head) comparisons between candidates as we learned in this section. This was one of the earliest demonstrations that democratic decision-making can have logical problems.

Condorcet was also a close associate of leading Enlightenment figures such as Voltaire and d'Alembert and was actively involved in the French Revolution, particularly in debates over democracy and education. These experiences shaped his strong advocacy for democratic government, human rights, and freedom of expression. Unusually for the eighteenth century, he supported equal political and civil rights for women, openly opposed slavery, and believed in universal public education grounded in reason and scientific knowledge.

Section 7.3: Condorcet Criterion [*In-Class Exercises*]

1. Consider the following preference table with Candidates X, Y, & Z:

   Preference ballot showing voter rankings for candidates X, Y, and Z

Number of Voters	5	4	2	1
First Choice	X	Y	X	Z
Second Choice	Y	X	Z	Y
Third Choice	Z	Z	Y	X

   a. Use the plurality method to determine the winner.
   b. Does the election violate the Condorcet criterion?
2. Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	7	6	6	5	3
First Choice	C	B	A	B	A
Second Choice	A	A	B	C	C
Third Choice	B	C	C	A	B

a. Use the plurality method to determine the winner.
b. Does the election violate the Condorcet criterion?

3. A group of 41 people is choosing where to eat lunch during their meeting. The four fast food delivery options are: Board & Brew, Naugles, T.K. Burgers, or In-N-Out Burgers. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Board & Brew, In-N-Out, Naugles, and T.K. Burgers

Number of Voters	11	9	8	6	4	3
First Choice	In-N-Out	T.K. Burgers	Naugles	Board & Brew	T.K.Burgers	In-N-Out
Second Choice	Naugles	Naugles	T.K. Burgers	T.K. Burgers	In-N-Out	T.K. Burgers
Third Choice	T.K. Burgers	Board & Brew	In-N-Out	In-N-Out	Naugles	Board & Brew
Fourth Choice	Board & Brew	In-N-Out	Board & Brew	Naugles	Board & Brew	Naugles

a. Use the plurality method to determine which restaurant is the winner.
b. Does the election violate the Condorcet criterion?

Answers available at the end of this section.

Answers

Section 7.3: Condorcet Criterion [*In-Class Exercises*]

1. a. Candidate X (7 votes)
   b. Candidate X wins by the plurality method and is also the Condorcet winner. Thus, there is no violation of the Condorcet criterion. **(NO)**
2. a. Candidate B (11 votes).
   b. Candidate B wins by the plurality method, yet there is no Condorcet winner. Thus, the method violates the Condorcet criterion. **(YES)**
3. a. In-N-Out (14 votes)
   b. In-N-Out wins by the plurality method, yet T.K. Burgers is the Condorcet winner. Thus, the method violates the Condorcet criterion. **(YES)**

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Section 7.3.H: Homework

Section 7.3: Condorcet Criterion - Homework Exercises

(Section 7.3) - Homework Exercise #1

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	6	5	2	1
First Choice	C	A	C	B
Second Choice	B	C	A	A
Third Choice	A	B	B	C

a. Use the plurality method to determine the winner.
b. Does the election violate the Condorcet criterion?

Answer available at the end of this section.

(Section 7.3) - Homework Exercise #2

Consider the following preference table with Candidates A, B, C, & D:

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	9	8	6	4	3
First Choice	D	C	A	A	C
Second Choice	B	B	C	C	D
Third Choice	A	A	D	B	A
Fourth Choice	C	D	B	D	B

a. Use the plurality method to determine the winner.
b. Does the election violate the Condorcet criterion?

Answer available at the end of this section.

(Section 7.3) - Homework Exercise #3

A group of 30 office coworkers are choosing where to eat lunch during their meeting. The restaurant options are: The Habit, Chick-Fil-A, and Panera Bread. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Chick-Fil-A, The Habit, and Panera Bread

Number of Voters	11	9	6	4
First Choice	The Habit	Chick-Fil-A	Panera Bread	Chick-Fil-A
Second Choice	Panera Bread	Panera Bread	Chick-Fil-A	The Habit
Third Choice	Chick-Fil-A	The Habit	The Habit	Panera Bread

a. Use the plurality method to determine the winner.
b. Does the election violate the Condorcet criterion?

Answer available at the end of this section.

(Section 7.3) - Homework Exercise #4

A geology professor surveyed his class of 19 students to determine which National Park in Utah they would prefer for a field study. The 5 National Parks in Utah, are Arches (A), Bryce Canyon (BC), Canyonlands (C), Capitol Reef (CR), and Zion (Z). The results are in the preference table below.

Preference ballot showing voter rankings for candidates Arches. Bryce Canyon, Canyonlands, Capitol Reef, and Zion

Number of Voters	4	4	3	2	2	1	1	1	1
First Choice	A	A	Z	B	A	Z	CR	C	B
Second Choice	Z	B	A	CR	Z	C	A	CR	C
Third Choice	B	Z	CR	Z	CR	A	C	Z	CR
Fourth Choice	C	C	B	A	C	CR	B	A	Z
Fifth Choice	CR	CR	C	C	B	B	Z	B	A

a. Use the plurality method to determine the winner.
b. Does the election violate the Condorcet criterion?

Answer available at the end of this section.

(Section 7.3) - Homework Exercise #5

A Los Angeles county Buffalo Wild Wings polled their customers and asked them to rank the local L.A. professional sports teams. Here are the results listed in a preference table.

Preference ballot showing voter rankings for candidates Chargers, Clippers, Dodgers, Kings, Lakers, and Rams

Number of Voters	15	13	9	9	4	3	3	1
First Choice	Dodgers	Lakers	Dodgers	Lakers	Rams	Lakers	Rams	Clippers
Second Choice	Lakers	Dodgers	Rams	Dodgers	Chargers	Rams	Chargers	Chargers
Third Choice	Rams	Kings	Lakers	Kings	Kings	Dodgers	Lakers	Kings
Fourth Choice	Kings	Rams	Clippers	Clippers	Dodgers	Kings	Clippers	Dodgers
Fifth Choice	Clippers	Chargers	Kings	Rams	Clippers	Chargers	Kings	Rams
Sixth Choice	Chargers	Clippers	Chargers	Chargers	Lakers	Clippers	Dodgers	Lakers

a. Use the plurality method to determine the winner.
b. Does the election violate the Condorcet criterion?

Answer available at the end of this section.

Answers

(Section 7.3) - Homework Exercise #1

a. Candidate C (7 votes).
b. Candidate C wins by the plurality method and is also the Condorcet winner. Thus, there is no violation of the Condorcet criterion.

(Section 7.3) - Homework Exercise #2

a. Candidate C (11 votes).
b. Candidate C wins by the plurality method, yet there is no Condorcet winner. Thus, the method violates the Condorcet criterion.

(Section 7.3) - Homework Exercise #3

a. Chick-Fil-A (13 votes).
b. Chick-Fil-A wins by the plurality method, yet there is no Condorcet winner. Thus, the method violates the Condorcet criterion.

(Section 7.3) - Homework Exercise #4

a. Arches (10 votes).
b. Arches wins by the plurality method and is also the Condorcet winner. Thus, there is no violation of the Condorcet criterion.

(Section 7.3) - Homework Exercise #5

a. Lakers (25 votes).
b. Lakers wins by the plurality method, yet Dodgers is the Condorcet winner. Thus, the method violates the Condorcet criterion

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Section 7.4: Borda Count Method

Learning Objectives

- Determine the winner of an election using the Borda count method

In Section 7.2, when the plurality method was used, it was simple. Count first-choice votes and whoever gets the most, wins. But, the major flaw was none of the second choices, third choices, etc., played any role in determining the winner.

Enter the Borda Count Method (named after French mathematician Jean-Charles de Borda, 1733-1799).

Major Voting Method #2 - Borda Count Method

The **Borda Count** is a voting method where voters rank candidates in order of preference, and points are awarded based on ranking position. The candidate with the most total points wins.

- **Step 1**: From the preference table, assign points based on position. With n candidates, a candidate receives:
  - (n) points for each 1st-place ranking
  - (n – 1) points for each 2nd-place ranking
  - (n – 2) points for each 3rd-place ranking
  - ... and so on ...
  - 1 point for each last-place ranking
- **Step 2**: Add up all the points each candidate receives
- **Step 3**: The candidate with the highest total wins

Point Assignment Examples

With 3 candidates (n = 3):

- 1st place = 3 points
- 2nd place = 2 points
- 3rd place = 1 points

With 4 candidates (n = 4):

- 1st place = 4 points
- 2nd place = 3 points
- 3rd place = 2 points
- 4th place = 1 points

With 5 candidates (n = 5):

- 1st place = 5 points
- 2nd place = 4 points
- 3rd place = 3 points
- 4th place = 2 points
- 5th place = 1 points

> And so on and so on.....

Example #7.4.1 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	7	5	2
First Choice	B	C	A
Second Choice	C	A	B
Third Choice	A	B	C

Use the Borda count method to determine a winner.

✅ **Solution:**

- **Step 1**: Assign the points based on position. Since there are 3 candidates, a candidate receives:
  - 3 points for each 1st-place ranking
  - 2 points for each 2nd-place ranking
  - 1 point for each 3rd-place ranking

- **Step 2**: Add up all the points each candidate receives.
  - Candidate A's Rankings:
    - **1st place rankings**: Column 3 → 2 votes × 3 points = **6 points**
    - **2nd place rankings**: Column 2 → 5 votes × 2 points = **10 points**
    - **3rd place rankings**: Column 1 → 7 votes × 1 point = **7 points**
      - Candidate A's total: **6 + 10 + 7 = 23 points**
  - Candidate B's Rankings:
    - **1st place rankings**: Column 1 → 7 votes × 3 points = **21 points**
    - **2nd place rankings**: Column 3 → 2 votes × 2 points = **4 points**
    - **3rd place rankings**: Column 2 → 5 votes × 1 point = **5 points**
      - Candidate B's total: **21 + 4 + 5 = 30 points**
  - Candidate C's Rankings:
    - **1st place rankings**: Column 2 → 5 votes × 3 points = **15 points**
    - **2nd place rankings**: Column 1 → 7 votes × 2 points = **14 points**
    - **3rd place rankings**: Column 3 → 2 votes × 1 point = **2 points**
      - Candidate C's total: **15 + 14 + 2 = 31 points**

- **Step 3**: The candidate with the highest total wins.
  - Candidate A's total score = **23 points**
  - Candidate B's total score = **30 points**
  - Candidate C's total score = **31 points**

Thus, Candidate C is the winner by the Borda count method.

Example #7.4.2 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	5	4	3
First Choice	A	B	C
Second Choice	B	A	A
Third Choice	C	C	B

Use the Borda count method to determine a winner.

✅ **Solution:**

- **Step 1**: Assign the points based on position. Since there are 3 candidates, a candidate receives:
  - 3 points for each 1st-place ranking
  - 2 points for each 2nd-place ranking
  - 1 point for each 3rd-place ranking

- **Step 2**: Add up all the points each candidate receives.
  - Candidate A's Rankings:
    - **1st place rankings**: Column 1 → 5 votes × 3 points = **15 points**
    - **2nd place rankings**: Columns 2 & 3 → (4 + 3) = 7 votes × 2 points = **14 points**
    - **3rd place rankings**: None → 0 votes × 1 point = **0 points**
      - Candidate A's total: **15 + 4 + 0 = 29 points**
  - Candidate B's Rankings:
    - **1st place rankings**: Column 2 → 4 votes × 3 points = **12 points**
    - **2nd place rankings**: Column 1 → 5 votes × 2 points = **10 points**
    - **3rd place rankings**: Column 3 → 3 votes × 1 point = **3 points**
      - Candidate B's total: **12 + 10 + 3 = 25 points**
  - Candidate C's Rankings:
    - **1st place rankings**: Column 3 → 3 votes × 3 points = **9 points**
    - **2nd place rankings**: None → 0 votes × 2 points = **0 points**
    - **3rd place rankings**: Columns 1 & 2 → (5 + 4) = 9 votes × 1 point = **9 points**
      - Candidate C's total: **9 + 0 + 9 = 18 points**

- **Step 3**: The candidate with the highest total wins.
  - Candidate A's total score = **29 points**
  - Candidate B's total score = **25 points**
  - Candidate C's total score = **18 points**

Thus, Candidate A is the winner by the Borda count method.

Example #7.4.3 🤔

A survey of 23 customers was conducted asking them to rank which protein they prefer at a local Chipotle restaurant. The results are listed below:

Preference ballot showing voter rankings for candidates Barbacoa, Carnitas, Chicken, and Steak

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas	Chicken	Steak

Number of Voters	8	7	5	3
Second Choice	Chicken	Barbacoa	Carnitas	Barbacoa
Third Choice	Barbacoa	Chicken	Steak	Chicken
Fourth Choice	Carnitas	Steak	Barbacoa	Carnitas

Use the Borda count method to determine a winner.

✅ **Solution:**

- **Step 1**: Assign the points based on position. Since there are 4 candidates, a candidate receives:
  - 4 points for each 1st-place ranking
  - 3 points for each 2nd-place ranking
  - 2 points for each 3rd-place ranking
  - 1 point for each 4th-place ranking

- **Step 2**: Add up all the points each candidate receives.
  - Steak's Rankings:
    - **1st place rankings**: Columns 1 & 4 → (8 + 3) = 11 votes × 4 points = **44 points**
    - **2nd place rankings**: None → 0 votes × 3 points = **0 points**
    - **3rd place rankings**: Column 3 → 5 votes × 2 points = **10 points**
    - **4th place rankings**: Column 2 → 7 votes × 1 point = **7 points**
      - Steak's total: **44 + 0 + 10 + 7 = 61 points**
  - Carnitas's Rankings:
    - **1st place rankings**: Column 2 → 7 votes × 4 points = **28 points**
    - **2nd place rankings**: Column 3 → 5 votes × 3 points = **15 points**
    - **3rd place rankings**: None → 0 votes × 2 points = **0 points**
    - **4th place rankings**: Columns 1 & 4 → (8 + 3) = 11 votes × 1 point = **11 points**
      - Carnitas's total: **28 + 15 + 0 + 11 = 54 points**
  - Chicken's Rankings:
    - **1st place rankings**: Column 3 → 5 votes × 4 point = **20 points**
    - **2nd place rankings**: Column 1 → 8 votes × 3 points = **24 points**
    - **3rd place rankings**: Columns 2 & 4 → (7 + 3) = 10 votes × 2 points = **20 points**
    - **4th place rankings**: None → 0 votes × 1 point = **0 points**
      - Chicken's total: **20 + 24 + 20 + 0 = 64 points**
  - Barbacoa's Rankings:
    - **1st place rankings**: None → 0 votes × 4 points = **0 points**
    - **2nd place rankings**: Column 2 & 4 → (7 + 3) = 10 votes × 3 points = **30 points**
    - **3rd place rankings**: Column 1 → 8 votes × 2 points = **16 points**
    - **4th place rankings**: Column 3 → 5 votes × 1 point = **5 points**
      - Barbacoa's total: **0 + 30 + 16 + 5 = 51 points**

- **Step 3**: The candidate with the highest total wins.
  - Steak's total score = **61 points**
  - Carnitas's total score = **54 points**
  - Chicken's total score = **64 points**
  - Barbacoa's total score = **51 points**

Thus, Chicken is the winner by the Borda count method.

Historical Note: Jean-Charles de Borda

The Borda count method is named after Jean-Charles de Borda (1733–1799), a French mathematician, physicist, and political scientist, like Condorcet, best known today for his work on voting methods.

He was a contemporary of Condorcet during the Enlightenment and a member of the French Academy of Sciences where he contributed to mathematics, navigation, physics, and social choice theory. His lasting legacy was the Borda count which was the first ranked-based voting method. Unlike Condorcet's approach, the Borda count aggregates overall rankings, not just head-to-head wins and does not always elect the Condorcet winner. This difference later became a major point of debate between Borda's and Condorcet's supporters, where Condorcet famously criticized the Borda count because it can reward compromise candidates and be manipulated by strategic ranking.

His other contributions included improvements to navigation instruments, research in fluid mechanics, and efforts to standardize scientific measurements during the French Revolution. He also strongly supported the application of reason and mathematical principles to public decision-making.

Section 7.4: Borda Count Method [*In-Class Exercises*]

1. Consider the following preference table with Candidates A, B, & C:

   Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	7	6	6	5	3
First Choice	C	B	A	B	A
Second Choice	A	A	B	C	C
Third Choice	B	C	C	A	B

   Use the Borda count to determine a winner.

2. A group of 41 people is choosing where to eat lunch during their meeting. The four fast food delivery options are: Board & Brew, Naugles, T.K. Burgers, or In-N-Out Burgers. Each voter ranks the restaurants from most preferred to least preferred.

   Preference ballot showing voter rankings for candidates Board & Brew, In-N-Out, Naugles, and T.K. Burgers

Number of Voters	11	9	8	6	4	3
First Choice	In-N-Out	T.K. Burgers	Naugles	Board & Brew	T.K.Burgers	In-N-Out
Second Choice	Naugles	Naugles	T.K.Burgers	T.K. Burgers	In-N-Out	T.K. Burgers
Third Choice	T.K. Burgers	Board & Brew	In-N-Out	In-N-Out	Naugles	Board & Brew
Fourth Choice	Board & Brew	In-N-Out	Board & Brew	Naugles	Board & Brew	Naugles

Use the Borda count method to determine a winner.

Answers available at the end of this section.

Answers

Section 7.4: Borda Count Method [*In-Class Exercises*]

1. Candidate A = 58 points, Candidate B = 55 points, Candidate C = 49 points. Candidate A wins.
2. In-N-Out = 105 pts; T.K. Burgers = 125 pts; Naugles = 109 pts; Board & Brew = 71 pts. T.K. Burgers wins.

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Section 7.4.H: Homework

Section 7.4: Borda Count Method - Homework Exercises

(Section 7.4) - Homework Exercise #1

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	4	3	2	2
First Choice	A	B	C	C
Second Choice	B	C	B	A
Third Choice	C	A	A	B

Use the Borda count method to determine a winner.

Answer available at the end of this section.

(Section 7.4) - Homework Exercise #2

Consider the following preference table with Candidates A, B, C, & D:

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	9	8	6	4	3
First Choice	D	C	A	A	C
Second Choice	B	B	C	C	D
Third Choice	A	A	D	B	A
Fourth Choice	C	D	B	D	B

Use the Borda count method to determine a winner.

Answer available at the end of this section.

(Section 7.4) - Homework Exercise #3

A group of 30 office coworkers are choosing where to eat lunch during their meeting. The restaurant options are: The Habit, Chick-Fil-A, and Panera Bread. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Chick-Fil-A, The Habit, and Panera Bread

Number of Voters	11	9	6	4
First Choice	The Habit	Chick-Fil-A	Panera Bread	Chick-Fil-A
Second Choice	Panera Bread	Panera Bread	Chick-Fil-A	The Habit
Third Choice	Chick-Fil-A	The Habit	The Habit	Panera Bread

Use the Borda count method to determine a winner.

Answer available at the end of this section.

(Section 7.4) - Homework Exercise #4

A class of 37 students is voting on a location for their end-of-year field trip. The four options are the Aquarium of the Pacific (A), the Butterfly Release at the Botanical Gardens (B), Disneyland (D), or the Griffith Observatory (G). The results are in the preference table below.

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	14	10	8	4	1
First Choice	A	B	D	D	B
Second Choice	G	G	B	G	D
Third Choice	B	D	A	B	G
Fourth Choice	D	A	G	A	A

Use the Borda count method to determine a winner.

Answer available at the end of this section.

(Section 7.4) - Homework Exercise #5

A Los Angeles county Buffalo Wild Wings polled their customers and asked them to rank the local L.A. professional sports teams. Here are the results listed in a preference table.

Preference ballot showing voter rankings for candidates Chargers, Clippers, Dodgers, Kings, Lakers, and Rams

Number of Voters	15	13	9	9	4	3	3	1
First Choice	Dodgers	Lakers	Dodgers	Lakers	Rams	Lakers	Kings	Clippers
Second Choice	Lakers	Dodgers	Rams	Dodgers	Chargers	Rams	Rams	Chargers
Third Choice	Rams	Kings	Lakers	Kings	Kings	Dodgers	Lakers	Kings
Fourth Choice	Kings	Rams	Clippers	Clippers	Dodgers	Kings	Clippers	Dodgers
Fifth Choice	Clippers	Chargers	Kings	Rams	Clippers	Chargers	Chargers	Rams
Sixth Choice	Chargers	Clippers	Chargers	Chargers	Lakers	Clippers	Dodgers	Lakers

Use the Borda count method to determine a winner.

Answer available at the end of this section.

Answers

(Section 7.4) - Homework Exercise #1

Candidate A = 21 points, Candidate B = 23 points, Candidate C = 22 points. Candidate B wins.

(Section 7.4) - Homework Exercise #2

Candidate A = 80 points, Candidate B = 68 points, Candidate C = 83 points, Candidate D = 69 points. Candidate C wins.

(Section 7.4) - Homework Exercise #3

The Habit = 56 points, Chick-Fil-A = 62 points, Panera Bread = 62 points. Tie between Chick-Fil-A and Panera Bread.

(Section 7.4) - Homework Exercise #4

Aquarium = 87 points, Botanical Gardens = 104 points, Disneyland = 85 points, Griffith Observatory = 94 points. Botanical Gardens wins.

(Section 7.4) - Homework Exercise #5

Dodgers = 284 points, Lakers = 278 points, Rams = 218 points, Kings = 198 points, Clippers = 123 points, Chargers = 96 points. Dodgers win.

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Section 7.5: Majority Criterion

Learning Objectives

- Decide if an election violates the Majority criterion

The Borda count method prioritizes ordinal position across the entire ranking rather than cardinal first-place dominance. Because of this, the Borda count method has its own serious flaws that can produce arguably unfair outcomes.

- A candidate with a clear majority of first-place votes can still lose.
- The "Compromise Candidate" Problem. Borda count systematically favors candidates who are nobody's favorite but everyone's acceptable second/third choice.
- Vulnerability to Strategic Manipulation. Strategic voting (also called tactical voting) occurs when voters rank candidates insincerely to achieve a better outcome for their preferred candidate. Voters can dramatically improve their favorite's chances by strategically ranking rivals lower than their true preferences.

These flaws mean that Borda count is inappropriate for many applications, particularly:

- High-stakes public elections where democratic legitimacy is essential
- Competitive environments where strategic voting incentives are strong
- Contexts where majority rule is considered non-negotiable

However, Borda count may still be suitable for:

- Small deliberative bodies seeking consensus
- Low-stakes preference surveys
- Situations where strategic voting is discouraged by transparency or social norms
- Contexts where compromise and broad acceptability are explicitly valued over majority preference

Borda count's simplicity and appealing use of complete preference information come at the cost of serious structural flaws. These flaws don't make it universally "bad," but they do make it contextually inappropriate for many applications and require careful consideration of whether its particular trade-offs align with the values and needs of the specific decision-making situation.

This brings us to our second fairness criterion.

Majority Criterion (Fairness Criterion #2)

If a candidate wins by a majority, then that candidate should win the election by another method. If no majority candidate exists, then the majority criterion does not apply.

The violation occurs because Borda count and the majority Criterion answer different questions:

- Borda count asks: "Which candidate has the highest average desirability across all voters?"
- Majority criterion asks: "Which candidate is the first choice of the most voters?"

These questions have different answers when:

- The most popular first choice is strongly unpopular among others

- A less popular first choice is moderately popular among everyone

Neither is objectively "correct". They represent different values about collective decision-making. The "violation" simply reveals that these values conflict.

Example #7.5.1 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	7	4	2
First Choice	C	B	B
Second Choice	B	A	C
Third Choice	A	C	A

Note: When using the Borda count method, Candidate B would be the winner with 32 points. (Candidate C = 29, Candidate A = 17). When using the plurality method, Candidate C is the winner.

a. Does the election violate the majority criterion when using the **Borda count method**?
b. Does the election violate the majority criterion when using the **plurality method**?

✅ **Solution:**

- Candidate A received 0 out of 13 first place votes or 0%
- Candidate B received 6 out of 13 first place votes or 46.2%
- Candidate C received 7 out of 13 first place votes or 53.8%

So, a majority exists here. Candidate C has a majority of first place votes.

a. Candidate B wins using the Borda count method, yet Candidate C has a majority of first-place votes. Thus, the method violates the majority criterion. **(YES)**
b. Candidate C wins by both the plurality method and by majority. Thus, the method is not a violation of the majority criterion. **(NO)**

Note: Plurality ALWAYS satisfies the majority criterion; if someone has a majority, they automatically have the most votes.

Example #7.5.2 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	5	4	3
First Choice	A	B	C
Second Choice	B	A	A
Third Choice	C	C	B

Note: When using the Borda Count method, Candidate A was the winner. When using the plurality method, Candidate A was also the winner.

a. Does the election violate the majority criterion when using the **Borda count method**?
b. Does the election violate the majority criterion when using the **plurality method**?

✅ **Solution:**

- Candidate A received 5 out of 12 first place votes or 41.7%
- Candidate B received 4 out of 12 first place votes or 33.3%
- Candidate C received 3 out of 12 first place votes or 25%

So, no majority exists here.

a. Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion. **(NO)**
b. Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion. **(NO)**

Example #7.5.3 🤔

A survey of 23 customers was conducted asking them to rank which protein they prefer at a local Chipotle restaurant. The results are listed below:

Preference ballot showing voter rankings for candidates Barbacoa, Carnitas, Chicken, and Steak

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas	Chicken	Steak
Second Choice	Chicken	Barbacoa	Carnitas	Barbacoa
Third Choice	Barbacoa	Chicken	Steak	Chicken
Fourth Choice	Carnitas	Steak	Barbacoa	Carnitas

Note: When using the Borda count method, Chicken would be the winner with 64 points. (Steak = 61, Carnitas = 54, Barbacoa = 53). When using the plurality method, Steak is the winner.

a. Does the election violate the majority criterion when using the **Borda count method**?
b. Does the election violate the majority criterion when using the **plurality method**?

✅ **Solution:**

- Steak received 11 out of 23 first place votes or 47.8%
- Carnitas received 7 out of 23 first place votes or 30.4%
- Chicken received 5 out of 23 first place votes or 21.7%
- Barbacoa received 0 out of 23 first place votes or 0%

So, no majority exists here.

- Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion. **(NO)**
- Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion. **(NO)**

Example #7.5.4 🤔

A student organization with 100 members is voting on which city to host their national conference. The results are listed below:

Preference ballot showing voter rankings for candidates Austin, Boston, Chicago, Denver, and Seattle

Number of Voters	52	18	15	10	5
First Choice	Austin	Boston	Chicago	Denver	Seattle
Second Choice	Boston	Chicago	Denver	Seattle	Denver
Third Choice	Chicago	Denver	Seattle	Boston	Boston
Fourth Choice	Denver	Seattle	Boston	Chicago	Chicago
Fifth Choice	Seattle	Austin	Austin	Austin	Austin

Does the election violate the Majority criterion when using the Borda count method?

✅ **Solution:**

- Austin received 52 out of 100 first place votes or 52%
- Boston received 18 out of 100 first place votes or 18%
- Chicago received 15 out of 100 first place votes or 15%
- Denver received 10 out of 100 first place votes or 10%
- Seattle received 5 out of 100 first place votes or 5%

So, a majority exists here. Austin has a majority of first place votes. Thus, Austin wins using the Majority criterion.

Now, use the Borda count method:

- **Step 1**: Assign the points based on position. Since there are 5 candidates, a candidate receives.
  - 5 points for each 1st-place ranking
  - 4 points for each 2nd-place ranking
  - 3 points for each 3rd-place ranking
  - 2 points for each 4th-place ranking
  - 1 point for each 5th-place ranking
- **Step 2**: Add up all the points each candidate receives.
  - Austin's Rankings:
    - **1st place rankings**: Column 2 → 52 votes × 5 points = **260 points**
    - **2nd place rankings**: Column 1 → 0 votes × 4 points = **0 points**
    - **3rd place rankings**: None → 0 votes × 3 points = **0 points**
    - **4th place rankings**: None → 0 votes × 2 points = **0 points**
    - **5th place rankings**: Columns 2, 3, 4, & 5 → (18 + 15 + 10 + 5) = 48 votes × 1 point = **48 points**
      - Austin's total: **260 + 0 + 0 + 0 + 48 = 308 points**
  - Boston's Rankings:
    - **1st place rankings**: Column 2 → 18 votes × 5 points = **90 points**
    - **2nd place rankings**: Column 1 → 52 votes × 4 points = **208 points**
    - **3rd place rankings**: Columns 4 and 5 → (10 + 5) = 15 votes × 3 points = **45 points**
    - **4th place rankings**: Column 3 → 15 votes × 2 points = **30 points**
    - **5th place rankings**: None → 0 votes × 1 point = **0 points**
      - Boston's total: **90 + 208 + 45 + 30 + 0 = 373 points**
  - Chicago's Rankings:

        - **1st place rankings**: Column 3 → 15 votes × 5 points = **75 points**
        - **2nd place rankings**: Column 2 → 18 votes × 4 points = **72 points**
        - **3rd place rankings**: Column 1 → 52 votes × 3 points = **156 points**
        - **4th place rankings**: Columns 4 and 5 → (10 + 5) = 15 votes × 2 points = **30 points**
        - **5th place rankings**: None → 0 votes × 1 point = **0 points**
            - Chicago's total: **75 + 72 + 156 + 30 + 0 = 333 points**
    - Denver's Rankings:
        - **1st place rankings**: Column 4 → 10 votes × 5 points = **50 points**
        - **2nd place rankings**: Columns 3 & 5 → (15 + 5) = 20 votes × 4 points = **80 points**
        - **3rd place rankings**: Column 2 → 18 votes × 3 points = **54 points**
        - **4th place rankings**: Column 1 → 52 votes × 2 points = **104 points**
        - **5th place rankings**: None → 0 votes × 1 point = **0 points**
            - Denver's total: **50 + 80 + 54 + 104 + 0 = 288 points**
    - Seattle's Rankings:
        - **1st place rankings**: Column 5 → 5 votes × 5 points = **25 points**
        - **2nd place rankings**: Column 4 → 10 votes × 4 points = **40 points**
        - **3rd place rankings**: Column 3 → 15 votes × 3 points = **45 points**
        - **4th place rankings**: Column 2 → 18 votes × 2 points = **36 points**
        - **5th place rankings**: Column 1 → 52 votes × 1 point = **52 points**
            - Seattle's total: **25 + 40 + 45 + 36 + 52 = 198 points**
- **Step 3**: The candidate with the highest total wins.
    - Austin's total score = **308 points**
    - Boston's total score = **373 points**
    - Chicago's total score = **333 points**
    - Denver's total score = **288 points**
    - Seattle's total score = **198 points**

Thus, Boston is the winner by the Borda Count method.

Boston wins using the Borda count method, yet Austin has a majority of first-place votes. Thus, the method violates the majority criterion. **(YES)**

Now, let's see if the Borda count method (Section 7.4) can violate the Condorcet criterion (Section 7.3).

Example #7.5.5 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	4	2	1
First Choice	B	A	A
Second Choice	A	C	B
Third Choice	C	B	C

a. Use the Borda count method to determine a winner.
b. Does the election violate the **Condorcet criterion**?

✅ **Solution:**

a. Borda count method:

- **Step 1**: Assign the points based on position. Since there are 3 candidates, a candidate receives:
  - 3 points for each 1st-place ranking
  - 2 points for each 2nd-place ranking
  - 1 point for each 3rd-place ranking

- **Step 2**: Add up all the points each candidate receives.
  - Candidate A's Rankings:
    - **1st place rankings**: Columns 2 & 3 → (2 + 1) = 3 votes × 3 points = **9 points**
    - **2nd place rankings**: Column 1 → 4 votes × 2 points = **8 points**
    - **3rd place rankings**: None → 0 votes × 1 point = **0 points**
      - Candidate A's total: **9 + 8 + 0 = 17 points**
  - Candidate B's Rankings:
    - **1st place rankings**: Column 1 → 4 votes × 3 points = **12 points**
    - **2nd place rankings**: Column 3 → 1 vote × 2 points = **2 points**
    - **3rd place rankings**: Column 2 → 2 votes × 1 point = **2 points**
      - Candidate B's total: **12 + 2 + 2 = 16 points**
  - Candidate C's Rankings:
    - **1st place rankings**: None → 0 votes × 3 point = **0 points**
    - **2nd place rankings**: Column 2 → 2 votes × 2 points = **4 points**
    - **3rd place rankings**: Columns 1 & 3 → (4 + 1) = 5 votes × 1 point = **5 points**
      - Candidate C's total: **0 + 4 + 5 = 9 points**

- **Step 3**: The candidate with the highest total wins.
  - Candidate A's total score = **17 points**
  - Candidate B's total score = **16 points**
  - Candidate C's total score = **9 points**

Thus, Candidate A is the winner by the Borda count method.

b. Condorcet criterion: We need to check the one-on-one scores for each possible pair of candidates. The match ups are A vs. B, A vs. C, and B vs. C.

Use a table for each match-up to record your results.

A	B
	4
2	
1	
3	**4**
B wins A	

A	C
4	
2	
1	
7	**0**
A wins C	

B	C
4	
	2
1	
5	**2**
B wins C	

Here are the head-to-head records:

A: 1 win (beat C only)
B: 2 wins (beat A and C)
C: 0 wins

Thus, Candidate B beats everyone head-to-head. Candidate B is the Condorcet winner.

Candidate A wins using the Borda count method, yet Candidate B is the Condorcet winner. Thus, the method violates the majority criterion. **(YES)**

Note: Looking at the first head-to-head matchup when B wins A, technically, there is no need to check the rest of the head-to-head matchups, since Candidate A cannot win from this point on. The violation has been established.

So, let's summarize what we have learned from the examples above.

Does the Borda count method violate the majority criterion?

YES. The Borda count violates the majority criterion, because a candidate with a majority of first-place votes may not have the highest total points. (As we saw in Examples #7.5.1a & 7.5.4).

Continuing the fairness criteria table from Section 7.3, we now examine how the previous voting method performs under the monotonicity criterion.

Does the plurality method violate the majority criterion?

NO. The plurality method **NEVER** violates the majority criterion, because a candidate with more than half of the first-place votes will have the highest vote total. (We did see a case in Example #7.5.1b, however there is not a case where a violation would exist).

Again, continuing the fairness criteria table from Section 7.3, we now examine how the Borda count method performs with respect to the previously studied criteria.

Does the Borda count method violate the Condorcet criterion?

YES. The Borda count violates the Condorcet criterion, because a candidate who wins all head-to-head comparisons may not receive the highest total point score. (As we saw in Example #7.5.5).

Thus, we can update our fairness criteria table with 3 more conclusions from Section 7.3:

Four Fairness Criteria Table (1/4 Complete)

Fairness Criterion ⟶ Major Voting Method ↓	Condorcet Criterion	Majority Criterion	Monotonicity Criterion	Independence of Irrelevant Alternatives Criterion
Plurality	**Violation Possible**	✓	**?**	**?**
Borda Count	**Violation Possible**	**Violation Possible**	**?**	**?**
Plurality-with-Elimination	**?**	**?**	**?**	**?**
Pairwise Comparison	**?**	**?**	**?**	**?**

So, we are still looking for a major voting method that satisfies all four fairness criterion's and unfortunately, we have determined that the plurality method and the Borda count method have at least one possible violation for each.

Section 7.5: Majority Criterion [*In-Class Exercises*]

1. Consider the following preference table with Candidates X, Y, & Z.

Preference ballot showing voter rankings for candidates X, Y, and Z

Number of Voters	5	4	2	1
First Choice	X	Y	X	Z
Second Choice	Y	X	Z	Y
Third Choice	Z	Z	Y	X

a. Use the Borda count method to determine the winner.
b. Does the election violate the **majority criterion**?
c. Does the election violate the **Condorcet criterion**?

2. A diner of 15 customers is voting for their favorite waiter/waitress. The candidates and ranked choices are listed below.

Preference ballot showing voter rankings for candidates Alice, Flo, Mel, and Vera

Number of Voters	8	4	3
First Choice	Flo	Mel	Vera
Second Choice	Alice	Vera	Mel
Third Choice	Vera	Alice	Alice
Fourth Choice	Mel	Flo	Flo

a. Use the Borda count method to determine the winner.
b. Does the election violate the **majority criterion**?
c. Does the election violate the **Condorcet criterion**?

Answers available at the end of this section.

Answers

Section 7.5: Majority Criterion [*In-Class Exercises*]

1. a. Candidate X = 30 points, Candidate Y = 26 points, Candidate Z = 16 points. Candidate X wins.
   b. Candidate X wins by both the plurality method and by majority. Thus, the method is not a violation of the majority criterion. **(NO)**
   c. Candidate X wins by both the Borda count method and is the Condorcet winner. Thus, the method is not a violation of the Condorcet criterion. **(NO)**
2. a. Alice = 38 points, Flo = 39 points, Mel = 33 points, Vera = 40 points. Vera wins.
   b. Vera wins using the Borda count method, yet Flo has a majority of first-place votes. Thus, the method violates the majority criterion. **(YES)**
   c. Vera wins using the Borda count method, yet Flo is the Condorcet winner. Thus, the method violates the majority criterion. **(YES)**

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Section 7.5.H: Homework

Section 7.5: Majority Criterion - Homework Exercises

(Section 7.5) - Homework Exercise #1

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	4	3	2	2
First Choice	A	B	C	C
Second Choice	B	C	B	A
Third Choice	C	A	A	B

a. Use the Borda count method to determine the winner.
b. Does the election violate the majority criterion?

Answer available at the end of this section.

(Section 7.5) - Homework Exercise #2

A recreational hockey team consisting of 15 players is selecting a new captain. The team narrowed the candidates



to the four youngest players—Alex, Brandon, Jacob, and Trevor—and asked each player to rank them from most to least preferred. The results are provided below.

Preference ballot showing voter rankings for candidates Alex, Brandon, Jacob, and Trevor

Number of Voters	8	7
First Choice	Brandon	Jacob
Second Choice	Alex	Trevor
Third Choice	Jacob	Alex
Fourth Choice	Trevor	Brandon

a. Use the Borda count method to determine the winner.
b. Does the election violate the majority criterion when using the Borda count method?
c. Use the plurality method to determine the winner.
d. Does the election violate the majority criterion when using the plurality method?

Answer available at the end of this section.

(Section 7.5) - Homework Exercise #3

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	6	5	2	1
First Choice	C	A	C	B
Second Choice	B	C	A	A
Third Choice	A	B	B	C

a. Use the Borda count method to determine the winner.
b. Does the election violate the majority criterion when using the Borda count method?
c. When using the plurality method, Candidate C wins with 8 first-place votes. Does the election violate the majority criterion when using the plurality method?

Answer available at the end of this section.

(Section 7.5) - Homework Exercise #4

A group of 30 office coworkers are choosing where to eat lunch during their meeting. The restaurant options are: The Habit, Chick-Fil-A, and Panera Bread. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Chick-Fil-A, The Habit, and Panera Bread

Number of Voters	11	9	6	4
First Choice	The Habit	Chick-Fil-A	Panera Bread	Chick-Fil-A
Second Choice	Panera Bread	Panera Bread	Chick-Fil-A	The Habit

Number of Voters	11	9	6	4
Third Choice	Chick-Fil-A	The Habit	The Habit	Panera Bread

a. Use the Borda count method to determine the winner.
b. Does the election violate the majority criterion when using the Borda count method?
c. When using the plurality method, Chick-Fil-A wins with 13 first-place votes. Does the election violate the majority criterion when using the plurality method?

Answer available at the end of this section.

(Section 7.5) - Homework Exercise #5

A high school is hosting a movie night, and 13 students vote on which movie to watch. The choices are:

- A = Action 💥
- An = Animated 🎨
- C = Comedy 😂
- M = Mystery 🕵️
- H = Horror 👻

Each student ranks all five options from favorite to least favorite.

Preference ballot showing voter rankings for candidates Action, Animated, Comedy, Mystery, and Horror

Number of Voters	7	2	2	1	1
First Choice	A	C	M	An	H
Second Choice	C	M	An	H	C
Third Choice	M	An	H	C	M
Fourth Choice	An	H	C	M	An
Fifth Choice	H	A	A	A	A

a. Use the Borda count method to determine the winner.
b. Does the election violate the majority criterion when using the Borda count method?
c. Use the plurality method to determine the winner.
d. Does the election violate the majority criterion when using the plurality method?

Answer available at the end of this section.

(Section 7.5) - Homework Exercise #6

Consider the following preference table with Candidates W, X, Y, & Z:

Preference ballot showing voter rankings for candidates W, X, Y, and Z

Number of Voters	7	5	3	2
First Choice	W	Z	Y	Z
Second Choice	X	X	X	W
Third Choice	Z	Y	W	X
Fourth Choice	Y	W	Z	Y

a. Use the Borda count method to determine the winner.
b. Does the election violate the majority criterion when using the Borda count method?
c. Does the election violate the Condorcet criterion when using the Borda count method?
d. Use the plurality method to determine the winner.
e. Does the election violate the majority criterion when using the plurality method?

Answer available at the end of this section.

Answers

(Section 7.5) - Homework Exercise #1

a. Candidate A = 21 points, Candidate B = 23 points, Candidate C = 22 points. Candidate B wins.
b. Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion.

(Section 7.5) - Homework Exercise #2

a. Alex = 38 points, Brandon = 39 points, Jacob = 44 points, Trevor = 29 points. Jacob wins.
b. Jacob wins using the Borda count method, yet Brandon has a majority of first-place votes. Thus, the method violates the majority criterion.
c. Brandon wins with 8 first-place votes.
d. Brandon wins by both the plurality method and by majority. Thus, the method is not a violation of the majority criterion.

(Section 7.5) - Homework Exercise #3

a. Candidate A = 27 points, Candidate B = 22 points, Candidate C = 35 points. Candidate C wins.
b. Candidate C wins by both the Borda count method and by majority. Thus, the method is not a violation of the majority criterion.
c. Candidate C wins by both the plurality method and by majority. Thus, the method is not a violation of the majority criterion.

(Section 7.5) - Homework Exercise #4

a. The Habit = 56 points, Chick-Fil-A = 62 points, Panera Bread = 62 points. Tie between Chick-Fil-A and Panera Bread.
b. Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion.
c. Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion.

(Section 7.5) - Homework Exercise #5

a. Action = 41 points, Animated = 35 points, Comedy = 49 points, Mystery = 44 points, Horror = 26 points. Comedy wins.
b. Comedy wins using the Borda count method, yet Action has a majority of first-place votes. Thus, the method violates the majority criterion.
c. Action wins with 7 first-place votes.
d. Action wins by both the plurality method and by majority. Thus, the method is not a violation of the majority criterion.

(Section 7.5) - Homework Exercise #6

a. Candidate W = 45 points, Candidate X = 47 points, Candidate Y = 31 points, Candidate Z = 45 points. Candidate X wins.
b. Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion.
c. Candidate W wins using the Borda count method, yet Candidate X is the Condorcet winner. Thus, the method violates the Condorcet criterion.
d. Candidate W wins with 7 first-place votes.
e. Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion.

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Section 7.6: Plurality-with-Elimination Method

Learning Objectives

- Determine the winner of an election using the plurality-with-elimination method

We've now studied two voting methods, and each has revealed serious flaws:

Plurality Method:

- ✗ Vulnerable to vote splitting and spoiler effects
- ✗ Ignores all preference information beyond first choice
- ✗ Can elect candidates opposed by large majorities
- ✗ Violates the Condorcet Criterion

Borda Count Method:

- ✗ Violates the Majority Criterion (can override clear majorities)
- ✗ Highly vulnerable to strategic manipulation
- ✗ Violates the Condorcet Criterion

Let's introduce another voting method called the plurality-with-elimination method:

Major Voting Method #3 - Plurality-with-Elimination

Plurality-with-Elimination (also called **Instant Runoff Voting** or **IRV**, or **Ranked Choice Voting** or **RCV**) attempts to simulate what would happen if we held multiple rounds of voting, eliminating the weakest candidate after each round, until someone achieves a majority.

- **Step 0: Collect Ranked Ballots (if a preference table is not given)**
  - Voters rank candidates in order of preference:
    - 1st choice (most preferred)
    - 2nd choice
    - 3rd choice
    - And so on...

Voters don't have to rank all candidates—partial rankings are allowed in most implementations.

- **Step 1: Count First-Place Votes (Round 1)**
  - Count how many voters ranked each candidate first.
    - **Check for majority winner**:
      - If any candidate has **more than 50%** of first-place votes → **That candidate wins immediately**
      - If no candidate has a majority → **Proceed to elimination**

- **Step 2: Eliminate the Weakest Candidate**
  - Identify the candidate with the **fewest first-place votes** and **eliminate** them from the race.

        - **Transfer their votes**:
            - Find all ballots that ranked the eliminated candidate first
            - Look at each ballot's **next preference** (second choice)
            - Transfer those votes to the next-ranked candidate still in the race
            - If a ballot's second choice is also eliminated, go to the third choice, and so on

---

- **Step 3: Recount with Transferred Votes (Round 2)**
    - Count the votes again, now including:
        - Original first-place votes for remaining candidates
        - Transferred votes from eliminated candidate's supporters
            - **Check for majority winner**:
                - If any remaining candidate now has **more than 50% → That candidate wins**
                - If no candidate has a majority → **Continue elimination process**

- **Step 4: Repeat Until a Winner Emerges**
    - Continue the cycle:
        1. Eliminate the candidate with fewest votes
        2. Transfer their votes to next preferences
        3. Recount
        4. Check for majority
            - **Continue until**: One candidate achieves more than 50% of the remaining votes.
            - That candidate is declared the winner.

Traditional runoff elections require a second election whenever no candidate receives a majority of the votes. However, holding a second election can be expensive, time-consuming, and often results in lower voter turnout. Ranked-choice voting addresses this issue by allowing voters to rank candidates in order of preference on a single ballot. Using these rankings, the election simulates multiple rounds of runoff voting by eliminating the candidate with the fewest votes and redistributing those ballots according to the voters' next choices. This process continues until one candidate receives a majority of the votes.

Example #7.6.1 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	6	5	3
First Choice	B	C	A
Second Choice	C	A	C
Third Choice	A	B	B

Use the plurality-with-elimination method to determine a winner..

✅ **Solution:**

- **Step 0**: A preference table was given, so preference ballots have been counted.

- **Step 1**: Count first-place votes. (Round 1)
    - Candidate A received 3 out of 14 first place votes or 21.4%
    - Candidate B received 6 out of 14 first place votes or 42.9%
    - Candidate C received 5 out of 14 first place votes or 35.7%

- **So, no majority winner.**

- **Step 2**: Eliminate the weakest candidate.

Candidate A received the fewest first place votes, so eliminate Candidate A by removing them from the table.

Preference ballot removing candidate A

Number of Voters	6	5	3
First Choice	B	C	
Second Choice	C		C
Third Choice		B	B

Then move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates B and C

Number of Voters	6	5	3
First Choice	B	C	C
Second Choice	C	B	B

- **Step 3**: Count first-place votes. (Round 2)
  - Candidate B received 6 out of 14 first place votes or 42.9%
  - Candidate C received (5 + 3) = 8 out of 14 first place votes or 57.1%
    - **So, a majority exists.**

Thus, Candidate C is the winner by the plurality-with-elimination method.

Example #7.6.2 🤔

A survey of 23 customers was conducted asking them to rank which protein they prefer at a local Chipotle restaurant. The results are listed below:

Preference ballot showing voter rankings for candidates Barbacoa, Carnitas, Chicken, and Steak

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas	Chicken	Steak
Second Choice	Chicken	Barbacoa	Carnitas	Barbacoa
Third Choice	Barbacoa	Chicken	Steak	Chicken
Fourth Choice	Carnitas	Steak	Barbacoa	Carnitas

Use the plurality-with-elimination method to determine a winner.

✅ **Solution:**

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Steak received (8 + 3) = 11 out of 23 first place votes or 47.8%
  - Carnitas received 7 out of 23 first place votes or 30.4%
  - Chicken received 5 out of 23 first place votes or 21.7%
  - Barbacoa received 0 out of 23 first place votes or 0%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Barbacoa received the fewest first place votes, so eliminate Barbacoa by removing them from the table.

Preference ballot removing candidate Barbacoa

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas	Chicken	Steak
Second Choice	Chicken		Carnitas	
Third Choice		Chicken	Steak	Chicken
Fourth Choice	Carnitas	Steak		Carnitas

Then move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Carnitas, Chicken, and Steak

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas	Chicken	Steak
Second Choice	Chicken	Chicken	Carnitas	Chicken
Third Choice	Carnitas	Steak	Steak	Carnitas

- **Step 3**: Count first-place votes. (Round 2)
  - Steak received (8 + 3) = 11 out of 23 first place votes or 47.8%
  - Carnitas received 7 out of 23 first place votes or 30.4%
  - Chicken received 5 out of 23 first place votes or 21.7%
    - **So, no majority winner.**
- **Step 4**: Repeat Until a Winner Emerges
  - Eliminate the weakest candidate.

Chicken received the fewest first place votes, so eliminate Chicken by removing them from the table.

Preference ballot removing candidate Chicken

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas		Steak
Second Choice			Carnitas	
Third Choice	Carnitas	Steak	Steak	Carnitas

Then move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Carnitas and Steak

Number of Voters	8	7	5	3
First Choice	Steak	Carnitas	Carnitas	Steak
Second Choice	Carnitas	Steak	Steak	Carnitas

- **Step 3 (Repeated)**: Count first-place votes. (Round 3)
  - Steak received (8 + 3) = 11 out of 23 first place votes or 47.8%
  - Carnitas received (7 + 5) = 12 out of 23 first place votes or 52.2%
    - **So, a majority exists.**

Thus, Carnitas is the winner by the the plurality-with-elimination method.

Example #7.6.3 🤔

City council members need to choose a new location for the city's community center. Four sites are being considered:

Preference ballot showing voter rankings for Downtown, Eastside, Northside, and Westside

Number of Voters	35	30	20	15
First Choice	Downtown	Northside	Eastside	Westside
Second Choice	Northside	Eastside	Westside	Eastside
Third Choice	Eastside	Westside	Northside	Northside
Fourth Choice	Westside	Downtown	Downtown	Downtown

Use the plurality-with-elimination method to determine a winner.

✅ **Solution:**

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Downtown received 35 out of 100 first place votes or 35%
  - Northside received 30 out of 100 first place votes or 30%
  - Eastside received 20 out of 100 first place votes or 20%
  - Westside received 15 out of 100 first place votes or 15%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Westside received the fewest first place votes, so eliminate Westside by removing them from the table.

Preference ballot removing candidate Westside

Number of Voters	35	30	20	15
First Choice	Downtown	Northside	Eastside	

Number of Voters	35	30	20	15
Second Choice	Northside	Eastside		Eastside
Third Choice	Eastside		Northside	Northside
Fourth Choice		Downtown	Downtown	Downtown

Then move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for Downtown, Eastside, and Northside

Number of Voters	35	30	20	15
First Choice	Downtown	Northside	Eastside	Eastside
Second Choice	Northside	Eastside	Northside	Northside
Third Choice	Eastside	Downtown	Downtown	Downtown

- **Step 3**: Count first-place votes. (Round 2)
  - Downtown received 35 out of 100 first place votes or 35%
  - Northside received 30 out of 100 first place votes or 30%
  - Eastside received (20 + 15) = 35 out of 100 first place votes or 35%
    - **So, no majority winner.**
- **Step 4**: Repeat Until a Winner Emerges
  - Eliminate the weakest candidate.

Northside received the fewest first place votes, so eliminate Northside by removing them from the table.

Preference ballot removing candidate Northside

Number of Voters	35	30	20	15
First Choice	Downtown		Eastside	Eastside
Second Choice		Eastside		
Third Choice	Eastside	Downtown	Downtown	Downtown

Then move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for Downtown and Eastside

Number of Voters	35	30	20	15
First Choice	Downtown	Eastside	Eastside	Eastside
Second Choice	Eastside	Downtown	Downtown	Downtown

- **Step 3 (Repeated)**: Count first-place votes. (Round 3)
  - Downtown received 35 out of 100 first place votes or 35%
  - Eastside received (30 + 20 + 15) = 65 out of 100 first place votes or 65%
    - **So, a majority exists.**

Thus, Eastside is the winner by the plurality-with-elimination method.

Section 7.6: Plurality-with-Elimination Method [*In-Class Exercises*]

1. Consider the following preference table with Candidates A, B, & C:

   Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	7	6	6	5	3
First Choice	C	B	A	B	A
Second Choice	A	A	B	C	C
Third Choice	B	C	C	A	B

   Use the plurality-with-elimination method to determine a winner.

2. A group of 41 people is choosing where to eat lunch during their meeting. The four fast food delivery options are: Board & Brew, Naugles, T.K. Burgers, or In-N-Out Burgers. Each voter ranks the restaurants from most preferred to least preferred.

   Preference ballot showing voter rankings for candidates Board & Brew, In-N-Out, Naugles, and T.K. Burgers

Number of Voters	11	9	8	6	4	3
First Choice	In-N-Out	T.K. Burgers	Naugles	Board & Brew	T.K.Burgers	In-N-Out
Second Choice	Naugles	Naugles	T.K.Burgers	T.K. Burgers	In-N-Out	T.K. Burgers
Third Choice	T.K. Burgers	Board & Brew	In-N-Out	In-N-Out	Naugles	Board & Brew
Fourth Choice	Board & Brew	In-N-Out	Board & Brew	Naugles	Board & Brew	Naugles

   Use the plurality-with-elimination method to determine a winner.

Answers available at the end of this section.

Answers

Section 7.6: Plurality-with-Elimination Method [*In-Class Exercises*]

1. Candidate A wins.
2. T.K. Burgers wins.

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Section 7.6.H: Homework

Section 7.6: Plurality-with-Elimination Method - Homework Exercises

(Section 7.6) - Homework Exercise #1

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	4	3	2	2
First Choice	A	B	C	C
Second Choice	B	C	B	A
Third Choice	C	A	A	B

Use the plurality-with-elimination method to determine a winner.

Answer available at the end of this section.

(Section 7.6) - Homework Exercise #2

Consider the following preference table with Candidates A, B, C, & D:

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	9	8	6	4	3
First Choice	D	C	A	A	C
Second Choice	B	B	C	C	D
Third Choice	A	A	D	B	A
Fourth Choice	C	D	B	D	B

Use the plurality-with-elimination method to determine a winner.

Answer available at the end of this section.

(Section 7.6) - Homework Exercise #3

A group of 30 office coworkers are choosing where to eat lunch during their meeting. The restaurant options are: The Habit, Chick-Fil-A, and Panera Bread. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Chick-Fil-A, The Habit, and Panera Bread

Number of Voters	11	9	6	4
First Choice	The Habit	Chick-Fil-A	Panera Bread	Chick-Fil-A
Second Choice	Panera Bread	Panera Bread	Chick-Fil-A	The Habit
Third Choice	Chick-Fil-A	The Habit	The Habit	Panera Bread

Use the plurality-with-elimination method to determine a winner.

Answer available at the end of this section.

(Section 7.6) - Homework Exercise #4

A class of 37 students is voting on a location for their end-of-year field trip. The four options are the Aquarium of the Pacific (A), the Butterfly Release at the Botanical Gardens (B), Disneyland (D), or the Griffith Observatory (G). The results are in the preference table below.

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	14	10	8	4	1
First Choice	A	B	D	D	B
Second Choice	G	G	B	G	D
Third Choice	B	D	A	B	G
Fourth Choice	D	A	G	A	A

Use the plurality-with-elimination method to determine a winner.

Answer available at the end of this section.

(Section 7.6) - Homework Exercise #5

A Los Angeles county Buffalo Wild Wings polled their customers and asked them to rank the local L.A. professional sports teams. Here are the results listed in a preference table.

Preference ballot showing voter rankings for candidates Chargers, Clippers, Dodgers, Kings, Lakers, and Rams

Number of Voters	15	13	9	9	4	3	3	1
First Choice	Dodgers	Lakers	Dodgers	Lakers	Rams	Lakers	Kings	Clippers
Second Choice	Lakers	Dodgers	Rams	Dodgers	Chargers	Rams	Rams	Chargers
Third Choice	Rams	Kings	Lakers	Kings	Kings	Dodgers	Lakers	Kings
Fourth Choice	Kings	Rams	Clippers	Clippers	Dodgers	Kings	Clippers	Dodgers
Fifth Choice	Clippers	Chargers	Kings	Rams	Clippers	Chargers	Chargers	Rams
Sixth Choice	Chargers	Clippers	Chargers	Chargers	Lakers	Clippers	Dodgers	Lakers

Use the plurality-with-elimination method to determine a winner.

Answer available at the end of this section.

(Section 7.6) - Homework Exercise #6

Consider the following preference table with Candidates A, B, C, & D:

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	7	6	5	4	3	2
First Choice	D	B	D	A	C	D
Second Choice	A	C	B	C	D	A
Third Choice	C	D	A	D	B	C
Fourth Choice	B	A	C	B	A	B

Use the plurality-with-elimination method to determine a winner.

Answer available at the end of this section.

Answers

(Section 7.6) - Homework Exercise #1

Candidate C.

(Section 7.6) - Homework Exercise #2

Candidate A.

(Section 7.6) - Homework Exercise #3

Chick-Fil-A.

(Section 7.6) - Homework Exercise #4

Disneyland wins.

(Section 7.6) - Homework Exercise #5

Dodgers. (Note: After the Chargers and Clippers were eliminated, there was a tie for the next least amount of votes when a majority was not obtained, which was Kings and Rams. Although in this particular problem it would not matter which one you would pick to be eliminate first, because the other would be next in line to be eliminated. Regardless, there should be a tie-breaking rule, since this could potentially create two different outcomes. Perhaps a head-to head matchup to see who would win, then the loser should be the next one eliminated in the process.)

(Section 7.6) - Homework Exercise #6

Candidate D wins. (Note: There is no elimination needed here, since Candidate D won the majority).

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Section 7.7: Monotonicity Criterion

Learning Objectives

- Decide if an election violates the Monotonicity criterion

Imagine this scenario: Election Day arrives. The votes are counted using the plurality with elimination method or Instant Runoff Voting (IRV), and Candidate A wins. Now imagine a parallel universe where everything is identical except:

- A small group of voters who originally ranked Candidate A in 3rd place decide they like A more
- They move A up to 1st place on their ballots
- Nothing else changes and all other rankings remain the same

Question: Should Candidate A still win? Obvious answer: Of course! If Candidate A won before, and then gained additional support, A should definitely still win. Getting more support should never hurt a candidate.

The shocking reality: Under IRV (and some other methods), Candidate A might now lose. This paradox, where gaining additional support causes a candidate to lose, is where we investigate our third fairness criterion called the Monotonicity criterion.

Monotonicity Criterion (Fairness Criterion #3)

Monotonicity Criterion (also called the **Monotonicity Fairness Criterion** or **Participation Criterion**):

If a candidate wins an election, and then in a re-election (or hypothetical scenario) some voters increase their support for that candidate (by ranking them higher) while everything else remains unchanged, that candidate should still win.

Receiving additional support or moving up on voters' preference rankings should never hurt a candidate's chances of winning.

Unpacking the Definition

Consider an election in which Candidate X wins under a given voting method. Now suppose that some voters modify their ballots so that Candidate X is ranked higher than before. These changes are strictly favorable to Candidate X: the candidate may be moved from third place to first, from second to first, or from a lower position to a higher one. Aside from the necessary adjustments required to move Candidate X upward, the relative ordering of all other candidates on those ballots remains unchanged. All other voters' ballots, as well as all other conditions of the election, remain exactly the same.

Under a voting system satisfying monotonicity, Candidate X should still win the election after these changes. A violation occurs if, despite receiving greater support in the form of higher rankings, Candidate X no longer wins.

The Intuitive Principle

The logic behind this requirement is straightforward. In a reasonable voting system, increased support for a candidate should never work to that candidate's disadvantage. If higher rankings represent stronger approval, then granting Candidate X more favorable rankings should improve, or at least not worsen, Candidate X's chances of winning.
This principle aligns with common ideas of causality and voter empowerment. Voters should be able to support their preferred candidate without fear that doing so might backfire. If Candidate X defeats another candidate with a certain level of support,

it would be logically inconsistent for Candidate X to lose after receiving even more support. A voting rule that violates this expectation creates perverse incentives by forcing voters to worry that honest and enthusiastic support could harm the very candidate they wish to help. In everyday terms, ranking Candidate X higher should help Candidate X, not hurt them.

Why This Matters: The Participation Paradox

Monotonicity violations have a troubling implication known as the participation paradox. If ranking Candidate X higher can cause Candidate X to lose, then it may be possible for a voter to better support Candidate X by abstaining from the election altogether or by ranking Candidate X lower. In extreme cases, staying home may be the most effective way to help Candidate X win.

For example, consider an election conducted under instant-runoff voting. Suppose that Candidate X would win if a particular voter did not participate. If that voter instead casts a ballot ranking Candidate X first, the added support may alter the elimination order in early rounds, ultimately causing Candidate X to lose. The voter's intention is to help Candidate X, but the effect of voting is the opposite. The rational strategy, paradoxically, is not to vote at all. This outcome is deeply counterintuitive and undermines the democratic ideal that participation and honest expression of preferences should be encouraged rather than punished.

Mathematical Explanation: How Violations Occur

Monotonicity violations arise from the internal mechanics of certain voting systems, particularly instant-runoff voting. These systems eliminate candidates sequentially, and the order of elimination plays a crucial role in determining the final outcome. Because the elimination process is path-dependent, small changes in the distribution of first-place votes can trigger entirely different elimination sequences, leading to different winners.

When a candidate is eliminated, their votes are transferred to other candidates according to the next preferences listed on those ballots. The destination of these transferred votes influences which candidate is eliminated next, and this process continues until a winner emerges. Changing how some voters rank Candidate X can redirect vote transfers in subtle but consequential ways, producing cascading effects throughout the successive rounds.

As a result, Candidate X may benefit from certain opponents being eliminated early and from receiving transferred votes later in the process. By increasing support for Candidate X early on, voters may inadvertently prevent a favorable elimination sequence from occurring. A different candidate may be eliminated instead, leading to a transfer pattern that ultimately deprives Candidate X of the support needed to win. Thus, although Candidate X gains direct support, the altered elimination order changes the flow of transferred votes in a way that harms Candidate X's overall chances.

In short, the additional support changes the path of the election, and the new path leads to a worse outcome. This counterintuitive result explains how a voting system can violate monotonicity even when voters act sincerely and consistently.

Example #7.7.1 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	5	4	3	2	1
First Choice	A	B	C	C	A
Second Choice	C	A	B	A	B
Third Choice	B	C	A	B	C

a. Use the plurality-with-elimination method to determine a winner.

b. Suppose that all 3 voters that voted C > B > A in the original election are persuaded to change their ballots to match the 5 voters that voted A > C > B and a re-election happens. Does this violate the monotonicity criterion?

✅ **Solution:**

a. Plurality-with-elimination method:

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Candidate A received 6 out of 15 first place votes or 40%
  - Candidate B received 4 out of 15 first place votes or 26.7%
  - Candidate C received 5 out of 15 first place votes or 33.3%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Candidate B received the fewest first place votes, so eliminate Candidate B by removing them from the table.

Preference ballot removing candidate B

Number of Voters	5	4	3	2	1
First Choice	A		C	C	A
Second Choice	C	A		A	
Third Choice		C	A		C

Then move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates A and C

Number of Voters	5	4	3	2	1
First Choice	A	A	C	C	A
Second Choice	C	C	A	A	C

- **Step 3**: Count first-place votes. (Round 2)
  - Candidate A received 10 out of 15 first place votes or 66.7%
  - Candidate C received 5 out of 15 first place votes or 33.3%
    - **So, a majority exists.**

Thus, Candidate A is the winner by the plurality-with-elimination method.

b. So, all 3 voters that voted C > B > A in the election are persuaded to change their ballots to match the 5 voters that voted A > C > B. In simplest terms, this means we eliminate the column of 3 voters that voted C > B > A and change the other column's number of voters that voted A > C > B from 5 to 8.

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	8	4	2	1
First Choice	A	B	C	A

Number of Voters	8	4	2	1
Second Choice	C	A	A	B
Third Choice	B	C	B	C

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the plurality-with-elimination method once again on the re-election.

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Candidate A received 9 out of 15 first place votes or 60%
  - Candidate B received 4 out of 15 first place votes or 26.7%
  - Candidate C received 2 out of 15 first place votes or 13.3%
    - **So, a majority exists.**

Candidate A wins the re-election.

Thus, Candidate A is the winner under the plurality-with-elimination method in the original election. Because Candidate A also wins the re-election after receiving increased support, the outcome does not violate the monotonicity criterion. Therefore, there is **NO** monotonicity violation.

Example #7.7.2 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	34	19	10	27
First Choice	A	B	B	C
Second Choice	B	C	A	A
Third Choice	C	A	C	B

a. Use the plurality-with-elimination method to determine a winner.
b. Suppose that 6 of the 10 voters that voted B > A > C in the election are persuaded to change their ballots to A > C > B and a re-election happens. Does this violate the monotonicity criterion?

✅ **Solution:**

a. Plurality-with-elimination method:

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Candidate A received 34 out of 90 first place votes or 37.8%
  - Candidate B received 29 out of 90 first place votes or 32.2%
  - Candidate C received 27 out of 90 first place votes or 30%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Candidate C received the fewest first place votes, so eliminate Candidate C by removing them from the table.

Preference ballot removing candidate C

Number of Voters	34	19	10	27
First Choice	A	B	B	
Second Choice	B		A	A
Third Choice		A		B

Then move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates A and B

Number of Voters	34	19	10	27
First Choice	A	B	B	A
Second Choice	B	A	A	B

- **Step 3**: Count first-place votes. (Round 2)
  - Candidate A received 61 out of 90 first place votes or 67.8%
  - Candidate B received 29 out of 90 first place votes or 32.2%
    - **So, a majority exists.**

Thus, Candidate A is the winner by the plurality-with-elimination method.

b. So, 6 of the 10 voters that voted B > A > C in the election are persuaded to change their ballots to vote A > C > B. In simplest terms, this means we add an extra column of 6 voters that vote A > C > B and change the other column's number of voters that voted B > A > C from 10 to 4.

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	34	19	4	27	6
First Choice	A	B	B	C	A
Second Choice	B	C	A	A	C
Third Choice	C	A	C	B	B

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the plurality-with-elimination method once again on the re-election.

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Candidate A received 40 out of 90 first place votes or 44.4%
  - Candidate B received 23 out of 90 first place votes or 25.6%
  - Candidate C received 27 out of 90 first place votes or 30%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Candidate B received the fewest first place votes, so eliminate Candidate B by removing them from the table.

Preference ballot removing candidate B

Number of Voters	34	19	4	27	6
First Choice	A			C	A
Second Choice		C	A	A	C
Third Choice	C	A	C		

Then move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates A and B

Number of Voters	34	19	4	27	6
First Choice	A	C	A	C	A
Second Choice	C	A	C	A	C

- **Step 3**: Count first-place votes. (Round 2)
  - Candidate A received 44 out of 90 first place votes or 48.9%
  - Candidate C received 46 out of 90 first place votes or 51.1%
    - **So, a majority exists.**
- Thus, Candidate C is the winner by the plurality-with-elimination method.

Candidate C wins the re-election.

Thus, Candidate A was the winner under the plurality-with-elimination method in the original election. Because Candidate C had won the re-election after Candidate A received increased support, the outcome does violate the monotonicity criterion. Therefore, **YES** there is a monotonicity violation.

Example #7.7.3 🤔

A city council is choosing one feature to build in a new public park. Residents rank the options based on preference. The city uses a plurality-with-elimination (instant-runoff) voting system.

Preference ballot showing voter rankings for candidates Dog Park, Garden, Pickle Ball Court, and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park	Garden	Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Playground	Garden	Pickle Ball Court
Fourth Choice	Garden	Dog Park	Playground	Garden

a. Use the plurality-with-elimination method to determine a winner.

b. Suppose that only 5 of the 10 voters that voted Dog Park > Playground > Pickle Ball Court > Garden in the election are persuaded to change their ballots to match the 36 voters that voted Playground > Dog Park > Pickle Ball Court > Garden and a re-election happens. Does this violate the monotonicity criterion?

✅ **Solution:**

a. Plurality-with-elimination method:

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Playground received 36 out of 100 first place votes or 36%
  - Pickle Ball Court received 30 out of 100 first place votes or 30%
  - Dog Park received 34 out of 100 first place votes or 34%
  - Garden received 0 out of 100 first place votes or 0%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Garden received the fewest first place votes, so eliminate Garden by removing them from the table.

Preference ballot removing candidate Garden

Number of Voters	36	30	24	10
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park		Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Playground		Pickle Ball Court
Fourth Choice		Dog Park	Playground	

Then move up all other features in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Dog Park, Pickle Ball Court, and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park	Playground	Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Dog Park	Playground	Pickle Ball Court

- **Step 3**: Count first-place votes. (Round 2)
  - Playground received 36 out of 100 first place votes or 36%
  - Pickle Ball Court received 30 out of 100 first place votes or 30%
  - Dog Park received 34 out of 100 first place votes or 34%
    - **So, no majority winner.**

- **Step 4**: Repeat Until a Winner Emerges
  - Eliminate the weakest candidate.

Pickle Ball Court received the fewest first place votes, so eliminate Pickle Ball Court by removing them from the table.

Preference ballot removing candidate Pickle Ball Court

Number of Voters	36	30	24	10
First Choice	Playground		Dog Park	Dog Park
Second Choice	Dog Park	Playground		Playground
Third Choice		Dog Park	Playground	

Then move up all other features in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Dog Park and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Playground	Dog Park	Dog Park
Second Choice	Dog Park	Dog Park	Playground	Playground

- **Step 3 (Repeated)**: Count first-place votes. (Round 3)
  - Playground received 66 out of 100 first place votes or 66%
  - Dog Park received 34 out of 100 first place votes or 34%
    - **So, a majority exists.**

Thus, Playground is the winner by the plurality-with-elimination method.

b. So, only 5 of the 10 voters that voted Dog Park > Playground > Pickle Ball Court > Garden in the election are persuaded to change their ballots to match the 36 voters that voted Playground > Dog Park > Pickle Ball Court > Garden. In simplest terms, this means we change the column of 10 voters that voted Dog Park > Playground > Pickle Ball Court > Garden from 10 to 5 and change the other column's of 36 voters that voted Playground > Dog Park > Pickle Ball Court > Garden from 36 to 41.

Preference ballot showing voter rankings for candidates Dog Park, Garden, Pickle Ball Court, and Playground

Number of Voters	41	30	24	5
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park	Garden	Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Playground	Garden	Pickle Ball Court
Fourth Choice	Garden	Dog Park	Playground	Garden

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the plurality-with-elimination method once again on the re-election.

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Playground received 41 out of 100 first place votes or 41%
  - Pickle Ball Court received 30 out of 100 first place votes or 30%
  - Dog Park received 29 out of 100 first place votes or 29%
  - Garden received 0 out of 100 first place votes or 0%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Garden received the fewest first place votes, so eliminate Garden by removing them from the table.

Preference ballot removing candidate Garden

Number of Voters	41	30	24	5
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park		Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Playground		Pickle Ball Court
Fourth Choice		Dog Park	Playground	

Then move up all other features in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Dog Park, Pickle Ball Court, and Playground

Number of Voters	41	30	24	5
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park	Playground	Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Dog Park	Playground	Pickle Ball Court

- **Step 3**: Count first-place votes. (Round 2)
  - Playground received 41 out of 100 first place votes or 41%
  - Pickle Ball Court received 30 out of 100 first place votes or 30%
  - Dog Park received 29 out of 100 first place votes or 29%
    - **So, no majority winner.**

- **Step 4**: Repeat Until a Winner Emerges
  - Eliminate the weakest candidate.

Dog Park received the fewest first place votes, so eliminate Dog Park by removing them from the table.

Preference ballot removing candidate Dog Park

Number of Voters	41	30	24	5
First Choice	Playground	Pickle Ball Court		
Second Choice		Playground	Pickle Ball Court	Playground
Third Choice	Pickle Ball Court		Playground	Pickle Ball Court

Then move up all other features in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Pickle Ball Court and Playground

Number of Voters	41	30	24	5
First Choice	Playground	Pickle Ball Court	Pickle Ball Court	Playground
Second Choice	Pickle Ball Court	Playground	Playground	Pickle Ball Court

- **Step 3 (Repeated)**: Count first-place votes. (Round 3)
  - Playground received 46 out of 100 first place votes or 46%
  - Pickle Ball Court received 54 out of 100 first place votes or 54%
    - **So, a majority exists.**

Thus, Pickle Ball Court is the winner by the plurality-with-elimination method.

Pickle Ball Court wins the re-election.

Thus, Playground was the winner under the plurality-with-elimination method in the original election. Because Pickle Ball Court had won the re-election after Playground received increased support, the outcome does violate the monotonicity criterion. Therefore, **YES** there is a monotonicity violation.

So, let's summarize what we have learned from the examples above.

Does the plurality-with-elimination method violate the monotonicity criterion?

YES. The plurality-with-elimination method violates the monotonicity criterion, because ranking a candidate higher can change the elimination order and cause them to lose. (As we saw in Examples #7.7.2 & 7.7.3).

Continuing the fairness criteria table from Section 7.3, we now examine how each previous voting method performs under the monotonicity criterion.

Does the Borda count method violate the monotonicity criterion?

The Borda count method **NEVER** violates the monotonicity criterion, because ranking a candidate higher increases their point total and cannot harm their chances of winning.

Does the plurality method violate the monotonicity criterion?

The plurality method **NEVER** violates the monotonicity criterion, because increasing a candidate's number of first-place votes cannot harm their chances of winning.

Again, continuing the fairness criteria table from Section 7.3, we now examine how the plurality-with-elimination method performs with respect to the previously studied criteria.

Does the plurality-with-elimination method violate the Majority criterion?

The plurality-with-elimination method **NEVER** violates the Majority criterion, because a candidate with a majority of first-place votes will not be eliminated and will win.

Does the plurality-with-elimination method violate the Condorcet criterion?

YES. The plurality-with-elimination method violates the Condorcet criterion, because a candidate who would win all head-to-head matchups may be eliminated early.

Thus, we can update our fairness criteria table with 5 more conclusions from Section 7.5:

Four Fairness Criteria Table (9/16 Complete)

Fairness Criterion ⟶ Major Voting Method ↓	Condorcet Criterion	Majority Criterion	Monotonicity Criterion	Independence of Irrelevant Alternatives Criterion
Plurality	**Violation Possible**	✓	✓	**?**
Borda Count	**Violation Possible**	**Violation Possible**	✓	**?**
Plurality-with-Elimination	**Violation Possible**	✓	**Violation Possible**	**?**
Pairwise Comparison	**?**	**?**	**?**	**?**

So, we are still looking for a major voting method that satisfies all four fairness criterion's and unfortunately, we have determined that the plurality-with-elimination method and the Borda count method have at least one possible violation for each.

Section 7.7: Monotonicity Criterion [*In-Class Exercises*]

1. Consider the following preference table with Candidates A, B, & C:

   Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	5	4	3	2
First Choice	A	B	C	C
Second Choice	B	A	A	B
Third Choice	C	C	B	A

   a. Use the plurality-with-elimination method to determine a winner.
   b. Suppose that 2 of the 3 voters that voted C > A > B in the original election are persuaded to change their ballots to A > C > B and a re-election happens. Does this violate the monotonicity criterion?
2. A survey of 36 college students were asked to rank their favorite series in order on some of the most watched series on Netflix. The results are summarized in a preference table:

   Preference ballot showing voter rankings for candidates Squid Games, Stranger Things, and Wednesday

Number of Voters	11	10	9	6
First Choice	Wednesday	Squid Games	Stranger Things	Stranger Things

Number of Voters	11	10	9	6
Second Choice	Stranger Things	Wednesday	Squid Games	Wednesday
Third Choice	Squid Games	Stranger Things	Wednesday	Squid Games

a. Use the plurality-with-elimination method to determine a winner.
b. Suppose that all 6 voters that voted Stranger Things > Wednesday > Squid Games in the original election are persuaded to change their ballots to match the 11 voters that voted Wednesday > Stranger Things > Squid Games and a re-election happens. Does this violate the monotonicity criterion?

3. A marine science department wants to vote on a department chair for the upcoming fiscal year. The results are summarized in a preference table:

Preference ballot showing voter rankings for candidates Dennis, Karen, Mary, and Rob

Number of Voters	7	6	5	3
First Choice	Dennis	Rob	Karen	Karen
Second Choice	Karen	Dennis	Rob	Rob
Third Choice	Rob	Karen	Mary	Dennis
Fourth Choice	Mary	Mary	Dennis	Mary

a. Use the plurality-with-elimination method to determine a winner.
b. Suppose that 3 of the 5 voters that voted Karen > Rob > Mary > Dennis in the election are persuaded to change their ballots to match the 7 voters that voted Dennis > Karen > Rob > Mary and a re-election happens. Does this violate the monotonicity criterion?

Answers available at the end of this section.

Answers

Section 7.7: Monotonicity Criterion [*In-Class Exercises*]

1. a. Candidate A
   b. Candidate A wins re-election; no, the monotonicity criterion is not violated
2. a. Wednesday
   b. Squid Games wins re-election; yes the monotonicity criterion is violated
3. a. Dennis
   b. Rob wins re-election; yes the monotonicity criterion is violated

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Section 7.7.H: Homework

Section 7.7: Monotonicity Criterion - Homework Exercises

(Section 7.7) - Homework Exercise #1

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	4	3	2	2
First Choice	A	B	C	C
Second Choice	B	C	B	A
Third Choice	C	A	A	B

a. Use the **plurality-with-elimination method** to determine a winner.
b. Suppose that all 3 voters that voted B > C > A in the original election are persuaded to change their ballots to match the 2 voters that voted C > A > B and a re-election happens. Does this violate the monotonicity criterion?

Answer available at the end of this section.

(Section 7.7) - Homework Exercise #2

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	14	12	11	8
First Choice	C	B	A	A
Second Choice	A	C	B	C
Third Choice	B	A	C	B

a. Use the **plurality-with-elimination method** to determine a winner.
b. Suppose that all 8 voters that voted A > C > B in the original election are persuaded to change their ballots to C > A > B and a re-election happens. Does this violate the monotonicity criterion?

Answer available at the end of this section.

(Section 7.7) - Homework Exercise #3

A school is voting for the winner of its talent show. The finalists are:

- D = Dancer 💃
- M = Magician 🎩
- S = Singer 🎤

A total of 30 students vote by ranking the performers from favorite to least favorite.

Preference ballot showing voter rankings for candidates Dancer, Magician, and Singer

Number of Voters	11	10	9
First Choice	S	M	D
Second Choice	D	D	M
Third Choice	M	S	S

a. Use the **plurality-with-elimination method** to determine a winner.
b. After seeing the performances again, 4 of the 11 students who originally preferred the Singer (S) decide they actually like the Musician (M) better and update their ballots from S > D > M to M > D > S. Now, a re-election then happens. Does this violate the monotonicity criterion when using the **plurality-with-elimination method**?
c. On the original election, use the **Borda count method** to determine a winner.
d. Suppose 3 of the 10 students who originally preferred the Musician (M) decide they actually like the Dancer (D) better and update their ballots from M > D > S to D > M > S. Now, a re-election then happens. Does this violate the monotonicity criterion when using the **Borda count**?
e. On the original election, use the **plurality method** to determine a winner.
f. Suppose 3 of the 10 students who originally preferred the Musician (M) decide they actually like the Singer (S) better and update their ballots from M > D > S to S > D > M. Now, a re-election then happens. Does this violate the monotonicity criterion when using the **plurality method**?

Answer available at the end of this section.

(Section 7.7) - Homework Exercise #4

A company with 200 employees is voting on a new office location:

- D = Downtown 🏙️
- S = Suburbs 🏡
- T = Tech Hub 💻
- W = Waterfront 🌊

Employees rank the options from most to least preferred.

Preference ballot showing voter rankings for candidates Downtown, Suburbs, Tech Hub, and Waterfront

Number of Voters	74	54	48	24
First Choice	D	W	T	T
Second Choice	W	D	W	D
Third Choice	T	T	S	W

Number of Voters	74	54	48	24
Fourth Choice	S	S	D	S

a. Use the **plurality-with-elimination method** to determine a winner.
b. After the results, the committee meets again and discusses any possible changes. Suppose 19 of the 24 staff members who voted T > D > W > S who originally preferred the Tech Hub (T) decide that Downtown (D) is a better option and update their ballots from T > D > W > S to D > W > T > S. Now, a re-election then happens. Does this violate the monotonicity criterion when using the **plurality-with-elimination method**?
c. In part b), what if only 17 (not 19) of the 24 staff members who voted T > D > W > S who originally preferred the Tech Hub (T) decide that Downtown (D) is a better option and update their ballots from T > D > W > S to D > W > T > S. Now, a re-election then happens. Does this violate the monotonicity criterion when using the **plurality-with-elimination method**?
d. On the original election, use the **plurality method** to determine a winner.
e. From the original election, suppose 19 of the 24 staff members who voted T > D > W > S who originally preferred the Tech Hub (T) decide that Downtown (D) is a better option and update their ballots from T > D > W > S to D > W > T > S. Now, a re-election then happens. Does this violate the monotonicity criterion when using the **plurality method**?

Answer available at the end of this section.

Answers

(Section 7.7) - Homework Exercise #1

a. Candidate C wins.
b. Candidate C wins using the plurality-with-elimination method in both elections. Since improving Candidate C's ranking does not change the outcome, there is no violation of the monotonicity criterion. **(NO)**

(Section 7.7) - Homework Exercise #2

a. Candidate C wins.
b. Candidate C wins using the plurality-with-elimination method in the original election, but after some voters rank Candidate C higher, Candidate B wins using the plurality-with-elimination method. Therefore, the method violates the monotonicity criterion. **(YES)**

(Section 7.7) - Homework Exercise #3

a. Musician (M) wins.
b. Musician (M) wins using the plurality-with-elimination method in the original election, but after some voters rank Musician (M) higher, Dancer (D) wins using the plurality-with-elimination method. Therefore, the method violates the monotonicity criterion. **(YES)**
c. Dancer (D) = 69 points, Musician (M) = 59 points, Singer (S) = 52 points. Dancer (D) wins.
d. Dancer (D) wins using the Borda count method in both elections. Since improving Dancer (D) ranking does not change the outcome, there is no violation of the monotonicity criterion. **(NO)**
e. Singer (S) wins.
f. Singer (S) wins using the plurality method in both elections. Since improving Singer's (S) ranking does not change the outcome, there is no violation of the monotonicity criterion. **(NO)**

(Section 7.7) - Homework Exercise #4

a. Downtown (D) wins.
b. Downtown (D) wins using the plurality-with-elimination method in the original election, but after some voters rank Downtown (D) higher, Waterfront (W) wins using the plurality-with-elimination method. Therefore, the method violates the monotonicity criterion. **(YES)**
c. Downtown (D) wins using the plurality-with-elimination method in both elections. Since improving Downtown's (D) ranking does not change the outcome, there is no violation of the monotonicity criterion. **(NO)**
d. Downtown (D) wins.
e. Downtown (D) wins using the plurality method in the original election, but after some voters rank Downtown (D) higher, Waterfront (W) wins using the plurality method. Therefore, the method violates the monotonicity criterion. **(YES)**

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Section 7.8: Pairwise Comparison Method

Learning Objectives

- Determine the winner of an election using the pairwise comparison method

Keep in mind that there is no fixed number of voting methods; in principle, infinitely many voting methods exist, although only a small number of representative methods are commonly studied. In the previous sections, we looked at three of those four commonly studied major voting methods. So, now let's introduce our last major voting method called the pairwise comparison method:

Major Voting Method #4 - Parwise Comparison

The **pairwise comparison method** is a voting procedure in which voters rank all candidates in order of preference, and every candidate is compared head-to-head with each of the others. In each comparison, the candidate preferred by a majority of voters receives 1 point, while a tie awards $\frac{1}{2}$ to each candidate. Any loss, the candidate receives 0 points. After all pairwise comparisons are completed, the candidate with the greatest total number of points is declared the winner. The winner is referred to as the Condorcet winner.

Note: The Condorcet winner should not be confused with the Condorcet criterion. The Condorcet criterion is a fairness standard used to evaluate voting methods, not a voting method itself. However, both the pairwise comparison method and the Condorcet criterion rely on the same head-to-head matchup framework, which is why these concepts are often confused.

Example #7.8.1 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	5	4	2
First Choice	B	C	C
Second Choice	A	A	B
Third Choice	C	B	A

Use the pairwise comparison method to determine a winner..

✅ **Solution:**

Use a table for every possible match-up to record your results.

A	B
	5
4	
	2
4	**7**

B wins A

A	C
5	
	4
	2
5	**6**

C wins A

B	C
5	
	4
	2
5	**6**

C wins B

Here are the head-to-head results:

Candidate A: 0 wins = 0 points
Candidate B: 1 win = 1 point
Candidate C: 2 wins = 2 points

Hence, Candidate C has the most points and wins the election by the pairwise comparison method.

Example #7.8.2 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	5	4	3	2
First Choice	B	C	C	A
Second Choice	A	B	A	B
Third Choice	C	A	B	C

Use the pairwise comparison method to determine a winner.

✅ **Solution:**

Use a table for every possible match-up to record your results.

A	B
	5
	4
3	
2	
5	**9**

B wins A

A	C
5	
	4
	3
2	
7	**7**

A ties C

B	C
5	
	4
	3
2	
7	**7**

B ties C

Here are the head-to-head results:

Candidate A: 0 wins & 1 tie = $\frac{1}{2}$ point = 0.5 points
Candidate B: 1 win & 1 tie = 1 + $\frac{1}{2}$=1.5 points
Candidate C: 0 wins & 2 ties = $\frac{1}{2} + \frac{1}{2}$ = 1 point

Hence, Candidate B has the most points and wins the election by the pairwise comparison method.

Example #7.8.3 🤔

A group of people is choosing where to eat lunch. The four fast-food options are: McDonald's, KFC, Subway, or Taco Bell. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates KFC, McDonald's, Subway, and Taco Bell

Number of Voters	6	5	4	3
First Choice	McDonald's	Subway	Taco Bell	KFC
Second Choice	Subway	Taco Bell	McDonald's	McDonald's
Third Choice	Taco Bell	McDonald's	KFC	Subway
Fourth Choice	KFC	KFC	Subway	Taco Bell

Use the pairwise comparison method to determine a winner.

✅ **Solution:**

Use a table for every possible match-up to record your results.

McDonald's	Subway	McDonald's	Taco Bell	McDonald's	KFC	Subway	Taco Bell	Subway	KFC	Taco Bell	KFC
6		6		6		6		6		6	
	5		5	5		5		5		5	
4			4	4			4		4	4	
3		3			3	3			3		3
13	**5**	**9**	**9**	**15**	**3**	**14**	**4**	**11**	**7**	**15**	**3**
McDonald's wins	Subway	McDonald's ties	Taco Bell	McDonald's wins	KFC	Subway wins	Taco Bell	Subway wins	KFC	Taco Bell wins	KFC

Here are the head-to-head results:

- McDonald's: 2 wins & 1 tie = 2 + $\frac{1}{2}$ = 2.5 points
- Subway: 2 wins = 2 points
- Taco Bell: 1 wins & 1 ties = 1 + $\frac{1}{2}$ = 1.5 points
- KFC: 0 wins = 0 points

Hence, McDonald's has the most points and wins the election by the pairwise comparison method.

Example #7.8.4 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	8	7	6
First Choice	A	B	C
Second Choice	B	C	A
Third Choice	C	A	B

Use the pairwise comparison method to determine a winner..

✅ **Solution:**

Use a table for every possible match-up to record your results.

A	B
8	
	7
6	
14	**7**
A wins B	

A	C
8	
	7
	6
8	**13**
C wins A	

B	C
8	
7	
	6
15	**6**
B wins C	

Here are the head-to-head records:

A: 1 win = 1 point
B: 1 win = 1 point
C: 1 win = 1 point

Hence, there is a 3-way tie. Candidate A, Candidate B, and Candidate C are all tied with 1 point each.

Despite its strengths, the pairwise comparison method is not without drawbacks. In some elections, as we see in Example #7.8.4, no candidate defeats all others in head-to-head contests. This situation, known as the Condorcet paradox, can produce circular preferences (for example, A beats B, B beats C, but C beats A). When this occurs, the method may rely on point totals that depend on how ties and wins are scored. Additionally, because the method requires many comparisons, it can become cumbersome when there are many candidates.

Section 7.8: Pairwise Comparison Method [*In-Class Exercises*]

1. Consider the following preference table with Candidates A, B, & C:

   Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	**7**	**6**	**6**	**5**	**3**
First Choice	C	B	A	B	A
Second Choice	A	A	B	C	C
Third Choice	B	C	C	A	B

   Use the pairwise comparison method to determine a winner.

2. A group of 41 people is choosing where to eat lunch during their meeting. The four fast food delivery options are: Board & Brew, Naugles, T.K. Burgers, or In-N-Out Burgers. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Board & Brew, In-N-Out, Naugles, and T.K. Burgers

Number of Voters	11	9	8	6	4	3
First Choice	In-N-Out	T.K. Burgers	Naugles	Board & Brew	T.K.Burgers	In-N-Out
Second Choice	Naugles	Naugles	T.K.Burgers	T.K. Burgers	In-N-Out	T.K. Burgers
Third Choice	T.K. Burgers	Board & Brew	In-N-Out	In-N-Out	Naugles	Board & Brew
Fourth Choice	Board & Brew	In-N-Out	Board & Brew	Naugles	Board & Brew	Naugles

Use the pairwise comparison method to determine a winner.

3. A 24-member committee is selecting a chairperson. The 3 candidates are David (D), Edward (E), and Fernando (F). Each committee member completely ranked the candidates on a separate ballot. The preference schedule is listed below.

Preference ballot showing voter rankings for candidates Davis, Edward, and Frank

Number of Voters	8	6	6	4
First Choice	D	E	F	E
Second Choice	E	F	D	D
Third Choice	F	D	E	F

Use the pairwise comparison method to determine a winner.

Answers available at the end of this section.

Answers

Section 7.8: Pairwise Comparison Method [*In-Class Exercises*]

1. Candidate A = 2 points, Candidate B = 1 point, Candidate C = 0 points. Candidate A wins.
2. In-N-Out = 2 pts; T.K. Burgers = 3 pts; Naugles = 1 pt; Board & Brew = 0 pts. T.K. Burgers wins.
3. David = 1.5 points, Edward = 1, Fernando = 0.5 points. David wins.

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Section 7.8.H: Homework

Section 7.8: Pairwise Comparison Method - Homework Exercises

(Section 7.8) - Homework Exercise #1

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	4	3	2	2
First Choice	A	B	C	C
Second Choice	B	C	B	A
Third Choice	C	A	A	B

Use the pairwise comparison method to determine a winner.

Answer available at the end of this section.

(Section 7.8) - Homework Exercise #2

Consider the following preference table with Candidates A, B, C, & D:

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	9	8	6	4	3
First Choice	D	C	A	A	C
Second Choice	B	B	C	C	D
Third Choice	A	A	D	B	A
Fourth Choice	C	D	B	D	B

Use the pairwise comparison method to determine a winner.

Answer available at the end of this section.

(Section 7.8) - Homework Exercise #3

A group of 30 office coworkers are choosing where to eat lunch during their meeting. The restaurant options are: The Habit, Chick-Fil-A, and Panera Bread. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Chick-Fil-A, The Habit, and Panera Bread

Number of Voters	11	9	6	4
First Choice	The Habit	Chick-Fil-A	Panera Bread	Chick-Fil-A
Second Choice	Panera Bread	Panera Bread	Chick-Fil-A	The Habit
Third Choice	Chick-Fil-A	The Habit	The Habit	Panera Bread

Use the pairwise comparison method to determine a winner.

Answer available at the end of this section.

(Section 7.8) - Homework Exercise #4

A class of 37 students is voting on a location for their end-of-year field trip. The four options are the Aquarium of the Pacific (A), the Butterfly Release at the Botanical Gardens (B), Disneyland (D), or the Griffith Observatory (G). The results are in the preference table below.

Preference ballot showing voter rankings for candidates A, B, C, and D

Number of Voters	14	10	8	4	1
First Choice	A	B	D	D	B
Second Choice	G	G	B	G	D
Third Choice	B	D	A	B	G
Fourth Choice	D	A	G	A	A

Use the pairwise comparison method to determine a winner.

Answer available at the end of this section.

(Section 7.8) - Homework Exercise #5

A Los Angeles county Buffalo Wild Wings polled their customers and asked them to rank the local L.A. professional sports teams. Here are the results listed in a preference table.

Preference ballot showing voter rankings for candidates Chargers, Clippers, Dodgers, Kings, Lakers, and Rams

Number of Voters	15	13	9	9	4	3	3	1
First Choice	Dodgers	Lakers	Dodgers	Lakers	Rams	Lakers	Kings	Clippers
Second Choice	Lakers	Dodgers	Rams	Dodgers	Chargers	Rams	Rams	Chargers
Third Choice	Rams	Kings	Lakers	Kings	Kings	Dodgers	Lakers	Kings
Fourth Choice	Kings	Rams	Clippers	Clippers	Dodgers	Kings	Clippers	Dodgers
Fifth Choice	Clippers	Chargers	Kings	Rams	Clippers	Chargers	Chargers	Rams
Sixth Choice	Chargers	Clippers	Chargers	Chargers	Lakers	Clippers	Dodgers	Lakers

Use the pairwise comparison method to determine a winner.

Answer available at the end of this section.

Answers

(Section 7.8) - Homework Exercise #1

A = 1 pt, B = 1 pt, C = 1 pt. Tie between Candidate A, Candidate B, and Candidate C. No single pairwise comparison winner exists.

(Section 7.8) - Homework Exercise #2

A = 2 pt, B = 1 pt, C = 2 pt, D = 1 pt. Tie between Candidate A and Candidate C. No single pairwise comparison winner exists.

(Section 7.8) - Homework Exercise #3

The Habit = 0.5 pt, Chick-Fil-A = 1 pt, Panera Bread = 1.5 pt. Panera Bread is the winner.

(Section 7.8) - Homework Exercise #4

Aquarium = 1 pt, Botanical Gardens = 3 pts, Disneyland = 1 pt, Griffith Observatory = 1 pt. Botanical Gardens wins.

(Section 7.8) - Homework Exercise #5

Dodgers = 5 pts, Lakers = 4 pts, Rams = 3 pts, Kings = 2 pts, Clippers = 1 pt, Chargers = 0 points. Dodgers win

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Section 7.9: Independence of Irrelevant Alternatives (IIA) Criterion

Learning Objectives

- Decide if an election violates the Independence of Irrelevant Alternatives criterion

So far, we have explored several voting methods, such as plurality, Borda count, plurality-with-elimination, and pairwise comparison—and evaluated them using different fairness criteria. Each criterion helps us determine whether a voting method produces results that align with our expectations of fairness. The Independence of Irrelevant Alternatives (IIA) criterion focuses on how the presence or absence of other candidates affects the outcome of an election.

Independence of Irrelevant Alternatives (IIA) (Fairness Criterion #4)

If a candidate is the winner of an election, then removing a non-winning candidate (an "irrelevant alternative") should not change the identity of the winner.

In other words, the final outcome should depend only on the relative preferences between the top candidates—not on candidates who are not in contention to win. If removing a losing candidate causes a different candidate to win, then the voting method violates the IIA criterion.

This criterion highlights an important question about fairness: Should a candidate who has no chance of winning be able to influence who does win?

As we examine this criterion, we will see that some commonly used voting methods can produce surprising results when "irrelevant" options are removed.

Example #7.9.1 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	5	4	3	2	1
First Choice	A	B	C	C	A
Second Choice	C	A	B	A	B
Third Choice	B	C	A	B	C

a. Use the pairwise comparison method to determine a winner.
b. Suppose Candidate B drops out and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

c. Suppose Candidate C drops out (instead of Candidate B) and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

✅ **Solution:**

a. Use a table for every possible match-up to record your results.

A	B
5	
	4
	3
2	
1	
8	**7**

A wins B

A	C
5	
4	
	3
	2
1	
10	**5**

A wins C

B	C
	5
4	
	3
	2
1	
5	**10**

C wins B

Here are the head-to-head results:

- Candidate A: 2 wins = 2 points
- Candidate B: 0 wins = 0 points
- Candidate C: 1 win = 1 point

Thus, Candidate A has the most points and wins the election by the pairwise comparison method.

---

b. So, Candidate B has dropped out. Then, we remove Candidate B and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates A and C

Number of Voters	5	4	3	2	1
First Choice	A	A	C	C	A
Second Choice	C	C	A	A	C

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the pairwise comparison method once again on the re-election.

A	C
5	
4	
	3
	2
1	
10	**5**

A wins C

Here are the head-to-head results:

- Candidate A: 1 wins = 1 point
- Candidate C: 0 win = 0 points

Thus, Candidate A has the most points and wins the re-election by the pairwise comparison method.

Candidate A is the winner under the plurality-with-elimination method in the original election. Since Candidate A also remains the winner after the irrelevant alternative, Candidate B, is removed, the outcome does not violate the independence of irrelevant alternatives criterion. **(NO)**

---

c. So, Candidate C has dropped out. Then, we remove Candidate C and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates A and B

Number of Voters	5	4	3	2	1
First Choice	A	B	B	A	A
Second Choice	B	A	A	B	b

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the pairwise comparison method once again on the re-election.

A B
5
4
3
2
1
8 **7**
A wins B

Here are the head-to-head results:

- Candidate A: 1 wins = 1 point
- Candidate B: 0 win = 0 points

Thus, Candidate A has the most points and wins the re-election by the pairwise comparison method.

Candidate A is the winner under the plurality-with-elimination method in the original election. Since Candidate A also remains the winner after the irrelevant alternative, Candidate C, is removed, the outcome does not violate the independence of irrelevant alternatives criterion. **(NO)**

Example #7.9.2 🤔

Starting in the 2026-27 school year, Canyon High School in Anaheim, CA, will change its mascot from Comanches to one of the following names:

- Cavaliers
- Cobras
- Commanders
- Crows

This change complies with new California legislation phasing out Native American nicknames. The Canyon Color Guard team gets to vote early before any of the other students and ranks their choices.The results are below:

Preference ballot showing voter rankings for candidates Cavaliers, Cobras, Commanders, and Crows

Number of Voters	6	5	5	4
First Choice	Cobras	Commanders	Cavaliers	Crows
Second Choice	Crows	Cavaliers	Cobras	Commanders
Third Choice	Commanders	Cobras	Crows	Cavaliers
Fourth Choice	Cavaliers	Crows	Commanders	Cobras

a. Based on these results, use the pairwise comparison method to determine a winner.
b. After further deliberation, administration decided to omit Crows from the choices and the votes were recounted. Does this violate the independence of irrelevant alternatives criterion?
c. Based on the original election, what if the administration decided to omit Cavaliers instead from the four choices and the votes were recounted. Does this violate the independence of irrelevant alternatives criterion?
d. Based on the original election, what if the administration decided to omit Commanders instead from the four choices and the votes were recounted. Does this violate the independence of irrelevant alternatives criterion?

✅ **Solution:**

a. Use a table for every possible match-up to record your results.

Cobras	Commanders
6	
	5
5	
	4
11	**9**
Cobras wins Commanders	

Cobras	Cavaliers
6	
	5
	5
	4
6	**14**
Cavaliers wins Cobras	

Cobras	Crows
6	
5	
5	
	4
16	**4**
Cobras wins Crows	

Commanders	Cavaliers
6	
5	
	5
4	
15	**5**
Commanders wins Cavaliers	

Commanders	Crows
	6
5	
	5
	4
5	**15**
Crows wins Commanders	

Cavaliers	Crows
	6
5	
5	
	4
10	**10**
Cavaliers ties Crows	

Here are the head-to-head results:

- Cobras: 2 wins = 2 points
- Cavaliers: 1 win, 1 tie = 1.5 points
- Commanders: 1 win = 1 point
- Crows: 1 win, 1 tie = 1.5 points

Thus, Cobras have the most points and wins the election by the pairwise comparison method.

---

b. So, Crows is being dropped as a possible choice. Then, we remove Crows and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Cavaliers, Cobras, and Commanders

Number of Voters	6	5	5	4
First Choice	Cobras	Commanders	Cavaliers	Commanders
Second Choice	Commanders	Cavaliers	Cobras	Cavaliers
Third Choice	Cavaliers	Cobras	Commanders	Cobras

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the pairwise comparison method once again on the re-election.

Cobras	Commanders
6	
	5
5	
	4
11	**9**

Cobras wins Commanders

Cobras	Cavaliers
6	
	5
	5
	4
6	**14**

Cavaliers wins Cobras

Commanders	Cavaliers
6	
5	
	5
4	
15	**5**

Commanders wins Cavaliers

Here are the head-to-head results:

- Cobras: 1 win = 1 point
- Cavaliers: 1 win = 1 point
- Commanders: 1 win = 1 point

Thus, there is a three-way tie, so there is no single mascot name that wins the re-election by the pairwise comparison method.

Cobras are the winner under the pairwise comparison method in the original election. However, there is a three-way tie in the re-election after the irrelevant alternative, Crows, was removed. Thus, the outcome does violate the independence of irrelevant alternatives criterion. **(YES)**

---

c. So, Cavaliers is being dropped as a possible choice. Then, we remove Cavaliers and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Cobras, Commanders, and Crows

Number of Voters	**6**	**5**	**5**	**4**
First Choice	Cobras	Commanders	Cobras	Crows
Second Choice	Crows	Cobras	Crows	Commanders
Third Choice	Commanders	Crows	Commanders	Cobras

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the pairwise comparison method once again on the re-election.

Cobras	Commanders
6	
	5
5	
	4
11	**9**

Cobras wins Commanders

Cobras	Crows
6	
5	
5	
	4
16	**4**

Cobras wins Crows

Commanders	Crows
	6
5	
	5
	4
5	**15**

Crows wins Commanders

Here are the head-to-head results:

- Cobras: 2 wins = 2 points
- Commanders: 0 win = 0 points
- Crows: 1 win = 1 point

Thus, Cobras have the most points and wins the election by the pairwise comparison method.

Cobras is the winner under the pairwise comparison method in the original election. Since Cobras also remains the winner after the irrelevant alternative, Cavaliers, is removed, the outcome does not violate the independence of irrelevant alternatives criterion. **(NO)**

---

d. So, Commanders is being dropped as a possible choice. Then, we remove Commanders and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Cavaliers, Cobras, and Crows

Number of Voters	6	5	5	4
First Choice	Cobras	Cavaliers	Cavaliers	Crows
Second Choice	Crows	Cobras	Cobras	Cavaliers
Third Choice	Cavaliers	Crows	Crows	Cobras

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the pairwise comparison method once again on the re-election.

Cobras	Cavaliers
6	
	5
	5
	4
6	**14**

Cavaliers wins Cobras

Cobras	Crows
6	
5	
5	
	4
16	**4**

Cobras wins Crows

Cavaliers	Crows
	6
5	
5	
	4
10	**10**

Cavaliers ties Crows

Here are the head-to-head results:

- Cobras: 1 win = 1 point
- Cavaliers: 1 win, 1 tie = 1.5 points
- Crows: 0 wins, 1 tie = 0.5 points

Thus, Cavaliers have the most points and wins the re-election by the pairwise comparison method.

Cobras are the winner under the pairwise comparison method in the original election. However, Cavaliers wins in the re-election after the irrelevant alternative, Commanders, was removed. Thus, the outcome does violate the independence of irrelevant alternatives criterion. **(YES)**

So, based off our last example.....

Does the pairwise comparison method violate the independence of irrelevant alternatives (IIA) criterion?

(As we saw in Examples #7.9.2b & 7.9.2d). **YES**. The pairwise comparison method (using point totals) violates the IIA criterion, because removing a non-winning candidate may change the total points and alter the election outcome.

Continuing the fairness criteria table from Section 7.3, we now examine how each voting method performs under the IIA criterion.

Does the plurality-with-elimination method violate the IIA criterion?

YES. The plurality-with-elimination method violates the IIA criterion because removing a non-winning candidate can change the order of elimination and alter the redistribution of votes, potentially leading to a different winner.

Does the Borda count method violate the IIA criterion?

YES. The Borda count method violates the IIA criterion, because removing a non-winning candidate changes the point distribution on each ballot, which can alter the total scores and lead to a different winner.

Does the plurality method violate the IIA criterion?

YES. The plurality method violates the IIA criterion, because removing a non-winning candidate can change the distribution of first-place votes and alter the winner.

Again, continuing the fairness criteria table from Section 7.3, we now examine how the pairwise comparison method performs with respect to the previously studied criteria.

Does the pairwise comparison method violate the monotonicity criterion?

NO. The pairwise comparison method **NEVER** violates the monotonicity criterion, because improving a candidate's position on some ballots can only increase or maintain their performance in head-to-head matchups. Therefore, a candidate cannot be harmed by being ranked higher.

Does the pairwise comparison method violate the majority criterion?

NO. The pairwise comparison method **NEVER** violates the majority criterion, because a candidate with a majority of first-place votes will defeat every other candidate in head-to-head comparisons and therefore receive the highest number of points.

The pairwise comparison method satisfies the majority criterion

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Does the pairwise comparison method violate the Condorcet criterion?

NO. The pairwise comparison method **NEVER** violates the Condorcet criterion, because a Condorcet winner, by definition, wins every head-to-head matchup and therefore earns the highest total number of points.

Thus, we can update our fairness criteria table with 7 more conclusions from Section 7.7 and the fairness criteria table is now complete:

Four Fairness Criteria Table (100% Complete)

Fairness Criterion ⟶ Major Voting Method ↓	Condorcet Criterion	Majority Criterion	Monotonicity Criterion	Independence of Irrelevant Alternatives Criterion
Plurality	**Violation Possible**	✓	✓	**Violation Possible**
Borda Count	**Violation Possible**	**Violation Possible**	✓	**Violation Possible**
Plurality-with-Elimination	**Violation Possible**	✓	**Violation Possible**	**Violation Possible**

Fairness Criterion → / Major Voting Method ↓	Condorcet Criterion	Majority Criterion	Monotonicity Criterion	Independence of Irrelevant Alternatives Criterion
Pairwise Comparison	✓	✓	✓	**Violation Possible**

After examining how different voting methods perform under each fairness criterion, we notice that no single method satisfies all of them. Each method has strengths in some areas and weaknesses in others. The **plurality method** is simple but limited in its ability to reflect voter preferences. The **Borda count** is more sensitive to rankings but can be influenced by changes in candidate sets. The **plurality-with-elimination method** is intuitive but can produce unstable outcomes in certain situations. The **pairwise comparison method** satisfies several fairness criteria, though it is not without its own shortcomings. This raises an important question: Is it possible to design a voting system that satisfies all fairness criteria at the same time? Arrow's Impossibility Theorem provides a surprising answer to this question.

Arrow's Impossibility Theorem

When there are three or more candidates, no voting method can satisfy all of the following fairness conditions at the same time:

- The system reflects voters' preferences (no dictatorship)
- The method works for all possible preference rankings
- The method satisfies reasonable fairness criteria such as independence of irrelevant alternatives

Arrow's theorem tells us that there is no perfect voting system. No matter which method we choose, some fairness criteria will be satisfied, while others will inevitably be violated.

There are many possible combinations of voting methods and fairness criteria to consider, and examining these relationships helps highlight the trade-offs involved in choosing an appropriate voting system. Here are a few more examples using a mix of the methods and criterion.

Historical Note: Kenneth Arrow

Kenneth Arrow developed Arrow's Impossible Theorem while studying voting systems and collective decision-making



in the early 1950's. He was trying to answer a fundamental question in social choice theory:

Can a voting system fairly combine individual preferences into one collective decision?

In his research, Arrow examined whether a ranked voting system could satisfy several reasonable fairness conditions at the same time. These conditions included ideas such as:

- if every voter prefers one option over another, the group ranking should reflect that preference,
- the voting system should not be controlled by a single dictator,
- and irrelevant alternatives should not unfairly affect the outcome.

Arrow proved that when there are three or more choices, no ranked voting system can satisfy all of these fairness conditions simultaneously. In other words, every voting method must involve some trade-offs or imperfections.

He published this result in 1951 in his book Social Choice and Individual Values. The theorem became one of the foundational results in economics, political science, mathematics, and voting theory.

Example #7.9.3 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	9	8	7	6	4
First Choice	A	B	C	C	B
Second Choice	B	C	B	A	A
Third Choice	C	A	A	B	C

a. Use the plurality method to determine a winner.
b. Suppose Candidate A drops out and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?
c. Suppose Candidate B drops out (instead of Candidate A) and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

✅ **Solution:**

a. Looking at the First Choice row,

- A = 9 first place votes
- B = 8 + 5 = 12 first place votes
- C = 7 + 6 = 13 first place votes

Thus, Candidate C is the winner by the plurality method.

---

b. So, Candidate A has dropped out. Then, we remove Candidate A and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates B and C

Number of Voters	9	8	7	6	4
First Choice	B	B	C	C	B

Number of Voters	9	8	7	6	4
Second Choice	C	C	B	B	C

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the plurality method once again on the re-election.

Looking at the First Choice row,

- B = 9 + 8 + 4 = 21 first place votes
- C = 7 + 6 = 13 first place votes

Thus, Candidate B is the winner by the plurality method. Candidate B wins the re-election.

Candidate C is the winner under the plurality method in the original election. However, Candidate B is the winner after the irrelevant alternative, Candidate A, is removed. Therefore, there is a violation of the independence of irrelevant alternatives criterion. **(YES)**

---

c. So, Candidate B has dropped out. Then, we remove Candidate B and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates A and C

Number of Voters	9	8	7	6	4
First Choice	A	C	C	C	A
Second Choice	C	A	A	A	C

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the plurality method once again on the re-election.

Looking at the First Choice row,

- A = 9 + 4 = 13 first place votes
- C = 8 + 7 + 6 = 21 first place votes

Thus, Candidate C is the winner by the plurality method. Candidate C wins the re-election.

Candidate C is the winner under the plurality method in the original election. Since Candidate C also remains the winner after the irrelevant alternative, Candidate B, is removed, the outcome does not violate the independence of irrelevant alternatives criterion. **(NO)**

Example #7.9.4 🤔

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	7	5	2
First Choice	B	C	A

Number of Voters	7	5	2
Second Choice	C	A	B
Third Choice	A	B	C

a. Use the Borda count method to determine a winner.
b. Suppose Candidate B drops out and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?
c. Suppose Candidate A drops out (instead of Candidate B) and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

✅ **Solution:**

- **Step 1**: Assign the points based on position. Since there are 3 candidates, a candidate receives:
  - 3 points for each 1st-place ranking
  - 2 points for each 2nd-place ranking
  - 1 point for each 3rd-place ranking

- **Step 2**: Add up all the points each candidate receives.
  - Candidate A's Rankings:
    - **1st place rankings**: Column 3 → 2 votes × 3 points = **6 points**
    - **2nd place rankings**: Column 2 → 5 votes × 2 points = **10 points**
    - **3rd place rankings**: Column 1 → 7 votes × 1 point = **7 points**
      - Candidate A's total: **6 + 10 + 7 = 23 points**
  - Candidate B's Rankings:
    - **1st place rankings**: Column 1 → 7 votes × 3 points = **21 points**
    - **2nd place rankings**: Column 3 → 2 votes × 2 points = **4 points**
    - **3rd place rankings**: Column 2 → 5 votes × 1 point = **5 points**
      - Candidate B's total: **21 + 4 + 5 = 30 points**
  - Candidate C's Rankings:
    - **1st place rankings**: Column 2 → 5 votes × 3 points = **15 points**
    - **2nd place rankings**: Column 1 → 7 votes × 2 points = **14 points**
    - **3rd place rankings**: Column 3 → 2 votes × 1 point = **2 points**
      - Candidate C's total: **15 + 14 + 2 = 31 points**

- **Step 3**: The candidate with the highest total wins.
  - Candidate A's total score = **23 points**
  - Candidate B's total score = **30 points**
  - Candidate C's total score = **31 points**

Thus, Candidate C is the winner by the Borda count method.

---

b. So, Candidate B is being dropped as a possible choice. Then, we remove Candidate B and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates A and C

Number of Voters	7	5	2
First Choice	C	C	A
Second Choice	A	A	C

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the pairwise comparison method once again on the re-election.

- **Step 1**: Assign the points based on position. Since there are 2 candidates, a candidate receives:
  - 2 points for each 1st-place ranking
  - 1 points for each 2nd-place ranking
- **Step 2**: Add up all the points each candidate receives.
  - Candidate A's Rankings:
    - **1st place rankings**: Column 3 → 2 votes × 2 points = **4 points**
    - **2nd place rankings**: Columns 1 & 2 → 12 votes × 1 points = **12 points**
      - Candidate A's total: **4 + 12 = 16 points**
  - Candidate C's Rankings:
    - **1st place rankings**: Columns 1 & 2 → 12 votes × 2 points = **24 points**
    - **2nd place rankings**: Column 3 → 2 votes × 1 points = **2 points**
      - Candidate C's total: **24 + 2 = 26 points**
- **Step 3**: The candidate with the highest total wins.
  - Candidate A's total score = **12 points**
  - Candidate C's total score = **26 points**

Thus, Candidate C is the winner by the Borda count method. Candidate C wins the re-election.

Candidate C is the winner under the Borda count method in the original election. Since Candidate C also remains the winner after the irrelevant alternative, Candidate B, is removed, the outcome does not violate the independence of irrelevant alternatives criterion. **(NO)**

---

c. So, Candidate A is being dropped as a possible choice. Then, we remove Candidate A and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates B and C

Number of Voters	7	5	2
First Choice	B	C	B
Second Choice	C	B	C

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the Borda count method once again on the re-election.

- **Step 1**: Assign the points based on position. Since there are 2 candidates, a candidate receives:
  - 2 points for each 1st-place ranking
  - 1 points for each 2nd-place ranking
- **Step 2**: Add up all the points each candidate receives.
  - Candidate B's Rankings:
    - **1st place rankings**: Columns 1 & 3 → 9 votes × 2 points = **18 points**
    - **2nd place rankings**: Column 2 → 5 votes × 1 points = **5 points**
      - Candidate B's total: **18 + 5 = 23 points**
  - Candidate C's Rankings:
    - **1st place rankings**: Column 2 → 5 votes × 2 points = **10 points**
    - **2nd place rankings**: Columns 1 & 3 → 9 votes × 1 points = **9 points**
      - Candidate C's total: **10 + 9 = 19 points**
- **Step 3**: The candidate with the highest total wins.
  - Candidate B's total score = **23 points**
  - Candidate C's total score = **19 points**

Thus, Candidate B is the winner by the Borda count method. Candidate B wins the re-election.

Candidate C is the winner under the Borda count method in the original election. However, Candidate B is the winner after the irrelevant alternative, Candidate A, is removed. Therefore, there is a violation of the independence of irrelevant alternatives criterion. **(YES)**

Example #7.9.5 🤔

A city council is choosing one feature to build in a new public park. Residents rank the options based on preference. The city uses a plurality-with-elimination (instant-runoff) voting system.

Preference ballot showing voter rankings for candidates Dog Park, Garden, Pickle Ball Court, and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park	Garden	Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Playground	Garden	Pickle Ball Court
Fourth Choice	Garden	Dog Park	Playground	Garden

a. Use the plurality-with-elimination method to determine a winner.
b. Suppose the city council has decided to remove the Garden from the possible choices and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?
c. Suppose the city council has decided to remove the Dog Park (instead of the Garden) from the possible choices and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

✅ **Solution:**

a. Plurality-with-elimination method:

- **Step 0**: A preference table was given, so preference ballots have been counted.
- **Step 1**: Count first-place votes. (Round 1)
  - Playground received 36 out of 100 first place votes or 36%
  - Pickle Ball Court received 30 out of 100 first place votes or 30%
  - Dog Park received 34 out of 100 first place votes or 34%
  - Garden received 0 out of 100 first place votes or 0%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Garden received the fewest first place votes, so eliminate Garden by removing them from the table.

Preference ballot removing candidate Garden

Number of Voters	36	30	24	10
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park		Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Playground		Pickle Ball Court
Fourth Choice		Dog Park	Playground	

Then move up all other features in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Dog Park, Pickle Ball Court, and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park	Playground	Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Dog Park	Playground	Pickle Ball Court

- **Step 3**: Count first-place votes. (Round 2)
  - Playground received 36 out of 100 first place votes or 36%
  - Pickle Ball Court received 30 out of 100 first place votes or 30%
  - Dog Park received 34 out of 100 first place votes or 34%
    - **So, no majority winner.**

- **Step 4**: Repeat Until a Winner Emerges
  - Eliminate the weakest candidate.

Pickle Ball Court received the fewest first place votes, so eliminate Pickle Ball Court by removing them from the table.

Preference ballot removing candidate Pickle Ball Court

Number of Voters	36	30	24	10
First Choice	Playground		Dog Park	Dog Park
Second Choice	Dog Park	Playground		Playground
Third Choice		Dog Park	Playground	

Then move up all other features in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Dog Park and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Playground	Dog Park	Dog Park
Second Choice	Dog Park	Dog Park	Playground	Playground

- **Step 3 (Repeated)**: Count first-place votes. (Round 3)
  - Playground received 66 out of 100 first place votes or 66%
  - Dog Park received 34 out of 100 first place votes or 34%
    - **So, a majority exists.**

Thus, Playground is the winner by the plurality-with-elimination method.

---

b. So, Garden has been dropped from the choices. Then, we remove Garden and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Dog Park, Pickle Ball Court, and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Pickle Ball Court	Dog Park	Dog Park
Second Choice	Dog Park	Playground	Pickle Ball Court	Playground
Third Choice	Pickle Ball Court	Dog Park	Playground	Pickle Ball Court

Now, we use the same method as we did in the original election to see if the same candidate would win. So, we will use the plurality-with-elimination method once again on the re-election.

- **Step 1**: Count first-place votes. (Round 1)
  - Playground received 36 out of 100 first place votes or 41%
  - Pickle Ball Court received 30 out of 100 first place votes or 30%
  - Dog Park received 34 out of 100 first place votes or 29%
    - **So, no majority winner.**
- **Step 2**: Eliminate the weakest candidate.

Pickle Ball Court received the fewest first place votes, so eliminate Pickle Ball Court by removing them from the table.

Preference ballot removing candidate Pickle Ball Court

Number of Voters	36	30	24	10
First Choice	Playground		Dog Park	Dog Park
Second Choice	Dog Park	Playground		Playground
Third Choice		Dog Park	Playground	

Then move up all other features in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Dog Park and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Playground	Dog Park	Dog Park
Second Choice	Dog Park	Dog Park	Playground	Playground

- **Step 3 (Repeated)**: Count first-place votes. (Round 3)
  - Playground received 66 out of 100 first place votes or 66%
  - Dog Park received 34 out of 100 first place votes or 34%
    - **So, a majority exists.**

Playground wins the re-election.

Thus, Playground is the winner under the plurality-with-elimination method in the original election. Since Playground also remains the winner after the irrelevant alternative, Garden, is removed, the outcome does not violate the independence of irrelevant alternatives criterion. **(NO)**

---

c. So, Dog Park has been dropped from the choices. Then, we remove Dog Park and move up all other candidates in the next available slots and remove the last row.

Preference ballot showing voter rankings for candidates Garden, Pickle Ball Court, and Playground

Number of Voters	36	30	24	10
First Choice	Playground	Pickle Ball Court	Pickle Ball Court	Playground
Second Choice	Pickle Ball Court	Garden	Garden	Pickle Ball Court
Third Choice	Garden	Playground	Playground	Garden

Now, we use the same method as we did in the original election to see if the same feature would win. So, we will use the plurality-with-elimination once again on the re-election.

- **Step 1**: Count first-place votes. (Round 1)
  - Playground received 46 out of 100 first place votes or 46%
  - Pickle Ball Court received 54 out of 100 first place votes or 54%
  - Garden received 0 out of 100 first place votes or 0%
    - **So, a majority exists.**

Pickle Ball Court wins the re-election.

Thus, Playground is the winner under the plurality-with-elimination method in the original election. However, Pickle Ball Court is the winner after the irrelevant alternative, Dog Park, is removed. Therefore, there is a violation of the independence of irrelevant alternatives criterion. **(YES)**

Section 7.9: Independence of Irrelevant Alternatives (IIA) Criterion [*In-Class Exercises*]

1. Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	6	3	3
First Choice	A	B	C
Second Choice	B	C	A
Third Choice	C	A	B

a. Use the pairwise comparison method to determine a winner.
b. After consideration, Candidate C has decided to withdraw from the election. The votes were recounted and a new election is held. Does this violate the independence of irrelevant alternatives criterion?
c. Suppose, Candidate B (instead of Candidate C) had decided to withdraw from the original election. The votes were recounted and a new election is held. Does this violate the independence of irrelevant alternatives criterion?

---

2. Starting in the 2026-27 school year, Canyon High School in Anaheim, CA, will change its mascot from Comanches to one of the following names: Cavaliers, Cobras, Commanders, or Crows. This change complies with new California legislation phasing out Native American nicknames. The Color Guard team along with all band members gets a special early vote and ranks their choices before any of the other students. The results are below:

Preference ballot showing voter rankings for candidates Cavaliers, Cobras, Commanders, and Crows

Number of Voters	19	17	8	6
First Choice	Cobras	Crows	Commanders	Cobras
Second Choice	Crows	Commanders	Cobras	Cavaliers
Third Choice	Commanders	Cobras	Cavaliers	Commanders
Fourth Choice	Cavaliers	Cavaliers	Crows	Crows

a. Based on these results, use the pairwise comparison method to determine a winner.
b. After further deliberation, administration decided to omit Crows from the choices and the votes were recounted. Does this violate the independence of irrelevant alternatives criterion?
c. Now, use the plurality method to determine a winner.
d. In part c, when using the plurality method, if Crows was omitted from the choices and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

Answers available at the end of this section.

Answers

Section 7.9: Independence of Irrelevant Alternatives (IIA) Criterion [*In-Class Exercises*]

1. a. Candidate A wins
   b. Candidate A wins re-election; no, the independence of irrelevant alternatives criterion is not violated
   c. Candidate A and Candidate C tie; yes, the independence of irrelevant alternatives criterion is violated
2. a. Cobras win
   b. Cobras and Commanders tie; yes, the independence of irrelevant alternatives criterion is violated
   c. Cobras win using the Plurality method
   d. Cobras and Commanders tie; yes, the independence of irrelevant alternatives criterion is violated

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Section 7.9.H: Homework

Section 7.9: Independence of Irrelevant Alternatives Criterion - Homework Exercises

(Section 7.9) - Homework Exercise #1

Consider the following preference table with Candidates A, B, & C:

Preference ballot showing voter rankings for candidates A, B, and C

Number of Voters	9	7	7	5
First Choice	B	C	C	A
Second Choice	C	A	B	C
Third Choice	A	B	A	B

a. Use the pairwise comparison method to determine a winner.
b. Suppose Candidate B drops out and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?
c. Suppose Candidate A drops out (instead of Candidate B) and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

Answer available at the end of this section.

(Section 7.9) - Homework Exercise #2

A recreational hockey league with four teams surveyed 36 players from a lower skill level, asking them to rank which team they would most like to join. The teams were ranked from most to least preferred, and the results are shown in the preference table below.

Preference ballot showing voter rankings for candidates Chupuckabras, Ice Hawks, Misfits HC, and Rubber Puckies

Number of Voters	11	9	7	6	3
First Choice	Rubber Puckies	Misfits HC	Ice Hawks	Chupuckabras	Ice Hawks
Second Choice	Misfits HC	Chupuckabras	Rubber Puckies	Misfits HC	Chupuckabras
Third Choice	Chupuckabras	Ice Hawks	Chupuckabras	Ice Hawks	Rubber Puckies
Fourth Choice	Ice Hawks	Rubber Puckies	Misfits HC	Rubber Puckies	Misfits HC

a. Use the pairwise comparison method to determine a winner.
b. Suppose Rubber Puckies decide to drop out of the league and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?
c. Suppose Ice Hawks decide to drop out of the league (instead of Rubber Puckies) and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

Answer available at the end of this section.

(Section 7.9) - Homework Exercise #3

A group of 30 office coworkers are choosing where to eat lunch during their meeting. The restaurant options are: The Habit, Chick-Fil-A, and Panera Bread. Each voter ranks the restaurants from most preferred to least preferred.

Preference ballot showing voter rankings for candidates Chick-Fil-A, The Habit, and Panera Bread

Number of Voters	11	9	6	4
First Choice	The Habit	Chick-Fil-A	Panera Bread	Chick-Fil-A
Second Choice	Panera Bread	Panera Bread	Chick-Fil-A	The Habit
Third Choice	Chick-Fil-A	The Habit	The Habit	Panera Bread

a. Use the pairwise comparison method to determine a winner.
b. Suppose the workers found out that the only Chick-Fil-A in town was closed due to renovation and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?
c. Use the plurality method method to determine a winner.
d. Based off your answer in part c., if Panera Bread was removed from the original 3 choices and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?
e. Based off your answer in part c., if The Habit was removed from the original 3 choices and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?

Answer available at the end of this section.

(Section 7.9) - Homework Exercise #4

A class of 37 students is voting on a location for their end-of-year field trip. The four options are the Aquarium of the Pacific (A), the Butterfly Release at the Botanical Gardens (B), Disneyland (D), or the Griffith Observatory (G). The results are in the preference table below.

Preference ballot showing voter rankings for candidates A, B, D, and G

Number of Voters	14	10	8	4	1
First Choice	A	B	D	D	B
Second Choice	G	G	B	G	D
Third Choice	B	D	A	B	G
Fourth Choice	D	A	G	A	A

a. Use the pairwise comparison method to determine a winner.

b. Suppose the Aquarium of the Pacific was too expensive for the school's budget and was removed from the possible choices and the votes were recounted, does this violate the independence of irrelevant alternatives criterion?
c. Would the original election violate the Condorcet criterion?
d. Would the re-election violate the monotonicity criterion, if the 4 people that voted D > G > B > A was convinced to change their vote to match the 10 voters that voted B > G > D > A?

Answer available at the end of this section.

(Section 7.9) - Homework Exercise #5

A Los Angeles county Buffalo Wild Wings polled their customers and asked them to rank the local L.A. professional sports teams. Here are the results listed in a preference table.

Preference ballot showing voter rankings for candidates Chargers, Clippers, Dodgers, Kings, Lakers, and Rams

Number of Voters	**15**	**13**	**9**	**9**	**4**	**3**	**3**	**1**
First Choice	Dodgers	Lakers	Dodgers	Lakers	Rams	Lakers	Kings	Clippers
Second Choice	Lakers	Dodgers	Rams	Dodgers	Chargers	Rams	Rams	Chargers
Third Choice	Rams	Kings	Lakers	Kings	Kings	Dodgers	Lakers	Kings
Fourth Choice	Kings	Rams	Clippers	Clippers	Dodgers	Kings	Clippers	Dodgers
Fifth Choice	Clippers	Chargers	Kings	Rams	Clippers	Chargers	Chargers	Rams
Sixth Choice	Chargers	Clippers	Chargers	Chargers	Lakers	Clippers	Dodgers	Lakers

When using the pairwise comparison method to determine a winner, here were the results: Dodgers = 5 pts, Lakers = 4 pts, Rams = 3 pts, Kings = 2 pts, Clippers = 1 pt, Chargers = 0 points. Dodgers win.

a. Would this election violate the Condorcet criterion?
b. Would this election violate the majority criterion?

Answer available at the end of this section.

Answers

(Section 7.9) - Homework Exercise #1

a. Candidate C is the pairwise comparison winner.
b. After Candidate A drops out, Candidate C is the pairwise comparison winner of the re-election. There is no violation of the independence of irrelevant alternatives criterion. **(NO)**
c. After Candidate B drops out, Candidate C is the pairwise comparison winner of the re-election. There is no violation of the independence of irrelevant alternatives criterion. **(NO)**

(Section 7.9) - Homework Exercise #2

a. Misfits HC is the pairwise comparison winner.
b. After the Rubber Puckies drop out, Misfits HC is the pairwise comparison winner of the re-election. There is no violation of the independence of irrelevant alternatives criterion. **(NO)**

c. After the Ice Hawks drop out, Rubber Puckies is the pairwise comparison winner of the re-election. There is a violation of the independence of irrelevant alternatives criterion. **(YES)**

(Section 7.9) - Homework Exercise #3

a. Panera Bread is the pairwise comparison winner.
b. After Chick-Fil-A was removed from the possible choices, there is a tie between The Habit and Panera Bread in the re-election. Thus, there is a violation of the independence of irrelevant alternatives criterion. **(YES)**
c. Chick-Fil-A is the plurality winner.
d. After Panera Bread was removed from the possible choices, Chick-Fil-A is the plurality winner of the re-election. There is no violation of the independence of irrelevant alternatives criterion. **(NO)**
e. After The Habit was removed from the possible choices, Panera Bread is the plurality winner of the re-election. Thus, there is a violation of the independence of irrelevant alternatives criterion. **(YES)**

(Section 7.9) - Homework Exercise #4

a. Botanical Gardens is the pairwise comparison winner.
b. After Aquarium of the Pacific was removed from the possible choices, Botanical Gardens is the pairwise comparison winner of the re-election. There is no violation of the independence of irrelevant alternatives criterion. **(NO)**
c. Since Botanical Gardens did win against every other candidate head-to-head, then they are the Condorcet winner. Thus, there is no violation of the Condorcet criterion. **(NO)** (Note: Pairwise comparison method always satisfies the Condorcet criterion, because if there was a Condorcet winner, then they always win every head-to-head matchup and therefore earns the highest total number of points.)
d. Botanical Gardens is the pairwise comparison winner. Because Botanical Gardens also wins the re-election after receiving increased support, the outcome does not violate the monotonicity criterion. **(NO)**

(Section 7.9) - Homework Exercise #5

a. Based off the results, the Dodgers are the Condorcet winner, because they won all head-to-head matchups. Thus, there is no violation of the Condorcet criterion **(NO)**
b. Since, no majority exists here, you cannot apply the majority criterion. Thus, there is no violation of the majority criterion. **(NO)**

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Section 7.10: Approval Voting

Learning Objectives

- Determine the winner of an election using approval voting

Up to this point, we have studied several voting methods (plurality, Borda count, plurality-with-elimination, and pairwise comparison) and evaluated them using various fairness criteria, including the Majority, Condorcet, Monotonicity, and IIA criteria. These methods are all ranked-choice systems, meaning voters must place candidates in a strict order of preference. While ranking candidates provides detailed information about voter preferences, it also introduces complexity and, in many cases, unavoidable fairness violations.

The examination of fairness criteria highlights an important limitation in voting theory: no ranked-choice voting method can satisfy all fairness criteria simultaneously. As a result, scholars have explored alternative approaches that relax some assumptions about how voters express preferences. One such approach is **approval voting**, which departs from ranked ballots entirely.

Approval Voting

In **approval voting**, each voter may cast a vote for every candidate he or she considers acceptable. After all votes are counted, the candidate who receives the greatest number of approvals is declared the winner. Under this system, voters may approve of any number of candidates, ranging from none to all of the candidates on the ballot.

Approval voting is introduced at this stage as a response to the shortcomings identified in earlier methods. Instead of ranking candidates, voters simply indicate which candidates they find acceptable. A voter may approve of one candidate, several candidates, or all candidates, without being required to express a strict ordering among them. The candidate receiving the greatest number of approvals is declared the winner.

This method exists because it shifts the focus away from ranking precision and toward broad acceptability. By allowing voters to support multiple candidates, approval voting avoids certain paradoxes and strategic complications that arise in ranked systems, such as vote splitting and non-monotonic behavior. Although approval voting introduces its own trade-offs, it provides a useful contrast to the ranked methods studied earlier and offers an alternative way to balance simplicity, fairness, and voter expression. In the following section, we examine how approval voting works and evaluate how it performs with respect to the same fairness criteria previously discussed.

Example #7.10.1 🤔

Consider the following table with Candidates A, B, & C:

Approval voting results for favorite Candidate

Number of Votes:	4	3	2	1
Candidate A		/	/	

Number of Votes:	4	3	2	1
Candidate B	/	/		/
Candidate C	/		/	/

Use approval voting to determine a winner.

✅ **Solution:**

From the table, we count the number of votes for each candidate:

- Candidate A: 3 + 2 = 5
- Candidate B: 4 + 3 + 1 = 8
- Candidate C: 4 + 2 + 1 = 7

Thus, Candidate B has the most votes with 8 and wins by approval voting.

Example #7.10.2 🤔

Several junior high schoolers were asked what their favorite Pokémon character is. The results are in the table below:

Approval voting results for favorite Pokémon character

Number of Votes:	11	6	4	2	1
Charizard	/	/		/	
Eevee	/	/	/		
Gengar		/	/	/	/
Meowth	/	/		/	/
Pikachu	/	/			/

Use approval voting to determine a winner.

✅ **Solution:**

From the table, we count the number of votes for each candidate:

- Charizard: 11 + 6 + 2 = 19
- Eevee: 11 + 6 + 4 = 21
- Gengar: 6 + 4 + 2 + 1 = 13
- Meowth: 11 + 6 + 2 + 1 = 20
- Pikachu: 11 + 6 + 1 = 18

Thus, Eevee has the most votes with 21 and wins by approval voting.

Example #7.10.3 🤔

Each year, the Anaheim Ducks of the National Hockey League (NHL) distribute a feedback survey to their season seat holders to better understand fan preferences such as promotional giveaways, food options, and the overall game-day experience. In this survey, participants were allowed to indicate all options in which they were interested, rather than being limited to a single choice. A random sample of 100 season seat holders was selected to complete the survey. The results of the survey are summarized in the table below.

Approval voting results for fan preferences

Number of Votes:	**58**	**11**	**10**	**5**	**5**	**3**	**2**	**2**	**1**	**1**	**1**	**1**
Unique Food Options	/		/		/		/		/			
Unique Beverage Options	/			/	/				/		/	
DJ/ Music Between Stoppages	/	/	/	/				/				
Easy Access to Wild Wing (Mascot)	/	/	/			/	/	/		/		/
50/ 50 Raffle	/								/			
Pre-Game Show	/	/	/		/		/				/	
Discounted Suite Rentals	/											/
Discounted// Affordable Items in Team Store		/	/			/		/	/	/		/
Hat/ Cap Giveaways	/	/	/			/		/		/	/	/

Number of Votes:	58	11	10	5	5	3	2	2	1	1	1	1
Bobblehead/ Giveaways			/	/	/	/		/	/	/	/	/

Use approval voting to determine a winner.

✅ **Solution:**

From the table, we count the number of votes for each candidate:

- Unique Food Options: 58 + 10 + 5 + 2 + 1 = 76
- Unique Beverage Options: 58 + 5 + 5 + 1 + 1 = 70
- Music Between Stoppages: 58 + 11 + 10 + 5 + 2 = 86
- Easy Access to Wild Wing (Mascot): 58 + 11 + 10 + 3 + 2 + 2 + 1 + 1 = 88
- 50/50 Raffle: 58 + 1 = 59
- Pre-Game Show: 58 + 11 + 10 + 5 + 2 = 86
- Discounted Suite Rentals: 58 + 1 = 59
- Discounted/Affordable Items in Team Store: 58 + 11 + 10 + 3 + 2 + 1 + 1 + 1 = 87
- Hat/Cap Giveaways: 58 + 11 + 10 + 3 + 2 + 1 + 1 + 1 = 87
- Bobblehead Giveaways: 58 + 10 + 5 + 5 + 3 + 2 + 1 + 1 + 1 + 1 = 87

Thus, Easy Access to Wild Wing (Mascot) has the most votes with 88 and wins by approval voting.

Section 7.10: Voting Approval [*In-Class Exercises*]

1. Several children were asked what their favorite treat was during Halloween. The results are in the table below:

Approval voting results for favorite treat

Number of Votes:	8	7	6	4	3	1
Butterfinger	/			/		
Kit-Kat	/			/	/	
Hershey's			/		/	
Snickers				/		/
Twix		/	/		/	
Milky Way			/		/	/
3-Musketeers					/	/
Tootsie Roll						/

Use approval voting to determine a winner.

2. Attendees at a fanfest convention are asked to vote for their favorite character from the Five Nights at Freddy's franchise. Each attendee may vote for as many characters as they desire. After collecting the ballots, the organizers will total the approvals to determine which character is the overall favorite among the attendees.

   Approval voting results for favorite Five Nights at Freddy's character

Number of Votes:	13	11	11	10	8	7	7	6	6	3
Bonnie	/	/	/		/					
Chica		/	/	/				/		/
Foxy	/		/			/		/	/	/
Freddy Fazbear			/		/	/	/			
Springtrap		/	/		/			/	/	

   Use approval voting to determine a winner.

Answers available at the end of this section.

Answers

Section 7.10: Voting Approval [*In-Class Exercises*]

1. Twix has the most votes with 16 and wins by approval voting.
2. Foxy has the most votes with 46 and wins by approval voting.

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Section 7.10.H: Homework

Section 7.10: Voting Approval - Homework Exercises

(Section 7.10) - Homework Exercise #1

A survey was conducted among students at a local elementary school to determine which breakfast cereals they enjoy the most. Rather than being limited to selecting only one cereal, students were allowed to vote for all cereals that they liked. This approach recognizes that many children enjoy more than one type of cereal. A group of students was randomly selected to participate in the survey, and the results are summarized in the table below.

Approval voting results for favorite cereal

Number of Votes:	4	3	2	2	1	1	1	1
Apple Jacks	/	/		/			/	
Cheerios	/			/		/		
Frosted Flakes	/	/			/		/	
Fruity Pebbles	/		/	/	/	/	/	
Grape Nuts								/
Lucky Charms	/	/		/		/		
Trix	/	/	/		/			

Use approval voting to determine a winner.

Answer available at the end of this section.

(Section 7.10) - Homework Exercise #2

Employees vote for “Employee of the Month.” They may approve more than one coworker if they feel multiple people deserve recognition.

Approval voting results for "Employee of the Month"

Number of Votes:	6	5	3	2	1
Huyvu	/		/	/	
Farimah		/	/	/	/
Aisha	/	/			
Dai	/		/		/
Thu-Kim		/	/	/	

Use approval voting to determine a winner.

Answer available at the end of this section.

(Section 7.10) - Homework Exercise #3

Thirteen coworkers are voting for a new department chair. Each coworker can vote for more than one candidate, if they prefer. The results are provided in the table below.

Approval voting results for department chair

Number of Votes:	3	3	2	2	1	1	1
Mitchell		/					/
Haydeh					/	/	
Riki	/						/
Tab				/		/	/
Greg							/
Bill			/		/	/	/
Marti		/			/		
Doug (Evan)			/		/		/
Melissa	/				/		

Use approval voting to determine a winner.

Answer available at the end of this section.

(Section 7.10) - Homework Exercise #4

Customers of a convenience store are asked to approve of any energy drinks they would consider purchasing regularly from the following choices. Each customer may vote for as many drinks as they approve of. The store will total the approvals and use the results to decide which products should receive the most prominent placement in the store.

Approval voting results for favorite energy drink

Number of Votes:	18	15	12	7	7	5	5	1
C4	/		/		/			/
Celsius		/		/	/	/	/	
Ghost				/		/	/	
Monster	/		/			/	/	
Red Bull		/	/		/		/	
Rock Star		/		/	/	/	/	

Use approval voting to determine a winner.

Answer available at the end of this section.

(Section 7.10) - Homework Exercise #5

A local Bronze level hockey team took a vote on “Most Improved Player” from the following season. Each member of the team can vote for as many players as they wish. The results are in the table below.

Approval voting results for "Most Improved Player"

Number of Votes:	2	2	1	1	1	1	1	1	1	1	1
Brian I.				/	/		/		/		/
Brian M.		/		/		/			/		/
Cody	/	/			/	/					/
Cooper	/			/	/		/		/		/
Greg	/	/					/		/		/
Jason			/		/	/		/			/
Joey											/
Justin	/				/	/	/	/		/	/

Number of Votes:	2	2	1	1	1	1	1	1	1	1	1
Kevin		/	/		/			/	/		/
Ricky			/		/		/			/	/
Scott	/	/		/					/		/
Skyler		/				/	/	/			/
Tim B.	/	/		/					/		/
Tim C.					/						

Use approval voting to determine a winner.

Answer available at the end of this section.

Answer available at the end of this section.

(Section 7.10) - Homework Exercise #6

An esports arena wants to determine which Call of Duty League team should be featured on banners and promotional displays during an upcoming tournament watch party. Since fans may support more than one team, the arena decides to use an approval voting system. Fans are asked to approve of any teams they would like featured from the following list:

- Vegas FaZe
- OpTic Texas
- Toronto KOI
- Los Angeles Thieves
- Miami Heretics

The results are in the table below:

Approval voting results for favorite Call of Duty League team

Number of Votes:	11	10	8	7	6	5	5	5	4	4	3	3	2	1
Vegas FaZe	/		/	/			/				/		/	/
OpTic Texas	/	/	/		/					/	/	/		/
Toronto KOI		/		/					/				/	/
Los Angeles Thieves	/	/	/			/				/	/	/	/	/

Number of Votes:	11	10	8	7	6	5	5	5	4	4	3	3	2	1
Miami Heretics		/	/	/				/			/		/	

Use approval voting to determine a winner.

Answer available at the end of this section.

Answers

(Section 7.10) - Homework Exercise #1

Fruity Pebbles has the most votes with 11 and wins by approval voting.

(Section 7.10) - Homework Exercise #2

Huyvu, Farimah, and Aisha have the most votes with 11 and a three-way tie exists by approval voting.

(Section 7.10) - Homework Exercise #3

Bill has the most votes with 5 and wins by approval voting.

(Section 7.10) - Homework Exercise #4

Monster has the most votes with 40 and wins by approval voting.

(Section 7.10) - Homework Exercise #5

Justin has the most votes with 8 and wins by approval voting.

(Section 7.10) - Homework Exercise #5

(Section 7.10) - Homework Exercise #6

Los Angeles Thieves has the most votes with 47 and wins by approval voting.

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Section 7.11: Apportionment - The Basics

Learning Objectives

- Find the standard divisor
- Find standard quotas
- Determine lower and upper quotas

Introduction to Apportionment and Indivisible Resources

In many contexts, a limited number of items must be distributed among several groups in a way that reflects their relative sizes. Situations such as assigning seats in a legislature based on population, distributing representatives to committees, or allocating resources among regions all involve this type of problem. These situations are known as apportionment problems, and they form an important area of applied mathematics.

One of the primary challenges in apportionment arises from the nature of what is being distributed. The items involved, such as seats, representatives, or positions, cannot be divided into fractional pieces. This constraint makes apportionment fundamentally different from simple proportional calculations and leads to the need for specialized methods to ensure fairness.

Apportionment

Apportionment is the process of distributing a fixed number of indivisible items among different groups according to their proportions, usually based on population or size.

Indivisible

An item is **indivisible** if it cannot be split into smaller fractional parts and must be assigned as a whole unit.

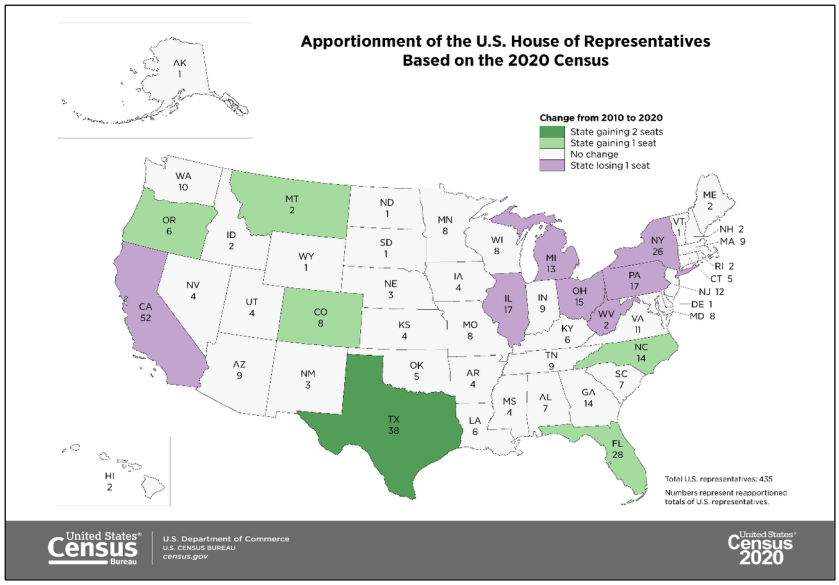
Because the items being apportioned are indivisible, a perfectly proportional distribution is often impossible. For example, if population ratios suggest that a group should receive 4.6 seats, that result cannot be implemented directly. Instead, careful procedures are required to assign whole numbers in a way that remains as close as possible to proportional fairness.

Apportionment of the United States House of Representatives

The most common example of apportionment is the United States Congress. In the Senate there are 100 senators where 2 senators are represented for each state. Pretty simple with an equal number of senators per state. However, in the House of Representatives, the number of representatives is based on the population of the state. There is a total of 435 seats in the House, and they're divided according to population. For example, after the 2020 census, California has 52 seats, yet Wyoming has only 1. Coincidentally, California is the most populous of all 50 states, whereas Wyoming is the least populous.

To be more concise, the United States congressional apportionment is the process by which seats in the United States House of Representatives are distributed among the 50 states according to the most recent decennial census mandated by the United States Constitution. Each state is apportioned a number of seats which approximately corresponds to its share of the aggregate population of the 50 states. However, every state is constitutionally guaranteed at least one seat. The size of a state's total

congressional delegation also determines the size of its representation in the U.S. Electoral College, which elects the U.S. president.



The actual process of how each state is apportioned will be discussed in the last section of this chapter.

For the remainder of this section, we will refer to the objects being assigned as “seats.” The individuals they are allocated to will be called “states” or “districts.”

Proportional Fairness and Ideal Allocations

Despite the restriction that items must be whole numbers, apportionment begins by considering what a perfectly proportional allocation would look like if fractional assignments were allowed. These ideal values provide a reference point for evaluating fairness, even though they cannot be used directly.

To compute these ideal values, we introduce several key concepts that form the foundation of all apportionment methods.

The Standard Divisor

The standard divisor represents the average population per item being allocated. It is calculated by dividing the total population by the total number of seats/items to be apportioned.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}}$$

The standard divisor provides a baseline for determining how many items each group should receive based purely on proportional reasoning.

Example #7.11.1 🤔

A large city has 20 seats to be allocated proportionally among five districts according to the population of each district. The populations of the districts are shown in the table.

Population distribution across five congressional districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000

a. If there are 20 seats and five districts, why would it be unfair to just assign 4 seats to each district?
b. Find the standard divisor and describe exactly what it tells us.

✅ **Solution:**

a. Assigning 4 seats to each district would be unfair because it ignores population differences, which defeats the entire purpose of proportional representation.
b. Let the number of seats = 20 and the total population = 150,000. Use the formula for standard divisor:

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{150{,}000}{20} = 7{,}500$$

This tells us that there will be on average 1 seat for every 7,500 people.

The Standard Quota

Once the standard divisor is known, each group's standard quota can be calculated. The standard quota is the ideal (and often fractional) number of items a group would receive if fractional assignments were allowed.

$$\text{Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}}$$

Standard quotas reflect perfect proportionality but typically result in decimal values, which cannot be assigned directly because the items are indivisible.

Example #7.11.2 🤔

Refer to the table in Example #7.11.1. Recall that the standard divisor was 7,500.

Population distribution across five congressional districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000

a. Find the standard quota for each of the districts.
b. Based on the standard quota, how many seats do you think District 3 should get?

✅ **Solution:**

a. Use the formula for standard quota. Take each of the five district's population and divide by the standard divisor.

$$\text{District 1: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{22{,}000}{7{,}500} \approx 2.93$$

$$\text{District 2: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{70{,}000}{7{,}500} \approx 9.33$$

$$\text{District 3: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{11{,}000}{7{,}500} \approx 1.47$$

$$\text{District 4: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{35{,}000}{7{,}500} \approx 4.67$$

$$\text{District 5: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{12{,}000}{7{,}500} = 1.6$$

b. Given that the standard quota for District 3 is 1.47, it should most definitely get either one or two seats.

Lower Quota and Upper Quota

To manage the transition from fractional quotas to whole numbers, two additional concepts are introduced.

- The lower quota of a group is the greatest integer less than or equal to its standard quota.
- The upper quota of a group is the smallest integer greater than or equal to its standard quota.

Together, the lower and upper quotas define a reasonable range within which a group's allocation should fall. Assigning an amount outside this range deviates significantly from proportional fairness.

Example #7.11.3 🤔

Refer to the results in Example #7.11.2a. Find the upper and lower quotas for the districts. What do they describe?

✅ **Solution:**

Start with each of the five district's standard quotas and simply find the two whole numbers that lie between each of the five district's standard quotas. The lower quota is the lower of those two whole numbers and the upper quota is the higher of those two whole numbers.

- District 1: 2.93; so lower quota is 2 and the upper quota is 3
- District 2: 9.33; so lower quota is 9 and the upper quota is 10
- District 3: 1.47; so lower quota is 1 and the upper quota is 2
- District 4: 4.67; so lower quota is 4 and the upper quota is 5
- District 5: 1.6; so lower quota is 1 and the upper quota is 2

The best way to organize all of your results is to extend the table as follows:

Population totals, standard quotas, lower quotas, and upper quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Standard Quota	2.93	9.33	1.47	4.67	1.6	20*
Lower Quota	2	9	1	4	1	17
Upper Quota	3	10	2	5	2	22

It is also important to compute and record the totals for the standard quotas, as well as the lower and upper quotas, since these values will be used in later topics. Notice the asterisk (*) next to the total of the standard quotas, which equals 20 in this example. This total will not always match the number of seats exactly. In this case, the sum happens to be exactly 20. However, in other situations, depending on rounding and calculation precision, the total of the standard quotas may be slightly greater than or less than 20, such as 20.01 or 19.99.

The lower and upper quotas are the only reasonable choices for the number of seats assigned to a given district. It remains to be seen how to decide whether to assign a certain district its upper or lower quota.

Why Apportionment Methods Are Necessary

Although standard quotas and quota bounds provide guidance, they do not by themselves solve apportionment problems. Simply rounding all quotas up or down almost always results in either too many or too few items being assigned overall.

This difficulty explains why formal apportionment methods are necessary. We need systematic procedures to convert standard quotas into whole-number allocations while remaining as faithful to proportionality as possible. These methods will be introduced in the remaining sections of this chapter. Before turning to those methods, however, we will examine a few additional examples to reinforce and clarify the foundational ideas discussed so far.

Example #7.11.4 🤔

A school district must assign 12 board members among four attendance zones. The zones along with the population of each zone is shown in the table below:

Population distribution across four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000

a. Find the standard divisor.
b. Find the standard quota for each of the districts.
c. Determine the lower and upper quotas for each district.

✅ **Solution:**

a. Let the number of seats = 12 and the total population = 6,000. Use the formula for standard divisor:

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{6{,}000}{12} = 500$$

This tells us that there will be on average 1 board member for every 500 people.

b. Use the formula for standard quota. Take each of the four zones population and divide by the standard divisor.

$$\text{North: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{2{,}400}{500} = 4.8$$

$$\text{South: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{1{,}800}{500} = 3.6$$

$$\text{East: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{1{,}200}{500} = 2.4$$

$$\text{West: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{600}{500} = 1.2$$

c. Start with each of the four zone's standard quotas and simply find the two whole numbers that lie between each of the four zone's standard quotas. The lower quota is the lower of those two whole numbers and the upper quota is the higher of those two whole numbers.

- North: 4.8; so lower quota is 4 and the upper quota is 5
- South: 3.6; so lower quota is 3 and the upper quota is 4
- East: 2.4; so lower quota is 2 and the upper quota is 3

- West: 1.2; so lower quota is 1 and the upper quota is 2

Organize all of your results in the table below:

Population totals, standard quotas, lower quotas, and upper quotas for four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	2	1	10
Upper Quota	5	4	3	2	14

Example #7.11.5 🤔

A company assigns 15 committee seats among five departments based on employee counts.

Population distribution across five departments

Department	Sales	Marketing	Engineering	HR	Support	Total
Population	101	65	18	23	41	248

a. Find the standard divisor.
b. Find the standard quota for each of the department.
c. Determine the lower and upper quotas for each department.

✅ **Solution:**

a. Let the number of seats = 15 and the total population = 6,000. Use the formula for standard divisor:

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{248}{15} \approx 16.53$$

This tells us that there will be on average 1 seat for every 16.53 people.

b. Use the formula for standard quota. Take each of the five department's population and divide by the standard divisor.

$\text{Sales: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{101}{16.53} \approx 6.11$

$\text{Marketing 2: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{65}{16.53} \approx 3.93$

$\text{Engineering: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{18}{16.53} \approx 1.09$

$\text{HR: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{23}{16.53} \approx 1.39$

$\text{Support: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{41}{16.53} \approx 2.48$

c. Start with each of the five department's standard quotas and simply find the two whole numbers that lie between each of the five department's standard quotas. The lower quota is the lower of those two whole numbers and the upper quota is the higher of those two whole numbers.

- Sales: 6.11; so lower quota is 6 and the upper quota is 7
- Marketing: 3.93; so lower quota is 3 and the upper quota is 4
- Engineering: 1.09; so lower quota is 1 and the upper quota is 2
- HR: 1.39; so lower quota is 1 and the upper quota is 2
- Support: 2.48; so lower quota is 2 and the upper quota is 3

Organize all of your results in the table below:

Population totals, standard quotas, lower quotas, and upper quotas for five departments

Department	Sales	Marketing	Engineering	HR	Support	Total
Population	101	65	18	23	41	248
Standard Quota	6.11	3.93	1.09	1.39	2.48	15
Lower Quota	6	3	1	1	2	13
Upper Quota	7	4	2	2	3	18

Example #7.11.6 🤔

A city council wants to allocate 10 council seats among four districts.

Population distribution across four districts

District	1	2	3	4	Total
Population	11,000	20,000	25,000	14,000	70,000

a. Find the standard divisor.
b. Find the standard quota for each of the districts.
c. Determine the lower and upper quotas for each district.

✅ **Solution:**

a. Let the number of seats = 10 and the total population = 70,000. Use the formula for standard divisor:

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{70{,}000}{10} = 7{,}000$$

This tells us that there will be on average 1 seat for every 7,000 people.

b. Use the formula for standard quota. Take each of the four district's population and divide by the standard divisor.

$$\text{District 1: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{11{,}000}{7{,}000} \approx 1.57$$

$$\text{District 2: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{20{,}000}{7{,}000} \approx 2.86$$

$$\text{District 3: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{25{,}000}{7{,}000} \approx 3.57$$

$$\text{District 4: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{14{,}000}{7{,}000} = 2$$

c. Start with each of the four district's standard quotas and simply find the two whole numbers that lie between each of the five district's standard quotas. The lower quota is the lower of those two whole numbers and the upper quota is the higher of those two whole numbers.

- District 1: 1.57; so lower quota is 1 and the upper quota is 2
- District 2: 2.86; so lower quota is 2 and the upper quota is 3
- District 3: 3.57; so lower quota is 3 and the upper quota is 4
- District 4: 2; so lower quota is 2 and the upper quota is 2

When a standard quota is already a whole number as in District 4, both the lower and upper quota equal that number, and there is no ambiguity in how many seats that group should receive.

Organize all of your results in the table below:

Population totals, standard quotas, lower quotas, and upper quotas for four districts

District	1	2	3	4	Total
Population	11,000	20,000	25,000	14,000	70,000
Standard Quota	1.57	2.86	3.57	2	10
Lower Quota	1	2	3	2	8
Upper Quota	2	3	4	2	11

Section 7.11: Apportionment: The Basics [*In-Class Exercises*]

1. Suppose a country has 4 states, and a 20-seat House of Representatives. The number of representatives assigned to each state is proportional to the population of the state. The population for each state is given in the following table.

   Population distribution across four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882

   a. Find the standard divisor.
   b. Find the standard quota for each state.
   c. Determine the lower and upper quotas for each state.
2. A company has 24 laptops to distribute among five offices within the region based on the number of active projects in each office.

   Population distribution across five offices within the region

Office	North County	West County	South County	East County	Midtown	Total
Active Projects	17	25	30	12	42	126

   a. Find the standard divisor.
   b. Find the standard quota for each office.
   c. Determine the lower and upper quotas for each state.

Answers available at the end of this section.

Answers

Section 7.11: Apportionment: The Basics [*In-Class Exercises*]

1.

Population totals, standard quotas, lower quotas, and upper quotas for four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Upper Quota	5	6	2	9	22

a. Standard Divisor = 594.1
b. Standard Quotas: 4.31, 5.58, 1.67, 8.44
c. Lower Quotas: 4, 5, 1, 8; Upper Quotas: 5, 6, 2, 9

2.

Population totals, standard quotas, lower quotas, and upper quotas for five offices within the region

Office	North County	West County	South County	East County	Midtown	Total
Active Projects	17	25	30	12	42	126
Standard Quota	3.24	4.76	5.71	2.29	8	24
Lower Quota	3	4	5	2	8	22
Upper Quota	4	5	6	3	8	26

a. Standard Divisor = 5.25
b. Standard Quotas: 3.24, 4.76, 5.71, 2.29, 8
c. Lower Quotas: 3, 4, 5, 2, 8; Upper Quotas: 4, 5, 6, 3, 8

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Section 7.11.H: Homework

Section 7.11: Apportionment - The Basics - Homework Exercises

(Section 7.11) - Homework Exercise #1

A university is assigning 20 student council seats across four departments based on enrollment:

Population distribution across four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000

a. Find the standard divisor.
b. Find the standard quota for each of the districts.
c. Determine the lower and upper quotas for each district.

Answer available at the end of this section.

(Section 7.11) - Homework Exercise #2

A gaming company is distributing 50 servers across regions based on the number of active players:

Active players distribution across six regions

Region	North America	South America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000

a. Find the standard divisor.
b. Find the standard quota for each of the regions.
c. Determine the lower and upper quotas for each region.

Answer available at the end of this section.

(Section 7.11) - Homework Exercise #3

A large music festival is assigning 40 performance slots to different genres based on ticket demand:

Ticket sales distribution across five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000

a. Find the standard divisor.
b. Find the standard quota for each of the genres.
c. Determine the lower and upper quotas for each genre.

Answer available at the end of this section.

(Section 7.11) - Homework Exercise #4

A logistics company is distributing 45 delivery trucks to its regional hubs based on the number of packages handled last month:

Package distribution across five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000

a. Find the standard divisor.
b. Find the standard quota for each of the genres.
c. Determine the lower and upper quotas for each genre.

Answer available at the end of this section.

(Section 7.11) - Homework Exercise #5

A streaming company is distributing 25 production slots across four genres based on viewer demand:

Viewer distribution across four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000

a. Find the standard divisor.
b. Find the standard quota for each of the genres.
c. Determine the lower and upper quotas for each genre.

Answer available at the end of this section.

Answers

(Section 7.11) - Homework Exercise #1

a. Standard divisor = 600
b. See third row:
c. See fourth and fifth row:

Population totals, standard quotas, lower quotas, and upper quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Standard Quota	7	5.5	4.17	3.33	20
Lower Quota	7	5	4	3	19
Upper Quota	7	6	5	4	22

(Section 7.11) - Homework Exercise #2

a. Standard divisor = 3,680
b. See third row:
c. See fourth and fifth row:

Active players totals, standard quotas, lower quotas, and upper quotas for six regions

Region	North America	South America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Standard Quota	14.13	3.53	10.33	1.36	17.66	2.99	50
Lower Quota	14	3	10	1	17	2	47
Upper Quota	15	4	11	2	18	3	53

(Section 7.11) - Homework Exercise #3

a. Standard divisor = 1,750
b. See third row:
c. See fourth and fifth row:

Tickets sales totals, standard quotas, lower quotas, and upper quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Lower Quota	13	10	7	5	3	38
Upper Quota	14	11	8	6	4	43

(Section 7.11) - Homework Exercise #4

a. Standard divisor = 2,177.78
b. See third row:
c. See fourth and fifth row:

Package totals, standard quotas, lower quotas, and upper quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Lower Quota	3	14	7	11	8	43
Upper Quota	4	15	8	12	9	48

(Section 7.11) - Homework Exercise #5

a. Standard divisor = 4,800
b. See third row:
c. See fourth and fifth row:

Viewer totals, standard quotas, lower quotas, and upper quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Standard Quota	6.65	11.08	4.54	2.73	25
Lower Quota	6	11	4	2	23
Upper Quota	7	12	5	3	27

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Section 7.12: Apportionment - Hamilton's Method

Learning Objectives

- Apportion seats using Hamilton's Method

Introduction to Hamilton's Method

In the previous section, we introduced the fundamental ideas behind apportionment, including standard divisors, standard quotas, and the concepts of lower and upper quotas. These tools allow us to measure what a fair, proportional distribution should look like if fractional allocations were possible. However, because the items being apportioned are indivisible, we must adopt a systematic way to convert these fractional quotas into whole-number assignments.

One of the earliest and most intuitive procedures for accomplishing this task is known as **Hamilton's Method**. This method provides a clear and structured approach to apportionment that builds directly on the ideas of standard quotas and remainders. Rather than rounding all quotas up or down arbitrarily, Hamilton's Method begins by assigning lower quotas to each group and then distributes any remaining items based on which groups have the largest fractional parts.

Hamilton's Method is sometimes called the method of largest remainders, since the remaining items are allocated to the groups whose standard quotas have the largest decimal remainders. This approach attempts to stay as close as possible to proportional fairness by giving extra items to the groups that were closest to earning an additional whole unit.

The appeal of Hamilton's Method lies in both its simplicity and its transparency. Each step can be explained and justified using the quantities already defined, and the reasoning behind the final allocation is easy to follow. For this reason, Hamilton's Method often serves as a natural first example of a formal apportionment procedure.

Here is the step-by-step process:

Hamilton's Method

To apply Hamilton's Method:

- **Step 1:** Find the standard divisor.
- **Step 2:** Compute the standard quota for each group.
- **Step 3:** Assign each group its lower quota. (Ignore upper quotas).
- **Step 4:** Add the lower quotas.
- **Step 5:** Distribute the remaining seats/items, one at a time, to the groups with the largest fractional parts of their standard quotas until all seats/items are assigned.

Example #7.12.1 🤔

Refer to the results from Example #7.11.3. Using Hamilton's Method, divide the 20 seats between the five districts.

Population totals, standard quotas, lower quotas, and upper quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Standard Quota	2.93	9.33	1.47	4.67	1.6	20*
Lower Quota	2	9	1	4	1	17
Upper Quota	3	10	2	5	2	22

✅ **Solution:**

Applying Hamilton's Method:

- **Step 1**: Find the standard divisor.
- **Step 2**: Compute the standard quota for each group.
- **Step 3**: Assign each group its lower quota.
- **Step 4**: Add the lower quotas.

So, these four steps were already done to get the results in the table above. You can ignore or remove the upper quota row.

- **Step 5**: Distribute the remaining seats, one at a time, to the groups with the largest fractional parts of their standard quotas until all seats are assigned.

So, 17 seats were assigned out of the 20 using the lower quota, thus creating a deficit of 3 seats. We need to assign 3 more seats. Writing the standard quota in descending order looking at the fractional part only, we have:

2.**93** - District 1 gets one more seat on top of the lower quota.
4.**67** - District 4 gets one more seat on top of the lower quota.
1.**60** - District 5 gets one more seat on top of the lower quota. **Stop.** We have the 3 assigned seats.
1.**47**
9.**33**

Now add a new row with the 3 assigned seats apportioned using Hamilton's Method:

Population totals, standard quotas, lower quotas, and seats apportioned for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Standard Quota	2.93	9.33	1.47	4.67	1.6	20*
Lower Quota	2	9	1	4	1	17
Seats apportioned using Hamilton's Method	**3**	9	1	**5**	**2**	**20**

Thus, the apportionment is as follows: 3 seats for District 1, 9 seats for District 2, 1 seat for District 3, 5 seats for District 4, and 2 seats for District 5.

Example #7.12.2 🤔

Refer to the results from Example #7.11.4. Using Hamilton's Method, divide the 12 board members among the four attendance zones:

Population totals, standard quotas, lower quotas, and upper quotas for four attendance zones

District:	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	2	1	10
Upper Quota	5	4	3	2	14

✅ **Solution:**

Applying Hamilton' Method:

- **Step 1**: Find the standard divisor.
- **Step 2**: Compute the standard quota for each group.
- **Step 3**: Assign each group its lower quota.
- **Step 4**: Add the lower quotas.

So, these four steps were already done to get the results in the table above. You can ignore or remove the upper quota row.

- **Step 5**: Distribute the remaining seats, one at a time, to the groups with the largest fractional parts of their standard quotas until all seats are assigned.

So, 10 seats were assigned out of the 12 using the lower quota, thus creating a deficit of 2 seats. We need to assign 2 more seats. Writing the standard quota in descending order looking at the fractional part only, we have:

4.**8** - North gets one more seat on top of the lower quota.
3.**6** - South gets one more seat on top of the lower quota. **Stop.** We have the 2 assigned seats.
2.**4**
1.**2**

Now add a new row with the 2 assigned seats apportioned using Hamilton's Method:

Population totals, standard quotas, lower quotas, and seats apportioned for four attendance zones

District:	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	2	1	10

District:	North	South	East	West	Total
Seats apportioned using Hamilton's Method	**5**	**4**	2	1	**12**

Thus, the apportionment is as follows: 5 seats for North, 4 seats for South, 2 seats for East, and 1 seat for West.

Example #7.12.3 🤔

A logistics company is allocating 15 delivery trucks among five distribution centers based on average daily deliveries. The number of average daily deliveries for each distribution center is given in the following table.

Population distribution across five distribution centers

Distribution Center	A	B	C	D	E	Total
Daily Deliveries	47	62	51	88	79	327

Use Hamilton's Method to assign the 15 delivery trucks among the five distribution centers.

✅ **Solution:**

Applying Hamilton' Method:

- **Step 1**: Find the standard divisor.

Let the number of trucks = 15 and the total population = 327. Use the formula for standard divisor:

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{327}{15} \approx 21.8$$

This tells us that there will be on average 1 truck for every 21.8 deliveries.

- **Step 2**: Compute the standard quota for each group.

Find the standard quotas: Use the formula for standard quota. Take each of the five distribution center's and divide by the standard divisor.

$$\text{A: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{47}{21.8} \approx 2.16$$

$$\text{B: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{62}{21.8} \approx 2.84$$

$$\text{C: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{51}{21.8} \approx 2.34$$

$$\text{D: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{88}{21.8} \approx 4.04$$

$$\text{E: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{79}{21.8} \approx 3.62$$

- **Step 3**: Assign each group its lower quota.

Start with each of the five distribution center's standard quotas and simply find the lower of the two whole numbers that lie between each of the five distribution center's. The lower quota is the lower of those two whole numbers. We do not need the upper quota here when using Hamilton's Method.

- A: 2.16; so lower quota is 2
- B: 2.84; so lower quota is 2
- C: 2.34; so lower quota is 2
- D: 4.04; so lower quota is 4
- E: 3.62; so lower quota is 3

- **Step 4**: Add the lower quotas.

The sum of the lower quotas is 2 + 2 + 2 + 4 + 3 = 13.

- **Step 5**: Distribute the remaining items, one at a time, to the groups with the largest fractional parts of their standard quotas until all items are assigned.

So, 13 seats were assigned out of the 15 using the lower quota, thus creating a deficit of 2 trucks. We need to assign 2 more trucks. Writing the standard quota in descending order looking at the fractional part only, we have:

2.**84** - B gets one more truck on top of the lower quota.
3.**62** - E gets one more truck on top of the lower quota. **Stop.** We have the 2 assigned trucks.
2.**34**
2.**16**
4.**04**

Organize all of your results in the finalized table below:

Population totals, standard quotas, lower quotas, and seats apportioned for five distribution centers

Distribution Center	A	B	C	D	E	Total
Daily Deliveries	47	62	51	88	79	327
Standard Quota	2.16	2.84	2.34	4.04	3.62	15
Lower Quota	2	2	2	4	3	13
Trucks apportioned using Hamilton's Method	2	**3**	2	4	**4**	**15**

Thus, the apportionment is as follows: 2 trucks for center A, 3 trucks for center B, 2 trucks for center C, 4 trucks for center D, and 4 trucks for center E.

Historical Note: Alexander Hamilton and Apportionment

Hamilton's Method is named after Alexander Hamilton, one of the Founding Fathers of the United States and the first Secretary of the Treasury. The method emerged during early debates over how to apportion seats in the United States House of Representatives according to state populations.

Following the first United States census in 1790, Congress debated several apportionment proposals. In 1792, Congress passed an apportionment plan based on Hamilton's proposal. However, President George Washington vetoed the bill, marking the first presidential veto in United States history. Although rejected at the time, Hamilton's Method continued to influence later discussions on proportional representation.

Hamilton's Method was eventually adopted and used by the United States Congress to apportion seats in the House of Representatives from 1852 to 1901. Despite its intuitive appeal and close connection to proportional fairness, the method was later abandoned because it can produce paradoxical outcomes. Most notably, Hamilton's Method can lead to the Alabama Paradox, in which increasing the total number of seats causes a state to lose a seat rather than gain one.

Today, Hamilton's Method is studied not only for its simplicity and fairness, but also because it highlights the subtle challenges inherent in apportionment. These challenges help motivate the development of alternative apportionment methods, which will be examined in subsequent chapters.

Alexander Hamilton is probably best known as the face on the U.S. $10 bill and for his



role in early American finance, but his accomplishments extend far beyond apportionment debates and monetary policy. Some of his major contributions include the creation of the National Financial System, founder of the First Bank of the United States as well as the U.S. Coast Guard, plus playing a role in the Revolutionary War.

Section 7.12: Hamilton's Method [*In-Class Exercises*]

1. Suppose a country has 4 states, and a 20-seat House of Representatives. The number of representatives assigned to each state is proportional to the population of the state. The population for each state is given in the following table.

   Population totals, standard quotas, lower quotas, and upper quotas for four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Upper Quota	5	6	2	9	22

Use Hamilton's Method to assign the 20 seats among the four states.

2. A company has 24 laptops to distribute among five offices within the region based on the number of active projects in each office.

Active projects totals, standard quotas, lower quotas, and upper quotas for four states

Office	North County	West County	South County	East County	Midtown	Total
Active Projects	17	25	30	12	42	126
Standard Quota	3.24	4.76	5.71	2.29	8	24
Lower Quota	3	4	5	2	8	22
Upper Quota	4	5	6	3	8	26

Use Hamilton's Method to assign the 24 laptops among the five offices.

Answers available at the end of this section.

Answers

Section 7.12: Hamilton's Method [*In-Class Exercises*]

1.

Population totals, standard quotas, lower quotas, and seats apportioned for four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Seats apportioned using Hamilton's Method	4	**6**	**2**	8	**20**

Thus, the apportionment is as follows: 4 seats for A, 6 seats for B, 2 seats for C, and 8 seats for D.

2.

Active projects totals, standard quotas, lower quotas, and laptops apportioned for four states

Office	North County	West County	South County	East County	Midtown	Total
Active Projects	17	25	30	12	42	126
Standard Quota	3.24	4.76	5.71	2.29	8	24

Office	North County	West County	South County	East County	Midtown	Total
Lower Quota	3	4	5	2	8	22
Laptops apportioned using Hamilton's Method	3	**5**	**6**	2	8	**24**

Thus, the apportionment is as follows: 3 seats for North County, 5 seats for West County, 6 seats for South County, 2 seats for East County, and 8 seats for Midtown.

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Section 7.12.H: Homework

Section 7.12: Apportionment - Hamilton's Method - Homework Exercises

(Section 7.12) - Homework Exercise #1

Refer to Homework Exercise #1 from Section 7.11.

A university is assigning 20 student council seats across four departments based on enrollment:

Population totals, standard quotas, lower quotas, and upper quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Standard Quota	7	5.5	4.17	3.33	20
Lower Quota	7	5	4	3	19
Upper Quota	7	6	5	4	22

The standard divisor was 600. Use Hamilton's Method to assign the 20 student council seats among the four departments.

Answer available at the end of this section.

(Section 7.12) - Homework Exercise #2

Refer to Homework Exercise #2 from Section 7.11.

A gaming company is distributing 50 servers across regions based on the number of active players:

Active players totals, standard quotas, lower quotas, and upper quotas for six regions

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players:	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Standard Quota:	14.13	3.53	10.33	1.36	17.66	2.99	50
Lower Quota:	14	3	10	1	17	2	47
Upper Quota:	15	4	11	2	18	3	53

The standard divisor was 3,680. Use Hamilton's Method to assign the 50 servers among the four regions.

Answer available at the end of this section.

(Section 7.12) - Homework Exercise #3

Refer to Homework Exercise #3 from Section 7.11.

A large music festival is assigning 40 performance slots to different genres based on ticket demand:

Tickets sales totals, standard quotas, lower quotas, and upper quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Lower Quota	13	10	7	5	3	38
Upper Quota	14	11	8	6	4	43

The standard divisor was 1,750. Use Hamilton's Method to assign the 40 performance slots among the five genres.

Answer available at the end of this section.

(Section 7.12) - Homework Exercise #4

Refer to Homework Exercise #4 from Section 7.11.

A logistics company is distributing 45 delivery trucks to its regional hubs based on the number of packages handled last month:

Package totals, standard quotas, lower quotas, and upper quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Lower Quota	3	14	7	11	8	43
Upper Quota	4	15	8	12	9	48

The standard divisor was 2,177.78. Use Hamilton's Method to assign the 45 delivery trucks among the five regional hubs.

Answer available at the end of this section.

(Section 7.12) - Homework Exercise #5

Refer to Homework Exercise #5 from Section 7.11.

A streaming company is distributing 25 production slots across four genres based on viewer demand:

Viewer totals, standard quotas, lower quotas, and upper quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Standard Quota	6.65	11.08	4.54	2.73	25
Lower Quota	6	11	4	2	23
Upper Quota	7	12	5	3	27

The standard divisor was 4,800. Use Hamilton's Method to assign the 25 production slots among the four genres.

Answer available at the end of this section.

(Section 7.12) - Homework Exercise #6

A company has 50 million ad impressions to distribute across four platforms based on monthly active users:

Active users across four platforms

Platform	Mobile App	Website	Smart TV	Tablet	Total
Users (millions)	92.4	74.8	53.6	29.2	250

Use Hamilton's Method to assign the 50 million ad impressions among the four platforms.

Answer available at the end of this section.

(Section 7.12) - Homework Exercise #7

A city is installing 60 public EV charging stations across four districts based on the number of registered electric vehicles:

Registered vehicles across four districts

District	North	South	West	East	Total
Registered Vehicles	25,374	31,512	16,841	41,890	115,617

Use Hamilton's Method to assign the 60 EV charging stations among the four districts.

Answer available at the end of this section.

Answers

(Section 7.12) - Homework Exercise #1

Population totals, standard quotas, lower quotas, and seats apportioned for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Standard Quota	7	5.5	4.17	3.33	20
Lower Quota	7	5	4	3	19
Seats apportioned using Hamilton's Method	7	**6**	4	3	20

(Section 7.12) - Homework Exercise #2

Active players totals, standard quotas, lower quotas, and servers apportioned for six regions

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players:	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Standard Quota:	14.13	3.53	10.33	1.36	17.66	2.99	50
Lower Quota:	14	3	10	1	17	2	47
Servers apportioned using Hamilton's Method	14	**4**	10	1	**18**	**3**	**50**

(Section 7.12) - Homework Exercise #3

Tickets sales totals, standard quotas, lower quotas, and slots apportioned for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Lower Quota	13	10	7	5	3	38

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Slots apportioned using Hamilton's Method	13	10	7	**6**	**4**	**40**

(Section 7.12) - Homework Exercise #4

Package totals, standard quotas, lower quotas, and trucks apportioned for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Lower Quota	3	14	7	11	8	43
Trucks apportioned using Hamilton's Method	**4**	14	7	11	**9**	**45**

(Section 7.12) - Homework Exercise #5

Viewer totals, standard quotas, lower quotas, and slots apportioned for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Standard Quota	6.65	11.08	4.54	2.73	25
Lower Quota	6	11	4	2	23
Slots apportioned using Hamilton's Method	**7**	11	4	**3**	**25**

(Section 7.12) - Homework Exercise #6

Standard Divisor = 5

Active user totals, standard quotas, lower quotas, and ad impressions apportioned for four platforms

Platform	Mobile App	Website	Smart TV	Tablet	Total
Users (millions)	92.4	74.8	53.6	29.2	250
Standard Quota (millions)	18.48	14.96	10.72	5.84	50

Platform	Mobile App	Website	Smart TV	Tablet	Total
Lower Quota(millions)	18	14	10	5	47
Ad impressions apportioned using Hamilton's Method (millions)	18	**15**	**11**	**6**	**50**

(Section 7.12) - Homework Exercise #7

Standard divisor = 1,926.95

Registered vehicle totals, standard quotas, lower quotas, and EV charging stations apportioned for four districts

District	North	South	West	East	Total
Registered Vehicles	25,374	31,512	16,841	41,890	115,617
Standard Quota	13.17	16.35	8.74	21.74	60
Lower Quota	13	16	8	21	58
EV charging stations apportioned using Hamilton's Method	13	16	**9**	**22**	**60**

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Section 7.13: Apportionment - Jefferson's Method

Learning Objectives

- Apportion seats using Jefferson's Method

Introduction to Jefferson's Method

In the previous section, we studied Hamilton's Method, an apportionment procedure that uses standard quotas and fractional remainders to distribute indivisible items as fairly as possible. While Hamilton's Method is intuitive and closely tied to proportional ideals, it is not without limitations. In some situations, Hamilton's Method can produce outcomes that seem counterintuitive, especially when small changes in population or in the total number of items lead to unexpected shifts in allocation.

These concerns motivated the development of alternative apportionment methods. One such method is **Jefferson's Method**, named after the third U.S. President, Thomas Jefferson, which approaches the apportionment problem differently by modifying the standard divisor rather than relying on fractional remainders. Jefferson's Method eliminates the need to compare decimal parts of standard quotas and instead focuses on adjusting the divisor until the total number of assigned items is correct.

Jefferson's Method is historically significant because it represents a deliberate attempt to avoid certain paradoxes associated with Hamilton's Method and to produce more stable outcomes. By using a modified divisor and rounding down, Jefferson's Method tends to favor larger groups slightly, leading to outcomes that some have argued better support political stability and administrative simplicity.

How Jefferson's Method Differs from Hamilton's Method

Hamilton's Method begins with the standard divisor, assigns lower quotas, and distributes remaining items based on largest remainders. In contrast, Jefferson's Method does not fix the standard divisor in advance. Instead, it adjusts the divisor downward until the total number of items assigned matches the required total.

Rather than comparing remainders, Jefferson's Method assigns seats by rounding down every adjusted quota. The divisor itself is changed to correct for under-allocation or over-allocation. Because of this approach, Jefferson's Method always assigns each group either its lower quota or fewer items, but never more than its upper quota. Here is the step-by-step process:

Jefferson's Method

Jefferson's Method is an apportionment procedure that assigns indivisible items by adjusting the divisor downward and rounding all resulting quotas down until the total number of items assigned is correct.

To apply Jefferson's Method:

- **Step 1:** Find the standard divisor.
- **Step 2:** Compute the standard quota for each group.
- **Step 3:** Assign each group its lower quota. (Ignore upper quotas).
- **Step 4:** Add the lower quotas.

- **Step 5:** If the total number of seats/items assigned is less than required, decrease the divisor slightly. Recalculate the quotas and round down again.
- **Step 6:** Repeat Step 5 until the total number of seats/items assigned equals the required total.

Jefferson's method avoids the problem of an apportionment resulting in a deficit of seats/items by using a divisor that will result in the correct number of seats being apportioned.

Example #7.13.1 🤔

Refer to the results from Example #7.11.3. The standard divisor was 7,500. Using Jefferson's Method, divide the 20 seats between the five districts.

Population totals, standard quotas, lower quotas, and upper quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Standard Quota	2.93	9.33	1.47	4.67	1.6	20
Lower Quota	2	9	1	4	1	17
Upper Quota	3	10	2	5	2	22

✅ **Solution:**

Applying Jefferson's Method:

- **Step 1**: Find the standard divisor.
- **Step 2**: Compute the standard quota for each group.
- **Step 3**: Assign each group its lower quota.
- **Step 4**: Add the lower quotas.

So, these four steps were already done to get the results in the table above. You can ignore or remove the upper quota row.

- **Step 5**: If the total number of items assigned is less than required, decrease the divisor slightly. Recalculate the quotas and round down again.

So, 17 seats were assigned out of the 20 using the lower quota, thus creating a deficit of 3 seats. We need to assign 3 more seats.

Recall that the standard divisor is 7,500. Since the sum of the lower quotas (17) is less than the total number of seats available (20), we will use a modified divisor and recalculate everything. Remember, if we are adding seats, then we need a divisor that is smaller than the standard divisor of 7,500.

Let's try 7,250 and calculate the modified quota and check to see if our rounded-down quotas have increased. ("Rounded-Down Quota" replaces "Lower Quota" since we are no longer working with a standard quota anymore).

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{22{,}000}{7{,}250} \approx 3.03$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{70{,}000}{7{,}250} \approx 9.66$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{11{,}000}{7{,}250} \approx 1.52$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{35{,}000}{7{,}250} \approx 4.83$$

$$\text{District 5: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{12{,}000}{7{,}250} \approx 1.66$$

Update the results using a table and also include the rounded-down quotas. Check the sum of those rounded-down quotas.

Population totals, modified quotas, and rounded-down quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Modified Quota (MD=7,250)	3.03	9.66	1.52	4.83	1.66	20
Rounded-Down Quota	**3**	9	1	4	1	**18**

Unfortunately, the rounded-down quotas only increased from 17 to 18. We need two more seats. So, we need to decrease the divisor some more. Let's try 7,000 now and recalculate again.

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{22{,}000}{7{,}000} \approx 3.14$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{70{,}000}{7{,}000} = 10$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{11{,}000}{7{,}000} \approx 1.57$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{35{,}000}{7{,}000} = 5$$

$$\text{District 5: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{12{,}000}{7{,}000} \approx 1.71$$

Update the results using a table and also include the rounded-down quotas. Check the sum of those rounded-down quotas.

Population totals, modified quotas, and rounded-down quotas for five districts

District	1	2	3	4	5	Total
Population:	22,000	70,000	11,000	35,000	12,000	150,000
Modified Quota (MD=7,000)	3.14	10	1.57	5	1.71	20
Rounded-Down Quota	**3**	**10**	1	**5**	1	**20**

So, now we have the 20 seats. However, we were fortunate that we didn't use too low of a number that created more seats than our 20 we needed. If/when that happens, you will have to increase your divisor.

Thus, the modified divisor using Jefferson's Method is 7,000 and the apportionment is as follows:

3 seats for District 1, 10 seats for District 2, 1 seat for District 3, 5 seats for District 4, and 1 seat for District 5.

Note: There are other modified divisors that would also work. In fact, any number from 6,367 through 7,004 inclusive would serve as the modified divisor to satisfy Jefferson's Method.

Example #7.13.2 🤔

Refer to the results from Example #7.11.4. The standard divisor was 500. Using Jefferson's Method, divide the 12 board members among the four attendance zones:

Population totals, standard quotas, lower quotas, and upper quotas for four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	2	1	10
Upper Quota	5	4	3	2	14

✅ **Solution:**

Applying Jefferson's method:

- **Step 1**: Find the standard divisor.
- **Step 2**: Compute the standard quota for each group.
- **Step 3**: Assign each group its lower quota.
- **Step 4**: Add the lower quotas.

So, these four steps were already done to get the results in the table above. You can ignore or remove the upper quota row.

- **Step 5**: If the total number of items assigned is less than required, decrease the divisor slightly. Recalculate the quotas and round down again.

So, 10 seats were assigned out of the 12 using the lower quota, thus creating a deficit of 2 seats. We need to assign 2 more seats.

Recall that the standard divisor is 500. Since the sum of the lower quotas (10) is less than the total number of seats available (12), we will use a modified divisor and recalculate everything. Remember, if we are adding seats, then we need a divisor that is smaller than the standard divisor of 500.

Let's try 450 and calculate the modified quota and check to see if our rounded-down quotas have increased.

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{2{,}400}{450} \approx 5.33$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{1{,}800}{450} = 4$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{1{,}200}{450} \approx 2.67$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{600}{450} \approx 1.33$$

Update the results using a table and also include the rounded-down quotas. Check the sum of those rounded-down quotas.

Population totals, modified quotas, and rounded-down quotas for four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000

District	North	South	East	West	Total
Modified Quota (MD=450)	5.33	4	2.67	1.33	12
Rounded-Down Quota	**5**	**4**	2	1	**12**

So, now we have the 12 seats. Here, we were fortunate to get a correct modified divisor on the first try.

Thus, the modified divisor using Jefferson's Method is 450 and the apportionment is as follows:

5 seats for North, 4 seats for South, 2 seats for East, and 1 seat for West.

Note: There are other modified divisors that would also work. In fact, any number from 401 through 450 inclusive would serve as the modified divisor to satisfy Jefferson's Method.

Historical Note: Thomas Jefferson and Apportionment

Thomas Jefferson was one of the most influential figures in early American history. He was the principal author of the Declaration of Independence, the third President of the United States, and a key architect of American political philosophy. Jefferson strongly believed in limited government, individual liberty, and the protection of states' rights.

Jefferson's involvement in apportionment debates arose from his concern with fair representation in the House of Representatives. He proposed what is now known as Jefferson's Method as an alternative to Hamilton's Method, favoring a process that used a modified divisor and rounded quotas downward. Jefferson believed this approach produced more practical and stable outcomes, particularly for larger states.

Jefferson's Method was the first apportionment method ever officially used by the United States to allocate seats in the House of Representatives. It was adopted beginning with the apportionment following the 1790 census and remained in use until 1842. Jefferson's method adjusts the divisor downward and rounds all quotas down in order to reach the required total number of seats. As a result, Jefferson's Method tends to favor states or groups with larger populations.

Although Jefferson's Method provided stable and predictable apportionment and avoided some issues later associated with Hamilton's Method, it was eventually replaced with Hamilton's Method almost 60 years later. Critics argued that the method consistently disadvantaged smaller states by systematically rounding their quotas down. Despite this, Jefferson's Method established the foundation for later divisor methods and had a significant influence on the development of apportionment theory in the United States.

Today, Jefferson's Method remains important in the study of apportionment, both for its mathematical properties and for its historical role in shaping the early representation system of the United States.

Thomas Jefferson is also featured on the U.S. $2 bill as well as the U.S. nickel.





Section 7.13: Jefferson's Method [*In-Class Exercises*]

1. Suppose a country has 4 states, and a 20-seat House of Representatives. The number of representatives assigned to each state is proportional to the population of the state. The population for each state is given in the following table.

   Population totals, standard quotas, lower quotas, and upper quotas for four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Upper Quota	5	6	2	9	22

Use Jefferson's Method to assign the 20 seats among the four states. The standard divisor is 594.1.

2. A company has 24 laptops to distribute among five offices within the region based on the number of active projects in each office.

Active project totals, standard quotas, lower quotas, and upper quotas for five offices within the region

Office	North County	West County	South County	East County	Midtown	Total
Active Projects	17	25	30	12	42	126
Standard Quota	3.24	4.76	5.71	2.29	8	24
Lower Quota	3	4	5	2	8	22
Upper Quota	4	5	6	3	8	26

Use Jefferson's Method to assign the 24 laptops among the five offices. The standard divisor is 5.25

Answers available at the end of this section.

Answers

Section 7.13: Jefferson's Method [*In-Class Exercises*]

1.

Population totals, modified quotas, and rounded-down quotas for four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Modified Quota (MD=550)	4.65	6.03	1.81	9.11	20
Rounded-Down Quota	4	**6**	1	**9**	**20**

Thus, the apportionment is as follows: 4 seats for A, 6 seats for B, 1 seat for C, and 9 seats for D. (The modified divisor using Jefferson's Method here is 550. There are other values that work as well).

2.

Active project totals, modified quotas, and rounded-down quotas for five offices within the region

Office	North County	West County	South County	East County	Midtown	Total
Active Projects	17	25	30	12	42	126
Modified Quota (MD=5)	3.4	5	6	2.4	8.4	24

Office	North County	West County	South County	East County	Midtown	Total
Rounded-Down Quota	3	**5**	**6**	2	8	**24**

Thus, the apportionment is as follows: 3 seats for North County, 5 seats for West County, 6 seats for South County, 2 seats for East County, and 8 seats for Midtown. (A modified divisor using Jefferson's Method here is 5. There are other values that work as well).

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Section 7.13.H: Homework

Section 7.13: Apportionment - Jefferson's Method - Homework Exercises

(Section 7.13) - Homework Exercise #1

Refer to Homework Exercise #1 from Section 7.11.

A university is assigning 20 student council seats based on enrollment:

Population totals, standard quotas, lower quotas, and upper quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Standard Quota	7	5.5	4.17	3.33	20
Lower Quota	7	5	4	3	19
Upper Quota	7	6	5	4	22

The standard divisor was 600. Use Jefferson's Method to assign the 20 student council seats among the four departments.

Answer available at the end of this section.

(Section 7.13) - Homework Exercise #2

Refer to Homework Exercise #2 from Section 7.11.

A gaming company is distributing 50 servers across regions based on the number of active players:

Active players totals, standard quotas, lower quotas, and upper quotas for six regions

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Standard Quota	14.13	3.53	10.33	1.36	17.66	2.99	50
Lower Quota	14	3	10	1	17	2	47
Upper Quota	15	4	11	2	18	3	53

The standard divisor was 3,680. Use Jefferson's Method to assign the 50 servers among the four regions.

Answer available at the end of this section.

(Section 7.13) - Homework Exercise #3

Refer to Homework Exercise #3 from Section 7.11.

A large music festival is assigning 40 performance slots to different genres based on ticket demand:

Tickets sales totals, standard quotas, lower quotas, and upper quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Lower Quota	13	10	7	5	3	38
Upper Quota	14	11	8	6	4	43

The standard divisor was 1,750. Use Jefferson's Method to assign the 40 performance slots among the five genres.

Answer available at the end of this section.

(Section 7.13) - Homework Exercise #4

Refer to Homework Exercise #4 from Section 7.11.

A logistics company is distributing 45 delivery trucks to its regional hubs based on the number of packages handled last month:

Package totals, standard quotas, lower quotas, and upper quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Lower Quota	3	14	7	11	8	43
Upper Quota	4	15	8	12	9	48

The standard divisor was 2,177.78. Use Jefferson's Method to assign the 45 delivery trucks among the five regional hubs.

Answer available at the end of this section.

(Section 7.13) - Homework Exercise #5

Refer to Homework Exercise #5 from Section 7.11.

A streaming company is distributing 25 production slots across four genres based on viewer demand:

Viewer totals, standard quotas, lower quotas, and upper quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Standard Quota	6.65	11.08	4.54	2.73	25
Lower Quota	6	11	4	2	23
Upper Quota	7	12	5	3	27

The standard divisor was 4,800. Use Jefferson's Method to assign the 25 production slots among the four genres.

Answer available at the end of this section.

Answers

(Section 7.13) - Homework Exercise #1

Using the lower quotas, there is a deficit of one seat. Standard divisor = 600; Modified divisor used = 550. (There are other modified divisors).The apportionment is assigned in the last row of the table.

Population totals, modified quotas, and rounded-down quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Modified Quota (MD=550)	7.64	6	4.55	3.64	20
Rounded-Down Quota	7	6	4	3	20

(Section 7.13) - Homework Exercise #2

Using the lower quotas, there is a deficit of three servers. Standard divisor = 3,680; Modified divisor used = 3,460. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Active players totals, modified quotas, and rounded-down quotas for six regions

Region	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Modified Quota (MD=3,460)	15.03	3.76	10.98	1.45	18.79	3.18	50

Region	N. America	S. America	Europe	Africa	Asia	Australia	Total
Rounded-Down Quota	**15**	3	10	1	**18**	**3**	**50**

(Section 7.13) - Homework Exercise #3

Using the lower quotas, there is a deficit of two performance slots. Standard divisor = 1,750; Modified divisor used = 1,625. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Tickets sales totals, modified quotas, and rounded-down quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Modified Quota (MD=1,625)	14.43	10.97	7.85	6.12	3.71	40
Rounded-Down Quota	**14**	10	7	**6**	3	**40**

(Section 7.13) - Homework Exercise #4

Using the lower quotas, there is a deficit of two delivery trucks. Standard divisor = 2,177.78; Modified divisor used = 2,075. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Package totals, modified quotas, and rounded-down quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Modified Quota (MD=2,075)	3.86	15.07	7.41	11.995	9.37	45
Rounded-Down Quota	3	**15**	7	11	**9**	**45**

(Section 7.13) - Homework Exercise #5

Using the lower quotas, there is a deficit of two production slots. Standard divisor = 4,800; Modified divisor used = 4,400. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Viewer totals, modified quotas, and rounded-down quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Modified Quota (MD=4,400)	7.25	12.09	4.95	2.98	25
Rounded-Down Quota	7	12	4	2	**25**

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Section 7.14: Apportionment - Adams' Method

Learning Objectives

- Apportion seats using Adams' Method

Introduction to Adams' Method

In the previous section, we examined Jefferson's Method, an apportionment procedure that adjusts the divisor downward and rounds all quotas down. Jefferson's Method was designed to provide stable and predictable allocations and to avoid certain issues associated with Hamilton's Method. However, Jefferson's Method has its own limitation: because it always rounds down, it tends to favor larger groups while disadvantaging smaller ones.

This shortcoming led to the development of another divisor-based approach known as Adams' Method, named after the sixth U.S. President, John Quincy Adams. Adams' Method addresses the imbalance created by Jefferson's Method by rounding quotas up rather than down. As a result, Adams' Method tends to favor smaller groups and ensures that no group receives fewer seats than its standard quota would suggest.

Adams' Method demonstrates that apportionment outcomes depend not only on the populations involved, but also on how rounding decisions are made. By comparing Jefferson's and Adams' methods, we see how two reasonable mathematical approaches can produce different, yet defensible, results.

How Adams' Method Differs from Jefferson's Method

Jefferson's Method adjusts the divisor and rounds all quotas down until the correct total is reached. Adams' Method also uses a modified divisor, but it takes the opposite rounding approach.

- Jefferson's Method always rounds quotas down.
- Adams' Method always rounds quotas up.

Because of this difference:

- Jefferson's Method generally benefits larger groups.
- Adams' Method generally benefits smaller groups.

Adams' Method exists to counterbalance the bias introduced by always rounding down and to ensure that smaller groups are not consistently underrepresented.

Why Adams' Method Exists

Adams' Method exists to provide an alternative perspective on fairness in apportionment. While Jefferson's Method focuses on stability and favors larger populations, Adams' Method emphasizes protection for smaller groups by ensuring they are not rounded down below their proportional share.

Together, Hamilton's Method, Jefferson's Method, and Adams' Method illustrate that apportionment is not about finding a single perfect answer, but about understanding the consequences of different mathematical choices. Each method reflects a different

interpretation of fairness and highlights the challenges inherent in distributing indivisible items proportionally.

Here is the step-by-step process:

Adams' Method

Adams' Method is an apportionment procedure that assigns indivisible items by adjusting the divisor upward and rounding all resulting quotas up until the total number of items assigned is correct.

To apply Adams' Method:

- **Step 1:** Find the standard divisor.
- **Step 2:** Compute the standard quota for each group.
- **Step 3:** Assign each group its upper quota. (Ignore lower quotas).
- **Step 4:** Add the upper quotas.
- **Step 5:** If the total number of seats/items assigned is greater than required, increase the divisor slightly. Recalculate the quotas and round up again.
- **Step 6:** Repeat Step 5 until the total number of seats/items assigned equals the required total.

Adams' method avoids the problem of an apportionment resulting in a surplus of seats/items by using a divisor that will result in the correct number of seats being apportioned.

Example #7.14.1 🤔

Refer to the results from Example #7.11.3. The standard divisor was 7,500. Using Adams' Method, divide the 20 seats between the five districts.

Population totals, standard quotas, lower quotas, and upper quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Standard Quota	2.93	9.33	1.47	4.67	1.6	20
Lower Quota	2	9	1	4	1	17
Upper Quota	3	10	2	5	2	22

✅ **Solution:**

Applying Adams' Method:

- **Step 1**: Find the standard divisor.
- **Step 2**: Compute the standard quota for each group.
- **Step 3**: Assign each group its upper quota.
- **Step 4**: Add the upper quotas.

So, these four steps were already done to get the results in the table above. You can ignore or remove the lower quota row.

- **Step 5**: If the total number of seats/items assigned is greater than required, increase the divisor slightly. Recalculate the quotas and round down again.

So, 22 seats were assigned out of the 20 using the upper quota, thus creating a surplus of 2 seats. We need to remove 2 seats.

Recall that the standard divisor is 7,500. Since the sum of the upper quotas (22) is greater than the total number of seats available (20), we will use a modified divisor and recalculate everything. Remember, if we are removing seats, then we need a divisor that is greater than the standard divisor of 7,500.

Let's try 8,000 and calculate the modified quota and check to see if our rounded-up quotas have decreased. ("Rounded-Up Quota" replaces "Upper Quota" since we are no longer working with a standard quota anymore).

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{22{,}000}{8{,}000} = 2.75$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{70{,}000}{8{,}000} = 8.75$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{11{,}000}{8{,}000} = 1.375$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{35{,}000}{8{,}000} = 4.375$$

$$\text{District 5: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{12{,}000}{8{,}000} = 1.5$$

Update the results using a table and also include the rounded-up quotas. Check the sum of those rounded-up quotas.

Population totals, modified quotas, and rounded-up quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Modified Quota (MD=8,000)	2.75	8.75	1.375	4.375	1.5	20
Rounded-Up Quota	3	**9**	2	5	2	**21**

Unfortunately, the rounded-up quotas only decreased from 22 to 21. We need to remove one more seat. So, we need to increase the divisor some more. Let's try 8,500 now and recalculate again.

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{22{,}000}{8{,}500} \approx 2.59$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{70{,}000}{8{,}500} \approx 8.24$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{11{,}000}{8{,}500} \approx 1.29$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{35{,}000}{8{,}500} \approx 4.12$$

$$\text{District 5: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{12{,}000}{8{,}500} \approx 1.41$$

Update the results using a table and also include the rounded-up quotas. Check the sum of those rounded-up quotas.

Population totals, modified quotas, and rounded-up quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000

District	1	2	3	4	5	Total
Modified Quota (MD=8,500)	2.59	8.24	1.29	4.12	1.41	20
Rounded-Up Quota	3	**9**	2	5	2	**21**

Unfortunately, the rounded-up quotas did not change. We still need to remove one more seat. So, we need to increase the divisor some more. Let's try 8,750 now and recalculate again. (Note: Look at the modified quotas for District's 3 and 4, they are ever so close to have the rounded-up quotas get changed, so that's why we didn't choose 9,000).

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{22{,}000}{8{,}750} \approx 2.51$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{70{,}000}{8{,}750} = 8$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{11{,}000}{8{,}750} \approx 1.26$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{35{,}000}{8{,}750} = 4$$

$$\text{District 5: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{12{,}000}{8{,}750} \approx 1.37$$

Update the results using a table and also include the rounded-up quotas. Check the sum of those rounded-up quotas.

Population totals, modified quotas, and rounded-up quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Modified Quota (MD=8,750)	2.51	8	1.26	4	1.37	20
Rounded-Up Quota	3	**8**	2	**4**	2	**19**

So, now we went down to 19 seats. However, notice that the modified quotas for District 2 and District 4 equal their respective rounded-up quotas. So, it appears that we have to decrease the modified divisor ever so slightly, since you have two modified quotas equaling their rounded-up quotas. So, let's try 8,740 and recalculate again.

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{22{,}000}{8{,}740} \approx 2.52$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{70{,}000}{8{,}740} \approx 8.01$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{11{,}000}{8{,}740} \approx 1.26$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{35{,}000}{8{,}740} \approx 4.00$$

$$\text{District 5: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{12{,}000}{8{,}740} \approx 1.37$$

Update the results using a table and include the rounded-up quotas. Check the sum of those rounded-up quotas.

Population totals, modified quotas, and rounded-up quotas for five districts

District	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Modified Quota (MD=8,740)	2.52	8.01	1.26	4.00	1.37	20
Rounded-Up Quota	3	**9**	2	**4**	2	**20**

At this point, all 20 seats have been assigned. The range of acceptable modified divisors was very small in this apportionment. For example, the modified quota for District 4 was approximately 4.0046, which rounds to 4.00 when rounded to two decimal places. Similarly, the modified quota for District 2 was about 8.01, which resulted in an rounded-up quota of 9. These values show that the apportionment was extremely close, with very small changes in the divisor affecting the final seat assignments.

Thus, the modified divisor using Adams' Method is 8,740 and the apportionment is as follows:

3 seats for District 1, 9 seats for District 2, 2 seats for District 3, 4 seats for District 4, and 2 seats for District 5.

Note: There are only a few other modified divisors that would also work. In fact, any number from 8,740 through 8,744 inclusive would serve as the modified divisor to satisfy Adams' method.

Example #7.14.2 🤔

Refer to the results from Example #7.11.4. The standard divisor was 500. Using Adams' Method, divide the 12 board members among the four attendance zones:

Population totals, standard quotas, lower quotas, and upper quotas for four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	2	1	10
Upper Quota	5	4	3	2	14

✅ **Solution:**

Applying Adams' Method:

- **Step 1**: Find the standard divisor.
- **Step 2**: Compute the standard quota for each group.
- **Step 3**: Assign each group its upper quota.
- **Step 4**: Add the upper quotas.

So, these four steps were already done to get the results in the table above. You can ignore or remove the lower quota row.

- **Step 5**: If the total number of items assigned is greater than required, increase the divisor slightly. Recalculate the quotas and round up again.

So, 14 seats were assigned out of the 12 using the upper quota, thus creating a surplus of 2 seats. We need to remove 2 seats.

Recall that the standard divisor is 500. Since the sum of the upper quotas (14) is greater than the total number of seats available (12), we will use a modified divisor and recalculate everything. Remember, if we are removing seats, then we need a divisor that is greater than the standard divisor of 500.

Let's try 550 and calculate the modified quota and check to see if our rounded-up quotas have increased.

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{2{,}400}{550} \approx 4.36$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{1{,}800}{550} \approx 3.27$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{1{,}200}{550} \approx 2.18$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{600}{550} \approx 1.09$$

Update the results using a table and also include the rounded-up quotas. Check the sum of those rounded-up quotas.

Population totals, modified quotas, and rounded-up quotas for four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Modified Quota (MD=550)	4.36	3.27	2.18	1.09	12
Rounded-Up Quota	5	4	3	2	**14**

Unfortunately, the rounded-up quotas did not change. We still need to remove two seats. So, we need to increase the divisor some more. Let's try 600 now and recalculate again.

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{2{,}400}{600} \approx 4$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{1{,}800}{600} \approx 3$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{1{,}200}{600} \approx 2$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{600}{600} \approx 1$$

Update the results using a table and also include the rounded-up quotas. Check the sum of those rounded-up quotas.

Population totals, modified quotas, and rounded-up quotas for four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Modified Quota (MD=600)	4	3	2	1	12

District	North	South	East	West	Total
Rounded-Up Quota	**4**	**3**	**2**	**1**	**10**

So, now we went down to 10 seats. However, notice that the modified quotas for all zones equal their respective rounded-up quotas. So, it appears that we have to decrease the modified divisor ever so slightly, since you have all four modified quotas equaling their rounded-up quotas. We will have to be very careful with our selection and make sure to round to the nearest hundredths place, because two of those modified quotas need to result in a whole number after rounding while the other two cannot in order to get the desired 12 seats. So, let's try 599 and recalculate again.

$$\text{District 1: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{2{,}400}{599} \approx 4.01$$

$$\text{District 2: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{1{,}800}{599} \approx 3.01$$

$$\text{District 3: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{1{,}200}{599} \approx 2$$

$$\text{District 4: Modified Quota} = \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{600}{599} \approx 1$$

Update the results using a table and also include the rounded-up quotas. Check the sum of those rounded-up quotas.

Population totals, modified quotas, and rounded-up quotas for four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Modified Quota (MD=599)	4.01	3.01	2.00	1.00	12
Rounded-Up Quota	**5**	**4**	2	1	**12**

So, now we have the 12 seats.

Thus, the modified divisor using Adams' Method is 599 and the apportionment is as follows:

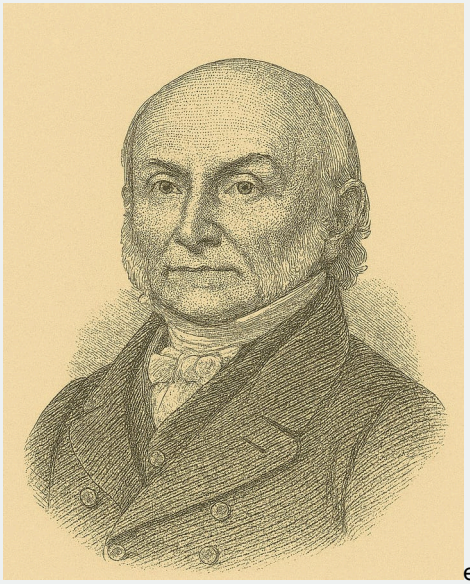
5 seats for North, 4 seats for South, 2 seats for East, and 1 seat for West.

Note: There are no other modified divisors that produce a valid apportionment in this case; the only value that works is 599. You may also observe that Hamilton's Method, Jefferson's Method, and Adams' Method all result in exactly the same seat assignment. This usually occurs when two or more of the populations, when divided by a modified quota, results in an exact whole number for each. This makes it difficult both to identify alternative modified divisors and to obtain different outcomes from the various apportionment methods.

Historical Note: John Quincy Adams and Apportionment

John Quincy Adams was the sixth President of the United States and one of the most accomplished public servants in American history. He was the son of John Adams, the second president, and spent much of his life in public service. Adams served as a diplomat, senator, Secretary of State, and member of the House of Representatives both before and after his presidency.

John Quincy Adams is also remembered for his involvement in debates over congressional representation. Concerned that



existing apportionment methods favored larger states, he proposed what is now known as Adams' Method of apportionment. This method rounds quotas upward and adjusts the divisor accordingly, offering greater protection for smaller states. Although Adams' Method was seriously considered by Congress, it was never officially adopted for apportioning seats in the House of Representatives. Lawmakers were concerned that the method could grant disproportionately high representation to smaller states. Nevertheless, Adams' Method remains an important theoretical contribution, illustrating how different rounding rules can significantly affect apportionment outcomes.

After leaving the presidency, Adams served in the House of Representatives, where he became a leading opponent of slavery and a strong defender of civil liberties. His dedication to public service and principled leadership left a lasting impact on both American politics and the mathematics of representation.

Section 7.14: Adams' Method [*In-Class Exercises*]

1. Suppose a country has 4 states, and a 20-seat House of Representatives. The number of representatives assigned to each state is proportional to the population of the state. The population for each state is given in the following table.

   Population totals, standard quotas, lower quotas, and upper quotas for four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Upper Quota	5	6	2	9	22

Use Adams' Method to assign the 20 seats among the four states. The standard divisor is 594.1.

2. A company has 24 laptops to distribute among five offices within the region based on the number of active projects in each office.

Population totals, standard quotas, lower quotas, and upper quotas for five offices within the region

Office	North County	West County	South County	East County	Midtown	Total
Population	17	25	30	12	42	126
Standard Quota	3.24	4.76	5.71	2.29	8	24
Lower Quota	3	4	5	2	8	22
Upper Quota	4	5	6	3	8	26

Use Adams' Method to assign the 24 laptops among the five offices. The standard divisor is 5.25.

Answers available at the end of this section.

Answers

Section 7.14: Adams' Method [*In-Class Exercises*]

1.

Population totals, modified quotas, and rounded-up quotas for four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Modified Quota (MD=650)	3.94	5.1	1.53	7.71	20
Rounded-Up Quota	**4**	6	2	**8**	**20**

Thus, the apportionment is as follows: 4 seats for A, 6 seats for B, 2 seats for C, and 8 seats for D. (The modified divisor using Adams' Method here is 650. There are other values that work as well).

2.

Population totals, modified quotas, and rounded-up quotas for five offices within the region

Office	North County	West County	South County	East County	Midtown	Total
Population	17	25	30	12	42	126
Modified Quota (MD=5.99)	2.84	4.17	5.01	2.00	7.01	24
Rounded-Up Quota	**3**	5	6	**2**	8	**24**

Thus, the apportionment is as follows: 3 seats for North County, 5 seats for West County, 6 seats for South County, 2 seats for East County, and 8 seats for Midtown. (A modified divisor using Adams' Method here is 5.99. There are no other values that will work).

Note: When two or more of the populations are divided by a modified divisor that results in an exact whole number for each, there will probably be only one or a few possible modified divisors that will work. If the modified divisor is 6, both South County and East County have whole numbers as modified quotas.

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Section 7.14.H: Homework

Section 7.14: Adams' Method - Homework Exercises

(Section 7.14) - Homework Exercise #1

Refer to Homework Exercise #1 from Section 7.11.

A university is assigning 20 student council seats based on enrollment:

Population totals, standard quotas, lower quotas, and upper quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Standard Quota	7	5.5	4.17	3.33	20
Lower Quota	7	5	4	3	19
Upper Quota	7	6	5	4	22

The standard divisor was 600. Use Adams' Method to assign the 20 student council seats among the four departments.

Answer available at the end of this section.

(Section 7.14) - Homework Exercise #2

Refer to Homework Exercise #2 from Section 7.11.

A gaming company is distributing 50 servers across regions based on the number of active players:

Active players totals, standard quotas, lower quotas, and upper quotas for six regions

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Standard Quota	14.13	3.53	10.33	1.36	17.66	2.99	50
Lower Quota	14	3	10	1	17	2	47
Upper Quota	15	4	11	2	18	3	53

The standard divisor was 3,680. Use Adams' Method to assign the 50 servers among the four regions.

Answer available at the end of this section.

(Section 7.14) - Homework Exercise #3

Refer to Homework Exercise #3 from Section 7.11.

A large music festival is assigning 40 performance slots to different genres based on ticket demand:

Tickets sales totals, standard quotas, lower quotas, and upper quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Lower Quota	13	10	7	5	3	38
Upper Quota	14	11	8	6	4	43

The standard divisor was 1,750. Use Adams' Method to assign the 40 performance slots among the five genres.

Answer available at the end of this section.

(Section 7.14) - Homework Exercise #4

Refer to Homework Exercise #4 from Section 7.11.

A logistics company is distributing 45 delivery trucks to its regional hubs based on the number of packages handled last month:

Package totals, standard quotas, lower quotas, and upper quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Lower Quota	3	14	7	11	8	43
Upper Quota	4	15	8	12	9	48

The standard divisor was 2,177.78. Use Adams' Method to assign the 45 delivery trucks among the five regional hubs.

Answer available at the end of this section.

(Section 7.14) - Homework Exercise #5

Refer to Homework Exercise #5 from Section 7.11.

A streaming company is distributing 25 production slots across four genres based on viewer demand:

Viewer totals, standard quotas, lower quotas, and upper quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Standard Quota	6.65	11.08	4.54	2.73	25
Lower Quota	6	11	4	2	23
Upper Quota	7	12	5	3	27

The standard divisor was 4,800. Use Adams' Method to assign the 25 production slots among the four genres.

Answer available at the end of this section.

Answers

(Section 7.14) - Homework Exercise #1

Using the upper quotas, there is a surplus of two seats. Standard divisor = 600; Modified divisor used = 666. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Population totals, modified quotas, and rounded-up quotas for four departments

District:	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Modified Quota (MD=666)	6.31	4.95	3.75	3.003	20
Rounded-Up Quota	7	5	**4**	**4**	**20**

(Section 7.14) - Homework Exercise #2

Using the upper quotas, there is a surplus of three servers. Standard divisor = 3,680; Modified divisor used = 3,980. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Active players totals, modified quotas, and rounded-up quotas for six regions

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Modified Quota (MD=4,000)	13.07	3.27	9.55	1.26	16.33	2.76	50

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Rounded-Up Quota	**14**	4	**10**	2	**17**	3	**50**

(Section 7.14) - Homework Exercise #3

Using the upper quotas, there is a surplus of three performance slots. Standard divisor = 1,750; Modified divisor used = 1,900. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Tickets sales totals, modified quotas, and rounded-up quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Modified Quota (MD=1,900)	12.34	9.38	6.72	5.23	3.17	40
Rounded-Up Quota	**13**	**10**	**7**	6	4	**40**

(Section 7.14) - Homework Exercise #4

Using the upper quotas, there is a surplus of three delivery trucks. Standard divisor = 2,177.78; Modified divisor used = 2,300. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Package totals, modified quotas, and rounded-up quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Modified Quota (MD=2,300)	3.48	13.60	6.68	10.82	8.46	45
Rounded-Up Quota	4	**14**	**7**	**11**	9	**45**

(Section 7.14) - Homework Exercise #5

Using the upper quotas, there is a surplus of two production slots. Standard divisor = 4,800; Modified divisor used = 5,318. (There are other modified divisors). The apportionment is assigned in the last row of the table.

Viewer totals, modified quotas, and rounded-up quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Modified Quota (MD=5,318)	5.998	10.004	4.10	2.46	25
Rounded-Up Quota	**6**	**11**	5	3	**25**

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Section 7.15: Apportionment - Webster's Method

Learning Objectives

- Apportion seats using Webster's Method

Introduction to Webster's Method

In the previous three sections, we examined Hamilton's Method, Jefferson's Method, and Adams' Method. Each of these approaches attempts to solve the apportionment problem in a systematic way, yet each emphasizes a different interpretation of fairness. Hamilton's Method relies on fractional remainders, Jefferson's Method rounds quotas down, and Adams' Method rounds quotas up. As a result, these methods can favor different groups depending on population size.

Webster's Method was developed as a compromise between Jefferson's and Adams' methods. Instead of consistently rounding quotas down or up, Webster's Method rounds quotas to the nearest whole number. This approach aims to balance representation more evenly, avoiding the systematic bias toward larger groups seen in Jefferson's Method and the bias toward smaller groups seen in Adams' Method.

Because of this balanced rounding approach, Webster's Method is often viewed as one of the most neutral apportionment procedures. It seeks to assign seats in a way that is closest, overall, to each group's standard quota.

Here is the step-by-step process:

Webster's Method

Webster's Method is an apportionment procedure that assigns indivisible seats/items by adjusting the divisor and rounding each resulting quota to the nearest whole number until the total number of seats/items assigned is correct.

To apply Webster's Method:

- **Step 1:** Find the standard divisor.
- **Step 2:** Compute the standard quota for each group.
- **Step 3:** Round each quota to the nearest whole number.
- **Step 4:** Add the rounded quotas.
- **Step 5:** If the initial total number of seats/items assigned
  - is equal to the required amount, then stop and assign those rounded quotas.
  - is less than required, decrease the divisor slightly. Recalculate the quotas and round to the nearest whole number again (Jefferson's Method).
  - is greater than required, increase the divisor slightly. Recalculate the quotas and round to the nearest whole number again (Adams' Method).
- **Step 6:** Repeat Step 5 until the total number of seats/items assigned equals the required total.

Why Webster's Method Exists

Webster's Method exists to address the limitations of the earlier divisor methods. Jefferson's Method favors larger groups

because it always rounds down, while Adams' Method favors smaller groups because it always rounds up. Webster's Method reduces these extremes by rounding to the nearest integer.

This makes Webster's Method especially attractive when proportional fairness is defined as minimizing overall rounding error across all groups. In many cases, Webster's Method produces results that feel more intuitive and equitable, particularly when population sizes do not differ dramatically.

Webster's Method can be seen as a midpoint between Jefferson's and Adams' methods. While all three adjust the divisor to reach the correct total, Webster's Method balances upward and downward rounding, often producing apportionment closer to the standard quotas overall.

Example #7.15.1 🤔

Refer to the results from Example #7.11.3. The standard divisor was 7,500. Using Webster's Method, divide the 20 seats between the five districts.

Population totals, standard quotas, lower quotas, and upper quotas for five districts

District:	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Standard Quota	2.93	9.33	1.47	4.67	1.6	20
Lower Quota	2	9	1	4	1	17
Upper Quota	3	10	2	5	2	22

✅ **Solution:**

Applying Webster's Method:

- **Step 1**: Find the standard divisor.
- **Step 2**: Compute the standard quota for each group.
- **Step 3**: Round each quota to the nearest whole number.
- **Step 4**: Add the rounded quotas.

So, Steps 1 and 2 were already done to get the results in the table above.

For Steps 3 and 4, simply add another row with these initial assignment quotas and the sum of the rounded quotas to the table.

Population totals, standard quotas, lower quotas, upper quotas, and initial assignment quotas for five districts

District:	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Standard Quota	2.93	9.33	1.47	4.67	1.6	20
Lower Quota	2	9	1	4	1	17
Upper Quota	3	10	2	5	2	22

District:	1	2	3	4	5	Total
Initial Assignment Quotas	**3**	**9**	**1**	**5**	**2**	**20**

- **Step 5:** If the initial total number of seats/items assigned is equal to the required amount, then stop and assign those rounded quotas.

Fortunately, the initial assignment quotas yielded our required 20 seats and we did not need to find a modified divisor for Webster's method.

Thus, the apportionment is as follows:

3 seats for District 1, 9 seats for District 2, 1 seat for District 3, 5 seats for District 4, and 2 seats for District 5.

Example #7.14.2 🤔

Refer to the results from Example #7.11.4. The standard divisor was 500. Using Webster's Method, divide the 12 board members among the four attendance zones:

Population totals, standard quotas, lower quotas, and upper quotas for four attendance zones

District	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	2	1	10
Upper Quota	5	4	3	2	14

✅ **Solution:**

Applying Webster's Method:

- **Step 1**: Find the standard divisor.
- **Step 2**: Compute the standard quota for each group.
- **Step 3**: Round each quota to the nearest whole number.
- **Step 4**: Add the rounded quotas.

So, Steps 1 and 2 were already done to get the results in the table above.

For Steps 3 and 4, simply add another row with these initial assignment quotas and the sum of the rounded quotas to the table.

Population totals, standard quotas, lower quotas, upper quotas, and initial assignment quotas for four attendance zones

District:	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Standard Quota	4.8	3.6	2.4	1.2	12

District:	North	South	East	West	Total
Lower Quota	4	3	2	1	10
Upper Quota	5	4	3	2	14
Initial Assignment Quotas	**5**	**4**	**2**	**1**	**12**

- **Step 5:** If the initial total number of seats/items assigned is equal to the required amount, then stop and assign those rounded quotas.

Fortunately, the initial assignment quotas yielded our required 12 seats and we did not need to find a modified divisor for Webster's method.

Thus, the apportionment is as follows:

5 seats for North, 4 seats for South, 2 seats for East, and 1 seat for West.

Example #7.15.3 🤔

A state must allocate 25 hospital beds among five regions based on population using Webster's Method.

Population distribution among five regions

Region:	North	South	West	East	Central	Total
Population	410,000	190,000	40,000	95,000	265,000	1,000,000

a. Compute the standard divisor and the initial standard quotas.
b. Round each quota to the nearest whole number to find the initial assignments.
c. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of beds. Indicate whether a modified divisor would be required, but do not continue beyond this point.

✅ **Solution:**

a. Compute the standard divisor and the initial standard quotas.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{1{,}000{,}000}{25} = 40{,}000$$

$$\text{North: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{410{,}000}{40{,}000} = 10.25$$

$$\text{South: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{190{,}000}{40{,}000} = 4.75$$

$$\text{West: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{40{,}000}{40{,}000} = 1$$

$$\text{East: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{95{,}000}{40{,}000} \approx 2.38$$

$$\text{Central: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{265{,}000}{40{,}000} \approx 6.63$$

b. Round each quota to the nearest whole number to find the initial assignments.

Use a table here and include all of your results.

Population totals, standard quotas, and initial assignment quotas for five regions

Region:	North	South	West	East	Central	Total
Population	410,000	190,000	40,000	95,000	265,000	1,000,000
Standard Quota	10.25	4.75	1	2.38	6.63	25
Initial Assignment Quotas	**10**	**5**	**1**	**2**	**7**	**25**

c. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of beds. Indicate whether a modified divisor would be required, but do not continue beyond this point.

After the initial assignment quotas, the total number of beds is correct at 25, so no divisor adjustment is needed.

Thus, the apportionment is as follows: 10 beds for North, 5 beds for South, 1 bed for West, 2 beds for East, and 7 beds for Central.

Example #7.15.4 🤔

A company has 18 autonomous robots to distribute among four warehouses based on shipping volume.

Shipping Volume distribution among four warehouses

Warehouse	A	B	C	D	Total
Shipping Volume	4,900	800	3,100	8,200	17,000

Using Webster's Method,

a. Compute the standard divisor and the initial standard quotas.
b. Round each quota to the nearest whole number to find the initial assignments.
c. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of autonomous robots. Indicate whether a modified divisor would be required, but do not continue beyond this point.

✅ **Solution:**

a. Compute the standard divisor and the initial standard quotas.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{17{,}000}{18} \approx 944.44$$

$$\text{A: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{4{,}900}{944.44} \approx 5.19$$

$$\text{B: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{800}{944.44} \approx 0.85$$

$$\text{C: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{3{,}100}{944.44} \approx 3.28$$

$$\text{D: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{8{,}200}{944.44} \approx 8.68$$

b. Round each quota to the nearest whole number to find the initial assignments.

Use a table here and include all of your results.

Shipping volume totals, standard quotas, and initial assignment quotas for four warehouses

Warehouse	A	B	C	D	Total
Shipping Volume	4,900	800	3,100	8,200	17,000
Standard Quota	5.19	0.85	3.28	8.68	18
Initial Assignment Quotas	**5**	**1**	**3**	**9**	**18**

c. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of autonomous robots. Indicate whether a modified divisor would be required, but do not continue beyond this point.

After the initial assignment quotas, the total number of robots is correct at 18, so no divisor adjustment is needed.

Thus, the apportionment is as follows: 5 autonomous robots for A, 1 autonomous robots for B, 3 autonomous robots for C, and 9 autonomous robots for D.

Example #7.15.5 🤔

A state assigns 14 delegates among five counties based on registered voters.

Registered voter distribution across five counties

County:	Alpha	Beta	Gamma	Delta	Epsilon	Total
Registered Voters	54,000	18,000	5,000	92,000	31,000	200,000

Using Webster's Method,

a. Compute the standard divisor and the initial standard quotas.
b. Round each quota to the nearest whole number to find the initial assignments.
c. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of delegates. Indicate whether a modified divisor would be required, and if so, find a modified divisor and the correct apportionment.

✅ **Solution:**

a. Compute the standard divisor and the initial standard quotas.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{200{,}000}{14} \approx 14{,}285.71$$

$$\text{Alpha: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{54{,}000}{14{,}285.71} \approx 3.78$$

$$\text{Beta: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{18{,}000}{14{,}285.71} \approx 1.26$$

$$\text{Gamma: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{5{,}000}{14{,}285.71} = 0.35$$

Delta: Standard Quota $= \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{92{,}000}{14{,}285.71} \approx 6.44$

Epsilon: Standard Quota $= \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{31{,}000}{14{,}285.71} \approx 2.17$

b. Round each quota to the nearest whole number to find the initial assignments.

Use a table here and include all of your results.

Registered voter totals, standard quotas, and initial assignment quotas for five counties

County:	Alpha	Beta	Gamma	Delta	Epsilon	Total
Registered Votes	54,000	18,000	5,000	92,000	31,000	200,000
Standard Quota	3.78	1.26	0.35	6.44	2.17	14
Initial Assignment Quotas	**4**	**1**	**0**	**6**	**2**	**13**

c. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of delegates. Indicate whether a modified divisor would be required, and if so, find a modified divisor and the correct apportionment.

After the initial assignment quotas, we have a deficit of one delegate, so we decrease the standard divisor and recalculate (Jefferson's Method).

The standard divisor is 14,285.71.

Let's try 13,000 and calculate the modified standard quota and check to see if our initial assignments have increased.

Alpha: Modified Quota $= \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{54{,}000}{13{,}000} \approx 4.15$

Beta: Modified Quota $= \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{18{,}000}{13{,}000} \approx 1.38$

Gamma: Modified Quota $= \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{5{,}000}{13{,}000} \approx 0.38$

Delta: Modified Quota $= \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{92{,}000}{13{,}000} \approx 7.08$

Epsilon: Modified Quota $= \frac{\text{Group Population}}{\text{Modified Divisor}} = \frac{31{,}000}{13{,}000} \approx 2.38$

Update the results using a table and also include the initial assignment of modified quotas. Check the sum of those initial assignment of modified quotas.

Registered voter totals, modified quotas, and initial assignment of modified quotas for five counties

County:	Alpha	Beta	Gamma	Delta	Epsilon	Total
Modified Quota (MD=13,000)	4.15	1.38	0.38	7.08	2.38	14

County:	Alpha	Beta	Gamma	Delta	Epsilon	Total
Initial Assignment of Modified Quotas (MD=13,000)	**4**	**1**	**0**	**7**	**2**	**14**

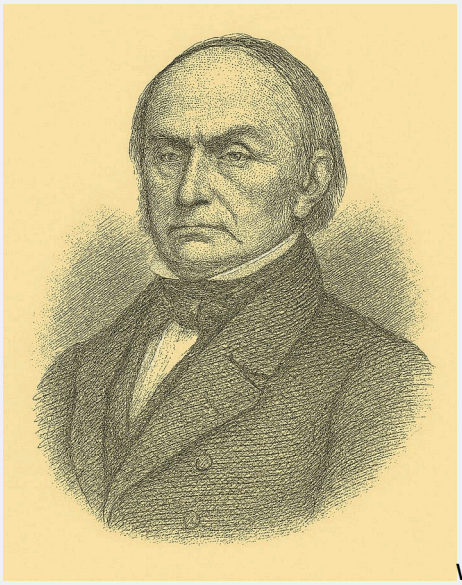
After the initial assignment of modified quotas, the total number of delegates is now correct at 14, so no other divisor adjustment is needed. (Delta was the only county that increased).

Thus, the apportionment is as follows: 4 delegates for Alpha, 1 delegate for Beta, 0 delegates for Gamma, 7 delegates for Delta, and 2 delegates for Epsilon.

Historical Note: Daniel Webster and Apportionment

Daniel Webster (1782—1852) was a prominent American statesman, lawyer, and orator during the early nineteenth century. He served as a United States representative, senator from Massachusetts, and later as Secretary of State under three different presidents. Webster was widely regarded as one of the most influential political figures of his time and a leading voice for the preservation of the Union.

Webster was known for his strong advocacy of national unity and a balanced federal system. He played a major role in key political debates of the era, including those involving states' rights, federal authority, and economic policy. His speeches and writings helped shape constitutional interpretation and strengthened the role of the federal government.



Webster's involvement in apportionment debates reflected his interest in fair and balanced representation. The apportionment procedure that bears his name, Webster's Method, was proposed as a compromise between Jefferson's Method, which favored larger states, and Adams' Method, which favored smaller states. Webster's Method rounds quotas to the nearest whole number, aiming to minimize overall rounding bias and produce results that align closely with proportional fairness.

Although Webster's Method was not immediately adopted for congressional apportionment, it later influenced the development of modern divisor methods and is valued today for its balanced approach. Daniel Webster's legacy lies not only in his political leadership but also in his lasting contribution to the mathematics of representation.

Section 7.15: Webster's Method [*In-Class Exercises*]

1. Suppose a country has 4 states, and a 20-seat House of Representatives. The number of representatives assigned to each state is proportional to the population of the state. The population for each state is given in the following table.

   Population totals, standard quotas, lower quotas, and upper quotas for four states

State:	A	B	C	D	Total
Population:	2,560	3,315	995	5,012	11,882
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Upper Quota	5	6	2	9	22

Use Webster's Method to assign the 20 seats among the four states. The standard divisor is 594.1.

2. A company has 24 laptops to distribute among five offices within the region based on the number of active projects in each office.

   Active project totals, standard quotas, lower quotas, and upper quotas for five offices within the region

Office:	North County	West County	South County	East County	Midtown	Total
Active Projects	17	25	30	12	42	126
Standard Quota	3.24	4.76	5.71	2.29	8	24
Lower Quota	3	4	5	2	8	22
Upper Quota	4	5	6	3	8	26

Use Webster's Method to assign the 24 laptops among the five offices. The standard divisor is 5.25.

3. A university has 22 internal research grants to distribute among six departments.

   Faculty distribution among six departments

Department:	Biology	Engineering	Chemistry	Economics	History	Art	Total
Faculty:	33	18	22	16	28	12	129

Use Webster's Method to assign the 22 grants among the six departments.

   a. Compute the standard divisor and the initial standard quotas.
   b. Round each quota to the nearest whole number to find the initial assignments.
   c. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of grants. Indicate whether a modified divisor would be required, and if so, find a modified divisor and the correct apportionment.

Answers available at the end of this section.

Answers

Section 7.15: Webster's Method [*In-Class Exercises*]

1.

Population totals, standard quotas, lower quotas, and upper quotas for four states

State:	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Upper Quota	5	6	2	9	22
Initial Assignments	**4**	**6**	**2**	**8**	**20**

Thus, the apportionment is as follows: 4 seats for A, 6 seats for B, 2 seats for C, and 8 seats for D. (No modified divisor needed).

2.

Active project totals, standard quotas, lower quotas, upper quotas, and initial assignments for five offices within the region

Office:	North County	West County	South County	East County	Midtown	Total
Active Projects	17	25	30	12	42	126
Standard Quota	3.24	4.76	5.71	2.29	8	24
Lower Quota	3	4	5	2	8	22
Upper Quota	4	5	6	3	8	26
Initial Assignments	**3**	**5**	**6**	**2**	**8**	**24**

Thus, the apportionment is as follows: 3 laptops for North County, 5 laptops for West County, 6 laptops for South County, 2 laptops for East County, and 8 laptops for Midtown. (No modified divisor needed).

3.

Faculty totals, standard quotas, initial assignments, modified quota, initial assignment of modified quotas for six departments

Department:	Biology	Engineering	Chemistry	Economics	History	Art	Total
Faculty	33	18	22	16	28	12	129
Standard Quota	5.63	3.07	3.75	2.73	4.78	2.05	22
Initial Assignments	**6**	**3**	**4**	**3**	**5**	**2**	**23**

Department:	Biology	Engineering	Chemistry	Economics	History	Art	Total
Modified Quota (MD=6.1)	5.41	2.95	3.61	2.62	4.59	1.97	22
Initial Assignment of Modified Quotas (MD=6.1)	**5**	**3**	**4**	**3**	**5**	**2**	**22**

a. Standard divisor = 5.86; Initial standard quotas in table above.
b. Initial assignments are in table above.
c. A surplus of one grant has been made based off the initial assignments. A modified divisor is required. One modified divisor that will work is 6.1. Thus, the apportionment is as follows: 5 grants for Biology, 3 grants for Engineering, 4 grants for Chemistry, 3 grants for Economics, 5 grants for History, and 2 grants for Art.

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Section 7.15.H: Homework

Section 7.15: Webster's Method - Homework Exercises

(Section 7.15) - Homework Exercise #1

Refer to Homework Exercise #1 from Section 7.11.

A university is assigning 20 student council seats based on enrollment:

Population totals, standard quotas, lower quotas, and upper quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Standard Quota	7	5.5	4.17	3.33	20
Lower Quota	7	5	4	3	19
Upper Quota	7	6	5	4	22

The standard divisor was 600.

a. Use Webster's Method to initially assign the 20 student council seats among the four departments.
b. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of seats.
c. Indicate whether a modified divisor would be required, and if so, find a modified divisor and the correct apportionment.

Answer available at the end of this section.

(Section 7.15) - Homework Exercise #2

Refer to Homework Exercise #2 from Section 7.11.

A gaming company is distributing 50 servers across regions based on the number of active players:

Active players totals, standard quotas, lower quotas, and upper quotas for six regions

Region	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Standard Quota	14.13	3.53	10.33	1.36	17.66	2.99	50

Region	N. America	S. America	Europe	Africa	Asia	Australia	Total
Lower Quota	14	3	10	1	17	2	47
Upper Quota	15	4	11	2	18	3	53

The standard divisor was 3,680.

a. Use Webster's Method to initially assign the 50 servers among the four regions.
b. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of seats.
c. Indicate whether a modified divisor would be required, and if so, find a modified divisor and the correct apportionment.

Answer available at the end of this section.

(Section 7.15) - Homework Exercise #3

Refer to Homework Exercise #3 from Section 7.11.

A large music festival is assigning 40 performance slots to different genres based on ticket demand:

Tickets sales totals, standard quotas, lower quotas, and upper quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Lower Quota	13	10	7	5	3	38
Upper Quota	14	11	8	6	4	43

The standard divisor was 1,750.

a. Use Webster's Method to initially assign the 40 performance slots among the five genres.
b. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of performance slots.
c. Indicate whether a modified divisor would be required, and if so, find a modified divisor and the correct apportionment.

Answer available at the end of this section.

(Section 7.15) - Homework Exercise #4

Refer to Homework Exercise #4 from Section 7.11.

A logistics company is distributing 45 delivery trucks to its regional hubs based on the number of packages handled last month:

Package totals, standard quotas, lower quotas, and upper quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Lower Quota	3	14	7	11	8	43
Upper Quota	4	15	8	12	9	48

The standard divisor was 2,177.78.

a. Use Webster's Method to initially assign the 45 delivery trucks among the four regions.
b. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of trucks.
c. Indicate whether a modified divisor would be required, and if so, find a modified divisor and the correct apportionment.

Answer available at the end of this section.

(Section 7.15) - Homework Exercise #5

Refer to Homework Exercise #5 from Section 7.11.

A streaming company is distributing 25 production slots across four genres based on viewer demand:

Viewer totals, standard quotas, lower quotas, and upper quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Standard Quota	6.65	11.08	4.54	2.73	25
Lower Quota	6	11	4	2	23
Upper Quota	7	12	5	3	27

The standard divisor was 4,800.

a. Use Webster's Method to initially assign the 25 production slots among the four genres.
b. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of production slots.
c. Indicate whether a modified divisor would be required, and if so, find a modified divisor and the correct apportionment.

Answer available at the end of this section.

(Section 7.15) - Homework Exercise #6

A state legislature has 30 seats to distribute among four districts based on population:

Population distribution across four districts

District	North	South	Central	Coastal	Total
Population	41,200	24,900	33,700	18,200	118,000

a. Use Webster's Method to initially assign the 30 seats among the four districts.
b. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of seats and indicate whether a modified divisor would be required. If a modified divisor is required, do not find a modified divisor for the correct apportionment. Only state the conditions needed.

Answer available at the end of this section.

(Section 7.15) - Homework Exercise #7

A state legislature has 15 seats to distribute among four districts based on population:

Population distribution across four districts

District	West	North	South	East	Total
Population	9,900	20,300	12,200	17,600	60,000

a. Use Webster's Method to initially assign the 15 seats among the four districts.
b. State whether these initial assignment quotas produce a surplus, a deficit, or the correct total number of seats and indicate whether a modified divisor would be required. If a modified divisor is required, do not find a modified divisor for the correct apportionment. Only state the conditions needed.

Answer available at the end of this section.

Answers

(Section 7.15) - Homework Exercise #1

a. The initial assignment quotas are in the last row of the table.

Population totals, standard quotas, and initial assignment quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Standard Quota	7	5.5	4.17	3.33	20
Initial Assignment Quotas	**7**	**6**	**4**	**3**	**20**

b. The initial assignment quotas produce a correct number of seats.
c. No modified divisor is needed. The correct apportionment are the initial assignment quotas assigned in the last row of the table.

(Section 7.15) - Homework Exercise #2

a. The initial assignment quotas are in the last row of the table.

Population totals, standard quotas, and initial assignment quotas for six regions

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Standard Quota	14.13	3.53	10.33	1.36	17.66	2.99	50
Initial Assignment Quotas	**14**	**4**	**10**	**1**	**18**	**3**	**50**

b. The initial assignment quotas produce a correct number of servers.
c. No modified divisor is needed. The correct apportionment are the initial assignment quotas assigned in the last row of the table.

(Section 7.15) - Homework Exercise #3

a. The initial assignment quotas are in the last row of the table.

Tickets sales totals, standard quotas, and initial assignment quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Initial Assignment Quotas	13	10	7	6	3	**39**

b. The initial assignment quotas produces a deficit of one slot.
c. A modified divisor is needed. Use Jefferson's method to find a modified divisor and then use Webster's method to assign the quotas. The modified divisor used is 1,735 (There are other modified divisors). The apportionment are the assignment modified quotas assigned in the last row of the table.

Tickets sales totals, modified quotas, and assignment of modified quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Modified Quota (MD=1,735)	13.52	10.27	7.35	5.73	3.48	40
Assignment of Modified Quotas (MD=1,735)	**14**	10	7	6	3	**40**

(Section 7.15) - Homework Exercise #4

a. The initial assignment quotas are in the last row of the table.

Package totals, standard quotas, and initial assignment quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Initial Assignment Quotas	4	14	7	11	9	45

b. The initial assignment quotas produce a correct number of trucks.
c. No modified divisor is needed. The correct apportionment are the initial assignment quotas assigned in the last row of the table.

(Section 7.15) - Homework Exercise #5

a. The initial assignment quotas are in the last row of the table.

Viewer totals, standard quotas, and initial assignment quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Standard Quota	6.65	11.08	4.54	2.73	25
Initial Assignment Quotas	7	11	5	3	**26**

b. The initial assignment quotas produces a surplus of one slot.
c. A modified divisor is needed. Use Adams' method to find a modified divisor and then use Webster's method to assign the quotas. The modified divisor used is 4,900 (There are other modified divisors). The apportionment are the assignment modified quotas assigned in the last row of the table.

Viewer totals, modified quotas, and assignment of modified quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Modified Quota (MD=4,900)	6.51	10.86	4.45	2.67	25
Assignment of Modified Quotas (MD=4,900)	7	11	**4**	3	**25**

(Section 7.15) - Homework Exercise #6

a. The initial assignment quotas are in the last row of the table. The standard divisor is 3,933.33.

Population totals, standard quotas, lower quotas, upper quotas, and initial assignment quotas for four districts

District	North	South	Central	Coastal	Total
Population	41,200	24,900	33,700	18,200	118,000
Standard Quota	10.47	6.33	8.57	4.63	30

District	North	South	Central	Coastal	Total
Lower Quota	10	6	8	4	28
Upper Quota	11	7	9	5	32
Initial Assignment Quotas	**10**	**6**	**9**	**5**	**30**

b. The initial assignment quotas produce a correct number of seats. No modified divisor is needed. The correct apportionment are the initial assignment quotas assigned in the last row of the table.

(Section 7.15) - Homework Exercise #7

a. The initial assignment quotas are in the last row of the table. The standard divisor is 4,000.

Population totals, standard quotas, lower quotas, upper quotas, and initial assignment quotas for four districts

District	West	North	South	East	Total
Population	9,900	20,300	12,200	17,600	60,000
Standard Quota	2.48	5.08	3.05	4.4	15
Lower Quota	2	5	3	4	14
Upper Quota	3	6	4	5	18
Initial Assignment Quotas	**2**	**5**	**3**	**4**	**14**

b. The initial assignment quotas produce a deficit of one seat. A modified divisor is required. Use Jefferson's method to find a modified divisor and then use Webster's method to assign the quotas.

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Section 7.16: Apportionment - Huntington-Hill Method

Learning Objectives

- Apportion seats using Huntington–Hill Method

Introduction to the Huntington–Hill Method

In the previous four sections, we examined Hamilton's Method and three divisor methods: Jefferson's Method, Adams' Method, and Webster's Method. Each of these approaches attempts to convert fractional quotas into whole-number assignments while preserving proportional fairness, yet each method embodies a different rounding strategy and therefore favors different groups.

The **Huntington–Hill Method**, also known as the method of equal proportions, was developed to address limitations observed in earlier apportionment methods. Rather than consistently rounding up, rounding down, or rounding to the nearest whole number, this method uses a geometric mean to determine rounding decisions. The goal is to equalize the ratios of representation between groups as closely as possible.

Because of this mathematical approach, the Huntington–Hill Method often produces apportionment that minimize relative differences in representation across groups. For this reason, it is widely regarded as one of the most balanced and mathematically justified apportionment procedures.

Here is the step-by-step process:

Huntington–Hill Method

The Huntington–Hill Method is an apportionment procedure that assigns indivisible seats/items by adjusting a divisor and rounding each quota according to whether it exceeds or falls short of the geometric mean of the two nearest whole numbers.

To apply Webster's Method:

- **Step 1:** Find the standard divisor.
- **Step 2:** Compute the standard quota for each group.
- **Step 3:** Find the lower and upper quotas for each group.
- **Step 4:** Calculate the geometric mean of the lower and upper quotas for each group.
- **Step 5:**
  - Initially assign the lower quota if the standard quota is less than the geometric mean of the lower and upper quotas; or
  - Initially assign the upper quota if the standard quota is greater than the geometric mean of the lower and upper quotas.
- **Step 6:** If the total number of seats/items assigned in Step 5
  - equals the number of seats/items available, then the assignment from that step is final.
  - does not equal the number of seats/items available, a modified divisor must be used. The seats/items are then reassigned using the modified divisor, and this process is repeated until the total number of assigned seats/items matches the number of seats/items available.

Geometric Mean

The geometric mean of any two positive numbers *a* and *b* is the square root of their product:

$$\sqrt{a \cdot b} \tag{7.16.1}$$

In apportionment methods such as the Huntington–Hill Method, the geometric mean of the lower quota (LQ) and the upper quota (UQ)is needed, so

$$\sqrt{(\mathrm{LQ}) \cdot (\mathrm{UQ})} \tag{7.16.2}$$

Historically, the geometric mean originates in Ancient Greek geometry and the theory of proportions, where it was used to compare quantities by their ratios rather than their differences. Mathematically, it balances relative sizes instead of absolute amounts, making it especially useful in settings where proportional fairness matters. Practically, this makes the geometric mean ideal for situations in which equity depends on relative size or share. Civically, this idea underlies the modern U.S. system of congressional apportionment. Overall, the geometric mean is used because it produces the fairest proportional balance by minimizing relative error rather than absolute error.

Why the Huntington–Hill Method Exists

Earlier apportionment methods demonstrated that rounding rules can systematically benefit certain groups. Jefferson's Method tends to favor larger groups, Adams' Method tends to favor smaller groups, and even Webster's Method can produce outcomes that appear uneven in some edge cases. The Huntington–Hill Method was introduced to reduce these biases by focusing on proportional equality rather than absolute rounding direction.

Specifically, the Huntington–Hill Method aims to minimize the percentage difference between how well groups are represented, rather than the absolute difference in seats. This principle of equal proportions leads to apportionment in which representation ratios are as uniform as possible across all groups.

The Huntington–Hill Method represents the culmination of historical efforts to find a fair and consistent approach to apportionment. Its mathematical structure avoids persistent bias toward either large or small groups and emphasizes proportional equality. Due to these properties, the Huntington–Hill Method was eventually adopted as the official apportionment method for the United States House of Representatives in 1911 and is still being used today.

Example #7.16.1 🤔

Refer to the results from Example #7.11.3. The standard divisor was 7,500. Using the Huntington–Hill Method, divide the 20 seats between the five districts.

Population totals, standard, lower, and upper quotas for five districts

District:	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Standard Quota	2.93	9.33	1.47	4.67	1.6	20
Lower Quota	2	9	1	4	1	17
Upper Quota	3	10	2	5	2	22

✅ **Solution:**

Applying the Huntington–Hill Method:

- **Step 1**: Find the standard divisor.
- **Step 2:** Compute the standard quota for each group.
- **Step 3:** Find the lower and upper quotas for each group.

So, Steps 1, 2, and 3 were already done to get the results in the table above.

- **Step 4:** Calculate the geometric mean of the lower and upper quotas for each group.

Using the formula for geometric mean of the lower quota and upper quota: $\sqrt{(\mathrm{LQ}) \cdot (\mathrm{UQ})}$

District 1: $\sqrt{2 \cdot 3} = \sqrt{6} \approx 2.45$

District 2: $\sqrt{9 \cdot 10} = \sqrt{90} \approx 9.49$

District 3: $\sqrt{1 \cdot 2} = \sqrt{2} \approx 1.41$

District 4: $\sqrt{4 \cdot 5} = \sqrt{20} \approx 4.47$

District 5: $\sqrt{1 \cdot 2} = \sqrt{2} \approx 1.41$

Now, put the results by adding a new row with the geometric means between the "Population" and the Standard Quota" rows as follows (or at the end would work as well):

Population totals, geometric means, standard, lower, and upper quotas for five districts

District:	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Geometric Mean	2.45	9.49	1.41	4.47	1.41	-----
Standard Quota	2.93	9.33	1.47	4.67	1.6	20
Lower Quota	2	9	1	4	1	17
Upper Quota	3	10	2	5	2	22

- **Step 5:**
  - Initially assign the lower quota if the standard quota is less than the geometric mean of the lower and upper quotas; or
  - Initially assign the upper quota if the standard quota is greater than the geometric mean of the lower and upper quotas.

District 1: Since the standard quota 2.93 is greater than the geometric mean 2.45, then round up. Use the upper quota.

District 2: Since the standard quota 9.33 is less than the geometric mean 9.49, then round down. Use the lower quota.

District 3: Since the standard quota 1.47 is greater than the geometric mean 1.41, then round up. Use the upper quota.

District 4: Since the standard quota 4.67 is greater than the geometric mean 4.47, then round up. Use the upper quota.

District 5: Since the standard quota 1.47 is greater than the geometric mean 1.41, then round up. Use the upper quota.

Now, add another row to the table with these initial assignment quotas.

Population totals, geometric means, standard, lower, upper and initial assignment quotas for five districts

District:	1	2	3	4	5	Total
Population	22,000	70,000	11,000	35,000	12,000	150,000
Geometric Mean	2.45	9.49	1.41	4.47	1.41	-----
Standard Quota	2.93	9.33	1.47	4.67	1.6	20
Lower Quota	2	**9**	1	4	1	17
Upper Quota	**3**	10	**2**	**5**	**2**	22
Initial Assignment Quotas	**3**	**9**	**2**	**5**	**2**	**21**

- **Step 6:** If the total number of seats/items assigned in Step 5
  - equals the number of seats/items available, then the assignment from that step is final.
  - does not equal the number of seats/items available, a modified divisor must be used. The seats/items are then reassigned using the modified divisor, and this process is repeated until the total number of assigned seats/items matches the number of seats/items available.

Thus, the total number of seats allocated does not match the number of seats available; instead, there is a surplus of one seat.

At this stage, we could assume that a student might apply a modified divisor, such as Adams' Method, to obtain the correct apportionment. However, for our purposes, we will conclude with the initial apportionment so that we can focus on the concept itself.

Therefore, the initial apportionment is as follows: 3 seats for District 1, 9 seats for District 2, 1 seat for District 3, 5 seats for District 4, and 2 seats for District 5.

Example #7.16.2 🤔

Refer to the results from Example #7.11.4. The standard divisor was 500. Using Huntington–Hill Method, divide the 12 board members among the four attendance zones:

Population distribution across four attendance zones

Zones	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	2	1	10
Upper Quota	5	4	3	2	14

✅ **Solution:**

Applying the Huntington–Hill Method:

- **Step 1**: Find the standard divisor.
- **Step 2:** Compute the standard quota for each group.
- **Step 3:** Find the lower and upper quotas for each group.

So, Steps 1, 2, and 3 were already done to get the results in the table above.

- **Step 4:** Calculate the geometric mean of the lower and upper quotas for each group.

Using the formula for geometric mean of the lower quota and upper quota: $\sqrt{(\mathrm{LQ}) \cdot (\mathrm{UQ})}$

North: $\sqrt{4 \cdot 5} = \sqrt{20} \approx 4.47$

South: $\sqrt{3 \cdot 4} = \sqrt{12} \approx 3.46$

East: $\sqrt{2 \cdot 3} = \sqrt{6} \approx 2.45$

West: $\sqrt{1 \cdot 2} = \sqrt{2} \approx 1.41$

Now, put the results by adding a new row with the geometric means between the "Population" and the Standard Quota" rows as follows (or at the end would work as well):

Population totals, geometric means, standard, lower, and upper quotas for four attendance zones

Zones	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000
Geometric Mean	4.47	3.46	2.45	1.41	-----
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	2	1	10
Upper Quota	5	4	3	2	14

- **Step 5:**
  - Initially assign the lower quota if the standard quota is less than the geometric mean of the lower and upper quotas; or
  - Initially assign the upper quota if the standard quota is greater than the geometric mean of the lower and upper quotas.

North: Since the standard quota 4.8 is greater than the geometric mean 4.47, then round up. Use the upper quota.

South: Since the standard quota 3.6 is greater than the geometric mean 3.46, then round up. Use the upper quota.

East: Since the standard quota 2.4 is less than the geometric mean 2.45, then round down. Use the lower quota.

West: Since the standard quota 1.2 is less than the geometric mean 1.41, then round down. Use the lower quota.

Now, add another row to the table with these initial assignment quotas.

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for four attendance zones

Zones	North	South	East	West	Total
Population	2,400	1,800	1,200	600	6,000

Zones	North	South	East	West	Total
Geometric Mean	4.47	3.46	2.45	1.41	-----
Standard Quota	4.8	3.6	2.4	1.2	12
Lower Quota	4	3	**2**	**1**	10
Upper Quota	**5**	**4**	3	2	14
Initial Assignment Quotas	**5**	**4**	**2**	**1**	**12**

- **Step 6:** If the total number of seats/items assigned in Step 5
  - equals the number of seats/items available, then the assignment from that step is final.
  - does not equal the number of seats/items available, a modified divisor must be used. The seats/items are then reassigned using the modified divisor, and this process is repeated until the total number of assigned seats/items matches the number of seats/items available.

Thus, the total number of board members allocated matches the number of board members available; we did not need to find a modified divisor.

Therefore, the initial apportionment is as follows: 5 board members for North, 4 board members for South, 2 board members for East, and 1 board member for West.

Example #7.16.3 🤔

A state must allocate 25 hospital beds among five regions based on population using the Huntington–Hill Method.

Population distribution across five regions

Region:	North	South	West	East	Central	Total
Population	410,000	190,000	40,000	95,000	265,000	1,000,000

✅ **Solution:**

- **Step 1**: Find the standard divisor.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{1{,}000{,}000}{25} = 40{,}000$$

- **Step 2:** Compute the standard quota for each group.

$$\text{North: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{410{,}000}{40{,}000} = 10.25$$

$$\text{South: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{190{,}000}{40{,}000} = 4.75$$

$$\text{West: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{40{,}000}{40{,}000} = 1$$

$$\text{East: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{95{,}000}{40{,}000} \approx 2.38$$

Central: $\text{Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{265{,}000}{40{,}000} \approx 6.63$

- **Step 3:** Find the lower and upper quotas for each group.
  - North: 10.25; so lower quota is 10 and the upper quota is 11
  - South: 4.75; so lower quota is 4 and the upper quota is 5
  - West: 1; so lower quota is 1 and the upper quota is 1
  - East: 2.38; so lower quota is 2 and the upper quota is 3
  - Central: 6.63; so lower quota is 6 and the upper quota is 7
- **Step 4:** Calculate the geometric mean of the lower and upper quotas for each group.

Using the formula for geometric mean of the lower quota and upper quota: $\sqrt{(\text{LQ}) \cdot (\text{UQ})}$

North: $\sqrt{10 \cdot 11} = \sqrt{20} \approx 10.48$

South: $\sqrt{4 \cdot 5} = \sqrt{12} \approx 4.47$

West: $\sqrt{1 \cdot 1} = \sqrt{1} = 1$

East: $\sqrt{2 \cdot 3} = \sqrt{6} \approx 2.45$

Central: $\sqrt{6 \cdot 7} = \sqrt{42} \approx 6.48$

Now, put all of the results in a table:

Population totals, geometric means, standard, lower, and upper quotas for five regions

Region:	North	South	West	East	Central	Total
Population	410,000	190,000	40,000	95,000	265,000	1,000,000
Geometric Mean	10.48	4.47	1	2.45	6.48	-----
Standard Quota	10.25	4.75	1	2.38	6.63	25
Lower Quota	10	4	1	2	6	23
Upper Quota	11	5	1	3	7	27

- **Step 5:**
  - Initially assign the lower quota if the standard quota is less than the geometric mean of the lower and upper quotas; or
  - Initially assign the upper quota if the standard quota is greater than the geometric mean of the lower and upper quotas.

North: Since the standard quota 10.25 is less than the geometric mean 10.48, then round down. Use the lower quota.

South: Since the standard quota 4.75 is greater than the geometric mean 4.47, then round up. Use the upper quota.

West: Since the standard quota 1 is equal to the geometric mean 1, then it stays at 1. Use either the lower quota or upper quota.

East: Since the standard quota 2.38 is less than the geometric mean 2.45, then round down. Use the lower quota.

Central: Since the standard quota 6.63 is greater than the geometric mean 6.48, then round up. Use the upper quota.

Now, add another row with these initial assignment quotas.

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for five regions

Region	North	South	West	East	Central	Total
Population	410,000	190,000	40,000	95,000	265,000	1,000,000
Geometric Mean	10.48	4.47	1	2.45	6.48	-----
Standard Quota	10.25	4.75	1	2.38	6.63	25
Lower Quota	**10**	4	**1**	**2**	6	23
Upper Quota	11	**5**	**1**	3	**7**	27
Initial Assignment Quotas	**10**	**5**	**1**	**2**	**7**	**25**

- **Step 6:** If the total number of seats/items assigned in Step 5
  - equals the number of seats/items available, then the assignment from that step is final.
  - does not equal the number of seats/items available, a modified divisor must be used. The seats/items are then reassigned using the modified divisor, and this process is repeated until the total number of assigned seats/items matches the number of seats/items available.

Thus, the total number of beds allocated matches the number of beds available; we did not need to find a modified divisor.

Therefore, the initial apportionment is as follows: 10 beds for North, 5 beds for South, 1 bed for West, 2 beds for East, and 7 beds for Central.

Example #7.16.4 🤔

A county library system has 19 computers to apportion among five branches based on average weekly visitors. Use the Huntington–Hill Method to apportion the 19 computers among the five branches:

Visitor distribution across five branches

Branch	A	B	C	D	E	Total
Visitors	785	428	971	579	1,237	4,000

✅ **Solution:**

- **Step 1**: Find the standard divisor.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{4{,}000}{19} = 210.53$$

- **Step 2:** Compute the standard quota for each group.

$$\text{A: Standard Quota} = \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{785}{210.53} \approx 3.73$$

B: Standard Quota $= \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{428}{210.53} \approx 2.03$

C: Standard Quota $= \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{971}{210.53} \approx 4.61$

D: Standard Quota $= \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{579}{210.53} \approx 2.75$

E: Standard Quota $= \frac{\text{Group Population}}{\text{Standard Divisor}} = \frac{1,237}{210.53} \approx 5.88$

- **Step 3:** Find the lower and upper quotas for each group.
  - A: 3.73; so lower quota is 3 and the upper quota is 4
  - B: 2.03; so lower quota is 2 and the upper quota is 3
  - C: 4.61; so lower quota is 4 and the upper quota is 5
  - D: 2.75; so lower quota is 2 and the upper quota is 3
  - E: 5.88; so lower quota is 5 and the upper quota is 6

- **Step 4:** Calculate the geometric mean of the lower and upper quotas for each group.

Using the formula for geometric mean of the lower quota and upper quota: $\sqrt{(\text{LQ}) \cdot (\text{UQ})}$

A: $\sqrt{3 \cdot 4} = \sqrt{12} \approx 3.46$

B: $\sqrt{2 \cdot 3} = \sqrt{6} \approx 2.45$

C: $\sqrt{4 \cdot 5} = \sqrt{20} = 4.47$

D: $\sqrt{2 \cdot 3} = \sqrt{6} \approx 2.45$

E: $\sqrt{5 \cdot 6} = \sqrt{30} \approx 5.48$

Now, put all of the results in a table:

Visitor totals, geometric means, standard, lower, and upper quotas for five branches

Branch	A	B	C	D	E	Total
Visitors	785	428	971	579	1,237	4,000
Geometric Mean	3.46	2.45	4.47	2.45	5.48	-----
Standard Quota	3.73	2.03	4.61	2.75	5.88	19
Lower Quota	3	2	4	2	5	16
Upper Quota	4	3	5	3	6	21

- **Step 5:**
  - Initially assign the lower quota if the standard quota is less than the geometric mean of the lower and upper quotas; or
  - Initially assign the upper quota if the standard quota is greater than the geometric mean of the lower and upper quotas.

A: Since the standard quota 3.73 is greater than the geometric mean 3.46, then round up. Use the upper quota.

B: Since the standard quota 2.03 is less than the geometric mean 2.45, then round down. Use the lower quota.

C: Since the standard quota 4.61 is greater than the geometric mean 4.47, then round up. Use the upper quota.

D: Since the standard quota 2.75 is greater than the geometric mean 2.45, then round up. Use the upper quota.

E: Since the standard quota 5.88 is greater than the geometric mean 5.48, then round up. Use the upper quota.

Now, add another row with these initial assignment quotas.

Visitor totals, geometric means, standard, lower, upper, and initial assignment quotas for five branches

Branch	A	B	C	D	E	Total
Visitors	785	428	971	579	1,237	4,000
Geometric Mean	3.46	2.45	4.47	2.45	5.48	-----
Standard Quota	3.73	2.03	4.61	2.75	5.88	19
Lower Quota	3	**2**	4	2	5	16
Upper Quota	**4**	3	**5**	**3**	**6**	21
Initial Assignment Quotas	**4**	**2**	**5**	**3**	**6**	**20**

- **Step 6:** If the total number of seats/items assigned in Step 5
  - equals the number of seats/items available, then the assignment from that step is final.
  - does not equal the number of seats/items available, a modified divisor must be used. The seats/items are then reassigned using the modified divisor, and this process is repeated until the total number of assigned seats/items matches the number of seats/items available.

Thus, the total number of computers allocated does not match the number of computers available; instead, there is a surplus of one computer.

At this stage, we could assume that a student might apply a modified divisor, such as Adams' Method, to obtain the correct apportionment. However, for our purposes, we will conclude with the initial apportionment so that we can focus on the concept itself.

Therefore, the initial apportionment is as follows: 4 computers for A, 2 computers for B, 5 computers for C, 3 computers for D, and 6 computers for E.

Example #7.16.5 🤔

The population of Arizona is 7,158,923 and the U.S. population is 331,108,434. There are 435 seats in the House of Representatives. What is the assigned seat total for Arizona using the Huntington-Hill method?

✅ **Solution:**

- **Step 1**: Find the standard divisor.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{331{,}108{,}434}{435} = 761{,}169$$

- **Step 2:** Compute the standard quota for Arizona.

$\text{Arizona: Standard Quota} = \frac{\text{Arizona Population}}{\text{Standard Divisor}} = \frac{7{,}158{,}923}{761{,}169} \approx 9.41$

- **Step 3:** Find the lower and upper quota.
  - Arizona: 9.41; so lower quota is 9 and the upper quota is 10
- **Step 4:** Calculate the geometric mean of the lower and upper quota.

Using the formula for geometric mean of the lower quota and upper quota: $\sqrt{(\text{LQ}) \cdot (\text{UQ})}$

Arizona: $\sqrt{9 \cdot 10} = \sqrt{90} \approx 9.49$

- **Step 5:**
  - Initially assign the lower quota if the standard quota is less than the geometric mean of the lower and upper quotas; or
  - Initially assign the upper quota if the standard quota is greater than the geometric mean of the lower and upper quotas.

Arizona: Since the standard quota 9.41 is less than the geometric mean 9.49, then round down. Use the lower quota.

Thus, the assigned seat total for Arizona using the Huntington-Hill method is 9.

Example #7.16.6 🤔

The population of Oregon is 4,241,500 and the U.S. population is 331,108,434. There are 435 seats in the House of Representatives. What is the assigned seat total for Arizona using the Huntington-Hill method?

✅ **Solution:**

- **Step 1**: Find the standard divisor.

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{Total Number of Seats/Items}} = \frac{331{,}108{,}434}{435} = 761{,}169$$

- **Step 2:** Compute the standard quota for Arizona.

$\text{Oregon: Standard Quota} = \frac{\text{Oregon Population}}{\text{Standard Divisor}} = \frac{4{,}241{,}500}{761{,}169} \approx 5.57$

- **Step 3:** Find the lower and upper quota.
  - Oregon: 5.57; so lower quota is 5 and the upper quota is 6
- **Step 4:** Calculate the geometric mean of the lower and upper quota.

Using the formula for geometric mean of the lower quota and upper quota: $\sqrt{(\text{LQ}) \cdot (\text{UQ})}$

Oregon: $\sqrt{5 \cdot 6} = \sqrt{30} \approx 5.48$

- **Step 5:**
  - Initially assign the lower quota if the standard quota is less than the geometric mean of the lower and upper quotas; or
  - Initially assign the upper quota if the standard quota is greater than the geometric mean of the lower and upper quotas.

Oregon: Since the standard quota 5.57 is higher than the geometric mean 5.48, then round up. Use the upper quota.

Thus, the assigned seat total for Oregon using the Huntington-Hill method is 6.

Historical Note: Edward V. Huntington & Joseph A. Hill and Apportionment

The Huntington–Hill method was developed in the early 20th century by mathematician Edward V. Huntington and statistician Joseph A. Hill during a period of intense debate over how to fairly apportion seats in the U.S. House of Representatives. After the 1920 census, disagreements over competing apportionment methods prevented Congress from reapportioning at all. In response, a committee of mathematicians reviewed available methods and recommended the Huntington–Hill method because it minimizes relative (percentage) differences in representation between states. Congress formally adopted the method in 1941, and it has been used for every congressional apportionment since, making it the longest-standing method in U.S. history.

Timeline of U.S. Apportionment Methods

1787–1790: Constitutional Framework

- The U.S. Constitution requires seats in the House of Representatives to be apportioned based on population, but does not specify a method.

1792–1832: Jefferson's Method

- Proposed by Thomas Jefferson and first used after the 1790 Census.
- Uses a modified divisor and always rounds down.
- Tended to favor larger states, which eventually led to criticism and abandonment.

1793 (Proposed but Not Used): Hamilton's Method

- Proposed by Alexander Hamilton and approved by Congress.
- Vetoed by President George Washington, marking the first presidential veto.
- Later reintroduced and used intermittently in the 19th century.

1842–1852: Webster's Method (First Use)

- Proposed by Daniel Webster as a compromise between Jefferson and Hamilton.
- Uses standard rounding at 0.5.
- Viewed as more balanced between large and small states.

1852–1911: Hamilton's Method (Primary Use)

- Hamilton's method became dominant because of its simplicity.
- Eventually abandoned due to apportionment paradoxes (e.g., Alabama paradox).

1911–1940: Webster's Method (Second Use)

- Returned as a way to avoid Hamilton's paradoxes.
- Still produced disagreements over fairness, especially after the 1920 Census.

1920: No Apportionment

- Congress failed to agree on a method after the 1920 Census.
- This crisis led to the creation of a committee of mathematicians to study apportionment methods.

1941–Present: Huntington–Hill Method

- Developed by Edward V. Huntington and Joseph A. Hill.
- Formally adopted by Congress in 1941 under the Apportionment Act of 1941.
- Minimizes relative (percentage) differences in representation using geometric means.
- Still used today for congressional apportionment.

Section 7.16: Huntington–Hill Method [*In-Class Exercises*]

1. Suppose a country has 4 states, and a 20-seat House of Representatives. The number of representatives assigned to each state is proportional to the population of the state. The population for each state is given in the following table.

   Population totals, standard quotas, lower quotas, and upper quotas for four states

State:	A	B	C	D	Total
Population:	2,560	3,315	995	5,012	11,882
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Upper Quota	5	6	2	9	22

Use the Huntington-Hill Method to assign the 20 seats among the four states. The standard divisor is 594.1.

2. A logistics company must assign 21 delivery drones among five hubs based on monthly deliveries.

   Delivery distribution across five hubs

Hub	A	B	C	D	E	Total
Deliveries	3,700	4,100	2,100	1,200	2,900	14,000

Use the Huntington–Hill Method to apportion the 21 delivery drones among the five hubs. If the initial assignment quotas produce a surplus or a deficit, do not find a modified divisor, only give the initial assignment quotas and indicate the surplus or deficit.

3. The population of California is 39,576,757 and the U.S. population is 331,108,434. There are 435 house seats in all. What is the assigned seat total for California using the Huntington-Hill method?

Answers available at the end of this section.

Answers

Section 7.16: Huntington–Hill Method [*In-Class Exercises*]

1.

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for four states

State	A	B	C	D	Total
Population	2,560	3,315	995	5,012	11,882
Geometric Mean	4.47	5.48	1.41	8.49	-----

State	A	B	C	D	Total
Standard Quota	4.31	5.58	1.67	8.44	20
Lower Quota	4	5	1	8	18
Upper Quota	5	6	2	9	22
Initial Assignment Quotas	**4**	**6**	**2**	**8**	**20**

Thus, the apportionment is as follows: 4 seats for A, 6 seats for B, 2 seats for C, and 8 seats for D. (No modified divisor needed).

2.

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for five hubs

Hub	A	B	C	D	E	Total
Deliveries	3,700	4,100	2,100	1,200	2,900	14,000
Geometric Mean	5.48	6.48	3.46	1.41	4.47	-----
Standard Quota	5.55	6.15	3.15	1.80	4.35	21
Lower Quota	5	**6**	**3**	1	**4**	19
Upper Quota	**6**	7	4	**2**	5	24
Initial Assignment Quotas	**6**	**6**	**3**	**2**	**4**	**21**

The standard divisor is 666.67. The apportionment is as follows: 6 drones for A, 6 drones for B, 3 drones for C, 2 drones for D, and 4 drones for E. (No modified divisor needed).

3. 52 seats

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Section 7.16.H: Homework

Section 7.16: Apportionment - Huntington-Hill Method - Homework Exercises

(Section 7.16) - Homework Exercise #1

Refer to Homework Exercise #1 from Section 7.11.

A university is assigning 20 student council seats based on enrollment:

Population totals, standard quotas, lower quotas, and upper quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Standard Quota	7	5.5	4.17	3.33	20
Lower Quota	7	5	4	3	19
Upper Quota	7	6	5	4	22

The standard divisor was 600. Use the Huntington-Hill Method to initially assign the 20 student council seats among the four departments. If the initial assignment quotas produce a surplus or a deficit, do not find a modified divisor for the correct apportionment.

Answer available at the end of this section.

(Section 7.16) - Homework Exercise #2

Refer to Homework Exercise #2 from Section 7.11.

A gaming company is distributing 50 servers across regions based on the number of active players:

Active players totals, standard quotas, lower quotas, and upper quotas for six regions

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Standard Quota	14.13	3.53	10.33	1.36	17.66	2.99	50

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Lower Quota	14	3	10	1	17	2	47
Upper Quota	15	4	11	2	18	3	53

The standard divisor was 3,680. Use the Huntington-Hill Method to initially assign the 50 servers among the four regions. If the initial assignment quotas produce a surplus or a deficit, do not find a modified divisor for the correct apportionment.

Answer available at the end of this section.

(Section 7.16) - Homework Exercise #3

Refer to Homework Exercise #3 from Section 7.11.

A large music festival is assigning 40 performance slots to different genres based on ticket demand:

Tickets sales totals, standard quotas, lower quotas, and upper quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Lower Quota	13	10	7	5	3	38
Upper Quota	14	11	8	6	4	43

The standard divisor was 1,750. Use the Huntington-Hill Method to initially assign the 40 performance slots among the five genres. If the initial assignment quotas produce a surplus or a deficit, do not find a modified divisor for the correct apportionment.

Answer available at the end of this section.

(Section 7.16) - Homework Exercise #4

Refer to Homework Exercise #4 from Section 7.11.

A logistics company is distributing 45 delivery trucks to its regional hubs based on the number of packages handled last month:

Package totals, standard quotas, lower quotas, and upper quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Lower Quota	3	14	7	11	8	43
Upper Quota	4	15	8	12	9	48

The standard divisor was 2,177.78. Use the Huntington-Hill Method to initially assign the 45 delivery trucks among the five regional hubs. If the initial assignment quotas produce a surplus or a deficit, do not find a modified divisor for the correct apportionment.

Answer available at the end of this section.

(Section 7.16) - Homework Exercise #5

Refer to Homework Exercise #5 from Section 7.11.

A streaming company is distributing 25 production slots across four genres based on viewer demand:

Viewer totals, standard quotas, lower quotas, and upper quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Standard Quota	6.65	11.08	4.54	2.73	25
Lower Quota	6	11	4	2	23
Upper Quota	7	12	5	3	27

The standard divisor was 4,800. Use the Huntington-Hill Method to initially assign the 25 production slots among the four genres. If the initial assignment quotas produce a surplus or a deficit, do not find a modified divisor for the correct apportionment.

Answer available at the end of this section.

(Section 7.16) - Homework Exercise #6

The population of New Jersey is 9,294,493 and the U.S. population is 331,108,434. There are 435 seats in the House of Representatives. What is the assigned seat total for New Jersey using the Huntington-Hill method?

Answer available at the end of this section.

(Section 7.16) - Homework Exercise #7

The population of Colorado is 5,782,171 and the U.S. population is 331,108,434. There are 435 seats in the House of Representatives. What is the assigned seat total for Colorado using the Huntington-Hill method?

Answer available at the end of this section.

Answer available at the end of this section.

(Section 7.16) - Homework Exercise #8

The population of Michigan is 10,084,442 and the U.S. population is 331,108,434. There are 435 seats in the House of Representatives. What is the assigned seat total for Michigan using the Huntington-Hill method?

Answer available at the end of this section.

Answers

(Section 7.16) - Homework Exercise #1

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for four departments

District	Arts	Science	Business	Engineering	Total
Population	4,200	3,300	2,500	2,000	12,000
Geometric Mean	7	5.48	4.47	3.46	20
Standard Quota	7	5.5	4.17	3.33	20
Lower Quota	**7**	5	**4**	**3**	19
Upper Quota	**7**	**6**	**5**	4	22
Initial Assignment Quotas	**7**	**6**	**4**	**3**	**20**

The initial assignment quotas produce a correct number of seats. A modified divisor is not required.

(Section 7.16) - Homework Exercise #2

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for six regions

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Active Players	52,000	13,000	38,000	5,000	65,000	11,000	184,000
Geometric Mean	14.49	3.46	10.49	1.41	17.49	2.45	50
Standard Quota	14.13	3.53	10.33	1.36	17.66	2.99	50
Lower Quota	**14**	3	**10**	**1**	17	2	47
Upper Quota	15	**4**	11	2	**18**	**3**	53

Region:	N. America	S. America	Europe	Africa	Asia	Australia	Total
Initial Assignment Quotas	**14**	**4**	**10**	**1**	**18**	**3**	**50**

The initial assignment quotas produce a correct number of servers. A modified divisor is not required.

(Section 7.16) - Homework Exercise #3

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for five genres

Genre	Rock	Pop	Hip-Hop	EDM	Country	Total
Ticket's Sold	23,450	17,820	12,760	9,940	6,030	70,000
Geometric Mean	13.49	10.49	7.48	5.48	3.46	40
Standard Quota	13.4	10.18	7.29	5.68	3.45	40
Lower Quota	**13**	**10**	**7**	5	**3**	38
Upper Quota	14	11	8	**6**	4	43
Initial Assignment Quotas	**13**	**10**	**7**	**6**	**3**	**39**

The initial assignment quotas produces a deficit of one slot. A modified divisor is required for the correct apportionment.

(Section 7.16) - Homework Exercise #4

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for five regional hubs

Regional Hubs	Northwest	Northeast	West	Midwest	South	Total
Packages	8,000	31,275	15,375	24,890	19,460	98,000
Geometric Mean	3.46	14.49	7.48	11.49	8.49	45
Standard Quota	3.67	14.36	7.06	11.43	8.94	45
Lower Quota	3	**14**	**7**	**11**	8	43
Upper Quota	**4**	15	8	12	**9**	48
Initial Assignment Quotas	**4**	**14**	**7**	**11**	**9**	**45**

The initial assignment quotas produce a correct number of trucks. A modified divisor is not required.

(Section 7.16) - Homework Exercise #5

Population totals, geometric means, standard, lower, upper, and initial assignment quotas for four genres

Genre	Comedy	Drama	Reality	Sci-Fi	Total
Viewers	31,900	53,200	21,800	13,100	120,000
Geometric Mean	6.48	11.49	4.47	2.45	25
Standard Quota	6.65	11.08	4.54	2.73	25
Lower Quota	6	**11**	4	2	23
Upper Quota	**7**	12	**5**	**3**	27
Initial Assignment Quotas	**7**	**11**	**5**	**3**	**26**

The initial assignment quotas produces a surplus of one slot. A modified divisor is required for the correct apportionment.

(Section 7.16) - Homework Exercise #6

- Standard divisor = 761,169
- Standard quota for New Jersey = 12.21
- Lower quota = 12
- Upper quota = 13
- Geometric mean = 12.49
- Assigned seat total = 12

(Section 7.16) - Homework Exercise #7

- Standard divisor = 761,169
- Standard quota for Colorado = 7.60
- Lower quota = 7
- Upper quota = 8
- Geometric mean = 7.48
- Assigned seat total = 8

(Section 7.16) - Homework Exercise #7

(Section 7.16) - Homework Exercise #8

- Standard divisor = 761,169
- Standard quota for Michigan = 13.25
- Lower quota = 13
- Upper quota = 14
- Geometric mean = 13.49
- Assigned seat total = 13

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CHAPTER 8: GRAPH THEORY

Section 8.1: Basic Concepts of Graph Theory

Section 8.1.H: Homework

Section 8.2: Euler Paths and Euler Circuits

Section 8.2.H: Homework

Section 8.3: Hamilton Paths and Hamilton Circuits

Section 8.3.H: Homework

Section 8.4: TSP - The Brute Force Method

Section 8.4.H: Homework

Section 8.5: TSP - The Nearest Neighbor Method

Section 8.5.H: Homework

Section 8.6: TSP - The Cheapest Link Algorithm

Section 8.6.H: Homework

Section 8.7: Trees

Section 8.7.H: Homework

Section 8.8: Kruskal's Algorithm

Section 8.8.H: Homework

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Section 8.1: Basic Concepts of Graph Theory

Learning Objectives

- Define basic graph theory terms
- Represent relationships with graphs
- Determine if two graphs are equivalent

Introduction to Graph Theory

Graph theory is a branch of mathematics that studies relationships between objects. These relationships are represented using simple structures called graphs, which consist of points connected by lines. Although graph theory is relatively young as a mathematical discipline, it has become one of the most important areas of modern mathematics due to its wide range of applications.

Graph theory began in 1736 and since then, it has grown into a powerful framework for modeling and solving problems involving networks, routes, connections, and interactions. Today, graph theory is used in fields such as computer science, transportation, communications, biology, social sciences, and economics.

Graph, Vertices, Edges, and Loops

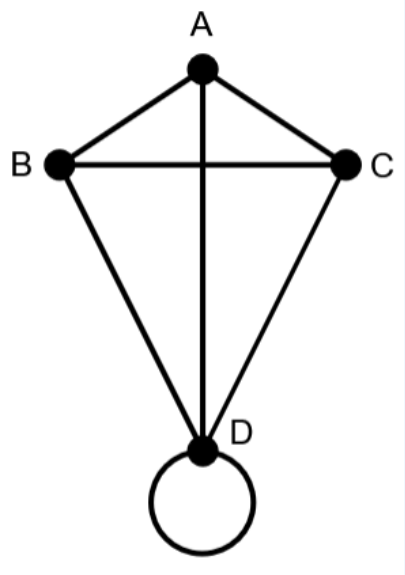
A **graph** is a mathematical structure consisting of two components:

- A set of **vertices** (also called nodes), and
- A set of **edges** (also called links) connecting pairs of vertices.
- A **loop** is an edge that begins and ends at the same vertex.

*Note: The singular form of vertices is **vertex**.*

Example #8.1.1 🤔

Consider the graph below.



a. Identify all vertices.
b. Identify all edges.
c. Identify any loops.

✅ **Solution:**

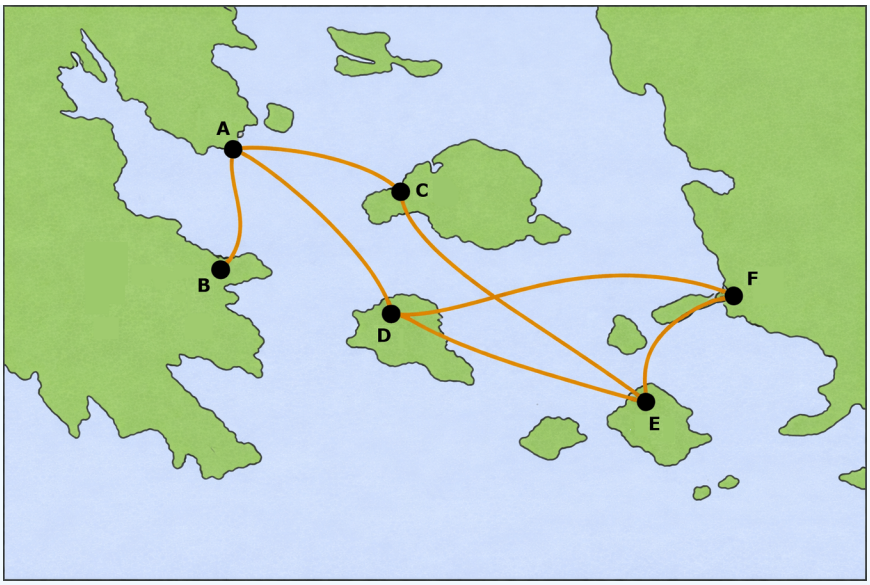
a. There are a total of 4 vertices. The vertices are A, B, C, and D.
b. There are a total of 7 edges. The edges are A-B, A-C, B-D, C-D, B-C, A-D and D-D. (Note: There is no vertex where A-C and B-D intersect, thus it is assumed that the edges are not connected).
c. There is 1 loop. The loop is D-D.

Representing Graph Theory

In applications of graph theory, a vertex represents an object or location. Some examples include cities in a transportation network, land masses in an island-bridge problem, or computers in a network. An edge represents a connection between those two vertices. Some examples include roads between cities, bridges between islands, and communication links between computers.

Example #8.1.2 🤔

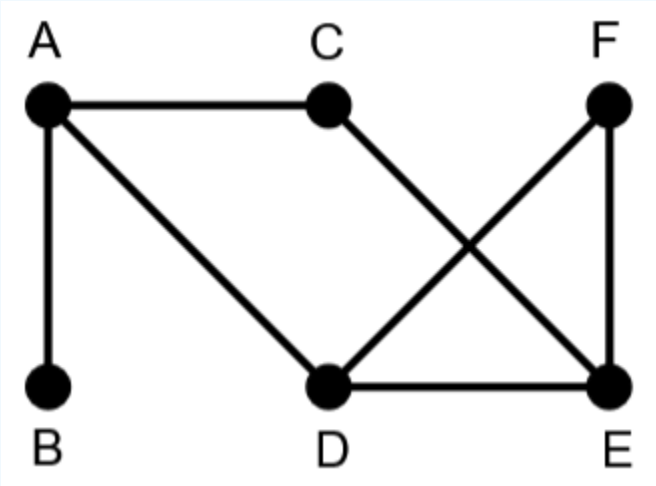
Consider the map of the mainlands and islands that shows all of the ferry/boat routes to the points of interest. Represent ALL mainland/islands using a graph.



✅ **Solution:**

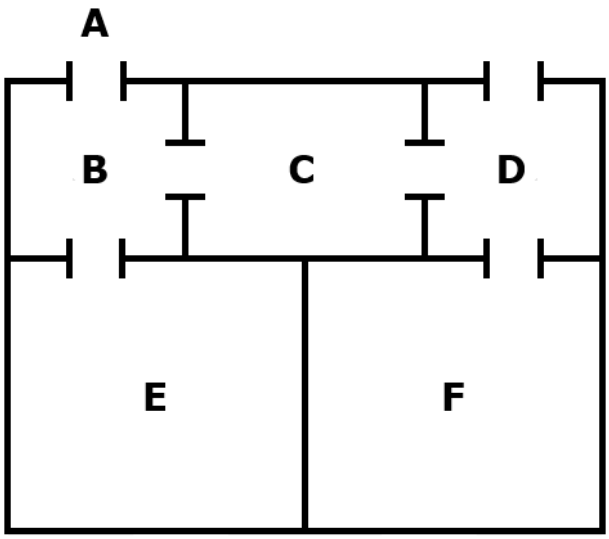
Vertex B has one ferry route connecting it to A. Vertex A has ferry routes connecting it to both C and D, with a route returning to A. Vertex C has ferry routes connecting it to E and back to A. Vertex D has ferry routes connecting it to E and F, as well as a route back to A. Vertex E has a ferry route connecting it to F and routes back to C and D. Finally, vertex F has ferry routes returning to D and E.

Each ferry route is represented by an edge, so the diagram below shows one possible graph representing this system. While multiple representations are possible, all valid graphs will share the same set of connecting points.



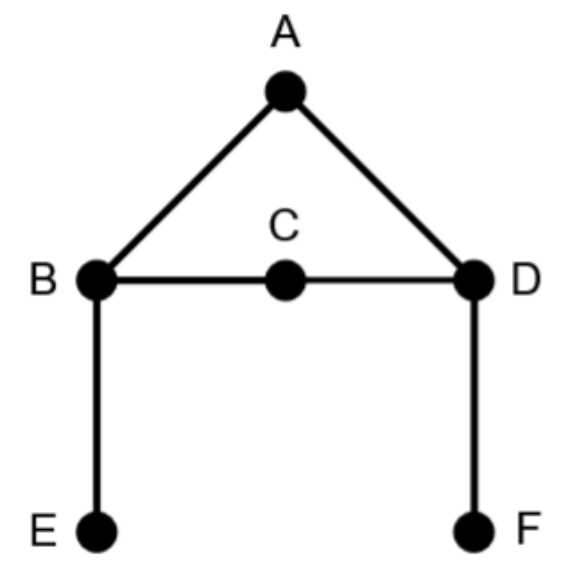
Example #8.1.3 🤔

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.



✅ **Solution:**

Each room and the exterior area should be represented by a vertex and each door opening should be represented by an edge, so the diagram below shows one possible graph representing this system. While multiple representations are possible, all valid graphs will share the same set of connecting points.



Note: Exterior A (outside) connects to both rooms B and D, so A should have two edges connecting them.

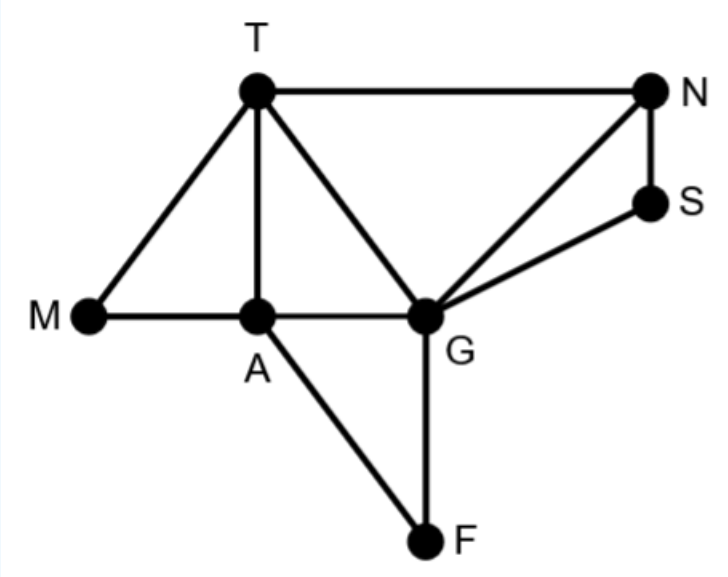
Example #8.1.4 🤔

Draw a graph for the map of the south-eastern U.S. states below using each state as a vertex and each common border as an edge.



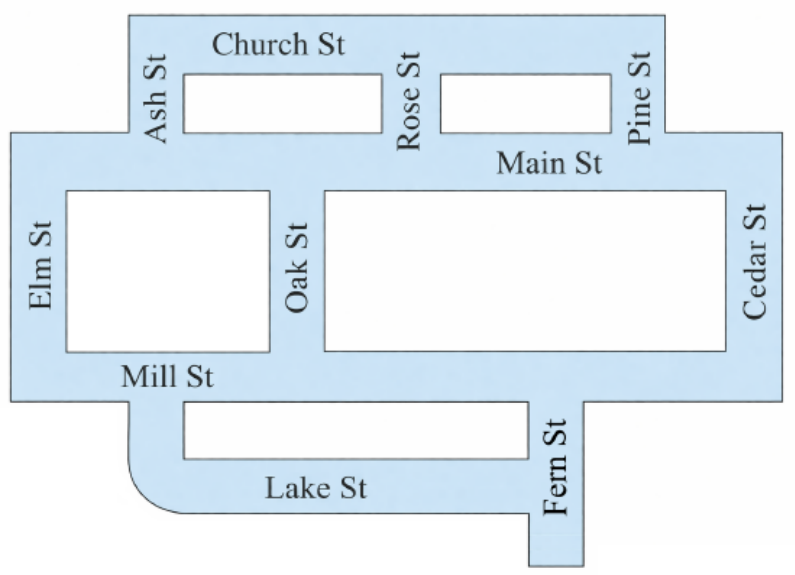
✅ **Solution:**

Each state should be represented by a vertex and each common border should be represented by an edge, so the diagram below shows one possible graph representing this system. While multiple representations are possible, all valid graphs will share the same set of connecting points.



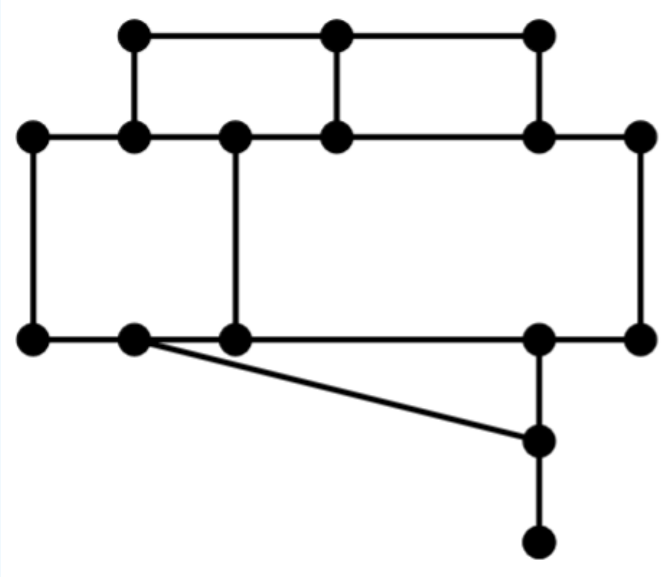
Example #8.1.5 🤔

Draw a graph that represents the street layout of this neighborhood. Let each intersection of two or more differently named streets be represented by a vertex. Also include a vertex for any dead end. Represent each street as an edge connecting the appropriate vertices.



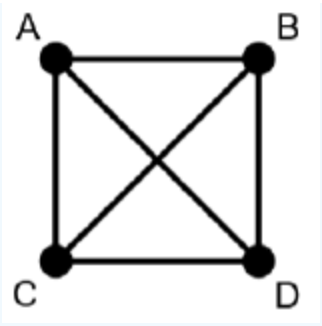
✅ **Solution:**

Here, it wouldn't matter if we labeled each vertex, because we would just assign random letters for each. So, the diagram below shows one possible graph representing the street layout. While multiple representations are possible, all valid graphs will share the same set of connecting points.



Example #8.1.6 🤔

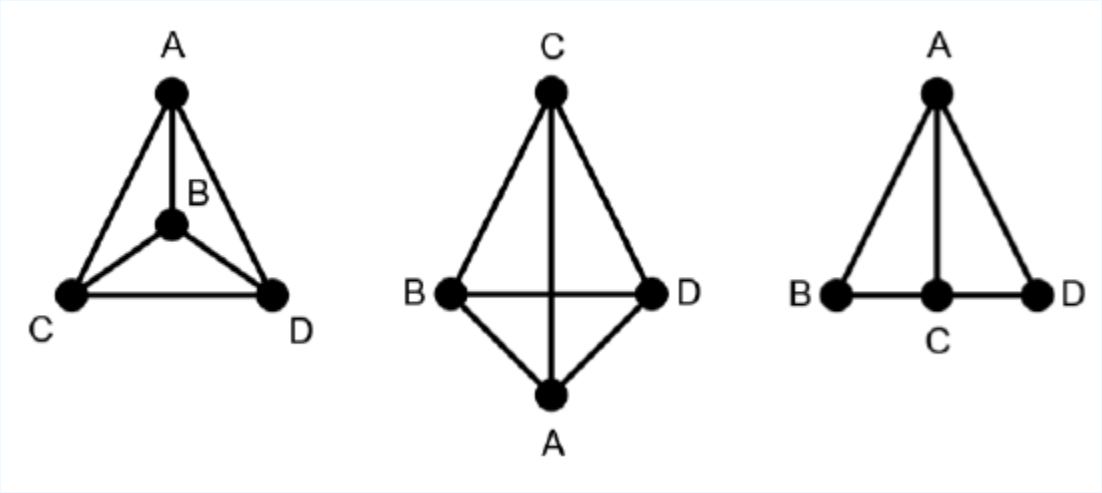
Consider the graph below.



Create three other equivalent graphs that visually look different.

✅ **Solution:**

Here are three possible graphs.



One helpful way to think about graph equivalence is to imagine each vertex as a tack and each edge as a rubber band on a corkboard. When a tack is moved, the attached rubber bands move with it, but the connections remain the same. In the first image, for example, vertex A was moved to the right, and vertex B was moved down and to the left, without changing any of the connections.

Equivalent graphs may look different on paper, but they describe the same relationships. Rotating, stretching, or redrawing a graph does not change its equivalence, as long as no connections are added or removed.

Degree (of a vertex)

The degree of a vertex is the number of edges connected to it.

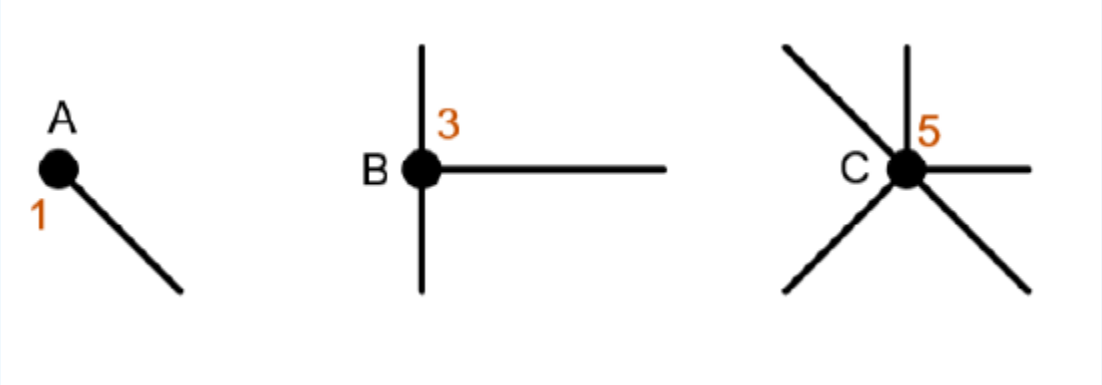
- A vertex with an even number of edges has even degree.
- A vertex with an odd number of edges has odd degree.

Example #8.1.7 🤔

Give examples of vertices with odd degrees.

✅ **Solution:**

Here are three odd vertices (from different graphs):



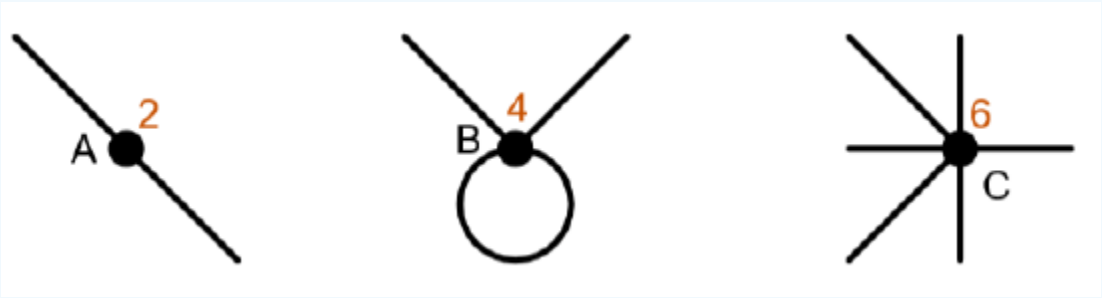
It will be helpful to indicate the degrees by labeling it at the vertex when needed.

Example #8.1.8 🤔

Give examples of vertices with even degrees.

✅ **Solution:**

Here are three even vertices (from different graphs):



It will be helpful to indicate the degrees by labeling it at the vertex when needed.

Note: For vertex B, a loop is a single edge that starts and ends at the same vertex. Although it is only one edge, it contributes 2 to the degree of that vertex. Degree counts edge ends, not edges themselves.

Adjacent

Two vertices are called **adjacent** if there is an edge connecting them.

Walk

A **walk** is a sequence of vertices in which consecutive vertices are connected by edges.

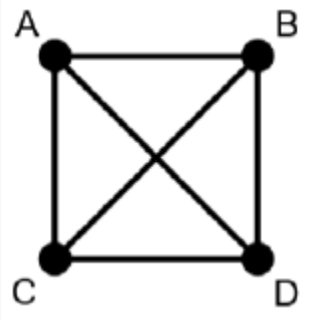
Path

A **path** is a sequence of vertices connected by edges (or a walk), with no edge used more than once.

To keep matters simple, a path is defined as a walk that does not repeat any edges. In general usage, a path may repeat vertices. If a path contains no repeated vertices, it is called a **simple path**. It is important to have a solid understanding of the definition of a path, as it forms the foundation for concepts introduced later. Many of the definitions that follow will focus directly on paths or build upon this idea.

Example #8.1.9 🤔

Give examples of different paths from the following graph used in Exercise #8.1.6.



✅ **Solution:**

So, the following directions would be considered a path:

A-B, A-B-C, A-B-C-D, A-B-C-D-A, and A-B-C-D-A-C are all paths. There are many other paths as well.

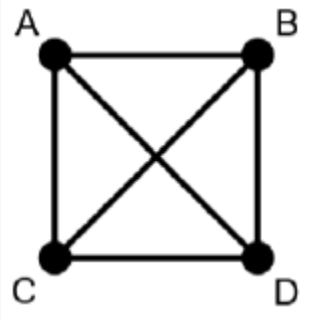
A-B-C-A-B is not a path, because edge A-B was used more than once.

Circuit

A **circuit** is a path that starts and ends at the same vertex.

Example #8.1.10 🤔

Give examples of different circuits from the following graph used in Exercise #8.1.6.



✅ **Solution:**

So, the following directions would be considered a circuit:

A-B-C-A, A-B-C-D-A, and A-B-D-C-A are all circuits. There are many other circuits as well.

A-B-C-D is not a circuit, because the path did not start and end at the same vertex.

A-B-C-D-B-A is not a circuit (nor a path), because edge A-B was used more than once.

Connected and Disconnected Graphs

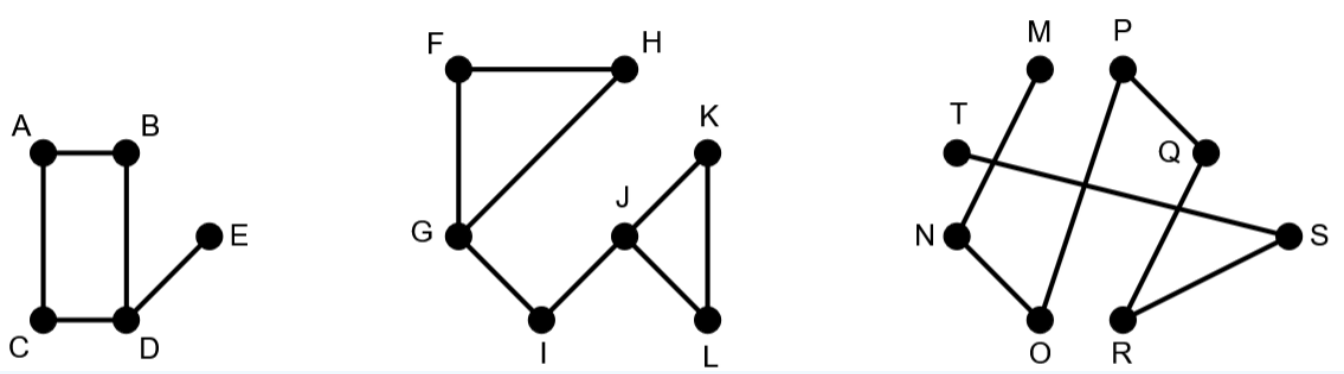
A **connected graph** is a graph in which there is a path between every pair of vertices. Equivalently, a graph is connected if it is possible to travel from any vertex to any other vertex by following edges in the graph.

A **disconnected graph** is a graph that is not connected; that is, there exists at least one pair of vertices for which no path exists between them.

Example #8.1.11 🤔

Give examples of connected graphs.

✅ **Solution:**

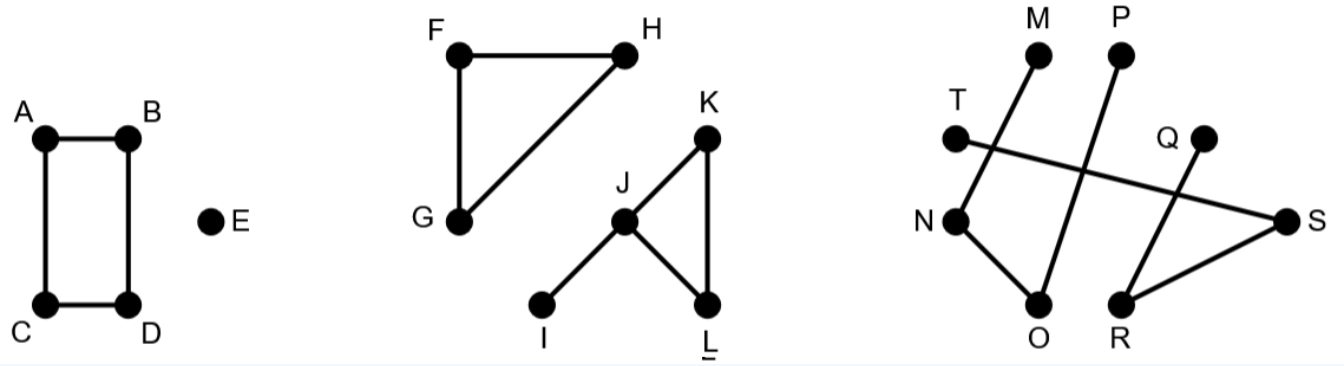


Each of the following three graphs are connected. Starting with any vertex, there is a path to any other vertex within that graph.

Example #8.1.12 🤔

Give examples of disconnected graphs.

✅ **Solution:**



Each of the following three graphs are disconnected. Starting with any vertex, there is a not path to every other vertex within that graph.

In the first graph, vertex E cannot be reached by any other vertex.

In the second graph, vertices F, G, and H, cannot be reached by vertices, I, J, K and L (and vice-versa).

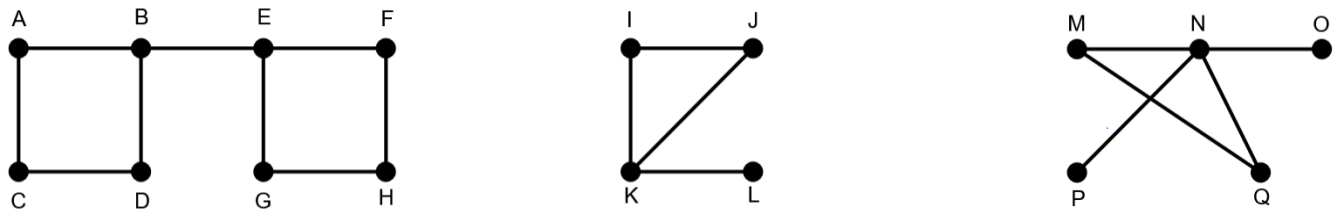
In the third graph, vertices M, N, O, and P, cannot be reached by vertices, Q, R, S and T (and vice-versa). Note that edge T-S does not intersect with edge M-N or P-Q.

Bridge

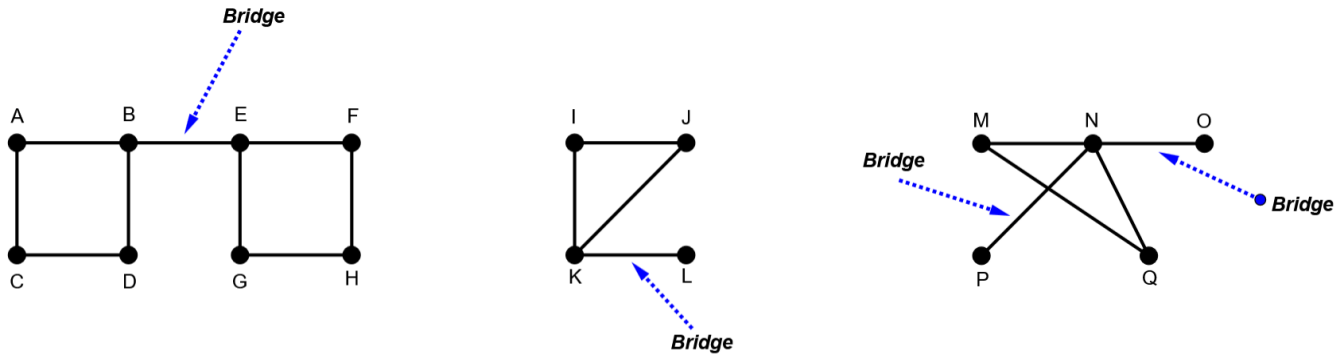
In a graph, a **bridge** is an edge whose removal disconnects the graph. In other words, a bridge is an edge that provides the only connection between two parts of a graph. If a bridge is removed, there is no longer a path between at least two vertices that were previously connected, which will result in a disconnected graph.

Example #8.1.13 🤔

For the following graphs, identify any bridges that exist.



✅ **Solution:**

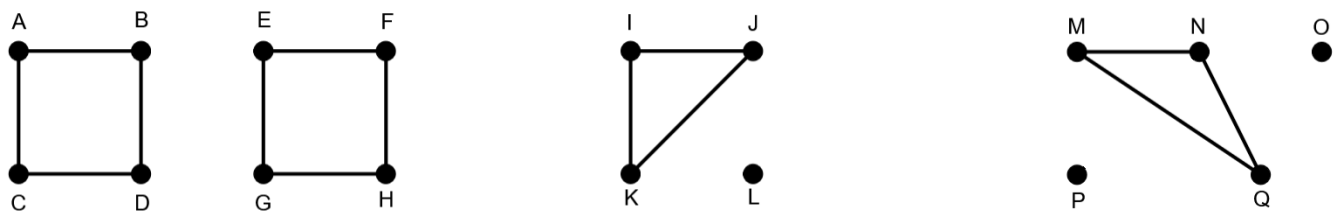


For the first graph, edge B-E is a bridge. If removed, vertices A, B, C, and D cannot be reached by vertices, E, F, G and H (and vice-versa). Then, the graph is disconnected.

For the second graph, edge K-L is a bridge. If removed, vertices I, J, and K cannot be reached by vertex L (and vice-versa). Then, the graph is disconnected.

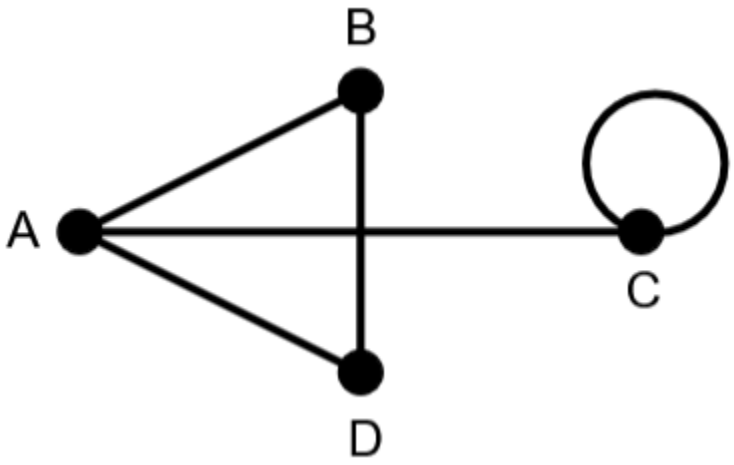
For the third, both edges P-N and N-O are bridges. If edge P-N is removed, vertices M, N, O, and Q cannot be reached by vertex P (and vice-versa). Then, the graph is disconnected. If edge N-O is removed, vertices M, N, Q, and P cannot be reached by vertex O (and vice-versa). Then, the graph is disconnected.

Here are the graphs without those bridges showing the disconnection of each of the graphs:



Section 8.1: Basic Concepts of Graph Theory [*In-Class Exercises*]

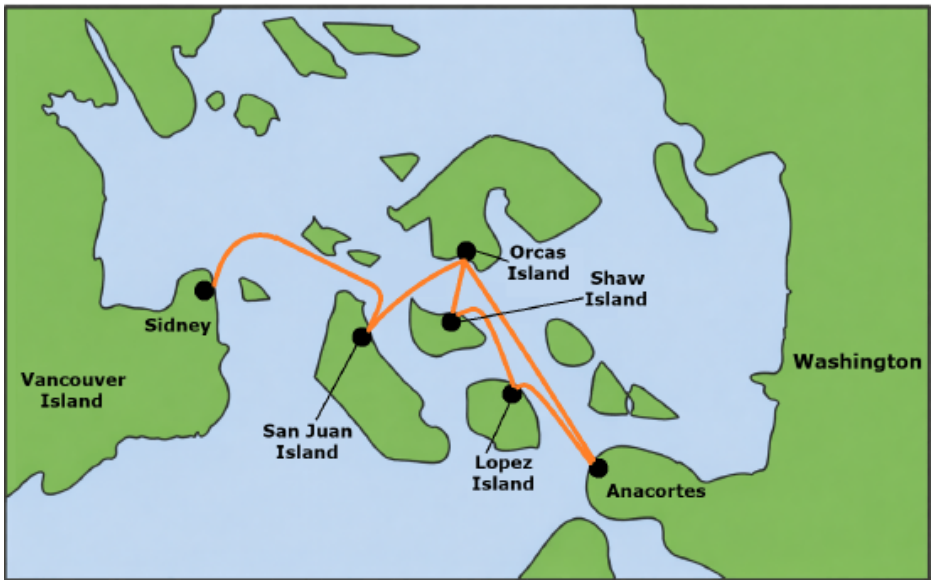
1. Consider the graph below:



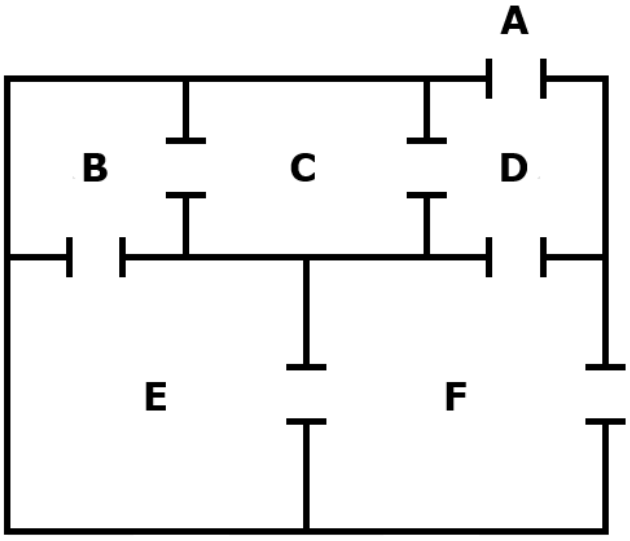
a. Identify all vertices.

b. Identify all edges.
c. Identify any loops.
d. Identify the degree of each vertex.
e. Indicate any bridges, if possible.
f. Is A-B-D a path? Why or why not?
g. Is A-C-C-A a path? Why or why not?
h. Is A-B-D a circuit? Why or why not?
i. Is D-A-B-D a circuit? Why or why not?
j. Is A-D-B-A-C-C a path? Why or why not?

2. Consider the map of the islands of the Puget Sound in northwest Washington State, USA. The map of the mainlands and islands shows all of the ferry/boat routes possible. Draw a graph using each named mainland or island as a vertex and each ferry route as an edge. Ignore all unnamed islands



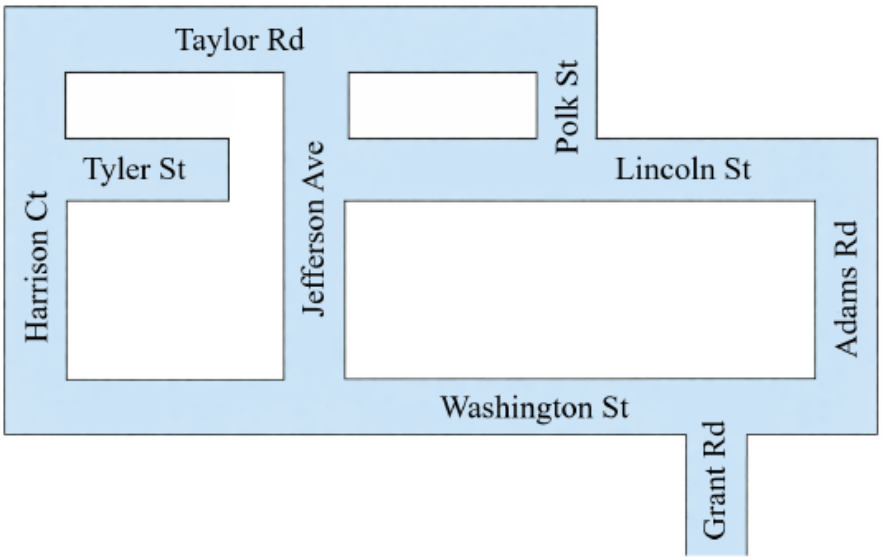
3. Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.



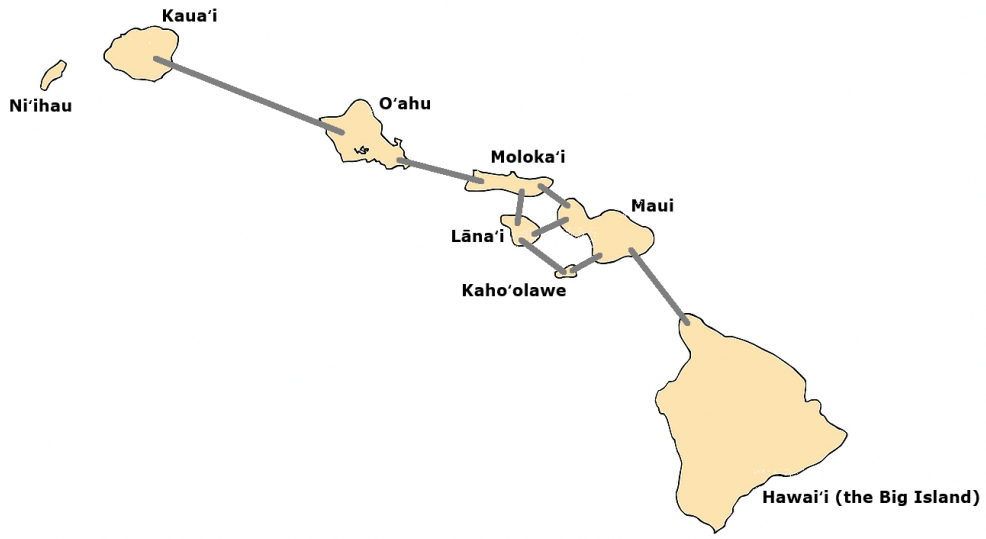
4. Draw a graph for the map of the seven cities in south Orange County using each city as a vertex and each common border as an edge. (Ignore any unincorporated areas).



5. Draw a graph that represents the street layout of this neighborhood. Let each intersection of two or more differently named streets be represented by a vertex. Also include a vertex for any dead end. Represent each street as an edge connecting the appropriate vertices.



6. The map of the Hawaiian Islands shows the ferry/boat routes between the islands. Draw a graph using each island as a vertex and each ferry route as an edge.

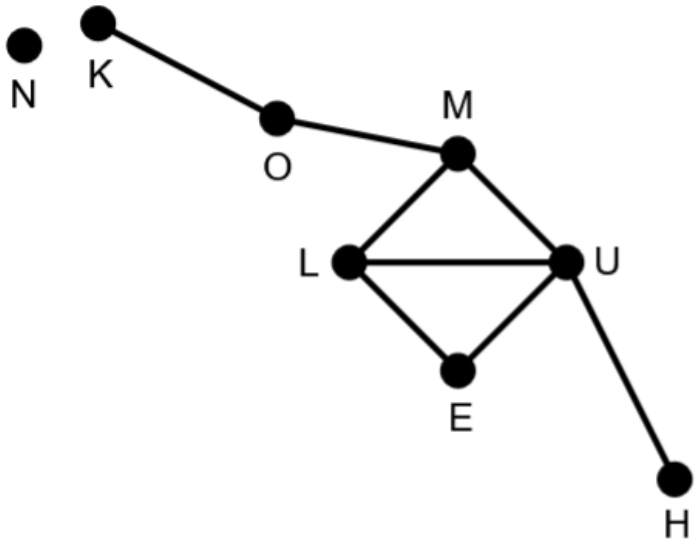


Answers available at the end of this section.

Answers

Section 8.1: Basic Concepts of Graph Theory [*In-Class Exercises*]

1. The image shows a graph with four labeled vertices connected by edges. Vertex A connects with vertices B, C, and D; vertex B connects with vertex D; and vertex C has a loop edge connected to itself.
   a. There are vertices A, B, C, and D.
   b. There are 5 edges, A-B, A-D, A-C, B-D, and C-C.
   c. There is 1 loop, C-C.
   d. A=3, B=2, D=2, C=2.
   e. A-C is a bridge.
   f. A-B-D is a path, because the walk crosses no edge more than once.
   g. A-C-C-A is not a path, because it crosses edge A-C more than once.
   h. A-B-D is not a circuit, because the path A-B-D does not start and end at the same vertex.
   i. D-A-B-D is a circuit, because the path D-A-B-D starts and ends at the same vertex D.
   j. A-D-B-A-C-C is a path, because the walk crosses no edge more than once.
2. The image shows a graph with six labeled vertices connected by edges. Vertex A connects with vertex B; vertex B connects with vertex C; vertex C connects with vertices D and F; vertex D connects with vertex E; and vertex E connects with vertex F.
3. The image shows a graph with six labeled vertices connected by edges. Vertex A connects with vertices B, D, and F; vertex B connects with vertices C and E; vertex C connects with vertex D; and vertex D connects with vertex F.
4. The image shows a graph with seven labeled vertices connected by edges. Vertex H connects with vertices F, C, and N; vertex F connects with vertices S and C; vertex C connects with vertices S, I, and N; vertex S connects with vertices T and I; vertex T connects with vertex I; and vertex I connects with vertex N.
5. The image shows an unlabeled graph with thirteen vertices connected by edges. The vertices are numbered from left to right, then top to bottom. The first vertex connects to the second and fourth vertices; the second vertex connects to the third and sixth vertices; the third vertex connects to the seventh vertex; the fourth vertex connects to the fifth and ninth vertices; the fifth vertex connects to the sixth and eighth vertices; the sixth vertex connects to the seventh and tenth vertices; the seventh vertex connects to the eighth vertex; the eighth vertex connects to the twelfth vertex; the ninth vertex connects to the tenth vertex; the tenth vertex connects to the eleventh vertex; and the eleventh vertex connects to the twelfth and thirteenth vertices.

6. 

(Note: U = Maui, E = Kaho'olawe)

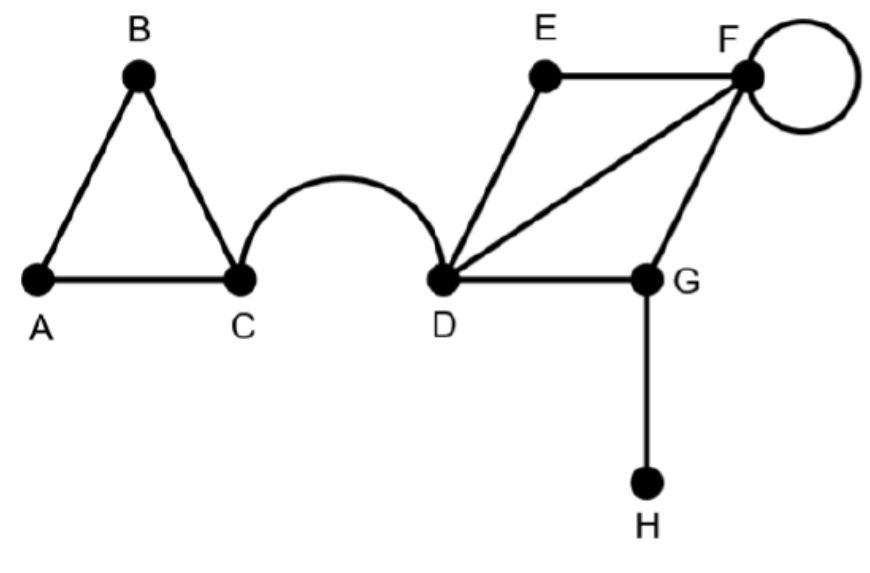
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Section 8.1.H: Homework

Section 8.1: Basic Concepts of Graph Theory - Homework Exercises

(Section 8.1) - Homework Exercise #1

Consider the graph below.



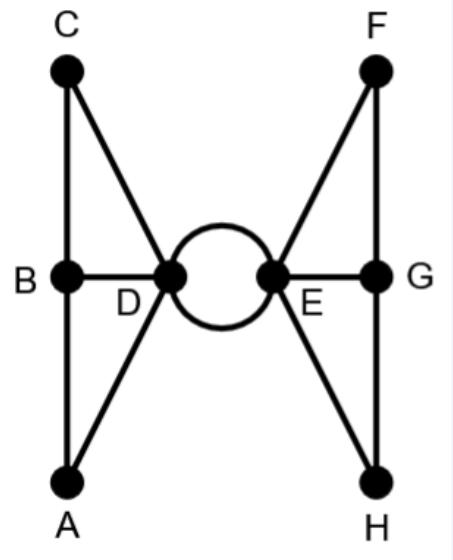
Vertex A connects with vertices B and C; vertex B connects with vertex C; vertex C connects with vertex D with a curved edge; vertex D connects with vertices E, F, and G; vertex F connects with vertices E and G and also has a loop edge; and vertex G connects with vertex H.

a. Identify all vertices.
b. Identify all edges.
c. Identify any loops.
d. Identify the degree of each vertex.
e. Indicate any bridges, if possible.
f. Is C-A-B-C-D-E-F-F-D-G-H a path? Why or why not?
g. Is D-G-F-D-E-F-F-G-H a path? Why or why not?
h. Is D-C-A-B-C-D a circuit? Why or why not?
i. Is F-G-D-E-F-F a circuit? Why or why not?

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #2

Consider the graph below.



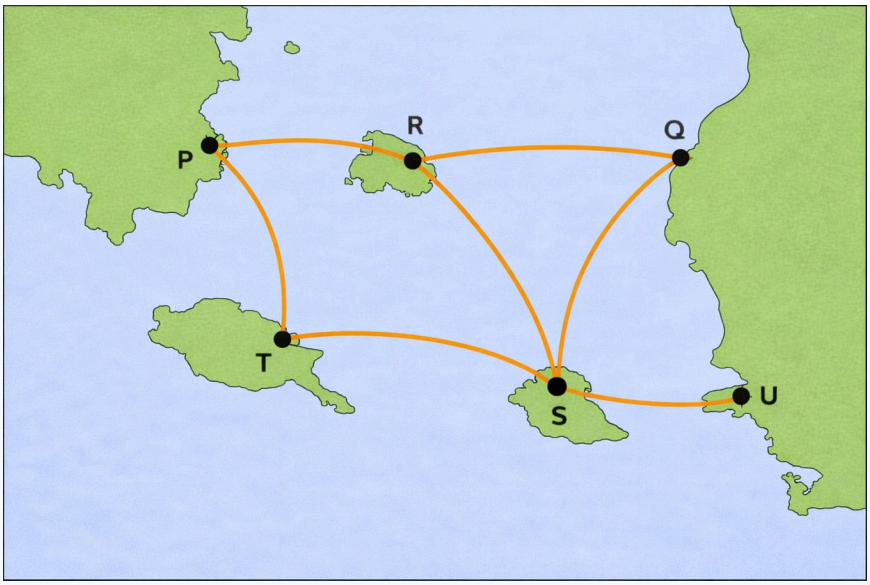
Vertex A connects with vertices B and D; vertex B connects with vertices C and D; vertex C connects with vertex D; vertex D connects with vertex E with two different curved edges; vertex E connects with vertices F, G, and H; and vertex G connects with vertices F and H.

a. Identify all vertices.
b. Identify all edges.
c. Identify any loops.
d. Identify the degree of each vertex.
e. Indicate any bridges, if possible.
f. Is A-B-C-D-B-A-D a path? Why or why not?
g. Is B-D-E-G-H-E-D a path? Why or why not?
h. Is B-D-E-G-H-E-D-B a circuit? Why or why not?
i. Is B-D-AF-G-D-E-F-F a circuit? Why or why not?
j. Can you find a circuit that includes all of the vertices?
k. Can you find a circuit that includes all of the edges?

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #3

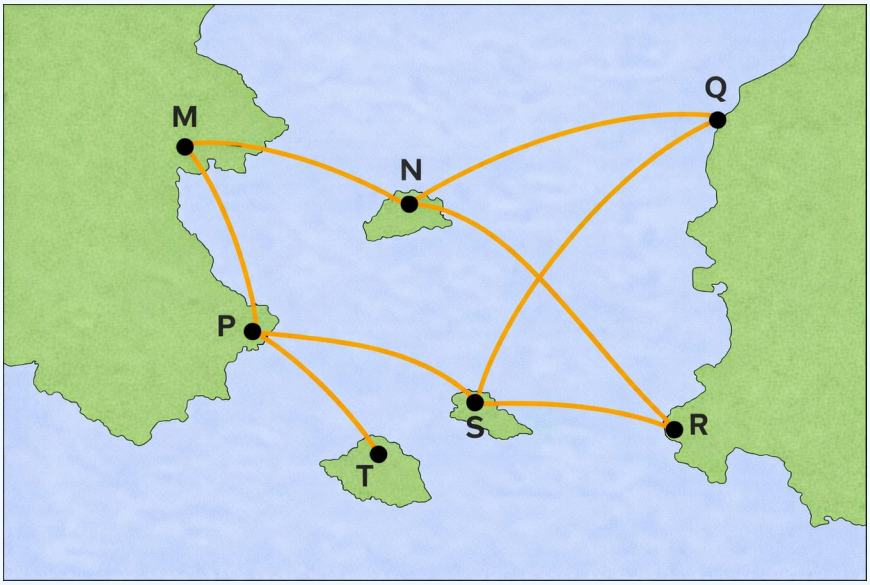
Consider the map of the mainlands and islands that shows all of the ferry/boat routes to the points of interest. Represent ALL mainland/islands using a graph.



Answer available at the end of this section.

(Section 8.1) - Homework Exercise #4

Consider the map of the mainlands and islands that shows all of the ferry/boat routes to the points of interest. Represent ALL mainland/islands using a graph.

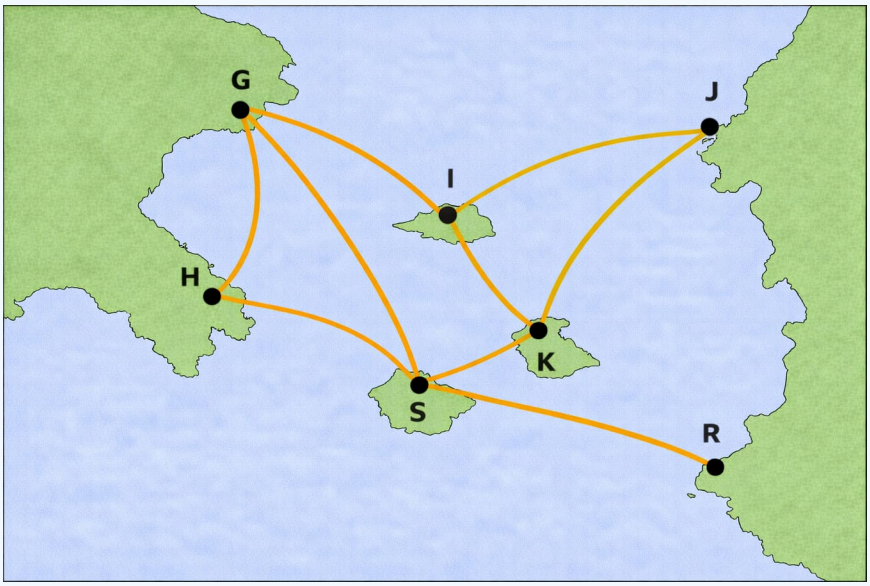


Vertex M connects with vertices N and P; vertex N connects with vertices Q and R; vertex P connects with vertices S and T; vertex Q connects with vertex S; and vertex S connects with vertex R.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #5

Consider the map of the mainlands and islands that shows all of the ferry/boat routes to the points of interest. Represent ALL mainland/islands using a graph.

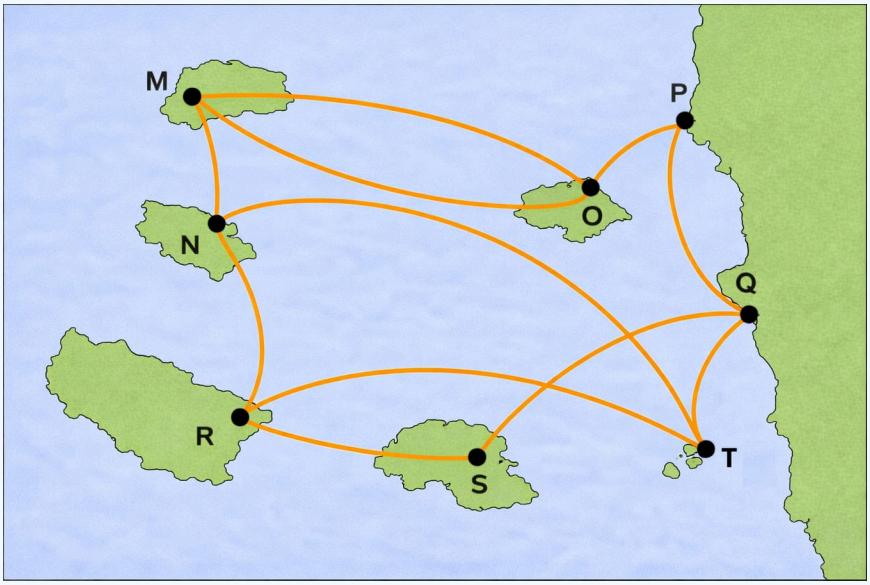


Vertex G connects with vertices H, I, and S; vertex I connects with vertices J and K; vertex J connects with vertex K; vertex H connects with vertex S; and vertex S connects with vertices K and R.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #6

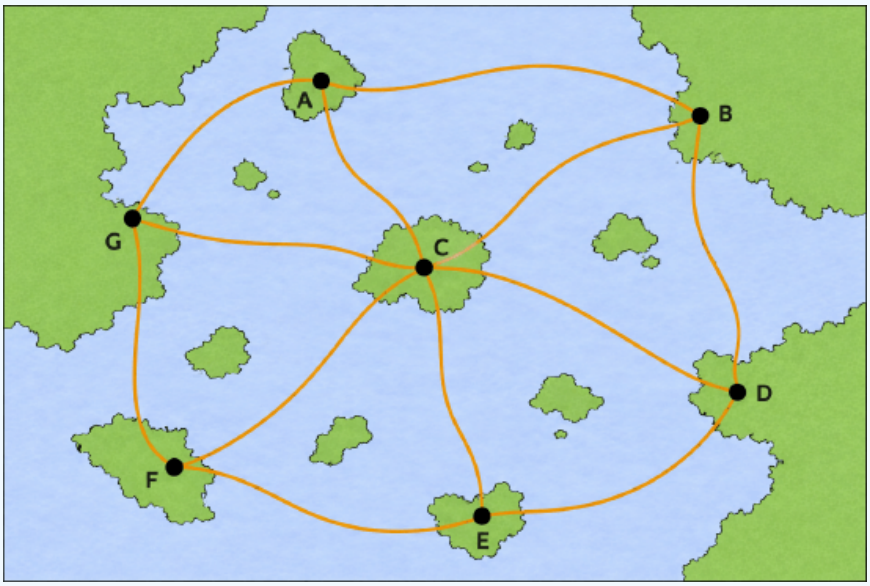
Consider the map of the mainlands and islands that shows all of the ferry/boat routes to the points of interest. Represent ALL mainland/islands using a graph.



Answer available at the end of this section.

(Section 8.2) - Homework Exercise #7

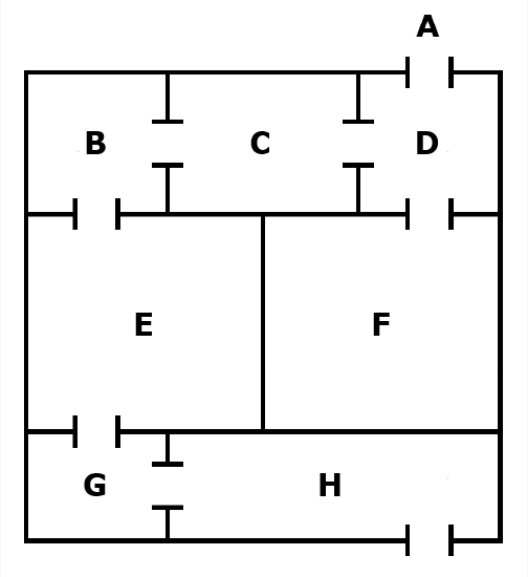
Consider the map of the mainlands and islands that shows all of the ferry/boat routes to the points of interest. Represent ALL mainland/islands using a graph.



Answer available at the end of this section.

(Section 8.1) - Homework Exercise #8

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.

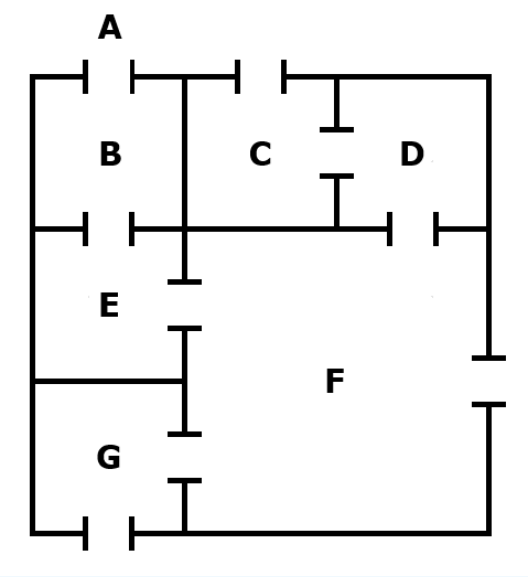


The image shows a floor plan divided into seven labeled rooms connected by doorway openings. Room B connects with room C and room E; room C connects with room D; room D connects with room F and outside area A; room E connects with room G; room G connects with room H; and room H connects with outside area A.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #9

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.

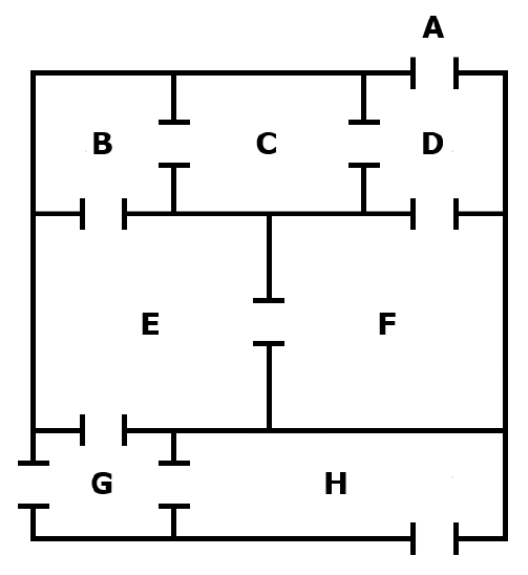


The image shows a floor plan divided into six labeled rooms connected by doorway openings. Room B connects with outside area A, room C and room E; room C connects with outside area A, and room D; room D connects with room F; room E connects with room F; room F connects to G, and outside area A; and room G connects with outside area A.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #10

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.

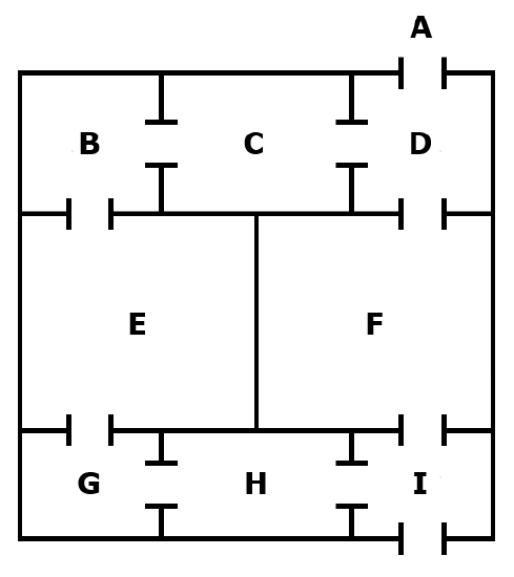


The image shows a floor plan divided into seven labeled rooms connected by doorway openings. Room B connects with room C and room E; room C connects with room D; D connects to room F, and outside area A; room E connects with room F, and with room G; room G connects with room H, and outside area A; and room H connects with outside area A.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #11

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.

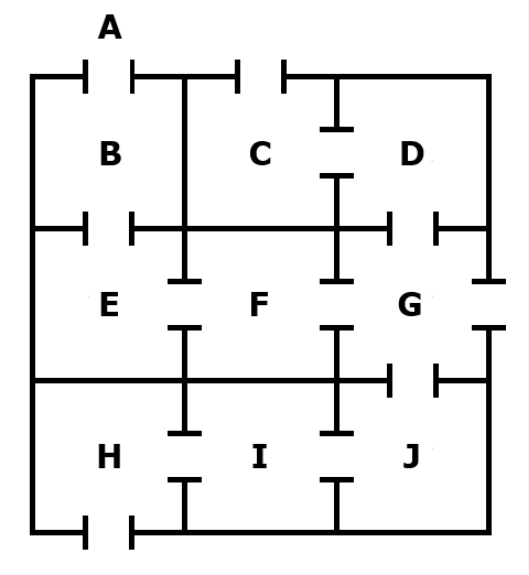


The image shows a floor plan divided into eight labeled rooms connected by doorway openings. Room B connects with room C and room E; room C connects with room D; room D connects with room F and outside area A; room E connects with room G; room F connects with room I; room G connects with room H; room H connects with room I; and room I connects with outside area A.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #12

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.

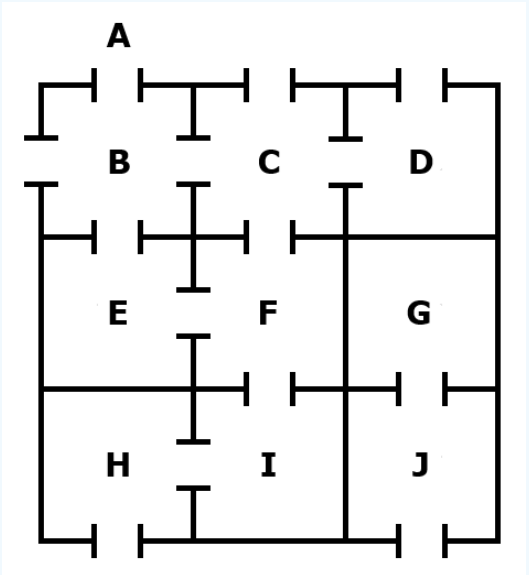


The image shows a floor plan divided into nine labeled rooms connected by doorway openings. Room B connects with outside area A, room C, and room E; room C connects with outside area A and room D; room D connects with room G; room E connects with room F; room F connects with room G; room G connects with room J and outside area A; room H connects with room I and outside area A; and room I connects with room J.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #13

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.

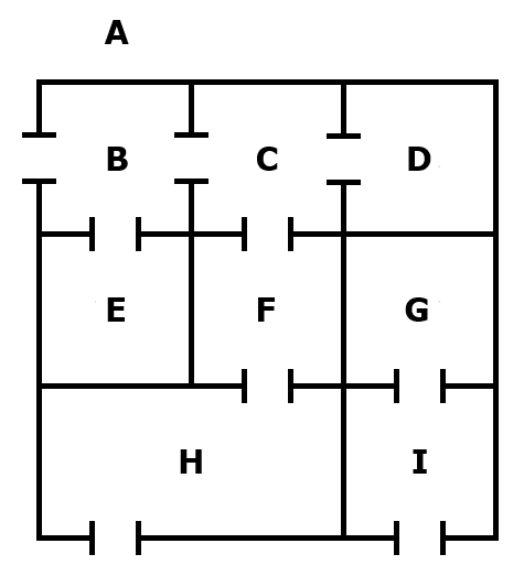


The image shows a floor plan divided into nine labeled rooms connected by doorway openings. Room B connects with outside area A, room C, and room E; room C connects with outside area A, room F, and room D; room D connects with outside area A; room E connects with room F; room F connects with room I; room G connects with room J; room H connects with room I and outside area A; and room J connects with outside area A.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #14

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge.

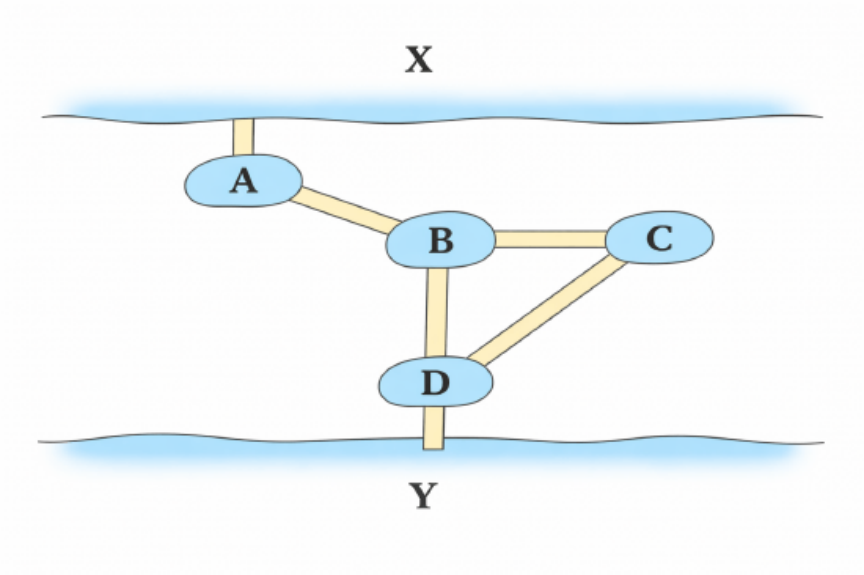


The image shows a floor plan divided into eight labeled rooms connected by doorway openings. Room B connects with outside area A, room C, and room E; room C connects with room D, and room F; room F connects with room H; room G connects with room I; room H connects with outside area A; and room I connects with outside area A.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #15

For the map of the islands and mainlands below, identify the degree of each island and mainland.

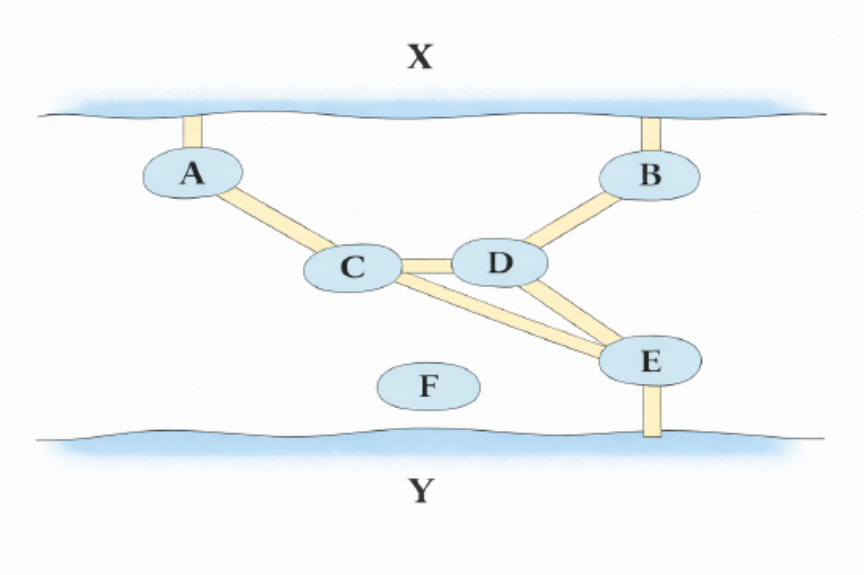


The image shows islands labeled A, B, C, and D located between regions X and Y. Island A connects with region X and island B; island B connects with islands C and D; island C connects with island D; and island D connects with region Y.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #16

For the map of the islands and mainlands below, identify the degree of each island and mainland.

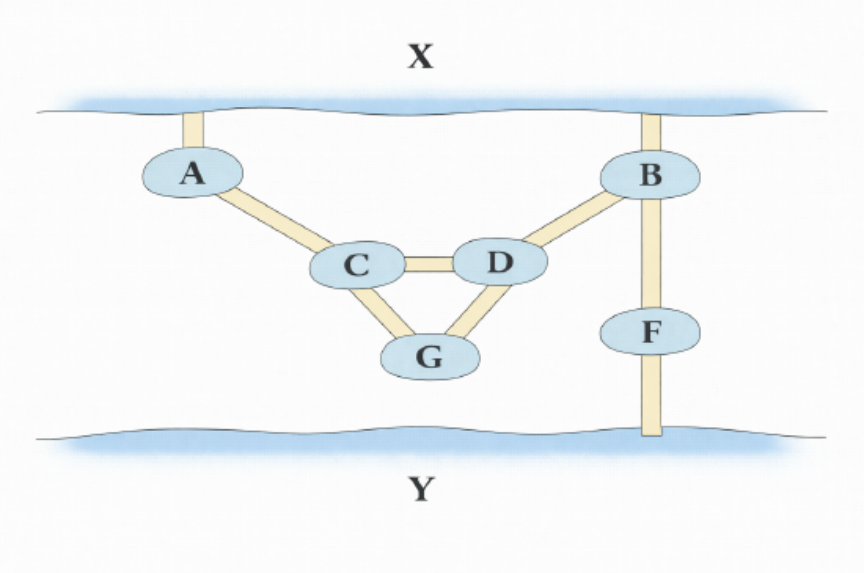


The image shows islands labeled A, B, C, D, E, and F located between regions X and Y. Island A connects with region X and island C; island B connects with region X and island D; island C connects with islands D and E; island D connects with island E; island E connects with region Y; and island F is isolated with no connections.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #17

For the map of the islands and mainlands below, identify the degree of each island and mainland.

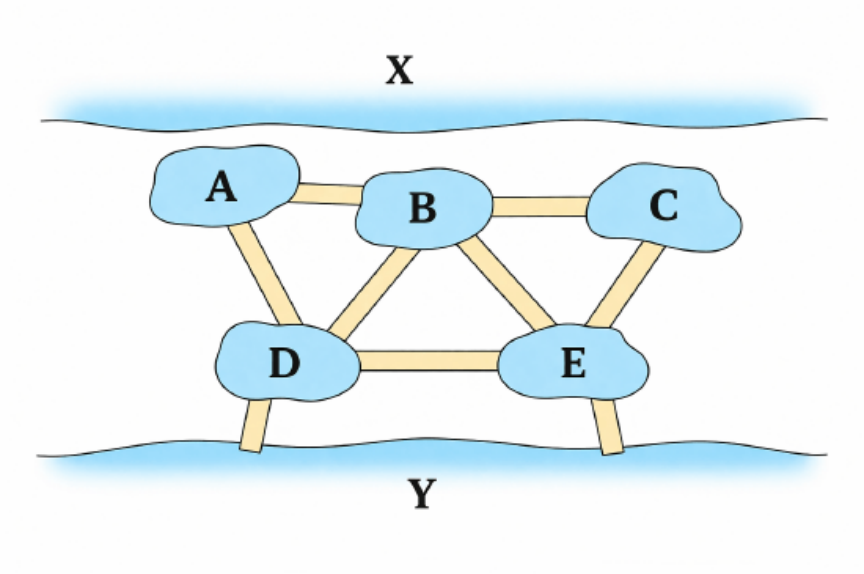


The image shows islands labeled A, B, C, D, F, and G located between regions X and Y. Island A connects with region X and island C; island B connects with islands D and F and also connects with region X; island C connects with islands D and G; island D connects with island G; and island F connects with region Y.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #18

For the map of the islands and mainlands below, identify the degree of each island and mainland.

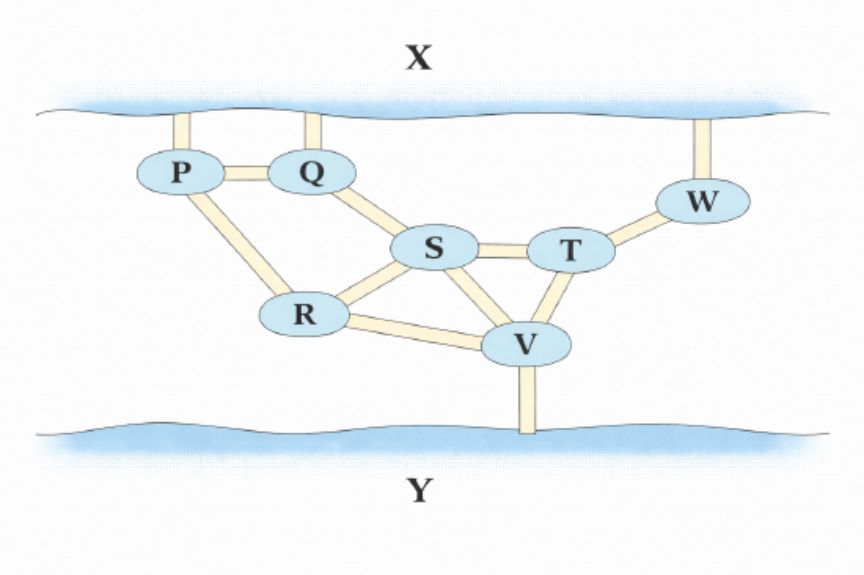


The image shows islands labeled A, B, C, D, and E located between regions X and Y. Island A connects with islands B and D; island B connects with islands C, D, and E; island C connects with island E; island D connects with island E and region Y; and island E connects with region Y.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #19

For the map of the islands and mainlands below, identify the degree of each island and mainland.



The image shows islands labeled P, Q, R, S, T, V, and W located between regions X and Y. Island P connects with region X and islands Q and R; island Q connects with region X and island S; island R connects with islands S and V; island S connects with islands T and V; island T connects with islands V and W; island V connects with region Y; island W connects with region X; and island V connects with region Y.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #20

A restaurant chain has the following locations in the states on the map below.



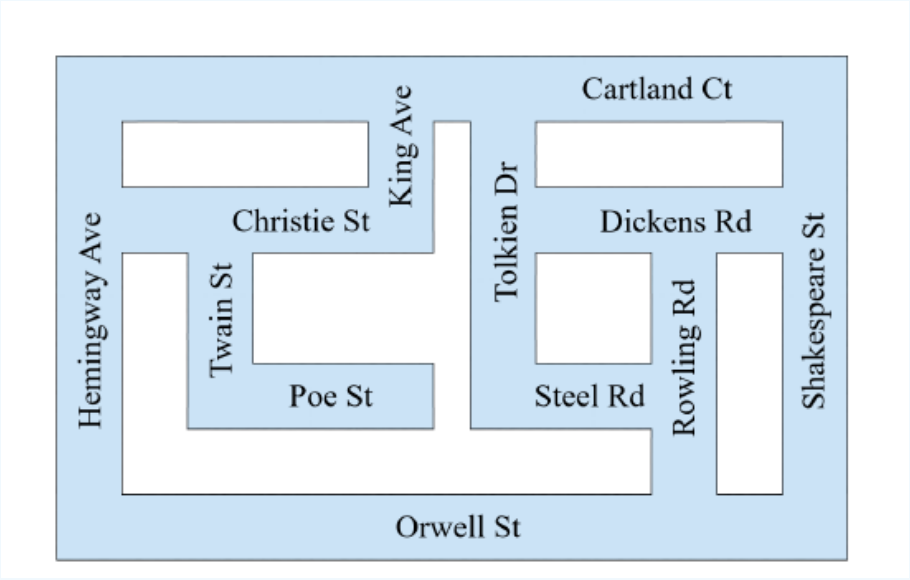
North Dakota borders South Dakota and Minnesota; South Dakota borders Nebraska, Minnesota, and Iowa; Nebraska borders Iowa, Missouri, and Kansas; Minnesota borders Wisconsin and Iowa; Iowa borders Wisconsin, Illinois, and Missouri; Wisconsin borders Michigan and Illinois; Michigan borders Indiana; Illinois borders Indiana, Missouri, and Kentucky; Indiana borders Kentucky; Missouri borders Kentucky and Tennessee; and Kentucky borders Tennessee.

Create a graph where each state is a vertex and each common border is an edge. (Note: Michigan is in two parts of land, the Upper Peninsula and the Lower Peninsula. Consider it all as one state, since both lands can be accessed by a bridge via Interstate 75).

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #21

Draw a graph that represents the street layout of this neighborhood. Let each intersection of two or more differently named streets be represented by a vertex. Also include a vertex for any dead end. Represent each street as an edge connecting the appropriate vertices.

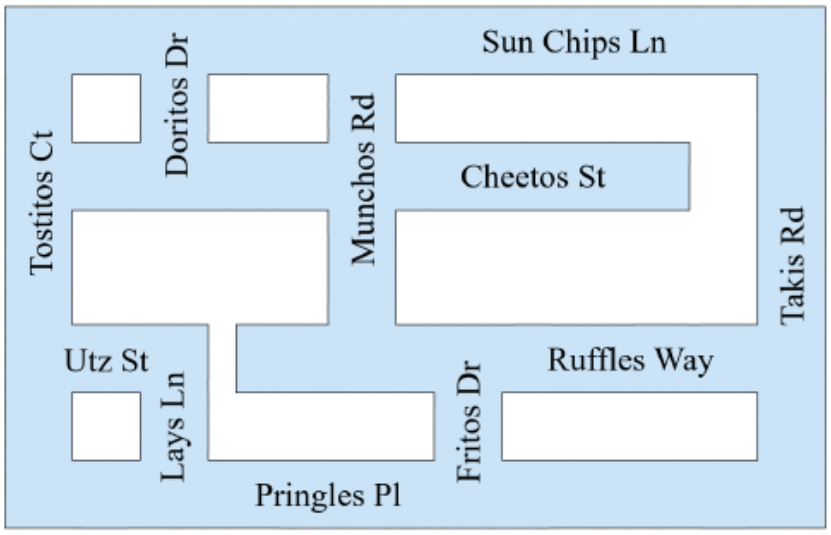


The image shows a street map with labeled streets connected by intersections. Cartland Court connects with Hemingway Avenue, King Avenue, Tolkien Drive, and Shakespeare Street; Christie Street connects with Hemingway Avenue, Twain Street, and King Avenue; Dickens Road connects with Tolkien Drive, Rowling Road, and Shakespeare Street; Poe Street connects with Twain Street; Steel Road connects with Tolkien Drive and Rowling Road; and Orwell Street connects with Hemingway Avenue, Rowling Road, and Shakespeare Street.

Answer available at the end of this section.

(Section 8.1) - Homework Exercise #22

Draw a graph that represents the street layout of this neighborhood. Let each intersection of two or more differently named streets be represented by a vertex. Also include a vertex for any dead end. Represent each street as an edge connecting the appropriate vertices.



The image shows a street map with labeled streets connected by intersections. Sun Chips Lane connects with Toe-stee-toes Court, Doritos Drive, Munchos Road, and Ta-keys Road; Cheetos Street connects with Toe-stee-toes Court, Doritos Drive, and Munchos Road; Utz Street connects with Toe-stee-toes Court, and Lays Lane; Ruffles Way connects with Munchos Road, Fritos Drive, and Ta-keys Road; and Pringles Place connects with Toe-stee-toes Court, Lays Lane, Fritos Drive and Ta-keys Road.

Answer available at the end of this section.

Answers

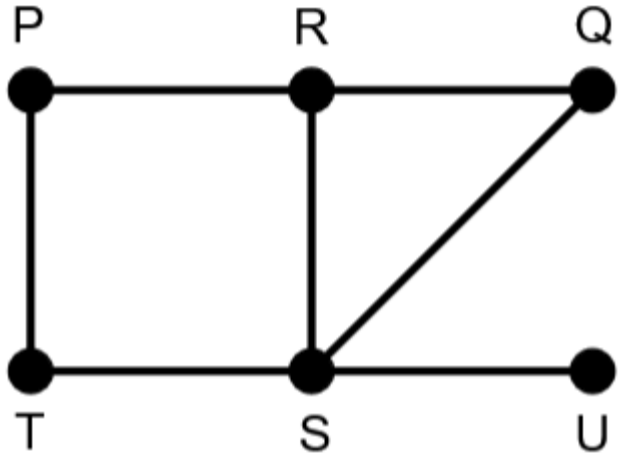
(Section 8.1) - Homework Exercise #1

a. Eight vertices: A, B, C, D, E, F, G, H.
b. Eleven edges: A-B, B-C, A-C, C-D, D-E, D-F, D-G, E-F, F-F, F-G, G-H.
c. F-F is a loop.
d. A=2, B=2, C=3, D=4, E=2, F=5, G=3, H=1.
e. Two bridges: C-D, G-H.
f. C-A-B-C-D-E-F-F-D-G-H is a path, because the walk crosses no edge more than once.
g. D-G-F-D-E-F-F-G-H is not a path, because it crosses edge D-G more than once.
h. D-C-A-B-C-D is not a circuit, because it is not a path. Edge D-C is crossed more than once.
i. F-G-D-E-F-F is a circuit, because the walk crosses no edge more than once and begins and ends at the same vertex F.

(Section 8.1) - Homework Exercise #2

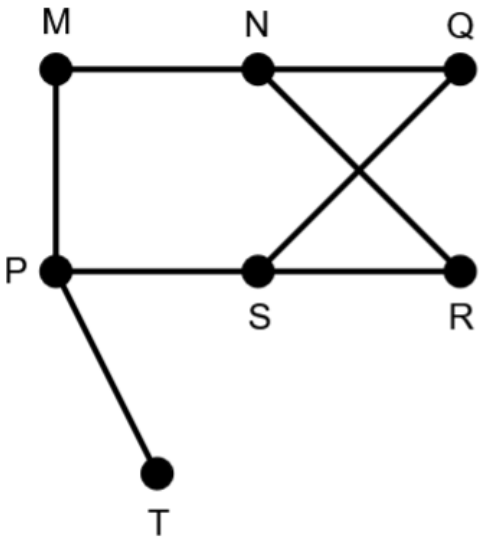
a. Eight vertices: A, B, C, D, E, F, G, H.
b. Twelve edges: A-B, A-D, B-C, B-D, C-D, D-E (top), D-E (bottom), E-F, E-G, E-H, F-G, G-H.
c. There are no loops.
d. A=2, B=3, C=2, D=5, E=5, F=2, G=3, H=2.
e. There are no bridges.
f. A-B-C-D-B-A-D is not a path, because it crosses edge A-B more than once. the walk crosses no edge more than once.
g. B-D-E-G-H-E-D is a path, because the walk crosses no edge more than once. (There is a top edge D-E and a bottom edge D-E; thus not the same edge).
h. B-D-E-G-H-E-D-B is not a circuit, because it is not a path. Edge B-D is crossed more than once.
i. G-H-E-D-E-F-G is a circuit, because the walk crosses no edge more than once and begins and ends at the same vertex G.
j. Yes; Answers will vary (one possible circuit that includes all vertices: A-B-C-D-E-F-G-H-E-D-A).
k. No

(Section 8.1) - Homework Exercise #3

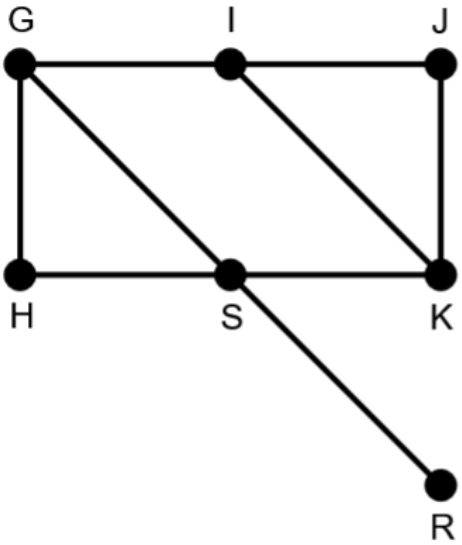


Vertex P connects with vertices R and T; vertex R connects with vertices Q and S; and vertex S connects with vertices Q, T, and U;.

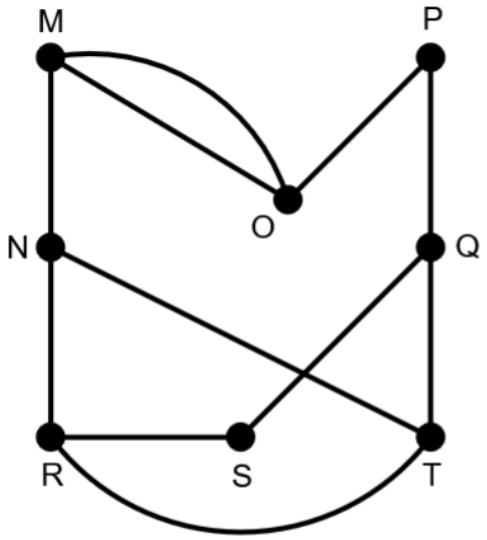
(Section 8.1) - Homework Exercise #4



(Section 8.1) - Homework Exercise #5

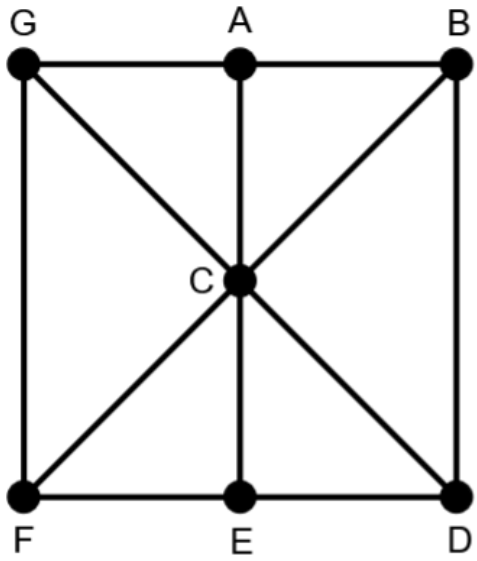


(Section 8.1) - Homework Exercise #6



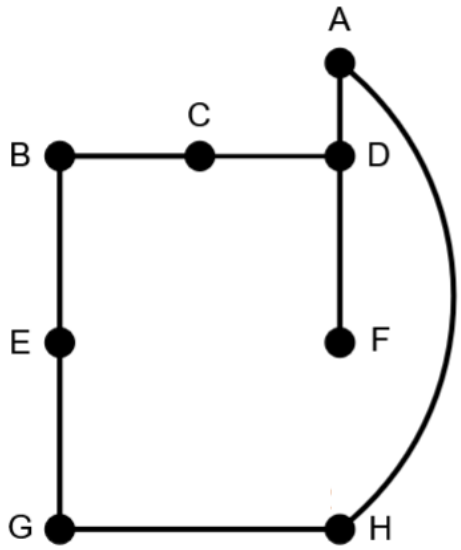
Vertex M connects with vertices N and O, and again with O by a curved edge; vertex O connects with vertex P; vertex P connects with vertex Q; vertex Q connects with vertices S and T; vertex N connects with vertices R and T; and vertex R connects with vertices S and T by a curved edge.

(Section 8.2) - Homework Exercise #7



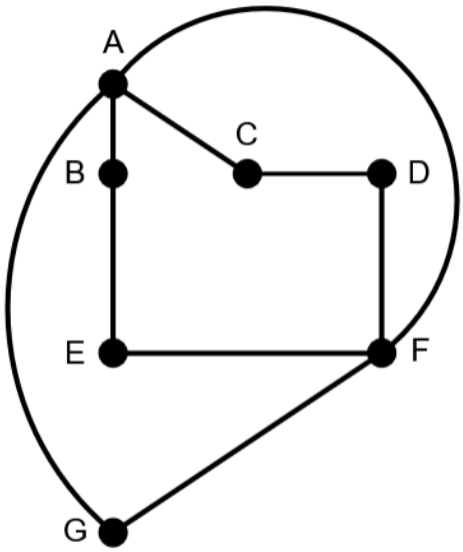
Vertex G connects with vertices A, F, and C; vertex A connects with vertices B and C; vertex B connects with vertices C and D; vertex C connects with vertices D, E, and F; vertex D connects with vertex E; and vertex E connects with vertex F.

(Section 8.1) - Homework Exercise #8



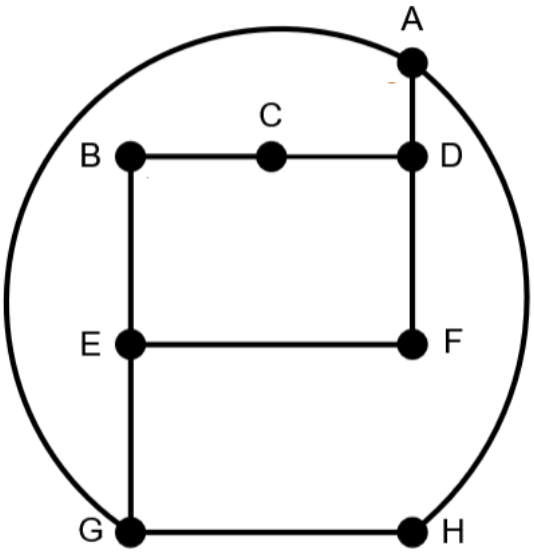
Vertex A connects with vertices D and H; vertex B connects with vertices C and E; vertex C connects with vertex D; vertex D connects with vertex F; vertex E connects with vertex G; and vertex G connects with vertex H.

(Section 8.1) - Homework Exercise #9



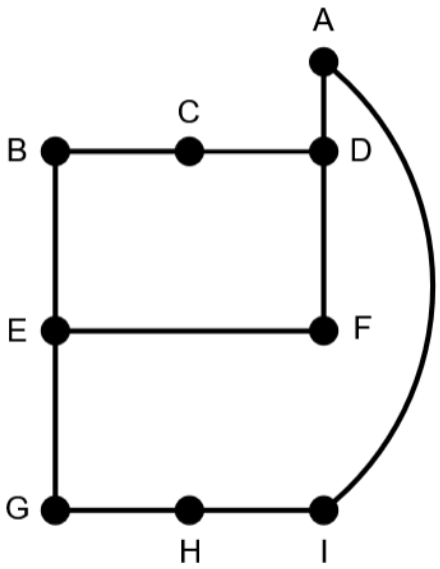
Vertex A connects with vertices B, C, F, and G; vertex B connects with vertex E; vertex C connects with vertex D; vertex D connects with vertex F; vertex E connects with vertex F; and vertex F connects with vertex G.

(Section 8.1) - Homework Exercise #10



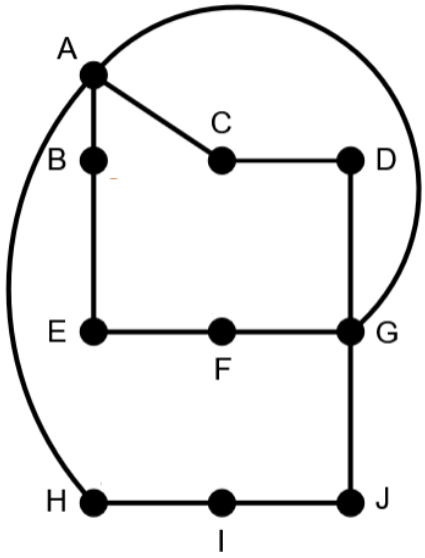
Vertex A connects with vertices D, G, and H; vertex B connects with vertices C and E; vertex C connects with vertex D; vertex D connects with vertex F; vertex E connects with vertices F and G; and vertex G connects with vertex H.

(Section 8.1) - Homework Exercise #11



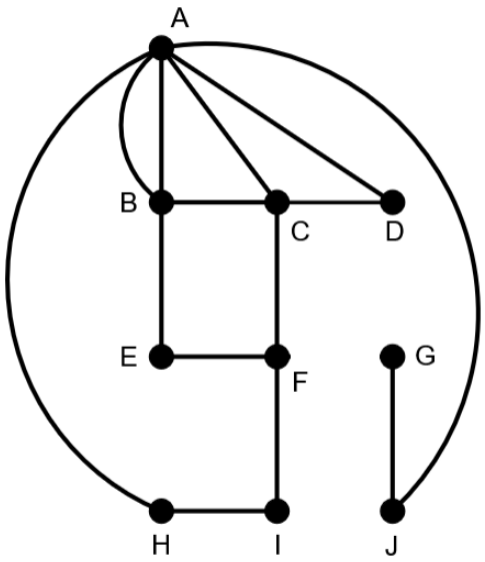
Vertex A connects with vertices D and I; vertex B connects with vertices C and E; vertex C connects with vertex D; vertex D connects with vertex F; vertex E connects with vertices F and G; vertex G connects with vertex H; and vertex H connects with vertex I.

(Section 8.1) - Homework Exercise #12



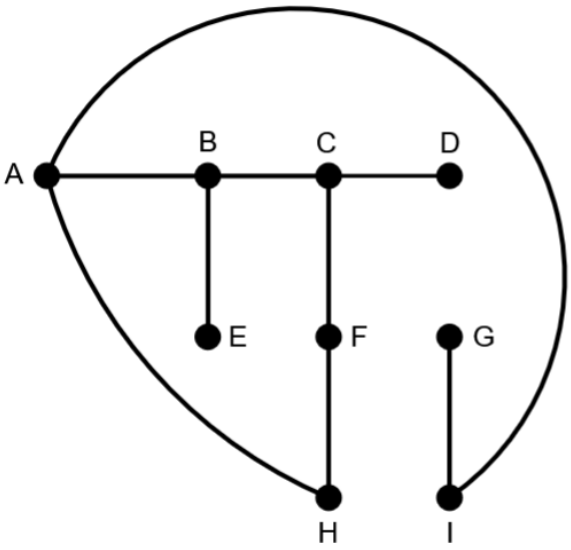
Vertex A connects with vertices B, C, G, and H; vertex B connects with vertex E; vertex C connects with vertex D; vertex D connects with vertex G; vertex E connects with vertex F; vertex F connects with vertex G; vertex G connects with vertex J; vertex H connects with vertex I; and vertex I connects to vertex J.

(Section 8.1) - Homework Exercise #13



Vertex A connects with vertices C, D, H, J, B, and with B again with a curved edge; vertex B connects with vertices C and E; vertex C connects with vertex D; vertex E connects with vertex F; vertex F connects with vertex I; vertex I connects with vertex H; and vertex J connects with vertex G.

(Section 8.1) - Homework Exercise #14



Vertex A connects with vertices B, H, and I; vertex B connects with vertices C and E; vertex C connects with vertices D and F; vertex F connects with vertex H; and vertex G connects with vertex I.

(Section 8.1) - Homework Exercise #15

Degree 1: X, Y; Degree 2: A, C; Degree 3: B, D.

(Section 8.1) - Homework Exercise #16

Degree 1: Y; Degree 2: X, A, B; Degree 3: C, D, E; F has no degree.

(Section 8.1) - Homework Exercise #17

Degree 1: Y; Degree 2: X, A, F, G; Degree 3: B, C, D, G.

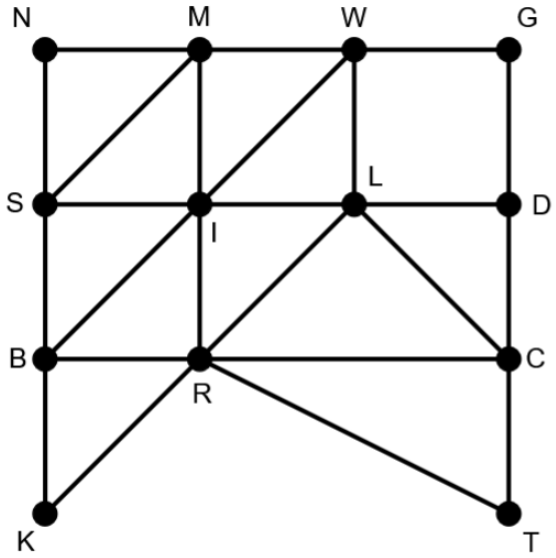
(Section 8.1) - Homework Exercise #18

Degree 2: Y, A, C; Degree 4: B, D, E; X has no degree

(Section 8.1) - Homework Exercise #19

Degree 1: Y; Degree 2: W; Degree 3: X, P, Q, R, T; Degree 4: S, V.

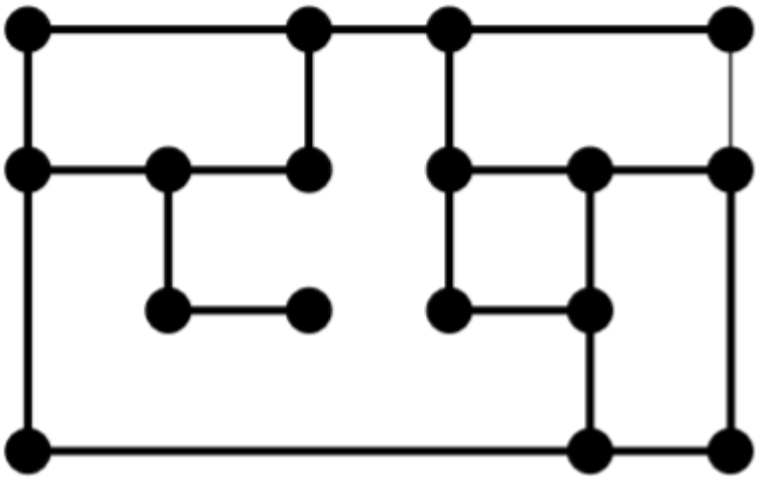
(Section 8.1) - Homework Exercise #20



Vertex N connects with vertices M and S; vertex M connects with vertices S, I, and W; vertex W connects with vertices G, L, and I; vertex G connects with vertex D; vertex S connects with vertices B and I; vertex I connects with vertices B, R, and L; vertex L connects with vertices D, C, and R; vertex D connects with vertex C; vertex B connects with vertices R and K; vertex R connects with vertices C, K, and T; and vertex C connects with vertex T.

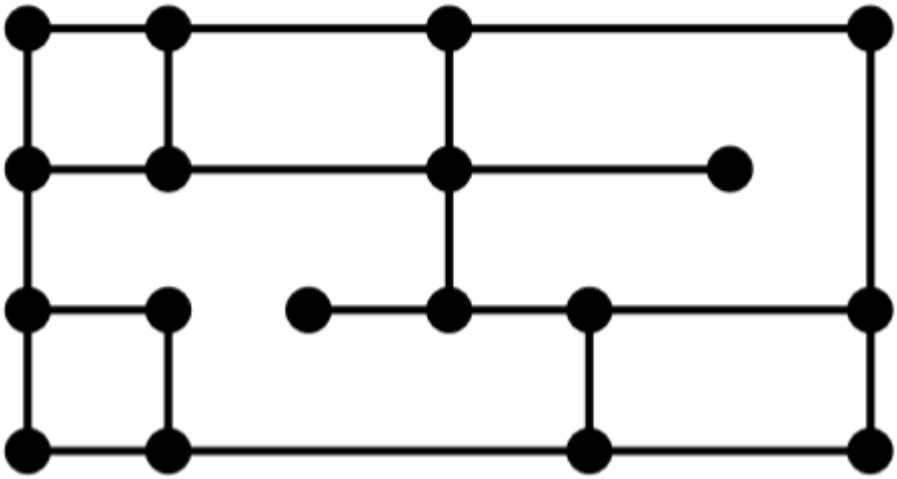
All states are represented by the first letter of their name for each vertex, except for the following:
B = Nebraska, R = Missouri, L = Illinois, G = Michigan, D = Indiana, C = Kentucky.

(Section 8.1) - Homework Exercise #21



The image shows an unlabeled graph with seventeen vertices connected by edges. The vertices are numbered from left to right, then top to bottom. The first vertex connects to the second and fifth vertices; the second vertex connects to the third and seventh vertices; the third vertex connects to the fourth and eigth vertices; the fourth vertex connects to the tenth vertex; the fifth vertex connects to the sixth and fifteenth vertices; the sixth vertex connects to the seventh and eleventh vertices; the eight vertex connects to the ninth and thirteenth vertices; the ninth vertex connects to the tenth and fourteenth vertices; the tenth vertex connects to the seventeenth vertex; the eleventh vertex connects to the twelfth vertex; the thirteenth vertex connects to the fourteenth vertex; the fourteenth vertex connects to the sixteenth vertex; the fifteenth vertex connects to the sixteenth vertex; and the sixteenth vertex connects to the seventeenth vertex;.

(Section 8.1) - Homework Exercise #22



The image shows an unlabeled graph with nineteen vertices connected by edges. The vertices are numbered from left to right, then top to bottom. The first vertex connects to the second and fifth vertices; the second vertex connects to the third and sixth vertices; the third vertex connects to the fourth and seventh vertices; the fourth vertex connects to the fourteenth vertex; the fifth vertex connects to the sixth and ninth vertices; the sixth vertex connects to the seventh vertex; the seventh vertex connects to the eighth and twelfth vertices; the ninth vertex connects to the tenth and fifteenth vertices; the tenth vertex connects to the sixteenth vertex; the eleventh vertex connects to the twelfth vertex; the twelfth vertex connects to the thirteenth vertex; the thirteenth vertex connects to the fourteenth and seventeenth vertices; the fourteenth vertex connects to the eighteenth vertex; the fifteenth vertex connects to the sixteenth vertex; the sixteenth vertex connects to the seventeenth vertex; and the seventeenth vertex connects to the eighteenth vertex.

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Section 8.2: Euler Paths and Euler Circuits

Learning Objectives

- Define an Euler path and an Euler circuit
- Use Euler's Theorem to determine if an Euler path or an Euler circuit exists
- Use Fluery's Algorithm to find an Euler path or an Euler circuit
- Solve application problems using Euler paths and Euler circuits

In graph theory, many problems involve traversing a graph in a systematic way. While earlier discussions of paths and circuits focus on sequences of vertices connected by edges, a different class of problems emphasizes the traversal of edges themselves. This leads to this section the concepts of **Euler paths** and **Euler circuits**, which require that every edge in a graph be used exactly once.

Euler Path

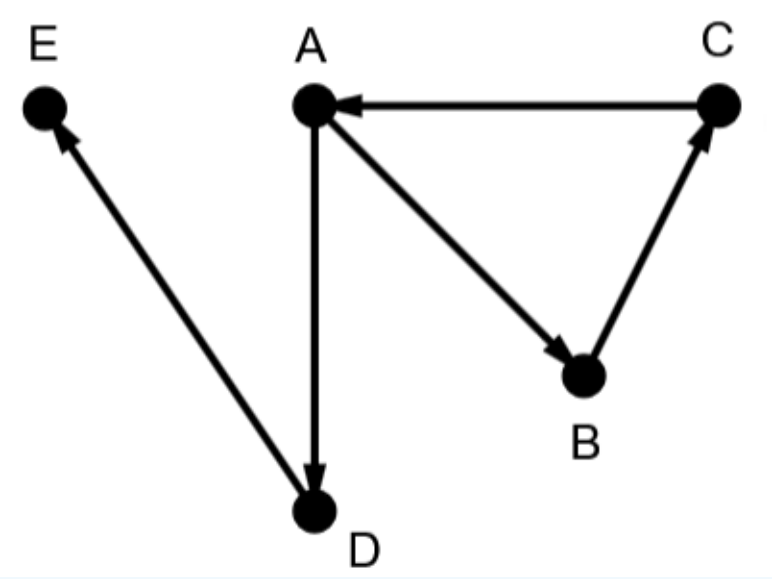
An **Euler path** (also called an Euler trail) is a path that uses every edge of a graph exactly once.

Euler Circuit

An **Euler circuit** (also called an Euler cycle) is an Euler path that starts and ends at the same vertex.

Example #8.2.1 🤔

Classify the path shown in the graph as an Euler path, Euler circuit, or neither.



The path is A-B-C-A-D-E.

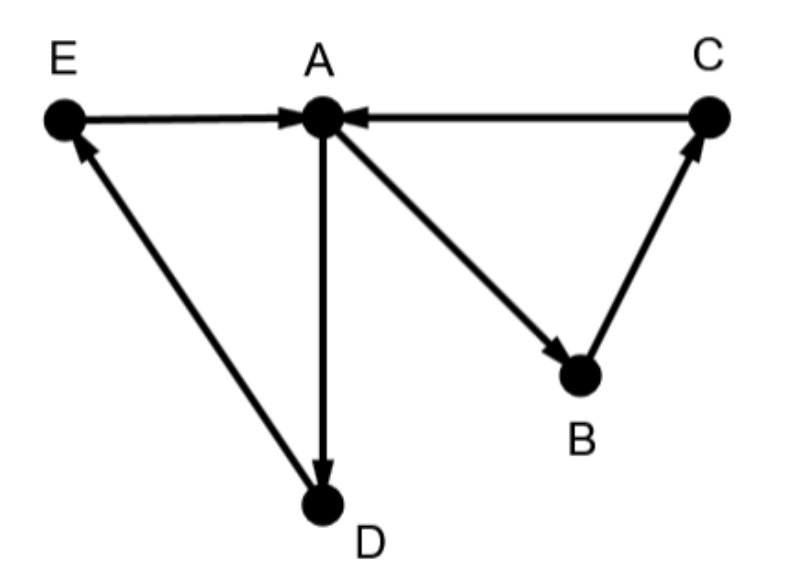
✅ **Solution:**

- Since the path A-B-C-A-D-E uses every edge exactly once, it is an Euler path.
- It is not an Euler circuit, because it does not begin and end at vertex A.

Euler path

Example #8.2.2 🤔

Classify the path shown in the graph as an Euler path, Euler circuit, or neither.



The path is A-B-C-A-D-E-A.

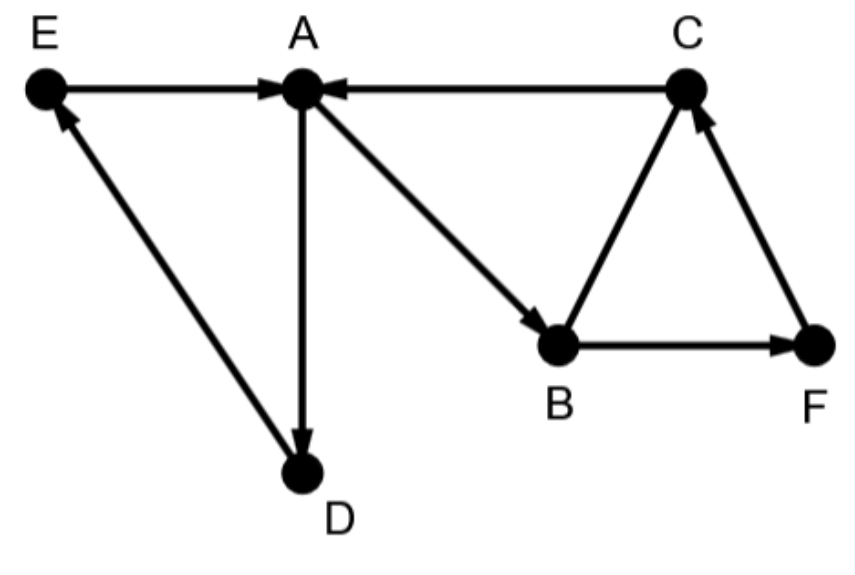
✅ **Solution:**

- Since the path A-B-C-A-D-E-A uses every edge exactly once, it is an Euler path.
- Also, since the path begins and ends at vertex A, it is also an Euler circuit. (Note: An Euler circuit means it is already an Euler path, so the best answer here is Euler Circuit).

Euler circuit.

Example #8.2.3 🤔

Classify the path shown in the graph as an Euler path, Euler circuit, or neither.



The path is A-B-F-C-A-D-E-A.

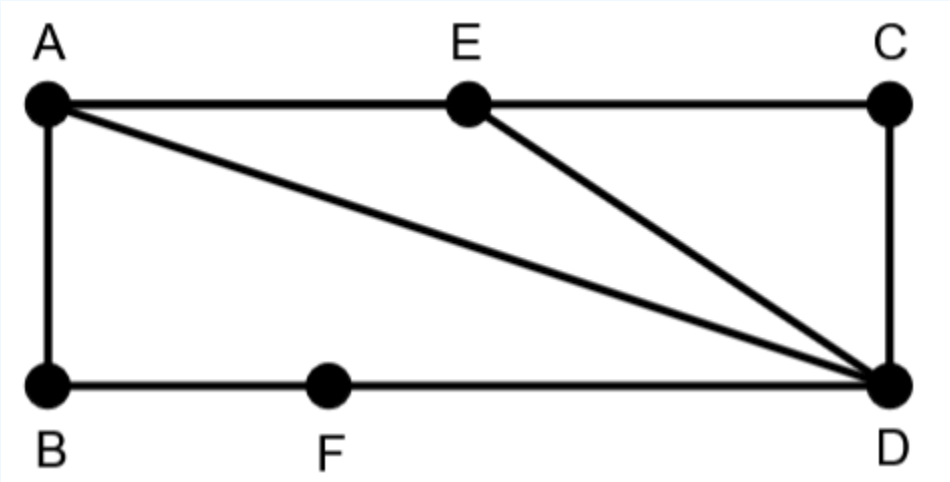
✅ **Solution:**

- Path A-B-F-C-A-D-E-A does not use every edge exactly once. Edge B-C was not used. Thus, it is not an Euler path. (Note: It is still considered a path).
- Because it is not an Euler path, it is not an Euler circuit, even though the path begins and ends at vertex A. (Note: It is still considered a circuit).

Neither

Example #8.2.4 🤔

Classify the path of E-D-C-E-A-D-F-B-A-E in the graph as an Euler path, Euler circuit, or neither.



✅ **Solution:**

- Path E-D-C-E-A-D-F-B-A-E does not use every edge exactly once. Edge A-E was used twice. Thus, it is not an Euler path. (Note: It is still considered a path).
- Because it is not an Euler path, it is not an Euler circuit, even though the path begins and ends at vertex A. (Note: It is still considered a circuit).

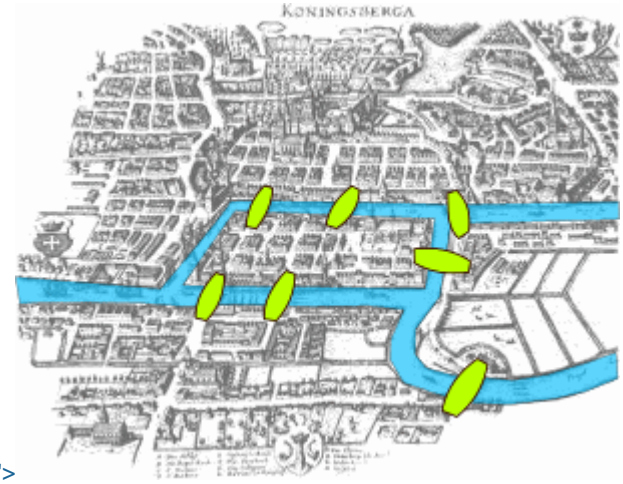
Neither

The Seven Bridges of Königsberg Problem

The Seven Bridges of Königsberg problem is one of the most famous problems in the history of mathematics and is widely regarded as the birth of graph theory. It was solved by the Swiss mathematician Leonhard Euler in 1736 and led directly to the development of the concepts as Euler paths and Euler circuits.

Königsberg (now Kaliningrad, Russia) was a city built around the Pregel River, which divided the city into several land regions connected by seven bridges. The city contained:

- Two large islands in the river, and
- Two mainland regions on either side of the river.



, via Wikimedia Commons">

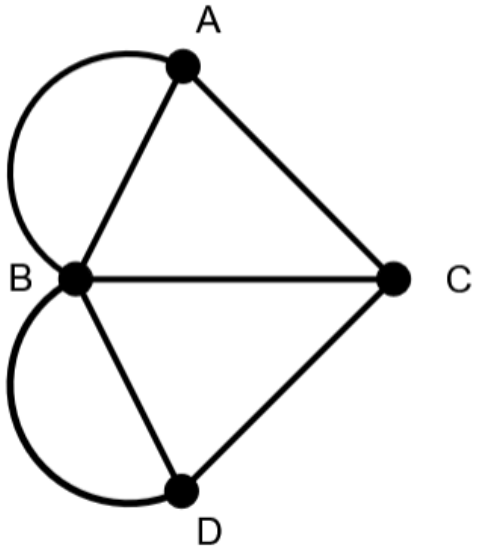
Residents of the city posed the following question: **Is it possible to take a walk that crosses each of the seven bridges exactly once and returns to the starting point?**

Variations of the problem also asked whether such a walk was possible without requiring a return to the starting point. Despite many attempts, no one had been able to find such a route, and the problem was considered a puzzle rather than a mathematical question until Euler analyzed it. Euler's breakthrough was realizing that the exact distances, shapes, and geography of the city were irrelevant. What mattered was only:

- Which land masses were connected, and
- How many bridges connected to each land mass.

The Königsberg Graph

Euler simplified the problem by representing each land region as a vertex, and representing each bridge as an edge between vertices. This abstraction turned a real-world situation into a graph, now called the Königsberg graph. Here is one variation of that graph:



When Euler examined this graph, he calculated the degree of each vertex (the number of edges incident to it) and found that:

- There were four vertices, corresponding to the four land regions.
- Each vertex had an odd degree.
- No vertex had an even degree.

Why the Walk Is Impossible! Euler's conclusion followed from a simple but powerful observation: Each time a traveler enters a vertex along one edge, they must leave the vertex along another edge. This means that edges at a vertex must occur in pairs, one edge for entering, and one for leaving, except possibly at the starting and ending points of a walk. From this reasoning, Euler established two fundamental conditions:

- A walk that starts and ends at the same vertex requires all vertices to have even degree.
- A walk that starts and ends at different vertices can have exactly two vertices of odd degree.

In the Königsberg graph:

- All four vertices have odd degree.
- This violates both conditions.

Therefore, Euler concluded that: **No walk exists that crosses each of the seven bridges exactly once, whether or not the walk returns to its starting point.**

Mathematical Significance

Euler's solution was revolutionary for several reasons:

- He did not search for routes or rely on trial and error.
- He introduced the idea of reasoning based on structure rather than geometry.
- He showed that certain problems can be answered definitively using logical conditions alone.

This marked a major shift away from classical geometry and toward what is now known as graph theory and discrete mathematics.

Connection to Euler Paths and Circuits

The Seven Bridges of Königsberg problem directly motivates the concepts of Euler paths and Euler circuits:

- An Euler circuit requires that every vertex in a graph have even degree.

- An Euler path (but not a circuit) requires exactly two vertices of odd degree.
- Any graph with more than two odd-degree vertices has no Euler traversal.

Since the Königsberg graph has four vertices of odd degree, it has neither an Euler path nor an Euler circuit.

Euler's analysis of the Seven Bridges of Königsberg problem led to a general result describing when a graph admits a walk that uses every edge exactly once. This result is now known as Euler's Theorem.

Euler's Theorem

For any connected, undirected graph:

1. **Euler Circuit**
   - The graph has an Euler circuit if and only if every vertex of the graph has even degree.
2. **Euler Path (but not a circuit)**
   - The graph has an Euler path, but not an Euler circuit if and only if exactly two vertices of the graph have odd degree.
3. **No Euler Path**
   - If the graph has more than two vertices of odd degree, then the graph has neither an Euler path nor an Euler circuit.

Interpretation of the Theorem

Euler's Theorem provides a complete classification of when an Euler traversal is possible. Remarkably, the existence of such a path or circuit depends only on the degrees of the vertices, not on the size, shape, or layout of the graph.

- Vertices of even degree allow edges to be paired naturally as "entering" and "leaving."
- Vertices of odd degree disrupt this pairing and can appear only at the start or end of a traversal.
- At most two odd-degree vertices can be accommodated in any path that uses every edge exactly once.

Consequences

From Euler's Theorem, several immediate conclusions follow:

- Every Euler circuit is an Euler path, but not every Euler path is an Euler circuit.
- A connected graph with no odd-degree vertices is Eulerian.
- A connected graph with exactly two odd-degree vertices is semi-Eulerian.
- Any graph with four or more odd-degree vertices cannot have an Euler traversal.

Euler's Theorem determines whether a graph contains an Euler path or Euler circuit, but it does not explain how to find such a traversal when one exists. One of the earliest systematic procedures for constructing an Euler traversal is known as **Fleury's Algorithm**.

Fleury's Algorithm provides a step-by-step method for producing an Euler path or Euler circuit in a connected graph that satisfies the degree conditions of Euler's Theorem.

Fleury's Algorithm

For any connected, undirected graph that contains an Euler path or Euler circuit, to construct such a traversal, proceed as follows:

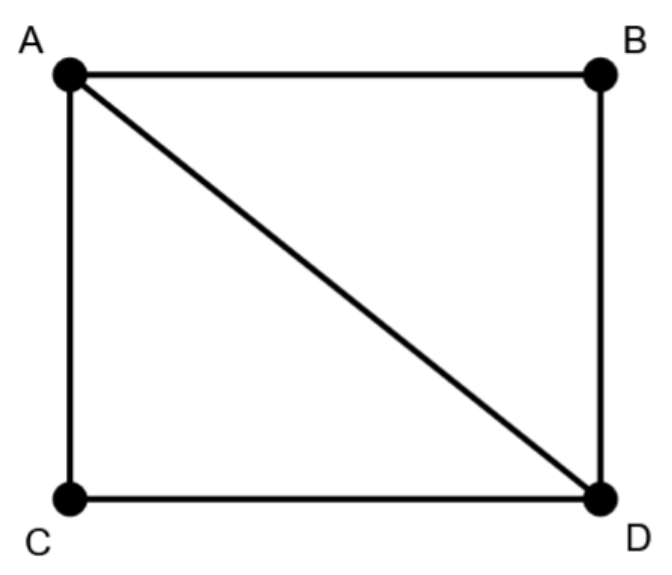
- **Step 1:** Choose a starting vertex.

- If the graph has an Euler circuit, start at any vertex.
- If the graph has an Euler path but not a circuit, start at one of the two vertices of odd degree.

- **Step 2:** At each step, from the current vertex:
  - Select an incident edge that has not yet been used.
  - Avoid choosing a bridge, unless there is no alternative.
- **Step 3:** Traverse the selected edge and move to the adjacent vertex. (Make sure to not use that particular edge again)
- **Step 4:** Repeat this process until **all** edges have been traversed.

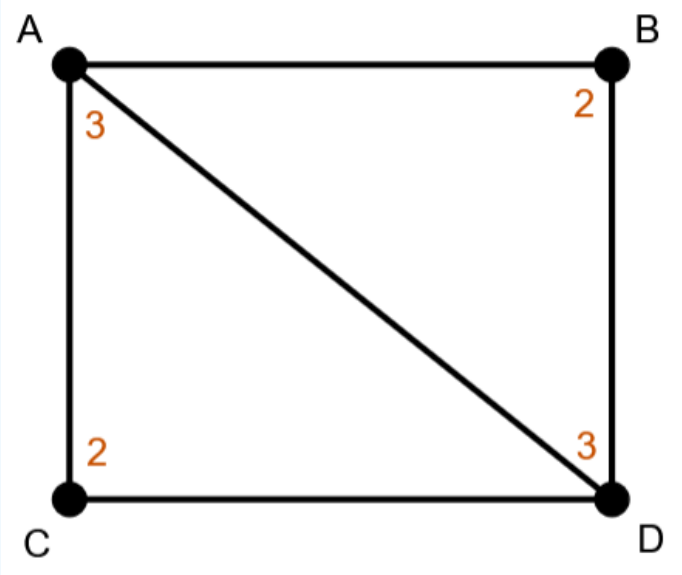
Example #8.2.5 🤔

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



✅ **Solution:**

Start by identifying the degree of each vertex and labeling each vertex with it's degree.



Since exactly two vertices of the graph have odd degree (A and D), from Euler's Theorem, the graph has an Euler path, but not an Euler circuit. To find at least one Euler path, we will use Fleury's Algorithm.

- **Step 1:** Choose a starting vertex.
  - If the graph has an Euler circuit, start at any vertex.
  - If the graph has an Euler path but not a circuit, start at one of the two vertices of odd degree.

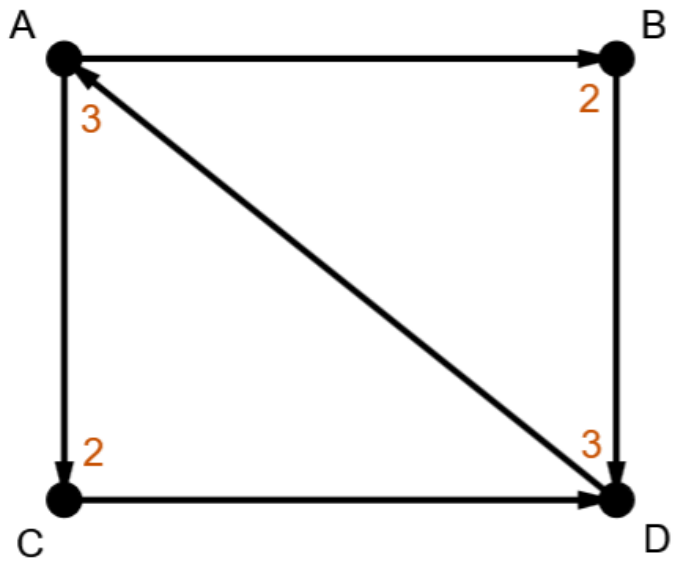
Since the graph has an Euler path but not a circuit, start at one of the two vertices of odd degree. Let's choose vertex A.

- **Step 2:** At each step, from the current vertex:
  - Select an incident edge that has not yet been used.
  - Avoid choosing a bridge, unless there is no alternative.

There are no bridges here, so it does not matter which edge to choose from next. Let's choose edge A-B.

- **Step 3:** Traverse the selected edge and move to the adjacent vertex. (Make sure to not use that particular edge again).
- **Step 4:** Repeat this process until **all** edges have been traversed.

From edge A-B, the next edge is forced to be B-D. From here, there are two choices for edges, so choose either edge. Let's choose D-A. The next two edges are forced to be A-C, then C-D. See figure below with the direction arrows.



Thus, we have an Euler path with the direction A-B-D-A-C-D.

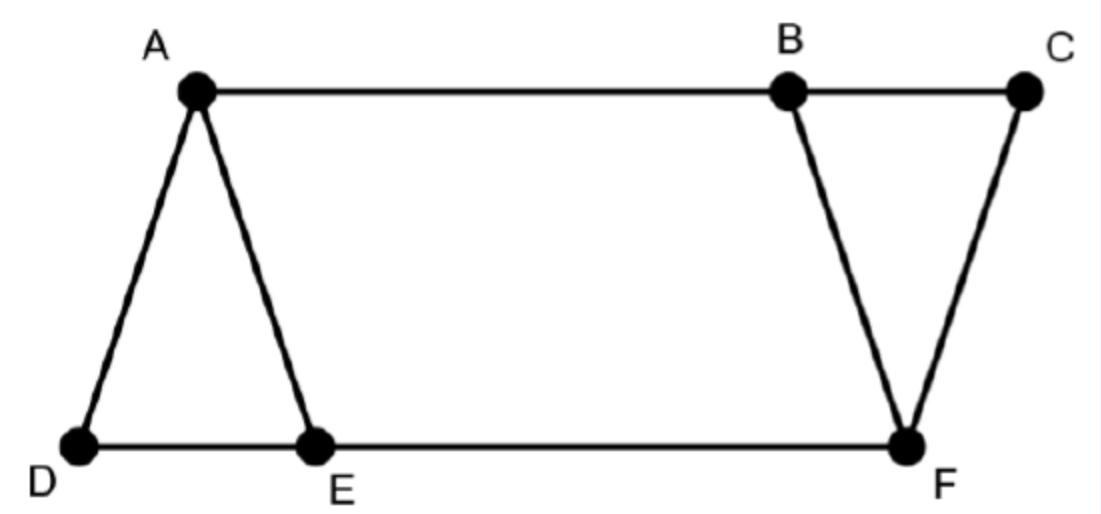
Note: There are many possible solutions, depending on the choice of starting vertex and the order in which edges are selected. However, our goal is not to enumerate all Euler paths or Euler circuits, but simply to determine whether one exists and to construct at least one example. Having said that here are actually 12 possible solutions. Here they are:

A-B-D-A-C-D, A-B-D-C-A-D, A-C-D-A-B-D, A-C-D-B-A-D, A-D-B-A-C-D, A-D-C-A-B-D,

D-C-A-D-B-A, D-A-C-D-B-A, D-B-A-D-C-A, D-A-B-D-C-A, D-C-A-B-D-A, D-B-A-C-D-A

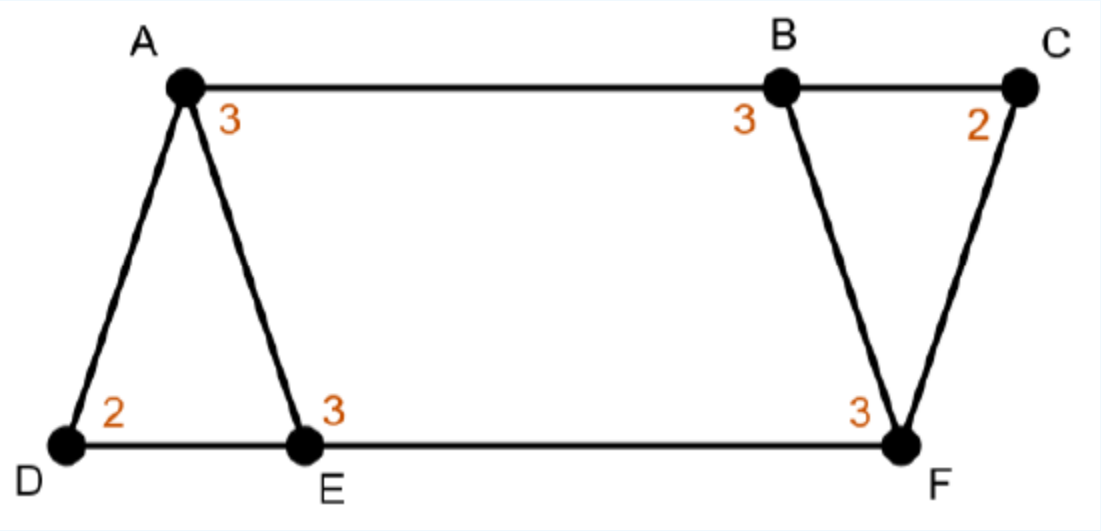
Example #8.2.6 🤔

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



✅ **Solution:**

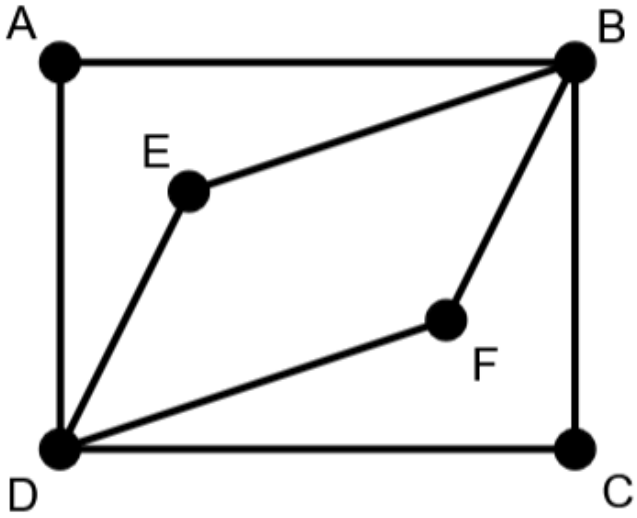
Start by identifying the degree of each vertex and labeling each vertex with it's degree.



Since the graph has more than two vertices of odd degree (A, B, E, and F), from Euler's Theorem, the graph has neither an Euler path nor an Euler circuit.

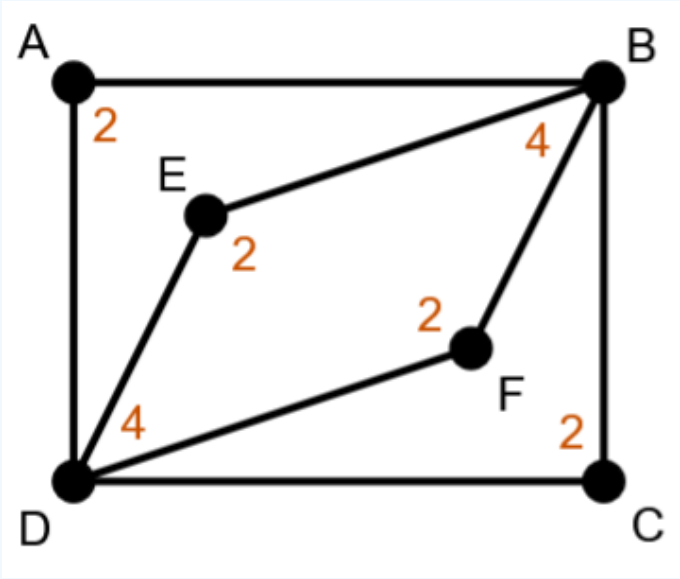
Example #8.2.7 🤔

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



✅ **Solution:**

Start by identifying the degree of each vertex and labeling each vertex with it's degree.



Since all vertices of the graph have even degree, from Euler's Theorem, the graph has an Euler circuit. To find at least one Euler circuit, we will use Fleury's Algorithm.

- **Step 1:** Choose a starting vertex.
  - If the graph has an Euler circuit, start at any vertex.
  - If the graph has an Euler path but not a circuit, start at one of the two vertices of odd degree.

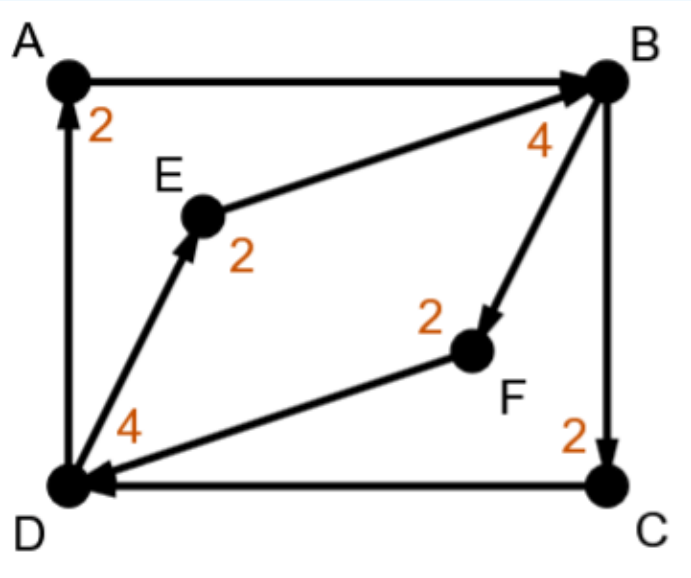
Since the graph has an Euler circuit, start at any vertex. However, it is a good strategy to go around the perimeter of the graph when starting with a vertex that has degree 4, if possible. Let's choose vertex B.

- **Step 2:** At each step, from the current vertex:
  - Select an incident edge that has not yet been used.
  - Avoid choosing a bridge, unless there is no alternative.

There are no bridges here, so it does not matter which edge to choose from next. But we will choose edge B-C and go clockwise around the graph.

- **Step 3:** Traverse the selected edge and move to the adjacent vertex. (Make sure to not use that particular edge again).
- **Step 4:** Repeat this process until **all** edges have been traversed.

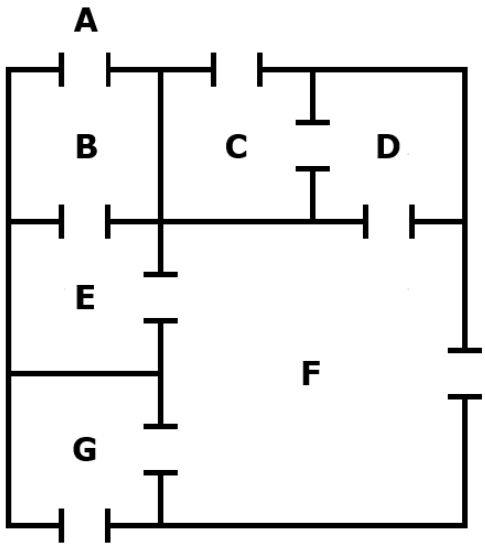
From edge B-C, the next edge is forced to be C-D. From here, there are three choices for edges, so just choose the edge going around the perimeter. Let's choose D-A. The next edge is forced to be A-B. Now, we will go around the parallelogram that was inscribed in the rectangle. Choose edge B-F, then F-D, then D-E, and finally E-B. See figure below with the direction arrows.



Thus, we have an Euler circuit with the direction B-C-D-A-B-F-D-E-B.

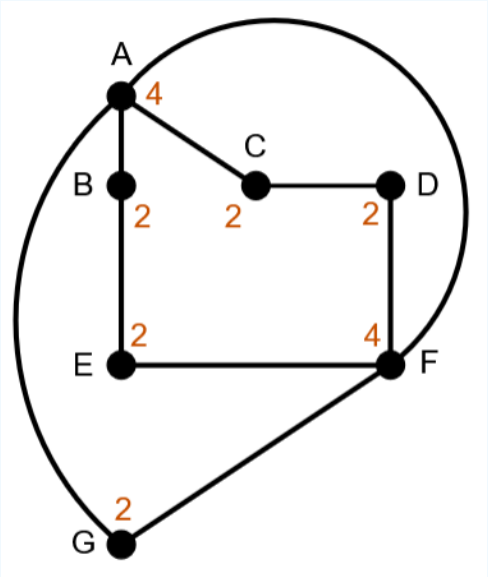
Example #8.2.8 🤔

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



✅ **Solution:**

Start by drawing a graph for the floor plan identifying the degree of each vertex and labeling each vertex with it's degree.



Since all vertices of the graph have even degree, from Euler's Theorem, the graph has an Euler circuit. To find at least one Euler circuit, we will use Fleury's Algorithm.

- **Step 1:** Choose a starting vertex.
  - If the graph has an Euler circuit, start at any vertex.
  - If the graph has an Euler path but not a circuit, start at one of the two vertices of odd degree.

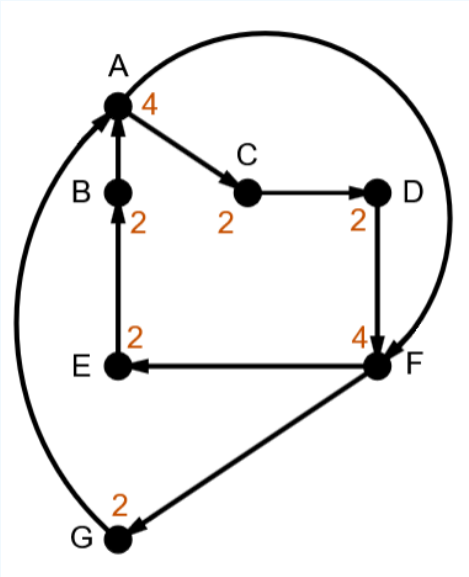
Since the graph has an Euler circuit, start at any vertex. However, it is a good strategy to go around the perimeter of the graph when starting with a vertex that has degree 4, if possible. Let's choose vertex A.

- **Step 2:** At each step, from the current vertex:
  - Select an incident edge that has not yet been used.
  - Avoid choosing a bridge, unless there is no alternative.

There are no bridges here, so it does not matter which edge to choose from next. But we will choose edge A-F and go clockwise around the graph.

- **Step 3:** Traverse the selected edge and move to the adjacent vertex. (Make sure to not use that particular edge again).
- **Step 4:** Repeat this process until **all** edges have been traversed.

From edge A-F, there are three choices for edges, so choose the edge going around the perimeter. Let's choose F-G. The next edge is forced to be G-A. Now, we will go around the pentagon (5-sided figure) inside. Choose edge A-C, then C-D, then D-F, then F-E, then E-B, and finally B-A. See figure below with the direction arrows.



Thus, we have an Euler circuit with the direction A-F-G-A-C-D-F-E-B-A.

Historical Note: Leonard Euler

Leonhard Euler (1707–1783) was one of the most prolific and influential mathematicians in history. Born in Basel, Switzerland, in 1707, Euler made fundamental contributions to nearly every branch of mathematics, as well as to physics, engineering, and astronomy. His work laid much of the foundation for modern mathematics.

Euler is perhaps best known for introducing and standardizing much of the notation that is still used today. Among his many contributions are the symbols e for the base of the natural logarithm, i for the imaginary unit, and the function notation $f(x)$. He also made significant advances in calculus, differential equations, number theory, mechanics, and mathematical analysis.

In 1736, Euler solved the Seven Bridges of Königsberg problem (as we learned earlier), a question that at first glance appeared to be a recreational puzzle. Euler's solution was revolutionary because it ignored the distances, shapes, and physical layout of the city, focusing instead on the pattern of connections between regions. This shift from geometric reasoning to abstract structure marked the beginning of graph theory and, more broadly, discrete mathematics.



Euler's approach demonstrated that certain problems can be solved purely through logical structure rather than measurement or visualization. His reasoning led to the concepts now known as Euler paths and Euler circuits, and to the degree-based criteria summarized in Euler's Theorem.

Despite losing sight in one eye early in adulthood and becoming almost completely blind later in life, Euler continued to work and publish extensively until his death in 1783. His productivity was extraordinary; it is estimated that his collected works fill dozens of volumes.

Today, Euler is remembered not only for the quantity of his contributions but also for their depth and lasting influence. His work on the Königsberg bridges remains one of the clearest examples of how abstraction and careful reasoning can transform a simple question into a new field of mathematics.

Historical Note: Michel Fleury

Fleury's Algorithm is named after the French mathematician Michel Fleury, who described the procedure in the late nineteenth century as a systematic way to construct Euler paths and Euler circuits in graphs that satisfy Euler's degree conditions.

Unlike Euler's original work, which focused on determining whether such traversals are possible, Fleury's contribution addressed a different question: how to explicitly find an Euler path or circuit once its existence is guaranteed. His algorithm reflects an early attempt to convert Euler's structural insight into a step-by-step method.

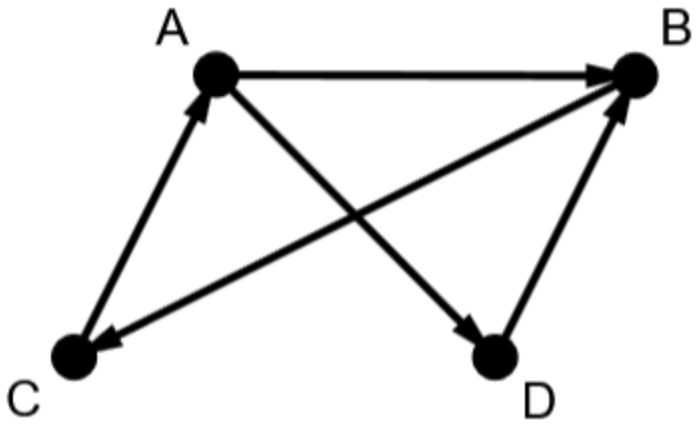
The key idea underlying Fleury's Algorithm is the careful treatment of bridges (edges) whose removal would disconnect the graph. Fleury observed that removing a bridge too early could prevent traversal of remaining edges. As a result, his algorithm prescribes avoiding bridges whenever possible, postponing them until no alternative edges remain.

Historically, Fleury's Algorithm occupies an important place in the development of graph theory because it illustrates a shift from purely existential proofs to constructive procedures. It highlighted the role of connectivity and bridges in edge-traversal problems and helped clarify the relationship between global graph structure and local decision-making.

Although Fleury's Algorithm is conceptually simple and well suited for classroom demonstrations, it is computationally inefficient for large graphs, since determining whether an edge is a bridge may require repeated connectivity checks. For this reason, it has largely been supplanted in practice by more efficient algorithms, such as Hierholzer's Algorithm. Nevertheless, Fleury's Algorithm remains an important educational tool and a natural complement to Euler's Theorem.

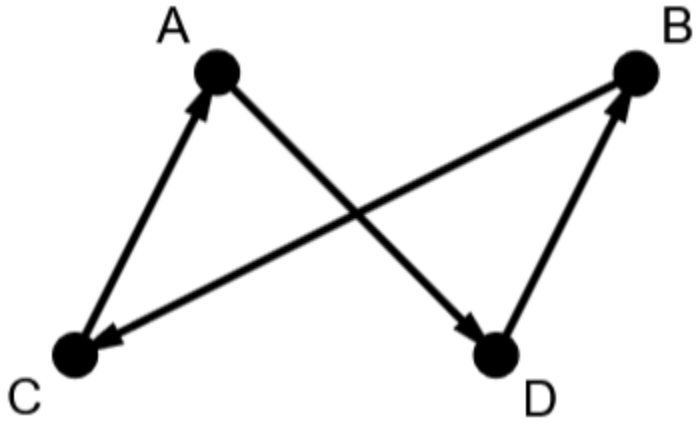
Section 8.2: Euler Paths and Euler Circuits [*In-Class Exercises*]

1. Classify the path shown in the graph as an Euler path, Euler circuit, or neither.



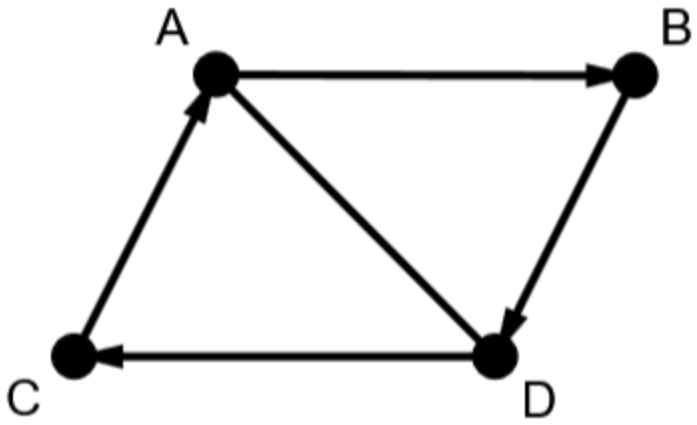
The path is A-B-C-A-D-B.

2. Classify the path shown in the graph as an Euler path, Euler circuit, or neither.

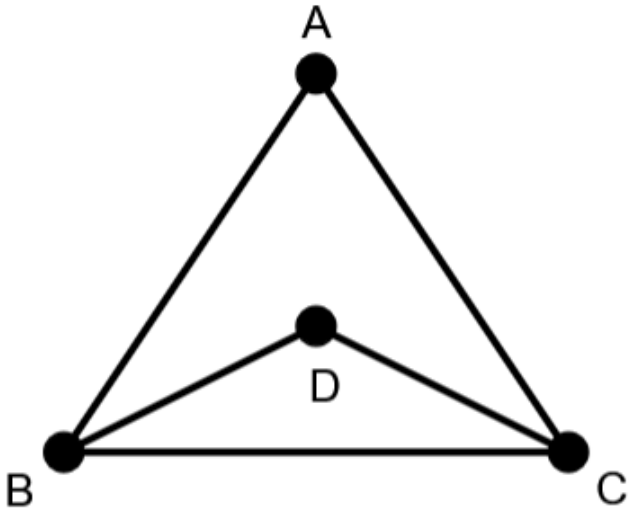


The path is A-D-B-C-A.

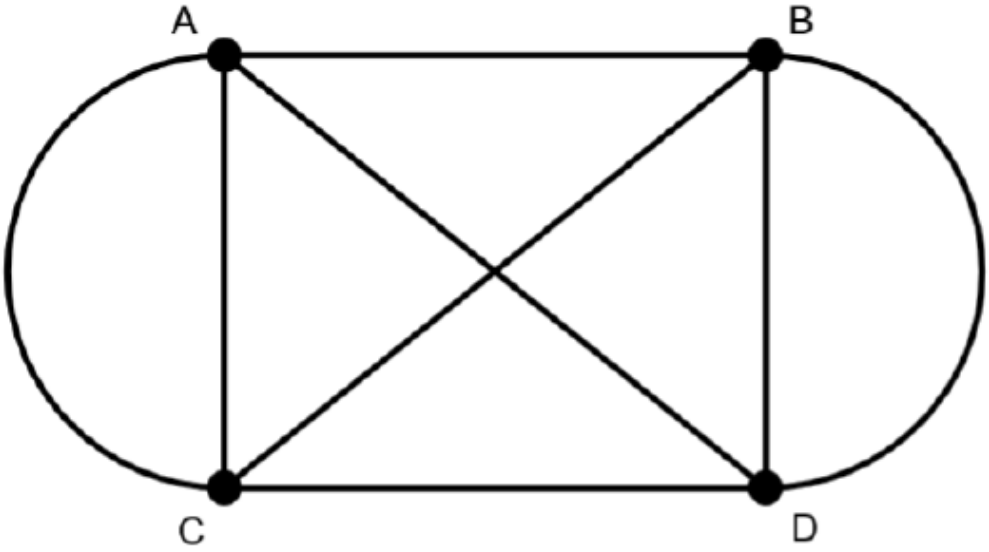
3. Classify the path shown in the graph as an Euler path, Euler circuit, or neither.



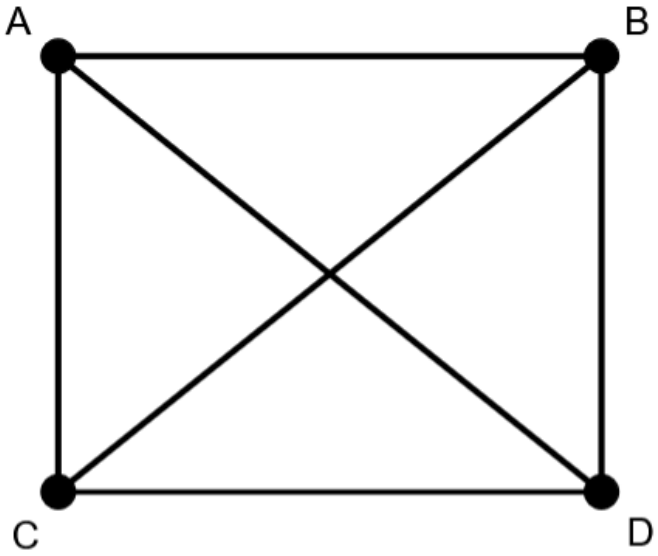
4. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



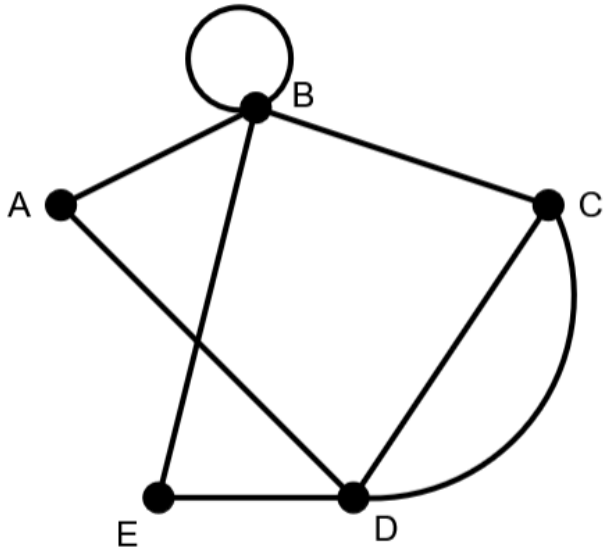
5. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



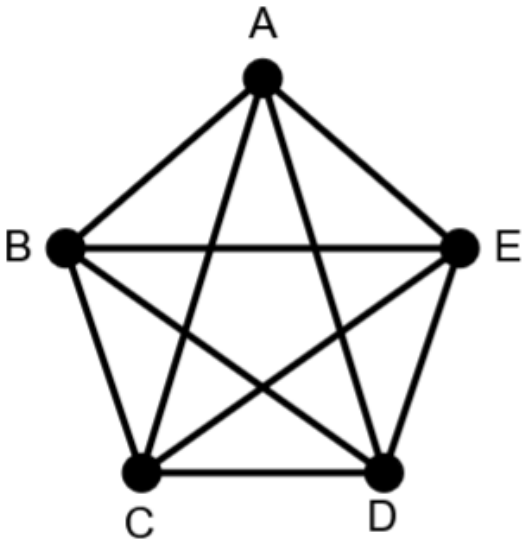
6. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



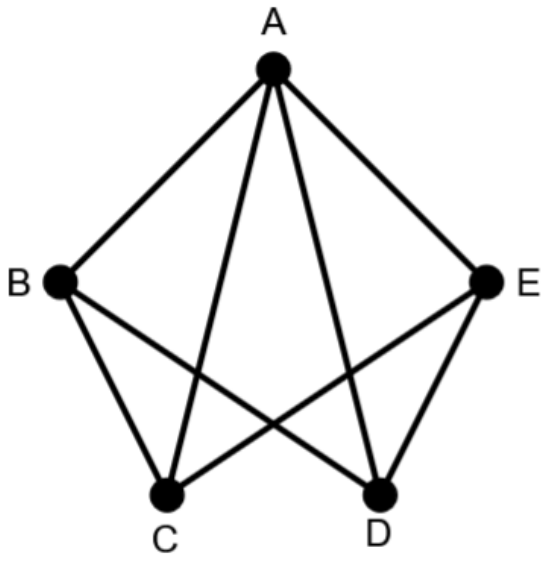
7. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



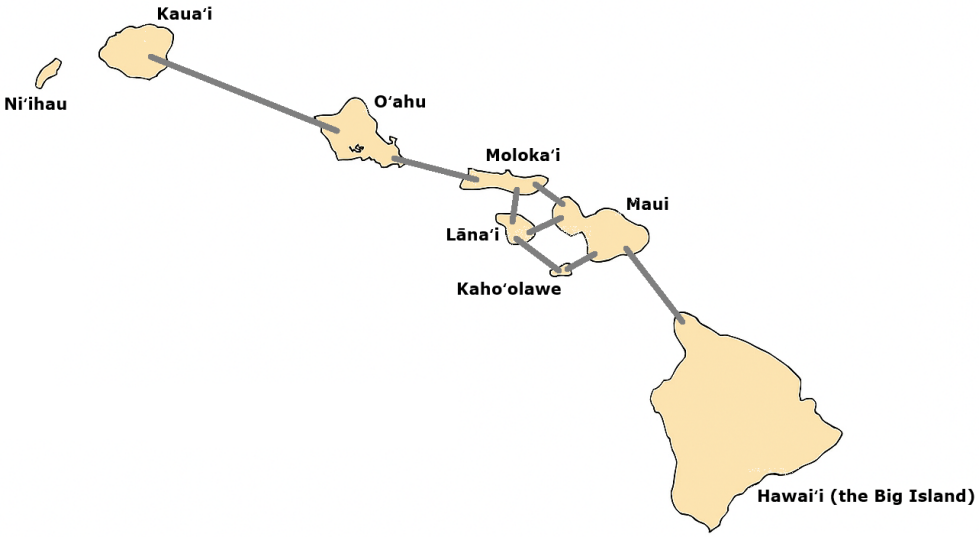
8. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



9. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



10. The map of the Hawaiian Islands shows the ferry/boat routes between the islands. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



Answers available at the end of this section.

Answers

Section 8.2: Euler Paths and Euler Circuits [*In-Class Exercises*]

1. Euler Path
2. Euler Circuit
3. Neither
4. Has an Euler path; Answers will vary (one possible Euler path: C-B-A-C-D-B)
5. Has an Euler circuit; Answers will vary (one possible Euler circuit: A-B-D-C-A-D-B-C-A)
6. Neither
7. Euler Path; Answers will vary (one possible Euler path: C-D-E-B-B-A-D-C-B) (Recall: One loop = One edge of degree 2)
8. Euler Circuit; Answers will vary (one possible Euler circuit: C-B-A-E-D-C-A-D-B-E-C)
9. Neither
10. Neither

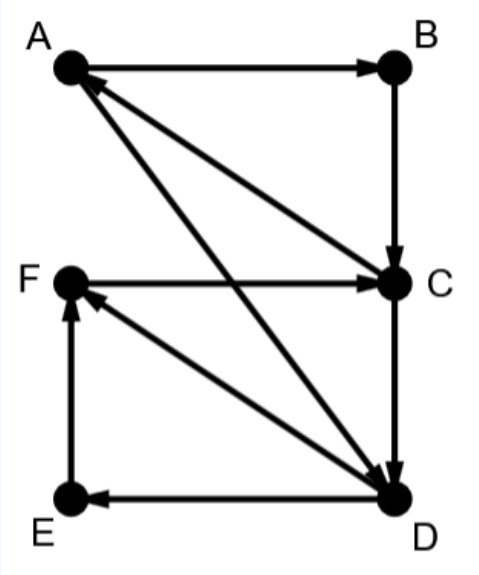
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Section 8.2.H: Homework

Section 8.2: Euler Paths and Euler Circuits - Homework Exercises

(Section 8.2) - Homework Exercise #1

Classify the following path shown in the graph as an Euler path, Euler circuit, or neither.



Vertex A points to vertex B; vertex B points to vertex C; vertex C points to vertex D; vertex D points to vertex E; vertex E points to vertex F; vertex F points to vertex C; vertex C points to vertex A; vertex A points to vertex D; and vertex D points to vertex F.

The path is A-B-C-D-E-F-C-A-D-F.

Answer available at the end of this section.

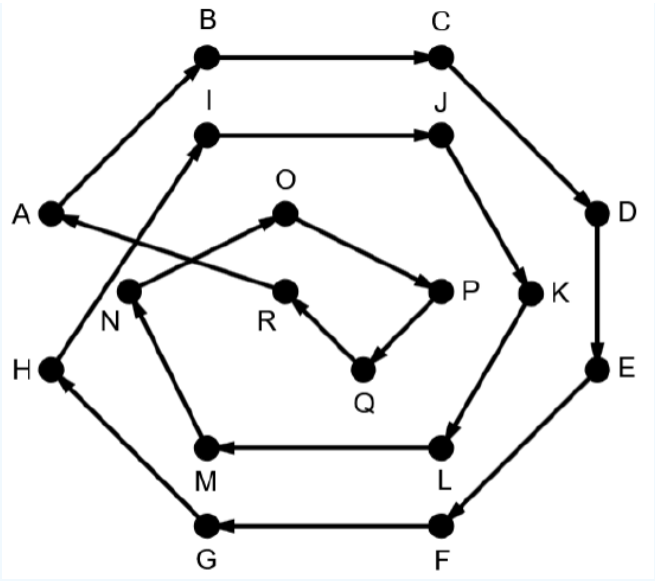
(Section 8.2) - Homework Exercise #2

In Homework Exercise #1, if the path was A-B-C-D-E-F-C-A-D-F-C-A, is the path an Euler path, Euler circuit, or neither?

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #3

Classify the following path shown in the graph as an Euler path, Euler circuit, or neither.



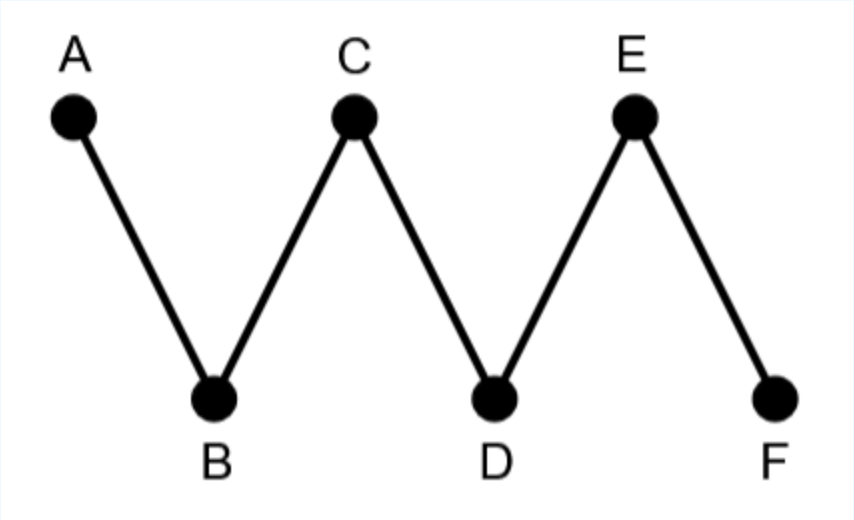
The image shows a graph with labeled nodes connected by line segments and directed line segments. A points to B; B points to C; C points to D; D points to E; E points to F; F points to G; G points to H; H points to I; I points to J; J points to K; K points to L; L points to M; M points to N; N points to O; O points to P; P points to Q; Q points to R; and R points to A.

The path is A-B-C-D-E-F-G-H-I-J-K-L-M-N-O-P-Q-R-A.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #4

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

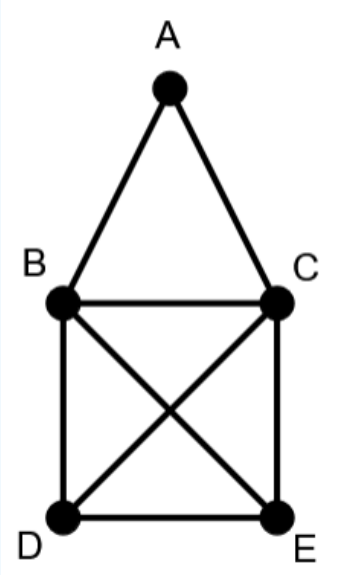


Vertex A connects to vertex B; vertex B connects to vertex C; vertex C connects to vertex D; vertex D connects to vertex E; and vertex E connects to vertex F.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #5

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

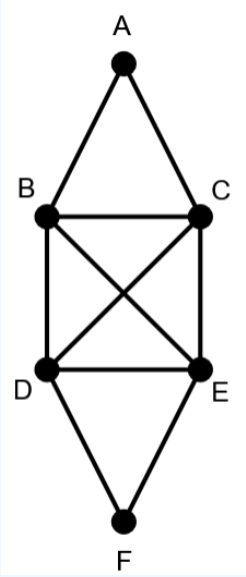


Vertex A connects to vertices B and C; vertex B connects to vertices C, D, and E; vertex C connects to vertices D and E; and vertex D connects to vertex E.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #6

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

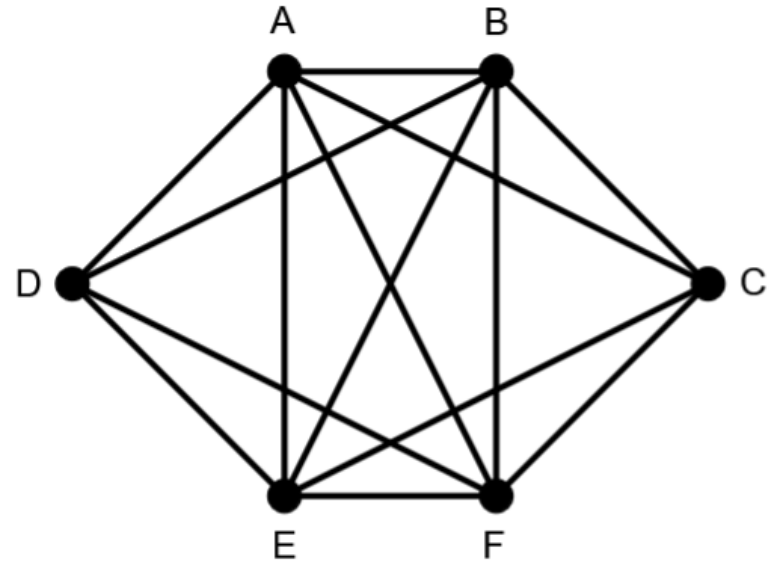


Vertex A connects to vertices B and C; vertex B connects to vertices C, D, and E; vertex C connects to vertices D and E; vertex D connects to vertices E and F; and vertex E connects to vertex F.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #7

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

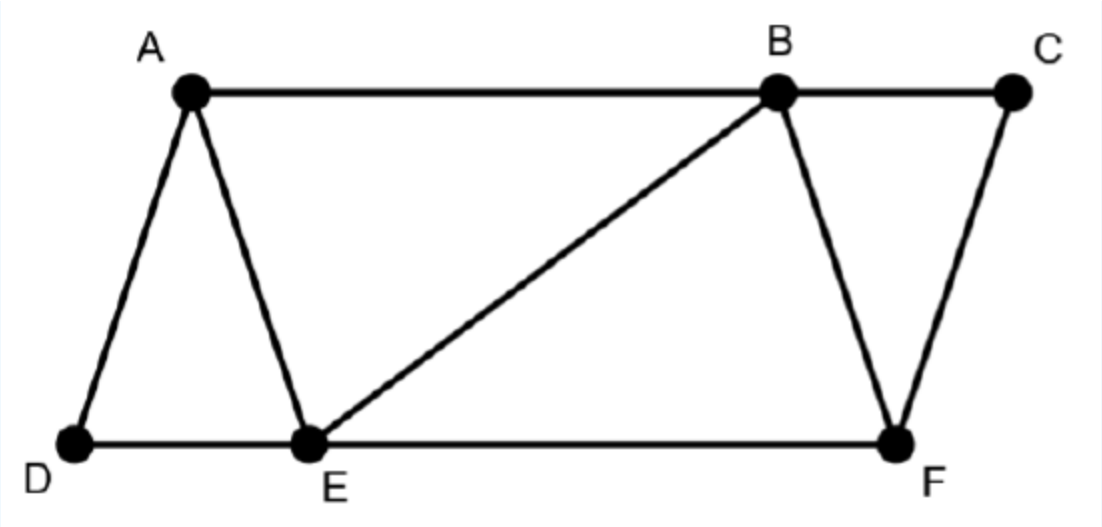


Vertex A connects to vertices B, C, D, E, and F; vertex B connects to vertices C, D, E, and F; vertex C connects to vertices E and F; vertex D connects to vertices E, and F; and vertex E connects to vertex F.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #8

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

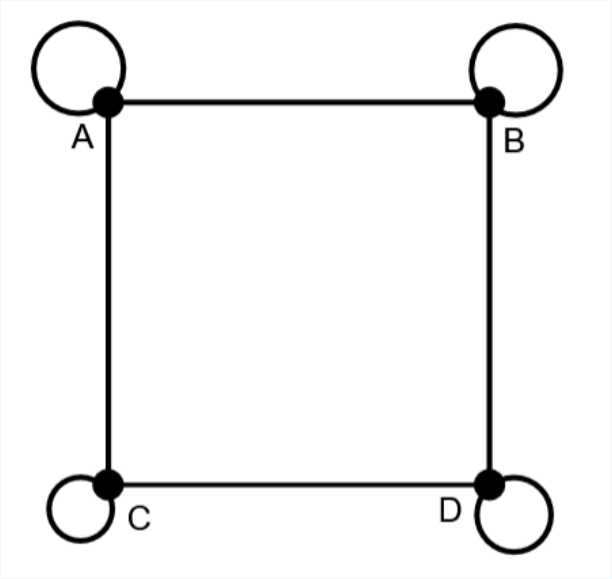


Vertex A connects to vertices B, D, and E; vertex B connects to vertices C, E, and F; vertex C connects to vertex F; vertex D connects to vertex E; and vertex E connects to vertex F.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #9

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

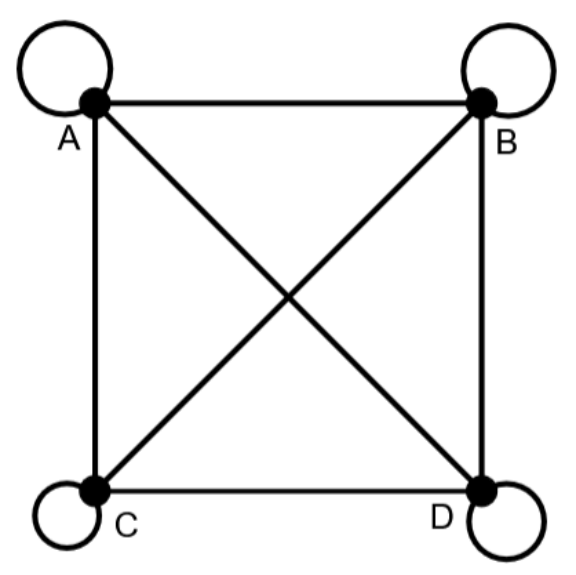


Vertex A connects to vertices B and C; and vertex D connects to vertices B and C. Vertices A, B, C, and D each also contain a loop.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #10

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

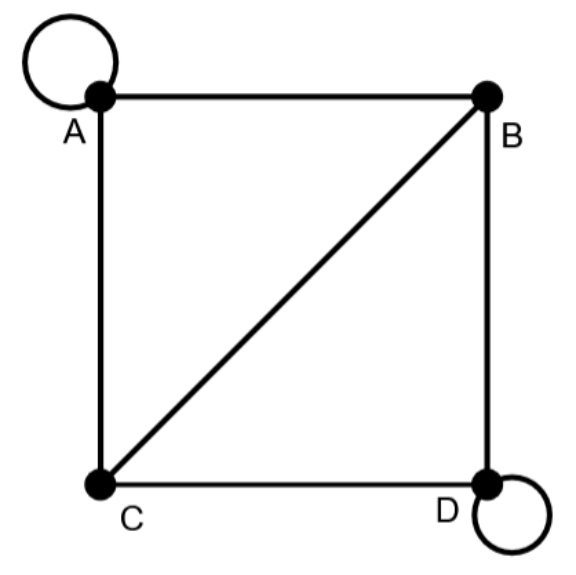


Vertex A connects to vertices B, C, and D; vertex B connects to vertices C and D; and vertex C connects to vertex D. Vertices A, B, C, and D each also contain a loop.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #11

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

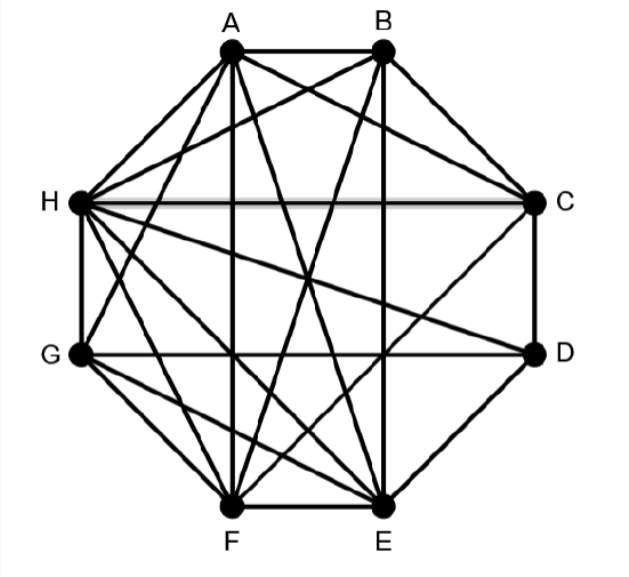


Vertex A connects to vertices B and C; vertex B connects to vertices C and D; and vertex C connects to vertex D. Vertices A and D each also contain a loop.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #12

Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



Vertex A connects to vertices B, C, E, F, G, and H; vertex B connects to vertices C, E, F, and H; vertex C connects to vertices D, F, and H; vertex D connects to vertices E, G, and H; vertex E connects to vertices F, G, and H; vertex F connects to vertices G and H, and G connects to vertex H.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #13

Draw a graph for the map below using each state as a vertex and each common border as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

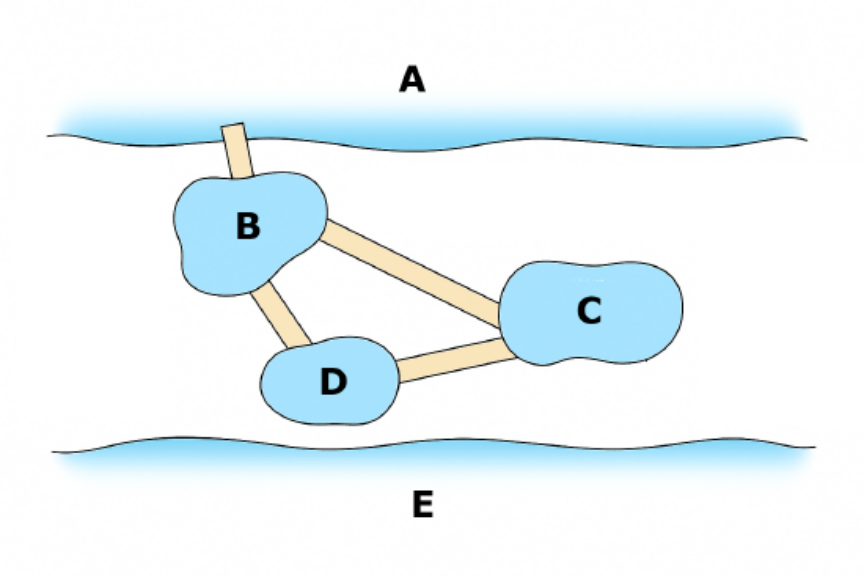


Washington borders Oregon and Idaho; Oregon borders Idaho, Nevada, and California; Idaho borders Nevada and Utah; Nevada borders Utah, Arizona, and California; and Arizona borders California and Utah.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #14

Draw a graph for the map of the islands below using each island or mainland as a vertex and each connecting bridge as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

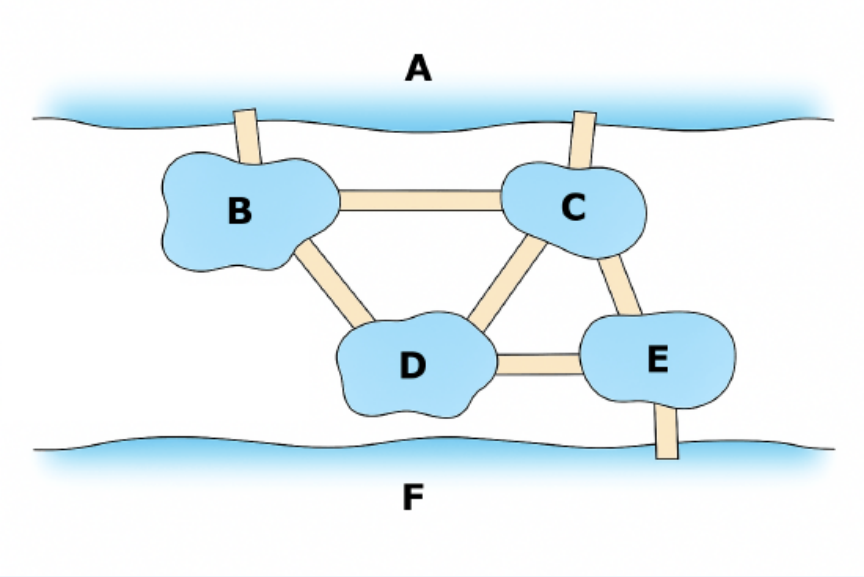


Island B connects to islands C and D and to mainland A; and island C connects to island D. There is no bridge to any island from mainland E.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #15

Draw a graph for the map of the islands below using each island or mainland as a vertex and each connecting bridge as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

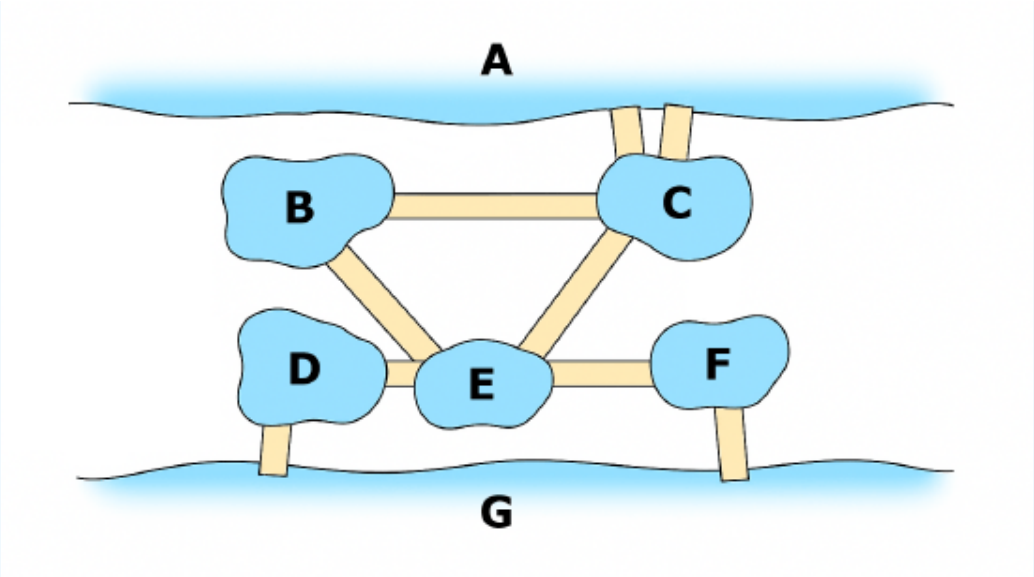


Island B connects to island C, island D, and mainland A; island C connects to islands D and E and to mainland A; island D connects to island E; and island E connects to mainland F.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #16

Draw a graph for the map of the islands below using each island or mainland as a vertex and each connecting bridge as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

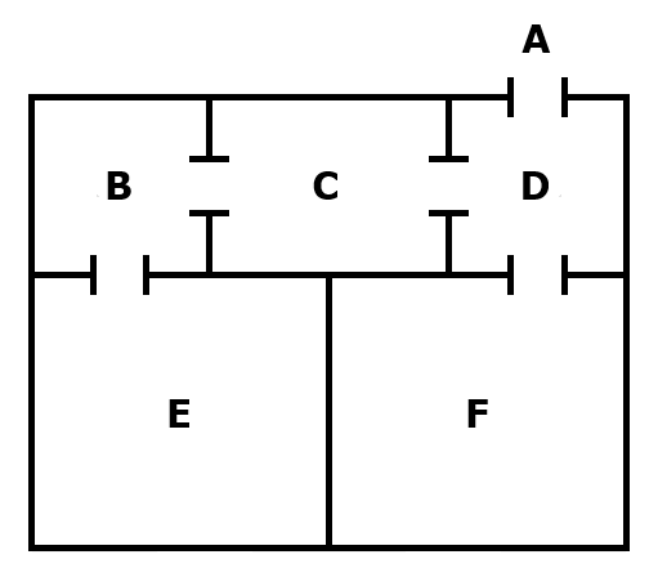


Island B connects to islands C and E; island C connects to island E and to mainland A by two bridges; island D connects to island E and to mainland G; island E connects to island F; and island F connects to mainland G.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #17

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

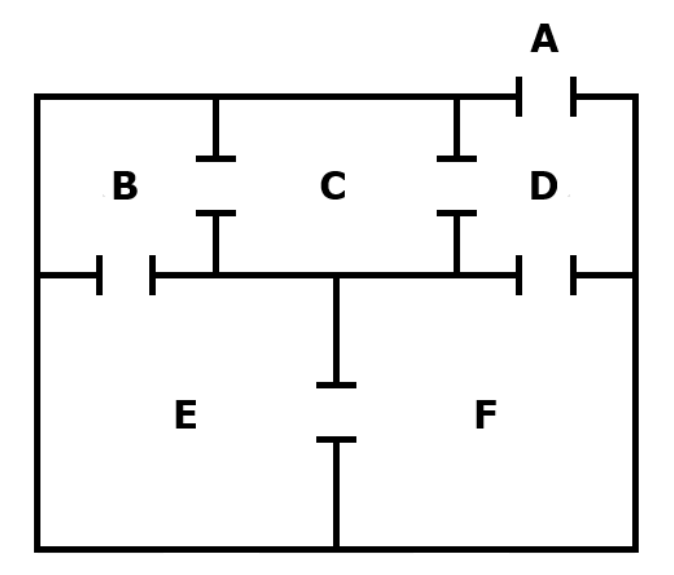


The image shows a floor plan divided into five labeled rooms connected by doorway openings. Room B connects with room C and room E; room C connects with room D; and room D connects with room F and outside area A.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #18

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

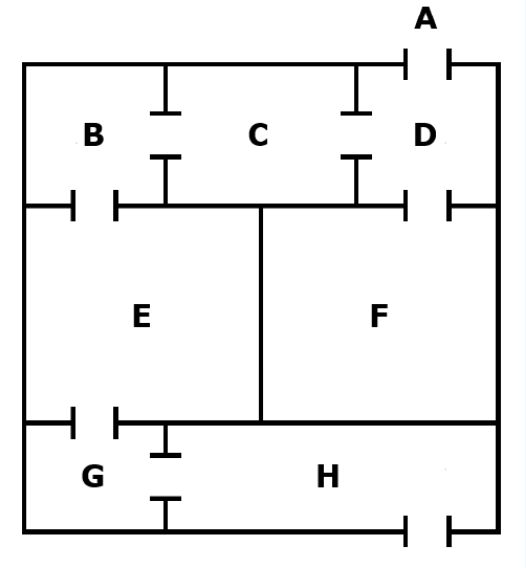


The image shows a floor plan divided into five labeled rooms connected by doorway openings. Room B connects with room C and room E; room C connects with room D; room E connects with room F; and room D connects with room F and outside area A.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #19

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

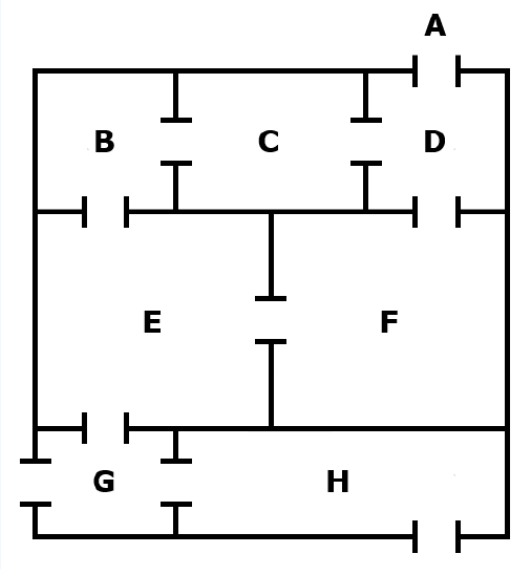


The image shows a floor plan divided into seven labeled rooms connected by doorway openings. Room B connects with room C and room E; room C connects with room D; room D connects with room F and outside area A; room E connects with room G; room G connects with room H; and room H connects with outside area A.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #20

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

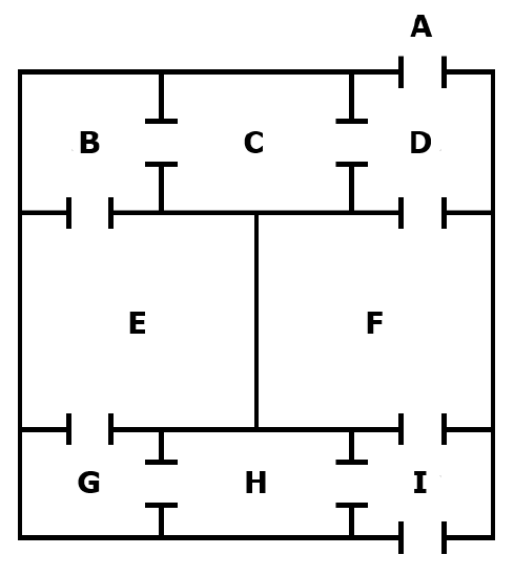


The image shows a floor plan divided into seven labeled rooms connected by doorway openings. Room B connects with room C and room E; room C connects with room D; D connects to room F, and outside area A; room E connects with room F, and with room G; room G connects with room H, and outside area A; and room H connects with outside area A.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #21

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

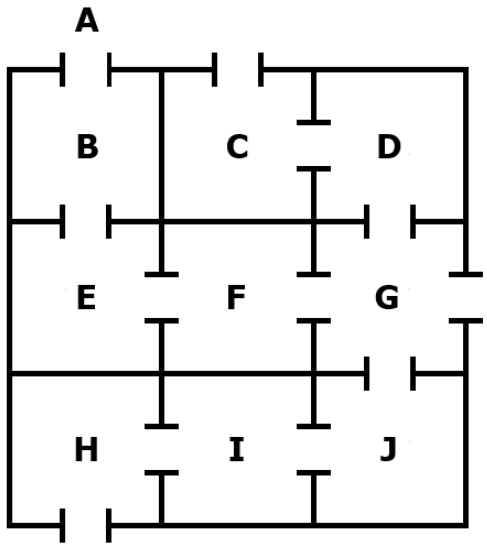


The image shows a floor plan divided into eight labeled rooms connected by doorway openings. Room B connects with room C and room E; room C connects with room D; room D connects with room F and outside area A; room E connects with room G; room F connects with room I; room G connects with room H; room H connects with room I; and room I connects with outside area A.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #22

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.

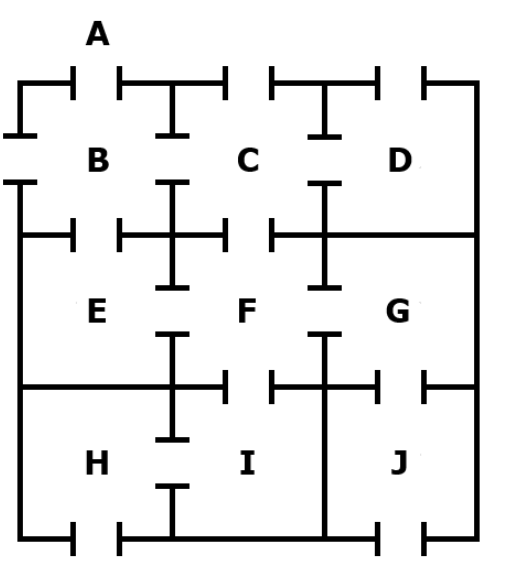


The image shows a floor plan divided into nine labeled rooms connected by doorway openings. Room B connects with outside area A, room C, and room E; room C connects with outside area A and room D; room D connects with room G; room E connects with room F; room F connects with room G; room G connects with room J and outside area A; room H connects with room I and outside area A; and room I connects with room J.

Answer available at the end of this section.

(Section 8.2) - Homework Exercise #23

Draw a graph for the floor plan using each room and the exterior area as a vertex and each door opening as an edge. Determine whether the graph has an Euler path, an Euler circuit, or neither. If the graph has an Euler path or an Euler circuit find one.



The image shows a floor plan divided into nine labeled rooms connected by doorway openings. Room B connects with outside area A, room C, and room E; room C connects with outside area A, room F, and room D; room D connects with outside area A; room E connects with room F; room F connects with room G and room I; room G connects with room J; room H connects with room I and outside area A; and room J connects with outside area A.

Answer available at the end of this section.

Answers

(Section 8.2) - Homework Exercise #1

Euler path.

(Section 8.2) - Homework Exercise #2

Neither; uses edges F-C and C-A more than once.

(Section 8.2) - Homework Exercise #3

Euler circuit.

(Section 8.2) - Homework Exercise #4

Euler path. There are exactly 2 vertices of odd degree, A and F. Possible Euler paths A-B-C-D-E-F or F-E-D-C-B-A.

(Section 8.2) - Homework Exercise #5

Has an Euler path. There are exactly 2 vertices of odd degree, D and E. Answers will vary (one possible Euler path: D-B-A-C-E-B-C-D-E).

(Section 8.2) - Homework Exercise #6

Has an Euler circuit. All vertices are of even degree, A and F. Answers will vary (one possible Euler circuit: A-B-D-F-E-C-B-E-D-C-A).

(Section 8.2) - Homework Exercise #7

Neither. There are more than 2 vertices of odd degree, A, B, E and F.

(Section 8.2) - Homework Exercise #8

Has an Euler path. There are exactly 2 vertices of odd degree, A and F. Answers will vary (one possible Euler path: A-B-C-F-E-D-A-E-B-F).

(Section 8.2) - Homework Exercise #9

Has an Euler circuit. All vertices are of even degree, A, B, C and D. Answers will vary (one possible Euler circuit: A-A-B-B-D-D-C-C-A).

(Section 8.2) - Homework Exercise #10

Neither. There are more than 2 vertices of odd degree, A, B, C and D.

(Section 8.2) - Homework Exercise #11

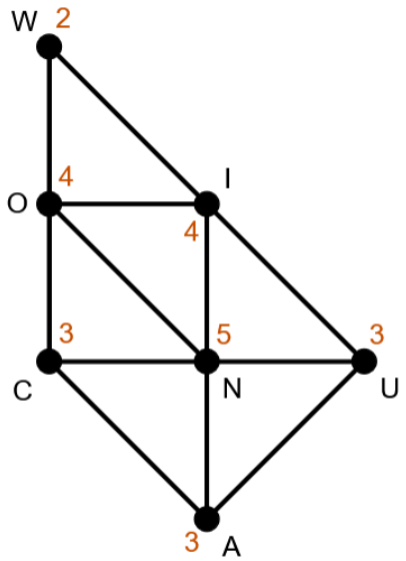
Has an Euler path. There are exactly 2 vertices of odd degree, B and C. Answers will vary (one possible Euler path: C-A-A-B-C-D-D-B).

(Section 8.2) - Homework Exercise #12

Neither. There are more than 2 vertices of odd degree, B, C, G and H.

(Section 8.2) - Homework Exercise #13

Neither; Graphs will vary (one possible graph below).

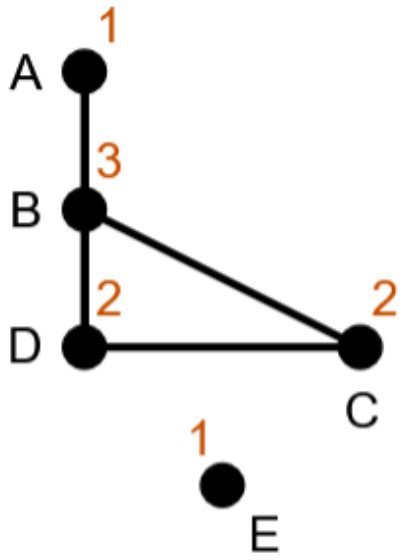


Vertex W connects to vertex O and vertex I; vertex O connects to vertex I, vertex C, and vertex N; vertex I connects to vertex N and vertex U; vertex C connects to vertex N and vertex A; vertex N connects to vertex U and vertex A; and vertex U connects to vertex A; vertex W has degree 2; vertices A, C, and U, have degree 3;

vertices I and O have degree 4, and vertex N has degree 5.

(Section 8.2) - Homework Exercise #14

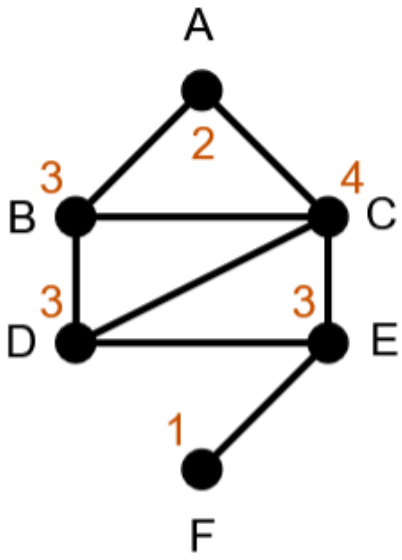
Has an Euler path; Answers will vary (one possible graph below with one possible Euler path: A-B-D-C-B).



Vertex A connects to vertex B; vertex B connects to vertices D and C; and vertex D connects to vertex C; vertex A has no edges connected. Vertices A and E have degree 1; vertices C and D have degree 2; and vertex B has degree 3.

(Section 8.2) - Homework Exercise #15

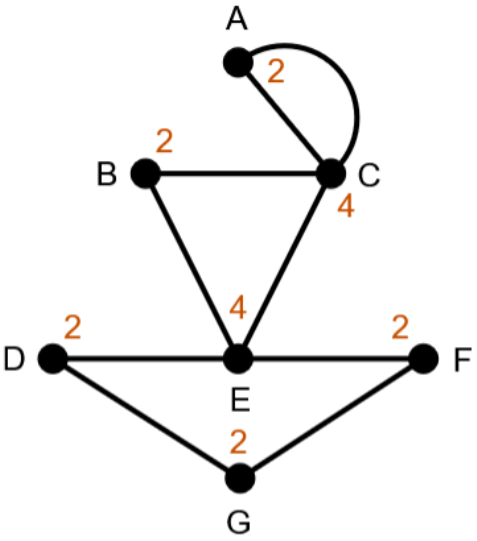
Neither; Graphs will vary (one possible graph below).



Vertex A connects to vertices B and C; vertex B connects to vertices C and D; vertex C connects to vertices D and E; vertex D connects to vertex E; and vertex E connects to vertex F. Vertex F has degree 1; vertex A has degree 2; vertices B, D, and E have degree 3; and vertex C has degree 4.

(Section 8.2) - Homework Exercise #16

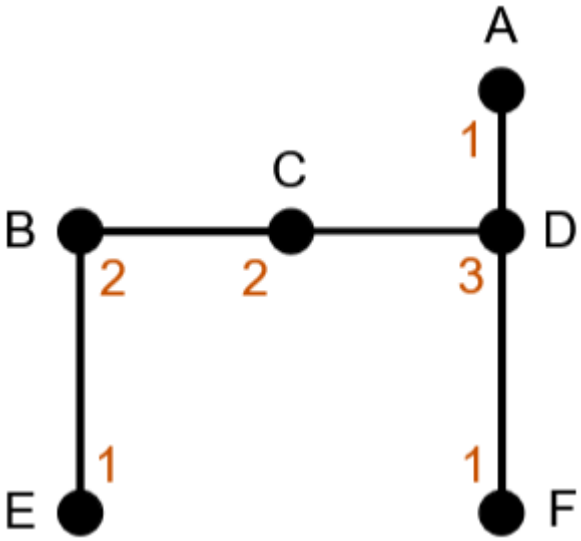
Has an Euler circuit; Answers will vary (one possible graph below with one possible Euler circuit: A-C-B-E-D-G-F-E-C-A).



Vertex A connects to vertex C, and vertex C again with a curved edge; vertex B connects to vertices C and E; vertex C connects to vertex E; vertex D connects to vertices E and G; vertex E connects to vertex F; and vertex F connects to vertex G. Vertices A, B, D, F, and G have degree 2; and vertices C and E have degree 4.

(Section 8.2) - Homework Exercise #17

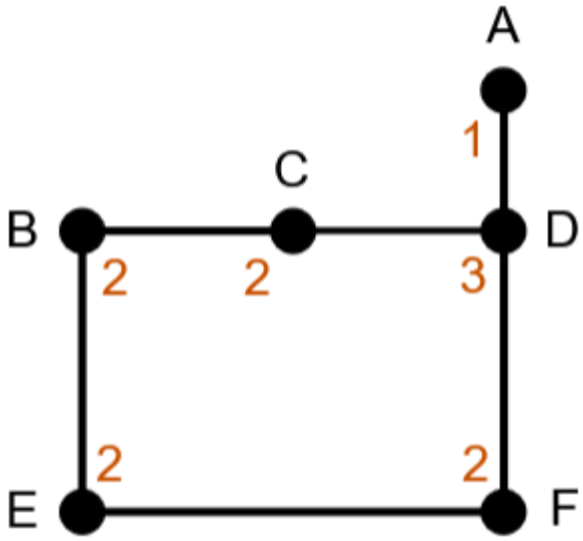
Neither; Graphs will vary (one possible graph below).



Vertex B connects to vertex C and vertex E; vertex C connects to vertex D; and vertex D connects to vertices A and F. Vertices A, E, and F have degree 1; vertices B and C have degree 2; and vertex D has degree 3.

(Section 8.2) - Homework Exercise #18

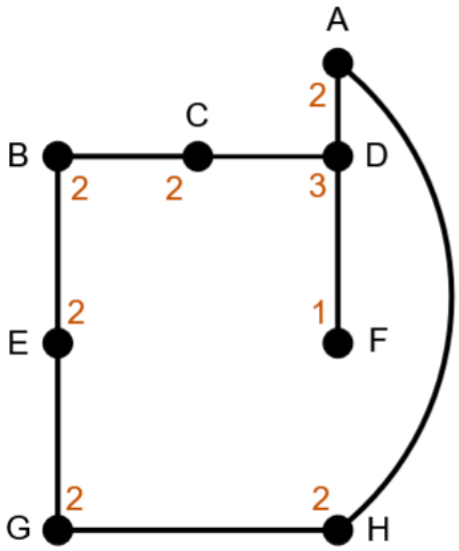
Has an Euler path; Answers will vary (one possible graph below with one possible Euler path: A-D-F-E-B-C-D).



Vertex B connects to vertex C and vertex E; vertex C connects to vertex D; vertex D connects to vertex A and vertex F; and vertex E connects to vertex F. Vertex A has degree 1; vertices B, C, E, and F, have degree 2; and vertex D has degree 3.

(Section 8.2) - Homework Exercise #19

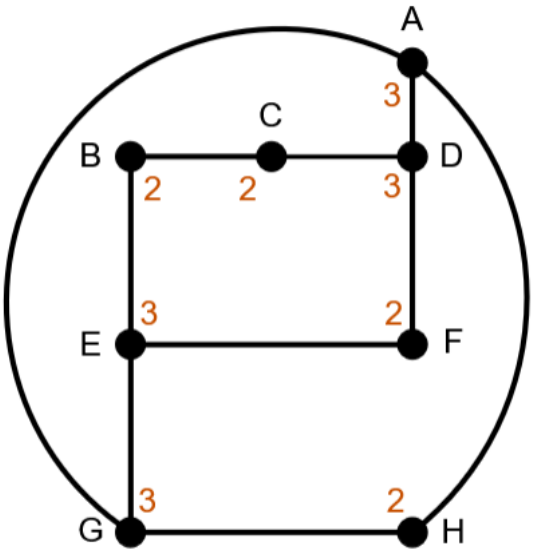
Has an Euler path; Answers will vary (one possible graph below with one possible Euler path: F-D-A-H-G-E-B-C-D).



Vertex B connects to vertex C and vertex E; vertex C connects to vertex D; vertex D connects to vertices A and F; vertex E connects to vertex G; vertex G connects to vertex H; and vertex H connects to vertex A. Vertex F has degree 1; vertices A, B, C, E, G, and H have degree 2; and vertex D has degree 3.

(Section 8.2) - Homework Exercise #20

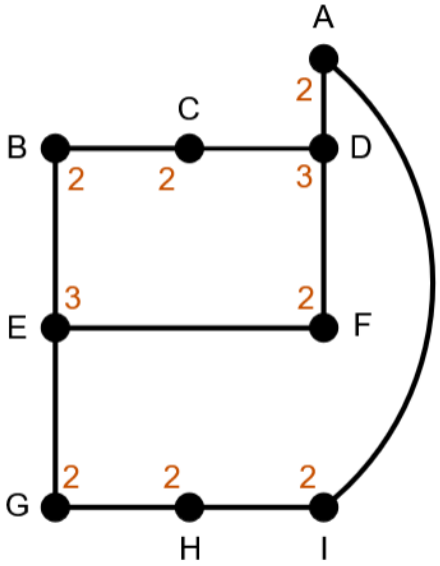
Neither; Graphs will vary (one possible graph below).



Vertex B connects to vertex C and vertex E; vertex C connects to vertex D; vertex D connects to vertices A and F; vertex E connects to vertex F and vertex G; vertex G connects to vertices A and H; and H connects to vertex A. Vertices B, C, F, and H have degree 2; and vertices A, D, and G have degree 3.

(Section 8.2) - Homework Exercise #21

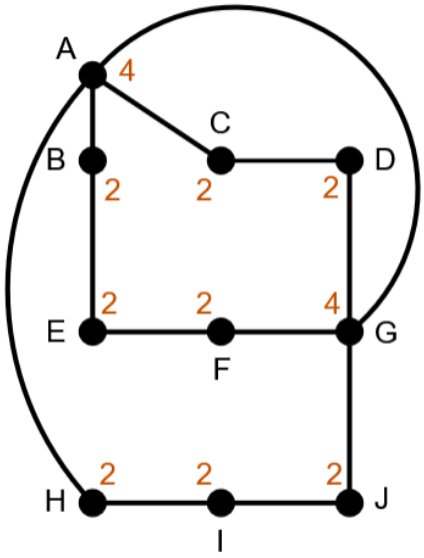
Has an Euler path; Answers will vary (one possible graph below with one possible Euler path: D-A-I-H-G-E-B-C-D-F-E).



Vertex B connects to vertex C and vertex E; vertex C connects to vertex D; vertex D connects to vertices A and F; vertex E connects to vertex F and vertex G; vertex G connects to vertex H; vertex H connects to vertex I; and vertex I connects to vertex A. Vertices A, B, C, F, G, H, and I have degree 2; and vertices D and E have degree 3.

(Section 8.2) - Homework Exercise #22

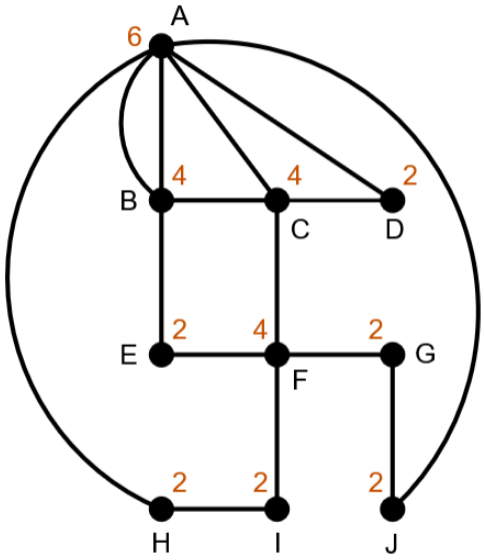
Has an Euler circuit; Answers will vary (one possible graph below with one possible Euler circuit: A-G-J-I-H-A-C-D-G-F-E-B-A).



Vertex A connects to vertices B, C, G, and H; vertex B connects to vertex E; vertex C connects to vertex D; vertex D connects to vertex G; vertex E connects to vertex F; vertex F connects to vertex G; vertex G connects to vertex J; vertex J connects to vertex I; and vertex I connects to vertex H. Vertices B, C, D, E, F, H, I, and J have degree 2; and vertices A and G have degree 4.

(Section 8.2) - Homework Exercise #23

Has an Euler circuit; Answers will vary (one possible graph below with one possible Euler circuit: A-H-I-F-E-B-A-B-C-A-D-C-F-G-J-A).



Vertex A connects to vertices B, C, D, H, J, and vertex A again with a curved edge; vertex B connects to vertices C and E; vertex C connects to vertex D and vertex F; vertex E connects to vertex F; vertex F connects to vertices G and I; vertex G connects to vertex J; and vertex I connects to vertex H. Vertices D, E, G, H, I, and J have degree 2; vertices B, C, and F have degree 4; and vertex A has degree 6.

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Section 8.3: Hamilton Paths and Hamilton Circuits

Learning Objectives

- Define a Hamilton path and a Hamilton circuit
- Define a complete graph

In the last section, we focused on whether it was possible to use every edge of a graph exactly once, regardless if we used a vertex more than once. Another important question in graph theory is whether it is possible to visit every vertex of a graph exactly once while following its edges. This idea leads to the concepts of **Hamilton paths** and **Hamilton circuits**, named after the mathematician William Rowan Hamilton.

Hamilton Path

A **Hamilton path** is a path in a graph that visits each vertex exactly once. The path does not need to end where it started; it simply must pass through every vertex without repetition.

- A graph that contains a Hamilton path is called a traceable graph.
- The focus is on visiting all vertices, not necessarily all edges.

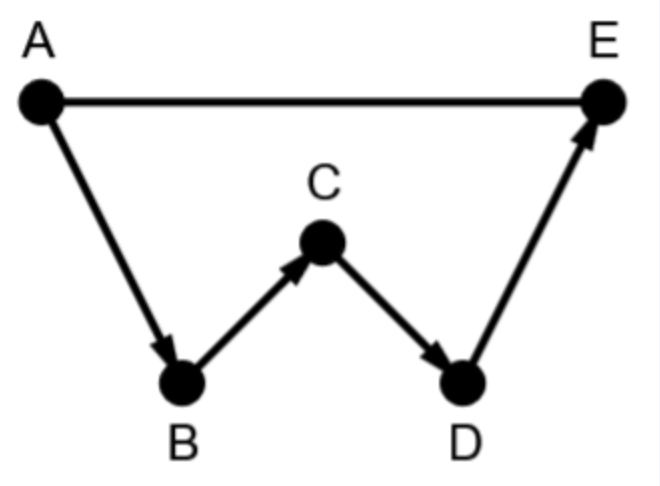
Hamilton Circuit

A **Hamilton circuit** (or Hamilton cycle) is a Hamilton path that starts and ends at the same vertex, forming a closed loop.

- A graph that contains a Hamilton circuit is called a Hamiltonian graph.
- Every vertex is visited exactly once, except for the starting vertex, which is also the ending vertex.

Example #8.3.1 🤔

Classify the path of A-B-C-D-E shown in the graph as a Hamilton Path, Hamilton circuit, or neither.



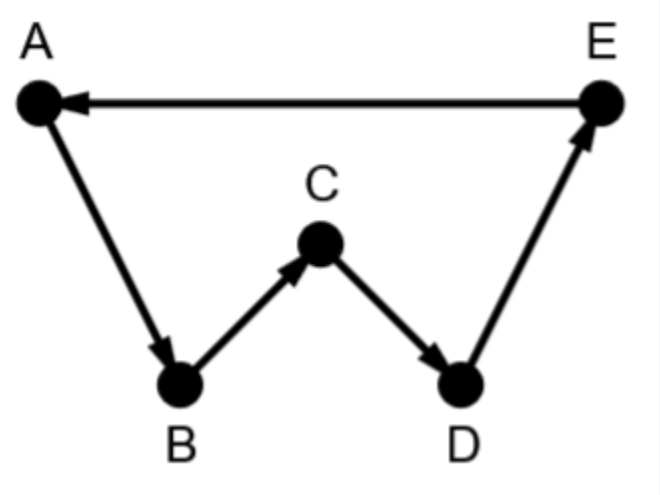
✅ **Solution:**

- Since the path A-B-C-D-E uses every vertex exactly once, it is a Hamilton path.
- It is not a Hamilton circuit, because it does not begin and end at vertex A.

Hamilton path

Example #8.3.2 🤔

Classify the path of A-B-C-D-E-A shown in the graph as a Hamilton Path, Hamilton circuit, or neither.



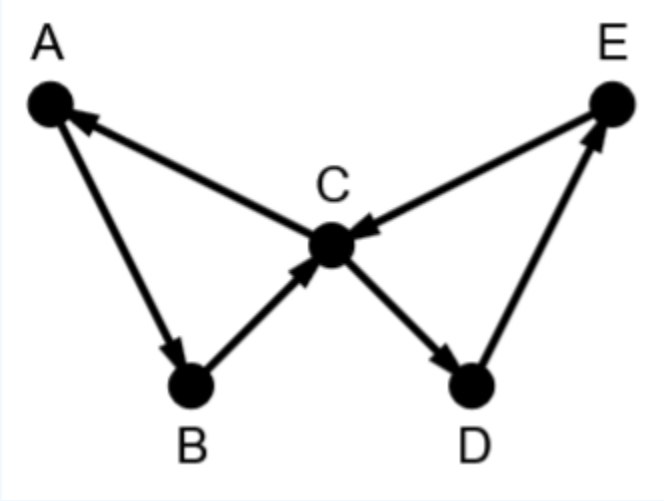
✅ **Solution:**

- Since the path A-B-C-D-E-A uses every vertex exactly once (except for the starting vertex), it is a Hamilton path.
- Also, since the path begins and ends at vertex A, it is also a Hamilton circuit. (Note: A Hamilton circuit means it is already a Hamilton path, so the best answer here is Hamilton Circuit).

Hamilton circuit.

Example #8.3.3 🤔

Classify the path of A-B-C-D-E-C-A shown in the graph as a Hamilton Path, Hamilton circuit, or neither.



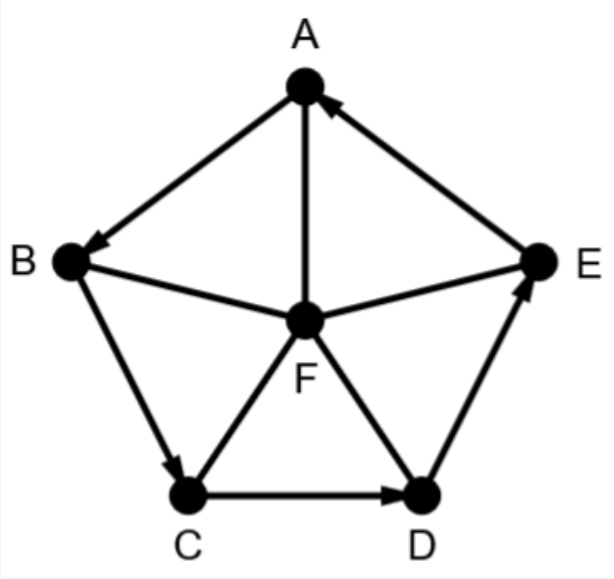
✅ **Solution:**

- Path A-B-C-D-E-C-A does not use every vertex exactly once (except for the starting vertex). Vertex C is used twice in the path. Thus, it is not a Hamilton path. (Note: It is an Euler path).
- Because it is not an Hamilton path, then it is not a Hamilton circuit, even though the path begins and ends at point A. (Note: It is an Euler circuit).

Neither

Example #8.3.4 🤔

Classify the path of A-B-C-D-E-A shown in the graph as a Hamilton Path, Hamilton circuit, or neither.



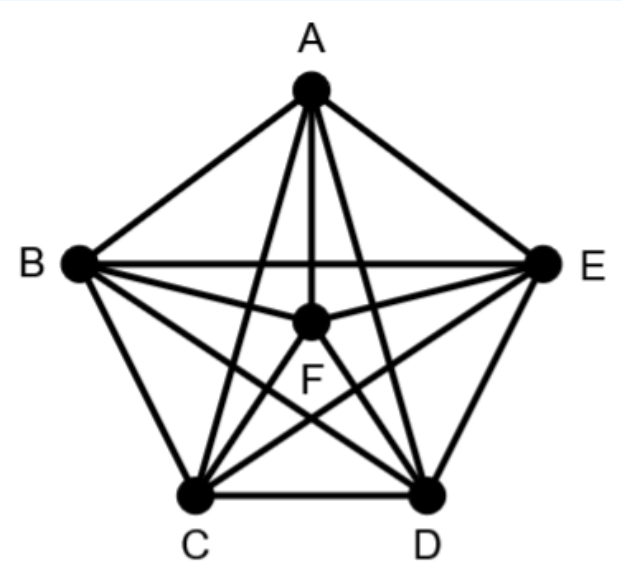
✅ **Solution:**

- Path A-B-C-D-E-A does not use every vertex exactly once (except for the starting vertex). Vertex F was not used in the path. Thus, it is not a Hamilton path.
- Because it is not a Hamilton path, then it is not a Hamilton circuit, even though the path begins and ends at point A. (Note: It is merely a circuit).

Neither

Example #8.3.5 🤔

Classify the path of A-B-C-D-E-F-A shown in the graph as a Hamilton Path, Hamilton circuit, or neither. (No directional arrows are given).



✅ **Solution:**

- Since the path A-B-C-D-E-F-A uses every vertex exactly once (except for the starting vertex), it is a Hamilton path.
- Also, since the path begins and ends at vertex A, it is also a Hamilton circuit. (Note: A Hamilton circuit means it is already a Hamilton path, so the best answer here is Hamilton Circuit).

Hamilton circuit.

Hamilton paths and circuits are concerned with vertices, whereas Euler paths and circuits are concerned with edges. Unlike the previous section involving Euler paths, there isn't a single theorem that allows us to tell if any graph does or does not have a Hamilton path. There is no simple rule (like degree conditions for Euler circuits) that guarantees whether a graph has a Hamilton circuit. However, there is one condition that guarantees the existence of Hamilton circuits; that is, if the graph is a **complete graph**.

Complete Graph

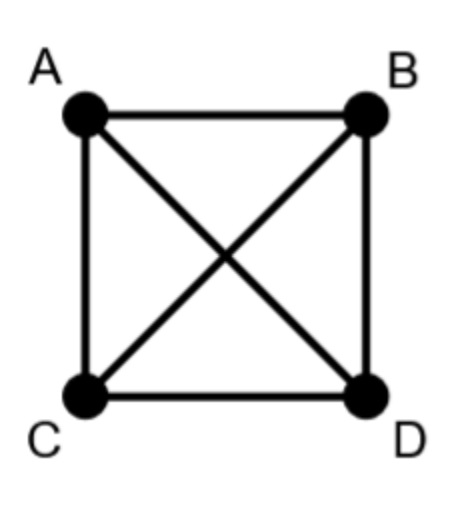
A **complete graph** is a graph in which every pair of distinct vertices is connected by an edge.

In other words, if you pick any two different vertices in the graph, there is always a direct connection between them.

If a graph has n vertices, then it is a complete graph if there are $n-1$ edges connecting to each and every vertex.

Example #8.3.6 🤔

Is the following graph a complete graph?

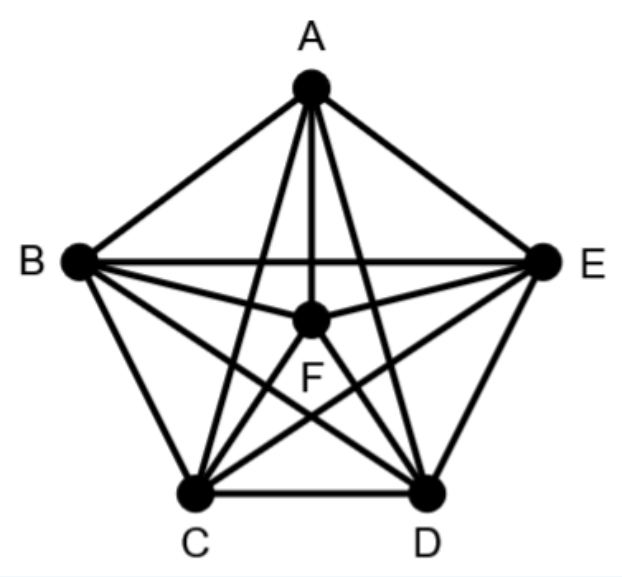


✅ **Solution:**

The graph is a complete graph, since each vertex is connected to each and every other vertex. Note there are 4 vertices, so there must be 3 edges connected to each and every vertex, which is indeed the case.

Example #8.3.7 🤔

Is the following graph a complete graph?

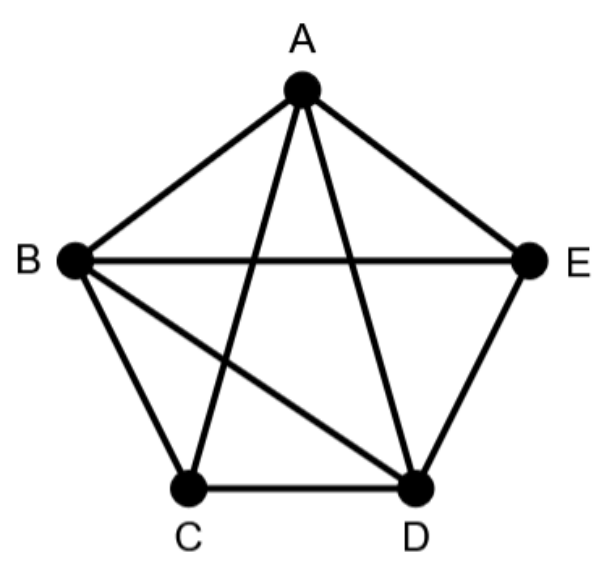


✅ **Solution:**

The graph is a complete graph, since each vertex is connected to each and every other vertex. Note there are 6 vertices, so there must be 5 edges connected to each and every vertex, which is indeed the case.

Example #8.3.8 🤔

Is the following graph a complete graph?



✅ **Solution:**

The graph is not a complete graph, since each vertex is not connected to each and every other vertex. Vertex C is not connected to vertex E, or simply there is no edge C-E. Note there are 5 vertices, so there must be 4 edges connected to each and every vertex. Vertices C and E only have 3 edges connected.

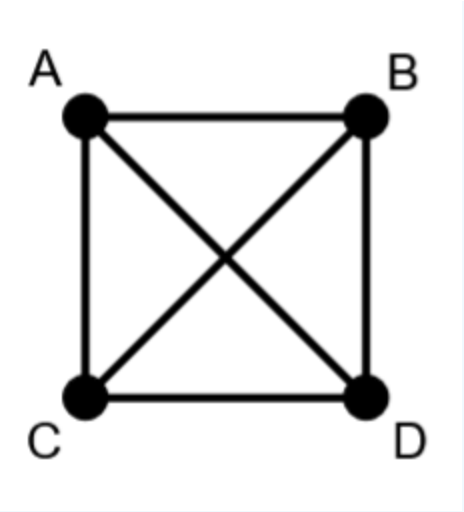
There is a relationship with complete graphs and the total number Hamilton circuits within each complete graph.

Complete Graphs and Hamilton Circuits

Every complete graph with more than two vertices has a Hamilton circuit. Furthermore, the number of Hamilton circuits in a complete graph with n vertices is $(n–1)!$.

Example #8.3.9 🤔

How many Hamilton circuits are in the complete graph below?



✅ **Solution:**

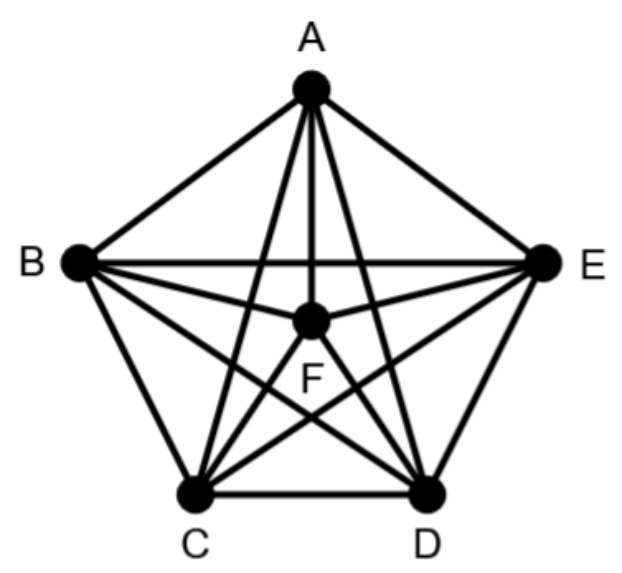
Since there are 4 vertices, then there are (4 – 1)! = 3! = 6 different Hamilton circuits.

Note: Even though it wasn't asked here, those 6 Hamilton circuits are listed below:

- A-B-C-D-A
- A-B-D-C-A
- A-C-B-D-A
- A-C-D-B-A
- A-D-B-C-A
- A-D-C-B-A

Example #8.3.10 🤔

How many Hamilton circuits are in the complete graph below?



✅ **Solution:**

Since there are 6 vertices, then there are (6 – 1)! = 5! = 120 different Hamilton circuits.

Note: Unlike in Example #8.3.9, we will not be listing all 120 different Hamilton circuits.

Complete graphs will play a significant role in the next three sections!

Historical Note: William Rowan Hamilton

William Rowan Hamilton (1805–1865) was an Irish mathematician, physicist, and astronomer whose work had a lasting influence on both mathematics and science. In graph theory, he is best known for inspiring the concepts of Hamilton paths and Hamilton circuits, although his contributions extend well beyond this area.

From a young age, Hamilton displayed extraordinary intellectual ability. By the age of thirteen, he had already studied several languages and advanced mathematics. At eighteen, he entered Trinity College Dublin, where he quickly distinguished himself as an exceptional student. By the age of twenty-two, he had been appointed Royal Astronomer of Ireland. His early research focused on optics and mechanics, and many of his ideas remain important in modern physics.

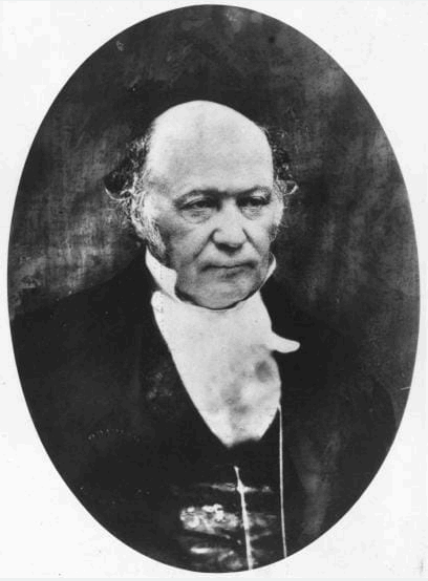
One of the most famous events in Hamilton's life occurred on October 16, 1843, at Broom Bridge in



Dublin. While walking with his wife, Hamilton suddenly realized a new mathematical system known as quaternions. In a moment of inspiration, he carved the fundamental formula into the stone of the bridge. This event, often referred to as "The Famous Bridge Discovery," is remembered as one of the most dramatic moments of insight in the history of mathematics.

Hamilton also contributed to recreational mathematics through the invention of the "Icosian Game," a puzzle that required finding a path through a graph that visits each vertex exactly once. This idea later became the foundation for what are now called Hamilton paths and Hamilton circuits. Unlike the work of Euler, which focuses on traversing edges, Hamilton's work emphasizes visiting each vertex exactly once. This distinction is fundamental in graph theory and has important applications in modern problem solving.

Hamilton's influence extends into many areas of mathematics and science. His work plays a key role in physics,

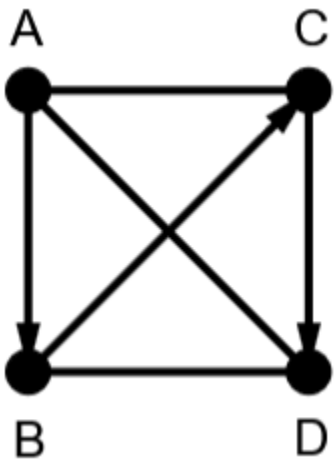


particularly in Hamiltonian mechanics, as well as in computer science, where it is applied to optimization and routing problems. His ideas are also central to graph theory and are used in fields such as 3D graphics and robotics, where quaternions are applied to represent rotations.

In summary, William Rowan Hamilton was a pioneering mathematician whose work continues to shape both theoretical and applied mathematics. His contributions to graph theory, especially the concept of visiting each vertex exactly once, remain fundamental to the study of networks and real-world systems.

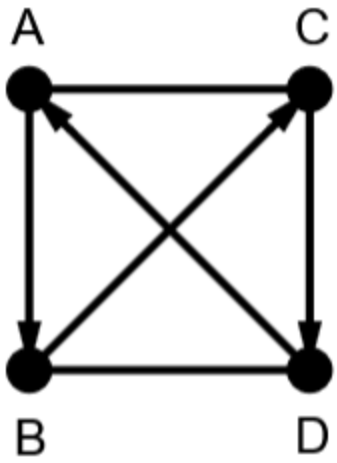
Section 8.3: Hamilton Paths and Hamilton Circuits [*In-Class Exercises*]

1. Classify the path of A-B-C-D shown in the graph as a Hamilton Path, Hamilton circuit, or neither.

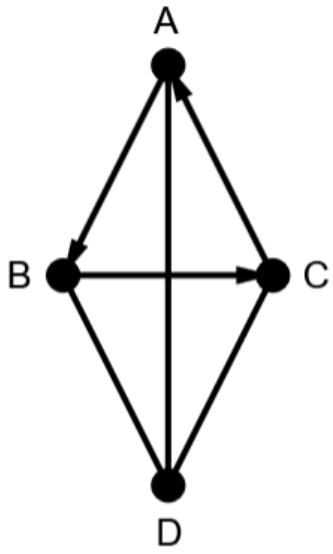


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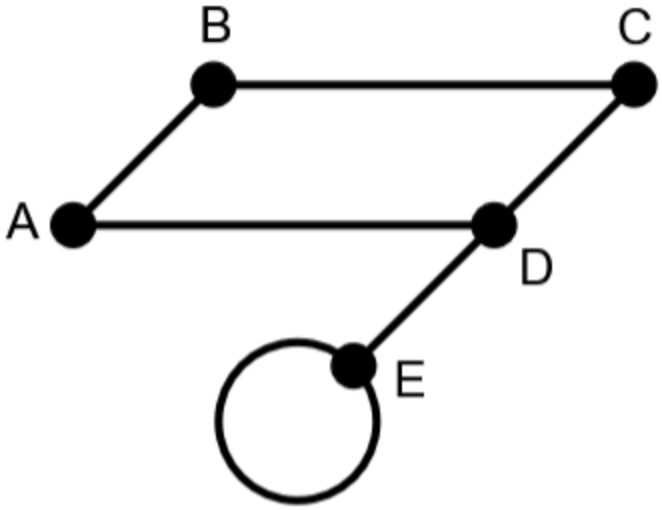
2. Classify the path of A-B-C-D-A shown in the graph as a Hamilton Path, Hamilton circuit, or neither.



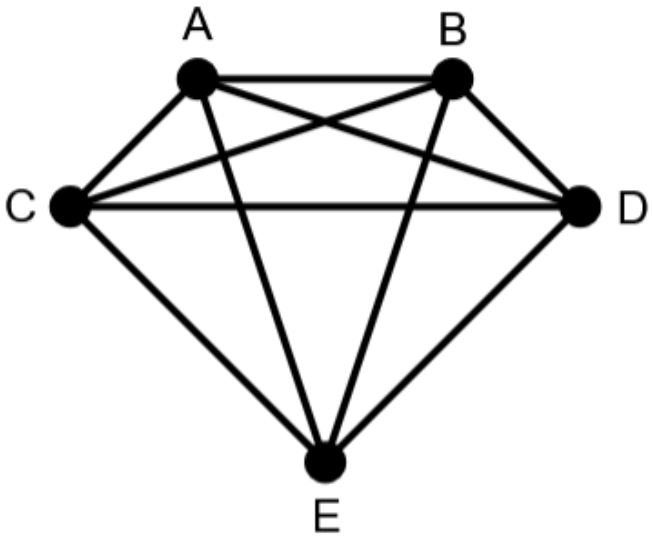
3. Classify the path of A-B-C-A shown in the graph as a Hamilton Path, Hamilton circuit, or neither.



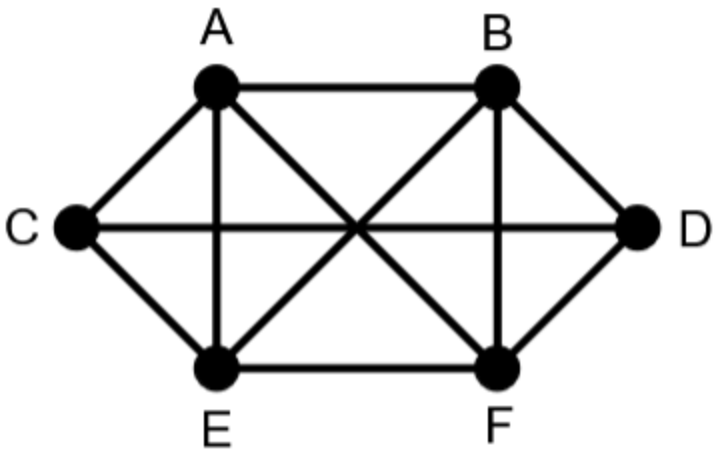
4. Classify the path of A-B-C-D-E-E shown in the graph as a Hamilton Path, Hamilton circuit, or neither. (No directional arrows are given).



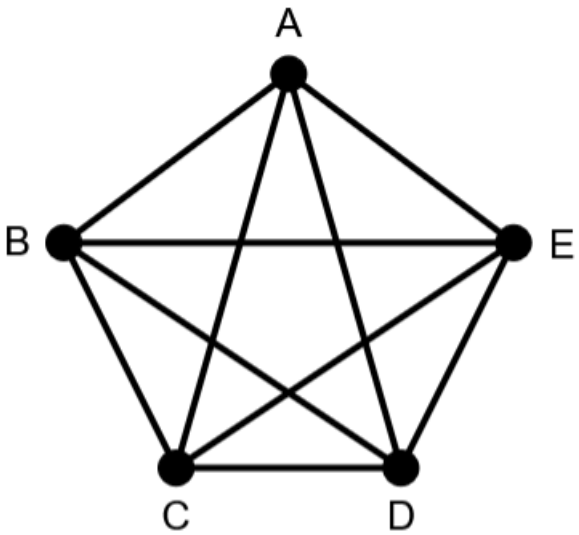
5. Is the following graph a complete graph?



6. Is the following graph a complete graph?



7. How many Hamilton circuits are in the complete graph?



Answers available at the end of this section.

Answers

Section 8.3: Hamilton Paths and Hamilton Circuits [*In-Class Exercises*]

1. Hamilton path
2. Hamilton circuit
3. Neither
4. Neither
5. Yes
6. No
7. 4! = 24

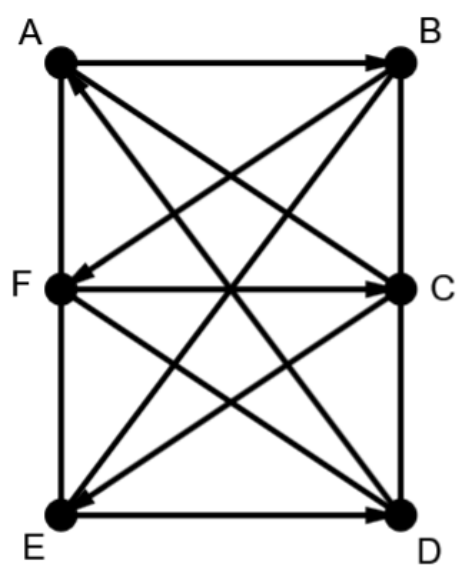
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Section 8.3.H: Homework

Section 8.3: Hamilton Paths and Hamilton Circuits - Homework Exercises

(Section 8.3) - Homework Exercise #1

Classify the path of A-B-F-C-E-D-A shown in the graph as a Hamilton path, Hamilton circuit, or neither.

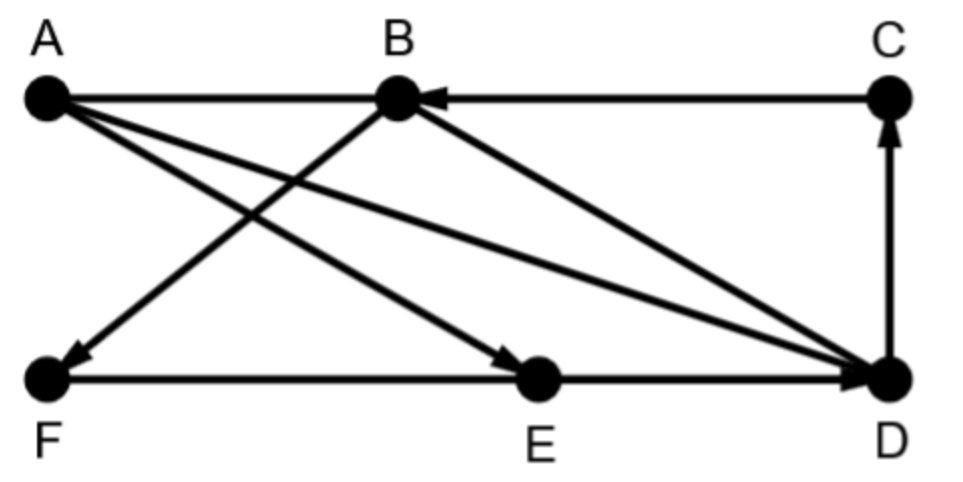


The image shows a graph with labeled nodes connected by line segments and directed line segments. A points to B; B points to F; F points to C; C points to E; E points to D; and D points to A. A is connected to F, and C; F is connected to D and E; B is connected to C, and E; and C is connected to D.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #2

Classify the path of A-E-D-C-B-F shown in the graph as a Hamilton path, Hamilton circuit, or neither.

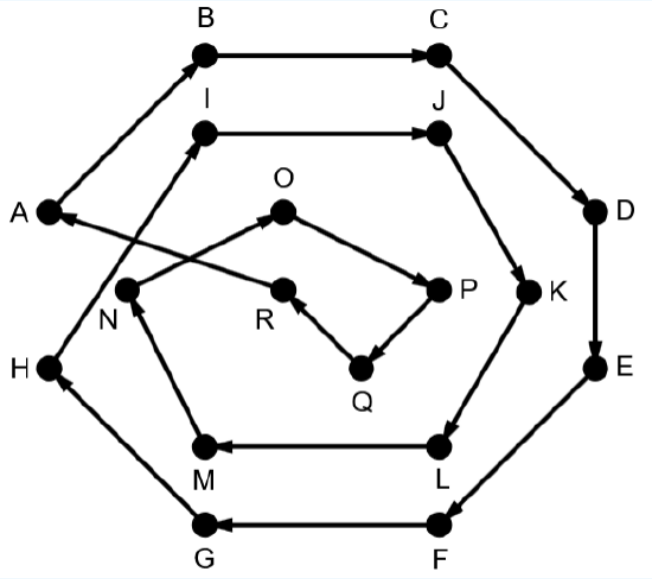


The image shows a graph with labeled nodes connected by line segments and directed line segments. A points to E; E points to D; D points to C; C points to B; and B points to F. A is connected to B, and D; B is connected to D; and E is connected to F.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #3

Classify the path of A-B-C-D-E-F-G-H-I-J-K-L-M-N-O-P-Q-R-A shown in the graph as a Hamilton path, Hamilton circuit, or neither.

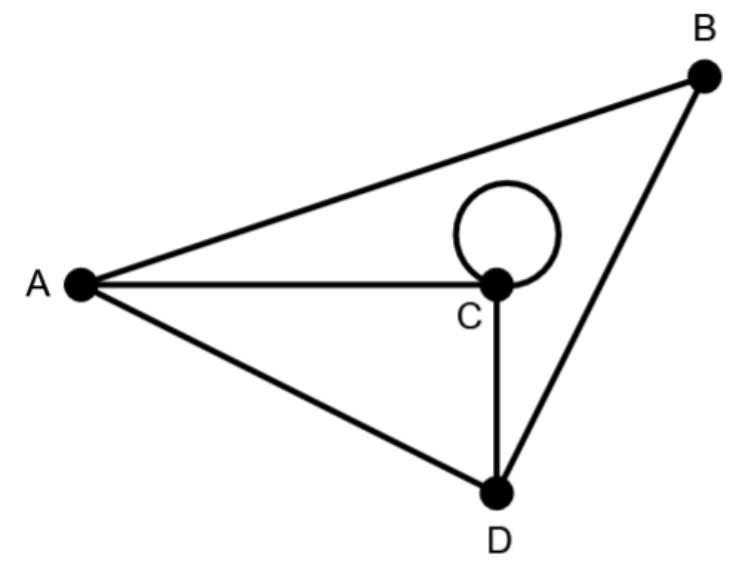


The image shows a graph with labeled nodes connected by line segments and directed line segments. A points to B; B points to C; C points to D; D points to E; E points to F; F points to G; G points to H; H points to I; I points to J; J points to K; K points to L; L points to M; M points to N; N points to O; O points to P; P points to Q; Q points to R; and R points to A.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #4

Classify the path of C-A-B-D-C-C shown in the graph as a Hamilton path, Hamilton circuit, or neither. (No directional arrows are given).

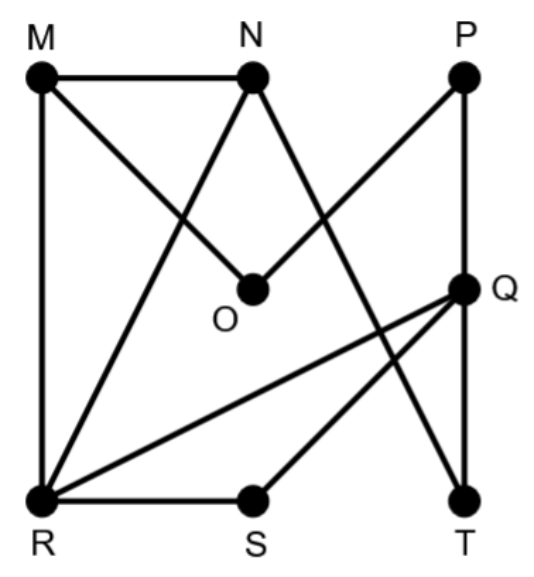


The image shows a graph with labeled nodes connected by line segments. A connects to B, C, and D; B connects to D; and C connects to D, and itself with a loop.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #5

Classify the path of N-T-Q-S-R-M-O-P shown in the graph as a Hamilton path, Hamilton circuit, or neither.

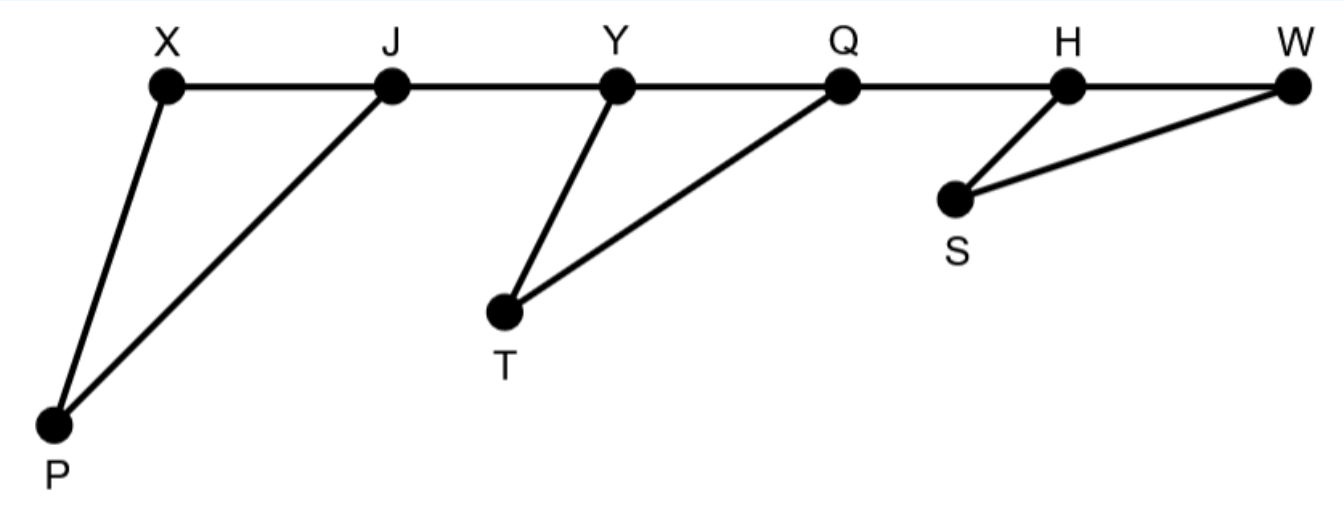


The image shows a graph with labeled nodes connected by line segments. M connects to N, O, and R; N connects to R, and T; O connects to P; P connects to Q; Q connects to R, S, and T; and R connects to S.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #6

Classify the path of X-P-J-Y-T-Q-H-S-W shown in the graph as a Hamilton path, Hamilton circuit, or neither.

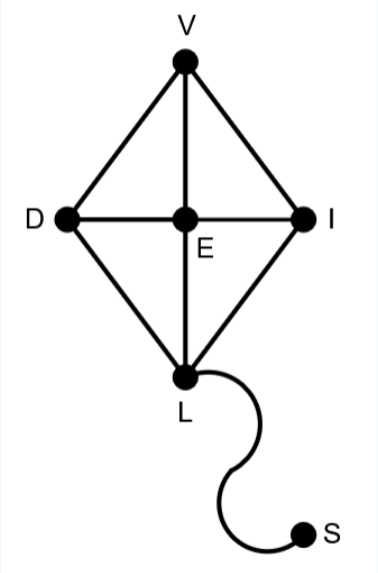


The image shows a graph with labeled nodes connected by line segments. P connects to J, and X; X connects to J; J connects to Y; Y connects to Q, and T; Q connects to H, and T; H connects to S, and W; and S connects to W.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #7

Classify the path of D-E-V-I-L-S shown in the graph as a Hamilton path, Hamilton circuit, or neither.

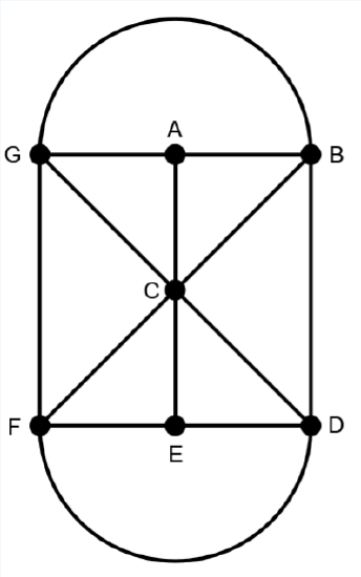


The image shows a graph with labeled nodes connected by line segments. V connects to D, E, and I; D connects to E, and L; E connects to I, and L; I connects to L; and L connects to S with a curved line.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #8

Classify the path of A-G-C-F-D-B-A shown in the graph as a Hamilton path, Hamilton circuit, or neither.

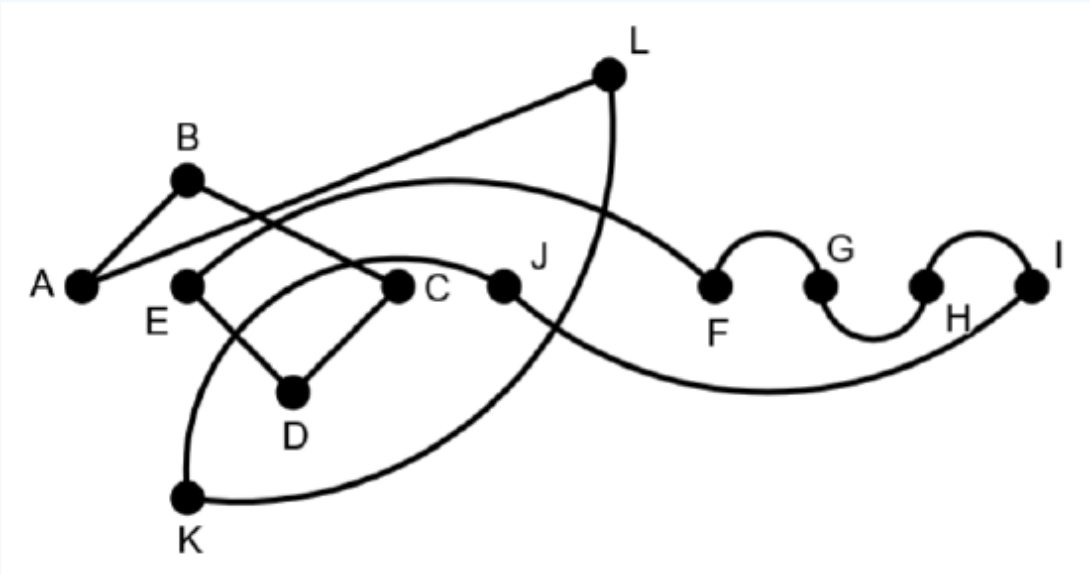


The image shows a graph with labeled nodes connected by line segments. G connects to A, C, and F; A connects to B, and C; B connects to C, and D; C connects to D, E, and F; D connects to E; and E connects to F. Curved outer edges connect G to B, and F to D.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #9

Find at least one Hamilton circuit in the graph below, if possible.

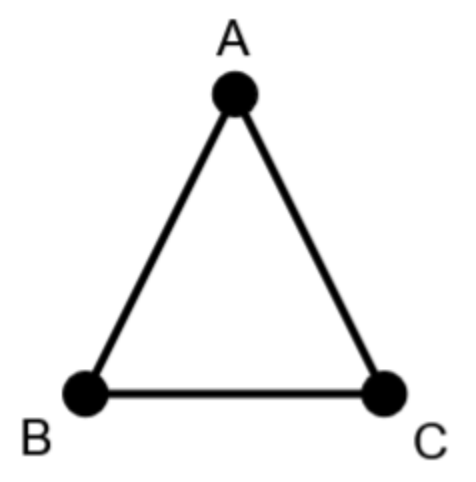


The image shows a graph with labeled nodes connected by line segments. A connects to B; B connects to C; C connects to D; D connects to E; E connects F with a curved edge; F connects G with a curved edge; G connects H with a curved edge; H connects I with a curved edge; I connects J with a curved edge; J connects K with a curved edge; K connects L with a curved edge; and L connects A.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #10

Is the following graph a complete graph?

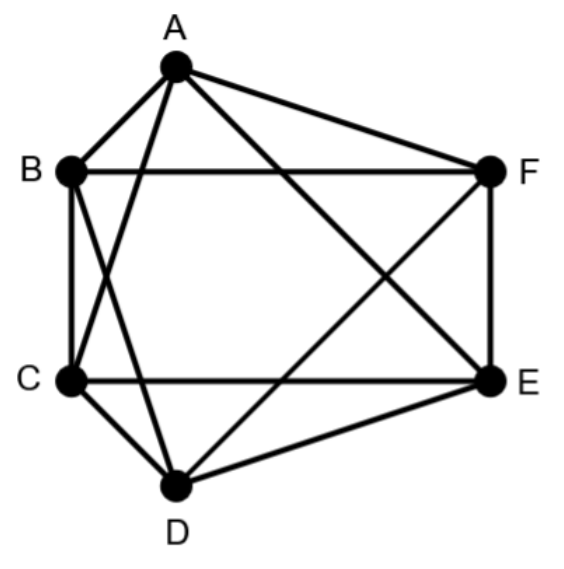


The image shows a graph with labeled nodes connected by line segments. A connects to B, and C; and B connects to C.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #11

Is the following graph a complete graph?

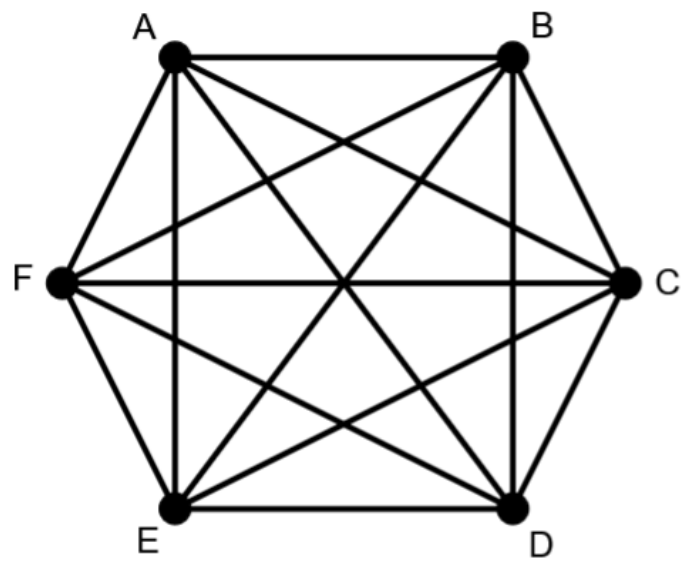


The image shows a graph with labeled nodes connected by line segments. A connects to B, C, E, and F; B connects to C, D, and F; C connects to D, and E; D connects to E, and F; and E connects to F.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #12

Is the following graph a complete graph?

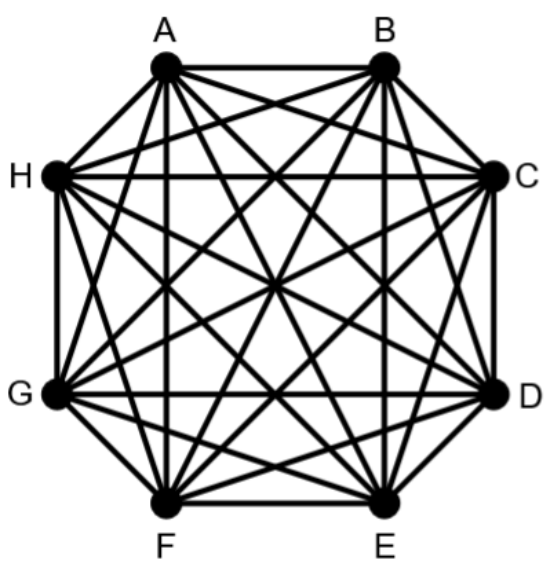


The image shows a graph with labeled nodes connected by line segments. A connects to B, C, D, E, and F; B connects to C, D, E, and F; C connects to D, E, and F; D connects to E, and F; and E connects to F.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #13

Is the following graph a complete graph?

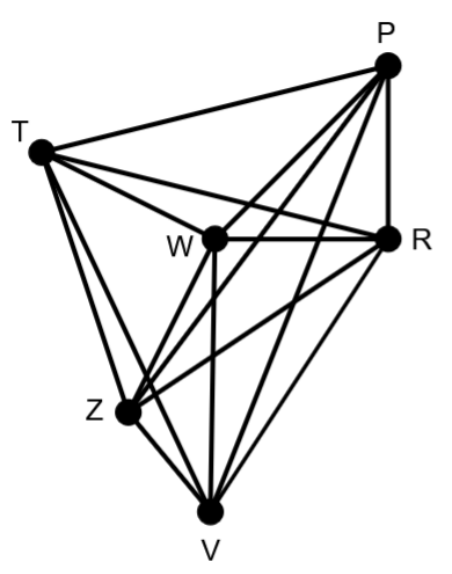


The image shows a graph with labeled nodes connected by line segments. A connects to B, C, D, E, F, G, and H; B connects to C, D, E, F, G, and H; C connects to D, E, F, G, and H; D connects to E, F, G, and H; E connects to F, G, and H; F connects to G, and H; and G connects to H.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #14

Is the following graph a complete graph?

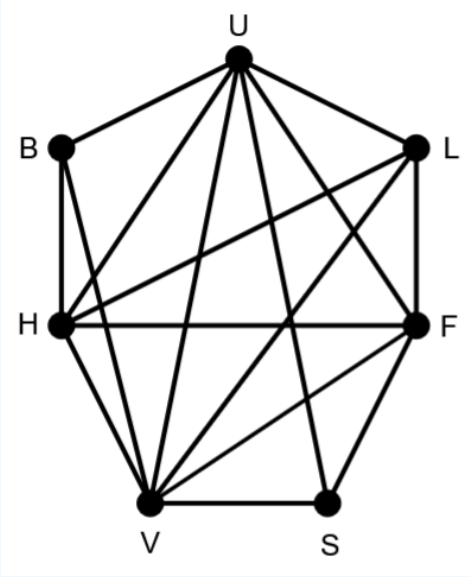


The image shows a graph with labeled nodes connected by line segments. P connects to R, T, V, W, and Z; R connects to T, V, W, and Z; T connects to V, W, and Z; V connects to W, and Z; and W connects to Z.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #15

The following graph is not complete. Fill in the missing edges to make the graph complete.



The image shows a graph with labeled nodes connected by line segments. B connects to H, U, and V; F connects to H, L, S, U, and V; H connects to L, U, and V; L connects to U, and V; S connects to U, and V; and U connects to V.

Answer available at the end of this section.

(Section 8.3) - Homework Exercise #16

Refer to Homework Exercises #10 through #14. For the graphs that were complete, how many Hamilton circuits are there in each complete graph.

Answer available at the end of this section.

Answers

(Section 8.3) - Homework Exercise #1
Hamilton circuit.

(Section 8.3) - Homework Exercise #2
Hamilton path.

(Section 8.3) - Homework Exercise #3
Hamilton circuit.

(Section 8.3) - Homework Exercise #4
Neither.

(Section 8.3) - Homework Exercise #5
Hamilton path.

(Section 8.3) - Homework Exercise #6
Hamilton path.

(Section 8.3) - Homework Exercise #7
Hamilton path.

(Section 8.3) - Homework Exercise #8
Neither.

(Section 8.3) - Homework Exercise #9

There are many possible circuits here. One possible circuit is A-B-C-D-E-F-G-H-I-J-K-L-A.

(Section 8.3) - Homework Exercise #10

Yes.

(Section 8.3) - Homework Exercise #11

No.

(Section 8.3) - Homework Exercise #12

Yes.

(Section 8.3) - Homework Exercise #13

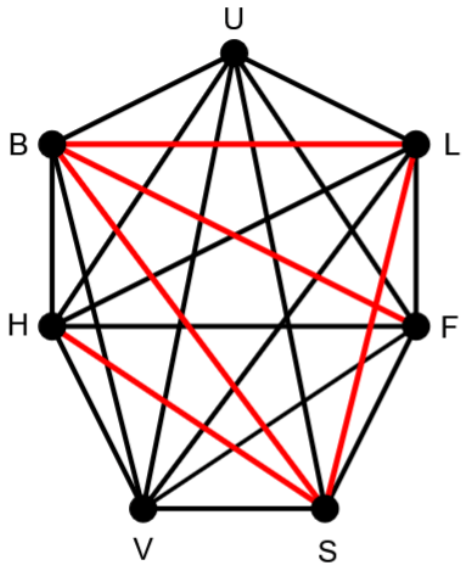
Yes.

(Section 8.3) - Homework Exercise #14

Yes.

(Section 8.3) - Homework Exercise #15

There are 5 missing edges that are needed to make the graph complete. They are B-L, B-F, B-S, H-S, and L-S.



The image shows a graph with labeled nodes connected by line segments. B connects to H, U, and V; F connects to H, L, S, U, and V; H connects to L, U, and V; L connects to U, and V; S connects to U, and V; and U connects to V. Highlighted red edges connect B to L, B to F, B to S, H to S, and L to S.

(Section 8.3) - Homework Exercise #16

- Homework Exercise #10: (3 – 1)! = 2! = 2
- Homework Exercise #11: Graph was not complete
- Homework Exercise #12: (6 – 1)! = 5! = 120
- Homework Exercise #13: (8 – 1)! = 7! = 5,040
- Homework Exercise #14: (6 – 1)! = 5! = 120

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Section 8.4: TSP - The Brute Force Method

Learning Objectives

- Solve The Traveling Salesperson Problem using the Brute Force Method

The Traveling Salesperson Problem

A natural application of Hamilton circuits arises in a well-known problem called the **Traveling Salesperson Problem (TSP)**.

Consider the following situation:

A saleswoman lives in Los Angeles and must travel by air to attend regional meetings in San Francisco, Denver, and Fort Worth. After completing all meetings, she must return to her home city of Los Angeles. Her goal is to minimize total in-air flight time.

She determines the average in-air flight time between each pair of cities as follows:

- Los Angeles $\longleftrightarrow$ San Francisco: 1 hour
- Los Angeles $\longleftrightarrow$ Denver: 2.5 hours
- Los Angeles $\longleftrightarrow$ Fort Worth: 3 hours
- San Francisco $\longleftrightarrow$ Denver: 2.5 hours
- San Francisco $\longleftrightarrow$ Fort Worth: 3.5 hours
- Denver $\longleftrightarrow$ Fort Worth: 2 hours

Question: What is the minimum total number of hours spent in the air if she must visit each city exactly once and return to Los Angeles?

While this example may be solved by inspection, the problem becomes significantly more difficult as the number of cities and distances increases. To better understand how to approach this problem, we begin with a formal definition.

The Traveling Salesperson Problem (TSP)

Given a set of cities and the distances between every pair of cities, determine the shortest possible route that visits each city exactly once and returns to the starting city.

At first glance, this problem resembles a Hamilton circuit. In both cases, each vertex is visited exactly once and the path returns to the starting point. However, the Traveling Salesperson Problem adds an important condition: the route must be optimal, meaning it has the smallest possible total distance (or cost).

To model this situation, each city is represented by a vertex, and each connection between cities is represented by an edge. Each edge is assigned a weight, representing distance, time, or cost. The goal is to find a Hamilton circuit of minimum weight, often called an optimal tour.

Complete Weighted Graph & Weights

A **complete weighted graph** is a graph in which every pair of vertices is connected by an edge, and each edge is assigned a numerical value called a **weight**. These weights may represent distance, time, or cost, depending on the context.

There are three common ways to approach the Traveling Salesperson Problem:

- Brute Force Method
- Nearest Neighbor Method
- Cheapest Link Algorithm

We begin with the most direct approach.

Brute Force Method (for Solving TSP)

The **brute force method** solves the Traveling Salesperson Problem by examining every possible tour, computing the total distance for each tour, and selecting the shortest one.

- **Step 1** - Draw a complete weighted graph representing the problem.
- **Step 2** - List all possible permutations of Hamilton circuits.
- **Step 3** - Find the sum of the weights of the edges for each circuit.
- **Step 4** - The circuit(s) with the smallest sum is the optimal solution.

This method guarantees that the optimal solution will be found. However, it quickly becomes impractical as the number of cities increases, since the number of possible routes grows rapidly.

Because of this complexity, the Traveling Salesperson Problem is considered one of the most important and challenging problems in mathematics and computer science. It has applications in areas such as logistics, transportation, manufacturing, and circuit design, where finding efficient routes or processes is essential.

Example #8.4.1 🤔

A saleswoman lives in Los Angeles and must travel by air to attend regional meetings in San Francisco, Denver, and Fort Worth. After completing all meetings, she must return to her home city of Los Angeles. Her goal is to minimize total in-air flight time.

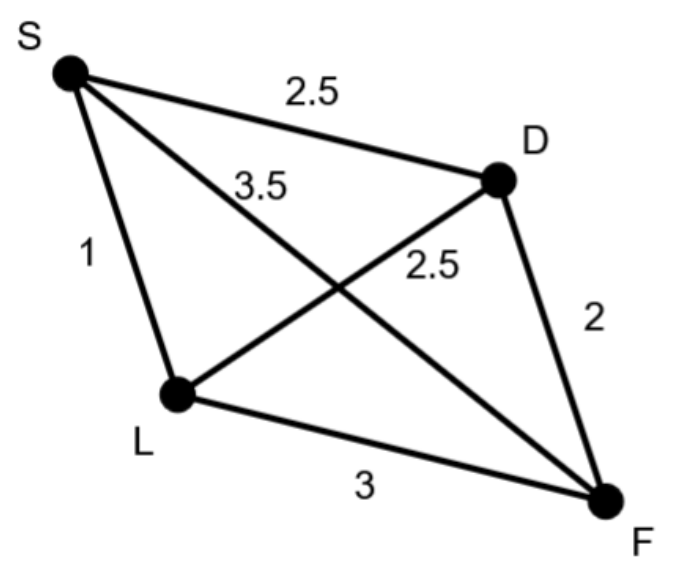
She determines the average in-air flight time between each pair of cities as follows:

- Los Angeles $\longleftrightarrow$ San Francisco: 1 hour
- Los Angeles $\longleftrightarrow$ Denver: 2.5 hours
- Los Angeles $\longleftrightarrow$ Fort Worth: 3 hours
- San Francisco $\longleftrightarrow$ Denver: 2.5 hours
- San Francisco $\longleftrightarrow$ Fort Worth: 3.5 hours
- Denver $\longleftrightarrow$ Fort Worth: 2 hours

Using the brute force method, what is the minimum total number of hours spent in the air, if she must visit each city exactly once and return to Los Angeles?

✅ **Solution:**

- **Step 1** - Draw a complete weighted graph representing the problem.



- **Step 2** - List all possible permutations of Hamilton circuits.

We start with vertex L (Los Angeles), because the saleswoman lives there. Since there are 4 vertices, then there are (4 – 1)! = 3! = 6 different Hamilton circuits. The circuits are listed on the left side of the table below.

- **Step 3** - Find the sum of the weights of the edges for each circuit.

Follow each circuit and add the weights of all the edges in that circuit. The total weight for each circuit is shown on the right side of the table below.

Table showing each Hamilton circuit and the sum of its weights

Hamilton Circuits	Sum of the Weights
L-D-F-S-L	2.5 + 2 + 3.5 + 1 = 9 hours
L-D-S-F-L	2.5 + 2.5 + 3.5 + 3 = 11.5 hours
L-F-D-S-L	3 + 2 + 2.5 + 1 = 8.5 hours
L-F-S-D-L	3 + 3.5 + 2.5 + 2.5 = 11.5 hours
L-S-D-F-L	1 + 2.5 + 2 + 3 = 8.5 hours
L-S-F-D-L	1 + 3.5 + 2 + 2.5 = 9 hours

- **Step 4** - The circuit(s) with the smallest sum is the optimal solution.

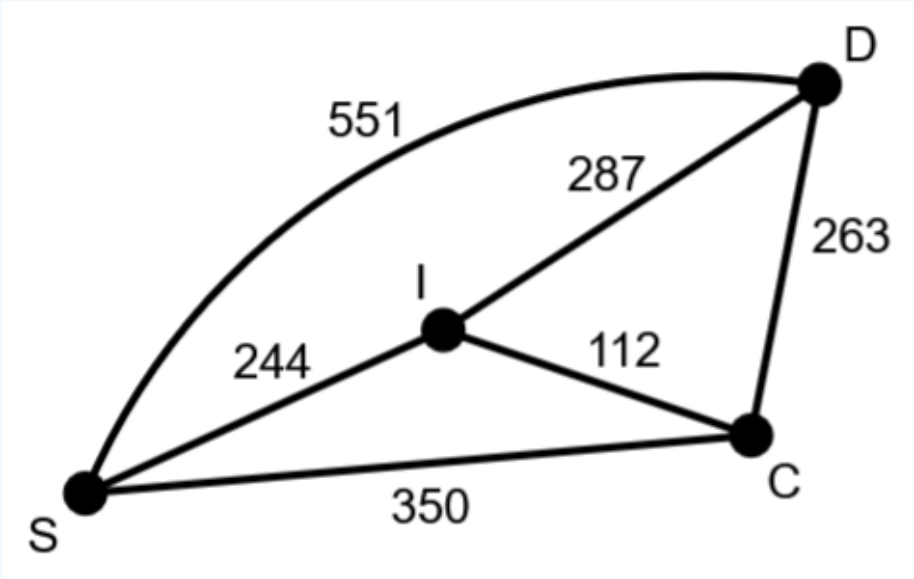
Thus, the two optimal solutions are the circuits L-F-D-S-L and L-S-D-F-L which both have a total weight of 8.5 hours each,

In the example above, when using the brute force method, there will always be at least two Hamilton circuits that produce the same optimal solution. This occurs because traveling a route in one direction and then traveling the same route in reverse result in two distinct circuits with identical total weights.

In fact, the six possible Hamilton circuits in this example can be grouped into three pairs. Each pair consists of a circuit and its reverse, and both have the same total distance.

Example #8.4.2 🤔

A salesman from Indianapolis has a regional manager's meeting in his hometown as well as Detroit, St. Louis, and Cincinnati. He wants to minimize the mileage as best as he can. Find the optimal solution for the traveling salesperson problem using the brute force method.



✅ **Solution:**

- **Step 1** - Draw a complete weighted graph representing the problem.

The complete weighted graph was given.

- **Step 2** - List all possible permutations of Hamilton circuits.

We start with vertex I (Indianapolis), because the salesman lives there. Since there are 4 vertices, then there are (4 – 1)! = 3! = 6 different Hamilton circuits. The circuits are listed on the left side of the table below.

- **Step 3** - Find the sum of the weights of the edges for each circuit.

Follow each circuit and add the weights of all the edges in that circuit. The total weight for each circuit is shown on the right side of the table below.

Table showing each Hamilton circuit and the sum of its weights

Hamilton Circuits	Sum of the Weights
I-C-D-S-I	112 + 263 + 551 + 244 = 1,170 miles
I-C-S-D-I	112 + 350 + 551 + 287 = 1,300 miles
I-D-C-S-I	287 + 263 + 350 + 244 = 1,144 miles
I-D-S-C-I	287 + 551 + 350 + 112 = 1,300 miles
I-S-C-D-I	244 + 350 + 263 + 287 = 1,144 miles

Hamilton Circuits	Sum of the Weights
I-S-D-C-I	244 + 551 + 263 + 112 = 1,170 miles

- **Step 4** - The circuit(s) with the smallest sum is the optimal solution.

Thus, the two optimal solutions are the circuits I-D-C-S-I and I-S-C-D-I which both have a total weight of 1,144 miles each.

Section 8.4: TSP - The Brute Force Method [*In-Class Exercises*]

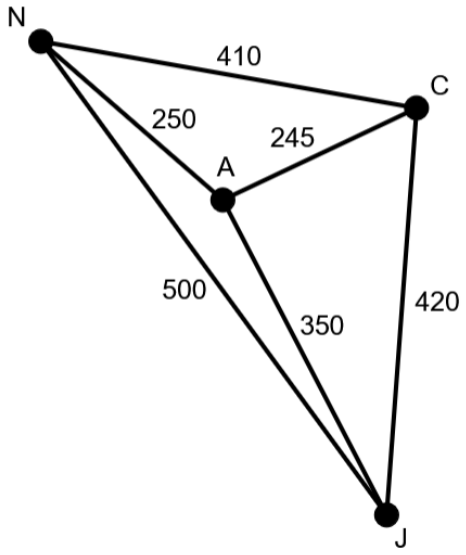
1. A saleswoman lives in Dallas and must travel by car to attend corporate meetings in Austin, Houston, and San Antonio. After completing all meetings, she must return to her home city of Dallas. Her goal is to minimize the total distance of travel. She determines the distance between each pair of cities as follows:

- Dallas $\longleftrightarrow$ Houston: 240 miles
- Dallas $\longleftrightarrow$ Austin: 195 miles
- Dallas $\longleftrightarrow$ San Antonio: 275 miles
- Houston $\longleftrightarrow$ Austin: 165 miles
- Houston $\longleftrightarrow$ San Antonio: 200 miles
- Austin $\longleftrightarrow$ San Antonio: 80 miles

Using the brute force method, what is the minimum total distance in miles, if she must visit each city exactly once and return to Dallas?

---

2. A regional sales manager based in Atlanta must travel to three cities, Charlotte, Nashville, and Jacksonville for client meetings. The complete weighted graph with the approximate driving distances (in miles) between the cities is given below:



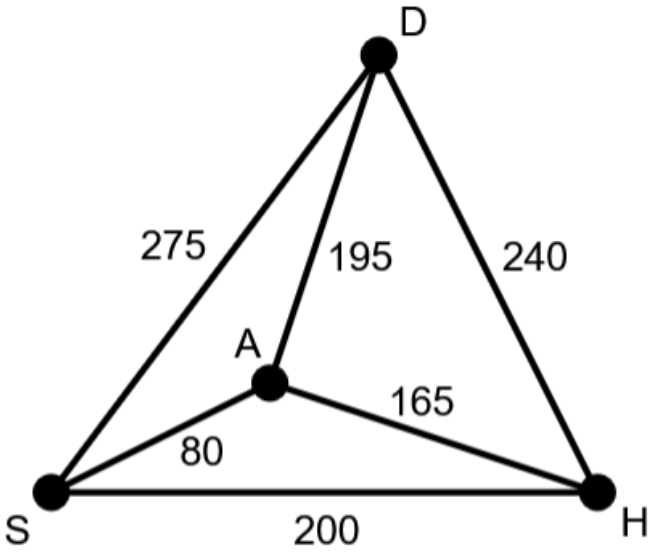
Using the brute force method, what is the minimum total distance in miles, if he must visit each city exactly once and return back to his home in Atlanta?

Answers available at the end of this section.

Answers

Section 8.4: TSP - The Brute Force Method [*In-Class Exercises*]

1. D-H-S-A-D & D-A-S-H-D; 715 miles each.



2. A-C-J-N-A & A-N-J-C-A; 1,415 miles each.

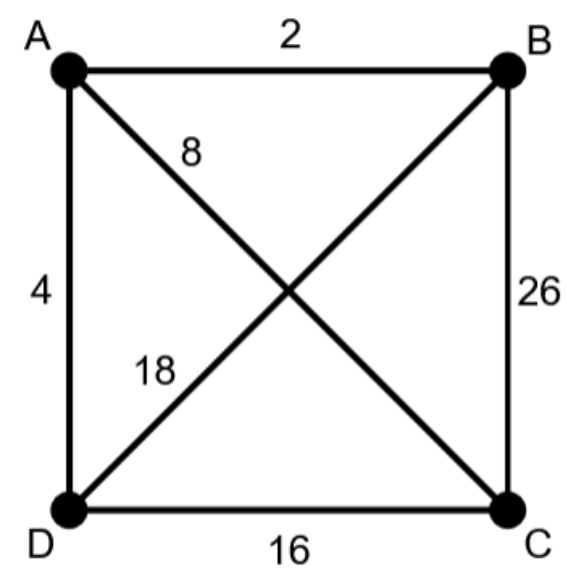
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Section 8.4.H: Homework

Section 8.4: TSP - The Brute Force Method - Homework Exercises

(Section 8.4) - Homework Exercise #1

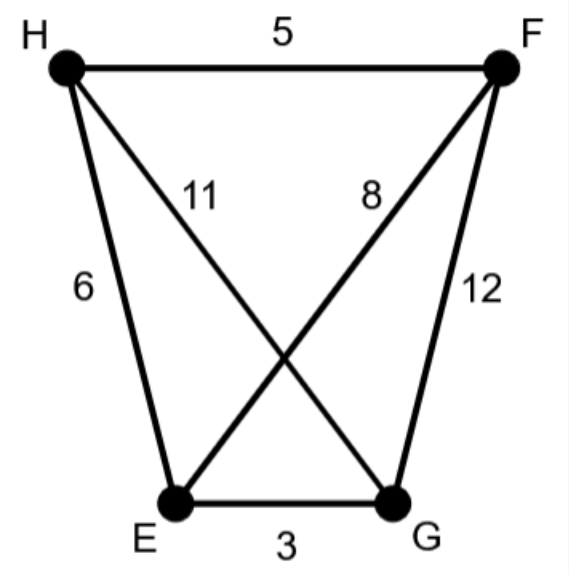
Use the brute force method to list all possible Hamilton circuits for the graph starting at vertex A. Then determine which circuit has the minimum total weight.



Answer available at the end of this section.

(Section 8.4) - Homework Exercise #2

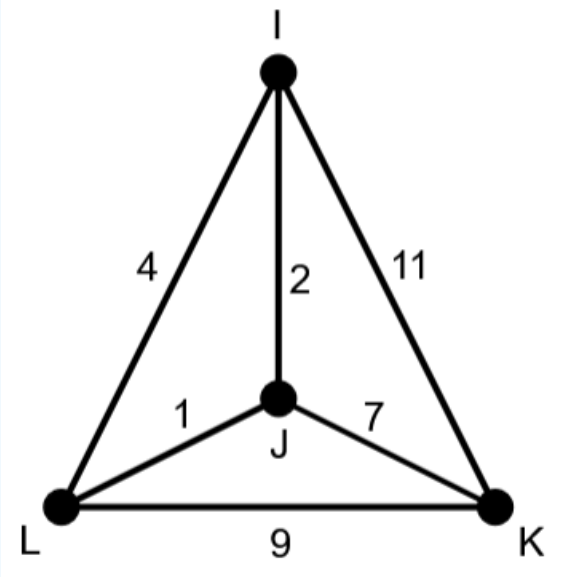
Use the brute force method to list all possible Hamilton circuits for the graph starting at vertex E. Then determine which circuit has the minimum total weight.



Answer available at the end of this section.

(Section 8.4) - Homework Exercise #3

Use the brute force method to list all possible Hamilton circuits for the graph starting at vertex I. Then determine which circuit has the minimum total weight.



Answer available at the end of this section.

(Section 8.4) - Homework Exercise #4

The distances in miles between four national monuments in Arizona are shown below.

- Casa Grande Ruins ⟷ Organ Pipe Cactus: 141
- Casa Grande Ruins ⟷ Tonto: 87

- Casa Grande Ruins ⟷ Tumacácori: 114
- Organ Pipe Cactus: ⟷ Tonto: 228
- Organ Pipe Cactus ⟷ Tumacácori: 185
- Tonto ⟷ Tumacácori: 181

Draw a complete weighted graph for the information. Use the brute force method to list all possible Hamilton circuits for the graph starting from Casa Grande Ruins. Then determine which circuit has the minimum total weight.

Answer available at the end of this section.

(Section 8.4) - Homework Exercise #5

A telecommunications company is planning to install high-speed fiber lines between four cities:

A: Albany
B: Boston
N: New York
P: Portland

The cost (in millions of dollars) to install direct network lines between each pair of cities is given below:

A–B = 7, A–N = 9, A–P = 8
B–N = 10, B–P = 6
N–P = 12

Draw a complete weighted graph for the information. Use the brute force method to list all possible Hamilton circuits for the graph starting from New York. Then determine which circuit has the minimum total cost (in millions of dollars).

Answer available at the end of this section.

(Section 8.4) - Homework Exercise #6

A medical courier starts at a hospital H and must deliver supplies to four clinics:

- A: Urgent Care
- B: Pediatric Clinic
- C: Surgical Center
- D: Rehabilitation Center

The costs (in dollars, representing fuel + time) are:

- H-A = 8, H-B = 9, H-C = 6, H–D = 7
- A–B = 6, A–C = 4, A–D = 5
- B–C = 5, B–9 = 18
- C–D = 7

Draw a complete weighted graph for the information. Using the brute force method, what is the minimum total cost (in dollars), if the medical courier must deliver supplies to all four clinics starting from and returning back to the hospital?

Answer available at the end of this section.

Answers

(Section 8.4) - Homework Exercise #1

Table showing each Hamilton circuit and the sum of its weights

Hamilton Circuits	Sum of the Weights
A-B-C-D-A	2 + 26 + 16 + 4 = 48
A-B-D-C-A	2 + 18 + 16 + 8 = 44
A-C-B-D-A	8 + 26 + 18 + 4 = 56
A-C-D-B-A	8 + 16 + 18 + 2 = 44
A-D-B-C-A	4 + 18 + 26 + 8 = 56
A-D-C-B-A	4 + 16 + 26 + 2 = 48

Optimal solutions are A-B-D-C-A and A-C-D-B-A both with a weight of 44.

(Section 8.4) - Homework Exercise #2

Table showing each Hamilton circuit and the sum of its weights

Hamilton Circuits	Sum of the Weights
E-F-G-H-E	8 + 12 + 11 + 6 = 37
E-F-H-G-E	8 + 5 + 11 + 3 = 27
E-G-F-H-E	3 + 12 + 5 + 6 = 26
E-G-H-F-E	3 + 11 + 5 + 8 = 27
E-H-F-G-E	6 + 5 + 12 + 3 = 26
E-H-G-F-E	6 + 11 + 12 + 8 = 37

Optimal solutions are E-G-F-H-E and E-H-G-F-E both with a weight of 26.

(Section 8.4) - Homework Exercise #3

Table showing each Hamilton circuit and the sum of its weights

Hamilton Circuits	Sum of the Weights
I-J-K-L-I	2 + 7 + 9 + 4 = 22
I-J-L-K-I	2 + 1 + 9 + 11 = 23
I-K-J-L-I	11 + 7 + 1 + 4 = 23
I-K-L-J-I	11 + 9 + 1 + 2 = 23
I-L-J-K-I	4 + 1 + 7 + 11 = 23
I-L-K-J-I	4 + 9 + 7 + 2 = 22

Optimal solutions are I-J-K-L-I and I-L-K-J-I both with a weight of 22.

(Section 8.4) - Homework Exercise #4

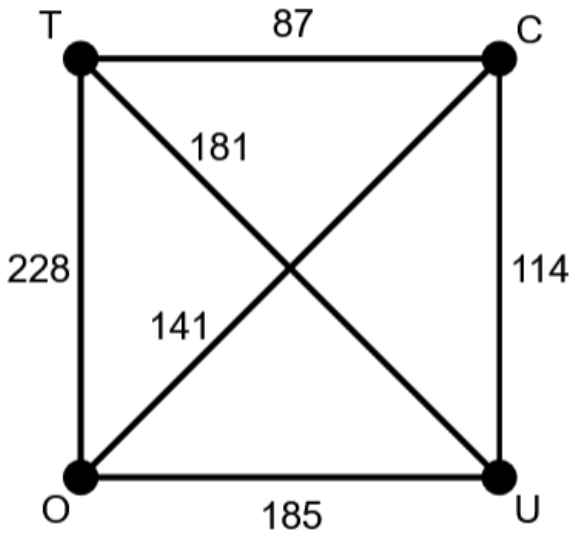


Table showing each Hamilton circuit and the sum of its weights

Hamilton Circuits	Sum of the Weights
C-O-T-U-C	141 + 228 + 181 + 114 = 664
C-O-U-T-C	141 + 185 + 181 + 87 = 594
C-T-O-U-C	87 + 228 + 185 + 114 = 614
C-T-U-O-C	87 + 181 + 185 + 141 = 594
C-U-O-T-C	114 + 185 + 228 + 87 = 614
C-U-T-O-C	114 + 181 + 228 + 141 = 664

Optimal solutions are C-O-U-T-C and C-T-U-O-C at 594 miles each.

(Section 8.4) - Homework Exercise #5

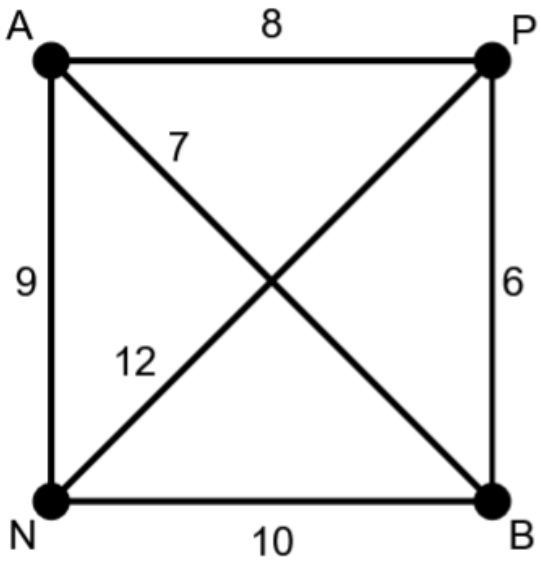


Table showing each Hamilton circuit and the sum of its weights

Hamilton Circuits	Sum of the Weights
N-A-B-P-N	9 + 7 + 6 + 12 = 34
N-A-P-B-N	9 + 8 + 6 + 10 = 33
N-B-A-P-N	10 + 7 + 8 + 12 = 37
N-B-P-A-N	10 + 6 + 8 + 9 = 33
N-P-A-B-N	12 + 8 + 7 + 10 = 37
N-P-B-A-N	12 + 6 + 7 + 9 = 34

Optimal solutions are N-A-P-B-N and N-B-P-A-N at $33 million each.

(Section 8.4) - Homework Exercise #6

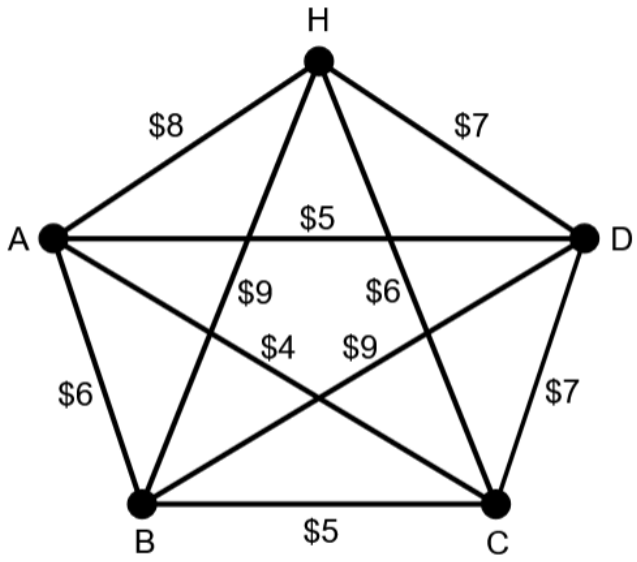


Table showing each Hamilton circuit and the sum of its weights

Hamilton Circuits	Sum of the Weights
H-A-B-C-D-H	$8 + $6 + $5 + $7 + $7 = $33
H-A-B-D-C-H	$8 + $6 + $9 + $7 + $6 = $36
H-A-C-B-D-H	$8 + $4 + $5 + $9 + $7 = $33
H-A-C-D-B-H	$8 + $4 + $7 + $9 + $9 = $37
H-A-D-B-C-H	$8 + $5 + $9 + $5 + $6 = $33
H-A-D-C-B-H	$8 + $5 + $7 + $5 + $9 = $34
H-B-A-C-D-H	$9 + $6 + $4 + $7 + $7 = $33
H-B-A-D-C-H	$9 + $6 + $5 + $7 + $6 = $33

Hamilton Circuits	Sum of the Weights
H-B-C-A-D-H	$9 + $5 + $4 + $5 + $7 = $30
H-B-C-D-A-H	$9 + $5 + $7 + $5 + $8 = $34
H-B-D-A-C-H	$9 + $9 + $5 + $4 + $6 = $33
H-B-D-C-A-H	$9 + $9 + $7 + $4 + $8 = $37
H-C-A-B-D-H	$6 + $4 + $6 + $9 + $7 = $32
H-C-A-D-B-H	$6 + $4 + $5 + $9 + $9 = $33
H-C-B-A-D-H	$6 + $5 + $6 + $5 + $7 = $29
H-C-B-D-A-H	$6 + $5 + $9 + $5 + $8 = $33
H-C-D-A-B-H	$6 + $7 + $5 + $6 + $9 = $33
H-C-D-B-A-H	$6 + $7 + $9 + $6 + $8 = $36
H-D-A-B-C-H	$7 + $5 + $6 + $5 + $6 = $29
H-D-A-C-B-H	$7 + $5 + $4 + $5 + $9 = $30
H-D-B-A-C-H	$7 + $9 + $6 + $4 + $6 = $32
H-D-B-C-A-H	$7 + $9 + $5 + $4 + $8 = $33
H-D-C-A-B-H	$7 + $7 + $4 + $6 + $9 = $33
H-D-C-B-A-H	$7 + $7 + $5 + $6 + $8 = $33

Optimal solutions are H-C-B-A-D-H and H-D-A-B-C-H at $29 each.

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Section 8.5: TSP - The Nearest Neighbor Method

Learning Objectives

- Solve The Traveling Salesperson Problem using the Nearest Neighbor Method

Although the brute force method ensures that the optimal solution will be found for the TSP, the number of possible routes grows rapidly as more cities are added. If 5 cities were involved, then there are (5 – 1)! = 4! = 24 different Hamilton circuits. If 6 cities were involved, then there are (6 – 1)! = 5! = 120 different Hamilton circuits. As a result, this method can become time-consuming and inefficient for TSPs involving 5 or more cities. To address this, mathematicians have developed alternative strategies that provide quicker, though not always optimal, solutions.

This leads us to our second method for solving the Traveling Salesperson Problem, known as the **nearest neighbor method**.

Nearest Neighbor Method (for Solving TSP)

The **nearest neighbor method** solves the Traveling Salesperson Problem and is designed to find a good solution quickly, though it does not always guarantee the optimal one.

- **Step 1** - Draw a complete weighted graph representing the problem.
- **Step 2** - Starting at a designated vertex, travel along the edge with the smallest weight to an unvisited vertex. (If there are two or more edges the same, pick any one).
- **Step 3** - From the current vertex, select the edge with the smallest weight that leads to a vertex that has not yet been visited. (If there are two or more edges the same, pick any one).
- **Step 4** - Continue this process until all vertices have been visited, then return to the starting vertex to complete the circuit.
- **Step 5** - The sum of the weights is an approximation to the optimal solution.

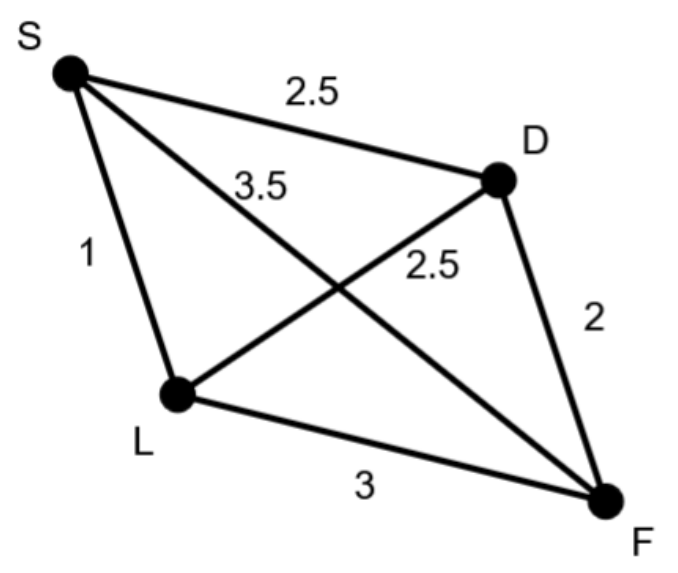
The nearest neighbor method offers a practical and efficient approach to solving the Traveling Salesperson Problem. Although it does not guarantee an optimal solution, its simplicity and speed make it especially useful for larger and more complex networks. This method is an example of a greedy algorithm, as it selects the best immediate (local) option at each step.

While this strategy can produce a reasonable solution quickly, it may result in a route that is longer than the optimal one. Therefore, the nearest neighbor method is best viewed as an approximation rather than a guaranteed solution.

In practice, it is often used when a quick solution is needed or as a starting point for more advanced optimization methods.

Example #8.5.1 🤔

A saleswoman lives in Los Angeles and must travel by air to attend regional meetings in San Francisco, Denver, and Fort Worth. After completing all meetings, she must return to her home city of Los Angeles. Her goal is to minimize total in-air flight time. The complete weighted graph with the in-air flight time (in hours) between the cities is given below:



Using the nearest neighbor method, what is the minimum total number of hours spent in the air, if she must visit each city exactly once and return to Los Angeles?

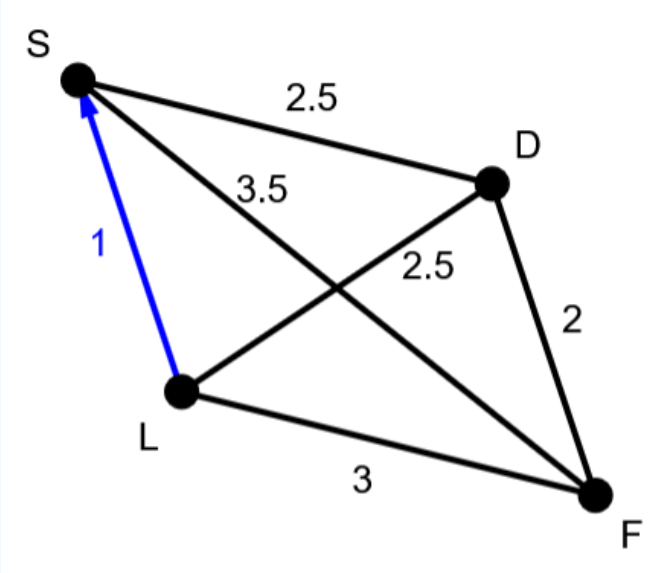
✅ **Solution:**

- **Step 1** - Draw a complete weighted graph representing the problem.

The complete weighted graph was given.

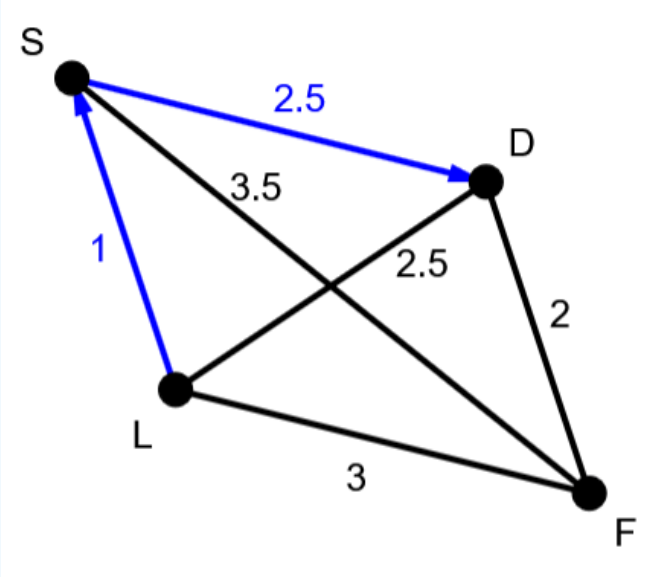
- **Step 2** - Starting at a designated vertex, travel along the edge with the smallest weight to an unvisited vertex. (If there are two or more edges the same, pick any one).

We start with vertex L (Los Angeles), because the saleswoman lives there. The edge with the smallest weight is L-S (1 hour) which means we travel to vertex S (San Francisco).



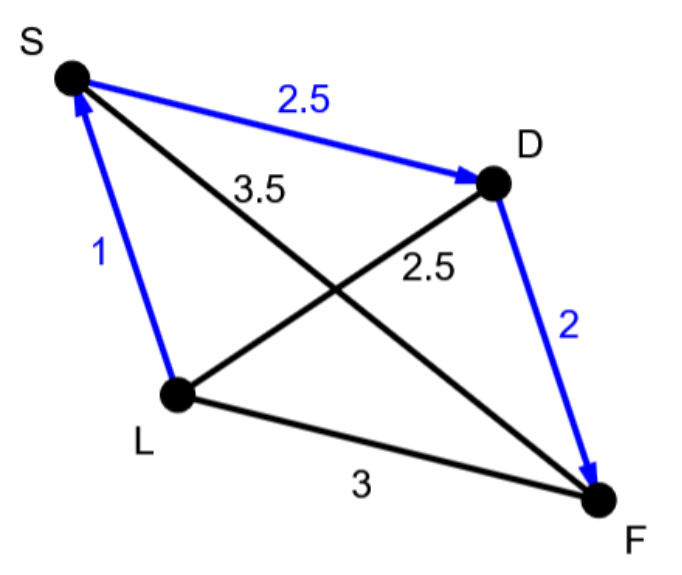
- **Step 3** - From the current vertex, select the edge with the smallest weight that leads to a vertex that has not yet been visited. (If there are two or more edges the same, pick any one).

From vertex S (San Francisco), the next edge with the smallest weight is S-D (2.5 hours) which means we travel to vertex D (Denver).

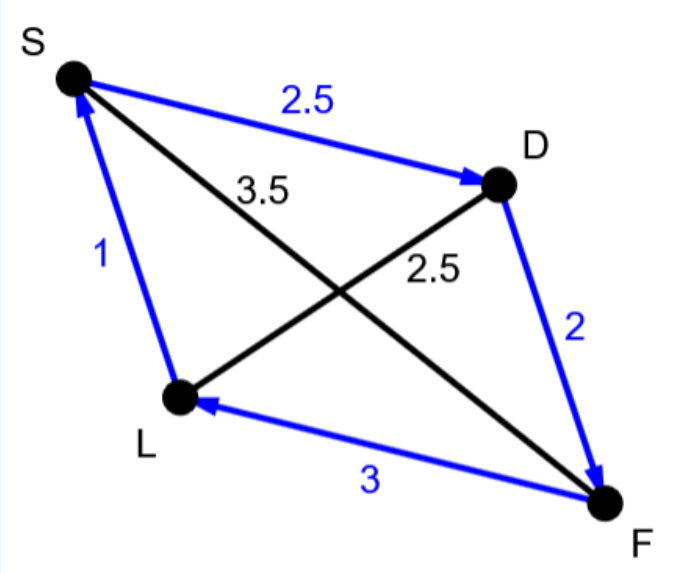


- **Step 4** - Continue this process until all vertices have been visited, then return to the starting vertex to complete the circuit.

From vertex D (Denver), there is only one other vertex that has not been visited, which is F (Fort Worth). Thus, there is no need to check for the smallest weight here. So, the next edge is D-F (2 hours).



From this point, all vertices have been visited, so we use the edge that takes us back to the starting vertex L (Los Angeles), which is F-L (3 hours).



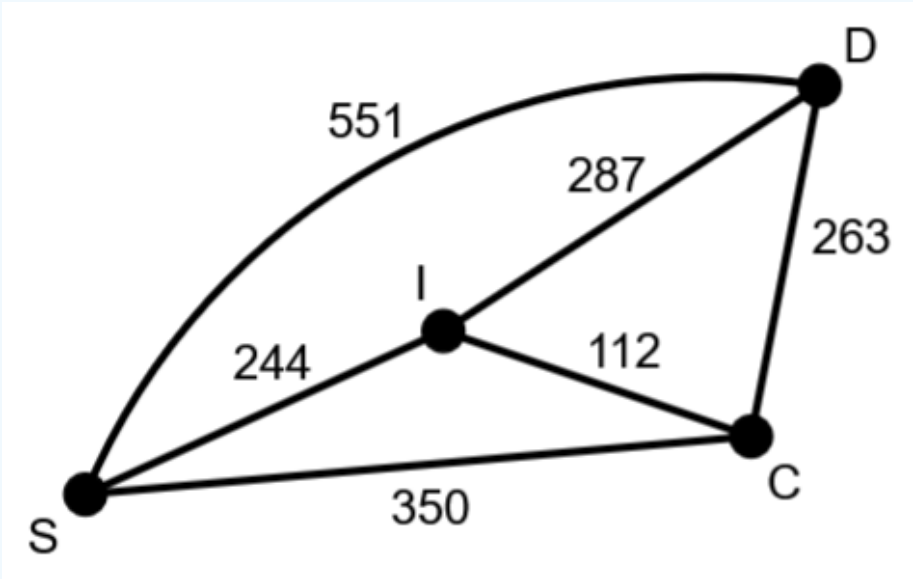
- **Step 5** - The sum of the weights is an approximation to the optimal solution.

Thus, the circuit L-S-D-F-L has a total weight of 1 + 2.5 + 2 + 3 = 8.5 hours. The circuit in reverse would have the same total weight. Therefore, the two approximate optimal solutions are L-S-D-F-L and L-F-D-S-L, each with a total travel time of 8.5 hours.

In the example above, the brute force method in Section 8.3 showed that the optimal solution is 8.5 hours. In this case, the nearest neighbor method produced the same result; however, this is not always guaranteed.

Example #8.5.2 🤔

A salesman from Indianapolis has a regional manager's meeting in his hometown as well as Detroit, St. Louis, and Cincinnati. He wants to minimize the mileage as best as he can. Find the optimal solution for the traveling salesperson problem using the nearest neighbor method.



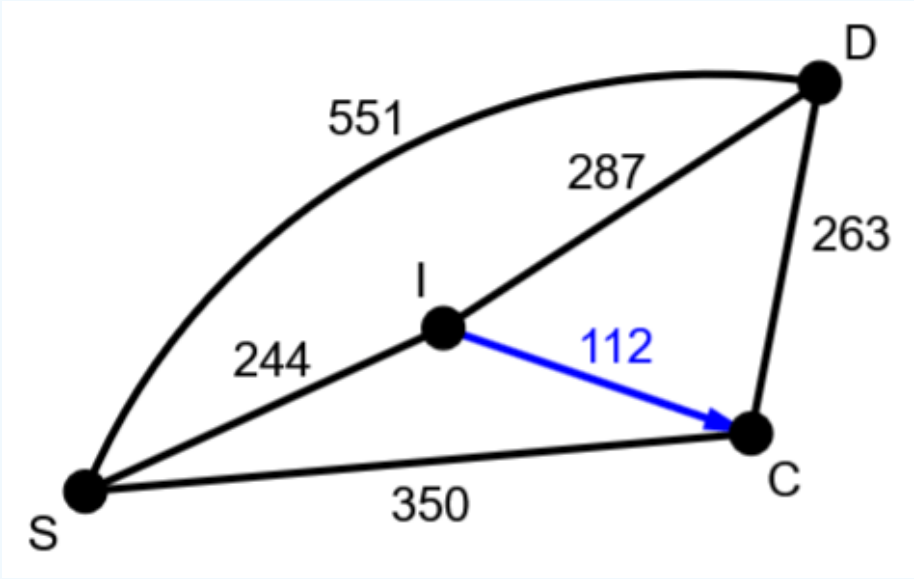
✅ **Solution:**

- **Step 1** - Draw a complete weighted graph representing the problem.

The complete weighted graph was given.

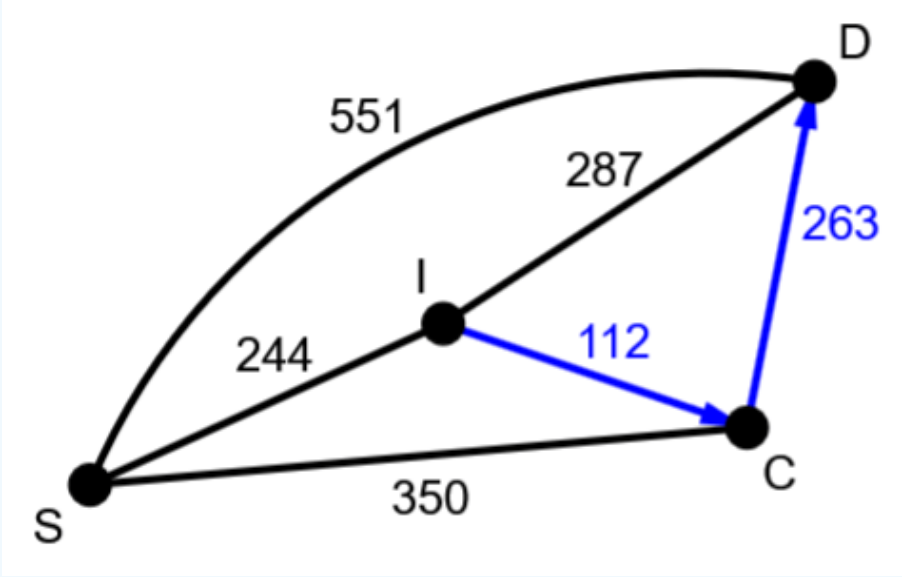
- **Step 2** - Starting at a designated vertex, travel along the edge with the smallest weight to an unvisited vertex. (If there are two or more edges the same, pick any one).

We start with vertex I (Indianapolis), because the salesman lives there. The edge with the smallest weight is I-C (112 miles) which means we travel to vertex C (Cincinnati).



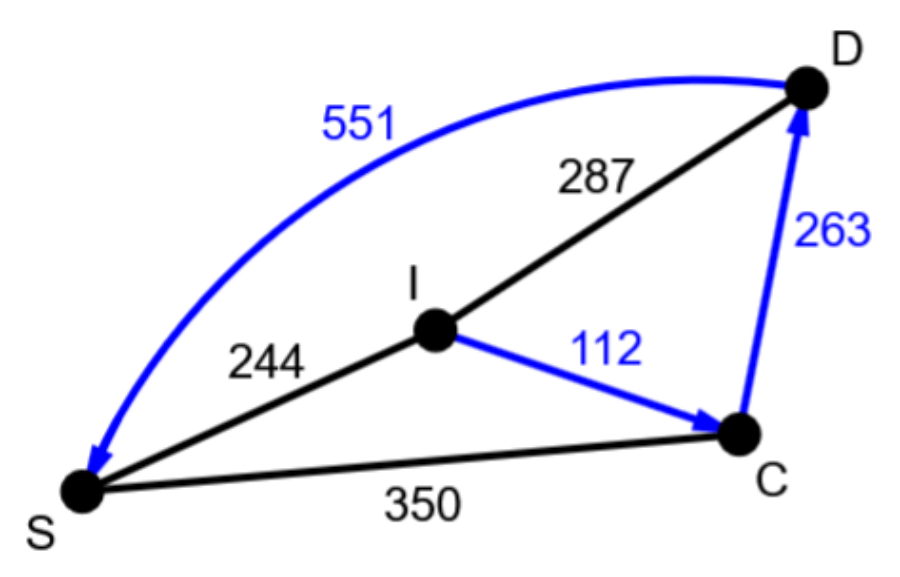
- **Step 3** - From the current vertex, select the edge with the smallest weight that leads to a vertex that has not yet been visited. (If there are two or more edges the same, pick any one).

From vertex C (Cincinnati), the next edge with the smallest weight is C-D (263 miles) which means we travel to vertex D (Detroit).

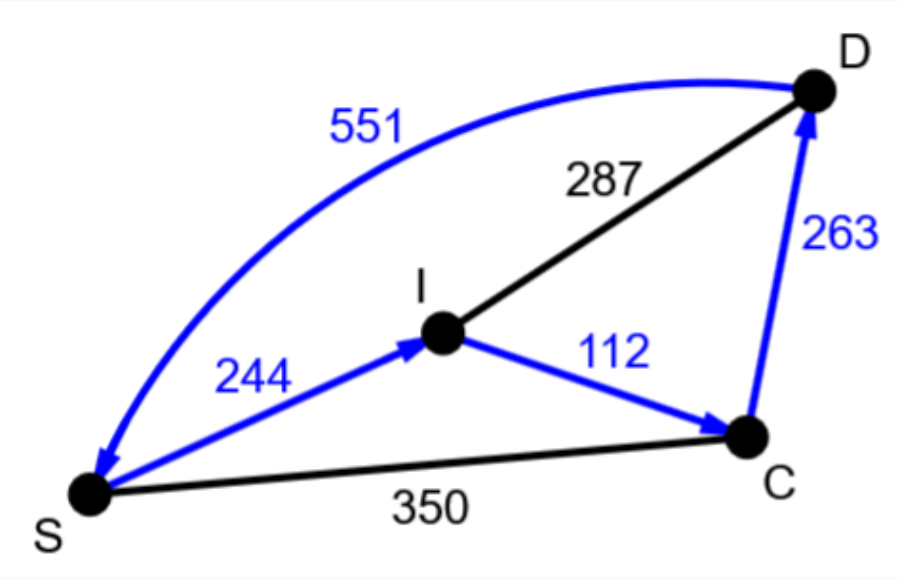


- **Step 4** - Continue this process until all vertices have been visited, then return to the starting vertex to complete the circuit.

From vertex D (Detroit), there is only one other vertex that has not been visited, which is S (St. Louis). Thus, there is no need to check for the smallest weight here. So, the next edge is D-S (551 miles).



From this point, all vertices have been visited, so we use the edge that takes us back to the starting vertex I (Indianapolis), which is S-I (244 miles).



- **Step 5** - The sum of the weights is an approximation to the optimal solution.

Thus, the circuit I-C-D-S-I has a total weight of 112 + 263 + 551 + 244 = 1,170 miles. The circuit in reverse would have the same total weight. Therefore, the two approximate optimal solutions are I-C-D-S-I and I-S-D-C-I, each with a total traveled distance of 1,170 miles.

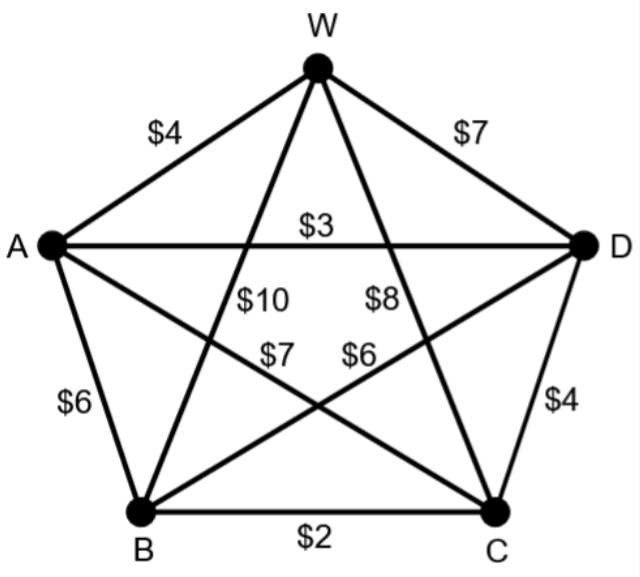
In the example above, the brute force method in Section 8.3 showed that the optimal solution is 1,144 miles hours. In this case, the nearest neighbor method did not produce the same result. So, this is why 1,170 miles is labeled as an approximate optimal solution.

Example #8.5.3 🤔

A small logistics company operates from a warehouse (W) and must deliver packages to four businesses:

- A: Grocery Store
- B: Pharmacy
- C: Electronics Store
- D: Clothing Store

The cost (in dollars) of traveling between each location is shown below:



Using the nearest neighbor method, if the company must deliver packages to each of the four businesses on a daily basis, what route is obtained and what is the total daily cost, assuming the trip begins and ends at the warehouse?

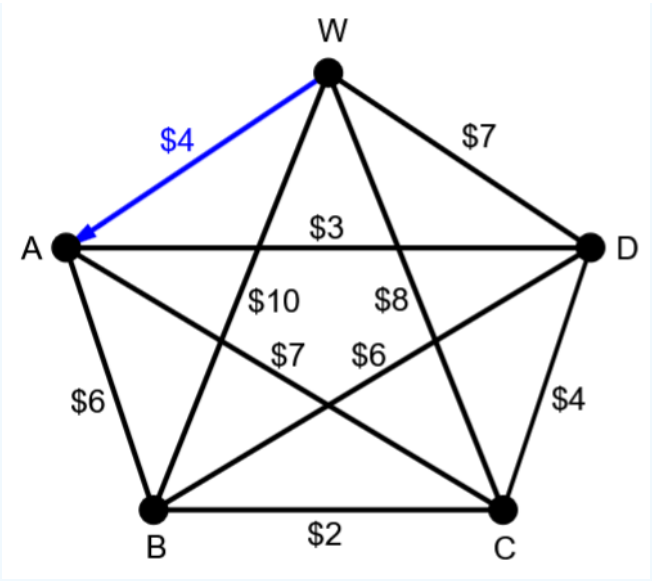
✅ **Solution:**

- **Step 1** - Draw a complete weighted graph representing the problem.

The complete weighted graph was given.

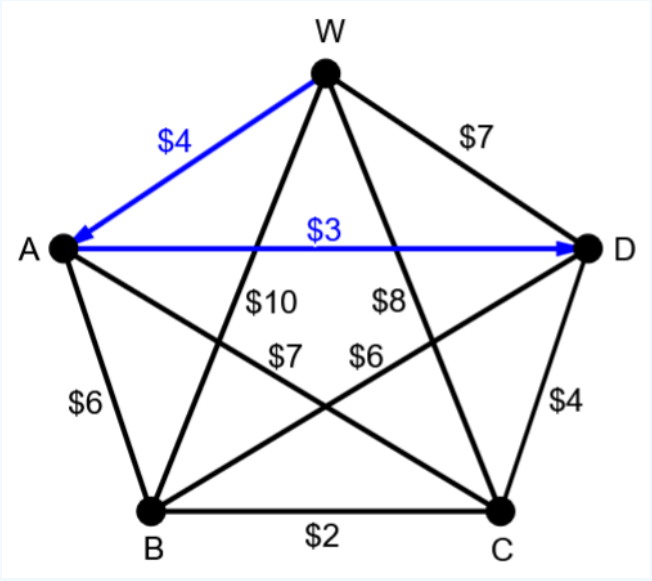
- **Step 2** - Starting at a designated vertex, travel along the edge with the smallest weight to an unvisited vertex. (If there are two or more edges the same, pick any one).

We start with vertex W (Warehouse), because that's where the packages originate from. The edge with the smallest weight is W-A ($4) which means we travel to vertex A (Grocery Store).



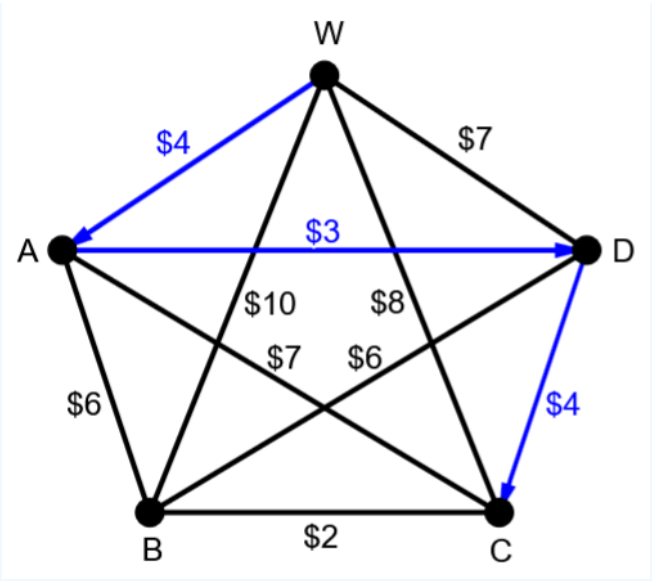
- **Step 3** - From the current vertex, select the edge with the smallest weight that leads to a vertex that has not yet been visited. (If there are two or more edges the same, pick any one).

From vertex A (Grocery Store), the next edge with the smallest weight is A-D ($3) which means we travel to vertex D (Clothing Store).

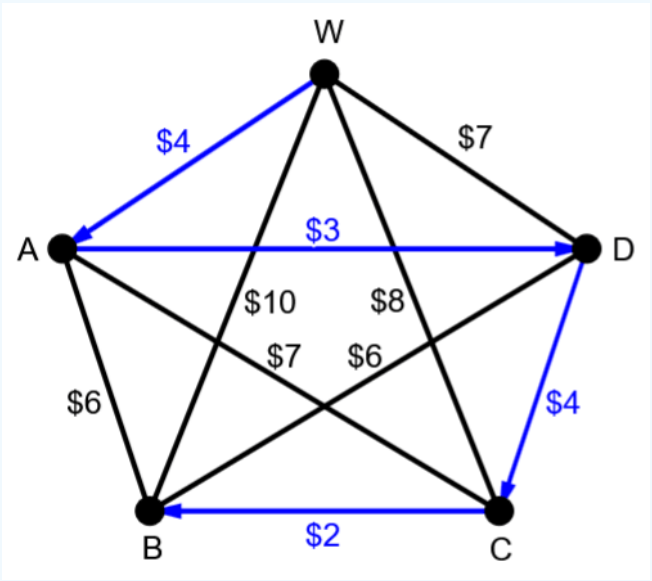


- **Step 4** - Continue this process until all vertices have been visited, then return to the starting vertex to complete the circuit.

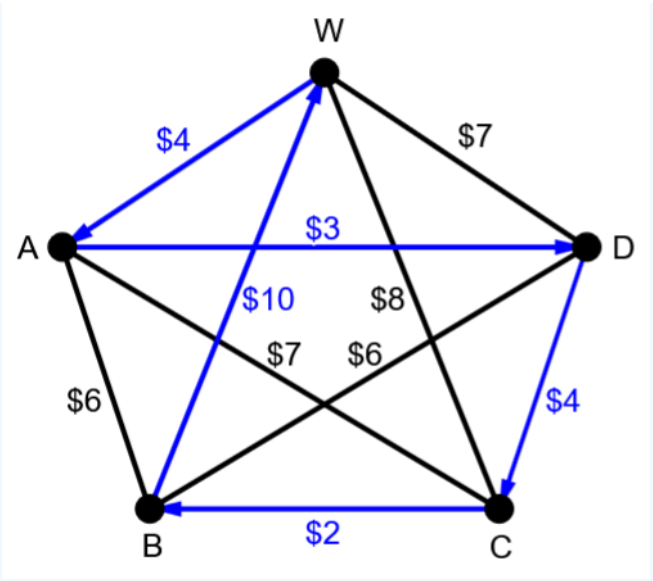
From vertex D (Clothing Store), the next edge with the smallest weight is D-C ($4) which means we travel to vertex C (Electronics Store).



From vertex C (Electronics Store), there is only one other vertex that has not been visited, which is B (Pharmacy). Thus, there is no need to check for the smallest weight here. So, the next edge is C-B ($2).



From this point, all vertices have been visited, so we use the edge that takes us back to the starting vertex W (Warehouse), which is B-W ($10).

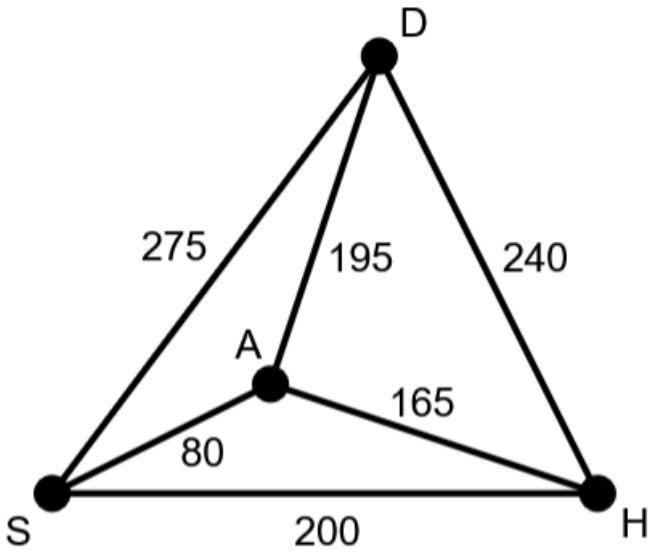


- **Step 5** - The sum of the weights is an approximation to the optimal solution.

Thus, the circuit W-A-D-C-B-W has a total weight of $4 + $3 + $4 + $2 + $10 = $23. The circuit in reverse would have the same total weight. Therefore, the two approximate optimal solutions are W-A-D-C-B-W and W-B-C-D-A-W, each with a total cost of $23.

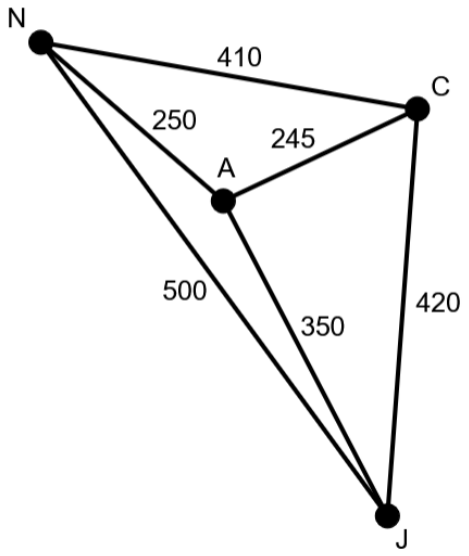
Section 8.5: TSP - The Nearest Neighbor Method [*In-Class Exercises*]

1. A saleswoman lives in Dallas and must travel by car to attend corporate meetings in Austin, Houston, and San Antonio. After completing all meetings, she must return to her home city of Dallas. Her goal is to minimize the total distance of travel. The complete weighted graph with the driving distances (in miles) between the cities is given below:



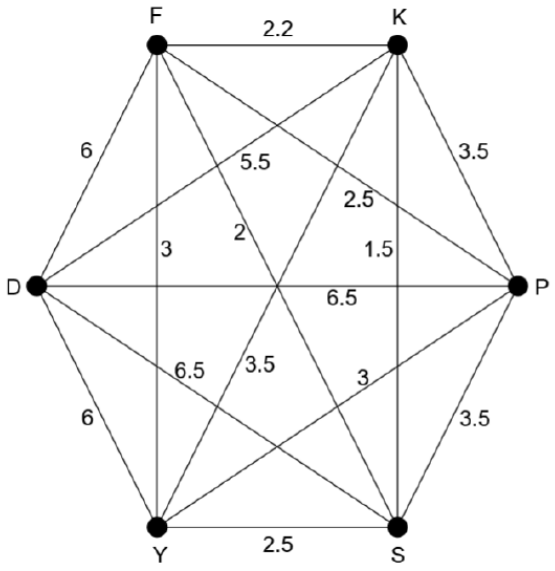
Using the nearest neighbor method, what is the minimum total distance in miles, if she must visit each city exactly once and return to Dallas? Compare this result with the result for In-Class Exercise #1 from Section 8.4.

2. A regional sales manager based in Atlanta must travel to three cities, Charlotte, Nashville, and Jacksonville for client meetings. The complete weighted graph with the approximate driving distances (in miles) between the cities is given below:



Using the nearest neighbor method, what is the minimum total distance in miles, if he must visit each city exactly once and return back to his home in Atlanta? Compare this result with the result for In-Class Exercise #2 from Section 8.4.

3. A family from Fresno is planning a big summer vacation. They want to visit the 5 National Parks that are closest to them. The parks are Death Valley, Kings Canyon, Pinnacles, Sequoia, and Yosemite. The complete weighted graph with the approximate driving times (in hours) between each of the National Parks and Fresno is given below:



Using the nearest neighbor method, what is the minimum time in hours, if the family visits each National Park exactly once and return back to their home in Fresno?

Answers available at the end of this section.

Answers

Section 8.5: TSP - The Nearest Neighbor Method [*In-Class Exercises*]

1. D-A-S-H-D & D-H-S-A-D; 715 miles each. Results were the same using the brute force method and the nearest neighbor method (bfm = 715 and nnm = 715).
2. A-C-N-J-A & A-J-N-C-A; 1,505 miles each. Results were different when using the brute force method and the nearest neighbor method (bfm = 1,415 and nnm = 1,505).
3. There are two different possible pairs of approximate solutions that can be obtained, either one is considered correct. F-S-K-P-Y-D-F & F-D-Y-P-K-S-F with 22.0 hours; or F-S-K-Y-P-D-F & F-D-P-Y-K-S-F with 22.5 hours. The reason for the 2 sets of answers is because after F-S-K there is a tie of 3 hours for the next vertex, P or Y. Taking the next vertex as P instead of Y gets a better time by 0.5 hours. Again, both answers are correct, since the nearest neighbor method may give multiple approximate solutions due to ties along the path.

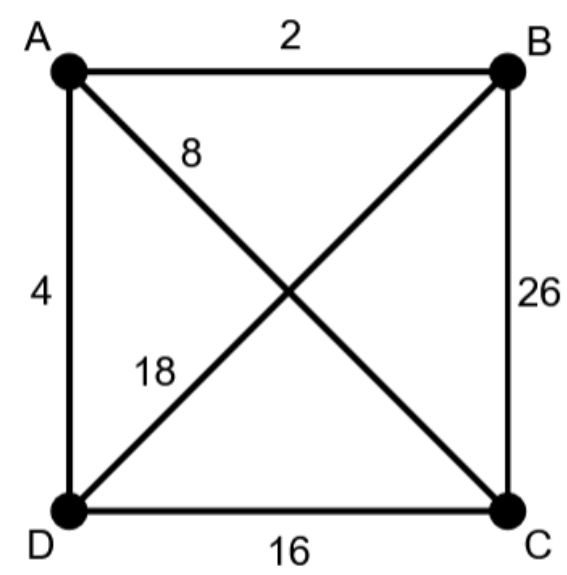
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Section 8.5.H: Homework

Section 8.5: TSP - The Nearest Neighbor Method - Homework Exercises

(Section 8.5) - Homework Exercise #1

Use the nearest neighbor method, starting at vertex A, to find a Hamilton circuit and its total weight.

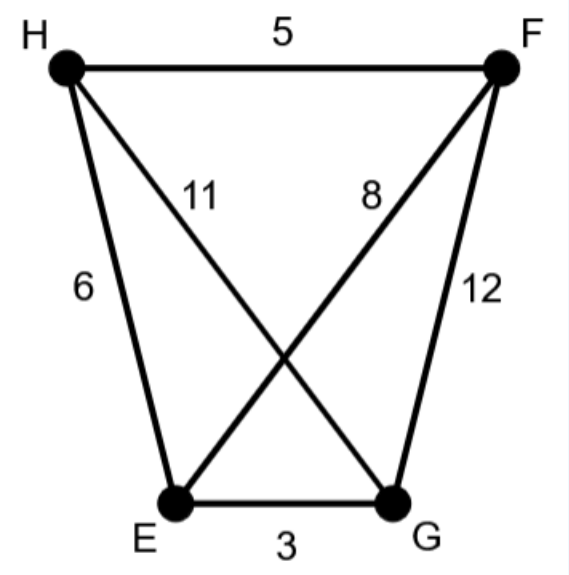


The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 2, C with weight 8, and D with weight 4; B connects to C with weight 26, and D with weight 18; and C connects to D with weight 16.

Answer available at the end of this section.

(Section 8.5) - Homework Exercise #2

Use the nearest neighbor method, starting at vertex E, to find a Hamilton circuit and its total weight.

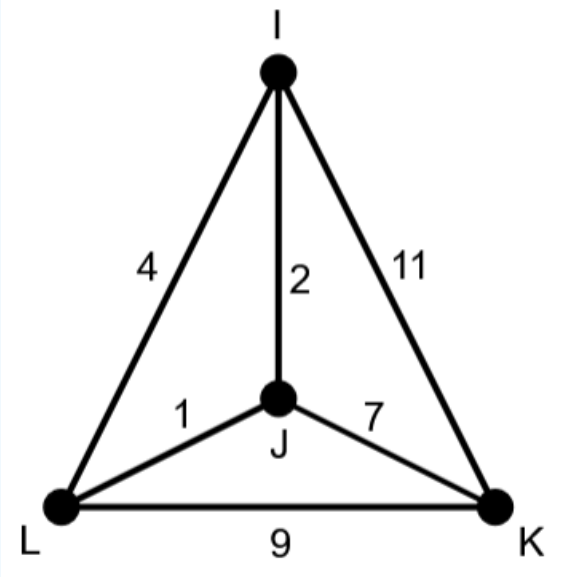


The image shows a weighted graph with labeled nodes connected by line segments. H connects to E with weight 6, F with weight 5, and G with weight 11; E connects to F with weight 8, and G with weight 3; and F connects to G with weight 12.

Answer available at the end of this section.

(Section 8.5) - Homework Exercise #3

Use the nearest neighbor method, starting at vertex I, to find a Hamilton circuit and its total weight.



The image shows a weighted graph with labeled nodes connected by line segments. I connects to J with weight 2, K with weight 11, and L with weight 4; J connects to K with weight 7, and L with weight 1; and K connects to L with weight 9.

Answer available at the end of this section.

(Section 8.5) - Homework Exercise #4

A food delivery driver begins at a central kitchen (K) and must deliver meals to four customers:

O: Office Building
A: Apartment Complex
H: Hospital
S: School

The cost (in dollars, representing time, fuel, and traffic) between locations is:

K–O = $9, K–A = $10, K–H = $4, K–S = $5
O–A = $8, O–H = $2, O–S = $7
A-H = $6, A–S = $6
H–S = $8

Use the nearest neighbor method, starting at K, to determine a route and its total cost.

Answer available at the end of this section.

(Section 8.5) - Homework Exercise #5

Sydney is flying into Calgary, Alberta from Anaheim, California and wants to do a tour of the National Hockey League's home teams within the area. She wants to see the Calgary Flames, the Edmonton Oilers, the Seattle Kraken, the Vancouver Canucks, and the Winnipeg Jets. After driving to all of the NHL stadiums in their cities, she will return back to Calgary for the return flight back home.

She determines the estimated driving distance in miles between each pair of cities as follows:

- Calgary, Alberta ⟷ Edmonton, Alberta: 186 miles
- Calgary, Alberta ⟷ Seattle, Washington: 709 miles
- Calgary, Alberta ⟷ Vancouver, British Columbia: 602 miles
- Calgary, Alberta ⟷ Winnipeg, Manitoba: 823 miles
- Edmonton, Alberta ⟷ Seattle, Washington: 789 miles
- Edmonton, Alberta ⟷ Vancouver, British Columbia: 719 miles
- Edmonton, Alberta ⟷ Winnipeg, Manitoba: 841 miles
- Seattle, Washington ⟷ Vancouver, British Columbia: 141 miles
- Seattle, Washington ⟷ Winnipeg, Manitoba: 1,395 miles
- Vancouver, British Columbia ⟷ Winnipeg, Manitoba: 1,436 miles

Using the nearest neighbor method, starting at Calgary, determine a route and its mileage for Sydney.

Answer available at the end of this section.

(Section 8.5) - Homework Exercise #6

A home healthcare nurse begins at a clinic (C) and must visit five patients before returning to the clinic:

- A: Patient in a retirement home
- B: Patient in an apartment
- D: Patient in a suburban house
- E: Patient in assisted living
- F: Patient in a rural area

The cost (in minutes of travel time) between locations is:

- C–A = 10, C–B = 14, C–D = 8, C–E = 12, C–F = 15
- A–B = 7, A–D = 6, A–E = 9, A–F = 11
- B–D = 10, B–E = 8, B–F = 9
- D–E = 5, D–F = 12
- E–F = 6

Using the nearest neighbor method, starting at the clinic, determine a route and its time for the nurse.

Answer available at the end of this section.

Answers

(Section 8.5) - Homework Exercise #1

Approximate optimal solutions are A-B-D-C-A and A-C-D-B-A both with a weight of 44.

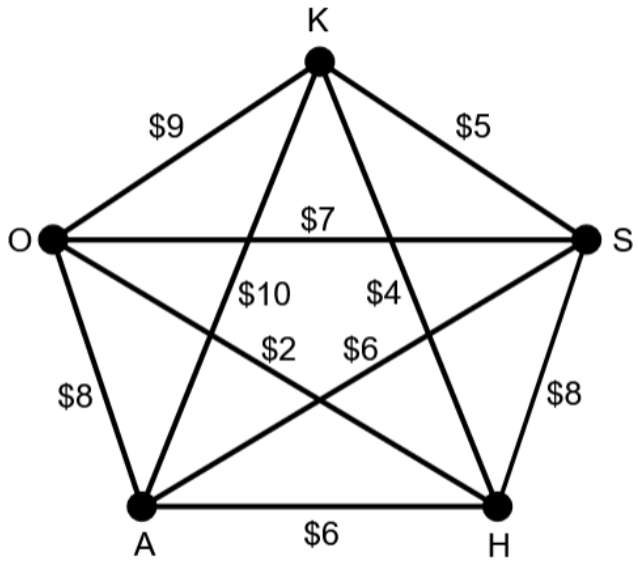
(Section 8.5) - Homework Exercise #2

Approximate optimal solutions are E-G-H-F-E and E-F-H-G-E both with a weight of 27.

(Section 8.5) - Homework Exercise #3

Approximate optimal solutions are I-J-L-K-I and I-K-L-J-I both with a weight of 23.

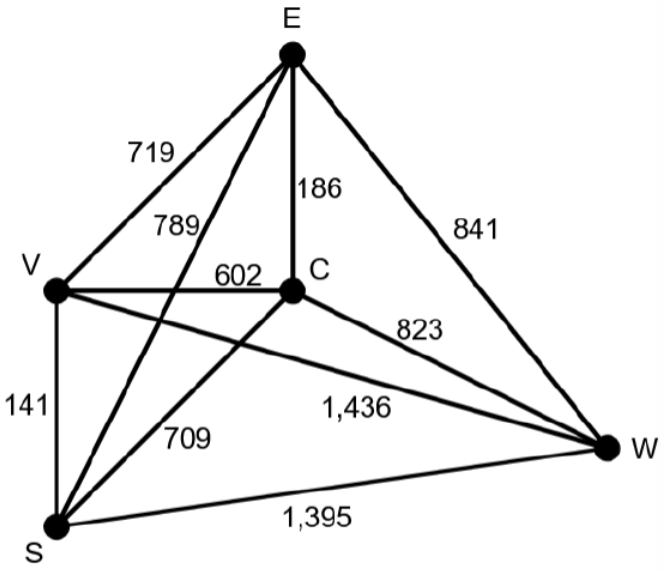
(Section 8.5) - Homework Exercise #4



The image shows a weighted graph with labeled nodes connected by line segments. O connects to A with weight 8 dollars, H with weight 2 dollars, K with weight 9 dollars, and S with weight 7 dollars; A connects to H with weight 6 dollars, K with weight 10 dollars, and S with weight 6 dollars; H connects to K with weight 4 dollars, and S with weight 8 dollars; and K connects to S with weight 5 dollars.

Approximate optimal solutions are K-H-O-S-A-K and K-A-S-O-H-K both with a weight of $29.

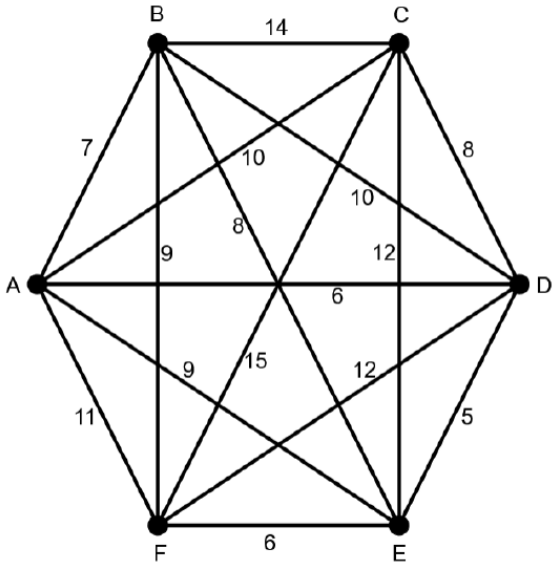
(Section 8.5) - Homework Exercise #5



The image shows a weighted graph with labeled nodes connected by line segments. V connects to C with weight 602, E with weight 719, S with weight 141, and W with weight 1436; C connects to E with weight 186, S with weight 709, and W with weight 823; E connects to S with weight 789, and W with weight 841; and S connects to W with weight 1395.

Approximate optimal solutions are C-E-V-S-W-C and C-W-S-V-E-C both with a weight of 3,264 miles.

(Section 8.5) - Homework Exercise #6



The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 7, D with weight 6, E with weight 9, F with weight 11, and C with weight 10; B connects to C with weight 14, D with weight 10, E with weight 8, and F with weight 9; C connects to D with weight 8, E with weight 12, and F with weight 15; D connects to E with weight 5, and F with weight 12; and E connects to F with weight 6.

Approximate optimal solutions are C-D-E-F-B-A-C and C-A-B-F-E-D-C both with a weight of 45 minutes.

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Section 8.6: TSP - The Cheapest Link Algorithm

Learning Objectives

- Solve The Traveling Salesperson Problem using the Cheapest Link Algorithm

While the nearest neighbor method provides a quick and intuitive way to construct a route, it does not always produce the best possible solution as we saw firsthand in a few examples. Because it makes decisions based only on the closest next vertex, it can overlook better overall routes. This limitation motivates the development of alternative methods that consider the graph more globally rather than step by step.

One such method is the **cheapest link algorithm**. Instead of building a route by moving from one vertex to the next, this approach focuses on selecting the least expensive edges in the graph, regardless of where they occur. By considering all edges at once and choosing the smallest available weights, the algorithm attempts to construct a low-cost circuit more efficiently.

Cheapest Link Algorithm (for Solving TSP)

The **cheapest link algorithm** solves the Traveling Salesperson Problem and is designed to find a good solution quickly, though it does not always guarantee the optimal one.

- **Step 1** - Draw a complete weighted graph representing the problem.
- **Step 2** - Choose an edge with the smallest weight. (If there are two or more edges the same, pick any one).
- **Step 3** - Continue selecting edges in order of increasing weight. (If there are two or more edges the same, pick any one). Only choose an edge if it:
  - does not create a smaller circuit that fails to include all vertices, and
  - does not result in more than two edges being connected to any single vertex.
- **Step 4** - Repeat Step 3 until a Hamilton circuit is formed that includes every vertex exactly once.
- **Step 5** - The total weight of this circuit is an approximation of the optimal solution.

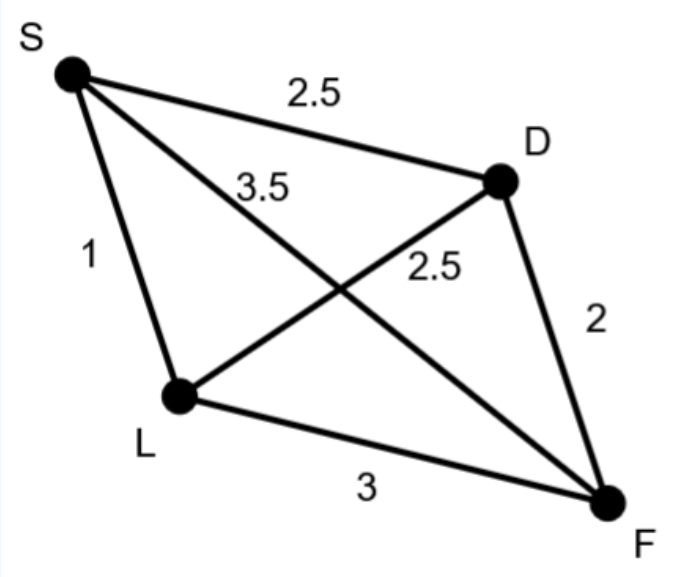
The cheapest link algorithm works by repeatedly selecting the edge with the smallest weight, provided that doing so does not create a smaller circuit prematurely or cause any vertex to have more than two edges. This process continues until a complete circuit is formed that includes every vertex exactly once.

Like the nearest neighbor method, the cheapest link algorithm is a heuristic. It is generally faster than the brute force method and often produces a good solution. However, it does not guarantee the optimal route in every case. Its advantage lies in balancing efficiency with a broader view of the graph, making it a useful alternative when solving larger Traveling Salesperson Problems.

The cheapest link algorithm was developed to improve upon some of the limitations of the nearest neighbor method by considering edge weights more globally. While it still does not ensure an optimal solution, it often provides a better approximation and is an important tool in solving real-world routing problems.

Example #8.6.1 🤔

A saleswoman lives in Los Angeles and must travel by air to attend regional meetings in San Francisco, Denver, and Fort Worth. After completing all meetings, she must return to her home city of Los Angeles. Her goal is to minimize total in-air flight time. The complete weighted graph with the in-air flight time (in hours) between the cities is given below:



Using the cheapest link algorithm, what is the minimum total number of hours spent in the air, if she must visit each city exactly once and return to Los Angeles?

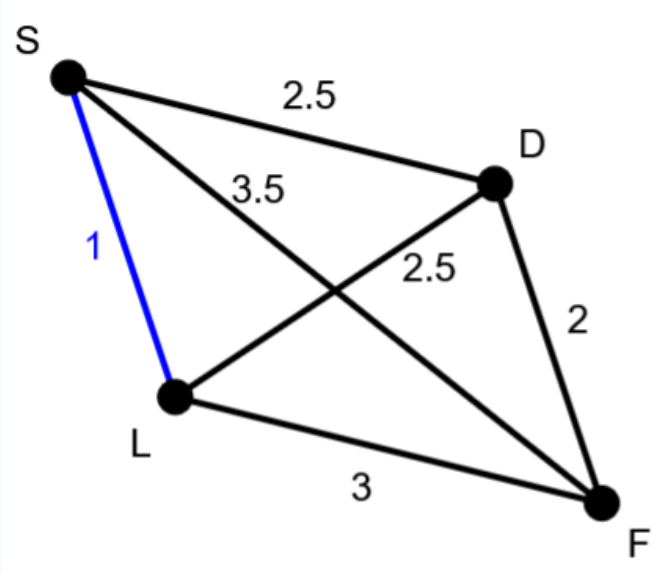
✅ **Solution:**

- **Step 1** - Draw a complete weighted graph representing the problem.

The complete weighted graph was given.

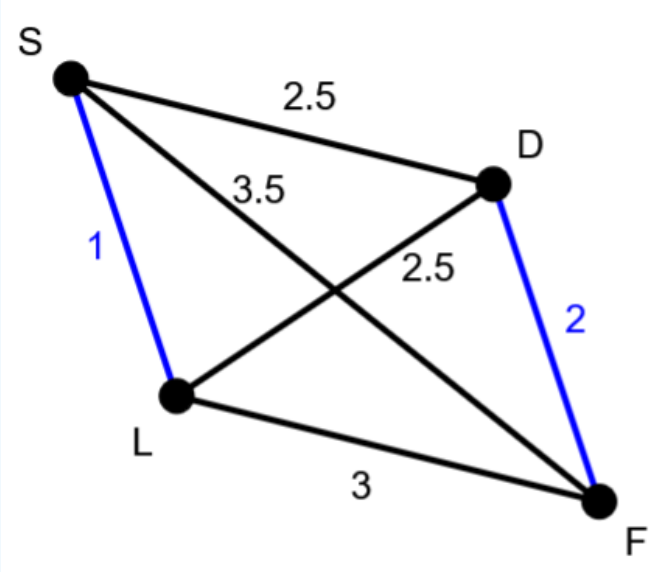
- **Step 2** - Choose an edge with the smallest weight. (If there are two or more edges the same, pick any one).

The edge with the smallest weight is L-S (1 hour). Now, highlight that edge.



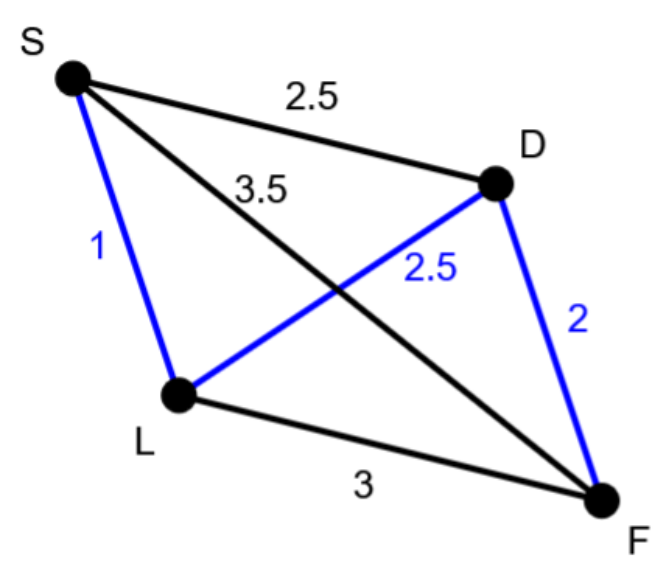
- **Step 3** - Continue selecting edges in order of increasing weight. (If there are two or more edges the same, pick any one). Only choose an edge if it:
  - does not create a smaller circuit that fails to include all vertices, and
  - does not result in more than two edges being connected to any single vertex.

The next edge with the smallest weight is D-F (2 hours). Now, highlight that edge.

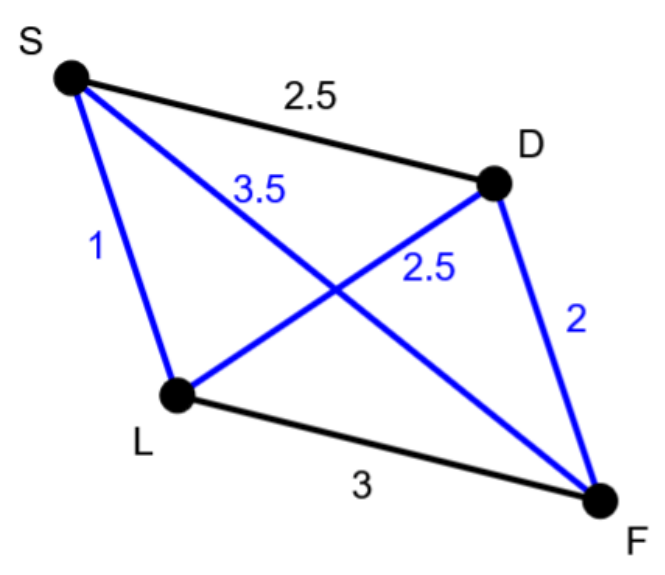


- **Step 4** - Repeat Step 3 until a Hamilton circuit is formed that includes every vertex exactly once.

The next edge with the smallest weight is either D-S (2.5 hours) or D-L (2.5 hours). Let's choose D-L. Now, highlight that edge.



The next edge with the smallest weight is D-S (2.5 hours). However, that creates 3 edges at vertex D. This means a circuit cannot be made. The next edge with the smallest weight is L-F (3 hours). However, adding this edge would create a smaller circuit that does not visit every vertex and does not start and end at vertex L. Thus, the only edge remaining is S-F (3.5 hours). Now, highlight that edge. (Technically, all vertices had already been visited, so it would have been easier to just choose the only edge that would create the circuit).



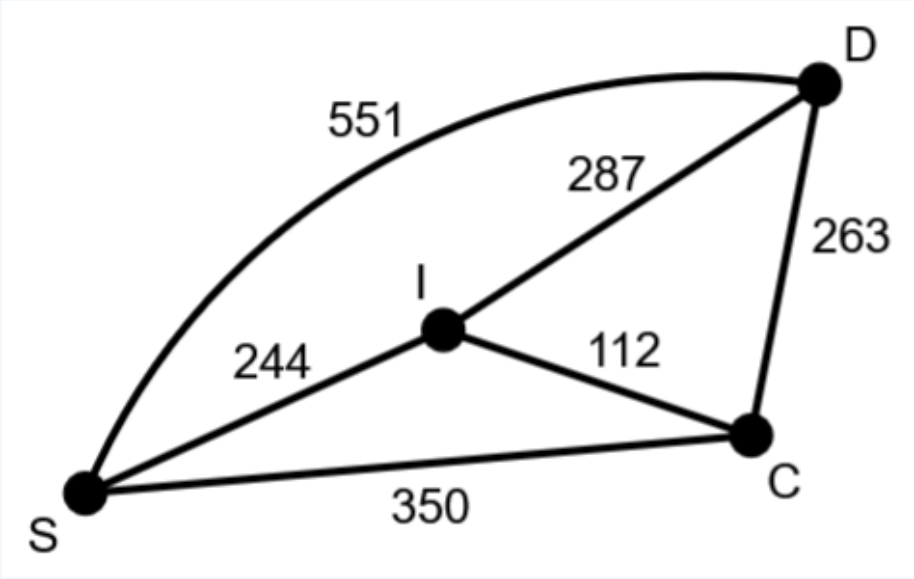
- **Step 5** - The total weight of this circuit is an approximation of the optimal solution.

Remember, we need to start with vertex L (Los Angeles). Thus, the circuit L-S-F-D-L has a total weight of 1 + 3.5 + 2.5 + 2 = 9 hours. The circuit in reverse would have the same total weight. Therefore, the two approximate optimal solutions are L-S-F-D-L and L-D-F-S-L, each with a total travel time of 9 hours.

Note: For the third edge from above where there was a tie, if you were to choose D-S instead of D-L, then the resulting circuits would have been L-S-D-F-L and L-F-D-S-L, each with a total of 8.5 hours, which happens to be the optimal solution. It is also the same solution you would get with both the brute force method and nearest neighbor method. Again, both answers are correct, since the cheapest link algorithm may give multiple approximate solutions due to ties along the path.

Example #8.6.2 🤔

A salesman from Indianapolis has a regional manager's meeting in his hometown as well as Detroit, St. Louis, and Cincinnati. He wants to minimize the mileage as best as he can. Find the optimal solution for the traveling salesperson problem using the cheapest link algorithm.



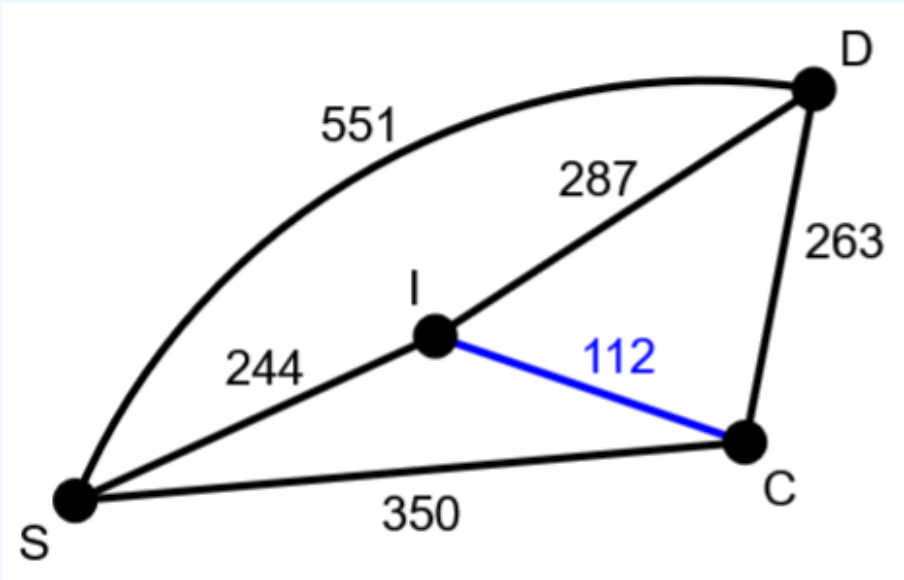
✅ **Solution:**

- **Step 1** - Draw a complete weighted graph representing the problem.

The complete weighted graph was given.

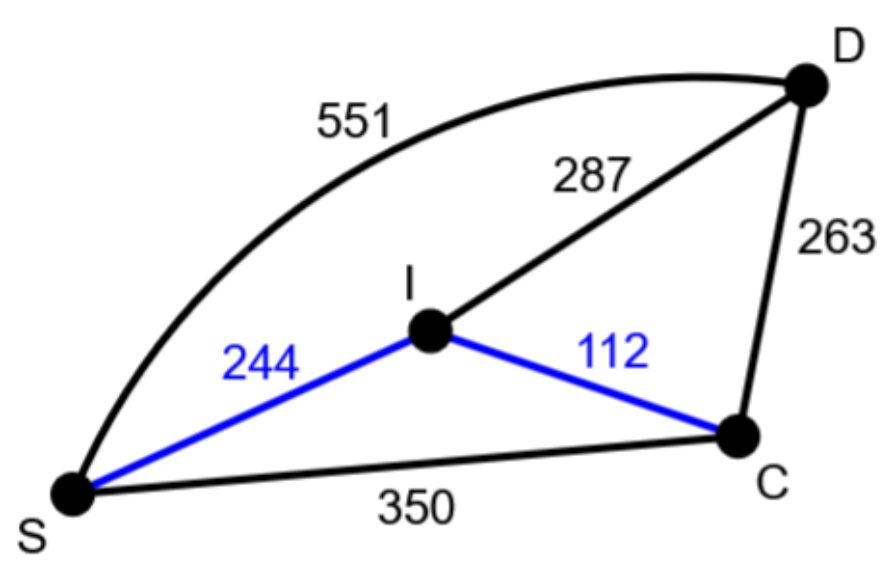
- **Step 2** - Choose an edge with the smallest weight. (If there are two or more edges the same, pick any one).

The edge with the smallest weight is I-C (112 miles). Now, highlight that edge.



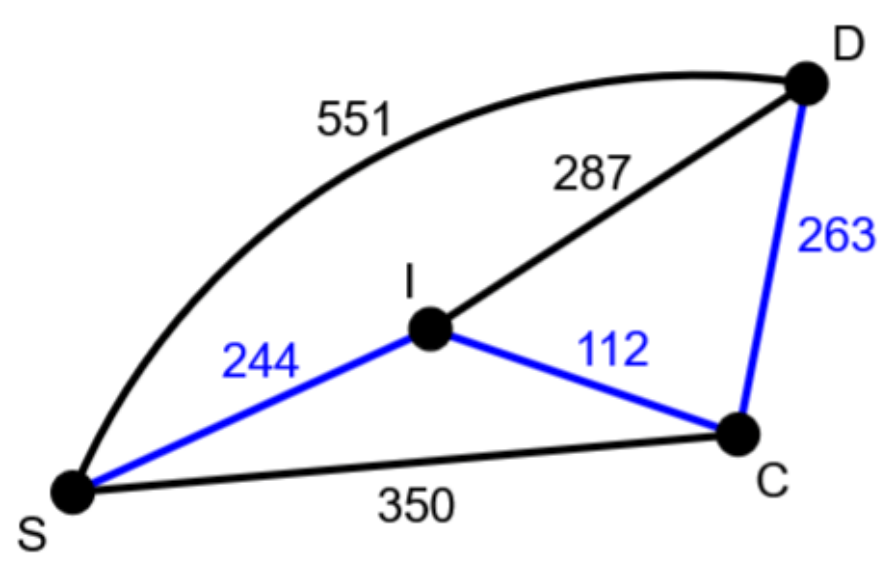
- **Step 3** - Continue selecting edges in order of increasing weight. (If there are two or more edges the same, pick any one). Only choose an edge if it:
  - does not create a smaller circuit that fails to include all vertices, and
  - does not result in more than two edges being connected to any single vertex.

The next edge with the smallest weight is S-I (244 miles). Now, highlight that edge.

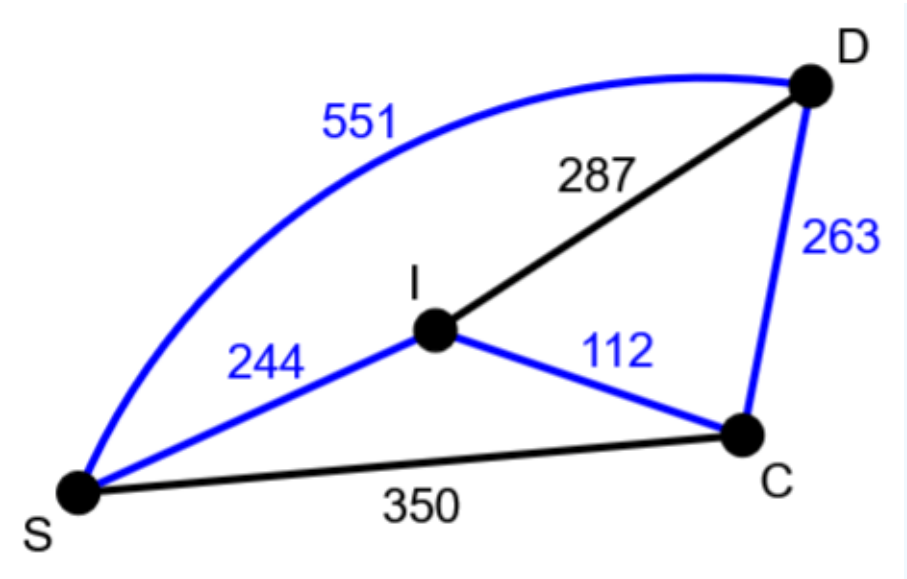


- **Step 4** - Repeat Step 3 until a Hamilton circuit is formed that includes every vertex exactly once.

The next edge with the smallest weight is S-I (263 miles). Now, highlight that edge.



The next edge with the smallest weight is I-D (287 miles). However, that creates 3 edges at vertex D. This means a circuit cannot be made. So, the next edge with the smallest weight is S-C (350 miles), but this would create a smaller circuit that fails to visit all vertices (missing vertex D). Thus, the only edge remaining is S-D (551 miles). Now, highlight that edge. (Technically, all vertices had already been visited, so it would have been easier to just choose the only edge that would create the circuit).



- **Step 5** - The total weight of this circuit is an approximation of the optimal solution.

Remember, we need to start with vertex I (Indianapolis). Thus, the circuit I-C-D-S-I has a total weight of 112 + 263 + 551 + 244 = 1,170 miles. The circuit in reverse would have the same total weight. Therefore, the two approximate optimal solutions are I-C-D-S-I and I-S-D-C-I, each with a total traveled distance of 1,170 miles.

In the example above, the brute force method in Section 8.4 showed that the optimal solution is 1,144 miles. In this case, the cheapest link algorithm did not produce the same result. So, this is why 1,170 miles is labeled as an approximate optimal solution. Recall, also, in Section 8.5, that the nearest neighbor method also produced the same result of 1,170 miles as well.

Example #8.6.3 🤔

A small logistics company operates from a warehouse (W) and must deliver packages to four businesses:

- A: Grocery Store
- B: Pharmacy
- C: Electronics Store
- D: Clothing Store

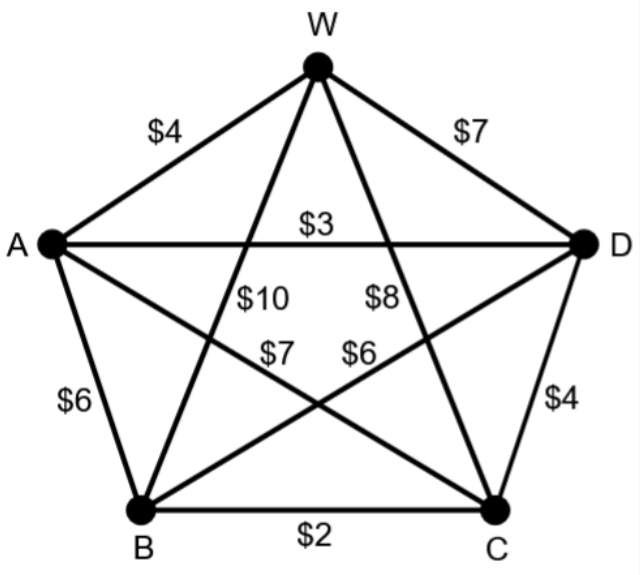
The cost (in dollars, representing time, fuel, and traffic) between locations is:

- W–A = $4, W–B = $10, W–C = $8, W–D = $7
- A–B = $6, A–C = $7, A–D = $3
- B-C = $2, B–D = $6
- C–D = $4

Using the cheapest link algorithm, if the company must deliver packages to each of the four businesses on a daily basis, what route is obtained and what is the total daily cost, assuming the trip begins and ends at the warehouse?

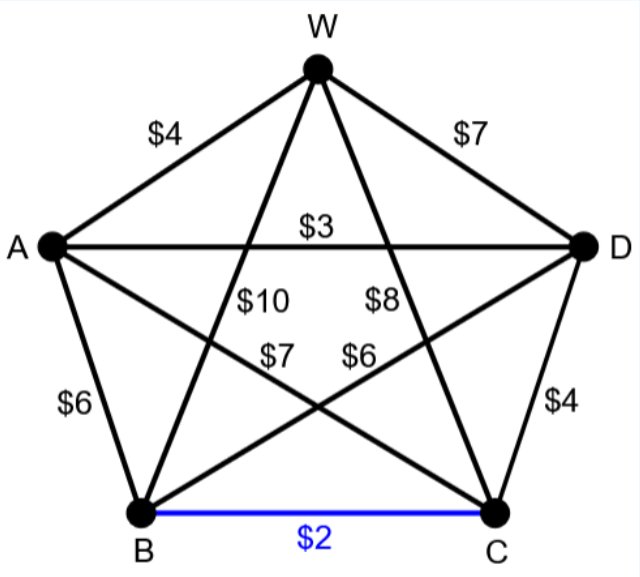
✅ **Solution:**

- **Step 1** - Draw a complete weighted graph representing the problem.



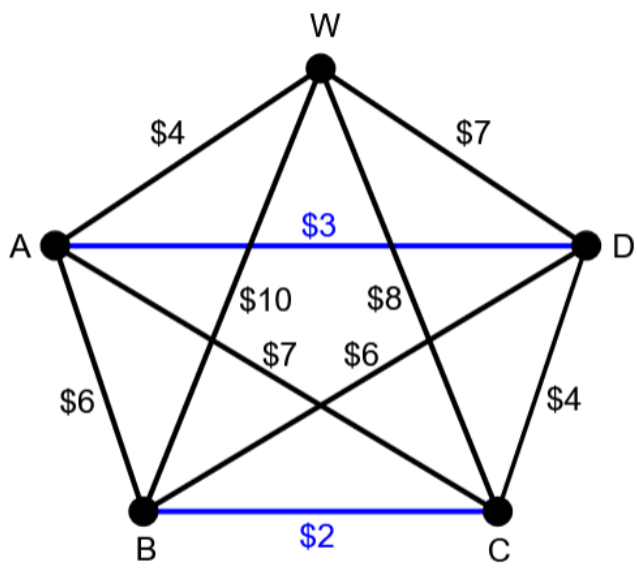
- **Step 2** - Choose an edge with the smallest weight. (If there are two or more edges the same, pick any one).

The edge with the smallest weight is B-C ($2). Now, highlight that edge.



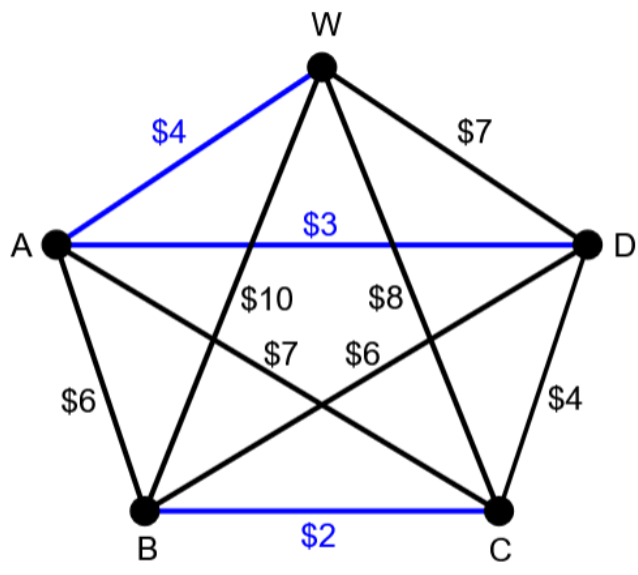
- **Step 3** - Continue selecting edges in order of increasing weight. (If there are two or more edges the same, pick any one). Only choose an edge if it:
  - does not create a smaller circuit that fails to include all vertices, and
  - does not result in more than two edges being connected to any single vertex.

The next edge with the smallest weight is A-D ($3). Now, highlight that edge.

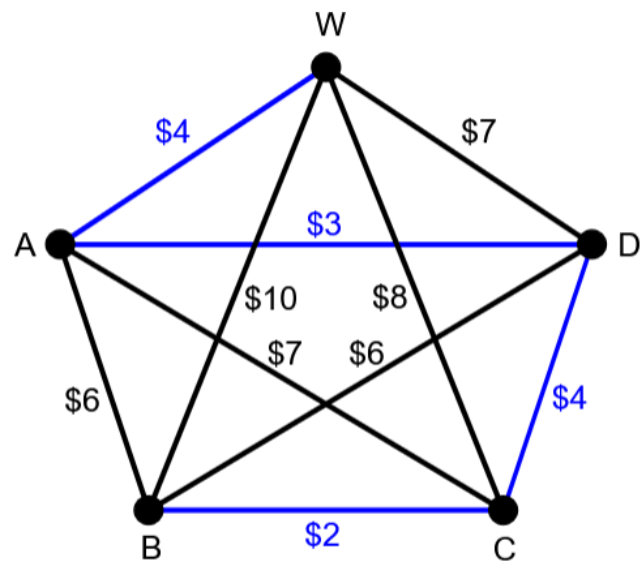


- **Step 4** - Repeat Step 3 until a Hamilton circuit is formed that includes every vertex exactly once.

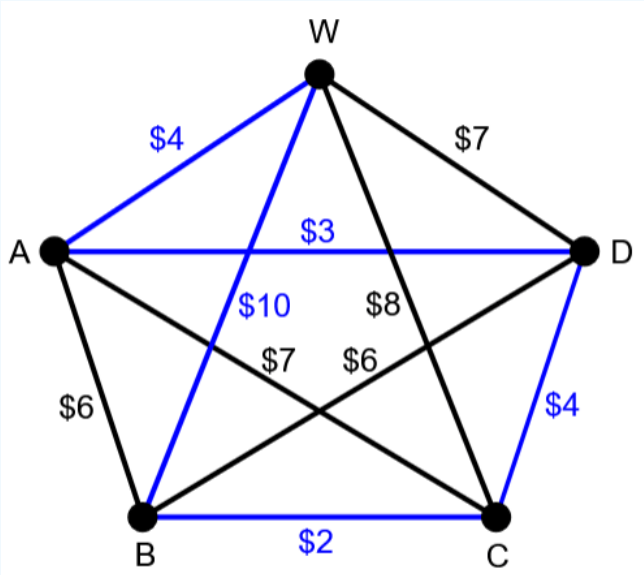
The next edge with the smallest weight is either A-W ($4) or D-C ($4). Let's choose A-W. Now, highlight that edge.



The next edge with the smallest weight is D-C ($4). Now, highlight that edge. (So, it did not matter whether to choose A-W or D-C first, since both edges were going to be used in the algorithm).



From this point, all vertices have been visited, so just choose the only edge that would create the circuit, which is B-W ($10). Now, highlight that edge.



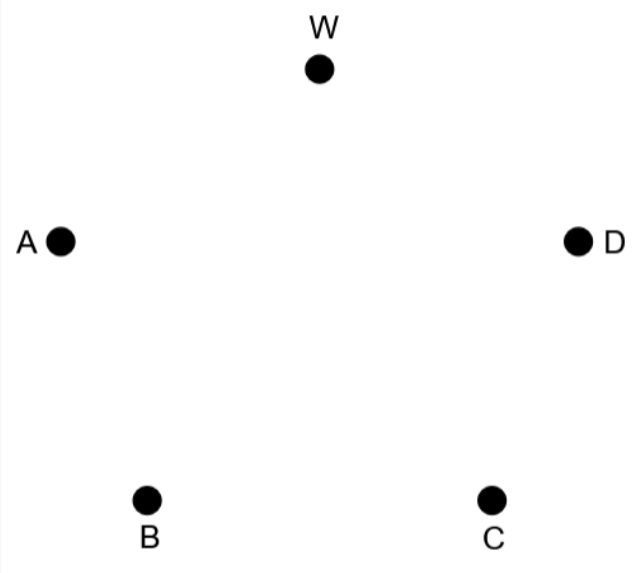
- **Step 5** - The total weight of this circuit is an approximation of the optimal solution.

Thus, the circuit W-A-D-C-B-W has a total weight of $4 + $3 + $4 + $2 + $10 = $23. The circuit in reverse would have the same total weight. Therefore, the two approximate optimal solutions are W-A-D-C-B-W and W-B-C-D-A-W, each with a total cost of $23.

✅ **Alternative Solution:**

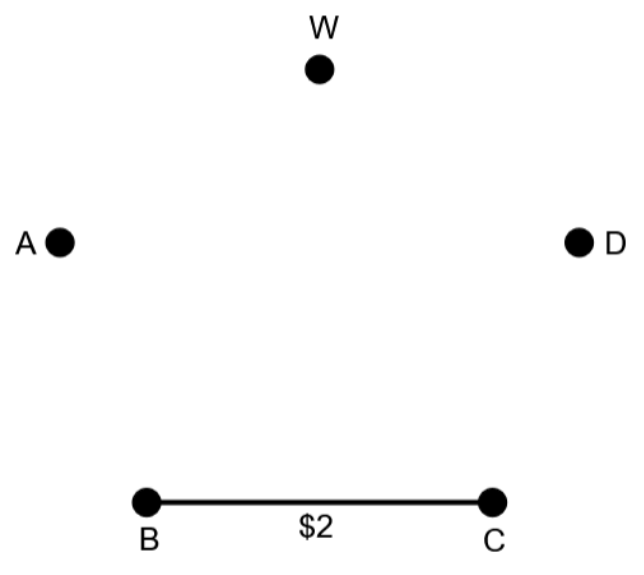
When there are several vertices, sometimes a completed weighted graph may be too time consuming. So, and alternate way is to start with the vertices going around equidistant from each other with no edges. From there, use the cheapest link algorithm and only add the edges that will be correct.

- **Alternate Step 1** - Draw a graph representing only the vertices.



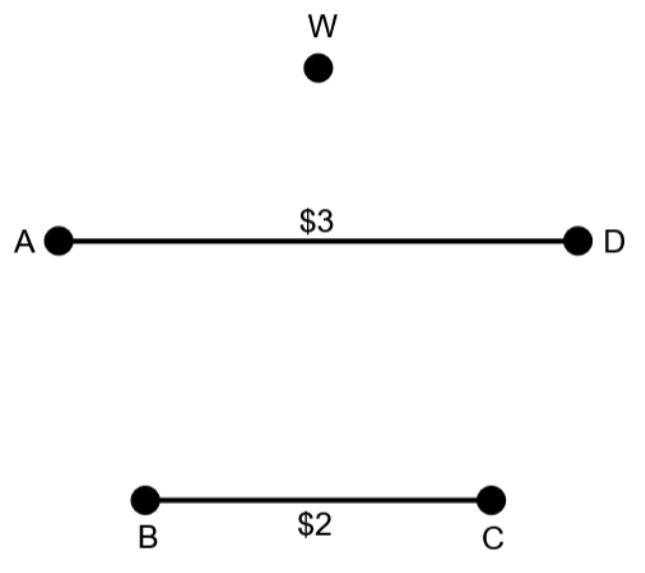
- **Step 2** - Choose an edge with the smallest weight. (If there are two or more edges the same, pick any one).

The edge with the smallest weight is B-C ($2). Now, create that edge.



- **Step 3** - Continue selecting edges in order of increasing weight. (If there are two or more edges the same, pick any one). Only choose/**create** an edge if it:
  - does not create a smaller circuit that fails to include all vertices, and
  - does not result in more than two edges being connected to any single vertex.

The next edge with the smallest weight is A-D ($3). Now, create that edge.

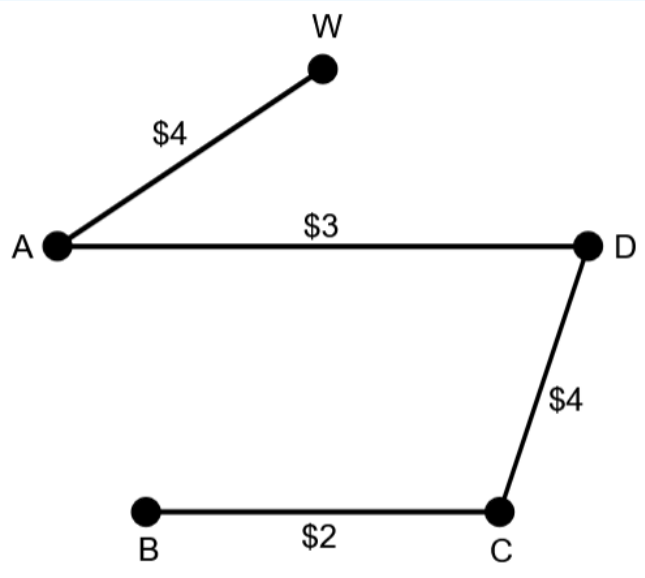


- **Step 4** - Repeat Step 3 until a Hamilton circuit is formed that includes every vertex exactly once.

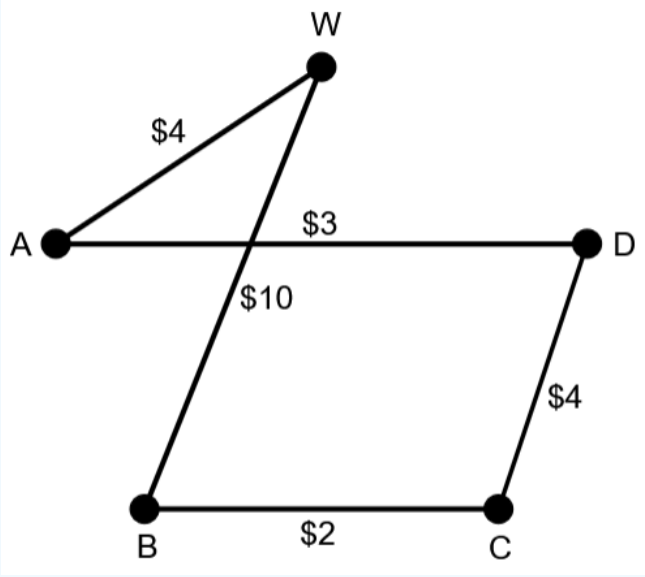
The next edge with the smallest weight is either A-W ($4) or D-C ($4). Let's choose A-W. Now, create that edge.

Weighted graph with nodes, A, B, C, D, and W;.

The next edge with the smallest weight is D-C ($4). Now, create that edge. (So, it did not matter whether to choose A-W or D-C first, since both edges were going to be used in the algorithm).



From this point, all vertices have been visited, so just choose the only edge that would create the circuit, which is B-W ($10). Now, create that edge.

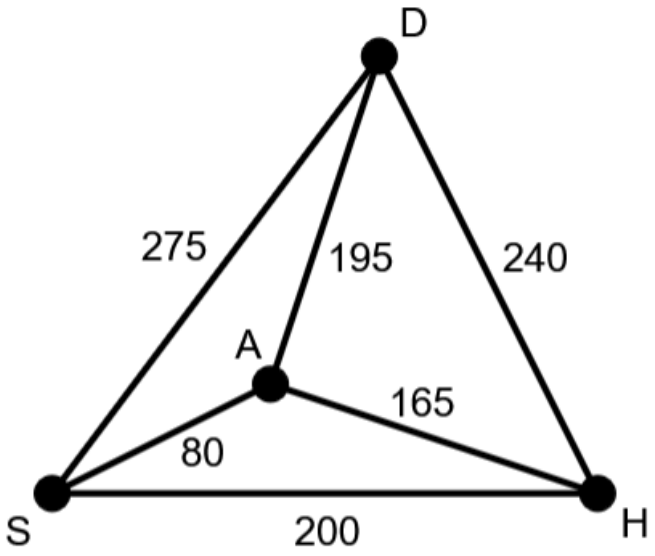


- **Step 5** - The total weight of this circuit is an approximation of the optimal solution.

Thus, the circuit W-A-D-C-B-W has a total weight of $4 + $3 + $4 + $2 + $10 = $23. The circuit in reverse would have the same total weight. Therefore, the two approximate optimal solutions are W-A-D-C-B-W and W-B-C-D-A-W, each with a total cost of $23.

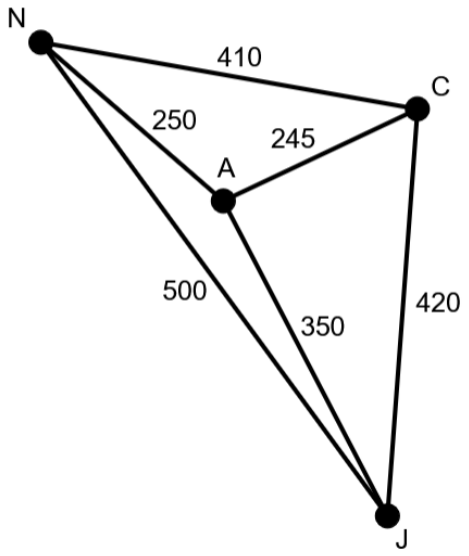
Section 8.6: TSP - The Cheapest Link Algorithm [*In-Class Exercises*]

1. A saleswoman lives in Dallas and must travel by car to attend corporate meetings in Austin, Houston, and San Antonio. After completing all meetings, she must return to her home city of Dallas. Her goal is to minimize the total distance of travel. The complete weighted graph with the driving distances (in miles) between the cities is given below:



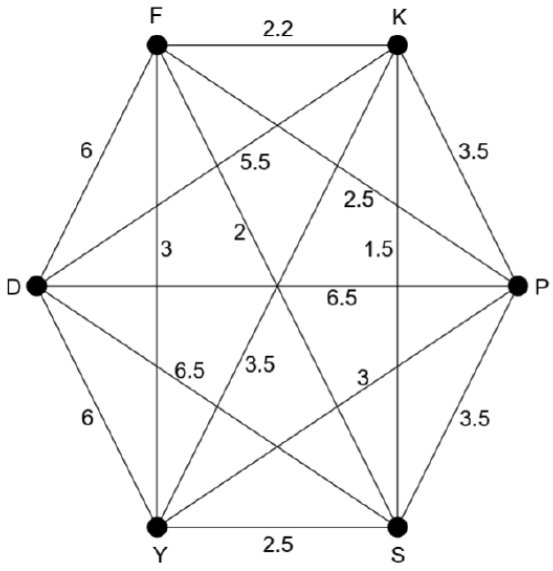
Using the cheapest link algorithm, what is the minimum total distance in miles, if she must visit each city exactly once and return to Dallas? Compare this result with the result for In-Class Exercise #1 from Section 8.4 and In-Class Exercise #1 from Section 8.5.

2. A regional sales manager based in Atlanta must travel to three cities, Charlotte, Nashville, and Jacksonville for client meetings. The complete weighted graph with the approximate driving distances (in miles) between the cities is given below:



Using the cheapest link algorithm, what is the minimum total distance in miles, if he must visit each city exactly once and return back to his home in Atlanta? Compare this result with the result for In-Class Exercise #2 from Section 8.4 and In-Class Exercise #2 from Section 8.5.

3. A family from Fresno is planning a big summer vacation. They want to visit the 5 National Parks that are closest to them. The parks are Death Valley, Kings Canyon, Pinnacles, Sequoia, and Yosemite. The complete weighted graph with the approximate driving times (in hours) between each of the National Parks and Fresno is given below:



Using the cheapest link algorithm, what is the minimum time in hours, if the family visits each National Park exactly once and return back to their home in Fresno? Compare this result with the result for In-Class Exercise #3 from Section 8.5.

Answers available at the end of this section.

Answers

Section 8.6: TSP - The Cheapest Link Algorithm [*In-Class Exercises*]

1. D-S-A-H-D & D-H-A-S-D; 760 miles each. Results were different using the brute force method, nearest neighbor method, and the cheapest link algorithm (bfm = 715, nnm = 715, and cla = 760).
2. A-C-J-N-A & A-N-J-C-A; 1,415 miles each. Results were different when using the nearest neighbor method, but were the same when using the brute force method and the cheapest link algorithm (bfm = 1,415, nnm = 1,505, and cla = 1,415).
3. F-S-K-D-Y-P-F & F-P-Y-D-K-S-F with 20.5 hours each. Results were different when using the nearest neighbor method and the cheapest link algorithm (nnm = 22.0/22.5, and cla = 20.5).

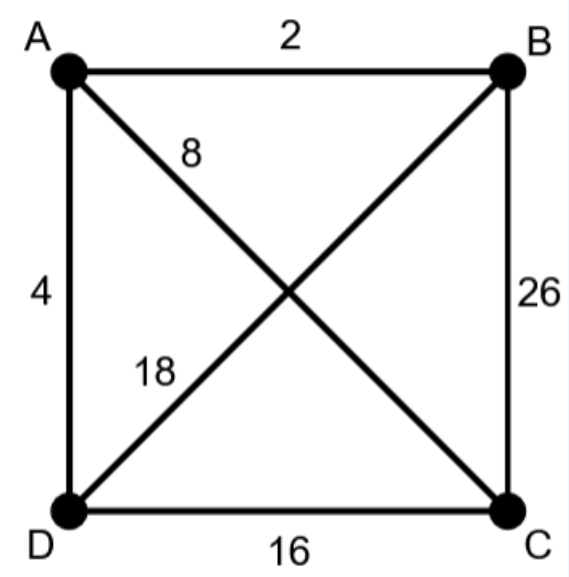
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Section 8.6.H: Homework

Section 8.6: TSP - The Cheapest Link Algorithm - Homework Exercises

(Section 8.6) - Homework Exercise #1

Use the cheapest link algorithm to find a Hamilton circuit and its total weight.

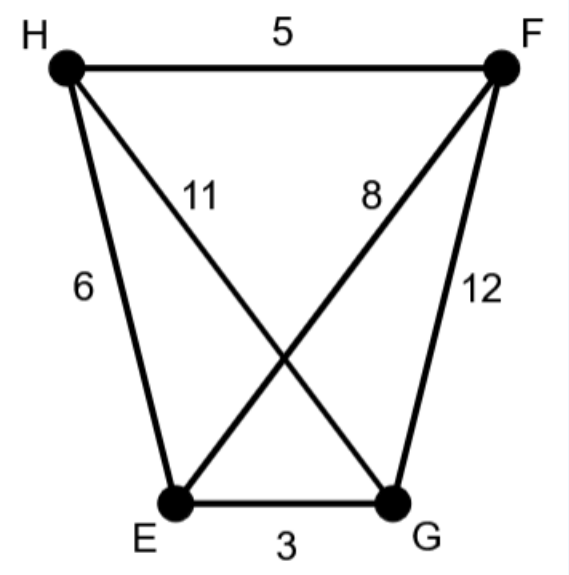


The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 2, C with weight 8, and D with weight 4; B connects to C with weight 26, and D with weight 18; and C connects to D with weight 16.

Answer available at the end of this section.

(Section 8.6) - Homework Exercise #2

Use the cheapest link algorithm to find a Hamilton circuit and its total weight.

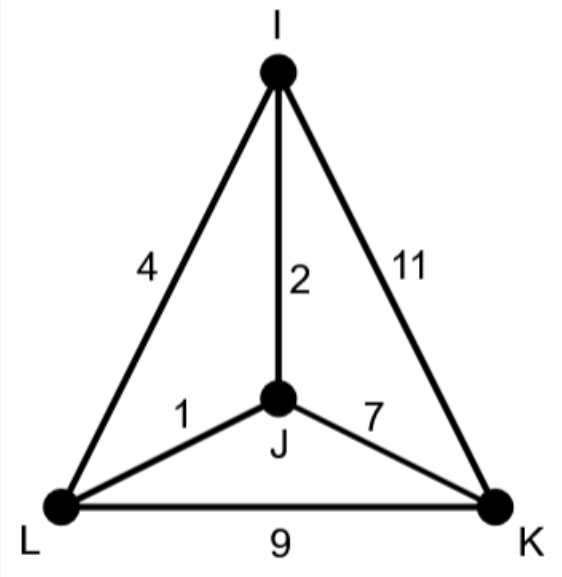


The image shows a weighted graph with labeled nodes connected by line segments. H connects to E with weight 6, F with weight 5, and G with weight 11; E connects to F with weight 8, and G with weight 3; and F connects to G with weight 12.

Answer available at the end of this section.

(Section 8.6) - Homework Exercise #3

Use the cheapest link algorithm to find a Hamilton circuit and its total weight.



The image shows a weighted graph with labeled nodes connected by line segments. I connects to J with weight 2, K with weight 11, and L with weight 4; J connects to K with weight 7, and L with weight 1; and K connects to L with weight 9.

Answer available at the end of this section.

(Section 8.6) - Homework Exercise #4

A food delivery driver begins at a central kitchen (K) and must deliver meals to four customers:

O: Office Building
A: Apartment Complex
H: Hospital
S: School

The cost (in dollars, representing time, fuel, and traffic) between locations is:

K–O = $9, K–A = $10, K–H = $4, K–S = $5
O–A = $8, O–H = $2, O–S = $7
A-H = $6, A–S = $6
H–S = $8

Using the cheapest link algorithm, determine a route and the cost for the driver.

Answer available at the end of this section.

(Section 8.6) - Homework Exercise #5

Sydney is flying into Calgary, Alberta from Anaheim, California and wants to do a tour of the National Hockey League's home teams within the area. She wants to see the Calgary Flames, the Edmonton Oilers, the Seattle Kraken, the Vancouver Canucks, and the Winnipeg Jets. After driving to all of the NHL stadiums in their cities, she will return back to Calgary for the return flight back home.

She determines the estimated driving distance in miles between each pair of cities as follows:

- Calgary, Alberta ⟷ Edmonton, Alberta: 186 miles
- Calgary, Alberta ⟷ Seattle, Washington: 709 miles
- Calgary, Alberta ⟷ Vancouver, British Columbia: 602 miles
- Calgary, Alberta ⟷ Winnipeg, Manitoba: 823 miles
- Edmonton, Alberta ⟷ Seattle, Washington: 789 miles
- Edmonton, Alberta ⟷ Vancouver, British Columbia: 719 miles
- Edmonton, Alberta ⟷ Winnipeg, Manitoba: 841 miles
- Seattle, Washington ⟷ Vancouver, British Columbia: 141 miles
- Seattle, Washington ⟷ Winnipeg, Manitoba: 1,395 miles
- Vancouver, British Columbia ⟷ Winnipeg, Manitoba: 1,436 miles

Using the cheapest link algorithm, determine a route and its mileage for Sydney.

Answer available at the end of this section.

(Section 8.6) - Homework Exercise #6

A home healthcare nurse begins at a clinic (C) and must visit five patients before returning to the clinic:

- A: Patient in a retirement home
- B: Patient in an apartment
- D: Patient in a suburban house
- E: Patient in assisted living
- F: Patient in a rural area

The cost (in minutes of travel time) between locations is:

- C–A = 10, C–B = 14, C–D = 8, C–E = 12, C–F = 15
- A–B = 7, A–D = 6, A–E = 9, A–F = 11
- B–D = 10, B–E = 8, B–F = 9
- D–E = 5, D–F = 12
- E–F = 6

Using the cheapest link algorithm, determine a route and its time for the nurse.

Answer available at the end of this section.

Answers

(Section 8.6) - Homework Exercise #1

Approximate optimal solutions are A-B-C-D-A and A-D-C-B-A both with a weight of 48.

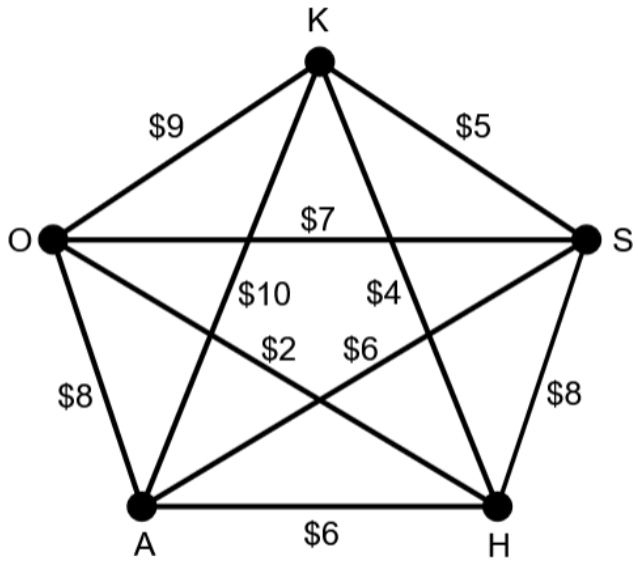
(Section 8.6) - Homework Exercise #2

Approximate optimal solutions are E-G-F-H-E and E-H-F-G-E both with a weight of 26.

(Section 8.6) - Homework Exercise #3

Approximate optimal solutions are I-J-L-K-I and I-K-L-J-I both with a weight of 23.

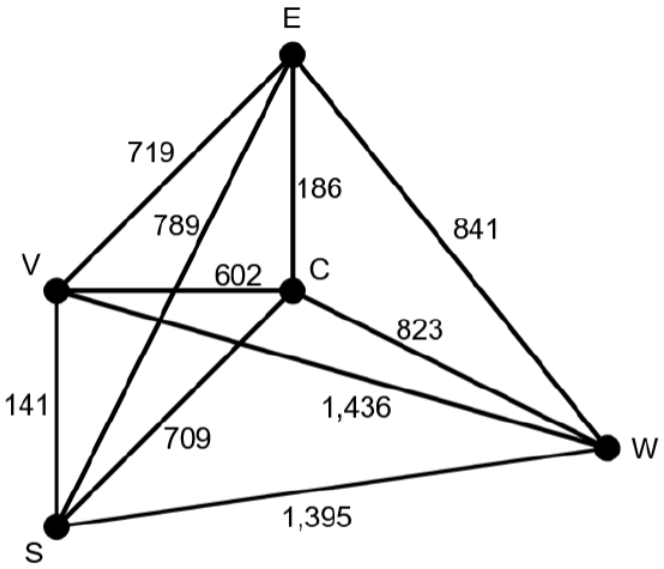
(Section 8.6) - Homework Exercise #4



The image shows a weighted graph with labeled nodes connected by line segments. O connects to A with weight 8 dollars, H with weight 2 dollars, K with weight 9 dollars, and S with weight 7 dollars; A connects to H with weight 6 dollars, K with weight 10 dollars, and S with weight 6 dollars; H connects to K with weight 4 dollars, and S with weight 8 dollars; and K connects to S with weight 5 dollars.

Approximate optimal solutions are K-H-O-A-S-K and K-S-A-O-H-K both with a weight of $25.

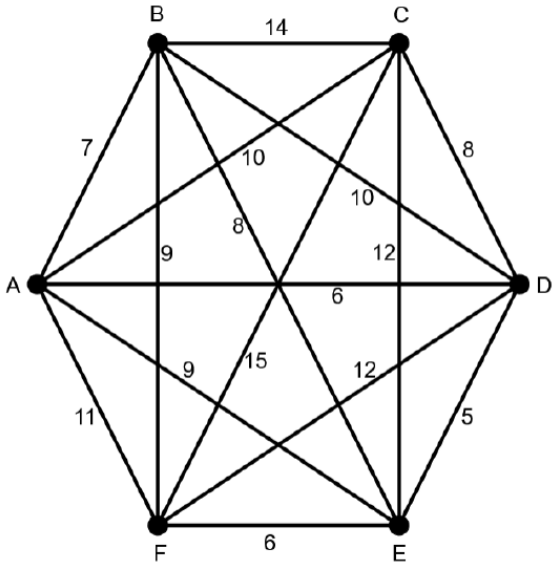
(Section 8.6) - Homework Exercise #5



The image shows a weighted graph with labeled nodes connected by line segments. V connects to C with weight 602, E with weight 719, S with weight 141, and W with weight 1436; C connects to E with weight 186, S with weight 709, and W with weight 823; E connects to S with weight 789, and W with weight 841; and S connects to W with weight 1395.

Approximate optimal solutions are C-V-S-W-E-C and C-E-W-S-V-C both with a weight of 3,165 miles.

(Section 8.6) - Homework Exercise #6



The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 7, D with weight 6, E with weight 9, F with weight 11, and C with weight 10; B connects to C with weight 14, D with weight 10, E with weight 8, and F with weight 9; C connects to D with weight 8, E with weight 12, and F with weight 15; D connects to E with weight 5, and F with weight 12; and E connects to F with weight 6.

Approximate optimal solutions are C-F-E-D-A-B-C and C-B-A-D-E-F-C both with a weight of 53 minutes.

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Section 8.7: Trees

Learning Objectives

- Determine whether a graph is a tree
- Find a spanning tree for a graph

In graph theory, many real-world systems, such as communication networks, transportation routes, and organizational structures, can be represented using graphs. In many of these situations, it is important to connect all points efficiently without creating unnecessary loops (circuits). This leads to an important type of graph known as a **tree**.

Tree

A **tree** is a connected graph that contains no circuits.

In other words, it is a graph in which there is exactly one path between any two vertices. This property ensures that every connection is essential; removing any edge would disconnect the graph, while adding any additional edge would create a circuit.

Trees are especially useful because they represent the simplest way to connect a set of vertices. For example, a company might want to connect several offices using the least amount of wiring, or a computer network may need to ensure efficient communication without redundancy. In each case, a tree structure provides a solution that is both connected and minimal.

Properties of Trees

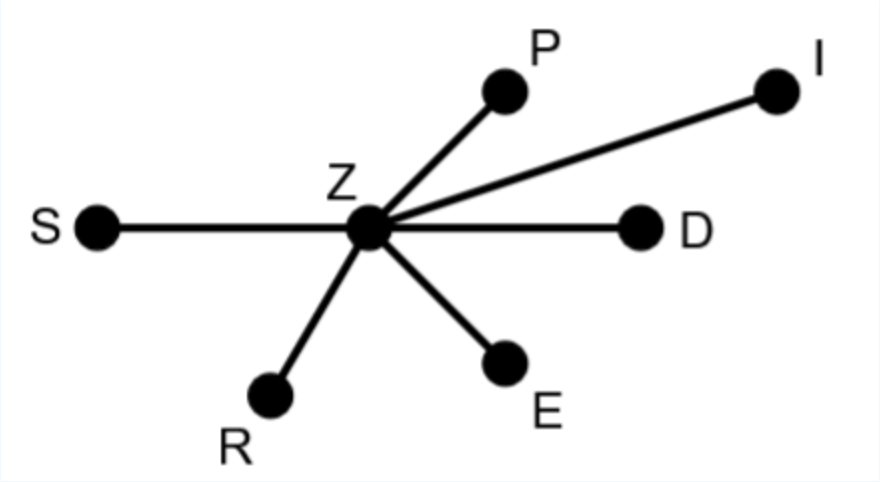
1. A tree is a connected graph (from definition).
2. A tree contains no circuits (from definition).
3. If a tree has n vertices, then it must have exactly $n-1$ edges.
4. Every edge is a bridge.
5. There is exactly one path between any two vertices.

The relationship between the number of vertices and the number of edges highlights the efficiency of trees, as they use the smallest possible number of edges while still maintaining connectivity.

Trees also play a central role in many areas of mathematics and computer science. They are used in designing networks, organizing data (such as file systems and search algorithms), and solving optimization problems. As we continue our study of graph theory, trees will provide a foundation for understanding more advanced concepts, including spanning trees and minimum spanning trees.

Example #8.7.1 🤔

Determine whether the given graph is a tree.



✅ **Solution:**

Let's check the properties for a tree:

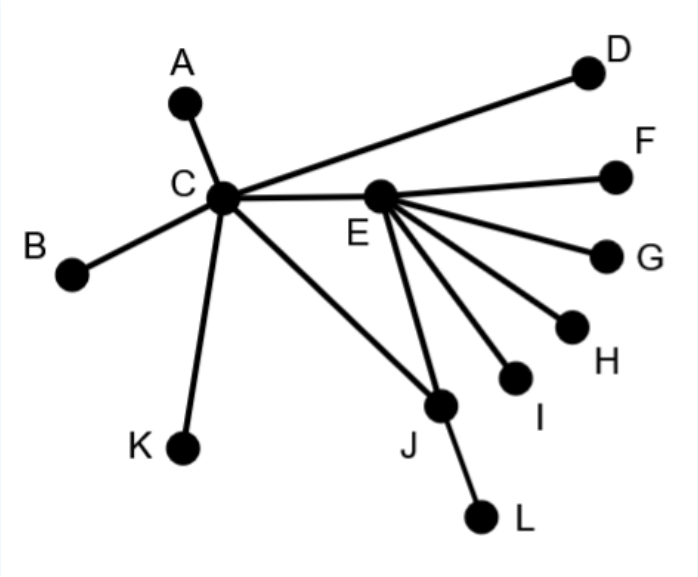
1. The graph is connected. ✓
2. The graph contains no circuits. ✓
3. The graph has 7 vertices and 6 edges. (Edges = Vertices – 1). ✓
4. Every edge is a bridge. ✓
5. There is exactly one path between any two vertices. ✓

Thus, the graph is a tree.

Note: If the first three properties are true, the fourth and fifth properties will always be true as well. Therefore, there is technically no need to check the last two properties separately. However, if you can immediately see that an edge is not a bridge, it may be easier to recognize that the graph is not a tree. There is no required order for checking the properties. So, once one property fails, the graph is not a tree.

Example #8.7.2 🤔

Determine whether the given graph is a tree.



✅ **Solution:**

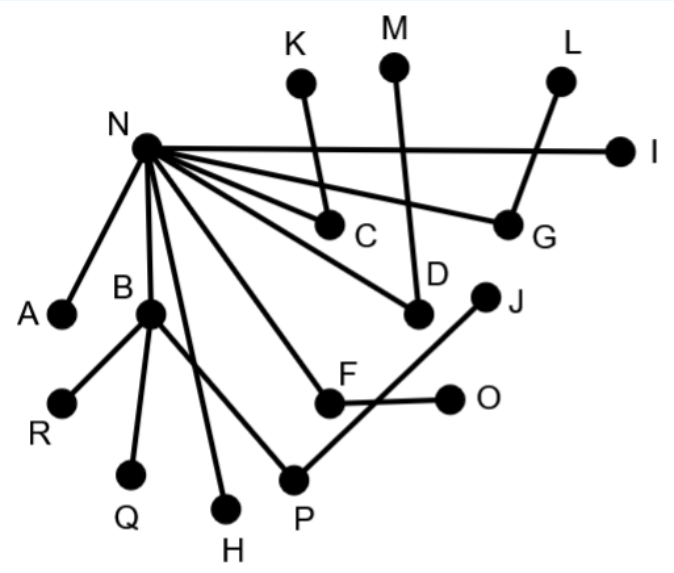
Let's check the properties for a tree:

1. The graph is connected. ✓
2. The graph contains no circuits. ✗.

Thus, the graph is not a tree, because C-E-J-C is a circuit. There is no need to verify any other properties from this point on.

Example #8.7.3 🤔

Determine whether the given graph is a tree.



✅ **Solution:**

Let's check the properties for a tree:

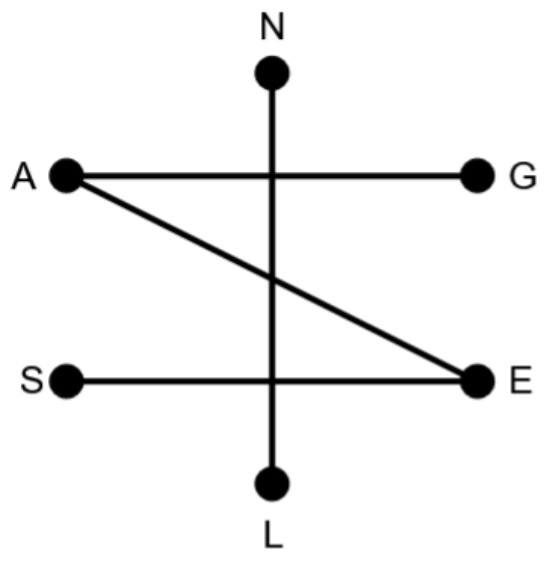
1. The graph is connected. ✓
2. The graph contains no circuits. ✓
3. The graph has 17 vertices and 16 edges. (Edges = Vertices – 1). ✓
4. Every edge is a bridge. ✓
5. There is exactly one path between any two vertices. ✓

Thus, the graph is a tree.

Note: If the first three properties are true, the fourth and fifth properties will always be true as well. Therefore, there is technically no need to check the last two properties separately. However, if you can immediately see that an edge is not a bridge, it may be easier to recognize that the graph is not a tree. There is no required order for checking the properties. So, once one property fails, the graph is not a tree.

Example #8.7.4 🤔

Determine whether the given graph is a tree.



✅ **Solution:**

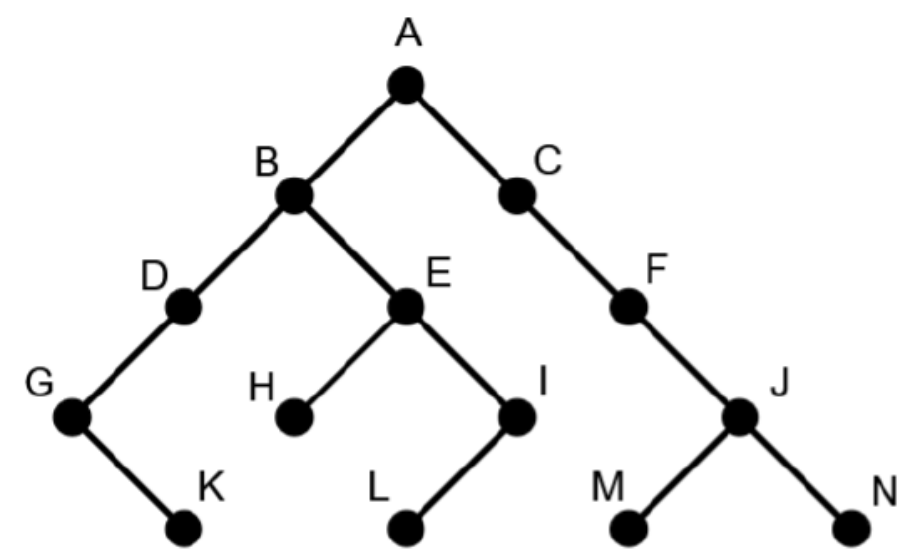
Let's check the properties for a tree:

1. The graph is connected. ✗.

Thus, the graph is not a tree, since edge N-L is disconnected from the rest of the graph and cannot be reached by any of the other vertices, G, A, E, and S. There is no need to verify any other properties from this point on.

Example #8.7.5 🤔

Determine whether the given graph is a tree.



✅ **Solution:**

Let's check the properties for a tree:

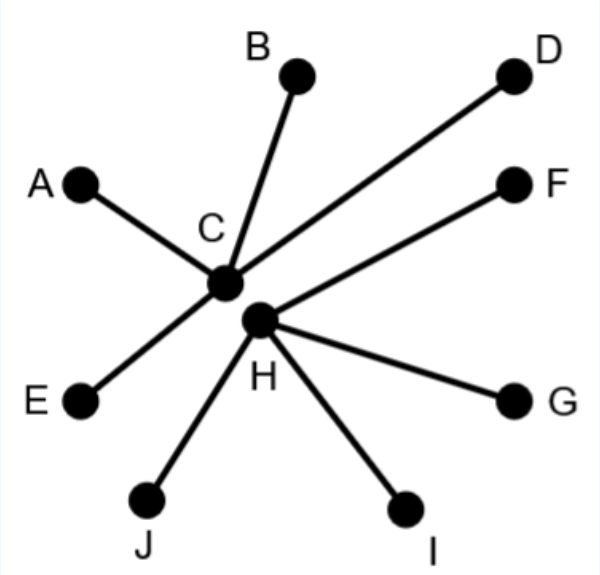
1. The graph is connected. ✓
2. The graph contains no circuits. ✓
3. The graph has 14 vertices and 13 edges. (Edges = Vertices – 1). ✓
4. Every edge is a bridge. ✓
5. There is exactly one path between any two vertices. ✓

Thus, the graph is a tree.

Note: If the first three properties are true, the fourth and fifth properties will always be true as well. Therefore, there is technically no need to check the last two properties separately. However, if you can immediately see that an edge is not a bridge, it may be easier to recognize that the graph is not a tree. There is no required order for checking the properties. So, once one property fails, the graph is not a tree.

Example #8.7.6 🤔

Determine whether the given graph is a tree.



✅ **Solution:**

Let's check the properties for a tree:

1. The graph is connected. ✗.

Thus, the graph is not a tree, since the graph is disconnected. Vertices A, B, C, D, and E cannot reach any of the vertices, F, G, H, I and J and vice-versa. There is no need to verify any other properties from this point on.

Trees provide a simple and efficient way to connect a set of vertices using the minimum number of edges. However, many real-world networks are more complex and contain circuits. This raises an important question: How can we simplify such networks while still maintaining connectivity? This leads to the concept of a **spanning tree**.

Spanning Tree

A **spanning tree** of a graph is a sub-graph that includes all the vertices of the original graph and forms a tree. In other words, it "spans" all the vertices while removing enough edges to eliminate any circuits. The result is a connected graph with no circuits that uses the minimum number of edges.

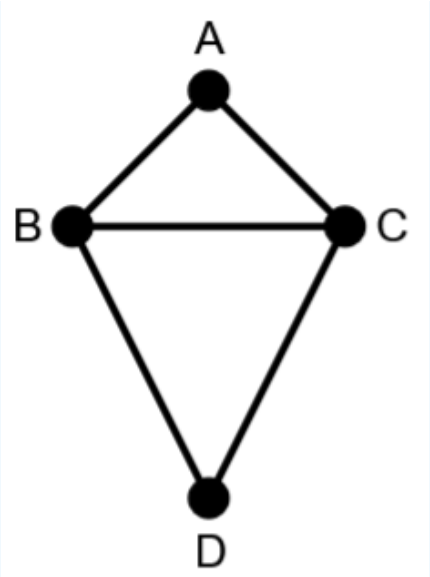
Spanning trees are important because they allow us to take a complex network and reduce it to its most efficient structure without losing connectivity. For example, in designing communication networks, electrical grids, or transportation systems, it is often desirable to connect all locations using the least amount of material or cost. A spanning tree provides a way to achieve this.

Every connected graph has at least one spanning tree, and a graph may have many different spanning trees. Each spanning tree will contain the same number of vertices as the original graph, but exactly $n-1$ edges, where n is the number of vertices.

In summary, the transition from trees to spanning trees allows us to apply the simplicity and efficiency of tree structures to more complex graphs. By extracting a spanning tree from a graph, we preserve connectivity while eliminating unnecessary edges, making it a powerful tool in both theory and real-world applications.

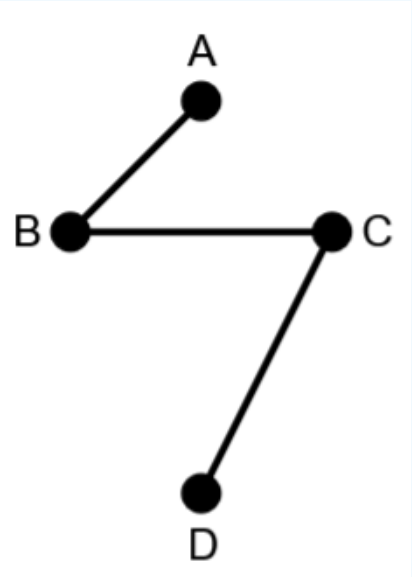
Example #8.7.7 🤔

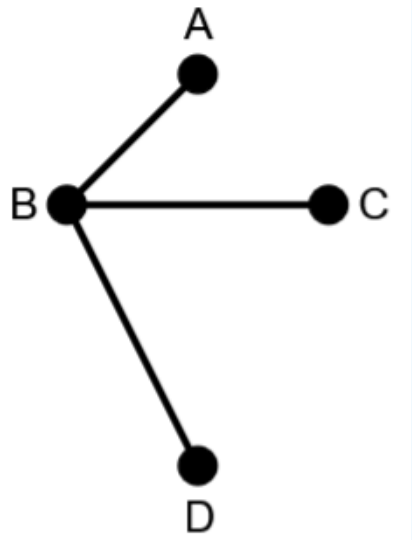
Find a spanning tree for the graph.

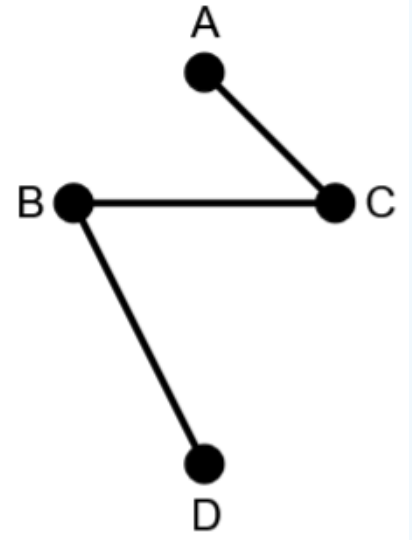


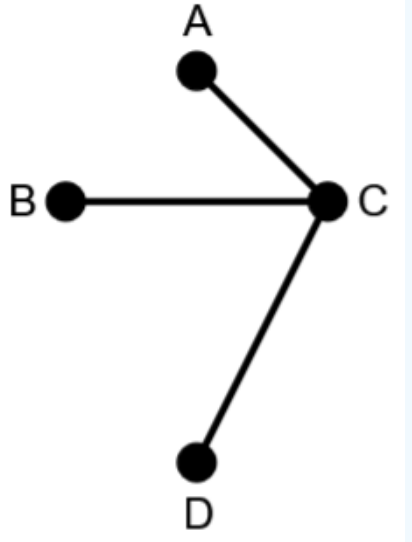
✅ **Solution:**

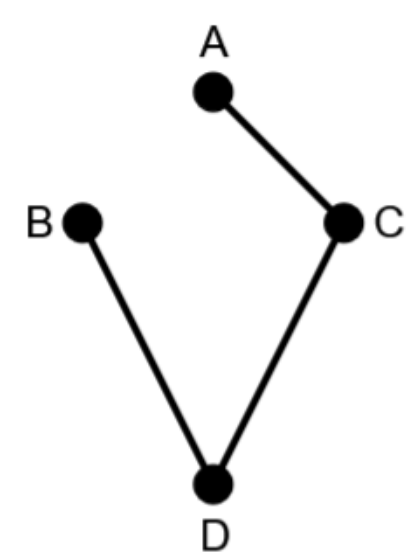
The graph is not a tree. It has 4 vertices and 5 edges, whereas a tree with 4 vertices must have exactly 3 edges. Therefore, 2 edges must be removed to form a tree. In this example, there will be 8 different possible spanning trees:

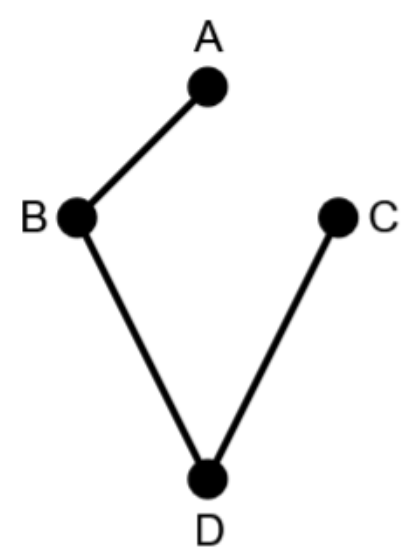


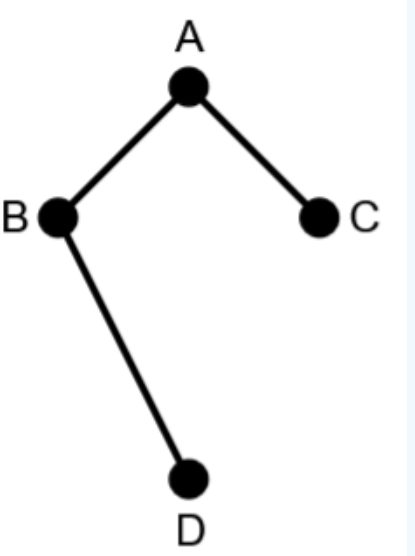

A
B
C
D

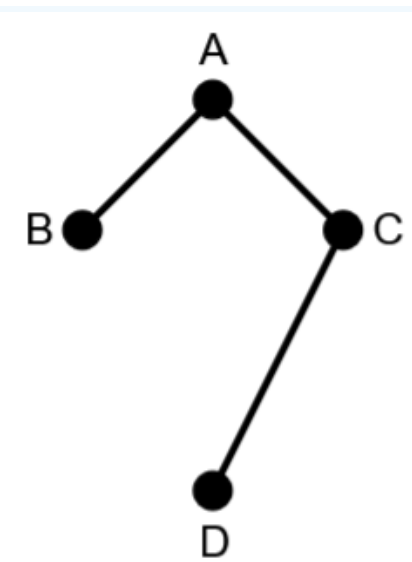

A
B
C
D


A
B
C
D


A
B
C
D


A
B
C
D

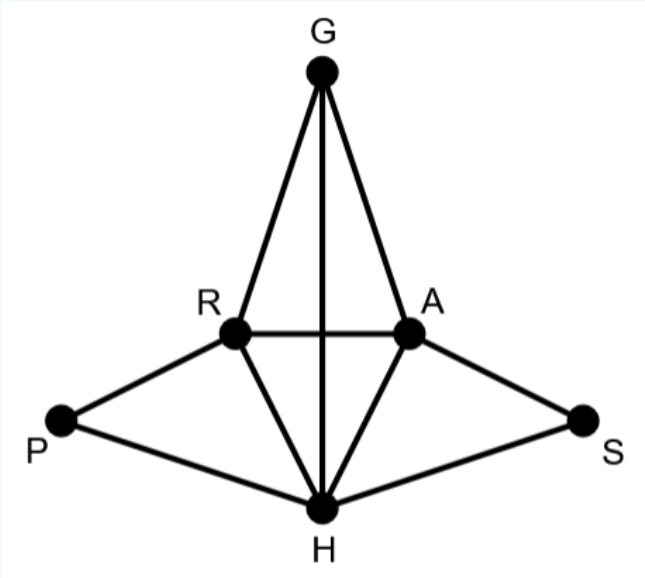

A
B
C
D


A
B
C
D

We are only interested in selecting one of these possible graphs to serve as a spanning tree. We are not concerned with determining how many spanning trees exist or identifying all of them.

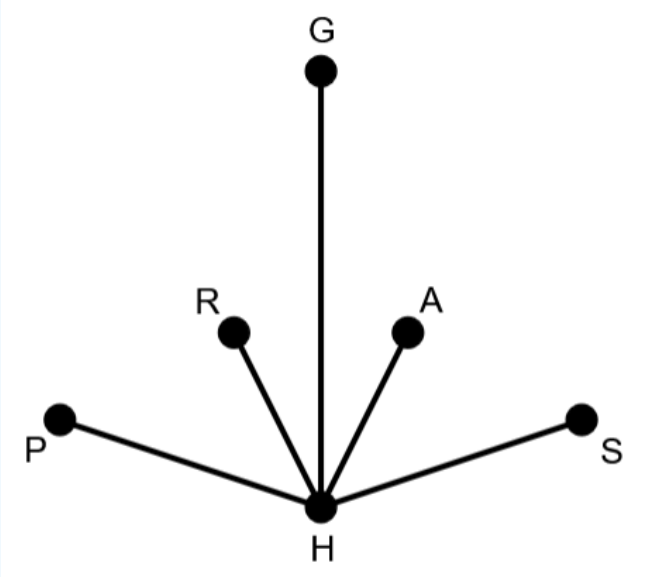
Example #8.7.8 🤔

Find a spanning tree for the graph.



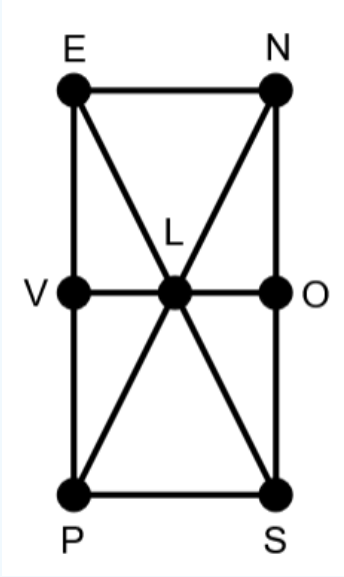
✅ **Solution:**

The graph is not a tree. It has 6 vertices and 10 edges, whereas a tree with 6 vertices must have exactly 5 edges. Therefore, 5 edges must be removed to form a tree. Answers will vary. One possible solution:



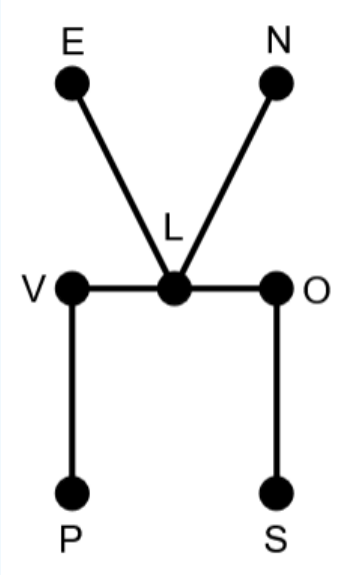
Example #8.7.9 🤔

Find a spanning tree for the graph.



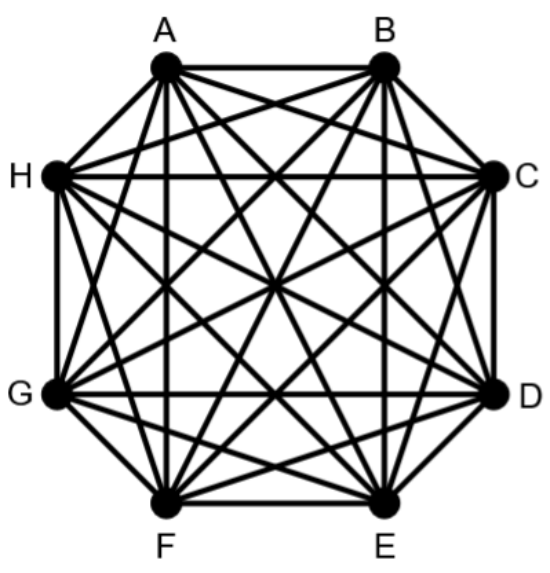
✅ **Solution:**

The graph is not a tree. It has 7 vertices and 12 edges, whereas a tree with 7 vertices must have exactly 6 edges. Therefore, 6 edges must be removed to form a tree. Answers will vary. One possible solution:



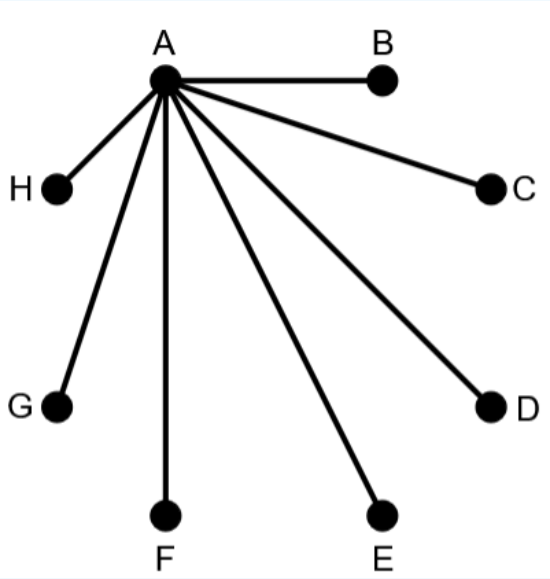
Example #8.7.10 🤔

Find a spanning tree for the graph.



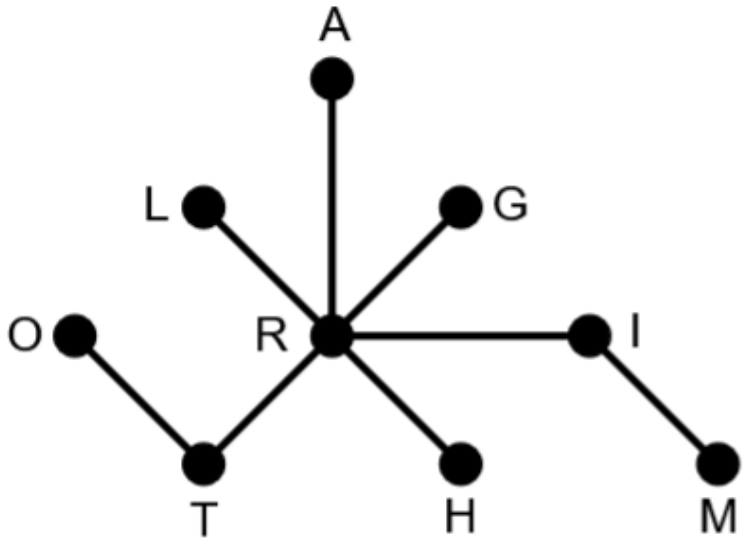
✅ **Solution:**

The graph is not a tree. It has 8 vertices and many edges (28), whereas a tree with 8 vertices must have exactly 7 edges. Therefore, remove as many edges until only 7 edges are remaining to form a tree. Answers will vary. One possible solution:

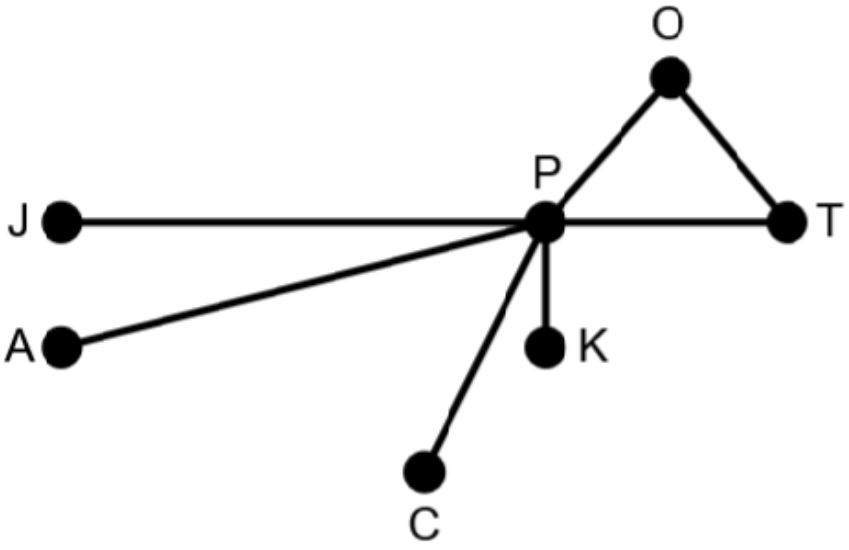


Section 8.7: Trees [*In-Class Exercises*]

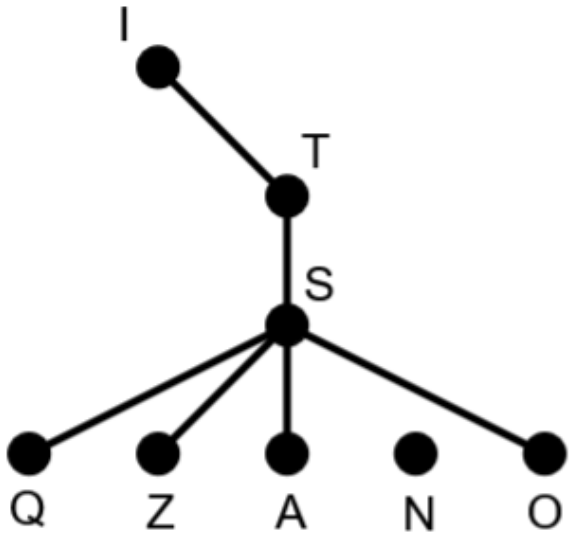
1. Determine whether the given graph is a tree.



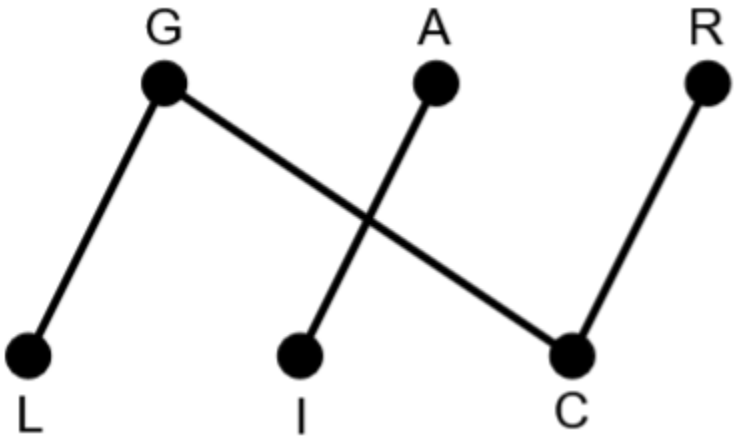
2. Determine whether the given graph is a tree.



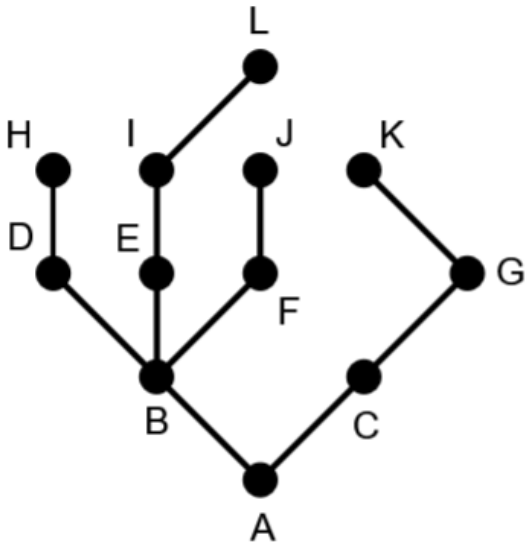
3. Determine whether the given graph is a tree.



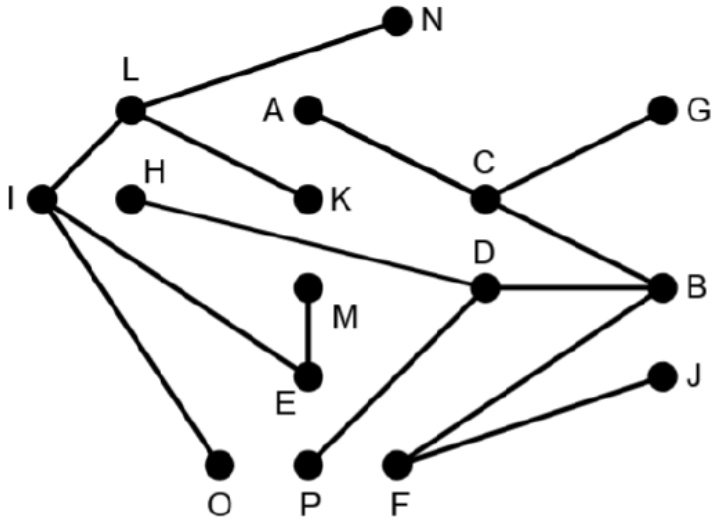
4. Determine whether the given graph is a tree.



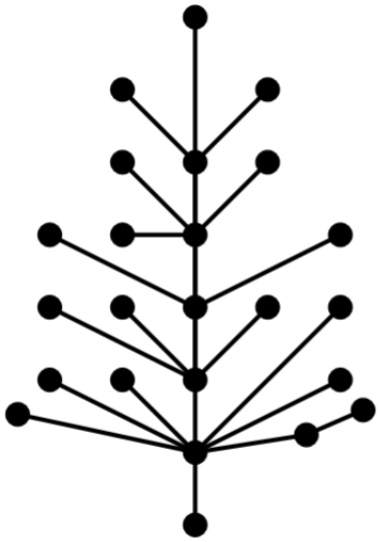
5. Determine whether the given graph is a tree.



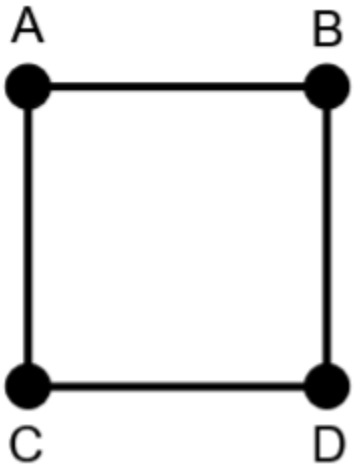
6. Determine whether the given graph is a tree.



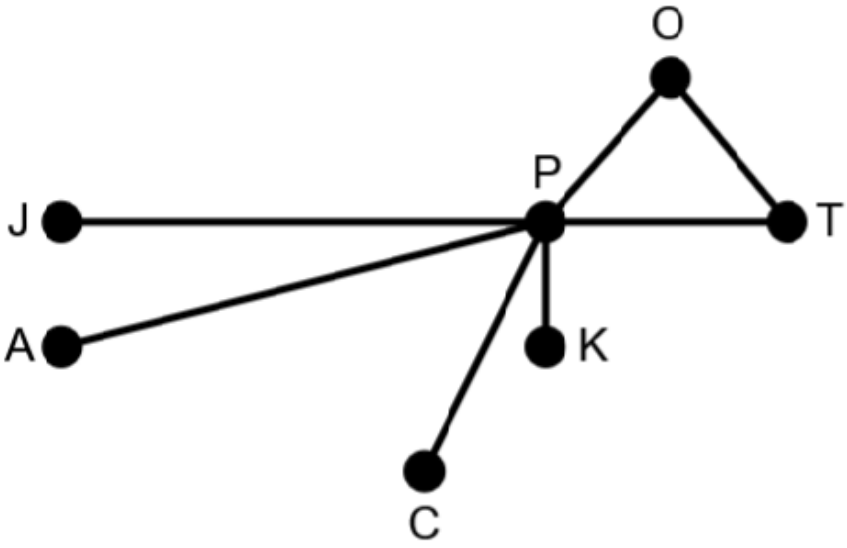
7. Determine whether the given graph is a tree.



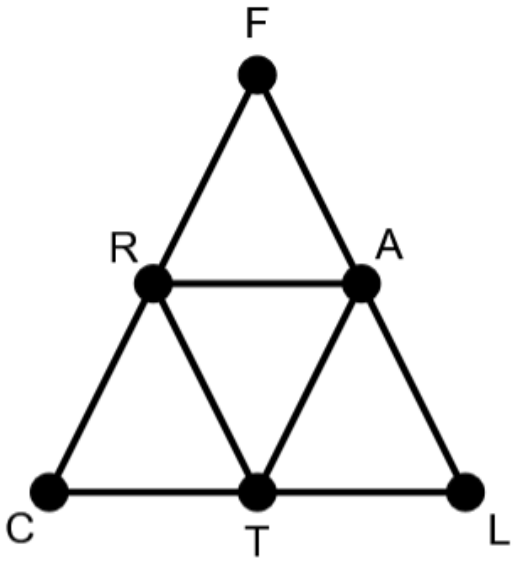
8. Find a spanning tree for the graph.



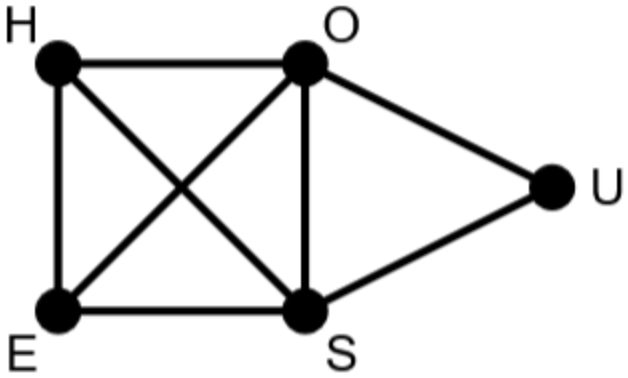
9. Find a spanning tree for the graph.



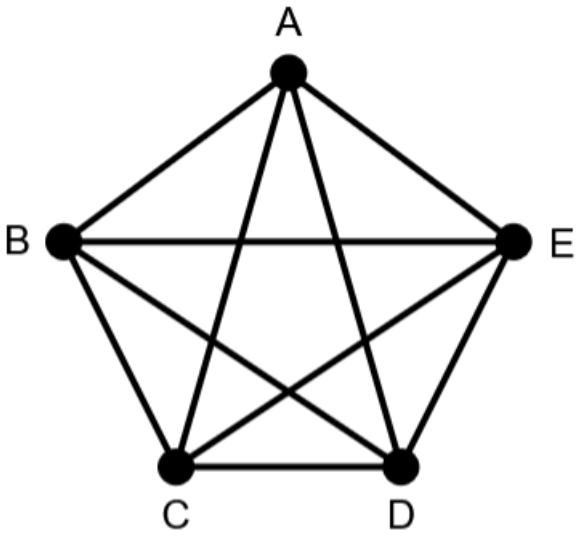
10. Find a spanning tree for the graph.



11. Find a spanning tree for the graph.



12. Find a spanning tree for the graph.

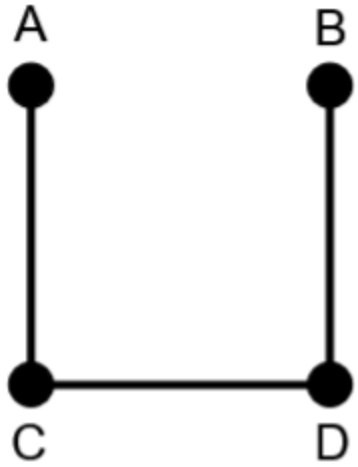


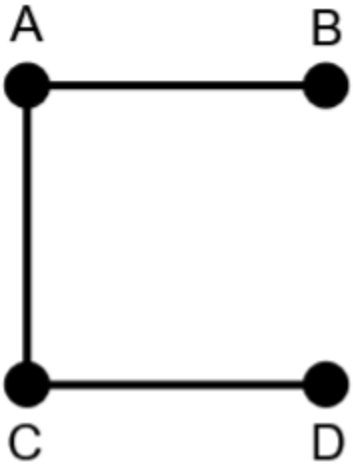
Answers available at the end of this section.

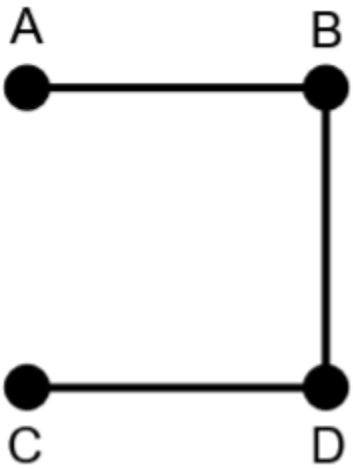
Answers

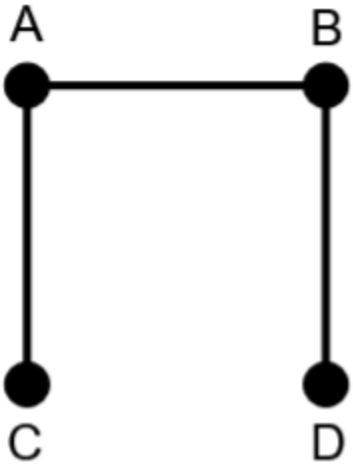
Section 8.7: Trees [*In-Class Exercises*]

1. Yes, it is a tree.
2. No, it is not a tree.
3. No, it is not a tree.
4. No, it is not a tree.
5. Yes, it is a tree.
6. No, it is not a tree.
7. Yes, it is a tree.
8. Any of these graphs will be a spanning tree:

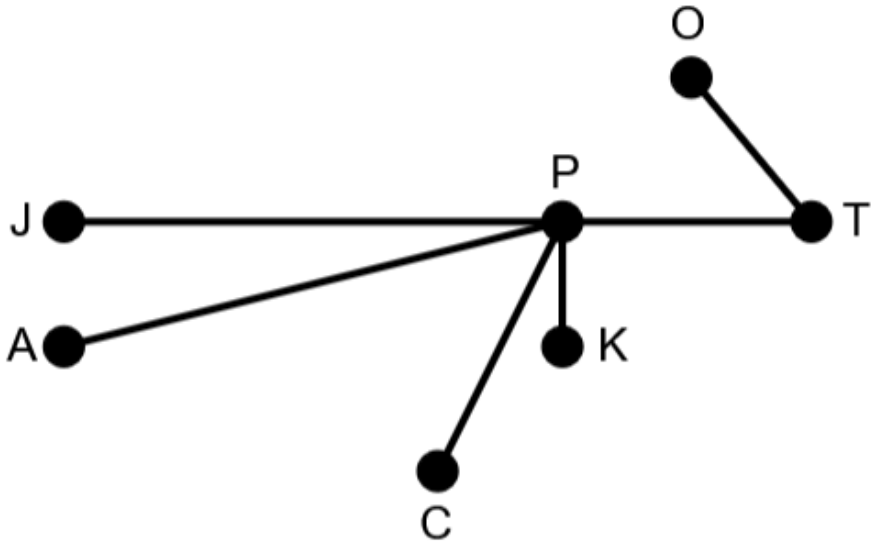


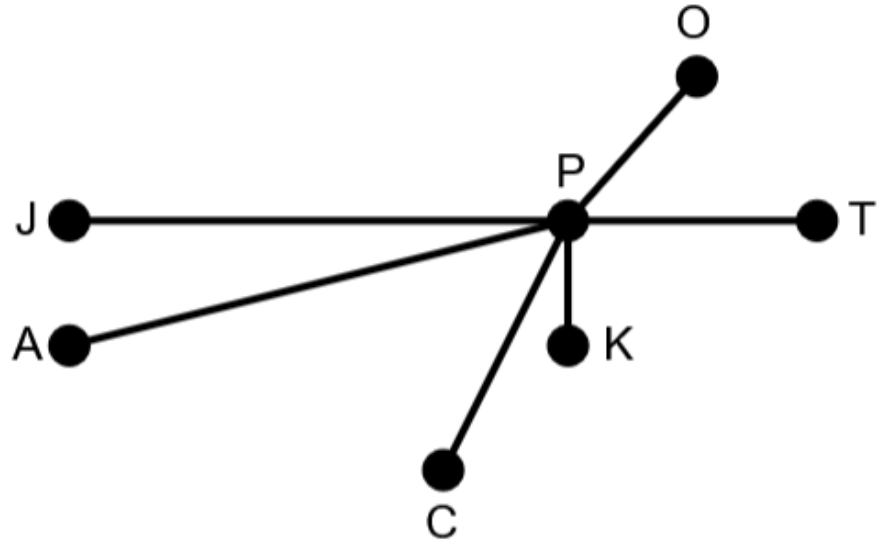




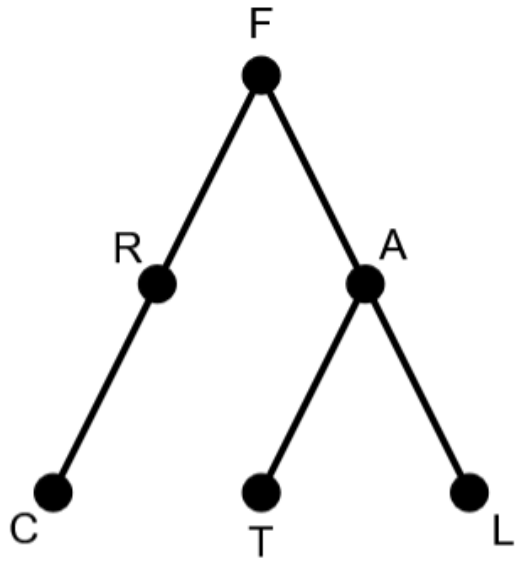


9. Either of these graphs will be a spanning tree:

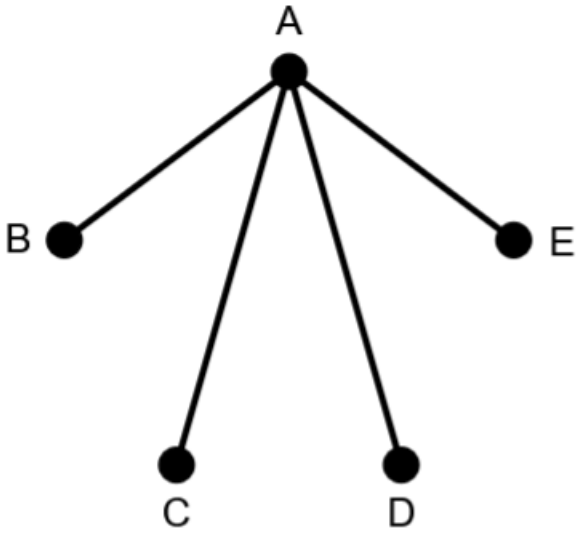




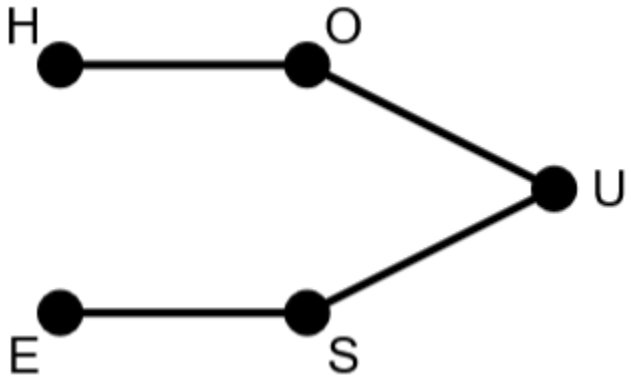
10. Answers will vary. One possible example:



11. Answers will vary. One possible example:



12. Answers will vary. One possible example:



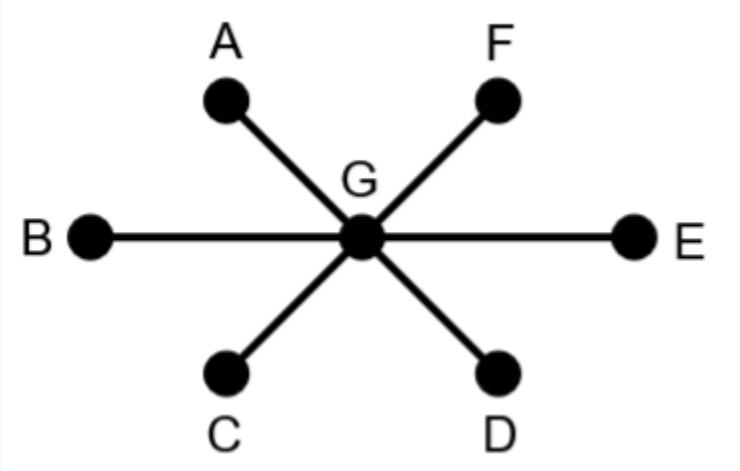
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Section 8.7.H: Homework

Section 8.7: Trees - Homework Exercises

(Section 8.7) - Homework Exercise #1

Determine whether the given graph is a tree.

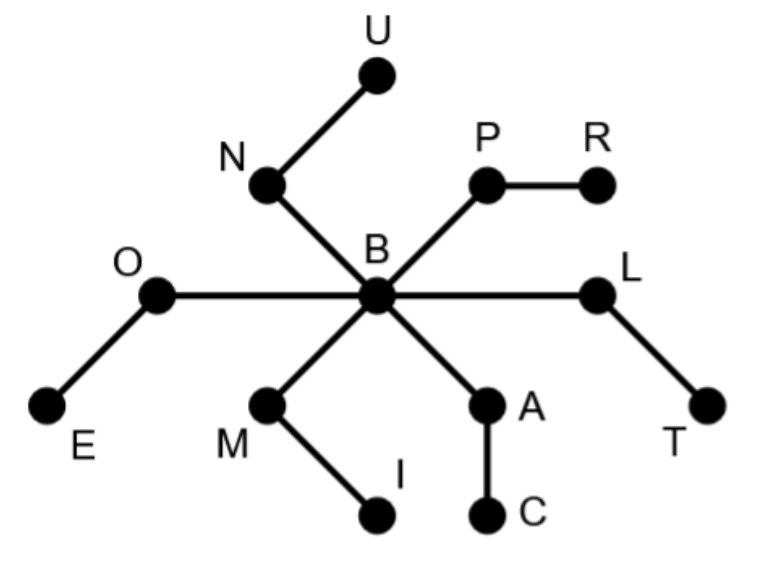


The image shows a graph with labeled nodes connected by line segments. G connects to A, B, C, D, E, and F.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #2

Determine whether the given graph is a tree.

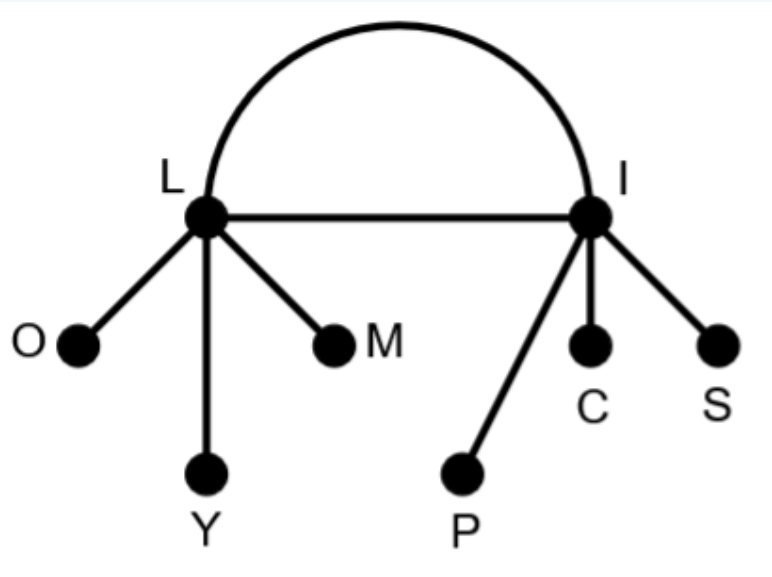


The image shows a graph with labeled nodes connected by line segments. B connects to A, L, M, N, O, and P; A connects to C; M connects to I; N connects to U; O connects to E; P connects to R; L connects to T.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #3

Determine whether the given graph is a tree.

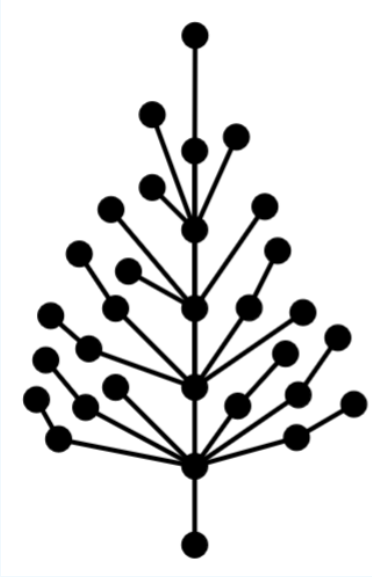


The image shows a graph with labeled nodes connected by line segments. L connects to I, M, O, and Y; I connects to C, P, and S. An additional curved edge also connects L and I.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #4

Determine whether the given graph is a tree.

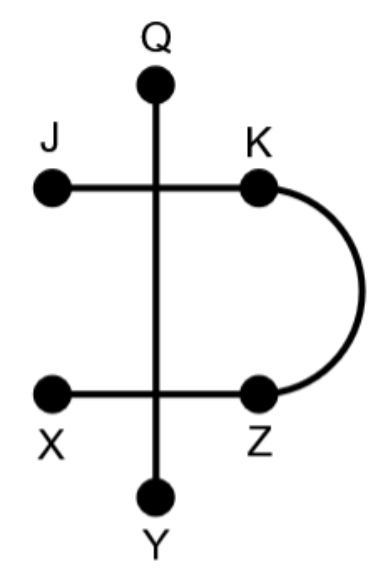


The image shows a graph made of many unlabeled nodes connected by line segments. A single central node near the bottom connects to several branches extending outward and upward. Many branches split into smaller sub-branches, creating a layered, tree-like structure. A vertical chain of connected nodes runs through the center from the bottom node to the topmost node. Additional branches extend diagonally left and right at several levels. All nodes are connected, and there are no visible cycles or closed loops.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #5

Determine whether the given graph is a tree.

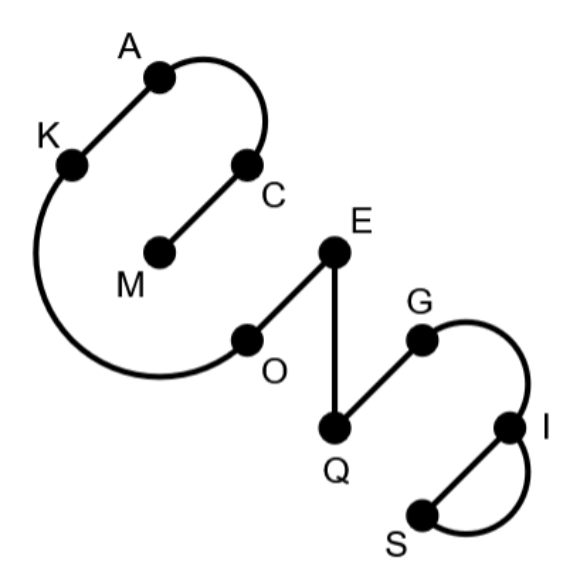


Accessibility detailed: The image shows a graph with labeled nodes connected by line segments. Q connects to Y; J connects to K; X connects to Z; K connects to Z with a curved edge. The vertical edge from Q to Y crosses the horizontal edges between J and K and between X and Z without forming additional nodes at the intersections.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #6

Determine whether the given graph is a tree.

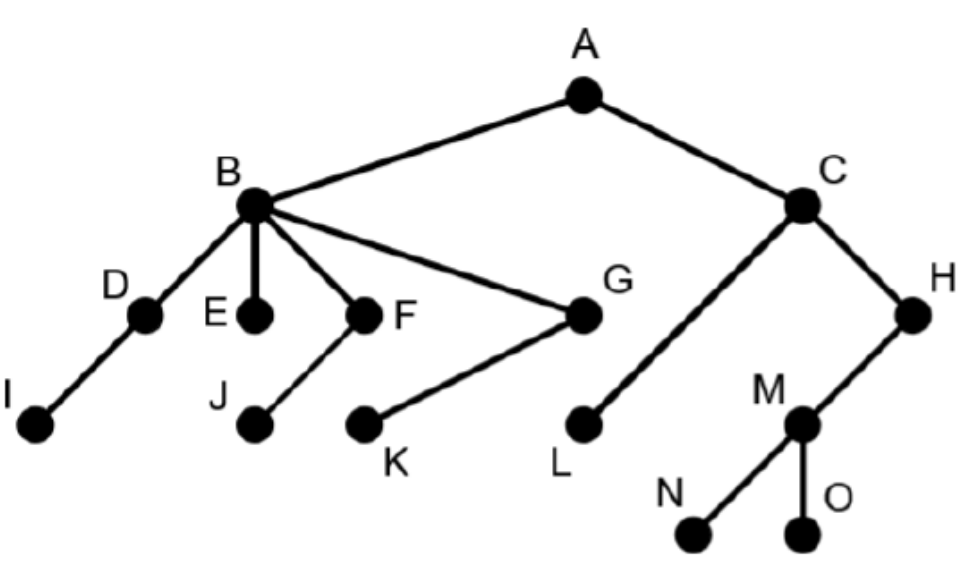


The image shows a graph with labeled nodes connected by line segments and curved edges. K connects to A and O with curved edges; A connects to C with a curved edge; C connects to M; O connects to E; E connects to Q; Q connects to G; G connects to I with a curved edge; I connects to S with a curved edge; and S connects back to I with another curved edge.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #7

Determine whether the given graph is a tree.

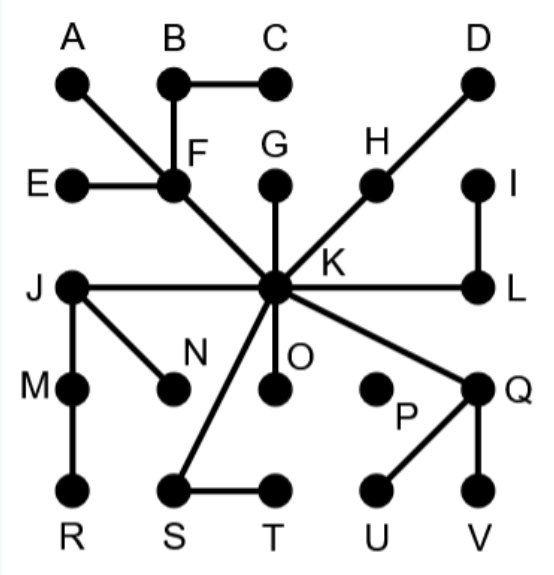


The image shows a graph with labeled nodes connected by line segments. A connects to B and C; B connects to D, E, F, and G; D connects to I; F connects to J; G connects to K; C connects to H and L; H connects to M; and M connects to N and O.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #8

Determine whether the given graph is a tree.

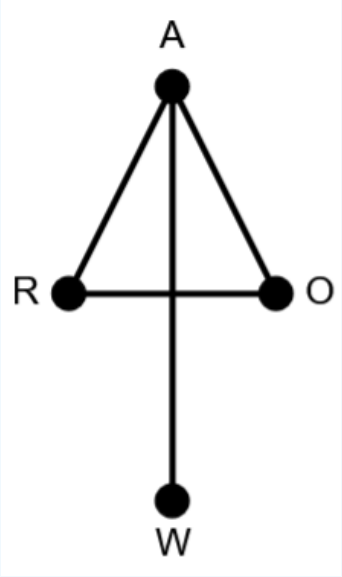


The image shows a graph with labeled nodes connected by line segments. F connects to A, B, E, and K; B connects to C; K connects to G, H, J, L, N, O, Q, and S; H connects to D; L connects to I; J connects to M and N; M connects to R; S connects to T; and Q connects to U and V; P is an isolated node with no connections.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #9

Determine whether the given graph is a tree.

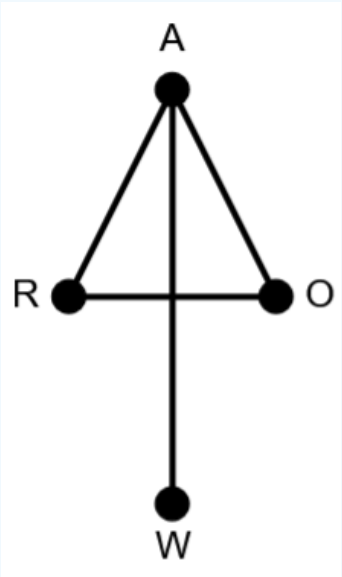


The image shows a graph with labeled nodes connected by line segments. A connects to O, R, and W; while R connects to O.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #10

Find all possible spanning trees for the graph in Homework Exercise #9.

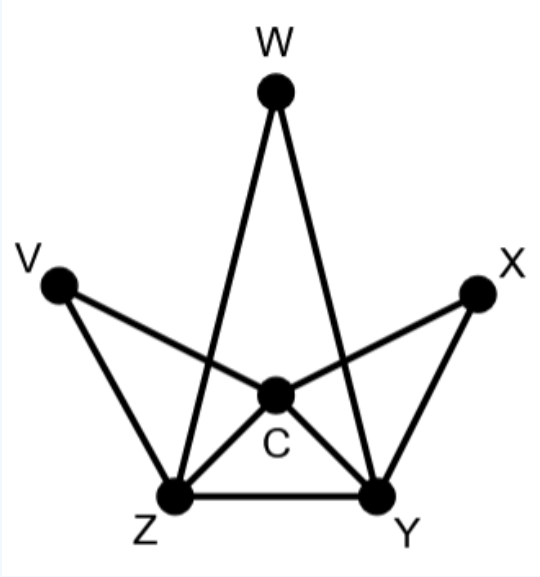


The image shows a graph with labeled nodes connected by line segments. A connects to O, R, and W; while R connects to O.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #11

Find a spanning tree for the graph.

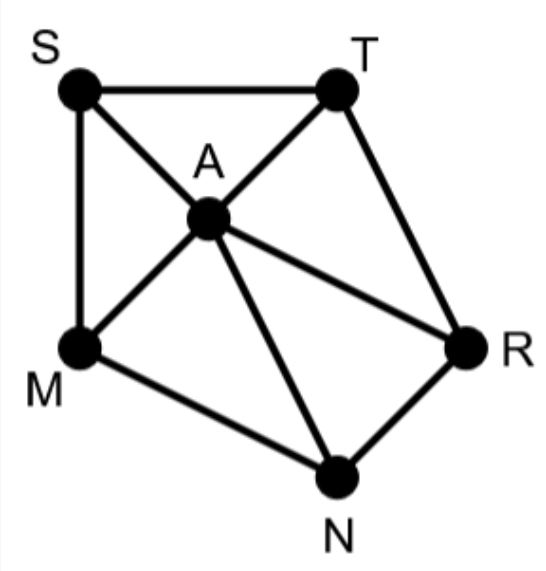


The image shows a graph with labeled nodes connected by line segments. C connects to V, X, Y, and Z; W connects to Y and Z; X connects to Y; V connects to Z; and Y connects to Z.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #12

Find a spanning tree for the graph.

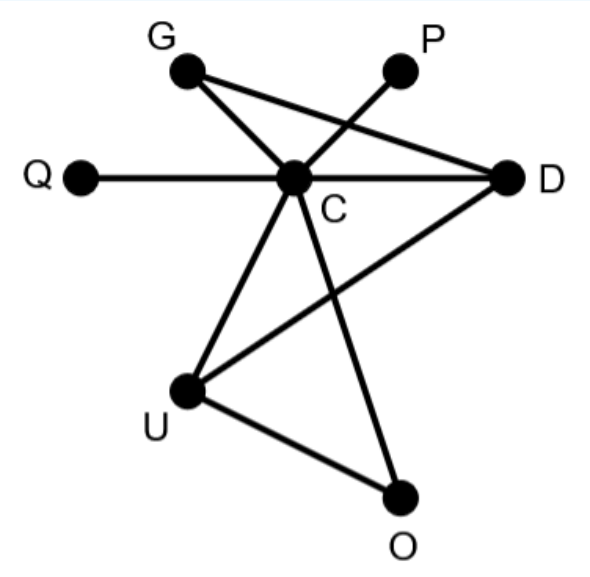


The image shows a graph with labeled nodes connected by line segments. A connects to M, N, R, S, and T; S connects to M and T; T connects to R; R connects to N; and M connects to N.

Answer available at the end of this section.

(Section 8.7) - Homework Exercise #13

Find a spanning tree for the graph.



The image shows a graph with labeled nodes connected by line segments. C connects to D, G, O, P, Q, and U; G connects to D; D connects to U; and U connects to O.

Answer available at the end of this section.

Answers

(Section 8.7) - Homework Exercise #1
Yes, the graph is a tree.

(Section 8.7) - Homework Exercise #2
Yes, the graph is a tree.

(Section 8.7) - Homework Exercise #3
No, the graph is not a tree.

(Section 8.7) - Homework Exercise #4
Yes, the graph is a tree.

(Section 8.7) - Homework Exercise #5
No, the graph is not a tree.

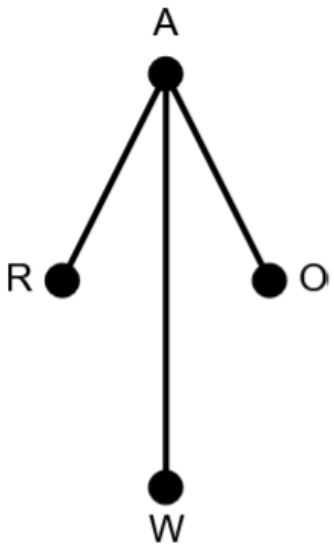
(Section 8.7) - Homework Exercise #6
No, the graph is not a tree.

(Section 8.7) - Homework Exercise #7
Yes, the graph is a tree.

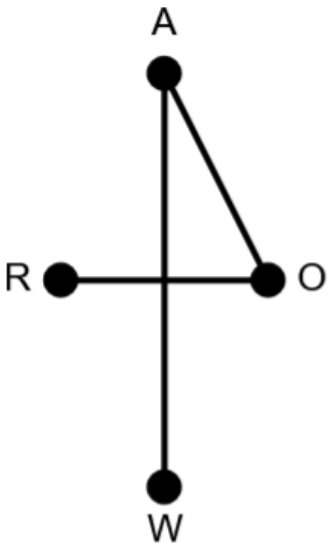
(Section 8.7) - Homework Exercise #8
No, the graph is not a tree.

(Section 8.7) - Homework Exercise #9
No, the graph is not a tree.

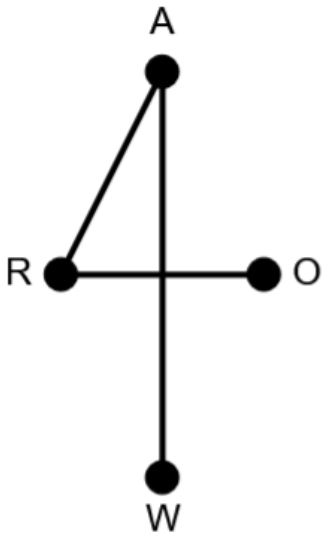
(Section 8.7) - Homework Exercise #10



The image shows a graph with labeled nodes connected by line segments. A connects to O, R, and W.



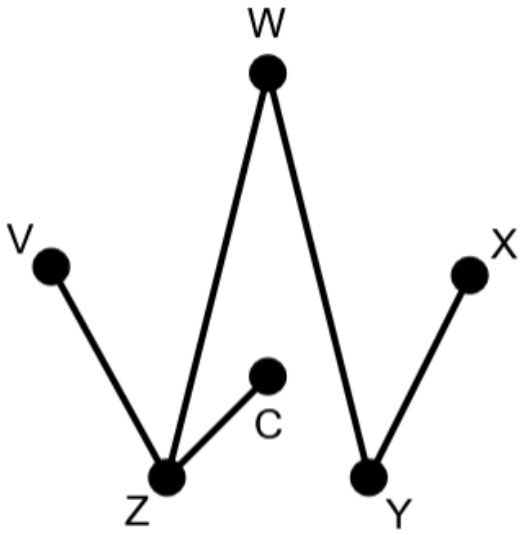
The image shows a graph with labeled nodes connected by line segments. A connects to O and W; and R connects to O.



The image shows a graph with labeled nodes connected by line segments. A connects to R and W; and R connects to O.

(Section 8.7) - Homework Exercise #11

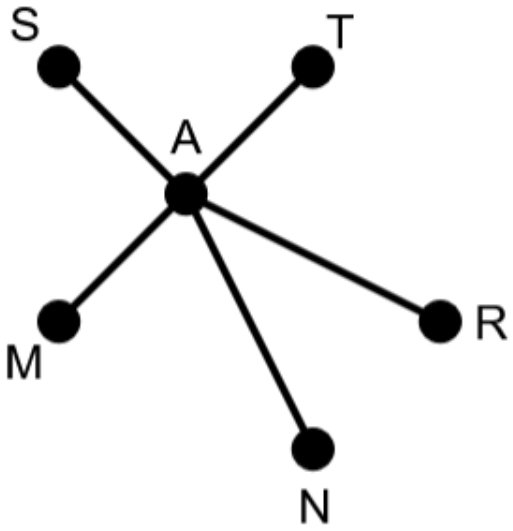
Answers will vary. One possible example:



The image shows a graph with labeled nodes connected by line segments. W connects to Y and Z; V connects to Z; X connects to Y; and Z connects to C.

(Section 8.7) - Homework Exercise #12

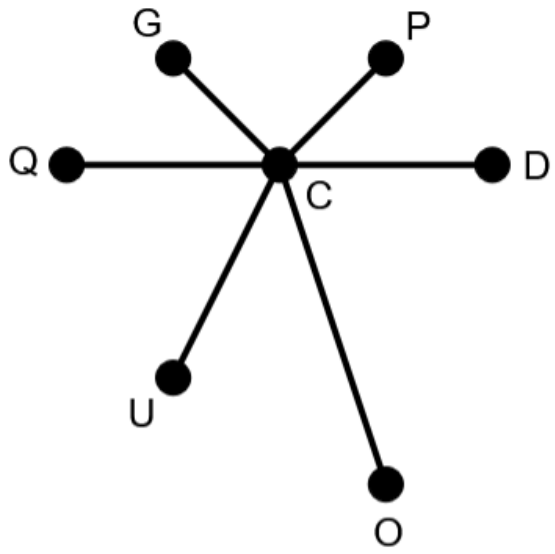
Answers will vary. One possible example:



The image shows a graph with labeled nodes connected by line segments. A connects to M, N, R, S, and T.

(Section 8.7) - Homework Exercise #13

Answers will vary. One possible example:



The image shows a graph with labeled nodes connected by line segments. C connects to D, G, O, P, Q, and U.

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Section 8.8: Kruskal's Algorithm

Learning Objectives

- Finding a minimum spanning tree for a weighted graph
- Apply minimum spanning trees to real world applications

In the previous section, we introduced spanning trees as a way to connect all vertices of a graph without creating circuits. However, in many real-world situations, the edges of a graph have weights that represent cost, distance, or time. In these cases, simply finding any spanning tree is not enough. Instead, we want a spanning tree with the smallest possible total weight. Such a tree is called a minimum spanning tree.

This leads to an important question: How can we efficiently find a minimum spanning tree from a weighted graph? One effective method is Kruskal's Algorithm.

Kruskal's Algorithm

To construct a minimum spanning tree for a weighted graph.

- **Step 1** - Choose an edge with the smallest weight and highlight it. (If there are two or more edges the same, pick any one).
- **Step 2** - Choose an unmarked edge with the next lowest weight that does not create a circuit and highlight it. (If there are two or more edges the same, pick any one).
- **Step 3** - Continue selecting edges in order of increasing weight that does not create a circuit and highlighting each one until all vertices have been connected. (If there are two or more edges the same, pick any one).
- **Step 4** - Once all vertices are connected, only the highlighted portion of your graph is the minimum spanning tree. Add the weights.

Note: Remember that the number of edges equals the number of vertices minus 1, so you will know the exact number of edges needed to obtain a minimum spanning tree.

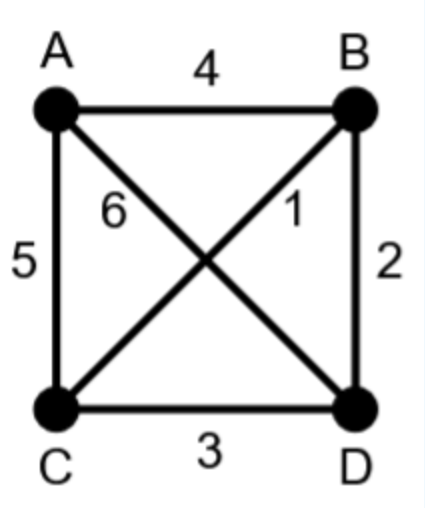
Rather than building a tree by starting at a particular vertex, Kruskal's Algorithm focuses on the edges of the graph. It works by repeatedly selecting the edge with the smallest weight and adding it to the graph, as long as doing so does not create a circuit. This process continues until all vertices are connected.

Kruskal's Algorithm is an example of a greedy algorithm, meaning it makes the best local choice at each step, like in this case, choosing the smallest available edge. While each decision is made independently, the result is a globally optimal solution which is a minimum spanning tree.

This method is especially useful in applications such as designing communication networks, laying electrical wiring, and building transportation systems, where minimizing total cost is essential. Kruskal's Algorithm provides a systematic and efficient way to move from the concept of spanning trees to finding the most cost-effective way to connect a network.

Example #8.8.1 🤔

Use Kruskal's algorithm to find a minimum spanning tree for the complete weighted graph. Also, include the total weight.

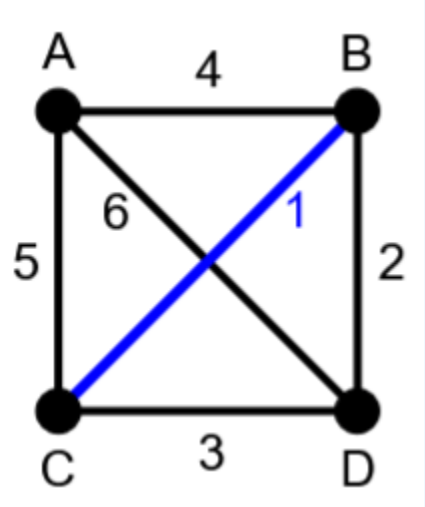


✅ **Solution:**

Using the steps of Kruskal's algorithm:

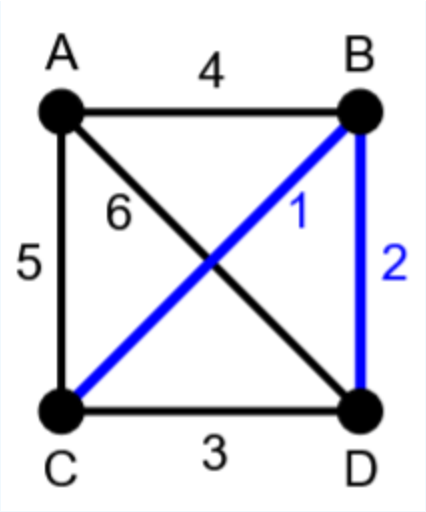
- **Step 1** - Choose an edge with the smallest weight and highlight it. (If there are two or more edges the same, pick any one).

The edge with the smallest weight is B-C (1 unit). Now, highlight that edge.



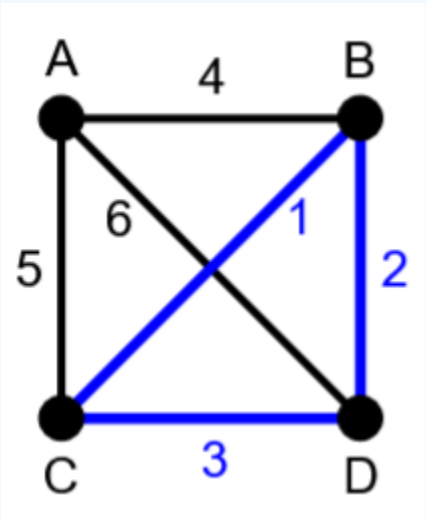
- **Step 2** - Choose an unmarked edge with the next lowest weight that does not create a circuit and highlight it. (If there are two or more edges the same, pick any one).

The next edge with the smallest weight is B-D (2 units). Now, highlight that edge.

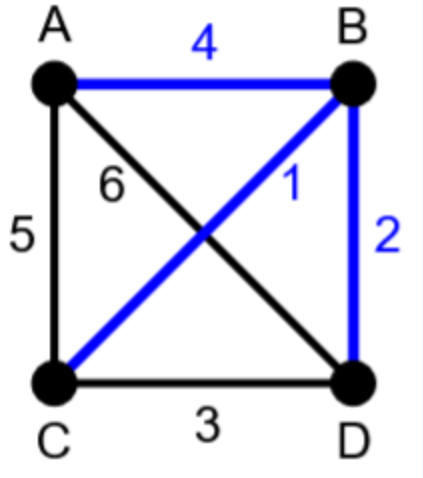


- **Step 3** - Continue selecting edges in order of increasing weight that does not create a circuit and highlighting each one until all vertices have been connected. (If there are two or more edges the same, pick any one).

The next edge with the smallest weight is C-D (3 units). However, that creates a circuit, which is not a tree as shown in the figure below.

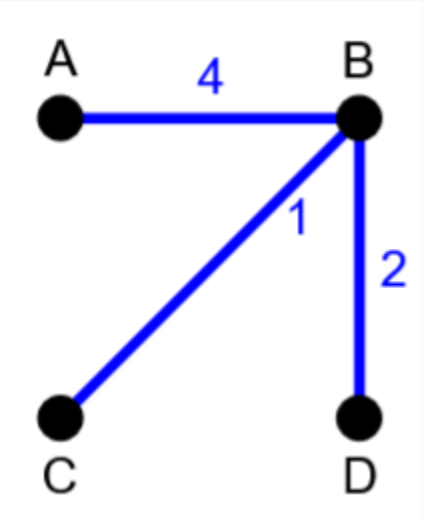


So, we skip that edge (do not highlight it). The next edge with the smallest weight is A-B (4 units).



- **Step 4** - Once all vertices are connected, only the highlighted portion of your graph is the minimum spanning tree. Add the weights.

So, all vertices are connected. To be certain that it is a tree, the number of vertices is 4 and the number of edges is 3. Thus, we have a minimum spanning tree.



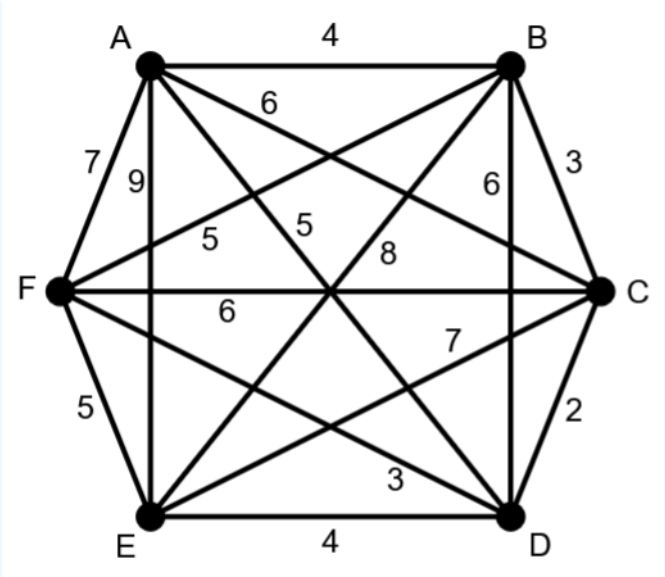
We include only the highlighted edges and sum up all of the weights which is 1 + 2 + 4 = 7 units.

Example #8.8.2 🤔

A city is planning to install water pipelines to connect six facilities:

- A: Water Treatment Plant
- B: Residential Area
- C: Industrial Park
- D: Hospital
- E: School
- F: Shopping Center

The cost (in millions of dollars) to build pipelines between locations is given in the complete weighted graph:



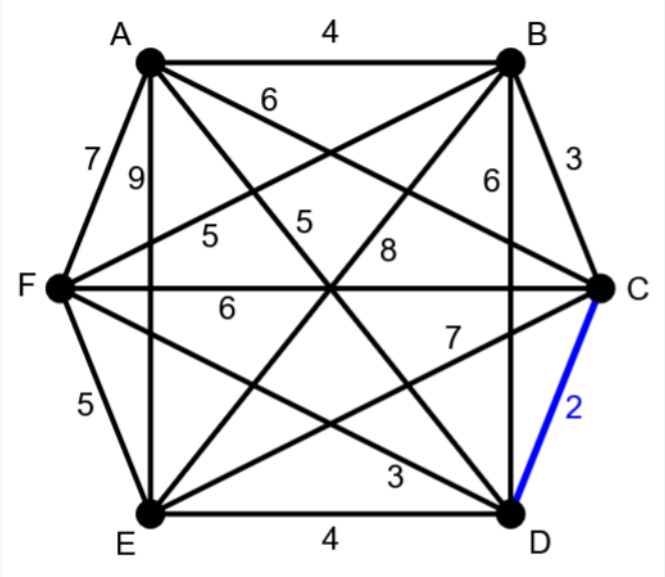
Use Kruskal's algorithm to find a minimum spanning tree for the graph. What is the total minimum cost?

✅ **Solution:**

Using the steps of Kruskal's algorithm:

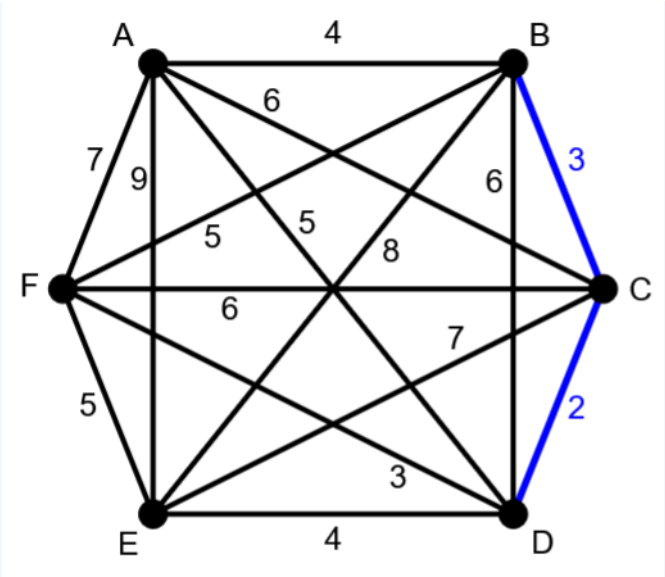
- **Step 1** - Choose an edge with the smallest weight and highlight it. (If there are two or more edges the same, pick any one).

The edge with the smallest weight is C-D ($2M). Now, highlight that edge.



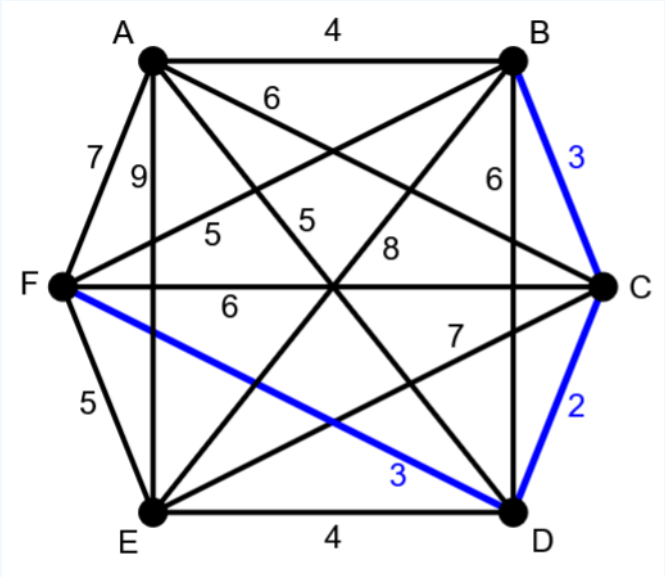
- **Step 2** - Choose an unmarked edge with the next lowest weight that does not create a circuit and highlight it. (If there are two or more edges the same, pick any one).

The next edge with the smallest weight is either B-C ($3M) or F-D ($3M). Let's choose B-C. Now, highlight that edge.

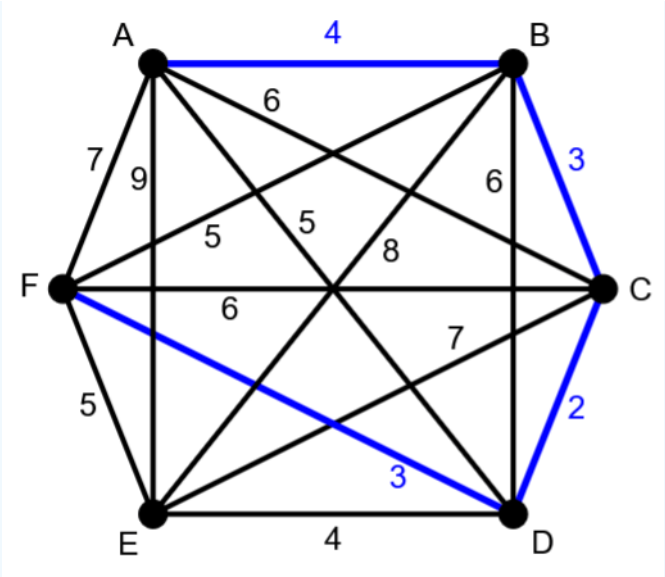


- **Step 3** - Continue selecting edges in order of increasing weight that does not create a circuit and highlighting each one until all vertices have been connected. (If there are two or more edges the same, pick any one).

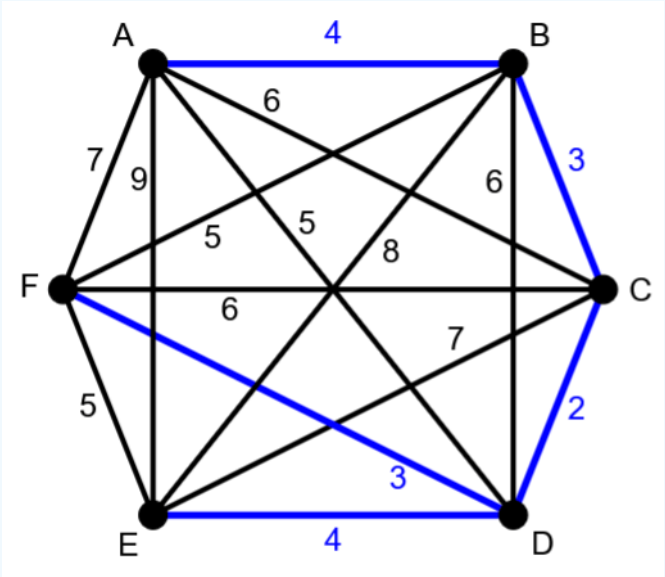
The next edge with the smallest weight is F-D ($3M), which was the other choice in the previous step. Now, highlight that edge.



The next edge with the smallest weight is either A-B ($4M) or E-D ($4M). Let's choose A-B. Now, highlight that edge.

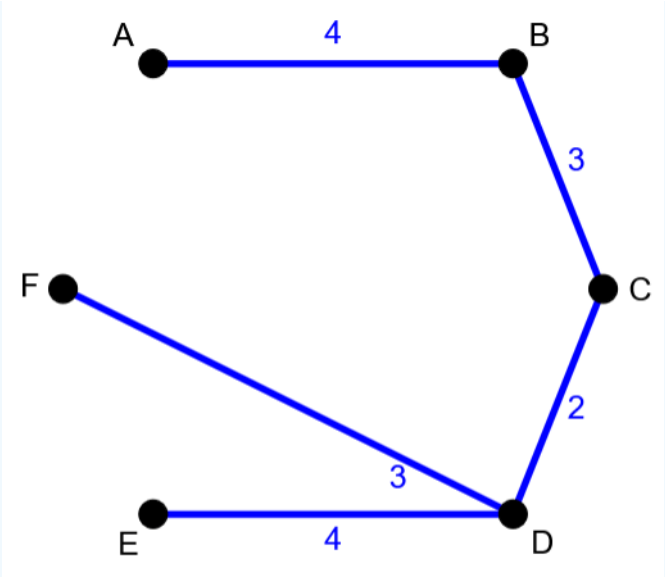


The next edge with the smallest weight is E-D ($4M), which was the other choice in the previous step. Now, highlight that edge.



- **Step 4** - Once all vertices are connected, only the highlighted portion of your graph is the minimum spanning tree. Add the weights.

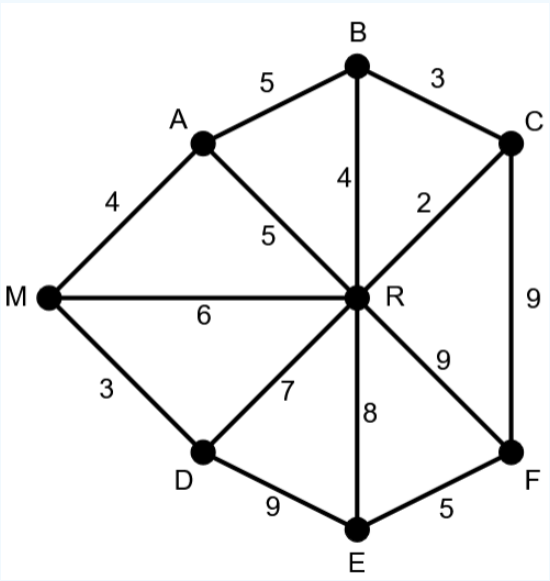
So, all vertices are connected. To be certain that it is a tree, the number of vertices is 6 and the number of edges is 5. Thus, we have a minimum spanning tree.



We include only the highlighted edges and sum up all of the weights which is 4 + 3 + 2 + 3 + 4 = 16. Thus, the minimum cost is $16 million.

Example #8.8.3 🤔

A postal service wants to connect their main post office (M) and their regional hub (R) to six distribution centers (A, B, C, D, E, F), so that mail can be transferred efficiently between all locations. The goal is to build routes with the minimum total cost. The cost (in thousands of dollars) to establish direct routes is shown in the weighted graph.



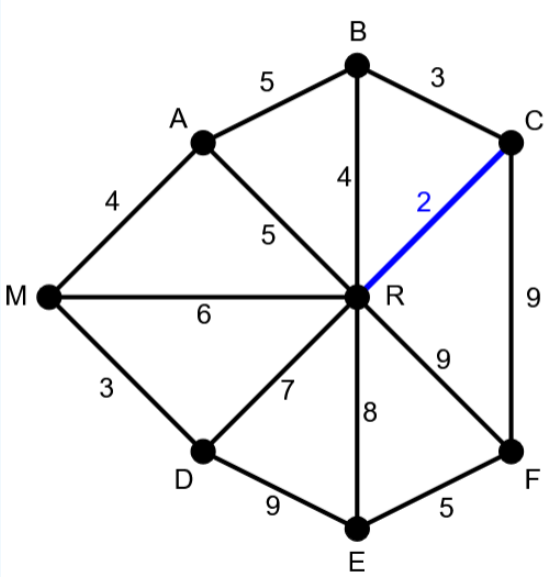
Find a minimum spanning tree for the graph. What is the total minimum cost?

✅ **Solution:**

Using the steps of Kruskal's algorithm:

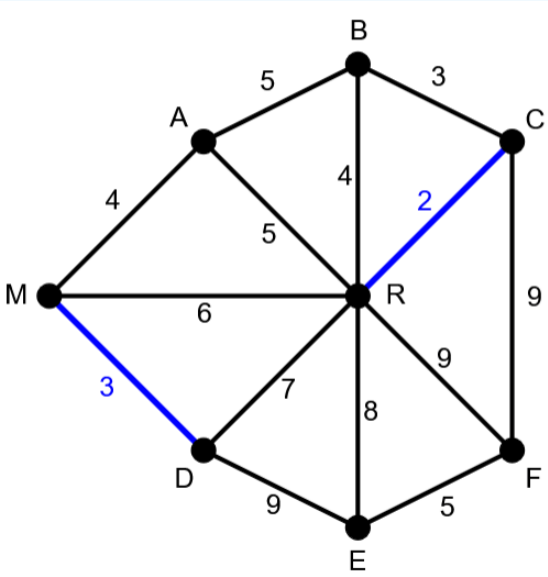
- **Step 1** - Choose an edge with the smallest weight and highlight it. (If there are two or more edges the same, pick any one).

The edge with the smallest weight is R-C (2K). Now, highlight that edge.



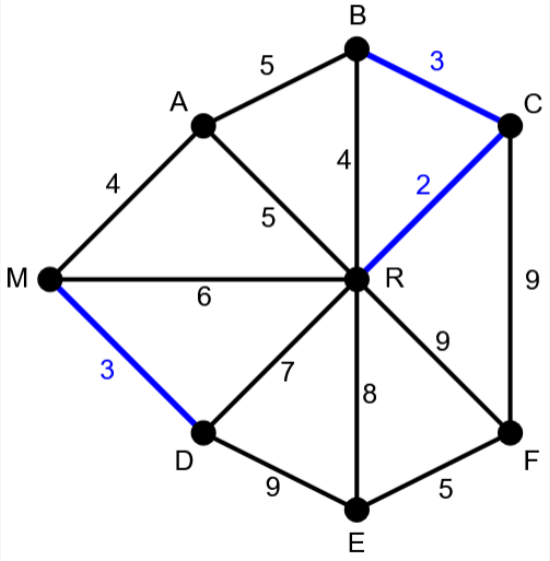
- **Step 2** - Choose an unmarked edge with the next lowest weight that does not create a circuit and highlight it. (If there are two or more edges the same, pick any one).

The next edge with the smallest weight is either M-D ($3K) or B-C ($3K). Let's choose M-D. Now, highlight that edge.

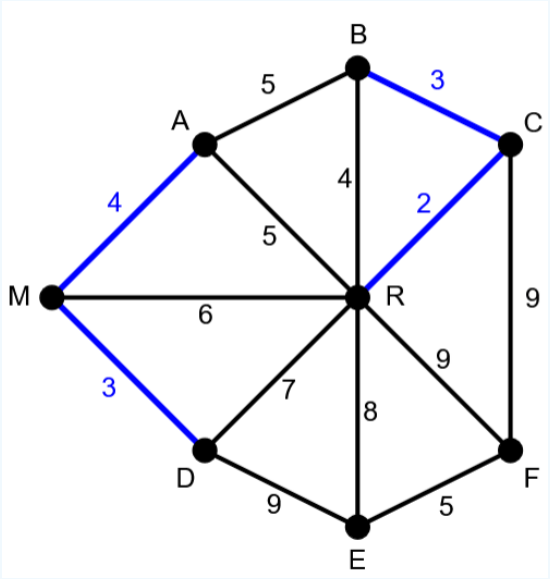


- **Step 3** - Continue selecting edges in order of increasing weight that does not create a circuit and highlighting each one until all vertices have been connected. (If there are two or more edges the same, pick any one).

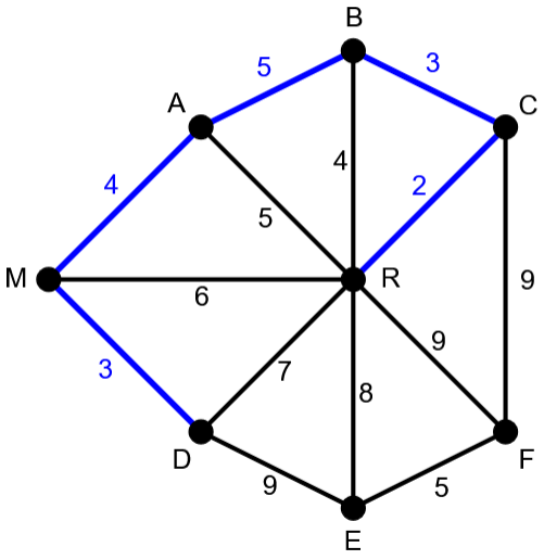
The next edge with the smallest weight is B-C ($3K), which was the other choice in the previous step. Now, highlight that edge.



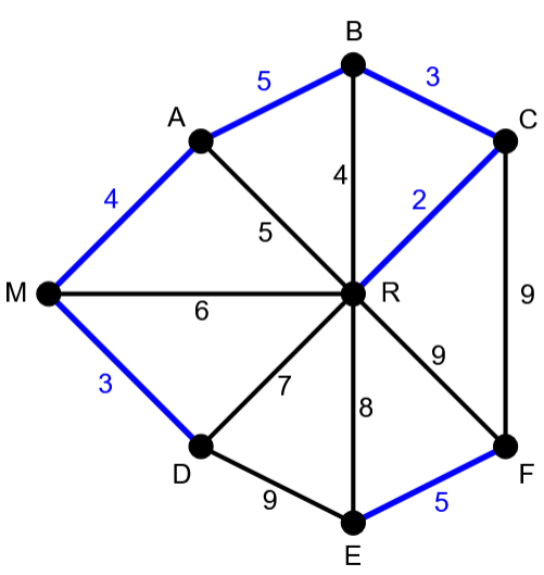
The next edge with the smallest weight is either M-A ($4K) or R-B ($4K). However, choosing R-B will create a circuit. So, choose M-A. Now, highlight that edge.



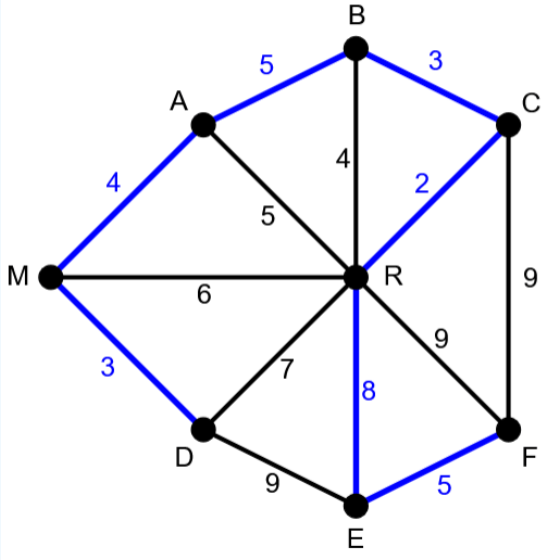
The next edge with the smallest weight is A-B ($5K), A-R ($5K), or E-F ($5K). Let's choose A-B. Now, highlight that edge.



The next edge with the smallest weight is A-R ($5K) or E-F ($5K). However, choosing A-R will create a circuit. So, choose E-F. Now, highlight that edge.

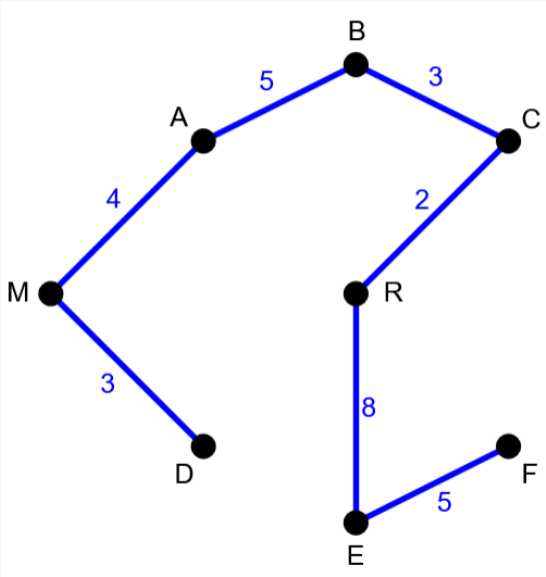


The next edge with the smallest weight is M-R ($6K). However, choosing M-R will create a circuit. The next edge with the smallest weight is D-R ($7K) which would also create a circuit. The next edge with the smallest weight is R-E ($8K). This edge does not create a circuit, so, choose R-E. Now, highlight that edge.



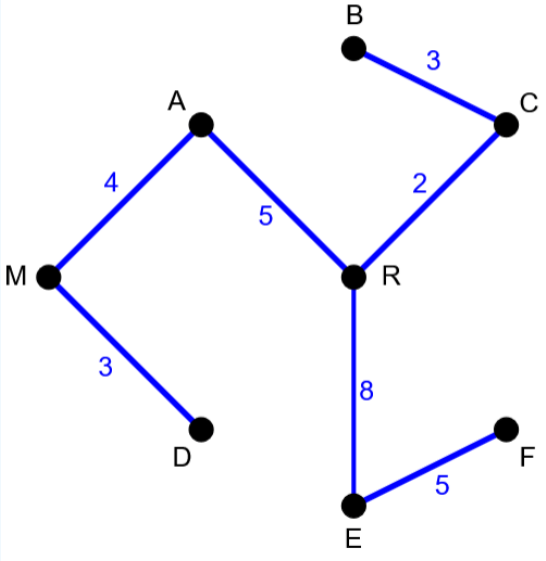
- **Step 4** - Once all vertices are connected, only the highlighted portion of your graph is the minimum spanning tree. Add the weights.

So, all vertices are connected. To be certain that it is a tree, the number of vertices is 8 and the number of edges is 7. Thus, we have a minimum spanning tree.



We include only the highlighted edges and sum up all of the weights which is 3 + 4 + 5 + 3 + 2 + 8 + 5 = 30. Thus, the minimum cost is $30 thousand.

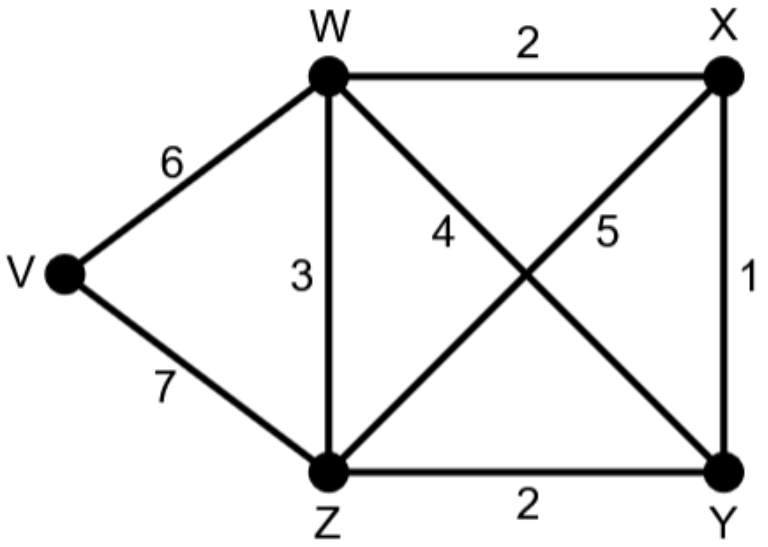
Note: There is one other spanning tree that would also be accepted. In the step where there was a choice of the two edges A-B ($5K) or A-R ($5K), choosing edge A-R instead of edge A-B would give the following spanning tree with the same minimum cost of $30 thousand.



Once again, we are only interested in selecting one of these possible graphs to serve as a spanning tree. We are not concerned with determining how many spanning trees exist or identifying all of them.

Section 8.8: Kruskal's Algorithm [*In-Class Exercises*]

1. Use Kruskal's algorithm to find a minimum spanning tree for each. Also, include the total weight.



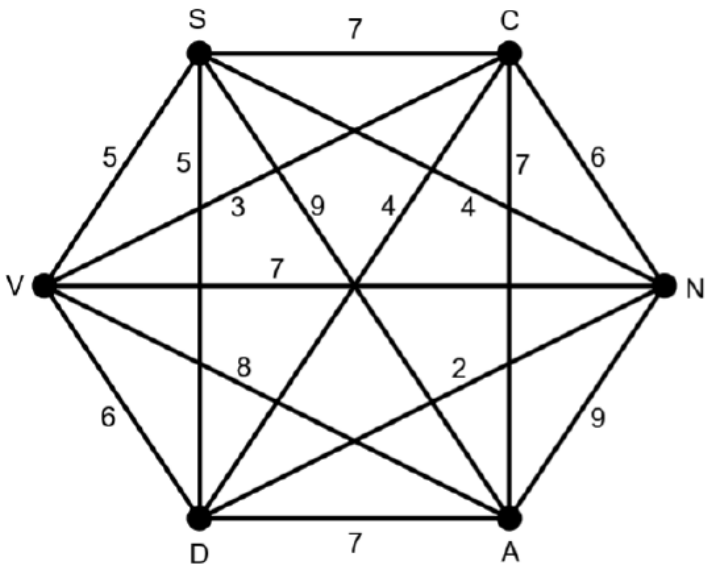
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2. An airline wants to establish direct flight routes between six airports so that all airports are connected while minimizing total operating costs. The airports are:

   - A: Atlanta
   - C: Chicago
   - D: Dallas
   - N: New York
   - S: Seattle

- V: Denver

The cost (in thousands of dollars) to establish each route is given in the complete weighted graph below:



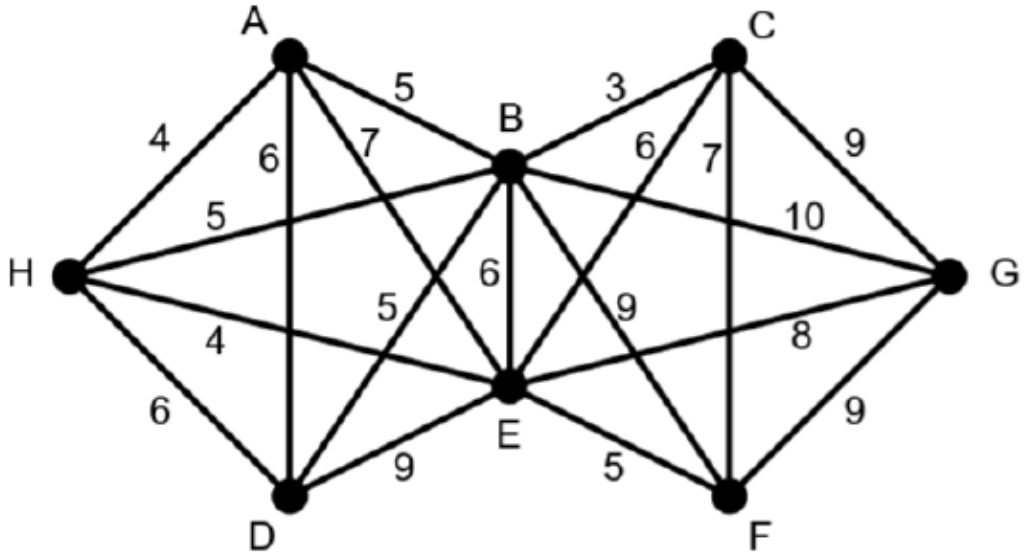
Use Kruskal's algorithm to find a minimum spanning tree for the graph. What is the total minimum cost?

---

3. A university wants to install Wi-Fi connections between eight buildings so that all buildings are connected at the lowest total cost. The buildings are:

- A: Library
- B: Math
- C: English
- D: Science Hall
- E: Student Center
- F: Cafeteria
- G: Gymnasium
- H: Administration Building

The cost (in thousands of dollars) to install connections is given in the graph below:



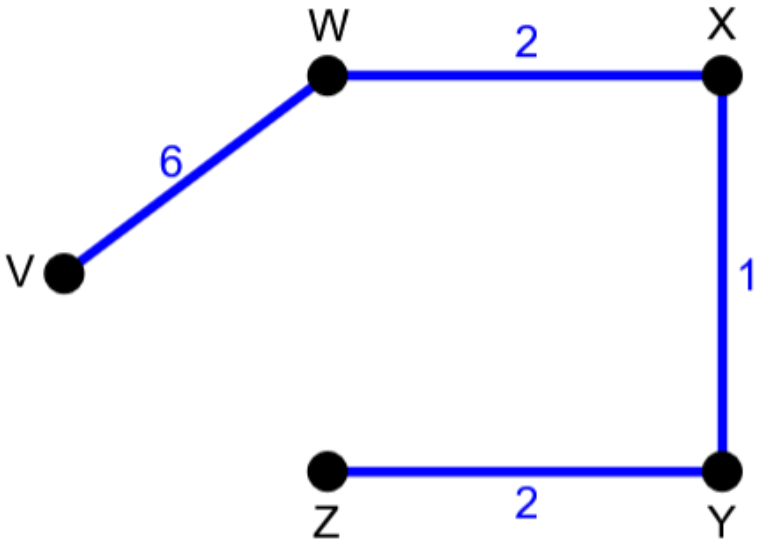
Use Kruskal's algorithm to find a minimum spanning tree for the graph. What is the total minimum cost?

Answers available at the end of this section.

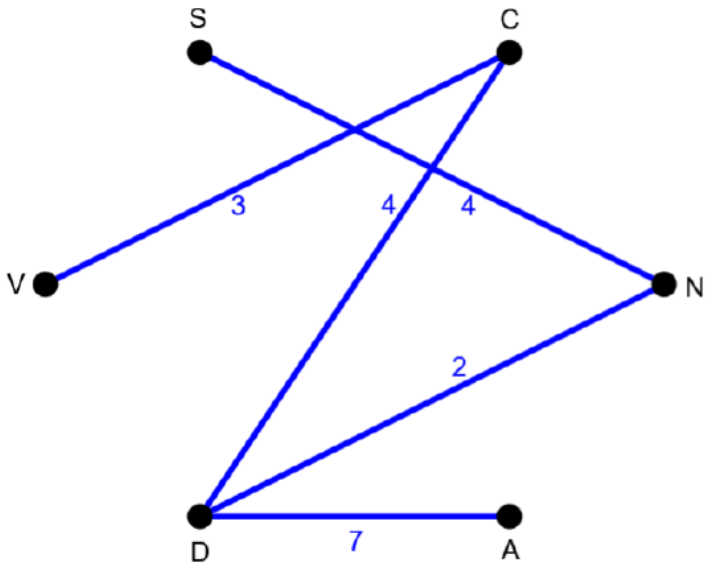
Answers

Section 8.8: Kruskal's Algorithm [*In-Class Exercises*]

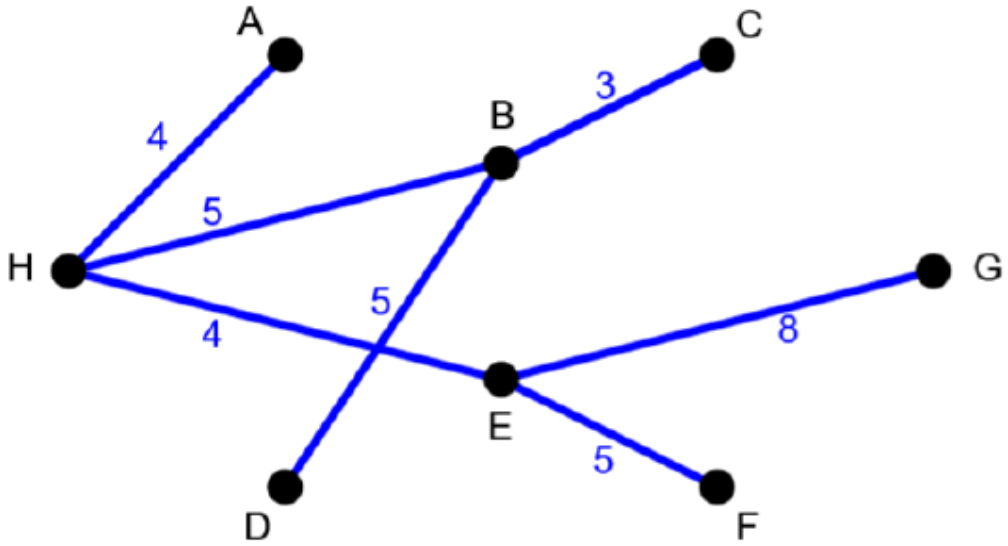
1.



Total weight = 11



2. Total weight = 20



3. Total weight = 34

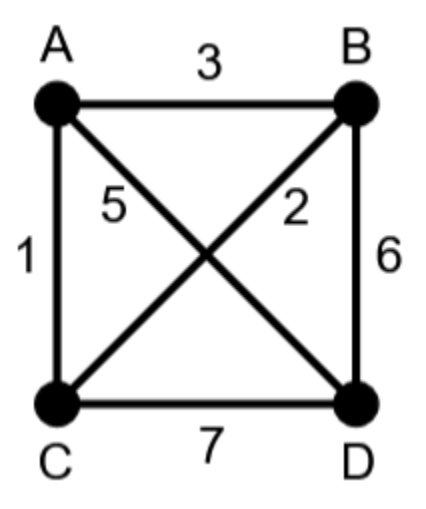
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Section 8.8.H: Homework

Section 8.8: Kruskal's Algorithm - Homework Exercises

(Section 8.8) - Homework Exercise #1

An energy company wants to connect four power stations using the least amount of wiring. The cost of connecting each pair of stations is given in thousands of dollars and the graph is given below.



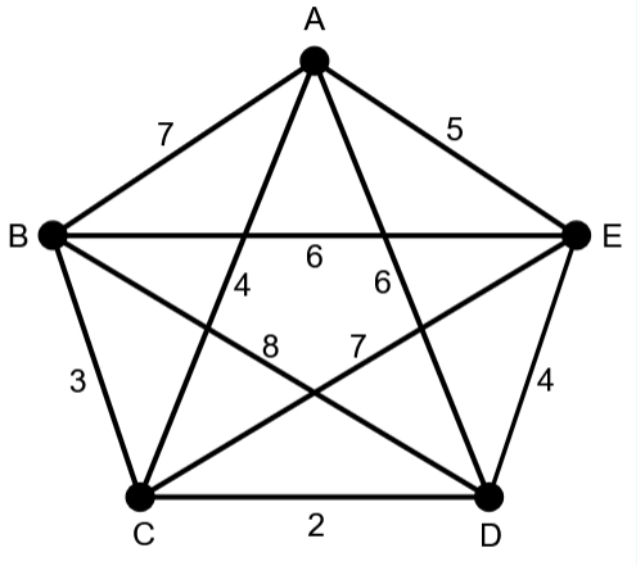
The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 3, C with weight 1, and D with weight 5; B connects to C with weight 2, and D with weight 6; and C connects to D with weight 7.

Use Kruskal's Algorithm to find a minimum spanning tree and determine the total cost of the network.

Answer available at the end of this section.

(Section 8.8) - Homework Exercise #2

A company is installing charging stations for delivery drones across five locations. The cost (in thousands) of connecting stations is given in the graph below.



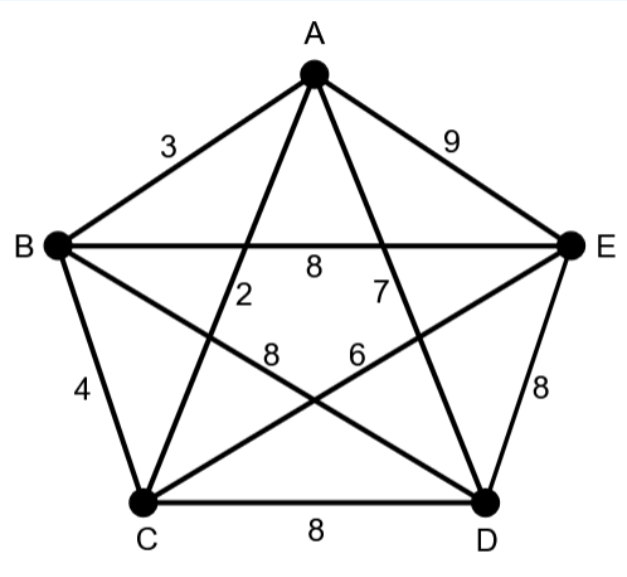
The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 7, C with weight 4, D with weight 6, and E with weight 5; B connects to C with weight 3, D with weight 8, and E with weight 6; C connects to D with weight 2, and E with weight 7; and D connects to E with weight 4.

Use Kruskal's Algorithm to determine the least expensive network that connects all stations. What is the total cost?

Answer available at the end of this section.

(Section 8.8) - Homework Exercise #3

A transportation agency plans to build railway lines connecting five cities. The cost (in millions) of constructing each rail line is given in the graph below.



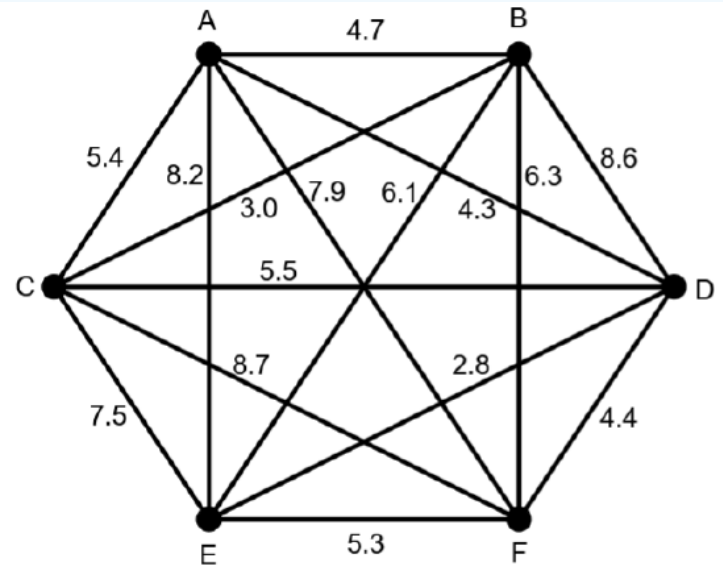
The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 3, C with weight 2, D with weight 7, and E with weight 9; B connects to C with weight 4, D with weight 8, and E with weight 8; C connects to D with weight 8, and E with weight 6; and D connects to E with weight 8.

Use Kruskal's Algorithm to determine which rail lines should be built so that all cities are connected at the minimum total cost. What is the total cost in millions of dollars?

Answer available at the end of this section.

(Section 8.8) - Homework Exercise #4

A region is setting up an emergency communication network connecting six sites. The cost of establishing communication lines is given in thousands of dollars and is given in the graph below.



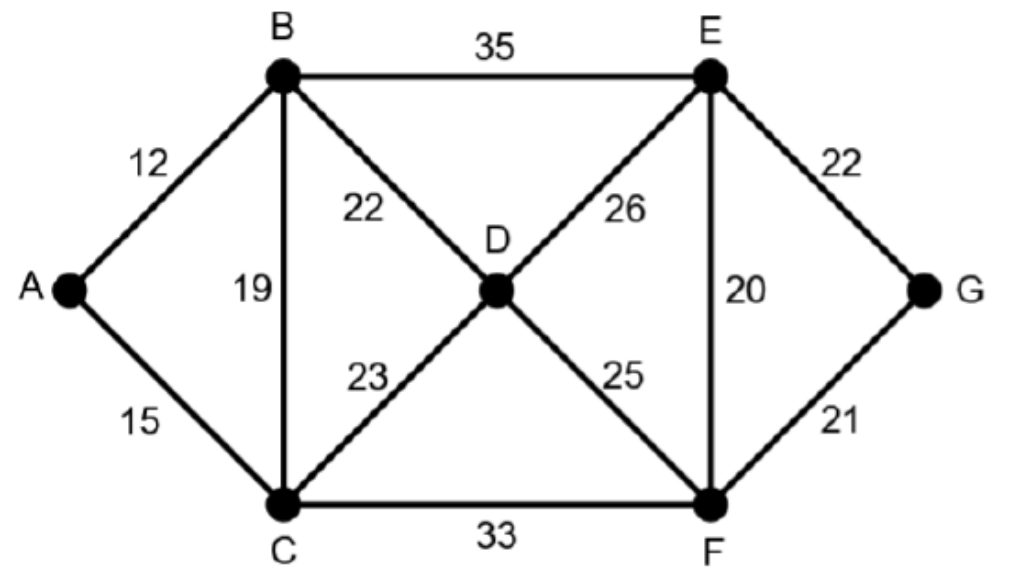
The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 4.7, C with weight 5.4, D with weight 4.3, E with weight 8.2, and F with weight 7.9; B connects to C with weight 3.0, D with weight 8.6, E with weight 6.1, and F with weight 6.3; C connects to D with weight 5.5, E with weight 7.5, and F with weight 8.7; D connects to E with weight 2.8, and F with weight 4.4; and E connects to F with weight 5.3.

Use Kruskal's Algorithm to determine the minimum spanning tree. What is the total cost of the network?

Answer available at the end of this section.

(Section 8.8) - Homework Exercise #5

A company is installing a network to connect seven office buildings. The cost (in hundreds of dollars) of connecting each pair of offices is given in the following graph.



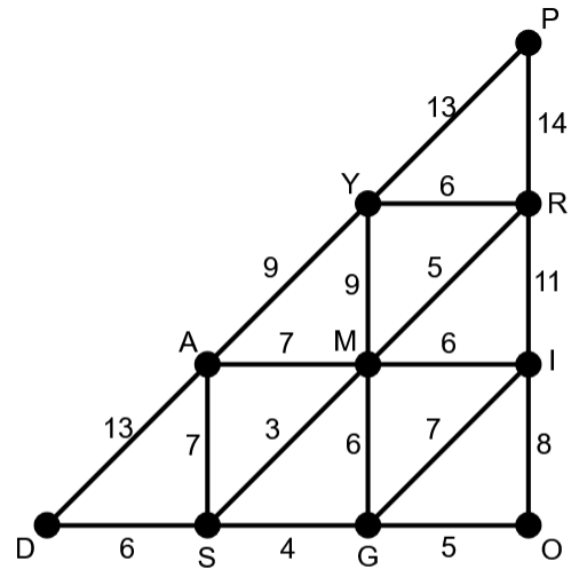
The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 12, and C with weight 15; B connects to C with weight 19, D with weight 22, and E with weight 35; C connects to D with weight 23, and F with weight 33; D connects to E with weight 26, and F with weight 25; E connects to F with weight 20, and G with weight 22; and F connects to G with weight 21.

Use Kruskal's Algorithm to determine the most cost-efficient network. What is the total cost in hundreds of dollars?

Answer available at the end of this section.

(Section 8.8) - Homework Exercise #6

A gaming center is installing Ethernet (Cat 6) cables to connect several gaming stations. The cost of running cable (in hundreds of dollars) between each pair of stations is given in the graph below.



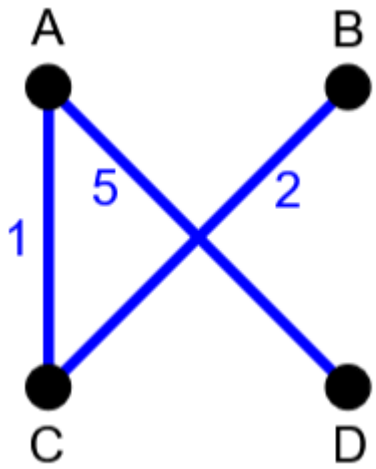
The image shows a weighted graph with labeled nodes connected by line segments. D connects to A with weight 13, and S with weight 6; A connects to M with weight 7, S with weight 7, and Y with weight 9; S connects to M with weight 3, and G with weight 4; M connects to Y with weight 9, R with weight 5, I with weight 6, and G with weight 6; G connects to I with weight 7, and O with weight 5; Y connects to P with weight 13, and R with weight 6; R connects to P with weight 14, and I with weight 11; and I connects to O with weight 8.

Determine which connections should be made so that all stations are connected at the lowest possible total cost.

Answer available at the end of this section.

Answers

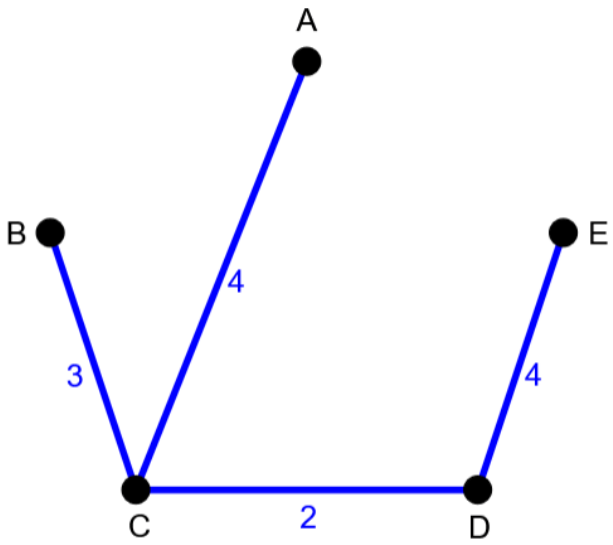
(Section 8.8) - Homework Exercise #1



Total cost = $8K.

The image shows a weighted graph with labeled nodes connected by line segments. A connects to C with weight 1, and D with weight 5; and B connects to C with weight 2.

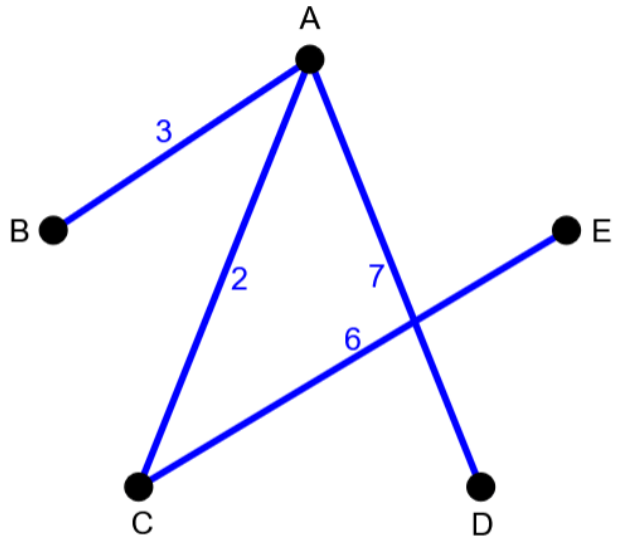
(Section 8.8) - Homework Exercise #2



Total cost = $13K.

The image shows a weighted graph with labeled nodes connected by line segments. B connects to C with weight 3; C connects to A with weight 4, and D with weight 2; and D connects to E with weight 4.

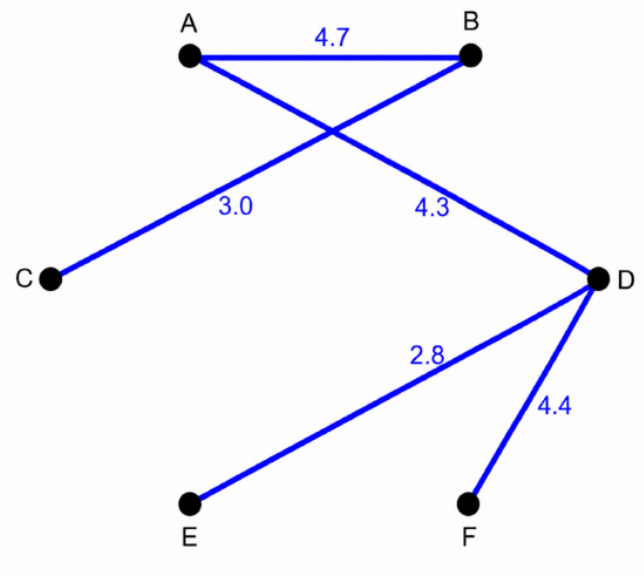
(Section 8.8) - Homework Exercise #3



Total cost = $18M.

The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 3, C with weight 2, and D with weight 7; and C connects to E with weight 6.

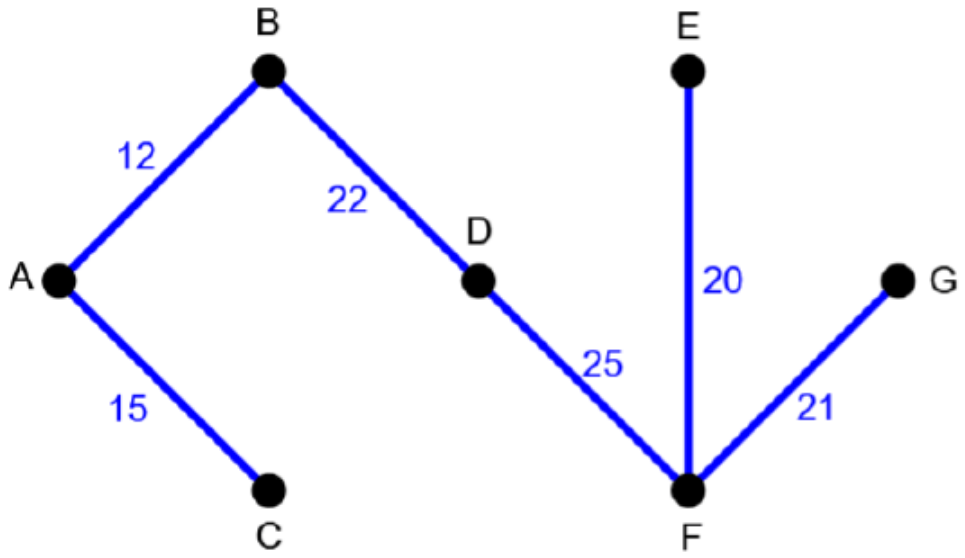
(Section 8.8) - Homework Exercise #4



Total cost = $19.2K.

The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 4.7, and D with weight 4.3; B connects to C with weight 3.0; D connects to E with weight 2.8, and F with weight 4.4.

(Section 8.8) - Homework Exercise #5

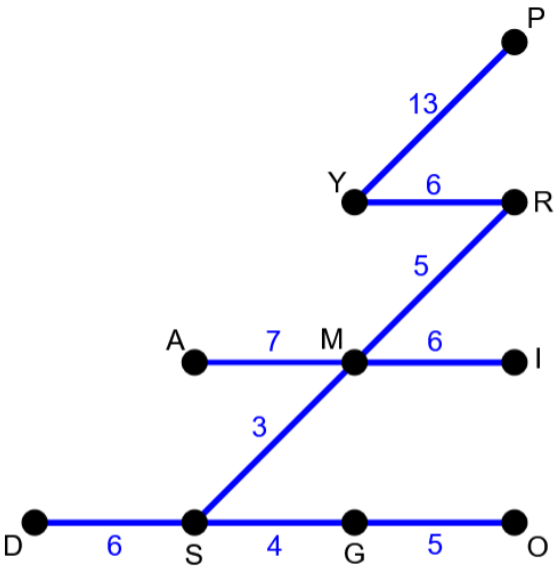


Total cost = $11,500 (115 x $100).

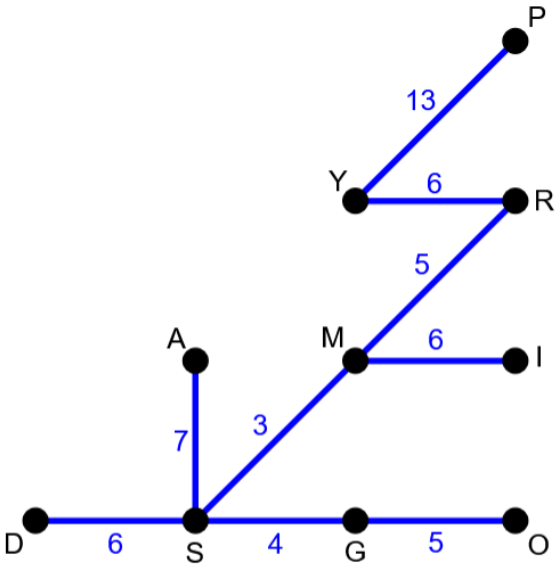
The image shows a weighted graph with labeled nodes connected by line segments. A connects to B with weight 12, and C with weight 15; B connects to D with weight 22; D connects to F with weight 25; F connects to E with weight 20, and G with weight 21.

(Section 8.8) - Homework Exercise #6

There are 2 possible spanning trees, both with a minimum cost of $5,500 (55 x $100).



The image shows a weighted graph with labeled nodes connected by line segments. D connects to S with weight 6; S connects to M with weight 3, and G with weight 4; M connects to M with weight 7, R with weight 5, and I with weight 6; G connects to O with weight 5; Y connects to P with weight 13, and R with weight 6.



The image shows a weighted graph with labeled nodes connected by line segments. D connects to S with weight 6; S connects to A with weight 7, M with weight 3, and G with weight 4; M connects to R with weight 5, and I with weight 6; G connects to O with weight 5; and Y connects to P with weight 13, and R with weight 6.

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S

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